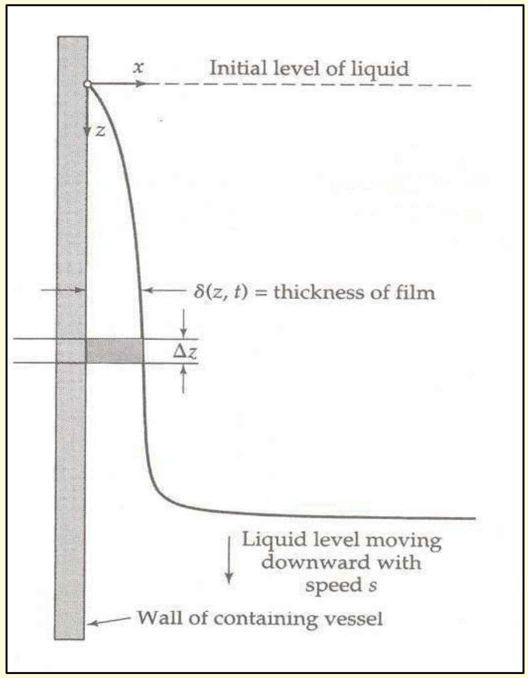
Advanced Fluid Mechanics

Let it be desired to find out how much liquid clings to the inside surface of a large vessel when it is drained. The local film thickness is a function of both z and t.

$$\delta(z,t)=\sqrt{\frac{\mu.z}{\rho.g.t}}$$



Advanced Fluid Mechanics

Consider mass balance on element.

Input = Output + Accumulation

$$\rho\bar{u}_x\delta\big|_x = \rho\bar{u}_x\delta\big|_{x+\delta x} + \rho\,\frac{\partial}{\partial t}[\delta.x]$$

With constant density:

$$-\frac{\partial\delta}{\partial t}=\frac{\left[\bar{u}_x\delta\big|_{x+\delta x}-\bar{u}_x\delta\big|_x\right]}{\Delta x}$$

Taking limit (Δx→0) gives:

$$-\frac{\partial\delta}{\partial t}=\frac{\partial}{\partial x}\left[\bar{u}_x\delta\right]$$

Now: $$\bar{u}_x=\frac{\int_0^{\delta}u_x\,dy}{\int_0^{\delta}dy}$$

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The velocity distribution must be determined. In general this requires solution of the N-S equations and it is not possible since the film thickness changes with x and t.
However a reasonable guess at the velocity distribution will be adequate since the mean velocity will not be sensitive to the guess made.
Assume the velocity profile for a parallel flow.

i.e. solution of $\nu \frac{d^2 u_x}{dy^2} = -g$ with B.C.s $\begin{cases} @\, y = 0 & u_x = 0 \\ @\, y = \delta & \frac{\partial u_x}{\partial y} = 0 \end{cases}$

Hence $u_x = \frac{\rho g}{\mu}\left[\delta y - \frac{y^2}{2}\right]$

$$\bar{u}_x = \frac{\rho g \delta^2}{3\mu}$$

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Substitute in mass balance:

$$\frac{\partial}{\partial x}\left[\frac{\rho g}{\mu}\frac{\delta^3}{3}\right] + \frac{\partial \delta}{\partial t} = 0$$

With $\eta = \frac{x}{t} \Rightarrow \delta = \delta(\eta)$

$$\frac{\partial \delta}{\partial x} = \frac{d\delta}{d\eta}\cdot\frac{\partial \eta}{\partial x} \qquad \frac{\partial \delta}{\partial t} = \frac{d\delta}{d\eta}\cdot\frac{\partial \eta}{\partial t}$$

Hence

$$\frac{\rho g \delta^2}{\mu}\cdot\frac{d\delta}{d\eta}\cdot\frac{1}{t} + \frac{d\delta}{d\eta}\left(-\frac{x}{t^2}\right) = 0$$

$$\left[\frac{\rho g \delta^2}{\mu} - \eta\right]\frac{d\delta}{d\eta} = 0$$

But $\frac{d\delta}{d\eta} \neq 0$ hence we must have $\boxed{\delta = \sqrt{\frac{\mu . x}{\rho . g . t}}}$

Advanced Fluid Mechanics

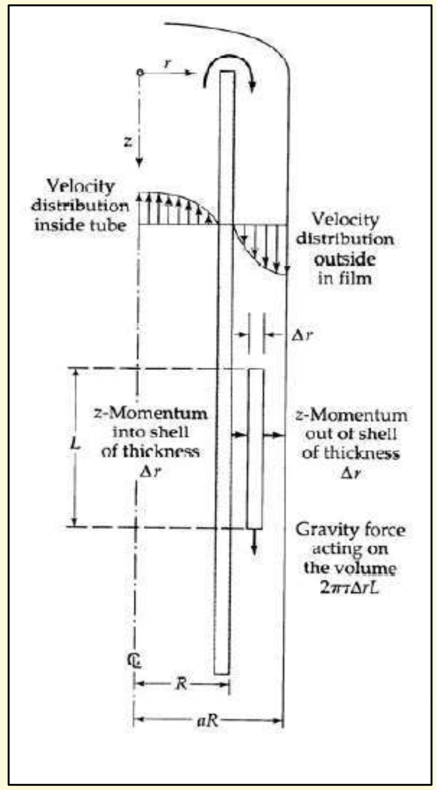
In a gas absorption experiment a viscous fluid flows upward thought a small circular tube and then downward on the outside.

1- show that the velocity distribution in the falling film is:

$$u_z = \frac{\rho g R^4}{4\mu}\left[1 - \frac{r^2}{R^2} + 2a^2 \ln\frac{r}{R}\right]$$

2- Obtain an expression for the volume rate of flow in the film.

3- Obtain the film thickness on the wall.



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(a) Navier Stokes equation for steady state laminar flow at low Reynolds numbers:

$$0 = \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right] + \rho g$$

Continuity equation:

$$\frac{\partial u_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r u_r) = 0$$

No radial flow: $\frac{\partial u_z}{\partial z} = 0 \quad \Rightarrow \quad u_z = u_z(r)$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = -\frac{\rho g}{\mu}$$

Integrate:

$$u_z = -\frac{\rho g r^4}{4\mu} + c_1 \ln r + c_2$$

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Boundary conditions:

$$u_z = 0 \qquad @ \quad r = R$$

$$\frac{\partial u_z}{\partial r} = 0 \qquad @ \quad r = aR$$

$$0 = -\frac{\rho g R^4}{4\mu} + c_1 \ln R + c_2$$

$$0 = -\frac{\rho g R a}{2\mu} + \frac{c_1}{Ra}$$

$$c_1 = \frac{\rho g R^2 a^2}{2\mu}$$

$$c_2 = \frac{\rho g R^4}{4\mu} - \frac{\rho g R^2 a^2}{2\mu} \ln R = \frac{\rho g R^4}{4\mu}\left\{1 - 2a^2 \ln R\right\}$$

Hence

$$u_z = \frac{\rho g R^4}{4\mu}\left[1 - \frac{r^2}{R^2} + 2a^2 \ln \frac{r}{R}\right]$$

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(b) Volumetric floe rate:

$$Q = 2\pi \int_R^{aR} r u_z dr$$

$$Q = 2\pi \int_R^{aR} \frac{\rho g R^4}{4\mu}\left\{1 - \frac{r^2}{R^2} + 2a^2 \ln \frac{r}{R}\right\} r dr$$

$$= 2\pi \frac{\rho g R^4}{4\mu}\left\{\frac{R^2}{2}\left(a^2 - 1\right) - \frac{R^4}{4R^2}\left(a^4 - 1\right) + 2a^2\left\{\frac{a^2 R^2}{2}\ln a + \frac{R^2}{4}\left(1 - a^2\right)\right\}\right\}$$

$$= 2\pi \frac{\rho g R^4}{4\mu}\left\{\frac{1}{2}\left(a^2 - 1\right) - \frac{1}{4}\left(a^4 - 1\right) + a^4 \ln a + \frac{a^2}{2}\left(1 - a^2\right)\right\}$$

$$Q = \frac{\pi \rho g R^4}{8\mu}\left\{-1 + 4a^2 - 3a^4 + 4a^4 \ln a\right\}$$

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(c) Film thickness:

$$h = aR - R = R(a-1)$$
$$a = 1 + \left(\frac{h}{R}\right)$$

{For a film flowing down a flat plate, $Q = \dfrac{\rho g h^3}{3\mu}$}

Flow / Unit periphery $= Q/2\pi r$

$$= \frac{\pi \rho g R^3}{16\mu}\left\{-1 + 4a^2 - 3a^4 + 4a^4 \ln a\right\}$$

Now:

$$a^2 = 1 + 2\left(\frac{h}{R}\right) + \left(\frac{h}{R}\right)^2$$
$$a^4 = 1 + 4\left(\frac{h}{R}\right) + 6\left(\frac{h}{R}\right)^2 + 4\left(\frac{h}{R}\right)^3 + 0\left(\frac{h}{R}\right)^4$$
$$a^4 \ln a = \left(\frac{h}{R}\right) + \frac{7}{2}\left(\frac{h}{R}\right)^2 + \frac{13}{3}\left(\frac{h}{R}\right)^3 + 0\left(\frac{h}{R}\right)^4$$

Hence

$$\frac{Q}{2\pi r} = \frac{\rho g R^3}{16\mu}\left\{\begin{matrix} -1 + 4 + 8\left(\frac{h}{R}\right) + 4\left(\frac{h}{R}\right)^2 - 3 - 12\left(\frac{h}{R}\right) - \\ 18\left(\frac{h}{R}\right)^2 - 12\left(\frac{h}{R}\right)^3 + 4\left(\frac{h}{R}\right) + 14\left(\frac{h}{R}\right)^2 + \frac{53}{3}\left(\frac{h}{R}\right)^3 \end{matrix}\right\}$$

$$\frac{Q}{2\pi r} = \frac{\rho g R^3}{16\mu}\frac{16}{3}\left(\frac{h}{R}\right)^3 = \frac{\rho g h^3}{3\mu}$$

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TURBULENT FLOW

Presented by:

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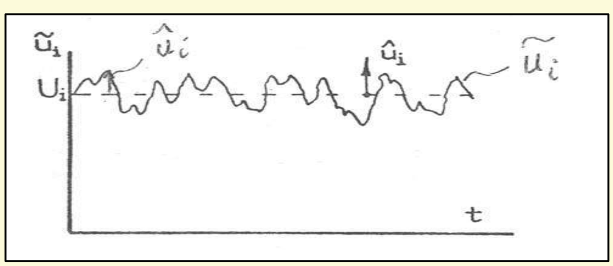
Advanced Fluid Mechanics

Turbulent transport of momentum

Turbulence of random velocity fluctuations- Use statistical methods

Turbulent velocity $\tilde{u}_i$

$$\underset{\text{Mean Velocity}}{\tilde{u}_i = U_i} \quad + \quad \underset{\text{Fluctuatin g Component}}{\hat{u}_i}$$



Interpret U_i as a time averaged velocity defined by:

$$U_i = \lim_{T\to\infty} \frac{1}{T} \int_0^T \tilde{u}_i dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T (U_i + \hat{u}_i) dt \equiv \bar{\tilde{u}}_i$$

$$\bar{\hat{u}}_i = \lim_{T\to\infty} \frac{1}{T} \int_0^T \hat{u}_i dt = U_i - \lim_{T\to\infty} \frac{1}{T} \int_0^T (U_i) dt = U_i - U_i = 0$$

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i.e. the mean value (or time average) of the fluctuating quantity is zero. Assume that U_i the mean flow is steady ($\partial U_i/\partial t = 0$)

Note: Time averaging commutes w.r.t. differentiation.

$$\overline{\frac{\partial \tilde{u}_i}{\partial x_j}} = \frac{1}{T}\int_0^T \frac{\partial \tilde{u}_i}{\partial x_j} dt = \frac{\partial}{\partial x_j}\left\{\frac{1}{T}\int_0^T \tilde{u}_i dt\right\} = \frac{\partial U_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\overline{\tilde{u}_i}\right)$$

The time average of the fluctuation $\overline{\hat{u}_i}$ is zero, but the average of the square of the fluctuation is not zero and the quantity $\frac{\overline{\hat{u}_i^2}}{U_i}$ is used as a convenient measure of the turbulent fluctuation-known as the "intensity of turbulence" and ranges from 0.01 to 0.1 for most turbulent flows.

$\sqrt{\left(\overline{\hat{u}_i^2}\right)}$ r.m.s. velocity.

Mean K.E./unit volume = $\overline{KE} = \frac{1}{2}\rho\overline{(U_i + \hat{u}_i)} = \frac{1}{2}\rho\left(U_i^2 + \overline{\hat{u}_i^2}\right)$

mean flow + fluctuations

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Equations for the mean flow

Consider the momentum and continuity equations. These apply to the instantaneous velocity in a turbulent field.

$$\tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j}$$

$$\frac{\partial \tilde{u}_i}{\partial x_i} = 0 \qquad (1)$$

The equations must apply on average

$$\tilde{u}_i = U_i + \hat{u}_i$$

Continuity

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{\partial \tilde{u}_i}{\partial x_i} dt = \overline{\frac{\partial \tilde{u}_i}{\partial x_i}} = \frac{\partial}{\partial x_i}\overline{(U_i + \hat{u}_i)} = \frac{\partial U_i}{\partial x_i} = 0 \qquad (2)$$

The mean value satisfies continuity. It is the mean value of velocity that we measure and require in applications.

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Momentum:

The equations of motion for the mean flow U_i are obtained by taking the time average of all terms in the resulting equation.
Consider each term:

$$\text{(i)}\quad \overline{\tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j}} = \overline{\frac{\partial}{\partial x_j}(\tilde{u}_j \tilde{u}_i)} - \overline{\tilde{u}_i\left(\frac{\partial \tilde{u}_j}{\partial x_j}\right)} = \frac{\partial}{\partial x_j}\left\{\overline{(U_j+\hat{u}_j)(U_i+\hat{u}_i)}\right\}$$

$$= \frac{\partial}{\partial x_j}\left\{\overline{U_j U_i} + \overline{\hat{u}_j \hat{u}_i} + \overline{U_i \hat{u}_j} + \overline{U_j \hat{u}_i}\right\}$$

$$= \frac{\partial}{\partial x_j}\left\{U_i U_j + \overline{\hat{u}_i \hat{u}_j}\right\} = U_j \frac{\partial U_i}{\partial x_j} + \frac{\partial}{\partial x_j}\overline{(\hat{u}_j \hat{u}_i)} \qquad (2.1)$$

$$\text{(ii)}\quad -\overline{\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i}} = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\overline{(P+\hat{p}_i)} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} \qquad (2.2)$$

$$\text{(iii)}\quad \nu \overline{\frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j}} = \nu \frac{\partial^2}{\partial x_j \partial x_j}\overline{(U_i+\hat{u}_i)} = \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} \qquad (2.3)$$

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Hence

$$U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}\overline{(\hat{u}_j \hat{u}_i)} \qquad (3)$$

Equation for mean flow has an additional term.(Drop the ^ $\Rightarrow \overline{\hat{u}_j \hat{u}_i} = \overline{u_j u_i}$)

Term $\frac{\partial \overline{u_j u_i}}{\partial x_j}$ is analogous to the convective term $U_j \frac{\partial U_i}{\partial x_j}$;

It represents the mean transport of fluctuating momentum by turbulent velocity fluctuations.

If $\hat{u}_i$ and $\hat{u}_j$ uncorrelated i.e. $\overline{u_j u_i} = 0$ - no turbulent momentum transfer but experience shows that $\overline{u_j u_i} \neq 0$ - momentum transfer is a key feature of turbulent motion.

Term $\frac{\partial}{\partial x_j}\overline{(\hat{u}_j \hat{u}_i)}$ thus exchanges momentum between the turbulence and the mean flow (equation 2.1)even though the mean momentumof the turbulent velocity fluctuations is zero ($\rho \overline{\tilde{u}_i} = 0$).

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Because of the decomposition $\tilde{u}_i = U_i + \hat{u}_i$, turbulent motion can be perceived as something which produces stresses in the mean flow. For this reason, equation (3) may be rearrange so that all stress can be put together.

$$\rho U_j \frac{\partial U_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(-P\delta_{ji} + \mu\left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right] - \rho\overline{\hat{u}_j\hat{u}_i}\right) = \frac{\partial}{\partial x_j}(T_{ji}) \text{ - mean stress tensor.}(\tilde{\tau} = T + \hat{\tau})$$

$$T_{ji} = -P\delta_{ji} + \sigma_{ji} - \rho\overline{\hat{u}_j\hat{u}_i} \quad ; \quad \sigma_{ji} = \mu\left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right]$$

(normal) (shear)

The contribution of the turbulent motion to the mean stress tensor is $\sigma_{ji}^T = -\rho\overline{u_j u_i}$ called the Reynolds stress tensor. Define, $\Omega_{ji} = \sigma_{ji} + \sigma_{ji}^T$

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Turbulent shearing stresses

Time averaging of the equations of motion leads to the Reynolds stress tensor, $\rho\overline{\hat{u}_j\hat{u}_i}$.
$\hat{u}_i$ and $\hat{u}_j$ are the velocity fluctuations in the $i \neq j$ directions at one point and $\overline{u_i u_j}$ is a measure of the "correlation" between the fluctuations.

Correlated variables

$$\overline{\tilde{u}_i\tilde{u}_j} = \overline{(U_i + \hat{u}_i)(U_j + \hat{u}_j)} = U_i U_j + \overline{u_i u_j}$$

If $\overline{u_i u_j} \neq 0$, $\hat{u}_i$ and $\hat{u}_j$ are said to be correlated i.e. dependent.
If $\overline{u_i u_j} = 0$, uncorrelated i.e. $\hat{u}_i$ and $\hat{u}_j$ are independent.

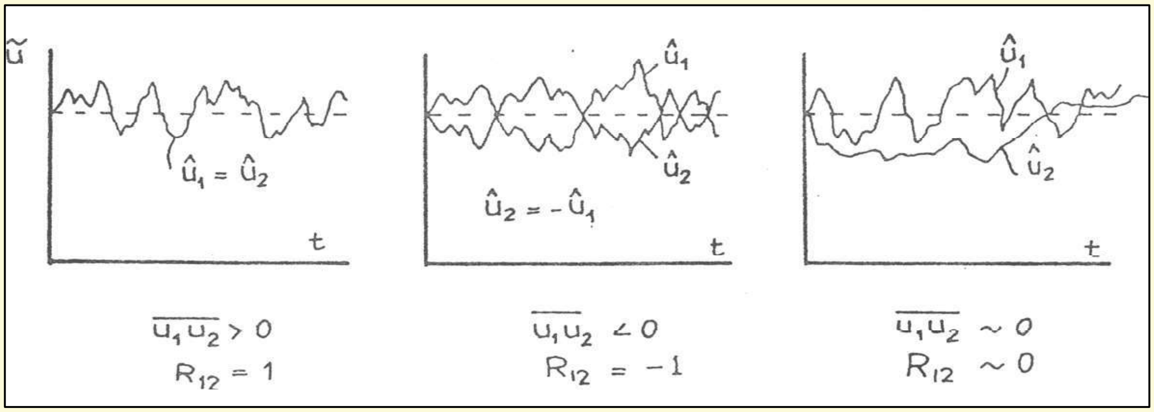

Fig2(a)	Fig2(b)	Fig2(c)
$\overline{u_1 u_2} > 0$	$\overline{u_1 u_2} < 0$	$\overline{u_1 u_2} \approx 0$
$R_{12} = 1$	$R_{12} = -1$	$R_{12} \approx 1$

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A measure of the degree of correlation between $\hat{u}_1$ and $\hat{u}_2$ is obtained from: $\dfrac{\overline{\hat{u}_1 \hat{u}_2}}{\left\{\overline{\hat{u}_1^2}\,\overline{\hat{u}_2^2}\right\}^{1/2}}$

$$R_{12} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{\hat{u}_1 \hat{u}_2}{u_1' u_2'}\,dt \quad : \quad u_i' = \sqrt{\overline{\hat{u}_i^2}} = \left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\hat{u}_i^2\right) dt\right\}^{1/2}$$

$$R_{12} = \frac{\overline{u_1 u_2}}{u_1' u_2'}$$

N.B. $(a-b)^2 \geq 0 \Rightarrow \frac{1}{2}(a^2+b^2) \geq |ab|$

Hence $$R_{12} \leq \lim_{T\to\infty} \frac{1}{T}\int_0^T \left\{\frac{\hat{u}_1^2}{\overline{\hat{u}_1^2}} + \frac{\hat{u}_2^2}{\overline{\hat{u}_2^2}}\right\} dt \leq 1$$

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Pure shear flow

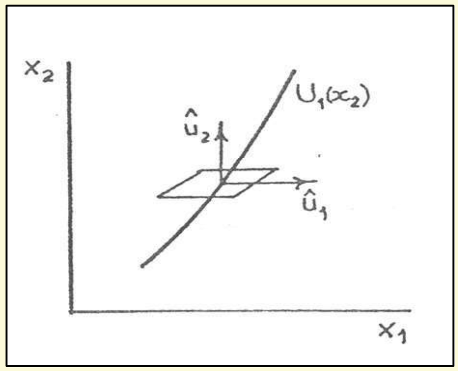
Consider a turbulent shear flow with $U_1(x_2)$ the only non-zero velocity component.

Ω_{12} is the only component of the mean stress tensor, $\Omega_{12} = \Omega\frac{\partial U_1}{\partial x_2} - \Omega\overline{\hat{u}_2\hat{u}_1}$

Ω_{12} – stress in 1 direction on face, normal in 2 direction and must result from molecular transport of momentum in the x_2 direction, and turbulent transport.

Assume $\frac{\partial U_1}{\partial x_2} > 0$.

A fluid particle with positive $\hat{u}_2$ is being carried by turbulence in positive x_2 direction. It is coming from a region where the mean velocity is smaller i.e. is likely to be moving downstream more slowly than its new environment. Thus $\hat{u}_1$ is negative. Similarly negative $\hat{u}_2$ associated with positive $\hat{u}_1$.



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{Momentum/unit volume of flow at A in 1-direction} $= \rho\tilde{u}_1 = \rho(U_1 + \hat{u}_1)$

The x_1-momentum is transported in the x_2-direction if u_1 and u_2 are correlated.

{Flux of x_1-momentum in x_2-direction} $= \rho(U_1 + \hat{u}_1)\hat{u}_2$

{Average flux of x_1-momentun in x_2-direction} $= \rho\overline{\hat{u}_1\hat{u}_2}$

$\hat{u}_1$ and $\hat{u}_2$ are negatively correlated: $\sigma_{12}^T = \sigma_{21}^T = -\rho\overline{u_2 u_1}$

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Turbulent Channel Flow

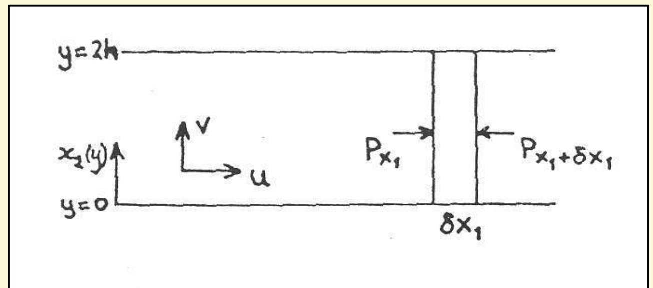
The Navier-Stokes equations in rectangular coordinates are

$$U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}(\overline{u_i u_j})$$

For parallel, fully developed, 2 D flow

$$\left.\begin{array}{l} U_2 = U_3 = 0 \\ \dfrac{\partial U_i}{\partial x_1} = 0\,;\ \dfrac{\partial U_i}{\partial x_3} = 0 \end{array}\right\} \quad \therefore\ L.H.S. = 0$$

$$\frac{\partial}{\partial x_1}(\overline{u_i u_j}) = 0\,; \qquad \frac{\partial}{\partial x_3}(\overline{u_i}\overline{u_3}) = 0$$



Parallel plates of separation 2h

Hence the equations can be written in the simplified form,

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu \frac{\partial^2 U}{\partial y^2} - \frac{\partial}{\partial y}(\overline{uv}) \qquad (1)$$

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial y} - \frac{\partial}{\partial y}(\overline{v^2}) \qquad (2)$$

At the walls $\overline{v^2} = 0$, P = P_0(x) . Hence form (2)

$$\frac{P_0}{\rho} = \frac{P}{\rho} + \overline{v^2} \qquad (3)$$

$$\frac{\partial P}{\partial x} = \frac{\partial P_0}{\partial x} = \frac{dP_0}{dx} \qquad (4)$$

Hence (1) can be integrated from y=0 to y with $\overline{uv}\big|_{y=0}=0$

$$0=-\frac{y}{\rho}\left(\frac{dP_0}{dx}\right)+v\frac{\partial U}{\partial y}-v\frac{\partial U}{\partial y}\bigg|_{y=0}-\overline{uv}$$

At y=h, uv=0, $\partial U/\partial y=0$ (zero velocity gradient, no correlation)

$$\frac{\tau_w}{\rho}=-\frac{h}{\rho}\left(\frac{dP_0}{dx}\right)=\frac{\mu}{\rho}\frac{\partial U}{\partial y}\bigg|_{y=0}$$

Defining a friction velocity u_*

$$\tau_w=\rho u_*^2$$

Substituting in (5)

$$-\overline{uv}+v\frac{\partial U}{\partial y}=u_*^2\left(1-\frac{y}{h}\right)$$

Equation (8) may be written in dimensionless form in 2 ways.

(I)
$$\frac{-\overline{uv}}{u_*^2}+\frac{v}{u_*h}\frac{d(U/u_*)}{d(y/h)}=\left(1-\frac{y}{h}\right)$$

R* =u_* h/v. As R* becomes large, (R* is of course a Reynolds number), the viscous stress is suppressed. Such a limit will not applied because viscous forces must always dominate near solid boundaries.

(II)
$$-\frac{\overline{uv}}{u_*^2}+\frac{d(U/u_*)}{d(yu_*/v)}=1-\frac{yu_*}{v}\cdot\frac{v}{hu_*}$$

In this case as R* becomes large the change in total stress becomes small.
Defining appropriate dimensionless variables

$$y+ = \frac{yu_*}{\nu}; \quad u+ = \frac{U}{u_*}; \quad \eta = \frac{y}{h}$$

Then

$$-\frac{\overline{uv}}{u_*^2} + \frac{1}{R^*}\frac{du+}{d\eta} = 1-\eta \qquad (11)$$

$$-\frac{\overline{uv}}{u_*^2} + \frac{du+}{dy+} = 1-\frac{y+}{R^*} \qquad (12)$$

Law of wall

For large R* (from 12)

$$-\frac{\overline{uv}}{u_*^2} + \frac{du+}{dy+} = 1 \qquad (13)$$

The solution of this equation must be of the form,

$$-\frac{\overline{uv}}{u_*^2} = g(y+); \quad u+ = f(y+) \quad \text{(law of the wall)} \qquad (14)$$

For sufficiently small y+, turbulent stress negligible.

$$\frac{du+}{dy+} = 1 \quad ; \quad \text{with} \quad u+(0)=0 \qquad (15)$$

u+ = y+

Core region

For large R* (from 11)

$$-\frac{\overline{uv}}{u_*^2} = (1-\eta)$$

This equation gives no information, about U itself. However h and u_* are the only feasible length and velocity scales, we can write

$$\frac{dU}{dy} = \frac{u_*}{h}\frac{dF}{d\eta} \qquad \text{Where } F(\eta) \text{ is some function of } \eta. \qquad (17)$$

Integration from the center where $U=U_0$

$$\frac{U-U_0}{u_*} = F(\eta) \qquad (18)$$

From equation (14),

$$\frac{U}{u_*} = f(y+); \qquad \frac{dU}{dy} = \frac{u_*^2}{\nu}\frac{df(y+)}{dy+} \qquad (19)$$

Matching (17) & (19),

$$\frac{u_*}{h}.\frac{dF}{d\eta} = \frac{u_*^2}{\nu}\frac{df}{dy+} \quad ; \quad \eta\frac{dF}{d\eta} = y^+\frac{df}{dy^+} = \frac{1}{K} \qquad (20)$$

$$F(\eta) = \frac{1}{K}\ln\eta + const. \qquad f(y+) = \frac{1}{K}\ln y^+ + const.$$

Hence

$$\boxed{\frac{U-U_0}{u_*} = \frac{1}{K}\ln\eta + const.} \qquad \boxed{\frac{U}{u_*} = \frac{1}{K}\ln y^+ + const.}$$

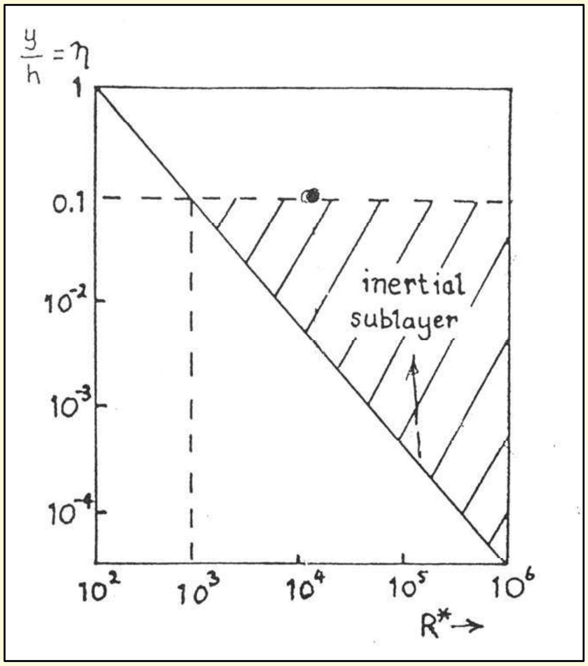
Discussion

To simplify (12) to (14) requires $\frac{y+}{R^*} = \eta \ll 1$ (a)

To simplify (11) to (16) requires $\frac{1}{R^*}\frac{du^+}{d\eta} \ll 1-\eta$ (b)

Matching only possible if $y+ \to \infty$, $\eta \to 0$

In practice it is found that $\begin{cases} y+ > 100 \\ \eta < 0.1 \end{cases}$ are sufficient



Now $\eta < 0.1$; $\quad \therefore \frac{y+}{R*} < 0.1$ (cf.(a))

And $y+ > 100$ $\quad \therefore \eta R* > 100$

$$\therefore R* > 100/\eta$$

$$\eta < 0.1$$

$$\therefore R* > 1000$$

Experimentally

$$\frac{du+}{d\eta} = \frac{2.5}{\eta}$$

$$\therefore \frac{1}{R*}\frac{du+}{d\eta} = \frac{2.5}{R*\eta}$$

Hence $\frac{1}{R*}\frac{du+}{d\eta} = \frac{2.5}{R*\eta} << (1-\eta)$ (cf . (b))

Also from (20) $\quad y^{+}\frac{df}{dy+} = \frac{1}{K}$

$$\therefore \quad f(y^{+}) = \frac{1}{K}\ln(y^{+}) + const. \qquad (21)$$

Experimentally

$$\frac{U - U_0}{u_*} = 2.5\ln\eta - 1.0$$

$$\frac{U}{u*} = 2.5\ln y^{+} + 5.0$$

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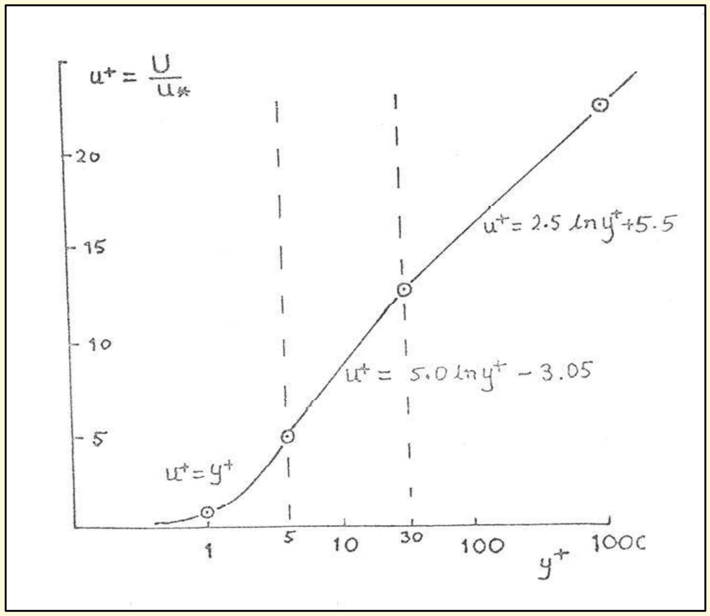
Application

1. For Engineering Purposes these equations have been used for $\eta > 0.1$, i.e. to describe the core region, and also for $\eta \to 0$. Note as $\eta \to 0,\ u^+ = U/u^* \to -\infty$

2. Sometimes the Universal Velocity profile is used.
Equn. (15) $u^+=y^+$ for $y^+ \le 5$

Equn. (21) $u^+=2.5\ln y^+ +5.0$ for $y^+ \ge 30$.

Limits determined experimentally.
A curve fit for $5<y^+<30$
Is $u^+=5.0 \ln y^+ - 3.05$



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Example of use of turbulent velocity profiles.

momentum transfer

Friction factor $$f = \frac{\tau_\omega}{\frac{1}{2}\rho \bar{U}^2} = \frac{2u_*^2}{\bar{U}^2}$$

Using the velocity defect law for flow in a tube

$$\frac{1}{\pi h^2}\int_{y=0}^{y=h}\left[\frac{U-U_0}{u_*}\right].2\pi r dy = \frac{\bar{U}-U_0}{u_*}$$

$$= \int_{\eta=0}^{\mu=1}\frac{2\pi r}{\pi h^2} h\{2.5\ln\eta - 1\}d\eta$$

Now $r = h - y$; $\eta = y/h \to dy = h d\eta$; $r = h(1-\eta)$

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Hence $$\frac{(U-U_0)}{u_*}=\int_{\eta=0}^{\eta=1}2(1-\eta)\{2.5\ln\eta-1\}d\eta$$

Also from experimental results

$$\frac{U_0}{u_*}=\frac{U}{u_*}-2.5\ln\eta+1.0$$
$$=2.5\ln y^+-2.5\ln\eta+1.0+5.0$$
$$=2.5\ln R^*+6.0$$

$$R^*=\frac{hu^*}{\nu}=\frac{h}{\nu}\sqrt{\frac{f\bar{u}^2}{2}}=\frac{\text{Re}}{2}\sqrt{\frac{f}{2}}$$

$$\frac{U_0}{u^*}=2.5\ln[\frac{\text{Re}}{2}\sqrt{\frac{f}{2}}]+6$$
$$\frac{\bar{U}}{u_*}=2.5\ln[\frac{\text{Re}}{2}\sqrt{\frac{f}{2}}]+6+[5\eta\ln\eta-5\eta-2\eta-\frac{5\eta^2}{2}\ln\eta+\frac{5\eta^2}{4}+\frac{\eta^2}{2}]_0^1$$

$$\boxed{\frac{1}{\sqrt{f}}=4.07\log_{10}\left\{\frac{\text{Re}}{2}\sqrt{\frac{f}{2}}\right\}+0.53}$$

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Mass transfer: Turbulent Taylor Analysis. Proc. Royal. Soc. (1954), A223, P446, for Axial Dispersion in turbulent pipe flow.

Consider diffusion equation in rectangular coordinates for simplicity.

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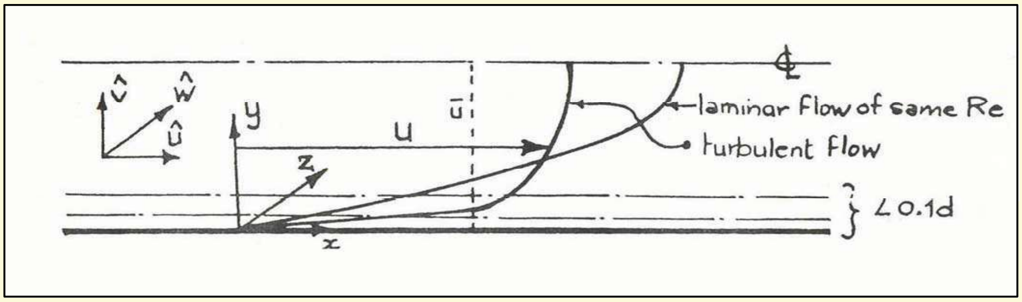
Turbulence in pipe flows

Scope of Turbulence

Most flows in nature: rivers, the atmosphere

Engineer: pipe flow, packed and plate column

Pipe Flow



Laminar sublayer - viscous forces dominate, very thin

Transition region - region of damped turbulence because of nearby wall, eddy size y.

Turbulent core - region of fully developed turbulence, eddies of size d, velocity nearly constant.

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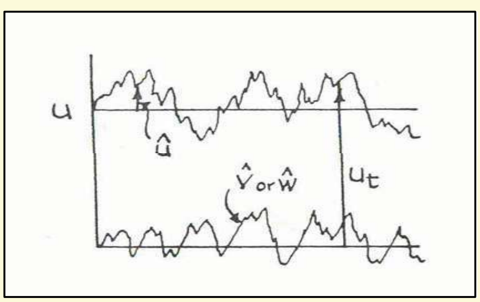
Turbulent Velocities

- local downstream velocity fluctuates due to turbulent eddies .decompose



$$u_t = u + \hat{u}$$

$$\begin{pmatrix} Instantane\ \ us \\ velocity \end{pmatrix} = \begin{pmatrix} localmean \\ velocity \end{pmatrix} + \begin{pmatrix} eddy \\ velocity \end{pmatrix}$$

- definition of u (mean velocity)

$$u = \frac{1}{T}\int_0^T u_t dt$$

- clearly the average of the eddy velocity is zero

$$u = \frac{1}{T}\int_0^T (u + \hat{u})dt = \frac{1}{T}\int_0^T udt + \frac{1}{T}\int_0^T \hat{u}dt$$

$$\frac{1}{T}\int_0^T \hat{u}dt = 0$$

- the magnitude of turbulent velocities is characterized by the RMS
-

$$u' = \left[\frac{1}{T}\int_0^T \hat{u}^2 dt\right]^{\frac{1}{2}}$$

(RMS fluctuating or eddy velocity.)

- the turbulence intensity is defined by,

  turbulent intensity = $\frac{u'}{u}$ (typically up to 0.1) i.e. the average eddy velocity may be 1/10 of the mean velocity.

Properties of turbulent flows (with particular reference to pipe flows)

1) Irregularity local velocities fluctuate in random manner. But all turbulent flows are irregular. E.g. smoke plume.

2) 3D Nature pipe flows are normally considered as 1 dimension in that downstream velocity depends only on radius. However in turbulent flows normal velocity components, though zero on average, have fluctuating components, ($\hat{V}$ and $\hat{W}$). These give rise to turbulent stresses (remember the mail bag example) and are important in turbulent energy processes. This 3D nature adds the mathematical difficulty.

3) Turbulence is a property of the flow not of the fluid writing Newton's law for a flow involving turbulent stresses.

$$\frac{\tau}{\rho} = -(\nu + \nu_T)\frac{du}{dy} \quad ; \quad [\text{divided by } \rho]$$

Where $\nu = \frac{\mu}{\rho}$ is the kinematics viscosity . $\frac{[L]^2}{[T]}$

$\nu_T =$ eddy viscosity

In laminar sub layer $\nu_T << \nu$

Transition region $\nu_T \sim \nu$

Turbulent core $\nu_T >> \nu$

Thus ν_T varies with environment and is a flow property.

4) Mixing in turbulent flows-diffusivity

Rewrite Newton's law in the form, explicit in shear stress.

$$\tau = -(\nu + \nu_T)\frac{d(\rho u)}{dy}$$

Dimensions :

$$\frac{[M][U]}{[L]^2[T]} = \frac{[L]^2}{T} \cdot \frac{[M][U]/[L]^3}{[L]}$$

i.e. (momentum flux) = (diffusivity) * (gradient of momentum / volume)

- this fundamental relation shows how transport (here of momentum) is related to the driving force (momentum gradient).the coefficient, υ , is the momentum diffusivity. It shows how large a flux is produced by a given gradient. Exactly analogous laws apply for heat transfer (Fourier's Law) and mass transfer (Flick's law).

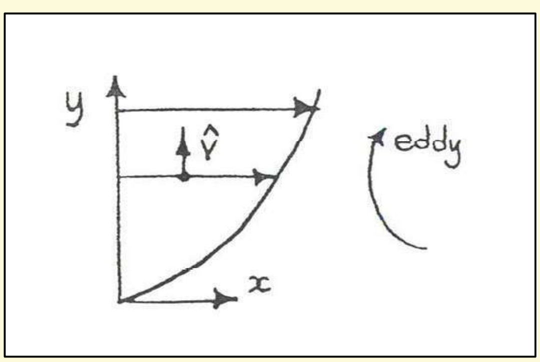
- Now $\nu_T \succ\succ \nu$ in turbulent flows.

Turbulent is a very effective mixer of momentum which accounts for the almost constant velocity of the core region will usually be of almost constant temperature and composition.

But $\nu \succ\succ \nu_T$ in laminar sublayer.

In laminar flow it is the molecular motion which transports momentum. (Remember mail bag example). Hence lower rates of transport for a given driving force. Alternatively if we consider heat transfer from the wall to bulk, heat conduction across the laminar sub layer dominates the process (Heat transfer comes later).

- Pictorially



Eddy gives rise to normal velocity $\hat{V}$. This transports x directional momentum in the y direction gives rise to a momentum flux, or shear, τ.

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5) Dissipative Nature of Turbulence.

- Turbulence comprises eddies of all sizes.
- The largest eddies are as big as the flow field. They extract energy from the flow but are not efficient at dissipating energy. In the absence of an energy source, however, turbulence dies away .
- There is an energy cascade from the large eddies, through eddies of progressively smaller size until a lower limit is reached. This lower limit is controlled by viscous dissipation of energy and Kinematics viscousity and the rate of energy supply are the important quantities. Based on dimensional analysis this lower limit of eddy size is given by:

$$\frac{\eta}{d}=\left[\frac{\nu}{u'd}\right]^{3/4}=\mathrm{Re}^{-3/4}$$

Where η = size of small eddies; ν = kinematics viscosity
d = size of largest eddies; u' = RMS turbulent velocity

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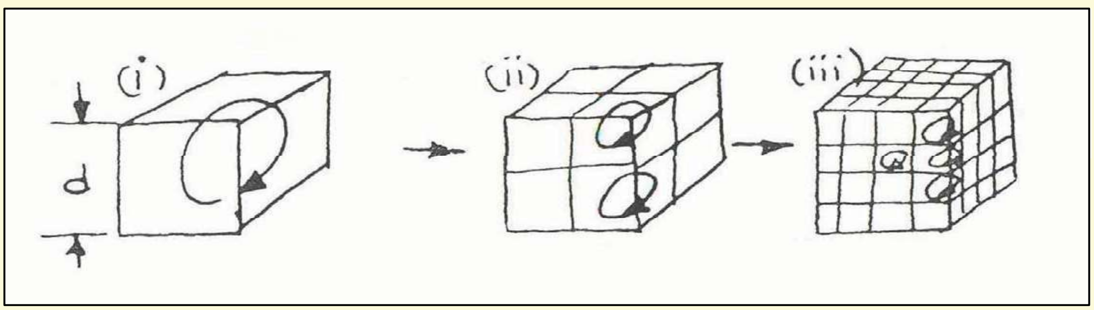

$$dissipation\ rate \propto \frac{surface}{volume}\quad\left(\frac{s}{v}\right)$$

$$(i)\ \left(\frac{s}{v}=\frac{6}{d}\right)\quad (ii)\ \left(\frac{s}{v}=8\times\frac{6d^2}{4}\Big/d^3=\frac{12}{d}\right)\quad (iii)\ \left(\frac{s}{v}=64\times\frac{6d^2}{16}\Big/d^3=\frac{24}{d}\right)$$

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High Reynolds number phenomenon

- Express Newton's Law of viscosity in dimensionless form.

$$\tau^{+} = \frac{\tau}{\rho \bar{u}^{2}} = -\frac{\mu d}{\rho \bar{u}^{2} d}\frac{du}{dy} = -\frac{1}{\text{Re}}\frac{d\left(u/\bar{u}\right)}{d\left(y/d\right)} = -\frac{1}{\text{Re}}\frac{du^{+}}{dy^{+}}$$

Reynolds's number arises in dimensionless form of Newton's Law.
- Similarity: compare two flows in similar geometries(same shape but different size)i.e. flows exhibiting geometrical similarity. Suppose Reynolds numbers of each flow are the same though d,u, ρ and μ of each flow may be individually different. Then as a consequence of the above equation each flow will have the same dimensionless distribution of stress and velocity gradient as a function of position,

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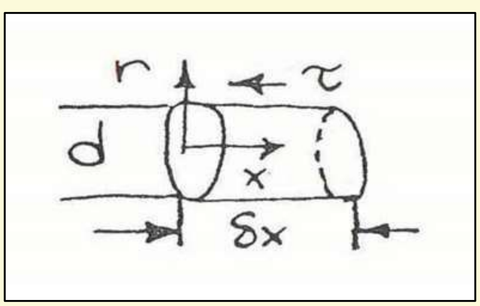
Stress and velocity gradient as a function of position, provided each has the same Reynolds number;

i.e. $u+ = f(y+)$ kinetic similarity
$\tau+ = g(y+)$ dynamic similarity.

The consequence is that friction factor (dimensionless wall shear stress) can be considered a unique function of Re.
Consider a cylindrical element of diameter d and length of δx



$$\text{Viscous forces} \propto \mu \frac{du}{dr}.\pi d \delta x$$

$$\text{Interia forces} \propto e\frac{\pi d^{2}}{4}\delta x \frac{du}{dt}$$

$$\frac{InteriaFor\ \ ces}{ViscousFor\ \ ces} \propto \frac{\rho d}{\mu}\frac{dr}{dt} \equiv \text{Re}$$

High Re-interia forces dominate → Turbulent flow
Low Re-viscous forces dominate → Laminar flow

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Summery Notes on Turbulence

- Most flows are turbulent both in nature and engineering.
- A turbulent pipe flow can be divided into three regions:
  - a) Laminar sublayer - no eddies.
  - b) Transition region – damped eddies (size y)
  - c) Turbulent core – undamped eddies (size d)

- Turbulent velocities:

$u_t = u + \hat{u}$ {Instantaneous = local mean + fluctuant}

$u = \frac{1}{T}\int_0^T u_t dt$ {T is a time long enough to include many eddies}

$u' = [\frac{1}{T}\int_0^T \hat{u}^2 dt]^{\frac{1}{2}}$ {RMS velocity characterizes turbulence}

$u' \cong v' \cong w'$ {Turbulence is homogenous}

$u' / \bar{u} \cong 0.1$ {Turbulence intensity}

- Newton's Law in turbulent flows. It is tempting to write

$$\tau / \rho = -(\nu + \nu_t)\frac{du}{dy} \qquad \begin{cases} \nu - Kinematic\ Viscosity\ \ for\ \ Molecular \\ \nu_t - Eddy\ Viscosity\ \ for\ Turbulence \end{cases}$$

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ν is a fluid property and constant.
ν_t is a flow property and depends on environment (eddy size)

- Rewrite above equation as
-

$$\tau = -(\nu + \nu_t)\frac{d(\rho u)}{dy} \qquad \text{dimensions} \quad \frac{[M]}{[L]^2}\frac{[L]}{[T]^2}$$

(momentum flux)=(momentum diffusivity)(gradient of mom/vol)

Large ν implies rapid mixing. Diffusivity has dim. $[L]^2/[T]$
$\nu_T >> \nu$: Turbulent flows are rapidly mixed due to eddies.

- Energy in turbulent flows: turbulent dissipates considerable energy. Large eddies take energy from mean floe, but are not efficient in dispersing energy. Small eddies do dissipate energy efficiently. There is a transfer of energy to the small eddies, which appears as heat due to frictional effects.
  Smallest eddy size, η, si given by (dimensional analysis)

  $\frac{\eta}{d} = (\frac{u'd}{\nu})^{\frac{-3}{4}}$ {η is also a good estimate of laminar sub-layer thickness}

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- Rynolds Number arises in Newton's Law in dimensionless form

$$\tau^+ = \frac{\tau}{\rho\bar{u}^2} = -\frac{\mu}{\rho\bar{u}^2}\frac{du}{dy} = -\frac{\mu}{\rho\bar{u}^2 d}\frac{d(u/\bar{u})}{d(y/d)} = -\frac{1}{\mathrm{Re}}\frac{du^+}{dy^+}$$

It may be interpreted as the ratio (interia forces / viscous forces).

- Large Re implies dominance of interia forces which promote turbulence. Small Re will dominance of friction (viscous) forces gives laminar flows. Similarity (Consider different flows of same Reynolds Number) If we have geometric similarity (e.g. two different pipe flows) then we will have kinematic similarity (same du^+/dy^+) and dynamic similarity (same τ^+).

Result

$$f = \tau_w^+ = f(\mathrm{Re})$$

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(Summer 2007)

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بردار چیست؟

کمیت های فیزیکی:

- جرم، حجم، درجه حرارت، فشار، حرارت مخصوص
  - تنها بوسیله یک عدد مشخص میشوند(scalar)
- سرعت، نیرو، مومنتوم، گرادیان درجه حرارت
  - بوسیله هم مقدار و هم جهت مشخص میشوند(vector)

$$\vec{V} = \delta_1 V_1 + \delta_2 V_2 + \delta_3 V_3$$

تنسور چیست؟

یک تنسور درجه دوم τ بوسیله 9 جزء از قبیل:

$$\tau_{11}, \tau_{12}, \tau_{13}, \tau_{21}, \tau_{22}, \tau_{23}, \tau_{31}, \tau_{32}, \tau_{33}$$

مشخص میشود، که فقط به خاطر bookkeeping بصورت زیر نشان داده میشود.

$$\tau = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

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المانهای τ_{11} و τ_{22} و τ_{33} را المانهای دیاگونال و المانهای دیگر از قبیل τ_{12} و τ_{13} و τ_{21} و ... را المانهای غیر دیاگونال گویند.

اگر در تنسوری $\tau_{12} = \tau_{21}$ و $\tau_{13} = \tau_{31}$ و $\tau_{23} = \tau_{32}$ باشد، تنسور را سیمتریک و در غیر اینصورت تنسور را آنتی سیمتریک گویند. ($\tau_{12} = -\tau_{21}$ و $\tau_{13} = -\tau_{31}$ و $\tau_{23} = -\tau_{32}$).

scalar و vector حالت خاصی از تنسور هستند.

تنسور درجه n دارای 3^n المان خواهد بود.

Scalar ~ tensor of order 0	$(3^0 = 1)$
Vector ~ tensor of order 1	$(3^1 = 3)$
Tensor of order 2	$(3^2 = 9)$
Tensor of order 3	$(3^3 = 27)$

Properties

1- Sum of two 2^{nd} order tensors is a 2^{nd} order tensor.
2- If m = scalar, $A_{ij} = 2^{nd}$ order tensor then mA_{ij} is a 2^{nd} order tensor.
3- Symmetric tensor:

$$A_{ij} = - A_{ji}$$

4- Antisymmetric tensor:

$$A_{ij} = - A_{ij}$$

5- $A_{ij} + A_{ji} \rightarrow 2^{nd}$ order symmetric tensor
6- Unit tensor δ_{ij} (2^{nd} order)

$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or KRONECKER DELTA}$$

در واقع ستون ها و ردیف های یک تنسور درجه دو واحد اجزاء سه بردار واحد یعنی δ_1 و δ_2 و δ_3 هستند.

Vector operations from an analytical viewpoint

خیلی از فرمولها و ترمها را میتوان با استفاده از تتنسور درجه دو واحد δ_{ij} (Kronecher delta) و تنسور درجه سه واحد، ϵ_{ijk} (The alternative unit tensor or Levi Civita tensor) که به صورت زیر تعریف میشوند، بشکل خلاصه نوشت:

$$\delta_{ij} = \begin{cases} +1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$$

$$\epsilon_{ijk} = \begin{cases} +1 & if \quad ijk = 123,231,312 \\ -1 & if \quad ijk = 321,213,132 \\ 0 & if \quad \text{any two indices are alike} \end{cases}$$

به عنوان مثال دترمینان 3×3 زیر را با استفاده از ϵ_{ijk} میتوان بصورت زیر نوشت:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{i}^{3} \sum_{j}^{3} \sum_{k}^{3} \epsilon_{ijk}\, a_{1i} a_{2j} a_{3k} \qquad or \qquad \epsilon_{ijk}\, a_{1i} a_{2j} a_{3k}$$

حاصلضرب scalar و cross بردارهای واحد را میتوان بصورت زیر نوشت:

$$\delta_i . \delta_j = \delta_{ij}$$

$$\delta_i \times \delta_j = \sum_{k=1}^{3} \epsilon_{ijk}\ \delta_k \qquad \text{or} \qquad \epsilon_{ijk}\ \delta_k$$

Summation convention

وقتی یک suffix (مثل i,j,k) در یک جمله یا ترم تکرار شود، آن جمله یا ترم باید به اندازه تکرار ممکن آن suffix تکرار شود.

$$e.g. \qquad x_j' = \sum_{i=1}^{3} \lambda_{ji}\, x_i \qquad ; (j = 1,2,3)$$

که میتوانم چنین بنویسیم:

$$x_j' = \lambda_{ji}\, x_i \quad ; \begin{cases} i : \text{Repeated suffix} \\ j : \quad \text{Free suffix} \end{cases}$$

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Definition of a vector and its magnitude the unit vector

$$\vec{V} = \delta_1 V_1 + \delta_2 V_2 + \delta_3 V_3 = \sum_{i=1}^{3} \delta_i V_i$$

$$= \delta_i V_i$$

که در آن δ_1 و δ_2 و δ_3 بردارهای واحد در جهت محورهای 1,2,3 (x,y,z) هستند.

$$|\vec{V}| = \sqrt{V_1^2 + V_2^2 + V_3^2} = \sqrt{\sum_{i=1}^{3} V_i V_i}$$

دو بردار $\vec{V}$ و $\vec{W}$ را مساوی گویند اگر:

$$V_1 = W_1 \quad , \quad V_2 = W_2 \quad , \quad V_3 = W_3$$

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در ارتباط با بردارهای واحد میتوان نوشت:

$$\delta_1.\delta_1 = \delta_2.\delta_2 = \delta_3.\delta_3 = 1$$

$$\delta_1.\delta_2 = \delta_2.\delta_3 = \delta_3.\delta_1 = 0$$

$$\delta_1 \times \delta_1 = \delta_2 \times \delta_2 = \delta_3 \times \delta_3 = 0$$

$\delta_1 \times \delta_2 = \delta_3$ و $\delta_2 \times \delta_3 = \delta_1$ و $\delta_3 \times \delta_1 = \delta_2$

$\delta_2 \times \delta_1 = -\delta_3$ و $\delta_3 \times \delta_2 = -\delta_1$ و $\delta_1 \times \delta_3 = -\delta_2$

Comment

1- A repeated suffix is known as a DUMMY suffix it may be replaced by any other suitable symbol

$$\text{e.g.} \quad \lambda_{ji}\, x_i = \lambda_{jk}\, x_k = \lambda_{jm}\, x_m$$

2- Don't use any suffix more than twise

$$e.g. \quad \lambda_{ji}\, \lambda_{ii} \quad \text{is meaningless}$$

3- $x_i x_i = x_1 x_1 + x_2 x_2 + x_3 x_3 = x_1^2 + x_2^2 + x_3^2$

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Applications

1- Equations of motion of a viscous fluid

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2} + u_3\frac{\partial u_1}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \nu\left(\frac{\partial^2 u_1}{\partial {x_1}^2} + \frac{\partial^2 u_1}{\partial {x_2}^2} + \frac{\partial^2 u_1}{\partial {x_3}^2}\right) + g_1$$

$$\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2} + u_3\frac{\partial u_2}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_2} + \nu\left(\frac{\partial^2 u_2}{\partial {x_1}^2} + \frac{\partial^2 u_2}{\partial {x_2}^2} + \frac{\partial^2 u_2}{\partial {x_3}^2}\right) + g_2$$

$$\frac{\partial u_3}{\partial t} + u_1\frac{\partial u_3}{\partial x_1} + u_2\frac{\partial u_3}{\partial x_2} + u_3\frac{\partial u_3}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_3} + \nu\left(\frac{\partial^2 u_3}{\partial {x_1}^2} + \frac{\partial^2 u_3}{\partial {x_2}^2} + \frac{\partial^2 u_3}{\partial {x_3}^2}\right) + g_3$$

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