Introduction to the Finite Element Method

Lecture 4: 1-D Elements: Beam

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Outline

- Stiffness Matrix for a Beam Element
- Assembling Stiffness Matrix
- Distributed Loading
- Comparison with exact solution
- Beam element with nodal hinge
- Potential Energy Approach

Beam Element

- A beam is a long, slender structural member generally subjected to transverse loading
- D.O.F. of each node: a transverse displacement and a rotation

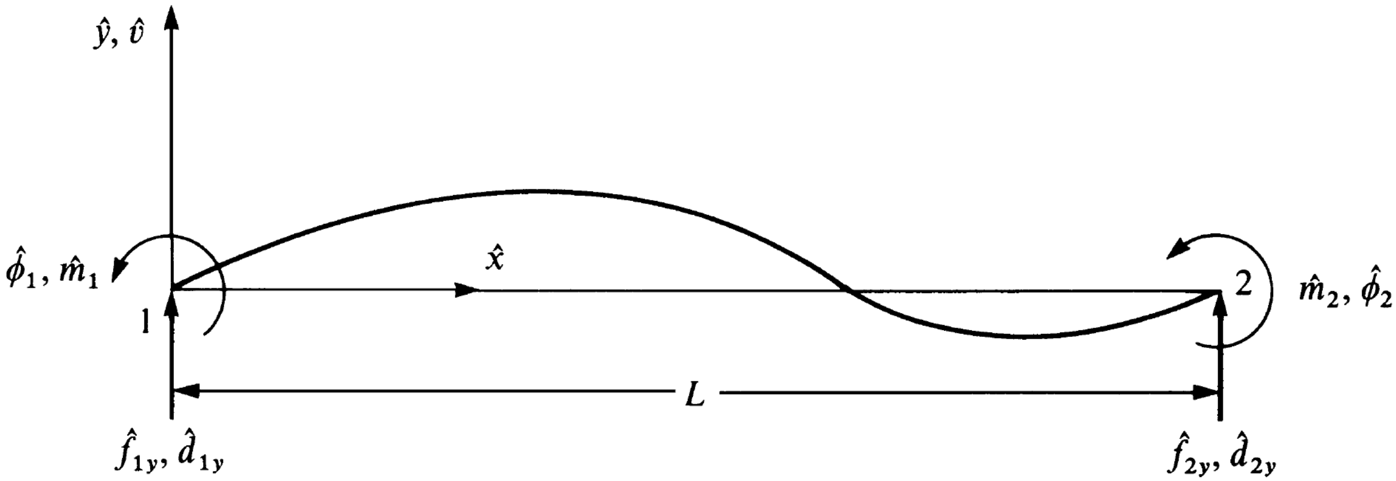


Figure 4–1 Beam element with positive nodal displacements, rotations, forces, and moments

Euler-Bernouli Beam Stiffness

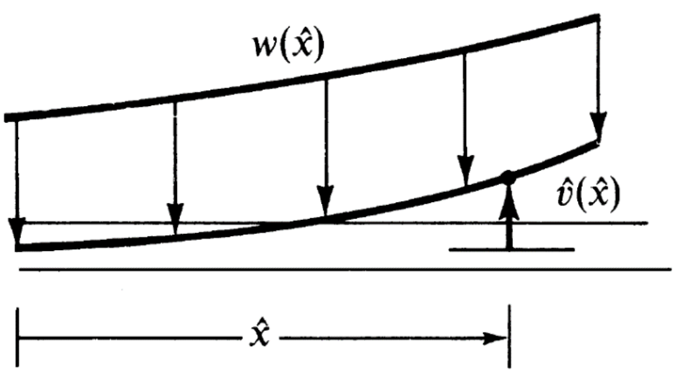
- plane cross sections perpendicular to the neutral axis remaining plane and perpendicular to it after bending.

$$\kappa = \frac{1}{\rho} = \frac{M}{EI} \qquad \kappa = \frac{d^2\hat{v}}{d\hat{x}^2}$$

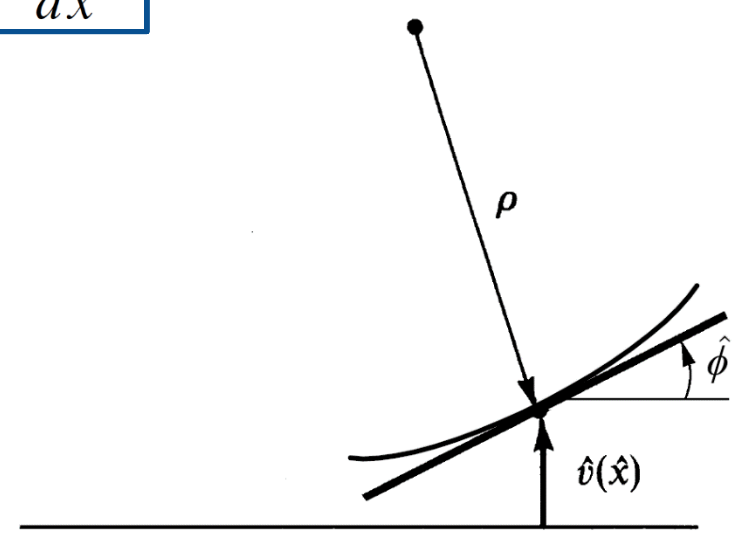
$$\Rightarrow \boxed{\frac{d^2\hat{v}}{d\hat{x}^2} = \frac{M}{EI}}$$

$$w = -\frac{dV}{d\hat{x}} \qquad V = \frac{dM}{d\hat{x}}$$

$$\Rightarrow \boxed{\frac{d^2}{d\hat{x}^2}\left(EI\frac{d^2\hat{v}}{d\hat{x}^2}\right) = -w(\hat{x})}$$



(a) Portion of deflected curve of beam



(b) Radius of deflected curve at $\hat{v}(\hat{x})$

Displacement Function

- For constant EI and only nodal forces and moments:

$$EI\frac{d^4\hat{v}}{d\hat{x}^4} = 0$$

- Transverse displacement function (cubic)

$$\hat{v}(\hat{x}) = a_1\hat{x}^3 + a_2\hat{x}^2 + a_3\hat{x} + a_4$$

$$\hat{v}(0) = \hat{d}_{1y} = a_4$$

$$\frac{d\hat{v}(0)}{d\hat{x}} = \hat{\phi}_1 = a_3$$

$$\hat{v}(L) = \hat{d}_{2y} = a_1L^3 + a_2L^2 + a_3L + a_4$$

$$\frac{d\hat{v}(L)}{d\hat{x}} = \hat{\phi}_2 = 3a_1L^2 + 2a_2L + a_3$$

$$\hat{v} = \left[\frac{2}{L^3}(\hat{d}_{1y} - \hat{d}_{2y}) + \frac{1}{L^2}(\hat{\phi}_1 + \hat{\phi}_2)\right]\hat{x}^3 + \left[-\frac{3}{L^2}(\hat{d}_{1y} - \hat{d}_{2y}) - \frac{1}{L}(2\hat{\phi}_1 + \hat{\phi}_2)\right]\hat{x}^2 + \hat{\phi}_1\hat{x} + \hat{d}_{1y}$$

Shape Functions

- In matrix form

$$\hat{v} = [N]\{\hat{d}\}$$

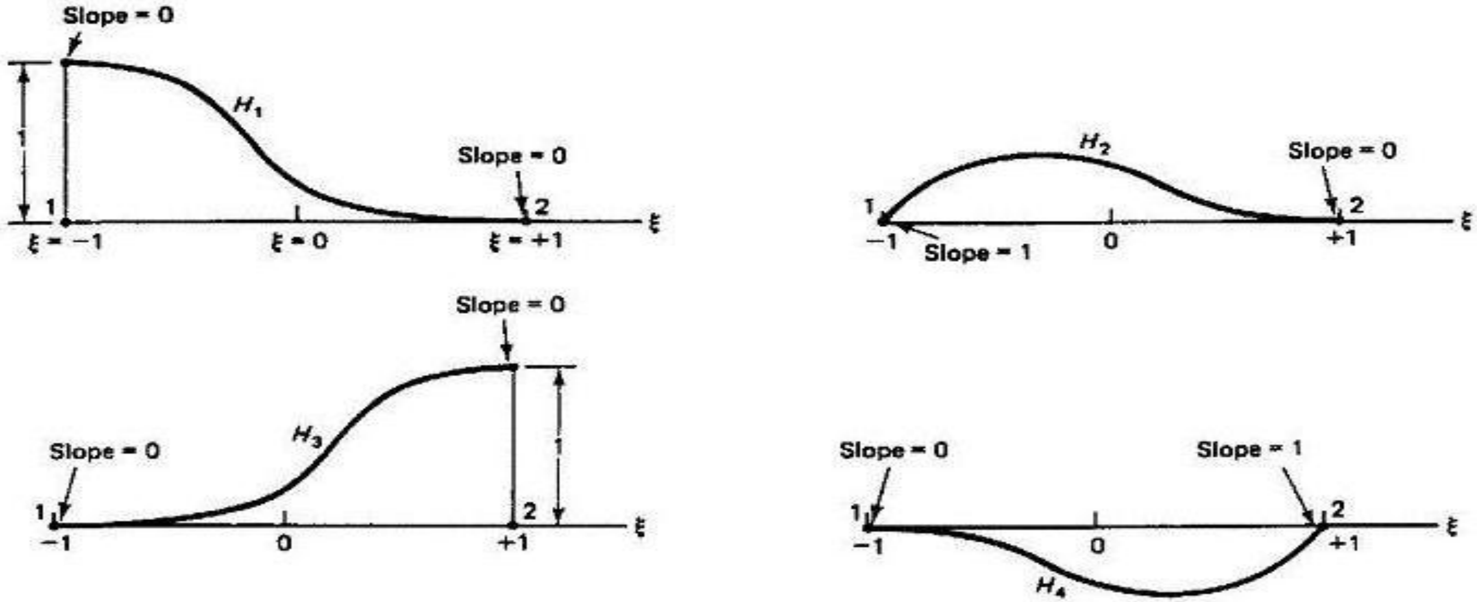
$$\{\hat{d}\} = \begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

$$[N] = [N_1 \quad N_2 \quad N_3 \quad N_4]$$

$$N_1 = \frac{1}{L^3}(2\hat{x}^3 - 3\hat{x}^2 L + L^3) \qquad N_2 = \frac{1}{L^3}(\hat{x}^3 L - 2\hat{x}^2 L^2 + \hat{x} L^3)$$

$$N_3 = \frac{1}{L^3}(-2\hat{x}^3 + 3\hat{x}^2 L) \qquad N_4 = \frac{1}{L^3}(\hat{x}^3 L - \hat{x}^2 L^2)$$

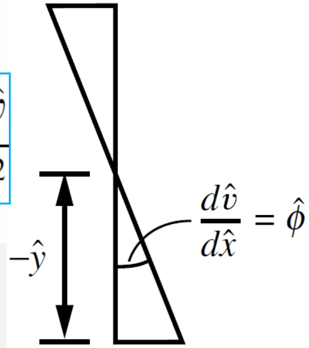
Shape Functions

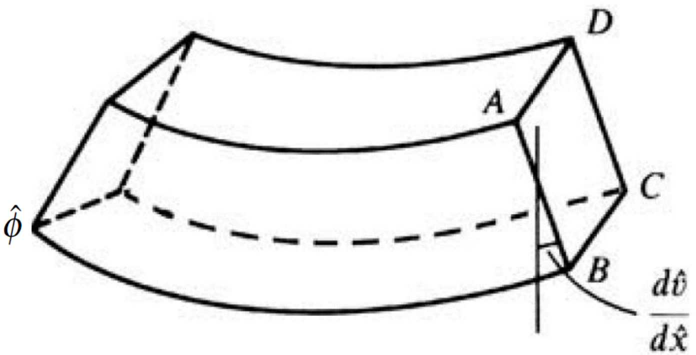


$$\varepsilon_x(\hat{x},\hat{y}) = \frac{d\hat{u}}{d\hat{x}}$$

$$\hat{u} = -\hat{y}\,\frac{d\hat{v}}{d\hat{x}}$$

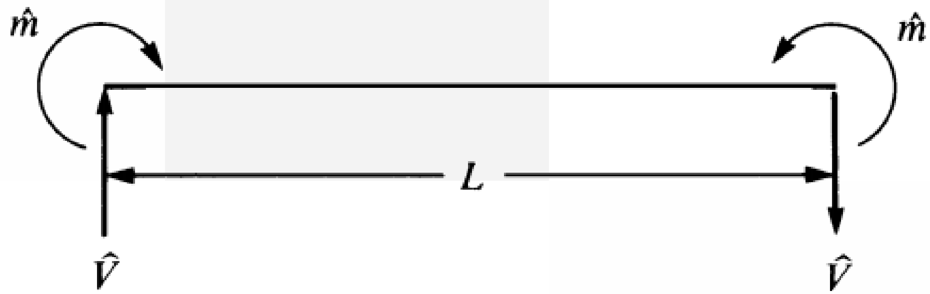
$$\Rightarrow \boxed{\varepsilon_x(\hat{x},\hat{y}) = -\hat{y}\,\frac{d^2\hat{v}}{d\hat{x}^2}}$$





Stiffness Matrix

$$\hat{m}(\hat{x}) = EI\frac{d^2\hat{v}}{d\hat{x}^2} \qquad \hat{V} = EI\frac{d^3\hat{v}}{d\hat{x}^3}$$



$$\hat{f}_{1y} = \hat{V} = EI\frac{d^3\hat{v}(0)}{d\hat{x}^3} = \frac{EI}{L^3}(12\hat{d}_{1y} + 6L\hat{\phi}_1 - 12\hat{d}_{2y} + 6L\hat{\phi}_2)$$

$$\hat{m}_1 = -\hat{m} = -EI\frac{d^2\hat{v}(0)}{d\hat{x}^2} = \frac{EI}{L^3}(6L\hat{d}_{1y} + 4L^2\hat{\phi}_1 - 6L\hat{d}_{2y} + 2L^2\hat{\phi}_2)$$

$$\hat{f}_{2y} = -\hat{V} = -EI\frac{d^3\hat{v}(L)}{d\hat{x}^3} = \frac{EI}{L^3}(-12\hat{d}_{1y} - 6L\hat{\phi}_1 + 12\hat{d}_{2y} - 6L\hat{\phi}_2)$$

$$\hat{m}_2 = \hat{m} = EI\frac{d^2\hat{v}(L)}{d\hat{x}^2} = \frac{EI}{L^3}(6L\hat{d}_{1y} + 2L^2\hat{\phi}_1 - 6L\hat{d}_{2y} + 4L^2\hat{\phi}_2)$$

Stiffness Matrix

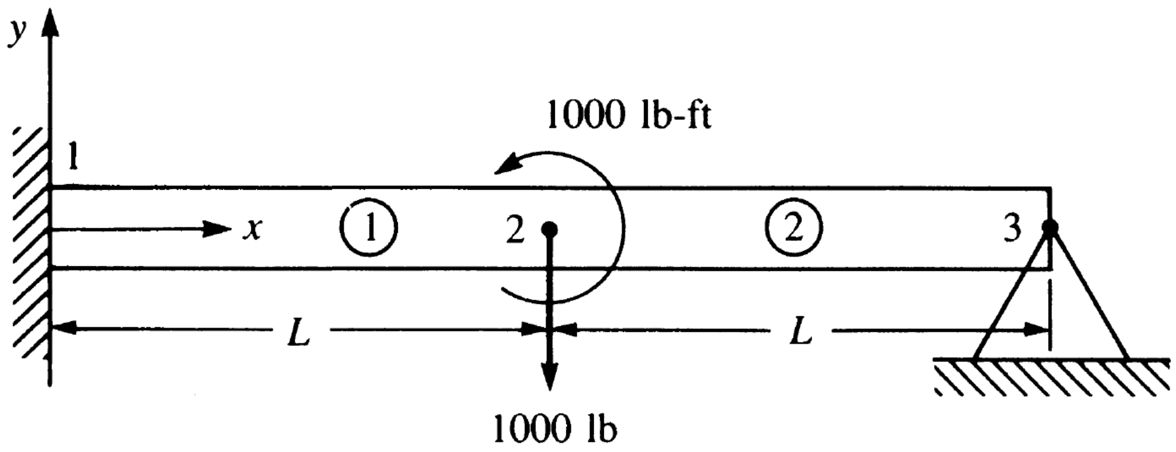
$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

- Stiffness matrix:

$$\underline{\hat{k}} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Example 1

- Assemblage of Beam Stiffness Matrices



Example 1

$$\underline{k}^{(1)} = \frac{EI}{L^3} \begin{array}{c} \begin{matrix} d_{1y} & \phi_1 & d_{2y} & \phi_2 \end{matrix} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array} \qquad \underline{k}^{(2)} = \frac{EI}{L^3} \begin{array}{c} \begin{matrix} d_{2y} & \phi_2 & d_{3y} & \phi_3 \end{matrix} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12+12 & -6L+6L & -12 & 6L \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix}$$

Example 1

- B.C.'s: $\phi_1 = 0 \qquad d_{1y} = 0 \qquad d_{3y} = 0$

$$\begin{Bmatrix} -1000 \\ 1000 \\ 0 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 24 & 0 & 6L \\ 0 & 8L^2 & 2L^2 \\ 6L & 2L^2 & 4L^2 \end{bmatrix} \begin{Bmatrix} d_{2y} \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

Example 2

Using the direct stiffness method, solve the problem of the propped cantilever beam subjected to end load P in Figure 4–8. The beam is assumed to have constant EI and length $2L$. It is supported by a roller at midlength and is built in at the right end.

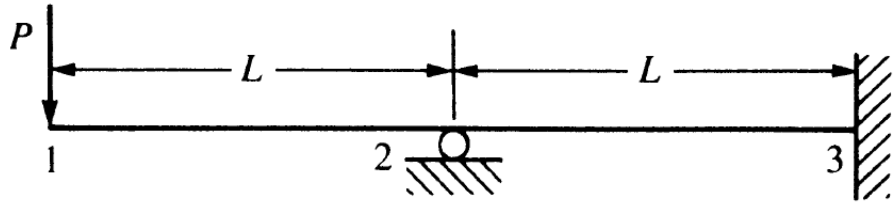


Figure 4–8 Propped cantilever beam

The $\underline{K}$ is

$$\begin{array}{c} \\ \underline{K} = \dfrac{EI}{L^3} \end{array}
\begin{array}{c}
\begin{array}{cccccc} d_{1y} & \phi_1 & d_{2y} & \phi_2 & d_{3y} & \phi_3 \end{array} \\
\begin{bmatrix}
12 & 6L & -12 & 6L & 0 & 0 \\
 & 4L^2 & -6L & 2L^2 & 0 & 0 \\
 & & 12+12 & -6L+6L & -12 & 6L \\
 & & & 4L^2+4L^2 & -6L & 2L^2 \\
 & & & & 12 & -6L \\
\text{Symmetry} & & & & & 4L^2
\end{bmatrix}
\end{array}$$

Example 2

The governing equations for the beam are then given by

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix}$$

On applying the boundary conditions

$$d_{2y} = 0 \qquad d_{3y} = 0 \qquad \phi_3 = 0$$

$$\begin{Bmatrix} -P \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & 6L \\ 6L & 4L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix} \begin{Bmatrix} d_{1y} \\ \phi_1 \\ \phi_2 \end{Bmatrix}$$

where $F_{1y} = P$, $M_1 = 0$, and $M_2 = 0$ have been used

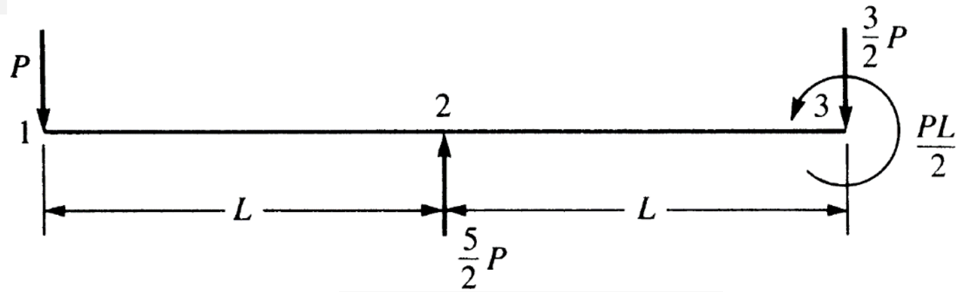
Example 2

$$d_{1y} = -\frac{7PL^3}{12EI} \qquad \phi_1 = \frac{3PL^2}{4EI} \qquad \phi_2 = \frac{PL^2}{4EI}$$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -\frac{7PL^3}{12EI} \\ \frac{3PL^2}{4EI} \\ 0 \\ \frac{PL^2}{4EI} \\ 0 \\ 0 \end{Bmatrix}$$

$$F_{1y} = -P \qquad M_1 = 0 \qquad F_{2y} = \tfrac{5}{2}P$$

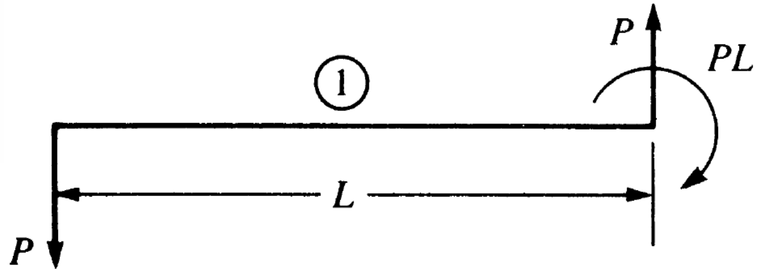
$$M_2 = 0 \qquad F_{3y} = -\tfrac{3}{2}P \qquad M_3 = \tfrac{1}{2}PL$$



Example 2

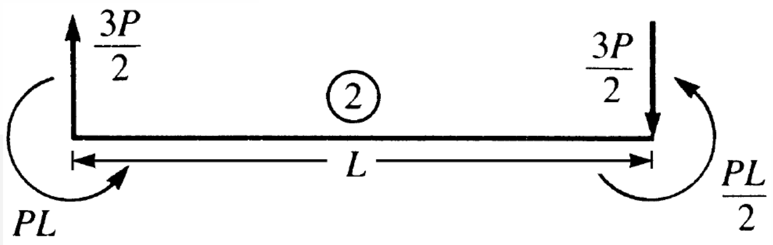
For element 1

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -\dfrac{7PL^3}{12EI} \\ \dfrac{3PL^2}{4EI} \\ 0 \\ \dfrac{PL^2}{4EI} \end{Bmatrix}$$



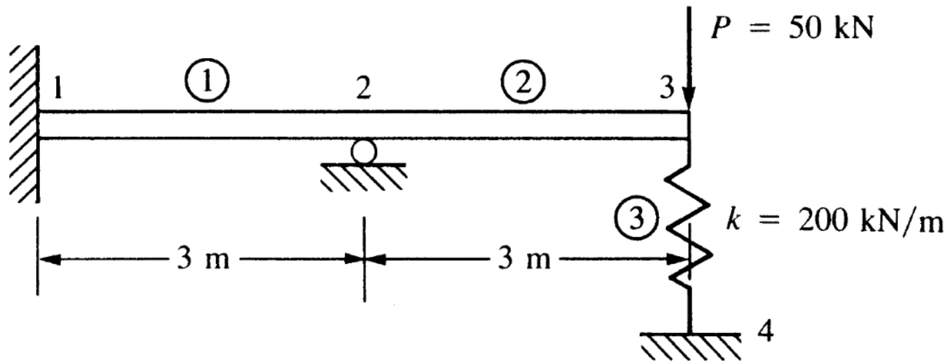
$$\hat{f}_{1y} = -P \qquad \hat{m}_1 = 0 \qquad \hat{f}_{2y} = P \qquad \hat{m}_2 = -PL$$

For element 2



Example 3

Determine the nodal displacements and rotations and the global and element forces. Let E = 210 GPa and $I = 2\times10^{4}\,m^{4}$ throughout the beam, and let k = 200 kN/m.



We obtain the structure stiffness matrix as

$$
\underline{K} = \frac{EI}{L^3}
\begin{array}{c}
\begin{array}{ccccccc} d_{1y} & \phi_1 & d_{2y} & \phi_2 & d_{3y} & \phi_3 & d_{4y} \end{array} \\
\begin{bmatrix}
12 & 6L & -12 & 6L & 0 & 0 & 0 \\
 & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\
 & & 24 & 0 & -12 & 6L & 0 \\
 & & & 8L^2 & -6L & 2L^2 & 0 \\
 & & & & 12 + \dfrac{kL^3}{EI} & -6L & -\dfrac{kL^3}{EI} \\
 & & & & & 4L^2 & 0 \\
\text{Symmetry} & & & & & & \dfrac{kL^3}{EI}
\end{bmatrix}
\end{array}
$$

Example 3

where the spring stiffness matrix k_s given below

$$\begin{array}{cc} & \begin{array}{cc} d_{3y} & d_{4y} \end{array} \\ \underline{k}_s = & \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{array}$$

The governing equations for the beam are then given by

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \\ F_{4y} \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 \\ & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\ & & 24 & 0 & -12 & 6L & 0 \\ & & & 8L^2 & -6L & 2L^2 & 0 \\ & & & & 12+k' & -6L & -k' \\ & & & & & 4L^2 & 0 \\ \text{Symmetry} & & & & & & k' \end{bmatrix} \begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \\ d_{4y} \end{Bmatrix}$$

where $k' = kL^3/(EI)$

Example 3

We now apply the B.C.'s:

$$d_{1y} = 0 \qquad \phi_1 = 0 \qquad d_{2y} = 0 \qquad d_{4y} = 0$$

$$\begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 8L^2 & -6L & 2L^2 \\ -6L & 12 + k' & -6L \\ 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix}$$

$$d_{3y} = -\frac{7PL^3}{EI}\left(\frac{1}{12 + 7k'}\right) = \frac{-7(50\text{ kN})(3\text{ m})^3}{(210 \times 10^6\text{ kN/m}^2)(2 \times 10^{-4}\text{ m}^4)}\left(\frac{1}{12 + 7(0.129)}\right) = -0.0174\text{ m}$$

$$\phi_2 = -\frac{3PL^2}{EI}\left(\frac{1}{12 + 7k'}\right) = -0.00249\text{ rad}$$

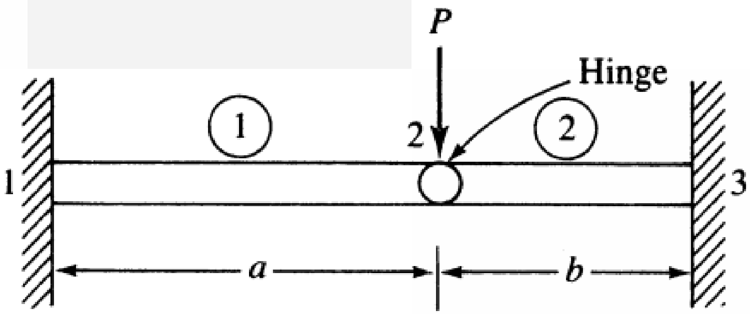
$$\phi_3 = -\frac{9PL^2}{EI}\left(\frac{1}{12 + 7k'}\right) = -0.00747\text{ rad}$$

$$\boxed{\begin{array}{ll} F_{1y} = -69.9\text{ kN} & M_1 = -69.7\text{ kN}\cdot\text{m} \\ F_{2y} = 116.4\text{ kN} & M_2 = 0.0\text{ kN}\cdot\text{m} \\ F_{3y} = -50.0\text{ kN} & M_3 = 0.0\text{ kN}\cdot\text{m} \end{array}}$$

$$\boxed{F_{4y} = -d_{3y}k = (0.0174)200 = 3.5\text{ kN}}$$

Example 4: Nodal Hinge

Determine the displacement and rotation at node 2 and the element forces for the uniform beam with an internal hinge at node 2 shown in Figure 4–34. Let EI be a constant.



$$k^{(1)} = \frac{EI}{a^3}\begin{array}{c} \begin{array}{cccc} d_{1y} & \phi_1 & d_{2y} & \phi_2 \end{array} \\ \begin{bmatrix} 12 & 6a & -12 & 6a \\ 6a & 4a^2 & -6a & 2a^2 \\ -12 & -6a & 12 & -6a \\ 6a & 2a^2 & -6a & 4a^2 \end{bmatrix} \end{array}$$

$$k^{(2)} = \frac{EI}{b^3}\begin{array}{c} \begin{array}{cccc} d_{2y} & \phi_2' & d_{3y} & \phi_3 \end{array} \\ \begin{bmatrix} 12 & 6b & -12 & 6b \\ 6b & 4b^2 & -6b & 2b^2 \\ -12 & -6b & 12 & -6b \\ 6b & 2b^2 & -6b & 4b^2 \end{bmatrix} \end{array}$$

$$\Rightarrow$$

$$k = EI\begin{array}{c} \begin{array}{ccc} d_{2y} & \phi_2 & \phi_2' \end{array} \\ \begin{bmatrix} \frac{12}{a^3}+\frac{12}{b^3} & \frac{-6}{a^2} & \frac{6}{b^2} \\ \frac{-6}{a^2} & \frac{4}{a} & 0 \\ \frac{6}{b^2} & 0 & \frac{4}{b} \end{bmatrix} \end{array}$$

Example 4: Nodal Hinge

$$EI\begin{bmatrix} \frac{12}{a^3}+\frac{12}{b^3} & \frac{-6}{a^2} & \frac{6}{b^2} \\ \frac{-6}{a^2} & \frac{4}{a} & 0 \\ \frac{6}{b^2} & 0 & \frac{4}{b} \end{bmatrix} \times \begin{Bmatrix} d_{2y} \\ \phi_2 \\ \phi_2' \end{Bmatrix} = \begin{Bmatrix} -P \\ 0 \\ 0 \end{Bmatrix} \Rightarrow$$

$$d_{2y} = \frac{-a^3b^3P}{3(b^3+a^3)EI}$$

$$\phi_2 = \frac{-a^2b^3P}{2(b^3+a^3)EI}$$

$$\phi_2' = \frac{a^3b^2P}{2(b^3+a^3)EI}$$

- For Element 2:

$$\begin{Bmatrix} \hat{f}_{2y} \\ \hat{m}_2 \\ \hat{f}_{3y} \\ \hat{m}_3 \end{Bmatrix} = \frac{EI}{b^3}\begin{bmatrix} 12 & 6b & -12 & 6b \\ 6b & 4b^2 & -6b & 2b^2 \\ -12 & -6b & 12 & -6b \\ 6b & 2b^2 & -6b & 4b^2 \end{bmatrix} \begin{Bmatrix} -\frac{a^3b^3P}{3(b^3+a^3)EI} \\ \frac{a^3b^2P}{2(b^3+a^3)EI} \\ 0 \\ 0 \end{Bmatrix} \Rightarrow$$

$$\hat{f}_{2y} = -\frac{a^3P}{b^3+a^3}$$

$$\hat{m}_2 = 0$$

$$\hat{f}_{3y} = \frac{a^3P}{b^3+a^3}$$

$$\hat{m}_3 = -\frac{ba^3P}{b^3+a^3}$$

Potential Energy Approach

The total potential energy for a beam is

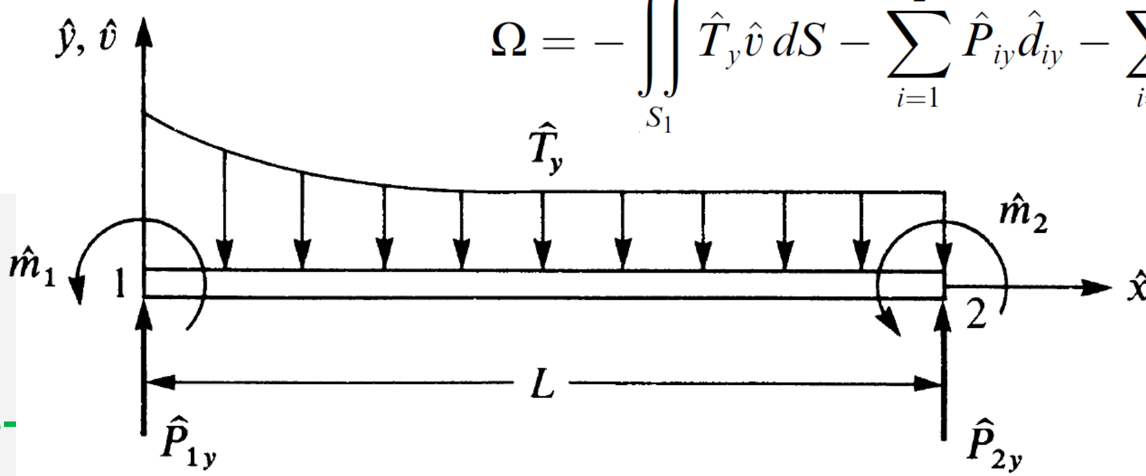
$$\pi_p = U + \Omega$$

where the general one-dimensional expression for the strain energy U for a beam is

$$U = \iiint_V \frac{1}{2}\sigma_x \varepsilon_x \, dV$$

and for a single beam element subjected to both distributed and concentrated nodal loads, the potential energy of forces is given by

$$\Omega = -\iint_{S_1} \hat{T}_y \hat{v} \, dS - \sum_{i=1}^{2} \hat{P}_{iy} \hat{d}_{iy} - \sum_{i=1}^{2} \hat{m}_i \hat{\phi}_i$$



Potential Energy Approach

$$\boxed{\hat{v} = [N]\{\hat{d}\}}$$

$$[N] = [N_1 \quad N_2 \quad N_3 \quad N_4]$$

$$N_1 = \frac{1}{L^3}(2\hat{x}^3 - 3\hat{x}^2 L + L^3) \qquad N_2 = \frac{1}{L^3}(\hat{x}^3 L - 2\hat{x}^2 L^2 + \hat{x} L^3)$$

$$\boxed{\varepsilon_x = -\hat{y}\frac{d^2\hat{v}}{d\hat{x}^2}}$$

$$N_3 = \frac{1}{L^3}(-2\hat{x}^3 + 3\hat{x}^2 L) \qquad N_4 = \frac{1}{L^3}(\hat{x}^3 L - \hat{x}^2 L^2)$$

$$\{\varepsilon_x\} = -\hat{y}\left[\frac{12\hat{x} - 6L}{L^3} \quad \frac{6\hat{x}L - 4L^2}{L^3} \quad \frac{-12\hat{x} + 6L}{L^3} \quad \frac{6\hat{x}L - 2L^2}{L^3}\right]\{\hat{d}\}$$

or

$$\boxed{\{\varepsilon_x\} = -\hat{y}[B]\{\hat{d}\}}$$

where we define

$$[B] = \left[\frac{12\hat{x} - 6L}{L^3} \quad \frac{6\hat{x}L - 4L^2}{L^3} \quad \frac{-12\hat{x} + 6L}{L^3} \quad \frac{6\hat{x}L - 2L^2}{L^3}\right]$$

Potential Energy Approach

$$\{\sigma_x\} = [D]\{\varepsilon_x\} = -\hat{y}[D][B]\{\hat{d}\}$$

$$[D] = [E]$$

The total potential energy is expressed in matrix notation as

$$\pi_p = \iiint_{\hat{x}\ A} \frac{1}{2}\{\sigma_x\}^T\{\varepsilon_x\}\, dA\, d\hat{x} - \int_0^L b\hat{T}_y[\hat{v}]^T\, d\hat{x} - \{\hat{d}\}^T\{\hat{P}\}$$

- Load per unit length $w = b\hat{T}_y$, and moment of inertia $I = \iint_A y^2\, dA$

$$\pi_p = \int_0^L \frac{EI}{2}\{\hat{d}\}^T[B]^T[B]\{\hat{d}\}\, d\hat{x} - \int_0^L w\{\hat{d}\}^T[N]^T\, d\hat{x} - \{\hat{d}\}^T\{\hat{P}\}$$

Potential Energy Approach

Differentiating π_p with respect to $\hat{d}_{1y}, \hat{\phi}_1, \hat{d}_{2y}$ and $\hat{\phi}_2$ and equating each term to zero to minimize π_p, we obtain four element equations, which are written in matrix form as

$$EI \int_0^L [B]^T [B]\, d\hat{x} \{\hat{d}\} - \int_0^L [N]^T w\, d\hat{x} - \{\hat{P}\} = 0$$

$$\{\hat{f}\} = \int_0^L [N]^T w\, d\hat{x} + \{\hat{P}\}$$

$$[\hat{k}] = EI \int_0^L [B]^T [B]\, d\hat{x}$$

$$\{\hat{f}\} = [\hat{k}]\{\hat{d}\}$$

$$[\hat{k}] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{Symmetry} & & & 4L^2 \end{bmatrix}$$

Distributed Load

- Equivalent nodal forces:

$$f_0 = \int_0^L N^T w \, d\hat{x} = \frac{1}{L^3} \int_0^L \begin{bmatrix} 2\hat{x}^3 - 3\hat{x}^2 L + L^3 \\ \hat{x}^3 L - 2\hat{x}^2 L^2 + \hat{x} L^3 \\ -2\hat{x}^3 + 3\hat{x}^2 L \\ \hat{x}^3 L - \hat{x}^2 L^2 \end{bmatrix} w \, d\hat{x}$$

- If w is constant

$$f_0 = \begin{Bmatrix} \dfrac{-wL}{2} \\ \dfrac{-wL^2}{12} \\ \dfrac{-wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix}$$

Distributed Load

In general, we can account for distributed loads or concentrated loads acting on beam elements by starting with the following formulation application for a general structure:

$$\underline{F} = \underline{K}\underline{d} - \underline{F}_o \tag{4.4.8}$$

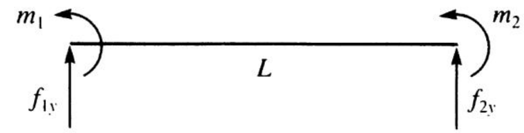
where $\underline{F}$ are the concentrated nodal forces and $\underline{F}_o$ are called the *equivalent nodal forces*, now expressed in terms of global-coordinate components, which are of such magnitude that they yield the same displacements at the nodes as would the distributed load. Using the table in Appendix D of equivalent nodal forces $\hat{\underline{f}}_o$ expressed in terms of local-coordinate components, we can express $\underline{F}_o$ in terms of global-coordinate components.

This concept can be applied on a local basis to obtain the local nodal forces $\hat{\underline{f}}$ in individual elements of structures by applying Eq. (4.4.8) locally as

$$\hat{\underline{f}} = \hat{\underline{k}}\hat{\underline{d}} - \hat{\underline{f}}_o \tag{4.4.11}$$

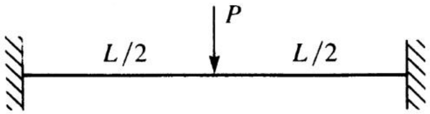
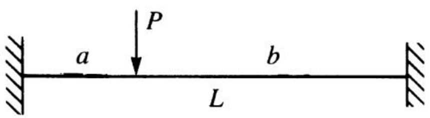
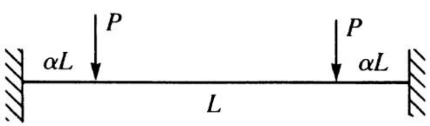
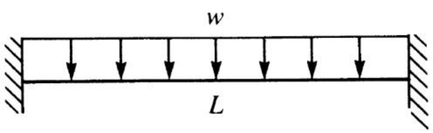
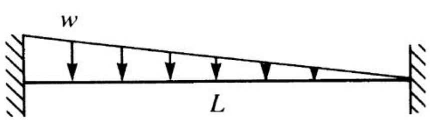
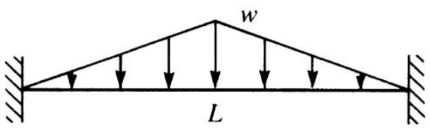
where $\hat{\underline{f}}_o$ are the equivalent local nodal forces.

Distributed Load



Positive nodal force conventions

Table D–1 Single element equivalent joint forces f_0 for different types of loads

	f_{1y}	m_1	Loading case	f_{2y}	m_2
1.	$\frac{-P}{2}$	$\frac{-PL}{8}$		$\frac{-P}{2}$	$\frac{PL}{8}$
2.	$\frac{-Pb^2(L+2a)}{L^3}$	$\frac{-Pab^2}{L^2}$		$\frac{-Pa^2(L+2b)}{L^3}$	$\frac{Pa^2b}{L^2}$
3.	$-P$	$-\alpha(1-\alpha)PL$		$-P$	$\alpha(1-\alpha)PL$
4.	$\frac{-wL}{2}$	$\frac{-wL^2}{12}$		$\frac{-wL}{2}$	$\frac{wL^2}{12}$
5.	$\frac{-7wL}{20}$	$\frac{-wL^2}{20}$		$\frac{-3wL}{20}$	$\frac{wL^2}{30}$
6.	$\frac{-wL}{4}$	$\frac{-5wL^2}{96}$		$\frac{-wL}{4}$	$\frac{5wL^2}{96}$

Example 1

For the cantilever beam subjected to the concentrated free-end load P and the uniformly distributed load w acting over the whole beam as shown in Figure 4–28, determine the free-end displacements and the nodal forces.

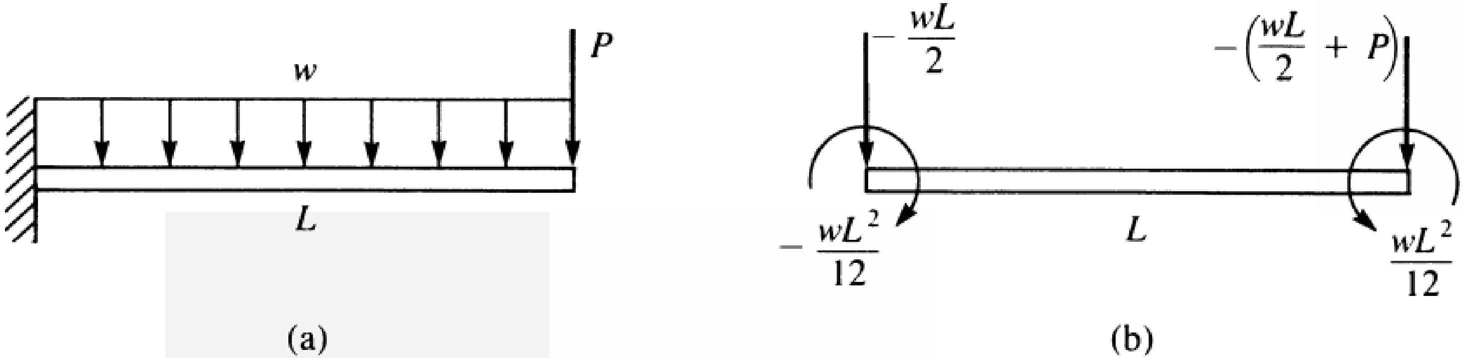


Figure 4–28 (a) Cantilever beam subjected to a concentrated load and a distributed load and (b) the equivalent nodal force replacement system

Example 1

Once again, the beam is modeled using one element with nodes 1 and 2, and the distributed load is replaced as shown in Figure 4–28(b) using appropriate loading case 4 in Appendix D. Using the beam element stiffness Eq. (4.1.14), we obtain

$$\frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{2y} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{-wL}{2} - P \\ \dfrac{wL^2}{12} \end{Bmatrix} \tag{4.4.24}$$

where we have applied the nodal forces from Figure 4–28(b) and the boundary conditions $d_{1y} = 0$ and $\phi_1 = 0$ to reduce the number of matrix equations for the usual longhand solution. Solving Eq. (4.4.24) for the displacements, we obtain

$$\begin{Bmatrix} d_{2y} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{-wL^4}{8EI} - \dfrac{PL^3}{3EI} \\ \dfrac{-wL^3}{6EI} - \dfrac{PL^2}{2EI} \end{Bmatrix} \begin{matrix} \downarrow \\ \circlearrowright \end{matrix} \tag{4.4.25}$$

Example 1

Next, we obtain the effective nodal forces using $\underline{F}^{(e)} = \underline{K}\underline{d}$ as

$$\begin{Bmatrix} F_{1y}^{(e)} \\ M_1^{(e)} \\ F_{2y}^{(e)} \\ M_2^{(e)} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \dfrac{-wL^4}{8EI} - \dfrac{PL^3}{3EI} \\ \dfrac{-wL^3}{6EI} - \dfrac{PL^2}{2EI} \end{Bmatrix} \tag{4.4.26}$$

Simplifying Eq. (4.4.26), we obtain

$$\begin{Bmatrix} F_{1y}^{(e)} \\ M_1^{(e)} \\ F_{2y}^{(e)} \\ M_2^{(e)} \end{Bmatrix} = \begin{Bmatrix} P + \dfrac{wL}{2} \\ PL + \dfrac{5wL^2}{12} \\ -P - \dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix} \tag{4.4.27}$$

Example 1

Finally, subtracting the equivalent nodal force matrix [see Figure 4–27(b)] from the effective force matrix of Eq. (4.4.27), we obtain the correct nodal forces as

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \end{Bmatrix} = \begin{Bmatrix} P + \dfrac{wL}{2} \\ PL + \dfrac{5wL^2}{12} \\ -P - \dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix} - \begin{Bmatrix} \dfrac{-wL}{2} \\ \dfrac{-wL^2}{12} \\ \dfrac{-wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix} = \begin{Bmatrix} P + wL \\ PL + \dfrac{wL^2}{2} \\ -P \\ 0 \end{Bmatrix} \qquad (4.4.28)$$

From Eq. (4.4.28), we see that F_{1y} is equivalent to the vertical reaction force, M_1 is the reaction moment at node 1, and F_{2y} is equal to the applied downward force P at node 2. [Remember that only the equivalent nodal force matrix is subtracted, not the original concentrated load matrix. This is based on the general formulation, Eq. (4.4.8).] ■

Example 2: Exact Solution

We will now compare the finite element solution to the exact classical beam theory solution for the cantilever beam shown in Figure 4–30 subjected to a uniformly distributed load. Both one- and two-element finite element solutions will be presented and compared to the exact solution obtained by the direct double-integration method. Let $E = 30 \times 10^6$ psi, $I = 100\ \text{in}^4$, $L = 100$ in., and uniform load $w = 20$ lb/in.

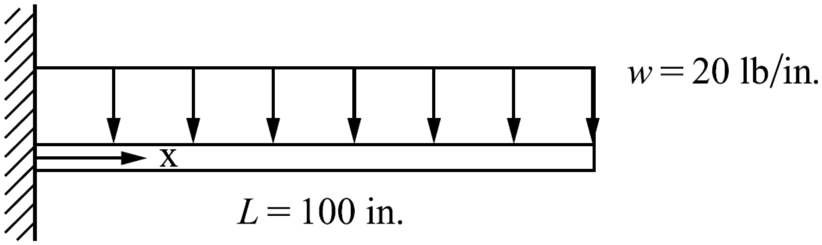
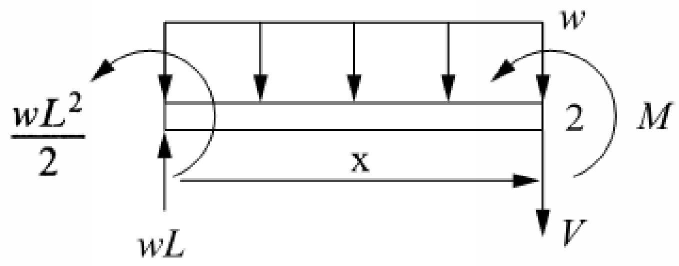


Figure 4–30 Cantilever beam subjected to uniformly distributed load

To obtain the solution from classical beam theory, we use the double-integration method [1]. Therefore, we begin with the moment-curvature equation

$$y'' = \frac{M(x)}{EI} \tag{4.5.1}$$

Example 2: Exact Solution



$$M(x) = \frac{-wL^2}{2} + wLx - (wx)\left(\frac{x}{2}\right)$$

$$y'' = \frac{1}{EI}\left(\frac{-wL^2}{2} + wLx - \frac{wx^2}{2}\right)$$

- Integrating and applying B.C.'s:

$$y = \frac{1}{EI}\left(\frac{-wx^4}{24} + \frac{wLx^3}{6} - \frac{wL^2x^2}{4}\right)$$

Example 2: FE Solution

- P=o in the previous example

$$\hat{\phi}_2 = \frac{-wL^3}{6EI} = \frac{-(20\ \text{lb/in.})(100\ \text{in.})^3}{6(30\times 10^6\ \text{psi})(100\ \text{in.}^4)} = -0.00111\ \text{rad}$$

$$\hat{d}_{2y} = \frac{-wL^4}{8EI} = \frac{-(20\ \text{lb/in.})(100\ \text{in.})^4}{8(30\times 10^6\ \text{psi})(100\ \text{in.}^4)} = -0.0833\ \text{in.}$$

$$\hat{v}(x) = \frac{1}{L^3}(-2x^3 + 3x^2L)\hat{d}_{2y} + \frac{1}{L^3}(x^3L - x^2L^2)\hat{\phi}_2$$

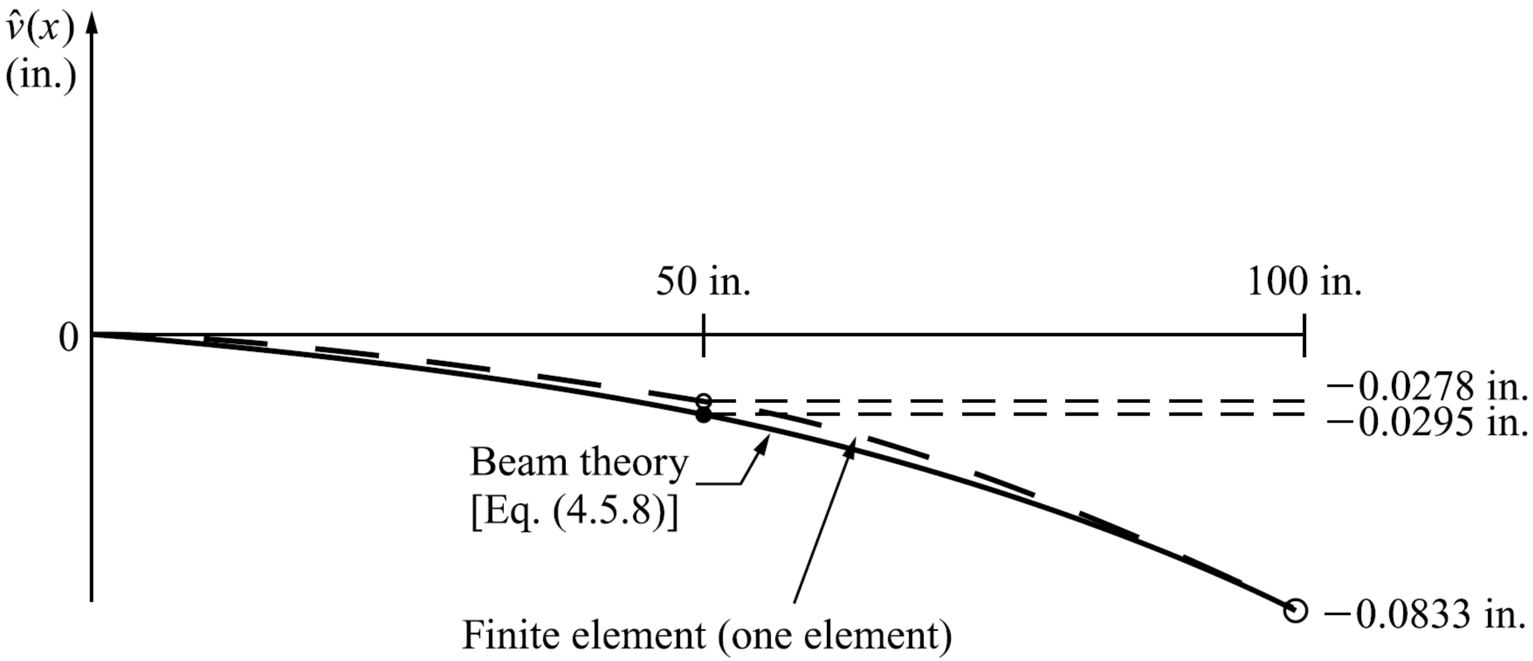
$$\hat{v}(x = 50\ \text{in.}) = \frac{1}{(100\ \text{in.})^3}[-2(50\ \text{in.})^3 + 3(50\ \text{in.})^2(100\ \text{in.})](-0.0833\ \text{in.})$$
$$+ \frac{1}{(100\ \text{in.})^3}[(50\ \text{in.})^3(100\ \text{in.}) - (50\ \text{in.})^2(100\ \text{in.})^2] \times (-0.00111\ \text{rad}) = -0.0278\ \text{in.}$$

$$y(x = 50\ \text{in.}) = \frac{20\ \text{lb/in.}}{30\times 10^6\ \text{psi}(100\ \text{in.}^4)}$$
$$\times \left[\frac{-(50\ \text{in.})^4}{24} + \frac{(100\ \text{in.})(50\ \text{in.})^3}{6} - \frac{(100\ \text{in.})^2(50\ \text{in.})^2}{4}\right] = -0.0295\ \text{in.}$$

Example 2: FE Solution

- Bending moment is derived by taking two derivatives on the displacement function.
- The shear force is derived by taking three derivatives on the displacement function.
- For uniform distributed load:
  - Bending moment: linear
  - Shear force: constant
- It then takes more elements to model Bending moment and shear force than displacement.

Example 2: Comparison



Example 2: Comparison