Introduction to the Finite Element Method

Lecture 6: 2-D Elements: Triangular element

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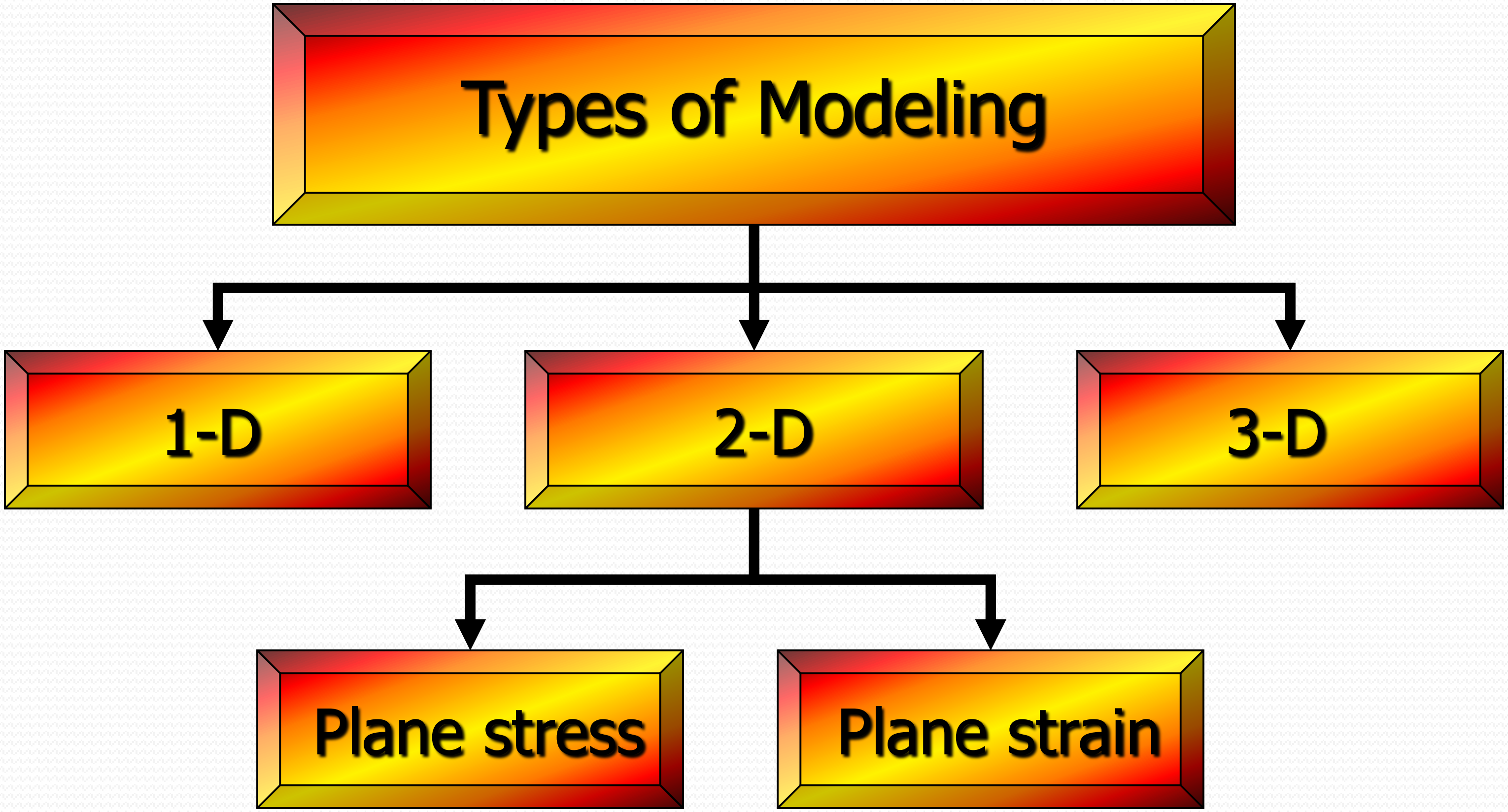
Outline

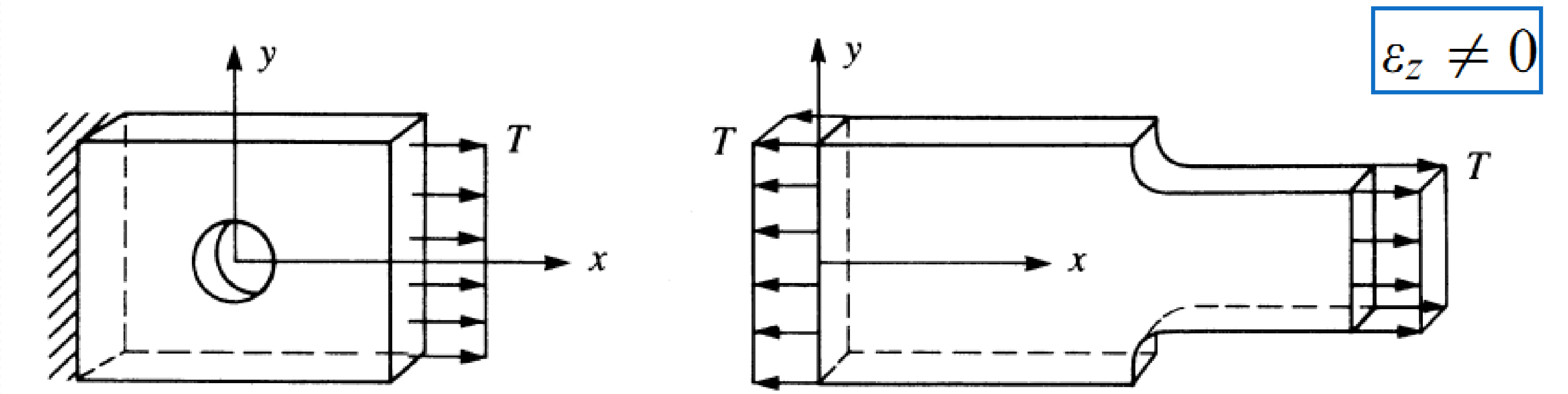
- 2-D Elements: Plain Stress and Plane Strain
- CST Element Stiffness
- Body and Surface Forces
- Examples
- LST Element Stiffness
- Comparison of Elements

Types of Modeling



2-D Elements

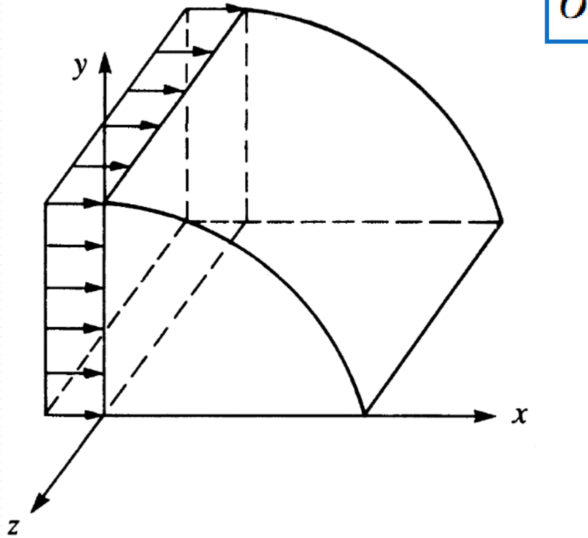
- 2-D elements:
  - Plane stress: plates with holes, fillets
  - Plane strain: a long underground box
- Plane Stress:
  - the normal stress (σ_z) and the shear stresses (τ_{xz} and τ_{yz}) perpendicular to the plane are assumed to be zero
  - For thin members when loads act only in the x-y plane

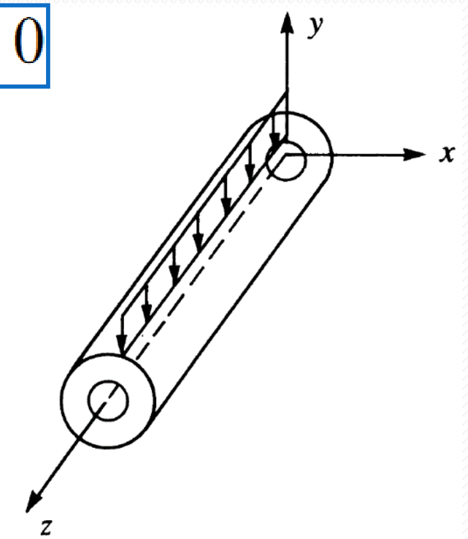


2-D Elements

- Plane Strain:
  - the strain normal to the x-y plane (ε_z) and the shear strains (γ_{xz} and γ_{yz}) are assumed to be zero
  - For long members with constant cross-sectional area subjected to loads that act only in the x and/or y directions and do not vary in the z direction.

$$\sigma_z \neq 0$$





2-D Elements

- General 3-D stress-strain relations

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

Strain vs. Stress

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x(1-\nu) + \nu\varepsilon_y + \nu\varepsilon_z]$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + (1-\nu)\varepsilon_y + \nu\varepsilon_z]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + \nu\varepsilon_y + (1-\nu)\varepsilon_z]$$

$$\tau_{xy} = G\gamma_{xy} \qquad \tau_{yz} = G\gamma_{yz} \qquad \tau_{zx} = G\gamma_{zx}$$

Stress vs. Strain

- For 2-D problems:

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \qquad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad \{\sigma\} = [D]\{\varepsilon\}$$

Plane Stress

$$\sigma_z = 0$$

$$\tau_{xz} = \tau_{yz} = 0$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \upsilon\sigma_y\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \upsilon\sigma_x\right]$$

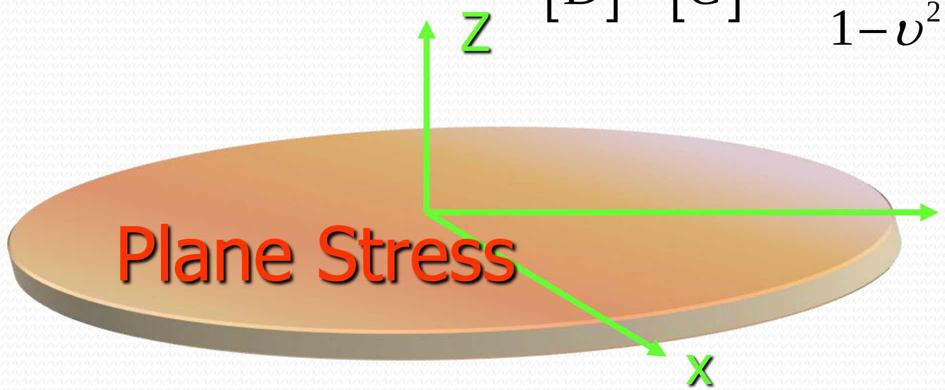
$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\varepsilon = [C]\sigma$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\upsilon & 0 \\ -\upsilon & 1 & 0 \\ 0 & 0 & 2(1+\upsilon) \end{bmatrix}\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$\sigma = [D]\varepsilon$$

$$[D] = [C]^{-1} = \frac{E}{1-\upsilon^2}\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix}$$



Plane Strain

$$\varepsilon_z = 0$$

$$\gamma_{xz} = \gamma_{yz} = 0$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \upsilon(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \upsilon(\sigma_x + \sigma_z)\right]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

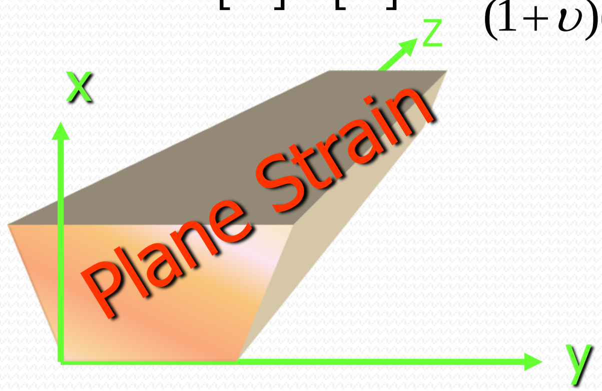
$$\sigma_z = \upsilon(\sigma_x + \sigma_y)$$

$$\varepsilon = [C]\sigma$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1+\upsilon}{E}\begin{bmatrix} 1-\upsilon & -\upsilon & 0 \\ -\upsilon & 1-\upsilon & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$\sigma = [D]\varepsilon$$

$$[D] = [C]^{-1} = \frac{E}{(1+\upsilon)(1-2\upsilon)}\begin{bmatrix} 1-\upsilon & \upsilon & 0 \\ \upsilon & 1-\upsilon & 0 \\ 0 & 0 & \frac{1-2\upsilon}{2} \end{bmatrix}$$



2-D Elements

- For plane stress:

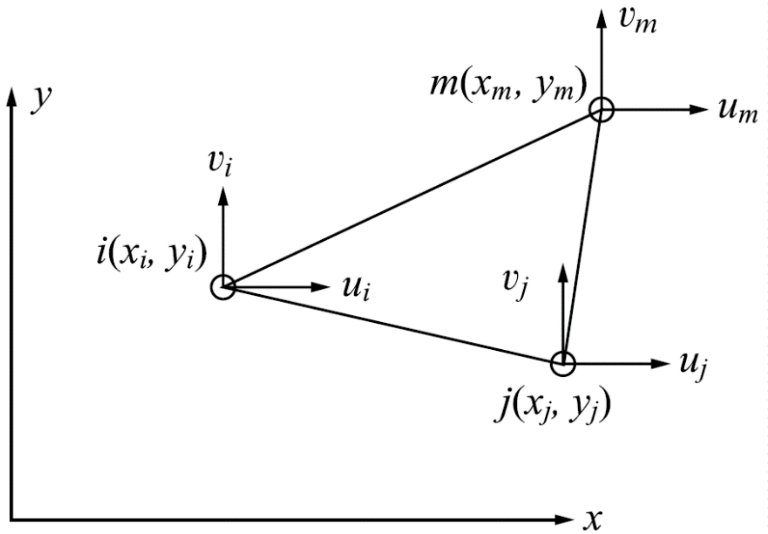
$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

- For plane strain:

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

CST Element Stiffness

- Counterclockwise node labeling (i, j, m)
- Linear displacement functions:



$$\{d\} = \begin{Bmatrix} \underline{d}_i \\ \underline{d}_j \\ \underline{d}_m \end{Bmatrix} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

$$u(x,y) = a_1 + a_2 x + a_3 y$$

$$v(x,y) = a_4 + a_5 x + a_6 y$$

$$\{\psi\} = \begin{Bmatrix} a_1 + a_2 x + a_3 y \\ a_4 + a_5 x + a_6 y \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

CST: displacement

$$u_i = u(x_i, y_i) = a_1 + a_2 x_i + a_3 y_i$$

$$u_j = u(x_j, y_j) = a_1 + a_2 x_j + a_3 y_j$$

$$u_m = u(x_m, y_m) = a_1 + a_2 x_m + a_3 y_m$$

$$v_i = v(x_i, y_i) = a_4 + a_5 x_i + a_6 y_i$$

$$v_j = v(x_j, y_j) = a_4 + a_5 x_j + a_6 y_j$$

$$v_m = v(x_m, y_m) = a_4 + a_5 x_m + a_6 y_m$$

$$\Rightarrow$$

$$\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} v_i \\ v_j \\ v_m \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

$$\{u\} = [x]\{a\} \Rightarrow \{a\} = [x]^{-1}\{u\}$$

$$[x]^{-1} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}$$

$$2A = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix}$$

$$= x_i(y_j - y_m) + x_j(y_m - y_i) + x_m(y_i - y_j)$$

CST: displacement

$$\alpha_j = y_i x_m - x_i y_m \qquad \alpha_m = x_i y_j - y_i x_j$$

$$\beta_j = y_m - y_i \qquad \beta_m = y_i - y_j$$

$$\gamma_j = x_i - x_m \qquad \gamma_m = x_j - x_i$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

$$\{u\} = [1 \quad x \quad y] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} [1 \quad x \quad y] \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

$$u(x, y) = \frac{1}{2A} \{(\alpha_i + \beta_i x + \gamma_i y) u_i + (\alpha_j + \beta_j x + \gamma_j y) u_j + (\alpha_m + \beta_m x + \gamma_m y) u_m\}$$

CST: displacement

Similarly,

$$v(x,y) = \frac{1}{2A}\{(\alpha_i + \beta_i x + \gamma_i y)v_i + (\alpha_j + \beta_j x + \gamma_j y)v_j + (\alpha_m + \beta_m x + \gamma_m y)v_m\}$$

To express u and v in simpler form, we define

$$N_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y)$$

$$N_j = \frac{1}{2A}(\alpha_j + \beta_j x + \gamma_j y)$$

$$N_m = \frac{1}{2A}(\alpha_m + \beta_m x + \gamma_m y)$$

$$\{\psi\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{Bmatrix} N_i u_i + N_j u_j + N_m u_m \\ N_i v_i + N_j v_j + N_m v_m \end{Bmatrix}$$

CST: displacement

$$\{\psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

$$\{\psi\} = [N]\{d\}$$

where $[N]$ is given by

$$[N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix}$$

CST: Shape Functions

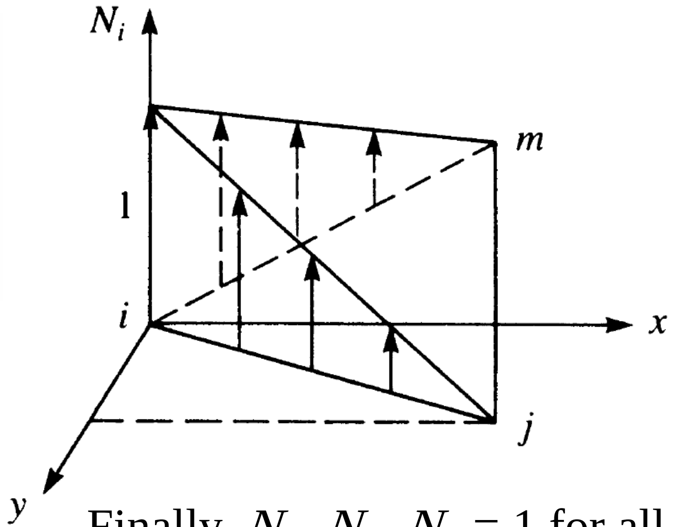
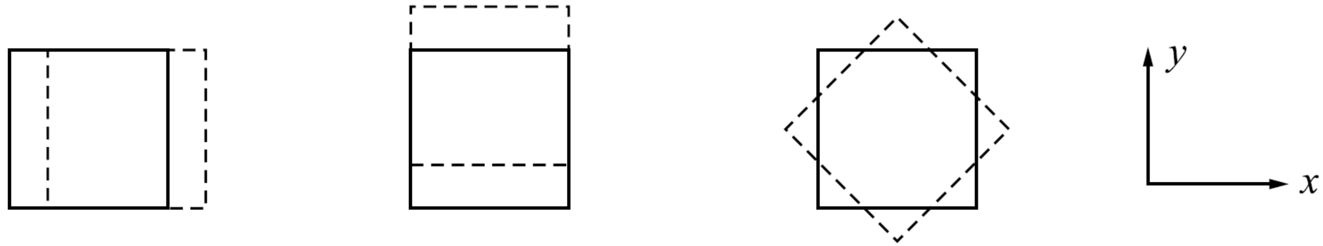


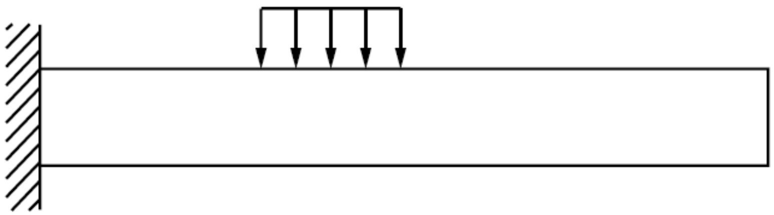
Figure 6–8 Variation of N_i over the *x-y* surface of a typical element

Finally, N_i , N_j , N_m= 1 for all x and y locations on the surface of the element so that u and v will yield a constant value when **rigid-body** displacement occurs.

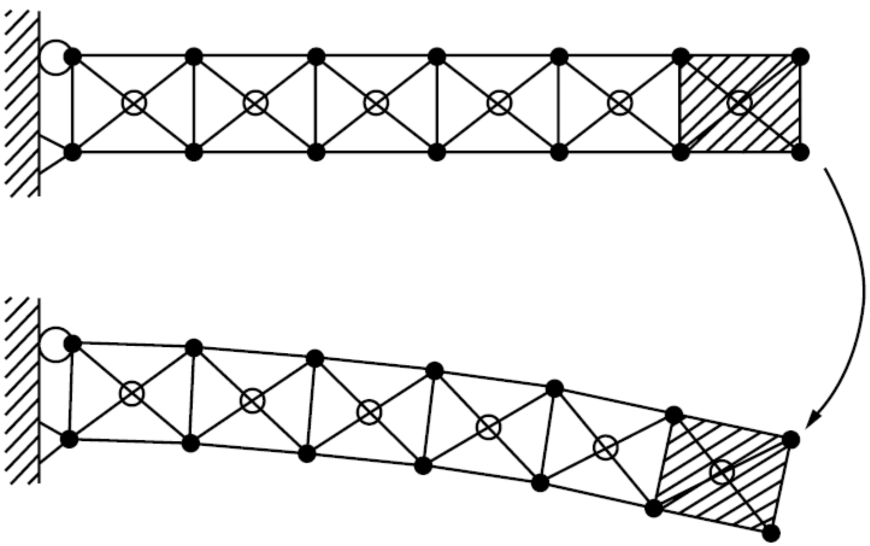


(a) Rigid-body modes of a plane stress element (from left to right, pure translation in *x* and *y* directions and pure rotation)

CST: Shape Functions



the beam elements beyond the loading are stress-free. Hence these elements must be free to translate and rotate without stretching or changing shape.



Rigid-body translation and rotation occurs for elements to right of load

Cantilever beam modeled using constant-strain triangle elements

CST: Shape Functions

The strains associated with the 2D element are given by

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

For the displacements, we have

$$\frac{\partial u}{\partial x} = u_{,x} = \frac{\partial}{\partial x}(N_i u_i + N_j u_j + N_m u_m)$$

$$u_{,x} = N_{i,x} u_i + N_{j,x} u_j + N_{m,x} u_m$$

The derivatives of the shape functions are

$$N_{i,x} = \frac{1}{2A}\frac{\partial}{\partial x}(\alpha_i + \beta_i x + \gamma_i y) = \frac{\beta_i}{2A}$$

Similarly, $N_{j,x} = \dfrac{\beta_j}{2A}$ and $N_{m,x} = \dfrac{\beta_m}{2A}$

CST: Element Strain

Therefore, we have

$$\frac{\partial u}{\partial x} = \frac{1}{2A}(\beta_i u_i + \beta_j u_j + \beta_m u_m)$$

Similarly, we can obtain

$$\frac{\partial v}{\partial y} = \frac{1}{2A}(\gamma_i v_i + \gamma_j v_j + \gamma_m v_m)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{2A}(\gamma_i u_i + \beta_i v_i + \gamma_j u_j + \beta_j v_j + \gamma_m u_m + \beta_m v_m)$$

Finally, we have

$$\{\varepsilon\} = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} = [\underline{B}_i \quad \underline{B}_j \quad \underline{B}_m]\begin{Bmatrix} \underline{d}_i \\ \underline{d}_j \\ \underline{d}_m \end{Bmatrix}$$

CST: Element Strain

where

$$[B_i] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \gamma_i & \beta_i \end{bmatrix} \qquad [B_j] = \frac{1}{2A}\begin{bmatrix} \beta_j & 0 \\ 0 & \gamma_j \\ \gamma_j & \beta_j \end{bmatrix} \qquad [B_m] = \frac{1}{2A}\begin{bmatrix} \beta_m & 0 \\ 0 & \gamma_m \\ \gamma_m & \beta_m \end{bmatrix}$$

Finally, in simplified matrix form, we have

$$\{\varepsilon\} = [B]\{d\}$$

$$[B] = [\underline{B}_i \quad \underline{B}_j \quad \underline{B}_m]$$

The B matrix is *independent* of the x and y coordinates. It depends solely on the element nodal coordinates. The strains are *constant*; hence, the element is called a ***Constant-Strain Triangle*** (CST).

CST: Stiffness

- Total Potential Energy

$$\pi_p = U + \Omega_b + \Omega_p + \Omega_s$$

- Strain energy

$$U = \frac{1}{2}\iiint_V \{\varepsilon\}^T \{\sigma\}\, dV \quad \xrightarrow{\{\sigma\}=[D]\{\varepsilon\}} \quad U = \frac{1}{2}\iiint_V \{\varepsilon\}^T [D]\{\varepsilon\}\, dV$$

- Potential energy of the body forces:

$$\Omega_b = -\iiint_V \{\psi\}^T \{X\}\, dV$$

- Potential energy of concentrated loads:

$$\Omega_p = -\{d\}^T \{P\}$$

- Potential energy of surface tractions:

$$\Omega_s = -\iint_S \{\psi_S\}^T \{T_S\}\, dS$$

CST: Stiffness

$$\pi_p = \frac{1}{2}\iiint_V \{d\}^T[B]^T[D][B]\{d\}\,dV - \iiint_V \{d\}^T[N]^T\{X\}\,dV$$

$$- \{d\}^T\{P\} - \iint_S \{d\}^T[N_S]^T\{T_S\}\,dS$$

$$\pi_p = \frac{1}{2}\{d\}^T \iiint_V [B]^T[D][B]\,dV\{d\} - \{d\}^T \iiint_V [N]^T\{X\}\,dV$$

$$- \{d\}^T\{P\} - \{d\}^T \iint_S [N_S]^T\{T_S\}\,dS$$

$$\pi_p = \frac{1}{2}\{d\}^T \iiint_V [B]^T[D][B]\,dV\{d\} - \{d\}^T\{f\}$$

$$\{f\} = \iiint_V [N]^T\{X\}\,dV + \{P\} + \iint_S [N_S]^T\{T_S\}\,dS$$

CST: Stiffness

- Derivation with respect to displacement

$$\frac{\partial \pi_p}{\partial \{d\}} = \left[\iiint_V [B]^T [D][B]\, dV \right] \{d\} - \{f\} = 0$$

$$\iiint_V [B]^T [D][B]\, dV \{d\} = \{f\}$$

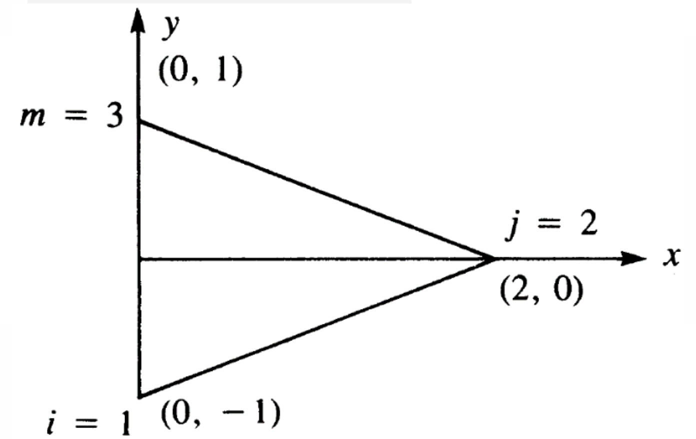
- Stiffness Matrix

$$[k] = \iiint_V [B]^T [D][B]\, dV$$

  - Constant thickness → $[k] = tA[B]^T [D][B]$
  - Integrand is not a function of x or y

Example 1

Evaluate the stiffness matrix for the plane stress element. Let thickness t = 1 in. Assume the element nodal displacements have been determined to be $u_1 = 0.0$, $v_1 = 0.0025$ in., $u_2 = 0.0012$ in., $v_2 = 0.0$, $u_3 = 0.0$, and $v_3 = 0.0025$ in. Determine the element stresses. $\left(E = 30\times10^6 \text{psi},\ \upsilon = 0.25\right)$



We first obtain the β's and γ's as follows:

$$\beta_i = y_j - y_m = 0 - 1 = -1 \qquad \gamma_i = x_m - x_j = 0 - 2 = -2$$

$$\beta_j = y_m - y_i = 1 - (-1) = 2 \qquad \gamma_j = x_i - x_m = 0 - 0 = 0$$

$$\beta_m = y_i - y_j = -1 - 0 = -1 \qquad \gamma_m = x_j - x_i = 2 - 0 = 2$$

Example 1

We obtain matrix $\underline{B}$ as

$$\underline{B} = \frac{1}{2(2)} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix}$$

For plane stress conditions

$$\underline{D} = \frac{30 \times 10^6}{1-(0.25)^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \dfrac{1-0.25}{2} \end{bmatrix} \text{psi}$$

$$\underline{k} = \frac{(2)30 \times 10^6}{4(0.9375)} \begin{bmatrix} -1 & 0 & -2 \\ 0 & -2 & -1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \frac{1}{2(2)} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix}$$

Example 1

$$\underline{k} = 4.0 \times 10^6 \begin{bmatrix} 2.5 & 1.25 & -2 & -1.5 & -0.5 & 0.25 \\ 1.25 & 4.375 & -1 & -0.75 & -0.25 & -3.625 \\ -2 & -1 & 4 & 0 & -2 & 1 \\ -1.5 & -0.75 & 0 & 1.5 & 1.5 & -0.75 \\ -0.5 & -0.25 & -2 & 1.5 & 2.5 & -1.25 \\ 0.25 & -3.625 & 1 & -0.75 & -1.25 & 4.375 \end{bmatrix} \frac{\text{lb}}{\text{in.}}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{30 \times 10^6}{1 - (0.25)^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \times \frac{1}{2(2)} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix} \begin{Bmatrix} 0.0 \\ 0.0025 \\ 0.0012 \\ 0.0 \\ 0.0 \\ 0.0025 \end{Bmatrix}$$

$\sigma_x = 19{,}200$ psi $\sigma_y = 4800$ psi $\tau_{xy} = -15{,}000$ psi

CST: Body force

- *Constant* body force:

$$\{f_b\} = \iiint_V [N]^T \{X\}\, dV$$

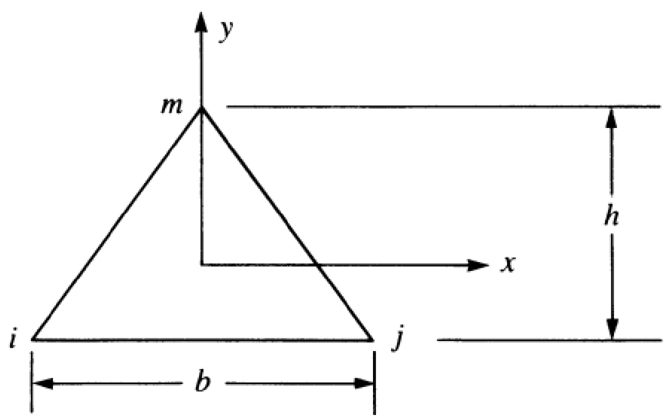
$$N_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y)$$

$$N_j = \frac{1}{2A}(\alpha_j + \beta_j x + \gamma_j y)$$

$$N_m = \frac{1}{2A}(\alpha_m + \beta_m x + \gamma_m y)$$

$$\iint \beta_i x\, dA = \iint \gamma_i y\, dA = 0$$

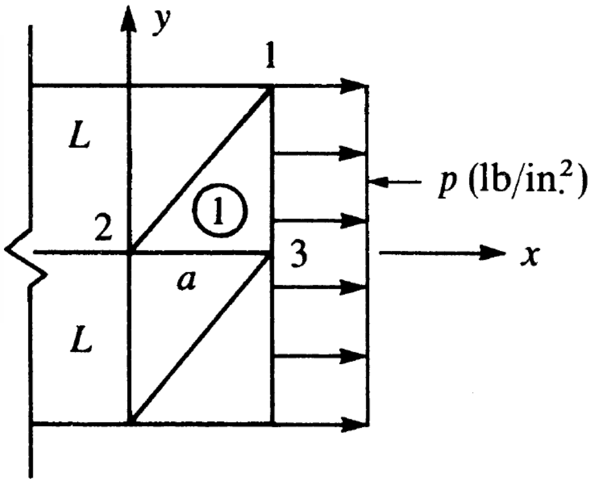
$$\alpha_i = \alpha_j = \alpha_m = \frac{2A}{3}$$

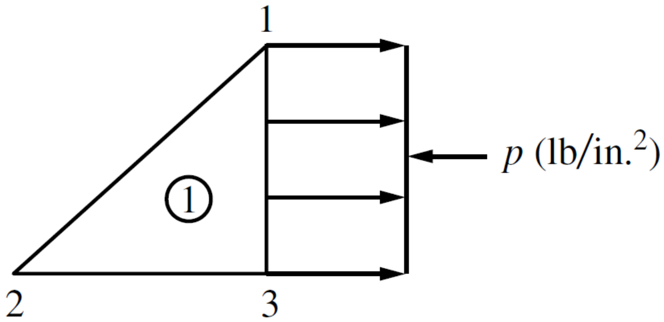


Element with centroidal coordinate axes

$$\{f_b\} = \begin{Bmatrix} f_{bix} \\ f_{biy} \\ f_{bjx} \\ f_{bjy} \\ f_{bmx} \\ f_{bmy} \end{Bmatrix} = \begin{Bmatrix} X_b \\ Y_b \\ X_b \\ Y_b \\ X_b \\ Y_b \end{Bmatrix} \frac{At}{3}$$

CST: Traction force





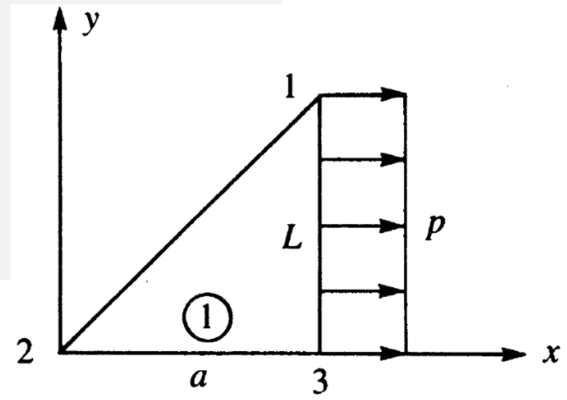
$$\{f_s\} = \iint_S [N_S]^T \{T_S\}\, dS \qquad \{T_S\} = \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} = \begin{Bmatrix} p \\ 0 \end{Bmatrix}$$

$$\{f_s\} = \int_0^t \int_0^L \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{Bmatrix} p \\ 0 \end{Bmatrix} dz\, dy$$

evaluated at $x = a,\ y = y$

CST: Body & traction force

$$\{f_s\} = t\int_0^L \begin{bmatrix} N_1 p \\ 0 \\ N_2 p \\ 0 \\ N_3 p \\ 0 \end{bmatrix} dy \quad \text{evaluated at } x = a,\ y = y$$



with $i = 1$, $j = 2$, *and* $m = 3$, $\alpha_1 = x_2 y_3 - y_2 x_3 = 0$

Similarly $\beta_1 = 0$ $\gamma_1 = a$

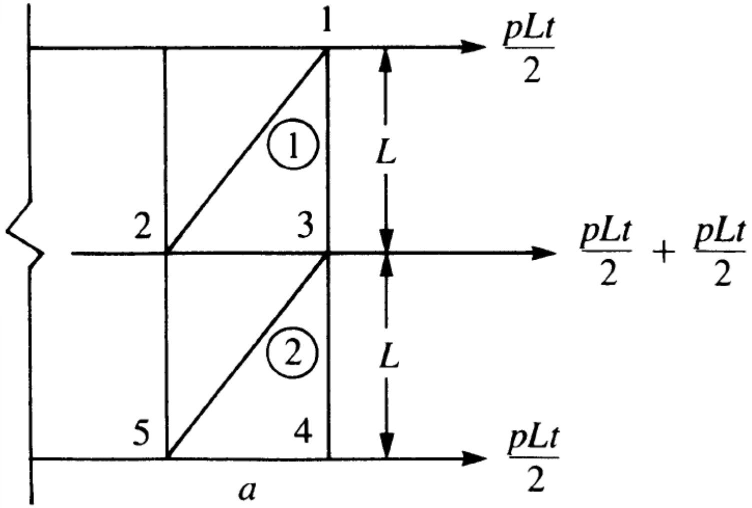
Therefore, we obtain

$$N_1 = \frac{ay}{2A} \qquad N_2 = \frac{L(a-x)}{2A} \qquad \text{and} \qquad N_3 = \frac{Lx - ay}{2A}$$

CST: Body & traction force

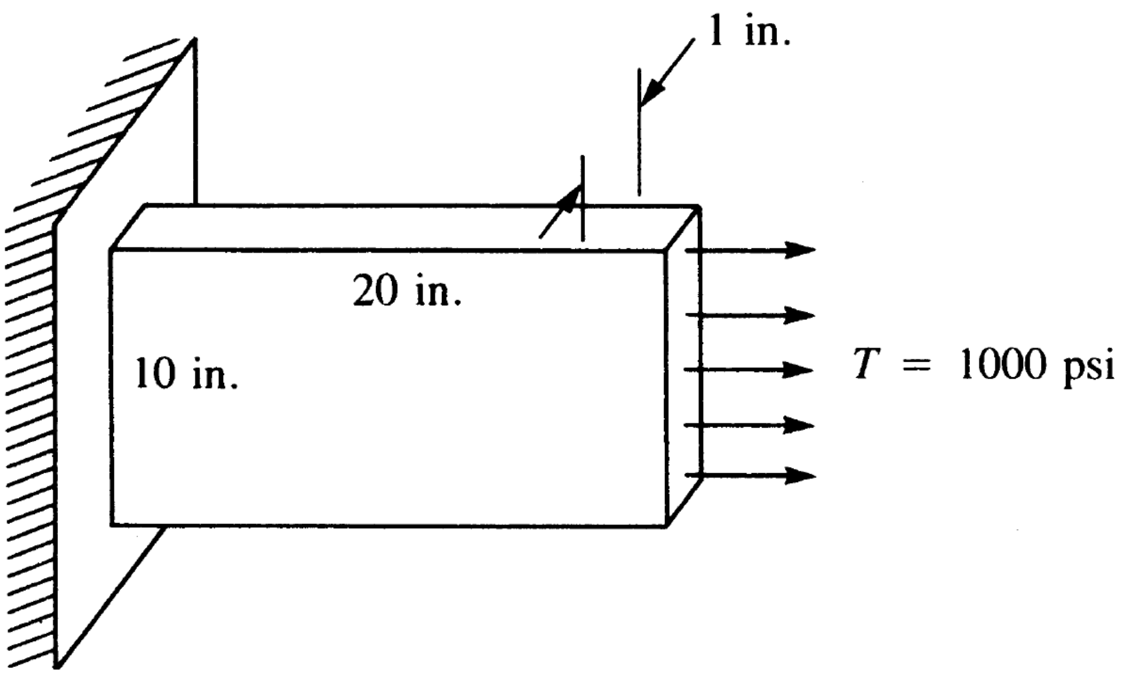
- Substituting $x=a$ and integrating

$$\{f_s\} = \frac{t}{2(aL/2)} \begin{Bmatrix} a\left(\frac{L^2}{2}\right)p \\ 0 \\ 0 \\ 0 \\ \left(L^2 - \frac{L^2}{2}\right)ap \\ 0 \end{Bmatrix} = \begin{Bmatrix} pLt/2 \\ 0 \\ 0 \\ 0 \\ pLt/2 \\ 0 \end{Bmatrix}$$



Example 2

For a thin plate subjected to the surface traction, determine the nodal displacements and the element stresses. The plate thickness t = 1 in., E = 30×10^6 psi, and $\upsilon = 0.30$.



CST Element Defects

- In bending problems, the mesh of CST elements will produce a model that is ***stiffer*** than the actual problem.

- As we will observe from the results shown for a beam-bending problem modeled by CST and LST elements, the CST model converges very ***slowly*** to the exact solution. This is partly due to the element predicting only constant stress within each element, when for a bending problem, the stress actually varies ***linearly*** through the depth of the beam.

CST Element Defects

- For a beam subjected to pure bending, the CST has a ***spurious*** or false shear stress and hence a spurious shear strain in parts of the model that should not have any shear stress or shear strain. This spurious shear strain absorbs energy; therefore, some of the energy that should go into bending is lost. The CST is then too stiff in bending, and the deformation is smaller than actually should be. This phenomenon developing in one or more modes of deformation is sometimes described as ***shear locking*** or ***parasitic shear***.

- In problems where plane strain conditions exist and the Poisson's ratio approaches 0.5, a mesh can actually ***lock***, which means the mesh then cannot deform at all.

Constant Strain Triangle

- Stiffness matrix for element $k = B^T E B \, tA$
- The CST gives good results in regions of the FE model where there is little strain gradient
  - Otherwise it does not work well.

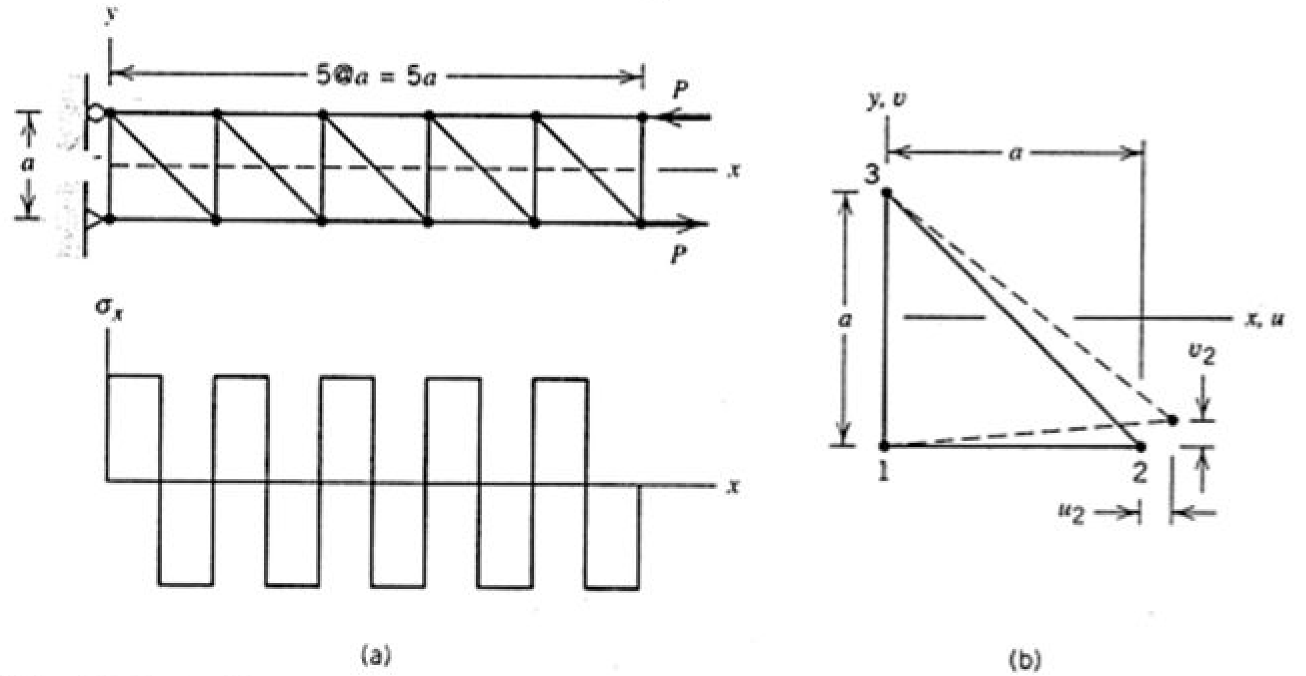


Fig. 3.2-2. (a) Stress σ_x along the x axis in a beam modeled by CSTs and loaded in pure bending. (b) Deformation of the lower-left CST in the model.

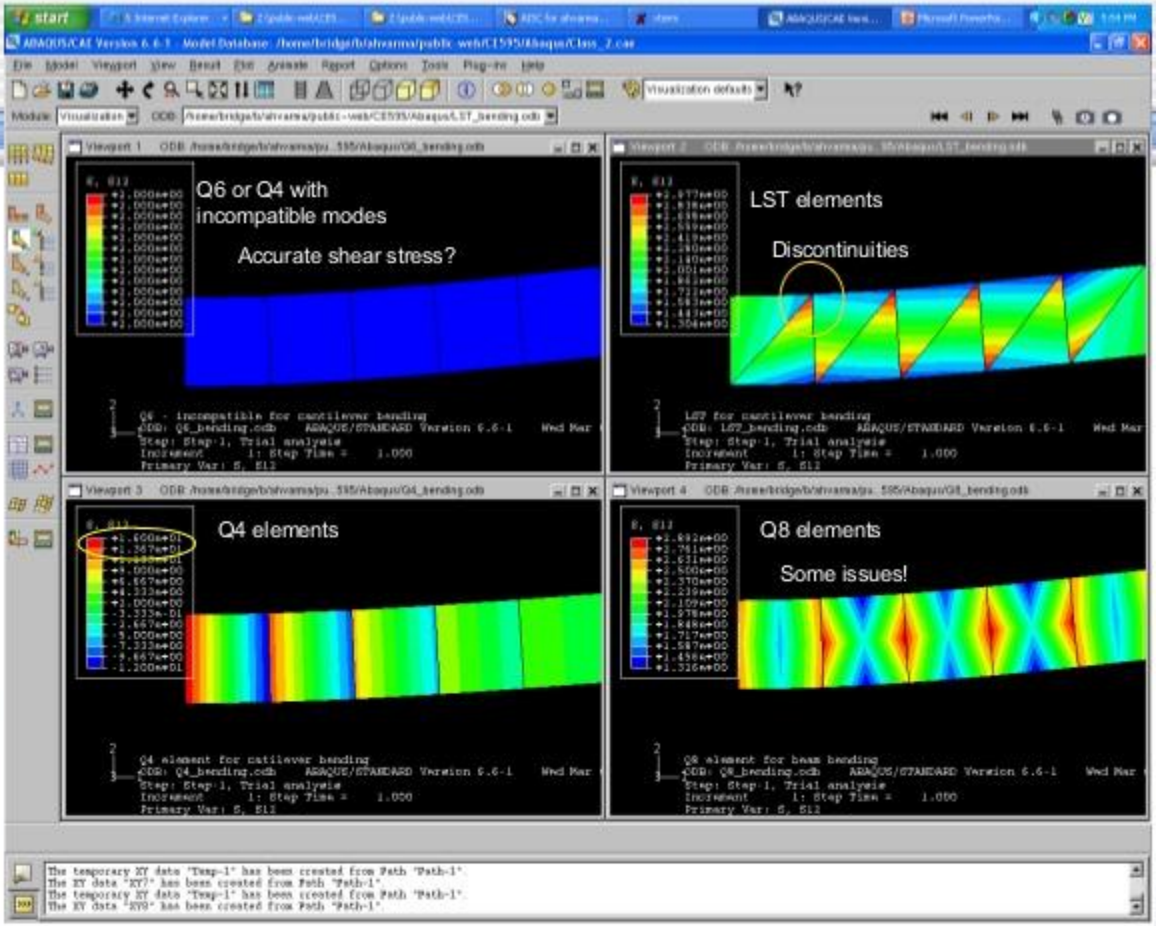
If you use CST to model bending.

See the stress along the x-axis - it should be zero.

The predictions of deflection and stress are poor

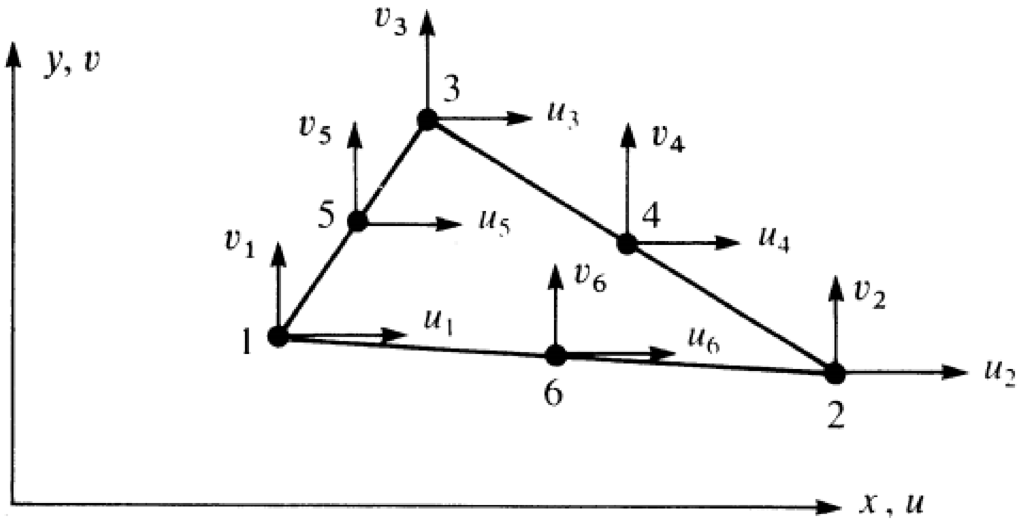
Spurious shear stress when bent

Mesh refinement will help.


Q6 or Q4 with incompatible modes
Accurate shear stress?
LST elements
Discontinuities
Q4 elements
Q8 elements
Some issues!

Linear-Strain Triangular Element

- LST Element



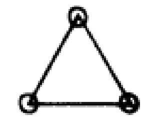
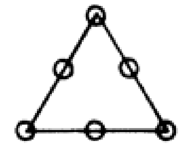
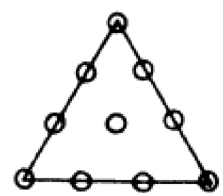
$$\{d\} = \begin{Bmatrix} \underline{d}_1 \\ \underline{d}_2 \\ \underline{d}_3 \\ \underline{d}_4 \\ \underline{d}_5 \\ \underline{d}_6 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{Bmatrix}$$

$$u(x, y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2$$

$$v(x, y) = a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} xy + a_{12} y^2$$

LST Element

- The number of coefficients equals the total number of degrees of freedom

Terms in Pascal Triangle	*Polynomial Degree*	*Number of Terms*	*Triangle*
1	0 (constant)	1	
$x \quad y$	1 (linear)	3	CST (Chap. 6) 
$x^2 \quad xy \quad y^2$	2 (quadratic)	6	LST (Chap. 8) 
$x^3 \quad x^2y \quad xy^2 \quad y^3$	3 (cubic)	10	QST 

LST Element

$$\{\psi\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{Bmatrix}$$

$$\{\psi\} = [M^*]\{a\}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \\ v_1 \\ \vdots \\ v_5 \\ v_6 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_5 & y_5 & x_5^2 & x_5y_5 & y_5^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \\ a_7 \\ \vdots \\ a_{11} \\ a_{12} \end{Bmatrix}$$

$$\{a\} = [X]^{-1}\{d\}$$

LST Element

$$\{\psi\} = [M^*]\{a\}$$

$$\{a\} = [X]^{-1}\{d\}$$

$$\{\psi\} = [N]\{d\}$$

$$[N] = [M^*][X]^{-1}$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \end{Bmatrix}$$

$$\{\varepsilon\} = \begin{bmatrix} 0 & 1 & 0 & 2x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & x & 2y \\ 0 & 0 & 1 & 0 & x & 2y & 0 & 1 & 0 & 2x & y & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{Bmatrix}$$

$$\{\varepsilon\} = [M']\{a\}$$

$$\{\varepsilon\} = [B]\{d\}$$

$$[B] = [M'][X]^{-1}$$

LST Element

$$[k] = \iiint_V [B]^T [D][B] \, dV$$

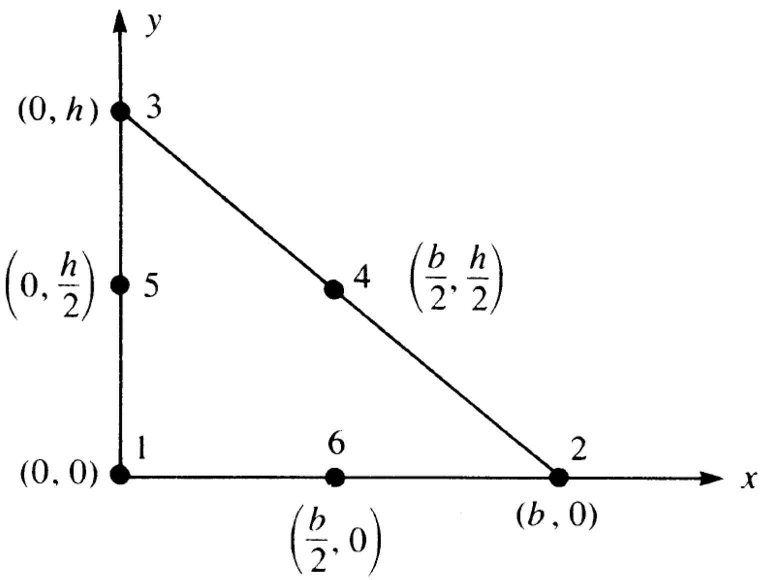
$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 & \beta_4 & 0 & \beta_5 & 0 & \beta_6 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 & 0 & \gamma_4 & 0 & \gamma_5 & 0 & \gamma_6 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 & \gamma_4 & \beta_4 & \gamma_5 & \beta_5 & \gamma_6 & \beta_6 \end{bmatrix}$$

where the β's and γ's are now functions of x and y as well as of the nodal coordinates.

$$\underset{(12 \times 1)}{\begin{Bmatrix} f_{1x} \\ f_{1y} \\ \vdots \\ f_{6y} \end{Bmatrix}} = \underset{(12 \times 12)}{\begin{bmatrix} k_{11} & \ldots & k_{1,12} \\ k_{21} & & k_{2,12} \\ \vdots & & \vdots \\ k_{12,1} & \ldots & k_{12,12} \end{bmatrix}} \underset{(12 \times 1)}{\begin{Bmatrix} u_1 \\ v_1 \\ \vdots \\ v_6 \end{Bmatrix}}$$

Example LST Stiffness Determination

To illustrate some of the procedures outlined in previous Section for deriving an LST stiffness matrix, consider the following example. Figure shows a specific LST and its coordinates. The triangle is of base dimension b and height h, with midside nodes.



Example LST Stiffness Determination

We calculate the coefficients a_1 through a_6 by evaluating the displacement u at each of the six known coordinates of each node as follows:

$$u_1 = u(0,0) = a_1$$

$$u_2 = u(b,0) = a_1 + a_2 b + a_4 b^2$$

$$u_3 = u(0,h) = a_1 + a_3 h + a_6 h^2$$

$$u_4 = u\left(\frac{b}{2},\frac{h}{2}\right) = a_1 + a_2\frac{b}{2} + a_3\frac{h}{2} + a_4\left(\frac{b}{2}\right)^2 + a_5\frac{bh}{4} + a_6\left(\frac{h}{2}\right)^2$$

$$u_5 = u\left(0,\frac{h}{2}\right) = a_1 + a_3\frac{h}{2} + a_6\left(\frac{h}{2}\right)^2$$

$$u_6 = u\left(\frac{b}{2},0\right) = a_1 + a_2\frac{b}{2} + a_4\left(\frac{b}{2}\right)^2$$

$$a_1 = u_1 \qquad a_2 = \frac{4u_6 - 3u_1 - u_2}{b} \qquad a_3 = \frac{4u_5 - 3u_1 - u_3}{h}$$

$$a_4 = \frac{2(u_2 - 2u_6 + u_1)}{b^2} \qquad a_5 = \frac{4(u_1 + u_4 - u_5 - u_6)}{bh}$$

$$a_6 = \frac{2(u_3 - 2u_5 + u_1)}{h^2}$$

Example LST Stiffness Determination

$$u = u_1 + \left[\frac{4u_6 - 3u_1 - u_2}{b}\right]x + \left[\frac{4u_5 - 3u_1 - u_3}{h}\right]y + \left[\frac{2(u_2 - 2u_6 + u_1)}{b^2}\right]x^2$$
$$+ \left[\frac{4(u_1 + u_4 - u_5 - u_6)}{bh}\right]xy + \left[\frac{2(u_3 - 2u_5 + u_1)}{h^2}\right]y^2$$

Similarly, for v we obtain

$$v = v_1 + \left[\frac{4v_6 - 3v_1 - v_2}{b}\right]x + \left[\frac{4v_5 - 3v_1 - v_3}{h}\right]y + \left[\frac{2(v_2 - 2v_6 + v_1)}{b^2}\right]x^2$$
$$+ \left[\frac{4(v_1 + v_4 - v_5 - v_6)}{bh}\right]xy + \left[\frac{2(v_3 - 2v_5 + v_1)}{h^2}\right]y^2$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \vdots \\ v_6 \end{Bmatrix}$$

Example LST Stiffness Determination

These shape functions are then given by

$$N_1 = 1 - \frac{3x}{b} - \frac{3y}{h} + \frac{2x^2}{b^2} + \frac{4xy}{bh} + \frac{2y^2}{h^2} \qquad N_2 = \frac{-x}{b} + \frac{2x^2}{b^2}$$

$$N_3 = \frac{-y}{h} + \frac{2y^2}{h^2} \qquad N_4 = \frac{4xy}{bh} \qquad N_5 = \frac{4y}{h} - \frac{4xy}{bh} - \frac{4y^2}{h^2}$$

$$N_6 = \frac{4x}{b} - \frac{4x^2}{b^2} - \frac{4xy}{bh}$$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 & \beta_4 & 0 & \beta_5 & 0 & \beta_6 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 & 0 & \gamma_4 & 0 & \gamma_5 & 0 & \gamma_6 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 & \gamma_4 & \beta_4 & \gamma_5 & \beta_5 & \gamma_6 & \beta_6 \end{bmatrix}$$

$$\underline{\varepsilon} = \underline{B}\underline{d}$$

Example LST Stiffness Determination

$$\beta_1 = -3h + \frac{4hx}{b} + 4y \qquad \beta_2 = -h + \frac{4hx}{b} \qquad \beta_3 = 0$$

$$\beta_4 = 4y \qquad \beta_5 = -4y \qquad \beta_6 = 4h - \frac{8hx}{b} - 4y$$

$$\gamma_1 = -3b + 4x + \frac{4by}{h} \qquad \gamma_2 = 0 \qquad \gamma_3 = -b + \frac{4by}{h}$$

$$\gamma_4 = 4x \qquad \gamma_5 = 4b - 4x - \frac{8by}{h} \qquad \gamma_6 = -4x$$

$$\varepsilon_x = \frac{1}{2A}[\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4 + \beta_5 u_5 + \beta_6 u_6]$$

$$\varepsilon_y = \frac{1}{2A}[\gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3 + \gamma_4 v_4 + \gamma_5 v_5 + \gamma_6 v_6]$$

$$\gamma_{xy} = \frac{1}{2A}[\gamma_1 u_1 + \beta_1 v_1 + \cdots + \beta_6 v_6]$$

The stiffness matrix for a constant-thickness element can now be obtained

$$[k] = \iiint_V [B]^T [D][B]\, dV$$

Comparison: LST & CST

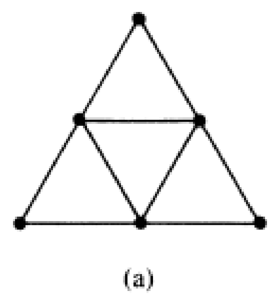


Figure 8–4 Basic triangular element: (a) four-CST and (b) one-LST

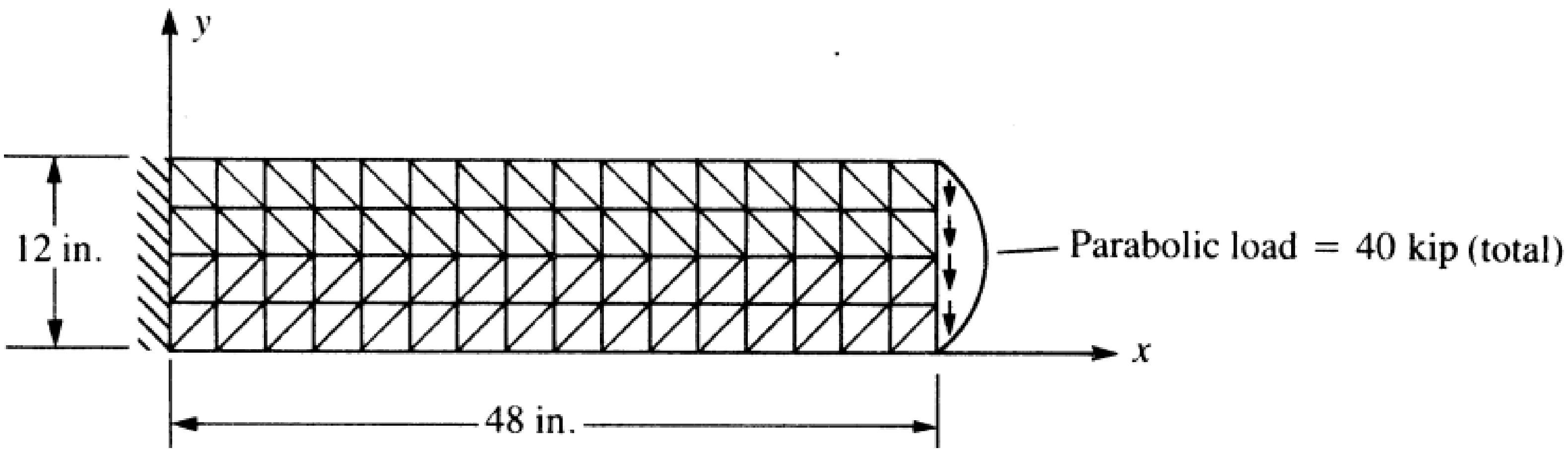


Figure 8–5 Cantilever beam used to compare the CST and LST elements with a 4 × 16 mesh

Comparison: LST & CST

Table 8–1 Models used to compare CST and LST results for the cantilever beam of Figure 8–5

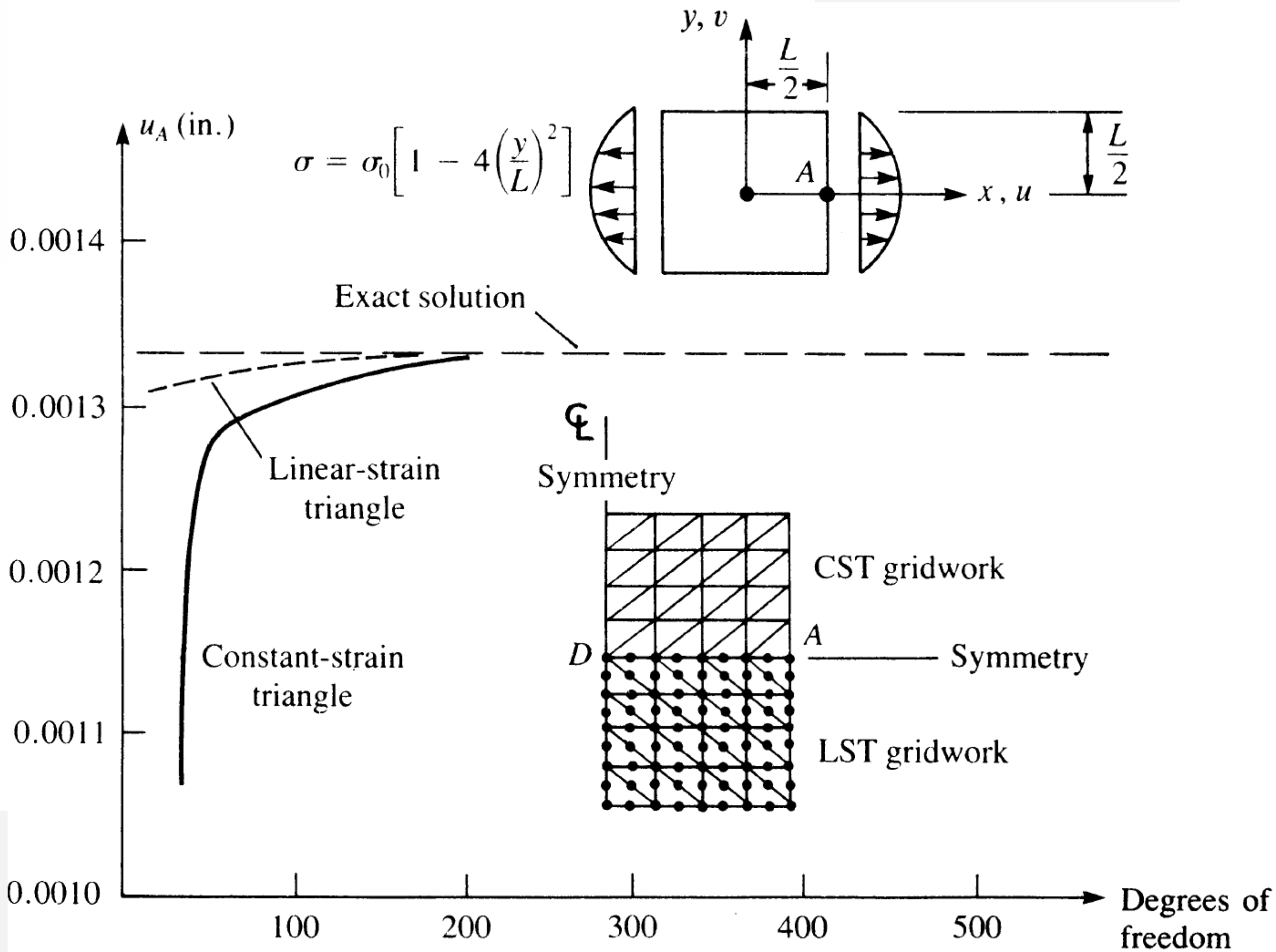
Series of Tests Run	Number of Nodes	Number of Degrees of Freedom, n_d	Number of Triangular Elements
A-1 4 × 16 mesh	85	160	128 CST
A-2 8 × 32	297	576	512 CST
B-1 2 × 8	85	160	32 LST
B-2 4 × 16	297	576	128 LST

Table 8–2 Comparison of CST and LST results for the cantilever beam of Figure 8–5

Run	n_d	Bandwidth[1] n_b	Tip Deflection (in.)	σ_x (ksi)	Location (in.), x, y
A-1	160	14	−0.29555	67.236	2.250, 11.250
A-2	576	22	−0.33850	81.302	1.125, 11.630
B-1	160	18	−0.33470	58.885	4.500, 10.500
B-2	576	22	−0.35159	69.956	2.250, 11.250
Exact solution			−0.36133	80.000	0, 12

[1] Bandwidth is described in Appendix B.4.

Comparison: LST & CST



Exercise

- 6.10 a,c
- 6.11
- 6.13
- 8.3
- 8.5