7 Control Charts for Attributes

Chapter Outline

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Supplemental Material for Chapter 7

    S7.1 Probability Limits on Control Charts

The supplemental material is on the textbook Website www.wiley.com/college/montgomery.

Learning Objectives

1. Understand the statistical basis of attributes control charts
2. Know how to design attributes control charts
3. Know how to set up and use the p chart for fraction nonconforming
4. Know how to set up and use the np control chart for the number of nonconforming items
5. Know how to set up and use the c control chart for defects
6. Know how to set up and use the u control chart for defects per unit
7. Use attributes control charts with variable sample size
8. Understand the advantages and disadvantages of attributes versus variables control charts
9. Understand the rational subgroup concept for attributes control charts
10. Determine the average run length for attributes control charts

7.2 The Control Chart for Fraction Nonconforming

The **fraction nonconforming** is defined as the ratio of the number of nonconforming items in a population to the total number of items in that population. The items may have *several* quality characteristics that are examined simultaneously by the inspector. If the item does not conform to standard on one or more of these characteristics, it is classified as nonconforming.

The statistical principles underlying the control chart for fraction nonconforming are based on the binomial distribution. Suppose the production process is operating in a stable manner, such that the probability that any unit will not conform to specifications is p, and that successive units produced are independent. Then each unit produced is a realization of a Bernoulli random variable with parameter p. If a random sample of n units of product is selected, and if D is the number of units of product that are nonconforming, then D has a binomial distribution with parameters n and p; that is,

$$P\{D = x\} = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \ldots, n \tag{7.1}$$

From Section 3.2.2 we know that the mean and variance of the random variable D are np and $np(1-p)$, respectively.

The **sample fraction nonconforming** is defined as the ratio of the number of nonconforming units in the sample D to the sample size n; that is,

$$\hat{p} = \frac{D}{n} \tag{7.2}$$

As noted in Section 3.2.2, the distribution of the random variable $\hat{p}$ can be obtained from the binomial. Furthermore, the mean and variance of $\hat{p}$ are

$$\mu_{\hat{p}} = p \tag{7.3}$$

and

$$\sigma_{\hat{p}}^2 = \frac{p(1-p)}{n} \tag{7.4}$$

respectively. We will now see how this theory can be applied to the development of a control chart for fraction nonconforming. Because the chart monitors the process fraction nonconforming p, it is also called the p chart.

7.2.1 Development and Operation of the Control Chart

In Chapter 5, we discussed the general statistical principles on which the Shewhart control chart is based. If w is a statistic that measures a quality characteristic, and if the mean of w is μ_w and the variance of w is σ_w^2, then the general model for the Shewhart control chart is as follows:

$$\begin{aligned} \text{UCL} &= \mu_w + L\sigma_w \\ \text{Center line} &= \mu_w \\ \text{LCL} &= \mu_w - L\sigma_w \end{aligned} \tag{7.5}$$

where L is the distance of the control limits from the center line, in multiples of the standard deviation of w. It is customary to choose $L = 3$.

Fraction Nonconforming Control Chart: Standard Given

$$
\begin{aligned}
\text{UCL} &= p + 3\sqrt{\frac{p(1-p)}{n}} \\
\text{Center line} &= p \\
\text{LCL} &= p - 3\sqrt{\frac{p(1-p)}{n}}
\end{aligned}
\tag{7.6}
$$

When the process fraction nonconforming p is not known, then it must be estimated from observed data. The usual procedure is to select m preliminary samples, each of size n. As a general rule, m should be at least 20 or 25. Then if there are D_i nonconforming units in sample i, we compute the fraction nonconforming in the ith sample as

$$\hat{p}_i = \frac{D_i}{n} \qquad i = 1,\ 2,\ \ldots,\ m$$

and the average of these individual sample fractions nonconforming is

$$\overline{p} = \frac{\sum_{i=1}^{m} D_i}{mn} = \frac{\sum_{i=1}^{m} \hat{p}_i}{m} \tag{7.7}$$

The statistic $\overline{p}$ estimates the unknown fraction nonconforming p. The center line and control limits of the control chart for fraction nonconforming are computed as follows:

Fraction Nonconforming Control Chart: No Standard Given

$$\begin{aligned} \text{UCL} &= \overline{p} + 3\sqrt{\frac{\overline{p}(1-\overline{p})}{n}} \\ \text{Center line} &= \overline{p} \\ \text{LCL} &= \overline{p} - 3\sqrt{\frac{\overline{p}(1-\overline{p})}{n}} \end{aligned} \tag{7.8}$$

This control chart is also often called the *p-chart*.

EXAMPLE 7.1 Construction and Operation of a Fraction Nonconforming Control Chart

Frozen orange juice concentrate is packed in 6-oz cardboard cans. These cans are formed on a machine by spinning them from cardboard stock and attaching a metal bottom panel. By inspection of a can, we may determine whether, when filled, it could possibly leak either on the side seam or around the bottom joint. Such a nonconforming can has an improper seal on either the side seam or the bottom panel. Set up a control chart to improve the fraction of nonconforming cans produced by this machine.

SOLUTION

To establish the control chart, 30 samples of $n = 50$ cans each were selected at half-hour intervals over a three-shift period in which the machine was in continuous operation. The data are shown in Table 7.1.

We construct a phase I control chart using this preliminary data to determine if the process was in control when these data were collected. Since the 30 samples contain $\sum_{i=1}^{30} D_i = 347$ nonconforming cans, we find from equation (7.7),

$$\overline{p} = \frac{\sum_{i=1}^{m} D_i}{mn} = \frac{347}{(30)(50)} = 0.2313$$

Using $\overline{p}$ as an estimate of the true process fraction nonconforming, we can now calculate the upper and lower control limits as

$$\begin{aligned}\overline{p} \pm 3\sqrt{\frac{\overline{p}(1-\overline{p})}{n}} &= 0.2313 \pm 3\sqrt{\frac{0.2313(0.7687)}{50}} \\ &= 0.2313 \pm 3(0.0596) \\ &= 0.2313 \pm 0.1789\end{aligned}$$

Therefore,

$$\text{UCL} = \overline{p} + 3\sqrt{\frac{\overline{p}(1-\overline{p})}{n}} = 0.2313 + 0.1789 = 0.4102$$

and

$$\text{LCL} = \overline{p} - 3\sqrt{\frac{\overline{p}(1-\overline{p})}{n}} = 0.2313 - 0.1789 = 0.0524$$

TABLE 7.1

Data for Trial Control Limits, Example 7.1, Sample Size $n = 50$

Sample Number	Number of Nonconforming Cans, D_i	Sample Fraction Nonconforming, $\hat{p}_i$	Sample Number	Number of Nonconforming Cans, D_i	Sample Fraction Nonconforming, $\hat{p}_i$
1	12	0.24	17	10	0.20
2	15	0.30	18	5	0.10
3	8	0.16	19	13	0.26
4	10	0.20	20	11	0.22
5	4	0.08	21	20	0.40
6	7	0.14	22	18	0.36
7	16	0.32	23	24	0.48
8	9	0.18	24	15	0.30
9	14	0.28	25	9	0.18
10	10	0.20	26	12	0.24
11	5	0.10	27	7	0.14
12	6	0.12	28	13	0.26
13	17	0.34	29	9	0.18
14	12	0.24	30	6	0.12
15	22	0.44		347	$\overline{p} = 0.2313$
16	8	0.16			

The control chart with center line at $\overline{p} = 0.2313$ and the above upper and lower control limits is shown in Fig. 7.1. The sample fraction nonconforming from each preliminary sample is plotted on this chart. We note that two points, those from samples 15 and 23, plot above the upper control limit, so the process is not in control. These points must be investigated to see whether an assignable cause can be determined.

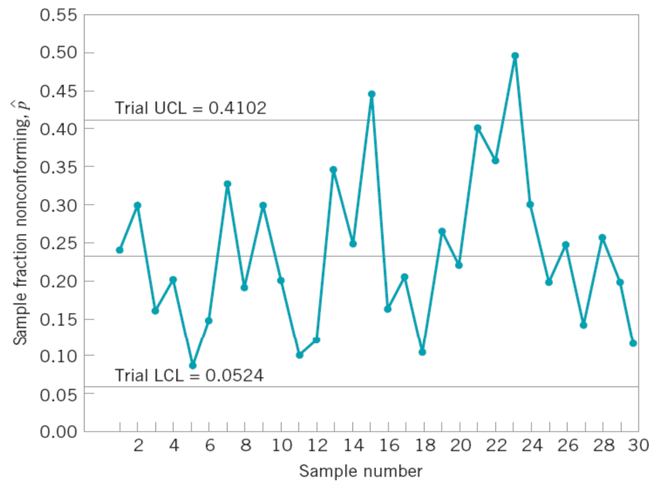


FIGURE 7.1 Initial phase I fraction nonconforming control chart for the data in Table 7.1.

Analysis of the data from sample 15 indicates that a new batch of cardboard stock was put into production during that half-hour period. The introduction of new batches of raw material sometimes causes irregular production performance, and it is reasonable to believe that this has occurred here. Furthermore, during the half-hour period in which sample 23 was obtained, a relatively inexperienced operator had been temporarily assigned to the machine, and this could account for the high fraction nonconforming obtained from that sample. Consequently, samples 15 and 23 are eliminated, and the new center line and revised control limits are calculated as

$$\overline{p} = \frac{301}{(28)(50)} = 0.2150$$

$$\text{UCL} = \overline{p} + 3\sqrt{\frac{\overline{p}(1-\overline{p})}{n}} = 0.2150 + 3\sqrt{\frac{0.2150(0.7850)}{50}} = 0.3893$$

$$\text{LCL} = \overline{p} - 3\sqrt{\frac{\overline{p}(1-\overline{p})}{n}} = 0.2150 - 3\sqrt{\frac{0.2150(0.7850)}{50}} = 0.0407$$

The revised center line and control limits are shown on the control chart in Fig. 7.2. Note that we have not dropped samples 15 and 23 from the chart, but they have been excluded from the control limit calculations, and we have noted this directly on the control chart. This annotation of the control chart to indicate unusual points, process adjustments, or the type of investigation made at a particular point in time forms a useful record for future process analysis and should become a standard practice in control chart usage.

Note also that the fraction nonconforming from sample 21 now exceeds the upper control limit. However, analysis of the data does not produce any reasonable or logical assignable cause for this, and we decide to retain the point. Therefore, we conclude that the new control limits in Fig. 7.2 can be used for future samples. Thus, we have concluded the control limit estimation phase (phase I) of control chart usage.

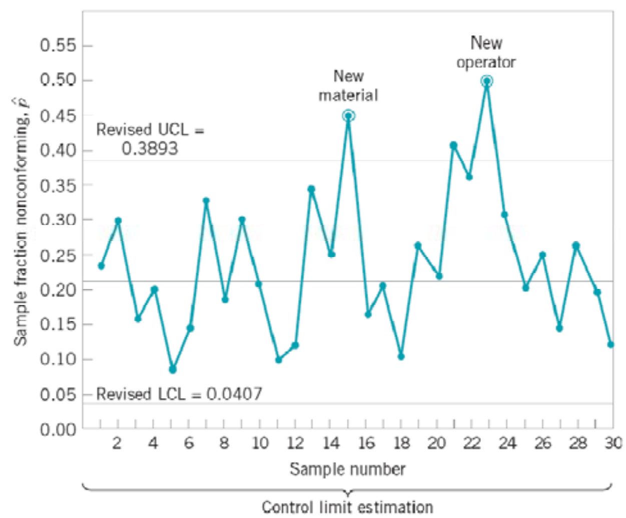


FIGURE 7.2 Revised control limits for the data in Table 7.1.

Sometimes examination of control chart data reveals information that affects other points that are not necessarily outside the control limits. For example, if we had found that the temporary operator working when sample 23 was obtained was actually working during the entire two-hour period in which samples 21–24 were obtained, then we should discard all four samples, even if only sample 21 exceeded the control limits, on the grounds that this inexperienced operator probably had some adverse influence on the fraction nonconforming during the entire period.

Before we conclude that the process is in control at this level, we could examine the remaining 28 samples for runs and other nonrandom patterns. The largest run is one of length 5 above the center line, and there are no obvious patterns present in the data. There is no strong evidence of anything other than a random pattern of variation about the center line.

We conclude that the process is in control at the level p = 0.2150 and that the revised control limits should be adopted for monitoring current production. However, we note that although the process is in control, the fraction nonconforming is much too high. That is, the process is operating in a stable manner, and no unusual **operator-controllable** problems are present. It is unlikely that the process quality can be improved by action at the workforce level. The nonconforming cans produced are **management controllable** because an intervention by management in the process will be required to improve performance. Plant management agrees with this observation and directs that, in addition to implementing the control chart program, the engineering staff should analyze the process in an effort to improve the process yield. This study indicates that several adjustments can be made on the machine that should improve its performance.

During the next three shifts following the machine adjustments and the introduction of the control chart, an additional 24 samples of $n = 50$ observations each are collected. These data are shown in Table 7.2, and the sample fractions nonconforming are plotted on the control chart in Fig. 7.3.

From an examination of Fig. 7.3, our immediate impression is that the process is now operating at a new quality level that is substantially better than the center line level of $\bar{p} = 0.2150$. One point, that from sample 41, is below the lower control limit. No assignable cause for this out-of-control signal can be determined. The only logical reasons for this ostensible change in process performance are the machine adjustments made by the engineering staff and, possibly, the operators themselves. It is not unusual to find that process performance improves following the introduction of formal statistical process-control procedures, often because the operators are more aware of process quality and because the control chart provides a continuing visual display of process performance.

We may formally test the hypothesis that the process fraction nonconforming in this current three-shift period differs

■ TABLE 7.2

Orange Juice Concentrate Can Data in Samples of Size $n = 50$

Sample Number	Number of Nonconforming Cans, D_i	Sample Fraction Nonconforming, $\hat{p}_i$	Sample Number	Number of Nonconforming Cans, D_i	Sample Fraction Nonconforming, $\hat{p}_i$
31	9	0.18	44	6	0.12
32	6	0.12	45	5	0.10
33	12	0.24	46	4	0.08
34	5	0.10	47	8	0.16
35	6	0.12	48	5	0.10
36	4	0.08	49	6	0.12
37	6	0.12	50	7	0.14
38	3	0.06	51	5	0.10
39	7	0.14	52	6	0.12
40	6	0.12	53	3	0.06
41	2	0.04	54	5	0.10
42	4	0.08		133	$\bar{p} = 0.1108$
43	3	0.06			

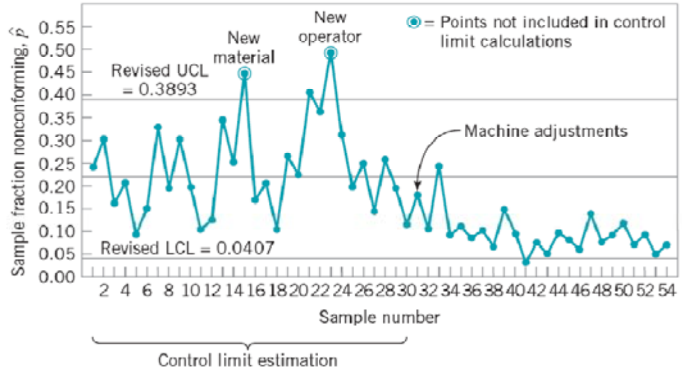


FIGURE 7.3 Continuation of the fraction nonconforming control chart, Example 7.1.

from the process fraction nonconforming in the preliminary data, using the procedure given in Section 4.3.4. The hypotheses are

$$H_0: \quad p_1 = p_2$$
$$H_1: \quad p_1 > p_2$$

where p_1 is the process fraction nonconforming from the preliminary data and p_2 is the process fraction nonconforming in the current period. We may estimate p_1 by $\hat{p}_1 = \overline{p} = 0.2150$, and p_2 by

$$\hat{p}_2 = \frac{\sum_{i=31}^{54} D_i}{(50)(24)} = \frac{133}{1200} = 0.1108$$

The (approximate) test statistic for the above hypothesis is, from equation (4.63),

$$Z_0 = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where

$$\hat{p} = \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2}$$

In our example, we have

$$\hat{p} = \frac{(1400)(0.2150) + (1200)(0.1108)}{1400 + 1200} = 0.1669$$

and

$$Z_0 = \frac{0.2150 - 0.1108}{\sqrt{(0.1669)(0.8331)\left(\frac{1}{1400} + \frac{1}{1200}\right)}} = 7.10$$

Comparing this to the upper 0.05 point of the standard normal distribution, we find that $Z_0 = 7.10 > Z_{0.05} = 1.645$. Consequently, we reject H_0 and conclude that there has been a significant decrease in the process fallout.

Based on the apparently successful process adjustments, it seems logical to revise the control limits again, using only the most recent samples (numbers 31–54). This results in the new control chart parameters:

$$\text{Center line} = \overline{p} = 0.1108$$

$$\text{UCL} = \overline{p} + 3\sqrt{\frac{\overline{p}(1-\overline{p})}{n}} = 0.1108 + 3\sqrt{\frac{(0.1108)(0.8892)}{50}} = 0.2440$$

$$\text{LCL} = \overline{p} - 3\sqrt{\frac{\overline{p}(1-\overline{p})}{n}} = 0.1108 - 3\sqrt{\frac{(0.1108)(0.8892)}{50}} = -0.0224 = 0$$

Figure 7.4 shows the control chart with these new parameters. Note that since the calculated lower control limit is less than zero, we have set LCL = 0. Therefore, the new control chart will have only an upper control limit. From inspection of Fig. 7.4, we see that all the points would fall inside the revised upper control limit; therefore, we conclude that the process is in control at this new level.

The continued operation of this control chart for the next five shifts is shown in Fig. 7.5. Data for the process during this period are shown in Table 7.3. The control chart does not indicate lack of control. Despite the improvement in yield following the engineering changes in the process and the introduction of the control chart, the process fallout of $\overline{p} = 0.1108$ is still too high. Further analysis and action will be required to improve the yield. These management interventions may be further adjustments to the machine. **Statistically designed experiments** (see Part IV) are an appropriate way to determine which machine adjustments are critical to further process improvement, and the appropriate magnitude and direction of these adjustments. The control chart should be continued during the period in which the adjustments are made. By marking the time scale of the control chart *when* a process change is made, the control chart becomes a **logbook** in which the timing of process interventions and their subsequent effect on process performance are easily seen. This logbook aspect of control chart usage is extremely important.

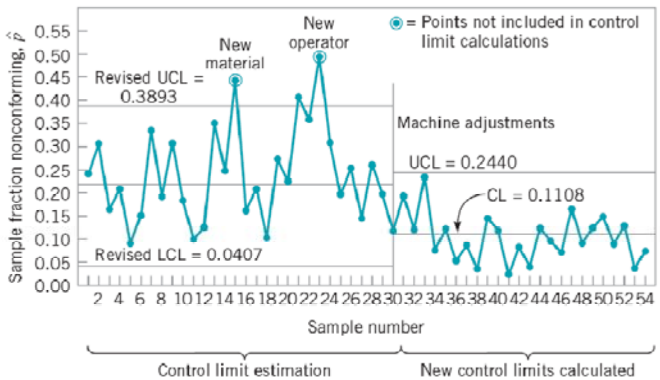


FIGURE 7.4 New control limits on the fraction nonconforming control chart, Example 7.1.

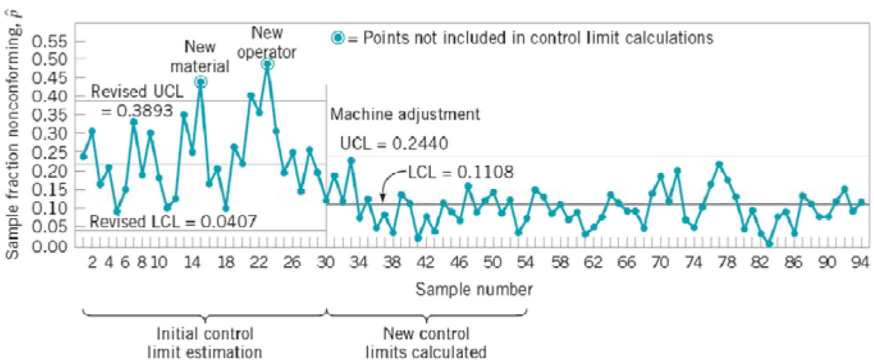


FIGURE 7.5 Completed fraction nonconforming control chart, Example 7.1.

■ TABLE 7.3

New Data for the Fraction Nonconforming Control Chart in Fig. 7.5, $n = 50$

Sample Number	Number of Nonconforming Cans, D_i	Sample Fraction Nonconforming, $\hat{p}_i$	Sample Number	Number of Nonconforming Cans, D_i	Sample Fraction Nonconforming, $\hat{p}_i$
55	8	0.16	75	5	0.10
56	7	0.14	76	8	0.16
57	5	0.10	77	11	0.22
58	6	0.12	78	9	0.18
59	4	0.08	79	7	0.14
60	5	0.10	80	3	0.06
61	2	0.04	81	5	0.10
62	3	0.06	82	2	0.04
63	4	0.08	83	1	0.02
64	7	0.14	84	4	0.08
65	6	0.12	85	5	0.10
66	5	0.10	86	3	0.06
67	5	0.10	87	7	0.14
68	3	0.06	88	6	0.12
69	7	0.14	89	4	0.08
70	9	0.18	90	4	0.08
71	6	0.12	91	6	0.12
72	10	0.20	92	8	0.16
73	4	0.08	93	5	0.10
74	3	0.06	94	6	0.12

Design of the Fraction Nonconforming Chart

- Three parameters must be specified
  1. The sample size
  2. The frequency of sampling
  3. The width of the control limits
- Common to base chart on 100% inspection of *all* process output over time
- Rational subgroups may also play role in determining sampling frequency

Interpretation of Points on the Control Chart for Fraction Nonconforming. Example 7.1 illustrated how points that plot beyond the control limits are treated, both in establishing the control chart and during its routine operation. Care must be exercised in interpreting points that plot *below* the lower control limit. These points often do not represent a real improvement in process quality. Frequently, they are caused by errors in the inspection process resulting from inadequately trained or inexperienced inspectors or from improperly calibrated test and inspection equipment. We have also seen cases in which inspectors deliberately passed nonconforming units or reported fictitious data. The analyst must keep these warnings in mind when looking for assignable causes if points plot below the lower control limits. Not all downward shifts in p are attributable to improved quality.

The* np *Control Chart. It is also possible to base a control chart on the number nonconforming rather than the fraction nonconforming. This is often called an **number nonconforming (*np*) control chart.** The parameters of this chart are as follows.

The *np* Control Chart

$$\begin{aligned} \text{UCL} &= np + 3\sqrt{np(1-p)} \\ \text{Center line} &= np \\ \text{LCL} &= np - 3\sqrt{np(1-p)} \end{aligned} \tag{7.13}$$

If a standard value for p is unavailable, then $\overline{p}$ can be used to estimate p. Many nonstatistically trained personnel find the np chart easier to interpret than the usual fraction nonconforming control chart.

7.2.2 Variable Sample Size

In some applications of the control chart for fraction nonconforming, the sample is a 100% inspection of process output over some period of time. Since different numbers of units could be produced in each period, the control chart would then have a variable sample size. There are three approaches to constructing and operating a control chart with a variable sample size.

Variable-Width Control Limits. The first and perhaps the most simple approach is to determine control limits for each individual sample that are based on the specific sample size. That is, if the ith sample is of size n_i, then the upper and lower control limits are $\overline{p} \pm 3\sqrt{\overline{p}(1-\overline{p})/n_i}$. Note that the width of the control limits is inversely proportional to the square root of the sample size.

To illustrate this approach, consider the data in Table 7.4. These data came from the purchasing group of a large aerospace company. This group issues purchase orders to the company's suppliers. The sample sizes in Table 7.4 are the total number of purchase orders issued each week. Obviously, this is not constant. A nonconforming unit is a purchase order with an error. Among the most common errors are specifying incorrect part numbers, wrong delivery dates, and wrong supplier information. Any of these mistakes can result in a purchase order change, which takes time and resources and may result in delayed delivery of material.

For the 25 samples, we calculate

$$\overline{p} = \frac{\sum_{i=1}^{25} D_i}{\sum_{i=1}^{25} n_i} = \frac{234}{2450} = 0.096$$

Consequently, the center line is at 0.096, and the control limits are

$$\text{UCL} = \overline{p} + 3\hat{\sigma}_{\hat{p}} = 0.096 + 3\sqrt{\frac{(0.096)(0.904)}{n_i}}$$

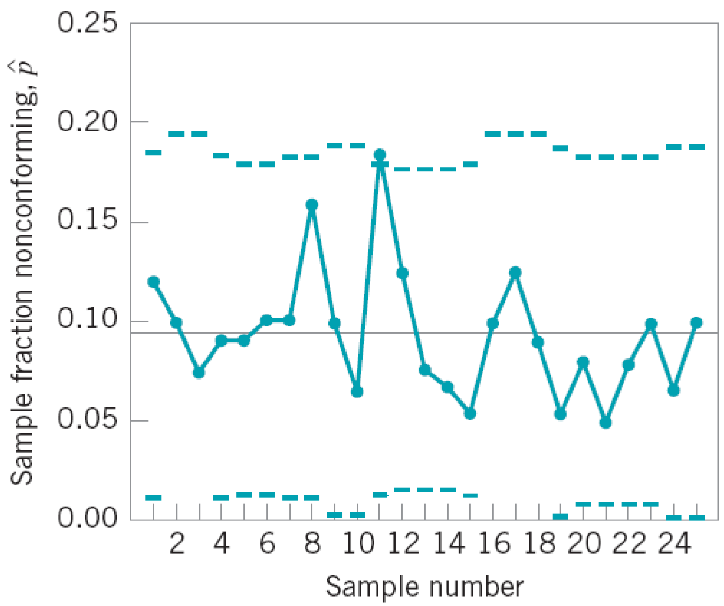
and

$$\text{LCL} = \overline{p} - 3\hat{\sigma}_{\hat{p}} = 0.096 - 3\sqrt{\frac{(0.096)(0.904)}{n_i}}$$

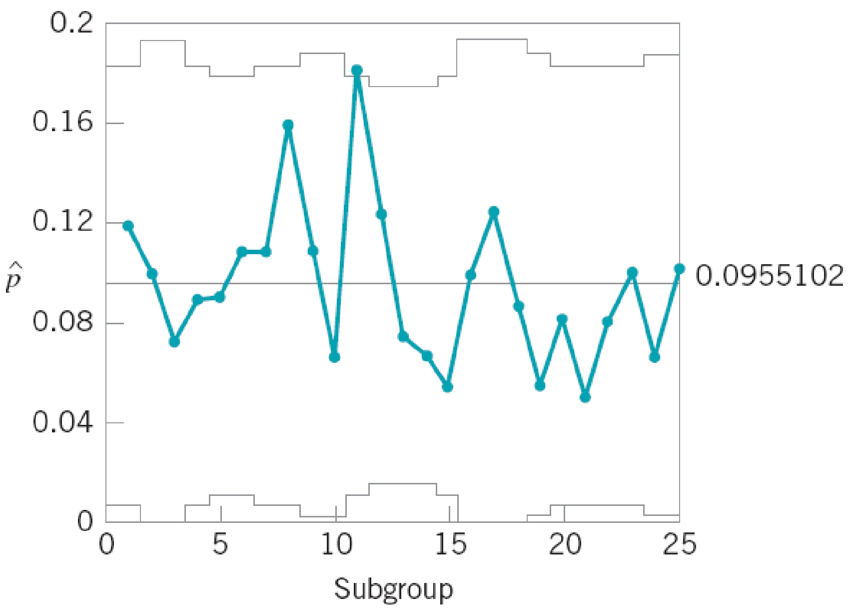
TABLE 7.4

Purchase Order Data for a Control Chart for Fraction Nonconforming with Variable Sample Size

Sample Number, i	Sample Size, n_i	Number of Nonconforming Units, D_i	Sample Fraction Nonconforming, $\hat{p}_i = D_i/n_i$	Standard Deviation $\hat{\sigma}_{\hat{p}} = \sqrt{\frac{(0.096)(0.904)}{n_i}}$	Control Limits LCL	Control Limits UCL
1	100	12	0.120	0.029	0.009	0.183
2	80	8	0.100	0.033	0	0.195
3	80	6	0.075	0.033	0	0.195
4	100	9	0.090	0.029	0.009	0.183
5	110	10	0.091	0.028	0.012	0.180
6	110	12	0.109	0.028	0.012	0.180
7	100	11	0.110	0.029	0.009	0.183
8	100	16	0.160	0.029	0.009	0.183
9	90	10	0.110	0.031	0.003	0.189
10	90	6	0.067	0.031	0.003	0.189
11	110	20	0.182	0.028	0.012	0.180
12	120	15	0.125	0.027	0.015	0.177
13	120	9	0.075	0.027	0.015	0.177
14	120	8	0.067	0.027	0.015	0.177
15	110	6	0.055	0.028	0.012	0.180
16	80	8	0.100	0.033	0	0.195
17	80	10	0.125	0.033	0	0.195
18	80	7	0.088	0.033	0	0.195
19	90	5	0.056	0.031	0.003	0.189
20	100	8	0.080	0.029	0.009	0.183
21	100	5	0.050	0.029	0.009	0.183
22	100	8	0.080	0.029	0.009	0.183
23	100	10	0.100	0.029	0.009	0.183
24	90	6	0.067	0.031	0.003	0.189
25	90	9	0.100	0.031	0.003	0.189
	2450	234	2.383			



■ FIGURE 7.6 Control chart for fraction nonconforming with variable sample size.



■ FIGURE 7.7 Control chart for fraction nonconforming with variable sample size using Minitab.

Average sample size approach

$$\bar{n} = \frac{\sum_{i=1}^{25} n_i}{25} = \frac{2450}{25} = 98$$

Therefore, the approximate control limits are

$$\text{UCL} = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{\bar{n}}} = 0.096 + 3\sqrt{\frac{(0.096)(0.904)}{98}} = 0.185$$

and

$$\text{LCL} = \bar{p} - \sqrt{\frac{\bar{p}(1-\bar{p})}{\bar{n}}} = 0.096 - 3\sqrt{\frac{(0.096)(0.904)}{98}} = 0.007$$

The resulting control chart is shown in Fig. 7.8. Note that $\hat{p}$ for sample 11 plots close to the approximate upper control limit, yet it appears to be in control. However, when compared to

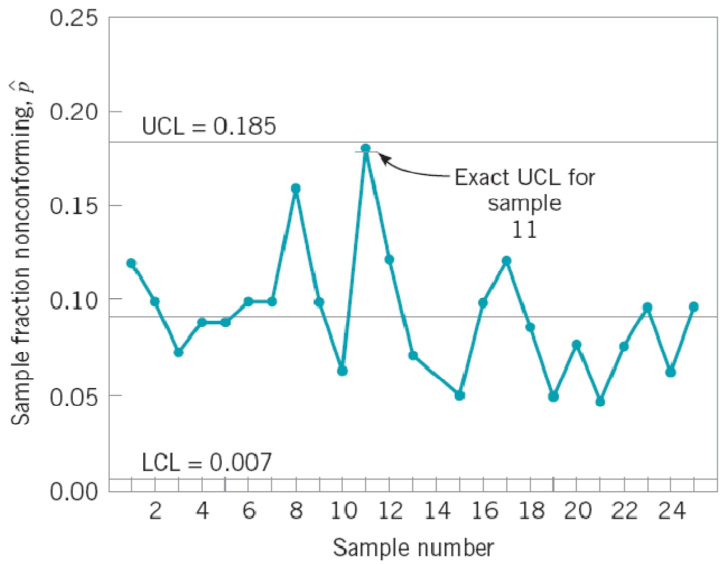


FIGURE 7.8 Control chart for fraction nonconforming based on average sample size.

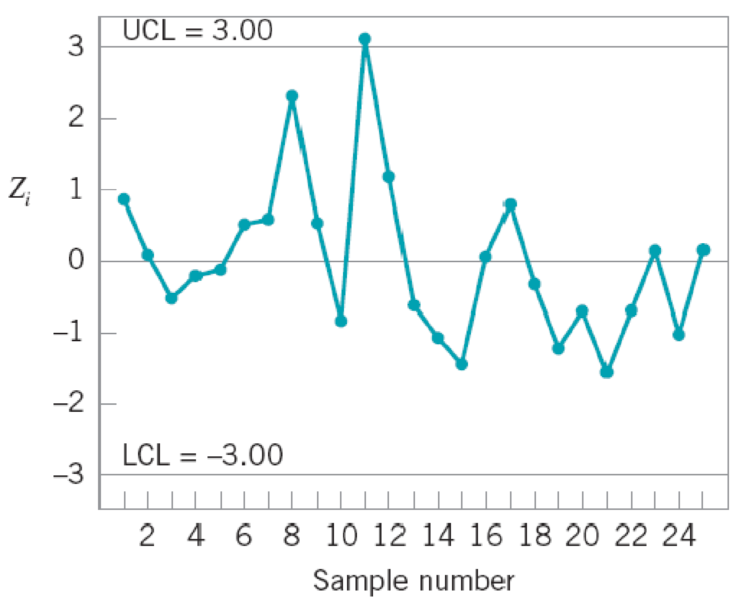
The Standardized Control Chart. The third approach to dealing with variable sample size is to use a **standardized control chart,** where the points are plotted in standard deviation units. Such a control chart has the center line at zero, and upper and lower control limits of +3 and −3, respectively. The variable plotted on the chart is

$$Z_i = \frac{\hat{p}_i - p}{\sqrt{\dfrac{p(1-p)}{n_i}}} \tag{7.14}$$

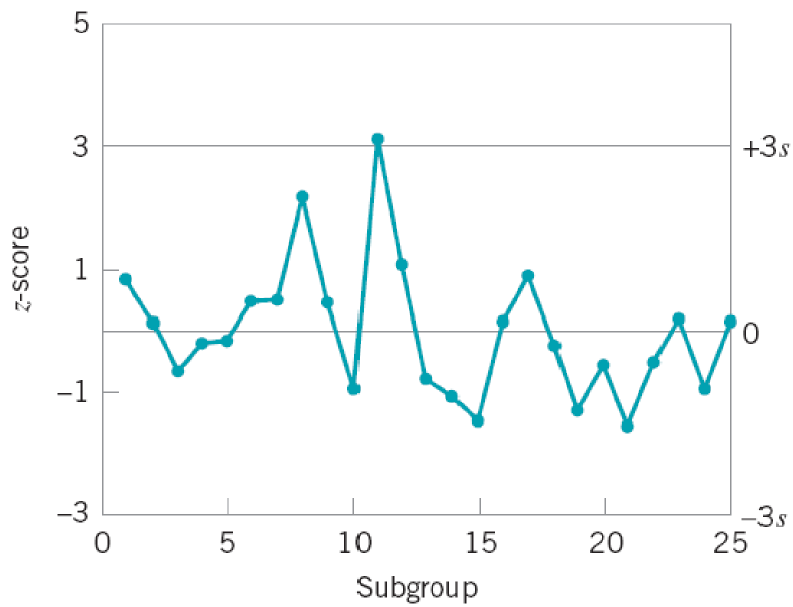
where p (or $\overline{p}$ if no standard is given) is the process fraction nonconforming in the in-control state. The standardized control chart for the purchase order data in Table 7.4 is shown in Fig. 7.9. The calculations associated with this control chart are shown in Table 7.5. Tests for runs and pattern-recognition methods could safely be applied to this chart, because the relative changes from one point to another are all expressed in terms of the same units of measurement.

The standardized control chart is no more difficult to construct or maintain than either of the other two procedures discussed in this section. In fact, many quality control software packages either automatically execute this as a standard feature or can be programmed to plot a standardized control chart. For example, the version of Fig. 7.9 shown in Fig. 7.10 was created using Minitab. Conceptually, however, it may be more difficult for operating personnel to understand and interpret, because reference to the actual process fraction defective has been "lost." However, if there is large variation in sample size, then runs and pattern-recognition methods can only be safely applied to the standardized control chart. In such a case, it might be advisable to maintain a control chart with individual control limits for each sample (as in Fig. 7.6) for the operating personnel, while simultaneously maintaining a standardized control chart for engineering use.

The standardized control chart is also recommended when the length of the production run is short, as in many job-shop settings. Control charts for short production runs are discussed in Chapter 9.



■ **FIGURE 7.9** Standardized control chart for fraction nonconforming.



■ **FIGURE 7.10** Standardized control chart from Minitab for fraction nonconforming, Table 7.4.

■ TABLE 7.5

Calculations for the Standardized Control Chart in Fig. 7.9, $\hat{p} = 0.096$

Sample Number, i	Sample Size, n_i	Number of Noncon-forming Units, D_i	Sample Fraction Noncon-forming, $\hat{p}_i = D_i/n_i$	Standard Deviation $\hat{\sigma}_p = \sqrt{\frac{(0.096)(0.904)}{n_i}}$	$z_i = \frac{\hat{p}_i - \bar{p}}{\sqrt{\frac{(0.096)(0.904)}{n_i}}}$
1	100	12	0.120	0.029	0.83
2	80	8	0.100	0.033	0.12
3	80	6	0.075	0.033	−0.64
4	100	9	0.090	0.029	−0.21
5	110	10	0.091	0.028	−0.18
6	110	12	0.109	0.028	0.46
7	100	11	0.110	0.029	0.48
8	100	16	0.160	0.029	2.21
9	90	10	0.110	0.031	0.45
10	90	6	0.067	0.031	−0.94
11	110	20	0.182	0.028	3.07
12	120	15	0.125	0.027	1.07
13	120	9	0.075	0.027	−0.78
14	120	8	0.067	0.027	−1.07
15	110	6	0.055	0.028	−1.46
16	80	8	0.100	0.033	0.12
17	80	10	0.125	0.033	0.88
18	80	7	0.088	0.033	−0.24
19	90	5	0.056	0.031	−1.29
20	100	8	0.080	0.029	−0.55
21	100	5	0.050	0.029	−1.59
22	100	8	0.080	0.029	−0.55
23	100	10	0.100	0.029	0.14
24	90	6	0.067	0.031	−0.94
25	90	9	0.100	0.031	0.13

7.2.4 The Operating-Characteristic Function and Average Run Length Calculations

The operating-characteristic (or OC) function of the fraction nonconforming control chart is a graphical display of the probability of incorrectly accepting the hypothesis of statistical control (i.e., a type II or β-error) against the process fraction nonconforming. The OC curve provides a measure of the **sensitivit**y of the control chart—that is, its ability to detect a shift in the process fraction nonconforming from the nominal value $\bar{p}$ to some other value p. The probability of type II error for the fraction nonconforming control chart may be computed from

$$\begin{aligned}\beta &= P\{\hat{p} < \mathrm{UCL}|p\} - P\{\hat{p} \le \mathrm{LCL}|p\} \\ &= P\{D < n\mathrm{UCL}|p\} - P\{D \le n\mathrm{LCL}|p\}\end{aligned} \tag{7.15}$$

Since D is a binomial random variable with parameters n and p, the β-error defined in equation (7.15) can be obtained from the cumulative binomial distribution. Note that when the LCL is negative, the second term on the right-hand side of equation (7.15) should be dropped.

TABLE 7.6

Calculations[a] for Constructing the OC Curve for a Control Chart for Fraction Nonconforming with $n = 50$, LCL = 0.0303, and UCL = 0.3697

p	$P\{D \le 18 \mid p\}$	$P\{D \le 1 \mid p\}$	$\beta = P\{D \le 18 \mid p\} - P\{D \le 1 \mid p\}$
0.01	1.0000	0.9106	0.0894
0.03	1.0000	0.5553	0.4447
0.05	1.0000	0.2794	0.7206
0.10	1.0000	0.0338	0.9662
0.15	0.9999	0.0029	0.9970
0.20	0.9975	0.0002	0.9973
0.25	0.9713	0.0000	0.9713
0.30	0.8594	0.0000	0.8594
0.35	0.6216	0.0000	0.6216
0.40	0.3356	0.0000	0.3356
0.45	0.1273	0.0000	0.1273
0.50	0.0325	0.0000	0.0325
0.55	0.0053	0.0000	0.0053

[a]The probabilities in this table were found by evaluating the cumulative binomial distribution. For small p ($p < 0.1$, say) the Poisson approximation could be used, and for larger values of p the normal approximation could be used.

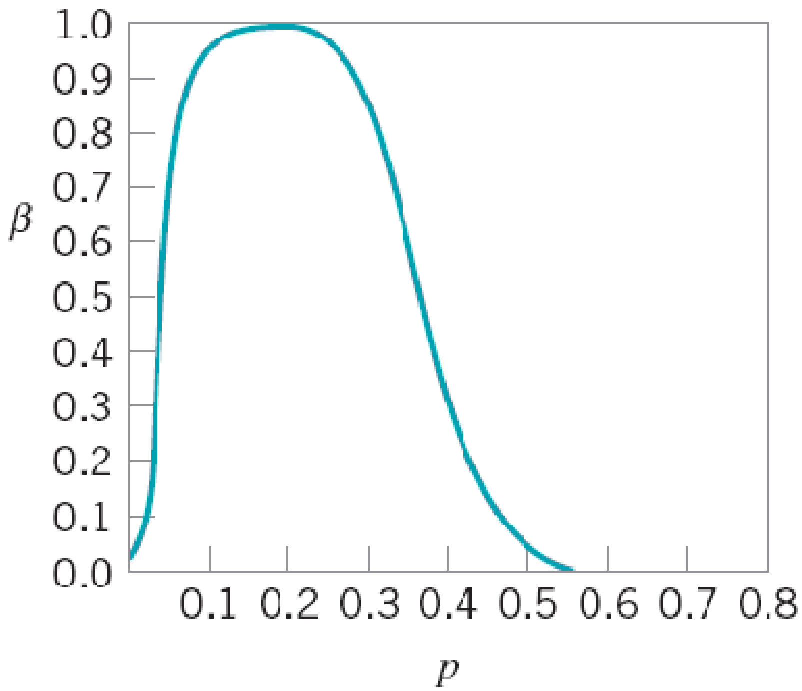


FIGURE 7.11 Operating-characteristic curve for the fraction nonconforming control chart with $\overline{p} = 0.20$, LCL = 0.0303, and UCL = 0.3697.

We may also calculate average run lengths (ARLs) for the fraction nonconforming control chart. Recall from Chapter 5 that for uncorrelated process data the ARL for any Shewhart control chart can be written as

$$\text{ARL} = \frac{1}{P(\text{sample point plots out of control})}$$

Thus, if the process is in control, ARL_0 is

$$\text{ARL}_0 = \frac{1}{\alpha}$$

and if it is out of control, then

$$\text{ARL}_1 = \frac{1}{1-\beta}$$

These probabilities (α, β) can be calculated directly from the binomial distribution or read from an OC curve.

To illustrate, consider the control chart for fraction nonconforming used in the OC curve calculations in Table 7.6. This chart has parameters $n = 50$, UCL = 0.3697, LCL = 0.0303, and the center line is $p = 0.20$. From Table 7.6 (or the OC curve in Fig. 7.11) we find that if the process is in control with $p = \overline{p}$, the probability of a point plotting in control is 0.9973. Thus, in this case $\alpha = 1 - \beta = 0.0027$, and the value of ARL_0 is

$$\text{ARL}_0 = \frac{1}{\alpha} = \frac{1}{0.0027} \simeq 370$$

Therefore, if the process is really in control, we will experience a false out-of-control signal about every 370 samples. (This will be approximately true, in general, for any Shewhart control chart with three-sigma limits.) This in-control ARL_0 is generally considered to be satisfactorily large. Now suppose that the process shifts out of control to $p = 0.3$. Table 7.6 indicates that if $p = 0.3$, then $\beta = 0.8594$. Therefore, the value of ARL_1 is

$$\text{ARL}_1 = \frac{1}{1-\beta} = \frac{1}{1-0.8594} \simeq 7$$

and it will take about seven samples, on the average, to detect this shift with a point outside of the control limits. If this is unsatisfactory, then action must be taken to reduce the out-of-control ARL_1. Increasing the sample size would result in a smaller value of β and a shorter out-of-control ARL_1. Another approach would be to reduce the interval *between* samples. That is, if we are currently sampling every hour, it will take about seven hours, on the average, to detect the shift. If we take the sample every half hour, it will require only three and a half hours, on the average, to detect the shift. Another approach is to use a control chart that is more responsive to small shifts, such as the cumulative sum charts in Chapter 9.

7.3 Control Charts for Nonconformities (Defects)

A nonconforming item is a unit of product that does not satisfy one or more of the specifications for that product. Each specific point at which a specification is not satisfied results in a **defect** or **nonconformity.** Consequently, a nonconforming item will contain at least one nonconformity. However, depending on their nature and severity, it is quite possible for a unit to contain several nonconformities and *not* be classified as nonconforming. As an example, suppose we are manufacturing personal computers. Each unit could have one or more very minor flaws in the cabinet finish, and since these flaws do not seriously affect the unit's functional operation, it could be classified as conforming. However, if there are too many of these flaws, the personal computer should be classified as nonconforming, since the flaws would be very noticeable to the customer and might affect the sale of the unit. There are many practical situations in which we prefer to work directly with the number of defects or nonconformities rather than the fraction nonconforming. These include the number of defective welds in 100 m of oil pipeline, the number of broken rivets in an aircraft wing, the number of functional defects in an electronic logic device, the number of errors on a document, and so forth.

It is possible to develop control charts for either the **total number of nonconformities** in a unit or the **average number of nonconformities per unit.** These control charts usually assume that the occurrence of nonconformities in samples of constant size is well modeled by the Poisson distribution. Essentially, this requires that the number of opportunities or potential locations for nonconformities be infinitely large and that the probability of occurrence of a nonconformity at any location be small and constant. Furthermore, the **inspection unit** must be the same for each sample. That is, each inspection unit must always represent an identical **area of opportunity** for the occurrence of nonconformities. In addition, we can count nonconformities of several different types on one unit, as long as the above conditions are satisfied for each class of nonconformity.

In most practical situations, these conditions will not be satisfied exactly. The number of opportunities for the occurrence of nonconformities may be finite, or the probability of occurrence of nonconformities may not be constant. As long as these departures from the assumptions are not severe, the Poisson model will usually work reasonably well. There are cases, however, in which the Poisson model is completely inappropriate. These situations are discussed in more detail at the end of Section 7.3.1.

7.3.1 Procedures with Constant Sample Size

Consider the occurrence of nonconformities in an inspection unit of product. In most cases, the inspection unit will be a single unit of product, although this is not necessarily always so. The inspection unit is simply an entity for which it is convenient to keep records. It could be a group of 5 units of product, 10 units of product, and so on. Suppose that defects or nonconformities occur in this inspection unit according to the Poisson distribution; that is,

$$p(x) = \frac{e^{-c} c^{x}}{x!} \qquad x = 0, 1, 2, \ldots$$

where x is the number of nonconformities and $c > 0$ is the parameter of the Poisson distribution. From Section 3.2.3 we recall that both the mean and variance of the Poisson distribution are the parameter c. Therefore, a **control chart for nonconformities, or c chart** with three-sigma limits would be defined as follows,[2]

Control Chart for Nonconformities: Standard Given

$$\begin{aligned} \text{UCL} &= c + 3\sqrt{c} \\ \text{Center line} &= c \\ \text{LCL} &= c - 3\sqrt{c} \end{aligned} \qquad (7.16)$$

Control Chart for Nonconformities: No Standard Given

$$
\begin{aligned}
\text{UCL} &= \bar{c} + 3\sqrt{\bar{c}} \\
\text{Center line} &= \bar{c} \\
\text{LCL} &= \bar{c} - 3\sqrt{\bar{c}}
\end{aligned}
\tag{7.17}
$$

When no standard is given, the control limits in equation (7.17) should be regarded as *trial* control limits, and the preliminary samples examined for lack of control in the usual phase I analysis. The control chart for nonconformities is also sometimes called the c chart.

EXAMPLE 7.3 Nonconformities in Printed Circuit Boards

Table 7.7 presents the number of nonconformities observed in 26 successive samples of 100 printed circuit boards. Note that, for reasons of convenience, the inspection unit is defined as 100 boards. Set up a c chart for these data.

TABLE 7.7

Data on the Number of Nonconformities in Samples of 100 Printed Circuit Boards

Sample Number	Number of Nonconformities	Sample Number	Number of Nonconformities
1	21	14	19
2	24	15	10
3	16	16	17
4	12	17	13
5	15	18	22
6	5	19	18
7	28	20	39
8	20	21	30
9	31	22	24
10	25	23	16
11	20	24	19
12	24	25	17
13	16	26	15

SOLUTION

Since the 26 samples contain 516 total nonconformities, we estimate c by

$$\bar{c} = \frac{516}{26} = 19.85$$

Therefore, the trial control limits are given by

$$\text{UCL} = \bar{c} + 3\sqrt{\bar{c}} = 19.85 + 3\sqrt{19.85} = 33.22$$

$$\text{Center line} = \bar{c} = 19.85$$

$$\text{LCL} = \bar{c} - 3\sqrt{\bar{c}} = 19.85 - 3\sqrt{19.85} = 6.48$$

The control chart is shown in Fig. 7.12. The number of observed nonconformities from the preliminary samples is plotted on this chart. Two points plot outside the control limits, samples 6 and 20. Investigation of sample 6 revealed that a new inspector had examined the boards in this sample and that he did not recognize several of the types of nonconformities that could have been present. Furthermore, the unusually large number of nonconformities in sample 20 resulted from a temperature control problem in the wave soldering machine, which was subsequently repaired. Therefore, it seems reasonable to exclude these two

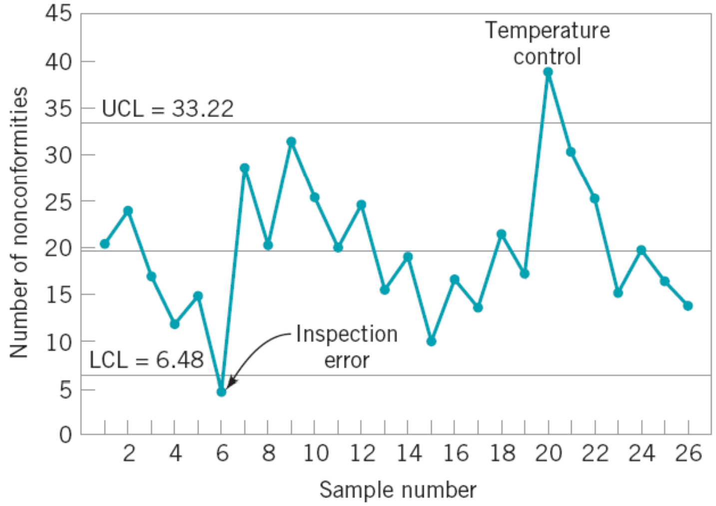


FIGURE 7.12 Control chart for nonconformities for Example 7.3.

samples and revise the trial control limits. The estimate of c is now computed as

$$\bar{c} = \frac{472}{24} = 19.67$$

and the revised control limits are

$$\begin{aligned} \text{UCL} &= \bar{c} + 3\sqrt{\bar{c}} = 19.67 + 3\sqrt{19.67} = 32.97 \\ \text{Center line} &= \bar{c} = 19.67 \\ \text{LCL} &= \bar{c} - 3\sqrt{\bar{c}} = 19.67 - 3\sqrt{19.67} = 6.36 \end{aligned}$$

These become the standard values against which production in the next period can be compared.

Twenty new samples, each consisting of one inspection unit (i.e., 100 boards), are subsequently collected. The number of nonconformities in each sample is noted and recorded in Table 7.8. These points are plotted on the control chart in Fig. 7.13. No lack of control is indicated; however, the number of nonconformities per board is still unacceptably high. Further action is necessary to improve the process.

TABLE 7.8

Additional Data for the Control Chart for Nonconformities, Example 7.3

Sample Number	Number of Nonconformities	Sample Number	Number of Nonconformities
27	16	37	18
28	18	38	21
29	12	39	16
30	15	40	22
31	24	41	19
32	21	42	12
33	28	43	14
34	20	44	9
35	25	45	16
36	19	46	21

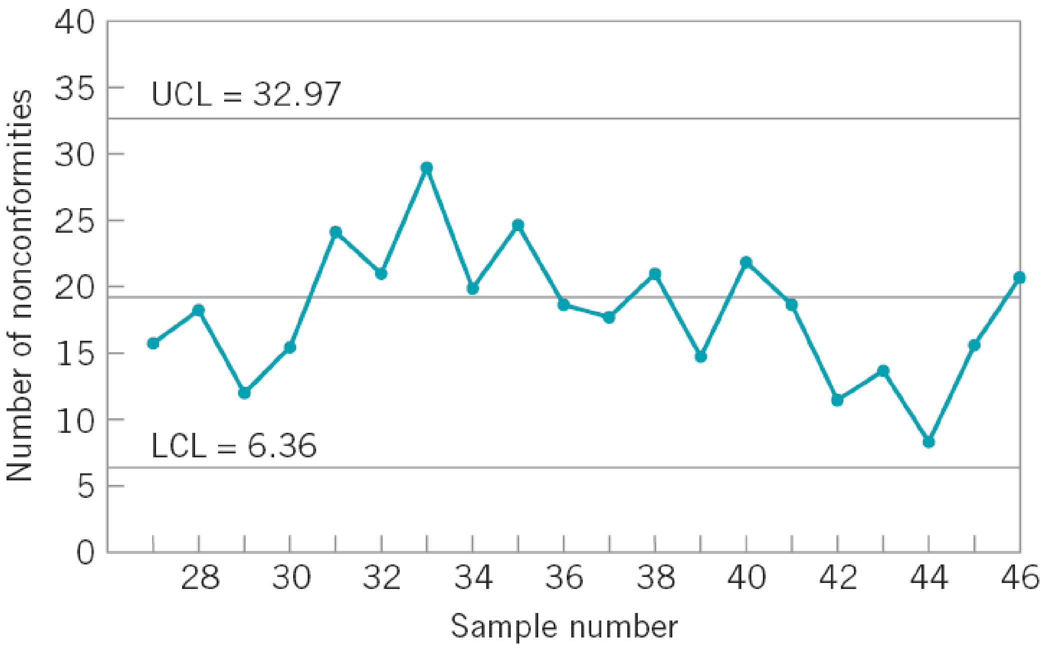


FIGURE 7.13 Continuation of the control chart for nonconformities. Example 7.3.

Further Analysis of Nonconformities. Defect or nonconformity data are always more informative than fraction nonconforming, because there will usually be several different *types* of nonconformities. By analyzing the nonconformities by type, we can often gain considerable insight into their cause. This can be of considerable assistance in developing the **out-of-control-action plans** (OCAPs) that must accompany control charts.

For example, in the printed circuit board process, there are sixteen different types of defects. Defect data for 500 boards are plotted on a Pareto chart in Fig. 7.14. Note that over 60% of the total number of defects is due to *two defect types:* solder insufficiency and solder cold joints. This points to further problems with the wave soldering process. If these problems can be isolated and eliminated, there will be a dramatic increase in process yield. Notice that the nonconformities follow the Pareto distribution; that is, most of the defects are attributable to a few (in this case, two) defect types.

This process manufactures several different types of printed circuit boards. Therefore, it may be helpful to examine the occurrence of defect type by type of printed circuit board (part number). Table 7.9 presents this information. Note that all 40 solder insufficiencies and all 20 solder cold joints occurred on the same part number, 0001285. This implies that this particular type of board is very susceptible to problems in wave soldering, and special attention must be directed toward improving this step of the process for this part number.

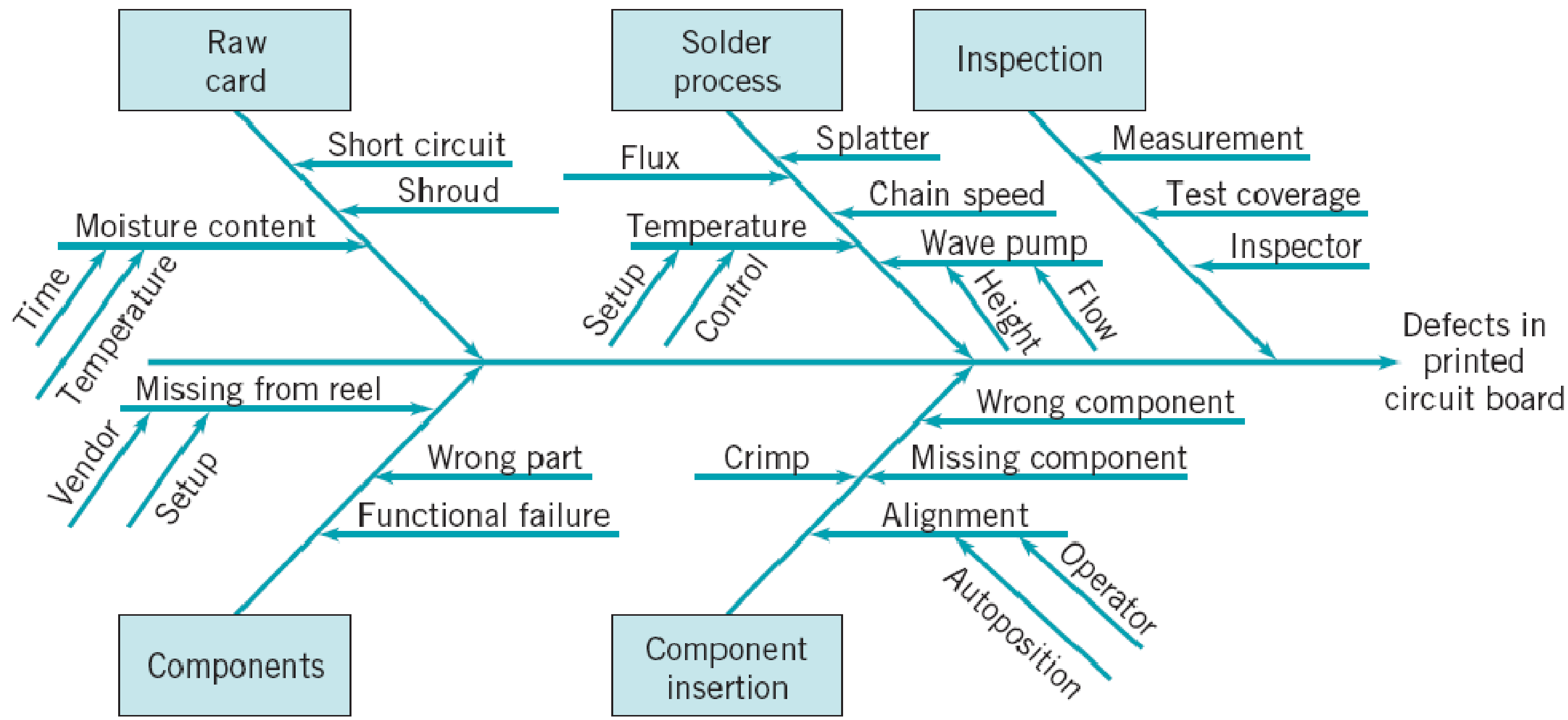
Defect code		Freq.	Cum. freq.	Percent	Cum. percent
Sold. Insufficie	**	40	40	40.82	40.82
Sold.cold joint	********************	20	60	20.41	61.23
Sold. opens/dewe	*******	7	67	7.14	68.37
Comp. improper 1	******	6	73	6.12	74.49
Sold. splatter/w	*****	5	78	5.10	79.59
Tst. mark ec mark	***	3	81	3.06	82.65
Tst. mark white m	***	3	84	3.06	85.71
Raw cd shroud re	***	3	87	3.06	88.78
Comp. extra part	**	2	89	2.04	90.82
Comp. damaged	**	2	91	2.04	92.86
Comp. missing	**	2	93	2.04	94.90
Wire incorrect s	*	1	94	1.02	95.92
Stamping oper id	*	1	95	1.02	96.94
Stamping missing	*	1	96	1.02	97.96
Sold. short	*	1	97	1.02	98.98
Raw cd damaged	*	1	98	1.02	100.00

1 10 20 30 40

Number of defects

FIGURE 7.14 Pareto analysis of nonconformities for the printed circuit board process.

Another useful technique for further analysis of nonconformities is the **cause-and-effect diagram** discussed in Chapter 5. The cause-and-effect diagram is used to illustrate the various sources of nonconformities in products and their interrelationships. It is useful in focusing the attention of operators, manufacturing engineers, and managers on quality problems. Developing a good cause-and-effect diagram usually advances the level of technological understanding of the process.



■ FIGURE 7.15 Cause-and-effect diagram.

Choice of Sample Size: The* u *Chart. Example 7.3 illustrates a control chart for nonconformities with the sample size exactly equal to one inspection unit. The inspection unit is chosen for operational or data-collection simplicity. However, there is no reason why the sample size must be restricted to one inspection unit. In fact, we would often prefer to use *several* inspection units in the sample, thereby increasing the area of opportunity for the occurrence of nonconformities. The sample size should be chosen according to statistical considerations, such as specifying a sample size large enough to ensure a positive lower control limit or to obtain a particular probability of detecting a process shift. Alternatively, economic factors could enter into sample-size determination.

Suppose we decide to base the control chart on a sample size of n inspection units. Note that n does not have to be an integer. To illustrate this, suppose that in Example 7.3 we were to specify a subgroup size of $n = 2.5$ inspection units. Then the sample size becomes $(2.5)(100) = 250$ boards. There are two general approaches to constructing the revised chart once a new sample size has been selected. One approach is simply to redefine a new inspection unit that is equal to n times the old inspection unit. In this case, the center line on the new control chart is $n\bar{c}$ and the control limits are located at $n\bar{c} \pm 3\sqrt{n\bar{c}}$, where $\bar{c}$ is the observed mean number of nonconformities in the *original* inspection unit. Suppose that in Example 7.3, after revising the trial control limits, we decided to use a sample size of $n = 2.5$ inspection units. Then the center line would have been located at $n\bar{c} = (2.5)(19.67) = 49.18$ and the control limits would have been $49.18 \pm 3\sqrt{49.18}$ or LCL = 28.14 and UCL = 70.22.

The second approach involves setting up a control chart based on the average number of nonconformities per inspection unit. If we find x *total* nonconformities in a sample of n inspection units, then the *average* number of nonconformities per inspection unit is

$$u = \frac{x}{n} \tag{7.18}$$

Note that x is a Poisson random variable; consequently, the parameters of the control chart for the average number of nonconformities per unit are as follows,

Control Chart for Average Number of Nonconformities per Unit

$$\begin{aligned} \text{UCL} &= \bar{u} + 3\sqrt{\frac{\bar{u}}{n}} \\ \text{Center line} &= \bar{u} \\ \text{LCL} &= \bar{u} - 3\sqrt{\frac{\bar{u}}{n}} \end{aligned} \tag{7.19}$$

where $\bar{u}$ represents the observed average number of nonconformities per unit in a preliminary set of data. Control limits found from equation (7.19) would be regarded as trial control limits. This per-unit chart often is called the **control chart for nonconformities, or *u* chart.**

EXAMPLE 7.4 Control Charts in Supply Chain Operations

A supply chain engineering group monitors shipments of materials through the company distribution network. Errors on either the delivered material or the accompanying documentation are tracked on a weekly basis. Fifty randomly selected shipments are examined and the errors recorded. Data for twenty weeks are shown in Table 7.10. Set up a u control chart to monitor this process.

SOLUTION

From the data in Table 7.10, we estimate the number of errors (nonconformities) per unit (shipment) to be:

$$\bar{u} = \frac{\sum_{i=1}^{20} u_i}{20} = \frac{1.48}{20} = 0.0740$$

Therefore, the parameters of the control chart are

$$\text{UCL} = \bar{u} + 3\sqrt{\frac{\bar{u}}{n}} = 0.0740 + 3\sqrt{\frac{0.0740}{50}} = 0.1894$$

$$\text{Center line } = \bar{u} = 1.93$$

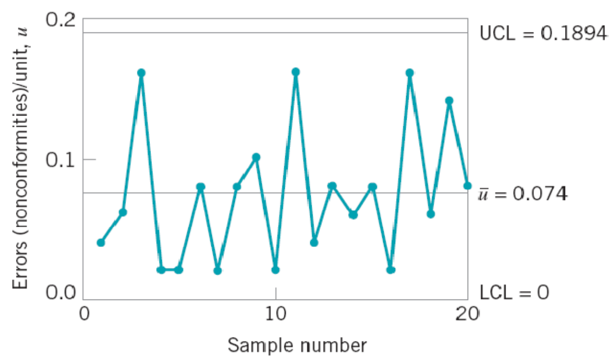
$$\text{LCL} = \bar{u} - 3\sqrt{\frac{\bar{u}}{n}} = 0.0740 - 3\sqrt{\frac{0.0740}{50}} = -0.0414$$

Since the LCL < 0, we would set LCL = 0 for the u chart. The control chart is shown in Fig. 7.16. The preliminary data do not exhibit lack of statistical control; therefore, the trial control limits given here would be adopted for phase II monitoring of future operations. Once again, note that, although the process is in control, the average number of errors per shipment is high. Action should be taken to improve the supply chain system.

■ TABLE 7.10

Data on Number of Shipping Errors in a Supply Chain Network

Sample Number (week), i	Sample Size, n	Total Number of Errors (Nonconformities), x_i	Average Number of Errors (Nonconformities) per Unit, $u_i = x_i/n$
1	50	2	0.04
2	50	3	0.06
3	50	8	0.16
4	50	1	0.02
5	50	1	0.02
6	50	4	0.08
7	50	1	0.02
8	50	4	0.08
9	50	5	0.10
10	50	1	0.02
11	50	8	0.16
12	50	2	0.04
13	50	4	0.08
14	50	3	0.06
15	50	4	0.08
16	50	1	0.02
17	50	8	0.16
18	50	3	0.06
19	50	7	0.14
20	50	4	0.08
		74	1.48



■ FIGURE 7.16 The control chart for nonconformities per unit from Minitab for Example 7.4.

As we noted in Section 3.2.4, the geometric distribution can also be useful as a model for count or "event" data. Kaminski et al. (1992) have proposed control charts for counts based on the geometric distribution. The probability model that they use for the geometric distribution is

$$p(x) = p(1-p)^{x-a} \text{ for } x = a, a+1, a+2, \ldots$$

where a is the known minimum possible number of events. Suppose that the data from the process is available as a subgroup of size n, say $x_1, x_2, \ldots . x_n$. These observations are independently and identically distributed observations from a geometric distribution when the process is stable (in control). The two statistics that can be used to form a control chart are the ***total*** number of events

$$T = x_1 + x_2 + \ldots + x_n$$

and the ***average*** number of events

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

It turns out that the sum of independently and identically distributed geometric random variables is a negative binomial random variable. This would be useful information in constructing OC curves or calculating ARLs for the control charts for T or $\bar{x}$.

The mean and variance of the total number of events T are

$$\mu_T = n\left(\frac{1-p}{p} + a\right)$$

and

$$\sigma_T^2 = \frac{n(1-p)}{p^2}$$

and the mean and variance of the average number of events are

$$\mu_{\bar{x}} = \frac{1-p}{p} + a$$

and

$$\sigma^2_{\bar{x}} = \frac{1-p}{np^2}$$

Consequently, the control charts can be constructed in the usual manner for Shewhart charts. Kaminski et al. (1992) refer to the control chart for the total number of events as a "g chart" and the control chart for the average number of events as an "h chart." The center lines and control limits for each chart are shown in the following display.

g and *h* Control Charts, Standards Given

	Total number of events chart, g chart	Average number of events chart, h chart
Upper control limit (UCL)	$n\left(\frac{1-p}{p} + a\right) + L\sqrt{\frac{n(1-p)}{p^2}}$	$\frac{1-p}{p} + a + L\sqrt{\frac{1-p}{np^2}}$
Center line (CL)	$n\left(\frac{1-p}{p} + a\right)$	$\frac{1-p}{p} + a$
Lower control limit (LCL)	$n\left(\frac{1-p}{p} + a\right) - L\sqrt{\frac{n(1-p)}{p^2}}$	$\frac{1-p}{p} + a - L\sqrt{\frac{1-p}{np^2}}$

While we have assumed that a is known, in most situations the parameter p will likely be unknown. The estimator for p is

$$\hat{p} = \frac{1}{\bar{\bar{x}} - a + 1}$$

where $\bar{\bar{x}}$ is the average of all of the count data. Suppose that there are m subgroups available, each of size n, and let the total number of events in each subgroup be $t_1, t_2, \ldots, t_m$. The average number of events per subgroup is

$$\bar{t} = \frac{t_1 + t_2 + \cdots + t_m}{m}$$

Therefore,

$$\bar{\bar{x}} = \frac{\bar{t}}{n} = \frac{1 - \hat{p}}{\hat{p}} + a$$

and

$$\frac{1 - \hat{p}}{\hat{p}^2} = \left(\frac{\bar{t}}{n} - a\right)\left(\frac{\bar{t}}{n} - a + 1\right)$$

The center line and control limits for the g chart and the h chart based on an estimate of p are shown below.

g and h Control Charts, No Standards Given

	Total number of events chart, g chart	Average number of events chart, h chart
Upper control limit (UCL)	$\bar{t} + L\sqrt{n\left(\frac{\bar{t}}{n} - a\right)\left(\frac{\bar{t}}{n} - a + 1\right)}$	$\frac{\bar{t}}{n} + \frac{L}{\sqrt{n}}\sqrt{\left(\frac{\bar{t}}{n} - a\right)\left(\frac{\bar{t}}{n} - a + 1\right)}$
Center line (CL)	$\bar{t}$	$\frac{\bar{t}}{n}$
Lower control limit (LCL)	$\bar{t} - L\sqrt{n\left(\frac{\bar{t}}{n} - a\right)\left(\frac{\bar{t}}{n} - a + 1\right)}$	$\frac{\bar{t}}{n} - \frac{L}{\sqrt{n}}\sqrt{\left(\frac{\bar{t}}{n} - a\right)\left(\frac{\bar{t}}{n} - a + 1\right)}$

7.3.2 Procedures with Variable Sample Size

Control charts for nonconformities are occasionally formed using 100% inspection of the product. When this method of sampling is used, the number of inspection units in a sample will usually not be constant. For example, the inspection of rolls of cloth or paper often leads to a situation in which the size of the sample varies, because not all rolls are exactly the same length or width. If a control chart for nonconformities (c chart) is used in this situation, both the center line and the control limits will vary with the sample size. Such a control chart would be very difficult to interpret. The correct procedure is to use a control chart for nonconformities per unit (u chart). This chart will have a constant center line; however, the control limits will vary inversely with the square root of the sample size n.

EXAMPLE 7.5 Constructing a *u* chart

In a textile finishing plant, dyed cloth is inspected for the occurrence of defects per 50 square meters. The data on ten rolls of cloth are shown in Table 7.11. Use these data to set up a control chart for nonconformities per unit.

■ TABLE 7.11

Occurrence of Nonconformities in Dyed Cloth

Roll Number	Number of Square Meters	Total Number of Nonconformities	Number of Inspection Units in Roll, n	Number of Nonconformities per Inspection Unit
1	500	14	10.0	1.40
2	400	12	8.0	1.50
3	650	20	13.0	1.54
4	500	11	10.0	1.10
5	475	7	9.5	0.74
6	500	10	10.0	1.00
7	600	21	12.0	1.75
8	525	16	10.5	1.52
9	600	19	12.0	1.58
10	625	23	12.5	1.84
		153	107.50	

SOLUTION

The center line of the chart should be the average number of nonconformities per inspection unit—that is, the average number of nonconformities per 50 square meters, computed as

$$\bar{u} = \frac{153}{107.5} = 1.42$$

Note that $\bar{u}$ is the ratio of the total number of observed nonconformities to the total number of inspection units.

The control limits on this chart are computed from equation (7.19) with n replaced by n_i. The width of the control limits will vary inversely with n_i, the number of inspection units in the roll. The calculations for the control limits are displayed in Table 7.12. Figure 7.17 plots the control chart constructed by Minitab.

TABLE 7.12
Calculation of Control Limits, Example 7.5

Roll Number, i	n_i	UCL $= \bar{u} + 3\sqrt{\bar{u}/n_i}$	LCL $= \bar{u} - 3\sqrt{\bar{u}/n_i}$
1	10.0	2.55	0.29
2	8.0	2.68	0.16
3	13.0	2.41	0.43
4	10.0	2.55	0.29
5	9.5	2.58	0.26
6	10.0	2.55	0.29
7	12.0	2.45	0.39
8	10.5	2.52	0.32
9	12.0	2.45	0.39
10	12.5	2.43	0.41

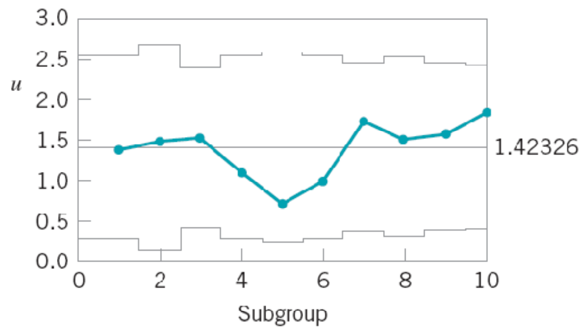


FIGURE 7.17 Computer-generated (Minitab) control chart for Example 7.5.

As noted previously, the u chart should always be used when the sample size is variable. The most common implementation involves variable control limits, as illustrated in Example 7.5. There are, however, two other possible approaches:

1. Use control limits based on an average sample size

$$\bar{n} = \sum_{i=1}^{m} n_i / m$$

2. Use a standardized control chart (this is the preferred option). This second alternative would involve plotting a standardized statistic

$$Z_i = \frac{u_i - \bar{u}}{\sqrt{\dfrac{\bar{u}}{n_i}}} \tag{7.20}$$

on a control chart with LCL = –3 and UCL = +3 and the center line at zero. This chart is appropriate if tests for runs and other pattern-recognition methods are to be used in conjunction with the chart. Figure 7.18 shows the standardized version of the control chart in Example 7.5. This standardized control chart could also be useful in the short production run situation (see Chapter 10, Section 10.1).

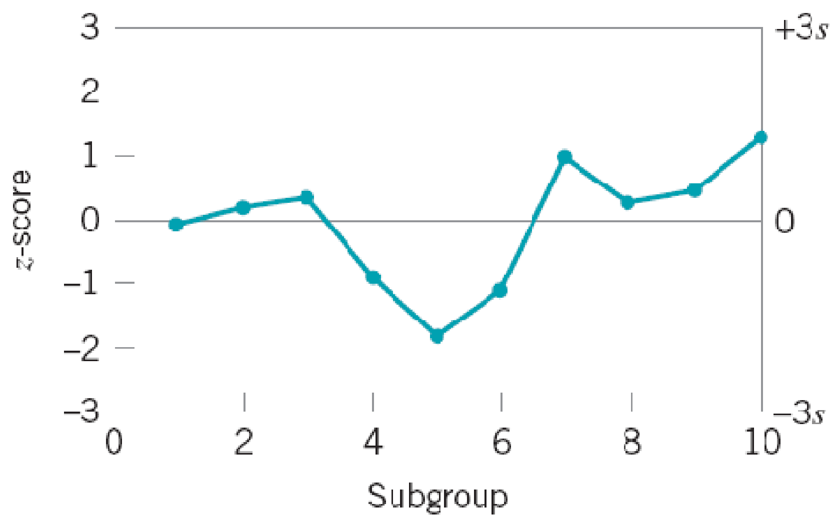


FIGURE 7.18 Standardized control chart for nonconformities per unit, Example 7.5.

7.3.3 Demerit Systems

With complex products such as automobiles, computers, or major appliances, we usually find that many different types of nonconformities or defects can occur. Not all of these types of defects are equally important. A unit of product having one very serious defect would probably be classified as nonconforming to requirements, but a unit having several minor defects might not necessarily be nonconforming. In such situations, we need a method to classify nonconformities or defects according to severity and to weight the various types of defects in a reasonable manner. **Demerit systems for attribute data** can be of value in these situations.

One possible demerit scheme is defined as follows.

Class A Defects—Very Serious. The unit is either completely unfit for service, or will fail in service in such a manner that cannot be easily corrected in the field, or will cause personal injury or property damage.

Class B Defects—Serious. The unit will possibly suffer a Class A operating failure, or will certainly cause somewhat less serious operating problems, or will certainly have reduced life or increased maintenance cost.

Class C Defects—Moderately Serious. The unit will possibly fail in service, or cause trouble that is less serious than operating failure, or possibly have reduced life or increased maintenance costs, or have a major defect in finish, appearance, or quality of work.

Class D Defects—Minor. The unit will not fail in service but has minor defects in finish, appearance, or quality of work.

Let c_{iA}, c_{iB}, c_{iC}, and c_{iD} represent the number of Class A, Class B, Class C, and Class D defects, respectively, in the ith inspection unit. We assume that each class of defect is independent, and the occurrence of defects in each class is well modeled by a Poisson distribution. Then we define the number of **demerits** in the inspection unit as

$$d_i = 100c_{iA} + 50c_{iB} + 10c_{iC} + c_{iD} \tag{7.21}$$

The demerit weights of Class A—100, Class B—50, Class C—10, and Class D—1 are used fairly widely in practice. However, any reasonable set of weights appropriate for a specific problem may also be used.

Suppose that a sample of n inspection units is used. Then the number of demerits per unit is

$$u_i = \frac{D}{n} \tag{7.22}$$

where $D = \sum_{i=1}^{n} d_i$ is the total number of demerits in all n inspection units. Since u_i is a linear combination of independent Poisson random variables, the statistics u_i could be plotted on a control chart with the following parameters:

$$\begin{aligned} \text{UCL} &= \overline{u} + 3\hat{\sigma}_u \\ \text{Center line} &= \overline{u} \\ \text{LCL} &= \overline{u} - 3\hat{\sigma}_u \end{aligned} \tag{7.23}$$

where

$$\overline{u} = 100\overline{u}_A + 50\overline{u}_B + 10\overline{u}_C + \overline{u}_D \tag{7.24}$$

and

$$\hat{\sigma}_u = \left[\frac{(100)^2\overline{u}_A + (50)^2\overline{u}_B + (10)^2\overline{u}_C + \overline{u}_D}{n}\right]^{1/2} \tag{7.25}$$

7.3.4 The Operating-Characteristic Function

The operating-characteristic (OC) curves for both the c chart and the u chart can be obtained from the Poisson distribution. For the c chart, the OC curve plots the probability of type II error β against the true mean number of defects c. The expression for β is

$$\beta = P\{x < \text{UCL}|c\} - P\{x \le \text{LCL}|c\} \tag{7.26}$$

where x is a Poisson random variable with parameter c. Note that if the LCL < 0 the second term on the right-hand side of equation (7.26) should be dropped.

We will generate the OC curve for the c chart in Example 7.3. For this example, since the LCL = 6.48 and the UCL = 33.22, equation (7.26) becomes

$$\beta = P\{x < 33.22|c\} - P\{x \le 6.48|c\}$$

Since the number of nonconformities must be integer, this is equivalent to

$$\beta = P\{x \le 33|c\} - P\{x \le 6|c\}$$

These probabilities are evaluated in Table 7.13. The OC curve is shown in Fig. 7.19.

For the u chart, we may generate the OC curve from

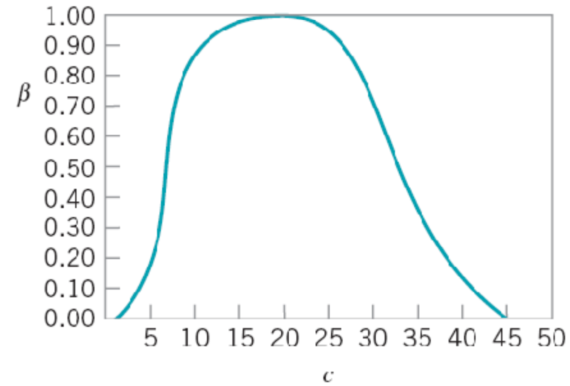
$$\begin{aligned}\beta &= P\{x < \text{UCL}|u\} - P\{x \le \text{LCL}|u\} \\ &= P\{c < n\text{UCL}|u\} - P\{c \le n\text{LCL}|u\} \\ &= P\{n\text{LCL} < x \le n\text{UCL}|u\} \\ &= \sum_{x=\langle n\text{LCL}\rangle}^{[n\text{UCL}]} \frac{e^{-nu}(nu)^x}{x!}\end{aligned} \tag{7.27}$$

where $\langle n\text{LCL}\rangle$ denotes the smallest integer greater than or equal to nLCL and $[n\text{UCL}]$ denotes the largest integer less than or equal to nUCL. The limits on the summation in equation (7.26) follow from the fact that the total number of nonconformities observed in a sample of n inspection units must be an integer. Note that n need not be an integer.

■ TABLE 7.13

Calculation of the OC Curve for a c Chart with UCL = 33.22 and LCL = 6.48

c	$P\{x \leq 33 \mid c\}$	$P\{x \leq 6 \mid c\}$	$\beta = P\{x \leq 33 \mid c\} - P\{x \leq 6 \mid c\}$
1	1.000	0.999	0.001
3	1.000	0.966	0.034
5	1.000	0.762	0.238
7	1.000	0.450	0.550
10	1.000	0.130	0.870
15	0.999	0.008	0.991
20	0.997	0.000	0.997
25	0.950	0.000	0.950
30	0.744	0.000	0.744
33	0.546	0.000	0.546
35	0.410	0.000	0.410
40	0.151	0.000	0.151
45	0.038	0.000	0.038



■ FIGURE 7.19 OC curve of a c chart with LCL = 6.48 and UCL = 33.22.

Low Defect Levels

One way to deal with this problem is adopt a **time between occurrence control chart,** which charts a new variable: the time between the successive occurrences of the count. The time-between-events control chart has been very effective as a process-control procedure for processes with low defect levels.

Suppose that defects or counts or "events" of interest occur according to a Poisson distribution. Then the probability distribution of the time between events is the exponential distribution. Therefore, constructing a time-between-events control chart is essentially equivalent to control charting an exponentially distributed variable. However, the exponential distribution is highly skewed, and as a result, the corresponding control chart would be very asymmetric. Such a control chart would certainly look unusual, and might present some difficulties in interpretation for operating personnel.

Nelson (1994) has suggested solving this problem by transforming the exponential random variable to a Weibull random variable such that the resulting Weibull distribution is well approximated by the normal distribution. If y represents the original exponential random variable, the appropriate transformation is

$$x = y^{1/3.6} = y^{0.2777} \tag{7.28}$$

One would now construct a control chart on x, assuming that x follows a normal distribution.

EXAMPLE 7.6

A chemical engineer wants to set up a control chart for monitoring the occurrence of failures of an important valve. She has decided to use the number of hours between failures as the variable to monitor. Table 7.14 shows the number of hours between failures for the last twenty failures of this valve. Figure 7.20 is a normal probability plot of the time between failures.

SOLUTION

Set up a time-between-events control chart for this process. Clearly, time between failures is not normally distributed. Table 7.14 also shows the values of the transformed time between events, computed from equation (7.27). Figure 7.21 is a normal probability plot of the transformed time between failures. Note that the plot indicates that the distribution of this transformed variable is well approximated by the normal.

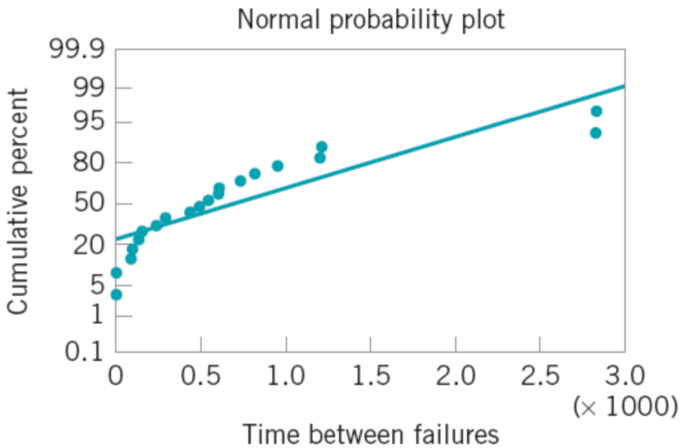


FIGURE 7.20 Normal probability plot of time between failures, Example 7.6.

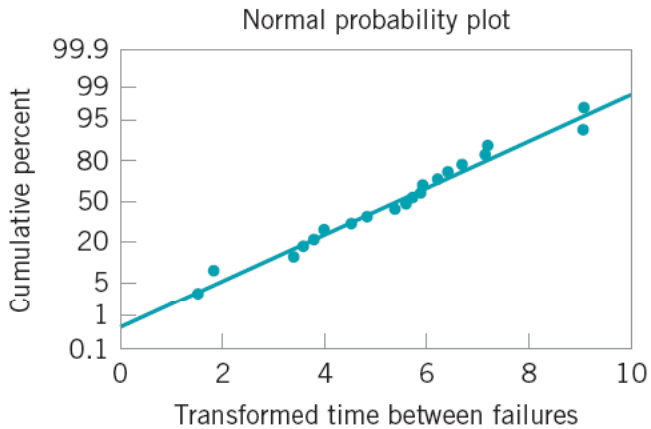
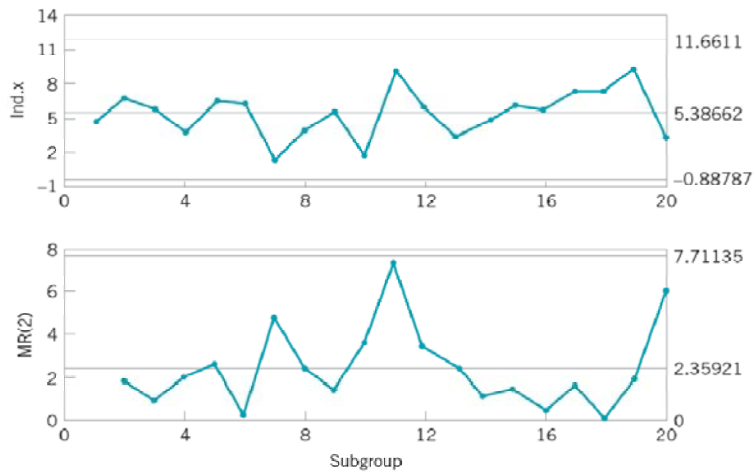


FIGURE 7.21 Normal probability plot for the transformed failure data.

■ TABLE 7.14

Time between Failure Data, Example 7.6

Failure	Time between Failures, y (hr)	Transformed Value of Time between Failures, $x = y^{0.2777}$
1	286	4.80986
2	948	6.70903
3	536	5.72650
4	124	3.81367
5	816	6.43541
6	729	6.23705
7	4	1.46958
8	143	3.96768
9	431	5.39007
10	8	1.78151
11	2837	9.09619
12	596	5.89774
13	81	3.38833
14	227	4.51095
15	603	5.91690
16	492	5.59189
17	1199	7.16124
18	1214	7.18601
19	2831	9.09083
20	96	3.55203



■ FIGURE 7.22 Control charts for individuals and moving-range control chart for the transformed time between failures, Example 7.6.

Figure 7.22 is a control chart for individuals and a moving range control chart for the transformed time between failures. Note that the control charts indicate a state of control, implying that the failure mechanism for this valve is constant. If a process change is made that improves the failure rate (such as a different type of maintenance action), then we would expect to see the mean time between failures get longer. This would result in points plotting above the upper control limit on the individuals control chart in Fig. 7.22.

The previous example illustrated the use of the individuals control chart with time-between-events data. In many cases, the cusum and EWMA control charts in Chapter 4 would be better alternatives, because they are more effective in detecting small shifts in the mean.

Kittlitz (1999) has also investigated transforming the exponential distribution for control charting purposes. He notes that a log transformation will stabilize the variance of the exponential distribution, but produces a rather negatively skewed distribution. Kittlitz suggests using the transformation $x = y^{0.25}$, noting that it is very similar to Nelson's recommendation and it is also very easy to compute (simply push the square root key on the calculator twice!).

7.4 Choice Between Attributes and Variables Control Charts

In many applications, the analyst will have to choose between using a variables control chart, such as the $\bar{x}$ and R charts, and an attributes control chart, such as the p chart. In some cases, the choice will be clear-cut. For example, if the quality characteristic is the color of the item, such as might be the case in carpet or cloth production, then attributes inspection would often be preferred over an attempt to quantify the quality characteristic "color." In other cases, the choice will not be obvious, and the analyst must take several factors into account in choosing between attributes and variables control charts.

Attributes control charts have the advantage that several quality characteristics can be considered jointly and the unit classified as nonconforming if it fails to meet the specification on any one characteristic. On the other hand, if the several quality characteristics are treated as variables, then each one must be measured, and either a separate $\bar{x}$ and R chart must be maintained on each or some multivariate control technique that considers all the characteristics must simultaneously be employed. There is an obvious simplicity associated with the attributes chart in this case. Furthermore, expensive and time-consuming measurements may sometimes be avoided by attributes inspection.

Variables control charts, in contrast, provide much more useful information about process performance than does an attributes control chart. Specific information about the process mean and variability is obtained directly. In addition, when points plot out of control on variables control charts, usually much more information is provided relative to the potential *cause* of that out-of-control signal. For a process capability study, variables control charts are almost always preferable to attributes control charts. The exceptions to this are studies relative to nonconformities produced by machines or operators in which there are a very limited number of sources of nonconformities, or studies directly concerned with process yields and fallouts.

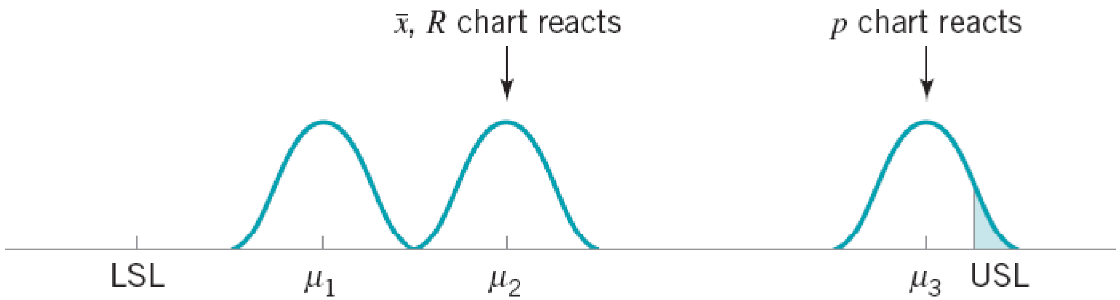


FIGURE 7.23 Why the $\bar{x}$ and R charts can warn of impending trouble.

Determining Which Characteristics to Control and Where to Put the Control Charts

At the start of a control chart program, it is usually difficult to determine which product or process characteristics should be controlled and at which points in the process to apply control charts. Some useful guidelines follow.

1. At the beginning of a control chart program, control charts should be applied to any product characteristics or manufacturing operations believed to be important. The charts will provide immediate feedback as to whether they are actually needed.
2. The control charts found to be unnecessary should be removed, and others that engineering and operator judgment indicates may be required should be added. More control charts will usually be employed at the beginning than after the process has stabilized.
3. Information on the number and types of control charts on the process should be kept current. It is best to keep separate records on the variables and attributes charts. In general, after the control charts are first installed, we often find that the number of control charts tends to increase rather steadily. After that, it will usually decrease. When the process stabilizes, we typically find that it has the same number of charts from one year to the next. However, they are not necessarily the same charts.

4. If control charts are being used effectively and if new knowledge is being gained about the key process variables, we should find that the number of $\bar{x}$ and R charts increases and the number of attributes control charts decreases.

5. At the beginning of a control chart program there will usually be more attributes control charts, applied to semifinished or finished units near the *end* of the manufacturing process. As we learn more about the process, these charts will be replaced with $\bar{x}$ and R charts applied *earlier* in the process to the critical parameters and operations that result in nonconformities in the finished product. Generally, **the earlier that process control can be established, the better.** In a complex assembly process, this may imply that process controls need to be implemented at the vendor or supplier level.

6. Control charts are an on-line, process-monitoring procedure. They should be implemented and maintained as close to the work center as possible, so that feedback will be rapid. Furthermore, the process operators and process engineering should have direct responsibility for collecting the process data, maintaining the charts, and interpreting the results. The operators and engineers have the detailed knowledge of the process required to correct process upsets and use the control chart to improve process performance. Microcomputers can speed up the feedback and should be an integral part of any modern, on-line, process-control procedure.

7. The out-of-control-action plan (OCAP) is a vital part of the control chart. Operating and engineering personnel should strive to keep OCAPs up-to-date and valid.

Choosing the Proper Type of Control Chart

A. $\bar{x}$ and R (or $\bar{x}$ and s) charts. Consider using variables control charts in these situations:

1. A new process is coming on stream, or a new product is being manufactured by an existing process.
2. The process has been in operation for some time, but it is chronically in trouble or unable to hold the specified tolerances.
3. The process is in trouble, and the control chart can be useful for diagnostic purposes (troubleshooting).
4. Destructive testing (or other expensive testing procedures) is required.
5. It is desirable to reduce acceptance-sampling or other downstream testing to a minimum when the process can be operated in control.
6. Attributes control charts have been used, but the process is either out of control or in control but the yield is unacceptable.
7. There are very tight specifications, overlapping assembly tolerances, or other difficult manufacturing problems.
8. The operator must decide whether or not to adjust the process, or when a setup must be evaluated.
9. A change in product specifications is desired.
10. Process stability and capability must be continually demonstrated, such as in regulated industries.

B. Attributes Charts (*p* charts, *c* charts, and *u* charts). Consider using attributes control charts in these situations:

1. Operators control the assignable causes, and it is necessary to reduce process fall-out.
2. The process is a complex assembly operation and product quality is measured in terms of the occurrence of nonconformities, successful or unsuccessful product function, and so forth. (Examples include computers, office automation equipment, automobiles, and the major subsystems of these products.)
3. Process control is necessary, but measurement data cannot be obtained.
4. A historical summary of process performance is necessary. Attributes control charts, such as p charts, c charts, and u charts, are very effective for summarizing information about the process for management review.
5. Remember that attributes charts are generally inferior to charts for variables. Always use $\bar{x}$ and R or $\bar{x}$ and s charts whenever possible.

C. Control Charts for Individuals. Consider using the control chart for individuals in conjunction with a moving-range chart in these situations:

1. It is inconvenient or impossible to obtain more than one measurement per sample, or repeat measurements will only differ by laboratory or analysis error. Examples often occur in chemical processes.
2. Automated testing and inspection technology allow measurement of every unit produced. In these cases, also consider the cumulative sum control chart and the exponentially weighted moving average control chart discussed in Chapter 7.
3. The data become available very slowly, and waiting for a larger sample will be impractical or make the control procedure too slow to react to problems. This often happens in nonproduct situations; for example, accounting data may become available only monthly.
4. Generally, once we are in phase II, individuals charts have poor performance in shift detection and can be very sensitive to departures from normality. Always use the EWMA and cusum charts of Chapter 8 in phase II instead of individuals charts whenever possible.

Actions taken to improve a process

		IS THE PROCESS CAPABLE? Yes	IS THE PROCESS CAPABLE? No
IS THE PROCESS IN CONTROL?	Yes	SPC	SPC Experimental design Investigate specifications Change process
	No	SPC	SPC Experimental design Investigate specifications Change process

FIGURE 7.27 Actions taken to improve a process.

The lower two boxes in Fig. 7.27 deal with the case of an out-of-control process. The southeast corner presents the case of a process that is out of control and not capable. (Remember our nontechnical use of the term *capability*.) The actions recommended here are identical to those for the box in the northeast corner, except that SPC would be expected to yield fairly rapid results now, because the control charts should be identifying the presence of assignable causes. The other methods of attack will warrant consideration and use in many cases, however. Finally, the southwest corner treats the case of a process that exhibits lack of statistical control but does not produce a meaningful number of defectives because the specifications are very wide. SPC methods should still be used to establish control and reduce variability in this case, for the following reasons:

1. Specifications can change without notice.
2. The customer may require both **control** and **capability.**
3. The fact that the process experiences assignable causes implies that unknown forces are at work; these unknown forces could result in poor capability in the near future.

Important Terms and Concepts

Attribute data
Average run length for attribute control charts
Cause-and-effect diagram
Choice between attributes and variables data
Control chart for defects or nonconformities per unit or *u* chart
Control chart for fraction nonconforming or *p* chart
Control chart for nonconformities or *c* chart
Control chart for number nonconforming or *np* chart
Defect
Defective
Demerit systems for attribute data
Design of attributes control charts
Fraction defective
Fraction nonconforming
Nonconformity
Operating characteristic curve for the *c* and *u* charts
Operating characteristic curve for the *p* chart
Pareto chart
Standardized control charts
Time between occurrence control charts
Variable sample size for attributes control chart

Learning Objectives

1. Understand the statistical basis of attributes control charts
2. Know how to design attributes control charts
3. Know how to set up and use the p chart for fraction nonconforming
4. Know how to set up and use the np control chart for the number of nonconforming items
5. Know how to set up and use the c control chart for defects
6. Know how to set up and use the u control chart for defects per unit
7. Use attributes control charts with variable sample size
8. Understand the advantages and disadvantages of attributes versus variables control charts
9. Understand the rational subgroup concept for attributes control charts
10. Determine the average run length for attributes control charts