In the name of God

Director of the course

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Mechanical Engineering

2-D Elements: Isoparametric Formulation

Outline

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- Isoparametric Formulation: Bar element
- Rectangular Element
- Isoparametric Formulation: Rectangular element
- Numerical Integration
- Higher-order shape functions

Bar element

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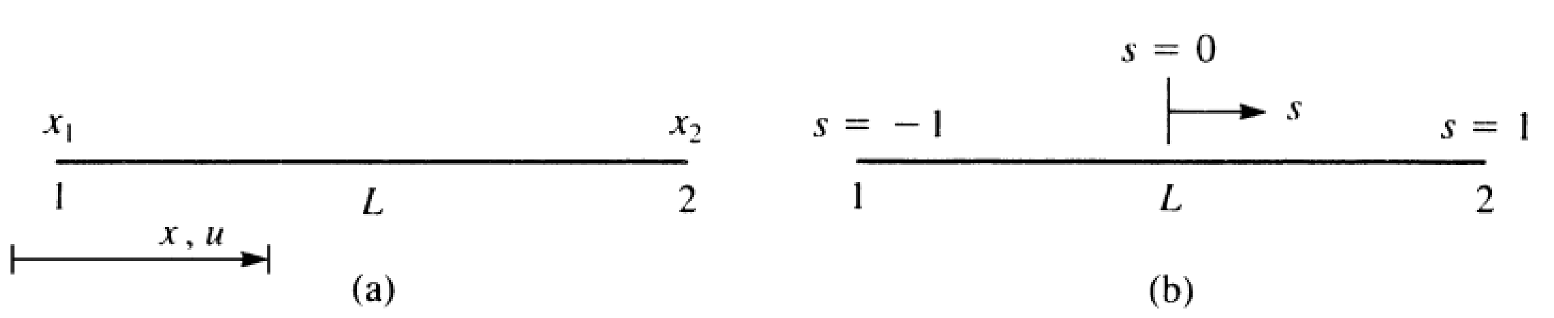
- Isoparametric element equations are formulated using a natural (or intrinsic) coordinate system *s*



- When the *s* and *x* axes are parallel to each other

$$x = x_c + \frac{L}{2}s$$

$$s = [x - (x_1 + x_2)/2](2/(x_2 - x_1))$$

- relating the natural coordinate to the global coordinate by:

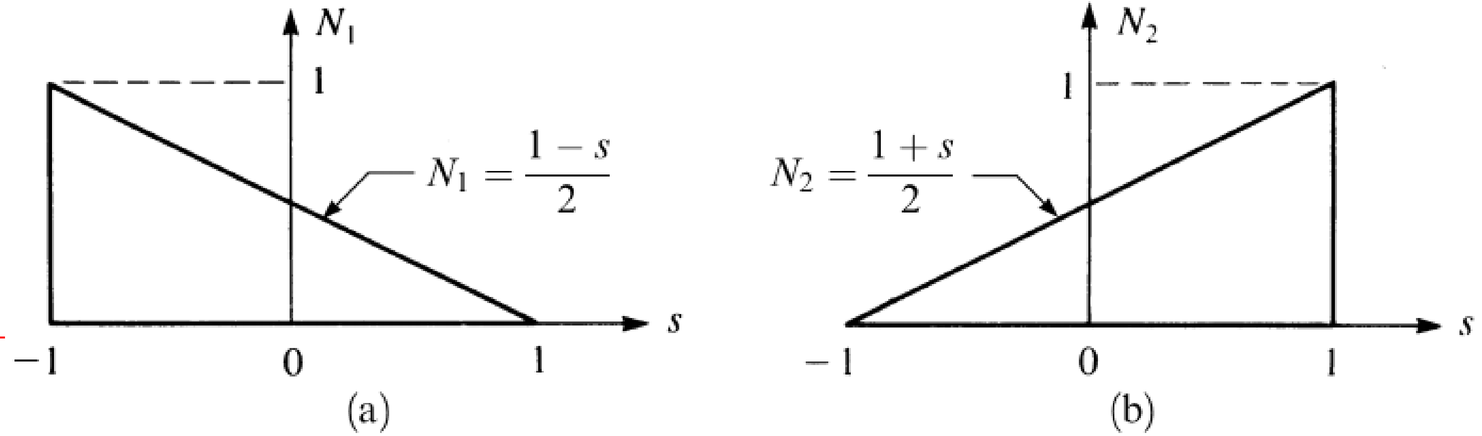
$$x = a_1 + a_2 s$$

- Solving for a_1 and a_2

$$x = \tfrac{1}{2}[(1-s)x_1 + (1+s)x_2]$$

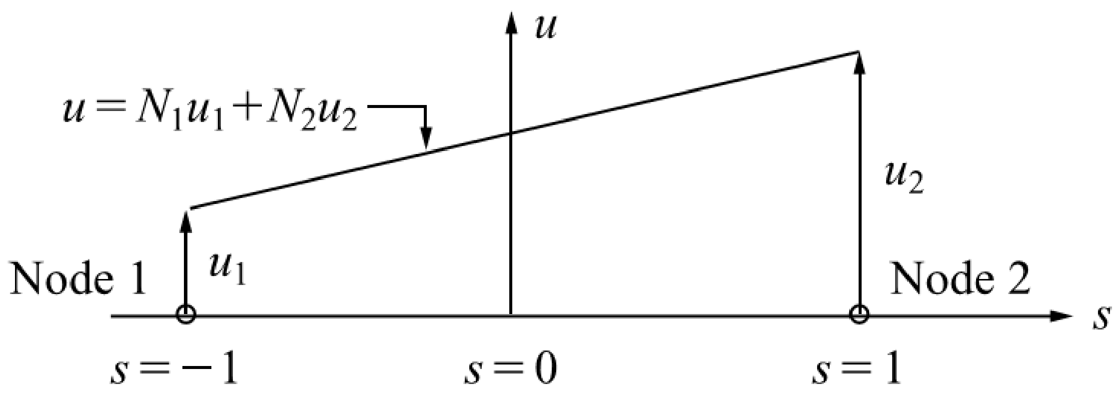
$$\{x\} = [N_1 \quad N_2]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$



- The displacement function within the bar is now defined by the same shape functions

$$\{u\} = [N_1 \quad N_2]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$



(c)

- Since *u* and *x* are defined by the same shape functions at the same nodes, the element is called isoparametric .

- To construct the element stiffness matrix, we must determine the strain
- Therefore, we should determine the derivative of the displacement (u)with respect to x
- however, u is now a function of s

$$\frac{du}{ds} = \frac{du}{dx}\frac{dx}{ds} \Rightarrow \frac{du}{dx} = \frac{\left(\frac{du}{ds}\right)}{\left(\frac{dx}{ds}\right)}$$

$$\frac{du}{ds} = \frac{u_2 - u_1}{2}$$

$$\frac{dx}{ds} = \frac{x_2 - x_1}{2} = \frac{L}{2}$$

$$\Rightarrow \{\varepsilon_x\} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$[B] = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$$

- [B] is ,in general, a function of *s*
- *stiffness matrix*

$$[k] = \int_0^L [B]^T [D][B] A \, dx$$

$$\int_0^L f(x)\,dx = \int_{-1}^{1} f(s) |\underline{J}| \, ds$$

- *|J| is called the Jacobian*

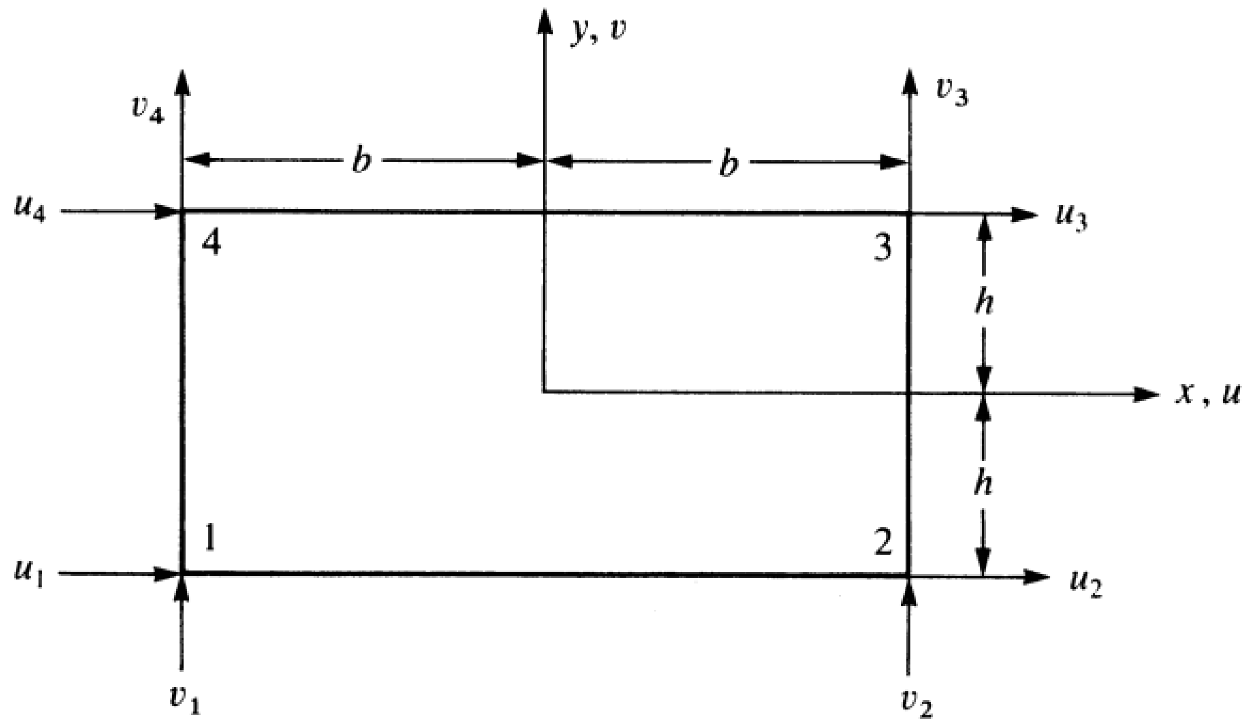
$$|\underline{J}| = \frac{dx}{ds} = \frac{L}{2}$$

$$[k] = \frac{L}{2} \int_{-1}^{1} [B]^T E [B] A \, ds$$

$$\{\hat{f}_b\} = A\int_0^L [N]^T\{\hat{X}_b\}\,dx \Rightarrow \{\hat{f}_b\} = A\int_{-1}^{1}\begin{Bmatrix}\dfrac{1-s}{2}\\[2ex] \dfrac{1+s}{2}\end{Bmatrix}\{\hat{X}_b\}\frac{L}{2}\,ds \Rightarrow \{\hat{f}_b\} = \frac{AL\hat{X}_b}{2}\begin{Bmatrix}1\\1\end{Bmatrix}$$

$$\{\hat{f}_s\} = \int_0^L [N_s]^T\{\hat{T}_x\}\,dx \Rightarrow \{\hat{f}_s\} = \int_{-1}^{1}\begin{Bmatrix}\dfrac{1-s}{2}\\[2ex] \dfrac{1+s}{2}\end{Bmatrix}\{\hat{T}_x\}\frac{L}{2}\,ds \Rightarrow \{\hat{f}_s\} = \hat{T}_x\frac{L}{2}\begin{Bmatrix}1\\1\end{Bmatrix}$$

Rectangular Element



$$\{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Basic four-node rectangular element with nodal degrees of freedom

$$u(x, y) = a_1 + a_2 x + a_3 y + a_4 xy$$

$$v(x, y) = a_5 + a_6 x + a_7 y + a_8 xy$$

$$u(x, y) = \frac{1}{4bh}[(b - x)(h - y)u_1 + (b + x)(h - y)u_2 + (b + x)(h + y)u_3 + (b - x)(h + y)u_4]$$

$$v(x, y) = \frac{1}{4bh}[(b - x)(h - y)v_1 + (b + x)(h - y)v_2 + (b + x)(h + y)v_3 + (b - x)(h + y)v_4]$$

$$\{\psi\} = [N]\{d\}$$

$$N_1 = \frac{(b-x)(h-y)}{4bh} \qquad N_2 = \frac{(b+x)(h-y)}{4bh}$$

$$N_3 = \frac{(b+x)(h+y)}{4bh} \qquad N_4 = \frac{(b-x)(h+y)}{4bh}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

$$[B] = \frac{1}{4bh} \left[\begin{matrix} -(h-y) & 0 & (h-y) & 0 \\ 0 & -(b-x) & 0 & -(b+x) \\ -(b-x) & -(h-y) & -(b+x) & (h-y) \end{matrix} \right.$$

$$\left. \begin{matrix} (h+y) & 0 & -(h+y) & 0 \\ 0 & (b+x) & 0 & (b-x) \\ (b+x) & (h+y) & (b-x) & -(h+y) \end{matrix} \right]$$

Isoparametric Formulation

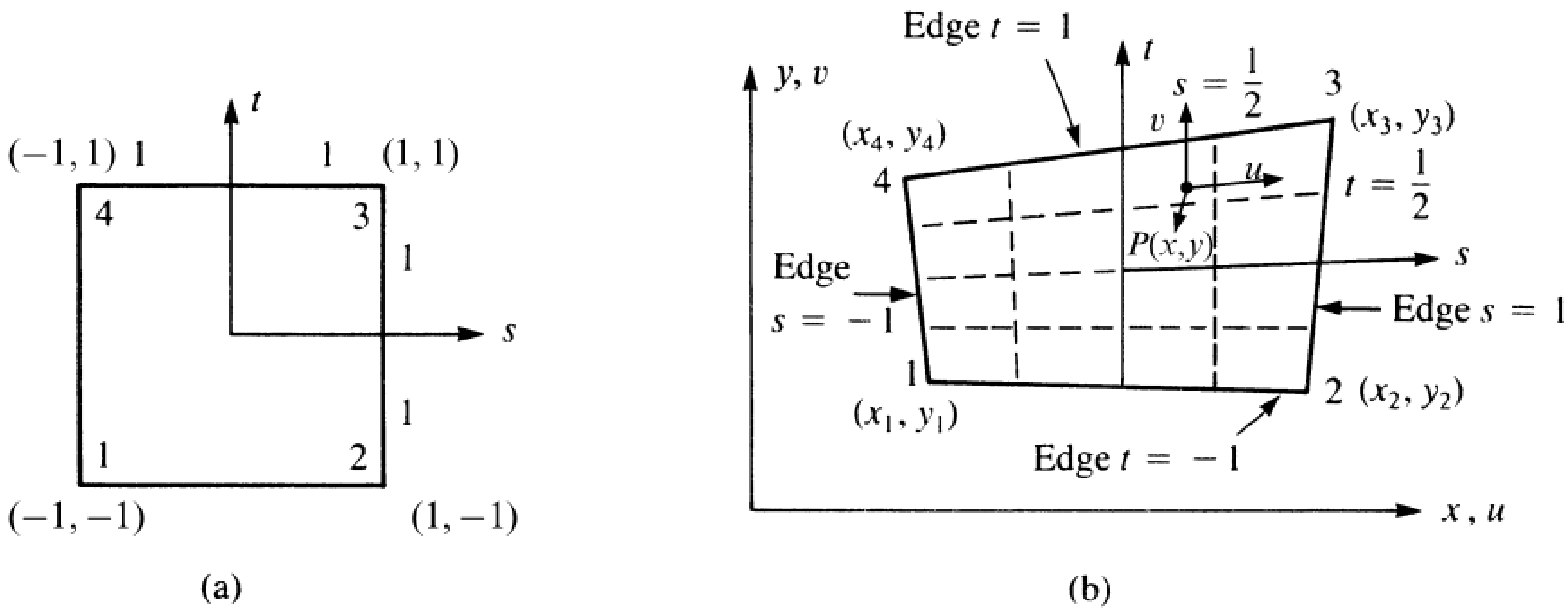


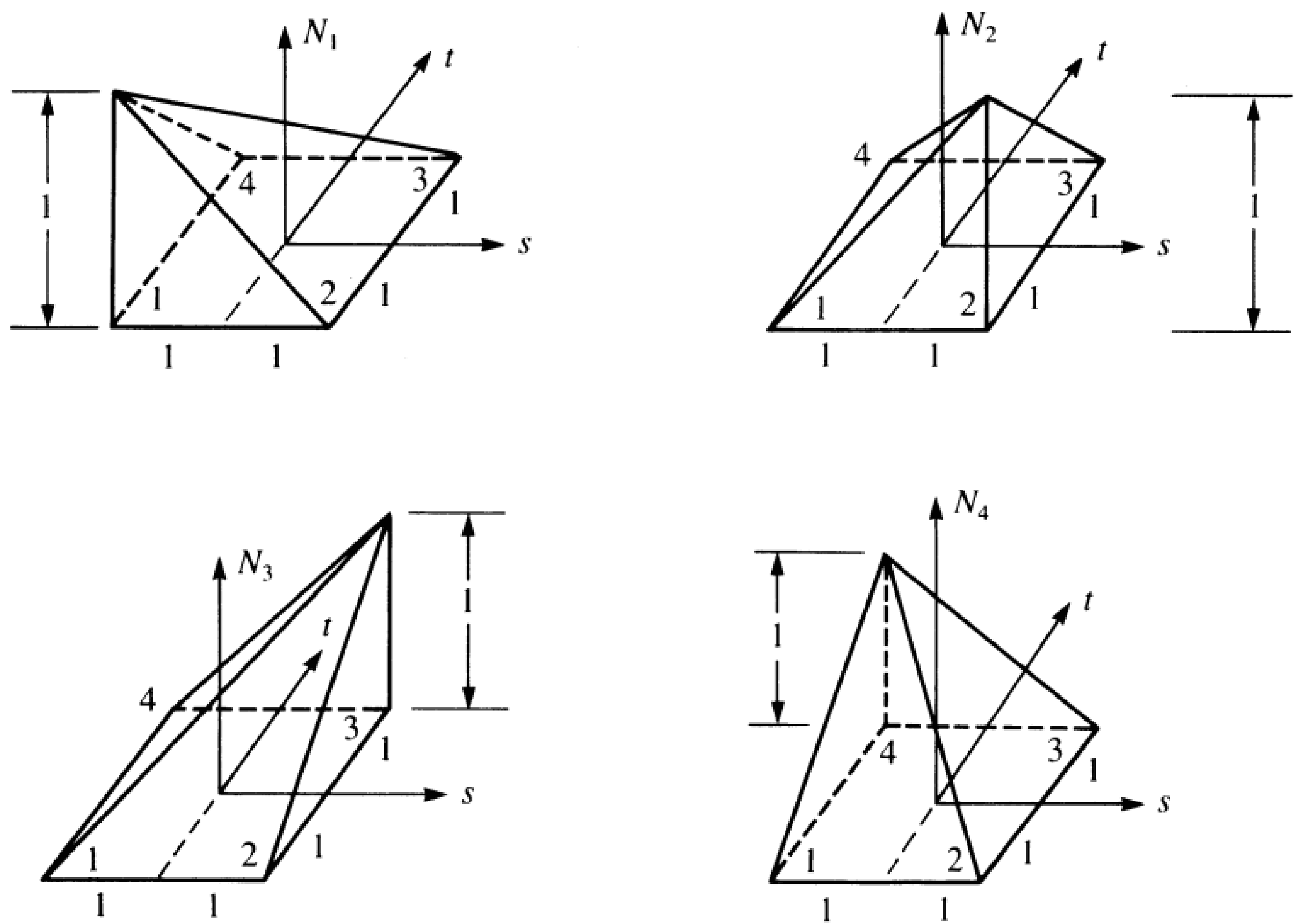
Figure 10–4 (a) Linear square element in *s*-*t* coordinates and (b) square element mapped into quadrilateral in *x*-*y* coordinates whose size and shape are determined by the eight nodal coordinates $x_1, y_1, \ldots, y_4$

$$x = a_1 + a_2 s + a_3 t + a_4 st$$
$$y = a_5 + a_6 s + a_7 t + a_8 st$$

$$x = \tfrac{1}{4}[(1-s)(1-t)x_1 + (1+s)(1-t)x_2 + (1+s)(1+t)x_3 + (1-s)(1+t)x_4]$$
$$y = \tfrac{1}{4}[(1-s)(1-t)y_1 + (1+s)(1-t)y_2 + (1+s)(1+t)y_3 + (1-s)(1+t)y_4]$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

$$N_1 = \frac{(1-s)(1-t)}{4} \qquad N_2 = \frac{(1+s)(1-t)}{4}$$
$$N_3 = \frac{(1+s)(1+t)}{4} \qquad N_4 = \frac{(1-s)(1+t)}{4}$$


N1
t
s
1
2
3
4
N2
N3
N4

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial f}{\partial x} = \frac{\begin{vmatrix} \frac{\partial f}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial f}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}} \qquad \frac{\partial f}{\partial y} = \frac{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial f}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial f}{\partial t} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial(\ )}{\partial x} & 0 \\ 0 & \frac{\partial(\ )}{\partial y} \\ \frac{\partial(\ )}{\partial y} & \frac{\partial(\ )}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$\frac{\partial(\ )}{\partial x} = \frac{1}{|\underline{J}|}\left[\frac{\partial y}{\partial t}\frac{\partial(\ )}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\ )}{\partial t}\right]$$

$$\frac{\partial(\ )}{\partial y} = \frac{1}{|\underline{J}|}\left[\frac{\partial x}{\partial s}\frac{\partial(\ )}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\ )}{\partial s}\right]$$

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{|\underline{J}|}\begin{bmatrix} \frac{\partial y}{\partial t}\frac{\partial(\ )}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\ )}{\partial t} & 0 \\ 0 & \frac{\partial x}{\partial s}\frac{\partial(\ )}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\ )}{\partial s} \\ \frac{\partial x}{\partial s}\frac{\partial(\ )}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\ )}{\partial s} & \frac{\partial y}{\partial t}\frac{\partial(\ )}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\ )}{\partial t} \end{bmatrix}\begin{Bmatrix} u \\ v \end{Bmatrix}$$

$\underline{D}'$

$$\underline{\varepsilon} = \underline{D}'\underline{N}\underline{d}$$

$$\underset{(3\times 8)}{\underline{B}} = \underset{(3\times 2)}{\underline{D}'} \; \underset{(2\times 8)}{\underline{N}}$$

$$\underline{B}(s,t) = \frac{1}{|\underline{J}|}[\underline{B}_1 \quad \underline{B}_2 \quad \underline{B}_3 \quad \underline{B}_4]$$

$$\underline{B}_i = \begin{bmatrix} a(N_{i,s}) - b(N_{i,t}) & 0 \\ 0 & c(N_{i,t}) - d(N_{i,s}) \\ c(N_{i,t}) - d(N_{i,s}) & a(N_{i,s}) - b(N_{i,t}) \end{bmatrix}$$

$$a = \tfrac{1}{4}[y_1(s-1) + y_2(-1-s) + y_3(1+s) + y_4(1-s)]$$

$$b = \tfrac{1}{4}[y_1(t-1) + y_2(1-t) + y_3(1+t) + y_4(-1-t)]$$

$$c = \tfrac{1}{4}[x_1(t-1) + x_2(1-t) + x_3(1+t) + x_4(-1-t)]$$

$$d = \tfrac{1}{4}[x_1(s-1) + x_2(-1-s) + x_3(1+s) + x_4(1-s)]$$

Jacobian

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$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \end{bmatrix}$$

$$x = \tfrac{1}{4}[(1-s)(1-t)x_1 + (1+s)(1-t)x_2 + (1+s)(1+t)x_3 + (1-s)(1+t)x_4]$$

$$y = \tfrac{1}{4}[(1-s)(1-t)y_1 + (1+s)(1-t)y_2 + (1+s)(1+t)y_3 + (1-s)(1+t)y_4]$$

$$|\underline{J}| = \tfrac{1}{8}\{X_c\}^T \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \{Y_c\}$$

$$\{X_c\}^T = [x_1 \quad x_2 \quad x_3 \quad x_4]$$

$$\{Y_c\} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

- B is a function of s and t in both the numerator and the denominator

$$[k] = \iint_A [B]^T [D][B] h \, dx \, dy$$

$$\iint_A f(x, y) \, dx \, dy = \iint_A f(s, t) |\underline{J}| \, ds \, dt$$

$$[k] = \int_{-1}^{1} \int_{-1}^{1} [B]^T [D][B] h |\underline{J}| \, ds \, dt$$

- The $|\underline{J}|$ and $\underline{B}$ are complicated expressions within the integral
- Therefore the integration to determine the element stiffness matrix is usually done numerically.

Body forces

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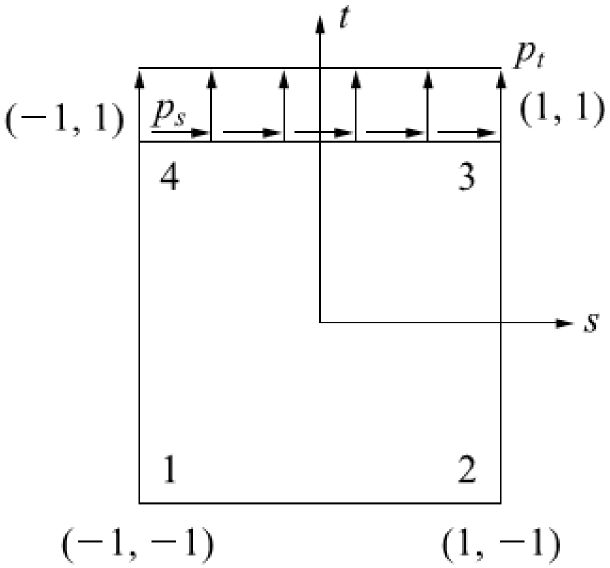
The element body-force matrix will now be determined from

$$\underset{(8\times 1)}{\{f_b\}} = \int_{-1}^{1}\int_{-1}^{1} \underset{(8\times 2)}{[N]^T} \underset{(2\times 1)}{\{X\}} \; h|\underline{J}|\, ds\, dt \qquad (10.3.28)$$

Like the stiffness matrix, the body-force matrix in Eq. (10.3.28) has to be evaluated by numerical integration.



Surface forces

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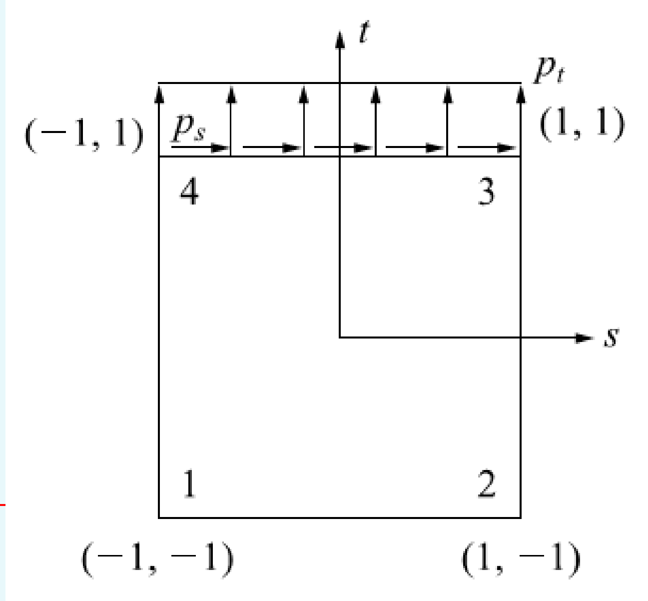
$$\underset{(4\times 1)}{\{f_s\}} = \int_{-1}^{1} \underset{(4\times 2)}{[N_s]^T} \underset{(2\times 1)}{\{T\}} \, h\frac{L}{2}ds$$

$$\begin{Bmatrix} f_{s3s} \\ f_{s3t} \\ f_{s4s} \\ f_{s4t} \end{Bmatrix} = \int_{-1}^{1} \begin{bmatrix} N_3 & 0 & N_4 & 0 \\ 0 & N_3 & 0 & N_4 \end{bmatrix}^T \Bigg|_{\substack{\text{evaluated} \\ \text{along } t=1}} \begin{Bmatrix} p_s \\ p_t \end{Bmatrix} h\frac{L}{2}ds$$

- $N_1=0$ and $N_2=0$ along edge t=1
- Therefore, no nodal forces exist at

nodes 1 and 2

$$\{f_s\} = h\frac{L}{2}[0 \quad 0 \quad 0 \quad 0 \quad p_s \quad p_t \quad p_s \quad p_t]^T$$



Example 10.1 : p. 461

Numerical Integration

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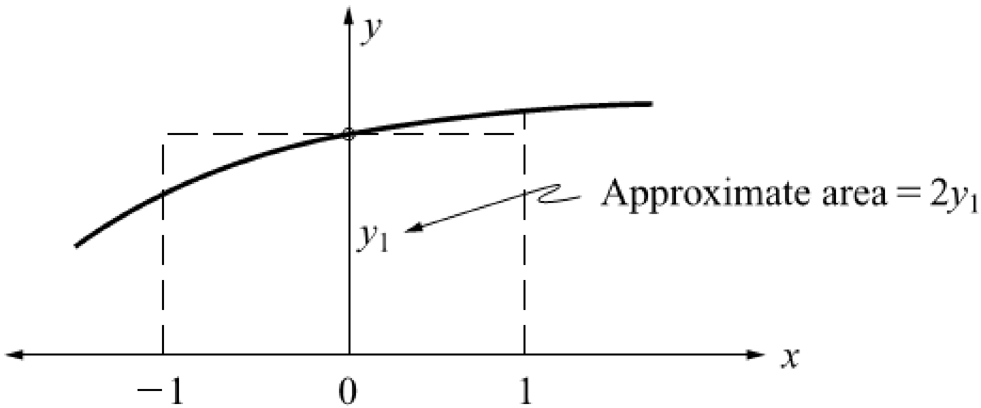
- **Gaussian Quadrature**: most useful for finite element work (evaluation of definite integrals)



$$I = \int_{-1}^{1} y(x)\,dx \cong 2y(0)$$

$$I = \int_{-1}^{1} y\,dx = \sum_{i=1}^{n} W_i y_i$$

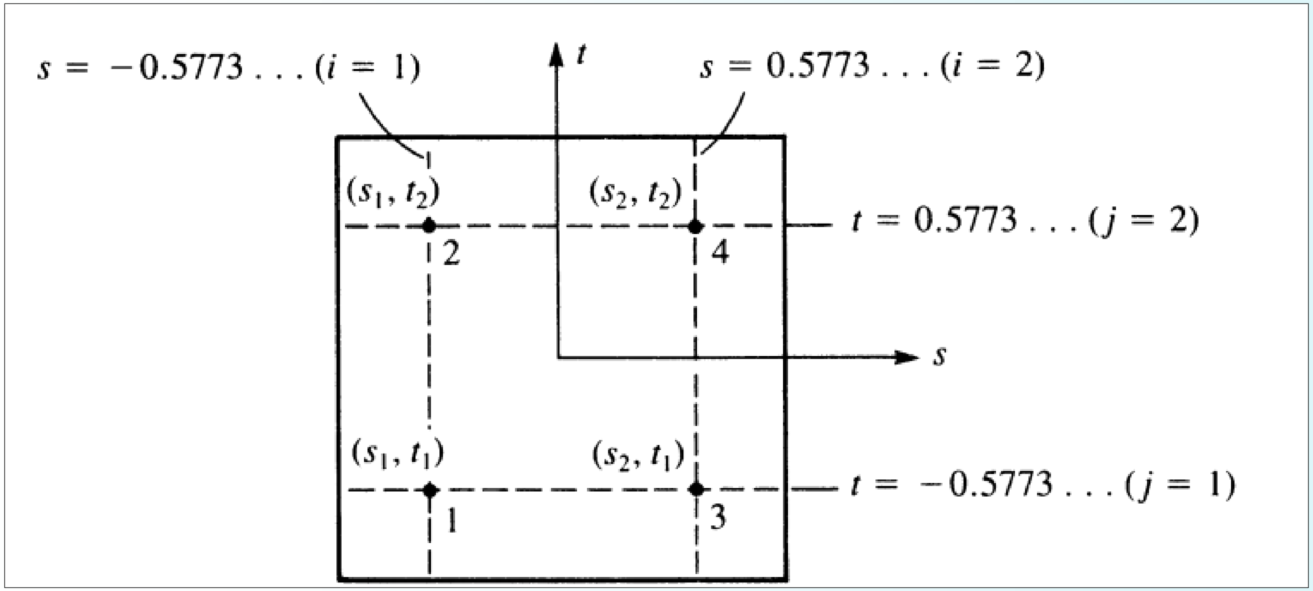
Table 10–1 Table for Gauss points for integration from minus one to one, $\int_{-1}^{1} y(x)\,dx = \sum_{i=1}^{n} W_i y_i$

Number of Points	Locations, x_i	Associated Weights, W_i
1	$x_1 = 0.000\ldots$	2.000
2	$x_1, x_2 = \pm 0.57735026918962$	1.000
3	$x_1, x_3 = \pm 0.77459666924148$	$\frac{5}{9} = 0.555\ldots$
	$x_2 = 0.000\ldots$	$\frac{8}{9} = 0.888\ldots$
4	$x_1, x_4 = \pm 0.8611363116$	0.3478548451
	$x_2, x_3 = \pm 0.3399810436$	0.6521451549

- In two dimensions:

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\,ds\,dt = \int_{-1}^{1}\left[\sum_i W_i f(s_i,t)\right] dt$$

$$= \sum_j W_j \left[\sum_i W_i f(s_i,t_j)\right] = \sum_i \sum_j W_i W_j f(s_i,t_j)$$



$$I = W_1 W_1 f(s_1,t_1) + W_1 W_2 f(s_1,t_2) + W_2 W_1 f(s_2,t_1) + W_2 W_2 f(s_2,t_2)$$

Stiffness Matrix: Gaussian

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$$\underline{k} = \int_{-1}^{1}\int_{-1}^{1} \underline{B}^T(s,t)\underline{D}\underline{B}(s,t)|\underline{J}|h\,ds\,dt$$

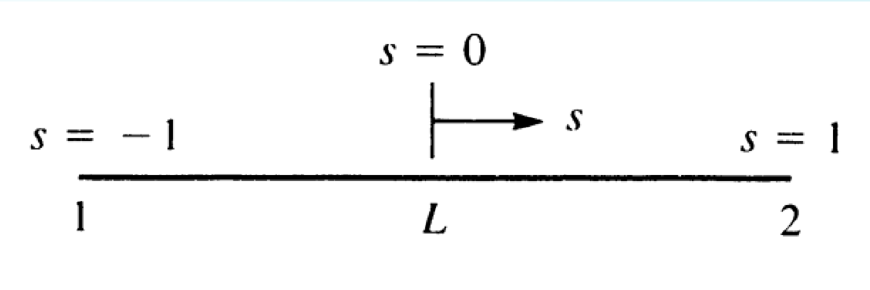
$$\begin{aligned}\underline{k} = {} & \underline{B}^T(s_1,t_1)\underline{D}\underline{B}(s_1,t_1)|\underline{J}(s_1,t_1)|hW_1W_1 \\ & + \underline{B}^T(s_2,t_2)\underline{D}\underline{B}(s_2,t_2)|\underline{J}(s_2,t_2)|hW_2W_2 \\ & + \underline{B}^T(s_3,t_3)\underline{D}\underline{B}(s_3,t_3)|\underline{J}(s_3,t_3)|hW_3W_3 \\ & + \underline{B}^T(s_4,t_4)\underline{D}\underline{B}(s_4,t_4)|\underline{J}(s_4,t_4)|hW_4W_4\end{aligned}$$

where $s_1 = t_1 = -0.5773$, $s_2 = -0.5773$, $t_2 = 0.5773$, $s_3 = 0.5773$, $t_3 = -0.5773$, and $s_4 = t_4 = 0.5773$ as shown in Figure 10–9, and $W_1 = W_2 = W_3 = W_4 = 1.000$.

Example 10.4: p.471

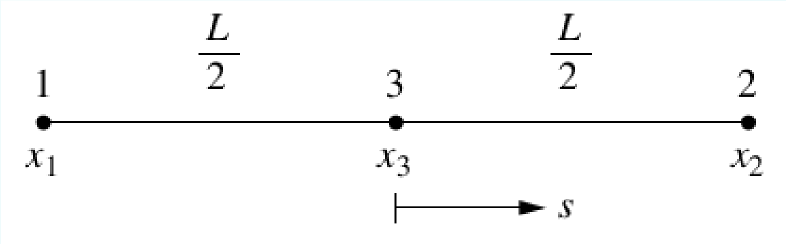
Higher-order elements

Linear bar element



$$N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$

Quadratic bar element



$$N_1 = \frac{s(s-1)}{2} \qquad N_2 = \frac{s(s+1)}{2} \qquad N_3 = (1-s^2)$$

$$\{\varepsilon_x\} = \frac{du}{dx} = \frac{du}{ds}\frac{ds}{dx} = [B]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$dx/ds = L/2 = [J]$$

$$\{\varepsilon_x\} = \frac{du}{dx} = \left[\frac{2s-1}{L} \quad \frac{2s+1}{L} \quad \frac{-4s}{L}\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [B]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$[B] = \left[\frac{2s-1}{L} \quad \frac{2s+1}{L} \quad \frac{-4s}{L}\right]$$

Exercise

- 10.8
- 10.15b
- 10.16a,e,f
- 10.18a