Sharif University of Technology
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Computational Fluid Dynamics

Numerical Methods for Model Equations

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Model Equations

- Here, we will study various methods for solving the followings:
  - First order linear wave equation: $$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \quad (a > 0)$$
  - Transient Diffusion: $$\frac{\partial u}{\partial t} - \alpha\frac{\partial^2 u}{\partial x^2} = 0 \quad (\alpha > 0)$$
  - Laplace Equation: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
  - Burger's Inviscid Equation: $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$
  - Burger's Viscous Equation: $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2}$$

Linear Wave Equation

- **We will study the following methods:**
  - **Euler explicit method**
  - **First order upwind method**
  - **Lax method**
  - **Euler implicit method**
  - **Leap frog method**
  - **Lax-Wendroff method (single step and 2-step)**
  - **Mac-Cormack method**
  - **2^{nd} order upwind method (Warming-Beam)**
  - **Trapezoidal method**
  - **Runge-Kutta method**

Diffusion Equation

- **We will study the following methods:**
    - **Simple explicit method**
    - **Simple implicit method**
    - **Crank-Nicholson method**
    - **Dufort-Frankel method**
    - θ **-methods**
    - **Keller-Box method**
    - **ADI method**

Inviscid Burger Equation

- **We will study the following methods:**
  - **Lax method**
  - **Lax-Wendroff method**
  - **Mac-Cormack method**
  - **Warming-Kutler-Lomax method**
  - **Beam-Warming method**

Linear Wave Equation

Euler Explicit Method

- **The method is ($c > 0$)**

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_{i+1}^n - u_i^n}{\Delta x} = 0 \qquad \text{1st order in t and x}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0 \qquad \text{1st order in t and 2nd order in x}$$

- **It is unconditionally unstable.**

First Order Upwind Method

- **The method is**

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0 \qquad \mathrm{c} > 0$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_i^n - u_{i+1}^n}{\Delta x} = 0 \qquad \mathrm{c} < 0$$

- **Or in general,**

$$u_i^{n+1} = u_i^n - \overbrace{\frac{c\,\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right)}^{\text{central difference}} + \overbrace{\frac{|c|\,\Delta t}{2\Delta x}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}^{\text{Artificial viscosity}}$$

- **The method is stable provided that**

$$0 \le \nu = \frac{c\,\Delta t}{\Delta x} \le 1$$

- **Remember that its modified equation is**

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \frac{c\Delta x}{2}(1-\nu)u_{xx} - \frac{c\Delta x^2}{6}(2\nu^2 - 3\nu + 1)u_{xxx} + O[\Delta x^3, \Delta x^2\Delta t, \Delta x\,\Delta t^2, \Delta t^3]$$

First Order Upwind Method…

- **The amplification factor and phase angle are:**

$$G = (1-\nu-\nu\cos\beta) - i(\nu\sin\beta) = |G|\exp(i\phi)$$
$$\phi = \tan^{-1}\frac{\mathrm{Im}(G)}{\mathrm{Re}(G)} = \tan^{-1}\left(\frac{-\nu\sin\beta}{1-\nu-\nu\cos\beta}\right)$$

- **The exact solution for the problem is** $u = \exp[i\kappa_m(x-ct)]$**, thus**

$$G \equiv \frac{u(t+\Delta t)}{u(t)} = \frac{\exp\{i\kappa_m[x-c(t+\Delta t)]\}}{\exp\{i\kappa_m[x-ct]\}} = \exp(-i\kappa_m c\,\Delta t) \equiv \exp(i\phi_e)$$
$$|G_e| = 1 \qquad \phi_e = -\kappa_m c\,\Delta t = -\beta\nu$$

- **This means that**
  - **If the initial wave amplitude is** A_0**, then after** N **steps the total dissipation error is given by** $\left(1-|G|^N\right)A_0$
  - **The total dispersion (phase) error is given by** $N(\phi_e - \phi)$
  - **Note that after one step, the relative shift error is**

$$\frac{\phi}{\phi_e} = \frac{-1}{\beta\nu}\tan^{-1}\left(\frac{-\nu\sin\beta}{1-\nu+\nu\cos\beta}\right)$$

First Order Upwind Method…

- For small wave numbers (small β) we can write

$$\frac{\phi}{\phi_e} \approx 1 - \frac{1}{6}\left(2\nu^2 - 3\nu + 1\right)\beta^2$$

- Note that

$$\frac{\phi}{\phi_e} > 1 \qquad \text{leading phase error}$$

$$\frac{\phi}{\phi_e} < 1 \qquad \text{lagging phase error}$$

- For the upwind method, we get

$$0 < \nu < 0.5 \qquad \text{leading phase error}$$

$$0.5 < \nu \qquad \text{lagging phase error}$$

Lax Method

- **The method reads**

$$\frac{1}{\Delta t}\left(u_i^{n+1} - \frac{u_{i+1}^n - u_{i-1}^n}{2}\right) + \frac{c}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) = 0$$

- **Where**

$$G = \cos\beta - i\nu\sin\beta$$

$$\frac{\phi}{\phi_e} = \frac{\tan^{-1}(-\nu\tan\beta)}{-\nu\beta}$$

Euler Implicit Method

- The method reads

$$\frac{1}{\Delta t}\left(u_i^{n+1} - u_i^n\right) + \frac{c}{2\Delta x}\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right) = 0$$

- The method is second order in space and first order in time.
- It is unconditionally stable.
- The modified equation is

$$u_t + c\,u_x = \left(\frac{c^2\,\Delta t}{2}\right)u_{xx} - \left(\frac{c\,\Delta x^2}{6} + \frac{c^3\,\Delta t}{3}\right)u_{xxx} + \ldots$$

- The amplification factor and phase error are

$$G = \frac{1 - i\nu\sin\beta}{1 + i\nu^2\sin^2\beta}$$

$$\frac{\phi}{\phi_e} = \frac{\tan^{-1}(-\nu\sin\beta)}{-\nu\beta}$$

Leap-Frog Method

- This 2nd order method is

$$\frac{1}{2\Delta t}\left(u_i^{n+1}-u_i^{n-1}\right)+\frac{c}{2\Delta x}\left(u_{i+1}^{n}-u_{i-1}^{n}\right)=0$$

- It is neutrally stable for $|\nu|\le 1$.

- The modified equation shows a <u>predominantly dispersive</u> behavior.

$$u_t+cu_x=\left(\frac{c\,\Delta x^2}{6}\right)(\nu^2-1)u_{xxx}-\left(\frac{c\,\Delta x^4}{120}\right)(9\nu^4-10\nu^2+1)u_{xxxxx}+\ldots$$

- The amplification factor and phase error are

$$G=\pm\sqrt{1-\nu^2\sin^2\beta}-i\nu\sin\beta$$

$$\frac{\phi}{\phi_e}=\frac{\tan^{-1}(\mp\nu\sin\beta/\sqrt{1-\nu^2\sin^2\beta})}{-\nu\beta}$$

Lax-Wendroff Method

- **The method (2nd order in time and space) is derived from the Taylor expansion in time:**

$$u_i^{n+1} = u_i^n - \overbrace{\frac{c\,\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right)}^{\text{central difference}} + \overbrace{\frac{c^2\,\Delta t^2}{2\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}^{\text{Artificial viscosity}}$$

- **It is stable for $|\nu| \le 1$.**

- **The modified equation is**

$$u_t + c\,u_x = \left(\frac{c\,\Delta x^2}{6}\right)(\nu^2 - 1)\,u_{xxx} - \left(\frac{c\,\Delta x^3}{8}\right)\nu\,(-\nu^2 + 1)\,u_{xxxx} + \ldots$$

- **The amplification and phase errors are:**

$$G = \left[1 - \nu^2\left(1 - \cos\beta\right)\right] - i\left(\nu \sin\beta\right)$$

$$\frac{\phi}{\phi_e} = \frac{\tan^{-1}\left\{-\nu\sin\beta / \left[1 - \nu^2\left(1 - \cos\beta\right)\right]\right\}}{-\nu\beta}$$

- **Note that the method has a <u>predominantly lagging phase error</u> except for large wave numbers with $\sqrt{0.5} < \nu < 1$**

Two-step Lax-Wendroff Method

- For nonlinear equations such as the inviscid flow equations, a 2-step variation of the original Lax-Wendroff method can be used:

$$\frac{1}{\Delta t/2}\left(u_{i+1/2}^{n+1/2}-\frac{u_{i+1}^{n}+u_{i}^{n}}{2}\right)+\frac{c}{\Delta x}\left(u_{i+1}^{n}-u_{i}^{n}\right)=0 \qquad \text{(step 1)}$$

$$\frac{u_{i}^{n+1}-u_{i}^{n}}{\Delta t}+\frac{c}{\Delta x}\left(u_{i+1/2}^{n+1/2}-u_{i-1/2}^{n+1/2}\right)=0 \qquad \text{(step 2)}$$

- The scheme is 2nd order in time and space.
- It is stable for $|\nu| \le 1$.
- For the linear wave equation, the method becomes exactly like the original Lax-Wendroff equation.

MacCormack Method

- This method is a predictor-corrector type method (1969):

$$\overline{u}_i^{n+1} = u_i^n - \frac{c\,\Delta t}{\Delta x}\left(u_{i+1}^n - u_i^n\right) \qquad \text{(predictor)}$$

$$u_i^{n+1} = \frac{1}{2}\left[u_i^n + \overline{u}_i^{n+1} - \frac{c\,\Delta t}{\Delta x}\left(\overline{u}_i^{n+1} - \overline{u}_{i-1}^{n+1}\right)\right] \qquad \text{(corrector)}$$

- For the linear wave equation, this method is equivalent to the original Lax-Wendroff method with the same accuracy and stability limit.

2^{nd} Order Upwind Method

- This is also called Warming and Beam method (1975):

$$\overline{u}_i^{n+1} = u_i^n - \frac{c\,\Delta t}{\Delta x}\left(u_i^n - u_{i-1}^n\right) \qquad \text{(predictor)}$$

$$u_i^{n+1} = \frac{1}{2}\left[u_i^n + \overline{u}_i^{n+1} - \frac{c\,\Delta t}{\Delta x}\left(\overline{u}_i^{n+1} - \overline{u}_{i-1}^{n+1}\right) - \frac{c\,\Delta t}{\Delta x}\left(u_i^n - 2u_{i-1}^n + u_{j-2}^n\right)\right] \qquad \text{(corrector)}$$

- For the linear wave equation, substituting the predictor into the corrector gives:

$$u_i^{n+1} = u_i^n - \nu\left(u_i^n - u_{i-1}^n\right) + \frac{\nu(\nu-1)}{2}\left(u_i^n - 2u_{i-1}^n + u_{i-2}^n\right)$$

- The modified equation is

$$u_t + c\,u_x = \left(\frac{c\,\Delta x^2}{6}\right)(1-\nu)(2-\nu)\,u_{xxx} - \left(\frac{\Delta x^4}{8\Delta t}\right)\nu(1-\nu)^2(2-\nu)\,u_{xxxx} + \ldots$$

- The stability condition is $0 \le \nu \le 2$
- Also

$$G = 1 - 2\nu\left[\nu + 2(1-\nu)\sin^2\frac{\beta}{2}\right]\sin^2\frac{\beta}{2} - i\left(\nu\sin\beta\right)\left[1 + 2(1-\nu)\sin^2\frac{\beta}{2}\right]$$

2^{nd} Order Upwind Method…

- **Note that**

$0 < \nu < 1$	a predominantly leading phase error
$1 < \nu < 2$	a predominantly lagging phase error

- **For $0 < \nu < 1$ the phase errors of the 2^{nd} order upwind method and the Lax-Wendroff scheme are opposite. Fromm's method (1968) uses this fact to produce a method free of dispersive error.**

Trapezoidal Differencing Method

- **The method is developed by using Taylor series for time and central differences for spatial derivatives: (time-centered & space-centered)**

$$u_i^{n+1} = u_i^n - \frac{\nu}{4}\left(u_{i+1}^{n+1} + u_{i+1}^n - u_{i-1}^{n+1} - u_{i-1}^n\right)$$

- **The method is second order in space and time.**
- **It is also unconditionally stable.**
- **It's modified equation is:**

$$u_t + c\,u_x = -\left(\frac{c^3\,\Delta t^2}{12} + \frac{c\,\Delta x^2}{6}\right)u_{xxx} - \left(\frac{c\,\Delta x^4}{120} + \frac{c^3\,\Delta t^2 \Delta x^2}{24} + \frac{c^4\,\Delta t^4}{80}\right)u_{xxxxx} + \ldots$$

- **Also**

$$G = \left(1 - \frac{i\nu}{2}\sin\beta\right) / \left(1 + \frac{i\nu}{2}\sin\beta\right)$$

Runge-Kutta Methods

- **These methods transform the PDE to an ODE as**

$$\frac{\partial u}{\partial t} = R(u) \qquad where \qquad R(u) = -c\frac{\partial u}{\partial x}$$

- **A 2nd order Runge-Kutta method becomes:**

$$u^{(1)} = u^n + \Delta t\, R^n \qquad \text{(step 1)}$$

$$u^{n+1} = u^n + \frac{\Delta t}{2}(R^n + R^{(1)}) \qquad \text{(step 2)}$$

where

$$R^{(1)} = -c\left(\frac{\partial u}{\partial x}\right)^{(1)} = -c\left[\left(\frac{\partial u}{\partial x}\right)^n + \Delta t\left(\frac{\partial R}{\partial x}\right)^n\right] = -c\left(\frac{\partial u}{\partial x}\right)^n + c^2\Delta t\left(\frac{\partial^2 u}{\partial x^2}\right)^n$$

- **Or simply we have**

$$u^{n+1} = u^n + \frac{\Delta t}{2}\left(-2c\,u_x^n + c^2\Delta t\,u_{xx}^n\right)$$

- **Using 2nd-order central differences, we obtain the 2nd order Lax-Wendroff scheme.**

Runge-Kutta Methods…

- **One of the most popular Runge-Kutta method is its 4-step version (2nd order in time and 4th order in space when 2nd order spatial differences are used):**

$$u^{(1)} = u^n + \frac{\Delta t}{2} R^n \qquad \text{(step 1)}$$

$$u^{(2)} = u^n + \frac{\Delta t}{2} R^{(1)} \qquad \text{(step 2)}$$

$$u^{(3)} = u^n + \Delta t\, R^{(2)} \qquad \text{(step 3)}$$

$$u^{n+1} = u^n + \frac{\Delta t}{6} (R^n + 2R^{(1)} + 2R^{(2)} + R^{(3)}) \qquad \text{(step 4)}$$

Diffusion Equation

Diffusion Equation

- **In 1D, we have**

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

- **This is a <u>parabolic</u> equation whose exact solution for an initial condition $u(x,0) = f(x)$ and boundary conditions $u(0,t) = u(1,t) = 0$ is given by**

$$u(x,t) = \sum_{n=1}^{\infty} A_n \exp(-\alpha\, k_n^2 t) \sin(k_n\, x)$$
$$where \quad (k_n = n\pi)$$
$$A_n = 2\int_0^1 f(x) \sin(k_n x)\, dx$$

- **The exact amplification is**

$$G_e = \frac{u(t+\Delta t)}{u(t)} = \exp(-\alpha k_m^2 \Delta t) = \exp(-r\beta^2)$$

Simple Explicit Method

- Using first order in time and second order in space, we get

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

- The scheme is stable for

$$0 \le r = \frac{\alpha \Delta t}{\Delta x^2} \le 1/2$$

- The amplification factor is

$$G = 1 + 2r(\cos\beta - 1)$$

- The modified equation becomes:

$$u_t - \alpha u_{xx} = \left(\frac{-\alpha^2 \Delta t}{2} + \frac{\alpha \Delta x^2}{12} \right) u_{xxxx} + \left(\frac{\alpha^3 \Delta t^2}{3} - \frac{\alpha^2 \Delta t \Delta x^2}{12} + \frac{\alpha \Delta x^4}{360} \right) u_{xxxxxx} + \ldots$$

Simple Implicit Method

- **This method is first order in time and second order in space:**

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}$$

- **It is unconditionally stable.**
- **The modified equation is**

$$u_t - \alpha u_{xx} = \left(\frac{\alpha^2 \Delta t}{2} + \frac{\alpha \Delta x^2}{12} \right) u_{xxxx} + \left(\frac{\alpha^3 \Delta t^2}{3} + \frac{\alpha^2 \Delta t \Delta x^2}{12} + \frac{\alpha \Delta x^4}{360} \right) u_{xxxxxx} + \ldots$$

- **The amplification factor is**

$$G = \frac{1}{1 + 2r(1 - \cos \beta)}$$

Crank-Nicolson Method

- **The method (1947) reads**

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\frac{\alpha}{2\Delta x^2}\left[\left(u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}\right)+\left(u_{i+1}^{n}-2u_i^{n}+u_{i-1}^{n}\right)\right]$$

- **Its modified equation is**

$$u_t-\alpha u_{xx}=\left(\frac{\alpha\Delta x^2}{12}\right)u_{xxxx}+\left(\frac{\alpha^3\Delta t^2}{12}+\frac{\alpha\Delta x^4}{360}\right)u_{xxxxxx}+\dots$$

- **The amplification factor is**

$$G=\frac{1-r(1-\cos\beta)}{1+r(1-\cos\beta)}$$

Dufort-Frankel Method

- This method is given by ($r = \alpha \Delta t / \Delta x^2$)

$$u_i^{n+1}(1+2r) = u_i^{n-1} + 2r(u_{i+1}^n - u_i^{n-1} + u_{i-1}^n)$$

- Remember that the method is not consistent with the diffusion equation if $\Delta t / \Delta x \to cte$.
- The modified equation becomes

$$u_t - \alpha u_{xx} = \left(\frac{\alpha \Delta x^2}{12} - \frac{\alpha^3 \Delta t^2}{\Delta x^2} \right) u_{xxxx} + \left(\frac{\alpha \Delta x^4}{360} - \frac{\alpha^3 \Delta t^2}{3} + \frac{2\alpha^5 \Delta t^4}{\Delta x^4} \right) u_{xxxxxx} + \ldots$$

- Also, the amplification factor is

$$G = \frac{2r\cos\beta \pm \sqrt{1 - 4r^2 \sin^2 \beta}}{1+2r}$$

- The method is unconditionally stable (for $r \geq 0$) and can be easily extended to 2D and 3D cases.

θ - Methods

- **By combining the formulation of the simple explicit, simple implicit and Crank-Nicolson methods, we obtain**

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\frac{\alpha}{\Delta x^2}\left[\theta\,\delta_x^2\,u_i^{n+1}+(1-\theta)\,\delta_x^2\,u_i^n\right] \qquad 0\le\theta\le 1$$

$$where$$

$$\delta_x^2\,u_i^n=u_i^{n+1}-2u_i^n+u_{i-1}^n$$

- **The method is 2nd order in time and space except for special cases:**

$Crank-Nicolson\ \ (\theta=1/2)$	$T.E.=O[\Delta t^2,\Delta x^2]$
$(\theta=\frac{1}{2}-\frac{\Delta x^2}{12\,\alpha\,\Delta t})$	$T.E.=O[\Delta t^2,\Delta x^4]$
$(\theta=\frac{1}{2}-\frac{\Delta x^2}{12\,\alpha\,\Delta t})\,,\ \frac{\Delta x^2}{\alpha\,\Delta t}=\sqrt{20}$	$T.E.=O[\Delta t^2,\Delta x^6]$

- **Its modified equation is**

$$u_t-\alpha\,u_{xx}=\left[\left(\theta-\frac{1}{2}\right)\alpha^2\Delta t+\frac{\alpha\Delta x^2}{12}\right]u_{xxxx}+\left[\left(\theta^2-\theta+\frac{1}{3}\right)\alpha^3\,\Delta t^2+\frac{1}{6}\left(\theta-\frac{1}{2}\right)\alpha^2\Delta t\,\Delta x^2+\frac{\alpha\Delta x^4}{360}\right]u_{xxxxxx}+\dots$$

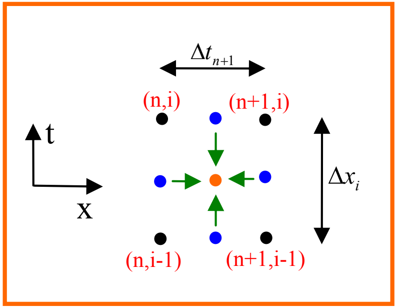
- **The method is unconditionally stable if** $1/2\le\theta\le 1$ **.**

- **When** $0\le\theta\le 1/2$ **the method is stable if** $0\le r\le(2-4\theta)^{-1}$

Keller-Box Method

- **In this method, we split the equation into the followings:**

$$\begin{cases} \dfrac{\partial u}{\partial x} = v \\ \dfrac{\partial u}{\partial t} = \alpha \dfrac{\partial v}{\partial x} \end{cases}$$



- **Then, the method becomes**

$$\frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x_i} = \frac{v_i^{n+1} + v_{i-1}^{n+1}}{2}$$

$$\frac{u_i^{n+1} + u_{i-1}^{n+1}}{\Delta t_{n+1}} = \frac{\alpha}{\Delta x_i}\left(v_i^{n+1} - v_{i-1}^{n+1}\right) + \frac{u_i^{n} + u_{i-1}^{n}}{\Delta t_{n+1}} + \frac{\alpha}{\Delta x_i}\left(v_i^{n} - v_{i-1}^{n}\right)$$

- **This method is 2^nd order in time and space.**

ADI Method

- The **A**lternating **D**irection **I**mplicit method consists of two steps:

$$\frac{u_{i,j}^{n+1/2} - u_{i,j}^{n}}{\Delta t/2} = \alpha\left(\bar{\delta}_x^2 u_{i,j}^{n+1/2} + \bar{\delta}_y^2 u_{i,j}^{n}\right) \qquad \text{(step 1)}$$

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n+1/2}}{\Delta t/2} = \alpha\left(\bar{\delta}_x^2 u_{i,j}^{n+1/2} + \bar{\delta}_y^2 u_{i,j}^{n+1}\right) \qquad \text{(step 2)}$$

where

$$\bar{\delta}_x^2 = \delta_x^2/\Delta x^2 \qquad \bar{\delta}_y^2 = \delta_y^2/\Delta y^2$$

- In the first step a tri-diagonal system is solved for each j row and during the 2nd step a tri-diagonal system is solved for each i row of grid points.

- The method is 2nd order in t, x, y. And

$$G = \frac{\left[1 - r_x(1-\cos\beta_x)\right]\left[1 - r_y(1-\cos\beta_y)\right]}{\left[1 + r_x(1-\cos\beta_x)\right]\left[1 + r_y(1-\cos\beta_y)\right]}$$

where

$$r_x = \frac{\alpha\,\Delta t}{\Delta x^2}, \qquad r_y = \frac{\alpha\,\Delta t}{\Delta y^2}, \qquad \beta_x = \kappa_m \Delta x, \qquad \beta_y = \kappa_m \Delta y$$

- The method is unconditionally stable.

ADI Method…

- **For a 3D problem, Douglas and Gunn suggest the following formulation:**

$$\left(1-\frac{r_x}{2}\delta_x^2\right)\Delta u^* = \left(r_x\delta_x^2 + r_y\delta_y^2 + r_z\delta_z^2\right)u^n \qquad \text{(step 1)}$$

$$\left(1-\frac{r_y}{2}\delta_y^2\right)\Delta u^{**} = \Delta u^* \qquad \text{(step 2)}$$

$$\left(1-\frac{r_z}{2}\delta_z^2\right)\Delta u = \Delta u^{**} \qquad \text{(step 3)}$$

where

$$\Delta u_{i,j} = u_{i,j}^{n+1} - u_{i,j}^n$$

Inviscid Burger Equation

General

- **The inviscid Burger equation**

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \quad with \quad F = u^2/2$$

- Is analog to the Euler equations for the flow of an inviscid fluid. It also represents a nonlinear wave equation, where each point on the wave front can propagate with a different speed. As a result this equation can show the coalescence of characteristics and therefore accept the formation of discontinuous solutions, similar to shock waves in fluid dynamics.

- **The characteristic of the Burger's equation is**

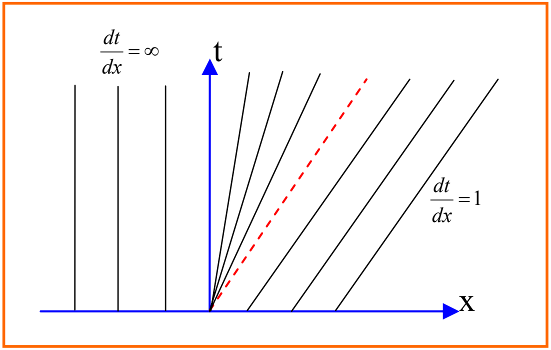
$$\frac{dt}{dx} = \frac{1}{u}$$

- **The solution to the Burger's equation under a specific initial condition is**

$$\begin{cases} u(x,0) = 0 & x < 0 \\ u(x,0) = 1 & 0 \le x \end{cases} \rightarrow \begin{cases} u = 0 & x \le 0 \\ u = x/t & 0 < x < t \\ u = 1 & t \le x \end{cases}$$

General…

- **The solution can be shown as below**



- **If the initial condition was**

$$\begin{cases} u(x,0) = u_1 & x \le a \\ u(x,0) = u_2 < u_1 & x > a \end{cases}$$

- **Then, a shock wave like discontinuity will be traveling in the domain at the average value of the $(u_1 + u_2)/2$ function across the wave front.**

Lax Method

- **This is a first order method:**

$$u_i^{n+1} = \left(\frac{u_{i+1}^n + u_{i-1}^n}{2} \right) - \frac{\Delta t}{2\Delta x}\left(F_{i+1}^n - F_{i-1}^n \right) = 0$$

- **Its amplification factor is**

$$G = \cos\beta - i\frac{\Delta t}{\Delta x}\frac{dF}{du}\sin\beta$$

- **The stability limit is**

$$\left| \frac{\Delta t}{\Delta x} u_{\max} \right| \leq 1$$

- **When using a finite volume method, a first order method in t is**

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n \right)$$

- **Using the Lax method, we get**

$$F_{i+1/2} = \frac{1}{2}\left[F_i^n + F_{i+1}^n - \frac{\Delta x}{\Delta t}\left(u_{i+1} - u_i \right) \right]$$

- **This flux function is consistent in the sense that**

$$F(u_i, u_{i+1}) = F(u_i) \qquad when \qquad u_i = u_{i+1}$$

Lax-Wendroff Method

- This method is developed using the Taylor series and is 2nd order accurate in space and time.
- The method reads: $\left(\partial F / \partial u\right)_{i+1/2} \equiv A_{i+1/2} = A\left((u_i + u_{i+1})/2\right)$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}\left(F_{i+1}^n - F_i^n\right) + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2\left[A_{i+1/2}^n\left(F_{i+1}^n - F_i^n\right) - A_{i-1/2}^n\left(F_i^n - F_{i-1}^n\right)\right]$$

- We also have

$$G = 1 - 2\left(A\frac{\Delta t}{\Delta x}\right)^2(1-\cos\beta) - 2i\,A\frac{\Delta t}{\Delta x}\sin\beta$$

- The stability requirement is

$$\left|\frac{\Delta t}{\Delta x}u_{\max}\right| \le 1$$

- As the Courant number $u\Delta t / \Delta x$ is reduced from 1.0, the quality of the solution is degraded and more oscillations are produced.

Lax-Wendroff Method...

- **Using the finite volume formulation, we get**

$$F_{i+1/2}=\frac{1}{2}\left(F_i+F_{i+1}\right)-\frac{\Delta t}{2\Delta x}\lambda_{i+1/2}^2\left(u_{i+1}-u_i\right)$$

- **Here $\lambda_{i+1/2}$ is the eigenvalue of the Jacobian $A_{i+1/2}$ which is $u_{i+1/2}$ for the Burger's equation.**

- **We also notice that**

$$F_{i+1/2}=\overbrace{\frac{1}{2}\left(F_i+F_{i+1}-\frac{\Delta x}{\Delta t}\left(u_{i+1}-u_i\right)\right)}^{\text{Lax method (first order)}}+\overbrace{\frac{\Delta x}{\Delta t}\left[1-\left(\frac{\Delta t}{\Delta x}\right)^2\lambda_{i+1/2}^2\right]\left(\frac{u_{i+1}-u_i}{2}\right)}^{\text{Extra term producing 2nd order accuracy}}$$

- **Since Lax method produces no oscillations, we can use a function ϕ to control the amount of the 2nd part to be added to the Lax method to minimize the oscillations.**

MacCormack Method

- **This method is a predictor-corrector version of the Lax-Wendroff scheme thus easier to implement.**

$$\overline{u}_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1}^n - F_i^n\right) \qquad \text{(predictor)}$$

$$u_i^{n+1} = \frac{1}{2}u_i^n + \overline{u}_i^{n+1} - \frac{\Delta t}{\Delta x}\left(\overline{F}_i^{n+1} - \overline{F}_{i-1}^{n+1}\right) \qquad \text{(corrector)}$$

- **The amplification factor and the stability limit are the same as those for the Lax-Wendroff method.**

Warming-Kutler-Lomax Method

- **This is a 3rd order scheme and uses the MacCormack method for the first two levels. The method (1973) reads**

$$u_i^{(1)} = u_i^n - \frac{2\Delta t}{3\Delta x}\left(F_{i+1}^n - F_i^n\right)$$

$$u_i^{(2)} = \frac{1}{2}\left[u_i^n + u_i^{(1)} - \frac{2\Delta t}{3\Delta x}\left(F_i^n - F_{i-1}^{(1)}\right)\right]$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{24\Delta x}\left(-2F_{i+2}^n + 7F_{i+1}^n - 7F_{i-1}^n + 2F_{i-2}^n\right)$$
$$- \frac{3\Delta t}{8\Delta x}\left(F_{i+1}^{(2)} - F_{i-1}^{(2)}\right) - \frac{\omega}{24}\left(u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n\right)$$

- **The stability limit for Burger's equation is given by**

$$|\nu| = \left|\frac{\Delta t}{\Delta x}u_{\max}\right| \le 1 \qquad and \qquad 4\nu^2 - \nu^4 \le \omega \le 3$$

Beam and Warming Method

- This is a 2nd order implicit method.
- First use the trapezoidal method,

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{2}\left[(u_t)^n + (u_t)^{n+1}\right] + O[\Delta t^3]$$

- Then, substitute the model equation into it

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2}\left[\left(\frac{\partial F}{\partial x}\right)^n + \left(\frac{\partial F}{\partial x}\right)^{n+1}\right]$$

- Beam and Warming (1976) suggested that

$$F^{n+1} \approx F^n + \left(\frac{\partial F}{\partial u}\right)^n \left(u^{n+1} - u^n\right) = F^n + A^n\left(u^{n+1} - u^n\right)$$

- Thus, the method becomes

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2}\left[2\left(\frac{\partial F}{\partial x}\right)^n + \frac{\partial}{\partial x}\left[A(u_i^{n+1} - u_i^n)\right]\right]$$

- Substitute x-derivatives with 2nd order central differences, we get

$$-\left(\frac{\Delta t\, A_{i-1}^n}{4\Delta x}\right)u_{i-1}^{n+1} + u_i^{n+1} + \left(\frac{\Delta t\, A_{i+1}^n}{4\Delta x}\right)u_{i+1}^{n+1} = -\left(\frac{\Delta t}{\Delta x}\right)\frac{F_{i+1}^n - F_{i-1}^n}{2} - \left(\frac{\Delta t\, A_{i-1}^n}{4\Delta x}\right)u_{i-1}^n + u_i^n + \left(\frac{\Delta t\, A_{i+1}^n}{4\Delta x}\right)u_{i+1}^n$$

Beam and Warming Method...

- **This leads to a tri-diagonal system and can be solved using the Thomas algorithm.**
- **This method is <u>stable</u> but produces oscillations.**
- **To reduce oscillations, an artificial smoothing can be added to the scheme** $0 < \omega \leq 1$**:**

$$\frac{-\omega}{8}\left(u_{i+2}^{n} - 4u_{i+1}^{n} + 6u_{i}^{n} - 4u_{i-1}^{n} + u_{i-2}^{n}\right)$$

- **An efficient form of the above method can be obtained using**

$$\Delta u_i = u_i^{n+1} - u_i^{n}$$

- **The trapezoidal formula with the following linearization**

$$F_i^{n+1} = F_i^{n} + A_i^{n} \Delta u_i$$

- **The final form of the scheme becomes**

$$-\left(\frac{\Delta t\, A_{i-1}^{n}}{4\Delta x}\right)\Delta u_{i-1} + \Delta u_i + \left(\frac{\Delta t\, A_{i+1}^{n}}{4\Delta x}\right)\Delta u_{i+1} = -\left(\frac{\Delta t}{\Delta x}\right)\frac{F_{i+1}^{n} - F_{i-1}^{n}}{2}$$

- **Which is simpler for computation. The method still needs a smoothing to produce an oscillation free solution.**