# 3 Modeling Process Quality

## Chapter Outline



**Supplemental Material for Chapter 3**



The supplemental material is on the textbook website www.wiley.com/college/montgomery.

# Learning Objectives

1. Construct and interpret visual data displays, including the stem-and-leaf plot, the histogram, and the box plot
2. Compute and interpret the sample mean, the sample variance, the sample standard deviation, and the sample range
3. Explain the concepts of a random variable and a probability distribution
4. Understand and interpret the mean, variance, and standard deviation of a probability distribution
5. Determine probabilities from probability distributions
6. Understand the assumptions for each of the discrete probability distributions presented
7. Understand the assumptions for each of the continuous probability distributions presented
8. Select an appropriate probability distribution for use in specific applications
9. Use probability plots
10. Use approximations for some hypergeometric and binomial distributions

# 3.1 Describing Variation

## Stem-and-Leaf Display

**TABLE 3.1**

**Cycle Time in Days to Pay Employee Health Insurance Claims**

| Claim | Days | Claim | Days | Claim | Days | Claim | Days |
|---|---|---|---|---|---|---|---|
| 1 | 48 | 11 | 35 | 21 | 37 | 31 | 16 |
| 2 | 41 | 12 | 34 | 22 | 43 | 32 | 22 |
| 3 | 35 | 13 | 36 | 23 | 17 | 33 | 33 |
| 4 | 36 | 14 | 42 | 24 | 26 | 34 | 30 |
| 5 | 37 | 15 | 43 | 25 | 28 | 35 | 24 |
| 6 | 26 | 16 | 36 | 26 | 27 | 36 | 23 |
| 7 | 36 | 17 | 56 | 27 | 45 | 37 | 22 |
| 8 | 46 | 18 | 32 | 28 | 33 | 38 | 30 |
| 9 | 35 | 19 | 46 | 29 | 22 | 39 | 31 |
| 10 | 47 | 20 | 30 | 30 | 27 | 40 | 17 |

Stem-and-Leaf Display: Days

```
Stem-and-leaf of Days
N = 40
Leaf Unit = 1.0
  3    1   677
  8    2   22234
 13    2   66778
 (8)   3   00012334
 19    3   555666677
 10    4   1233
  6    4   56678
  1    5
  1    5   6
```

**FIGURE 3.1** Stem-and-left plot for the health insurance claim data.

Easy to find percentiles of the data; see page 65

# Plot of Data in Time Order

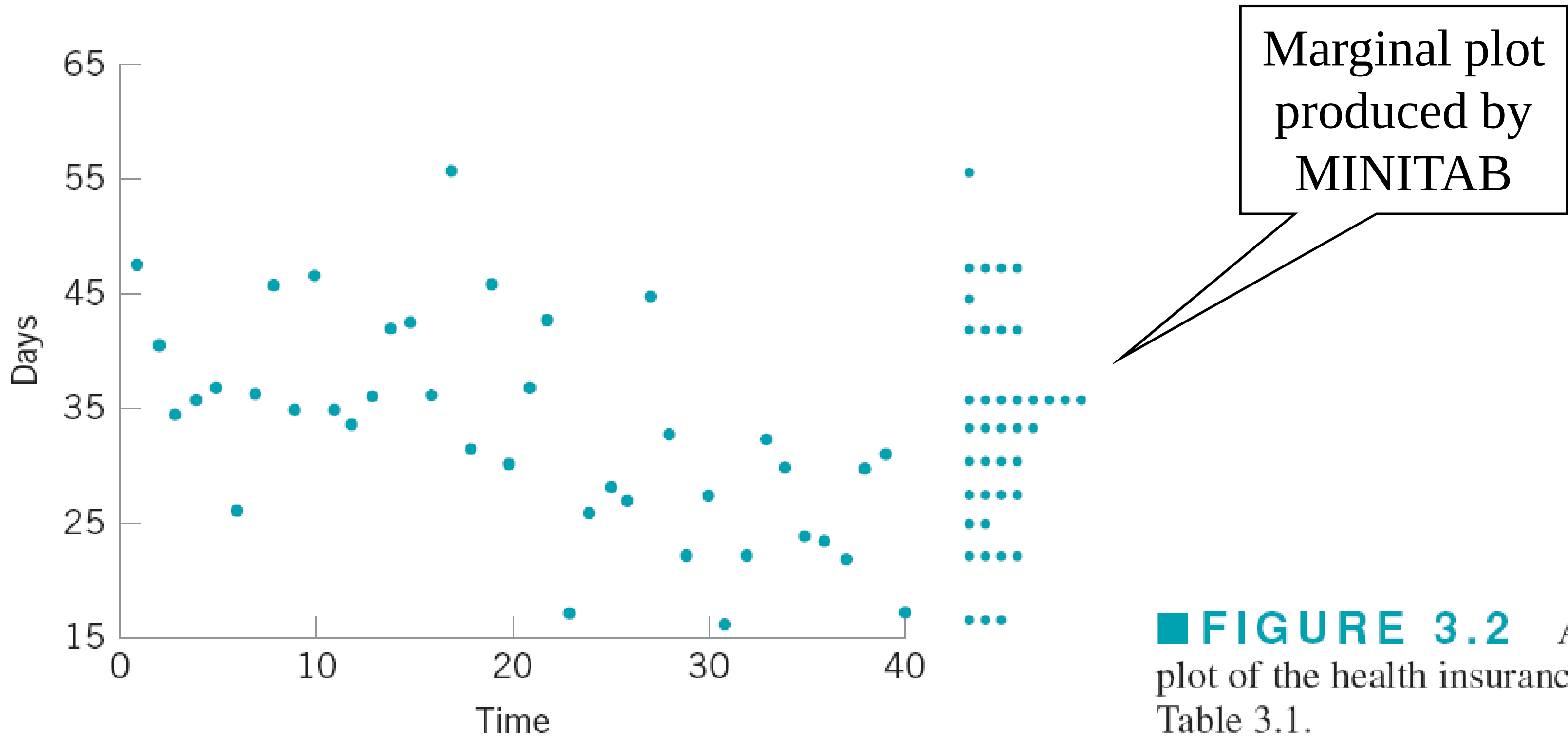



**FIGURE 3.2** A time series plot of the health insurance data in Table 3.1.

Also called a *run chart*

# Histograms – Useful for large data sets

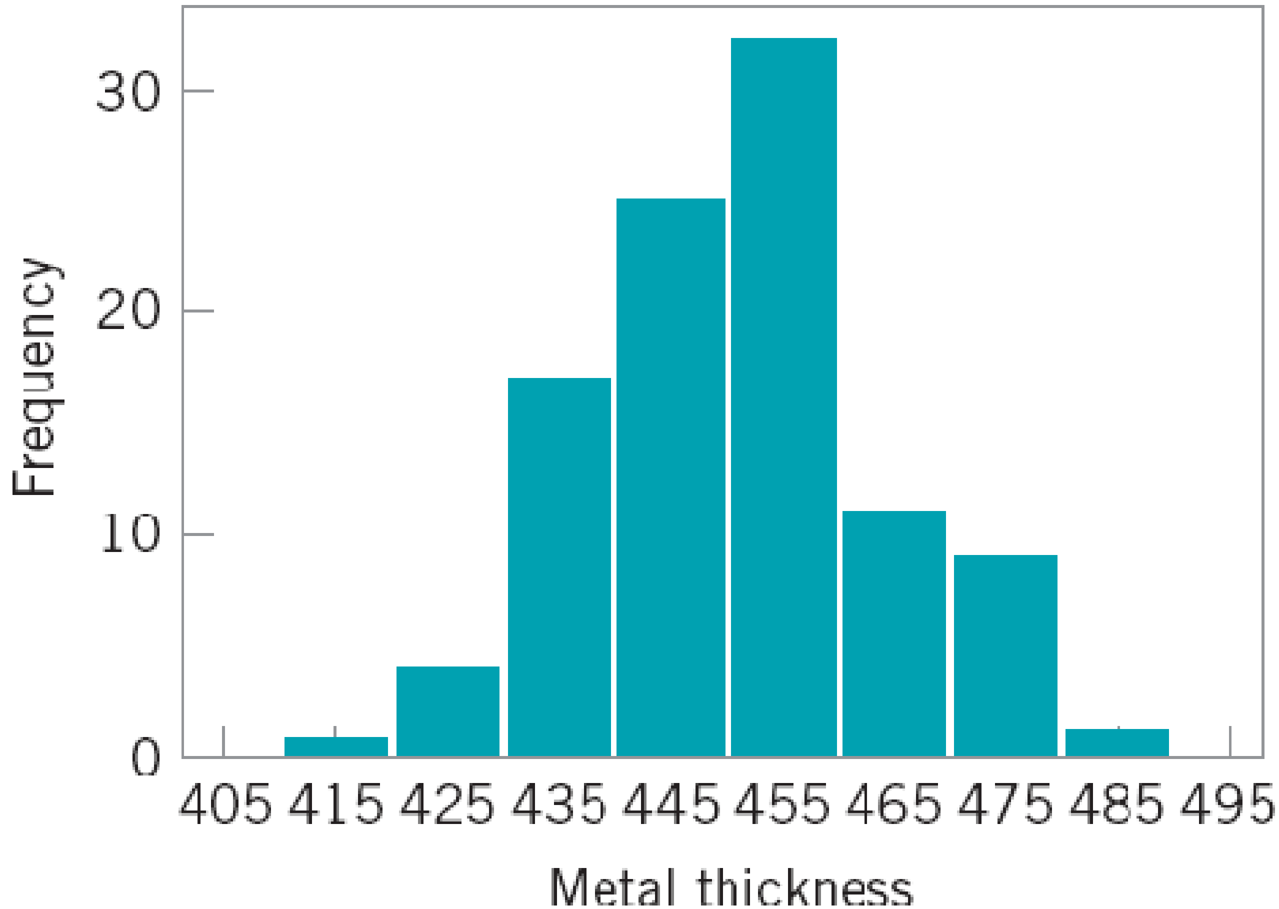
**TABLE 3.2**

**Layer Thickness (Å) on Semiconductor Wafers**

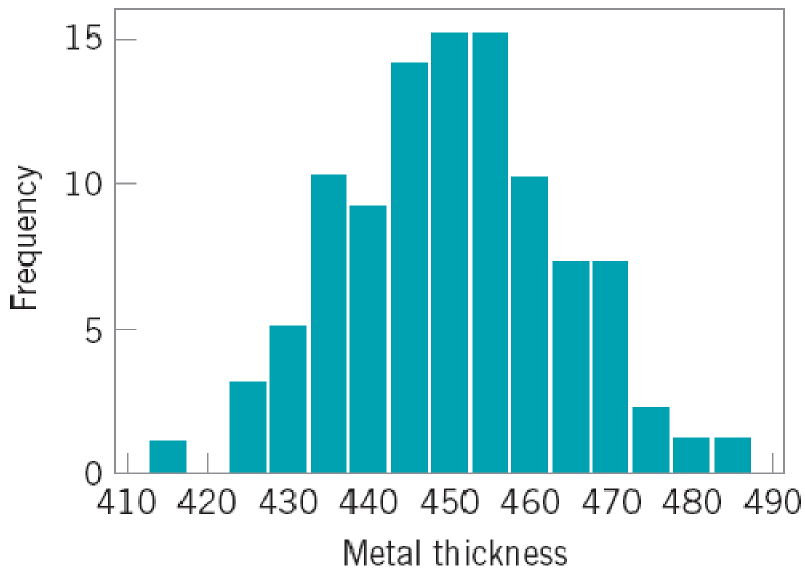
| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 438 | 450 | 487 | 451 | 452 | 441 | 444 | 461 | 432 | 471 |
| 413 | 450 | 430 | 437 | 465 | 444 | 471 | 453 | 431 | 458 |
| 444 | 450 | 446 | 444 | 466 | 458 | 471 | 452 | 455 | 445 |
| 468 | 459 | 450 | 453 | 473 | 454 | 458 | 438 | 447 | 463 |
| 445 | 466 | 456 | 434 | 471 | 437 | 459 | 445 | 454 | 423 |
| 472 | 470 | 433 | 454 | 464 | 443 | 449 | 435 | 435 | 451 |
| 474 | 457 | 455 | 448 | 478 | 465 | 462 | 454 | 425 | 440 |
| 454 | 441 | 459 | 435 | 446 | 435 | 460 | 428 | 449 | 442 |
| 455 | 450 | 423 | 432 | 459 | 444 | 445 | 454 | 449 | 441 |
| 449 | 445 | 455 | 441 | 464 | 457 | 437 | 434 | 452 | 439 |

Group values of the variable into bins, then count the number of observations that fall into each bin

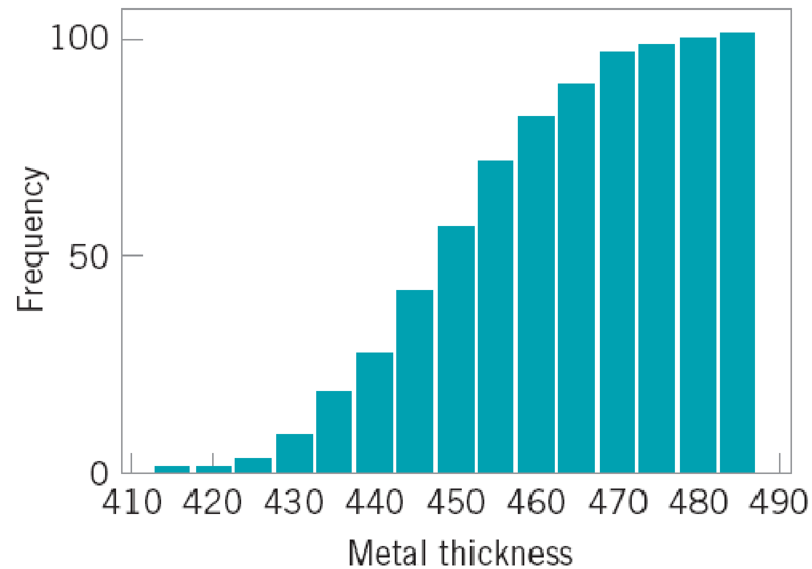
Plot frequency (or relative frequency) versus the values of the variable




■ **FIGURE 3.3** Minitab histogram for the metal layer thickness data in Table 3.2.




■ **FIGURE 3.4** Minitab histogram with 15 bins for the metal layer thickness data.




■ **FIGURE 3.5** A cumulative frequency plot of the metal thickness data from Minitab.

## ■ TABLE 3.3

**Surface Finish Defects in Painted Automobile Hoods**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 6 | 1 | 5 | 7 | 8 | 6 | 0 | 2 | 4 | 2 |
| 5 | 2 | 4 | 4 | 1 | 4 | 1 | 7 | 2 | 3 |
| 4 | 3 | 3 | 3 | 6 | 3 | 2 | 3 | 4 | 5 |
| 5 | 2 | 3 | 4 | 4 | 4 | 2 | 3 | 5 | 7 |
| 5 | 4 | 5 | 5 | 4 | 5 | 3 | 3 | 3 | 12 |

Figure 3.6 is the histogram of the defects. Notice that the number of defects is a discrete variable. From either the histogram or the tabulated data we can determine

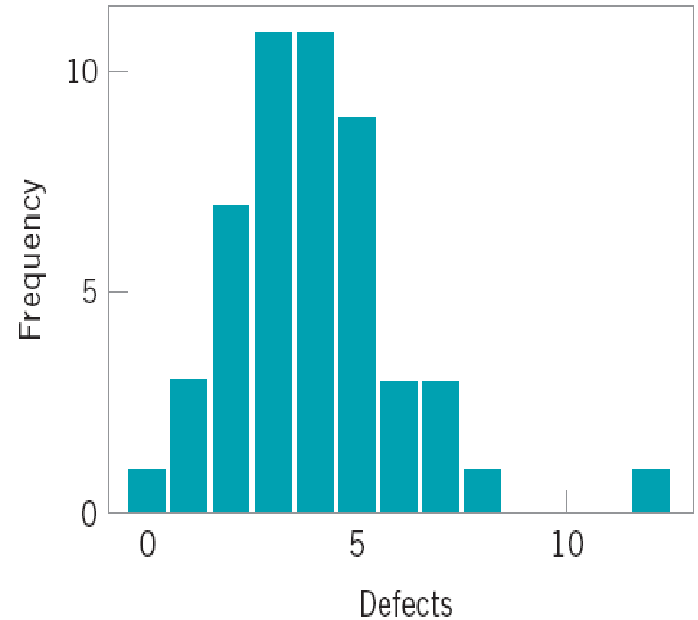
$$\text{Proportions of hoods with at least 3 defects} = \frac{39}{50} = 0.78$$

and

Proportions of hoods with between 0 and

$$\text{2 defects} = \frac{11}{50} = 0.22$$

These proportions are examples of relative frequencies.




■ **FIGURE 3.6** Histogram of the number of defects in painted automobile hoods (Table 3.3).

# Numerical Summary of Data

Sample average:

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n} \tag{3.1}$$

Note that the sample average $\bar{x}$ is simply the arithmetic mean of the $n$ observations. The sample average for the metal thickness data in Table 3.2 is

$$\bar{x} = \frac{\sum_{i=1}^{100} x_i}{100} = \frac{45.001}{100} = 450.01\,\text{Å}$$

Refer to Fig. 3.3 and note that the sample average is the point at which the histogram exactly "balances." Thus, the sample average represents the center of mass of the sample data.

The variability in the sample data is measured by the **sample variance,**

$$s^2 = \frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n-1} \qquad (3.2)$$

Note that the sample variance is simply the sum of the squared deviations of each observation from the sample average $\bar{x}$, divided by the sample size minus one. If there is no variability in the sample, then each sample observation $x_i = \bar{x}$, and the sample variance $s^2 = 0$. Generally, the larger is the sample variance $s^2$, the greater is the variability in the sample data.

The units of the sample variance $s^2$ are the square of the original units of the data. This is often inconvenient and awkward to interpret, and so we usually prefer to use the square root of $s^2$, called the **sample standard deviation** $s$**,** as a measure of variability.

It follows that

$$s = \sqrt{\frac{\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2}{n-1}} \qquad (3.3)$$

The primary advantage of the sample standard deviation is that it is expressed in the original units of measurement. For the metal thickness data, we find that

$$s^2 = 180.2928\,\text{Å}^2$$

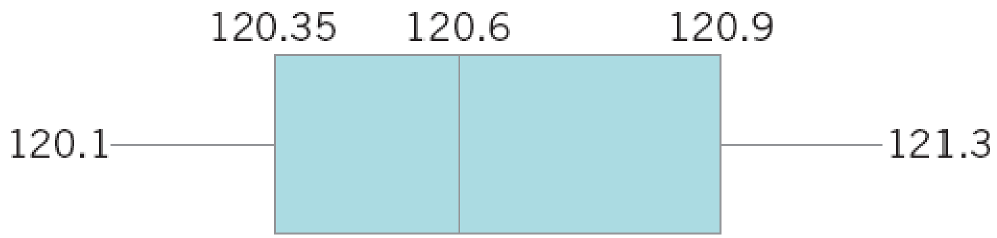
and

$$s = 13.43\,\text{Å}$$

# The Box Plot
# (or Box-and-Whisker Plot)

■ **TABLE 3.4**
**Hole Diameters (in mm) in Wing Leading Edge Ribs**

| | | |
|---|---|---|
| 120.5 | 120.4 | 120.7 |
| 120.9 | 120.2 | 121.1 |
| 120.3 | 120.1 | 120.9 |
| 121.3 | 120.5 | 120.8 |




■ **FIGURE 3.7** Box plot for the aircraft wing leading edge hole diameter data in Table 3.4.

# Comparative Box Plots

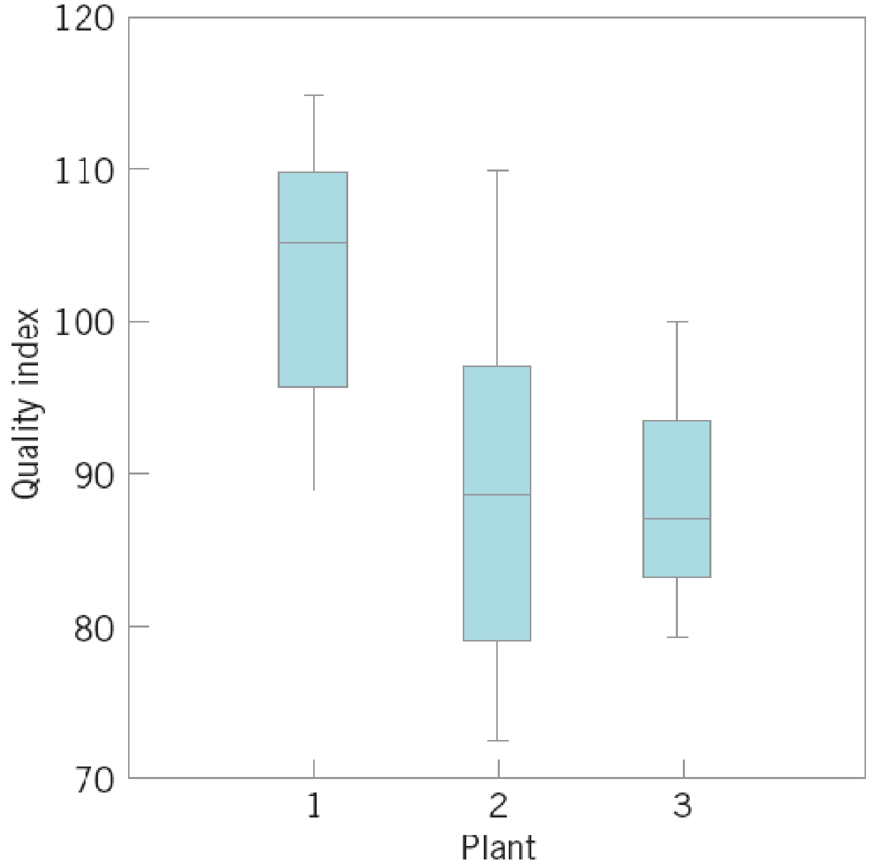



**FIGURE 3.8** Comparative box plots of a quality index for products produced at three plants.

# Probability Distributions

The histogram (or stem-and-leaf plot, or box plot) is used to describe *sample* data. A **sample** is a collection of measurements selected from some larger source or **population.** For example, the measurements on layer thickness in Table 3.2 are obtained from a sample of wafers selected from the manufacturing process. The population in this example is the collection of all layer thicknesses produced by that process. By using statistical methods, we may be able to analyze the sample layer thickness data and draw certain conclusions about the process that manufactures the wafers.

A **probability distribution** is a mathematical model that relates the value of the variable with the probability of occurrence of that value in the population. In other words, we might visualize layer thickness as a **random variable,** because it takes on different values in the population according to some random mechanism, and then the probability distribution of layer thickness describes the probability of occurrence of any value of layer thickness in the population. There are two types of probability distributions.

## Definition

1. **Continuous distributions.** When the variable being measured is expressed on a continuous scale, its probability distribution is called a *continuous distribution.* The probability distribution of metal layer thickness is continuous.

2. **Discrete distributions.** When the parameter being measured can only take on certain values, such as the integers 0, 1, 2, . . . , the probability distribution is called a *discrete distribution.* For example, the distribution of the number of nonconformities or defects in printed circuit boards would be a discrete distribution.

Sometimes called a
**probability mass function**

Sometimes called a
**probability density function**

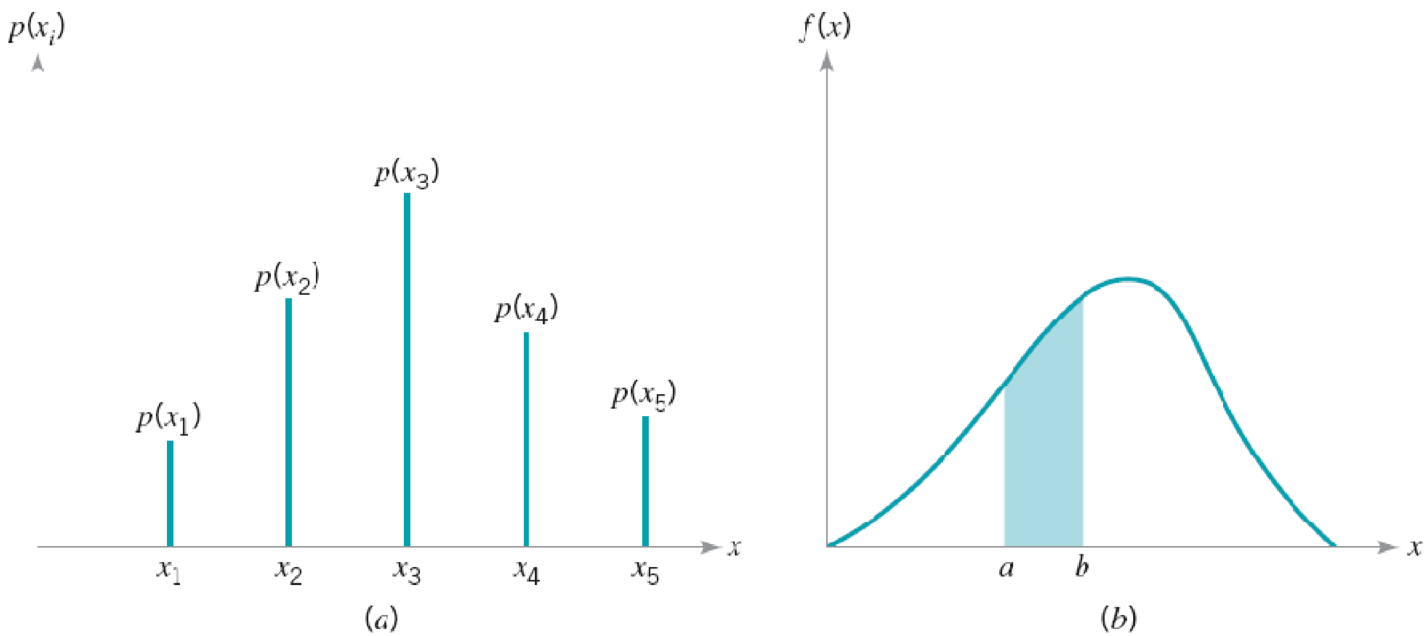



**FIGURE 3.9** Probability distributions. (a) Discrete case. (b) Continuous case.

Will see many examples in the text

# EXAMPLE 3.5 A Discrete Distribution

A manufacturing process produces thousands of semiconductor chips per day. On the average, 1% of these chips do not conform to specifications. Every hour, an inspector selects a random sample of 25 chips and classifies each chip in the sample as conforming or nonconforming. If we let $x$ be the random variable representing the number of nonconforming chips in the sample, then the probability distribution of $x$ is

$$p(x) = \binom{25}{x}(0.01)^x(0.99)^{25-x} \qquad x = 0,1,2,\ldots,25$$

where $\binom{25}{x} = 25!/[x!\,(25-x)!]$. This is a *discrete* distribution, since the observed number of nonconformances is $x = 0, 1, 2, \ldots, 25$, and is called the **binomial distribution.** We may calculate the probability of finding one or fewer nonconforming parts in the sample as

$$\begin{aligned} P(x \le 1) &= P(x = 0) + P(x = 1) \\ &= p(0) + p(1) \\ &= \sum_{x=0}^{1} \binom{25}{x}(0.01)^x(0.99)^{25-x} \\ &= \frac{25!}{0!25!}(0.99)^{25}(0.01)^0 + \frac{25!}{1!24!}(0.99)^{24}(0.01)^1 \\ &= 0.7778 + 0.1964 = 0.9742 \end{aligned}$$

# EXAMPLE 3.6 A Continuous Distribution

Suppose that $x$ is a random variable that represents the actual contents in ounces of a 1-lb bag of coffee beans. The probability distribution of $x$ is assumed to be

$$f(x) = \frac{1}{1.5} \qquad 15.5 \le x \le 17.0$$

This is a *continuous* distribution, since the range of $x$ is the interval [15.5, 17.0]. This distribution is called the **uniform distribution,** and it is shown graphically in Fig. 3.10. Note that the area under the function $f(x)$ corresponds to probability, so that the probability of a bag containing less than 16.0 oz is

$$P\{x \le 16.0\} = \int_{15.5}^{16.0} f(x)dx = \int_{15.5}^{16.0} \frac{1}{1.5}dx$$

$$= \left.\frac{x}{1.5}\right|_{15.5}^{16.0} = \frac{16.0-15.5}{1.5} = 0.3333$$

This follows intuitively from inspection of Fig. 3.9.

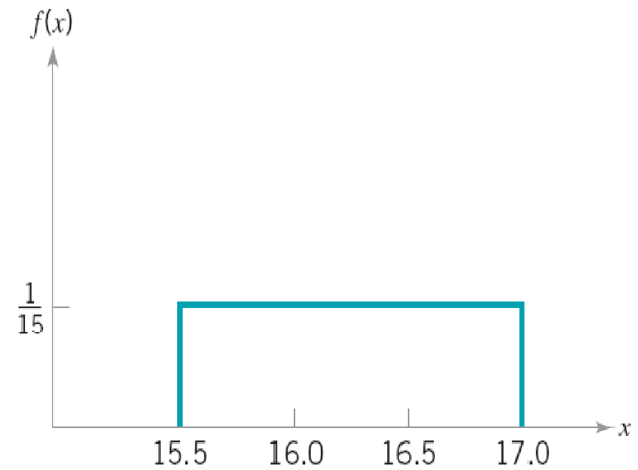



**FIGURE 3.10** The uniform distribution for Example 3.6.

The **mean** $\mu$ of a probability distribution is a measure of the **central tendency** in the distribution, or its **location.** The mean is defined as

$$\mu = \begin{cases} \int_{-\infty}^{\infty} xf(x)\,dx, \; x \text{ continuous} & (3.5a) \\ \sum_{i=1}^{\infty} x_i p(x_i), \; x \text{ discrete} & (3.5b) \end{cases}$$

For the case of a discrete random variable with exactly $N$ equally likely values [that is, $p(x_i) = 1/N$], then equation 3.5b reduces to

$$\mu = \frac{\sum_{i=1}^{N} x_i}{N}$$

Note the similarity of this last expression to the sample average $\bar{x}$ defined in equation 3.1. The mean is the point at which the distribution exactly "balances" (see Fig. 3.11).

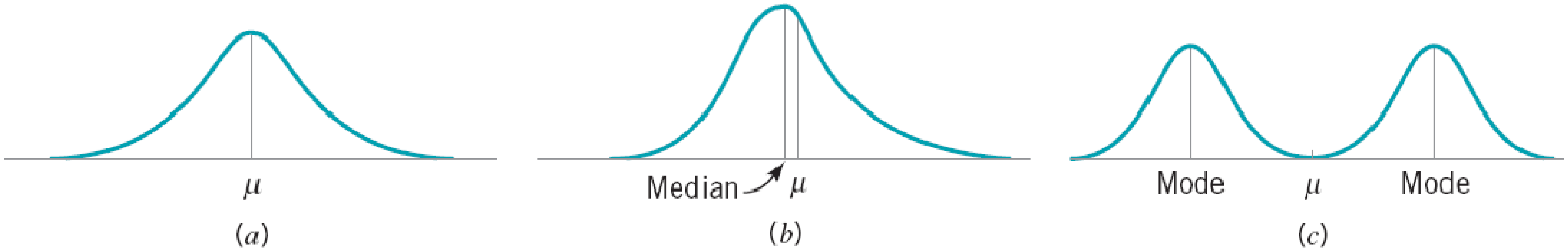



**FIGURE 3.11** The mean of a distribution.

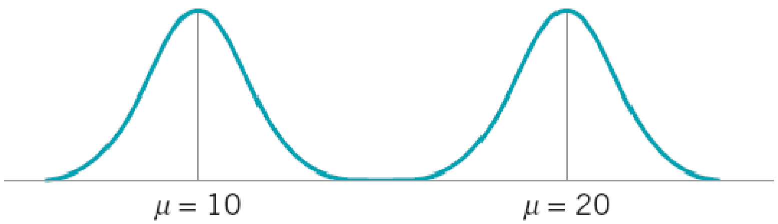



**FIGURE 3.12** Two probability distributions with different means.

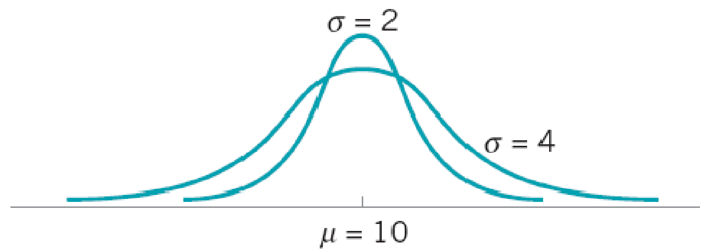



**FIGURE 3.13** Two probability distributions with the same mean but different standard deviations.

The mean is not necessarily the 50$^{th}$ percentile of the distribution (that's the median)

The mean is not necessarily the most likely value of the random variable (that's the mode)

The scatter, spread, or variability in a distribution is expressed by the **variance** $\sigma^2$. The definition of the variance is

$$\sigma^2 = \begin{cases} \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx, x \text{ continuous} & (3.6a) \\ \sum_{i=1}^{\infty} (x_i-\mu)^2 p(x_i), x \text{ discrete} & (3.6b) \end{cases}$$

when the random variable is discrete with $N$ equally likely values, then equation (3.6b) becomes

$$\sigma^2 = \frac{\sum_{i=1}^{N} (x_i-\mu)^2}{N}$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{\sum_{i=1}^{N}\left(x_i - \mu\right)^2}{N}} \tag{3.7}$$

The standard deviation is a measure of spread or scatter in the population expressed in the original units. Two distributions with the same mean but different standard deviations are shown in Fig. 3.13.

# 3.2 Important Discrete Distributions

## The Hypergeometric Distribution

**Definition**

The **hypergeometric probability distribution** is

$$p(x) = \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}} \qquad x = 0, 1, 2, \ldots, \min(n, D) \qquad (3.8)$$

The mean and variance of the distribution are

$$\mu = \frac{nD}{N} \qquad (3.9)$$

and

$$\sigma^2 = \frac{nD}{N}\left(1 - \frac{D}{N}\right)\left(\frac{N-n}{N-1}\right) \qquad (3.10)$$

The hypergeometric distribution is the appropriate probability model for selecting a random sample of $n$ items without replacement from a lot of $N$ items of which $D$ are nonconforming or defective. By a random sample, we mean a sample that has been selected in such a way that all possible samples have an equal chance of being chosen. In these applications, $x$ usually represents the number of nonconforming items found in the sample. For example, suppose that a lot contains 100 items, 5 of which do not conform to requirements. If 10 items are selected at random without replacement, then the probability of finding one or fewer nonconforming items in the sample is

$$P\{x \le 1\} = P\{x = 0\} + P\{x = 1\}$$
$$= \frac{\binom{5}{0}\binom{95}{10}}{\binom{100}{10}} + \frac{\binom{5}{1}\binom{95}{9}}{\binom{100}{10}} = 0.923$$

Discrete distributions are used frequently in designing acceptance sampling plans – see Chapter 15

Some computer programs can perform these calculations. The display below is the output from Minitab for calculating cumulative hypergeometric probabilities with $N = 100$, $D = 5$ (note that Minitab uses the symbol $M$ instead of $D$, and $n = 5$). Minitab will also calculate the individual probabilities for each value of $x$.

**Cumulative Distribution Function**

Hypergeometric with N = 100, M = 5, and n = 10

| x | P(X <= x) | x | P(X <= x) |
|---|---|---|---|
| 0 | 0.58375 | 6 | 1.00000 |
| 1 | 0.92314 | 7 | 1.00000 |
| 2 | 0.99336 | 8 | 1.00000 |
| 3 | 0.99975 | 9 | 1.00000 |
| 4 | 1.00000 | 10 | 1.00000 |
| 5 | 1.00000 | | |

## The Binomial Distribution

Basis is in **Bernoulli trials**

**Definition**

The **binomial distribution** with parameters $n \geq 0$ and $0 < p < 1$ is

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \ldots, n \tag{3.11}$$

The mean and variance of the binomial distribution are

$$\mu = np \tag{3.12}$$

and

$$\sigma^2 = np(1-p) \tag{3.13}$$

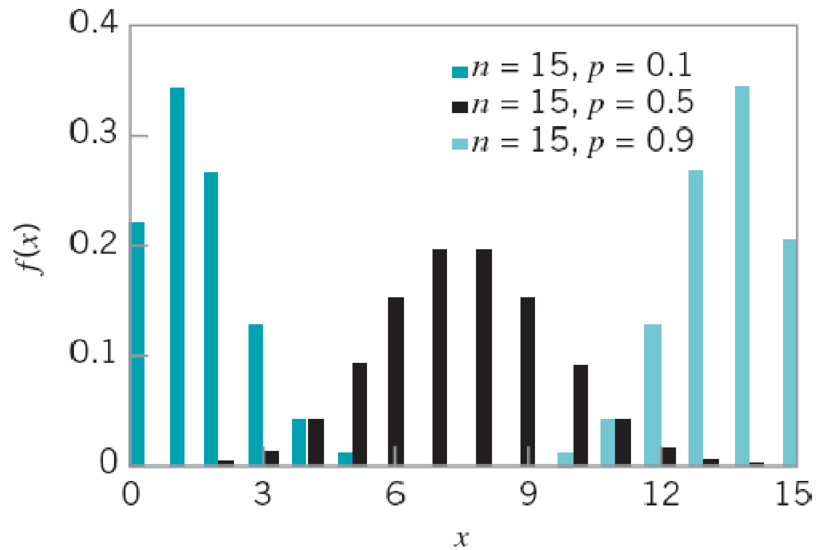
The random variable *x* is the number of successes out of *n* Bernoulli trials with constant probability of success *p* on each trial

The binomial distribution is used frequently in quality engineering. It is the appropriate probability model for sampling from an infinitely large population, where $p$ represents the fraction of defective or nonconforming items in the population. In these applications, $x$ usually represents the number of nonconforming items found in a random sample of $n$ items. For example, if $p = 0.10$ and $n = 15$, then the probability of obtaining $x$ nonconforming items is computed from equation 3.11 as follows:

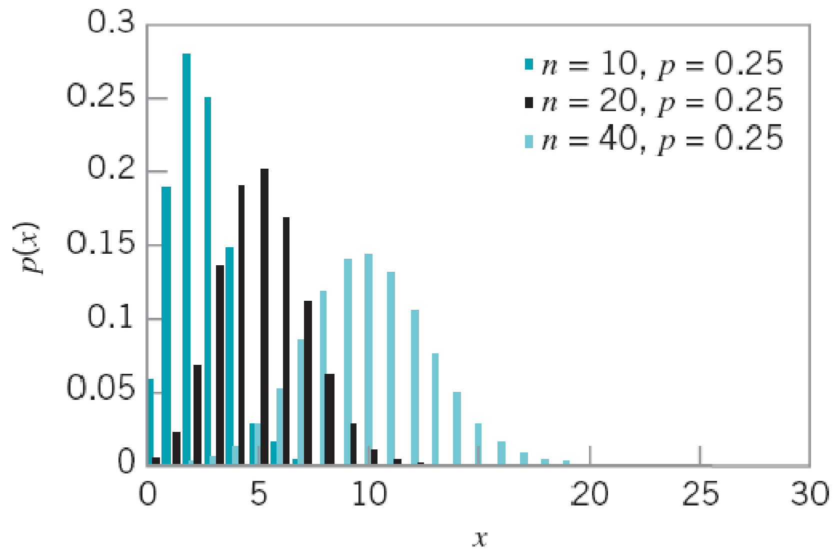
**Probability Density Function**

Binomial with n = 15 and p = 0.1

| x | P(X = x) | x | P(X = x) |
|---|---|---|---|
| 0 | 0.205891 | 6 | 0.001939 |
| 1 | 0.343152 | 7 | 0.000277 |
| 2 | 0.266896 | 8 | 0.000031 |
| 3 | 0.128505 | 9 | 0.000003 |
| 4 | 0.042835 | 10 | 0.000000 |
| 5 | 0.010471 | | |




($a$) Binomial distributions for different values of $p$ with $n$ = 15.




($b$) Binomial distributions for different values of $n$ with $p$ = 0.25.

**FIGURE 3.14** Binomial distributions for selected values of $n$ and $p$.

A random variable that arises frequently in statistical quality control is

$$\hat{p} = \frac{x}{n} \tag{3.14}$$

where $x$ has a binomial distribution with parameters $n$ and $p$. Often $\hat{p}$ is the ratio of the observed number of defective or nonconforming items in a sample ($x$) to the sample size ($n$) and this is usually called the **sample fraction defective** or **sample fraction nonconforming.** The "ˆ" symbol is used to indicate that $\hat{p}$ is an estimate of the true, unknown value of the binomial parameter $p$. The probability distribution of $\hat{p}$ is obtained from the binomial, since

$$P\{\hat{p} \le a\} = P\left\{\frac{x}{n} \le a\right\} = p\{x \le na\} = \sum_{x=0}^{[na]} \binom{n}{x} p^x (1-p)^{n-x}$$

where $[na]$ denotes the largest integer less than or equal to $na$. It is easy to show that the mean of $\hat{p}$ is $p$ and that the variance of $\hat{p}$ is

$$\sigma_{\hat{p}}^2 = \frac{p(1-p)}{n}$$

## The Poisson Distribution

**Definition**

The **Poisson distribution** is

$$p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad x = 0, 1, \ldots \tag{3.15}$$

where the parameter $\lambda > 0$. The **mean** and **variance** of the Poisson distribution are

$$\mu = \lambda \tag{3.16}$$

and

$$\sigma^2 = \lambda \tag{3.17}$$

Frequently used as a model for count data

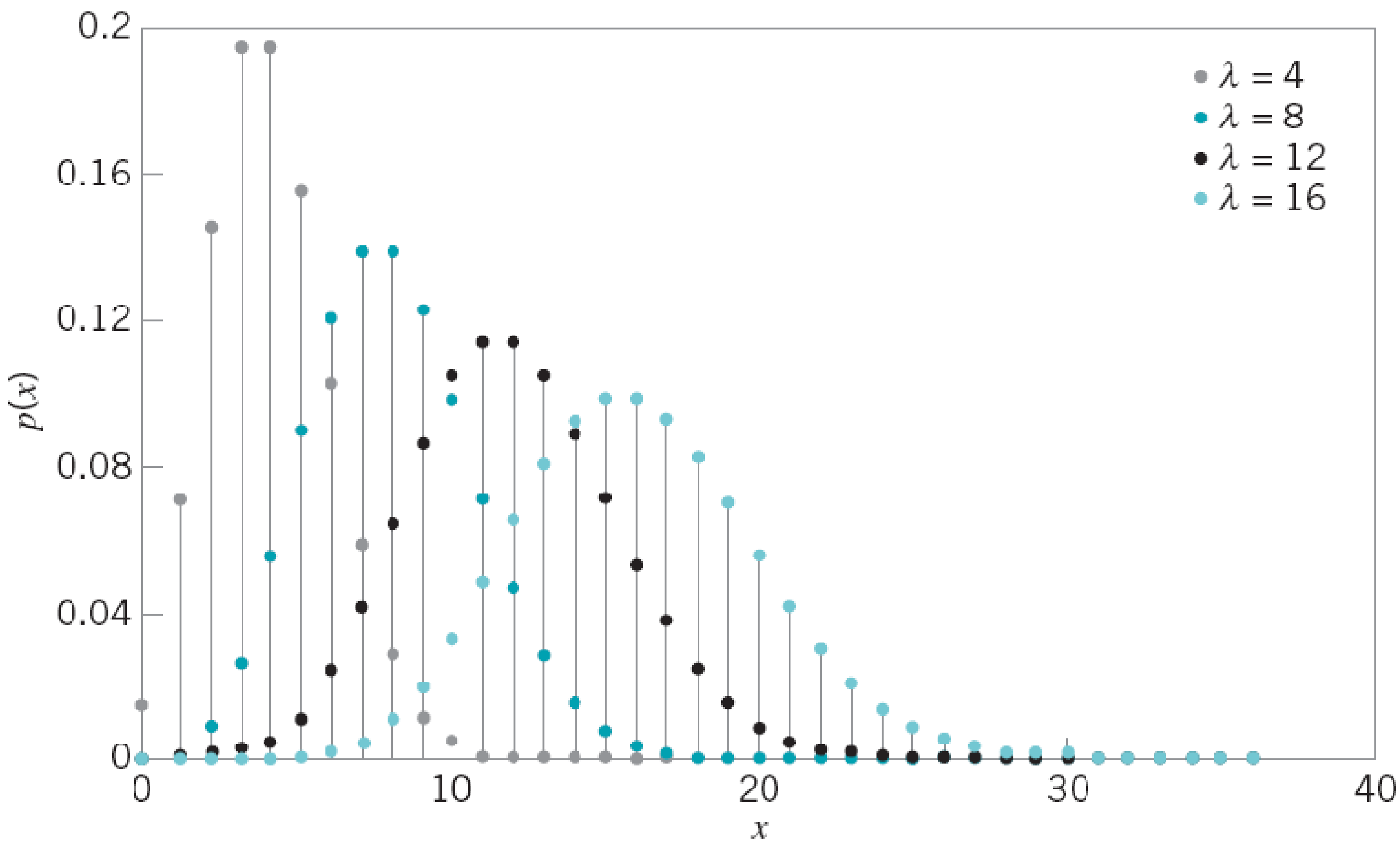



**FIGURE 3.15** Poisson probability distributions for selected values of $\lambda$.

A typical application of the Poisson distribution in quality control is as a model of the number of defects or nonconformities that occur in a unit of product. In fact, any random phenomenon that occurs on a per unit (or per unit area, per unit volume, per unit time, etc.) basis is often well approximated by the Poisson distribution. As an example, suppose that the number of wire-bonding defects per unit that occur in a semiconductor device is Poisson distributed with parameter $\lambda = 4$. Then the probability that a randomly selected semiconductor device will contain two or fewer wire-bonding defects is

$$P\{x \leq 2\} = \sum_{x=0}^{2} \frac{e^{-4}4^{x}}{x!} = 0.018316 + 0.073263 + 0.146525 = 0.238104$$

**Probability Density Function**

Poisson with mean = 4

| x | P(X = x) |
|---|---|
| 0 | 0.018316 |
| 1 | 0.073263 |
| 2 | 0.146525 |

## The Pascal Distribution

**Definition**

The **Pascal distribution** is

$$p(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r} \qquad x = r, r+1, r+2, \ldots \tag{3.18}$$

where $r \geq 1$ is an integer. The ***mean*** and ***variance*** of the Pascal distribution are

$$\mu = \frac{r}{p} \tag{3.19}$$

and

$$\sigma^2 = \frac{r(1-p)}{p^2} \tag{3.20}$$

respectively.

The random variable *x* is the number of Bernoulli trials upon which the *r*th success occurs

- When $r = 1$ the Pascal distribution is known as the **geometric** distribution

- The geometric distribution has many useful applications in SQC

# 3.3 Important Continuous Distributions

## The Normal Distribution

**Definition**

The **normal distribution** is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty \qquad (3.21)$$

The mean of the normal distribution is $\mu$ $(-\infty < \mu < \infty)$ and the variance is $\sigma^2 > 0$.

The normal distribution is used so much that we frequently employ a special notation, $x \sim N(\mu, \sigma^2)$, to imply that $x$ is normally distributed with mean $\mu$ and variance $\sigma^2$. The visual appearance of the normal distribution is a symmetric, unimodal or **bell-shaped** curve and is shown in Fig. 3.16.

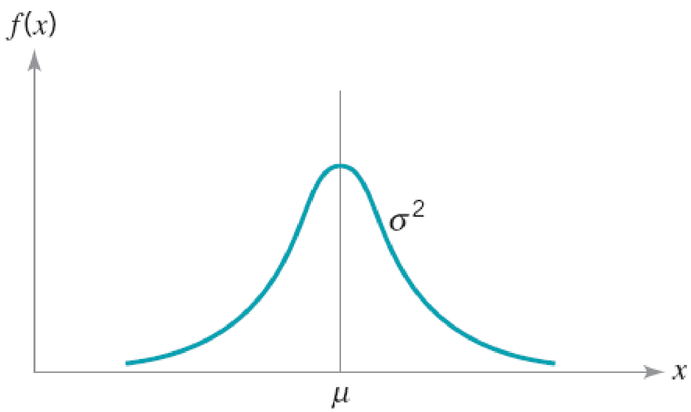



**FIGURE 3.16** The normal distribution.

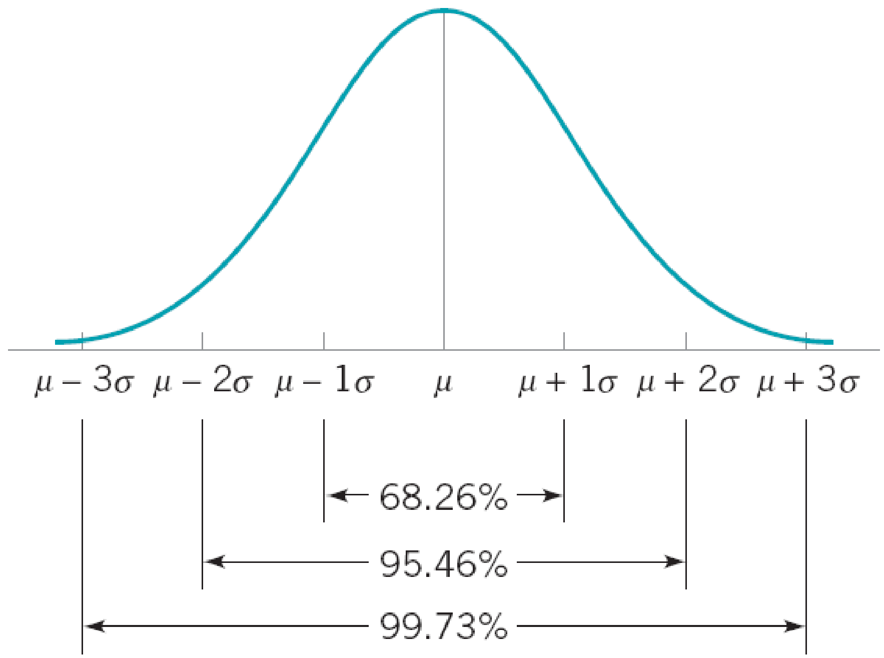



**FIGURE 3.17** Areas under the normal distribution.

The cumulative normal distribution is defined as the probability that the normal random variable $x$ is less than or equal to some value $a$, or

$$P\{x \le a\} = F(a) = \int_{-\infty}^{a} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \qquad (3.22)$$

This integral cannot be evaluated in closed form. However, by using the change of variable

$$z = \frac{x-\mu}{\sigma} \qquad (3.23)$$

the evaluation can be made independent of $\mu$ and $\sigma^2$. That is,

$$P\{x \le a\} = P\left\{z \le \frac{a-\mu}{\sigma}\right\} \equiv \Phi\left(\frac{a-\mu}{\sigma}\right)$$

where $\Phi(\cdot)$ is the cumulative distribution function of the **standard normal distribution** (mean = 0, standard deviation = 1). A table of the cumulative standard normal distribution is given in Appendix Table II. The transformation (3.23) is usually called **standardization,** because it converts a $N(\mu, \sigma^2)$ random variable into an $N(0, 1)$ random variable.

## EXAMPLE 3.7 Tensile Strength of Paper

The tensile strength of paper used to make grocery bags is an important quality characteristic. It is known that the strength—say, $x$—is normally distributed with mean $\mu = 40$ lb/in$^2$ and standard deviation $\sigma = 2$ lb/in$^2$, denoted $x \sim N(40, 2^2)$. The purchaser of the bags requires them to have a strength of at least 35 lb/in$^2$. Calculate the probability that bags produced from this paper will meet or exceed the specification.

## SOLUTION

The probability that a bag produced from this paper will meet or exceed the specification is $P\{x \geq 35\}$. Note that

$$P\{x \geq 35\} = 1 - P\{x \leq 35\}$$

To evaluate this probability from the standard normal tables, we standardize the point 35 and find

$$P\{x \leq 35\} = P\left\{z \leq \frac{35-40}{2}\right\} =$$

$$P\{z \leq -2.5\} = \Phi(-2.5) = 0.0062$$

Consequently, the desired probability is

$$P\{x \geq 35\} = 1 - P\{x \leq 35\} = 1 - 0.0062 = 0.9938$$

Figure 3.18 shows the tabulated probability for both the $N(40, 2^2)$ distribution and the standard normal distribution. Note that the shaded area to the left of 35 lb/in$^2$ in Fig. 3.18 represents the fraction nonconforming or "fallout" produced by the bag manufacturing process.

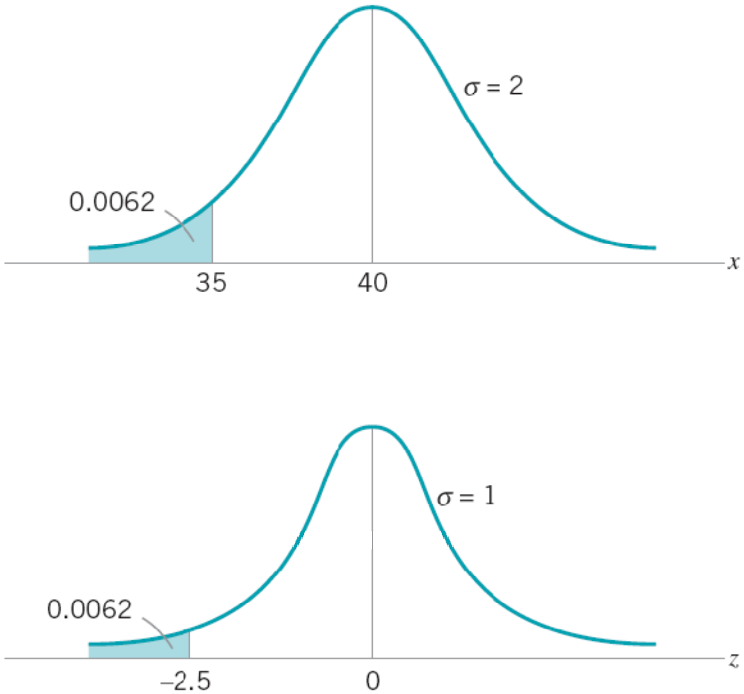



**FIGURE 3.18** Calculation of $P\{x \leq 35\}$ in Example 3.7.

## EXAMPLE 3.9 Another Use of the Standard Normal Table

Sometimes instead of finding the probability associated with a particular value of a normal random variable, we find it necessary to do the opposite—find a particular value of a normal random variable that results in a given probability. For example, suppose that $x \sim N(10, 9)$. Find the value of $x$—say, $a$—such that $P\{x > a\} = 0.05$.

## SOLUTION

From the problem statement, we have

$$P\{x > a\} = P\left\{z > \frac{a-10}{3}\right\} = 0.05$$

or

$$P\left\{z \leq \frac{a-10}{3}\right\} = 0.95$$

From Appendix Table II, we have $P\{z \leq 1.645\} = 0.95$, so

$$\frac{a-10}{3} = 1.645$$

or

$$a = 10 + 3(1.645) = 14.935$$

The normal distribution has many useful properties. One of these is relative to **linear combinations** of normally and independently distributed random variables. If $x_1$, $x_2$ . . . , $x_n$ are normally and independently distributed random variables with means $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, respectively, then the distribution of

$$y = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

is normal with mean

$$\mu_y = a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n \tag{3.27}$$

and variance

$$\sigma_y^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n^2\sigma_n^2 \tag{3.28}$$

where $a_1, a_2, \ldots, a_n$ are constants.

## Definition

**The Central Limit Theorem** if $x_1, x_2, \ldots, x_n$ are independent random variables with mean $\mu_i$ and variance $\sigma_i^2$, and if $y = x_1 + x_2 + \cdots + x_n$, then the distribution of

$$\frac{y - \sum_{i=1}^{n} \mu_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}}$$

approaches the $N(0, 1)$ distribution as $n$ approaches infinity.

Practical interpretation – the sum of independent random variables is approximately normally distributed regardless of the distribution of each individual random variable in the sum

## The Lognormal Distribution

**Definition**

Let $w$ have a normal distribution mean $\theta$ and variance $\omega^2$; then $x = \exp(w)$ is a **lognormal random variable,** and the lognormal distribution is

$$f(x) = \frac{1}{x\omega\sqrt{2\pi}} \quad \exp\left[-\frac{(\ln(x)-\theta)^2}{2\omega^2}\right] \quad 0 < x < \infty \tag{3.29}$$

The mean and variance of $x$ are

$$\mu = e^{\theta+\omega^2/2} \quad \text{and} \quad \sigma^2 = e^{2\theta+\omega^2}\left(e^{\omega^2}-1\right) \tag{3.30}$$

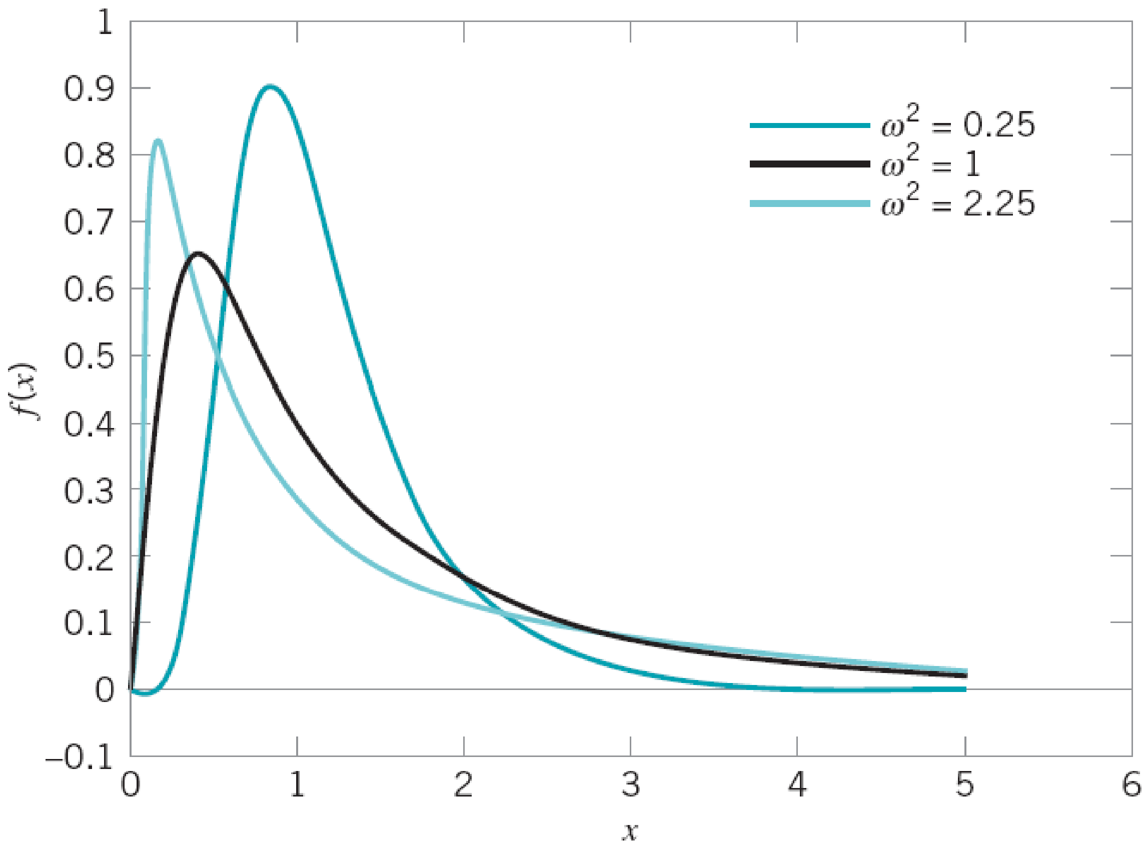



**FIGURE 3.20** Lognormal probability density functions with $\theta = 0$ for selected values of $\omega^2$.

# EXAMPLE 3.10 Medical Laser Lifetime

The lifetime of a medical laser used in ophthalmic surgery has a lognormal distribution with $\theta = 6$ and $\omega = 1.2$ hours. What is the probability the lifetime exceeds 500 hours?

## SOLUTION

From the cumulative distribution function for the lognormal random variable

$$P(x > 500) = 1 - P[\exp(w) \le 500] = 1 - P[w \le \ln(500)]$$

$$= \Phi\left(\frac{\ln(500) - 6}{1.2}\right) = 1 - \Phi(0.1788)$$

$$= 1 - 0.5710 = 0.4290$$

What lifetime is exceeded by 99% of lasers? Now the question is to determine $a$ such that $P(x > a) = 0.99$. Therefore,

$$P(x > a) = P[\exp(w) > a] = P[w > \ln(a)]$$

$$= 1 - \Phi\left(\frac{\ln(a) - 6}{1.2}\right) = 0.99$$

From Appendix Table II, $1 - \Phi(a) = 0.99$ when $a = -2.33$. Therefore,

$$\frac{\ln(a) - 6}{1.2} = -2.33 \quad \text{and} \quad a = \exp(3.204) = 24.63 \text{ hours}$$

Determine the mean and standard deviation of lifetime. Now,

$$\mu = e^{\theta + \omega^2/2} = \exp(6 + 0.72) = 828.82 \text{ hours}$$

$$\sigma^2 = e^{2\theta + \omega^2}\left(e^{\omega^2} - 1\right) = \exp(12 + 1.44)[\exp(1.44) - 1]$$

$$= 2,212,419.85$$

so the standard deviation of the lifetime is 1487.42 hours. Notice that the standard deviation of the lifetime is large relative to the mean.

# The Exponential Distribution

**Definition**

The **exponential distribution** is

$$f(x) = \lambda e^{-\lambda x} \qquad x \geq 0 \tag{3.31}$$

where $\lambda > 0$ is a constant. The ***mean*** and ***variance*** of the exponential distribution are

$$\mu = \frac{1}{\lambda} \tag{3.32}$$

and

$$\sigma^2 = \frac{1}{\lambda^2} \tag{3.33}$$

respectively.

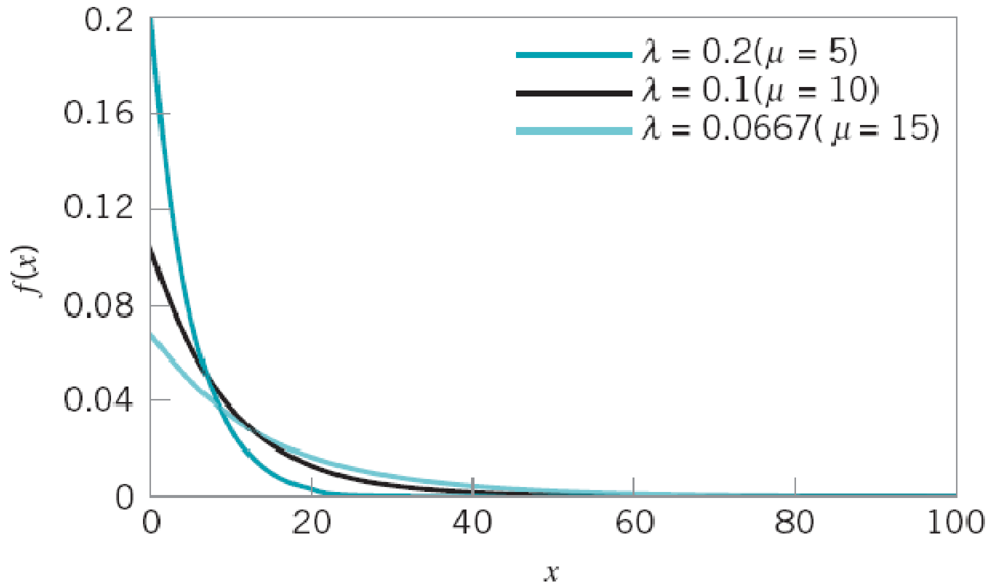



**FIGURE 3.21** Exponential distributions for selected values of $\lambda$.

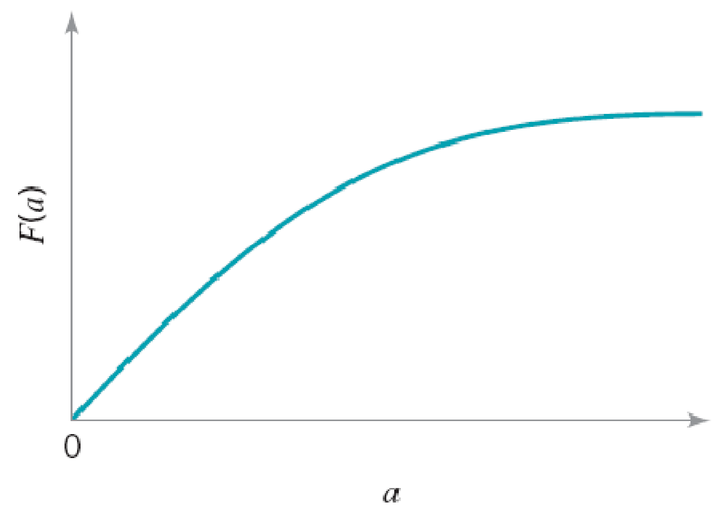



**FIGURE 3.22** The cumulative exponential distribution function.

Relationship between the Poisson and exponential distributions

## The Gamma Distribution

**Definition**

The **gamma distribution** is

$$f(x) = \frac{\lambda}{\Gamma(r)}(\lambda x)^{r-1}e^{-\lambda x} \qquad x \geq 0 \tag{3.36}$$

with **shape parameter** $r > 0$ and **scale parameter** $\lambda > 0$. The **mean** and **variance** of the gamma distribution are

$$\mu = \frac{r}{\lambda} \tag{3.37}$$

and

$$\sigma^2 = \frac{r}{\lambda^2} \tag{3.38}$$

respectively.[3]

[3] $\Gamma(r)$ in the denominator of equation 3.36 is the gamma function, defined as $\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}dx$, $r > 0$. If $r$ is a positive integer, then $\Gamma(r) = (r-1)!$

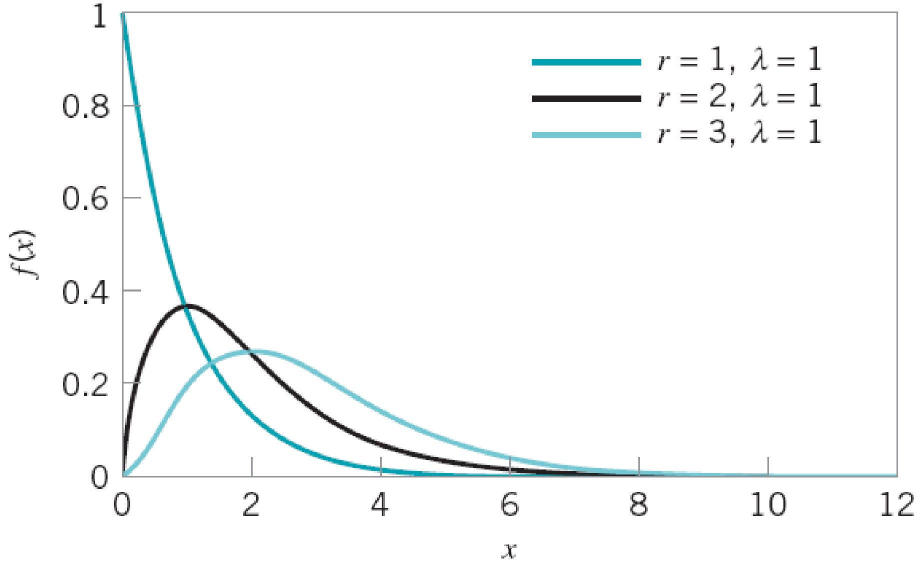



**FIGURE 3.23** Gamma distributions for selected values or $r$ and $\lambda = 1$.

When $r$ is an integer, the gamma distribution is the result of summing $r$ independently and identically exponential random variables each with parameter $\lambda$.

The gamma distribution has many applications in reliability engineering.

## EXAMPLE 3.11 A Standby Redundant System

Consider the system shown in Fig. 3.24. This is called a **standby redundant system,** because while component 1 is on, component 2 is off, and when component 1 fails, the switch automatically turns component 2 on. If each component has a life described by an exponential distribution with $\lambda = 10^{-4}/\text{h}$, say, then the system life is gamma distributed with parameters $r = 2$ and $\lambda = 10^{-4}$. Thus, the mean time to failure is $\mu = r/\lambda = 2/10^{-4} = 2 \times 10^4$ h.

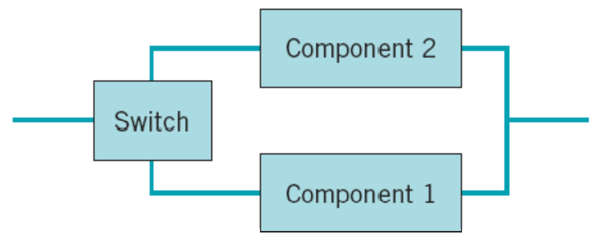



**FIGURE 3.24** The standby redundant system for Example 3.11.

# The Weibull Distribution

**Definition**

The **Weibull distribution** is

$$f(x) = \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\theta}\right)^{\beta}\right] \qquad x \geq 0 \tag{3.41}$$

where $\theta > 0$ is the **scale parameter,** and $\beta > 0$ is the **shape parameter.** The **mean** and **variance** of the Weibull distribution are

$$\mu = \theta\,\Gamma\left(1+\frac{1}{\beta}\right) \tag{3.42}$$

and

$$\sigma^2 = \theta^2\left[\Gamma\left(1+\frac{2}{\beta}\right) - \left\{\Gamma\left(1+\frac{1}{\beta}\right)\right\}^2\right] \tag{3.43}$$

respectively.

When β = 1, the Weibull reduces to the exponential

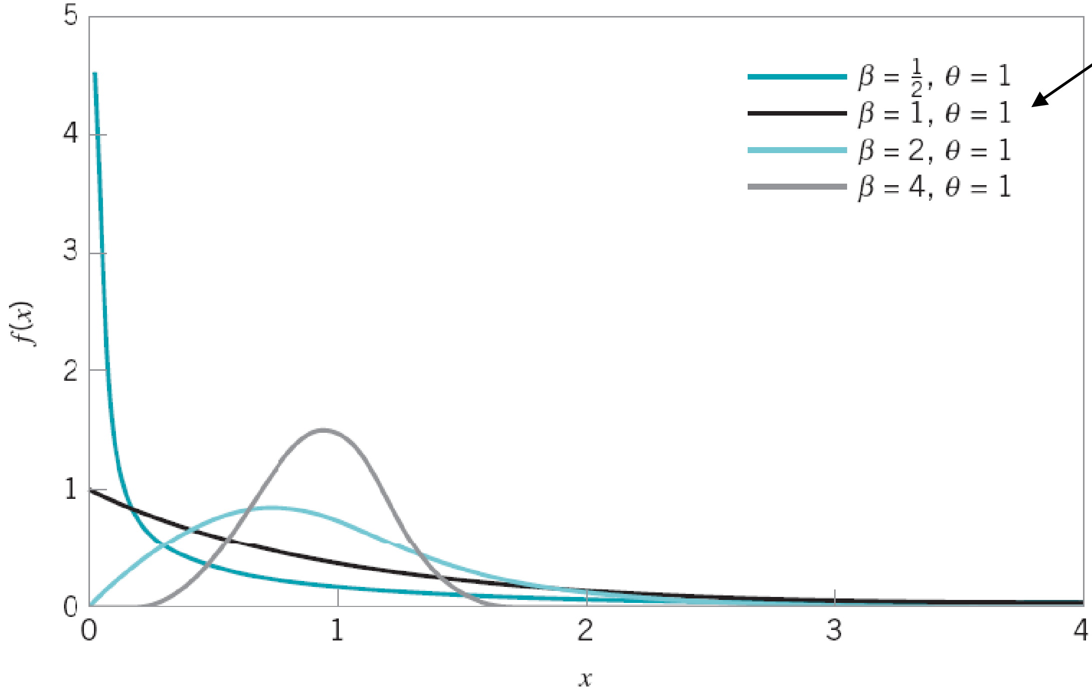



**FIGURE 3.25** Weibull distributions for selected values of the shape parameter $\beta$ and scale parameter $\theta = 1$.

# 3.4 Probability Plots

- Determining if a sample of data might reasonably be assumed to come from a specific distribution
- Probability plots are available for various distributions
- Easy to construct with computer software (MINITAB)
- Subjective interpretation

# Normal Probability Plot

## EXAMPLE 3.13 A Normal Probability Plot

Observations on the road octane number of ten gasoline blends are as follows: 88.9, 87.0, 90.0, 88.2, 87.2, 87.4, 87.8, 89.7, 86.0, and 89.6. We hypothesize that octane number is adequately modeled by a normal distribution. Is this a reasonable assumption?

## SOLUTION

To use probability plotting to investigate this hypothesis, first arrange the observations in ascending order and calculate their cumulative frequencies $(j - 0.5)/10$ as shown in the following table.

| $j$ | $x_{(j)}$ | $(j - 0.5)/10$ |
|---|---|---|
| 1 | 86.0 | 0.05 |
| 2 | 87.0 | 0.15 |
| 3 | 87.2 | 0.25 |
| 4 | 87.4 | 0.35 |
| 5 | 87.8 | 0.45 |
| 6 | 88.2 | 0.55 |
| 7 | 88.9 | 0.65 |
| 8 | 89.6 | 0.75 |
| 9 | 89.7 | 0.85 |
| 10 | 90.0 | 0.95 |

The pairs of values $x_{(j)}$ and $(j - 0.5)/10$ are now plotted on normal probability paper. This plot is shown in Fig. 3.26. Most normal probability paper plots $100(j - 0.5)/n$ on the left vertical scale (and some also plot $100[1 - (j - 0.5)/n]$ on the right vertical scale), with the variable value plotted on the horizontal scale. A straight line, chosen subjectively as a "best fit" line, has been drawn through the plotted points. In drawing the straight line, you should be influenced more by the points near the middle of the plot than by the extreme points. A good rule of thumb is to draw the line approximately between the twenty-fifth and seventy-fifth percentile points. This is how the line in Fig. 3.26 was determined. In assessing the systematic deviation of the points from the straight line, imagine a fat pencil lying along the line. If all the points are covered by this imaginary pencil, a normal distribution adequately describes the data. Because the points in Fig. 3.26 would pass the fat pencil test, we conclude that the normal distribution is an appropriate model for the road octane number data.

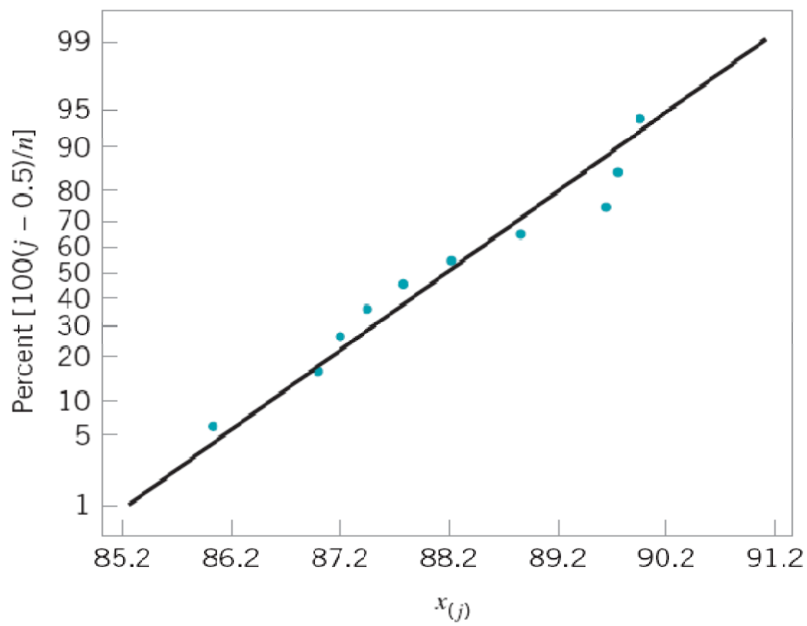
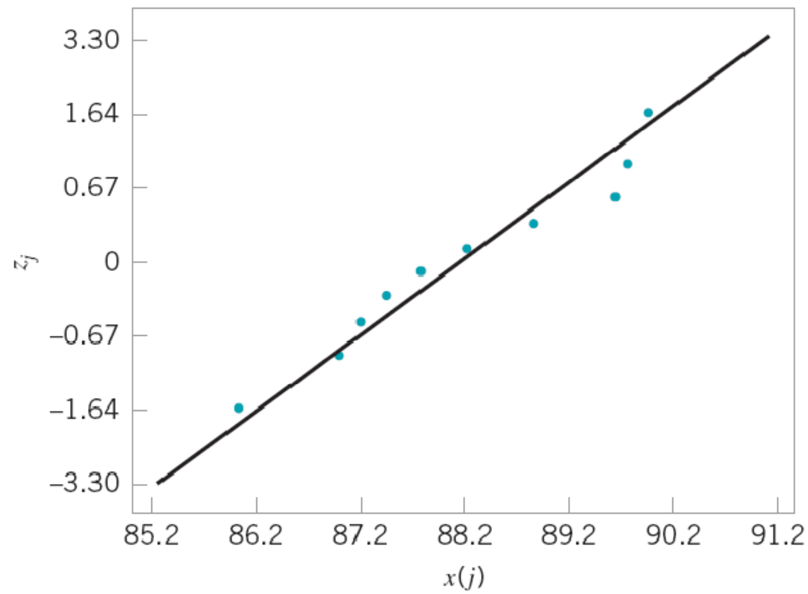



**FIGURE 3.26** Normal probability plot of the road octane number data.




■ **FIGURE 3.27** Normal probability plot of the road octane number data with standardized scores.

| $j$ | $x_{(j)}$ | $(j - 0.5)/10$ | $z_j$ |
|---|---|---|---|
| 1 | 86.0 | 0.05 | –1.64 |
| 2 | 87.0 | 0.15 | –1.04 |
| 3 | 87.2 | 0.25 | –0.67 |
| 4 | 87.4 | 0.35 | –0.39 |
| 5 | 87.8 | 0.45 | –0.13 |
| 6 | 88.2 | 0.55 | 0.13 |
| 7 | 88.9 | 0.65 | 0.39 |
| 8 | 89.6 | 0.75 | 0.67 |
| 9 | 89.7 | 0.85 | 1.04 |
| 10 | 90.0 | 0.95 | 1.64 |

# Other Probability Plots

- What is a reasonable choice as a probability model for these data?

**TABLE 3.5**

**Aluminum Contamination (ppm)**

| | | | | | | |
|---|---|---|---|---|---|---|
| 30 | 30 | 60 | 63 | 70 | 79 | 87 |
| 90 | 101 | 102 | 115 | 118 | 119 | 119 |
| 120 | 125 | 140 | 145 | 172 | 182 | |
| 183 | 191 | 222 | 244 | 291 | 511 | |

From "The Lognormal Distribution for Modeling Quality Data When the Mean Is Near Zero," *Journal of Quality Technology*, 1990, pp. 105–110.

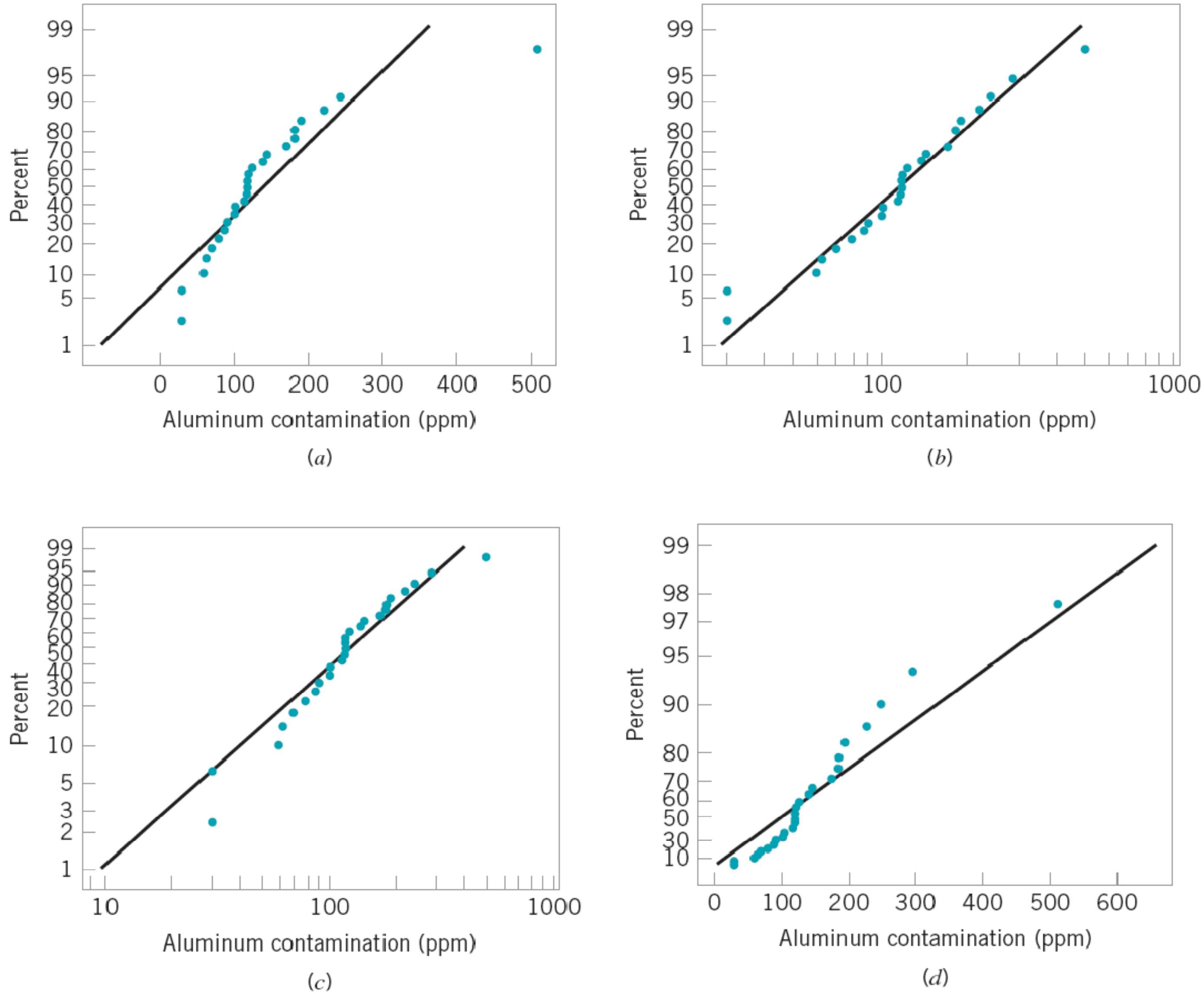



**FIGURE 3.28** Probability plots of the aluminum contamination data in Table 3.5. (*a*) Normal. (*b*) Lognormal. (*c*) Weibull. (*d*) Exponential.

# 3.6 Some Useful Approximations

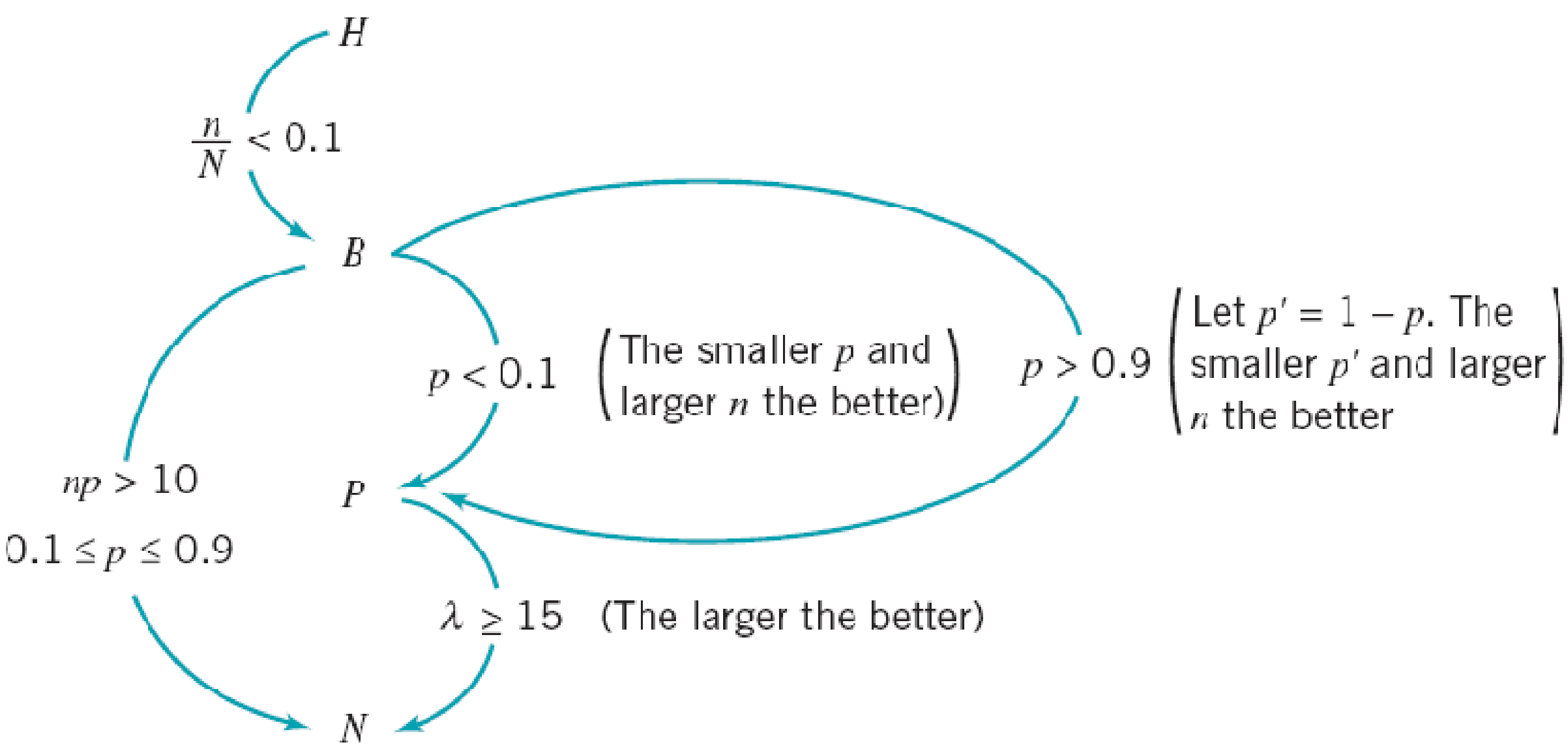



**FIGURE 3.29** Approximations to probability distributions.

## Important Terms and Concepts

Approximations to probability distributions
Binomial distribution
Box plot
Central limit theorem
Continuous distribution
Control limit theorem
Descriptive statistics
Discrete distribution
Exponential distribution
Gamma distribution
Geometric distribution
Histogram
Hypergeometric probability distribution
Interquartile range
Lognormal distribution
Mean of a distribution
Median
Negative binomial distribution
Normal distribution
Normal probability plot
Pascal distribution
Percentile
Poisson distribution
Population
Probability distribution
Probability plotting
Quartile
Random variable
Run chart
Sample
Sample average
Sample standard deviation
Sample variance
Standard deviation
Standard normal distribution
Statistics
Stem-and-leaf display
Time series plot
Uniform distribution
Variance of a distribution
Weibull distribution

# Learning Objectives

1. Construct and interpret visual data displays, including the stem-and-leaf plot, the histogram, and the box plot
2. Compute and interpret the sample mean, the sample variance, the sample standard deviation, and the sample range
3. Explain the concepts of a random variable and a probability distribution
4. Understand and interpret the mean, variance, and standard deviation of a probability distribution
5. Determine probabilities from probability distributions
6. Understand the assumptions for each of the discrete probability distributions presented
7. Understand the assumptions for each of the continuous probability distributions presented
8. Select an appropriate probability distribution for use in specific applications
9. Use probability plots
10. Use approximations for some hypergeometric and binomial distributions