Introduction to the Finite Element Method

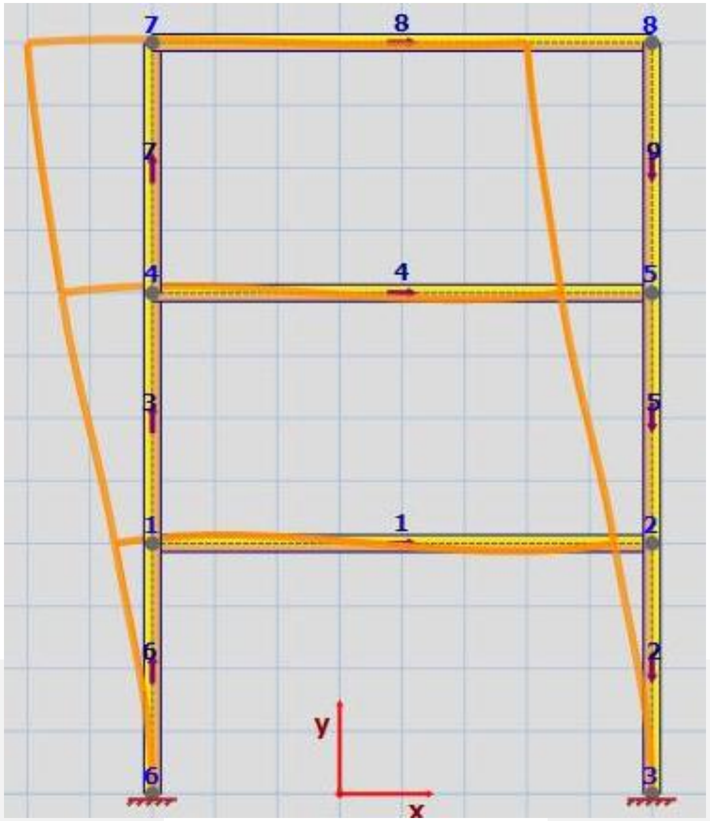
Chapter 5: 2-D Elements: Frame

Mohammad Javad Ashrafi, PhD

Mechanical Engineering Department

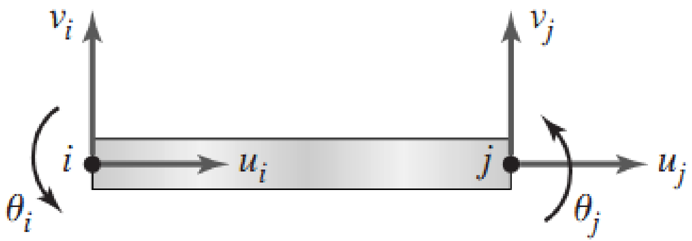
2-D Beam Element

- Many structures, such as buildings and bridges, are composed of frames



2-D Beam Element

- beam element with axial stiffness



2-D Beam Element

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

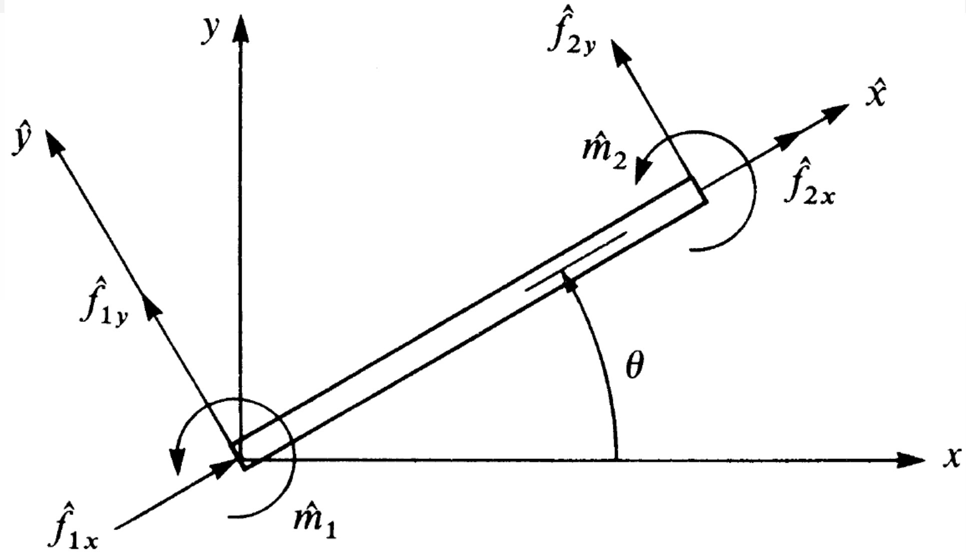
$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

Combining axial and bending effects

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2x} \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6C_2L & 0 & -12C_2 & 6C_2L \\ 0 & 6C_2L & 4C_2L^2 & 0 & -6C_2L & 2C_2L^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6C_2L & 0 & 12C_2 & -6C_2L \\ 0 & 6C_2L & 2C_2L^2 & 0 & -6C_2L & 4C_2L^2 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2x} \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

where $C_1 = \dfrac{AE}{L}$ and $C_2 = \dfrac{EI}{L^3}$

2D Arbitrarily Oriented Beam Element



$$\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2x} \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ \phi_1 \\ d_{2x} \\ d_{2y} \\ \phi_2 \end{Bmatrix}$$

2-D Beam Element

$$\underline{T} = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{k} = \underline{T}^T \underline{\hat{k}} \underline{T}$$

$$\underline{\hat{k}} = \begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6C_2L & 0 & -12C_2 & 6C_2L \\ 0 & 6C_2L & 4C_2L^2 & 0 & -6C_2L & 2C_2L^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6C_2L & 0 & 12C_2 & -6C_2L \\ 0 & 6C_2L & 2C_2L^2 & 0 & -6C_2L & 4C_2L^2 \end{bmatrix}$$

$$C_1 = \frac{AE}{L}$$

$$C_2 = \frac{EI}{L^3}$$

2-D Beam Element

$$\underline{k} = \frac{E}{L} \times$$

$$\begin{bmatrix} AC^2 + \frac{12I}{L^2}S^2 & \left(A - \frac{12I}{L^2}\right)CS & -\frac{6I}{L}S & -\left(AC^2 + \frac{12I}{L^2}S^2\right) & -\left(A - \frac{12I}{L^2}\right)CS & -\frac{6I}{L}S \\ & AS^2 + \frac{12I}{L^2}C^2 & \frac{6I}{L}C & -\left(A - \frac{12I}{L^2}\right)CS & -\left(AS^2 + \frac{12I}{L^2}C^2\right) & \frac{6I}{L}C \\ & & 4I & \frac{6I}{L}S & -\frac{6I}{L}C & 2I \\ & & & AC^2 + \frac{12I}{L^2}S^2 & \left(A - \frac{12I}{L^2}\right)CS & \frac{6I}{L}S \\ & & & & AS^2 + \frac{12I}{L^2}C^2 & -\frac{6I}{L}C \\ \text{Symmetry} & & & & & 4I \end{bmatrix}$$

2-D Beam Element

- **Rigid plane frame**
  - a series of beam elements rigidly connected to each other
  - the original angles made between elements at their joints remain unchanged after the deformation due to applied loads or applied displacements.
  - moment continuity exists at the rigid joints
  - the element centroids, as well as the applied loads, lie in a common plane (x-y plane).

Example 1

The frame is fixed at nodes 1 and 4 and subjected to a positive horizontal force of 10,000 lb applied at node 2 and to a positive moment of 5000 lb-in. applied at node 3.

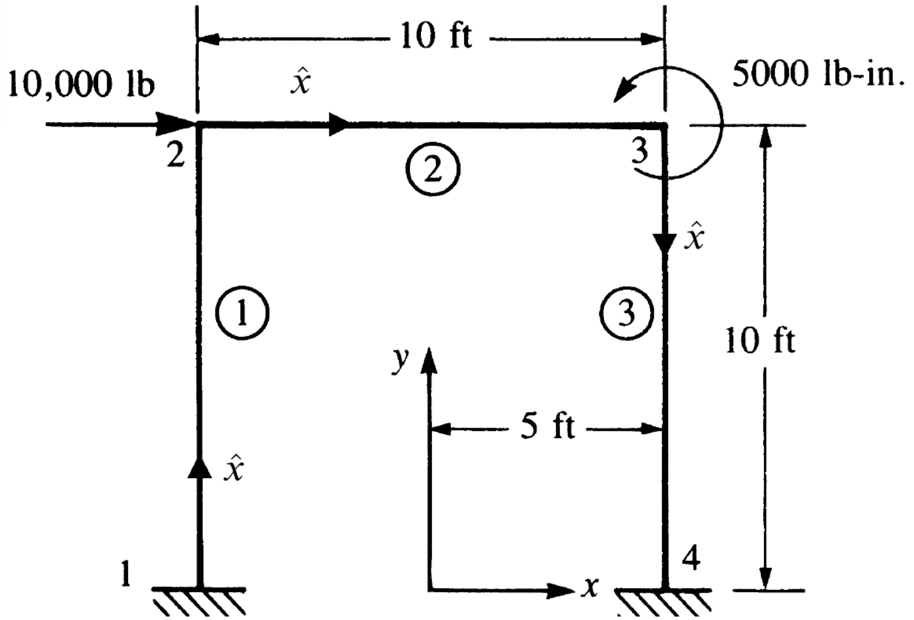


Figure 5–4 Plane frame for analysis, also showing local $\hat{x}$ axis for each element

Let $E = 30 \times 10^6$ psi and $A = 10$ in^2 for all elements, and let $I = 200$ in^4 for elements 1 and 3, and $I = 100$ in^4 for element 2.

Example 1

Element 1

$$C = \cos 90^\circ = \frac{x_2 - x_1}{L^{(1)}} = \frac{-60 - (-60)}{120} = 0$$

$$S = \sin 90^\circ = \frac{y_2 - y_1}{L^{(1)}} = \frac{120 - 0}{120} = 1$$

Also,

$$\frac{12I}{L^2} = \frac{12(200)}{(10 \times 12)^2} = 0.167 \text{ in}^2 \tag{5.2.1}$$

$$\frac{6I}{L} = \frac{6(200)}{10 \times 12} = 10.0 \text{ in}^3$$

$$\frac{E}{L} = \frac{30 \times 10^6}{10 \times 12} = 250{,}000 \text{ lb/in}^3$$

Then, using Eqs. (5.2.1) to help in evaluating Eq. (5.1.11) for element 1, we obtain the element global stiffness matrix as

$$\underline{k}^{(1)} = 250{,}000 \begin{array}{c} \begin{array}{cccccc} d_{1x} & d_{1y} & \phi_1 & d_{2x} & d_{2y} & \phi_2 \end{array} \\ \begin{bmatrix} 0.167 & 0 & -10 & -0.167 & 0 & -10 \\ 0 & 10 & 0 & 0 & -10 & 0 \\ -10 & 0 & 800 & 10 & 0 & 400 \\ -0.167 & 0 & 10 & 0.167 & 0 & 10 \\ 0 & -10 & 0 & 0 & 10 & 0 \\ -10 & 0 & 400 & 10 & 0 & 800 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}} \tag{5.2.2}$$

Example 1

Element 2

For element 2, the angle between x and $\hat{x}$ is zero because $\hat{x}$ is directed from node 2 to node 3. Therefore,

$$C = 1 \qquad S = 0$$

Also,

$$\frac{12I}{L^2} = \frac{12(100)}{120^2} = 0.0835 \text{ in}^2$$

$$\frac{6I}{L} = \frac{6(100)}{120} = 5.0 \text{ in}^3 \tag{5.2.3}$$

$$\frac{E}{L} = 250{,}000 \text{ lb/in}^3$$

Using the quantities obtained in Eqs. (5.2.3) in evaluating Eq. (5.1.11) for element 2, we obtain

$$\underline{k}^{(2)} = 250{,}000 \begin{array}{c} \begin{array}{cccccc} d_{2x} & d_{2y} & \phi_2 & d_{3x} & d_{3y} & \phi_3 \end{array} \\ \begin{bmatrix} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.0835 & 5 & 0 & -0.0835 & 5 \\ 0 & 5 & 400 & 0 & -5 & 200 \\ -10 & 0 & 0 & 10 & 0 & 0 \\ 0 & -0.0835 & -5 & 0 & 0.0835 & -5 \\ 0 & 5 & 200 & 0 & -5 & 400 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}} \tag{5.2.4}$$

Example 1

Element 3

For element 3, the angle between x and $\hat{x}$ is 270° (or −90°) because $\hat{x}$ is directed from node 3 to node 4. Therefore,

$$C = 0 \qquad S = -1$$

Therefore, evaluating Eq. (5.1.11) for element 3, we obtain

$$\underline{k}^{(3)} = 250{,}000 \begin{array}{c} \begin{array}{cccccc} d_{3x} & d_{3y} & \phi_3 & d_{4x} & d_{4y} & \phi_4 \end{array} \\ \begin{bmatrix} 0.167 & 0 & 10 & -0.167 & 0 & 10 \\ 0 & 10 & 0 & 0 & -10 & 0 \\ 10 & 0 & 800 & -10 & 0 & 400 \\ -0.167 & 0 & -10 & 0.167 & 0 & -10 \\ 0 & -10 & 0 & 0 & 10 & 0 \\ 10 & 0 & 400 & -10 & 0 & 800 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}} \qquad (5.2.5)$$

Superposition of Eqs. (5.2.2), (5.2.4), and (5.2.5) and application of the boundary conditions $d_{1x} = d_{1y} = \phi_1 = 0$ and $d_{4x} = d_{4y} = \phi_4 = 0$ at nodes 1 and 4 yield the reduced

Example 1

set of equations for a longhand solution as

$$\begin{Bmatrix} 10{,}000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5000 \end{Bmatrix} = 250{,}000 \begin{bmatrix} 10.167 & 0 & 10 & -10 & 0 & 0 \\ 0 & 10.0835 & 5 & 0 & -0.0835 & 5 \\ 10 & 5 & 1200 & 0 & -5 & 200 \\ -10 & 0 & 0 & 10.167 & 0 & 10 \\ 0 & -0.0835 & -5 & 0 & 10.0835 & -5 \\ 0 & 5 & 200 & 10 & -5 & 1200 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{2y} \\ \phi_2 \\ d_{3x} \\ d_{3y} \\ \phi_3 \end{Bmatrix} \tag{5.2.6}$$

Solving Eq. (5.2.6) for the displacements and rotations, we have

$$\begin{Bmatrix} d_{2x} \\ d_{2y} \\ \phi_2 \\ d_{3x} \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0.211 \text{ in.} \\ 0.00148 \text{ in.} \\ -0.00153 \text{ rad} \\ 0.209 \text{ in.} \\ -0.00148 \text{ in.} \\ -0.00149 \text{ rad} \end{Bmatrix} \tag{5.2.7}$$

Example 1

The element forces can now be obtained using $\underline{f} = \underline{k}\,\underline{T}\,\underline{d}$ for each element, as was previously done in solving truss and beam problems. We will illustrate this procedure only for element 1. For element 1, on using Eq. (5.1.10) for $\underline{T}$ and Eq. (5.2.7) for the displacements at node 2, we have

$$\underline{T}\,\underline{d} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} = 0 \\ d_{1y} = 0 \\ \phi_1 = 0 \\ d_{2x} = 0.211 \\ d_{2y} = 0.00148 \\ \phi_2 = -0.00153 \end{Bmatrix} \tag{5.2.8}$$

On multiplying the matrices in Eq. (5.2.8), we obtain

$$\underline{T}\,\underline{d} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.00148 \\ -0.211 \\ -0.00153 \end{Bmatrix} \tag{5.2.9}$$

Example 1

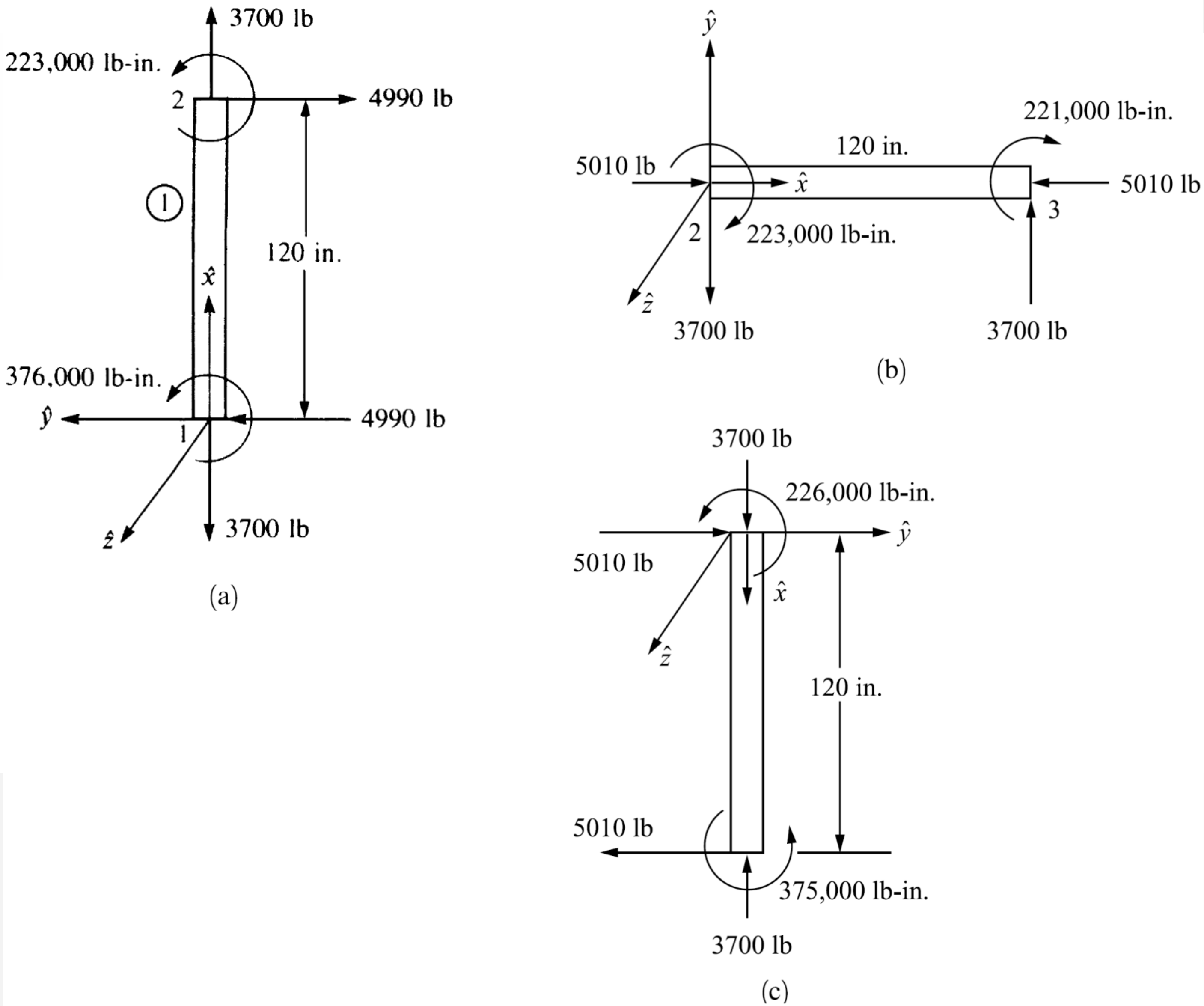
Then using $\underline{\hat{k}}$ from Eq. (5.1.8), we obtain element 1 local forces as

$$\underline{\hat{f}} = \underline{\hat{k}}\,\underline{T}\,\underline{d} = 250{,}000 \begin{bmatrix} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.167 & 10 & 0 & -0.167 & 10 \\ 0 & 10 & 800 & 0 & -10 & 400 \\ -10 & 0 & 0 & 10 & 0 & 0 \\ 0 & -0.167 & -10 & 0 & 0.167 & -10 \\ 0 & 10 & 400 & 0 & -10 & 800 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.00148 \\ -0.211 \\ -0.00153 \end{Bmatrix} \tag{5.2.10}$$

Simplifying Eq. (5.2.10), we obtain the local forces acting on element 1 as

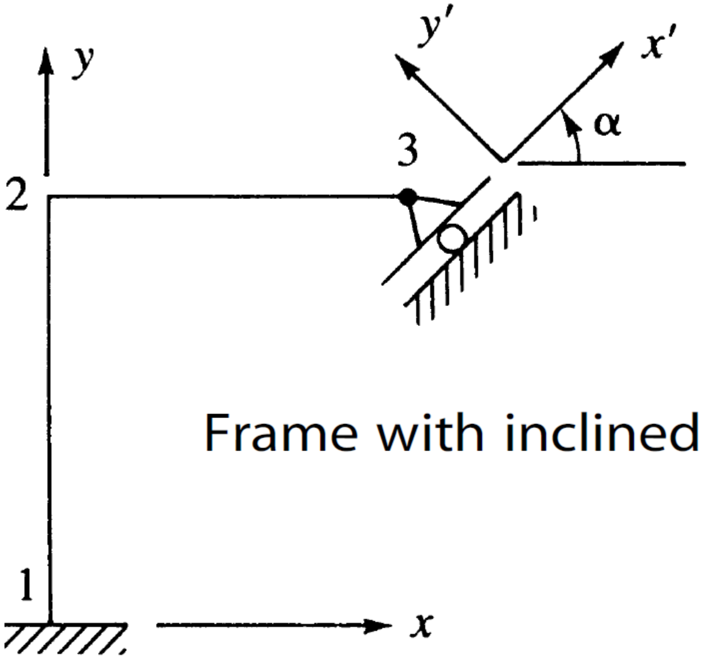
$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2x} \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \begin{Bmatrix} -3700 \text{ lb} \\ 4990 \text{ lb} \\ 376{,}000 \text{ lb-in.} \\ 3700 \text{ lb} \\ -4990 \text{ lb} \\ 223{,}000 \text{ lb-in.} \end{Bmatrix} \tag{5.2.11}$$

Example 1



3700 lb
223,000 lb-in.
2
4990 lb
1
120 in.
x̂
376,000 lb-in.
ŷ
1
4990 lb
ẑ
3700 lb
(a)
ŷ
221,000 lb-in.
120 in.
5010 lb
x̂
5010 lb
3
2
223,000 lb-in.
ẑ
3700 lb
3700 lb
(b)
3700 lb
226,000 lb-in.
ŷ
5010 lb
x̂
ẑ
120 in.
5010 lb
375,000 lb-in.
3700 lb
(c)

Inclined or Skewed supports: Frame Element



Frame with inclined support

$$\begin{Bmatrix} d'_{3x} \\ d'_{3y} \\ \phi'_3 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{3y} \\ \phi_3 \end{Bmatrix}$$

Inclined or Skewed supports: Frame Element

The same steps as given in Section 3.9 then follow for the plane frame. The resulting equations for the plane are:

$$[T_i]\{f\} = [T_i][K][T_i]^T\{d\}$$

or

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ M_1 \\ F_{2x} \\ F_{2y} \\ M_2 \\ F'_{3x} \\ F'_{3y} \\ M_3 \end{Bmatrix} = [T_i][K][T_i]^T \begin{Bmatrix} d_{1x} = 0 \\ d_{1y} = 0 \\ \phi_1 = 0 \\ d_{2x} \\ d_{2y} \\ \phi_2 \\ d'_{3x} \\ d'_{3y} = 0 \\ \phi'_3 = \phi_3 \end{Bmatrix}$$

where

$$[T_i] = \begin{bmatrix} [I] & [0] & [0] \\ [0] & [I] & [0] \\ [0] & [0] & [t_3] \end{bmatrix}$$

and

$$[t_3] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$