مرتضی

In the Name of God

Advanced Fluid Mechanics

Project #3

Instability of Narrow-Gap Taylor-Couette Flow

Due Date = 1389/3/29

Value = 4 out of 20

The linearized system of equations governing axially-symmetric, non-propagating, small perturbations superimposed on the base flow in the narrow-gap circular Couette flow have been shown in the class to be:

$$[(D^2 - a^2)]^3 \hat{V} = -a^2 \mathbf{T} f(x) \hat{V}$$

where $\hat{V}$ is the amplitude of the perturbation velocity v'_r, a is the wave number, $\mathbf{T}$ is the Taylor number. Also, in this equation we have $f(x) = 1 - (1-\mu)\left(x + \frac{1}{2}\right)$ with x and μ defined by:

$$x = \frac{r - (R_2 + R_1)/2}{R_2 - R_1} \; ; \quad \mu = \frac{\Omega_2}{\Omega_1}$$

Try to numerically solve the above system of equations subject to the following boundary conditions: @ $x = \pm\frac{1}{2}$ we have: $\hat{V} = (D^2 - a^2)\hat{V} = D(D^2 - a^2)\hat{V} = 0$

طول موج بحرانی

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Objectives:

1) Compute the critical wave number and also the critical Taylor number.
2) Reproduce Fig. 3.13 as reported in the book by Drazin.

Method of Solution:

Rely **only** on the spectral method. Use **only** the Fourier series as the base functions.

Requirements:

1) The final report should have the structure of a paper.
2) The report should include flow chart together with the listing of the compter code.
3) The code should be well written, structured, and executable.
4) Reproduce Fig. 3.13 in Drazin and Reid's book. Plot the secondary flow formed in the gap.
5) CD comprising the executable program and also the report (in Word) should be handed in.

Note: The performance of the code will be checked by running the code in front of the student for an arbitrary set of geometrical parameters and/or rotational speeds.

Good Luck.

سه‌شنبه : ۱۳۸۸/۱۲/۱۸

$$u_x = F(x, u(x))$$

$$u_x = F(x, u^{(i)}) + \frac{\partial F}{\partial u}(u - u^{(i)}) + O[(u - u^{(i)})^2]$$

$$\Rightarrow u_x^{(i+1)} - \hat{F}_u^{i}\, u^{(i+1)} = \hat{F}^{i} - \hat{F}_u^{i}\, u^{(i)}$$ برای Iteration جدید ①

$$\Delta = u^{(i+1)} - u^{(i)}$$ ②

①, ② $\Rightarrow \Delta_x - F_u \Delta = F - u_x^{(i)}$ ③

$$u = \sum a_n T_n \Rightarrow u^{(i)} = \sum a_n^{(i)} T_n(x)$$

$$u^{(i+1)} = \sum a_n^{(i+1)} T_n(x)$$

$$\Delta = \sum \overbrace{(a_n^{(i+1)} - a_n^{(i)})}^{\delta_n} T_n(x)$$

$$\Delta = \sum \delta_n T_n(x)$$

روش حل

حدس : $a_n^{(0)} \longrightarrow \delta_n^{(0)}$ ✓

$$a_n^{(1)} = a_n^{(0)} + \delta_n^{(0)}$$

اعمال شرایط مرزی : a_nها طوری اند که شرایط مرزی را ارضا کنند و δ_nها هم
طوری که Δ شرایط مرزی را صفر کند.

$$u_{xx} = F(x, u, u_x)$$

$$\Delta_{xx} - F_{u_x}\Delta_x - F_u \Delta = F - u_{xx}^{(i)}$$

مثال:

$$u_{xx} = \alpha (u_x)^2 + \exp(xu)$$

$$F_{u_x} = 2\alpha u_x$$

$$F_u = x \exp(xu)$$

$$\Rightarrow \Delta_{xx} - 2\alpha u_x^{(i)} \Delta_x - x\exp(xu^{(i)})\,\Delta = \alpha (u_x^{(i)})^2 + \exp(xu^{(i)})$$
$$- u_{xx}^{(i)}$$

$$u_{xx} = F(x, u, u_x)$$

$$u_{xx}^{(i+1)} = \overset{i}{\widehat{F}}(x, u^{(i)}, u_x^{(i)}) + \overset{i}{\widehat{F_u}}(u - u^{(i)}) + \overset{i}{\widehat{F_{u_x}}}(u_x^{(i+1)} - u_x^{(i)}) + \cdots$$

$$\Rightarrow u_{xx}^{(i+1)} = F + F_u \underbrace{(u^{(i+1)} - u^{(i)})}_{\Delta} + F_{u_x} \underbrace{(u_x^{(i+1)} - u_x^{(i)})}_{\Delta_x} + \cdots$$

$$\Rightarrow \Delta_{xx} = F + F_u \Delta + \overline{F_{u_x}}\,\overline{\Delta_x} - u_{xx}^{(i)}$$

$$\Rightarrow \Delta_{xx} - F_{u_x}\Delta_x - F_u \Delta = F - u_{xx}^{(i)}$$

(4.124), or, equivalently, (4.132). This latter equation gives

$$\Pi_1 - \Pi_2 - \frac{2\gamma}{a} + \left((\rho_1 - \rho_2)ga - \frac{3\mu_2 U(2+3\lambda)}{2a(1+\lambda)}\right)\cos\theta = 0,$$

which, by virtue of (4.139), reduces to

$$\Pi_1 - \Pi_2 = \frac{2\gamma}{a}, \tag{4.140}$$

but this would be the condition to be satisfied were the droplet maintained in a spherical shape by the balance of hydrostatic pressures and surface tension. Intuition suggests that the spherical shape of a droplet in motion can be maintained with a sufficiently large surface tension since, if $\gamma \gg \mu U$, the normal stresses due to the droplet motion are negligible compared with those due to hydrostatic pressure and surface tension. Equation (4.140) is remarkable in that it demonstrates that, if the fluid motion is governed by the Stokes equations, then the spherical shape is maintained *whatever* the value of γ. This result is confirmed by observations of air bubbles rising through highly viscous liquids such as glycerine; the bubbles remain spherical even when their radii are so large that the effect of surface tension cannot be significant in maintaining their spherical shape.

4.18 THREE-DIMENSIONAL STOKESLET AND ROTLET

Consider the solution obtained in section 4.14 for a solid sphere of radius a translating with velocity $U\mathbf{k}$ in unbounded fluid at rest at infinity. Let us now suppose that $U \to \infty$ and $a \to 0$ in such a way that the product Ua tends to a finite quantity which we shall write as $F/6\pi\mu$ with F a constant and μ the coefficient of (dynamic) viscosity of the fluid. Accordingly, the flow in this limiting situation is defined by the stream function

$$\Psi = -\frac{F}{8\pi\mu} r \sin^2\theta, \tag{4.141}$$

and the pressure

$$p = \frac{F}{4\pi\mu}\frac{\cos\theta}{r^2} + p_\infty,$$

on evaluating the appropriate limits in (4.102) and (4.108), respectively. It follows from (4.96) that the components of $\mathbf{q}$ in spherical polar coordinates are

$$q_r = \frac{F}{4\pi\mu}\frac{\cos\theta}{r}, \qquad q_\theta = -\frac{F}{8\pi\mu}\frac{\sin\theta}{r}, \qquad q_\phi = 0. \tag{4.142}$$

Keeping the viscosity μ_2 of the ambient fluid fixed in value, we see that, by letting $\lambda \to \infty$, the droplet becomes effectively solid, and in this limit we recover Stokes's law from (4.138). If, however, $\lambda \to 0$ and $\rho_2 \gg \rho_1$, we then have a model of a gas bubble in a liquid, and (4.138) gives $4\pi\mu_2 Ua$ for the hydrodynamic force acting on the bubble, while its terminal velocity is $-2ga^2/3\nu_2$, the minus sign indicating that the bubble is *rising* through the ambient fluid.

The Reynolds numbers for the flow within the droplet and ambient fluid are

$$(Re)_1 = \frac{Ua}{\nu_1}, \qquad (Re)_2 = \frac{Ua}{\nu_2}.$$

For a water droplet falling in air, we have

$$\begin{aligned} &\rho_1 = 1\ \mathrm{g/cm^3}, && \nu_1 = 0.01\ \mathrm{cm^2/s}, && \mu_1 = 0.01\ \mathrm{P},\\ &\rho_2 = 1.3 \times 10^{-3}\ \mathrm{g/cm^3}, && \nu_2 = 0.15\ \mathrm{cm^2/s}, && \mu_2 = 1.95 \times 10^{-4}\ \mathrm{P}. \end{aligned}$$

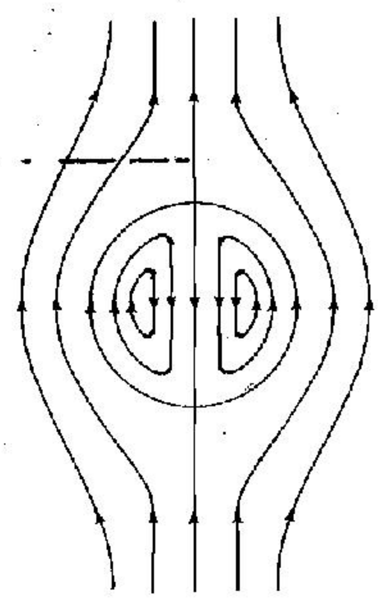
Thus $\lambda = \mu_1/\mu_2 = 51.28$, $\rho_1/\rho_2 = 769.23$, giving

$$(Re)_1 = 1.13 \times 10^8 a^3, \qquad (Re)_2 = 7.50 \times 10^6 a^3.$$

Accordingly, the Stokes flow approximation is valid provided that $(Re)_1 \ll 1$, which requires that $a \ll 2.07 \times 10^{-3}$ cm/s. The terminal settling velocity U of droplets of this size is then about $1.13 \times 10^6 a^2$ cm/s.

The streamlines for the flows within the droplet and ambient fluid are sketched in Fig. 4.7 in a frame of reference at rest relative to the droplet.

The above solution has been found *without* using the normal stress condition



The pressures p_1 and p_2 within the droplet and ambient fluid respectively may be obtained by solving

$$\frac{\partial p_i}{\partial r}=-\frac{\mu_i}{r^2\sin\theta}\frac{\partial}{\partial\theta}(E^2\Psi_i)-\rho_i g\cos\theta,$$

$$\frac{1}{r}\frac{\partial p_i}{\partial\theta}=\frac{\mu_i}{r\sin\theta}\frac{\partial}{\partial r}(E^2\Psi_i)+\rho_i g\sin\theta,$$

for $i=1,2$. After substitution of Ψ_i, it is found that

$$p_1=\frac{\mu_1 U}{a^2}\frac{5}{(1+\lambda)}r\cos\theta-\rho_1 gr\cos\theta+\Pi_1, \tag{4.135}$$

$$p_2=-\frac{\mu_2 U}{2}\frac{(2+3\lambda)}{(1+\lambda)}\frac{\cos\theta}{r^2}-\rho_2 gr\cos\theta+\Pi_2, \tag{4.136}$$

where the constants Π_1 and Π_2 represent the hydrostatic pressures within the droplet and ambient fluid in the absence of motion.

The force acting on the droplet may be calculated from the general expression (4.86), which gives in this case

$$\mathbf{F}=\pi\mu_2\mathbf{k}a^4\int_0^\pi\sin^3\theta\left[\frac{\partial}{\partial r}\left(\frac{E^2\Psi_2}{r^2\sin^2\theta}\right)\right]_{r=a}\mathrm{d}\theta. \tag{4.137}$$

From (4.99).

$$E^2\Psi_2=-2Dr^{-1}\sin^2\theta,$$

and since $\int_0^\pi\sin^3\theta\,\mathrm{d}\theta=\frac{4}{3}$, (4.137) gives $\mathbf{F}=F\mathbf{k}$ with

$$F=\frac{2\pi\mu_2 Ua(2+3\lambda)}{1+\lambda}. \tag{4.138}$$

However, we assume that the droplet falls with its terminal velocity. Thus F is balanced by the weight of the droplet minus the buoyancy force. Hence

$$\frac{\mu_2 Ua(2+3\lambda)}{1+\lambda}=\tfrac{2}{3}(\rho_1-\rho_2)ga^3,$$

showing that the terminal velocity of the droplet is

$$U=\frac{2ga^2(1+\lambda)}{3\nu_2(2+3\lambda)}\left(\frac{\rho_1}{\rho_2}-1\right). \tag{4.139}$$

and thus (4.127) may be replaced by

$$\Psi_1 = \Psi_2 = 0 \qquad (r = a). \tag{4.130}$$

On using the results of Example 2.2, it can be shown that (4.123) is equivalent to

$$\mu_1 r \frac{\partial}{\partial r}\left(\frac{v_1}{r}\right) + \frac{\mu_1}{r}\frac{\partial u_1}{\partial \theta} = \mu_2 r \frac{\partial}{\partial r}\left(\frac{v_2}{r}\right) + \frac{\mu_2}{r}\frac{\partial u_2}{\partial \theta} \qquad (r = a),$$

which, by virtue of (4.120), reduces to

$$\lambda \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \Psi_1}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \Psi_2}{\partial r}\right) \qquad (r = a), \tag{4.131}$$

with $\lambda = \mu_1/\mu_2$. The normal stress condition (4.124) gives

$$p_1 - 2\mu_1 \frac{\partial u_1}{\partial r} - p_2 + 2\mu_2 \frac{\partial u_2}{\partial r} = \frac{2\gamma}{a} \qquad (r = a)$$

which, by (4.129), reduces to

$$p_1 + \frac{2\mu_1}{r^2 \sin\theta}\frac{\partial^2 \Psi_1}{\partial r\,\partial \theta} - p_2 - \frac{2\mu_2}{r^2 \sin\theta}\frac{\partial^2 \Psi_2}{\partial r\,\partial \theta} = \frac{2\gamma}{a} \qquad (r = a). \tag{4.132}$$

From section 4.14, suitable forms for Ψ_1 and Ψ_2 which satisfy (4.120) and (4.126) are evidently

$$\Psi_1 = (Ar^2 + Br^4)\sin^2\theta \qquad (r \leqslant a), \tag{4.133}$$

$$\Psi_2 = \left(\frac{C}{r} + Dr - \tfrac{1}{2}Ur^2\right)\sin^2\theta \qquad (r \geqslant a). \tag{4.134}$$

Thus there are *four* constants to be determined but *five* equations to be satisfied. It must be remembered that the assumption that the droplet remains spherical has imposed the additional boundary condition (4.129) which is not one of the conditions (4.122)–(4.124) imposed purely by the kinematics and dynamics of the problem. On solving the four equations given by (4.128), (4.131) and (4.132), we obtain

$$A = \frac{U}{4(1+\lambda)}, \qquad B = -\frac{U}{4a^2(1+\lambda)},$$

$$C = -\frac{Ua^3\lambda}{4(1+\lambda)}, \qquad D = \frac{Ua(2+3\lambda)}{4(1+\lambda)}.$$

the interfacial surface tension at the droplet surface, the equations governing the flows in the droplet and ambient fluid are

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$$\nabla p_i = \mu_i \nabla^2 \mathbf{q}_i - \rho_i g \mathbf{k}, \tag{4.118}$$

q ← سرعت

$$\nabla \cdot \mathbf{q}_i = 0, \tag{4.119}$$

where $i = 1, 2$. The two Stokes equations (4.118) assume that the Reynolds numbers Ua/ν_1 and Ua/ν_2 are very small. Taking Cartesian axes *fixed relative to the droplet* and (r, θ, ϕ) spherical polar coordinates with origin at the centre of the droplet, the boundary conditions are

$$\mathbf{q}_2 \to U\mathbf{k} \qquad (r \to \infty), \tag{4.120}$$

$$|\mathbf{q}_1| \text{ is bounded} \qquad (r = 0), \tag{4.121}$$

$$\mathbf{q}_1 = \mathbf{q}_2 \qquad (r = a), \tag{4.122}$$

شرط عدم لغزش

$$(p_{r\theta})_2 = (p_{r\theta})_1 \qquad (r = a), \tag{4.123}$$

برابری تنش در مرز

$$(p_{rr})_2 - (p_{rr})_1 = \frac{2\gamma}{a} \qquad (r = a). \tag{4.124}$$

Since both droplet and ambient fluid motions are symmetric about the z axis, we may introduce stream functions Ψ_1 and Ψ_2 to satisfy (4.119). The non-zero components of velocity are accordingly

$$\begin{aligned} (q_r)_i = u_i &= -\frac{1}{r^2 \sin\theta} \frac{\partial \Psi_i}{\partial \theta} \qquad (i = 1, 2), \\ (q_\theta)_i = v_i &= \frac{1}{r \sin\theta} \frac{\partial \Psi_i}{\partial r} \qquad (i = 1, 2). \end{aligned} \tag{4.125}$$

Equation (4.120) implies that

$$\Psi_2 \sim \tfrac{1}{2} U r^2 \sin^2\theta \qquad (r \to \infty), \tag{4.126}$$

while (4.122) gives

$$\frac{\partial \Psi_1}{\partial \theta} = \frac{\partial \Psi_2}{\partial \theta} \qquad (r = a), \tag{4.127}$$

$$\frac{\partial \Psi_1}{\partial r} = \frac{\partial \Psi_2}{\partial r} \qquad (r = a). \tag{4.128}$$

It should be noted that the assumption that the droplet *remains* spherical in shape as it falls means that

$$u_1 = u_2 = 0 \qquad (r = a), \tag{4.129}$$

also the *kinematic* condition

$$(\hat{\mathbf{n}}\cdot\mathbf{q})_2 = (\hat{\mathbf{n}}\cdot\mathbf{q})_1$$

at P, expressing the mutual impenetrability of the interface, as well as continuity of velocity tangential to S:

$$(\hat{\mathbf{t}}\cdot\mathbf{q})_2 = (\hat{\mathbf{t}}\cdot\mathbf{q})_1$$

at P, which follows from the assumption that the two immiscible fluids cannot slip over each other because of viscosity.

4.17 LIQUID DROPLET FALLING UNDER GRAVITY

We shall make the following assumptions.

(1) The droplet is falling steadily under gravity in an infinite fluid which does not mix with that of the droplet, typically a water droplet in air.
(2) The droplet remains spherical during its motion.

Assumption (1) implies that the droplet is falling with its terminal velocity, defined generally in section 4.14. Assumption (2) can be shown to be valid *a posteriori* if the motions of droplet and ambient fluid are Stokes flows. Experimental observation also confirms that negligible deformation of the droplet shape occurs in those flow situations for which the Stokes equations are a good approximation to the Navier–Stokes equations.

Let the spherical droplet have radius a and its terminal velocity be $-U\mathbf{k}$, with the direction of $\mathbf{k}$ vertically upwards (Fig. 4.6). Letting subscripts 1 and 2 relate to quantities associated with the droplet fluid and ambient fluid respectively, and γ be

UTAH

Laminar Premix Flame

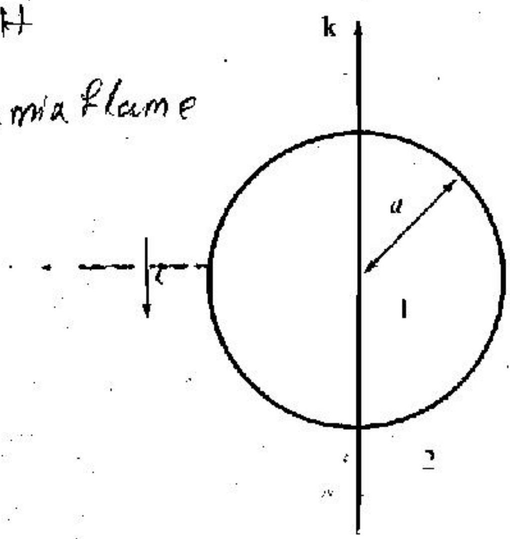

Fig. 4.6

Polar: $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right) + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in spherical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 - v_\phi^2}{r}\right)$$
$$= -\frac{\partial P}{\partial r} + \mu\left(\nabla^2 v_r - \frac{2v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2v_\theta\cot\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right) + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right)$$
$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta\cot\theta}{r}\right)$$
$$= -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi} + \mu\left(\nabla^2 v_\phi - \frac{v_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \phi}\right) + \rho g_\phi$$

where, in these equations, $\nabla \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)$.

Reference: R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, Wiley: NY, 1960.

The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

CM3110 Fall 2007 Faith A. Morrison

Continuity Equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

Continuity Equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\phi)}{\partial \phi} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial P}{\partial y} - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} - \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho g_z$$

Equation of Motion for an incompressible fluid, 3 components in cylindrical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r} - \left(\frac{1}{r}\frac{\partial(r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z}\right) + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} - \left(\frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} - \left(\frac{1}{r}\frac{\partial(r\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho g_z$$

Equation of Motion for an incompressible fluid, 3 components in spherical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2+v_\phi^2}{r}\right)$$
$$= -\frac{\partial P}{\partial r} - \left(\frac{1}{r^2}\frac{\partial(r^2\tau_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\tau_{r\theta}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{r\phi}}{\partial \phi} - \frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r}\right) + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right)$$
$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} - \left(\frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\tau_{\theta\theta}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\theta\phi}}{\partial \phi} + \frac{\tau_{r\theta}}{r} - \frac{(\cot\theta)\tau_{\phi\phi}}{r}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta\cot\theta}{r}\right)$$
$$= -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi} - \left(\frac{1}{r^2}\frac{\partial(r^2\tau_{r\phi})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{\tau_{r\phi}}{r} + \frac{(2\cot\theta)\tau_{\theta\phi}}{r}\right) + \rho g_\phi$$

A change of variables on the Cartesian equations will yield[11] the following momentum equations for r, φ, and z:

$$r:\quad \rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\phi^2}{r}\right)=-\frac{\partial p}{\partial r}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \phi^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi}\right]+\rho g_r$$

$$\phi:\quad \rho\left(\frac{\partial u_\phi}{\partial t}+u_r\frac{\partial u_\phi}{\partial r}+\frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi}+u_z\frac{\partial u_\phi}{\partial z}+\frac{u_r u_\phi}{r}\right)=-\frac{1}{r}\frac{\partial p}{\partial \phi}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial \phi^2}+\frac{\partial^2 u_\phi}{\partial z^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \phi}-\frac{u_\phi}{r^2}\right]+\rho g_\phi$$

$$z:\quad \rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\phi}{r}\frac{\partial u_z}{\partial \phi}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \phi^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+\rho g_z.$$

The gravity components will generally not be constants, however for most applications either the coordinates are chosen so that the gravity components are constant or else it is assumed that gravity is counteracted by a pressure field (for example, flow in horizontal pipe is treated normally without gravity and without a vertical pressure gradient). The continuity equation is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r u_r\right)+\frac{1}{r}\frac{\partial u_\phi}{\partial \phi}+\frac{\partial u_z}{\partial z}=0.$$

This cylindrical representation of the incompressible Navier–Stokes equations is the second most commonly seen (the first being Cartesian above). Cylindrical coordinates are chosen to take advantage of symmetry, so that a velocity component can disappear. A very common case is axisymmetric flow with the assumption of no tangential velocity ($u_\varphi = 0$), and the remaining quantities are independent of φ:

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)+\frac{\partial^2 u_r}{\partial z^2}-\frac{u_r}{r^2}\right]+\rho g_r$$

polar

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{\partial^2 u_z}{\partial z^2}\right]+\rho g_z$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r u_r\right)+\frac{\partial u_z}{\partial z}=0.$$

Spherical coordinates

In spherical coordinates, the r, θ, and φ momentum equations are[11] *(note the convention used: θ is colatitude*[13]*):*

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\phi}{r\sin(\theta)}\frac{\partial u_r}{\partial \phi}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\phi^2+u_\theta^2}{r}\right)=-\frac{\partial p}{\partial r}+\rho g_r+$$

$$\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_r}{\partial r}\right)+\frac{1}{r^2\sin(\theta)^2}\frac{\partial^2 u_r}{\partial \phi^2}+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial u_r}{\partial \theta}\right)-2\overbrace{\frac{u_r+\frac{\partial u_\theta}{\partial \theta}+u_\theta\cot(\theta)}{r^2}}^{?}+\frac{2}{r^2\sin(\theta)}\frac{\partial u_\phi}{\partial \phi}\right]$$

ϕ mom

$$\rho\left(\frac{\partial u_\phi}{\partial t}+u_r\frac{\partial u_\phi}{\partial r}+\frac{u_\phi}{r\sin(\theta)}\frac{\partial u_\phi}{\partial \phi}+\frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta}+\frac{u_r u_\phi+u_\phi u_\theta\cot(\theta)}{r}\right)=-\frac{1}{r\sin(\theta)}\frac{\partial p}{\partial \phi}+\rho g_\phi+$$

$$\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\phi}{\partial r}\right)+\frac{1}{r^2\sin(\theta)^2}\frac{\partial^2 u_\phi}{\partial \phi^2}+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial u_\phi}{\partial \theta}\right)+\frac{2\frac{\partial u_r}{\partial \phi}+2\cos(\theta)\frac{\partial u_\theta}{\partial \phi}-u_\phi}{r^2\sin(\theta)^2}\right]$$

spherical

θ mom

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\phi}{r\sin(\theta)}\frac{\partial u_\theta}{\partial \phi}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta-u_\phi^2\cot(\theta)}{r}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\rho g_\theta+$$

$$\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right)+\frac{1}{r^2\sin(\theta)^2}\frac{\partial^2 u_\theta}{\partial \phi^2}+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial u_\theta}{\partial \theta}\right)+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}-\overbrace{\frac{u_\theta+2\cos(\theta)\frac{\partial u_\phi}{\partial \phi}}{r^2\sin(\theta)^2}}^{?}\right].$$

Takehome,

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Mass continuity will read:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right)+\frac{1}{r\sin(\theta)}\frac{\partial u_\phi}{\partial \phi}+\frac{1}{r\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)u_\theta\right)=0.$$

These equations could be (slightly) compacted by, for example, factoring $1/r^2$ from the viscous terms. However, doing so would undesirably alter the structure of the Laplacian and other quantities.

Stream function formulation

Taking the curl of the Navier–Stokes equation results in the elimination of pressure. This is especially easy to see if 2D Cartesian flow is assumed ($w=0$ and no dependence of anything on z), where the equations reduce to: