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گزارش ←   4/1   3/18   روش اجزاء محدود 1

اسم + پروژه F.E.M تیک

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F.E.A :

Introduction

Any kind of field problems → D.E + B.C + I.C → B.V.P

B.V.P → Approximate Solution / Exact Solution

Approximate Solution ↓ F.E.A

F.E.A → 2 Variational method / weighted residual method

D.E صورت دیگری از → D.E شرط لازم داشتن → Variational method (حساب تغییرها)

functional → را حل می‌کند.

↳ function of a function

(optimization problems)

weighted residual method: خود D.E را حل می‌کند.

- 1 Galerkin method ✓ F.E.A Contex
- Least square method
- subdomain Collocation (تجمع زیرمجموعه‌ای)
- point Collocation (تجمع نقطه‌ای)

← در اغلب مسائل جامدات، نیازی به داشتن D.E وجود ندارد.

other Methods → Mechanical Eng (solid)

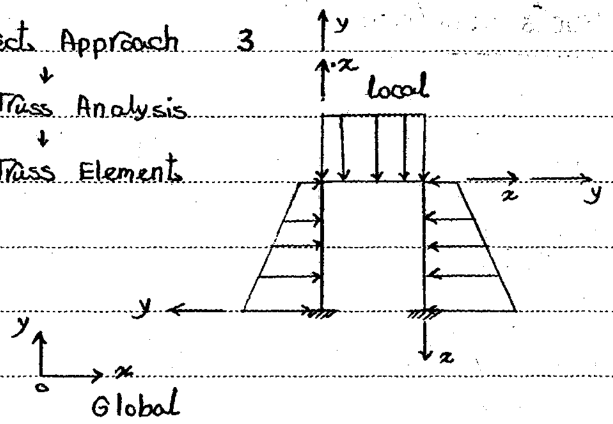
• → Direct Approach 3

↓

Truss Analysis

↓

Truss Elements



3 D.E.

+ Common B.C's

system of D.E

• → Min potential energy method 4 → برای طبق مسائلی که در حالت پایداری قرار دارند.
(برای مثال در حالت Buckling می‌توان از این روش استفاده کرد.)

• → Virtual work : Force & Displacement 5

Force	Displacement	
Real	Virtual	→ Displacement Method → اکثر مسائل
Virtual	Real	→ Force Method

← در المان محدود هیچ مساله‌ای خاصی نیست. چرا که با Matrix Analysis سروکار داریم.

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Displacement Method → توابعی بدست می‌دهد که بازتابی از → Shape Functions
میدان جابجایی هستند.

"Discretisation of The structure by the use of proper elements." → Number of Elements (N.E) + Number of Nodes (N.N)

"تکه تکه کردن سازه با المان های مناسب."

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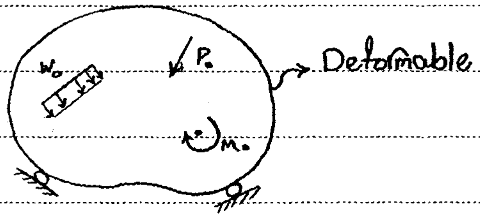
Minimum Potential Energy Method : "M.P.E.M" 1

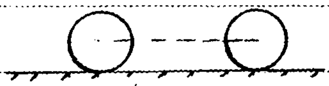


Π = total potential of the system

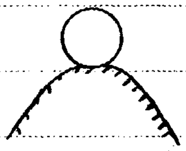
$\Pi = U + W_{Int}$

For Any kind of Equilibrium : $\delta\Pi = \delta U + \delta W_{Int} = 0$ 1

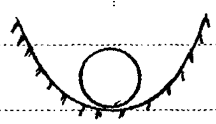
Π تغییرات دوم : $\delta^2\Pi = 0$: "Neutral"



$\delta^2\Pi < 0$: "Unstable"



$\delta^2\Pi > 0$: "stable" 2



$\delta W_{Int} + \delta W_{Ext} = \delta W_{acc}$ 3

$\longrightarrow \delta W_{Int} = \delta W_{acc} - \delta W_{Ext}$

برای المان عام

$$\xrightarrow{1} \sum_{e=1}^{N.E} \left(\underset{}{\delta\Pi^e} = \underset{\downarrow\,[k]^e}{\delta U^e} + \underset{\downarrow\,[m]^e}{\delta W_{acc}^e} - \underset{\downarrow\,[F]^e}{\delta W_{Ext}^e} = 0 \right)$$

در این روش از damping effects صرف نظر کرده ام.

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In static case : $\delta W^e_{acc} = 0$

$$\longrightarrow \boxed{\delta \Pi^e = \delta U^e - \delta W^e_{ext} = 0}$$

(element No "e") Ω^e , Γ^e

حجم ← ; مساحت سطح ←

$\underline{u}^e \longrightarrow \underline{\sigma}^e , \underline{\varepsilon}^e$

$\longrightarrow u_0 = \underline{\varepsilon}^T \cdot \underline{\sigma}$

$$U = \int_{\Omega^e} \underline{\varepsilon}^T \cdot \underline{\sigma} \cdot d\Omega^e \quad \text{strain Energy} \quad \left(= \sum_{e=1}^{N.E} u^e\right)$$

→ True strain

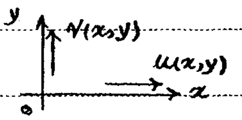
$$\underline{\sigma} = \underset{\substack{\downarrow \\ \text{Residual stress}}}{\underline{\sigma_0}} + \underset{\substack{\downarrow \\ \text{Matrix of Material properties}}}{\underline{D}} (\underline{\varepsilon} - \underline{\varepsilon_0})$$

$\varepsilon^T \cdot \sigma$

↳ strain due to Temperature changes

$$\delta(\underline{\sigma} \cdot \underline{\varepsilon}) = \underbrace{\underline{\sigma} \cdot \delta\underline{\varepsilon}}_{\text{displacement method}} + \underbrace{\delta\underline{\sigma} \cdot \underline{\varepsilon}}_{\text{Forced method}}$$

$\varepsilon \Rightarrow P$



$$\delta U = \int_{\Omega^e} \delta(\underline{\sigma} \cdot \underline{\varepsilon})\, d\Omega^e = \int_{\Omega^e} \underline{\sigma} \cdot \delta\underline{\varepsilon}\, d\Omega^e$$

$$\varepsilon_x = \frac{\partial u}{\partial x} , \quad \varepsilon_y = \frac{\partial v}{\partial y} , \quad \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad G = \frac{E}{2(1+\nu)}$$

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \underline{L} \cdot \underline{u} \qquad \boxed{\underline{\varepsilon} = \underline{L} \cdot \underline{u}}$$

$\downarrow$ $\underline{L}$: matrix of partial differential operators

$\downarrow$ $\underline{u}$: displacement vector

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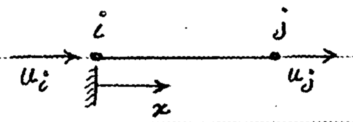

$0 \leq x \leq l$

مقدار گره‌ای را در دو جا (ابتدا و انتها) داریم.

در ساده‌ترین حالت: $u(x) = C_0 + C_1 x$

$$u_i = C_0$$

$$u_j = u_i + C_1 l$$

$$C_1 = \frac{1}{l}(u_j - u_i)$$

$$\longrightarrow u(x) = u_i + \frac{1}{l}(u_j - u_i)x = \left(1 - \frac{x}{l}\right) \cdot u_i + \frac{x}{l} \cdot u_j$$

$$= \begin{bmatrix} \underbrace{\left(1 - \frac{x}{l}\right)}_{N_i(x)} & \underbrace{\frac{x}{l}}_{N_j(x)} \end{bmatrix} \underbrace{\begin{bmatrix} u_i \\ u_j \end{bmatrix}}_{\underline{a}}$$

$\underline{N}$ توابع شکل shape functions (اغلب بی‌بعد هستند.)

$$u(x) = \underline{N} \cdot \underline{a}$$

$$\underline{u} = \underline{N} \cdot \underline{a}$$

مجموع توابع شکل بی‌بعد برابر یک می‌باشد.

توابع شکل بی‌بعد روی گرهٔ خودشان یک هستند و روی سایر گره‌ها صفر هستند.

$$\longrightarrow \underline{\varepsilon} = \underline{L} \cdot \underline{u} = \underline{L} \cdot \underline{N} \cdot \underline{a} = \underline{B} \cdot \underline{a} \qquad \underline{B} = \underline{L} \cdot \underline{N}$$

$$\underline{\sigma}^e = \underline{\sigma_0}^e + \underline{D}(\underline{B} \cdot \underline{a} - \underline{\varepsilon_0})$$

$$\delta \underline{\varepsilon} = \delta(\underline{B} \cdot \underline{a}) = \underline{B} \cdot \delta \underline{a} + \delta \underline{B} \cdot \underline{a}$$

$$\delta \underline{\varepsilon} = \underline{B} \cdot \delta \underline{a} = \delta \underline{a}^T \cdot \underline{B}^T$$

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$$\delta U^e = \int_{\Omega^e} \delta \underline{a}^T \cdot \underline{B}^T \cdot [\underline{\sigma}_0^e + \underline{D}(\underline{B} \cdot \underline{a} - \underline{\varepsilon}_0)] \, d\Omega^e$$

$$\delta W_{Ext} \begin{cases} \text{Body Forces} & \underline{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \\ \text{Surface Forces} & \underline{S} = \begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix} \end{cases}$$

$$\delta W_{Ext} = \delta W_{b.F} + \delta W_{S.F}$$

$$\delta W_{b.f} = \int_{\Omega^e} (b_x \cdot \delta u_x + b_y \cdot \delta u_y + b_z \cdot \delta u_z) \, d\Omega^e$$

$$b_x \cdot \delta u_x + b_y \cdot \delta u_y + b_z \cdot \delta u_z = \delta \underline{u}^T \cdot \underline{b} = \delta \underline{a}^T \cdot \underline{N}^T \cdot \underline{b}$$

$$u = \underline{N} \cdot \underline{a} \qquad \delta u = \underline{N} \cdot \delta \underline{a}$$

$$\delta W_{b.f} = \int_{\Omega^e} \delta \underline{a}^T \cdot \underline{N}^T \cdot \underline{b} \cdot d\Omega^e$$

$$\delta W_{s.f} = \int_{\Gamma^e} \delta \underline{a}^T \cdot \underline{N}^T \cdot \underline{S} \cdot d\Gamma^e$$

$$\delta \Pi^e = \delta U^e - \delta W^e_{B.f} - \delta W^e_{s.f} = 0 \rightarrow F.L$$

$$\int_{\Omega^e} \delta \underline{a}^T \cdot \underline{B}^T \cdot [\underline{\sigma}_0^e + \underline{D}(\underline{B} \cdot \underline{a} - \underline{\varepsilon}_0)] \, d\Omega^e - \int_{\Omega^e} \delta \underline{a}^T \cdot \underline{N}^T \cdot \underline{b} \, d\Omega^e$$

$$- \int_{\Gamma^e} \delta \underline{a}^T \cdot \underline{N}^T \cdot \underline{S} \, d\Gamma^e = 0$$

$$\underbrace{\cancel{\delta \underline{a}^T}}_{L} \cdot \Big\{ \underbrace{\int_{\Omega^e} B^T \cdot \sigma_0 \, d\Omega^e}_{-F^e_{\sigma_0}} + \underbrace{\Big(\int_{\Omega^e} \underline{B}^T \cdot \underline{D} \cdot \underline{B} \cdot d\Omega^e \Big)}_{K^{(e)}} \underbrace{\underline{a}^e}_{L} - \int_{\Omega^e} \underbrace{(\underline{B}^T \cdot \underline{D} \cdot \underline{\varepsilon}_0)}_{F^e_{\varepsilon_0}} \, d\Omega^e$$

stiffness Matrix $k(e)$ ← F/L

دیناسیون

$$- \int_{\Omega^e} \underbrace{\underline{N}^T \cdot \underline{b}}_{F^e_b} \cdot d\Omega^e - \int_{\Gamma} \underbrace{\underline{N}^T \cdot \underline{S}}_{F^e_S} \cdot d\Gamma^e \Big\} = 0$$

$$\sum_{e=1}^{N.E}\left(\underline{k}_{(e)} \cdot \underline{a}^{e} = \underline{F}_{\varepsilon_0}^{e} + \underline{F}_{\sigma_0}^{e} + \underline{F}_{b}^{e} + \underline{F}_{s}^{e}\right)$$

$$\underline{k}_{(e)} = \int_{\Omega^e} \underline{B}^T \cdot \underline{D} \cdot \underline{B} \, d\Omega^e$$

$$\underline{B} = \underline{L} \cdot \underline{N}$$

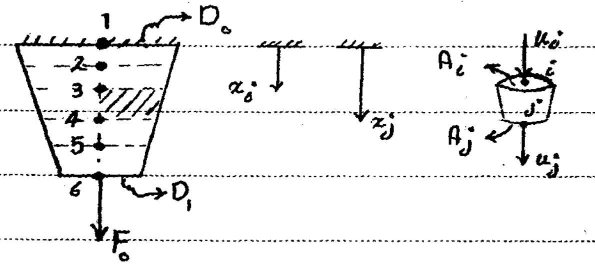
$$\underline{F}_{\varepsilon_0}^{e} = \int_{\Omega^e} \underline{B}^T \cdot \underline{D} \cdot \underline{\varepsilon}_0^{e} \, d\Omega^e$$

$$\underline{F}_{\sigma_0}^{e} = -\int_{\Omega^e} \underline{B}^T \cdot \underline{\sigma}_0^{e} \, d\Omega^e$$

$$\underline{F}_{b}^{e} = \int_{\Omega^e} \underline{N}^T \cdot \underline{b} \, d\Omega^e$$

$$\underline{F}_{s}^{e} = \int_{\Gamma^e} \underline{N}^T \cdot \underline{S} \, d\Gamma^e$$

Tapered Uniaxial stress structure :



$$x_{ji} = x_j - x_i$$

$$\underline{a} = \begin{bmatrix} u_i \\ u_j \end{bmatrix} \qquad \begin{matrix} u(x) = C_0 + C_1 x \\ u_j \quad\quad\quad x_j \\ u_i \quad\quad\quad x_i \end{matrix} = \begin{bmatrix} 1 & x \end{bmatrix}\begin{bmatrix} C_0 \\ C_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & x \end{bmatrix}\begin{bmatrix} 1 & x_i \\ 1 & x_j \end{bmatrix}^{-1}}_{N}\begin{bmatrix} u_i \\ u_j \end{bmatrix}$$

$$u_i = C_0 + C_1 \cdot x_i$$
$$u_j = C_0 + C_1 \cdot x_j$$

$$\begin{bmatrix} u_i \\ u_j \end{bmatrix} = \begin{bmatrix} 1 & x_i \\ 1 & x_j \end{bmatrix}\begin{bmatrix} C_0 \\ C_1 \end{bmatrix} \longrightarrow [\quad] = [\quad]^{-1}[\quad]$$

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$$u(x) = \begin{bmatrix} N_i(x) & N_j(x) \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix}$$

$$N_i(x) = \frac{1}{x_{ji}} (x_j - x) \qquad N_j(x) = \frac{1}{x_{ji}} (x - x_i)$$

$$\underline{B} = \underline{L} \cdot \underline{N} = \frac{d}{dx} \begin{bmatrix} N_i(x) & N_j(x) \end{bmatrix} = \frac{1}{x_{ji}} \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$\underline{k}(e) = \underbrace{\int_{x_i}^{x_j} \int_A \frac{1}{x_{ji}} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \cdot E \cdot \frac{1}{x_{ji}} \begin{bmatrix} -1 & +1 \end{bmatrix} dA}_{\bar{A}^e} \cdot dx \qquad \bar{A} = \frac{1}{2}(A_i + A_j)$$

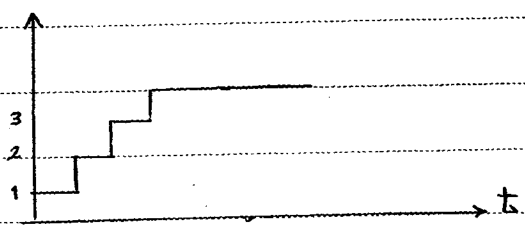
$$= \bar{A} \cdot E \cdot \int_{x_i}^{x_j} \frac{1}{x_{ji}^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx$$

$$\underline{k}(e) = \frac{\bar{A} \cdot E}{x_{ji}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\underline{F}^e_{\varepsilon_0} = \int_{x_i}^{x_j} \underbrace{\underline{B}^T}_{\frac{1}{x_{ji}} \begin{bmatrix} -1 \\ +1 \end{bmatrix}} \cdot \underbrace{\underline{D}}_{E} \cdot \underbrace{\varepsilon_0}_{\alpha_t \Delta T} \cdot \cancel{d\Omega} \rightarrow \bar{A} \cdot dx$$

$$\underline{F}^e_{\varepsilon_0} = \int_{x_i}^{x_j} \frac{1}{x_{ji}} \begin{bmatrix} -1 \\ +1 \end{bmatrix} \cdot E \cdot \alpha_t \cdot \Delta T \cdot \bar{A} \cdot dx$$

$$\underline{F}^e_{\varepsilon_0} = \bar{A} \cdot E \cdot \alpha_t \cdot \Delta T \cdot \begin{array}{c} \begin{bmatrix} -1 \\ +1 \end{bmatrix} \end{array} \begin{array}{l} \rightarrow i \\ \rightarrow j \end{array}$$



Incremental Loady → نیرو حرارتی اضافه می‌شود، سیستم به تعادل می‌رسد سپس نیرو حرارتی دیگر اضافه می‌شود و ...

کل سیستم Nonlinear را می‌توان به صورت segmental و Linear در نظر گرفت.

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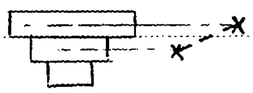
Non linearities:

1) Material

2) Geometrically
- total Lagrangian approach — دائم به زمان صفر به عنوان مرجع نگاه می‌کنیم.
- updated Lagrangian approach — به یک step قبل به عنوان مرجع برای حال نگاه می‌کنیم.

کتاب "Bathe"

تنش در هر المان ثابت است. پس در محل تماس دو المان پیوستگی در تنش نداریم. برای حل این مسأله، تنش را به مرکز هر المان نسبت می‌دهیم، و آن را interpolate می‌کنیم.

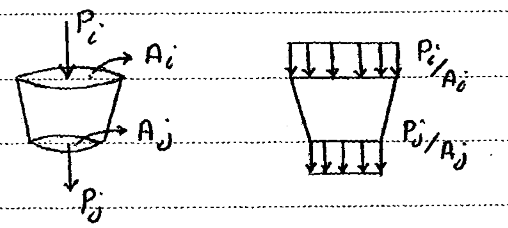


$$\underline{F}^e_{\sigma_0} = -\int_{x_i}^{x_j} \frac{1}{x_{ji}} \begin{bmatrix}-1\\+1\end{bmatrix} \sigma_0^e \cdot \bar{A}\, dx \qquad \underline{F}^e_{\sigma_0} = \bar{A}\cdot\sigma_0^e \cdot \begin{bmatrix}1\\-1\end{bmatrix}$$

$$\underline{b} = b_x = \rho g \qquad \underline{F}_b^e = \int_{x_i}^{x_j} \frac{1}{x_{ji}}\begin{bmatrix}x_j - x\\ x - x_i\end{bmatrix}\rho g\, \bar{A}\, dx$$

$$\underline{F}_b^e = \frac{1}{2} \cdot x_{ji}\cdot\bar{A}\cdot\rho g\cdot\begin{bmatrix}1\\1\end{bmatrix}$$

نصف جرم روی گره i، و نصف دیگر روی گره j.



$$F_s = \frac{1}{x_{ji}}\begin{bmatrix}x_j - x_i\\ x_i - x_i\end{bmatrix}\frac{P_i}{A_i}A_i + \frac{1}{x_{ji}}\begin{bmatrix}x_j - x_j\\ x_j - x_i\end{bmatrix}\frac{P_j}{A_j}A_j$$

$$F_s = \begin{bmatrix}1\\0\end{bmatrix}P_i + \begin{bmatrix}0\\1\end{bmatrix}P_j = \begin{bmatrix}P_i\\P_j\end{bmatrix}$$

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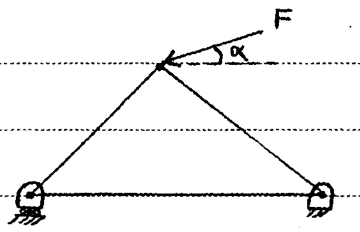
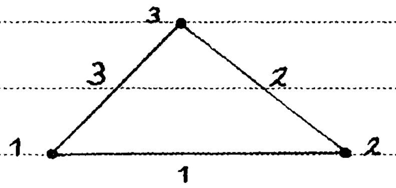
Direct Approach 2: Truss Analysis

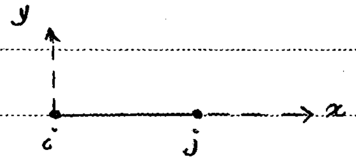
- Truss Element

or - bar Element / Rod Element

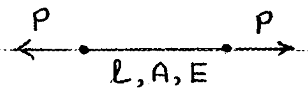
or - 2 Force member Element





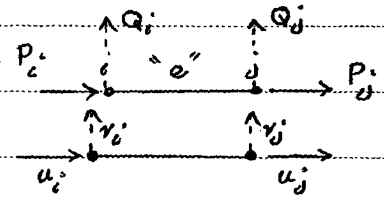


Each node has 1 D.O.F in local coordinate.



$$\Delta = \frac{Pl}{AE} \qquad P = \frac{A \cdot E}{l} \cdot \Delta$$

e	i	j
1	1	2
2	2	3
3	1	3



$$P_i = \frac{AE}{l}(u_i - u_j) \qquad P_j = \frac{AE}{l}(u_j - u_i)$$

$$\begin{bmatrix} P_i \\ P_j \end{bmatrix} = \frac{A \cdot E}{l} \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{k_{ij}|_{local}} \begin{bmatrix} u_i \\ u_j \end{bmatrix}$$

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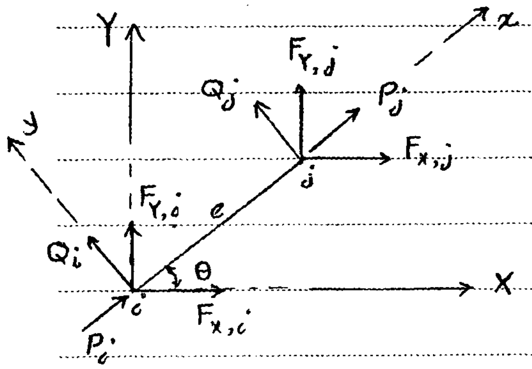
$$\begin{bmatrix} P_i \\ Q_i \\ P_j \\ Q_j \end{bmatrix} = \frac{A.E}{l} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{bmatrix}$$

$$\underline{f} = \underline{S} \cdot \underline{d}$$

برای حالت سه بعدی، ماتریس S، 6x6 می‌شود. (4 ستون صفر و 4 ردیف صفر)

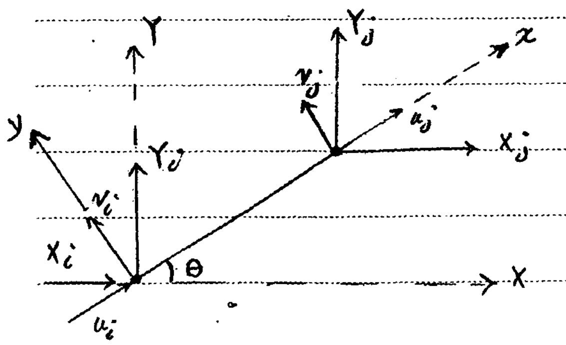


$$P_i = F_{X,i}\,\overset{\alpha}{\cos}(\theta) + F_{Y,i}\,\overset{\beta}{\sin}(\theta)$$

$$Q_i = -F_{X,i}\,\sin(\theta) + F_{Y,i}\,\cos(\theta)$$

$$\begin{bmatrix} P_i \\ Q_i \\ P_j \\ Q_j \end{bmatrix} = \left[\begin{array}{cc|cc} \alpha & \beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ \hline 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{array}\right] \begin{bmatrix} F_{X,i} \\ F_{Y,i} \\ F_{X,j} \\ F_{Y,j} \end{bmatrix}$$

$$\underline{f} = \underline{R} \cdot \underline{F}$$



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$u_i = X_i \cos(\theta) + Y_i \sin(\theta)$

$v_i = -X_i \sin(\theta) + Y_i \cos(\theta)$

$$\begin{bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{bmatrix} = \begin{bmatrix} \alpha & \beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{bmatrix} \begin{bmatrix} X_i \\ Y_i \\ X_j \\ Y_j \end{bmatrix} \qquad \underline{d} = \underline{R} \cdot \underline{a}$$

$\underline{f} = \underline{S} \cdot \underline{d} \qquad R^{-1} (\underline{R} \cdot \underline{F} = \underline{S} \cdot \underline{R} \cdot \underline{a})$

$\underline{F} = (\underline{R}^{-1} \cdot \underline{S} \cdot \underline{R}) \cdot \underline{a}$

R : Rotational Matrix $\qquad R^{-1} = R^{T}$

$$\underbrace{\underline{F}}_{\text{Nodal Force vector / Global Coord}} = \underbrace{(\underline{R}^{T} \cdot \underline{S} \cdot \underline{R})}_{\underline{k}_{ij} :\ \text{stiffness matrix / G.C.}} \cdot \underbrace{\underline{a}}_{\text{Nodal displacement vector / G.C.}}$$

$\underline{K}_{ij}\,(e) = \underline{R}^{T} \cdot \underline{S} \cdot \underline{R}$

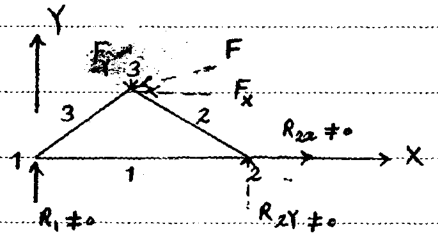
$$\underline{K}_{ij}\,(e) = \begin{bmatrix} \alpha & -\beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & -\beta \\ 0 & 0 & \beta & \alpha \end{bmatrix} \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha & \beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{bmatrix}$$

$$\begin{array}{cc} & \quad\overbrace{\qquad\qquad}^{i}\quad\overbrace{\qquad\qquad}^{j} \\ K_{ij}(e) = \dfrac{AE}{l} & \begin{bmatrix} \alpha^2 & \alpha.\beta & -\alpha^2 & -\alpha.\beta \\ \alpha.\beta & \beta^2 & -\alpha.\beta & -\beta^2 \\ -\alpha^2 & -\alpha.\beta & \alpha^2 & \alpha.\beta \\ -\alpha.\beta & -\beta^2 & \alpha.\beta & \beta^2 \end{bmatrix} \begin{array}{l} \}\, i \\ \\ \}\, j \end{array} \end{array}$$

i ——— j

r : root

$$K_{ij}(e) = \begin{bmatrix} r_{ii} & r_{ij} \\ r_{ji} & r_{jj} \end{bmatrix}$$



1	X_1, $Y_1 = 0$
2	$X_2 = 0$, $Y_2 = 0$
3	X_3, Y_3

total degree of freedoms = No. of nodes x D.O.F/Node = 3 x 2 = 6

$$\begin{bmatrix} \cancel{R_{1X}}\ 0 \\ R_{1Y}\,? \\ R_{2X}\,? \\ R_{2Y}\,? \\ F_X \\ F_Y \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \underbrace{\left[\begin{array}{c|c|c} r_{11}^{①} + r_{11}^{③} & r_{12}^{①} & r_{13}^{③} \\ \hline r_{21}^{①} & r_{22}^{①} + r_{22}^{②} & r_{23}^{②} \\ \hline r_{31}^{③} & r_{32}^{②} & r_{33}^{③} + r_{33}^{②} \end{array}\right]}_{K_T} \end{array} \begin{bmatrix} X_1\,? \\ \cancel{Y_1}\ 0 \\ \cancel{X_2}\ 0 \\ \cancel{Y_2}\ 0 \\ X_3\,? \\ Y_3\,? \end{bmatrix}$$

* هیچ دو مجهولی روی یک ردیف نباید باشند.

تمرین: stiffness matrix را برای المان خرپای سه بعدی بدست آورید.

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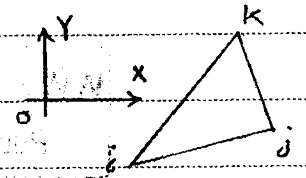
2 Dimensional Elements :

Triangular Elements

Quadrilateral Elements

Triangular Elements :



T_i, T_j, T_k : Nodal Temprature Values

$T(x,y) = C_0 + C_1 . x + C_2 . y$

$T_i = C_0 + C_1 . x_i + C_2 . y_i$

$T_j = C_0 + C_1 . x_j + C_2 . y_j$

$T_k = C_0 + C_1 . x_k + C_2 . y_k$

$$T(x,y) = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix}$$

$$\begin{bmatrix} T_i \\ T_j \\ T_k \end{bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix}$$

$$T(x,y) = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}^{-1} \begin{bmatrix} T_i \\ T_j \\ T_k \end{bmatrix}$$

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$\rightarrow$ tempreture

$$T_{(x,y)} = \begin{bmatrix} N_i(x,y) & N_j(x,y) & N_k(x,y) \end{bmatrix} \underbrace{\begin{bmatrix} T_i \\ T_j \\ T_k \end{bmatrix}}_{\underline{a}}$$

$\downarrow$ Temperature

$$T(x,y) = \underline{N} \cdot \underline{a}$$

$$T(x,y) = N_i T_i + N_j T_j + N_k T_k \qquad T(x,y) = \sum_{i=1}^{3} N_i \cdot T_i$$

$$N_i(x,y) = \frac{1}{2A} \left[a_i + b_i x + c_i y \right]$$

$$2A = \mathrm{DET} \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}$$

$$a_i = x_j y_k - x_k y_j$$

$$b_i = y_j - y_k$$

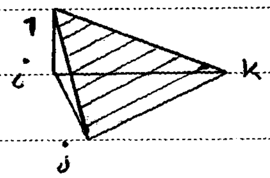
$$c_i = x_k - x_j$$

$k \rightarrow i \rightarrow j \rightarrow k$ Permutation

S_i : $a_j = x_k y_i - x_i y_k$

$b_j = y_k - y_i$

$c_j = x_i - x_k$



$$N_i + N_j + N_k = \frac{1}{2A} \left[a_i + a_j + a_k \right] = 1$$

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"M.P.E.M" → 2 Dimensional Solid Cases:

_plane stress

_plane strain

_plane stress:

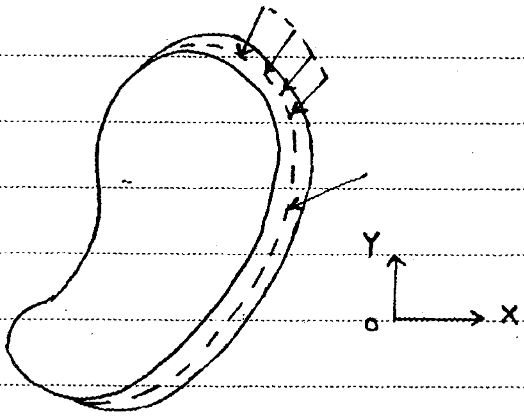
$$\underline{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \varepsilon_z \neq 0$$

$$3D \begin{cases} 2D & XOY \\ & \text{موقعیت } XOY \text{ در ربط } t \text{ می‌باشد.} \\ 1D & z \to t \end{cases}$$

بار در صفحه XOY قرار گرفته است.

هیچ گونه بار غیرصفحه‌ای نداریم.

بار می‌تواند متمرکز یا گسترده باشد.



$$\varepsilon_z = \frac{1}{E}\left[\underbrace{\cancel{\sigma_z}}_{\text{صفر}} - \nu(\sigma_x + \sigma_y)\right] \qquad \varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

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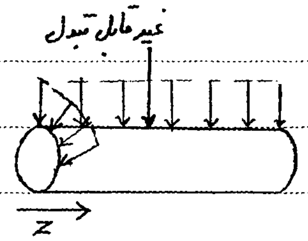
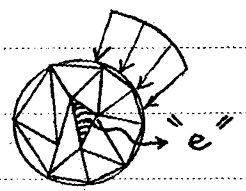
plane strain :

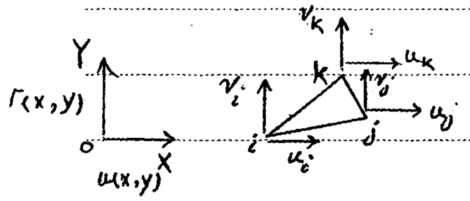
$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \varepsilon_z = 0$$

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix} \qquad \sigma_z \neq 0 \qquad \sigma_z = \nu(\sigma_x + \sigma_y)$$



در صفحهٔ YOZ می‌توان بار متمرکز داشت.
XOZ





$$i \left| \begin{matrix} x_i \\ y_i \end{matrix} \right. \qquad j \left| \begin{matrix} x_j \\ y_j \end{matrix} \right. \qquad k \left| \begin{matrix} x_k \\ y_k \end{matrix} \right.$$

$\underline{a}^e$, Nodal displacement vector

$$\underline{a}^e = \begin{bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{bmatrix}$$

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$$\underline{k}(e) \cdot \underline{a}^e = \underline{F}^e_{\sigma_0} + \underline{F}^e_{\epsilon_0} + \underline{F}^e_b + \underline{F}^e_s$$

$$\underline{k}(e) = \int_{\Omega^e} \underline{B}^T \cdot \underline{D} \cdot \underline{B} \cdot d\Omega^e$$

$$\underline{B} = \underline{L} \cdot \underline{N} \qquad u(x,y) = c_0 + c_1 x + c_2 y = \begin{bmatrix} N_i & N_j & N_k \end{bmatrix} \begin{bmatrix} u_i \\ u_j \\ u_k \end{bmatrix}$$

$$v(x,y) = c'_0 + c'_1 x + c'_2 y = \begin{bmatrix} N_i & N_j & N_k \end{bmatrix} \begin{bmatrix} v_i \\ v_j \\ v_k \end{bmatrix}$$

$$N_i(x,y) = \frac{1}{2A^e}\left[a_i + b_i x + c_i y\right]$$

$$2A^e = \mathrm{DET} \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}$$

$$a_i = (x_j \cdot y_k - x_k \cdot y_j)$$
$$b_i = (y_j - y_k)$$
$$c_i = (x_k - x_j)$$

$$\underline{u} = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_k & 0 \\ 0 & N_i & 0 & N_j & 0 & N_k \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{bmatrix} = \underline{N} \cdot \underline{a}$$

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \qquad \epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

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$$\underline{\varepsilon} = (\underset{3\times 2}{\underline{L}} \cdot \underset{2\times 6}{\underline{N}}) \cdot \underline{a} = \underset{3\times 6}{\underline{B}} \cdot \underline{a}$$

$$\underline{B} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_i & 0 & N_j & 0 & N_k & 0 \\ 0 & N_i & 0 & N_j & 0 & N_k \end{bmatrix}$$

$$\underline{B} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & \frac{\partial N_j}{\partial x} & 0 & \frac{\partial N_k}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 & \frac{\partial N_j}{\partial y} & 0 & \frac{\partial N_k}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial y} & \frac{\partial N_k}{\partial x} \end{bmatrix}$$

$$\underline{B} = \frac{1}{2A^e} \begin{bmatrix} b_i & 0 & b_j & 0 & b_k & 0 \\ 0 & c_i & 0 & c_j & 0 & c_k \\ c_i & b_i & c_j & b_j & c_k & b_k \end{bmatrix} \longrightarrow \text{ثابت}$$

$$\underline{k}(e) = \int_{\Omega^e} \overset{\text{cte Matrix}}{\underline{B}^T} \cdot \underline{D} \cdot \underbrace{\underline{B}}_{\text{cte Matrix}} \, d\Omega^e$$

plane stress:

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \overset{\text{صفر}}{\sigma_z})\right] \qquad \sigma_x - \nu\sigma_y = E \cdot \varepsilon_x$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \overset{\text{صفر}}{\sigma_z})\right] \qquad \sigma_y - \nu\sigma_x = E \cdot \varepsilon_y$$

$$\rightarrow (1-\nu^2)\,\sigma_x = E\varepsilon_x + \nu E\varepsilon_y \qquad \tau_{xy} = G \cdot \varepsilon_{xy} \qquad G = \frac{E}{2(1+\nu)}$$

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$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{bmatrix}$$

$\downarrow$

$D|_{\text{plane stress}}$

plane strain:

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \cancel{\sigma_z})\right] \qquad \nu(\sigma_x+\sigma_y) \qquad E.\varepsilon_x = \sigma_x(1-\nu^2) - \nu(1+\nu)\sigma_y$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] \qquad E.\varepsilon_y = \sigma_y(1-\nu^2) - \nu(1+\nu)\sigma_x$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] = 0 \qquad \sigma_z = \nu(\sigma_x + \sigma_y)$$

$$(1-\nu)E\varepsilon_x + \nu E\varepsilon_y = (1+\nu)\left[(1-\nu)^2 - \nu^2\right]\sigma_x \qquad \sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\varepsilon_y\right]$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{bmatrix} \qquad \tau_{xy} = G.\varepsilon_{xy} \qquad G = \frac{E}{2(1+\nu)}$$

$\downarrow$

$D|_{\text{plane strain}}$

$$\underline{K}(e) = \underline{B}^T . \underline{D} . \underline{B} * t * A^e$$

$$\underline{\varepsilon} = \underline{B}.\underline{a} \longrightarrow \underline{\sigma}^e = cte$$

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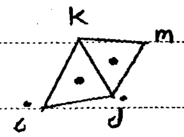
پیوستگی در کرنش وجود دارد، ولی در تنش خیر. به همین دلیل تنش برای هر المان را به مرکز همان المان نسبت می‌دهیم. جایی که shape function ها هر کدام $1/3$ هستند.



تنش میانه نیز مربوط به مرکز سطح است ولی برای یک time step قبل!

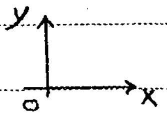
$$\underline{F}^e_{\sigma_0} = -\int_{\Omega^e} \underset{6\times3}{\underline{B}^T} \cdot \underset{3\times1}{\underline{\sigma}_0} * d\Omega^e \qquad \underline{\sigma}_0 = \begin{bmatrix} \sigma_{0x} \\ \sigma_{0y} \\ \tau_{0xy} \end{bmatrix}$$

$$\longrightarrow \underline{F}^e_{\sigma_0} = -\underline{B}^T \cdot \underline{\sigma}_0 * t * A^e$$

$$\underline{F}^e_{\varepsilon_0} = \int_{\Omega^e} \underset{6\times3}{\underline{B}^T} \cdot \underset{3\times3}{\underline{D}} \cdot \underset{3\times1}{\underline{\varepsilon}_0} * d\Omega^e \qquad \underline{\varepsilon}_0 = \begin{bmatrix} \varepsilon_{x_0} \\ \varepsilon_{y_0} \\ \varepsilon_{xy_0} \end{bmatrix} = \begin{bmatrix} \alpha_t \cdot \Delta T \\ \alpha_t \cdot \Delta T \\ 0 \end{bmatrix}$$

$$\longrightarrow \underline{F}^e_{\varepsilon_0} = \underline{B}^T \cdot \underline{D} \cdot \underline{\varepsilon}_0 * (t * A^e)$$

$$\underline{F}^e_b = \int_{\Omega^e} \underline{N}^T \cdot \underline{b} * \overset{t*dA}{\cancel{d\Omega^e}}$$



$$\underline{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

$$\underline{N}^T \downarrow \begin{bmatrix} N_i & 0 \\ 0 & N_i \\ N_j & 0 \\ 0 & N_j \\ N_k & 0 \\ 0 & N_k \end{bmatrix}$$

$$\longrightarrow \underline{F}^e_b = t \cdot A^e \cdot \begin{bmatrix} b_x/3 \\ b_y/3 \\ b_x/3 \\ b_y/3 \\ b_x/3 \\ b_y/3 \end{bmatrix}$$

uniform distribution
↓
body force $1/3$ برای هر گره

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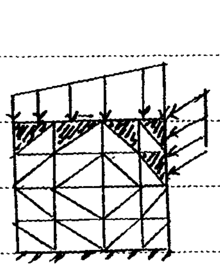
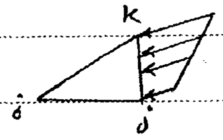
نیروی سطحی را فقط برای المان هایی داریم که در مرز قرار دارند و
آن هم در صورتی که روی external boundary ، نیرو وجود داشته باشد.





اگر بار متغیر باشد،
میانگین می‌گیریم.

$$\underline{F}_s^e = \int_{\Gamma^e} \underline{N}^T \cdot \underline{S} * \underbrace{d\Gamma^e}_{t*dl} = \int_0^{l_{jk}} \underline{N}^T \cdot \underline{S} * t * dl, \qquad \underline{S} = \begin{bmatrix} s_x \\ s_y \end{bmatrix}$$

روی خط jk : $\overset{\text{صفر}}{N_i} + N_j + N_k = 1$

$$F_s^e = t \cdot l_{jk} \cdot \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} s_x \\ \frac{1}{2} s_y \\ \frac{1}{2} s_x \\ \frac{1}{2} s_y \end{bmatrix}$$

نیروی سطحی در مرکز ضلع jk بسط داده می‌شود.
جایی که $N_j = N_k = \frac{1}{2}$

$$\longrightarrow (\cancel{t} * A^e * \underline{B}^T \cdot \underline{D} \cdot \underline{B})\, \underline{a}^e = -\cancel{t} * A^e * \underline{B}^T \underline{\sigma}_0^e + \cancel{t} * A^e * \underline{B}^T \cdot \underline{D} \cdot \varepsilon_0$$

$$+ \cancel{t} * \frac{A^e}{3} \begin{bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{bmatrix} + \cancel{t} * \frac{l_{jk}}{2} \begin{bmatrix} 0 \\ 0 \\ s_x \\ s_y \\ s_x \\ s_y \end{bmatrix}$$

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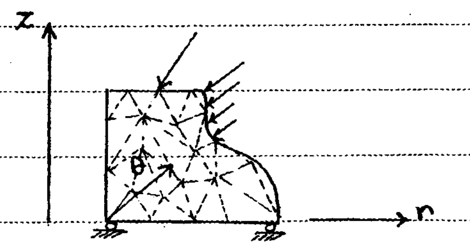
Axi _ symmetric :



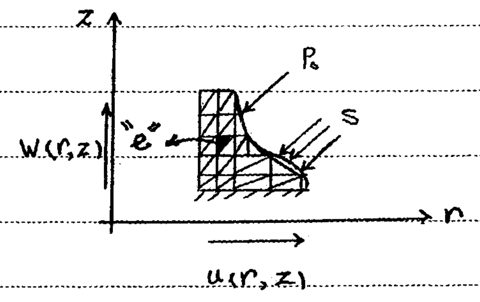
Symmetry :
- shape
- External loading
- B.C.
- Material Property

$\longrightarrow$ برای همهٔ θ ها یکسان باشد.

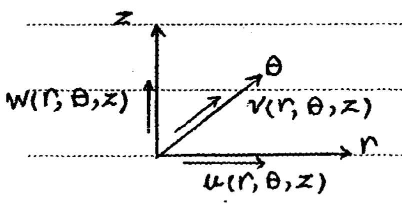
ring

در این صورت یک سطح مقطع از جسم را مورد تحلیل قرار می‌دهیم.

polar coordinate



$$\underline{k}(e) \cdot \underline{\alpha}^e = \underline{F}^e_{\varepsilon_0} + \underline{F}^e_{\sigma_0} + \underline{F}^e_{b} + \underline{F}^e_{s}$$



$\underline{\varepsilon} \longrightarrow f(u, v, w)$

$\checkmark \varepsilon_r$, $\checkmark \varepsilon_\theta$, $\checkmark \varepsilon_z$

$\checkmark \varepsilon_{rz}$, $\varepsilon_{\theta z}$, $\varepsilon_{r\theta}$

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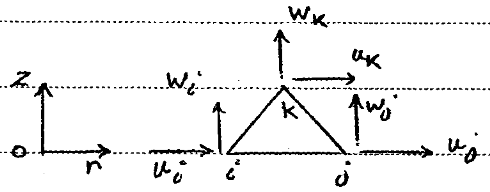
$$\varepsilon_r = \frac{\partial u}{\partial r} \qquad \varepsilon_z = \frac{\partial w}{\partial z} \qquad \varepsilon_\theta = \frac{u}{r} \qquad \varepsilon_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$



$$\underline{a}^e = \begin{bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_k \\ w_k \end{bmatrix} \qquad u(r,z) = \begin{bmatrix} N_i(r,z) & N_j(r,z) & N_k(r,z) \end{bmatrix} \begin{bmatrix} u_i \\ u_j \\ u_k \end{bmatrix}$$

$$W(r,z) = \begin{bmatrix} N_i(r,z) & N_j(r,z) & N_k(r,z) \end{bmatrix} \begin{bmatrix} W_i \\ W_j \\ W_k \end{bmatrix}$$

$$\underset{2\times1}{\underline{u}} = \begin{bmatrix} u(r,z) \\ \\ W(r,z) \end{bmatrix} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_k & 0 \\ \\ 0 & N_i & 0 & N_j & 0 & N_k \end{bmatrix} \begin{bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_k \\ w_k \end{bmatrix} = \underline{N} \cdot \underline{a}^e$$

$$\underline{K}_{(e)} = \int_{\Omega^e} \underline{B}^T \cdot \underline{D} \cdot \underline{B} \cdot d\Omega^e \qquad \underline{B} = \underline{L} \cdot \underline{N}$$

$$\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \varepsilon_{rz} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ \frac{1}{r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix}}_{\underline{L}} \begin{bmatrix} u(r,z) \\ w(r,z) \end{bmatrix} \qquad \underline{B} = \underset{4\times2}{\underline{L}} \cdot \underset{2\times6}{\underline{N}}$$

$$N_i(r,z) = \frac{1}{2A^e}(a_i + b_i r + c_i z)$$

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$$\frac{1}{2A^e} b_i$$

$$\underline{B} = \begin{bmatrix} \frac{\partial N_i}{\partial r} & 0 & \frac{\partial N_j}{\partial r} & 0 & \frac{\partial N_k}{\partial r} & 0 \\ \frac{N_i}{r}\bullet & 0 & \frac{N_j}{r}\bullet & 0 & \frac{N_k}{r}\bullet & 0 \\ 0 & \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_j}{\partial z} & 0 & \frac{\partial N_k}{\partial z} \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial r} & \frac{\partial N_j}{\partial z} & \frac{\partial N_j}{\partial r} & \frac{\partial N_k}{\partial z} & \frac{\partial N_k}{\partial r} \end{bmatrix}$$

$\bullet \leftarrow$ ترم های غیر ثابت

$$\frac{c_i}{2A^e}$$

به جای ماتریس $\underline{B}$ که متغیر است، ترم های غیر ثابت را به مرکز سطح نسبت می دهیم، ماتریس ثابت حاصل را با $\underline{\bar{B}}$ نشان می دهیم.

$$\frac{N_i}{r} = \frac{1}{2A}\left(\frac{a_i}{r} + b_i + c_i\frac{z}{r}\right)$$

$$\longrightarrow \frac{N_i}{\bar{r}} = \frac{N_j}{\bar{r}} = \frac{N_k}{\bar{r}} = \frac{1}{3\bar{r}} = \frac{1}{r_i + r_j + r_k} \qquad \bar{r} = \frac{1}{3}(r_i + r_j + r_k)$$

$$\underline{\sigma} = \begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{bmatrix} = \underset{4\times4}{\underline{D}} \cdot \begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \varepsilon_{rz} \end{bmatrix}$$

$$\underline{D} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}$$

$$\underline{\bar{k}}_{(e)} = \int_{\Omega^e \to A^e} \underset{6\times4}{\underline{\bar{B}}^T} \cdot \underset{4\times4}{\underline{D}} \cdot \underset{4\times6}{\underline{\bar{B}}} \cdot \underbrace{d\Omega^e}_{2\pi\cdot\bar{r}\cdot dr\cdot dz}$$

$$\underline{\bar{k}}_{(e)} = 2\pi\cdot\bar{r}\cdot A^e\cdot(\underline{\bar{B}}^T\cdot\underline{D}\cdot\underline{\bar{B}})$$

$$\underline{F}^e_{\varepsilon_0} = \int_{\Omega^e \to A^e} \underline{\bar{B}}^T\cdot\underline{D}\cdot\underline{\varepsilon}_0\cdot \overbrace{d\Omega^e}^{2\pi\cdot\bar{r}\cdot dr\cdot dz} \longrightarrow \underline{\bar{F}}^e_{\varepsilon_0} = 2\pi\cdot\bar{r}\cdot A^e(\underline{\bar{B}}^T\cdot\underline{D}\cdot\underline{\varepsilon}_0)$$

$$\begin{bmatrix} \varepsilon_{r_0} = \alpha_t\cdot\Delta T \\ \varepsilon_{\theta_0} = \alpha_t\cdot\Delta t \\ \varepsilon_{z_0} = \alpha_t\cdot\Delta t \\ 0 \end{bmatrix}$$

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$$\underline{\bar{F}}_{\sigma_0} = -\int \underline{\bar{B}}^T \cdot \underline{\sigma_0} \cdot d\Omega^e \qquad d\Omega^e \rightarrow 2\pi \cdot \bar{r} \cdot dr \cdot dz$$

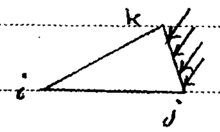
$$\underline{\bar{F}}^e_{\sigma_0} = -(2\pi \bar{r} \cdot A^e)(\underline{\bar{B}}^T \cdot \underline{\sigma_0})$$

$$\underline{\sigma_0} = \begin{bmatrix} \sigma_{r_0} \\ \sigma_{\theta_0} \\ \sigma_{z_0} \\ \sigma_{rz_0} \end{bmatrix}$$

constant vector

$$\underline{\bar{F}}^e_b = \int_{\Omega^e} \underline{N}^T \cdot \underline{b} \cdot d\Omega^e \qquad d\Omega^e \rightarrow 2\pi \cdot \bar{r} \cdot dr \cdot dz \qquad \underline{b} = \begin{bmatrix} b_r \\ b_z \end{bmatrix}$$

$$\begin{bmatrix} \cancel{N_i} \to \frac{1}{3} & 0 \\ 0 & N_i \to \frac{1}{3} \\ \cancel{N_j} \to \frac{1}{3} & 0 \\ 0 & N_j \\ \cancel{N_k} \to \frac{1}{3} & 0 \\ 0 & N_k \end{bmatrix} \qquad \underline{\bar{F}}^e_b = (2\pi \cdot \bar{r} \cdot A^e) \begin{bmatrix} b_r/3 \\ b_z/3 \\ b_r/3 \\ b_z/3 \\ b_r/3 \\ b_z/3 \end{bmatrix}$$



$$\underline{S} = \begin{bmatrix} \bar{S}_r \\ \bar{S}_z \end{bmatrix}$$

$$\underline{F}^e_s = \int_{\Gamma^e} \underline{N}^T \cdot \underline{S} \cdot d\Gamma^e \qquad \Gamma^e \to l_{jk}$$

روی مرکز طول

$$\begin{bmatrix} \cancel{N_i} \to 0 & 0 \\ 0 & N_i \\ \cancel{N_j} \to \frac{1}{2} & 0 \\ 0 & N_j \to \frac{1}{2} \\ \cancel{N_k} \to \frac{1}{2} & 0 \\ 0 & N_k \end{bmatrix} \begin{bmatrix} S_r \\ S_z \end{bmatrix} 2\pi \bar{r} \cdot dl \qquad 2\bar{r} = (r_j + r_k)$$

$$\underline{F}^e_s = 2\pi \cdot \bar{r}\, l_{ik} \begin{bmatrix} 0 \\ 0 \\ S_r/2 \\ S_z/2 \\ S_r/2 \\ S_z/2 \end{bmatrix}$$

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تمرین: استخراج معادلات دیفرانسیل حالت های:

1) تنش صفحه‌ای — plane stress

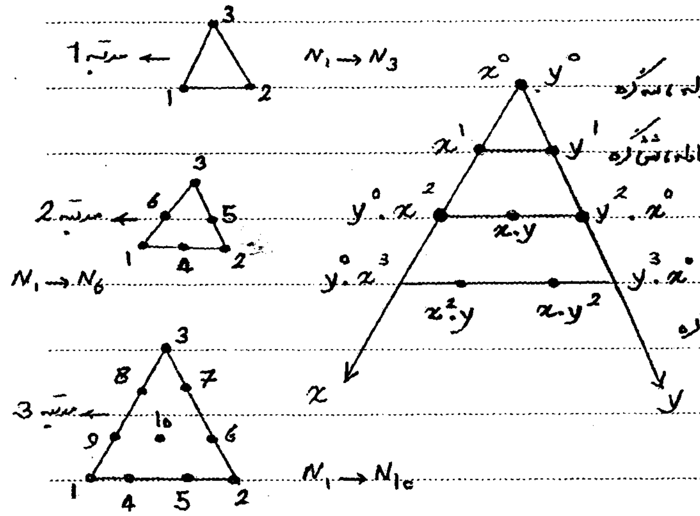
2) کرنش صفحه‌ای — plane strain ✓

3) تقارن محوری — Axi_symmetric

Tri_angular Shape Functions:
↓
Elements

Different Types of Triangular Elements:

دو جایی خیام:



سه معادله، سه مجهول

$$\varphi(x,y) = C_0 + C_1 x + C_2 y$$

شش معادله، شش مجهول

$$\varphi(x,y) = C_0 + C_1 x + C_2 y + C_3 x^2 + C_4 xy + C_5 y^2$$

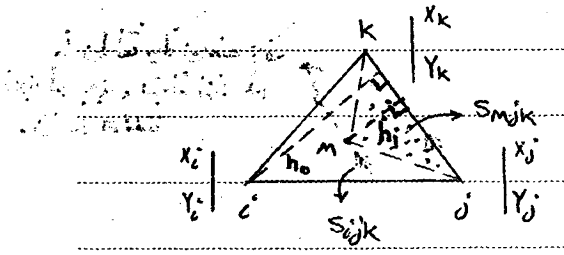
$$\varphi(x,y) = C_0 + C_1 x + C_2 y + C_3 x^2 + C_4 xy + C_5 y^2 + C_6 x^3 + C_7 x^2 y + C_8 x y^2 + C_9 y^3$$

ده معادله، ده مجهول

_ length coordinate — 1D المانهای

_ Area coordinate — 2D المانهای مثلثی

_ Volume coordinate — 3D المانهای هرمی — tetrahedral Elements



$$L_i = \frac{S_{mjk}}{S_{ijk}} = \frac{2S_{mjk}}{2S_{ijk}} = \frac{1}{2A}\,\mathrm{DET}\begin{bmatrix} 1 & x & y \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}$$

$$= \frac{1}{2A}\Big[\underbrace{(x_j y_k - x_k y_j)}_{a_i} + \underbrace{(y_j - y_k)}_{b_i}x + \underbrace{(x_k - x_j)}_{c_i}y\Big] = N_i(x,y)$$

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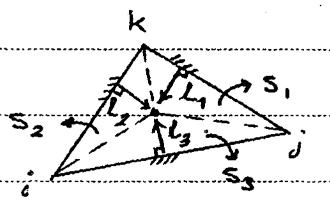
$$L_1 = \frac{l_{jk} * \frac{1}{2} h_1}{l_{jk} * \frac{1}{2} h_0} = \frac{h_1}{h_0}$$

$$L_1 = \frac{S_1}{S_0} \quad L_2 = \frac{S_2}{S_0} \quad L_3 = \frac{S_3}{S_0}$$



$$N_i = L_1 \ , \ N_j = L_2 \ , \ N_k = L_3$$

$$L_2 = \frac{S_{mki}}{S_{ijk}}$$

$$L_1 + L_2 + L_3 = 1 \qquad L_3 = 1 - L_1 - L_2$$

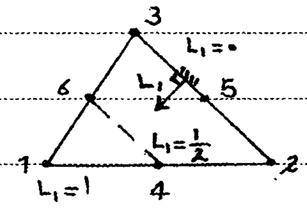
$$L_3 = \frac{S_{mij}}{S_{ijk}}$$

L_1 ، L_2 مستقل از یکدیگر تغییر می‌کنند. پس می‌توانند جایگزین x و y شوند

$$N_1 = \frac{1}{2A}(a_1 + b_1 x + c_1 y) = L_1$$

$$N_2 = \frac{1}{2A}(a_2 + b_2 x + c_2 y) = L_2$$

$$N_3 = \frac{1}{2A}(a_3 + b_3 x + c_3 y) = L_3 = 1 - L_1 - L_2$$



$$N_1 = C_0 (L_1)(L_1 - \tfrac{1}{2}) \qquad 1 = C(1)(\tfrac{1}{2}) \longrightarrow C = 2$$

$$N_1 = 2L_1(L_1 - \tfrac{1}{2})$$

و به طور مشابه $N_2 = 2L_2(L_2 - \frac{1}{2})$

$$N_3 = 2L_3(L_3 - \tfrac{1}{2}) \qquad L_3 = 1 - L_1 - L_2$$

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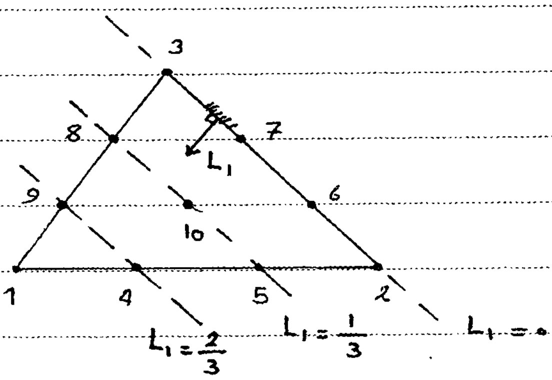
$N_4 = C \cdot L_1 \cdot L_2 \qquad 1 = \frac{1}{2} * \frac{1}{2} * C \qquad C = 4$

$N_4 = 4 L_1 L_2$

به طور مشابه : $N_5 = 4 L_2 L_3 \qquad L_3 = 1 - L_1 - L_2$

$N_6 = 4 L_1 L_3 \qquad L_3 = 1 - L_1 - L_2$



$N_1 = C L_1 (L_1 - \frac{1}{3})(L_1 - \frac{2}{3}) \qquad 1 = C(1)(\frac{2}{3})(\frac{1}{3}) \qquad C = \frac{9}{2}$

$N_1 = \frac{9}{2} L_1 (L_1 - \frac{1}{3})(L_1 - \frac{2}{3})$

و به طور مشابه $N_2 = \frac{9}{2} L_2 (L_2 - \frac{1}{3})(L_2 - \frac{2}{3})$

$N_3 = \frac{9}{2} L_3 (L_3 - \frac{1}{3})(L_3 - \frac{2}{3}) \qquad L_3 = 1 - L_1 - L_2$

$N_6 = C L_2 L_3 (L_2 - \frac{1}{3}) \qquad 1 = C(\frac{2}{3})(\frac{1}{3})(\frac{1}{3}) \qquad C = \frac{27}{2}$

$N_6 = \frac{27}{2} L_2 L_3 (L_2 - \frac{1}{3}) \qquad L_3 = 1 - L_1 - L_2$

به طور مشابه برای گره‌های 4 , 5 , 7 , 8 , 9

$N_{10} = C L_1 L_2 L_3 \qquad 1 = C \frac{1}{3} \frac{1}{3} \frac{1}{3} \qquad C = 27$

$N_{10} = 27 L_1 L_2 L_3 \qquad L_3 = 1 - L_1 - L_2$

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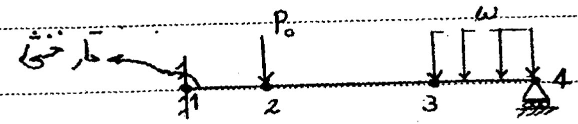
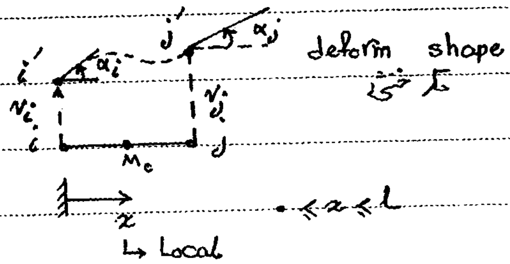
Virtual Work:

Euler Beam: No shear stress effect





2DOF: – deflection

– slope

$0 \leq x \leq l$

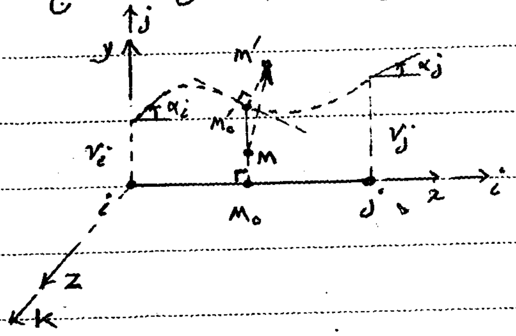
deflection: $v(x)$

slope: $\alpha(x) = \dfrac{dv(x)}{dx}$

پیوستگی تا مشتق مرتبه 1 و C^1 Continuity

تا به حال بحث ما محدود به C^0 continuty بود. چون slope را در نظر نمی‌گرفتیم

تنش روی تار خنثی صفر است. ← طول تار خنثی تغییر نمی‌کند. باید سایر نقاط خارج از تار خنثی ببریم:



چون تیر ما اویلری است، عمود بودن محورها بعد از تغییر شکل نیز باید حفظ شود.

$$\overrightarrow{MM'} = \begin{vmatrix} 0 \\ v(x) \\ 0 \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \\ \dfrac{dv(x)}{dx} \end{vmatrix} \wedge \begin{vmatrix} 0 \\ y \\ 0 \end{vmatrix} \qquad \overrightarrow{MM'} = \begin{vmatrix} u(x,y) = -y\,\dfrac{dv(x)}{dx} \\ v(x,y) = v(x) \\ 0 \end{vmatrix}$$

$$\frac{dv(x)}{dx}\,\underline{k} \times y\,\underline{j}$$

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برای تیر تیموشنکو : slope : $\theta(x) \neq \dfrac{dv(x)}{dx}$

$$\longrightarrow \overrightarrow{MM'} = \left| \begin{array}{l} u(x,y) = -y\,.\,\theta(x) \\ v(x,y) = v(x) \\ 0 \end{array} \right.$$

هر گره دو متغیر گرهای دارد : v_i , α_i ⟵ برای هر المان : v_i , α_i , v_j , α_j

$$\xrightarrow[\text{در نظر بگیریم.}]{\text{می توانیم چهار مجهول}} \quad v(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$\alpha(x) = \frac{dv(x)}{dx} = a_1 + 2a_2 x + 3a_3 x^2$$

$$a_0 = v_i \quad , \quad a_1 = \alpha_i$$

$$v_j = v_i + \alpha_i l + a_2 l^2 + a_3 l^3$$

$$\alpha_j = \alpha_i + 2.a_2 l + 3a_3 l^2$$

دو معادله، دو مجهول :

$$2(v_j - v_i - \alpha_i l = a_2 l^2 + a_3 l^3) \qquad 2(v_j - v_i) - 2l\alpha_i - l(\alpha_j - \alpha_i) = -a_3 l^3$$

$$l(\alpha_j - \alpha_i = 2a_2 l + 3a_3 l^2) \qquad 2(v_j - v_i) - l\alpha_j - l\alpha_i = -a_3 l^3$$

$$\longrightarrow a_3 = \frac{2}{l^3}(v_i - v_j) + \frac{1}{l^2}(\alpha_i + \alpha_j)$$

$$\longrightarrow a_2 = \frac{3}{l^2}(v_j - v_i) - \frac{2}{l}\alpha_i - \frac{1}{l}\alpha_j$$

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$$v(x) = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$v(x) = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{l^2} & -\frac{2}{l} & \frac{3}{l^2} & -\frac{1}{l} \\ \frac{2}{l^3} & \frac{1}{l^2} & -\frac{2}{l^3} & \frac{1}{l^2} \end{bmatrix} \begin{bmatrix} v_i \\ \alpha_i \\ v_j \\ \alpha_j \end{bmatrix} \begin{matrix} \rightarrow q_1 \\ \rightarrow q_2 \\ \rightarrow q_3 \\ \rightarrow q_4 \end{matrix}$$

بدون بعد shape functions ← N

Hermittian shape functions: $H(x)$, $\varphi(x)$

$$v(x) = \begin{bmatrix} \varphi_1(x) & \varphi_2(x) & \varphi_3(x) & \varphi_4(x) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

$$v(x) = \sum_{i=1}^{4} \varphi_i(x) \cdot q_i$$

$$\varphi_1(x) = 1 - 3\left(\frac{x}{l}\right)^2 + 2\left(\frac{x}{l}\right)^3$$

$$\varphi_2(x) = x\left(1 - \frac{x}{l}\right)^2$$

$$\varphi_3(x) = 3\left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right)^3$$

$$\varphi_4(x) = x\left(-\frac{x}{l} + \left(\frac{x}{l}\right)^2\right)$$

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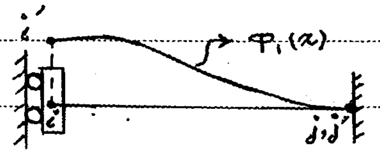
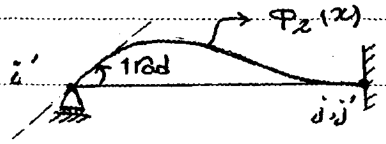
let $q_1 = 1$ unit displacement + $q_2 = q_3 = q_4 = 0$



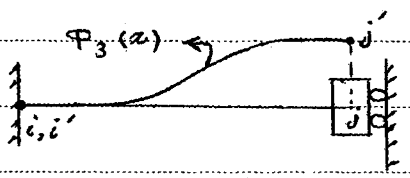
Concentrated Force

let $q_1 = 0$, $q_2 = 1$ unit rotation $q_3 = q_4 = 0$



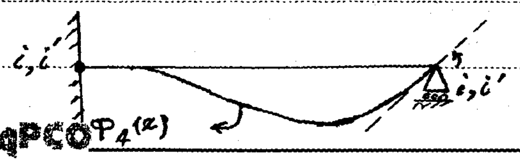
Concentrated moment

let $q_1 = q_2 = 0$, $q_3 = 1$ unit displacement , $q_4 = 0$



$\varphi_1 + \varphi_3 = 1$ در هر نقطه :

let $q_1 = q_2 = 0$, $q_3 = 0$, $q_4 = 1$ unit rotation



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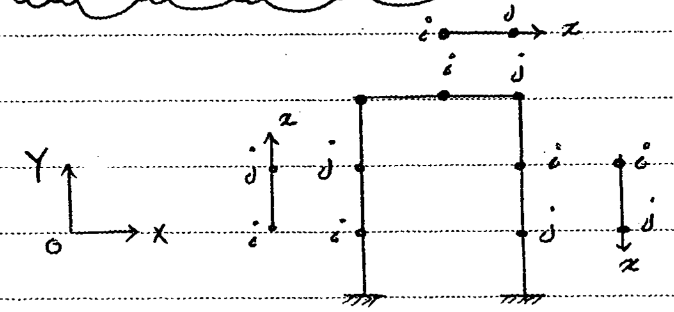
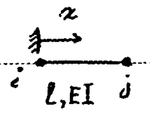
$$\boxed{\delta W^*_{Int} + \delta W^*_{ext} = \delta W^*_{acc}}$$

$W^* \longrightarrow$ کار مجازی



N.E , N.N

$$\delta W^*_{Int}\Big|_{total} = \sum_{e=1}^{N.E} \delta W^*_{Int}(e)$$



$0 \leq x \leq l$

$$\text{work} = \underset{\substack{A \\ \downarrow \\ \sigma}}{\text{Force}} \times \underset{\substack{l \\ \downarrow \\ \varepsilon}}{\text{disp}}$$

$$\delta W^*_{Int} = -\int_{\Omega^e} \sigma \cdot \underset{\substack{\downarrow \\ \text{کرنش مجازی}}}{\varepsilon^*} \cdot d\Omega^e$$

$$\delta W^*_{Int}(e) = -\int_{\Omega^e} \sigma_{ij}\, \delta\varepsilon^*_{ij}\, d\Omega^e \qquad \delta\varepsilon^* = \varepsilon^*_{New} - \cancel{\varepsilon^*_{old}}$$

$$\longrightarrow \delta W^*_{Int}(e) = -\int_{\Omega^e} \sigma_{ij}\, \varepsilon^*_{ij}\, d\Omega^e$$

$$u_x = -y \, \frac{dv(x)}{dx} \qquad u_y = v(x) = \sum_{i=1}^{4} \varphi_i(x)\, q_i$$

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$$\varepsilon_x = \frac{\partial u_x}{\partial x} = -y\,\frac{d^2 v(x)}{dx^2}$$

$$\varepsilon_x \neq 0$$

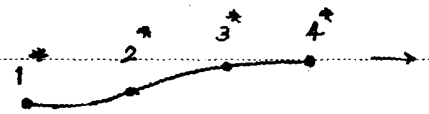
$$\varepsilon_y = \frac{\partial u_y}{\partial y} = 0$$

$$\varepsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = -\frac{dv(x)}{dx} + \frac{dv(x)}{dx} = 0$$

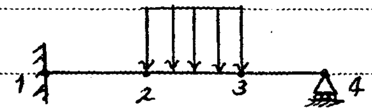
$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\varepsilon_x \longrightarrow \sigma_x \qquad \sigma_x = E \cdot \varepsilon_x$$

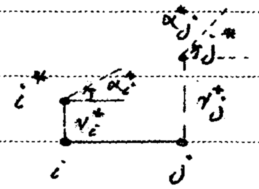
$$\sigma_x = -E \cdot y \cdot \frac{d^2 v(x)}{dx^2}$$



تصویر مجازی ← هیچگونه قیدی نمی‌بینیم.
هیچ ارتباطی با تصویر حقیقی ندارد.



$v_1 = 0 \;,\; \alpha_1 = 0 \;,\; v_4 = 0$



$$\overrightarrow{MM}^* = \left| \begin{array}{l} u_x^* = -y \cdot \dfrac{dv^*(x)}{dx} \\[2ex] u_y^* = v^*(x) = \displaystyle\sum_{j=1}^{4} \varphi_j(x)\, q_j^* \end{array} \right.$$

$$\varepsilon_x^* = \frac{\partial u_x^*}{\partial x} = -y\,\frac{d^2 v^*(x)}{dx^2}$$

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$$\varepsilon_x^* = -y \frac{d^2}{dx^2} \left(\sum_{j=1}^{4} \varphi_j(x)\, q_j^* \right)$$

$$\varepsilon_x^* = -y \sum_{j=1}^{4} \varphi_j''(x)\, q_j^*$$

$$\sigma_x = -E.y \frac{d^2}{dx^2} \sum_{i=1}^{4} \varphi_i(x)\, q_i$$

$$\sigma_x = -E.y \sum_{i=1}^{4} \varphi_i''(x)\, q_i$$

$$\delta W_{Int}^*(e) = -\int_{\Omega^e} \sigma_{ij}\, \varepsilon_{ij}^*\, d\Omega^e$$

$$\delta W_{Int}^*(e) = -\int_{\Omega^e} \sigma_x\, \varepsilon_x^*\, d\Omega^e$$

$$\delta W_{Int}^*(e) = -\int_0^{l}\int_A \left(-E.y. \sum_{i=1}^{4} \varphi_i''(x)\, q_i\right)\left(-y \sum_{j=1}^{4} \varphi_j''(x)\, q_j^*\right) dA\, dx$$

$$= -\int_0^{l}\int_A \left(E y^2 \sum_{j=1}^{4} \sum_{i=1}^{4} \varphi_i''(x)\, \varphi_j''(x)\, q_i\, q_j^*\right) dA\, dx$$

$$\delta W_{Int}^*(e) = -\sum_{i=1}^{4}\sum_{j=1}^{4} \left(\int_0^{l} E.\, \varphi_i''(x)\, \varphi_j''(x) \underbrace{\int_A y^2 dA}_{I(x)} \right) q_i\, q_j^*\, dx$$

$\bar{I}$: که اگر تابع x باشد، روی المان میانگین می‌گیریم

$$\underset{F.L}{\delta W_{Int}^*(e)} = -\sum_{j=1}^{4}\sum_{i=1}^{4} \left(\int_0^{l} E.I.\, \varphi_i''(x)\, \varphi_j''(x)\, dx \right) q_i\, q_j^*$$

$K_{ij} \leftarrow \frac{F}{l}$

$$\delta W_{Int}^*(e) = -\underset{1\times4}{\{q^*\}^T}\, \underset{4\times4}{[K_{ij}(e)]}\, \underset{4\times1}{\{q\}}$$

$$\{q\} = \begin{bmatrix} v_i \\ \alpha_i \\ v_j \\ \alpha_j \end{bmatrix} \qquad \{q^*\} = \begin{bmatrix} v_i^* \\ \alpha_i^* \\ v_j^* \\ \alpha_j^* \end{bmatrix}$$

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$$\boxed{k_{(e)} = \int_0^{l} EI \cdot \varphi_i''(x) \cdot \varphi_j''(x)\, dx}$$

$$\varphi_1(x) = 1 - 3(x/l)^2 + 2(x/l)^3$$

$$\varphi_2(x) = x(1 - x/l)^2$$

$$\varphi_3(x) = 1 - \varphi_1(x)$$

$$\varphi_4(x) = x(-x/l + (x/l)^2)$$

$$\delta W^{*(e)}_{Int} = \begin{array}{c} L \\ \\ \\ F.L \end{array} \begin{bmatrix} \gamma_i^* \\ \alpha_i^* \\ \gamma_j^* \\ \alpha_j^* \end{bmatrix}^T \begin{array}{c} \begin{array}{cc} F/L & F \end{array} \\ \left[\begin{array}{cc|c} k_{11} & k_{12} & \\ \multicolumn{2}{c|}{r_{ii}} & r_{ij} \\ k_{21} & k_{22} & \\ F & F.L & \\ \hline \multicolumn{2}{c|}{r_{ji}} & r_{jj} \end{array}\right] \end{array} \begin{array}{c} L \\ \begin{bmatrix} \gamma_i \\ \alpha_i \\ \gamma_j \\ \alpha_j \end{bmatrix} \end{array}$$

$$k_{11} = EI \int_0^{l} (\varphi_1''(x))^2 \cdot dx \qquad F/L$$

$$k_{22} = EI \int_0^{L} (\varphi_2''(x))^2 \cdot dx \qquad F.L$$

$$k_{12} = k_{21} = EI \int_0^{L} \varphi_1''(x)\, \varphi_2''(x)\, dx \qquad F$$

$$\varphi_1'(x) = -6\frac{x}{l^2} + 6\frac{x^2}{l^3}$$

$$\varphi_1''(x) = -\frac{6}{l^2} + 12\frac{x}{l^3}$$

$$k_{11} = EI \cdot \int_0^{l} \frac{1}{l^4}\left(12\frac{x}{l} - 6\right)^2 \cdot dx = \frac{EI}{l^3}\int_0^1 (12u - 6)^2 du = \frac{EI}{l^3}\left[\frac{1}{36}(12u - 6)^3\right]_0^1$$

$$k_{11} = \frac{12EI}{l^3}$$

$$\varphi_2(x) = x\left(1 - 2\frac{x}{l} + \frac{x^2}{l^2}\right) = x - 2\frac{x^2}{l} + \frac{x^3}{l^2}$$

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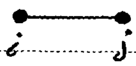
$$\varphi_2'(x) = \left(1 - 4\,x/L + 3\,x^2/L^2\right)$$

$$\varphi_2''(x) = -\frac{4}{L} + 6\,\frac{x}{L^2} = \frac{1}{L}\left(-4 + 6\,\frac{x}{L}\right)$$

$$k_{22} = EI\int_0^1 \frac{1}{L^2}\,(-4+6u)^2\,L\,.du = \frac{EI}{L}\,\frac{1}{3*6}\,(-4+6u)^3\Big|_0^1$$

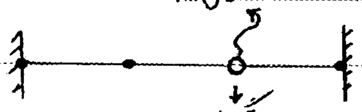
$$k_{22} = \frac{4EI}{L} \longrightarrow F.L$$

$$k_{ij} = \frac{EI}{L^3}\left[\begin{array}{cc|cc} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ \hline -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{array}\right]$$

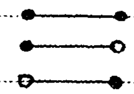


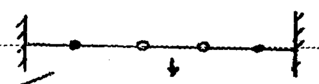
هرگره دارای دو درجه آزادی است.

گره‌های تیر ، C^1 Continuity



پیوستگی هست در
خیز (در شیب
پیوستگی نداریم.)





bending را تحمل نمی‌کند . C^0

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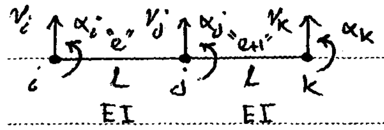
$$\delta W^*_{Int} + \delta W^*_{Ext} = \cancel{\delta W^*_{acc}} \quad \text{صفر}$$



$$\delta W^*_{Int}\,(\text{"}e\text{"} + \text{"}e+1\text{"}) = -\begin{bmatrix} v_i^* \\ \alpha_i^* \\ v_j^* \\ \alpha_j^* \end{bmatrix}^T \begin{bmatrix} r_{ii}^e & r_{ij}^e \\ r_{ji}^e & r_{jj}^e \end{bmatrix} \begin{bmatrix} v_i \\ \alpha_i \\ v_j \\ \alpha_j \end{bmatrix}$$

$$-\begin{bmatrix} v_j^* \\ \alpha_j^* \\ v_k^* \\ \alpha_k^* \end{bmatrix}^T \begin{bmatrix} r_{jj}^{e+1} & r_{jk}^{e+1} \\ r_{kj}^{e+1} & r_{kk}^{e+1} \end{bmatrix} \begin{bmatrix} v_j \\ \alpha_j \\ v_k \\ \alpha_k \end{bmatrix}$$

$$\delta W^*_{Int}\Big|_{(e)+(e+1)} = -\begin{bmatrix} v_i^* \\ \alpha_i^* \\ v_j^* \\ \alpha_j^* \\ v_k^* \\ \alpha_k^* \end{bmatrix}^T \left[\begin{array}{c|c|c} R_{ii}^e & R_{ij}^e & \\ \hline R_{ji}^e & R_{jj}^e + R_{jj}^{e+1} & R_{jk}^{e+1} \\ \hline & R_{kj}^{e+1} & R_{kk}^{e+1} \end{array}\right] \begin{bmatrix} v_i \\ \alpha_i \\ v_j \\ \alpha_j \\ v_k \\ \alpha_k \end{bmatrix} \qquad \delta W^*_{Int}\Big|_t = -q_t^{*T} \cdot k_t \cdot q_t$$

$$k_{total} = \frac{EI}{L^3} \left[\begin{array}{cc|cc|cc} 12 & 6L & -12 & 6L & & \\ 6L & 4L^2 & -6L & 2L^2 & & 0 \\ \hline -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ \hline & & -12 & -6L & 12 & -6L \\ & 0 & 6L & 2L^2 & -6L & 4L^2 \end{array}\right]$$

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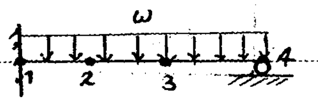
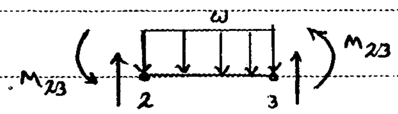


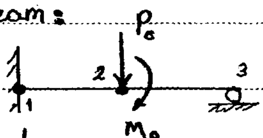


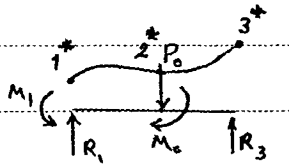
I. E Terms

Discretize در مورد بار خارجی ← ظاهر می‌شوند Inter element Terms ←
کردن بی معنی است. باید کل سازه را مورد بررسی قرار دهیم.

Continous beam:

1 حالت)

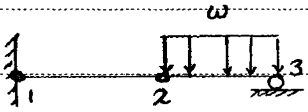


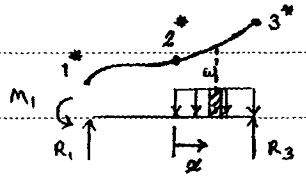


$v_1 = 0 \; , \; \alpha_1 = 0 \; , \; v_3 = 0$

Concentrated load on nodes.

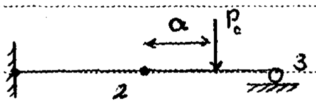
2 حالت)

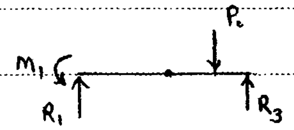




Distributed load between 2 nodes

3 حالت)





Concentrated load between 2 nodes

(برای حالت 1)
$$\delta W^*_{Ext} = R_1 v_1^* + M_1 \alpha_1^* - P_0 \cdot v_2^* - M_0 \alpha_2^* + R_3 v_3^*$$

$$= \begin{bmatrix} v_1^* \\ \alpha_1^* \\ v_2^* \\ \alpha_2^* \\ v_3^* \\ \alpha_3^* \end{bmatrix}^T \begin{bmatrix} R_1 \\ M_1 \\ -P_0 \checkmark \\ -M_0 \checkmark \\ R_3 \\ 0 \end{bmatrix} = \underline{q}^{*T} \cdot \underline{F}$$

Equivalents Concentrated Nodal Force vector

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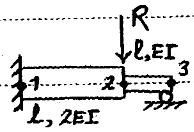
$$\delta W^*_{Int} + \delta W^*_{Ext} = 0$$

$$-\underline{q}^{*T} \cdot \underline{k}_T \cdot \underline{q} + \underline{q}^{*T} \cdot \underline{F} = 0$$

$$\underbrace{\underline{q}^{*T}}_{\neq 0} \left(-\underline{k}_T \cdot \underline{q} + \underline{F} \right) = 0 \longrightarrow k_T \cdot \underbrace{\underline{q}}_{\underline{\alpha}} = \underline{F}$$



تمرین ۵

$$k_{total} = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l & & \\ 6l & 4l^2 & -6l & 2l^2 & & \\ -12 & -6l & 24 & 0 & -12 & 6l \\ 6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\ & & -12 & -6l & 12 & -6l \\ & & 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ V_2 \\ \alpha_2 \\ 0 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ M_1 \\ -P_0 \\ -M_0 \\ R_3 \\ 0 \end{bmatrix}$$

شش معادله
شش مجهول

(برای حالت 2 $\quad \delta W^*_{Ext} = R_1 V_1^* + M_1 \alpha_1^* + R_3 V_3^* + \int_0^l -w \, dx \, V^*(x)$

$\int_0^l m(x) \, dx \, \frac{dV^*(x)}{dx}$ و اگر ممان گسترده داشتیم

$$V^*(x) = \sum_{i=1}^{4} \varphi_i(x) \, q_i^*$$

$$\longrightarrow \int_0^l -w \, dx \, V^*(x) = \sum_{i=1}^{4} \underbrace{\left(\int_0^l -w \cdot \varphi_i(x) \, dx \right)}_{F_i} q_i^*$$

$q_1^* = V_2^* \longrightarrow$ نیرو در گره 2 $\; F_1$

$q_2^* = \alpha_2^* \longrightarrow$ ممان در گره 2 $\; F_2$

$q_3^* = V_3^* \longrightarrow$ نیرو در گره 3 $\; F_3$

$q_4^* = \alpha_3^* \longrightarrow$ ممان در گره 3 $\; F_4$

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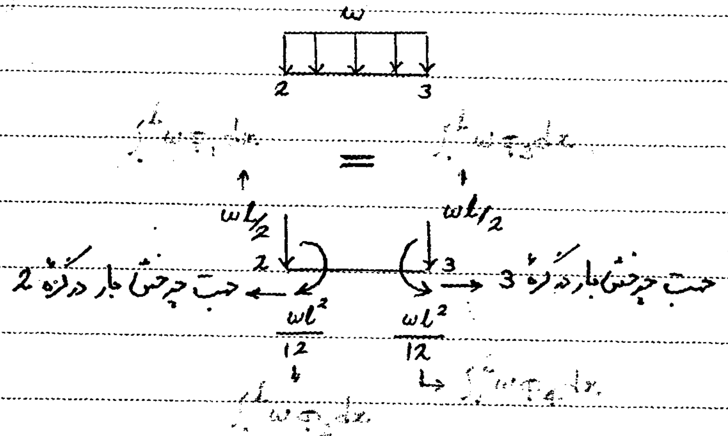
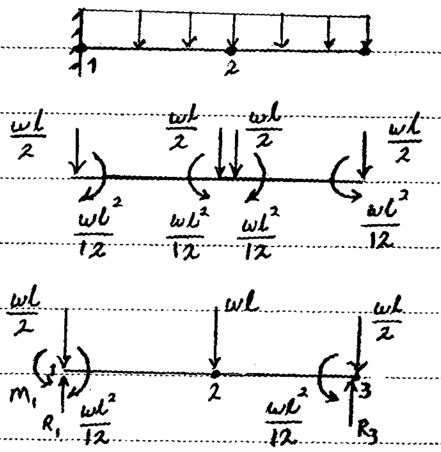
$$\varphi_1(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$

$$\varphi_2(x) = x\left(1 - \frac{x}{L}\right)^2$$

$$\varphi_3(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$$

$$\varphi_4(x) = x\left(-\frac{x}{L} + \left(\frac{x}{L}\right)^2\right)$$





$$\underline{F} = \begin{bmatrix} R_1 - \frac{wl}{2} \\ M_1 - \frac{wl^2}{12} \\ -wl \\ 0 \\ R_3 - \frac{wl}{2} \\ \frac{wl^2}{12} \end{bmatrix}$$

equivalent nodal force vector

3 (برای حالت)
$$\delta W^*_{ext} = R_1 \cdot \gamma_1^* + M_1 \cdot \alpha_1^* + R_3 \cdot \gamma_3^* - P_0 \, \underbrace{\gamma^*(x=a)}_{\sum_{i=1}^{4} \varphi_i(x)\big|_{x=4} \; q_i^*}$$

$$q_i^* \begin{cases} q_1^* = \gamma_2^* \\ q_2^* = \alpha_2^* \\ q_3^* = \gamma_3^* \\ q_4^* = \alpha_3^* \end{cases}$$

$$\delta W^*_{ext} = R_1 \cdot \gamma_1^* + M_1 \cdot \alpha_1^* + R_3 \cdot \gamma_3^* + \sum_{i=1}^{4} \underbrace{\left(-P_0 \, \varphi_i(x=a)\right)}_{F_i} \cdot q_i^*$$

$$f_i = -P_0 \, \varphi_i(x=4)$$

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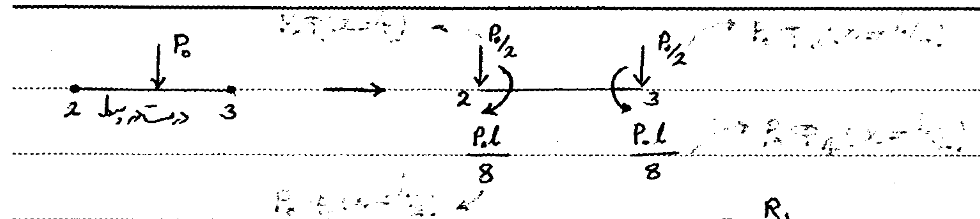
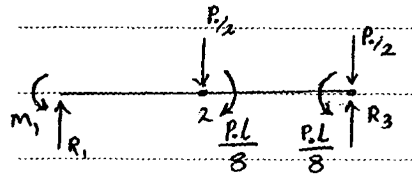
وقتی P_0 در وسط تیر باشد:





$$\underline{F} = \begin{bmatrix} R_1 \\ M_1 \\ -P_0/2 \\ -P_0 l/8 \\ R_3 - P_0/2 \\ P_0 l/8 \end{bmatrix}$$

$$\delta W^*_{Int}\Big|_{total} + \delta W^*_{Ext}\Big|_{total} = 0$$

$$-\underline{q}^{*T} K_T\, \underline{q} + \underline{q}^{*T} \cdot \underline{F} = 0$$

$$\underset{\neq 0}{\underline{q}^{*T}} \left(-K_T \cdot \underline{q} + \underline{F}\right) = 0$$

$$\underset{n \times n}{K_T} \cdot \underline{q} = \underline{F} \;\left(+ \text{B.C Terms} \right)$$

اعمال شرایط مرزی

کل درجات آزادی

n equation + n unknown

$$n = \underset{\text{number of nodes}}{(N.N)} \times \underset{\text{degree of freedom of each node}}{\left(\frac{D.O.F}{Node}\right)}$$

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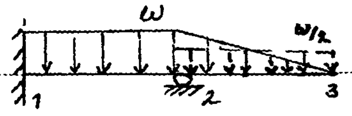
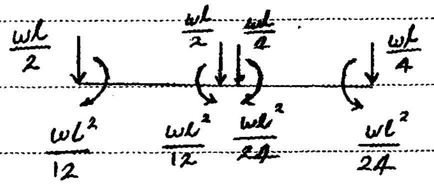
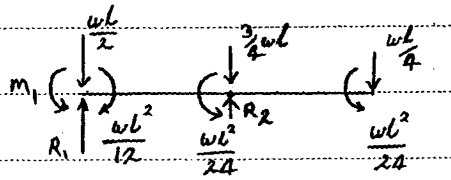
مثال :







$$\underline{k_T} \cdot \underline{q} = \underline{F}$$

$$\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ 6l & 4l^2 & -6l & 2l^2 & 0 & 0 \\ -12 & -6l & 24 & 0 & -12 & 6l \\ 6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\ 0 & 0 & -12 & -6l & 12 & -6l \\ 0 & 0 & 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{bmatrix} \cancel{v_1}^{\,0} \\ \cancel{\alpha_1}^{\,0} \\ \cancel{v_2}^{\,0} \\ \alpha_2 \\ v_3 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} R_1 - \frac{\omega l}{2} \\ M_1 - \frac{\omega l^2}{12} \\ R_2 - \frac{3}{4}\omega l \\ \frac{1}{24}\omega l^2 \\ -\omega l/4 \\ \frac{1}{24}\omega l^2 \end{bmatrix}$$

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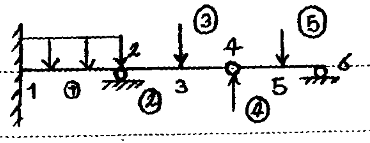
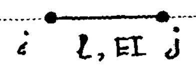


به غیر از گره 4 بقیه گره‌ها C^1 continuity هستند.



$$k_{ij} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

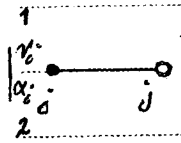
$$\varphi_1(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$

$$\varphi_2(x) = x\left(1 - \frac{x}{L}\right)^2$$

$$\varphi_3(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$$

$$\varphi_4(x) = x\left(-\frac{x}{L} + \left(\frac{x}{L}\right)^2\right)$$

$$k_{ij} = \int_0^l EI\, \varphi_i''\, \varphi_j''\, dx$$



$m(x=L)=0$

$\hookrightarrow m(x) = EI \left.\frac{\partial^2 v}{\partial x^2}\right|_{x=L} = 0$

4

در گره j شیب را مورد نظر قرار نمی‌دهیم.

4 شرط $\longrightarrow$ $v(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$$\alpha(x) = a_1 + 2a_2 x + 3a_3 x^2$$

$$\frac{d^2 v(x)}{dx^2} = 2a_2 + 6a_3 x$$

$a_0 = v_i \quad , \quad a_1 = \alpha_i$

$v_j = v_i + \alpha_i l + a_2 l^2 + a_3 l^3$

$2a_2 + 6a_3 l = 0$

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$$\alpha_2 = -3\alpha_3 l$$

$$-3\alpha_3 l^3 + \alpha_3 l^3 = v_j - v_i - \alpha_i l$$

$$\alpha_3 = \frac{1}{2l^3}(v_i - v_j) + \frac{1}{2l^2}\alpha_i \quad , \quad \alpha_2 = \frac{-3}{2l^2}(v_i - v_j) - \frac{3}{2l}\alpha_i$$

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-3}{2l^2} & \frac{-3}{2l} & \frac{3}{2l^2} & 0 \\ \frac{1}{2l^3} & \frac{1}{2l^2} & -\frac{1}{2l^3} & 0 \end{bmatrix} \begin{bmatrix} v_i \\ \alpha_i \\ v_j \\ \alpha_j \end{bmatrix}$$

$$v(x) = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$v(x) = \begin{bmatrix} \varphi_1(x) & \varphi_2(x) & \varphi_3(x) & \varphi_4(x) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

$$\varphi_1(x) = 1 - \frac{3}{2}(x/l)^2 + \frac{1}{2}(x/l)^3$$

$$\varphi_2(x) = x\left(1 - \frac{3}{2}(x/l) + \frac{1}{2}(x/l)^2\right)$$

$$\varphi_3(x) = \frac{3}{2}(x/l)^2 - \frac{1}{2}(x/l)^3$$

$$\varphi_4(x) = 0$$

$$\underline{k}_{ij} = \frac{EI}{l^3}\begin{bmatrix} 3 & 3l & -3 & 0 \\ 3l & 3l^2 & -3l & 0 \\ -3 & -3l & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$\left.\frac{d^2 v}{dx^2}\right|_{x=0} = 0$$

$$a_0 = v_i \qquad a_2 = 0$$

$$v_j = v_i + a_1 l + a_3 l^3$$

$$\alpha_j = a_1 + 3a_3 l^2$$

$$a_1 l + a_3 l^3 = v_j - v_i$$

$$a_1 l + 3a_3 l^3 = \alpha_j l$$

$$\longrightarrow \quad 2a_3 l^3 = \alpha_i l - (v_j - v_i)$$

$$a_3 = \frac{1}{2l^2}\alpha_j - \frac{1}{2l^3}(v_j - v_i)$$

$$2a_1 l = 3(v_j - v_i) - \alpha_j l$$

$$a_1 = \frac{3}{2l}(v_j - v_i) - \frac{1}{2}\alpha_j$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{3}{2l} & 0 & \frac{3}{2l} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ \frac{1}{2l^3} & 0 & -\frac{1}{2l^3} & \frac{1}{2l^2} \end{bmatrix} \begin{bmatrix} v_i \\ \alpha_i \\ v_j \\ \alpha_j \end{bmatrix}$$

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$$v(x) = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{3}{2l} & 0 & 3/2l & -1/2 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2l^3} & 0 & -1/2l^3 & 1/2l^2 \end{bmatrix} \begin{bmatrix} v_i \\ \alpha_i \\ v_j \\ \alpha_j \end{bmatrix}$$

$$v(x) = \begin{bmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

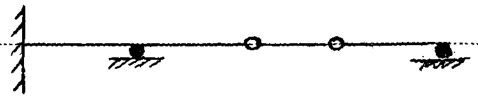
$$\varphi_1(x) = 1 - \frac{3}{2}(x/l) + \frac{1}{2}(x/l)^3$$

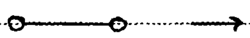
$$\varphi_2(x) = 0$$

$$\varphi_3(x) = \frac{3}{2}(x/l) - \frac{1}{2}(x/l)^3$$

$$\varphi_4(x) = x\left(-\frac{1}{2} + \frac{1}{2}(x/l)^2\right)$$

$$k_{ij} = \frac{EI}{l^3} \begin{bmatrix} 3 & 0 & -3 & 3l \\ 0 & 0 & 0 & 0 \\ -3 & 0 & 3 & -3l \\ 3l & 0 & -3l & 3l^2 \end{bmatrix}$$



 هم عضو با المان خرپا (المان خرپا بار در راستای
طول را تحمل می کند در صورتی که
این المان بار عرضی را تحمل می کند)

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$\varphi_1(x) = 1 - \frac{x}{L}$

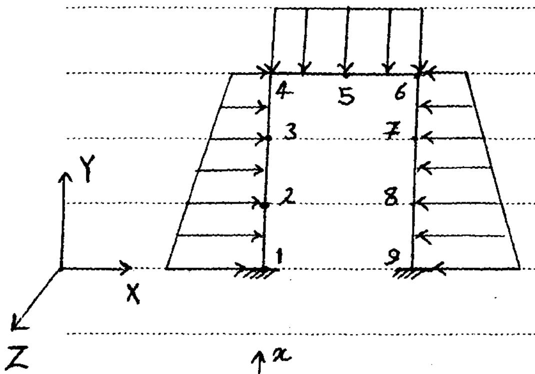
$\varphi_2(x) = 0$

$\varphi_3(x) = \frac{x}{L}$

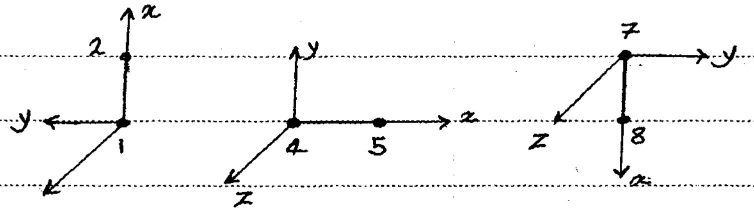
$\varphi_4(x) = 0$

$k_{ij} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right] \longrightarrow$ یعنی باری تحمل نمی‌کند!

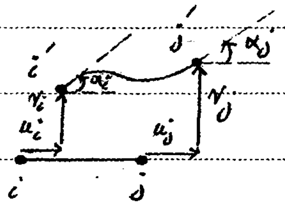
Frame :



Combined loading : Bending + axial loading



local coordinates



$k\big|_{6\times6} = k_{4\times4}\big|_{Bend} + k_{2\times2}\big|_{axial}$.

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If $\varphi_1 \longrightarrow \varphi_4$: Bending $\longrightarrow k_{4\times4_{\text{bending}}}$ تیر ـ تیر

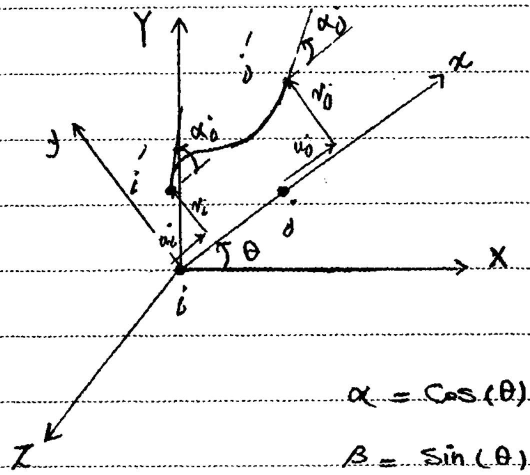
$\varphi_5 = 1 - x/l$

$\varphi_6 = x/l$ $\quad k_{\text{axial}} = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

هیچ coupling بین Axial , Bending نداریم.

$$k_{ij}\Big|_{\text{local}} = \begin{array}{c} \begin{array}{cc} \text{Bending} & \text{Axial} \end{array} \\ \left[\begin{array}{c|c} 4\times4 & 0 \\ \hline 0 & 2\times2 \end{array}\right] \end{array} \begin{bmatrix} v_i \\ \alpha_i \\ v_j \\ \alpha_j \\ u_i \\ u_j \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & 12\frac{EI}{l^3} & 6l\frac{EI}{l^3} & 0 & -12\frac{EI}{l^3} & 6l\frac{EI}{l^3} \\ 0 & 6l\frac{EI}{l^3} & 4l^2\frac{EI}{l^3} & 0 & -6l\frac{EI}{l^3} & 2l^2\frac{EI}{l^3} \\ \hline -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -12\frac{EI}{l^3} & -6l\frac{EI}{l^3} & 0 & 12\frac{EI}{l^3} & -6l\frac{EI}{l^3} \\ 0 & 6l\frac{EI}{l^3} & 2l^2\frac{EI}{l^3} & 0 & -6l\frac{EI}{l^3} & 4l^2\frac{EI}{l^3} \end{array}\right] \begin{bmatrix} u_i \\ v_i \\ \alpha_i \\ u_j \\ v_j \\ \alpha_j \end{bmatrix}$$



$\alpha = \cos(\theta)$

$\beta = \sin(\theta)$

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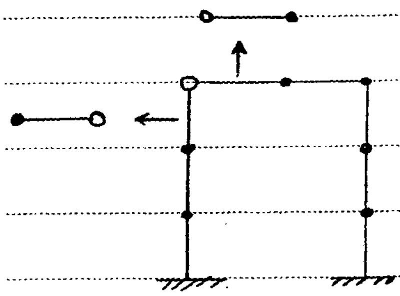
$$\underline{R} = \left[\begin{array}{ccc|ccc} \alpha & \beta & 0 & & & \\ -\beta & \alpha & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ \hline & & & \alpha & \beta & 0 \\ & 0 & & -\beta & \alpha & 0 \\ & & & 0 & 0 & 1 \end{array}\right]$$

$$k_{ij}\Big|_{Global} = \underline{R}^T \cdot k_{ij}\Big|_{local} \cdot R$$

اگر المانی با گره توخالی داشتیم، سهم Bending را در ماتریس K، از ماتریس مناسب محاسبه شده قبلی (o—• یا •—o) انتخاب می‌کنیم.



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"Weighted Residual Method" روش باقی مانده وزنی

"D.E" $\longrightarrow$ Integral Equation

$$\frac{d^2T}{dx^2} - 6T + 10 = 0$$

$$0 \leqslant x \leqslant 1 \qquad \text{B.C.} \begin{cases} T(x=0) = 0 \\ T(x=1) = 0 \end{cases}$$

Approximate Solution

Trial Solution $\longrightarrow$ Confirmation with respect to Geometrial B.C.

$\varphi_1(x) = x(1-x)$

$\varphi_2(x) = x^2(1-x)$

$\vdots$

$\varphi_n(x)$

Final Trial Solution $= \sum_{i=1}^{n} a_i \varphi_i(x)$

مستر نیت

$$T'(x, a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i \varphi_i(x)$$

$$\frac{d^2T'}{dx^2} - 6T' + 10 \neq 0 \qquad \frac{d^2T'}{dx^2} - 6T' + 10 = R(x, a_1, \ldots, a_n)$$

برای هر تابع حدسی، یک تابع وزنی در نظر می‌گیریم:

$\varphi_1(x) = x(1-x) \longrightarrow w_1(x)$

$\varphi_2(x) = x^2(1-x) \longrightarrow w_2(x)$

$\vdots$

$\varphi(x) = \qquad \longrightarrow w_n(x)$

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$$\int_0^1 w_1(x)\, R(x, a_1, \dots, a_n)\, dx = 0$$

$$\int_0^1 w_2(x)\, R(x, a_1, \dots, a_n)\, dx = 0 \longrightarrow$$ n معادله ـ n مجهول

$\vdots$ ضرایب $a_1, a_2, \dots, a_n$

$$\int_0^1 w_n(x)\, R(x, a_1, \dots, a_n)\, dz = 0$$ بدست می‌آیند.

$$T'(x, a_1, a_2) = a_1(x - x^2) + a_2(x^2 - x^3)$$

$$\frac{dT'}{dx} = a_1(1 - 2x) + a_2(2x - 3x^2)$$

$$\frac{d^2T'}{dx^2} = a_1(-2) + a_2(2 - 6x)$$

$$R(x, a_1, a_2) = a_1(-2) + a_2(2 - 6x) - 6\left[a_1(x - x^2) + a_2(x^2 - x^3)\right] + 10$$

$w_1(x)$, $w_2(x)$

- point Collocation
- sub_domain Collocation
- Least square Method
- Galerkin Method

point Collocation :

تابع خطا را در n نقطه دلخواه صفر می‌کنیم:

مثلاً :

$$\begin{cases} R(x = 1/3, a_1, a_2) = 0 \\ R(x = 2/3, a_1, a_2) = 0 \end{cases}$$

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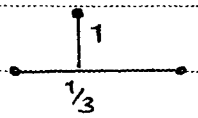
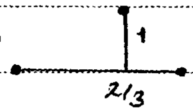
$$\int_0^1 w_1(x)\, R(x, a_1, a_2)\, dx = 0$$

$w_1(x)$  ارزش یک نقطه ($x = 1/3$) برابر یک، و ارزش بقیه نقاط صفر است.

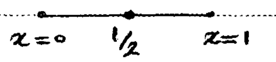
$$\int_0^1 w_2(x)\, R(x, a_1, a_2)\, dx = 0$$

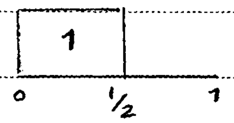
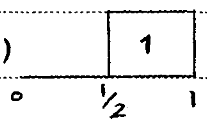
$w_2(x)$  ارزش یک نقطه ($x = 2/3$) برابر یک، و ارزش بقیه نقاط صفر است.

→ No Unique Solution

Subdomain Collocation:

domain را بطور دلخواه به دو قسمت تقسیم می‌کنیم:

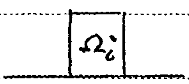
مثال: 

$w_1(x)$  $w_2(x)$ 

$$\int_0^{1/2} R(x, a_1, a_2)\, dx = 0 \qquad \int_{1/2}^1 R(x, a_1, a_2)\, dx = 0$$ دو معادله، دو مجهول

$R(x, a_1, \ldots, a_n)$

$$\Omega = \sum_{i=1}^{n} \Omega_i$$ ↳ sub-domain

$\forall x \in \Omega_i : w_i(x) = 1$ 

$\forall x \notin \Omega_i : w_i(x) = 0$ → No Unique solution

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Least square Method:

$$I(a_1, \dots, a_n) = \int_0^1 R^2(x, a_1, \dots, a_n)\, dx$$

مقدار I را مینیمم می‌کنیم:

$$\frac{\partial I}{\partial a_1} = 0 \quad , \quad \frac{\partial I}{\partial a_2} = 0 \quad , \quad \dots \quad , \quad \frac{\partial I}{\partial a_n} = 0$$

$$i = 1, \dots, n \qquad \frac{\partial I}{\partial a_i} = 0$$

$$\frac{\partial I}{\partial a_i} = \frac{\partial}{\partial a_i} \int_0^1 R^2(x, a_1, \dots, a_n)\, dx = 0$$

$$= 2 \int_0^1 \underbrace{\frac{\partial R(x, a_1, \dots, a_n)}{\partial a_i}}_{w_i(x)} R(x, a_1, \dots, a_n)\, dx$$

$\longrightarrow$ Unique Solution

4

Galerkin Method:

$w_i(x) = \varphi_i(x)$

weighting Function = Trial Function

$$\int_0^1 w_i(x)\, R(x, a_1, \dots, a_n)\, dx = 0$$

$\varphi_1(x)$ $\varphi_2(x)$ $\varphi_n(x)$

$\longrightarrow$ Unique Solution

weighted Residual Method

D.E $\longrightarrow$ Trial Solution

$\downarrow$

O.B.C

Error Function

Residual Function $\left| \begin{array}{l} R(x, a_1, \dots, a_n) \\ R(x, y, a_1, \dots, a_n) \\ R(x, y, z, a_1, \dots, a_n) \end{array} \right.$

Weighted Residual Method in Finite Element Context :

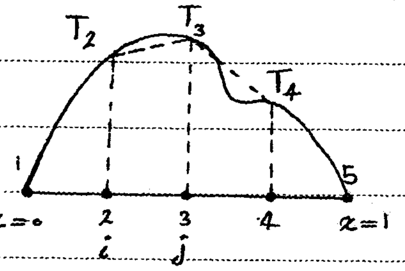
$\longrightarrow$ Galerkin Method

Finite Element : Trial Functions $\longrightarrow$ shape Functions

1 Dimentional Analysis

$$\frac{d^2T}{dx^2} - 6T + 10 = 0$$

$0 \leqslant x \leqslant 1$ B.C. $\left| \begin{array}{l} T(x=0) = 0 \\ T(x=1) = 0 \end{array} \right.$



domain را تکه تکه می‌کنیم :

$\left| \begin{array}{l} N.N = 5 \\ N.E = 4 \end{array} \right.$ B.C. $\left| \begin{array}{l} T_1 = 0 \\ T_5 = 0 \end{array} \right.$

اگر تعداد تکه‌ها را بیشتر کنیم ، می‌توانیم تغییرات دما در هر المان را بصورت خطی تقریب بزنیم.

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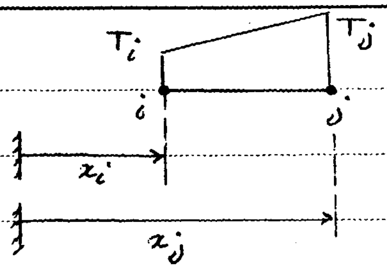

$$x_j - x_i = x_{ji}$$

C^0 continuity

$$T_e(x) = C_0 + C_1 x$$

$$T_i(x) = C_0 + C_1 x_i$$

$$T_j(x) = C_0 + C_1 x_j$$

$$T_e(x) = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \end{bmatrix}$$

$$\begin{bmatrix} T_i \\ T_j \end{bmatrix} = \begin{bmatrix} 1 & x_i \\ 1 & x_j \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \end{bmatrix}$$

$$T_e(x) = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} 1 & x_i \\ 1 & x_j \end{bmatrix}^{-1} \begin{bmatrix} T_i \\ T_j \end{bmatrix}$$

$$T_e(x) = \underbrace{\begin{bmatrix} N_i(x) & N_j(x) \end{bmatrix}}_{N} \begin{bmatrix} T_i \\ T_j \end{bmatrix}$$

$$T'(x, a_1, a_2) = a_1 N_1(x) + a_2 N_2(x)$$

$$T^e = T'(x, T_i, T_j) = T_i N_i(x) + T_j N_j(x)$$

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$$\frac{d^2T^e}{dx^2} - 6T^e + 10 = R(x, T_i, T_j) \neq 0$$

چون دما را خطی در نظر گرفتیم، این مقدار . حذف می‌شود. اما نمی‌توان آن را حذف کرد. (چون
بخشی از مسأله حذف می‌شود.)

$$\int_{x_i}^{x_j} N_i(x) \cdot R(x, T_i, T_j)\, dx = 0$$

$$\int_{x_i}^{x_j} N_j(x) \cdot R(x, T_i, T_j)\, dx = 0$$

$$\longrightarrow \int_{x_i}^{x_j} N^T \cdot R(x, T_i, T_j)\, dx = 0$$

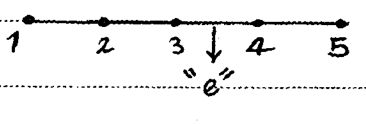
$$\int_{x_i}^{x_j} N^T \left(\frac{d^2T^e}{dx^2} - 6T^e + 10 \right) dx = 0$$

General form of a P.D.E :

$T(x,t)$

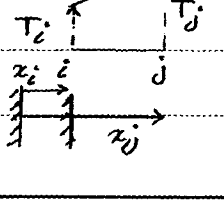
$$A \cdot \frac{d^2T}{dx^2} - B \cdot T(x) + Q = 0 \qquad 0 \leq x \leq l$$

$$\text{B.C.} \left| \begin{array}{l} T(x=0) = \checkmark \text{ or } \left.\frac{dT}{dx}\right|_{x=0} = \checkmark \\ T(x=l) = \checkmark \text{ or } \left.\frac{dT}{dx}\right|_{x=l} = \checkmark \end{array} \right.$$



$$\left. T(x) \right|_e = \underline{N} \cdot a^e = \begin{bmatrix} N_j(x) & N_j(x) \end{bmatrix} \begin{bmatrix} T_i \\ T_j \end{bmatrix}$$

$$= N_i(x) T_i + N_j(x) T_j$$



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$$A \cdot \frac{d^2 T^e}{dx^2} - B \cdot T^e_{(x)} + Q = R(x, T_i, T_j) \neq 0$$

$$\int_{x_i}^{x_j} N_i(x)\, R(x, T_i, T_j)\, dx = 0$$

$$\int_{x_j}^{x_j} N_j(x)\, R(x, T_i, T_j)\, dx = 0$$

$$\int_{x_i}^{x_j} \underline{N}^T R(x, T_i, T_j)\, dx = 0 \qquad \underline{N} = \begin{bmatrix} N_i(x) & N_j(x) \end{bmatrix}$$

$$\int_{x_i}^{x_j} \underline{N}^T \left(A \cdot \frac{d^2 T^e}{dx^2} - B \cdot T^e(x) + Q\right) dx = 0$$

$\underline{1}$ $$A \cdot \int_{x_i}^{x_j} \underset{u}{\underline{N}^T} \cdot \underset{v'}{\frac{d^2 T}{dx^2}}\, dx = A \cdot \underline{N}^T \cdot \frac{dT}{dx}\bigg|_{x_i}^{x_j} - A \cdot \int_{x_i}^{x_j} \frac{d\underline{N}^T}{dx} \frac{dT}{dx} \cdot dx$$

$\underline{2}$ $$-B \cdot \int_{x_i}^{x_j} N^T \cdot T^e_{(x)}\, dx = -B\left(\int_{x_i}^{x_j} \underbrace{\underline{N}^T \cdot \underline{N}}_{\underline{k}_B} \cdot dx\right) \alpha^e$$

$\underline{3}$ $$\int_{x_i}^{x_j} Q \cdot \underline{N}^T dx = Q \cdot \int_{x_i}^{x_j} N^T \cdot dx = \underline{F}^e$$

$$T^e_{(x)} = N \cdot \underline{\alpha}^e \qquad \frac{dT}{dx} = \frac{d\underline{N}}{dx} \cdot \underline{\alpha}$$

$\underline{1} \longrightarrow$ $$A \cdot \int_{x_i}^{x_j} \underline{N}^T \cdot \underbrace{\frac{d^2 T}{dx^2}}_{\text{"Force"}} dx = A \cdot \underline{N}^T \cdot \underbrace{\frac{dT}{dx}}_{\text{"Force"}}\bigg|_{x_i}^{x_j} - A\Bigg(\underbrace{\int_{x_i}^{x_j} \frac{d\underline{N}^T}{dx} \frac{d\underline{N}}{dx} \cdot dx}_{K_a^e}\Bigg) \cdot \underbrace{\alpha^e}_{\text{disp}}$$

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$$\left(A\cdot\int_{x_i}^{x_j}\frac{dN^T}{dx}\cdot\frac{dN}{dx}\cdot dx\right)\underline{\alpha}^e+B\left(\int_{x_i}^{x_j}N^T\cdot N\cdot dx\right)\underline{\alpha}^e=Q\int_{x_i}^{x_j}N^T\cdot dx+A\cdot N^T\cdot\frac{dT}{dx}\Big|_{x_i}^{x_j}$$

$$(\underline{k}_A^e+\underline{k}_B^e)\,\underline{\alpha}^e=\underline{F}^e+\underset{\text{inter element term}}{\underset{\downarrow}{I.E^e}}$$

$$\underline{k}_A^e=A\cdot\int_{x_i}^{x_j}\frac{d\underline{N}^T}{dx}\,\frac{d\underline{N}}{dx}\,dx$$

$$\underline{N}=\begin{bmatrix}N_i(x) & N_j(x)\end{bmatrix}$$

$$N_i(x)=\frac{1}{x_{ji}}(x_j-x)\ ,\quad N_j(x)=\frac{1}{x_{ji}}(x-x_i)$$

$$\frac{dN}{dx}=\begin{bmatrix}\frac{dN_i(x)}{dx} & \frac{dN_j(x)}{dx}\end{bmatrix}=\frac{1}{x_{ji}}\begin{bmatrix}-1 & 1\end{bmatrix}$$

$$\underline{k}_A^e=A\int_{x_i}^{x_j}\frac{1}{x_{ji}}\begin{bmatrix}-1\\+1\end{bmatrix}\frac{1}{x_{ji}}\begin{bmatrix}-1 & +1\end{bmatrix}dx=A\int_{x_i}^{x_j}\frac{1}{x_{ji}^2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\underset{x_{ji}=x_j-x_i}{\underset{\downarrow}{dx}}$$

$$\boxed{\underline{k}_A^e=\frac{A}{x_{ji}}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}}$$

$$\underline{k}_B^e=B\cdot\int_{x_i}^{x_j}\underline{N}^T\cdot N\cdot dx$$

$$\underline{k}_B^e=B\cdot\int_{x_i}^{x_j}\underbrace{\frac{1}{x_{ji}}\begin{bmatrix}x_j-x\\x-x_i\end{bmatrix}}_{\underline{N}^T}\underbrace{\frac{1}{x_{ji}}\begin{bmatrix}(x_j-x) & (x-x_i)\end{bmatrix}}_{\underline{N}}dx$$

$$\underline{k}_B^e=\frac{B}{x_{ji}^2}\int_{x_i}^{x_j}\begin{bmatrix}(x_j-x)^2 & (x_j-x)(x-x_i)\\(x_j-x)(x-x_i) & (x-x_i)^2\end{bmatrix}dx$$

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$$\int_{x_i}^{x_j} (x_j - x)^2 dx = -\frac{1}{3}(x_j - x)^3 \Big|_{x_i}^{x_j} = \frac{1}{3} x_{ji}^3$$

$$\int_{x_i}^{x_j} (x_j - x)(x - x_i)\, dx = \frac{1}{6} x_{ji}^3$$

$$\underline{K}_B^e = B \cdot x_{ji} \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} = \frac{B \cdot x_{ji}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

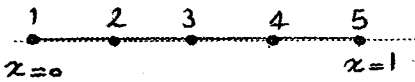
$$\underline{F}^e = Q \cdot \int_{x_i}^{x_j} \frac{1}{x_{ji}} \begin{bmatrix} x_j - x \\ x - x_i \end{bmatrix} dx = \frac{Q}{x_{ji}}\, x_{ji}^2 \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = Q \cdot x_{ji} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$I.E^e = A \cdot N^T \cdot \frac{dT}{dx} \Big|_{x_i}^{x_j} = A \frac{1}{x_{ji}} \begin{bmatrix} x_j - x \\ x - x_i \end{bmatrix} \frac{dT}{dx} \Big|_{x_i}^{x_j}$$

$$= A \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{dT_j^e}{dx} - A \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{dT_i^e}{dx}$$

$$I.E^e = A \cdot \begin{bmatrix} \frac{dT_i^e}{dx} \\ \frac{dT_j^e}{dx} \end{bmatrix}$$

$$\frac{d^2T}{dx^2} - 6T + 10 = 0 \qquad A = 1, \quad B = 6, \quad Q = 10$$



$T(0) = 0$ $\qquad$ $T(1) = 0$

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$$k_A^e = \frac{1}{1/4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 4\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$k_B^e = 6 * 1/4 * \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$\underline{F}^e = 10 * \frac{1}{4}\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \frac{5}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$I.E^e = \begin{bmatrix} -\frac{dT_i^e}{dx} \\ \frac{dT_j^e}{dx} \end{bmatrix}$$

$$\underline{k}_{total}^e = \underline{k}_A^e + \underline{k}_B^e = \begin{bmatrix} 4\frac{1}{2} & -3\frac{3}{4} \\ -3\frac{3}{4} & 4\frac{1}{2} \end{bmatrix}$$

$$\sum_{e=1}^{4}\left(\underline{k}_{total}^e \cdot \underline{a}^e = \underline{F}^e + I.E^e\right)$$

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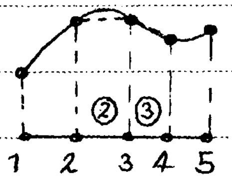
$$\begin{bmatrix} 4\frac{1}{2} & -3\frac{3}{4} & 0 & 0 & 0 \\ -3\frac{3}{4} & 9 & -3\frac{3}{4} & 0 & 0 \\ 0 & -3\frac{3}{4} & 9 & -3\frac{3}{4} & 0 \\ 0 & 0 & -3\frac{3}{4} & 9 & -3\frac{3}{4} \\ 0 & 0 & 0 & -3\frac{3}{4} & 4\frac{1}{2} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2^{1} \\ T_3^{2} \\ T_4^{3} \\ T_5 \end{bmatrix} = \frac{5}{4} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{dT_1^{(1)}}{dx} \\ \frac{dT_2^{(1)}}{dx} \quad \frac{dT_2^{(2)}}{dx} \\ \frac{dT_3^{(2)}}{dx} \quad \frac{dT_3^{(3)}}{dx} \\ \frac{dT_4^{(3)}}{dx} \quad \frac{dT_4^{(4)}}{dx} \\ \frac{dT_5^{(4)}}{dx} \end{bmatrix}$$

5 معادله، 5 مجهول!



اصلاً $\frac{dT_3^{(2)}}{dx} - \frac{dT_3^{(3)}}{dx} \neq 0$

امّا ما این دو ترم را مثل عمل و علی‌العمل گرفته و

حاصل عبارت فوق را صفر فرض می‌کنیم ← خطا ← چون C^0 continuity

داریم.

quadradic element :

$x=0 \quad x=\ell/2 \quad x=\ell$

1 3 2

$\underline{N} = [N_1 \quad N_2 \quad N_3]$

$N_1 = c(x-\ell)(x-\ell/2) \qquad 1 = c(-\ell)(-\ell/2) \qquad c = \frac{2}{\ell^2}$

$N_1 = \frac{2}{\ell^2}(x-\ell)(x-\ell/2)$

$N_2 = c(x)(x-\ell/2) \qquad 1 = c(\ell)(\ell/2) \qquad c = \frac{2}{\ell^2}$

$N_2 = \frac{2}{\ell^2}(x)(x-\ell/2)$

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$N_3 = C(x)(x-l) \qquad 1 = C(l/2)(-l/2) \qquad C = -\frac{4}{l^2}$

$N_3 = -\frac{4}{l^2}(x)(x-l)$

→ 2 Dimensional field problem
→ Beam Analysis → Dynamic case / Euler Beam
→ Timoshenko Beam
→ Beam Analysis with Displacement D.O.F only
→ Iso / sub / super parametric Analysis
→ Variational method

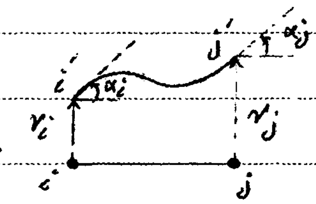
Timoshenko beam → Thick beam → shear Effect

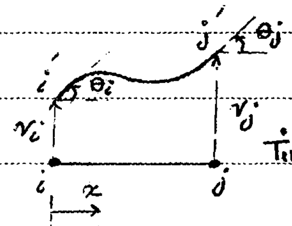
$k: \quad [k_b] , [k_s]$

"prevent" ← shear locking ، اگر k_s بر k_b غالب باشد
(k_b: bending, k_s: shear)

Method of solution:
Energy Method

$U = U_b + U_s$

Euler 

 Timoshenko

$\alpha_i = \left.\frac{dv(x)}{dx}\right|_{x_i}$

$\overrightarrow{MM'} = \begin{vmatrix} u(x,y) = -y \cdot \theta(x) \\ v(x) \end{vmatrix}$

$MM' = \begin{vmatrix} u(x,y) = -y\frac{dv(x)}{dx} \\ v(x) \end{vmatrix}$

PAPCO

$$\varepsilon_x = \frac{\partial u}{\partial x} = -y\,\frac{d\theta(x)}{dx}$$

$$\varepsilon_y = \frac{\partial v(x)}{\partial y} = 0$$

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\theta(x) + \frac{dv(x)}{dx} \neq 0$$

k_b منبع ←

$$\varepsilon = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & 0 \\ \varepsilon_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

k_s منبع ←

$$\sigma_b = E \cdot \varepsilon_b$$

$$\sigma_s = G \cdot \varepsilon_{xy} = G \cdot \varepsilon_s$$

از توابع شکل خطی استفاده می‌کنیم ← طول المان باید کوتاه باشد ← تعداد المان‌ها زیاد می‌شود.

$$v^e(x) = \begin{bmatrix} (1 - x/l) & (x/l) \end{bmatrix} \begin{bmatrix} v_i \\ v_j \end{bmatrix}$$

$$\theta^e(x) = \begin{bmatrix} (1 - x/l) & (x/l) \end{bmatrix} \begin{bmatrix} \theta_i \\ \theta_j \end{bmatrix}$$

$$\rightarrow \varepsilon_x = -y \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix} \begin{bmatrix} \theta_i \\ \theta_j \end{bmatrix} = -y \begin{bmatrix} 0 & -1/l & 0 & 1/l \end{bmatrix} \begin{bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{bmatrix} = \underline{B_b} \cdot \underline{a}$$

ε_x مستقیماً به شیب ربط پیدا کرده و مستقل از مقادیر خیز است ← خطا!

$$\rightarrow \varepsilon_{xy} = -\begin{bmatrix} N_i(x) & N_j(x) \end{bmatrix} \begin{bmatrix} \theta_i \\ \theta_j \end{bmatrix} + \begin{bmatrix} -1/l & +\frac{1}{l} \end{bmatrix} \begin{bmatrix} v_i \\ v_j \end{bmatrix}$$

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$$\varepsilon_{xy} = \underline{B}_s \cdot \underline{a} = \left[-\tfrac{1}{l} \quad -(1-\tfrac{x}{l}) \quad \tfrac{1}{l} \quad -\tfrac{x}{l}\right] \begin{bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{bmatrix}$$

$$U_b = \frac{1}{2} \int_{\Omega^e} \sigma_b^T \cdot \varepsilon_b \cdot d\Omega^e = \frac{1}{2} \int_{\Omega^e} \underline{a}^T \cdot \underline{B}_b^T \cdot E \cdot \underline{B}_b \cdot \underline{a} \cdot d\Omega$$

$$\varepsilon_b = \underline{B}_b \cdot \underline{a} \quad , \quad \sigma_b = E \cdot \varepsilon_b = E \cdot \underline{B}_b \cdot \underline{a}$$

$$U_b = \frac{1}{2} \cdot a^T \Big(\underbrace{\int_{\Omega^e} \underline{B}_b^T \cdot E \cdot \underline{B}_b \cdot d\Omega}_{K_b} \Big) \underline{a}$$

$$\underline{K}_b = \int_0^l \int_A -y \begin{bmatrix} 0 \\ -1/l \\ 0 \\ 1/l \end{bmatrix} \cdot E \cdot (-y) \left[0 \quad -1/l \quad 0 \quad 1/l\right] dA\, dx \qquad \left(\int_A y^2 dA = I\right)$$

$$K_b = E \cdot I \int_0^l \begin{bmatrix} 0 \\ -1/l \\ 0 \\ 1/l \end{bmatrix} \left[0 \quad -1/l \quad 0 \quad 1/l\right] dx$$

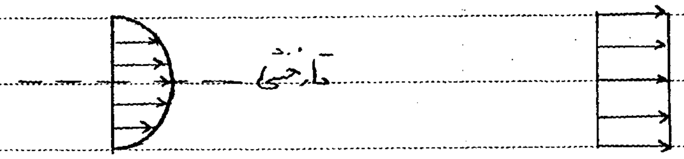
$$K_b = E \cdot I \int_0^l \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/l^2 & 0 & -1/l^2 \\ 0 & 0 & 0 & 0 \\ 0 & -1/l^2 & 0 & 1/l^2 \end{bmatrix} dx$$

$$K_b = \frac{EI}{l} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & +1 \end{bmatrix}$$

تمرینی ⑥

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\theta(x) + \frac{dv}{dx} = \begin{bmatrix} -1/l & -(1-x/l) & 1/l & -x/l \end{bmatrix} \begin{bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{bmatrix} = \underline{B}_s \cdot \underline{a}$$

$$\tau_{xy} = G \cdot \varepsilon_{xy} = G \cdot \underline{B}_s \cdot \underline{a}$$



نیروی برشی در واقعیت

نیروی برشی که تا اینجا حاصل شده است.

برای یکسان کردن تأثیر : $k = \frac{5}{6}$ shear stress correction Factor

$$\underbrace{\underline{a}^T \cdot \underline{B}_s^T \cdot G}_{\sigma_s^T}$$

$$u_s = \frac{1}{2} \underbrace{k}_{5/6} \int \sigma_s^T \cdot \underbrace{\varepsilon_s}_{\underline{B}_s \cdot \underline{a}} \cdot d\Omega^e$$

$$u_s = \frac{1}{2} \underline{a}^T \underbrace{\left(k \int_0^l \int_A \underline{B}_s^T \cdot G \cdot \underline{B}_s \cdot \underbrace{d\Omega^e}_{dA \cdot dx} \right)}_{k_{shear}} \underline{a}$$

$$k_{shear} = k \cdot G \cdot A \int_0^l \begin{bmatrix} -1/l \\ -(1-x/l) \\ 1/l \\ -x/l \end{bmatrix} \begin{bmatrix} -1/l & -(1-x/l) & 1/l & -x/l \end{bmatrix} \cdot dx$$

$$k_{shear} = \frac{5}{6} AG \int_0^l \begin{bmatrix} 1/l^2 & 1/l(1-x/l) & -1/l^2 & x/l^2 \\ 1/l(1-x/l) & (1-x/l)^2 & -1/l(1-x/l) & x/l(1-x/l) \\ -1/l^2 & -1/l(1-x/l) & 1/l^2 & -x/l^2 \\ x/l^2 & x/l(1-x/l) & -x/l^2 & (x/l)^2 \end{bmatrix} \cdot dx$$

Subject
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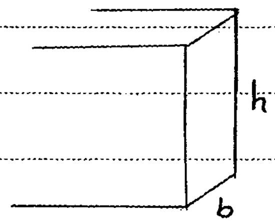
shear locking prevention : $k_{22}|_b \gg k_{22}|_{shear}$

$$k_{22}|_{shear} = \frac{5}{6}\cdot A\cdot G\cdot \int_0^l (1-x/l)^2\, dx = \frac{5}{6}\, A\cdot G \underbrace{\int_0^1 (1-u)^2\, l\, du}_{-\frac{1}{3}(1-u)^3\big|_0^1} = \frac{5}{6} A\cdot G\cdot l\cdot\frac{1}{3}$$

$$\frac{EI}{l} \gg \frac{5}{6}\cdot A\cdot G\cdot l\cdot \frac{1}{3}$$



$I = \frac{1}{12}\, bh^3$

$$E\cdot\frac{1}{12}\, bh^3 \gg \frac{5}{18}\, A\, \frac{E}{2(1+\nu)}\, l^2$$

$$\longrightarrow\ l^2 \ll \frac{3}{5}(1+\underbrace{\nu}_{0.3})\, h^2 \quad\longrightarrow\ \underbrace{l}_{\text{طول المان}} \ll \sqrt{0.78}\; h$$

برای K_{44} نیز بصورت مشابه می باشد.

"Reduced Numerical Integration Technique"

PaPCO

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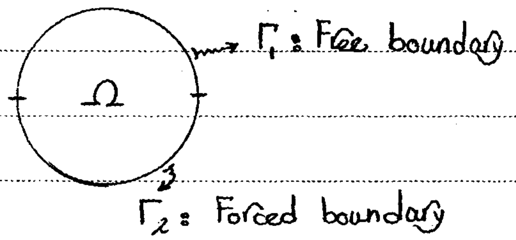
2 Dimensional Field Problems :

- heat Transfer
  - 2D
    - Conduction + Convection
- Solid Mechanics
  - Torsion of Non circular shafts
- Fluid Mechanics
  - irrotational Fluid Flow

$\varphi(x, y)$

$$D_x \frac{\partial^2 \varphi}{\partial x^2} + D_y \frac{\partial^2 \varphi}{\partial y^2} - G\varphi + Q = 0$$

که ظاهرش field نیز partial drivative مطابقت دارد.



← المان سه گرهی یا المان چهار گرهی

$$\varphi^e = \underline{N} \cdot \underline{a}$$

المان سه گرهی :
$$\varphi^e(x, y) = \begin{bmatrix} N_i & N_j & N_k \end{bmatrix} \begin{bmatrix} \varphi_i \\ \varphi_j \\ \varphi_k \end{bmatrix}$$

$$N_i = L_1 = \frac{1}{2A}(a_i + b_i x + c_i y)$$

$$D_x \cdot \frac{\partial^2 \varphi^e}{\partial x^2} + D_y \frac{\partial^2 \varphi^e}{\partial y^2} - G\varphi^e + Q = R(x, y, \underline{a}) \neq 0$$

D.E $\longrightarrow$ Integral Equation $\longrightarrow \int_{\Omega^e} \underline{N}^T \cdot R(x, y, \underline{a})\, dA = 0$

Integral Equation → Golerkin → $N_i = w_i$, $N_k = w_k$, $N_j = w_j$

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$$\int_{A^e} N^T \left(D_x \frac{\partial^2 \varphi^e}{\partial x^2} + D_y \frac{\partial^2 \varphi^e}{\partial y^2} - G\varphi^e + Q \right) dA = 0$$

توابع فرمی با بعد نظیر ← C^0 continuity

$$D_x \int_{A^e} N^T \frac{\partial^2 \varphi^e}{\partial x^2}\, dA = D_x \int_{A^e} \frac{\partial}{\partial x}\left(N^T \cdot \frac{\partial \varphi}{\partial x}\right) dA - D_x \int_{A^e} \frac{\partial N^T}{\partial x} \frac{\partial \varphi}{\partial x}\, dA$$

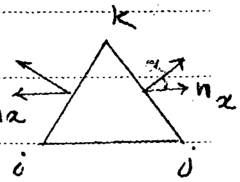
چراکه:

$$\frac{\partial}{\partial x}\left(N^T \cdot \frac{\partial \varphi}{\partial x}\right) - \frac{\partial N^T}{\partial x}\frac{\partial \varphi}{\partial x} = N^T \frac{\partial^2 \varphi}{\partial x^2}$$

Force

مطابق تقسیم گاوس:

1. $D_x \int_{A^e} N^T \frac{\partial^2 \varphi^e}{\partial x^2} dA =$



$$= D_x \int_{\Gamma^e} \underset{\cos\alpha}{\underset{\downarrow}{n_x}} \cdot N^T \frac{\partial \varphi}{\partial x}\, d\Gamma - D_x \Big(\underbrace{\int_{A^e} \frac{\partial N^T}{\partial x} \cdot \frac{\partial N}{\partial x} \cdot dA}_{\underline{k}_{D_x}} \Big) \underbrace{\underline{\alpha}^e}_{\text{Displacement}}$$

2. $$D_y \int_{A^e} \underline{N}^T \frac{\partial^2 \varphi^e}{\partial y^2}\, dA = D_y \int_{\Gamma^e} n_y\, N^T \frac{\partial \varphi}{\partial y}\, d\Gamma - D_y \Big(\underbrace{\int_{A^e} \frac{\partial N^T}{\partial y} \frac{\partial N}{\partial y}\, dA}_{\underline{k}_{D_y}} \Big) \underline{\alpha}^e$$

3. $$-G \int_{A^e} \underline{N}^T \underset{\underline{N} \cdot \underline{\alpha}}{\underset{\downarrow}{\varphi^e}}\, dA = -G \Big(\underbrace{\int_{A^e} N^T \cdot N \cdot dA}_{\underline{k}_G} \Big) \underline{\alpha}^e$$

4. $$\int_{A^e} N^T \cdot Q \cdot dA = \underline{F}^e \quad \text{equivalent force}$$

$$\longrightarrow \Big(\underbrace{D_x \int_{A^e} \frac{\partial N^T}{\partial x} \frac{\partial N}{\partial x} dA}_{\underline{k}_{D_x}} + \underbrace{D_y \int_{A^e} \frac{\partial N^T}{\partial y}\frac{\partial N}{\partial y} dA}_{\underline{k}_{D_y}} + \underbrace{G \int_{A^e} N^T N\, dA}_{\underline{k}_G} \Big) \underline{\alpha}^e$$

$$= \underbrace{\int_{A^e} \underline{N}^T \cdot Q \cdot dA}_{\underline{F}^e} + \underbrace{D_x \int_{\Gamma^e} n_x \cdot N^T \frac{\partial \varphi}{\partial x}\, d\Gamma + D_y \int_{\Gamma^e} n_y \cdot N^T \frac{\partial \varphi}{\partial y}\, d\Gamma}_{I.E^e}$$

$I.E^e$: inter Element Terms

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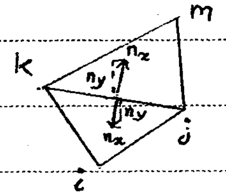

I.E : را صفر فرض می‌کنیم
و فقط برای المان‌هایی که حداقل یک ضلعشان روی
forced boundary قرار داشته باشد، آن را در نظر می‌گیریم

n_x , n_y برای دو المان مجاور هم
عکس شده است.

$$\underline{k}_{D_x} = D_x \int_{A^e} \frac{\partial N^T}{\partial x} \frac{\partial N}{\partial x} \, dA$$

$$\frac{\partial N}{\partial x} \rightarrow \frac{1}{2A} \begin{bmatrix} b_i & b_j & b_k \end{bmatrix}$$

$$\underline{N} = \begin{bmatrix} N_i & N_j & N_k \end{bmatrix}$$

$$N_i = \frac{1}{2A^e}(a_i + b_i x + c_i y)$$

$$\underline{k}_{D_x} = D_x \cdot \int_{A^e} \frac{1}{2A} \begin{bmatrix} b_i \\ b_j \\ b_k \end{bmatrix} \frac{1}{2A} \begin{bmatrix} b_i & b_j & b_k \end{bmatrix} dA$$

$$\underline{k}_{D_x} = \frac{D_x}{4A} \begin{bmatrix} b_i^2 & b_i b_j & b_i b_k \\ b_j b_i & b_j^2 & b_j b_k \\ b_k b_i & b_k b_j & b_k^2 \end{bmatrix}$$

$$\underline{k}_{D_y} = D_y \cdot \int_{A^e} \frac{\partial N^T}{\partial y} \frac{\partial N}{\partial y} \, dA \longrightarrow$$ مشابه k_{D_x} فقط به جای b ، c ظاهر می‌شود.

$$\underline{k}_G = G \int_{A^e} \begin{bmatrix} \overset{L_1}{\cancel{N_i}} \\ \overset{L_2}{\cancel{N_j}} \\ \overset{L_3}{\cancel{N_k}} \end{bmatrix} \begin{bmatrix} \overset{L_1}{\cancel{N_i}} & \overset{L_2}{\cancel{N_j}} & \overset{L_3}{\cancel{N_k}} \end{bmatrix} dA$$

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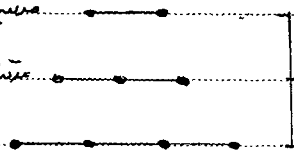
با استفاده از انتگرال زیر :

$$\begin{array}{c} \qquad\qquad\quad A/6 \qquad\quad A/12 \\ k_G = G\int_A \begin{bmatrix} L_1^2 & L_1L_2 & L_1L_3 \\ L_1L_2 & L_2^2 & L_2L_3 \\ L_1L_3 & L_2L_3 & L_3^2 \end{bmatrix} dA \end{array}$$

$$\int_A L_1^{\alpha} L_2^{\beta} L_3^{\gamma}\, dA = \frac{\alpha!\,\beta!\,\gamma!}{(\alpha+\beta+\gamma+2)!}\cdot 2A$$ Area coordinate

مرتبه 1
مرتبه 2
مرتبه 3



هر سه C^0 continuity

$$\int_L L_1^{\alpha} L_2^{\beta}\, dl = \frac{\alpha!\,\beta!}{(\alpha+\beta+1)!}\, L$$

Length coord

هرچه مرتبه بالاتر ← تعداد المان ها کمتر ← نرخ همگرایی بیشتر

$$\longrightarrow \underline{k}_G = \frac{AG}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\underline{F}^e = \int_A \underbrace{\underline{N}^T}_{\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}} \cdot Q \cdot dA = \frac{Q \cdot A}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \longrightarrow$$

در واقع N روی مرکز سطح عمل کرده است.
یعنی جایی که : $L_1 = L_2 = L_3 = \frac{1}{3}$

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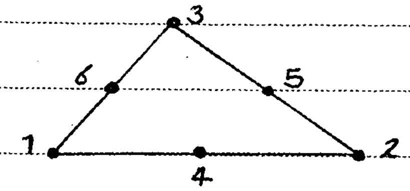
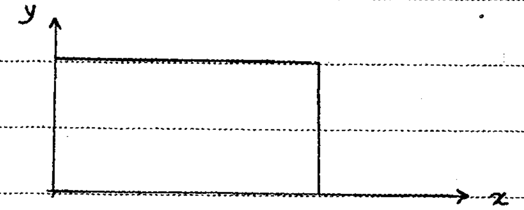
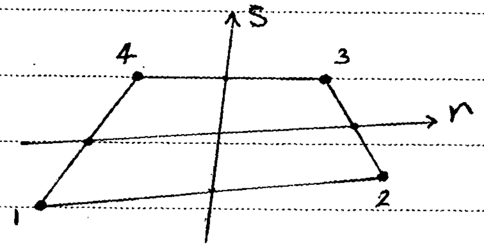


$N_1 \longrightarrow N_6 \ (L_1, L_2)$

$L_3 = 1 - L_2 - L_3$



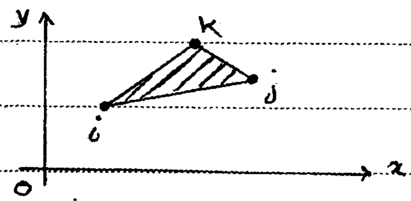


$N_1 \longrightarrow N_4 \ (r, s)$

$\underline{N}$: physical shape function matrix

$x, y \longrightarrow$ Geometry

$\underline{N'}$: Geometrical shape function matrix



$$x = \sum_{i=1}^{3} N'_i(x, y)\, x_i$$

$x = N'_1 x_1 + N'_2 x_2 + N'_3 x_3$ $\qquad$ $y = N'_1 y_1 + N'_2 y_2 + N'_3 y_3$

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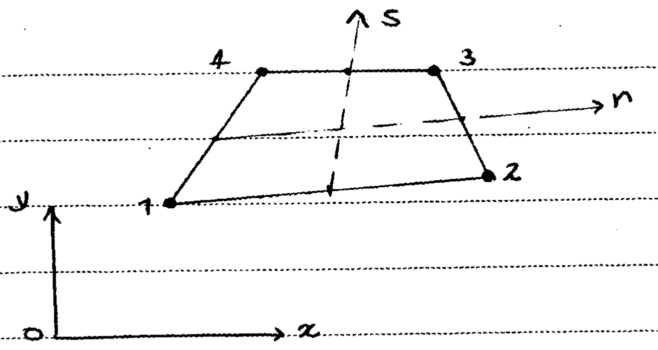


$$x(r,s) = \sum_{i=1}^{k} N'_i(r,s)\, x_i = N'_1 x_1 + N'_2 x_2 + N'_3 x_3 + N'_4 x_4$$

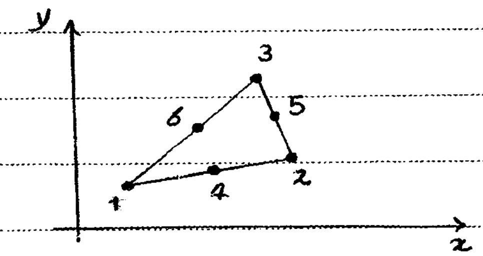
$$y(r,s) = \sum_{i=1}^{4} N_i(r,s)\, y_i$$

$$N'_1(r,s) = \frac{1}{4}(1-r)(1-s)$$

$$N'_2(r,s) = \frac{1}{4}(1+r)(1-s)$$

$$N'_3(r,s) = \frac{1}{4}(1+r)(1+s)$$

$$N'_4(r,s) = \frac{1}{4}(1-r)(1+s)$$



بعداز تغییر شکل ، قیدی مبنی بر اینکه گره‌های 2 ، 5 ، 3
در یک خط باقی بمانند ، وجود ندارد.

$$x = \sum_{i=1}^{6} N'_i \cdot x_i$$

$$y = \sum_{i=1}^{6} N'_i \cdot y_i$$

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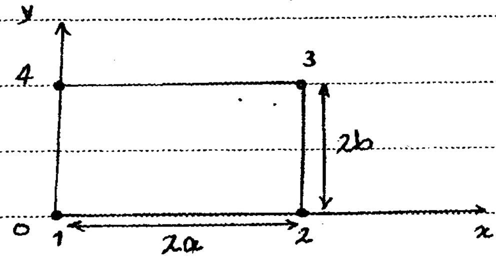
Quadrilateral elements :



! parent حالت
↳ undeforme

$$\varphi(x,y) = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 xy$$

bi_linear rectangular element → نسبت به x , y خطی

$1\left|\begin{matrix}0\\0\end{matrix}\right.$ $\quad 2\left|\begin{matrix}2a\\0\end{matrix}\right.$ $\quad 3\left|\begin{matrix}2a\\2b\end{matrix}\right.$ $\quad 4\left|\begin{matrix}0\\2b\end{matrix}\right.$

$$\varphi_1 = \alpha_0 \quad , \quad \varphi_2 = \varphi_1 + \alpha_1(2a) \longrightarrow \alpha_1 = \frac{1}{2a}(\varphi_2 - \varphi_1)$$

$$\varphi_4 = \varphi_1 + \alpha_2(2b) \longrightarrow \alpha_2 = \frac{1}{2b}(\varphi_4 - \varphi_1).$$

$$\varphi_3 = \varphi_1 + (\varphi_2 - \varphi_1) + (\varphi_4 - \varphi_1) + \alpha_3 \cdot 4ab$$

$$\alpha_3 = \frac{1}{4ab}(\varphi_3 + \varphi_1 - \varphi_2 - \varphi_4)$$

$$\varphi(x,y) = \begin{bmatrix} 1 & x & y & xy \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{-1}{2a} & \frac{1}{2a} & 0 & 0 \\ -\frac{1}{2b} & 0 & 0 & \frac{1}{2b} \\ \frac{1}{4ab} & \frac{-1}{4ab} & \frac{1}{4ab} & \frac{-1}{4ab} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix}$$

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$$\varphi(x,y) = \begin{bmatrix} N_1(x,y) & N_2(x,y) & N_3(x,y) & N_4(x,y) \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix}$$

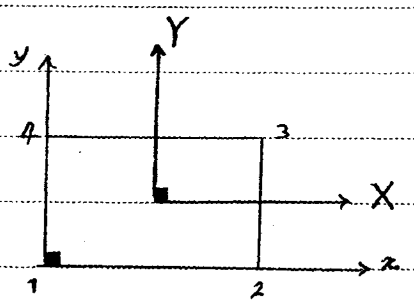
$$N_1(x,y) = 1 - \frac{x}{2a} - \frac{y}{2b} + \frac{x}{2a} \cdot \frac{y}{2b}$$

$$N_1(x,y) = \left(1 - \frac{x}{2a}\right)\left(1 - \frac{y}{2b}\right)$$

$$N_2(x,y) = \frac{x}{2a}\left(1 - \frac{y}{2b}\right)$$

$$N_3(x,y) = \frac{x}{2a} \cdot \frac{y}{2b}$$

$$N_4(x,y) = \frac{y}{2b}\left(1 - \frac{x}{2a}\right)$$



$$x = a + X \qquad \frac{x}{2a} = \frac{1}{2} + \frac{X}{2a} \qquad \frac{x}{2a} = \frac{1}{2}\left(1 + \frac{X}{a}\right) \qquad 1 - \frac{x}{2a} = \frac{1}{2}\left(1 - \frac{X}{a}\right)$$

$$y = b + Y \qquad \frac{y}{2b} = \frac{1}{2}\left(1 + \frac{Y}{b}\right) \qquad 1 - \frac{y}{2b} = \frac{1}{2}\left(1 - \frac{Y}{b}\right)$$

$$N_1(x,y) = \frac{1}{4}\left(1 - \frac{X}{a}\right)\left(1 - \frac{Y}{b}\right)$$

$$N_2(x,y) = \frac{1}{4}\left(1 + \frac{X}{a}\right)\left(1 - \frac{Y}{b}\right)$$

$$N_3(x,y) = \frac{1}{4}\left(1 + \frac{X}{a}\right)\left(1 + \frac{Y}{b}\right)$$

$$N_4(x,y) = \frac{1}{4}\left(1 - \frac{X}{a}\right)\left(1 + \frac{Y}{b}\right)$$

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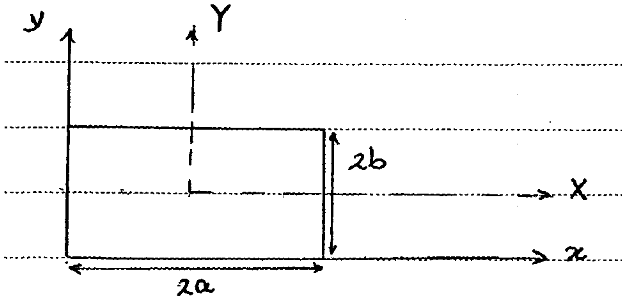


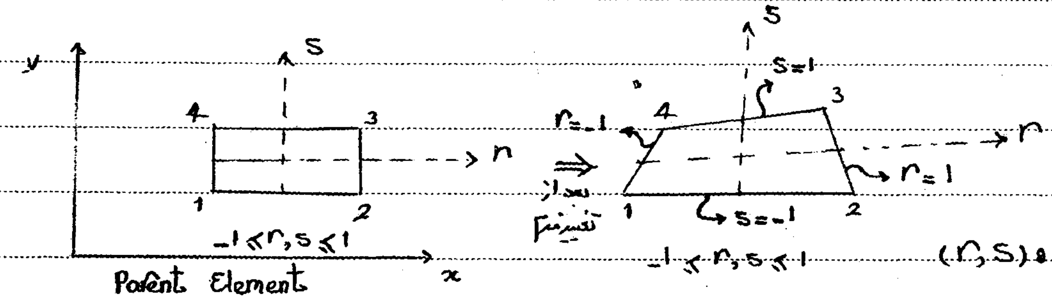
$$-1 \leqslant \underbrace{\frac{X}{a}}_{r} \leqslant 1 \quad , \quad -1 \leqslant \underbrace{\frac{Y}{b}}_{s} \leqslant 1$$

$$N_1(r,s) = \frac{1}{4}(1-r)(1-s)$$

$$N_2(r,s) = \frac{1}{4}(1+r)(1-s)$$

$$N_3(r,s) = \frac{1}{4}(1+r)(1+s)$$

$$N_4(r,s) = \frac{1}{4}(1-r)(1+s)$$



(r,s): Natural Coord

در فضا رابطه یک به یک با یکدیگر دارند، اما dr , ds uniform تبدیل به dr , ds ununiform

شده‌اند. برای هر دو حالت: (در تابع شکل المان parent و تغییر شکل یافته در مختصات طبیعی (r,s) تفاوتی وجود ندارد.)

$$N_1(r,s) = C(1-r)(1-s) \qquad 1 = C(2)(2) \longrightarrow C = \frac{1}{4}$$

در المان‌های چهارضلعی، مختصات کارتزین فقط در المان parent ظاهر می‌شود.

در مسائلی که Displacement field مورد تحلیل است، Mesh refinement اهمیت دارد،

در مسائل انتقال حرارت اهمیتی ندارد.

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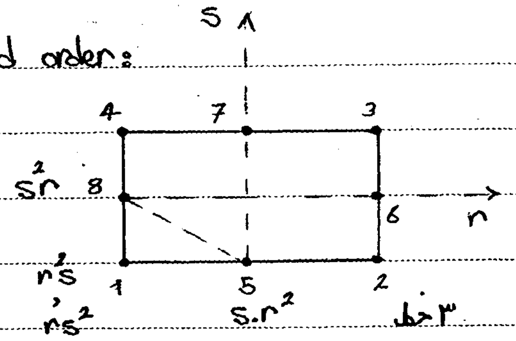
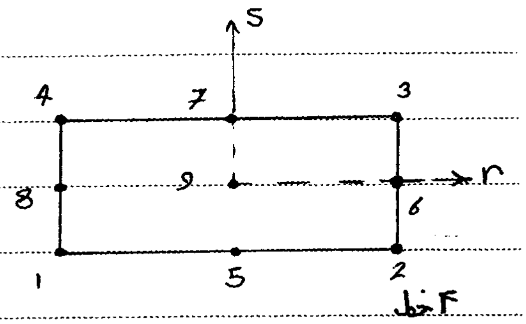
second order:





له بعد از تغییر نرم ← هر ضلع اجزای مرتبه 2 — المان لاگرانژ چهار وجهی مرتبه 2

$N_1 = (1-r)(1-s)(c_0 + c_1 r + c_2 s)$

$0 = 1 * 2 * (c_0 - c_2) \longrightarrow c_0 = c_2$

$0 = 2 * 1 * (c_0 - c_1) \longrightarrow c_0 = c_1$

$\longrightarrow N_1 = c(1-r)(1-s)(1+r+s)$

$1 = c(2)(2)(1-1-1) \longrightarrow c = \frac{-1}{4}$

$N_1 = -\frac{1}{4}(1-r)(1-s)(1+r+s)$

$N_5 = c(1-r^2)(1-s)$

$N_6 = c(1-s^2)(1-r)$

$\varphi(r,s) = c_0 + c_1 r + c_2 s + c_3 rs$ برای مرتبه 1

$\varphi(r,s) = c_0 + c_1 r + c_2 s + c_3 rs + c_4 r^2 + c_5 s^2 + c_6 r^2 s + c_7 r s^2$ برای مرتبه 2

$\varphi(r,s) = c_0 + c_1 r + c_2 s + c_3 rs + c_4 r^2 + c_5 s^2 + c_6 r^2 s + c_7 r s^2 + c_8 r^2 s^2$ برای مرتبه 2 لاگرانژی

$N_1(r,s) = c(r)(1-r)(s)(1-s) = c.r.s(1-s)(1-r)$ $\qquad c_8 \neq 0$

$1 = c(-1)(-1)(2)(2) \longrightarrow c = 1/4$

$N_5(r,s) = c(1-r^2)(s)(1-s) \longrightarrow 1 = c(1)(-1)(2) \longrightarrow c = -\frac{1}{2}$

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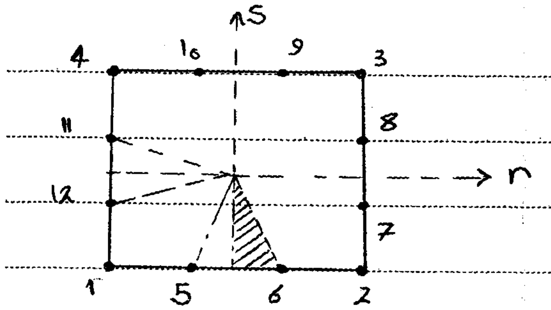
مرتبه 3 :



$5\left|\begin{matrix}-\frac{1}{3}\\ -1\end{matrix}\right.$ $\quad 6\left|\begin{matrix}\frac{1}{3}\\ -1\end{matrix}\right.$ $\quad 11\left|\begin{matrix}-1\\ +\frac{1}{3}\end{matrix}\right.$ $\quad 12\left|\begin{matrix}-1\\ -\frac{1}{3}\end{matrix}\right.$

برای گره 5، 6، 11، 12، 8، 7، 10، 9 : $r^2+s^2=\frac{1}{9}+1=\frac{10}{9}$

$$r^2+s^2-\frac{10}{9}=0$$

$$N_1=C(1-r)(1-s)\left(r^2+s^2-\frac{10}{9}\right)$$

جمع توان ها 4 می شود، ولی ترم r^2s^2 را نداریم ←

$$N_2=C(1+r)(1-s)\left(r^2+s^2-\frac{10}{9}\right)$$

در واقع r^2s^2 مخصوص المان لاگرانژ چهار گرهی مرتبه 2 است.

$$\varphi(r,s)=C_0+C_1r+C_2s+C_3rs+C_4r^2+C_5s^2+C_6r^2s+C_7rs^2+C_9r^3+C_{10}s^3+C_{11}r^3s+C_{12}rs^3$$

$C_{11}\neq 0$ ← برای 5، 6، 9، 10 گره های گوشه ای

$C_{12}\neq 0$ ← برای 7، 8، 11، 12 گره های گوشه ای

$$N_5=C(1-r^2)(1-s)\left(r-\frac{1}{3}\right)$$

$$N_{11}=C(1-s^2)(1-r)\left(s+\frac{1}{3}\right)$$

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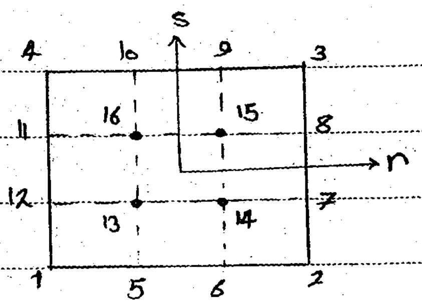


مرتبه 3 لاگرانژی :

$N_1 = c(1-r)(1-s)(\frac{1}{9}-r^2)(\frac{1}{9}-s^2)$ علاوه بر r^2s^2 ، r^3s^3 اینم دارد.

$N_{16} = c(1-r^2)(r-\frac{1}{3})(1-s^2)(s+\frac{1}{3})$

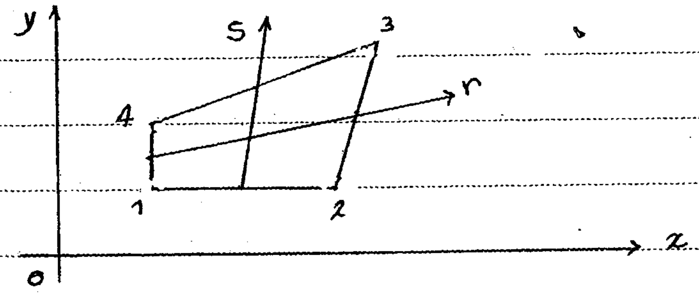
$14 \left| \begin{matrix} r = \frac{1}{3} \\ s = -\frac{1}{3} \end{matrix} \right.$ $\quad 15 \left| \begin{matrix} r = \frac{1}{3} \\ s = \frac{1}{3} \end{matrix} \right.$ $\quad 16 \left| \begin{matrix} r = -\frac{1}{3} \\ s = \frac{1}{3} \end{matrix} \right.$ $\quad 13 \left| \begin{matrix} r = \frac{1}{3} \\ s = -\frac{1}{3} \end{matrix} \right.$

3rd Lagrange Quadrilateral Elements :

$$\varphi(r,s) = c_0 + c_1 r + c_2 s + c_3 rs + c_4 r^2 + c_5 s^2 + c_6 r^2 s + c_7 r s^2 + c_9 r^3 + c_{10} s^3 - c_{11} r^3 s + c_{12} r s^3 + c_{13} r^2 s^2 + c_{14} r^2 s^3 + c_{15} r^3 s^2 + c_{16} r^3 s^3$$

Iso / sub / super parametric Analysis :

Geometry $\downarrow$ N' physic $\downarrow$ N



$1 \left| \begin{matrix} 1 \\ 1 \end{matrix} \right.$ $\quad 2 \left| \begin{matrix} 2 \\ 1 \end{matrix} \right.$ $\quad 3 \left| \begin{matrix} 3 \\ 4 \end{matrix} \right.$ $\quad 4 \left| \begin{matrix} 1 \\ 2 \end{matrix} \right.$

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$$N_1 = \frac{1}{4}(1-r)(1-s)$$

$$N_2 = \frac{1}{4}(1+r)(1-s)$$

$$N_3 = \frac{1}{4}(1+r)(1+s)$$

$$N_4 = \frac{1}{4}(1-r)(1+s)$$

: Geometrical Node

$$x = x(r,s) = \sum_{i=1}^{4} N_i'(r,s)\,X_i = N_1 X_1 + N_2 X_2 + N_3 X_3 + N_4 X_4$$

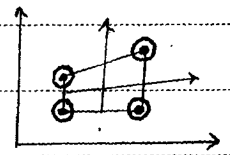
$$y = y(r,s) = \sum_{i=1}^{4} N_i' Y_i$$

Geometrical Node → N' گره‌های توخالی o

physical Node → N گره‌های توپر •

اگر N و N' باهم برابر باشد، کلاس فیزیک مسأله و

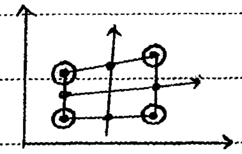
کلاس هندسه مسأله باهم یکی است: ← Iso parametric



$N' = N$ امت N، N' یکی است.

اگر تعداد گره‌های فیزیکی بیشتر از گره‌های هندسی باشد، ارزش

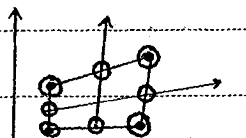
فیزیک مسأله بیشتر است: ← sub parametric



$N' < N$ امت N از N' بیشتر است.

اگر تعداد گره‌های هندسی بیشتر از گره‌های فیزیکی باشد، ارزش

هندسی مسأله بیشتر است: ← super parametric



$N' > N$ امت N' از N بیشتر است.

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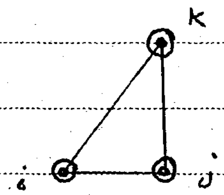
$$D_x \frac{\partial^2 \varphi}{\partial x^2} + D_y \frac{\partial^2 \varphi}{\partial y^2} - G\varphi + Q = 0$$

$\varphi(x, y)$: physical parameter

$$\underline{k}_{D_x} = D_x \int_A \frac{\partial \underline{N}^T}{\partial x} \frac{\partial \underline{N}}{\partial x} \, dA$$

$$\underline{k}_{D_y} = D_y \int_A \frac{\partial \underline{N}^T}{\partial y} \frac{\partial \underline{N}}{\partial y} \, dA$$

$$\underline{k}_G = G \int_A \underline{N}^T \cdot \underline{N} \, dA$$



در تحلیل های فوق از المان مثلثی مرتبه اول Iso parametric استفاده شده است.

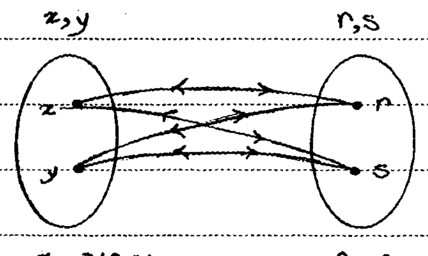
$$N_i = \frac{1}{2A}(a_i + b_i x + c_i y)$$

1) Quadrilateral Elements :

$N = N(r, s)$

$x = x(r, s)$

$y = y(r, s)$ → بین مختصات Natural و کارتزین رابطه یک به یک وجود دارد.



$x = x(r,s)$ $\quad$ $r = r(x,y)$

$y = y(r,s)$ $\quad$ $s = s(x,y)$

$$\frac{\partial \underline{N}}{\partial r} = \frac{\partial \underline{N}}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial \underline{N}}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial \underline{N}}{\partial s} = \frac{\partial \underline{N}}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial \underline{N}}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\begin{bmatrix} \dfrac{\partial \underline{N}}{\partial r} \\[2ex] \dfrac{\partial \underline{N}}{\partial s} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\[2ex] \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \underline{N}}{\partial x} \\[2ex] \dfrac{\partial \underline{N}}{\partial y} \end{bmatrix}$$

فیزیکی　　　　　　　　　　فیزیکی

$\downarrow$

$\underline{J}$ = Jacobian matrix

به تعداد گره‌های تداخلی

$$x = x(r,s) = \sum_{i=1}^{n} N_i' \cdot X_i$$

$$y = y(r,s) = \sum_{i=1}^{n} N_i' \cdot Y_i$$

$$J_{11} = \frac{\partial x}{\partial r} = \sum_{i=1}^{n} \frac{\partial N_i'}{\partial r} \cdot X_i$$

$$J_{12} = \frac{\partial y}{\partial r} = \sum_{i=1}^{n} \frac{\partial N_i'}{\partial r} \cdot Y_i$$

$$J_{21} = \frac{\partial x}{\partial s} = \sum_{i=1}^{n} \frac{\partial N_i'}{\partial s} \cdot X_i$$

$$J_{22} = \frac{\partial y}{\partial s} = \sum_{i=1}^{n} \frac{\partial N_i'}{\partial s} \cdot Y_i$$

ماتریس ژاکوبین، ماتریس متقارن نیست!

let $\underline{J}^{-1} = \underline{A}$

$$\longrightarrow \begin{bmatrix} \dfrac{\partial \underline{N}}{\partial x} \\[2ex] \dfrac{\partial \underline{N}}{\partial y} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\[2ex] A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \underline{N}}{\partial r} \\[2ex] \dfrac{\partial \underline{N}}{\partial s} \end{bmatrix}$$

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$$\frac{\partial \underline{N}}{\partial x} = A_{11}\frac{\partial \underline{N}}{\partial r} + A_{12}\frac{\partial \underline{N}}{\partial s} = \underline{c}(r,s)$$

$$\frac{\partial \underline{N}}{\partial y} = A_{21}\frac{\partial \underline{N}}{\partial r} + A_{22}\frac{\partial \underline{N}}{\partial s} = \underline{d}(r,s)$$

$$dA = dx\,.\,dy = \det J\,.\,dr\,ds$$

$$k_{xx} = D_x \int_{-1}^{1}\int_{-1}^{1} \overbrace{\underbrace{\underline{c}^T(r,s)\,.\,\underline{c}(r,s)}_{n\times n}\,.\,\det J}^{\underline{H}(r,s)}\,.\,dr\,.\,ds$$

تعداد گره‌های توپر

با فرض اینکه هر گره تک درجه آزادی داشته باشد.

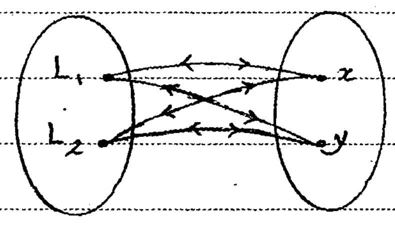
باید از عبارت فوق انتگرال عددی بگیریم.

$$k_{yy} = D_y \int_{-1}^{1}\int_{-1}^{1} \overbrace{\underline{d}^T(r,s)\,.\,\underline{d}(r,s)\,.\,\det J}^{\underline{k}(r,s)}\,.\,dr\,.\,ds$$

$$k_{xy} = \int_A \frac{\partial \underline{N}^T}{\partial x}\frac{\partial \underline{N}}{\partial y}\,dA = \int_{-1}^{1}\int_{-1}^{1} \underline{c}^T(r,s)\,.\,\underline{d}(r,s)\,.\,\det J\,.\,dr\,.\,ds$$

Triangular Elements :

Area Coord : L_1 , L_2 , $L_3 = 1 - L_1 - L_2$



$L_1 = L_1(x,y)$ $\qquad x = x(L_1, L_2)$

$L_2 = L_2(x,y)$ $\qquad y = y(L_1, L_2)$

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$$\underline{N} = \underline{N}(L_1, L_2)$$

$$\frac{\partial \underline{N}}{\partial L_1} = \frac{\partial \underline{N}}{\partial x}\frac{\partial x}{\partial L_1} + \frac{\partial \underline{N}}{\partial y}\frac{\partial y}{\partial L_1}$$

$$\frac{\partial \underline{N}}{\partial L_2} = \frac{\partial \underline{N}}{\partial x}\frac{\partial x}{\partial L_2} + \frac{\partial \underline{N}}{\partial y}\frac{\partial y}{\partial L_2}$$

$$\begin{bmatrix} \frac{\partial \underline{N}}{\partial L_1} \\ \\ \frac{\partial \underline{N}}{\partial L_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial L_1} & \frac{\partial y}{\partial L_1} \\ \\ \frac{\partial x}{\partial L_2} & \frac{\partial y}{\partial L_2} \end{bmatrix} \begin{bmatrix} \frac{\partial \underline{N}}{\partial x} \\ \\ \frac{\partial \underline{N}}{\partial y} \end{bmatrix}$$

$$\downarrow \underline{J}$$

$$\begin{bmatrix} \frac{\partial \underline{N}}{\partial x} \\ \\ \frac{\partial \underline{N}}{\partial y} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial \underline{N}}{\partial L_1} \\ \\ \frac{\partial \underline{N}}{\partial L_2} \end{bmatrix}$$

$$\frac{\partial \underline{N}}{\partial x} = A_{11}\frac{\partial \underline{N}}{\partial L_1} + A_{12}\frac{\partial \underline{N}}{\partial L_2} = \underline{c}(L_1, L_2)$$

$$\frac{\partial \underline{N}}{\partial y} = A_{21}\frac{\partial \underline{N}}{\partial L_1} + A_{22}\frac{\partial \underline{N}}{\partial L_2} = \underline{d}(L_1, L_2)$$

$$k_{xx} = \int_A \frac{\partial \underline{N}^T}{\partial x}\frac{\partial \underline{N}}{\partial x}\, dA \quad \rightarrow dx\,dy$$

$$k_{xx} = \int_0^1 \int_0^{1-L_2} \overbrace{\underline{c}^T(L_1, L_2)\, \underline{c}(L_1, L_2)\, \det \underline{J}}^{\underline{H}(L_1, L_2)}\, dL_1\, dL_2$$

$$k_{xx} = \int_0^1 \int_0^{1-L_2} \underline{H}(L_1, L_2)\, dL_1\, dL_2$$

$$k_{yy} = \int_A \frac{\partial \underline{N}^T}{\partial y}\frac{\partial N}{\partial y}\, dA$$

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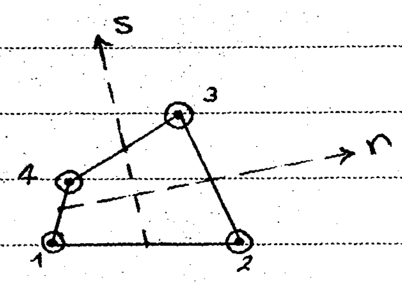
$$k_{yy}=\int_0^1\int_0^{1-L_2}\overbrace{\underline{d}^T(L_1,L_2)\,\underline{d}(L_1,L_2)\,\det J}^{k(L_1,L_2)}\,dL_1\,dL_2$$

$$k_{yy}=\int_0^1\int_0^{1-L_2}\underline{k}(L_1,L_2)\,dL_1\,dL_2$$



$$\underline{J}=\begin{bmatrix}\dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r}\\[2ex] \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s}\end{bmatrix}$$

$$x=\sum_{i=1}^{4}N_i'(r,s)\cdot X_i$$

$$J_{11}=\frac{\partial x}{\partial r}=\sum_{i=1}^{4}\frac{\partial N_i'}{\partial r}X_i$$

$$\underline{N}'=\frac{1}{4}\left[(1-r)(1-s)\;\vdots\;(1+r)(1-s)\;\vdots\;(1+r)(1+s)\;\vdots\;(1-r)(1+s)\right]$$

تا مدعی که با 4 گرهٔ توخالی سر و کار داریم (گره‌های توپر می‌توانند بیشتر باشند)

$$J_{11}=\frac{1}{4}\left[-(1-s)X_1+(1-s)X_2+(1+s)X_3-(1+s)X_4\right]$$

$$J_{11}=\frac{1}{4}\left[(1-s)(X_2-X_1)+(1+s)(X_3-X_4)\right]$$

$$J_{12}=\frac{\partial y}{\partial r}=\frac{1}{4}\left[(1-s)(Y_2-Y_1)+(1+s)(Y_3-Y_4)\right]$$

$$J_{21}=\frac{\partial x}{\partial s}=\frac{1}{4}\left[-(1-r)X_1-(1+r)X_2+(1+r)X_3+(1-r)X_4\right]$$

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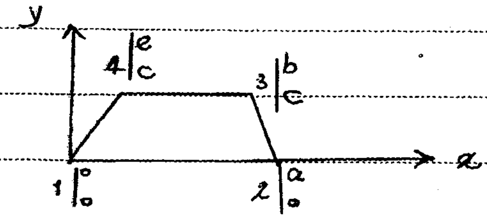
$$J_{21} = \frac{1}{4}\left[(1-r)(X_4 - X_1) + (1+r)(X_3 - X_2)\right]$$

$$J_{22} = \frac{1}{4}\left[(1-r)(Y_4 - Y_1) + (1+r)(Y_3 - Y_2)\right]$$

حالت خاص ذوزنقه



$$J_{11} = \frac{1}{4}\left[a(1-s) + (1+s)(b-c)\right]$$

$$J_{12} = \frac{1}{4}\left[(1-s)(0) + (1+s)(0)\right] = 0$$

$$J_{21} = \frac{1}{4}\left[(1-r)(e) + (1+r)(b-a)\right]$$

$$J_{22} = \frac{1}{4}\left[(1-r)(c) + (1+r)(c)\right] = \frac{c}{2}$$

$J \leftarrow$ ، inverse پذیر است. (مستقیماً)

$$\underline{J} = \begin{bmatrix} f_1(s) & f_2(s) \\ f_3(r) & f_4(r) \end{bmatrix} \qquad J^{-1} = \underline{A}$$

$$\begin{bmatrix} \dfrac{\partial \underline{N}}{\partial x} \\ \dfrac{\partial \underline{N}}{\partial y} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \underline{N}}{\partial r} \\ \dfrac{\partial \underline{N}}{\partial s} \end{bmatrix}$$

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$$k_{xx} = \int_{A^e} \frac{\partial N^T}{\partial x} \frac{\partial N}{\partial x}\, dA$$

$\hookrightarrow (A_{11} \frac{\partial N}{\partial r} + A_{12} \frac{\partial N}{\partial s})$

$\downarrow$

$c(r,s)$ 1^{st} order polynomial Function.

خطی خطی

به تعداد گره‌های [illegible]

4×1 $\quad$ 1×4

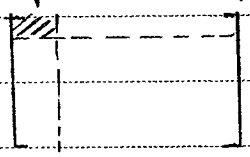
$$\longrightarrow k_{xx} = \int_{-1}^{1}\int_{-1}^{1} \underbrace{\underline{c}^T(r,s)\ \underline{c}(r,s)\ \det J}_{H(r,s)}\, dr\, ds$$

برای المان 4 گرهی : N

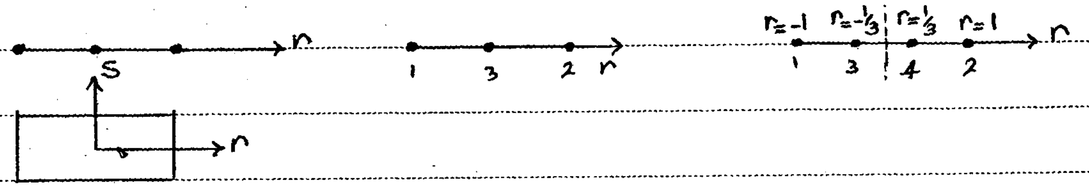
$$N = \begin{cases} \frac{1}{4}(1-r)(1-s) \\ \frac{1}{4}(1+r)(1-s) \\ \frac{1}{4}(1+r)(1+s) \\ \frac{1}{4}(1-r)(1+s) \end{cases}$$

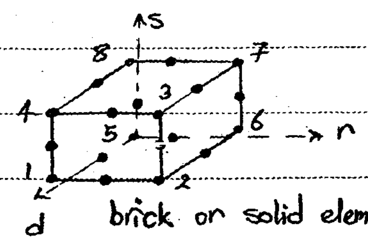
$\rightarrow 2^{nd}$ order polynomial

$H(r,s) =$



Numerical Integration in natural coord :





brick or solid element $\quad$ 1^{st} order $\quad$ 2^{nd} order $\rightarrow$ 20 گره

$-1 \leq (r,s,t) \leq 1 \qquad N_1 = \frac{1}{8}(1-r)(1-s)(1+t) \qquad 1 = c(2)(2)(2) \quad c = \frac{1}{8}$

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$H(r)$: 1D $\quad f(r)\,dr$

$H(r,s)$: 2D $\quad f(r,s)\,dr\,ds$

$H(r,s,t)$: 3D $\quad f(r,s,t)\,dr\,ds\,dt$

$$\int_a^b f(x)\,dx$$

sample points : $a \leq x \leq b$

No. of sample points $= n$

برای هر sample point ، ضریب تأثیر Coefficient of influence در نظر میگیریم:

$n=1 \rightarrow \omega_1$

$n=2 \rightarrow \omega_2$

$n=i \rightarrow \omega_i$

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \omega_i f(x_i)$$

$$\int_{-1}^{1} f(r)\,dr = \sum_{i=1}^{n} \omega_i f(r_i)$$

$f(r)$ be an odd polynomial Function of $(2n-1)$ order.

According to the Gauss_Legendre Theorem: we need n sample points for the Numerical Integration.

$\rightarrow$ and the summation of the whole weighting coefficients is 2.

$$\sum_{i=1}^{n} \omega_i = 2$$

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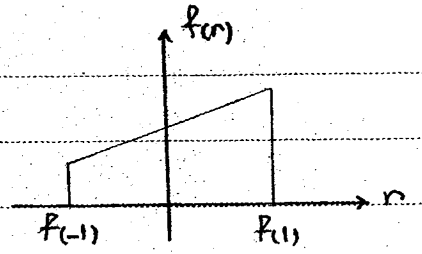

مساحت ذوزنقه: $\underbrace{\frac{f_{(1)}+f_{(-1)}}{2}}_{f_{(0)}} \cdot 2 \longrightarrow \underbrace{2}_{\omega} f_{(0)} \quad : (2n-1=1)$

$2n-1=1 \longrightarrow n=1$

$2n-1=2 \longrightarrow n=1.5 \longrightarrow n=2$ (مثل $2n-1=3$). برای صفتهٔ زوج، یک مرتبه بالاتر می‌رویم

$2n-1=3 \longrightarrow n=2 \longrightarrow r=\pm 0.577 \qquad \omega_1=\omega_2=1$

$2n-1=5 \longrightarrow n=3 \longrightarrow$
$r=0 \; : \; \omega=0.888$
$r=+0.774 \; : \; \omega=0.555$
$r=-0.774 \; : \; \omega=0.555$

$\underline{c}(r,s) \longrightarrow I=\int_{-1}^{1}\int_{-1}^{1} f(\underset{2n-1}{r},\underset{2m-1}{s})\,dr\,ds$

$$I=\int_{-1}^{1}\sum_{i=1}^{n}\omega_i \cdot f(r_i,\underset{2m-1}{s})\,ds$$

$$I=\sum_{j=1}^{m}\omega_j\sum_{i=1}^{n}\omega_i\, f(r_i,s_j)=\sum_{j=1}^{m}\sum_{i=1}^{n}\omega_i\,\omega_j\, f(r_i,s_j)$$

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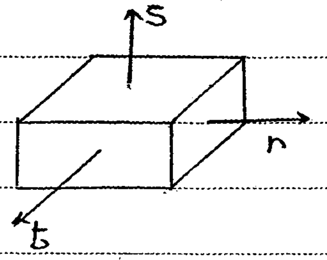
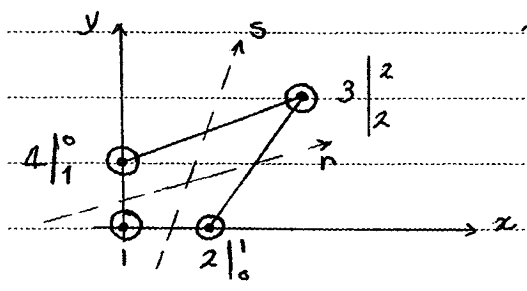


$$I = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\underset{2n-1}{r}, \overset{2m-1}{s}, \underset{2p-1}{t})\, dr\, ds\, dt$$

$$= \sum_{k=1}^{p}\sum_{j=1}^{m}\sum_{i=1}^{n} \omega_i \omega_j \omega_k f(r_i, s_j, t_k)$$

مثال:

$$K_{xy} = \int \frac{\partial \underline{N}^T}{\partial x} \frac{\partial \underline{N}}{\partial y}\, dA$$



* ماتریس J، Iso , sub با هم برابر است. (تا وقتی که تعداد نقاط گوسی تغییر نکند، یکی است.)

$\underline{N} = \underline{N}'$

$$N_1 = \frac{1}{4}(1-r)(1-s)$$

$$N_2 = \frac{1}{4}(1+r)(1-s)$$

$$N_3 = \frac{1}{4}(1+r)(1+s)$$

$$N_4 = \frac{1}{4}(1-r)(1+s)$$

$x(r,s) \quad , \quad y(r,s) \longrightarrow \quad r(x,y) \; , \; s(x,y)$

$$x(r,s) = \sum_{i=1}^{4} N_i' X_i \qquad y(r,s) = \sum_{i=1}^{4} N_i' Y_i$$

$N(r,s)$:
$$\frac{\partial \underline{N}}{\partial r} = \frac{\partial \underline{N}}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \underline{N}}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial \underline{N}}{\partial s} = \frac{\partial \underline{N}}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial \underline{N}}{\partial y}\frac{\partial y}{\partial s}$$

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$$\begin{bmatrix} \dfrac{\partial N}{\partial r} \\ \\ \dfrac{\partial N}{\partial s} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{bmatrix} \begin{bmatrix} \dfrac{\partial N}{\partial x} \\ \\ \dfrac{\partial N}{\partial y} \end{bmatrix} \qquad \underline{J}^{-1} = \underline{A}$$

$$\begin{bmatrix} \dfrac{\partial N}{\partial x} \\ \\ \dfrac{\partial N}{\partial y} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \dfrac{\partial N}{\partial r} \\ \\ \dfrac{\partial N}{\partial s} \end{bmatrix}$$

$$\frac{\partial N}{\partial y} = A_{21} \frac{\partial N}{\partial r} + A_{22} \frac{\partial N}{\partial s} = \underset{1\times 4}{\underline{d}(r,s)}$$

$$\frac{\partial N}{\partial x} = A_{11} \frac{\partial N}{\partial r} + A_{12} \frac{\partial N}{\partial s} = \underset{1\times 4}{\underline{c}(r,s)}$$

$$k_{xy} = \int_{-1}^{1}\int_{-1}^{1} \underline{c}^T(r,s) \cdot \underline{d}(r,s) \det J \, dr\, ds$$

$$k_{xy} = \int_{-1}^{1}\int_{-1}^{1} H(r,s)\, dr\, ds$$

$$H(r,s) = c^T(r,s)\, d(r,s) \det J$$

$$\underline{c}(r,s) = A_{11} \frac{1}{4} \begin{bmatrix} -(1-s) & +(1-s) & (1+s) & -(1+s) \end{bmatrix}$$

polynomial, odd, 1^{st} order w.r.t (r,s)

$$+ A_{12} \frac{1}{4} \begin{bmatrix} -(1-r) & -(1+r) & (1+r) & (1-r) \end{bmatrix}$$

$$\underline{d}(r,s) = A_{21} \frac{1}{4} \begin{bmatrix} -(1-s) & +(1-s) & (1+s) & -(1+s) \end{bmatrix}$$

polynomial, odd, 1^{st} order w.r.t (r,s)

$$+ A_{22} \frac{1}{4} \begin{bmatrix} -(1-r) & -(1+r) & (1+r) & (1-r) \end{bmatrix}$$

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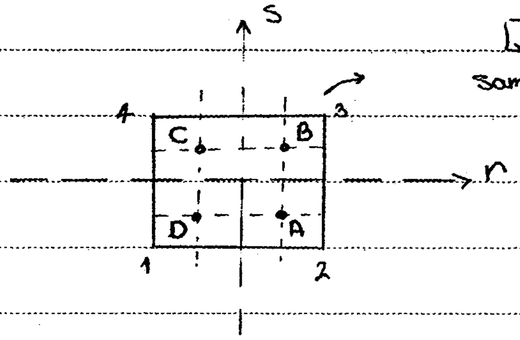
→ $H(r,s)$: polynomial of even function, 2^{nd} order, w.r.t. r, s

$$\begin{cases} 2n-1=2 \quad \rightarrow r \quad n=2 \\ 2m-1=2 \quad \uparrow s \quad m=2 \end{cases}$$



4 sample points

$$k(x,y) = \sum_{i=1}^{2} \sum_{j=1}^{2} \omega_i \, \omega_j \, \underline{H}(r_i, s_j) \qquad (*)$$

$$J_{11} = \frac{\partial x}{\partial r} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial r} X_i$$

$$J_{11} = \frac{1}{4}\left(-(1+s)X_1 + (1-s)X_2 + (1+s)X_3 - (1+s)X_4\right)$$

$$J_{11} = \frac{1}{4}\left[(1-s)(X_2 - X_1) + (1+s)(X_3 - X_4)\right]$$

$$J_{12} = \frac{1}{4}\left[(1-s)(Y_2 - Y_1) + (1+s)(Y_3 - Y_4)\right] = \frac{\partial y}{\partial r}$$

$$J_{21} = \frac{\partial x}{\partial s} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial s} X_i$$

$$J_{21} = \frac{1}{4}\left[-(1-r)X_1 - (1+r)X_2 + (1+r)X_3 + (1-r)X_4\right]$$

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$$J_{21} = \frac{1}{4}\left[(1-r)(x_4 - x_1) + (1+r)(x_3 - x_2)\right]$$

$$J_{22} = \frac{1}{4}\left[(1-r)(Y_4 - Y_1) + (1+r)(Y_3 - Y_2)\right]$$

$$J = \frac{1}{4}\begin{bmatrix} 3+s & 1+s \\ 1+r & 3+r \end{bmatrix}$$

sample point "A" $\left| \begin{array}{l} r = 0.577 \\ s = -0.577 \end{array} \right.$

calculate $J\big|_A$

calculate $J^{-1}\big|_A$

calculate $\underline{c}(r,s)\big|_A$, $\underline{d}(r,s)\big|_A$

Finally calculate $\underline{H}(r,s)\big|_{S.P.\ A}$

برای سه نقطهٔ دیگر نیز بصورت مشابه عمل می کنیم ، از رابطهٔ (*) ، مقدار k را حساب می کنیم.

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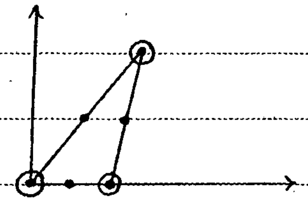

$\longrightarrow$ n برابر تعداد نقاط گوسی برای $L_1^2 L_2 \longrightarrow n=3$ است.

$$k_{xy} = \int_0^1 \int_0^{1-L_2} \underline{H}(L_1, L_2)\, dL_1\, dL_2 \longrightarrow \text{Quadratic}$$

مجموع ضرایب وزنی برای المان مثلثی برابر یک است.

Timoshenko beam / bending

$$\begin{array}{ccc} & \swarrow \quad \searrow & \\ u_b & & u_s \\ | & & | \\ k_b & \ll & k_s \\ | & & | \\ \text{Exact Integration} & & \text{Numerical Reduced Integration} \end{array}$$

$f(r) \longrightarrow$ polynomial $(2n-1) \longrightarrow$ Then we need "n" sample points $\longrightarrow$ Numerical Integration

$\quad \hookrightarrow$ if $2n-1 \geq 3 \longrightarrow$ "n-1" sample points

$\downarrow$

Numerical Reduced Integration

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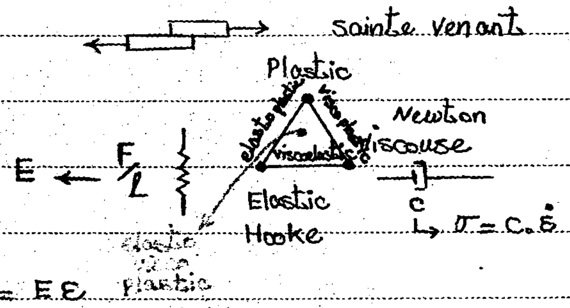
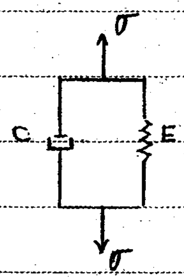


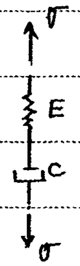
$\sigma = E\varepsilon$

وقتی بصورت Lumped مساله را بررسی کنیم ← 1D stress field داریم.

$$\sigma = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

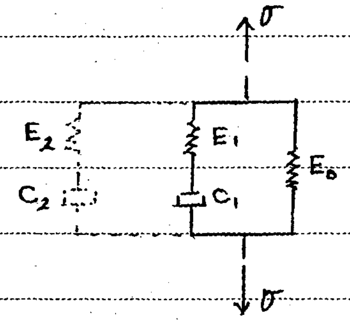


kelvin model



maxwell model

که این دو مدل ، مدل پایه هستند.



مدل پیروی مرتبه 1 مدل پیروی مرتبه 2

برای مدل کلوین :

$$\varepsilon = \varepsilon_e = \varepsilon_v \qquad 1$$

$$\sigma = \sigma_e + \sigma_v \qquad 2 \longrightarrow \sigma = E.\varepsilon + C.\dot{\varepsilon}$$

$$\sigma_e \downarrow E.\varepsilon_e \downarrow E.\varepsilon \qquad \sigma_v \downarrow C.\dot{\varepsilon}_v \downarrow C\dot{\varepsilon}$$

$$\dot{\varepsilon} = \frac{d}{dt}\varepsilon = \underbrace{S}_{\frac{d}{dt}}\varepsilon \longrightarrow \sigma = E.\varepsilon + C.S.\varepsilon$$

$$\sigma = (E + C.S)\,\varepsilon$$

برای مدل ماکسول :

$$\sigma = \sigma_e = \sigma_v \qquad 1$$

$$\varepsilon = \varepsilon_e + \varepsilon_v \qquad 2$$

$$\sigma_e = E.\varepsilon_e \longrightarrow \varepsilon_e = \frac{1}{E}\sigma_e = \frac{\sigma}{E}$$

$$\sigma_v = C.\dot{\varepsilon}_v \longrightarrow \dot{\varepsilon}_v = \frac{1}{C}\sigma_v = \frac{\sigma}{C}$$

$$\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_v \qquad \dot{\varepsilon} = \frac{1}{E}\dot{\sigma} + \frac{1}{C}\sigma$$

$$\longrightarrow S.\varepsilon = \frac{1}{E}S.\sigma + \frac{1}{C}\sigma$$

$$S.\varepsilon = \left(\frac{1}{C} + \frac{1}{E}S\right)\sigma$$

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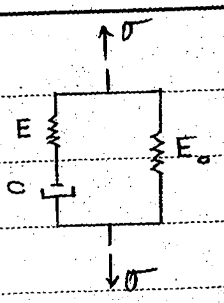

$\sigma = \underset{\text{hook}}{\sigma_h} + \underset{\text{mazwell}}{\sigma_m}$

$\varepsilon = \varepsilon_h = \varepsilon_m$

$(\frac{1}{C} + \frac{1}{E} S)\, \sigma_m = S \cdot \varepsilon$

$\sigma_m = \frac{S}{1/C + 1/E\, S} \cdot \varepsilon$

$\sigma_h = E_0 \cdot \varepsilon_h = E_0 \cdot \varepsilon$

$\sigma = E_0 \varepsilon + \frac{S}{1/C + 1/E\, S} \varepsilon$

$\longrightarrow \underbrace{(1/C + 1/E\, S)}_{P(S)}\, \sigma = [\underbrace{E_0 \cdot (\frac{1}{C} + \frac{1}{E} S) + S}_{Q(S)}]\, \varepsilon$

درحالت کلی :

$\boxed{P(S) \cdot \sigma = Q(S) \cdot \varepsilon}$

hook: $P(S) = 1$, $Q(S) = E$

Newton: $P(S) = 1$, $Q(S) = C \cdot S$

kelvin: $P(S) = 1$, $Q(S) = E + C \cdot S$

Mazwell: $P(S) = \frac{1}{C} + \frac{1}{E} S$, $Q(S) = S$

Prony 1st order: $P(S) = \frac{1}{C} + \frac{1}{E} S$, $Q(S) = E_0 (\frac{1}{C} + \frac{1}{E} \cdot S) + S$

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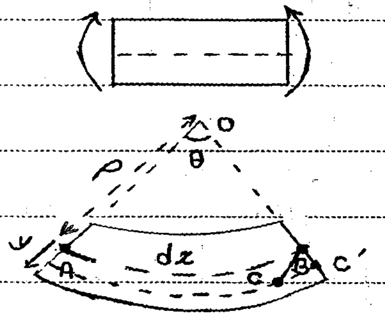
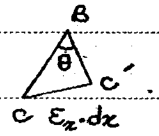
D.E of the Euler beam made from visco_Elastic material

Remark: [illegible]

1 $M = \int_A y \cdot \sigma \cdot dA$





$$\frac{\overline{BC'}}{\overline{OB}} = \frac{\overline{CC'}}{\overline{AB}}$$

$$\frac{y}{\rho} = \frac{\varepsilon_x dx}{dx} \qquad \varepsilon_x = \frac{1}{\rho} y$$

2 $\varepsilon_x = \frac{1}{\rho} y = \underbrace{\chi}_{\text{kappa}} \cdot y$

$$p(s) \cdot M = p(s) \int_A y \cdot \sigma \cdot dA$$

$$p(s) \cdot M = \int_A y \, p(s) \cdot \sigma \cdot dA$$

$$p(s) \cdot M = \int_A y \, Q(s) \cdot \varepsilon \, dA$$

$$p(s) \cdot M = Q(s) \int_A y^2 \cdot \chi \cdot dA = Q(s) \, \chi \int_A y^2 \, dA$$

3 $p(s) \cdot M = Q(s) \, \chi \cdot I$

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$$4 \quad \chi = -\frac{d^2 v}{dx^2} = -v''$$

$$4+3: \quad p(s).M = Q(s).I.(-v'')$$

$$v'' = -\frac{P(s).M}{Q(s).I}$$

$$Q(s).I.v'' = -p(s).M$$

$$Q(s).I\,v^{IV} = -p(s)\frac{d^2M}{dx^2} = p(s)P_n$$

$$\frac{d^2M}{dx^2} = -P_n \rightarrow$$ بار گسترده

$$Q(s).I.\frac{d^4v}{dx^4} - p(s)P_n = 0$$

pure elastic $\longrightarrow$ Hook : $P(s) = 1$, $Q(s) = E$

$$\longrightarrow E.I\frac{d^4v}{dx^4} - P_n = 0$$ در حالت الاستیک

kelvin : $$(E + C.S)I\frac{d^4v}{dx^4} - P_n = 0$$

Euler beam
- Dynamic
- Galerkin

$$\rho\frac{d^2v}{dt^2} + \frac{d^2}{dx^2}\left(E.I.\frac{d^2v}{dx^2}\right) = P_n \quad v(x,t)$$

$v(x,t)$: elastic deflection → Dynamic

$$Q(s).I.\frac{d^4v}{dx^4} - P(s).P_n = 0$$

$v(x,t)$: Visco-elastic deflection

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1. Euler Beam $\rightarrow$ Dynamic Loading

2. Viscoelastic Euler Beam

$$\text{D.E.}: \quad \rho \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(E.I.\frac{\partial^2 v}{\partial x^2}\right) = q(x,t)$$

خیز دینامیکی $\quad v^e(x,t) = \sum_{i=1}^{4} \varphi_i(x)\, q_i(t)$

$$\begin{cases} q_1(t) = v_i(t) \\ q_2(t) = \alpha_i(t) \\ q_3(t) = v_j(t) \\ q_4(t) = \alpha_j(t) \end{cases}$$

$$\rho \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(E.I.\frac{\partial^2 v^e}{\partial x^2}\right) - q(x,t) \neq 0 \quad R(x, q_i(t))$$

از روش گالرکین :

$$v^e(x,t) = \underline{\varphi}_{1\times 4} \cdot \underline{q}(t)_{4\times 1} = [\underset{w_1}{\varphi_1} \rightarrow \underset{w_4}{\varphi_4}] \begin{bmatrix} q_1(t) \\ \downarrow \\ q_4(t) \end{bmatrix} \qquad \dot{v}^e(x,t) = \underline{\varphi} \cdot \underline{\dot{q}}^e(t)$$

$$\ddot{v}^e(x,t) = \underline{\varphi} \cdot \underline{\ddot{q}}^e(t)$$

$$\int_0^l \underline{\varphi}^T \cdot R(x, q_i(t))\, dx = 0$$

$$\int_0^l \varphi^T \left(\rho \frac{\partial^2 v^e}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI.\frac{\partial^2 v^e}{\partial x^2}\right) - q(x,t)\right) dx = 0 \qquad 1$$

1.1: $\displaystyle \int_0^l \underline{\varphi}^T \rho \frac{\partial^2 v^e}{\partial t^2}\, dx = \int_0^l \rho \cdot \underline{\varphi}^T \cdot \underline{\varphi} \cdot \underline{\ddot{q}}^e(t)\, dx = \left(\int_0^l \rho\, \underset{4\times 1}{\underline{\varphi}^T} \cdot \underset{1\times 4}{\underline{\varphi}}\, dx\right) \underline{\ddot{q}}^e(t)$

$$= \underline{M}(e) \cdot \underline{\ddot{q}}^e(t)$$

1.2: $\displaystyle \int_0^l \underbrace{\varphi^T}_{u} \cdot \underbrace{\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 v^e}{\partial x^2}\right)}_{v'} dx = \underline{\varphi}^T \cdot \frac{\partial}{\partial x}\left(EI.\frac{\partial^2 v}{\partial x^2}\right)\Big|_0^l - \int_0^l \frac{\partial \varphi^T}{\partial x} \cdot \frac{\partial}{\partial x}\left(EI\frac{\partial^2 v}{\partial x^2}\right) dx$

 [illegible] از weak form باعث ← C' continuity

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1.2 : $= \underline{\varphi}^T \cdot \frac{\partial}{\partial x}(M(x,t))\Big|_0^l - \int_0^l \underbrace{\frac{\partial \varphi^T}{\partial x}}_{u} \underbrace{\frac{\partial}{\partial x}(EI \frac{\partial^2 v}{\partial x^2})}_{v'} \cdot dx$

$$- \frac{\partial \underline{\varphi}^T}{\partial x}(EI \frac{\partial^2 v}{\partial x^2})\Big|_0^l + \int_0^l \frac{\partial^2 \underline{\varphi}^T}{\partial x^2} EI \frac{\partial^2 v}{\partial x^2} dx$$

نیروی برشی

1.2 : $= -\underline{\varphi}^T \cdot T(x,t)\Big|_0^l - \frac{\partial \underline{\varphi}^T}{\partial x} \cdot M(x,t)\Big|_0^l + \underbrace{(\int_0^l EI \frac{\partial^2 \underline{\varphi}^T}{\partial x^2} \frac{\partial^2 \underline{\varphi}}{\partial x^2} \cdot dx)}_{k(e)} \underline{q}^e(t)$

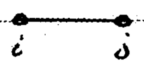
1.3 : $\int_0^l \underline{\varphi}^T \cdot q(x,t) \cdot dx = F^e(t)$

$$\boxed{\underline{M}(e) \cdot \ddot{\underline{q}}(t) + k(e) q(t) = \underline{F}^e(t) + \underline{I}.\underline{E}^e}$$

$\underline{M}(e) = \int_0^l \rho \cdot \underline{\varphi}^T \cdot \underline{\varphi} \cdot dx \qquad \underline{F}^e(t) = \int_0^l \underline{\varphi}^T \cdot q(x,t)\, dx$

$\underline{I}.\underline{E}^e = \underline{\varphi}^T \cdot T(x,t)\Big|_0^l + \frac{\partial \underline{\varphi}^T}{\partial x} M(x,t)\Big|_0^l$

$$\underline{k}(e) = \int_0^l EI \frac{\partial^2 \underline{\varphi}^T}{\partial x^2} \frac{\partial^2 \underline{\varphi}}{\partial x^2} dx = \int_0^l EI\, \varphi_i'' \cdot \varphi_j'' dx = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$



$\varphi_1(x) = 1 - 3(x/l)^2 + 2(x/l)^3$

$\varphi_2(x) = x(1 - x/l)^2$

$\varphi_3(x) = 3(x/l)^2 - 2(x/l)^3$

$\varphi_4(x) = x(-x/l + (x/l)^2)$

$$M_{ij}(e)=\int_0^{l}\rho\,\varphi_i\,\varphi_j\,dx$$

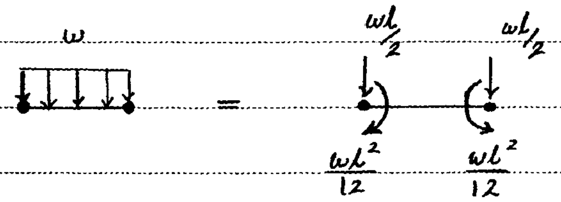
$$M_{11}=\rho\int_0^{l}\varphi_1^2\,dx \qquad \frac{x}{l}=u \qquad dx=l\,du$$

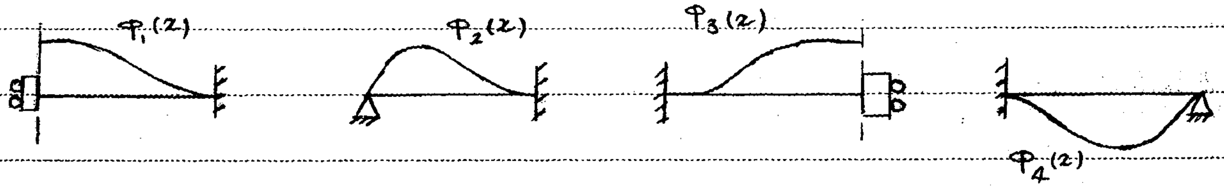
$$M_{11}=\rho.l\int_0^1(1-3u^2+2u^3)^2\,du$$

$$=\rho.l\int_0^1(1+9u^4+4u^6-6u^2+4u^3-12u^5)\,dx$$

$$\longrightarrow M_{11}=\frac{13}{35}\rho.l$$

اگر $q(x,t)=w$ باشد
ثابت





$$\begin{bmatrix}\varphi_1\\\varphi_2\\\varphi_3\\\varphi_4\end{bmatrix}T(x,t)\Big|_0^{l}=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}T(l,t)-\begin{bmatrix}1\\0\\0\\0\end{bmatrix}T(0,t)=\begin{bmatrix}-T(0,t)\\0\\T(l,t)\\0\end{bmatrix}$$

$$\frac{\partial\Phi^T}{\partial x}.M(x,t)\Big|_0^{l}=\begin{bmatrix}\varphi_1'\\\varphi_2'\\\varphi_3'\\\varphi_4'\end{bmatrix}M(x,t)\Big|_0^{l}=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}M(l,t)-\begin{bmatrix}0\\1\\0\\0\end{bmatrix}M(0,t)=\begin{bmatrix}0\\-M(0,t)\\0\\M(l,t)\end{bmatrix}$$

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$$\rightarrow I.E = \begin{bmatrix} -T(0,t) \\ -M(0,t) \\ T(l,t) \\ M(l,t) \end{bmatrix}$$

بعد از assemble کردن:

$$\underset{n\times n}{\underline{M}_t}\cdot\underset{n\times 1}{\ddot{\underline{q}}_t} + \underset{n\times n}{\underline{K}_t}\cdot\underset{n\times 1}{\underline{q}_t} = \underline{F}_t(t) + \underbrace{I.E}_{G.B.C.}\Big|_{total}$$

2n unknowns + n equations

معادلات دیفرانسیل به روش حل عددی است.
(n معادله)

$t-2\Delta t \quad t-\Delta t \quad t \quad t+\Delta t \quad t+2\Delta t$

one step backward — two step forward

برای بدست آوردن فرکانس های طبیعی: $\underline{F}_t(t)=0$

$$\underline{q}(t) = \underline{q}\,e^{i\omega t} \qquad \ddot{\underline{q}}(t) = -\omega^2\underline{q}\,e^{i\omega t}$$

$$(\underline{K}_t - \omega^2 \cdot \underline{M}_t)\cdot q\,e^{i\omega t} = 0 \qquad q \neq 0 \longrightarrow DET(\underline{K}_t - \omega^2 \underline{M}_t) = 0$$

$\longleftarrow$ ω های bending بدست می آید.

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D.E Visco _ Elastic Euler Beam

$$S = \frac{d}{dt}$$

$$p(s) \cdot \sigma = Q(s) \cdot \varepsilon$$

$$M = \int_A y \cdot \sigma \cdot dA$$

$$p(s) \cdot M = \int_A y \cdot p(s) \cdot \sigma \cdot dA = Q(s) \cdot \int_A y \cdot \overset{\chi \cdot y}{\overset{\uparrow}{\varepsilon}} \cdot dA = Q(s) \cdot \chi \cdot \int_A y^2 dA$$

$$= Q(s) \cdot \chi \cdot I$$

$$p(s) \cdot M = Q(s) \cdot \chi \cdot I$$

$$\chi = \frac{1}{\rho} = \frac{P(s) \cdot M}{Q(s) \cdot I} = -\nu'' \longrightarrow p(s) \cdot M = -Q(s)\, I\, \frac{d^2\nu}{dx^2}$$

$$\frac{d^2}{dx^2}\left[p(s) \cdot M = Q(s) \cdot I \cdot (-\nu'') \right]$$

$$p(s) \cdot \underbrace{\frac{d^2 M}{dx^2}}_{\text{بار گسترده}} = Q(s)\, I\, \frac{d^2}{dx^2}(\nu'')$$

$$P(s) \cdot (-P_n) = -Q(s) \cdot I \cdot \frac{\partial^4 \nu}{\partial x^4}$$

$$Q(s) \cdot I \cdot \frac{\partial^4 \nu}{\partial x^4} - p(s) \cdot P_n = 0$$

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$$\rho \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 v}{\partial x^2}\right) = q(x,t)$$

• Elastic Node

$$Q(s) \cdot I \cdot \frac{d^4 v^e}{dx^4} - p(s) \cdot P_n \neq 0$$

$$\int_0^l \underline{\varphi}^T \cdot \left[Q(s) \cdot I \cdot \frac{d^4 v}{dx^4} - p(s) \cdot P_n\right] \cdot dx = 0$$

$\underline{1}$ $$Q(s) \cdot I \cdot \int_0^l \underline{\varphi}^T \cdot \frac{d^4 v}{dx^4} dx$$

$\underline{2}$ $$-p(s) \cdot \int \underline{\varphi}^T \cdot P_n \, dx$$

$$\int_0^l \underbrace{\underline{\varphi}^T}_{u} \cdot \underbrace{\frac{d^4 v}{dx^4}}_{v'} \cdot dx = \underline{\varphi}^T \cdot \frac{d}{dx}\left(\frac{d^2 v}{dx^2}\right)\Big|_0^l - \int_0^l \underbrace{\frac{d\underline{\varphi}^T}{dx}}_{u} \underbrace{\frac{d^3 v^e}{dx^3}}_{v'} dx$$

$$\rightarrow \underline{1} = Q(s) \cdot I \cdot \left(\underline{\varphi}^T \cdot \frac{d}{dx}\left(\frac{d^2 v}{dx^2}\right)\Big|_0^l - \frac{d\underline{\varphi}^T}{dx} \frac{d^2 v^e}{dx^2}\Big|_0^l + \int_0^l \frac{d^2\underline{\varphi}^T}{dx^2} \frac{d^2 v^e}{dx^2} dx\right)$$

$$\underline{1} + \underline{2} = \underline{\varphi}^T \frac{d}{dx}\big(\overbrace{Q(s) \cdot I \frac{d^2 v}{dx^2}}^{-p(s) \cdot M}\big)\Big|_0^l - \frac{d\underline{\varphi}^T}{dx}\big(\overbrace{Q(s) \cdot I \cdot \frac{d^2 v}{dx^2}}^{-p(s) \cdot M}\big)\Big|_0^l$$

$$+ Q(s) I \int_0^l \frac{d^2\underline{\varphi}^T}{dx^2} \frac{d^2 v}{dx^2} dx - p(s) \cdot \int_0^l \underline{\varphi}^T \cdot P_n \cdot dx$$

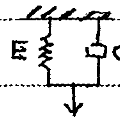
$$= -\underline{\varphi}^T \cdot p(s) \overbrace{\frac{dM}{dx}}^{-T}\Big|_0^l + p(s) \cdot \frac{d\underline{\varphi}^T}{dx} M\Big|_0^l + Q(s) I \left(\int_0^l \frac{d^2\underline{\varphi}^T}{dx^2} \frac{d^2\underline{\varphi}}{dx^2} dx\right) \underline{q}$$

$$- p(s) \cdot \underline{F}^e$$

let $\underline{k} = I \cdot \int_0^{\ell} \varphi_i'' \varphi_j'' \, dx$

$$\boxed{Q(s) \cdot \underline{k} \cdot \underline{q} = p(s) \cdot \underline{F} - p(s) \cdot \left(\underline{\varphi}^T \cdot T \Big|_0^{\ell} + \frac{d\varphi^T}{dx} \cdot M \Big|_0^{\ell} \right)}$$

مدل کلوین:



$$\sigma = \sigma_e + \sigma_v = E \cdot \varepsilon + C \cdot \dot{\varepsilon} = \left(E + C \frac{d}{dt} \right) \varepsilon$$
$$= (E + C \cdot S)\, \varepsilon$$

$\varepsilon = \varepsilon_e = \varepsilon_v \qquad p(s) = 1 \qquad Q(s) = E + C \cdot S$

$\sigma_e = E \cdot \varepsilon_e = E \cdot \varepsilon \qquad \sigma_v = C \cdot \dot{\varepsilon}_v = C \cdot \dot{\varepsilon}$

$$(E + C \cdot s)\, \underline{k} \cdot \underline{q} = \underline{F} - \underline{I} \cdot \underline{E}$$

$$\underbrace{\left(E \cdot I \int_0^{\ell} \varphi_i'' \varphi_j'' \, dx \right)}_{\underline{k}} \cdot \underline{q} + \underbrace{\left(C \cdot I \int_0^{\ell} \varphi_i'' \varphi_j'' \, dx \right)}_{\underline{C}} \underline{\dot{q}} = \underline{F} - \underline{I} \cdot \underline{E}$$

$$\boxed{\underline{C} \cdot \underline{\dot{q}} + \underline{K} \cdot \underline{q} = \underline{F} - \underline{I} \cdot \underline{E}}$$

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Variational Method:

1) General Concept

2) variational Method In F.E.A Contex.

با معادلهٔ دیفرانسیل به طور مستقیم کار نمی کنیم:

Function of a Function ⟶ Functional

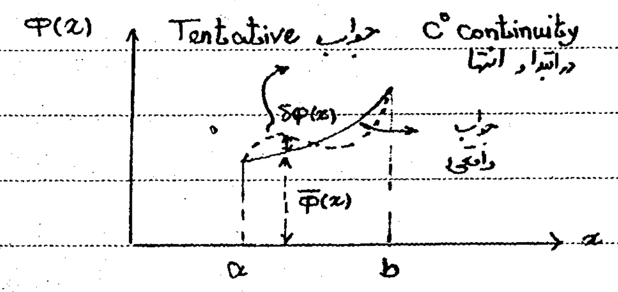
رابطهٔ یک به یک

$D.E. \longrightarrow F(x, \varphi, \varphi_x, \varphi_{xx})$ $\qquad \varphi_x = \dfrac{\partial \varphi}{\partial x} \qquad \varphi_{xx} = \dfrac{\partial^2 \varphi}{\partial x^2}$

$a \leqslant x \leqslant b$: Domain Variable

$$\underset{\text{functional}}{\underset{\downarrow}{I}} = \int_a^b \underset{\text{functional}}{\underset{\downarrow}{F}}(x, \varphi, \varphi_x, \varphi_{xx})\, dx$$

I ← مساحت زیر نمودار F.



$\delta\varphi_a = \delta\varphi_b = 0$

$$\underset{\substack{\text{Tentative}\\\text{Solution}}}{\underset{\downarrow}{\varphi(x)}} = \underset{\substack{\text{Exact}\\\text{solution}}}{\underset{\downarrow}{\bar{\varphi}(x)}} + \underset{\substack{\text{variation}\\\text{of } \varphi}}{\underset{\downarrow}{\delta\varphi(x)}}$$

$$\delta I = \delta \int_a^b F(x, \varphi, \varphi_x, \varphi_{xx})\, dx$$

$$\delta I = \int_a^b \delta F(x, \varphi, \varphi_x, \varphi_{xx})\, dx = 0$$

$$\delta F = \frac{\partial F}{\partial \varphi}\delta\varphi + \frac{\partial F}{\partial \varphi_x}\delta\varphi_x + \frac{\partial F}{\partial \varphi_{xx}}\delta\varphi_{xx}$$

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1) $\int_a^b \frac{\partial F}{\partial \varphi}\,\delta\varphi\,dx \qquad \left(\delta\varphi_x = \delta\left(\frac{\partial\varphi}{\partial x}\right) = \frac{\partial}{\partial x}(\delta\varphi)\right)$

2) $\int_a^b \frac{\partial F}{\partial \varphi_x}\,\delta\varphi_x\,dx = \int_a^b \underbrace{\frac{\partial F}{\partial \varphi_x}}_{u}\,\underbrace{\frac{\partial}{\partial x}(\delta\varphi)}_{v'}\,dx = \frac{\partial F}{\partial \varphi_x}\,\delta\varphi\Big|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial \varphi_x}\right)\delta\varphi\,dx$

3) $\int_a^b \frac{\partial F}{\partial \varphi_{xx}}\,\delta\varphi_{xx}\,dx = \int_a^b \frac{\partial F}{\partial \varphi_{xx}}\,\delta\left(\frac{\partial\varphi_x}{\partial x}\right)dx = \int_a^b \underbrace{\frac{\partial F}{\partial \varphi_{xx}}}_{u}\,\underbrace{\frac{\partial}{\partial x}(\delta\varphi_x)}_{v'}\,dx$

$= \frac{\partial F}{\partial \varphi_{xx}}\,\delta\varphi_x\Big|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial \varphi_{xx}}\right)\delta\varphi_x\,dx$

$= \frac{\partial F}{\partial \varphi_{xx}}\,\delta\varphi_x\Big|_a^b - \int_a^b \underbrace{\frac{d}{dx}\left(\frac{\partial F}{\partial \varphi_{xx}}\right)}_{u}\cdot\underbrace{\frac{d}{dx}(\delta\varphi)}_{v'}\,dx$

$\longrightarrow \int_a^b \frac{\partial F}{\partial \varphi_{xx}}\,\delta\varphi_{xx}\,dx = \frac{\partial F}{\partial \varphi_{xx}}\,\delta\varphi_x\Big|_a^b - \frac{d}{dx}\left(\frac{\partial F}{\partial \varphi_{xx}}\right)\cdot\delta\varphi\Big|_a^b + \int_a^b \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial \varphi_{xx}}\right)\delta\varphi\,dx$

$I = \int_a^b F(x, \varphi, \varphi_x, \varphi_{xx})\,dx \qquad \delta I = \int_a^b \delta F\cdot dx = 0$

$\delta I = \int_a^b \left(\frac{\partial F}{\partial \varphi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \varphi_x}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial \varphi_{xx}}\right)\right)\delta\varphi\,dx$

$+ \left(\frac{\partial F}{\partial \varphi_x} - \frac{d}{dx}\left(\frac{\partial F}{\partial \varphi_{xx}}\right)\right)\delta\varphi\Big|_a^b + \frac{\partial F}{\partial \varphi_{xx}}\,\delta\varphi_x\Big|_a^b = 0$

1) $\frac{\partial F}{\partial \varphi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \varphi_x}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial \varphi_{xx}}\right) = 0 \longrightarrow$ "D.E" $\rightarrow$ Euler Lagrange

مشتق مرتبه صفر   مشتق مرتبه 2   مشتق مرتبه 4

با توجه به شکل صفحه قبل $\delta\varphi$ در a و b برابر صفر است.

Forced boundary condition : $\delta\varphi\Big|_{a,b} = 0 \quad , \quad \delta\varphi_x\Big|_{a,b} = 0$

C^1 continuity C^1 continuity //

Free boundary condition : $\left(\frac{\partial F}{\partial \varphi_x} - \frac{d}{dx}\left(\frac{\partial F}{\partial \varphi_{xx}}\right)\right) = 0 \quad , \quad \frac{\partial F}{\partial \varphi_{xx}} = 0$

D.E. مرتبه 4 مورد توجه قرار می‌گیرند.

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$$\underbrace{\frac{d^2\varphi}{dx^2}}_{-\frac{d}{dx}\left(\frac{\partial F}{\partial \varphi_x}\right)} + \underbrace{\varphi + x}_{\frac{\partial F}{\partial \varphi}} = 0 \xrightarrow{\text{مشتق مرتبه } 2} F(x, \varphi, \varphi_x)$$

$$\longrightarrow F = \frac{1}{2}\varphi^2 + x\cdot\varphi$$

$$\longrightarrow -\varphi_x = \frac{\partial F}{\partial \varphi_x} \rightarrow F = -\frac{1}{2}\varphi_x^2$$

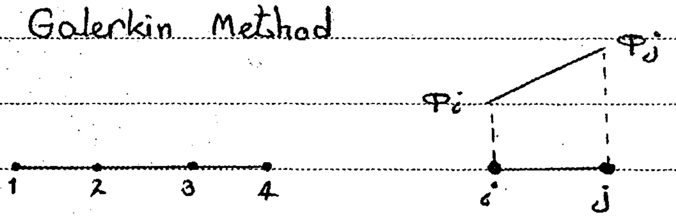
$$\longrightarrow F(x, \varphi, \varphi_x) = \frac{1}{2}(\varphi^2 + 2\cdot x\cdot\varphi - \varphi_x^2)$$

F ← مشروط به این بدست می‌آید که در D.E. مشتق مرتبهٔ فرد نسبت به مکان نداشته باشیم ← محدودیت

مثلاً برای تیر روی بستر الاستیک ← D.E. مرتبهٔ 5 ← روش variational نقض می‌شود

$$\frac{d^2\varphi}{dx^2} + (\varphi + x) = 0 \qquad 0 \leq x \leq 1 \qquad \varphi(0) = 0 \;,\; \varphi(1) = 0$$

1) W.R.M → Galerkin Method



$$\varphi^e(x) = \underline{N}\cdot\underline{a} = [N_i \quad N_j]\begin{bmatrix}\varphi_i \\ \varphi_j\end{bmatrix} \qquad N_i = \frac{1}{x_{ji}}(x_j - x) \;,\; N_j = \frac{1}{x_{ji}}(x - x_i)$$

$$\frac{d^2\varphi^e}{dx^2} + (\varphi^e + x) \neq 0$$
$$= R(x, \varphi, \varphi_x)$$

$$\int_{x_i}^{x_j} N^T \cdot \left[\frac{d^2\varphi^e}{dx^2} + \varphi^e + x\right] dx = 0 \longrightarrow \underline{k}\cdot\underline{a} = \underline{F}^e + \underline{I.E}$$

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$$1\quad \int_{x_i}^{x_j} \underbrace{\underline{N}^T}_{u} \underbrace{\frac{d^2\varphi}{dx^2}}_{v'} dx = N^T \cdot \frac{d\varphi}{dx}\Bigg|_{x_i}^{x_j} - \left(\int_{x_i}^{x_j} \frac{dN^T}{dx}\frac{dN}{dx}\, dx\right)\underline{a}$$

$$2\quad \int_{x_i}^{x_j} \underline{N}^T \cdot \varphi^e \cdot dx = \left(\int_{x_i}^{x_j} \underline{N}^T \cdot \underline{N} \cdot dx\right)\underline{a}$$

$$3\quad \int_{x_i}^{x_j} x \cdot \underline{N}^T \cdot dx$$

$$\underline{K}^e = \int_{x_i}^{x_j} \frac{d\underline{N}^T}{dx}\frac{d\underline{N}}{dx}\, dx - \int_{x_i}^{x_j} \underline{N}^T \cdot \underline{N} \cdot dx$$

$$\underline{F}^e = \int_{x_i}^{x_j} x \cdot \underline{N}^T \cdot dx$$

$$\underline{I \cdot E} = N^T \cdot \frac{d\varphi}{dx}\Bigg|_{x_i}^{x_j} \longrightarrow$$ در گره‌های 1 و 4 معنی پیدا کند.

2) Variational Method

$$F(x, \varphi, \varphi_x) = \frac{1}{2}\left[\varphi^2 + 2 \cdot x \cdot \varphi - \varphi_x^2\right]$$

$$I = \int_0^1 F(x, \varphi, \varphi_x)\, dx = \sum_{e=1}^{N.E.} I(e)$$

$$I(e) = \int_{x_i}^{x_j} F(x, \varphi, \varphi_x)\, dx = \frac{1}{2}\int_{x_i}^{x_j} (\varphi^2 + 2x\varphi - \varphi_x^2)\, dx$$

$$\varphi^e = [N]\{a\} \longrightarrow \varphi^e = \{a\}^T [N]^T \longrightarrow \varphi^2 = \underline{a}^T \cdot \underline{N}^T \cdot \underline{N} \cdot \underline{a}$$

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$$\varphi_x = \frac{d\underline{N}}{dx} \cdot \underline{a}$$

$$2 \cdot x \cdot \varphi = (2 \cdot x) \cdot \underline{a}^T \cdot \underline{N}^T$$

$$\varphi_x^e = \underline{a}^T \cdot \frac{d\underline{N}^T}{dx} \cdot \frac{d\underline{N}}{dx} \cdot \underline{a}$$

$$I(e) = \frac{1}{2} \int_{x_i}^{x_j} \left(\underline{a}^T \cdot \underline{N}^T \cdot \underline{N} \cdot \underline{a} + \underline{a}^T (2x \cdot \underline{N}^T) - \underline{a}^T \cdot \frac{d\underline{N}^T}{dx} \frac{d\underline{N}}{dx} \underline{a} \right) dx$$

$$I(e) = I^e(\underline{a}) \qquad \frac{\partial I(e)}{\partial \underline{a}} = 0$$

$$\frac{\partial I(e)}{\partial \underline{a}} = 0 \longrightarrow \left(\int_{x_i}^{x_j} \underline{N}^T \cdot \underline{N} \cdot dx \right) \underline{a} + \int_{x_i}^{x_j} x \cdot \underline{N}^T \cdot dx - \left(\int_{x_i}^{x_j} \frac{d\underline{N}^T}{dx} \frac{d\underline{N}}{dx} dx \right) \underline{a} = 0$$

$$F(e) = \underline{k}(e) \cdot a \qquad \left(\left(\int_{x_i}^{x_j} \frac{d\underline{N}^T}{dx} \frac{d\underline{N}}{dx} \cdot dx \right) - \int_{x_i}^{x_j} \underline{N}^T \cdot \underline{N} \cdot dx \right) \underline{a} = \int_{x_i}^{x_j} x \cdot \underline{N}^T \cdot dx$$

$$\frac{\partial F}{\partial \varphi_x} \cdot \delta\varphi \Big|_{x=0}^{x=1} = 0$$

که دو روش دقیقاً یک جواب می‌دهند.

Displacement D.O.F. only :

گره‌هایی که تا الان گرفتیم روی تار خنثی واقع است. اما می‌توان گره‌های دیگر تحلیل کرد.