Now $q_x = -k \frac{\partial T}{\partial y}\bigg|_{y=0} = h(T_W - T_\infty)$

$$\frac{h}{k} = \frac{\partial \theta}{\partial y}\bigg|_{y=0} = \frac{3}{2\delta_T}$$

$$\frac{d}{dx}\int_0^\infty u_\infty \tilde{u}(1-\theta)\,dy = \alpha\left(\frac{\partial \theta}{\partial y}\right)_{y=0}$$ Since $T_\infty - T = (T_\infty - T_W)(1-\theta)$

i) For $\delta \rangle \delta_T$, $\int_0^\infty = \int_0^{\delta_T}$ because $(1-\theta)$ is zero for $y \rangle \delta_T$

ii) For $\delta \langle \delta_T$, $\int_0^\infty = \int_0^{\delta_T} = \int_0^{\delta} + \int_\delta^{\delta_T}$

In either case integration gives:

$$\frac{d}{dx}\left(\frac{3u_\infty \delta_T^2}{20\,\delta}\right) = \frac{3\alpha}{2\delta_T}$$

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Assuming that $\frac{\delta_T}{\delta} = \lambda$

$$\frac{d}{dx}\left(\frac{3u_\infty}{20}\lambda^2\delta\right) = \frac{3\alpha}{2\lambda\delta} \qquad u_\infty, \lambda \quad \text{const .}$$

$$\lambda^3\delta\frac{d\delta}{dx} = \frac{10\alpha}{u_\infty} \quad \Rightarrow \quad \lambda^3\delta^2 = \frac{20\alpha x}{u_\infty} \quad \text{since} \quad \delta = 0, \quad x = 0$$

Momentum boundary layer analysis gave $\delta = 4.64\sqrt{\frac{\nu x}{u_\infty}}$

$$\lambda = \frac{\delta_T}{\delta} = \left(\frac{20}{4.64^2}\frac{\alpha}{\nu}\right)^{\frac{1}{3}}$$

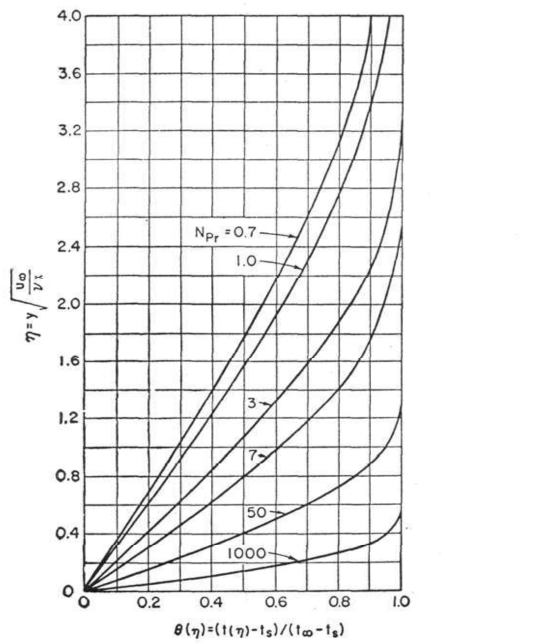
$$\frac{\delta}{\delta_T} = 1.025 Pr^{\frac{1}{3}} \qquad \left[Pr = \frac{\nu}{\alpha}\right]$$

Heat transfer coefficient.

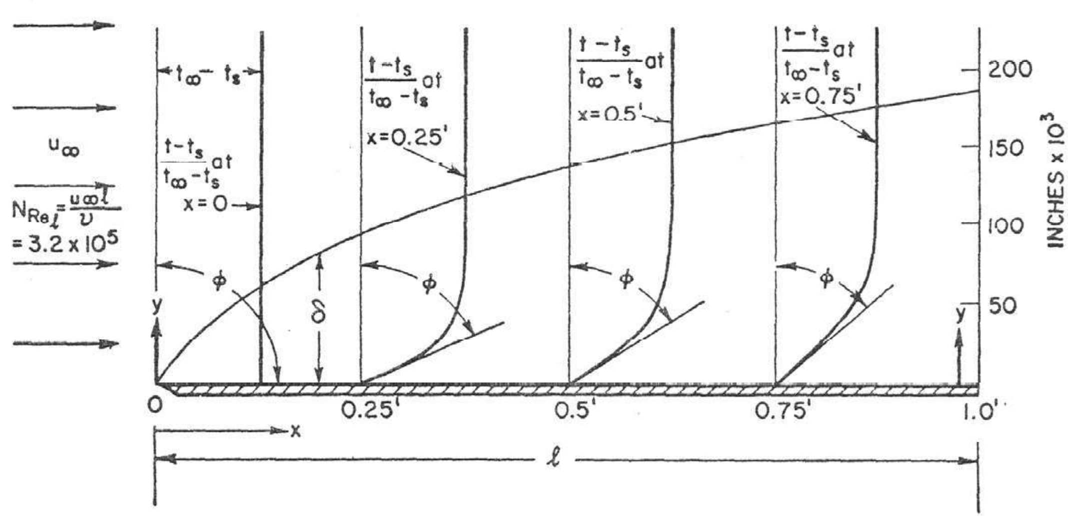
$$h/k = \frac{3}{2\delta_T} = \frac{3}{2\delta} 1.025\,\mathrm{Pr}^{\frac{1}{3}}$$

$$Nu_x = \frac{hx}{k} = \frac{3}{2}\left(\frac{1.025}{4.64}\right)\left(\frac{u_\infty}{\nu x}\right)^{\frac{1}{2}} x\,\mathrm{Pr}^{\frac{1}{3}}$$

$$Nu_x = 0.332\ \mathrm{Re}_x^{\frac{1}{2}}\ \mathrm{Pr}^{\frac{1}{3}}$$



Temperature distributions in the laminar boundary layer of a flat plate (frictional heat neglected).



Temperature distributions in the boundary layer along a flat plate ($\tan\phi = \partial t/\partial y$ at the surface).

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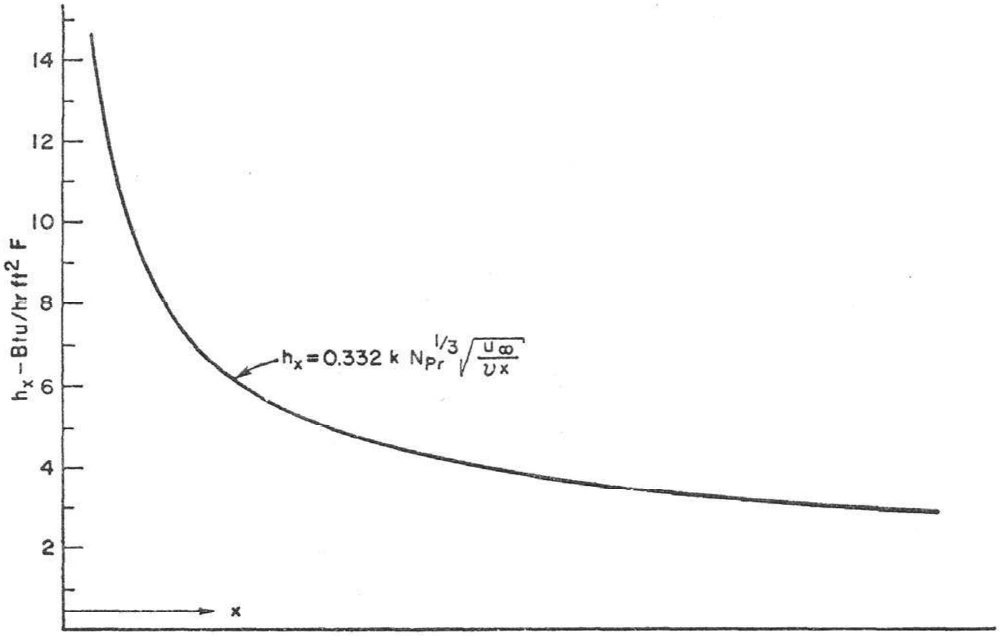

Variation of local heat transfer coefficient along flat plate in Fig. ($N_{Pr} = 1$ and $k = 0.0145$ Btu/hr ft F).

Advanced Fluid Mechanics

Integral Methods in Convective Mass Transfer

Diffusion Equation:
$$u\frac{\partial C_A}{\partial x} + v\frac{\partial C_A}{\partial y} = D_{AB}\frac{\partial^2 C_A}{\partial y^2}$$

Integrate normal to the plate:

$$\int_0^\infty \left[u\frac{\partial C_A}{\partial x} + v\frac{\partial C_A}{\partial y}\right]dy = \int_0^\infty D_{AB}\frac{\partial^2 C_A}{\partial y^2}dy$$

Now

$$\frac{\partial}{\partial y}(vC_A) = v\frac{\partial C_A}{\partial y} + C_A\frac{\partial v}{\partial y} = v\frac{\partial C_A}{\partial y} - C_A\frac{\partial u}{\partial x} \qquad \text{(From continuity)}$$

$$\int_0^\infty \left[u\frac{\partial C_A}{\partial x} + \frac{\partial}{\partial y}(vC_A) + C_A\frac{\partial u}{\partial x}\right]dy = \int_0^\infty \frac{\partial}{\partial y}\left(D_{AB}\frac{\partial C_A}{\partial y}\right)dy$$

$$\int_0^\infty \frac{\partial}{\partial x}(u\,C_A)dy + \left[v\,C_A\right]_0^\infty = D_{AB}\left[\frac{\partial C_A}{\partial y}\right]_0^\infty$$

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But $$[C_A v]_0^\infty = [C_A v]_\infty = -C_A \int_0^\infty \frac{\partial u}{\partial x} dy$$ (Continuity)

$$\int_0^\infty \frac{\partial}{\partial x}[u(C_A - C_{A\infty})]dy = -D_{AB}\left[\frac{\partial C_A}{\partial y}\right]_{y=0} \qquad if \qquad \left.\frac{\partial C_A}{\partial y}\right|_\infty = 0$$

$$\frac{d}{dx}\left[\int_0^{\delta_c} u(C_A - C_{A\infty})\right]dy = -D_{AB}\left[\frac{\partial C_A}{\partial y}\right]_{y=0}$$

Since $C_A = C_{A\infty} \quad for \quad y > \delta_c$

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Consider the soluble wall problem treated earlier.

Velocity profile: $$u = \frac{\rho g h^2}{\mu}\left[\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^2\right]$$

$$u \approx \frac{\rho g h^2}{\mu}\frac{y}{h} = \alpha y \qquad for \qquad y \ll h$$

Within concentration boundary limit:

$$C_A|_{y=0} = C_{A0} \quad : \quad C_A|_{y\to\infty}$$

$$\left.\frac{\partial^2 C_A}{\partial y^2}\right|_{y=0} = 0$$

$$\left.\frac{\partial^2 C_A}{\partial y^2}\right|_\infty = 0 \quad : \quad \left.\frac{\partial C_A}{\partial y}\right|_\infty = 0$$

Assuming the form of the concentration profile it is possible to calculate the rate of mass transfer. As with all integral methods the result is relatively insensitive to the guess.

Boundary conditions:

$$At \quad \eta = 0 \quad ; \quad C_A = C_{A0} \quad , \quad \frac{\partial^2 C_A}{\partial \eta^2} = 0$$

$$At \quad \eta = 1 \quad ; \quad C_A = C_{A0} \quad , \quad \frac{\partial C_A}{\partial y} = \frac{\partial^2 C_A}{\partial \eta^2} = 0$$

Then $a = -2 \quad , \quad b = 0 \quad , \quad c = 2 \quad , \quad d = -1$

$$\frac{C_A - C_{A\infty}}{C_{A0} - C_{A\infty}} = 1 - 2\eta + 2\eta^3 - \eta^4$$

$$\frac{d}{dx}\left[\int_0^1 \alpha \delta_c^2 (C_{A0} - C_{A\infty})\eta\left(1 - 2\eta + 2\eta^3 - \eta^4\right) d\eta\right] = \frac{2D_{AB}(C_{A0} - C_{A\infty})}{\delta_c}$$

Etc. to give $\delta_c = \sqrt[3]{\dfrac{9 D_{AB} x}{0.2\alpha}}$

$$N_A = \frac{D_{AB}(C_{A0} - C_{A\infty})}{0.855 \sqrt[3]{9 D_{AB} \dfrac{x}{\alpha}}}$$

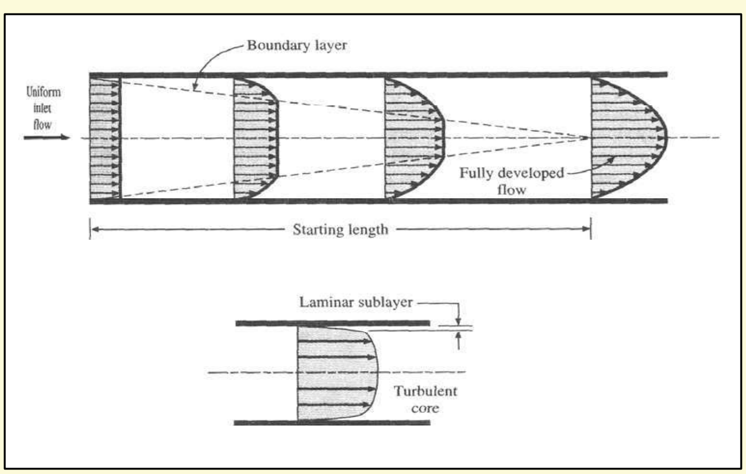
Compare both results with exact similarity solution

$$N_A = \frac{D_{AB}(C_{A0} - C_{A\infty})}{0.855 \sqrt[3]{9 D_{AB} \dfrac{x}{\alpha}}}$$

Boundary Layer Theory- Conclusion

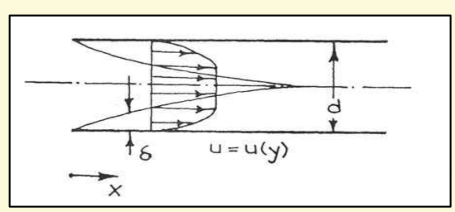
Typical Applications of Analysis

(a) Momentum Transfer: Entrance region in pipes

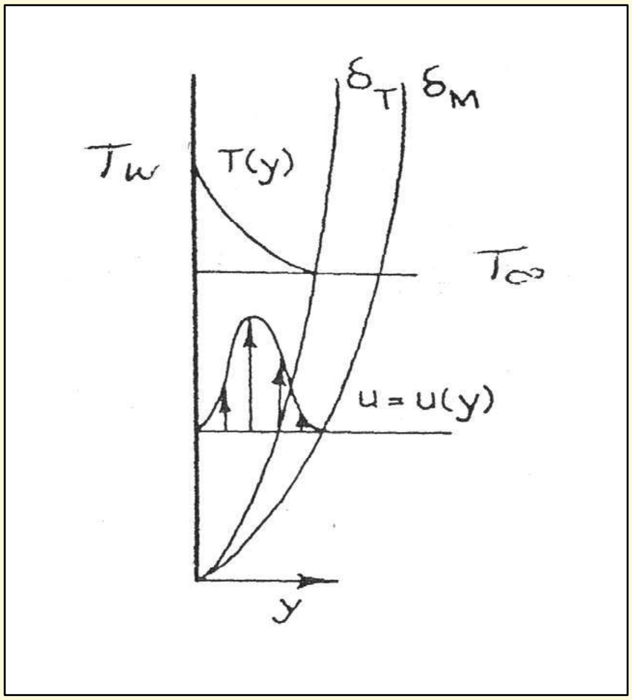


$\delta = \delta(x)$

When $2\delta = d$
Boundary layers combine
defines entry region



(b) Heat transfer: Natural convection



$\delta_M \rangle \delta_T$

$u = 0 \quad at \quad y = 0$

$u \to 0 \quad as \quad y \to \infty$

u Passes through maximum

T is a maximum at wall
Heats adjacent fluid
Fluid becomes less dense and rises.

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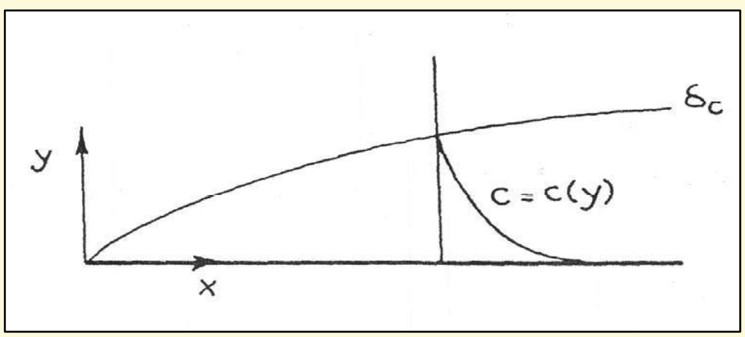
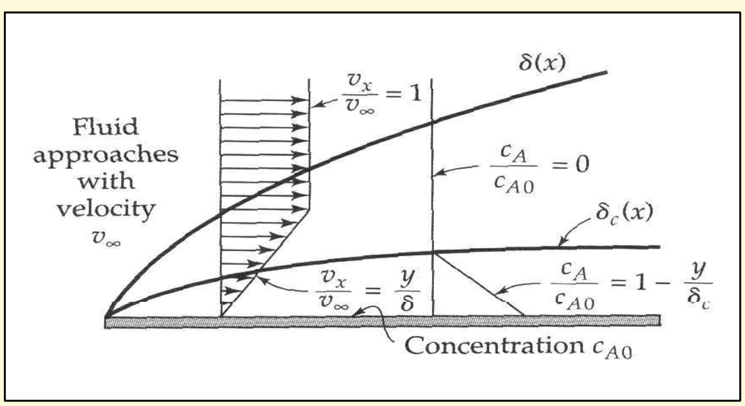
(c) Mass Transfer: Evaporation from ponds





Sharif University of Technology
Chemical & Petroleum Engineering Department

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Results of analysis

$$\frac{f}{2} = 0.332\ \mathrm{Re}_x^{-\frac{1}{2}} = \frac{hx}{k\,\mathrm{Re}_x\,\mathrm{Pr}^{1/3}} = \frac{k_c x}{cD_{AB}\,\mathrm{Re}_x\mathrm{Sc}^{1/3}}$$

$$\boxed{\frac{f}{2} = \frac{h}{\rho C_p U_\infty}\mathrm{Pr}^{\frac{1}{3}} = \frac{k_c}{U_\infty}\mathrm{Sc}^{\frac{1}{3}}}$$ (Chilton Colburn Analogy)

Expect this analogy to be accurate in developing flows.

Advanced Fluid Mechanics

Dimensionless group in convective Heat and Mass transfer

$$\mathrm{Pr} = \frac{\text{Coefficient of momentum transport}}{\text{Coefficient of thermal transport}} = \frac{\nu}{\alpha}$$

$$\mathrm{Sc} = \frac{\text{Coefficient of momentum transport}}{\text{Coefficient of mass transport}} = \frac{\nu}{\mathrm{D_{AB}}}$$

For water $\mathrm{Pr} \approx 10$ $\mathrm{Sc} \approx 1000$

Hence:

momentum transpor 〉 thermal transport 〉 mass transpor

$$\delta \quad \rangle \quad \delta_H \quad \rangle \quad \delta_M$$

$$\{\delta \approx 2\delta_H \approx 10\,\delta_M\}$$

$$Nu = \frac{hL}{k} = \frac{-\left(\frac{\partial T}{\partial y}\right)_{y=0}}{-(T_\infty - T_W)/L} = \frac{\text{Temperature gradient at wall}}{\text{Reference temperature gradient}}$$

$$Sh = \frac{kL}{D_{AB}} = \frac{-\left(\frac{\partial C}{\partial y}\right)_{y=0}}{-(C_{A\infty} - C_{AW})/L} = \frac{\text{Conc. gradient at wall}}{\text{Reference conc. gradient}}$$

At same Result:

$$Nu = Sh/5 \qquad \{ Sh = \frac{k_y L}{cD_{AB}} \}$$

FLOW AROUND SPHERE

Presented by:

Prof. D.Rashtchian

Now remember $\omega_i = \varepsilon_{ijk} \dfrac{\partial u_k}{\partial x_j} = 0$ for irrotational flow.

Hence if we can find a function Φ where $u_k = -\dfrac{\partial \Phi}{\partial x_k}$ then $\omega_i = 0$ and the flow is necessarily irrotational and (2) is satisfied.

Substituting for u_k into the continuity equation

$$\frac{\partial u_k}{\partial x_k} = -\frac{\partial^2 \Phi}{\partial x_k \partial x_k} = 0 \quad \text{or} \quad \nabla^2 \Phi = 0 \quad ; \quad \nabla^2 \equiv \frac{\partial^2}{\partial x_i \partial x_i}$$

i.e. Laplace's Equation

$$\frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_2^2} + \frac{\partial^2 \Phi}{\partial x_3^2} = 0$$

Thus the determination of the velocity profile requires the solution of laplace's equation with appropriate b.c.'s

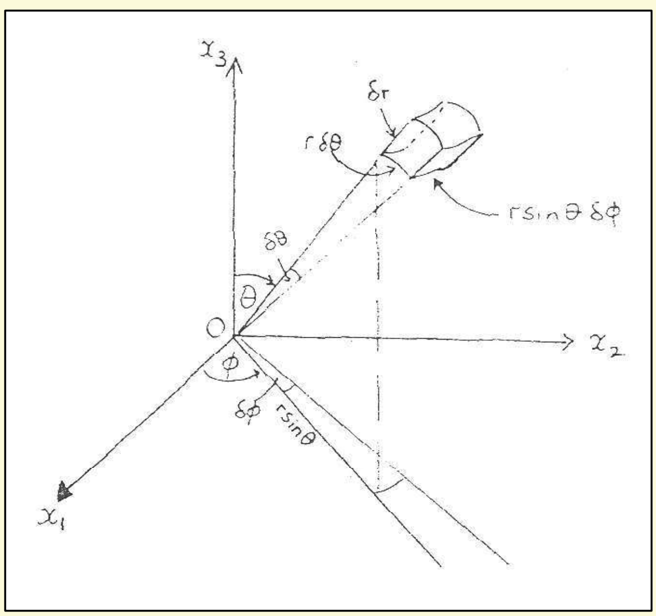
(i) at stationary solid surfaces $u_n = -\dfrac{\partial \Phi}{\partial x_n} = 0$

(normal component of velocity).

(ii) far from solid surface $(u_i)_\infty = (\dfrac{\partial \Phi}{\partial x_i})_\infty$

(main stream velocity)

Nothing has been said about the no slip condition and this is generally not satisfied by potential flows. The pressure distribution is determined by Bernoulli's equation.



Sphere, radius a, situated at O.

The velocity potential Φ satisfies Laplaces's equation $\nabla^2 \Phi = 0$

c.f. $\partial^2 \Phi / \partial x_j^2 = 0$ for Cartesian co-ordinates.

Flow is symmetrical about the x_3-axis, no rotational component i.e. $\partial / \partial \phi = 0$; $U_\phi = 0$.

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i.e. any tracer-marked fluid approaching the sphere is displaced radially but remains in the same plane w.r.t.Ox_3.
In spherical co-ordinates, Laplace's Equation is (B.S.L. p740(B))

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{\partial\Phi}{\partial r}\right\}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial\Phi}{\partial\theta}\right\}=0 \qquad (1)$$

In addition the velocity components are

$$U_r=-\frac{\partial\Phi}{\partial r}$$

$$U_\theta=-\frac{1}{r}\frac{\partial\Phi}{\partial\theta}$$

c.f. $$U_i=-\frac{\partial\Phi}{\partial x_i}$$

Boundary conditions:

(i) $U_r=0$ at $r=a$

(ii) $-\frac{\partial\Phi}{\partial r}=U_r=-U_0\cos\theta$ at $r\to\infty$

(iii) $-\frac{1}{r}\frac{\partial\Phi}{\partial\theta}=U_\theta=-U_0\sin\theta$ at $r\to\infty$

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Boundary conditions suggest a solution of the form (Separation of Variables)

$$\Phi=U_0 f(r)\cos\theta \qquad (2)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{\partial\Phi}{\partial r}\right\}=2U_0\frac{f'(r)}{r}\cos\theta+U_0 f''(r)\cos\theta$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial\Phi}{\partial\theta}\right\}=-2U_0\frac{f(r)}{r^2}\cos\theta$$

Hence $r^2 f''(r)+2rf'(r)-2f(r)=0$ (3)
This is a second order, homogeneous O.D.E.

Using the substitution $r=e^t$ gives a linear equation

$$f''(t)+f'(t)-2f=0 \qquad (4)$$

Which has the solution

$$f=Ae^{-2t}+Be^t$$
$$=\frac{A}{r^2}+Br$$

Using the Boundary Conditions $\Rightarrow A=a^3/2 : B=1$

Hence $$\Phi=U_0\left\{r+\frac{a^3}{2r^2}\right\}\cos\theta \qquad \ldots (5)$$

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Pressure difference around sphere

Applying Bernoulli's equation in free stream

$$u_r = -U_0 \cos\theta$$
$$u_\theta = U_0 \cos\theta$$
$$u_\phi = 0$$
$$\mathscr{P} = \mathscr{P}_0 \qquad \text{(say)}$$

Hence $$\rho\frac{\mathbf{u}\cdot\mathbf{u}}{2} + \mathscr{P} = constant = \rho\frac{U_0^2}{2} + \mathscr{P}_0$$

At the sphere surface

$$u_r = -\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = -U_0\left\{1 - \frac{a^3}{r^2}\right\}\cos\theta = 0$$

$$u_\theta = -\left.\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right|_{r=a} = -\frac{U_0}{r}\left\{r + \frac{a^3}{2r^2}\right\}\sin\theta = \frac{3}{2}U_0\sin\theta$$

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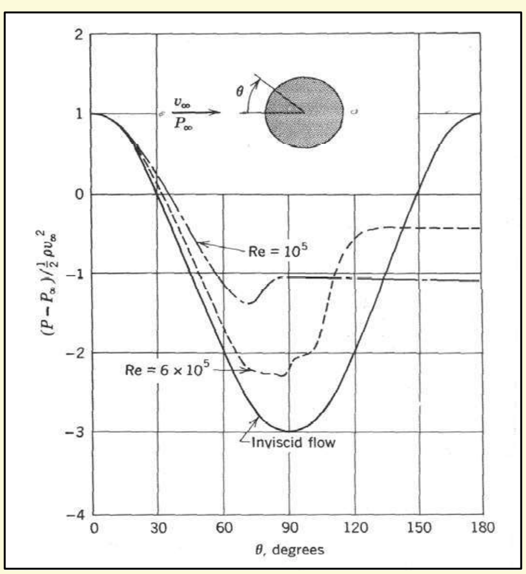
Hence
$$\left\{\frac{P - P_0}{\rho U_0^2/2}\right\} = \left\{1 - \frac{9}{4}\sin^2\theta\right\}$$

Diagram shows form of pressure distribution, compared with experiment.

Ideal Flow Around Spheres



For potential flow, dimensionless pressure around sphere,

$$\left\{\frac{P - P_0}{\rho U_0^2/2}\right\} = (1 - \frac{9}{4}\sin^2\theta)$$

Note:

(i) Improves with Re

(ii) No pressure loss as with real fluids

(iii) Pressure distribution symmetrical.
No drag force on sphere.

(iv) No fractional force on sphere.

Viscous flow around sphere

We study this particular subject as a link between potential flow theory and boundary layer theory.

Analytical

At very low Re, the creeping flow assumption may be made:

Viscous terms >> Inertia terms

$$\frac{\mu U}{L^2} >> \rho \frac{U^2}{L} \qquad \text{i.e.} \qquad \text{Re} << 1.0$$

A solution similar to the one for ideal flow is possible in terms of the stream function (introduced later).

Result of analysis, which is valid up to Re = 1.0, is that resultant force on sphere is due,

(i) partly to fractional effects
(ii) Partly to drag effects

Total Force, F, can be shown to be given by

$F = 6\pi a \mu U_0$ Stoke's Law
(Known as drug coefficient)

Define: f = friction factor
= dimensionless force per unit area acting on sphere

$$= \frac{F}{A_p[\rho U_0^2/2]} = \frac{6\pi a \mu U_0 .2}{\pi a^2(\rho U_0^2)} = \frac{24}{\mathbf{Re}}$$

Dimensional Analysis

Normalization of N-S equation gives:

$$\frac{Du_i'}{Dt'} = \frac{1}{\text{Re}} \frac{\partial^2 u_i'}{\partial x_j' \partial x_j'} - \frac{\partial P'}{\partial x_i'}$$

Solution of this P.D.E will certainly have Re as parameter.

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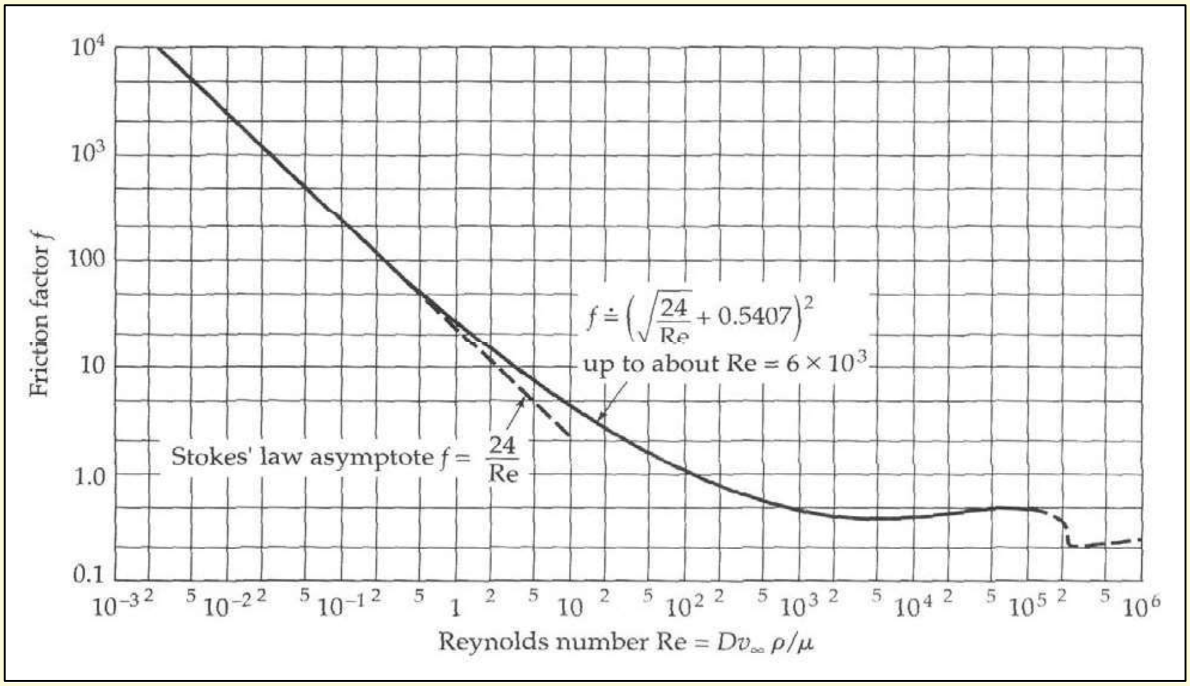
f = f (Re)



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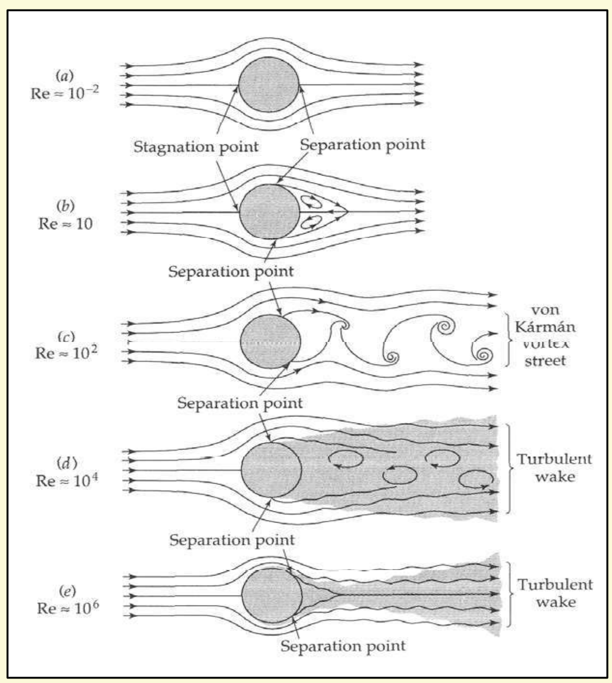
(a) $Re << 1$ (Stokes Regime)
- Analytical solution possible.
- No flow separation

(b) $Re \approx 10$
- A pair of vortex appears behind the cylinder.

(c) $Re \approx 10^2$ (Intermediate Region)
- Von Karman Vortex Street.
- Vortices shed from alternate sides at regular intervals.

(d) $Re \approx 10^4$ (high Re)
- Boundary layer develops on upstream face.
- Breaks away at separation point. This is associated with adverse pressure gradient on the downstream face.

(e) $Re \approx 10^6$
- Turbulence appears upstream of separation point



INVISCID FLUID FLOW

Presented by:

Prof. D.Rashtchian

Fluid rotation at a point

$$\omega_z = \frac{d}{dt}\left(\frac{\alpha+\beta}{2}\right)$$

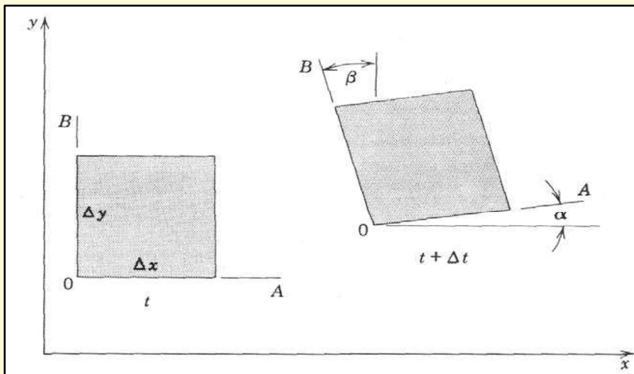


Figure 1 Rotation of a fluid element.

Where the counterclockwise sense is positive

From Figure (1) we see that:

$$\omega_z = \lim_{\Delta x,\Delta y,\Delta z,\Delta t\to 0}\frac{1}{2}\left(\frac{\arctan\left\{\left[\left(v_y|_{x+\Delta x}-v_y|_x\right)\Delta t\right]/\Delta x\right\}}{\Delta t}+\frac{\arctan\left\{-\left[\left(v_x|_{y+\Delta y}-v_x|_y\right)\Delta t\right]/\Delta y\right\}}{\Delta t}\right)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)$$

The subscript z indicates that the rotation is about the z axis.
In the xz and yz planes the rotation at a point is given by

$$\omega_y = \frac{1}{2}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)$$

and

$$\omega_x = \frac{1}{2}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)$$

The rotation at a point is related to the vector cross product of the velocity. As the student may verify,

$$\nabla \times \mathrm{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)e_x + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)e_y + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)e_z$$

And thus

$$\nabla \times \mathrm{v} = 2\omega$$

$$\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}$$

$$\omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}$$

$$\omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

i.e. $\underline{\omega} = \omega_i . e_i$

Components ω_i can be conveniently remembered using:

$$\begin{vmatrix} \omega_1 & \omega_2 & \omega_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix}$$

These quantities are the components of the vorticity vector. It is a measure of the rotational character of the flow about the i-axis.

$\omega_i \neq 0$ Paddle wheel at P will rotate. i.e. rotation flow.

$\omega_i = 0$ all i, irrotational flow.

Using our suffix notation:

$$\omega_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} = \left(\frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} \right) \qquad (j \neq k)$$

Where $\varepsilon_{ijk} = +1$ if i,j,k different and cyclic 1,2,3 / 2,3,1 / 3,1,2

$\varepsilon_{ijk} = -1$ if i,j,k different but not cyclic 1,3,2 / 2,1,3 / 3,2,1

$\varepsilon_{ijk} = 0$ if any of i,j,k same

The stream function

For a two-dimensional, incompressible flow, the continuity equation is

$$\nabla . v = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$\frac{\partial v_y}{\partial y} = -\frac{\partial F}{\partial x} \qquad \text{or} \qquad v_y = -\int \frac{\partial F}{\partial x} dy$$

$$v_x = \frac{\partial \psi}{\partial y}$$

As $\frac{\partial v_x}{\partial x} = -\frac{\partial v_y}{\partial y}$, we may write

$$\frac{\partial v_y}{\partial y} = -\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y} \right) \qquad \text{or} \qquad \frac{\partial}{\partial y}\left(v_y + \frac{\partial \psi}{\partial x} \right) = 0$$

For this to be true in general

$$v_y = -\frac{\partial \psi}{\partial x}$$

$$\psi = \psi(x, y)$$

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

Also,

$$\frac{\partial \psi}{\partial x} = -v_y \quad \text{and} \quad \frac{\partial \psi}{\partial y} = v_x$$

And thus

$$d\psi = -v_y dx + v_x dy \qquad (5)$$

Consider a path in the xy plane such that ψ = constant. Along this path, dψ = 0, and thus equation (5) becomes

$$\left.\frac{dy}{dx}\right|_{\psi = const.} = \frac{v_y}{v_x}$$

The slope of the path ψ = constant is seen to be the same as the slope of a streamline as discussed previously. The function $\psi\ (x, y)$ thus represents the streamlines. The following figure illustrates the streamlines and velocity components for flow about an airfoil.

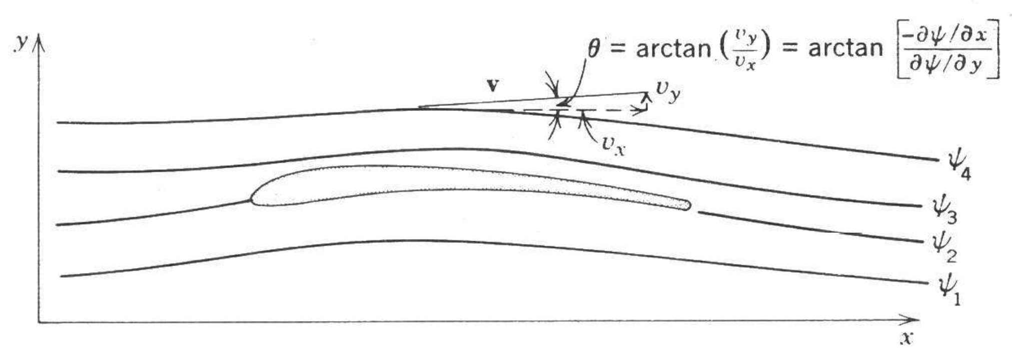


Figure 2 Streamlines and the stream function.

The differential equation which governs ψ is obtained by consideration of the fluid rotation, ω, at a point. In a two-dimensional flow, $\omega_z = \frac{1}{2}\left[\left(\partial v_y/\partial x\right)-\left(\partial v_x/\partial y\right)\right]$, and thus if the velocity components v_y and v_x are expressed in terms of the stream function ψ, we obtain, for an incompressible, steady flow,

$$-2\omega_z = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}$$

When the flow is irrotational equation becomes Laplace's Equation:

$$\nabla^2\Psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

Irrotational flow, the velocity potential

In a two-dimensional irrotational flow $\nabla\times\mathbf{v}=0$, and thus $\partial v_x/\partial y=\partial v_y/\partial x$. the similarity of this equation to the continuity equation suggests that the type of relation used to obtain the stream function may be used again. Note, however, that the order of differentiation is reversed from the continuity equation. If we let $v_x=\partial\phi(x,y)/\partial x$, we observe that

$$\frac{\partial v_x}{\partial y} = \frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial v_y}{\partial x}$$

or

$$\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y} - v_x\right) = 0$$

and for the general case

$$v_y = \frac{\partial\phi}{\partial y}$$

The velocity vector is given by

$$\mathbf{V} = v_x\,\boldsymbol{e}_x + v_y\,\boldsymbol{e}_y + v_z\,\boldsymbol{e}_z = \frac{\partial\phi}{\partial x}e_x + \frac{\partial\phi}{\partial y}e_y + \frac{\partial\phi}{\partial z}e_z$$

and thus, in vector notation,

$$\mathbf{v} = \nabla\phi$$

Closure

In this chapter we have examined potential flow. A short summary of the properties of the stream function and the velocity potential is given below.

Stream function

1. A stream function ψ(x,y) exists for each and every two-dimensional steady, incompressible flow, whether viscous or inviscid.
2. Lines for which ψ(x,y)=constant are streamlines.
3. in cartesian coordinates,

$$v_x = \frac{\partial \psi}{\partial y} \qquad v_y = -\frac{\partial \psi}{\partial x}$$

and in general

$$v_s = \frac{\partial \psi}{\partial n}$$

Where n is 90° counterclockwise from s.

4. The stream function identically satisfies the continuity equation.
5. for an irrotational, steady, incompressible flow,

$$\nabla^2 \psi = 0$$

Velocity potential

1. The velocity potential exists if and only if the flow is irrotational. No other restrictions are required.
2. $\nabla\phi = \mathbf{v}$.
3. For steady, incompressible flow, $\nabla^2\phi = 0$.
4. For steady, incompressible two dimensional flows, lines of constant velocity potential are perpendicular to the streamlines.

The differential equation defining ϕ is obtained from the continuity equation. Considering a steady, incompressible flow, we have $\nabla \cdot v = 0$, thus, using equation for **v**, we obtain:

$$\nabla \cdot \nabla \phi = \nabla^2 \phi = 0$$

Which is again Laplace's equation; this time dependent variable is ϕ. Clearly, ψ and ϕ must be relatd. This relation may be illustrated by a consideration of isolines of Ψ and ϕ. An isoline of is of course a streamline. Along the isolines:

$$d\Psi = \frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial y} dy \quad \text{Or} \quad \left.\frac{dy}{dx}\right|_{\Psi = \cos t.} = \frac{v_y}{v_x}$$

And $$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \qquad \left.\frac{dy}{dx}\right|_{d\phi = 0} = -\frac{v_x}{v_y}$$

Accordingly

$$\left.\frac{dy}{dx}\right|_{\phi = const} = -\left.\frac{1}{dy/dx}\right|_{\Psi = const.}$$

And thus Ψ and ϕ are orthogonal. The orthogonality of stream function and the velocity potential is a useful property, particularly when graphical solutions to equations are employed.

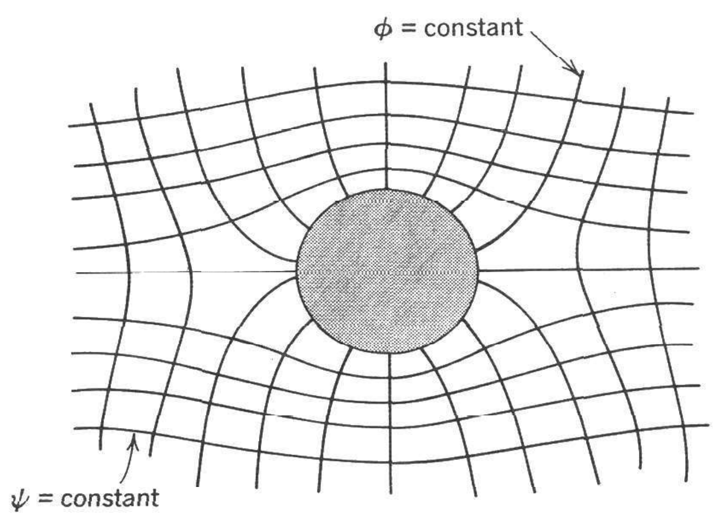


Figure 3 Streamlines and constant velocity potential lines for steady, incompressible, irrotational, inviscid flow about a cylinder.

Advanced Fluid Mechanics

(Bernoulli's Equation)

Navier-Stokes equation for steady flow: $u_j \dfrac{\partial u_i}{\partial x_j} = \nu \dfrac{\partial^2 u_i}{\partial x_j \partial x_j} - \dfrac{1}{\rho}\dfrac{\partial p}{\partial x_i} + g_i$

$$u_j\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) + u_j\left(\frac{\partial u_j}{\partial x_i}\right) = \nu \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) + \nu \frac{\partial^2 u_j}{\partial x_i \partial x_j} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i$$

i) ρ constant ==> By continuity $\dfrac{\partial^2 u_j}{\partial x_i \partial x_j} = \dfrac{\partial}{\partial x_i}\left(\dfrac{\partial u_j}{\partial x_j}\right) = 0$

ii) Rewrite $g_i = \dfrac{\partial}{\partial x_i}\left(g_j x_j\right)$

iii)

Irrotation al flow $\Rightarrow \left(\dfrac{\partial u_i}{\partial x_j} - \dfrac{\partial u_j}{\partial x_i}\right) = 0$ for i + j

Continuity $\Rightarrow \left(\dfrac{\partial u_i}{\partial x_j} - \dfrac{\partial u_j}{\partial x_i}\right) = 0$ for i = j

Hence for irrotational flow of a fluid of constant viscosity (or negligible viscosity) and constant density:

Advanced Fluid Mechanics

$$u_j \frac{\partial u_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\left(g_j x_j\right)$$

$$\frac{\partial}{\partial x_i}\left[\frac{u_j u_j}{2} + \frac{p}{\rho} - g_j x_j\right] = 0$$

Put $g_i = (0, 0, -g)$ and $\mathscr{P} = p + \rho g x_3$

Hence $\dfrac{u_j u_j}{2} + \dfrac{\mathscr{P}}{\rho} = const.$ (Bernoulli's Equation)

For i) ρ = constant and
ii) irrotational flow

consider the quantity , $\varepsilon_{ijk} \dfrac{\partial}{\partial x_j}\left(\dfrac{\partial \Phi}{\partial x_k}\right)$

$$\varepsilon_{ijk} \frac{\partial}{\partial x_j}\left(\frac{\partial \Phi}{\partial x_k}\right) \equiv \frac{\partial}{\partial x_j}\left(\frac{\partial \Phi}{\partial x_k}\right) - \frac{\partial}{\partial x_k}\left(\frac{\partial \Phi}{\partial x_j}\right) = 0$$

where Φ is a scalar function of position.

Inviscid Irrotational Flow about an Infinite Cylinder

In order to illustrate the use of the stream function, the inviscid, irrotational flow pattern about a cylinder of infinite length will be obtained by solving previous equations. The physical situation is illustrated in the figure below. A stationary circular cylinder of radius a is situated in uniform, parallel flow in the x direction. Making use of the cylindrical symmetry, we shall employ polar coordinates. In polar coordinates we have:

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0$$

Where the velocity components (v_r and v_θ) are given by

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v_\theta = -\frac{\partial \psi}{\partial r} \qquad (1)$$

1. The circle r=a must be a streamline. Science the velocity normal to a streamline is zero, $v_r\big|_{r=a} = 0$ or $\partial \Psi/\partial \theta\big|_{r=a} = 0$
2. from symmetry, the line θ=0 must also be a streamline. Hence $v_\theta\big|_{\theta=0} = 0$ or $\partial \Psi/\partial r\big|_{\theta=0} = 0$.
3. as $r \to \infty$ the velocity must be finite.
4. the magnitude of the velocity as $r \to \infty$ is v_∞, a constant.

$$\Psi(r,\theta) = F(r)G(\theta)$$

$$\frac{F''(r)}{F(r)} + r\frac{F'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)}$$

$$G''(\theta) + \lambda^2 G(\theta) = 0 \qquad (2)$$

$$r^2 F''(r) + rF'(r) - \lambda^2 F(r) = 0 \qquad (3)$$

$$G(\theta) = A\sin(\lambda\theta) + B\cos(\lambda\theta)$$

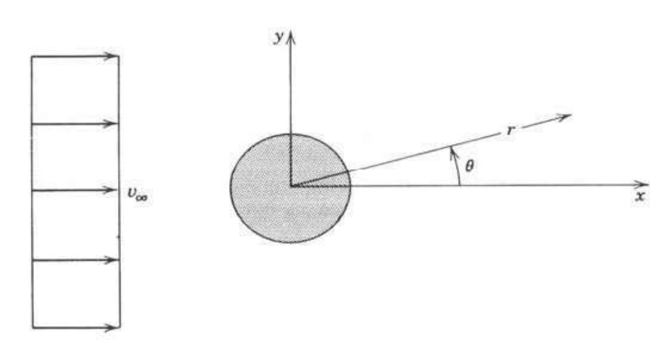


Figure 4 Cylinder in a uniform flow.

Equation (2) is known as an Euler equation and has the solution $F(r) = Cr^{\lambda} + Dr^{-\lambda}$
The boundary conditions listed above will determine the constants. From boundary condition (1) we have

$$G(\theta) = A\sin(\lambda\theta) + B\cos(\lambda\theta)$$

$$\left.\frac{\partial\Psi}{\partial\theta}\right|_{r=a} = \left(Ca^{\lambda} + Da^{-\lambda}\right)\lambda\left(A\cos(\lambda\theta) - B\sin(\lambda\theta)\right) = 0$$

And thus $D = -Ca^{2\lambda}$

Hence

$$\Psi(r,\theta) = \left(r^{\lambda} - \frac{a^{2\lambda}}{r^{\lambda}}\right)\left(A'\sin(\lambda\theta) + B'\cos(\lambda\theta)\right)$$

Where $A' = AC, B' = BC$

Boundary condition (2) states that at (θ=0) we have $\partial\Psi/\partial r = 0$. As sinθ=0, the only way this requirement can be met is to have $B' = 0$, yielding

$$\Psi(r,\theta) = A'\sin(\lambda\theta)\left(r^{\lambda} - \frac{a^{2\lambda}}{r^{\lambda}}\right)$$

Finally, conditions 3 and 4 require that the limit $\left(v_r^2 + v_\theta^2\right) = v_\infty^2$. As

$$v_r^2 + v_\theta^2 = A'^2\frac{\lambda^2\cos^2\lambda\theta}{r^2}\left(r^{\lambda} - \frac{a^{2\lambda}}{r^{\lambda}}\right)^2 + A'^2\lambda^2\sin^2\lambda\theta\left(r^{\lambda-1} + \frac{a^2}{r^{\lambda\lambda 1}}\right)^2$$

$$v_r^2 + v_\theta^2 = A'^2\lambda^2\left[\left(\cos^2\lambda\theta\right)\left(r^{\lambda-1} - \frac{a^{2\lambda}}{r^{\lambda+1}}\right)^2 + \left(\sin^2\lambda\theta\right)\left(r^{\lambda-1} + \frac{a^{2\lambda}}{r^{\lambda+1}}\right)^2\right]$$

The only value of λ for which the velocity will be finite as $r \to \infty$ is unity. Using λ=1 requires $A' = v_\infty$, and the stream function becomes

$$\Psi(r,\theta) = v_\infty r\sin\theta\left(1 - \frac{a^2}{r^2}\right)$$

The velocity components v_r and v_θ are:

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = v_\infty\cos\theta\left(1 - \frac{a^2}{r^2}\right)$$

$$v_\theta = -\frac{\partial\psi}{\partial r} = -v_\infty\sin\theta\left(1 + \frac{a^2}{r^2}\right)$$

$$v_r = 0$$
$$v_\theta = -2v_\infty\sin\theta$$

Utilization of potential flow

Potential flow has great utility in engineering for the prediction of pressure fields, forces, and flow rates in the field of aerodynamics, for example , potential flow solutions are used to predict force and momentum distributions on wings and other bodies.

An illustration of the determination of the pressure distribution from a potential flow solution may be obtained from the solution for the flow about a circular cylinder presented in the previous section. From the Bernoulli equation:

$$\frac{P}{\rho}+\frac{v^2}{2}=const\,.$$

We have deleted the potential energy term in accordance with the original assumption of uniform velocity in the x direction. At a great distance from the cylinder the pressure is P_∞, and the velocity is v_∞, so the above equation becomes:

$$P+\frac{\rho v^2}{2}=P_\infty+\frac{\rho v_\infty^2}{2}=P_0$$

Where P_0 is designated the stagnation pressure (i.e., the pressure at which the velocity is zero). In accordance with this equation the stagnation pressure is constant throughout the field in an irrotational flow. The velocity at the surface of the body is $v_0 = -2v_\infty \sin\theta$, thus the surface pressure is

$$P=P_0-2\rho v_\infty^2\sin^2\theta$$

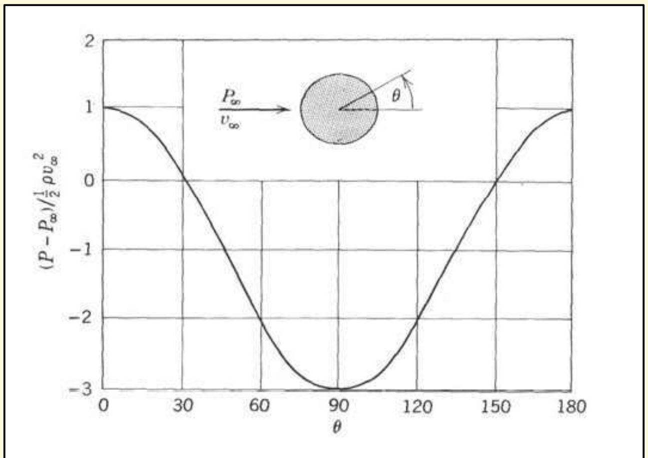


Figure 5 Pressure distribution on a cylinder in an inviscid, incompressible, steady flow.

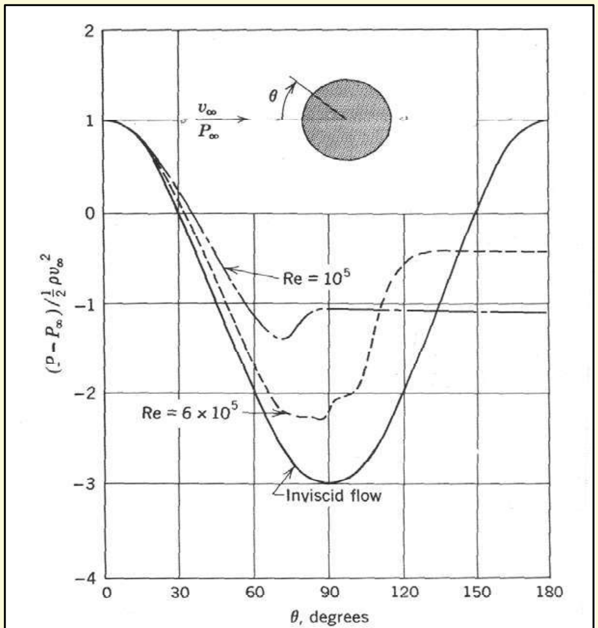


Figure 6 Pressure distribution on a circular cylinder at various Reynolds numbers.

PROBLEMS

مسأله ۱)در جریان توربولنت و توسعه یافته سیالی به جرم مخصوص ۷۵ پوند بر فوت مکعب در داخل لوله‌ای به قطر یک اینچ در جریان است. اگر افت فشار در طولی از لوله که صد برابر قطر آن است مساوی 0.1 psi باشد و تغییرات سرعت در مقطع لوله از رابطه زیر تبعیت نماید:

$$U^+ = \frac{1}{k}\ln y^+ + c$$

$$c = 5.67 \quad \mathrm{k} = 0.407$$

الف – نشان دهید رابطه زیر بین سرعت متوسط $\overline{U}$ و سرعت ماکزیمم $U_{\max}$ صادق است.

$$\frac{\overline{U}}{U_{\max}} = \frac{1}{1 + \frac{3}{2k}\sqrt{f/2}}$$

ب – دبی جرمی جریان در داخل لوله را به دست آورید.

$$\int \ln\eta = \eta\ln\eta - \eta$$

$$\int \eta\ln\eta = \frac{\eta^2}{2}\ln\eta - \frac{\eta^2}{4}$$

$$U^+ = \frac{1}{k}\ln y^+ + c$$

$$\frac{U}{U^*} = \frac{1}{k}\ln\frac{yU^*}{\nu} + c$$

$$\frac{U_{max}}{U^*} = \frac{1}{k}\ln\frac{r_\circ U^*}{\nu} + c$$

$$\frac{U_{max} - U}{U^*} = -\frac{1}{k}\ln\frac{y}{r_\circ}$$

$$\frac{U_{max} - \overline{U}}{U^*} = \frac{1}{\pi r_\circ^2}\int_0^{r_0}\left(\frac{U_{max} - U}{U^*}\right)2\pi r dr$$

Advanced Fluid Mechanics

$$h_f = \frac{P_1 - P_2}{\rho} = 4f\frac{L}{d}\frac{\overline{U}^2}{2g_c}$$

$$\sqrt{f}\,\overline{U} = \left[\left(\frac{P_1 - P_2}{\rho}\right)\left(\frac{d}{L}\right)\left(\frac{g_c}{2}\right)\right]^{1/2} = \left[\left(\frac{0.1*144}{75}\right)\left(\frac{1}{100}\right)\left(\frac{32.174}{2}\right)\right]^{1/2} = 0.18$$

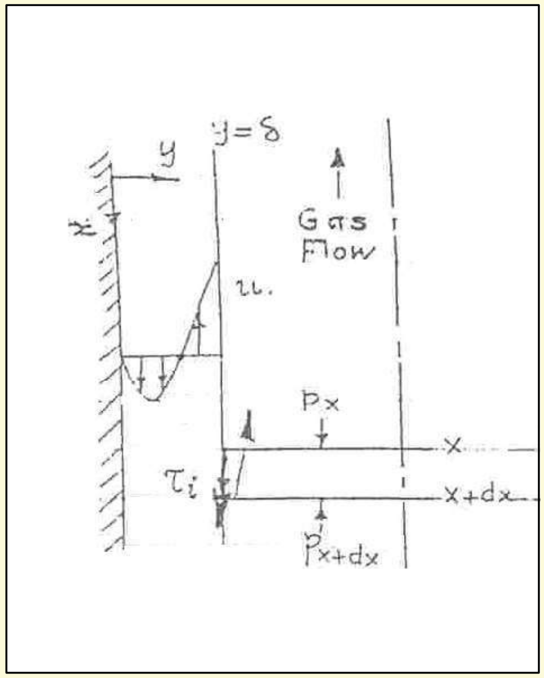
$$\overline{U} + \frac{3}{2k}\sqrt{f/2}\,\overline{U} = U_{max}$$

$$\overline{U} + (0.18)(2.61) = 2.5 \Rightarrow \overline{U} = 2.04\,\frac{ft}{s}$$

$$m = \rho\overline{U}\left(\frac{\pi d^2}{4}\right) = 0.834\ lb/s$$

Sharif University of Technology
Chemical & Petroleum Engineering Department

In a countercurrent gas-liquid contactor the maximum gas throughput is limited by flooding. To investigate this, experiments were carried out in a wetted wall column apparatus. Measurement of liquid holdup and pressure gradient were recorded. Show that the pressure gradient at flooding point is given by:



$$\frac{dp}{dx} = \frac{2\rho g H}{3 A g_c} \quad \text{where} \quad \frac{H}{2A} << 1.0$$

Advanced Fluid Mechanics

$$\frac{H}{2A} = \frac{2\pi R\delta}{2\pi R^2} = \frac{\delta}{R} << 1.0$$

Hence rectangular coordinates may be used.

$$\upsilon\frac{\partial^2 u}{\partial y} = -g + \frac{g_c}{\rho}\left(\frac{\partial p}{\partial x}\right) \qquad \frac{\delta}{R} << 1.0$$

$$B.C.\begin{cases} @\, y = 0 \rightarrow u = 0 \\ @\, y = \delta \rightarrow \tau_i g_c = \mu\dfrac{\partial u_x}{\partial y} \end{cases}$$

$$\text{integrating}\begin{cases} \nu\dfrac{\partial u}{\partial y} = \left[-g + \dfrac{g_c}{\rho}\left(-\dfrac{\partial p}{\partial x}\right)\right]y + c_1 \\ \nu u = \left[-g + \dfrac{g_c}{\rho}\left(\dfrac{\partial p}{\partial x}\right)\right]y^2 + c_1 y + c_2 \end{cases}$$

Applying B.C.1 $\Rightarrow c_2 = 0$

Sharif University of Technology
Chemical & Petroleum Engineering Department

Advanced Fluid Mechanics

Applying B.C.2 with force balance in gas:

$$(-\tau_i)2\pi R.dx = [P_{x+dx} - P_x]\pi R^2$$

$$\tau_i = \frac{R}{2}\left(-\frac{dp}{dx}\right)$$

$$\frac{\tau_i g_c}{\rho} = \nu\frac{\partial u_x}{\partial y} = \frac{g_c R}{2\rho}\left(-\frac{dp}{dx}\right)$$

$$c_1 = \frac{g_c R}{2\rho}\left(-\frac{dp}{dx}\right) - \left[-g + \frac{g_c}{\rho}\left(\frac{\partial P}{\partial x}\right)\right]\delta$$

Hence

$$\nu.u = \left[-g + \frac{g_c}{\rho}\left(\frac{dP}{dx}\right)\right]\left[\frac{y^2}{2} - y\delta\right] + \frac{g_c R}{2\rho}\left(-\frac{dp}{dx}\right)y$$

Advanced Fluid Mechanics

Averaging over δ:

$$\nu.\bar{u} = \frac{1}{\delta}\left\{\left[-g + \frac{g_c}{\rho}\left(\frac{dP}{dx}\right)\right]\left[\int_0^\delta \left(\frac{y^2}{2} - y\delta\right)dy\right] + \frac{g_c R}{2\rho}\left(-\frac{dp}{dx}\right)\int_0^\delta y dy\right\}$$

$$\nu.\bar{u} = \frac{1}{\delta}\left\{\left[-g + \frac{g_c}{\rho}\left(\frac{dP}{dx}\right)\right]\left(-\frac{\delta^3}{3}\right) + \frac{g_c}{4\rho}\left(-\frac{dp}{dx}\right)R\delta^2\right\}$$

Now $R\delta^2 >> \delta^3 \Rightarrow \frac{g_c}{4\rho}\left(\frac{dp}{dx}\right)R\delta^2 >> \frac{g_c}{\rho}\left(\frac{dP}{dx}\right)\left(\frac{\delta^3}{3}\right)$

Hence $\nu.\bar{u} = g\frac{\delta^2}{3} + \frac{g_c}{4\rho}\left(-\frac{dp}{dx}\right)R\delta$

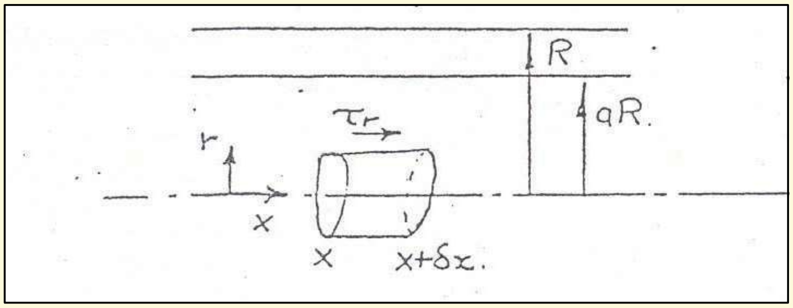
At flooding: $\bar{u} = 0$

Hence $g\frac{\delta^2}{3} = \frac{g_c}{4\rho}\left(\frac{dp}{dx}\right)R\delta \Rightarrow \boxed{\frac{dp}{dx} = \frac{2\rho g H}{3A g_c}}$

Advanced Fluid Mechanics

In pumping a high viscosity oil through a pipe, excessive pressure drop may prevent economic operation. It has been suggested that the pressure drop can be reduced by introducing a continuous water film around the tube periphery. What is the maximum factor by which the oil flow rate can be increased by a given pressure gradient if the viscosity ratio of the fluids is given by:

$$\frac{\mu_{oil}}{\mu_{water}} = 500$$



Advanced Fluid Mechanics

$$\mu_{oil} >> \mu_{water} \Rightarrow \mu_1 >> \mu_2$$

Force balance on element:

$$\pi r^2 \left\{ P\big|_x - P\big|_{x+\delta x} \right\} + \tau_r 2\pi r \delta x = 0$$

$$\therefore \frac{2\tau_r}{r} = \left(\frac{dp}{dx}\right) \Rightarrow \frac{\tau_r}{\tau_w} = \frac{r}{R}$$

$$\tau_r g_c = \mu_1 \frac{du}{dr}$$

B.C. $@r = aR \Rightarrow u = u_i$

Hence $$\frac{\tau_w g_c r}{R} = \mu_1 \frac{du}{dr} \Rightarrow u = \frac{\tau_w g_c r^2}{2\mu_1 R} + C$$

Hence $u = u_i + \frac{\tau_w g_c}{2\mu_1 R}\left[r^2 - aR^2\right]$

In Water similarly with $u\big|_{r=R} = 0$

$$u = \frac{\tau_w g_c}{2\mu_2 R}\left[r^2 - R^2\right] \Rightarrow u_i = \frac{\tau_w g_c}{2\mu_2 R}\left[a^2R^2 - R^2\right]$$

Hence if V is the volumetric flow rate of oil:

$$V = \int_0^{ar} 2\pi r u dr$$

$$= \int_0^{aR} -\frac{\pi\tau_w g_c}{R}\left[\left(\frac{a^2R^2 - r^2}{\mu_1}\right) + \left(\frac{R^2 - a^2R^2}{\mu_2}\right)\right] r dr$$

$$= -\pi\tau_w g_c R^3\left[\left(\frac{2a^4 - a^4}{4\mu_1}\right) + \left(\frac{a^2 - a^4}{2\mu_2}\right)\right]$$

$$\therefore \frac{\text{dv}}{\text{da}} = -\pi\tau_w g_c R^3\left[\left(\frac{a^3}{\mu_1}\right) + \left(\frac{a - 2a^3}{\mu_2}\right)\right]$$

Now $\mu_1 >> \mu_2$ Hence $a^2 = 1/2$

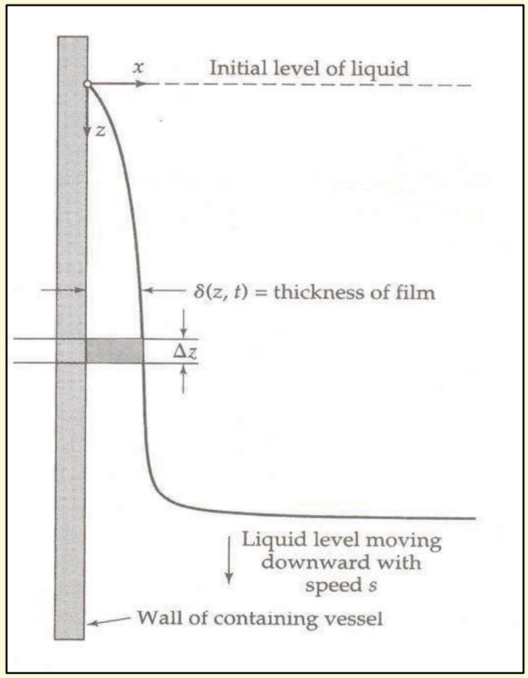
$$V_{\max} = -\pi\tau_w g_c R^3\left[\left(\frac{1}{8\mu_2}\right)\right]$$

$$V_{a=1} = -\pi\tau_w g_c R^3\left[\left(\frac{1}{4\mu_1}\right)\right]$$

Ratio $= \frac{\mu_1}{2\mu_2} = 250$

Let it be desired to find out how much liquid clings to the inside surface of a large vessel when it is drained. The local film thickness is a function of both z and t.

$$\delta(z,t)=\sqrt{\frac{\mu.z}{\rho.g.t}}$$



Consider mass balance on element.

Input = Output + Accumulation

$$\rho\bar{u}_x\delta\big|_x = \rho\bar{u}_x\delta\big|_{x+\delta x} + \rho\frac{\partial}{\partial t}[\delta.x]$$

With constant density:

$$-\frac{\partial\delta}{\partial t}=\frac{\left[\bar{u}_x\delta\big|_{x+\delta x}-\bar{u}_x\delta\big|_x\right]}{\Delta x}$$

Taking limit (Δx→0) gives:

$$-\frac{\partial\delta}{\partial t}=\frac{\partial}{\partial x}[\bar{u}_x\delta]$$

Now: $$\bar{u}_x=\frac{\int_0^{\delta}u_x dy}{\int_0^{\delta}dy}$$