در ارتباط با بردارهای واحد میتوان نوشت:

$$\delta_1 . \delta_1 = \delta_2 . \delta_2 = \delta_3 . \delta_3 = 1$$

$$\delta_1 . \delta_2 = \delta_2 . \delta_3 = \delta_3 . \delta_1 = 0$$

$$\delta_1 \times \delta_1 = \delta_2 \times \delta_2 = \delta_3 \times \delta_3 = 0$$

$\delta_1 \times \delta_2 = \delta_3$ و $\delta_2 \times \delta_3 = \delta_1$ و $\delta_3 \times \delta_1 = \delta_2$

$\delta_2 \times \delta_1 = -\delta_3$ و $\delta_3 \times \delta_2 = -\delta_1$ و $\delta_1 \times \delta_3 = -\delta_2$

Comment

1- A repeated suffix is known as a DUMMY suffix it may be replaced by any other suitable symbol

$$\text{e.g.} \quad \lambda_{ji}\, x_i = \lambda_{jk}\, x_k = \lambda_{jm}\, x_m$$

2- Don't use any suffix more than twise

$$e.g. \quad \lambda_{ji}\, \lambda_{ii} \quad \text{is meaningless}$$

3- $x_i x_i = x_1 x_1 + x_2 x_2 + x_3 x_3 = x_1^2 + x_2^2 + x_3^2$

Applications

1- Equations of motion of a viscous fluid

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} + u_3 \frac{\partial u_1}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \nu\left(\frac{\partial^2 u_1}{\partial {x_1}^2} + \frac{\partial^2 u_1}{\partial {x_2}^2} + \frac{\partial^2 u_1}{\partial {x_3}^2}\right) + g_1$$

$$\frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} + u_3 \frac{\partial u_2}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_2} + \nu\left(\frac{\partial^2 u_2}{\partial {x_1}^2} + \frac{\partial^2 u_2}{\partial {x_2}^2} + \frac{\partial^2 u_2}{\partial {x_3}^2}\right) + g_2$$

$$\frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x_1} + u_2 \frac{\partial u_3}{\partial x_2} + u_3 \frac{\partial u_3}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_3} + \nu\left(\frac{\partial^2 u_3}{\partial {x_1}^2} + \frac{\partial^2 u_3}{\partial {x_2}^2} + \frac{\partial^2 u_3}{\partial {x_3}^2}\right) + g_3$$

1- $d\phi = \frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3$

$d\phi = \frac{\partial \phi}{\partial x_i} dx_i$

2- $\frac{dx_k}{dt} = \frac{\partial x_k}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial x_k}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial x_k}{\partial x_3}\frac{\partial x_3}{\partial t}$

$\frac{dx_k}{dt} = \frac{\partial x_k}{\partial x_j}\frac{\partial x_j}{\partial t}$

3- $x_1^2 + x_2^2 + x_3^2 + \cdots + x_k^2$

$x_k . x_k \qquad (k = 1,2, \ldots . n)$

4- $a_{jk} . x_k = a_{j1} . x_1 + a_{j2} . x_2 + a_{j3} . x_3$

5- $A_{pq} . A_{qr} = A_{p1} . A_{1r} + A_{p2} . A_{2r} + A_{p3} . A_{3r}$

The vector differential operation

∇ known as “nobla” or “del”

$$\nabla = \delta_1 \frac{\partial}{\partial x_1} + \delta_2 \frac{\partial}{\partial x_2} + \delta_3 \frac{\partial}{\partial x_3}$$

$$\nabla = \delta_i \frac{\partial}{\partial x_i} = \delta_j \frac{\partial}{\partial x_j}$$

The symbol ∇ is a vector operator (the gradient of a scalar field)

$$\nabla S = \delta_1 \frac{\partial S}{\partial x_1} + \delta_2 \frac{\partial S}{\partial x_2} + \delta_3 \frac{\partial S}{\partial x_3}$$

$$\nabla S = \delta_i \frac{\partial S}{\partial x_i}$$

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The divergence of a vector field

$$\nabla . v = \delta_i \frac{\partial}{\partial x_i} . \left(\delta_j v_j\right)$$

$$= \left(\delta_i \delta_j\right) . \frac{\partial v_j}{\partial x_i} = \left(\delta_{ij}\right) . \frac{\partial v_j}{\partial x_i}$$

$$= \frac{\partial v_j}{\partial x_j}$$

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Product involving δ_{ij}

$$\delta_{ij} T_i = T_j$$

$$\begin{aligned} \delta_{ij} T_i &= \delta_{1j} T_1 + \delta_{2j} T_2 + \delta_{3j} T_3 \\ &= \delta_{11} T_1 + \delta_{21} T_2 + \delta_{31} T_3 = T_1 \\ &= \delta_{12} T_1 + \delta_{22} T_2 + \delta_{32} T_3 = T_2 \\ &= \delta_{13} T_1 + \delta_{23} T_2 + \delta_{33} T_3 = T_3 \end{aligned}$$

The gradient of a vector field

The gradient of a vector field is a tensor

$$\nabla\vec{v} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} & \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_1}{\partial x_3} & \frac{\partial v_2}{\partial x_3} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

The Laplacian of a scalar field

$$\nabla.\nabla S = \left(\delta_i \frac{\partial}{\partial x_i}\right).\left(\delta_j \frac{\partial}{\partial x_j}\right)$$

$$= \delta_{ij} \frac{\partial}{\partial x_i}\frac{\partial S}{\partial x_j}$$

$$= \delta_{ij} \frac{\partial^2 S}{\partial x_i \partial x_j}$$

$$= \frac{\partial^2 S}{\partial x_i \partial x_i} = \frac{\partial^2 S}{\partial x_j \partial x_j}$$

The collection of differential operators which is operating on S (scalar) in the last line is given the symbol ∇^2,

hence in the rectangular coordinate:

$$\nabla^2 = \frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + \frac{\partial^2}{\partial {x_3}^2}$$

In the cylindrical coordinate:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

In the spherical coordinate:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

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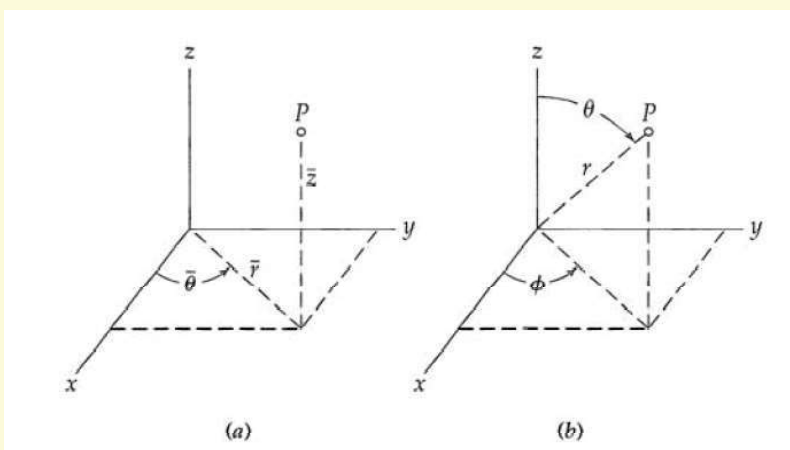
Curvilinear Coordinates



$$\begin{cases} x = r \cos\theta \\ y = r \sin\theta \\ z = z \end{cases}$$

$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r \cos\theta \end{cases}$$

$$r = +\sqrt{x^2 + y^2}$$
$$\theta = \arctan(y/x)$$
$$z = z$$

$$r = +\sqrt{x^2 + y^2 + z^2}$$
$$\theta = \arctan(\sqrt{x^2 + y^2}/z)$$
$$\phi = \arctan(y/x)$$

Description of a fluid motion

The development of an analytical description of fluid flow is based upon the expression of the physical laws related to fluid flow in a suitable mathematical form.

To discuss the motion of a fluid we need to look at its prosperities. If S represents some scalar property of the fluid (concentration, temperature, etc.) then it will be a function of its co-ordinates x_1, x_2, x_3 written as x_i (i=1, 2, 3) and time t.

$$S(x_1, x_2, x_3, t) \equiv S(x_i, t)$$

Time derivatives

The total time derivative, using the chain rule, is

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial S}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial S}{\partial x_3}\frac{dx_3}{dt}$$

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x_i}\frac{dx_i}{dt}$$

$\frac{dS}{dt}$ being the time rate of change of S.

The derivatives $\frac{dx_i}{dt}$ (i =1, 2, 3) have no meaning without definition since both x_i and t are independent variable.

If taken to represent the velocity of the fluid, $u_i = \frac{dx_i}{dt}$ and the time rate of change of S relative to the moving fluid is

$$\frac{\partial S}{\partial t} + u_i\frac{\partial S}{\partial x_i} \text{ written as } \frac{DS}{Dt}$$

Where the operator $\frac{D}{Dt} = \frac{\partial}{\partial t} + u_i\frac{\partial}{\partial x_i}$ is described as differentiation following the fluid motion or "substantial derivative"- imagine an observer riding on the fluid particle and continuously measuring S. however if S is temperature it is not practical to measure the temperature of the fluid with some device, which moves with the fluid.

The temperature would be measured at some fixed in space, x_i held constant. In this case $\frac{dx_i}{dt} = 0$ and the local change measured $\frac{\partial T}{\partial t}$.

Three kinds of time derivatives

1- The partial time derivatives; $\frac{\partial C}{\partial t}$

2- Total time derivatives; $\frac{dC}{dt}$

3- Substantial time derivative; $\frac{DC}{Dt}$

$$\frac{DC}{Dt} = \frac{\partial C}{\partial t} + v.\nabla C$$

$$= \frac{\partial C}{\partial t} + (\delta_i v_i).\left(\delta_j \frac{\partial C}{\partial x_j}\right)$$

$$= \frac{\partial C}{\partial t} + \delta_{ij}.v_i \frac{\partial C}{\partial x_j}$$

$$\frac{DC}{Dt} = \frac{\partial C}{\partial t} + v_j \frac{\partial C}{\partial x_j}$$

$$\boldsymbol{\frac{Du_j}{Dt} = \frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i}}$$

15 terms; 3 equations, i : repeated suffix, j : free suffix

مثال- یک میدان جریان بوسیله معادله زیر ارائه شده است:

$$v(x,y,t) = 2x^2 i + 4xytj$$

سرعت و شتاب یک ذره را در مختصات (1,2,3) بدست آورید؟

$$v(1,2,3) = 2i + 24j$$

$$a = a(x,y,z) = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$$

$$= 4xyj + 2x^2(4xi + 4ytj) + 4xy + 4xtj$$

$$a(1,2,3) = 8i + 252j$$

The differential momentum balance

Newton's second law applied to an element of fluid

$$F_x = \frac{1}{g_c} m a_x \qquad ; F_x, F_y, F_z \text{ components of force vector}$$

رابطه فوق را میتوان بصورت rate of change of momentum نوشت:

$$F_x = \frac{1}{g_c} m \frac{d(m v_x)}{dt}$$

$$F_x = \frac{m}{g_c} \frac{D v_x}{Dt} \qquad ; m = \rho dxdydz$$

$$= \frac{1}{g_c} \rho dxdydz \frac{D v_x}{Dt}$$

اگر F_x نشان دهنده منتج تمام نیروهای موثر در جهت x بر المان سیال باشد، مناسب خواهد بود که آنرا به دو نیرو به صورت زیر تقسیم کنیم:

1- Body forces
2- Mechanical stresses

Body forces

نیروی بدنی (body force) بر کل اعمال میشود و آنرا میتوان بصورت زیر نشان داد:

$$F_{xb} = x.\rho dxdydz \quad ; \quad x = \frac{Lb_f}{Lb_m} = \frac{g}{g_c} \cos\beta$$

که β زاویه بین جهت x و جهت اثر نیروی ثقل است

Stress

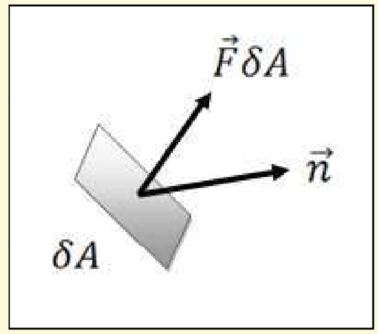
Linear momentum principle

The time rate of change of momentum of a body = force acting on the body

= body forces + surface forces

The forces acting on a body are either internal or external. To discuss the internal forces consider a small element of area δA. Define the outward pointing normal to be positive.

Assume that the outside material touching the surface exerts a forces on the inside material across an element of area, where is a vector which is a function of position, time and orientation of the element. is called the stress force/unit area or simply the stress. is a vector so called stresses act on an area that may in general have components in each direction. has components normal to (normal stresses) and parallel to (shear stresses).

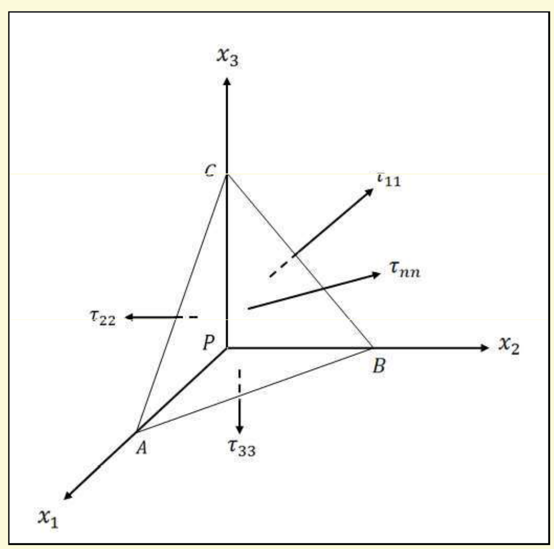


STSTIC STRESS IN A FLUID IS ISOTROPIC

Consider a small tetrahedron with three of it's faces through P and normal to the co-ordinate axes, and the fourth face normal to a given direction n_i

STSTIC STRESS ON A TETRAHEDRON

Plane	Area	outward normal	stress vector
ABC	δA_n	$\vec{n}$	$\tau_{nn}\vec{n}$
BCP	δA_1	$-\vec{e}_1$	$-\tau_{11}\vec{e}_1$
ACP	δA_2	$-\vec{e}_2$	$-\tau_{22}\vec{e}_2$
ABP	δA_3	$-\vec{e}_3$	$-\tau_{33}\vec{e}_3$



Momentum Balance

Rate of change of momentum=Body Forces + Surface Forces

$$0 = \rho g \delta V + \tau_{nn} n \delta\delta_n - e_1\tau_{11}\delta A_1 - e_2\tau_{22}\delta A_2 - e_3\tau_{33}\delta A_3$$

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Areas of co-ordinate planes are projection of the oblique plane onto co-ordinate planes, so $n_1.\delta A_1 = \delta A_1$ and $n_i.\delta A_n = \delta A_i$

Also, $\delta V / \delta A_n \longrightarrow 0$ as $\delta A_n \longrightarrow 0$

$$0=\tau_{nn} n - e_1 \tau_{11} \delta A_1 - e_2 \tau_{22} \delta A_2 - e_3 \tau_{33} \delta A_3$$

But $\underline{n} = n_1 \underline{e}_1 + n_2 \underline{e}_2 + n_3 \underline{e}_3$

Hence $\underline{0} = \underline{e}_1 n_1 (\tau_{nn} - \tau_{11}) + \ldots$

$$\tau_{nn} = \tau_{11} = \tau_{22} = \tau_{33}$$

i.e. the normal stress acting on a surface is independent of the orientation of the plane for fluid at rest.

We associate this stress with the static pressure. As the stress acting the element in the direction of the outward normal (i.e. tension) we write

$$\tau_{11} = \tau_{22} = \tau_{33} = -P$$

i.e. $\tau_{ij} = -P\delta_{ij}$ for fluids at rest.

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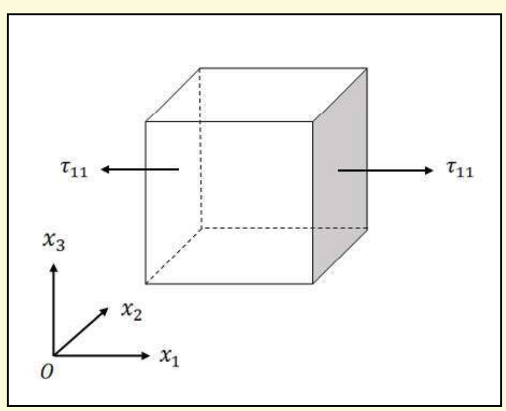
FLUID STATIC'S

Definition

A fluid deforms continuously under the action of a shear stress.
Thus no shear stress can act normally to any surface within the fluid at rest.
All stresses act normally to any surface within the fluid.
Thus $\tau_{ij} = 0 (i \neq j)$ (shear stress zero)
It can be shown (see handout) that, "the normal stress acting on a surface at a point is independent of the orientation of the surface including that point for a fluid at rest".

i.e. $\tau_{11} = \tau_{22} = \tau_{33} = -P$

$$\tau_{ij} = -P\delta_{ij} \qquad (\delta_{ij} = 1, i = j; \delta_{ij} = 0, i \neq j)$$



Definition

The pressure at the point in a fluid at rest is stress that acts on a surface in a direction opposite to the normal vector.

Newton's law

(applied to a parallelepiped of sides $\delta x_1, \delta x_2, \delta x_3$).
{rate of change of momentum}={body forces}

In 1 direction

$$0 = \rho g_1 \delta x_1 \delta x_2 \delta x_3 + \tau_{11} \delta x_2 \delta x_3 \,|_{x_1 + \delta x_1} - \tau_{11} \delta x_2 \delta x_3 \,|_{x_1}$$

Hence in the limit $\delta x_1 \longrightarrow 0$

$$\rho g_1 = \frac{-\partial \tau_{11}}{\partial x_1} = \frac{\partial p}{\partial x_1}$$

$$\rho g_j = \frac{\partial p}{\partial x_j}$$

This is a vector equation. It summaries 3 equations, 1 for each coordinate drin.

Advanced Fluid Mechanics

Fluid Dynamics

This is concerned with fluids in motion, in particular the application of Newton's Second Law to an element of fluid. Any element will deform, in general, and change shape. The rate of change of momentum must be for a specific element, no matter how it deforms subsequently. The appropriate derivative must follow the fluid motion-substantial time derivative.

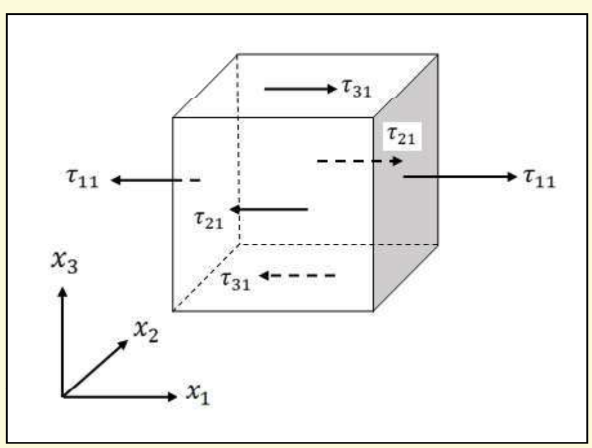
Local + Convective = body force + pressure force + viscous force

Mathematically

$$U_1 = U_1(x_1, x_2, x_3, t)$$

$$dU_1 = \frac{\partial u_1}{\partial x_i} dx_i + \frac{\partial u_1}{\partial t} dt$$

$$\frac{du_1}{dt} = \frac{\partial u_1}{\partial t} + u_i \frac{\partial u_1}{\partial x_i}$$



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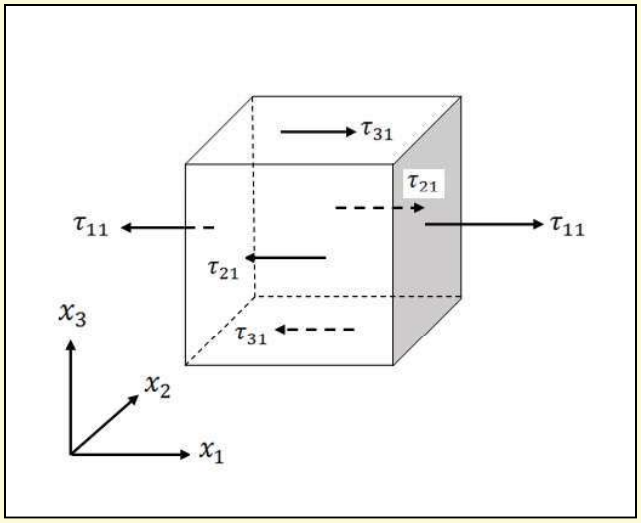
Consider a parallelepiped $\delta x_1, \delta x_2, \delta x_3$ are stress forces acting in 1 direction of this fluid element in a fluid motion are shown in figure

Apply Newton's Second law to the element in the 1 direction

$$\frac{D}{Dt}\{\rho \delta x_1 \delta x_2 \delta x_3 . u_1\} = X_1 \delta x_1 \delta x_2 \delta x_3 + \text{ Surface force}$$

The mass considered in the element = $\rho \delta x_1 \delta x_2 \delta x_3$. is constant.



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$$\rho \frac{Du_1}{Dt} = X_1 + \frac{\text{surface force}}{\text{unit volume}}$$

$$\frac{\text{surface force}}{\text{unit volume}} =$$

$$\frac{\left(\tau_{11}\big|_{x_1+\delta x_1} - \tau_{11}\big|_{x_1}\right)\delta x_2 \delta x_3 + \left(\tau_{21}\big|_{x_2+\delta x_2} - \tau_{21}\big|_{x_2}\right)\delta x_1 \delta x_3 + \left(\tau_{31}\big|_{x_3+\delta x_3} - \tau_{31}\big|_{x_3}\right)\delta x_1 \delta x_2}{\delta x_1 \delta x_2 \delta x_3}$$

$$= \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \tau_{31}}{\partial x_1} = \frac{\partial}{\partial x_j}\left(\tau_{j1}\right)$$

$$\rho \frac{Du_i}{Dt} = x_i + \frac{\partial}{\partial x_j}\left(\tau_{ji}\right)$$

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Body force

Usually due to gravity $X_i = \rho g_i$

Stress force

Stress forces include a contribution from a static pressure force and the rate of strain due to the flow. Stress may be 'decomposed' into these two contributions as shown in the following equation.

$\tau_{ji} = -P\delta_{ji} + \sigma_{ji}$ where $P\delta_{ji}$ = the pressure contribution to the stress tensor(normal)

$\sigma_{ji} =$ the dynamic (flow) contribution (normal and shear) for a Newtonian flow, the dynamic contribution is given by Newton's Law of viscosity (the derivation of which is beyond the scope of this course).

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$$\sigma_{ji} = \mu\left\{\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right\}$$ for constant ρ

It is interesting to calculate the trace of the tensor τ_{ji} by the contraction j=I.

$\tau_{ii} = -P\delta_{ii} + 2\mu\frac{\partial u_i}{\partial x_i}$ and $\frac{\partial u_i}{\partial x_i} = 0$ for constant ρ from continuity

Eq. $\tau_{ii}/3 = -P$

The trace of a tensor is a scalar, in this case of magnitude -3p.
Hence substitution in above equation,

$$\rho\frac{Du_i}{Dt} = \rho g_i + \frac{\partial}{\partial x_j}\left\{-P\delta_{ji} + \mu\left[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right]\right\}$$

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Now $\frac{\partial}{\partial x_j}\left\{-P\delta_{ji}\right\} = -\frac{\partial p}{\partial x_i}$ $[\delta_{ji} = 1$ only if j=i]

Assume constant density ; $\mu \frac{\partial^2 u_j}{\partial x_i \partial x_j} = \mu \frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial x_j}\right) = 0$ [see cons. Of mass]

Hence $\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = g_i - \frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{v\partial^2 u_i}{\partial x_j \partial x_j}$

(Local +convective) acceleration = $\frac{body\ \ force + pressure\ \ force + viscouse\ \ force}{unit\ mass}$

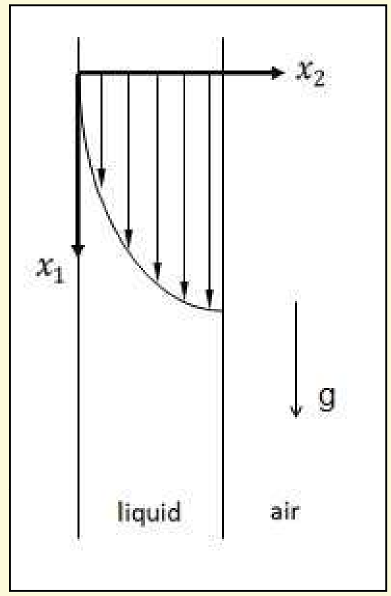
GENERAL PROBLEM IN FLUID FLOW

NO. Unknown	Unknown
1	Pressure: P
3	Velocity: u_i
9	Stresses: τ_{ij}
-9	Newtonian equation and viscosity
+ --------------------	
4	

Thus we need four equations for determining P and u_i, continuity equation and three motion equations.

Simple Solution of the N-S Equations

Consider laminar flow down a vertical wall,



(i) Steady flow $\dfrac{\partial u_i}{\partial t} = 0$

(ii) Two dimensional $\dfrac{\partial u_i}{\partial x_3} = 0, \quad u_3 = 0$

(iii) Suppose the flow is fully developed

$$\frac{\partial u_i}{\partial x_1} = 0, \quad \frac{\partial u_1}{\partial x_1} = 0$$

(iv) By continuity $\dfrac{\partial u_2}{\partial x_2} = -\dfrac{\partial u_1}{\partial x_1} = 0$

Integrating $u_2 = u_2(x_1)$

At $x_2 = 0$ $u_2 = 0$

$u_2 = 0$ everywhere

(v) Momentum equation in 3 direction is zero.

(vi) Momentum equation in 2 direction

$$\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2} = -\frac{1}{\rho}\frac{\partial p}{\partial x_2} + \nu\left\{\frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_1^2}\right\}$$

Steady developed $u_2=0$ $u_2 = 0$ developed

Hence $\dfrac{\partial p}{\partial x_2} = 0$ $\quad p = p(x_1)$

(vii) At liquid surface (still air) $p = p_0 + \rho_{air} g x_1$

(viii) Momentum equation in 1 direction.

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \upsilon\left\{\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x^2{}_2}\right\} + g$$

Steady developed $u_2=0$ developed

With $\dfrac{\partial p}{\partial x_1} = \rho_{air} g$

$$\upsilon \frac{d^2 u_1}{dx_2^2} = -g + \frac{\rho_{air}}{\rho} g = -g\left\{1 - \frac{\rho_{air}}{\rho}\right\}$$

But $\rho_{air} \langle\langle \rho_{liquid} = \rho$

$$\nu \frac{d^2 u_1}{dx^2} = -g$$

Boundary conditions; at $x_2 = 0$ $u_1 = 0$

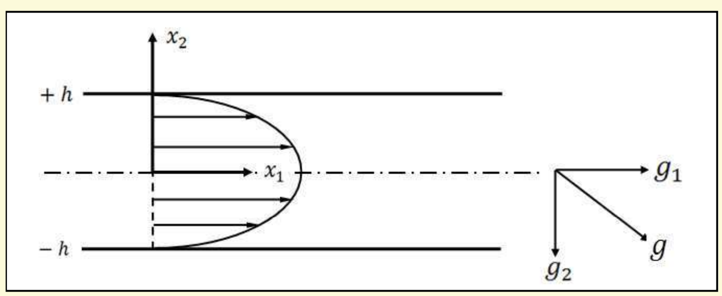
at $x_2 = \delta$ $\dfrac{du_1}{dx_2} = 0$ No interfacial shear

Integrating $\upsilon \dfrac{du_1}{dx_2} = -g x_2 + c$ $c = g\delta$

$$\upsilon u_1 = -g\left\{\delta x_2 - \frac{x_2^2}{2}\right\} = g\delta^2\left\{\left[\frac{x_2}{\delta}\right] - \frac{1}{2}\left[\frac{x_2}{\delta}\right]^2\right\}$$

Volumetric flow rate $Q = \displaystyle\int_0^\delta u_1\, dx_2 = \frac{g\delta^3}{3\nu}$

Flow Between Parallel Plates



At $x_2 = \pm$ h $u_1 = 0$

Pressure gradient:

$\left[\dfrac{\partial p}{\partial x_1}\right]$ is assumed constant in the x_1 direction.

Steady flow: $\dfrac{\partial u_i}{\partial t} = 0;$

2 dimensional: $\dfrac{\partial u_i}{\partial x_3} = 0;$ $u_3 = 0$

Fully developed flow, $\dfrac{\partial u_i}{\partial x_1} = 0; \dfrac{\partial u_1}{\partial x_1} = 0; \dfrac{\partial u_2}{\partial x_2} = 0$ $u_2 = 0$ (as before)

Continuity is satisfied,
Momentum in 3 direction is satisfied,
Momentum in 2 direction,

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x_2} + g_2 = -\frac{1}{\rho}\frac{\partial P}{\partial x_2} \quad \text{where} \quad P = p - \rho g_i x_i$$

$P = P(x_1)$

(N.B. Manometers connected to wall pressure tapping measure Δp)

Momentum in 1 direction

$$0=-\frac{1}{\rho}\frac{\partial P}{\partial x_1}+g_1+\upsilon\frac{\partial^2 u_1}{\partial x_2^2} \qquad \text{where} \quad P=p-\rho\, g_i\, x_i$$

$$\upsilon\frac{d^2 u_1}{dx_2^2}=\frac{1}{\rho}\frac{dP}{dx_1}$$

$$\upsilon\frac{du_1}{dx_2}=\frac{1}{\rho}\left[\frac{dP}{dx_1}\right]\{x_2+c\}$$

$$\upsilon\, u_1=\frac{1}{\rho}\left[\frac{dP}{dx_1}\right]\{\frac{x_2^2}{2}+cx_2+D\}$$

$$u_1=\frac{1}{2u}\left[\frac{dP}{dx_1}\right]\{x_2^2-h^2\}$$

3. Suppose top plate moves with velocity U_T
   Bottom plate with velocity U_B

   Substituting in (1)

$$\frac{\mu U_T}{\rho}=\frac{1}{\rho}\left[\frac{dp}{dx_1}\right]\left\{\frac{h^2}{2}+Ch+D\right\}$$

$$\frac{\mu U_B}{\rho}=\frac{1}{\rho}\left[\frac{dp}{dx_1}\right]\left\{\frac{h^2}{2}-Ch+D\right\}$$

$$\mu(U_T+U_B)=\left[\frac{dp}{dx_1}\right]\{h^2+2D\}$$

$$\mu(U_T-U_B)=\left[\frac{dp}{dx_1}\right]\{2Ch\}$$

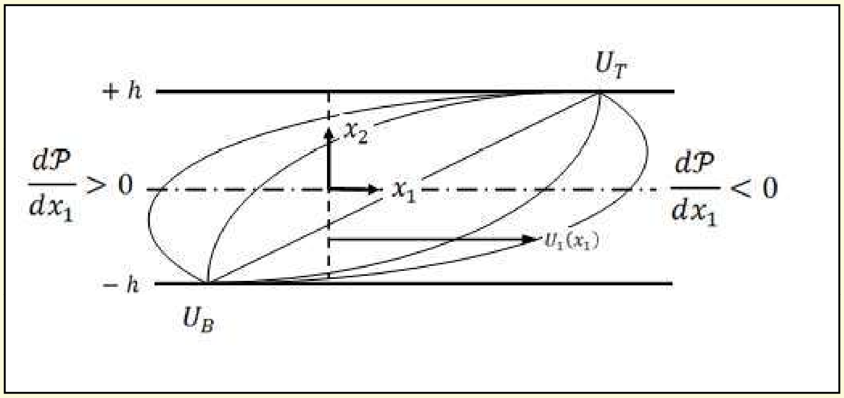
$$\left\{u_1-\frac{(U_T+U_B)}{2}\right\}=\frac{1}{\mu}\left[\frac{dp}{dx_1}\right]\left\{\frac{x_2^2-h^2}{2}\right\}+\left\{\frac{(U_T-U_B)x_2}{2h}\right\}$$

Check when $U_T = U_B = 0$ simplifies to previous expression
when $x_2 = +h$ $u_1 = U_T$ satisfies B.C.'s
$x_2 = -h$ $u_1 = U_B$

when $\left[\frac{dp}{dx}\right]=0$ $\qquad u_1=\left\{\frac{U_T+U_B}{2}\right\}+\frac{(U_T-U_B)x_2}{2h}$

i.e. linear velocity profile
known as Couette flow

The following velocity profiles are obtained with various (dp/dx)



Consider flow relative to lower plate.

If p decreases in direction of flow (normal pressure gradient)
Velocity is always downstream relative to lower plate

If p increases in direction of flow (adverse pressure gradient)
Velocity may become upstream relative to lower plate

This has importance in lubrication theory.

Plate suddenly brought into motion

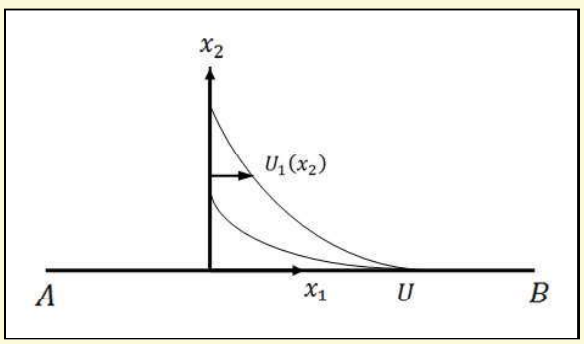


Plate AB initially stationary. Motion with velocity U starts at time t=0

B.C's $t < 0$ $u_1 = 0$ at $x_2 > 0$

$t \geq 0$ $u_1 = U$ at $x_2 = 0$

$u_1 \to 0$ at $x_2 \to \infty$

Momentum equation in 2 directions

$$\frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} = -\frac{1}{\rho}\frac{\partial p}{\partial x_2} + \nu\left\{\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2}\right\}$$

Assuming an infinite plate and 2D motion

$$\frac{\partial u_1}{\partial x_1} = 0 \qquad \therefore \frac{\partial u_2}{\partial x_2} = 0 \qquad \therefore u_2 = 0$$

$$\therefore \frac{\partial p}{\partial x_2} = 0$$

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i.e. pressure remains hydrostatic

$$\frac{\partial p}{\partial x_1} = \frac{\partial P}{\partial x_1} = 0$$

Momentum equation in 1 direction

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \upsilon\left\{\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2}\right\}$$

$$\frac{\partial u_1}{\partial x_1}=0 \qquad u_2 = 0 \qquad \frac{\partial P}{\partial x_1}=0 \qquad \frac{\partial u_1}{\partial x_1}=0$$

$$\frac{\partial u_1}{\partial t} = \upsilon \frac{\partial^2 u_1}{\partial x_2^2}$$

c.f. equation in maths course

$$\frac{u_1}{U} = 1 - erf\left\{\frac{x_2}{2\sqrt{\upsilon t}}\right\}$$

erf (0)=0.0 erf(∞)=1.0

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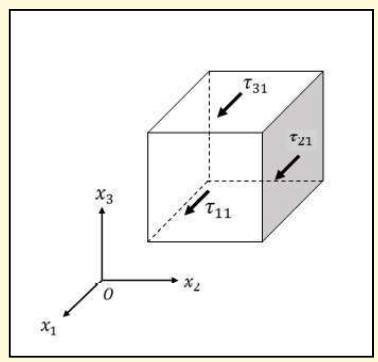
Conservation of Momentum



Consider parallel piped $\delta x_1, \delta x_2, \delta x_3$ fixed in space.

Carry out a momentum balance in the i direction. (Say 1 direction)

$$\left\{\frac{\text{Rate of accumulation of i directional momentum}}{\text{unit volume}}\right\} = \frac{\frac{\partial}{\partial t}\{\rho u_i \,\delta x_1 \delta x_2 \delta x_3\}}{\delta x_1 \delta x_2 \delta x_3}$$

$$\left\{\frac{\text{Net Rate of inflow of i directional momentum by convection}}{\text{unit volume}}\right\} = \frac{\rho u_1 \delta x_2 \delta x_3 u_i|_{x_i} - \rho u_1 \delta x_2 \delta x_3 u_i|_{x_1+\delta x_1}}{\delta x_1 \delta x_2 \delta x_3}$$

$$+ \ldots +$$

$$= -\frac{\partial}{\partial x_j}(\rho u_i u_j)$$

$$\left\{\frac{\text{Force acting on element due to gravity}}{\text{unit volume}}\right\} = \frac{\rho g_i \,\delta x_1 \delta x_2 \delta x_3}{\delta x_1 \delta x_2 \delta x_3}$$

$$\left\{\frac{\text{Force acting on element in i direction due to stresses}}{\text{unit volume}}\right\} = \frac{\tau_{1i}\,\delta x_2 \delta x_3|_{x_i+\delta x_1} - \tau_{1i}\,\delta x_2 \delta x_3|_{x_1}}{\delta x_1 \delta x_2 \delta x_3}$$

$$+ \ldots +$$

$$= -\frac{\partial}{\partial x_j}(\tau_{ji})$$

$$= -\frac{\partial}{\partial x_j}(-p\delta_{ji} + \sigma_{ji})$$

$$= -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(\sigma_{ji})$$

Hence. $$\frac{\partial}{\partial t}(\rho u_i)+\frac{\partial}{\partial x_j}(\rho u_i u_j)=\rho g_i-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}(\sigma_{ji})$$

$$\rho\frac{\partial u_i}{\partial t}+\rho u_i\frac{\partial u_j}{\partial x_j}+\{u_i\frac{\partial p}{\partial t}+u_i\frac{\partial}{\partial x_j}(\rho u_j)\}=\rho g_i-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}(\sigma_{ji})$$

$$\rho\frac{Du_i}{Dt}=-\frac{\partial p}{\partial x_i}+\rho g_i+\frac{\partial}{\partial x_j}(\sigma_{ji})$$

Multiply by u_i

$$\rho\frac{D}{Dt}(\frac{1}{2}u_i u_i)=-\frac{\partial}{\partial x_i}(u_i P)+P\frac{\partial u_i}{\partial x_i}=\rho g_i u_i+\frac{\partial}{\partial x_j}(u_i\sigma_{ji})-\sigma_{ji}\frac{\partial u_i}{\partial x_j}$$

Equation of mechanical energy (scalar)

Energy Equations

Momentum Equation

$$\frac{\partial}{\partial t}(\rho u_i)+\frac{\partial}{\partial x_j}(\rho u_i u_j)=\rho g_i-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}(\sigma_{ji})$$

$$\rho\frac{\partial u_i}{\partial t}+\rho u_i\frac{\partial u_j}{\partial x_j}+\{u_i\frac{\partial p}{\partial t}+u_i\frac{\partial}{\partial x_j}(\rho u_j)\}=\rho g_i-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}(\sigma_{ji})$$

Continuity Equation

$$\frac{\partial p}{\partial t}+\frac{\partial}{\partial x_j}(\rho u_i)=0 \qquad i.e. \quad \frac{D\rho}{Dt}=-\rho\frac{\partial u_j}{\partial x_j}$$

$$\boxed{\rho\frac{Du_i}{Dt}=-\frac{\partial p}{\partial x_i}+\rho g_i+\frac{\partial}{\partial x_j}(\sigma_{ji})}$$

Multiply by u_i:

$$\rho \frac{D}{Dt}(\frac{1}{2}u_iu_i) = -\frac{\partial}{\partial x_i}(u_i p) + p\frac{\partial u_i}{\partial x_i} + \rho g_i u_i + \frac{\partial}{\partial x_j}(u_i\sigma_{ji}) - \sigma_{ji}\frac{\partial u_i}{\partial x_j}$$

Physical Interpretation

(1) $$\rho \frac{D}{Dt}(\frac{1}{2}u_iu_i) = \rho\{\frac{\partial}{\partial t}(\frac{1}{2}u_iu_i) + u_j\frac{\partial}{\partial x_j}(\frac{1}{2}u_iu_i)\} + \frac{1}{2}u_iu_i\{\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\}$$
$$= \frac{\partial}{\partial t}(\frac{1}{2}\rho u_iu_i) + \frac{\partial}{\partial x_j}(\frac{1}{2}\rho u_iu_iu_j)$$

= {Rate of increase of kinetic energy/ unit volume} + {Net rate of kinetic energy/volume}

(2) $-\frac{\partial}{\partial x_i}(u_i p) =$ {Rate of work done on element by pressure forced per unit volume}

(3) $$p\frac{\partial u_i}{\partial x_i} = -\frac{p}{\rho}\left[\frac{D\rho}{Dt}\right] = p\rho\frac{D(\frac{1}{\rho})}{Dt} = \frac{p}{\hat{v}}\frac{D\hat{v}}{Dt}$$ $\hat{v}$=specific volume (i.e. per unit mass)

$=\frac{1}{\hat{v}}\frac{D\hat{U}}{Dt}$=[rate of conversion to internal energy / unit volume]

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(4) $\rho g_i u_i$ = [Rate of work done by gravity forces per unit volume]

(5) $\frac{\partial}{\partial x_j}(u_i\sigma_{ji}) =$ [Rate of work done by stress forces per unit volume]

By difference

(6) $$-\sigma_{ji}\frac{\partial u_i}{\partial x_j} = -\mu[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}]\frac{\partial u_i}{\partial x_j}$$ (see tutorial 1)

= [Rate of conversion to internal energy due to viscous dissipation / unit volume]

This is the equation of mechanical energy (No.2)
It states that

[Rate of accumulation of kinetic energy / unit volume]= [Sum of rates of work done by pressure, gravity and stresses / unit volume] – [Rate of conversion to internal energy / unit volume]

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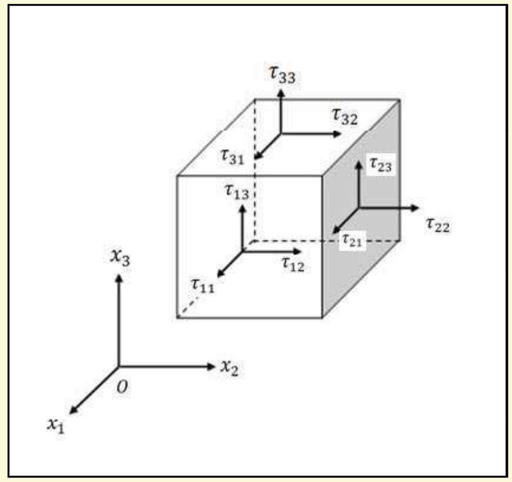
Conservation of Energy



Consider the parallelepiped of side $\partial x_1 \; \partial x_2 \; \partial x_3$

The first law of thermodynamics will be applied

[Rate of accumulation of internal and kinetic energy] = [Net rate of internal energy gained by convection and conduction] + [Net rate of work done on the system by surrounds]

[Rate of accumulation of internal and kinetic energy / unit volume] =

$$\frac{\partial}{\partial t}[\rho\, \delta x_1\, \delta x_2\, \delta x_3\, \{\hat{U} + \frac{1}{2}V^2\}] / \delta x_1\, \delta x_2\, \delta x_3$$

= [net rate of internal and kinetic energy in by convection / unit volume]

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[Rate of accumulation of internal and kinetic energy / unit volume] = $\frac{\partial}{\partial t}[\rho\, \delta x_1\, \delta x_2\, \delta x_3\, \{\hat{U}+\frac{1}{2}V^2\}]/\delta x_1\, \delta x_2\, \delta x_3$

$$= \begin{bmatrix} \text{net rate of internal and} \\ \text{kinetic energy in by} \\ \text{convection / unit volume} \end{bmatrix} = \lim_{\delta x_i \to 0} \left[\frac{\rho u_1\, \delta x_2\, \delta x_3 \{\hat{U}+\frac{1}{2}V^2\}\big|_{x_1} - \rho u_1\, \delta x_2\, \delta x_3 \{\hat{U}+\frac{1}{2}V^2\}\big|_{x_1+\delta x_1}}{\delta x_1 \delta x_2\, \delta x_3} \right]$$

$$+ \begin{bmatrix} \text{net rate of heat energy in by} \\ \text{conduction / unit volume} \end{bmatrix} = \lim_{\delta x_i \to 0} \left[\frac{(q_1\big|_{x_1} - q_1\big|_{x_1+\delta x_1})\delta\delta_2\, \delta x_3}{\delta x_1 \delta x_2\, \delta x_3} + \ldots + \right]$$

$$+ \begin{bmatrix} \text{net rate of work done by gravity} \\ \text{forces / unit volume} \end{bmatrix} = \lim_{\delta x_i \to 0} \left[\frac{u_1\, g_1\, \delta x_1\, \delta x_2\, \delta x_3}{\delta x_1 \delta x_2\, \delta x_3}\rho + \ldots + \right]$$

$$+ \begin{bmatrix} \text{net rate of workl done} \\ \text{by stress forces / unit volume} \end{bmatrix} = \lim_{\delta x_i \to 0} \left[\frac{u_1 \tau_{11}\, \delta x_2\, \delta x_3\big|_{x_1+\delta x_1} - u_1 \tau_{11}\, \delta x_2\, \delta x_3\big|_{x_1}}{\delta x_1 \delta x_2\, \delta x_3} \right.$$

$$+ \frac{u_2 \tau_{12}\, \delta x_2\, \delta x_3\big|_{x_1+\delta x_1} - u_2 \tau_{12}\, \delta x_2\, \delta x_3\big|_{x_1}}{\delta x_1 \delta x_2\, \delta x_3}$$

$$+ \frac{u_1 \tau_{21}\, \delta x_1\, \delta x_3\big|_{x2+\delta x2} - u_1 \tau_{21}\, \delta x_1\, \delta x_3\big|_{x2}}{\delta x_1 \delta x_2\, \delta x_3}$$

$$+ \ldots +$$

$$\left. + \ldots + \ldots + \quad \right]$$

$$\frac{\partial}{\partial t}\{\rho\{\hat{U}+\frac{1}{2}V^2\}\} + \frac{\partial}{\partial x_i}\{\rho u_i (\hat{U}+\frac{1}{2}V^2)\} = -\frac{\partial q_i}{\partial x_i} + \rho u_i\, g_i + \frac{\partial}{\partial x_j}\{u_i \tau_{ji}\}$$

$$(\hat{U}+\frac{1}{2}V^2)\{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i)\} + \rho\{\frac{D}{Dt}(\hat{U}+\frac{1}{2}V^2)\} = -\frac{\partial q_i}{\partial x_i} + \rho u_i\, g_i - \frac{\partial}{\partial x_i}\{u_i p\} + \frac{\partial}{\partial x_j}(u_i \sigma_{ji})$$

$$\rho\{\frac{D}{Dt}(\hat{U}+\frac{1}{2}V^2)\} = -\frac{\partial q_i}{\partial x_i} + \rho u_i\, g_i - \frac{\partial}{\partial x_i}\{u_i p\} + \frac{\partial}{\partial x_i}(u_i \sigma_{ji})$$

Equation of total energy

Subtract equation (2) from equation (3) to get
Equation of Thermal Energy

$$\rho\frac{D\hat{U}}{Dt}=-\frac{\partial q_i}{\partial x_i}-p[\frac{\partial u_i}{\partial x_i}]+\sigma_{ji}\frac{\partial u_i}{\partial x_j}$$

Physical Interpretation
[Rate of change of internal energy / unit volume] = [Rate of input of internal energy by conduction / vol.] [Rate of change of internal energy by compression / vol.] [Rate of gain of internal energy by viscous dissipation / vol.]

N.B. the last two terms represent interconversion of thermal and mechanical energy

$$-p\frac{\partial u_i}{\partial x_i}=-\frac{p}{\hat{V}}\frac{D\hat{V}}{Dt}$$ i.e. compression work

This term is reversible because $\frac{D\hat{V}}{Dt}$ may be true or –ve

$$\sigma_{ji}\frac{\partial u_i}{\partial x_j}=\mu\frac{\partial u_i}{\partial x_j}[\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}]$$

and may be written as a sum of squared terms

This term is irreversible because it is always positive.
It represents degradation of mechanical energy to internal energy (+ve in 4)

In this course we seek solution of equation (4).

Thermodynamic Relationships

The simplification of equation (4) required the following

(1) $d\hat{U}=Td\hat{S}-Pd\hat{V}$; $d\hat{H}=Td\hat{S}-\hat{V}dp$

$$\left[\frac{\partial\hat{U}}{\partial\hat{V}}\right]_T=T\left[\frac{\partial\hat{S}}{\partial\hat{V}}\right]_T-P \quad ; \quad \left[\frac{\partial\hat{H}}{\partial P}\right]_T=T\left[\frac{\partial\hat{S}}{\partial P}\right]_T+\hat{V}$$

$$=T[\frac{\partial P}{\partial T}]_{\hat{V}}-P \quad ; \quad =-T[\frac{\partial\hat{V}}{\partial T}]_P+\hat{V} \qquad \text{... (a)}$$

(2) $d\hat{U}=[\frac{\partial\hat{U}}{\partial\hat{V}}]_T d\hat{V}+[\frac{\partial\hat{U}}{\partial T}]_{\hat{V}}dT$; $d\hat{H}=[\frac{\partial\hat{H}}{\partial P}]_T dP+[\frac{\partial\hat{H}}{\partial T}]_P dT$

$$=[-p+T[\frac{\partial P}{\partial T}]_{\hat{V}}]d\hat{V}+\hat{C}_V dT \quad ; \quad =[-T[\frac{\partial\hat{V}}{\partial T}]_P+\hat{V}]dP+\hat{C}_p dT \qquad \text{... (b)}$$

(3) $d\hat{H}=Td\hat{S}+Vd\hat{P}$

$$\hat{C}_p=\left(\frac{\partial\hat{H}}{\partial T}\right)_p=T\left(\frac{\partial\hat{S}}{\partial T}\right)_p \quad \Rightarrow \quad \hat{C}_p dT=Td\hat{S}$$

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Hence at constant pressure

$$d\hat{U} = -p d\hat{V} + \hat{C}_p dT \quad \text{... (c)}$$

(4) $$\rho \frac{D\hat{V}}{Dt} = \rho \frac{D(\frac{1}{\rho})}{Dt} = -\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{\partial u_i}{\partial x_i} \quad \text{... (d)}$$

(5) $$d\hat{H} = d\hat{U} + p d\hat{V} + \hat{V} dp \quad \text{... (e)}$$

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$$\rho \frac{D\hat{H}}{Dt} = -\frac{\partial q_i}{\partial x_i} + \sigma_{ji}\frac{\partial u_i}{\partial x_j} + \frac{Dp}{Dt}$$

And

$$d\hat{H} = \hat{C}_p dT + [\hat{V} - T(\frac{\partial \hat{V}}{\partial T})_p] dp$$

Or

$$\frac{D\hat{H}}{Dt} = \hat{C}_p \frac{DT}{Dt} + [\hat{V} - T(\frac{\partial \hat{V}}{\partial T})_p]\frac{Dp}{Dt}$$

Hence

$$\rho[\hat{C}_p \frac{DT}{Dt} + \{\hat{V} - T(\frac{\partial \hat{V}}{\partial T})_p\}\frac{DP}{Dt}] = -\frac{\partial}{\partial x_i}(-K\frac{\partial T}{\partial x_i}) + \sigma_{ji}\frac{\partial u_i}{\partial x_j} + \frac{DP}{Dt}$$

$$\rho\hat{C}_p \frac{DT}{Dt} + \rho\hat{V}\frac{DP}{Dt} - \rho T(\frac{\partial \hat{V}}{\partial T})_p]\frac{DP}{Dt} = K\frac{\partial^2 T}{\partial x_i \partial x_i} + \sigma_{ji}\frac{\partial u_i}{\partial x_j} + \frac{DP}{Dt}$$

$$\because \rho\hat{V} = 1$$

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$$\rho \hat{C}_p \frac{DT}{Dt} = K \frac{\partial^2 T}{\partial x_i \partial x_i} + \sigma_{ji} \frac{\partial u_i}{\partial x_j} + \frac{T}{\hat{V}} \left(\frac{\partial \hat{V}}{\partial T}\right)_p \frac{DP}{Dt}$$

Or

$$\rho \hat{C}_p \frac{DT}{Dt} = K \frac{\partial^2 T}{\partial x_i \partial x_i} + \sigma_{ji} \frac{\partial u_i}{\partial x_j} + \left[\frac{\partial \ln(\hat{V})}{\partial \ln(T)}\right]_p \frac{DP}{Dt}$$

حال در رابطه فوق ترم آخر را مورد بررسی قرار می دهیم. در این ترم اگر دانسیته ثابت باشد $\partial \ln(\hat{V})$ صفر می شود و اگر شار ثابت باشد DP/Dt صفر می شود.

در نتیجه کل ترم $\frac{\partial \ln(\hat{V})}{\partial \ln(T)} \frac{DP}{Dt}$ در فشار یا دانسیته ثابت صفر است.

رابطه فوق را می توان بصورت زیر نوشت:

$$\rho \hat{C}_p \left[\frac{\partial T}{\partial t} + U_j \frac{\partial T}{\partial x_j}\right] = K \frac{\partial^2 T}{\partial x_i \partial x_i} + \sigma_{ji} \frac{\partial u_i}{\partial x_j}$$

اگر سیال در حالت سکون باشد U_j و σ_{ji} برابر صفر می شوند.

معادله فوریه:

$$\frac{\partial T}{\partial t} = \frac{K}{\rho \hat{C}_p} \frac{\partial^2 T}{\partial x_i \partial x_i}$$