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Computational Fluid Dynamics

Basic Governing Equations

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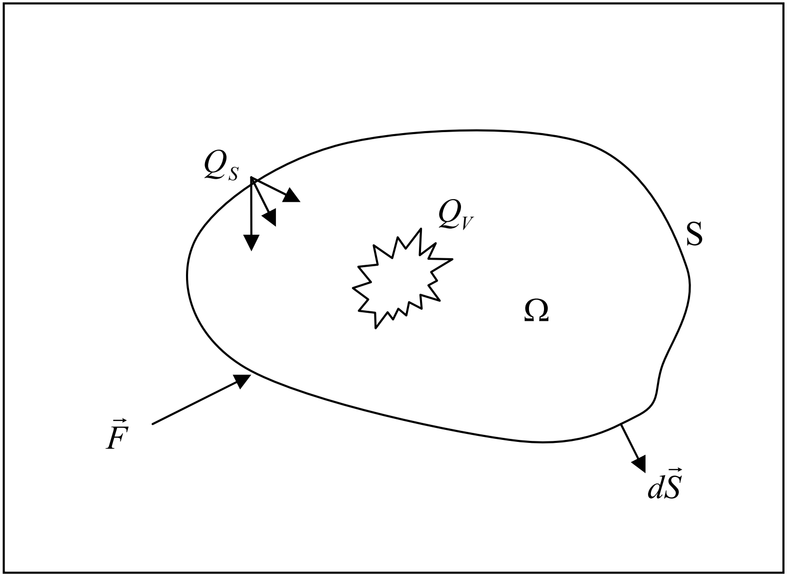
Conservation laws

- **Many of physical laws are described as conservation laws.**



Conservation laws

- For a typical scalar quantity, U , conservation law can be written as:

$$\boxed{\overbrace{\frac{\partial}{\partial t}\int_{\Omega}\mathbf{U}\,d\Omega}^{\text{variation per unit time}} + \overbrace{\oint_{S}\mathbf{F}.d\mathbf{S}}^{\text{incoming surface flux}} = \overbrace{\int_{\Omega}Q_V\,d\Omega}^{\text{volumetric source}} + \overbrace{\oint_{S}\mathbf{Q}_S.d\mathbf{S}}^{\text{Surface Tractions}}}$$

- Using Gauss theorem, $\oint_{S}\mathbf{A}.d\mathbf{S} = \int_{\Omega}\nabla.\mathbf{A}\,d\Omega$, we get

$$\boxed{\frac{\partial}{\partial t}\int_{\Omega}\mathbf{U}\,d\Omega + \int_{\Omega}\nabla.\mathbf{F}\,d\Omega = \int_{\Omega}Q_V\,d\Omega + \int_{\Omega}\nabla.\mathbf{Q}_S\,d\Omega}$$

- Or in differential form, we get

$$\boxed{\frac{\partial\mathbf{U}}{\partial t} + \nabla.\mathbf{F} = Q_V + \nabla.\mathbf{Q}_S}$$

Conservation laws…

- If $\mathbf{U}$ is a vector, then the flux $\mathbf{F}$ becomes a tensor and we get

$$\frac{\partial}{\partial t}\int_{\Omega}\mathbf{U}\,d\Omega+\int_{S}\overset{tensor}{\mathbf{F}}\,.\,d\mathbf{S}=\int_{\Omega}\mathbf{Q}_V\,d\Omega+\oint_{S}\overset{tensor}{\mathbf{Q}_S}\,.\,d\mathbf{S}$$

- Or in differential form:

$$\boxed{\frac{\partial\mathbf{U}}{\partial t}+\nabla.\overset{tensor}{\mathbf{F}}=\overset{vector}{\mathbf{Q}_V}+\nabla.\overset{tensor}{\mathbf{Q}_S}}$$

- Conservation of mass:

$$\frac{\partial}{\partial t}\int_{\Omega}\rho\,d\Omega+\oint_{S}\rho\,\mathbf{V}.d\mathbf{S}=0 \qquad \Rightarrow \qquad \frac{\partial}{\partial t}+\nabla.(\rho\mathbf{V})=0$$

- Conservation of momentum:

$$\frac{\partial}{\partial t}\int_{\Omega}\rho\mathbf{V}\,d\Omega+\oint_{S}\rho\mathbf{V}(\mathbf{V}.d\mathbf{S})=\int_{\Omega}\rho\,\mathbf{f}_e\,d\Omega+\oint_{S}\overset{tensor}{\sigma}\,dS$$

$$\frac{\partial}{\partial t}(\rho\mathbf{V})+\nabla.(\rho\mathbf{V}\otimes\mathbf{V}-\sigma)=\rho\,\mathbf{f}_e$$

Conservation laws…

- **Conservation of Energy**: $E = e + \mathbf{V}^2/2$

$$\boxed{\frac{\partial}{\partial t}\int_{\Omega} \rho E \, d\Omega + \overbrace{\oint_{S} \rho E \, \mathbf{V}.d\mathbf{S}}^{\text{convection}} = \overbrace{\oint_{S} K \, \nabla T . d\mathbf{S}}^{\text{conduction}} + \int_{\Omega} \left(\overbrace{\rho \, \mathbf{f}_e . \mathbf{V}}^{\substack{\text{work of} \\ \text{volume force}}} + \overbrace{\mathbf{q}_H}^{\substack{\text{heat} \\ \text{source}}} \right) d\Omega + \overbrace{\oint_{S} (\boldsymbol{\sigma}.\mathbf{V}) d\mathbf{S}}^{\substack{\text{work done by} \\ \text{internal stresses}}}}$$

- **Its differential form becomes**

$$\boxed{\frac{\partial}{\partial t}(\rho E) + \nabla.(\rho \mathbf{V} E) = \nabla.(K \nabla T) + \nabla.(\boldsymbol{\sigma}.\mathbf{V}) + \rho \, \mathbf{f}_e . \mathbf{V} + \mathbf{q}_H}$$

- **Or by introducing** $H = e + P/\rho + \mathbf{V}^2/2 = E + P/\rho$ **, we have**

$$\boxed{\frac{\partial}{\partial t}(\rho E) + \nabla.(\rho \mathbf{V} H - K \nabla T - \boldsymbol{\tau}.\mathbf{V}) = \rho \, \mathbf{f}_e . \mathbf{V} + \mathbf{q}_H}$$

Conservation Forms

- The conservation of equations can be written as

$$\frac{\partial \overbrace{\mathbf{U}}^{\substack{Conservative\\Variable}}}{\partial t}+\frac{\overbrace{\partial \mathbf{F}^i}^{Inviscid\ \ fluxes}}{\partial x_i}=\frac{\overbrace{\partial \mathbf{G}^i}^{viscous\ \ fluxes}}{\partial x_i}+\overbrace{\mathbf{S}}^{source}$$

- If a non-conservative form is used for a numerical scheme, it can easily lead to violation of conservation laws.
- For example, the mass balance may not be satisfied.
- Lax has shown that the use of non-conservative form will lead to in-accurate jump relations through a discontinuity.
- In general, there are 3 form for equations; conservative form, primitive form and characteristic form.

Euler Equations

- For an inviscid flow, we have

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla . \mathbf{F} = \mathbf{S}$$

- A Quasi-linear form of the equation is

$$\frac{\partial \mathbf{U}}{\partial t} + \overbrace{\frac{\partial \mathbf{F}}{\partial \mathbf{U}}}^{\mathbf{A}} . \nabla \mathbf{U} = \mathbf{S}$$

- For a Cartesian coordinate system (x, y, z) we get

$$\boxed{\begin{array}{l} \dfrac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\dfrac{\partial \mathbf{U}}{\partial x} + \mathbf{B}\dfrac{\partial \mathbf{U}}{\partial y} + \mathbf{C}\dfrac{\partial \mathbf{U}}{\partial z} = \mathbf{Q} \\ \textit{with the Jacobians given by} \\ \mathbf{A} = \dfrac{\partial \mathbf{F}_x}{\partial \mathbf{U}} \qquad \mathbf{B} = \dfrac{\partial \mathbf{F}_y}{\partial \mathbf{U}} \qquad \mathbf{C} = \dfrac{\partial \mathbf{F}_z}{\partial \mathbf{U}} \end{array}}$$

Euler Equations…

- For a fluid with the constitutive relation as $p = \rho f(e)$ where e is the internal energy, $\mathbf{F}_x$, $\mathbf{F}_y$ and $\mathbf{F}_z$ are homogenous functions of degree one of the conservative variables, $\mathbf{U}$:

$$\mathbf{F}(\lambda \mathbf{U}) = \lambda \mathbf{F}(\mathbf{U}) \qquad \textit{for any } \lambda$$

- By differentiating w.r.t λ and setting $\lambda = 1$, we get (prove!)

$$\boxed{\mathbf{F}(\mathbf{U}) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} \mathbf{U} = \mathbf{A}\,\mathbf{U} \quad \textit{or} \quad \mathbf{F}_x = \mathbf{AU}, \mathbf{F}_y = \mathbf{BU}, \mathbf{F}_z = \mathbf{CU}}$$

- In other words, the Euler equation for such fluids can be written as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial (\mathbf{AU})}{\partial x} + \frac{\partial (\mathbf{BU})}{\partial y} + \frac{\partial (\mathbf{CU})}{\partial z} = 0$$

Euler Equations…

- This form and the quasi-linear form can be equivalent as long as the functions are continuous.
- However, from the numerical point of view, the two formulations do not lead to identical discretization.
- For a perfect gas, we have:

$$\boxed{\begin{aligned}
&\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0 \\
&\qquad \textit{where} \\
&\qquad p = (\gamma - 1)\,\rho\, e \\
&\qquad \mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u H \end{bmatrix} \\
&\qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ (\gamma - 3)u^2/2 & (3-\gamma)u & \gamma - 1 \\ (\gamma - 1)u^3 - \gamma u E & \gamma E - 3(\gamma - 1)u^2/2 & \gamma u \end{bmatrix}
\end{aligned}}$$

Primitive Variables

- Using the vector $V = \lfloor \rho \quad u \quad v \quad w \quad p \rfloor^T$ as the vector of primitive variables, we get

$$\frac{\partial \mathbf{V}}{\partial t} + (\widetilde{\mathbf{A}}.\nabla)\,\mathbf{V} = \widetilde{\mathbf{Q}}$$

$$or$$

$$\frac{\partial \mathbf{V}}{\partial t} + \widetilde{\mathbf{A}}\frac{\partial \mathbf{V}}{\partial x} + \widetilde{\mathbf{B}}\frac{\partial \mathbf{V}}{\partial y} + \widetilde{\mathbf{C}}\frac{\partial \mathbf{V}}{\partial z} = \widetilde{\mathbf{Q}}$$

$$with \;\; for \; example$$

$$\widetilde{\mathbf{A}} = \begin{bmatrix} u & \rho & 0 & 0 & 0 \\ 0 & u & 0 & 0 & 1/\rho \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & \rho c^2 & 0 & 0 & u \end{bmatrix}$$

- We note that this Jacobian is in a much simpler form.

Primitive Variables…

- The Jacobian of transformation from the conservative to primitive variables is defined by ($\mathbf{M}$ is a similarity transformation)

$$\mathbf{M} = \frac{\partial \mathbf{U}}{\partial \mathbf{V}}, \qquad |\mathbf{M}| = \frac{\rho}{\gamma - 1}$$

- Which results in

$$\mathbf{M}\frac{\partial \mathbf{V}}{\partial t} + \mathbf{A}\,\mathbf{M}.\nabla\mathbf{V} = \mathbf{Q}$$

- Multiplying by M^{-1} gives:

$$\frac{\partial \mathbf{V}}{\partial t} + \overbrace{\mathbf{M}^{-1}\mathbf{A}\,\mathbf{M}}^{\tilde{\mathbf{A}}}.\nabla\mathbf{V} = \overbrace{\mathbf{M}^{-1}\mathbf{Q}}^{\tilde{\mathbf{Q}}}$$

Primitive Variables…

- **For a 1D case, we get:**

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ u^2/2 & \rho u & 1/(\gamma - 1) \end{bmatrix}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -u/\rho & 1/\rho & 0 \\ (\gamma - 1)u^2/2 & (1-\gamma)u & \gamma - 1 \end{bmatrix}$$

$$\widetilde{\mathbf{A}} = \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho c^2 & u \end{bmatrix}$$

Characteristic Variables

- **Define the left and right eigen-values of $\widetilde{\mathbf{A}}$ in a direction $\widetilde{\kappa}$: (i.e. $\widetilde{\mathbf{K}} = \widetilde{\mathbf{A}}.\widetilde{\kappa}$)**

$$\boxed{\begin{array}{l} \left|\lambda\,\mathbf{I} - \widetilde{\mathbf{A}}.\widetilde{\kappa}\right| = 0 \qquad \Rightarrow \\ \\ \qquad \widetilde{l}^{\,(j)}\widetilde{\mathbf{K}} = \lambda_j\,\widetilde{l}^{\,(j)} \qquad \text{(no summation on j)} \\ \qquad\qquad or \\ \widetilde{l}_i^{\,(j)}\left(\widetilde{\mathbf{A}}.\widetilde{\kappa}\right)_{ik} = \lambda_j\,\widetilde{l}_k^{\,(j)} \qquad i,j,k = 1,\ldots,5 \qquad \text{(summation on i only)} \end{array}}$$

- **A <u>wave like solution</u> will exist if the eigen-values of $\widetilde{\mathbf{K}}$ for arbitrary $\widetilde{\kappa}$ are <u>real</u> with <u>linear independence</u> of the corresponding left eigen-vectors.**

Characteristic Variables…

- **The Jacobian matrix can be diagonalized using L matrix whose rows are the left eigen-vectors of $\tilde{l}^{(j)}$:**

$$\mathbf{L}^{-1}\widetilde{\mathbf{K}} = \boldsymbol{\Lambda}\,\mathbf{L}^{-1} \quad \textit{or} \quad \boldsymbol{\Lambda} = \mathbf{L}^{-1}(\widetilde{\mathbf{A}}.\boldsymbol{\kappa})\,\mathbf{L}$$

where λ_j *are functions of* $\boldsymbol{\kappa}$ *and*

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

- **Therefore, we can decouple the equations in the direction $\tilde{\kappa}$.**
- **Since $\widetilde{K}$ is not symmetric, there exists a set of right eigen-vectors:**

$$\widetilde{\mathbf{K}}\,\tilde{r}^{(j)} = \lambda_j\,\tilde{r}^{(j)} \qquad \text{(no summation on j)}$$

$$\widetilde{\mathbf{K}} = \mathbf{R}\boldsymbol{\Lambda}\mathbf{R}^{-1} \quad \text{(R is a matrix whose columns are } \tilde{r}^{(j)}\text{)}$$

Characteristic Variables…

- Multiplying the eigen-vectors $\tilde{l}^{(j)}$ in the primitive equation, we get:

$$\boxed{\tilde{l}^{(j)} \frac{\partial \mathbf{V}}{\partial t} + \tilde{l}^{(j)} (\tilde{\mathbf{A}}.\nabla)\,\mathbf{V} = \tilde{l}^{(j)}\,\tilde{\mathbf{Q}} \qquad \text{(compatibility equation for } \lambda_{\mathrm{j}}\text{)}}$$

- Or after grouping these equations for all eigen-values, we obtain:

$$\left(\mathbf{L}^{-1}\partial_{\mathbf{t}} + \mathbf{L}^{-1}\,\tilde{\mathbf{A}}.\nabla\right)\mathbf{V} = \mathbf{L}^{-1}\,\tilde{\mathbf{Q}}$$

- Substituting $\mathbf{V} = \mathbf{MU}$ we see:

$$\mathbf{L}^{-1}\mathbf{M}^{-1}(\partial_{\mathbf{t}} + \mathbf{A}.\nabla)\,\mathbf{U} = \mathbf{L}^{-1}\mathbf{M}^{-1}\mathbf{Q}$$

- Defining

$$\mathbf{P}^{-1} = \mathbf{L}^{-1}\mathbf{M}^{-1} \qquad \textit{and} \qquad \mathbf{P} = \mathbf{ML}$$

- We get

$$\tilde{\mathbf{A}} = \mathbf{M}^{-1}\mathbf{AM}$$

- So that $\mathbf{P}$ will diagonalize $\mathbf{K} = \mathbf{A}.\boldsymbol{\kappa}$:

$$\boldsymbol{\Lambda} = \mathbf{L}^{-1}\mathbf{M}^{-1}\left(\mathbf{A}.\boldsymbol{\kappa}\right)\mathbf{ML} = \mathbf{P}^{-1}\mathbf{KP}$$

Characteristic Variables…

- Note that the rows of $\mathbf{P}^{-1}$ are the left eigen-vectors of $\mathbf{K}$ while the columns of $\mathbf{P}$ are the right eigen-vectors of the same matrix.
- Defining a new set of characteristic variables as

$$\delta \mathbf{W} = \mathbf{L}^{-1}\,\delta \mathbf{V} \qquad (\delta \equiv \partial_t \;\; or \;\; \nabla)$$

- Or explicitly

$$\delta w_k = \sum_i l_i^{(k)}\,\delta v_i$$

- Then, we have $\delta \mathbf{V} = \mathbf{L}\,\delta \mathbf{W}$.
- The compatibility equation becomes

$$\boxed{\begin{aligned} &\mathbf{L}^{-1}\frac{\partial \mathbf{V}}{\partial t} + \left(\mathbf{L}^{-1}\tilde{\mathbf{A}}\mathbf{L}\right)\mathbf{L}^{-1}\nabla \mathbf{V} = \mathbf{L}^{-1}\,\tilde{\mathbf{Q}} \\ &or \\ &\frac{\partial \mathbf{W}}{\partial t} + \left(\mathbf{L}^{-1}\tilde{\mathbf{A}}\mathbf{L}\right)\nabla \mathbf{W} = \mathbf{L}^{-1}\,\tilde{\mathbf{Q}} \end{aligned}}$$

Characteristic Variables…

- Note that the characteristic variables are now associated with a given direction of propagation $\boldsymbol{\kappa}$ and therefore are a function of $\boldsymbol{\kappa}$.

- Also, note that except when eigen-values are constant, we cannot directly evaluate $\mathbf{W}$ as $\mathbf{W} = \mathbf{L}^{-1}\mathbf{V}$ and in other cases we only work with $\delta\mathbf{W}$.

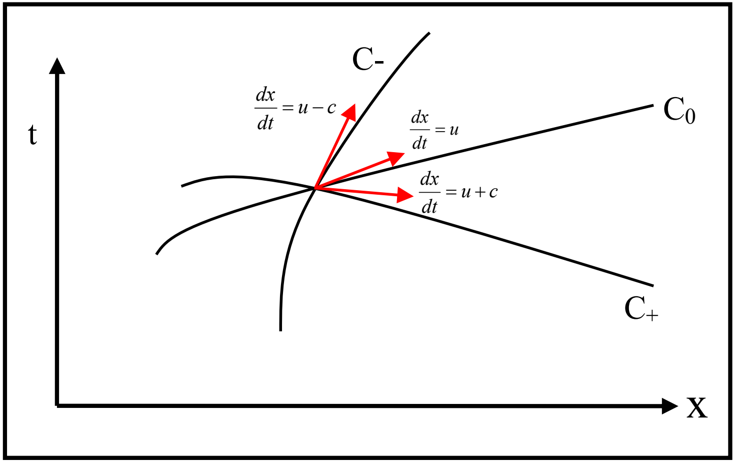
- The variables $\mathbf{W}$ are also called **Riemann variables**.

- Whenever, $\mathbf{W}$ remains constant, it is called **Riemann invariant**.

Characteristic Lines

- At each point in the domain, we can draw three lines corresponding to the three characteristic directions.

- C_- and C_+ are called Mach lines.



Shock and Contact Discontinuity

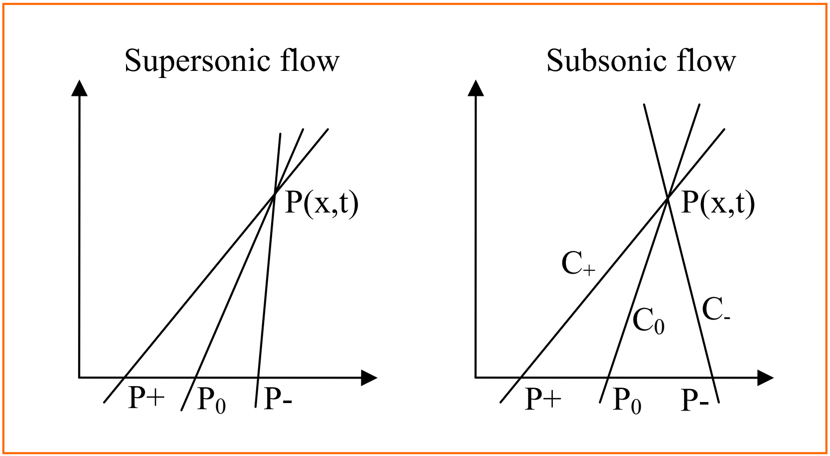
- **In the case of an isentropic flow we have**

$$p = k\,\rho^{\gamma} \qquad c^2 = k\,\gamma\,\rho^{\gamma-1}$$

- **This gives**

$$w_2 = u + \frac{2c}{\gamma - 1} \qquad \text{for C} +$$

$$w_3 = u - \frac{2c}{\gamma - 1} \qquad \text{for C -}$$



Shock…

- The area between the characteristic lines shows the **region of dependence** of the solution at point P.

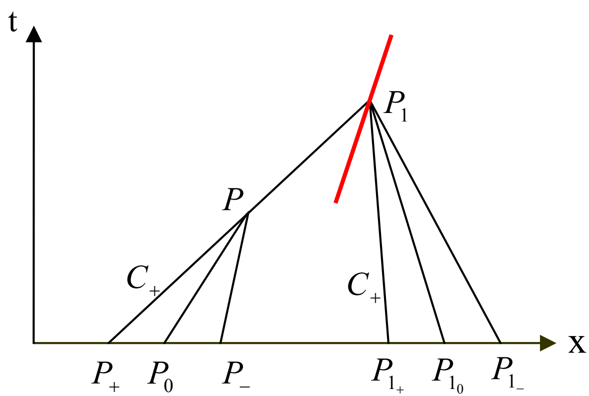
- The region between the characteristics emanating from P is called the **domain of influence** of P.

- We also have

$$
\begin{cases}
\left(u+\dfrac{2c}{\gamma-1}\right)_P = \left(u+\dfrac{2c}{\gamma-1}\right)_{P_+} & along\ C_+ \\
\left(u-\dfrac{2c}{\gamma-1}\right)_P = \left(u-\dfrac{2c}{\gamma-1}\right)_{P-} & along\ C_- \\
S_P = S_{P_0} & along\ C_0
\end{cases}
$$

Shock

- **Due to the nonlinearity of the flow equations, the streamline slopes may decrease (in particular if $\partial(u+c)/\partial x < 0$) and we have the situation where the C_+ characteristic emanating from P_{1_+} intersect the C_+ characteristic from P_+ and multi-valued quantities would occur in P_1.**



Shock

- **In this case we have**

$$\left(u+\frac{2c}{\gamma-1}\right)_{P_1} = \left(u+\frac{2c}{\gamma-1}\right)_{P+}$$

$$\left(u+\frac{2c}{\gamma-1}\right)_{P_1} = \left(u+\frac{2c}{\gamma-1}\right)_{P_{1+}}$$

$$\left(u+\frac{2c}{\gamma-1}\right)_{P_+} \neq \left(u+\frac{2c}{\gamma-1}\right)_{P_{1+}}$$

- **This impossible situation leads to a discontinuous flow behaviour called a shock wave.**
- **From this we find that for a shock to occur the following relations holds**

$$(u+c)_{P_{1_+}} < c < (u+c)_{P_+}$$

$$(\rho)_{P_+} > (\rho)_{P_{1+}} \qquad (u)_{P_+} > (u)_{P_{1+}}$$

$$(p)_{P_+} > (p)_{P_{1+}} \qquad (c)_{P_+} > (c)_{P_{1+}}$$

Shock…

- If $(P)_{P_+} < (P)_{P_{1+}}$ then an expansion fan will occur.

- A contact discontinuity is an interface between two fluid regions of different densities but equal pressures. The velocity is continuous over a contact discontinuity. (like a free moving piston)

- A hypothetical Expansion Shock would lead to a situation where

$$(u+c)_{P_+} < c < (u+c)_{P_{1+}}$$

And characteristics carry information away from the discontinuity.

Physical Boundary Conditions

- The number of boundary conditions to be imposed will depend on the way the information transported along the characteristics interacts with the boundaries.

	Subsonic	Supersonic
Inlet	2 conditions W_1 *and* W_2 *given*	3 conditions W_1, W_2 *and* W_3 *given*
Outlet	one conditions W_3 *given*	None

- As numerical schemes require the values of all flow variables on the boundaries, additional conditions called Numerical Boundary Conditions must be given in order to define the numerical problem completely.

Physical Boundary Conditions…

- The choice of boundary conditions has a significant effect on the accuracy, stability and convergence rate of many schemes.

- Many implicit schemes which are linearly, unconditionally stable, appear to be only conditionally stable in practice if an improper boundary treatment is introduced.

Characteristic Boundary Conditions

- In many numerical schemes we work with the primitive variables, and need to know which combinations of the primitive variables may be applied as physical boundary conditions that do not lead to an ill-posed problem.

- We know that $\Delta w = L^{-1} \Delta V$, therefore

$$\Delta w = \begin{bmatrix} \Delta w^P \\ \Delta w^N \end{bmatrix} = \begin{bmatrix} \left(L^{-1}\right)_I^P & \left(L^{-1}\right)_{II}^P \\ \left(L^{-1}\right)_I^N & \left(L^{-1}\right)_{II}^N \end{bmatrix} \begin{bmatrix} \Delta V^I \\ \Delta V^{II} \end{bmatrix}$$

- Here, P denotes physical and N denotes numerical boundary conditions.

- The group of variables V^I represents the imposed conditions while the group V^{II} represents the free variables to be defined by the <u>numerical or internal</u> information.

Boundary Conditions…

- The condition for well-posedness of the choice of variables is that V^{II} can be recovered from the information carried by the characteristics w^N which intersect the boundary from the interior of the flow domain.
- If we write

$$\Delta w^N = (L^{-1})_I^N \, \Delta V^I + (L^{-1})_{II}^N \, \Delta V^{II}$$

- The free variables V^{II} are defined by

$$\Delta V^{II} = \frac{1}{(L^{-1})_{II}^N}\left[\Delta w^N - (L^{-1})_I^N \, \Delta V^I\right]$$

- Hence, the condition for well-posedness is that the matrix $(L^{-1})_{II}^N$ is non-singular, i.e.:

$$\left|(L^{-1})_{II}^N\right| \neq 0$$

1D Euler Equation

- **For this equation, we have**

$$\Delta \mathbf{w} = \begin{bmatrix} \Delta w_1 \\ \Delta w_2 \\ \Delta w_3 \end{bmatrix} = \begin{bmatrix} \Delta\rho - \dfrac{\Delta p}{c^2} \\ \Delta u + \dfrac{\Delta p}{\rho c} \\ \Delta u - \dfrac{\Delta p}{\rho c} \end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix} \qquad \begin{bmatrix} \Delta w_3 \\ \Delta w_1 \\ \Delta w_2 \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{\rho c} & 0 & 1 \\ \dfrac{-1}{c^2} & 1 & 0 \\ \dfrac{1}{\rho c} & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta\rho \\ \Delta u \end{bmatrix}$$

- **Let us consider a subsonic outflow. Then,**

$$\mathbf{w}^P = w_3 \qquad \mathbf{w}^N = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad \mathbf{V}^I = p \qquad \mathbf{V}^{II} = \begin{bmatrix} \rho \\ u \end{bmatrix}$$

$$\rightarrow \left(\mathbf{L}^{-1}\right)_I^P = \frac{-1}{\rho c} \qquad \left(\mathbf{L}^{-1}\right)_{II}^P = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad \left(\mathbf{L}^{-1}\right)_I^N = \begin{bmatrix} \dfrac{-1}{c^2} \\ \dfrac{1}{\rho c} \end{bmatrix} \qquad \left(\mathbf{L}^{-1}\right)_{II}^N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1D Euler Equation…

- We can see that <u>any three variables</u> ρ, u and p can be chosen as a physical boundary condition, since none of the sub-matrices w^n is zero.

- For a **subsonic inlet**, the choice (u, p) as a physical boundary condition is not well-posed. Any other combination involving ρ as a physical condition is well-posed.

- For a steady-state **subsonic** nozzle flow with equal inlet and outlet areas leads to non-unique solutions if the same variable is specified at outlet and inlet.

Boundary Conditions and Accuracy

- **Gustafsson proved that, for linear equations, the boundary scheme can be one order lower than the interior scheme without reducing the global order of accuracy of the complete scheme.**

- **Important types of Boundary Conditions are:**
  - **Far field**
  - **Inviscid wall (slip b.c.)**
  - **Viscous wall (no-slip b.c.)**
  - **Symmetry plane**
  - **Adiabatic wall**
  - **Isothermal wall**
  - **Injection or suction wall**
  - **Moving wall**

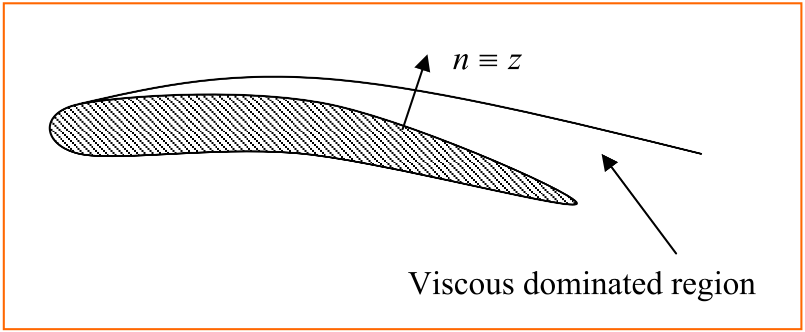
Simplified Forms of the Navier-Stokes Equations

- **Several simplified forms of the Navier-Stokes equations exist. These forms help to get a better mathematical behaviour from the governing equations.**

- **This normally results in using a simplified and specific numerical method for the solution.**

- **Here, we introduce the following approximation types:**
  - **Thin Shear Layer (TSL)**
  - **Parabolized Navier-Stokes (PNS)**
  - **Boundary layer**
  - **Inviscid Flow Model (Euler)**

Thin Shear Layer Equations



- When the viscous layers (**wall shear layer, wake or free shear layers**) are of limited size, then the influence of the shear stresses will come essentially from the gradients transverse to the main flow direction.
- This approximation is normally reasonable for **high Reynolds** numbers.
- In this case, the contribution of other gradients on the stresses is neglected.

Thin Shear Layer Equations

- For example, if the direction normal to the wall is z , we get:

$$\tau_{ij} \approx \mu\left(\delta_{3i}\frac{\partial v_j}{\partial z}+\delta_{3j}\frac{\partial v_i}{\partial z}\right)-\frac{2}{3}\mu\delta_{ij}\left(\frac{\partial v_z}{\partial z}\right)$$

- **Then, momentum equation becomes:**

$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho u^2+p)+\frac{\partial}{\partial y}(\rho uv)+\frac{\partial}{\partial z}(\rho uw)=\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)$$

$$\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}(\rho uv)+\frac{\partial}{\partial y}(\rho v^2+p)+\frac{\partial}{\partial z}(\rho vw)=\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)$$

$$\frac{\partial}{\partial t}(\rho w)+\frac{\partial}{\partial x}(\rho uw)+\frac{\partial}{\partial y}(\rho vw)+\frac{\partial}{\partial z}(\rho w^2+p)=\frac{4}{3}\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right)$$

- In a flow with large separation or wake, TSL cannot be used.

Parabolized Navier-Stokes Equations

- **This is similar to TSL but is applied to the steady-state formulations.**

- **In this case, the following conditions exist:**

  - **A pre-dominated main flow direction (such as channel flow) exists such that the cross flow components are of lower order of magnitude.**

  - **Along the solid boundaries the viscous regions are assumed to be dominated by the normal gradients, hence, the stream-wise diffusion of momentum and energy can be neglected.**

Parabolized Navier-Stokes Equations…

- Consider a flow with dominated velocity in the direction x while the flow is 3D x, y and z. The x-derivatives in the shear stress terms are all neglected compared to other terms.

- We get (for x -direction) the following parabolic equation:

$$\boxed{\frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw) = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right)}$$

- Here, if the term $\partial(\mu \partial u / \partial x) / \partial x$ were present, we had an elliptic equation in (x, y, z) space.

- However, now, the equation ca be solved for x as a pseudo-time variable.

- The equation can be integrated by advancing in x - direction, while solving an elliptic problem in $y - z$ plane.

Parabolized Navier-Stokes Equations…

- In the same way, we get:

$$\frac{\partial}{\partial x}(\rho uv)+\frac{\partial}{\partial y}(\rho v^2+p)+\frac{\partial}{\partial z}(\rho vw)=\frac{4}{3}\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)-\frac{2}{3}\frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial z}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial y}\right)$$

$$\frac{\partial}{\partial x}(\rho uH)+\frac{\partial}{\partial y}(\rho vH)+\frac{\partial}{\partial z}(\rho wH)=\frac{\partial}{\partial y}\left(\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial z}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

Boundary Layer & Inviscid Equations

Boundary Layer:

- The idea was introduced by L. Prandtl and in this method the flow is solved as a viscous region separated from an adjacent inviscid region.
- Pressure is normally computed from outside the boundary layer using a potential flow model.

Inviscid Flow:

- All viscous terms are neglected in the whole flow regions.
- In addition all heat conduction terms are dropped from the energy equation.