Advanced Fluid Mechanics

# BOUNDARY LAYER SEPARATION

Presented by:

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## Flow with a Pressure Gradient

In Blasius' solution for laminar flow over a flat plate, the pressure gradient was zero. A much more common flow situation involved flow with a pressure gradient.

$$\rho(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y})=-\frac{\partial P}{\partial x}+\mu\frac{\partial^2 v_x}{\partial y^2}$$

The pressure gradient plays a major role in flow separation, as can be seen with the aid of the boundary-layer equation. If we make use of the boundary conditions at the wall $v_x = v_y = 0$ , at y=0 equation becomes

$$\mu\frac{\partial^2 v_x}{\partial y^2}\bigg|_{y=0}=\frac{dP}{dx}$$

which relates the curvature of the velocity profile at the surface to the pressure gradient. Figure illustrates the variation in $v_x, \frac{\partial v_x}{\partial y}$ and $\frac{\partial^2 v_x}{\partial y^2}$ across the boundary layer for the case of a zero pressure gradient.

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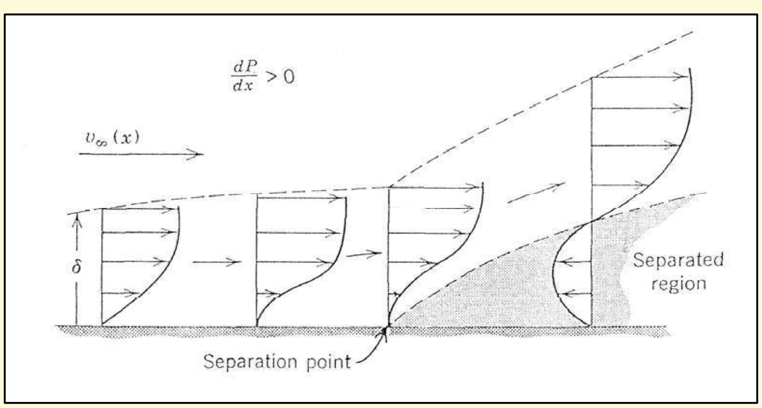
When dP/dx=0, the second derivative of the velocity at the wall must also be zero; hence the velocity profile is linear near the wall. Further out in the boundary layer, the velocity gradient becomes smaller and gradually approaches zero. The decrease in the velocity gradient means that the second derivatives of the velocity must be negative. The derivative $\frac{\partial^2 v_x}{\partial y^2}$ is shown as being zero at the wall, negative within the boundary layer , and approaching zero at the outer edge of the boundary layer. It is important to note that the second derivative must approach zero from the negative side as $y \to \delta$ .for values of $dP/dx \neq 0$, the variation in $v_x$ and its derivatives is shown in figure.

A negative pressure gradient is seen to produce a velocity variation somewhat similar to that of the zero-pressure-gradient case. A positive value of dP/dx, however requires a positive value of $\partial^2 v_x / \partial y^2$ at the wall. Since this derivative must approaches zero from the negative side, at some point within the boundary layer the second derivative must equal zero. A zero second derivative, it will be recalled, is associated with an inflection point.We may know turn our attention to the subject of low separation.





In order for flow separation to occur, the velocity in the layer of fluid adjacent to the wall must be zero or negative, as shown in figure below. This type of velocity profile is seem to require a point of inflection. As the only type of boundary layer flow that has an inflection point is flow with a positive pressure with gradient, it may be concluded that a positive pressure gradient is necessary for separation. For this reason a positive pressure gradient is called *an adverse pressure gradient.* Flow can remain un-separated with an adverse pressure gradient, thus dP/dx>0 is a necessary but not a sufficient condition for separation. In contrast a negative pressure gradient, in the absence of sharp corners, can not cause flow separation. Therefore a negative pressure gradient is called a favorable pressure gradient



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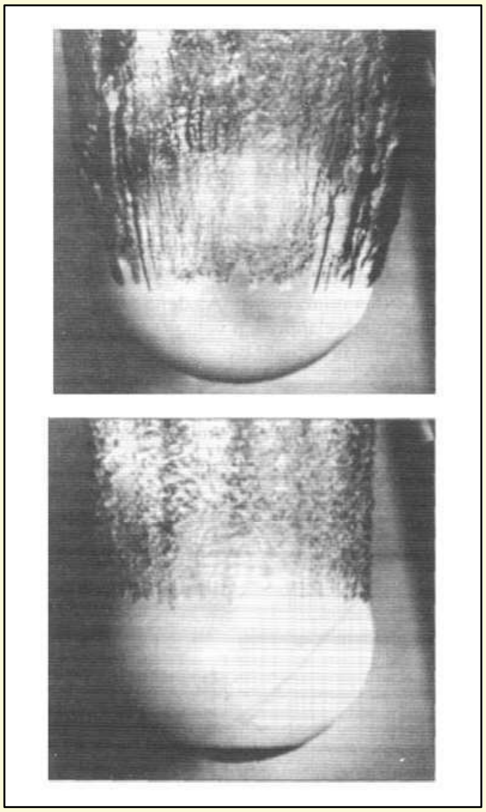

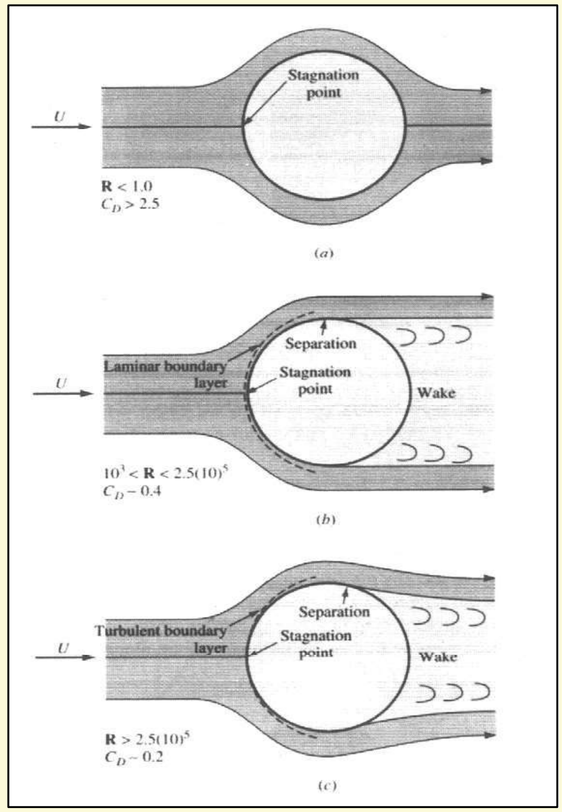




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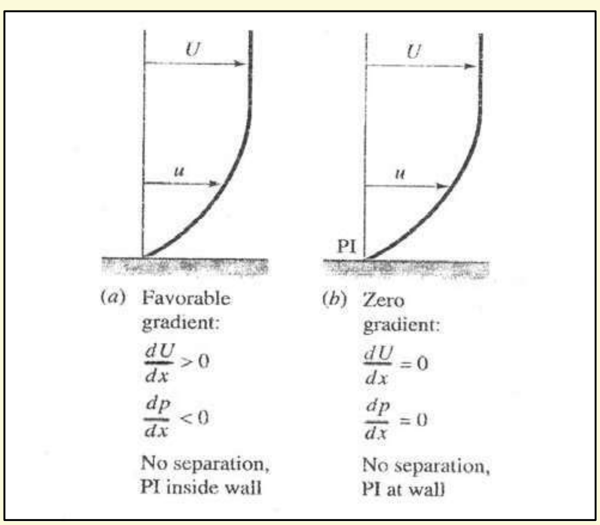


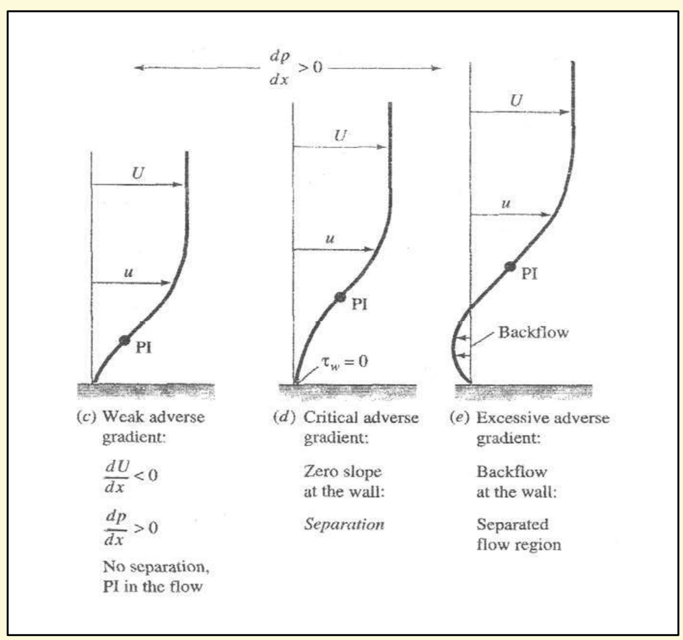

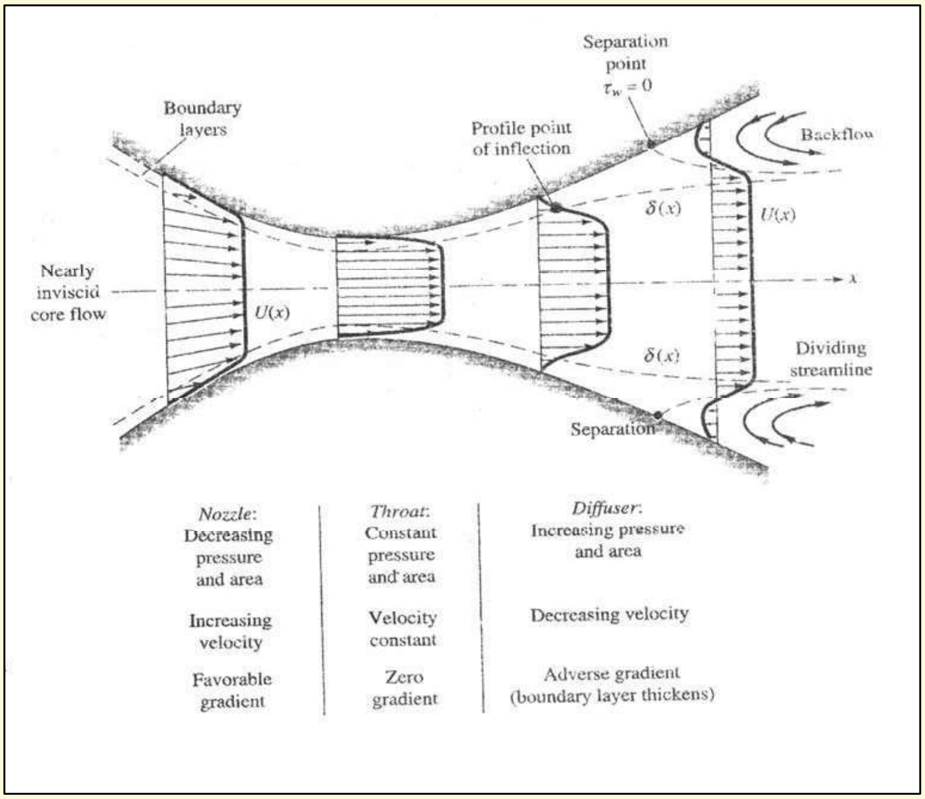




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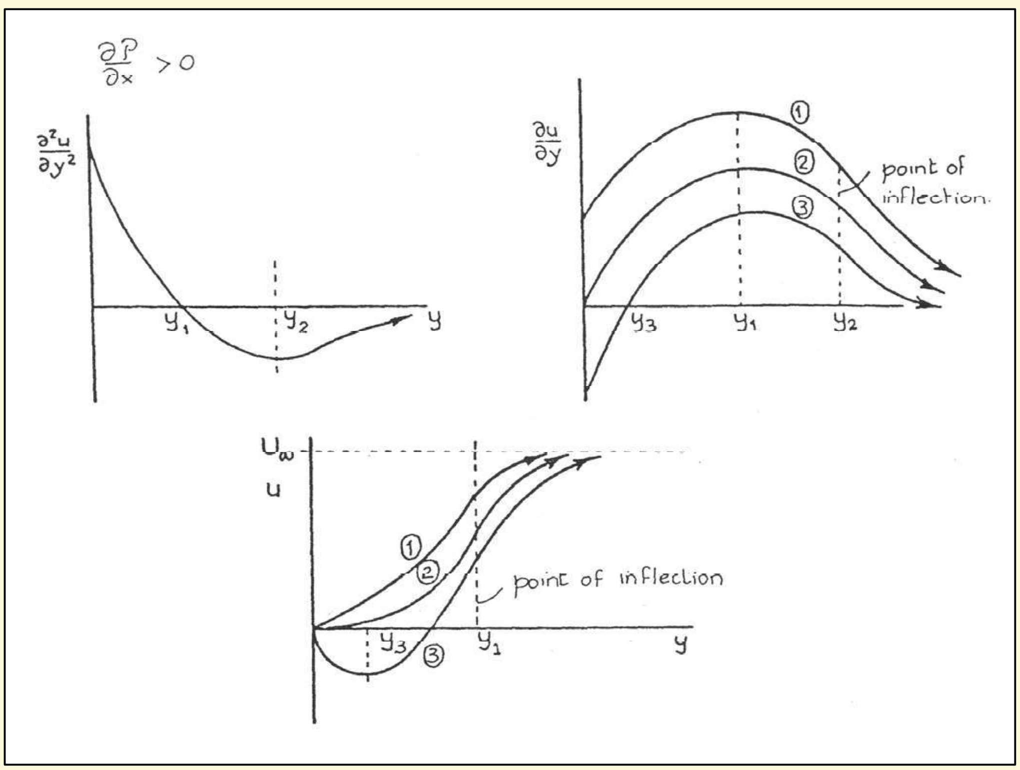


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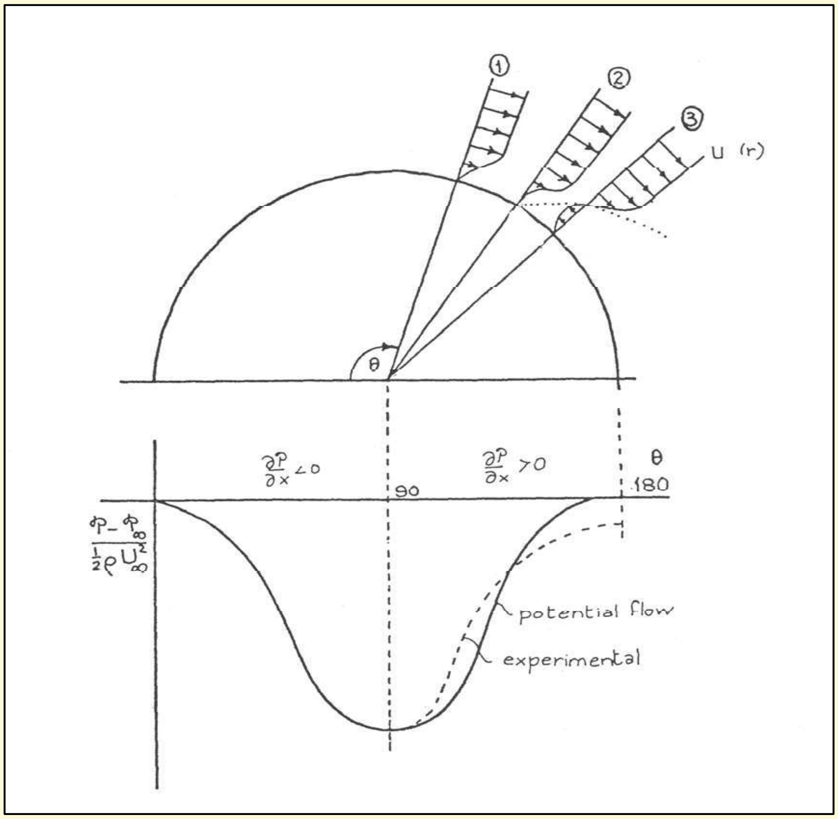




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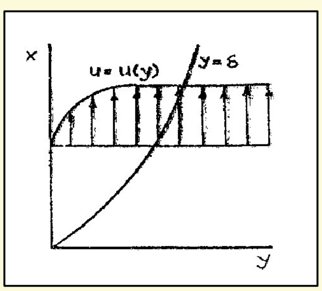
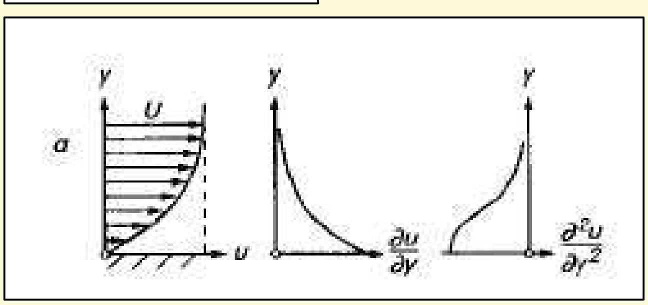
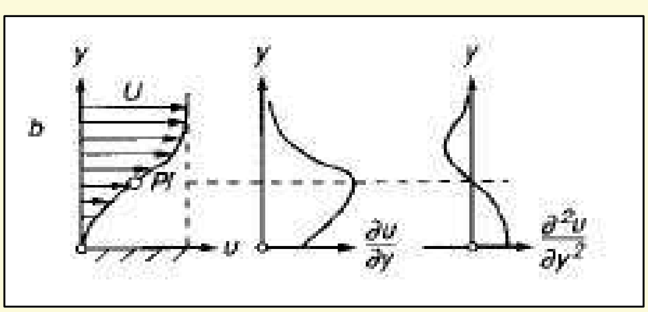



$$y = 0 \qquad u = 0$$

$$v = 0$$

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{1}{e}\left(\frac{\partial p}{\partial x}\right)\Big|_\infty$$

$$y \to \infty \quad u \to U_\infty \quad \textit{from below}$$

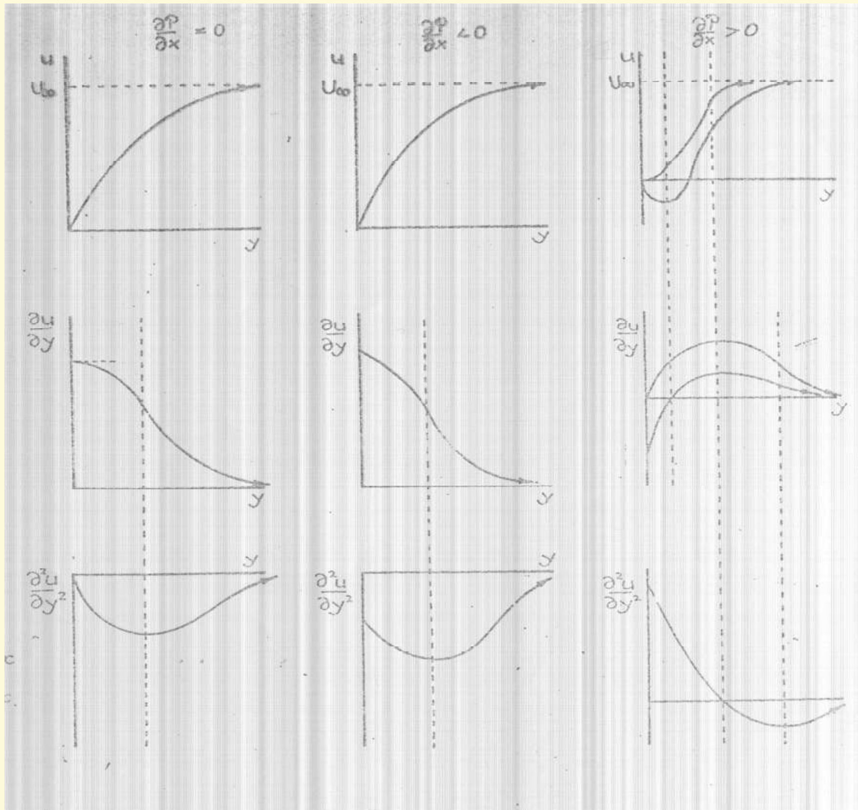
$$\frac{\partial u}{\partial y} \to 0 \quad \textit{from above}$$

$$\frac{\partial^2 u}{\partial y^2} \to 0 \quad \textit{from below}$$

$$\frac{eU_\infty^2}{2} + p = const.$$

$$\therefore \ \ if \quad \left(\frac{\partial p}{\partial x}\right)_\infty > 0 \ , \ U_\infty \uparrow with \ x$$

$$\left(\frac{\partial p}{\partial x}\right)_\infty < 0 \ , \ U_\infty \uparrow with \ x$$



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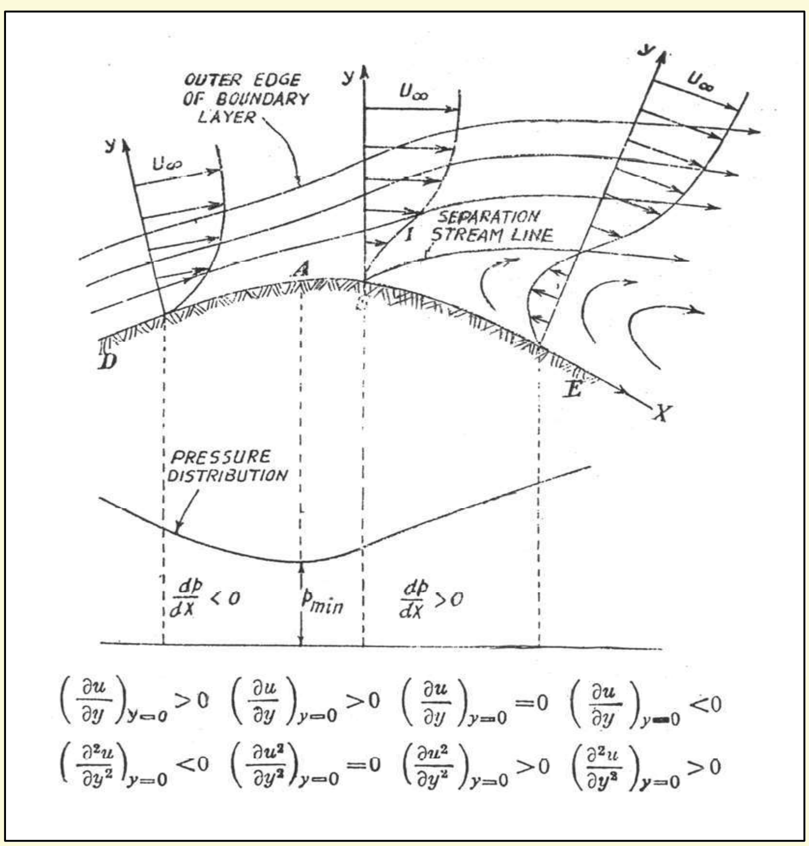


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# BOUNDARY LAYER THEORY

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# Boundary Layer Theory

The solution to problems involving flow around immersed bodies will be considered. The solution depends on the fact that the viscous effects are confined to the region near solid surfaces. In fact one can always find predominance of all molecular processes close enough to solid walls.

The region of flow is subdivided into two:

i) Far from solid surfaces – viscous effects negligible.
   i.e. Potential Flow Theory (ideal flow, irrotational)

ii) Near solid surface – viscous effects important.
   i.e. Boundary Layer Theory.

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# Boundary Layer Theory

## Link with Potential Flow Theory

We have shown that near solid surfaces

(Viscous terms) ~ (Inertia terms)

$$\frac{\mu U}{l^2} \sim \rho \frac{U^2}{L}$$

i.e. $$\frac{l}{L} \sim \mathrm{Re}_L^{-\frac{1}{2}}$$

$l$ may be identified as thickness of region where viscous effects are important. All significant velocity changes will be concentrated in this region.



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# Description of Boundary Layer over Flat Plate

For flow over a flat plate.

The locus of point where $u=0.99U_\infty$ has been measured.

The locus $\delta(X)_1$ defines the boundary layer thickness

It has been verified experimentally that

$$\left[\frac{\delta}{X}\right] = 5.5\left[\frac{U_\infty X}{\nu}\right]^{-\frac{1}{2}} \; or \; \frac{\delta}{X}\mathrm{Re}_X^{\frac{1}{2}} = 5.5$$

Provided $\mathrm{Re}_X < 3.2 * 10^5$

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i) $\delta$ increases with x and $\nu$ decreases with $U_\infty$

ii) All velocity change in b.1.

iii) Turbulent b.1. formed if plate long enough

## Engineering Application

i) Flow near solid surface

a. Entrance region in pipes-pressure drop
b. Natural convection from vertical surfaces – heat transfer
c. Evaporation from ponds – mass transfer



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## Order of Magnitude Analysis of N-S Equations for flow near a solid boundary

Consider the 2-D Navier Stoke's equations for steady,

$$u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \mathscr{P}}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

Define: L – a characteristic length in flow direction
$\delta$ – boundary layer thickness. The characteristic length normal to the boundary in which inertia and viscous forces have the same order of magnitude.
$U_\infty$ – free stream velocity
$\rho U_\infty^2$ – characteristic pressure

Define: $u_i' = \frac{u_i}{U_\infty}$ ; $x_i' = \frac{x_i}{L}$ ; $\mathscr{P}' = \frac{\mathscr{P}}{\rho U_\infty^2}$

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Hence the dimensionless equations foe steady flow are:

$$u'_j \frac{\partial u'_i}{\partial x'_j} = -\frac{\partial \mathscr{P}'}{\partial x'_i} + \frac{1}{\mathrm{Re}} \frac{\partial^2 u'_i}{\partial x'_j \partial x'_j} \qquad \mathrm{Re} = \frac{LU_\infty}{\nu} \sim O(\frac{L}{\delta})^2$$

Order

i=1 $\quad 1 \quad 1 \quad \left[\frac{\delta}{L}\right]^2 \quad \left[1+\left[\frac{L}{\delta}\right]^2\right]$

i=2 $\quad \frac{\delta}{L} \quad \frac{L}{\delta} \quad \left[\frac{\delta}{L}\right]^2 \quad \left[\frac{\delta}{L}+\frac{L}{\delta}\right]$

Since

$$\left.\begin{array}{l} \mathscr{P}' \sim 1 \\ u'_1 \sim 1 \\ x'_1 \sim 1 \\ x'_2 \sim \frac{\delta}{L} \end{array}\right\} \Rightarrow \frac{\partial u'_1}{\partial x'_1} \sim 1 \Rightarrow \frac{\partial u'_2}{\partial x'_2} \sim 1$$

i.e. $\quad u_2' \sim x_2' \sim \frac{\delta}{L}$

$$u'_j \frac{\partial}{\partial x'_j} \sim 1$$

i.e. $\quad u'_j \frac{\partial u'_1}{\partial x'_j} \sim 1 \quad ; \quad u'_j \frac{\partial u'_2}{\partial x'_j} \sim \frac{\delta}{L} \quad ; \quad \frac{\partial^2}{\partial x'_j \partial x'_j} \sim 1+\left[\frac{L}{\delta}\right]^2$





In engineering applications we are interested in high $\mathrm{Re} = \frac{LU_\infty}{\nu}$

$$\mathrm{Re} = \frac{Intertia\ \ Forces}{Viscous\ \ Forces}\left[\frac{L}{\delta}\right]^2 \qquad \delta << L$$

Hence $\quad \frac{\partial \mathscr{P}'}{\partial x'_2} = 0 \quad ; \quad u'_j \frac{\partial u'_1}{\partial x'_j} = -\frac{\partial \mathscr{P}'}{\partial x'_1} + \frac{1}{\mathrm{Re}} \frac{\partial^2 u'_1}{\partial x'^2_2}$

$$\mathscr{P}' = \mathscr{P}'(x'_1) \qquad \frac{\partial \mathscr{P}'}{\partial x_1} = \left[\frac{\partial \mathscr{P}'}{\partial x_1}\right]_\infty = \left[\frac{d\mathscr{P}'}{dx_1}\right]_\infty$$

i.e. External pressure is imposed on boundary layer.

u = O (l)

Now x can vary from zero to l, and thus the maximum change in x is also of the order of unity. The change in u can be from zero to unity, and writing the derivatives in finite -difference form

$$\frac{\partial u}{\partial x} \cong \frac{\Delta u}{\Delta x} = \frac{O(l)}{O(l)} = 1$$

Likewise we can readily show that the order of magnitude of the second derivative is

$$\frac{\partial^2 u}{\partial x^2} = O(1)$$

Examining the y direction, the extreme value of y is the very thin boundary-layer thickness δ. From the continuity equation with $\frac{\partial u}{\partial x} = O(1)$, we conclude that $\frac{\partial v}{\partial y} = O(1)$. Since the extreme value of Δ y is O(δ), we require that v=O(δ)

Then the derivatives of u with respect to y are

$$\frac{\partial u}{\partial y} = O(\frac{1}{\delta}) \quad , \quad \frac{\partial^2 u}{\partial y^2} = O(\frac{1}{\delta^2})$$





Proceeding with similar arguments, we obtain

$$\frac{\partial^2 v}{\partial y^2} = O(\frac{1}{\delta}) \quad , \quad \frac{\partial v}{\partial x} = O(\delta) \quad , \quad \frac{\partial^2 v}{\partial x^2} = O(\delta)$$

Assuming the extreme change of time to be of the order of unity, we have

$$\frac{\partial u}{\partial t} = O(1) \quad , \quad \frac{\partial v}{\partial t} = O(\delta)$$

$\mathscr{P}$ does not exceed unity and assuming it to be $O(1)$, we see that

$$\frac{\partial \mathscr{P}}{\partial x} = O(1)$$

For the pressure term to be significant. By the same argument w.r.t. equation of motion in the y-direction:

$$\frac{\partial \mathscr{P}}{\partial y} = O(\delta)$$

# Flow over Flat Plate

The Navier-Stokes equations for incompressible flow reduce to

x-direction:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\rho\frac{\partial \mathcal{P}}{\partial x}+\nu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})$$

1 1 1 δ 1/δ δ² 1 1/δ²

y-direction:

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\rho\frac{\partial \mathcal{P}}{\partial y}+\nu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2})$$

δ 1 δ δ 1 δ² δ 1/δ

Continuity:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

1 1

Quantities 1, δ, etc. result from an order-of-magnitude analysis.



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$$\frac{\partial^2 u}{\partial x^2} << \frac{\partial^2 u}{\partial y^2}$$

We have:

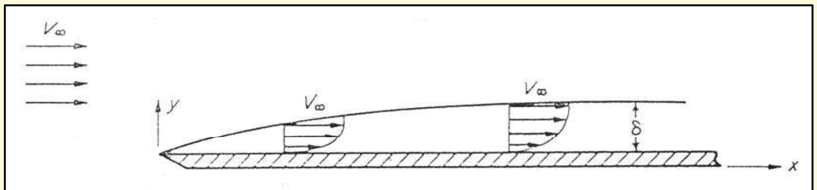
i) $$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial \mathcal{P}}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}$$

ii) $$\frac{\partial \mathcal{P}}{\partial y}=0$$

iii) $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

These are known as Prandtl's boundary-layer equations.

Blasius solution




Similarity profiles in boundary-layer flow over a flat plate.

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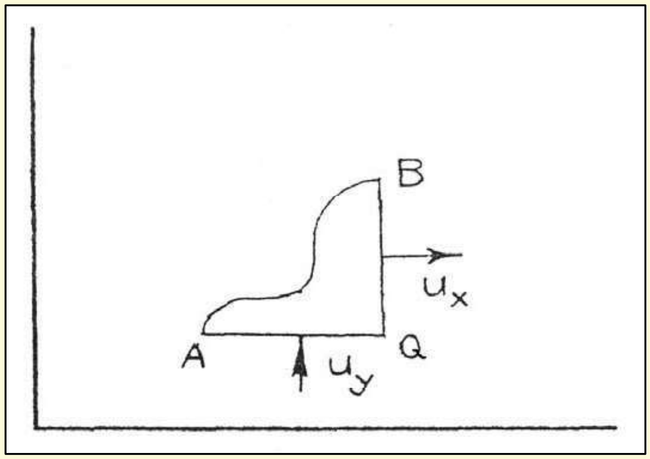
## The Stream Function (2 dimensional)




Consider points A and B in a flow field.
Join A and B along any path.
Draw AQ and BQ parallel to x and y axes.
Define $\delta\psi$ as the flow rate/unit width into the volume ABQ from left to right across AB.





Then $\delta\psi = u_x \delta y - u_y \delta x$

And $\delta\psi$ is path independent. Hence $\psi$ a function of x and y alone (scalar function of position)

Taking Limit A→B

$$d\psi = u_x dy - u_y dx$$

$$= \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx$$

Hence: $u_x = \frac{\partial \psi}{\partial y}$ ; $u_y = -\frac{\partial \psi}{\partial x}$

Properties of $\psi$

(1) If we have

$$d\psi = 0$$

$$\left[\frac{dy}{dx}\right]_{\psi = const.} = \frac{u_y}{u_x}$$

Hence $\psi$ = constant is a streamline

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## Flow over a flat plate: Analytic solution

Boundary Layer equations must be solved subject to B.C.'s.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\left[\frac{\partial \mathscr{P}}{\partial x}\right]_{\infty}+\nu\frac{\partial^2 u}{\partial y^2} \qquad (1)$$

B.C (i) At $y=0, \quad u=v=0$

(ii) At $y\rightarrow\infty, \quad u\rightarrow u_{\infty}$

Further assume flow in y direction is infinite.
Then from Bernoullis equation

$$\rho\frac{u.u}{2}+\mathscr{P}=constant$$

$$u=u_{\infty}=constant$$

$$\left[\frac{\partial \mathscr{P}}{\partial x}\right]_{\infty}=0$$



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Hence equations simplify to:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (2)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \qquad (3)$$

Problem is to solve equations (2) & (3) subject to B.C.'s (1).

Use stream function to obtain a single equation

$$u=\frac{\partial \psi}{\partial y} \qquad v=-\frac{\partial \psi}{\partial x}$$

Continuity is automatically satisfied.

Momentum Equation becomes:

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(ii) Consider $(\nabla\Phi . \nabla\psi)$ consider $\left\{\frac{\partial\Phi}{\partial x_i} \quad \frac{\partial\psi}{\partial x_i}\right\}$

$$= \left[\frac{\partial\Phi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\Phi}{\partial y}\frac{\partial\psi}{\partial y}\right] = -u_x y_y + u_x u_y = 0$$

Streamlines & Equipotential lines orthogonal
N.B. δψ is the flowrate/unit depth between streamlines ψ and ψ+δψ

(iii) $\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial y} = \frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial y\partial x} = 0$

Continuity automatically satisfied by stream function

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$$\frac{\partial\psi}{\partial y}\cdot\frac{\partial^2\psi}{\partial x\partial y} + \left[-\frac{\partial\psi}{\partial x}\right]\left[\frac{\partial^2\psi}{\partial y^2}\right] = \nu\left[\frac{\partial^3\psi}{\partial y^3}\right] \qquad (4)$$

If an analytic solution is possible, combination of variables suggested by open range in y.

Must have $u' = \frac{u}{U_\infty} = u'(x, y) = u'(\eta)$

Now at different distances downstream the velocity at corresponding points in boundary layer (constant y/δ) must be equal. This suggests

$$\eta \propto \frac{y}{\delta}$$

The form of the downstream dependence is suggested by the equation of motion and confirmed by experiment

Hence $\eta = y\sqrt{\frac{U_\infty}{\nu\, x}}$

Later a simpler method of obtaining the combined variable will be discussed.

Now $u' = \frac{u}{U_\infty} = \frac{1}{U_\infty}\left[\frac{\partial\psi}{\partial y}\right]_x$

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In general

$$\psi = \psi(x, y) = \psi(x, \eta)$$

$$d\psi = \left[\frac{\partial \psi}{\partial x}\right]_\eta dx + \left[\frac{\partial \psi}{\partial \eta}\right]_x d\eta$$

$$\left[\frac{\partial \psi}{\partial y}\right]_x = \left[\frac{\partial \psi}{\partial \eta}\right]_x \left[\frac{\partial \eta}{\partial y}\right]_x = \left[\frac{\partial \psi}{\partial \eta}\right]_x \sqrt{\frac{U_\infty}{\nu x}}$$

$$u'(\eta) = \frac{\partial}{\partial \eta}\left[\frac{\psi}{\sqrt{U_\infty \nu x}}\right] = \frac{\partial f}{\partial \eta}$$

Thus ψ cannot be written as a function of η alone.

However $f = \left[\frac{\psi}{\sqrt{U_\infty \nu x}}\right] = f(\eta)$ alone.

Thus equation (4) must be written with f as the dependent variable .



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Equation reduced to

$$\frac{2d^3 f}{d\eta^3} + \frac{f d^2 f}{d\eta^2} = 0$$

With boundary conditions

At $\eta = 0$ $\frac{df}{d\eta} = 0$, $f = 0$ At $n \to \infty$ $\frac{df}{d\eta} \to 1.0$

The general solution of this equation is not possible analytically.

Blasius solved the equation using a series expansion in η.

$$\frac{\partial \psi}{\partial y} \cdot \frac{\partial^2 \psi}{\partial x \partial y} + \left[-\frac{\partial \psi}{\partial x}\right]\left[\frac{\partial^2 \psi}{\partial y^2}\right] = \nu \left[\frac{\partial^3 \psi}{\partial y^3}\right]$$

If an analytic solution is possible, combination of variables suggested by open range in y must have:

$$u' = \frac{u}{U_\infty} = u'(x, y) = u'(\eta)$$

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$$\frac{u}{V_\infty} = \phi_1\left(\frac{y}{\delta}\right) \quad , \quad \frac{u}{V_\infty} = \phi_2\left(y\sqrt{\frac{V_\infty}{\nu\, x}}\right)$$

Where the term in parentheses is a nondimensional coordinate, is given by the symbol η

$$\eta = y\sqrt{\frac{V_\infty}{\nu\, x}}$$

It is convenient to express the velocity components and their derivatives in terms of the stream function ψ. A dimensionless stream function in the η coordinate system which satisfies the continuity equation is $f(\eta) = \psi / \sqrt{\nu x V_\infty}$, and consequently

$$\psi = \sqrt{\nu\, x V_\infty}\, f(\eta)$$

The velocity components are related to the stream function by

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$



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$$u = \frac{\partial}{\partial y}\left[\sqrt{\nu\, x V_\infty}\, f(\eta)\right] = \sqrt{\nu\, x V_\infty}\,\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$u = \sqrt{\nu\, x V_\infty}\sqrt{\frac{V_\infty}{\nu\, x}}\frac{\partial f}{\partial \eta} = V_\infty f'$$

In a similar fashion we obtain

$$\frac{\partial u}{\partial x} = -\frac{\eta V_\infty}{2x} f'' \quad , \quad \frac{\partial u}{\partial y} = V_\infty\sqrt{\frac{V_\infty}{\nu\, x}} f'' \quad , \quad \frac{\partial^2 u}{\partial y^2} = \frac{V_\infty{}^2}{\nu\, x} f'''$$

$$v = \frac{1}{2}\sqrt{\frac{\nu\, V_\infty}{x}}(\eta\, f' - f)$$

$$2f''' + ff'' = 0$$

(1) at $y = 0$: $u = v = 0$

(2) at $y \to \infty$: $u = V_\infty$ , $v = 0$

$$f = f' = 0 \quad @ \quad \eta = 0$$
$$f' = 1 \quad @ \quad \eta \to \infty$$

Thus ψ cannot be written as a function of η alone.

However

$$f = \left[ \frac{\psi}{\sqrt{u_\infty \nu x}} \right] = f(\eta) \qquad \text{alone.}$$

Thus equation (4) must be written with f as the dependence variable (see handout)

Equation reduced to

$$\frac{2d^3 f}{d\eta^3} + \frac{f d^2 f}{d\eta^2} = 0$$

With boundary conditions

$$@ \quad \eta = 0 \qquad \frac{df}{d\eta} = 0$$

$$f = 0$$

$$@ \quad \eta \to \infty \qquad \frac{df}{d\eta} \to 1.0$$





## The boundary-layer thickness δ

The boundary-layer thickness D is a arbitrarily defined as the location where the velocity is 99 percent of free-stream value, i.e.,

$$\frac{u}{V_\infty} = 0.99$$

Using this, we find from Table 12-3 that $\eta \approx 5.0$. Thus,

$$\eta = \delta \sqrt{\frac{V_\infty}{\nu x}} \cong 5.0$$

Or

$$\frac{\delta}{x} \approx \frac{5.0}{\sqrt{\mathrm{Re}_x}}$$

Which expresses the boundary-layer thickness along a flat plate in steady incompressible laminar flow of a viscous fluid.

The general solution of this equation is not possible analytically.
Blasius solved the equation using a series expansion in η.

$$f(\eta) = A_0 + A_1\eta + \frac{A_2}{2!}\eta^2 + \frac{A_3}{3!}\eta^3 \quad \text{etc.}$$

More recently numerical solutions have been carried out. Tables of f, f', f'', f'''
as a function of η are variable.

{Howard Proc. Royal Soc. Al64, 547(1938)}



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## The boundary-layer thickness δ

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# Use of exact analysis

$$\text{Drag force/unit width} = \int_0^L \tau_W \, dx$$

$$\tau_W = \mu \frac{\partial u}{\partial y}\Big|_{y=0} = \mu \frac{\partial u}{\partial \eta}\Big|_{\eta=0} \frac{\partial \eta}{\partial y}$$

$$= \mu . u_\infty \frac{d^2 f}{d\eta 2}\Big|_{\eta=0} \sqrt{\frac{u_\infty}{\nu . x}}$$

$$\text{Friction factor} \quad f = \frac{\tau_W}{\frac{1}{2}\rho . u_\infty^{\ 2}} = 2(0.332)\sqrt{\frac{\nu}{u_\infty . x}} = 0.664 \, \mathrm{Re}_x^{-\frac{1}{2}}$$

$$\text{Drag force} \quad \frac{F}{W} = \int_0^L \tau_W \, dx = \int_0^L 0.664 \sqrt{\frac{\nu}{u_\infty . x}} \, \frac{1}{2} \rho . u_\infty^{\ 2} . dx$$

$$\frac{F}{W} = 0.664 \, . u_\infty^{\frac{1}{2}} . L^{\frac{1}{2}} . \rho . \nu^{\frac{1}{2}}$$

$$\text{Frictional drag force} \quad \alpha \quad u_\infty^{\frac{3}{2}}$$

$$\alpha \quad L^{\frac{1}{2}}$$



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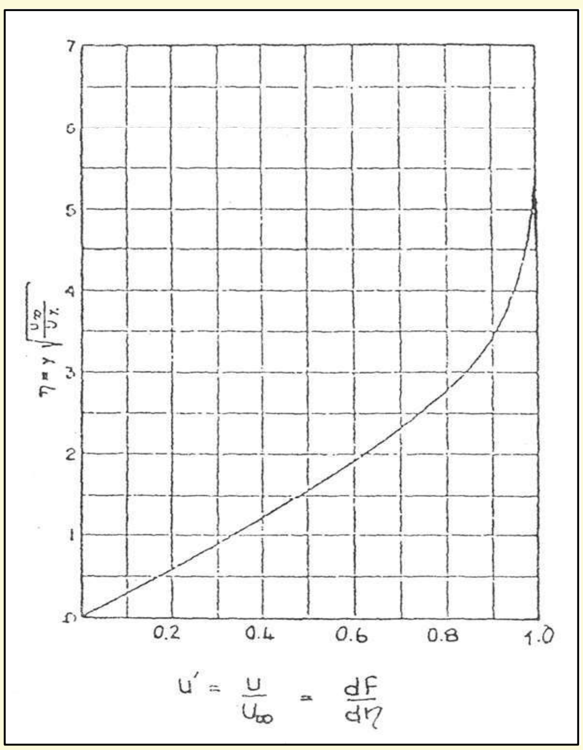


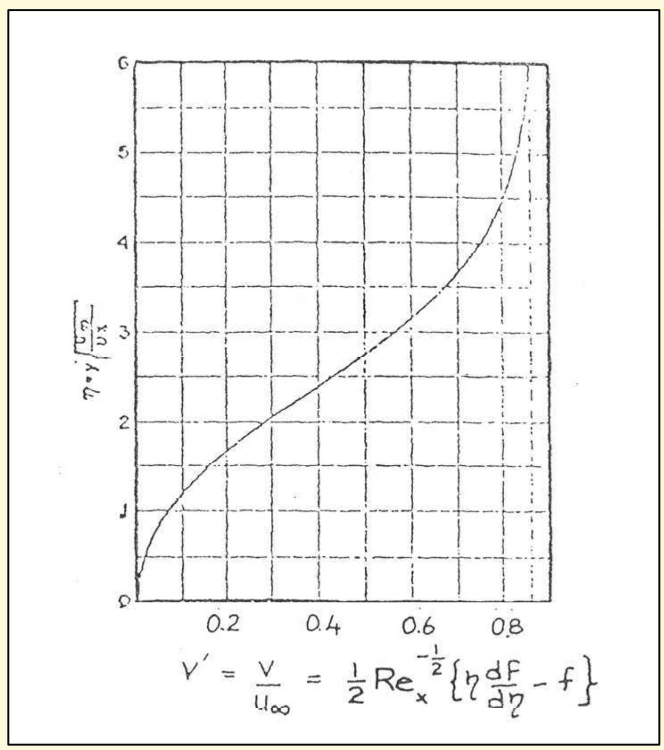


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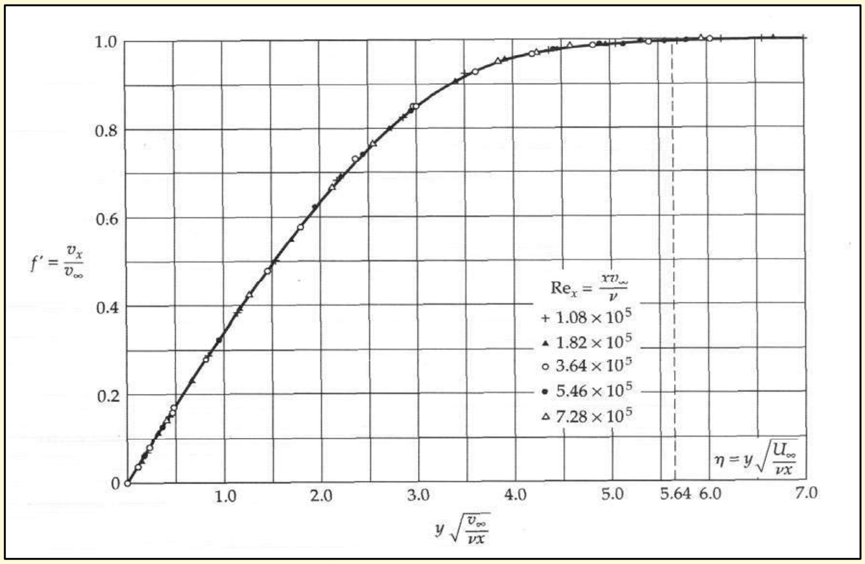
The function $f(\eta)$ and its derivatives†

| $\eta = y\sqrt{\frac{V_\infty}{\nu x}}$ | $f$ | $f' = \frac{u}{V_\infty}$ | $f''$ |
|---|---|---|---|
| 0 | 0 | 0 | 0.33206 |
| 0.2 | 0.00664 | 0.06641 | 0.33199 |
| 0.4 | 0.02656 | 0.13277 | 0.33147 |
| 0.6 | 0.05974 | 0.19894 | 0.33008 |
| 0.8 | 0.10611 | 0.26471 | 0.32739 |
| 1.0 | 0.16557 | 0.32979 | 0.32301 |
| 1.2 | 0.23795 | 0.39378 | 0.31659 |
| 1.4 | 0.32298 | 0.45627 | 0.30787 |
| 1.6 | 0.42032 | 0.51676 | 0.29667 |
| 1.8 | 0.52952 | 0.57477 | 0.28293 |
| 2.0 | 0.65003 | 0.62977 | 0.26675 |
| 2.2 | 0.78120 | 0.68132 | 0.24835 |
| 2.4 | 0.92230 | 0.72899 | 0.22809 |
| 2.6 | 1.07252 | 0.77246 | 0.20646 |
| 2.8 | 1.23099 | 0.81152 | 0.18401 |
| 3.0 | 1.39682 | 0.84605 | 0.16136 |
| 3.2 | 1.56911 | 0.87609 | 0.13913 |
| 3.4 | 1.74696 | 0.90177 | 0.11788 |
| 3.6 | 1.92954 | 0.92333 | 0.09809 |
| 3.8 | 2.11605 | 0.94112 | 0.08013 |
| 4.0 | 2.30576 | 0.95552 | 0.06424 |
| 4.2 | 2.49806 | 0.96696 | 0.05052 |
| 4.4 | 2.69238 | 0.97587 | 0.03897 |
| 4.6 | 2.88826 | 0.98269 | 0.02948 |
| 4.8 | 3.08534 | 0.98779 | 0.02187 |
| 5.0 | 3.28329 | 0.99155 | 0.01591 |
| 5.2 | 3.48189 | 0.99425 | 0.01134 |
| 5.4 | 3.68094 | 0.99616 | 0.00793 |
| 5.6 | 3.88031 | 0.99748 | 0.00543 |
| 5.8 | 4.07990 | 0.99838 | 0.00365 |
| 6.0 | 4.27964 | 0.99898 | 0.00240 |
| 6.2 | 4.47948 | 0.99937 | 0.00155 |
| 6.4 | 4.67938 | 0.99961 | 0.00098 |
| 6.6 | 4.87931 | 0.99977 | 0.00061 |
| 6.8 | 5.07928 | 0.99987 | 0.00037 |
| 7.0 | 5.27926 | 0.99992 | 0.00022 |
| 7.2 | 5.47925 | 0.99996 | 0.00013 |
| 7.4 | 5.67924 | 0.99998 | 0.00007 |
| 7.6 | 5.87924 | 0.99999 | 0.00004 |
| 7.8 | 6.07923 | 1.00000 | 0.00002 |
| 8.0 | 6.27923 | 1.00000 | 0.00001 |
| 8.2 | 6.47923 | 1.00000 | 0.00001 |
| 8.4 | 6.67923 | 1.00000 | 0.00000 |
| 8.6 | 6.87923 | 1.00000 | 0.00000 |
| 8.8 | 7.07923 | 1.00000 | 0.00000 |




Predicted and observed velocity profiles for tangential laminar flow along a flat plate. The solid line represents the solution of equation:

$$\frac{\partial \psi}{\partial y} \cdot \frac{\partial^2 \psi}{\partial x \partial y} + \left[-\frac{\partial \psi}{\partial x}\right]\left[\frac{\partial^2 \psi}{\partial y^2}\right] = \upsilon \left[\frac{\partial^3 \psi}{\partial y^3}\right]$$



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# Integral Methods

Integral methods provide the engineer with a rapid and fairly accurate method of estimating transfer rates in boundary layers.

The method is as follows:

(i) Express the boundary layer equation in integral form.

(ii) Guess a velocity distribution; this allows the integral equation to be solved for the boundary layer thickness. The result is found to be insensitive to the assumed velocity distribution.

## Boundary layer equation

$$\int_0^\infty [u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}]dy = \int_0^\infty [-\frac{1}{\rho}\left(\frac{\partial u}{\partial x}\right)_\infty + \nu\frac{\partial^2 u}{\partial y^2}]dy$$

But

$$\frac{\partial(u.v)}{\partial y} = v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y} = v\frac{\partial u}{\partial v} - u\frac{\partial u}{\partial x} \qquad \text{(continuity)}$$

Using Bernoulli's equation for the flow outside the boundary layer:

$$\frac{1}{2}\rho.u_i^{\,2} + \mathscr{P} = const \qquad \mathscr{P} = p_0 + \rho g y$$

$$-\frac{1}{\rho}\left(\frac{\partial \mathscr{P}}{\partial x}\right)_\infty = u_\infty \frac{du_\infty}{dx}$$

Substituting in equation (1)

$$\int_0^\infty [2u\frac{\partial u}{\partial x} + \frac{\partial (uv)}{\partial y}]dy = \int_0^\infty [u_\infty \frac{du_\infty}{dx} + \frac{\partial}{\partial y}\left(\nu \frac{\partial u}{\partial y}\right)]\, dy$$

Integrating

$$\int_0^\infty 2u\frac{\partial u}{\partial x}dy + u.v\,|_0^\infty = \int_0^\infty u_\infty \frac{du_\infty}{dx}dy + \nu \frac{\partial u}{\partial y}\,|_0^\infty$$





Boundary conditions

$$u\,|_0 = v\,|_0 = 0 \quad ; \quad v\,|_\infty = \int_0^\infty \frac{\partial v}{\partial y}dy = \int_0^\infty -\frac{\partial u}{\partial x}dy \quad ;$$

$$\frac{\mu}{\rho}\frac{\partial u}{\partial y}\,|_0 = \frac{\tau_0}{\rho} \quad ; \quad \frac{\partial u}{\partial y}\,|_\infty = 0$$

Substituting the boundary conditions in equation (3),

$$\int_0^\infty [2u\frac{\partial u}{\partial x} - u_\infty \frac{\partial u}{\partial x} - u_\infty \frac{du_\infty}{dx}].dy = -\frac{\tau_0}{\rho}$$

Regrouping,

$$\int_0^\infty [\frac{\partial}{\partial x}(u^2 - u.u_\infty) + \frac{du_\infty}{dx}(u - u_\infty)].dy = -\frac{\tau_0}{\rho}$$

Which can be written:

$$\frac{\partial}{\partial x}\left\{u_\infty^2 \int_0^\infty \frac{u}{u_\infty}[1-\frac{u}{u_\infty}].dy\right\} + u_\infty \frac{du_\infty}{dx}\int_0^\infty \left[1-\frac{u}{u_\infty}\right].dy = \frac{\tau_0}{\rho}$$

This is von Karman's Integral equation:

Putting $$\theta = \int_0^\infty \frac{u}{u_\infty}\left[1-\frac{u}{u_\infty}\right].dy$$

And $$\delta^* = \int_0^\infty \left[1-\frac{u}{u_\infty}\right].dy$$

The integral equation may be written:

$$\frac{\partial}{\partial x}\left[u_\infty^2\theta\right] + u_\infty \frac{du_\infty}{dx}\delta^* = \frac{\tau_0}{\rho}$$

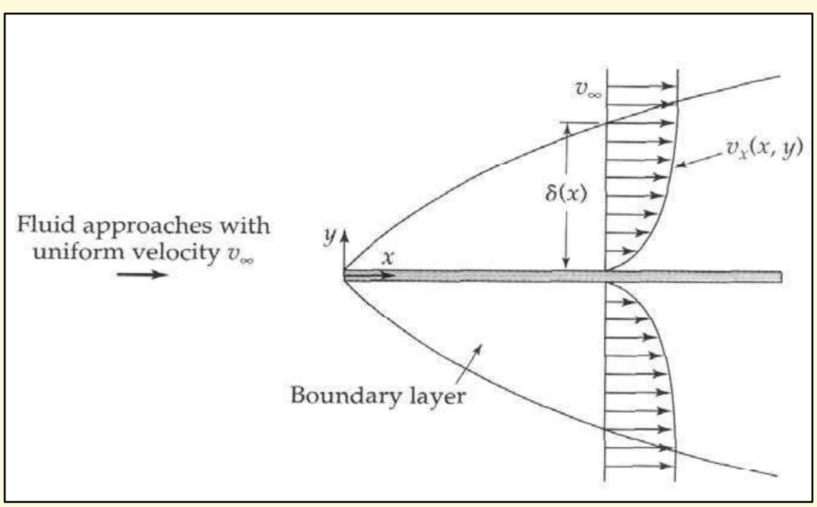




## Solution for Flow over a flat plate

The Von Karman integral equation is:

$$\frac{\partial}{\partial x}\left\{u_\infty^2 \int_0^\infty \frac{u}{u_\infty}[1-\frac{u}{u_\infty}].dy\right\} + u_\infty \frac{du_\infty}{dx}\int_0^\infty \left[1-\frac{u}{u_\infty}\right].dy = \frac{\tau_0}{\rho}$$



Guess velocity profile

(i) $\tilde{u} = \dfrac{u}{u_\infty} = f\left(\dfrac{y}{\delta}\right) = f(\eta)$

(ii) $\tilde{u} = a + b\eta + c\eta^2 + d\eta^3 + e\eta^4 + \ldots$

Boundary conditions

At

$\eta = 0,\quad \tilde{u} = 0,\quad \tilde{u}'' = 0$

$\eta = 1,\quad \tilde{u} = 1,\quad \tilde{u}' = 0$

Hence

$$\tilde{u} = \frac{3}{2}\eta - \frac{1}{2}\eta^3$$





Evaluating the terms of equation (1):

(i) $$\int_0^\delta \frac{u}{u_\infty}\left[1 - \frac{u}{u_\infty}\right].dy = \int_0^\delta \tilde{u}\,(1 - \tilde{u})dy$$

$$= \int_0^1 \left(\frac{3}{2}\eta - \frac{\eta^3}{2}\right)\left(1 - \frac{3\eta}{2} + \frac{\eta^3}{2}\right).\delta.d\eta$$

$$= \frac{39}{280}\delta$$

(ii) $$\frac{\tau_0}{\rho} = \nu\frac{\partial u}{\partial y}\Big|_{y=0} = \nu\left(\frac{du}{d\eta}\right)_{\eta=0}\left(\frac{\partial\eta}{\partial y}\right)$$

$$= \nu\frac{u_\infty}{\delta}\left(\frac{3}{2}\right)$$

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Substituting in equation (1):

$$u_\infty^2 \frac{d}{dx}\left[\frac{39\,\delta}{280}\right] = \frac{3}{2}\frac{\nu.u_\infty}{\delta}$$

$$\int \delta.d\delta = \frac{3}{2}\frac{280}{39}\frac{\nu}{u_\infty}\int dx$$

$$\delta = \left(\frac{3\times 280}{39}\frac{\nu.x}{u_\infty}\right)^{1/2} = 4.64\sqrt{\frac{\nu.x}{u_\infty}}$$

Also, from (4b):

$$\frac{\tau_0}{\frac{1}{2}\rho.u_\infty^2} = \frac{3}{2}\frac{\nu.u_\infty}{\delta}.\frac{1}{\frac{1}{2}u_\infty^2} = \frac{3\nu}{u_\infty}.\frac{u_\infty^{\frac{1}{2}}}{4.64\,(\nu.x)^{\frac{1}{2}}}$$

$$= 0.646\ \mathrm{Re}_x^{-\frac{1}{2}}$$

$$\frac{F_D}{W} = \int_0^L \tau_0.dx = 0.646\ u_\infty^{\frac{3}{2}}.L^{\frac{1}{2}}.\rho.\nu^{\frac{1}{2}}$$

These results should be compared with those of the exact solution.



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## Note on boundary layer thickness

### Boundary layer thickness

Thickness of the narrow region near a solid surface where the velocity is less than some arbitrary fraction of the free stream velocity (usually 99%)

Then with $\frac{df}{d\eta} = u' = \frac{u}{u_\infty} = 0.99$

From Howarth $\eta = \delta\sqrt{\frac{u_\infty}{\nu.x}} = 5.5$ $\qquad \delta = 5.5\sqrt{\frac{\nu.x}{u_\infty}}$

### Displacement thickness

The distance, normal to the surface, by which the external flow is displaced as a result of reduced flow in the boundary layer. i.e. the distance by which a stream line is displaced.

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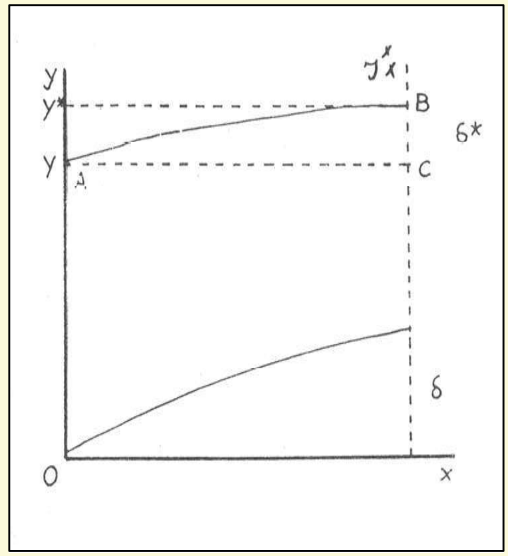
Consider the streamline AB from the point Y on the OY axis.

BC= $\delta^*$ is the displacement thickness.

{mass flow rate across AO} =
{mass flow rate across BX}

$$\delta W \int_0^Y \rho . u_\infty . dy = \delta W \int_0^{Y^*} \rho . u . dy$$

Where $\delta W$ is the width of flow considered.



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Hence $$\int_0^Y \rho . u_\infty . dy = \int_0^Y \rho . u . dy + \int_Y^{Y^*} \rho . u . dy$$

$$Lim Y \rightarrow \infty , u \rightarrow u_\infty \quad \int_0^Y u_\infty . dy = \int_0^\infty u_\infty . dy = \int_0^\infty u . dy + u_\infty . \delta^*$$

$$\delta^* = \int_0^\infty \left\{ 1 - \frac{u}{u_\infty} \right\} . dy = \int_0^\infty \left[ 1 - \frac{df}{d\eta} \right] \sqrt{\frac{\nu . x}{u_\infty}} d\eta$$

From Howarth's Table

$$\int_0^\infty \left[ 1 - \frac{df}{d\eta} \right] . d\eta = 1.72 \qquad \delta^* = 1.72 \sqrt{\frac{\nu . x}{u_\infty}} = \frac{\delta}{3}$$

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## Momentum thickness

The distance in the free stream, normal to the surface, across which rate of flow of momentum is equal to the rate of loss of momentum in the boundary layer to the same downstream coordinate.
Since

(Mass flow rate across AO) = (Mass flow rate across BX)

$$\text{Rate of loss of momentum} = \left( u_\infty \int_0^{Y^*} \rho . u . dy - \int_0^{Y^*} \rho . u^2 . dy \right)$$

In the B.L. over [O, X] / unit width
Rate of flow of momentum / unit width
Across distance $\theta$ in free stream $= \rho . u_\infty^2 . \theta$



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Further as $Y \rightarrow \infty , u \rightarrow u_\infty$

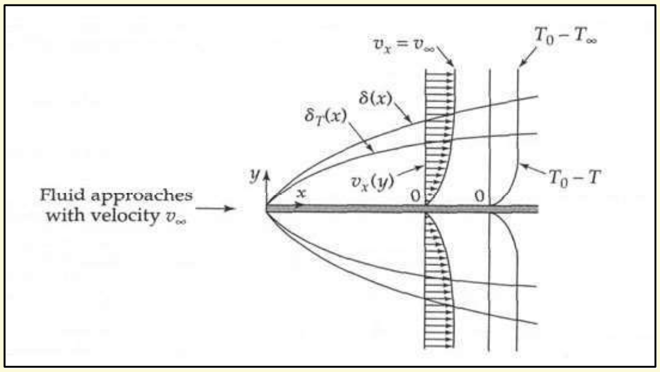
Hence $$\rho . u_\infty^2 \theta = \int_0^\infty \rho . u_\infty^2 \frac{u}{u_\infty} \left\{ 1 - \frac{u}{u_\infty} \right\} dy$$

Then $$\theta = \int_0^\infty \frac{u}{u_\infty} \left\{ 1 - \frac{u}{u_\infty} \right\} dy = \sqrt{\frac{\nu . x}{u_\infty}} \int_0^\infty \frac{df}{d\eta} \left\{ 1 - \frac{df}{d\eta} \right\} d\eta$$

From Howrth's Tables $$\int_0^\infty \frac{df}{d\eta} \left\{ 1 - \frac{df}{d\eta} \right\} d\eta = 0.66$$

$$\theta = 0.66 \sqrt{\frac{\nu . x}{u_\infty}} \cong \frac{\delta}{8}$$

# Heat transfer in laminar boundary layers




Consider flow over a flat plate at zero angle of incidence. A boundary layer will be set up in the manner that has been discussed. Consider now heat transfer from a stream of temperature $T_\infty$ to a wall having constant temperature $T_W$.

## Assumptions:

(i) the temperature difference $T_\infty - T_W$ is mush greater than any temperature change due to viscous dissipation. (Reasonable for gases and liquids of low viscosity)

(ii) Fluid properties invariant with temperature.

The energy equation reduces to:



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$$\upsilon\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \qquad \text{Where} \qquad \alpha = \frac{k}{\rho C_p}$$

Compare the momentum equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

The equations are identical in form. This suggests an analogous treatment of heat transfer. A quantity $\delta_T$, the thermal boundary layer thickness, is defined as that region near the solid boundary where all the major temperature changes occur.

Defining $$\theta = \frac{(T - T_W)}{(T_\infty - T_W)} \quad ; \quad \text{and using} \quad f' = \frac{u}{u_\infty}$$

The above equations become

$$u\frac{\partial f'}{\partial x} + v\frac{\partial f'}{\partial y} = \nu\frac{\partial^2 f'}{\partial y^2} \quad ; \quad u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \alpha\frac{\partial^2 \theta}{\partial y^2}$$

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Consider now the special case of $\upsilon = \alpha$. The dimensionless groups $\theta$ and $f'$ become interchangeable in the equations. Hence the velocity and temperature profiles are identical. Thus

$$\delta = \delta_T$$

The thermal boundary layer has the same thickness as the momentum boundary layer. This result is of great importance in flow of gases since $\alpha \approx \nu$.

If $\nu \rangle \alpha$ , momentum transport occurs more readily. Hence the momentum boundary layer is thicker $\delta \rangle \delta_T$

$\nu \langle \alpha$ , momentum transport occurs less readily $\delta \langle \delta_T$

Thus the Prantle Number, given by $Pr = \frac{\nu}{\alpha}$ is an important group.

$$Pr = \frac{\text{Coefficent of momentum transport}}{\text{Coefficient of thermal transport}}$$



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## Integral Treatment of the Thermal Boundary Layer

Energy Equation (Integral form)

$$\frac{d}{dx}\int_0^\infty u\left(T_\infty - T\right)dy = \alpha \left.\frac{\partial T}{\partial y}\right|_{y=0}$$

Guess velocity and temperature profiles in boundary layers:

$$\tilde{u}(\eta) = \tilde{u} = \frac{u}{u_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \quad \text{(See momentum eqn.)}$$

Assume temperature profile is similar:

$$\theta = \frac{T - T_W}{T_\infty - T_W} = \frac{3}{2}\left(\frac{y}{\delta_T}\right) - \frac{1}{2}\left(\frac{y}{\delta_T}\right)^3 \quad ; \quad \delta \neq \delta_T$$

Now $q_x = -k \left.\frac{\partial T}{\partial y}\right|_{y=0} = h(T_W - T_\infty)$

$$\frac{h}{k} = \left.\frac{\partial \theta}{\partial y}\right|_{y=0} = \frac{3}{2\delta_T}$$

$$\frac{d}{dx}\int_0^\infty u_\infty \tilde{u}(1-\theta)\,dy = \alpha\left(\frac{\partial \theta}{\partial y}\right)_{y=0}$$ Since $T_\infty - T = (T_\infty - T_W)(1-\theta)$

**i)** For $\delta \rangle \delta_T$ , $\int_0^\infty = \int_0^{\delta_T}$ because $(1-\theta)$ is zero for $y \rangle \delta_T$

**ii)** For $\delta \langle \delta_T$ , $\int_0^\infty = \int_0^{\delta_T} = \int_0^{\delta} + \int_\delta^{\delta_T}$

In either case integration gives:

$$\frac{d}{dx}\left(\frac{3u_\infty \delta_T^2}{20\,\delta}\right) = \frac{3\alpha}{2\delta_T}$$





Assuming that $\frac{\delta_T}{\delta} = \lambda$

$$\frac{d}{dx}\left(\frac{3u_\infty}{20}\lambda^2\delta\right) = \frac{3\alpha}{2\lambda\delta}$$ $u_\infty, \lambda$ const .

$$\lambda^3\delta\frac{d\delta}{dx} = \frac{10\alpha}{u_\infty} \Rightarrow \lambda^3\delta^2 = \frac{20\alpha x}{u_\infty}$$ since $\delta = 0, \quad x = 0$

Momentum boundary layer analysis gave $\delta = 4.64\sqrt{\frac{\nu x}{u_\infty}}$

$$\lambda = \frac{\delta_T}{\delta} = \left(\frac{20}{4.64^2}\frac{\alpha}{\nu}\right)^{\frac{1}{3}}$$

$$\frac{\delta}{\delta_T} = 1.025 Pr^{\frac{1}{3}} \qquad \left[Pr = \frac{\nu}{\alpha}\right]$$