8 Process and Measurement System Capability Analysis

CHAPTER OUTLINE

Supplemental Material for Chapter 8

Learning Objectives

1. Investigate and analyze process capability using control charts, histograms, and probability plots
2. Understand the difference between process capability and process potential
3. Calculate and properly interpret process capability ratios
4. Understand the role of the normal distribution in interpreting most process capability ratios
5. Calculate confidence intervals on process capability ratios
6. Know how to conduct and analyze a measurement systems capability (or gauge R & R) experiment
7. Know how to estimate the components of variability in a measurement system
8. Know how to set specifications on components in a system involving interaction components to ensure that overall system requirements are met
9. Estimate the natural limits of a process from a sample of data from that process

Process Capability

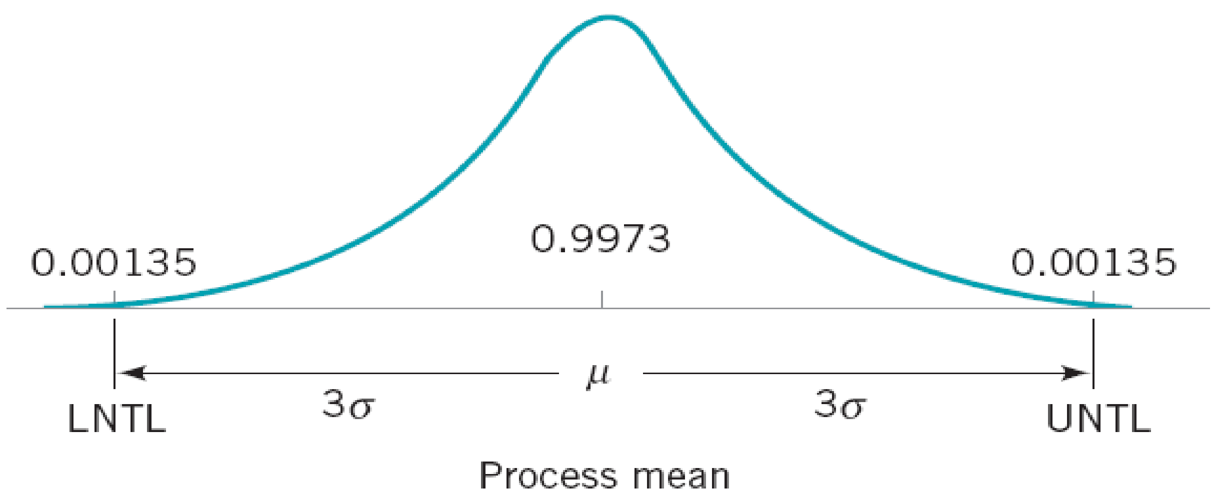
Process capability refers to the **uniformity** of the process. Obviously, the variability of critical-to-quality characteristics in the process is a measure of the uniformity of output. There are two ways to think of this variability:

1. The natural or inherent variability in a critical-to-quality characteristic at a specified time; that is, "instantaneous" variability
2. The variability in a critical-to-quality characteristic over time

Natural tolerance limits are defined as follows:

$$\text{UNTL} = \mu + 3\sigma$$

$$\text{LNTL} = \mu - 3\sigma$$



■ FIGURE 8.1 Upper and lower natural tolerance limits in the normal distribution.

We define **process capability analysis** as an engineering study to estimate process capability. The estimate of process capability may be in the form of a probability distribution having a specified shape, center (mean), and spread (standard deviation). For example, we may determine that the process output is normally distributed with mean $\mu = 1.0$ cm and standard deviation $\sigma = 0.001$ cm. In this sense, a process capability analysis may be performed **without regard to specifications on the quality characteristic.** Alternatively, we may express process capability as a percentage outside of specifications. However, specifications are not *necessary* to process capability analysis.

Uses of process capability data:

1. Predicting how well the process will hold the tolerances
2. Assisting product developers/designers in selecting or modifying a process
3. Assisting in establishing an interval between sampling for process monitoring
4. Specifying performance requirements for new equipment
5. Selecting between competing suppliers and other aspects of supply chain management
6. Planning the sequence of production processes when there is an interactive effect of processes on tolerances
7. Reducing the variability in a manufacturing process

Reasons for Poor Process Capability

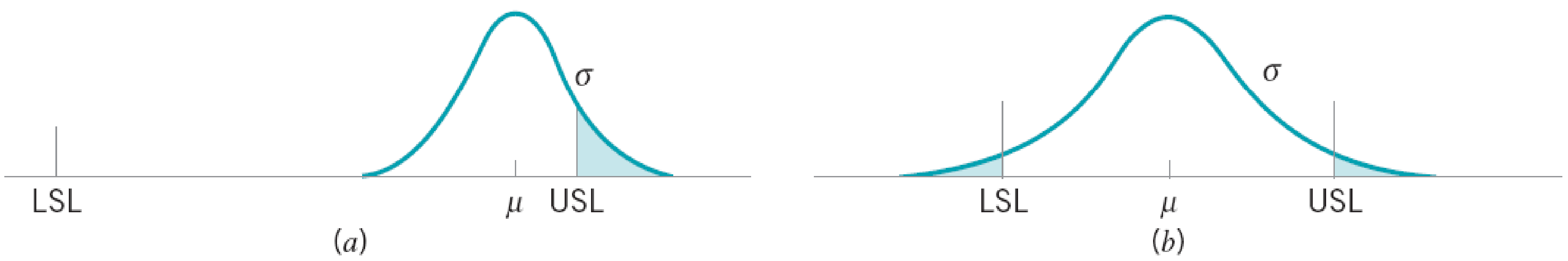


FIGURE 8.3 Some reasons for poor process capability. (*a*) Poor process centering. (*b*) Excess process variability.

Process may have good potential capability

8.2 Process Capability Analysis Using a Histogram or a Probability Plot

8.2.1 Using the Histogram

The histogram can be helpful in estimating process capability. Alternatively, a stem-and-leaf plot may be substituted for the histogram. At least 100 or more observations should be available for the histogram (or the stem-and-leaf plot) to be moderately stable so that a reasonably reliable estimate of process capability may be obtained. If the quality engineer has access to the process and can control the data-collection effort, the following steps should be followed prior to data collection:

1. Choose the machine or machines to be used. If the results based on one (or a few) machines are to be extended to a larger population of machines, the machine selected should be representative of those in the population. Furthermore, if the machine has multiple workstations or heads, it may be important to collect the data so that head-to-head variability can be isolated. This may imply that designed experiments should be used.
2. Select the process operating conditions. Carefully define conditions, such as cutting speeds, feed rates, and temperatures, for future reference. It may be important to study the effects of varying these factors on process capability.
3. Select a representative operator. In some studies, it may be important to estimate *operator* variability. In these cases, the operators should be selected at random from the population of operators.
4. Carefully monitor the data-collection process, and record the time order in which each unit is produced.

The histogram, along with the sample average $\bar{x}$ and sample standard deviation s, provides information about process capability. You may wish to review the guidelines for constructing histograms in Chapter 3.

EXAMPLE 8.1 Estimating Process Capability with a Histogram

Figure 8.2 presents a histogram of the bursting strength of 100 glass containers. The data are shown in Table 8.1. What is the capability of the process?

SOLUTION

Analysis of the 100 observations gives

$$\bar{x} = 264.06 \qquad s = 32.02$$

Consequently, the process capability would be estimated as

$$\bar{x} \pm 3s$$

or

$$264.06 \pm 3(32.02) \simeq 264 \pm 96 \text{ psi}$$

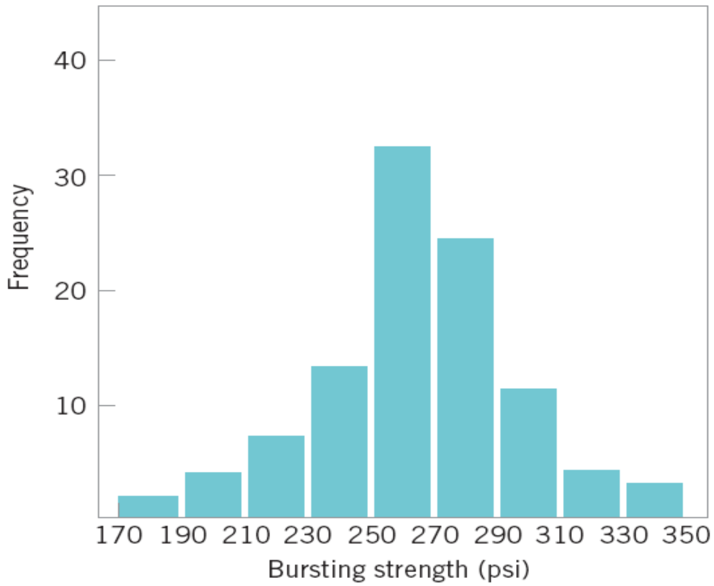


FIGURE 8.2 Histogram for the bursting-strength data.

Furthermore, the shape of the histogram implies that the distribution of bursting strength is approximately normal. Thus, we can estimate that approximately 99.73% of the bottles manufactured by this process will burst between 168 and 360 psi. Note that we can estimate process capability *independent of the specifications on bursting strength.*

TABLE 8.1

Bursting Strengths for 100 Glass Containers

265	197	346	280	265	200	221	265	261	278
205	286	317	242	254	235	176	262	248	250
263	274	242	260	281	246	248	271	260	265
307	243	258	321	294	328	263	245	274	270
220	231	276	228	223	296	231	301	337	298
268	267	300	250	260	276	334	280	250	257
260	281	208	299	308	264	280	274	278	210
234	265	187	258	235	269	265	253	254	280
299	214	264	267	283	235	272	287	274	269
215	318	271	293	277	290	283	258	275	251

8.2.2 Probability Plotting

Probability plotting is an alternative to the histogram that can be used to determine the shape, center, and spread of the distribution. It has the advantage that it is unnecessary to divide the range of the variable into class intervals, and it often produces reasonable results for moderately small samples (which the histogram will not). Generally, a probability plot is a graph of the ranked data versus the sample cumulative frequency on special paper with a vertical scale chosen so that the cumulative distribution of the assumed type is a straight line. In Chapter 3 we discussed and illustrated **normal probability plots.** These plots are very useful in process capability studies.

To illustrate the use of a normal probability plot in a process capability study, consider the following 20 observations on glass container bursting strength: 197, 200, 215, 221, 231, 242, 245, 258, 265, 265, 271, 275, 277, 278, 280, 283, 290, 301, 318, and 346. Figure 8.4 is the normal probability plot of strength. Note that the data lie nearly along a straight line, implying that the distribution of bursting strength is normal. Recall from Chapter 4 that the mean of the normal distribution is the fiftieth percentile, which we may estimate from Fig. 8.4 as approximately 265 psi, and the standard deviation of the distribution is the *slope* of the straight line. It is convenient to estimate the standard deviation as the difference between the eighty-fourth and the fiftieth percentiles. For the strength data shown above and using Fig. 8.4, we find that

$$\hat{\sigma} = 84\text{th percentile} - 50\text{th percentile} = 298 - 265 \text{ psi} = 33 \text{ psi}$$

Note that $\hat{\mu} = 265$ psi and $\hat{\sigma} = 33$ psi are not far from the sample average $\bar{x} = 264.06$ and standard deviation $s = 32.02$.

Probability Plotting

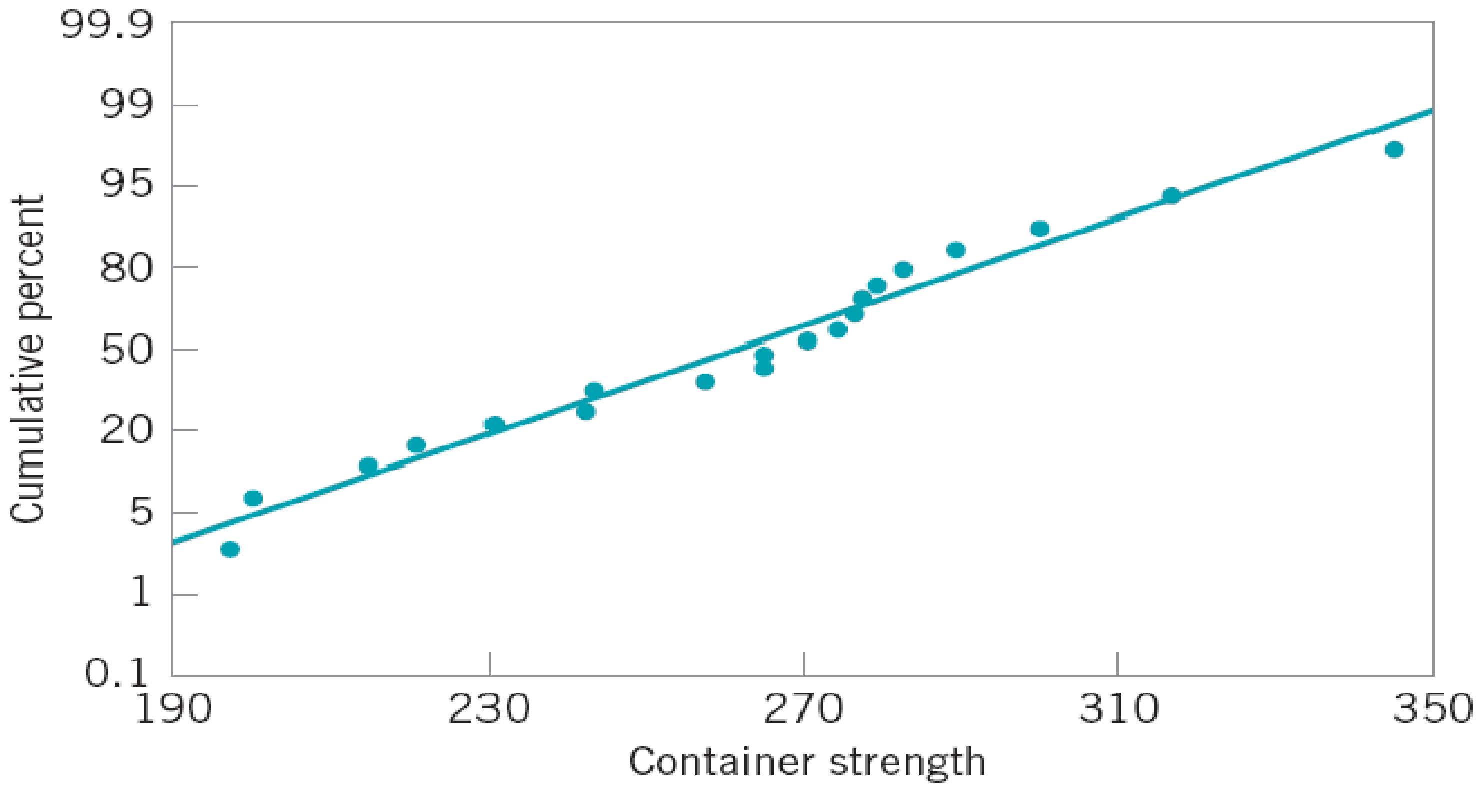


FIGURE 8.4 Normal probability plot of the container-strength data.

Care should be exercised in using probability plots. If the data do not come from the assumed distribution, inferences about process capability drawn from the plot may be seriously in error. Figure 7-5 presents a normal probability plot of times to failure (in hours) of a valve in a chemical plant. From examining this plot, we can see that the distribution of failure time is not normal.

An obvious disadvantage of probability plotting is that it is not an objective procedure. It is possible for two analysts to arrive at different conclusions using the same data. For this reason, it is often desirable to supplement probability plots with more formal statistically based goodness-of-fit tests. A good introduction to these tests is in Shapiro (1980). Augmenting the interpretation of a normal probability plot with the Shapiro–Wilk test for normality can make the procedure much more powerful and objective.

- The distribution may not be normal; other types of probability plots can be useful in determining the appropriate distribution.

8.3 Process Capability Ratios

8.3.1 Use and Interpretation of C_p

It is frequently convenient to have a simple, quantitative way to express process capability. One way to do so is through the **process capability ratio (PCR)** C_p first introduced in Chapter 6. Recall that

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma} \tag{8.4}$$

where USL and LSL are the upper and lower specification limits, respectively. C_p and other process capability ratios are used extensively in industry. They are also widely *misused*. We will point out some of the more common abuses of process capability ratios. An excellent recent book on process capability ratios that is highly recommended is Kotz and Lovelace (1998). There is also extensive technical literature on process capability analysis and process capability ratios. The review paper by Kotz and Johnson (2002) and the bibliography by Spiring, Leong, Cheng, and Yeung (2003) are excellent sources.

In a practical application, the process standard deviation σ is almost always unknown and must be replaced by an estimate σ. To estimate σ we typically use either the *sample standard deviation s* or $\overline{R}/d_2$ (when variables control charts are used in the capability study). This results in an estimate of C_p—say,

$$\hat{C}_p = \frac{\text{USL} - \text{LSL}}{6\hat{\sigma}} \tag{8.5}$$

To illustrate the calculation of C_p, recall the semiconductor hard-bake process first analyzed in Example 6.1 using $\bar{x}$ and R charts. The specifications on flow width are USL = 1.00 microns and LSL = 2.00 microns, and from the R chart we estimated $\sigma = \bar{R}/d_2 = 0.1398$. Thus, our estimate of the PCR C_p is

$$\hat{C}_p = \frac{\text{USL} - \text{LSL}}{6\hat{\sigma}} = \frac{2.00 - 1.00}{6(0.1398)} = 1.192$$

In Chapter 6, we assumed that flow width is approximately normally distributed (a reasonable assumption, based on the histogram in Fig. 8.7) and the cumulative normal distribution table in the Appendix was used to estimate that the process produces approximately 350 ppm (parts per million) defective.

The PCR C_p in equation (8.4) has a useful practical interpretation—namely,

$$P = \left(\frac{1}{C_p}\right)100 \tag{8.6}$$

For the hard bake process:

$$P = \left(\frac{1}{1.192}\right)100 = 83.89$$

One-Sided PCR

$$C_{pu} = \frac{\text{USL} - \mu}{3\sigma} \qquad \text{(upper specification only)} \qquad (8.7)$$

$$C_{pl} = \frac{\mu - \text{LSL}}{3\sigma} \qquad \text{(lower specification only)} \qquad (8.8)$$

Interpretation of the PCR

■ **TABLE 8.2**

Values of the Process Capability Ratio (C_p) and Associated Process Fallout for a Normally Distributed Process (in Defective ppm) That Is in Statistical Control

	Process Fallout (in defective ppm)	
PCR	**One-Sided Specifications**	**Two-Sided Specifications**
0.25	226,628	453,255
0.50	66,807	133,614
0.60	35,931	71,861
0.70	17,865	35,729
0.80	8,198	16,395
0.90	3,467	6,934
1.00	1,350	2,700
1.10	484	967
1.20	159	318
1.30	48	96
1.40	14	27
1.50	4	7
1.60	1	2
1.70	0.17	0.34
1.80	0.03	0.06
2.00	0.0009	0.0018

Assumptions for Interpretation of Numbers in Table 8.2

1. The quality characteristic has a normal distribution.
2. The process is in statistical control.
3. In the case of two-sided specifications, the process mean is centered between the lower and upper specification limits.

- Violation of these assumptions can lead to big trouble in using the data in Table 8.2.

■ TABLE 8.3

Recommended Minimum Values of the Process Capability Ratio

	Two-Sided Specifications	One-Sided Specifications
Existing processes	1.33	1.25
New processes	1.50	1.45
Safety, strength, or critical parameter, existing process	1.50	1.45
Safety, strength, or critical parameter, new process	1.67	1.60

- C_p does not take process centering into account
- It is a measure of *potential* capability, not *actual* capability

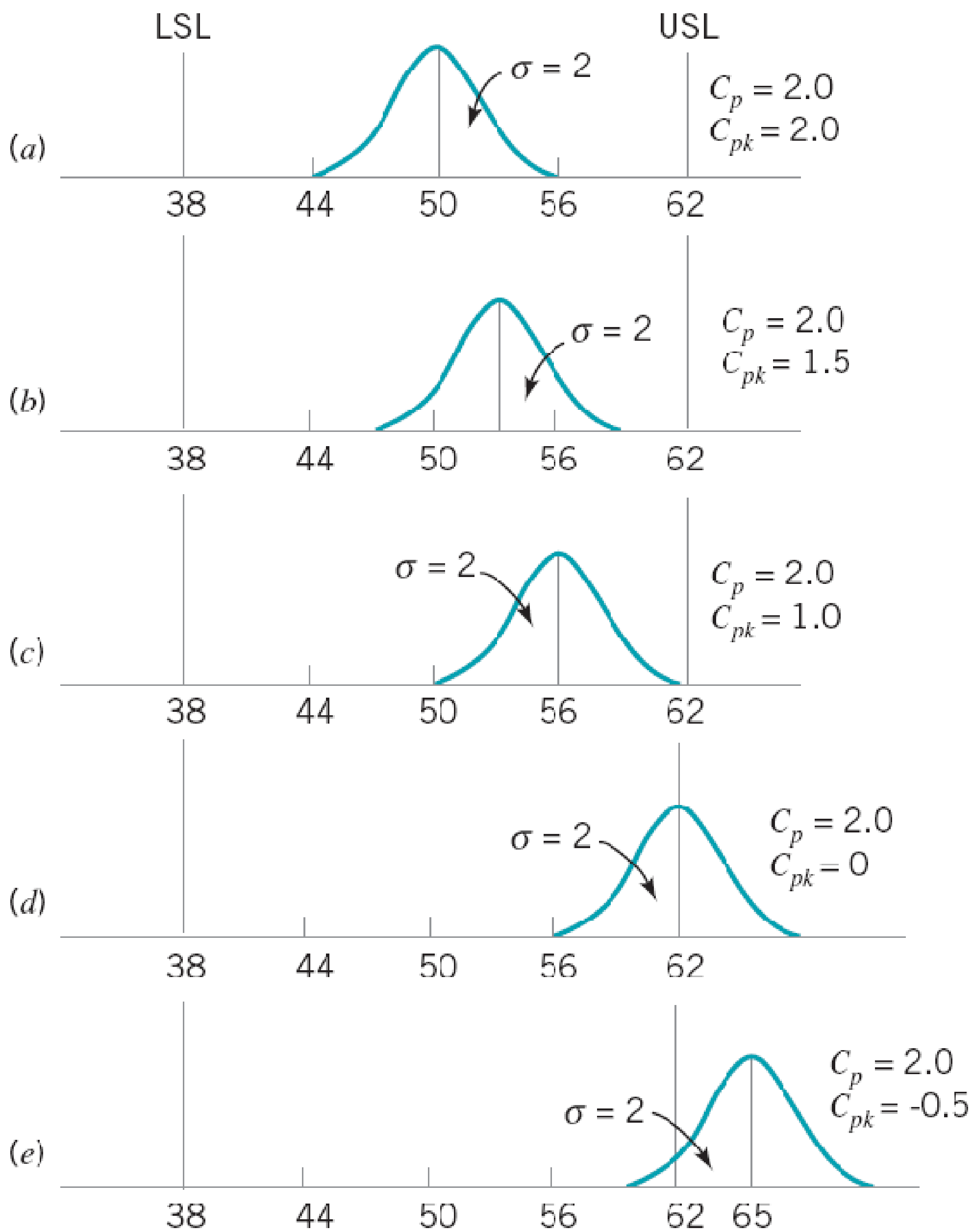


FIGURE 8.8 Relationship of C_p and C_{pk}.

A Measure of **Actual** Capability

$$C_{pk} = \min\left(C_{pu}, C_{pl}\right) \qquad (8.9)$$

Note that C_{pk} is simply the one-sided PCR for the specification limit nearest to the process average. For the process shown in Fig. 8.8*b*, we would have

$$\begin{aligned} C_{pk} &= \min\left(C_{pu}, C_{pl}\right) \\ &= \min\left(C_{pu} = \frac{\text{USL} - \mu}{3\sigma}, C_{pl} = \frac{\mu - \text{LSL}}{3\sigma}\right) \\ &= \min\left(C_{pu} = \frac{62 - 53}{3(2)} = 1.5, C_{pl} = \frac{53 - 38}{3(2)} = 2.5\right) \\ &= 1.5 \end{aligned}$$

Normality and Process Capability Ratios

- The assumption of normality is critical to the usual interpretation of these ratios (such as Table 8.2)
- For non-normal data, options are

1. Transform non-normal data to normal
2. Extend the usual definitions of PCRs to handle non-normal data
3. Modify the definitions of PCRs for general families of distributions

Other Types of Process Capability Ratios

- First generation
- Second generation
- Third generation
- Lots of research has been done to develop ratios that overcome some of the problems with the basic ones
- Not much evidence that these ratios are used to any significant extent in practice

Confidence Intervals on Process Capability Ratios. Much of the industrial use of process capability ratios focuses on computing and interpreting the **point estimate** of the desired quantity. It is easy to forget that $\hat{C}_p$ or $\hat{C}_{pk}$ (for examples) are simply point estimates, and, as such, are subject to statistical fluctuation. An alternative that should become standard practice is to report **confidence intervals for process capability ratios.**

It is easy to find a confidence interval for the "first generation" ratio C_p. If we replace σ by s in the equation for C_p, we produce the usual point estimator $\hat{C}_p$. If the quality characteristic follows a normal distribution, then a $100(1 - \alpha)\%$ confidence interval on C_p is obtained from

$$\frac{\text{USL} - \text{LSL}}{6s}\sqrt{\frac{\chi^2_{1-\alpha/2,n-1}}{n-1}} \le C_p \le \frac{\text{USL} - \text{LSL}}{6s}\sqrt{\frac{\chi^2_{\alpha/2,n-1}}{n-1}} \tag{8.19}$$

or

$$\hat{C}_p\sqrt{\frac{\chi^2_{1-\alpha/2,n-1}}{n-1}} \le C_p \le \hat{C}_p\sqrt{\frac{\chi^2_{\alpha/2,n-1}}{n-1}} \tag{8.20}$$

where $\chi^2_{1-\alpha/2,n-1}$ and $\chi^2_{\alpha/2,n-1}$ are the lower $\alpha/2$ and upper $\alpha/2$ percentage points of the chi-square distribution with $n - 1$ degrees of freedom. These percentage points are tabulated in Appendix Table III.

EXAMPLE 8.4 A Confidence Interval in C_p

Suppose that a stable process has upper and lower specifications at USL = 62 and LSL = 38. A sample of size $n = 20$ from this process reveals that the process mean is centered approximately at the midpoint of the specification interval and that the sample standard deviation $s = 1.75$. Find a 95% confidence interval on C_p.

SOLUTION

A point estimate of C_p is

$$\hat{C}_p = \frac{\text{USL} - \text{LSL}}{6s} = \frac{62-38}{6(1.75)} = 2.29$$

The 95% confidence interval on C_p is found from equation (8.20) as follows:

$$\hat{C}_p \sqrt{\frac{\chi^2_{1-0.025,n-1}}{n-1}} \le C_p \le \hat{C}_p \sqrt{\frac{\chi^2_{0.025,n-1}}{n-1}}$$

$$2.29\sqrt{\frac{8.91}{19}} \le C_p \le 2.29\sqrt{\frac{32.85}{19}}$$

$$1.57 \le C_p \le 3.01$$

where $\chi^2_{0.975,19} = 8.91$ and $\chi^2_{0.025,19} = 32.85$ were taken from Appendix Table III.

For more complicated ratios such as C_{pk} and C_{pm}, various authors have developed approximate confidence intervals; for example, see Zhang, Stenbeck, and Wardrop (1990), Bissell (1990), Kushler and Hurley (1992), and Pearn et al. (1992). If the quality characteristic is normally distributed, then an approximate $100(1 - \alpha)\%$ confidence interval on C_{pk} is given as follows.

$$\hat{C}_{pk}\left[1 - Z_{\alpha/2}\sqrt{\frac{1}{9n\hat{C}_{pk}^2} + \frac{1}{2(n-1)}}\right] \le C_{pk} \le \hat{C}_{pk}\left[1 + Z_{\alpha/2}\sqrt{\frac{1}{9n\hat{C}_{pk}^2} + \frac{1}{2(n-1)}}\right] \qquad (8.21)$$

Kotz and Lovelace (1998) give an extensive summary of confidence intervals for various PCRs.

EXAMPLE 8.5 A Confidence Interval in C_{pk}

A sample of size $n = 20$ from a stable process is used to estimate C_{pk}, with the result that $\hat{C}_{pk} = 1.33$. Find an approximate 95% confidence interval on C_{pk}.

SOLUTION

Using equation (8.21), an approximate 95% confidence interval on C_{pk} is

$$\hat{C}_{pk}\left[1 - Z_{\alpha/2}\sqrt{\frac{1}{9n\hat{C}_{pk}^2} + \frac{1}{2(n-1)}}\right] \leq C_{pk} \leq \hat{C}_{pk}\left[1 + Z_{\alpha/2}\sqrt{\frac{1}{9n\hat{C}_{pk}^2} + \frac{1}{2(n-1)}}\right]$$

$$1.33\left[1 - 1.96\sqrt{\frac{1}{9(20)(1.33)^2} + \frac{1}{2(19)}}\right] \leq C_{pk} \leq 1.33\left[1 + 1.96\sqrt{\frac{1}{9(20)(1.33)^2} + \frac{1}{2(19)}}\right]$$

or

$$0.88 \leq C_{pk} \leq 1.78$$

This is an extremely wide confidence interval. Based on the sample data, the ratio C_{pk} could be less than 1 (a very bad situation), or it could be as large as 1.78 (a very good situation). Thus, we have learned very little about actual process capability, because C_{pk} is very imprecisely estimated. The reason for this, of course, is that a very small sample ($n = 20$) has been used.

Process Performance Indices. In 1991, the Automotive Industry Action Group (AIAG) was formed and consists of representatives of the "big three" (Ford, General Motors, and Chrysler) and the American Society for Quality Control (now the American Society for Quality). One of their objectives was to standardize the reporting requirements from suppliers and in general of their industry. The AIAG recommends using the process capability indices C_p and C_{pk} when the process is in control, with the process standard deviation estimated by $\hat{\sigma} = \overline{R}/d_2$. When the process is *not* in control, the AIAG recommends using **process performance indices P_p and P_{pk},** where, for example,

$$\hat{P}_p = \frac{\text{USL} - \text{LSL}}{6s}$$

and s is the usual sample standard deviation $s = \sqrt{\Sigma_{i=1}^{n}(x_i - \overline{x})^2/(n-1)}$. Even the American National Standards Institute in ANSI Standard Z1 on Process Capability Analysis (1996) states that P_p and P_{pk} should be used when the process is not in control.

Now it is clear that when the process is normally distributed and in control, $\hat{P}_p$ is essentially $\hat{C}_p$ and $\hat{P}_{pk}$ is essentially $\hat{C}_{pk}$ because for a stable process the difference between s and $\hat{\sigma} = \overline{R}/d_2$ is minimal. However, please note that if the process is **not** in control, the indices P_p and P_{pk} have no meaningful interpretation relative to process capability, because they cannot predict process performance. Furthermore, their statistical properties are not determinable, and so no valid inference can be made regarding their true (or population) values. Also, P_p and P_{pk} provide no motivation or incentive to the companies that use them to bring their processes into control.

Kotz and Lovelace (1998) strongly recommend **against** the use of P_p and P_{pk}, indicating that these indices are actually a step **backwards** in quantifying process capability. They refer to the mandated use of P_p and P_{pk} through quality standards or industry guidelines as undiluted **statistical terrorism** (i.e., the use or misuse of statistical methods along with threats and/or intimidation to achieve a business objective).

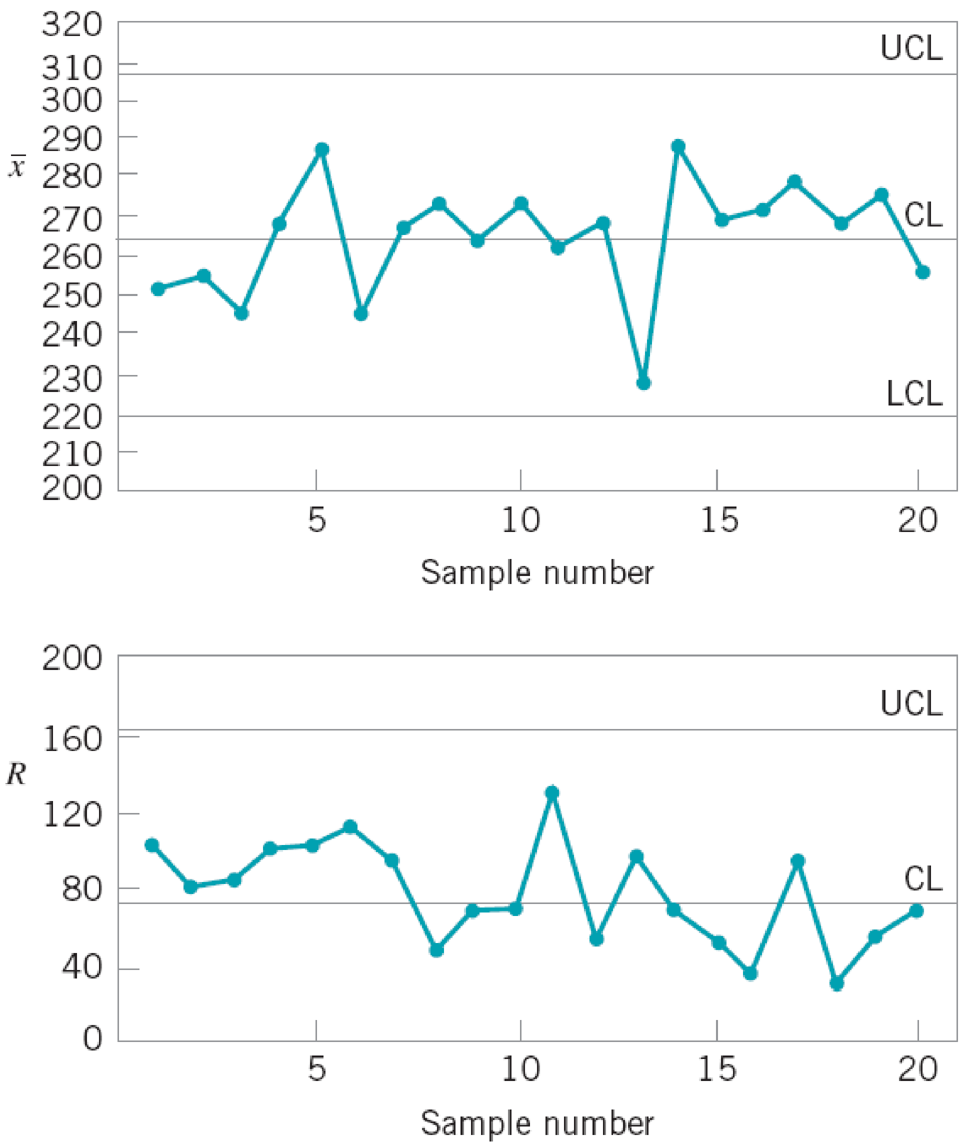
This author agrees completely with Kotz and Lovelace. The process performance indices P_p and P_{pk} are actually more than a step backwards. **They are a waste of engineering and management effort—they tell you nothing.** Unless the process is stable (in control), no index is going to carry useful predictive information about process capability or convey any information about future performance. Instead of imposing the use of meaningless indices, organizations should devote effort to developing and implementing an effective process characterization and control plan. The U.S. semiconductor industry did this in the late 1980s (at Sematech) with great success. This is a much more reasonable and effective approach to process improvement.

Process Capability Analysis using Control Charts

■ TABLE 8.5

Glass Container Strength Data (psi)

Sample	Data					$\bar{x}$	R
1	265	205	263	307	220	252.0	102
2	268	260	234	299	215	255.2	84
3	197	286	274	243	231	246.2	89
4	267	281	265	214	318	269.0	104
5	346	317	242	258	276	287.8	104
6	300	208	187	264	271	246.0	113
7	280	242	260	321	228	266.2	93
8	250	299	258	267	293	273.4	49
9	265	254	281	294	223	263.4	71
10	260	308	235	283	277	272.6	73
11	200	235	246	328	296	261.0	128
12	276	264	269	235	290	266.8	55
13	221	176	248	263	231	227.8	87
14	334	280	265	272	283	286.8	69
15	265	262	271	245	301	268.8	56
16	280	274	253	287	258	270.4	34
17	261	248	260	274	337	276.0	89
18	250	278	254	274	275	266.2	28
19	278	250	265	270	298	272.2	48
20	257	210	280	269	251	253.4	70
						$\bar{\bar{x}} = 264.06$	$\bar{R} = 77.3$



$$\hat{\mu} = \bar{\bar{x}} = 264.06$$

$$\hat{\sigma} = \frac{\bar{R}}{d_2} = \frac{77.3}{2.326} = 33.23$$

Since LSL = 200

$$\hat{C}_{pl} = \frac{\mu - \text{LSL}}{3\hat{\sigma}} = \frac{264.06 - 200}{3(33.23)} = 0.64$$

FIGURE 8.12 $\bar{x}$ and R charts for the bottle-strength data.

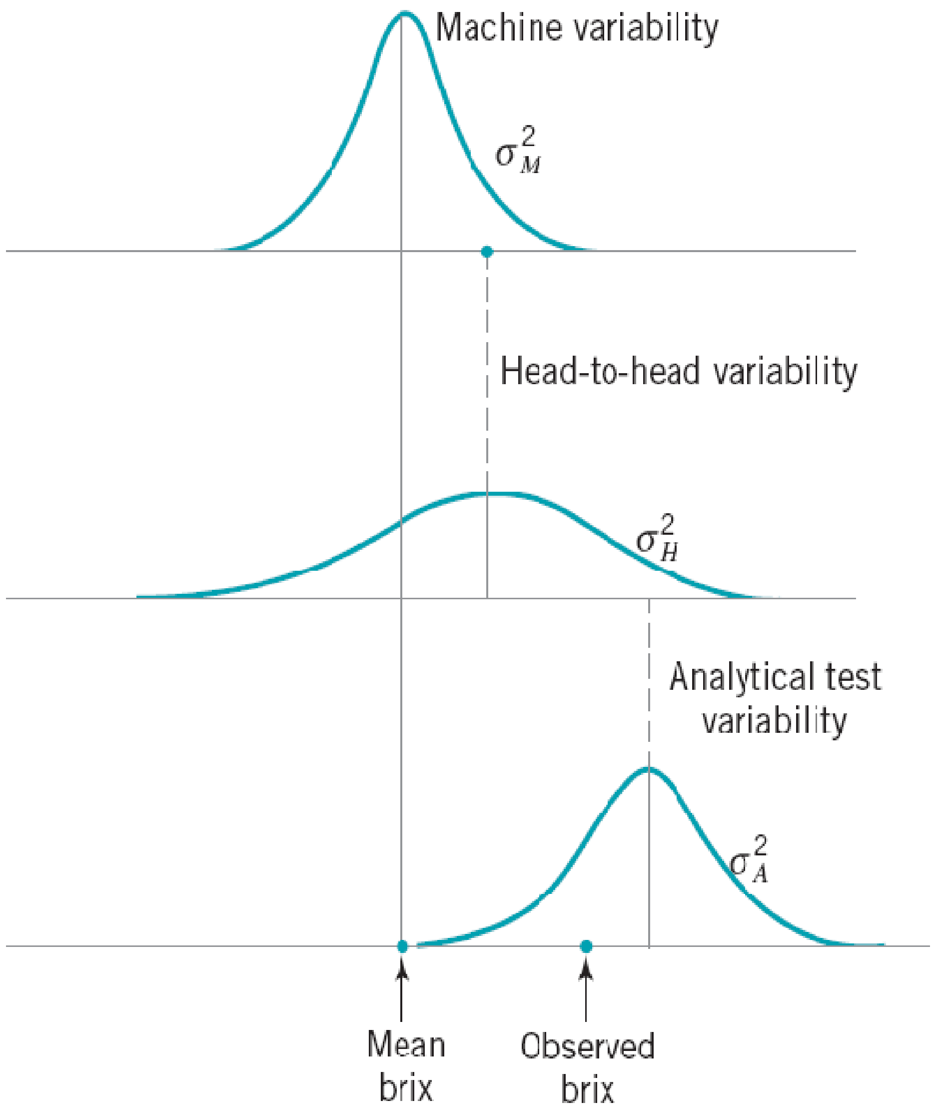
8.5 Process Capability Analysis Using Designed Experiments

A designed experiment is a systematic approach to varying the input **controllable** variables in the process and analyzing the effects of these process variables on the output. Designed experiments are also useful in discovering *which* set of process variables is influential on the output, and at what levels these variables should be held to optimize process performance. Thus, design of experiments is useful in more general problems than merely estimating process capability. For an introduction to design of experiments, see Montgomery (2005). Part V of this textbook provides more information on experimental design methods and on their use in process improvement.

One of the major uses of designed experiments is in isolating and estimating the **sources of variability** in a process. For example, consider a machine that fills bottles with a soft-drink beverage. Each machine has a large number of filling heads that must be independently adjusted. The quality characteristic measured is the syrup content (in degrees brix) of the finished product. There can be variation in the observed brix (σ_B^2) because of machine variability (σ_M^2), head variability (σ_H^2), and analytical test variability (σ_A^2). The variability in the observed brix value is

$$\sigma_B^2 = \sigma_M^2 + \sigma_H^2 + \sigma_A^2$$

An experiment can be designed, involving sampling from several machines and several heads on each machine, and making several analyses on each bottle, which would allow estimation of the variances (σ_M^2), (σ_H^2), and (σ_A^2). Suppose that the results appear as in Fig. 8.13. Since a substantial portion of the total variability in observed brix is due to variability between heads, this indicates that the process can perhaps best be improved by reducing the head-to-head variability. This could be done by more careful setup or by more careful control of the operation of the machine.



■ **FIGURE 8.13** Sources of variability in the bottling line example.

8.6 Process Capability Analysis with Attribute Data

Often process performance is measured in terms of **attribute** data; that is, nonconforming units or defectives, or nonconformities or defects. When a fraction nonconforming is the measure of performance, it is typical to use the parts per million (ppm) defective as a measure of process capability. In some organizations, this ppm defective is converted to an equivalent sigma level. For example, a process producing 2700 ppm defective would be equivalent to a three-sigma process (without the "usual" 1.5 σ shift in the mean that many six-sigma organizations employ in the calculations taken into account).

When dealing with nonconformities or defects, a defects per unit (DPU) statistic is often used as a measure of capability, where

$$\text{DPU} = \frac{\text{Total number of defects}}{\text{Total number of units}}$$

Here the unit is something that is delivered to a customer and can be evaluated or judged as to its suitability. Some examples include:

1. An invoice
2. A shipment
3. A customer order
4. An enquiry or call

The defects or nonconformities are anything that does not meet the customer requirements, such as:

1. An error on an invoice
2. An incorrect or incomplete shipment
3. An incorrect or incomplete customer order
4. A call that is not satisfactorily completed

Obviously, these quantities are estimated from sample data. Large samples need to be used to obtain reliable estimates.

The DPU measure does not directly take the complexity of the unit into account. A widely used way to do this is the defect per million opportunities (DPMO) measure

$$\text{DPMO} = \frac{\text{Total number of defects}}{\text{Number of units} \times \text{Number of opportunities}}$$

Opportunities are the number of potential chances within a unit for a defect to occur. For example, on a purchase order, the number of opportunities would be the number of fields in which information is recorded times two, because each field can either be filled out incorrectly or blank (information is missing). It is important to be consistent about how opportunities are defined, as a process may be artificially improved simply by increasing the number of opportunities over time.

8.7 Gauge and Measurement Systems Capability Studies

- Determine how much of the observed variability is due to the gauge or measurement system
- Isolate the components of variability in the measurement system
- Assess whether the gauge is capable (suitable for the intended application)

To introduce some of the basic ideas of measurement systems analysis (MSA) consider a simple but reasonable model for measurement system capability studies

$$y = x + \varepsilon \tag{8.23}$$

where y is the total observed measurement, x is the true value of the measurement on a unit of product, and ε is the measurement error. We will assume that x and ε are normally and independently distributed random variables with means μ and 0 and variances (σ_P^2) and $(\sigma_{\text{Gauge}}^2)$, respectively. The variance of the total observed measurement, y, is then

$$\sigma_{\text{Total}}^2 = \sigma_P^2 + \sigma_{\text{Gauge}}^2 \tag{8.24}$$

Control charts and other statistical methods can be used to separate these components of variance, as well as to give an assessment of gauge capability.

EXAMPLE 8.7 Measuring Gauge Capability

An instrument is to be used as part of a proposed SPC implementation. The quality-improvement team involved in designing the SPC system would like to get an assessment of gauge capability. Twenty units of the product are obtained, and the process operator who will actually take the measurements for the control chart uses the instrument to measure each unit of product twice. The data are shown in Table 8.6.

SOLUTION

Figure 8.14 shows the $\bar{x}$ and R charts for these data. Note that the x chart exhibits many out-of-control points. This is to be expected, because in this situation the $\bar{x}$ chart has an interpretation that is somewhat different from the usual interpretation. The $\bar{x}$ chart in this example shows the **discriminating power** of the instrument—literally, the ability of the gauge to distinguish between units of product. The R chart directly shows the magnitude of measurement error, or the gauge capability. The R values represent the difference between measurements made on the same unit using the same instrument. In this example, the R chart is in control. This indicates that the operator is having no difficulty in making consistent measurements. Out-of-control points on the R chart could indicate that the operator is having difficulty using the instrument.

The standard deviation of measurement error, σ_{Gauge}, can be estimated as follows:

$$\hat{\sigma}_{\text{Gauge}} = \frac{\bar{R}}{d_2} = \frac{1.0}{1.128} = 0.887$$

The distribution of measurement error is usually well approximated by the normal. Thus, $\hat{\sigma}_{\text{Guage}}$ is a good estimate of gauge capability.

(continued)

TABLE 8.6

Parts Measurement Data

Part Number	Measurements 1	Measurements 2	$\bar{x}$	R
1	21	20	20.5	1
2	24	23	23.5	1
3	20	21	20.5	1
4	27	27	27.0	0
5	19	18	18.5	1
6	23	21	22.0	2
7	22	21	21.5	1
8	19	17	18.0	2
9	24	23	23.5	1
10	25	23	24.0	2
11	21	20	20.5	1
12	18	19	18.5	1
13	23	25	24.0	2
14	24	24	24.0	0
15	29	30	29.5	1
16	26	26	26.0	0
17	20	20	20.0	0
18	19	21	20.0	2
19	25	26	25.5	1
20	19	19	19.0	0
			$\bar{\bar{x}} = 22.3$	$\bar{R} = 1.0$

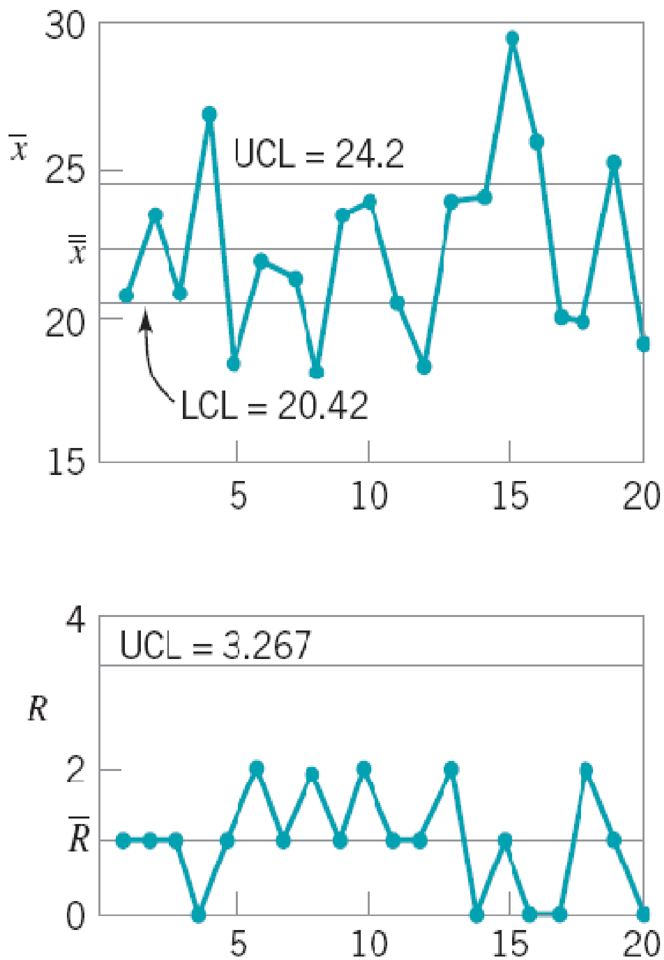


FIGURE 8.14 Control charts for the gauge capability analysis in Example 8.7.

The *P/T (precision-to-tolerance)* ratio:

$$P/T = \frac{k\hat{\sigma}_{\text{Guage}}}{\text{USL} - \text{LSL}} \qquad (8.25)$$

In equation (8.25), popular choices for the constant k are $k = 5.15$ and $k = 6$. The value $k = 5.15$ corresponds to the limiting value of the number of standard deviations between bounds of a 95% tolerance interval that contains at least 99% of a normal population, and $k = 6$ corresponds to the number of standard deviations between the usual natural tolerance limits of a normal population.

The part used in Example 8.7 has USL = 60 and LSL = 5. Therefore, taking $k = 6$ in equation (8.25), an estimate of the *P/T* ratio is

$$P/T = \frac{6(0.887)}{60 - 5} = \frac{5.32}{55} = 0.097$$

Values of the estimated ratio P/T of 0.1 or less often are taken to imply adequate gauge capability. This is based on the generally used rule that requires a measurement device to be calibrated in units one-tenth as large as the accuracy required in the final measurement. However, we should use **caution** in accepting this general rule of thumb in all cases. A gauge must be sufficiently capable to measure product accurately enough and precisely enough so that the analyst can make the correct decision. This may not necessarily require that $P/T \leq 0.1$.

Estimating the Variance Components

$$\hat{\sigma}^2_{\text{Total}} = s^2 = (3.17)^2 = 10.05$$

Since from equation (8.24) we have

$$\sigma^2_{\text{Total}} = \sigma^2_P + \sigma^2_{\text{Gauge}}$$

and because we have an estimate of $\hat{\sigma}^2_{\text{Gauge}} = (0.887)^2 = 0.79$, we can obtain an estimate of σ^2_P as

$$\hat{\sigma}^2_P = \hat{\sigma}^2_{\text{Total}} - \hat{\sigma}^2_{\text{Gauge}} = 10.05 - 0.79 = 9.26$$

Therefore, an estimate of the standard deviation of the product characteristic is

$$\hat{\sigma}_p = \sqrt{9.26} = 3.04$$

There are other measures of gauge capability that have been proposed. One of these is the ratio of process (part) variability to total variability:

$$\rho_P = \frac{\sigma_P^2}{\sigma_{\text{Total}}^2} \tag{8.26}$$

and another is the ratio of measurement system variability to total variability:

$$\rho_M = \frac{\sigma_{\text{Gauge}}^2}{\sigma_{\text{Total}}^2} \tag{8.27}$$

Obviously, $\rho_P = 1 - \rho_M$. For the situation in Example 8.7 we can calculate an estimate of ρ_M as follows:

$$\hat{\rho}_M = \frac{\hat{\sigma}_{\text{Gauge}}^2}{\hat{\sigma}_{\text{Total}}^2} = \frac{0.79}{10.05} = 0.0786$$

Thus the variance of the measuring instrument contributes about 7.86% of the total observed variance of the measurements.

Another measure of measurement system adequacy is defined by the AIAG (1995) [note that there is also on updated edition of this manual, AIAG (2002)] as the **signal-to-noise ratio (*SNR*)**:

$$SNR = \sqrt{\frac{2\rho_P}{1 - \rho_P}} \tag{8.28}$$

AIAG defined the *SNR* as the number of distinct levels or categories that can be reliably obtained from the measurements. A value of five or greater is recommended, and a value of less than two indicates inadequate gauge capability. For Example 8.7 we have $\hat{\rho}_M = 0.0786$, and using $\hat{\rho}_P = 1 - \hat{\rho}_M$ we find that $\hat{\rho}_P = 1 - \hat{\rho}_M = 1 - 0.0786 = 0.9214$, so an estimate of the *SNR* in equation (8.28) is

$$SNR = \sqrt{\frac{2\hat{\rho}_P}{1 - \hat{\rho}_P}} = \sqrt{\frac{2(0.9214)}{1 - 0.9214}} = 4.84$$

The gauge is not capable by this criterion

Discrimination Ratio

$$DR = \frac{1 + \rho_P}{1 - \rho_P} \tag{8.29}$$

Some authors have suggested that for a gauge to be capable the *DR* must exceed four. This is a very arbitrary requirement. For the situation in Example 8.7, we would calculate an estimate of the discrimination ratio as

$$\widehat{DR} = \frac{1 + \rho_P}{1 - \rho_P} = \frac{1 + 0.9214}{1 - 0.9214} = 24.45$$

Clearly by this measure, the gauge is capable.

Accuracy and Precision

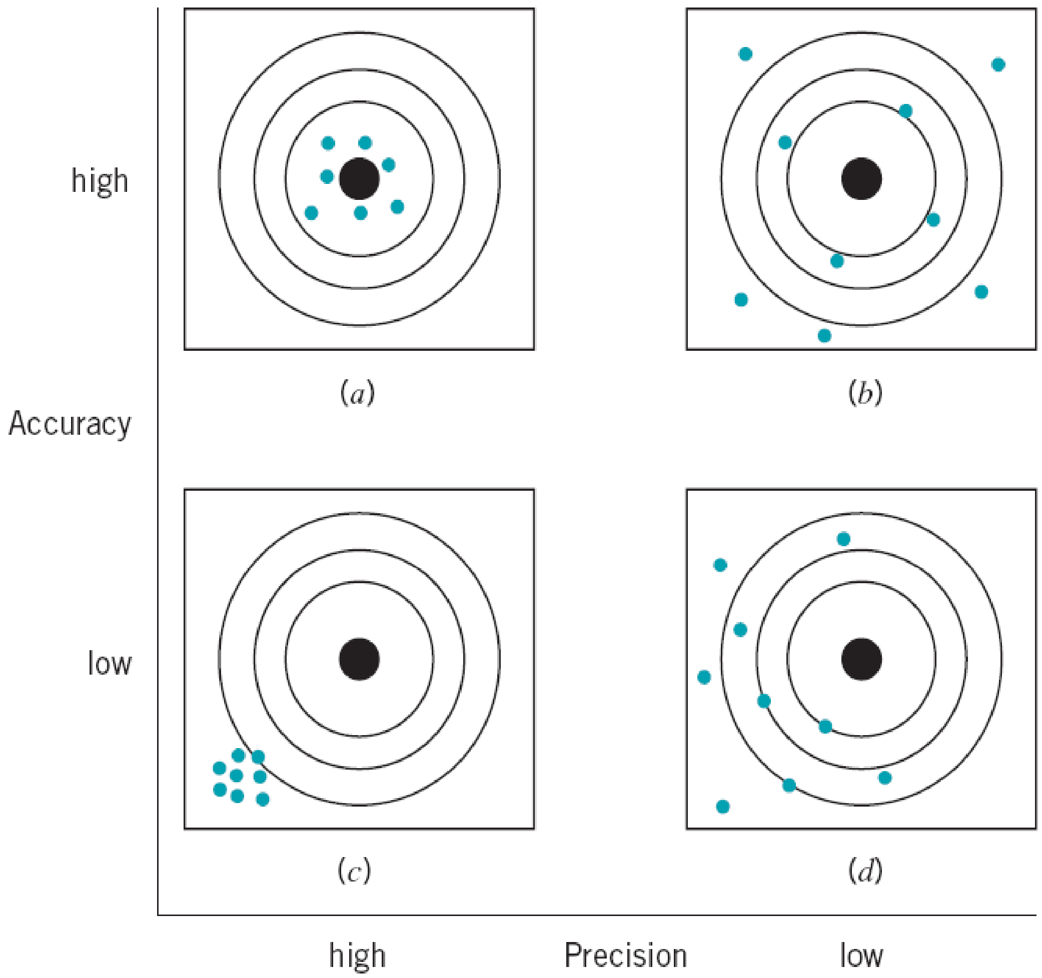


FIGURE 8.15 The concepts of accuracy and precision. (*a*) The gauge is accurate and precise. (*b*) The gauge is accurate but not precise. (*c*) The gauge is not accurate but it is precise. (*d*) The gauge is neither accurate nor precise.

We have focused only on precision

Gauge R&R Studies

It is also possible to design measurement systems capability studies to investigate two components of measurement error, commonly called the **repeatability** and the **reproducibility** of the gauge. We define reproducibility as the variability due to different operators using the gauge (or different time periods, or different environments, or in general, different conditions) and repeatability as reflecting the basic inherent precision of the gauge itself. That is,

$$\sigma^2_{\text{Measurement Error}} = \sigma^2_{\text{Gauge}} = \sigma^2_{\text{Repeatability}} + \sigma^2_{\text{Reproducibility}} \qquad (8.30)$$

The experiment used to measure the components of σ^2_{Gauge} is usually called a gauge R & R study, for the two components of σ^2_{Gauge}. We now show how to analyze **gauge R & R experiments**.

Gauge R&R Studies Are Usually Conducted with a Factorial Experiment

If there are a randomly selected parts and b randomly selected operators, and each operator measures every part n times, then the measurements (i = part, j = operator, k = measurement) could be represented by the model

$$y_{ijk} = \mu + P_i + O_j + (PO)_{ij} + \varepsilon_{ijk} \quad \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, o \\ k = 1, 2, \ldots, n \end{cases}$$

where the model parameters P_i, O_j, $(PO)_{ij}$, and ε_{ijk} are all independent random variables that represent the effects of parts, operators, the interaction or joint effects of parts and operators, and random error. This is a **random effects model analysis of variance (ANOVA)**. It is also sometimes called the standard model for a gauge R & R experiment. We assume that the random variables P_i, O_j, $(PO)_{ij}$, and ε_{ijk} are normally distributed with mean zero and variances given by $V(P_i) = \sigma_P^2$, $V(O_j) = \sigma_O^2$, $V[(PO)_{ij}] = \sigma_{PO}^2$, and $V(\varepsilon_{ijk}) = \sigma^2$. Therefore, the variance of any observation is

$$V(y_{ijk}) = \sigma_P^2 + \sigma_O^2 + \sigma_{PO}^2 + \sigma^2 \qquad (8.31)$$

and σ_P^2, σ_O^2, σ_{PO}^2, and σ^2 are the **variance components.** We want to estimate the variance components.

■ TABLE 8.7

Thermal Impedance Data (°C/$W \times 100$) for the Gauge R & R Experiment

Part Number	Inspector 1			Inspector 2			Inspector 3		
	Test 1	Test 2	Test 3	Test 1	Test 2	Test 3	Test 1	Test 2	Test 3
1	37	38	37	41	41	40	41	42	41
2	42	41	43	42	42	42	43	42	43
3	30	31	31	31	31	31	29	30	28
4	42	43	42	43	43	43	42	42	42
5	28	30	29	29	30	29	31	29	29
6	42	42	43	45	45	45	44	46	45
7	25	26	27	28	28	30	29	27	27
8	40	40	40	43	42	42	43	43	41
9	25	25	25	27	29	28	26	26	26
10	35	34	34	35	35	34	35	34	35

This is a two-factor factorial experiment

ANOVA methods are used to analyze the data and to estimate the variance components

$$SS_{\text{Total}} = SS_{\text{Parts}} + SS_{\text{Operators}} + SS_{P\times O} + SS_{\text{Error}} \tag{8.32}$$

$$MS_P = \frac{SS_{\text{Parts}}}{p-1}$$

$$MS_O = \frac{SS_{\text{Operators}}}{o-1}$$

$$MS_{PO} = \frac{SS_{P\times O}}{(p-1)(o-1)}$$

$$MS_E = \frac{SS_{\text{Error}}}{po(n-1)}$$

$$E(MS_P) = \sigma^2 + n\sigma_{PO}^2 + bn\sigma_P^2$$

$$E(MS_O) = \sigma^2 + n\sigma_{PO}^2 + an\sigma_O^2$$

$$E(MS_{PO}) = \sigma^2 + n\sigma_{PO}^2$$

$$E(MS_E) = \sigma^2$$

$$\hat{\sigma}^2 = MS_E$$

$$\hat{\sigma}_{PO}^2 = \frac{MS_{PO} - MS_E}{n}$$

$$\hat{\sigma}_O^2 = \frac{MS_O - MS_{PO}}{pn}$$

$$\hat{\sigma}_P^2 = \frac{MS_P - MS_{PO}}{on}$$

TABLE 8.8

ANOVA: Thermal Impedance versus Part Number, Operator

Factor	Type	Levels	Values						
Part Num	random	10	1	2	3	4	5	6	7
			8	9	10				
Operator	random	3	1	2	3				

Analysis of Variance for Thermal

Source	DF	SS	MS	F	P
Part Num	9	3935.96	437.33	162.27	0.000
Operator	2	39.27	19.63	7.28	0.005
Part Num*Operator	18	48.51	2.70	5.27	0.000
Error	60	30.67	0.51		
Total	89	4054.40			

Source	Variance component	Error term	Expected Mean Square for Each Term (using unrestricted model)
1 Part Num	48.2926	3	(4) + 3(3) + 9(1)
2 Operator	0.5646	3	(4) + 3(3) + 30(2)
3 Part Num*Operator	0.7280	4	(4) + 3(3)
4 Error	0.5111		(4)

$$\hat{\sigma}_P^2 = \frac{437.33 - 2.70}{(3)(3)} = 48.29$$

$$\hat{\sigma}_O^2 = \frac{19.63 - 2.70}{(10)(3)} = 0.56$$

$$\hat{\sigma}_{PO}^2 = \frac{2.70 - 0.51}{3} = 0.73$$

$$\sigma^2 = 0.51$$

$$\hat{\sigma}^2 = MS_E$$

$$\hat{\sigma}_{PO}^2 = \frac{MS_{PO} - MS_E}{n}$$

$$\hat{\sigma}_O^2 = \frac{MS_O - MS_{PO}}{pn}$$

$$\hat{\sigma}_P^2 = \frac{MS_P - MS_{PO}}{on}$$

- Negative estimates of a variance component would lead to filling a reduced model, such as, for example:

$$y_{ijk} = \mu + P_i + O_j + \varepsilon_{ijk}$$

Typically we think of σ^2 as the **repeatability** variance component, and the gauge **reproducibility** as the sum of the operator and the part × operator variance components,

$$\sigma^2_{\text{Reproducibility}} = \sigma^2_O + \sigma^2_{PO}$$

Therefore

$$\sigma^2_{\text{Gauge}} = \sigma^2_{\text{Reproducibility}} + \sigma^2_{\text{Repeatability}}$$

For this Example

$$\hat{\sigma}^2_{\text{Gauge}} = \hat{\sigma}^2 + \hat{\sigma}^2_O + \hat{\sigma}^2_{PO}$$
$$= 0.51 + 0.56 + 0.73$$
$$= 1.80$$

The lower and upper specifications on this power module are $LSL = 18$ and $USL = 58$. Therefore the P/T ratio for the gauge is estimated as

$$\widehat{P/T} = \frac{6\hat{\sigma}_{\text{Gauge}}}{\text{USL} - \text{LSL}} = \frac{6(1.34)}{58 - 18} = 0.27$$

By the standard measures of gauge capability, this gauge would not be considered capable because the estimate of the P/T ratio exceeds 0.10.

Other Topics in Gauge R&R Studies

- Section 8.7.3 provides a description of methods to obtain confidence intervals on the variance components and measures of gauge R&R
- Section 8.7.4 presents a new measure of gauge capability, the probabilities of misclassification of parts
  - Rejecting good units (producer's risk)
  - Passing bad units (consumer's risk)
  - Methods for calculating these two probabilities are given

8.7.5 Attribute Gauge Capability

- Sometimes the output of a gauge isn't numerical – it's just pass/fail
- Nominal or ordinal data is also common
- Occurs frequently in service businesses
- Common situation – do operating personnel consistently make the same decisions regarding the units they are inspecting or analyzing
- Example – a bank uses manual underwriting of mortgage loans
- The underwriter uses information to classify the applicant into one of four categories; decline or category 1, 2, 3 – categories 2 & 3 are low-risk and 1 is high risk
- Compare underwriters performance relative to a "consensus" evaluation determined by a panel of "experts"

TABLE 8.13

Loan Evaluation Data for Attribute Gauge Capability Analysis

Application	Classification	Sue1	Sue2	Fred1	Fred2	John1	John2
1	Fund-1	Fund-3	Fund-3	Fund-2	Fund-2	Fund-1	Fund-3
2	Fund-3	Fund-3	Fund-3	Fund-3	Fund-3	Fund-3	Fund-1
3	Fund-1	Fund-3	Fund-3	Fund-2	Fund-2	Fund-1	Fund-1
4	Fund-1	Fund-1	Fund-1	Fund-2	Fund-1	Fund-1	Fund-1
5	Fund-2	Fund-1	Fund-2	Fund-2	Fund-2	Fund-2	Fund-1
6	Fund-3	Fund-3	Fund-3	Fund-1	Fund-3	Fund-3	Fund-1
7	Fund-3	Fund-3	Fund-3	Fund-3	Fund-3	Fund-3	Fund-3
8	Fund-3	Fund-3	Fund-3	Fund-1	Fund-3	Fund-3	Fund-3
9	Fund-1	Fund-3	Fund-3	Fund-1	Fund-1	Fund-1	Fund-1
10	Fund-2	Fund-1	Fund-2	Fund-2	Fund-2	Fund-2	Fund-1
11	Decline	Decline	Decline	Fund-3	Fund-3	Decline	Decline
12	Fund-2	Fund-3	Fund-1	Fund-2	Fund-2	Fund-2	Fund-2
13	Fund-2	Fund-2	Fund-2	Fund-2	Fund-2	Fund-2	Fund-1
14	Fund-2	Fund-2	Fund-2	Fund-2	Fund-2	Fund-2	Fund-2
15	Fund-1	Fund-1	Fund-1	Fund-1	Fund-1	Fund-1	Fund-1
16	Fund-2	Fund-2	Fund-2	Fund-2	Fund-2	Fund-2	Fund-1
17	Fund-3	Decline	Fund-3	Fund-1	Fund-1	Fund-3	Fund-3
18	Fund-3	Fund-3	Fund-1	Fund-3	Fund-3	Fund-3	Fund-1
19	Decline	Fund-3	Fund-3	Fund-3	Decline	Decline	Decline
20	Fund-1	Fund-1	Fund-1	Fund-1	Fund-1	Fund-1	Fund-1
21	Fund-2	Fund-2	Fund-2	Fund-1	Fund-2	Fund-2	Fund-1
22	Fund-2	Fund-1	Fund-2	Fund-2	Fund-2	Fund-2	Fund-2
23	Fund-1	Fund-1	Fund-1	Fund-1	Fund-1	Fund-1	Fund-1
24	Fund-3	Decline	Fund-3	Fund-1	Fund-2	Fund-3	Fund-1
25	Fund-3	Fund-3	Fund-3	Fund-3	Fund-3	Fund-3	Fund-3
26	Fund-3	Fund-3	Fund-3	Fund-3	Fund-3	Fund-3	Fund-1
27	Fund-2	Fund-2	Fund-2	Fund-2	Fund-1	Fund-2	Fund-2
28	Decline	Decline	Decline	Fund-3	Decline	Decline	Decline
29	Decline	Decline	Decline	Fund-3	Decline	Decline	Fund-3
30	Fund-2	Fund-2	Fund-2	Fund-2	Fund-2	Fund-2	Fund-2

Thirty applicants, three underwriters

Each underwriter evaluates each application twice

The applications are “blinded” by removing names, SSNs, addresses, and other identifying information

Attribute Gauge Capability

- Determine the proportion of time that the underwriter agrees with him/herself – this measures repeatability
- Determine the proportion of time that the underwriter agrees with the correct classification – this measures bias
- Minitab performs the analysis – using the attribute agreement analysis routine

TABLE 8.14

Minitab Attribute Agreement Analysis for the Loan Evaluation Data in Table 8.13

Attribute Agreement Analysis for Sue1, Sue2, Fred1, Fred2, John1, John2

Within Appraisers

Assessment Agreement

Appraiser	# Inspected	# Matched	Percent	95% CI
Sue	30	23	76.67	(57.72, 90.07)
Fred	30	21	70.00	(50.60, 85.27)
John	30	18	60.00	(40.60, 77.34)

Matched: Appraiser agrees with him/herself across trials.

Each Appraiser vs Standard

Assessment Agreement

Appraiser	# Inspected	# Matched	Percent	95% CI
Sue	30	19	63.33	(43.86, 80.07)
Fred	30	17	56.67	(37.43, 74.54)
John	30	18	60.00	(40.60, 77.34)

Matched: Appraiser's assessment across trials agrees with the known standard.

Between Appraisers

Assessment Agreement

# Inspected	# Matched	Percent	95% CI
30	7	23.33	(9.93, 42.28)

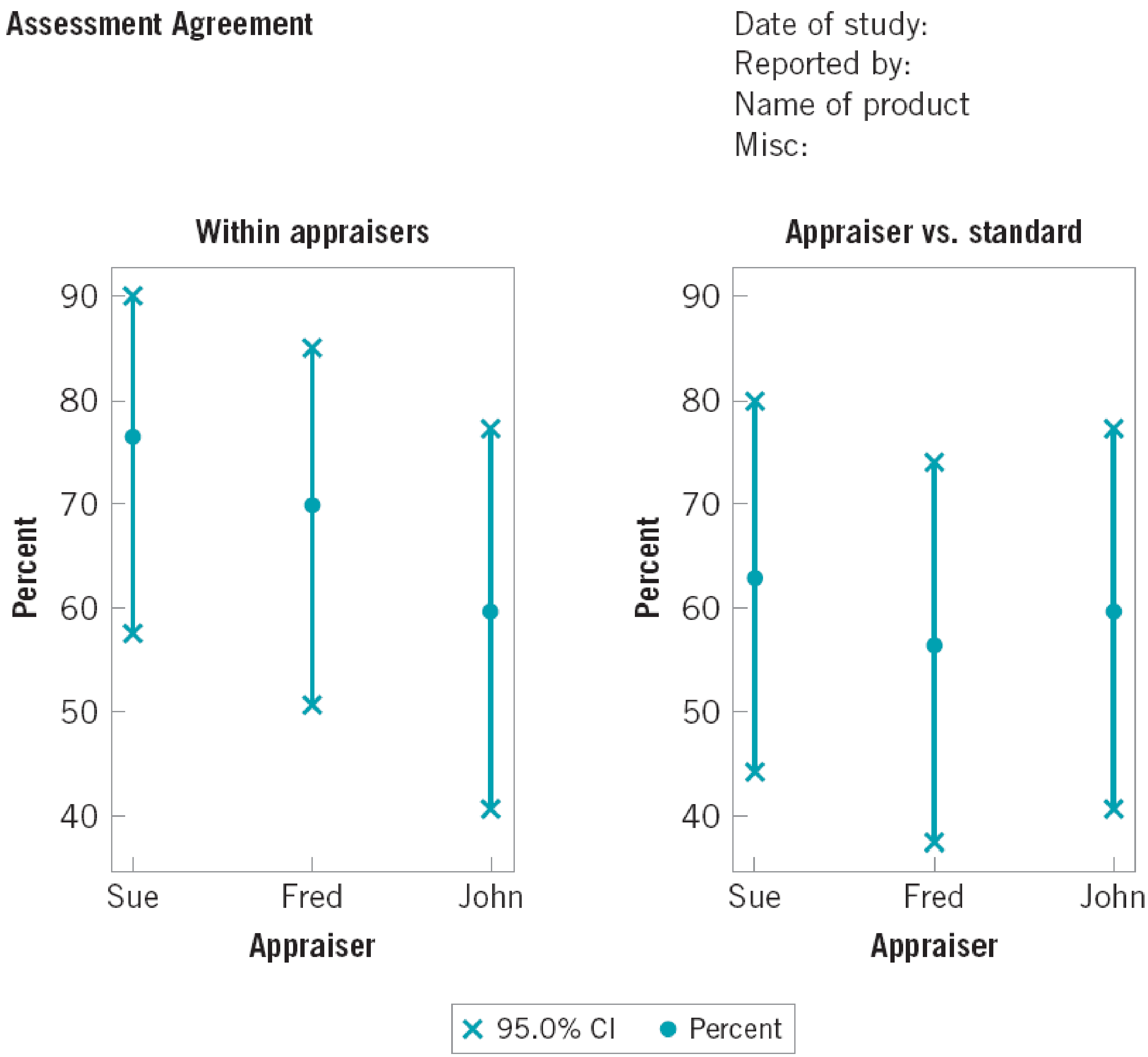
Matched: All appraisers' assessments agree with each other.

All Appraisers vs Standard

Assessment Agreement

# Inspected	# Matched	Percent	95% CI
30	7	23.33	(9.93, 42.28)

Matched: All appraisers' assessments agree with the known standard.



■ FIGURE 8.17 Confidence intervals for the attribute agreement analysis.

8.8 Setting Specifications on Discrete Components

- Components interact with other components
- Complex assemblies
- Tolerance stack-up problems
- Linear combinations
- Nonlinear combinations

8.8.1 Linear Combinations

In many cases, the dimension of an item is a linear combination of the dimensions of the component parts. That is, if the dimensions of the components are $x_1, x_2, \ldots, x_n$, then the dimension of the final assembly is

$$y = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \tag{8.38}$$

where the a_c are constants.

If the x_i are normally and independently distributed with mean μ_i and variance σ_i^2, then y is normally distributed with mean $\mu_y = \Sigma_{n=1}^{i} a_i \mu_i$ and variance $\sigma_y^2 = \Sigma_{n=1}^{i} a_i^2 \sigma_i^2$. Therefore, if μ_i and σ_i^2 are known for each component, the fraction of assembled items falling outside the specifications can be determined.

EXAMPLE 8.8 Meeting Customer Specifications

A linkage consists of four components as shown in Fig. 8.18. The lengths of x_1, x_2, x_3, and x_4 are known to be $x_1 \sim N(2.0, 0.0004)$, $x_2 \sim N(4.5, 0.0009)$, $x_3 \sim N(3.0, 0.0004)$, and $x_4 \sim N(2.5, 0.0001)$. The lengths of the components can be assumed independent, because they are produced on different machines. All lengths are in inches. Determine the proportion of linkages that meet the customer specification on overall length of 12 ± 0.10.

SOLUTION

To find the fraction of linkages that fall within design specification limits, note that y is normally distributed with mean

$$\mu_y = 2.0 + 4.5 + 3.0 + 2.5 = 12.0$$

and variance

$$\sigma_y^2 = 0.0004 + 0.0009 + 0.0004 + 0.0001 = 0.0018$$

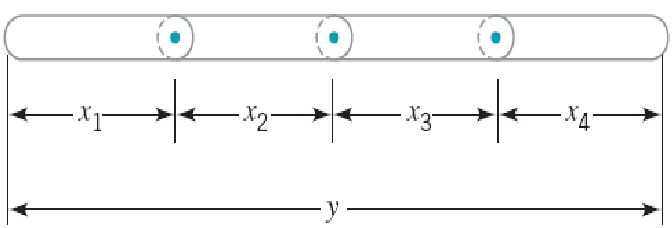


FIGURE 8.18 A linkage assembly with four components.

To find the fraction of linkages that are within specification, we must evaluate

$$\begin{aligned} P\{11.90 \le y \le 12.10\} &= P\{y \le 12.10\} - P\{y \le 11.90\} \\ &= \Phi\left(\frac{12.10 - 12.00}{\sqrt{0.0018}}\right) - \Phi\left(\frac{11.90 - 12.00}{\sqrt{0.0018}}\right) \\ &= \Phi(2.36) - \Phi(-2.36) \\ &= 0.99086 - 0.00914 \\ &= 0.98172 \end{aligned}$$

Therefore, we conclude that 98.172% of the assembled linkages will fall within the specification limits. This is not a six-sigma product.

8.8.2 Nonlinear Combinations

In some problems, the dimension of interest may be a **nonlinear function** of the n component dimensions $x_1, x_2, \ldots x_n$—say,

$$y = g(x_1, x_2, \ldots, x_n) \tag{8.41}$$

In problems of this type, the usual approach is to approximate the nonlinear function g by a linear function of the x_i in the region of interest. If $\mu_1, \mu_2, \ldots \mu_n$ are the nominal dimensions associated with the components $x_1, x_2, \ldots x_n$, then by expanding the right-hand side of equation (8.41) in a Taylor series about $\mu_1, \mu_2, \ldots \mu_n$, we obtain

$$\begin{aligned} y &= g(x_1, x_2, \ldots, x_n) \\ &= g(\mu_1, \mu_2, \ldots, \mu_n) + \sum_{i=1}^{n} (x_i - \mu_i) \frac{\partial g}{\partial x_i}\bigg|_{\mu_1, \mu_2, \ldots, \mu_n} + R \end{aligned} \tag{8.42}$$

where R represents the higher-order terms. Neglecting the terms of higher order, we can apply the expected value and variance operators to obtain

$$\mu_y \simeq g(\mu_1, \mu_2, \ldots, \mu_n) \tag{8.43}$$

and

$$\sigma_y^2 \simeq \sum_{i=1}^{n} \left(\frac{\partial g}{\partial x_i}\bigg|_{\mu_1, \mu_2, \ldots, \mu_n} \right)^2 \sigma_i^2 \tag{8.44}$$

This procedure to find an approximate mean and variance of a nonlinear combination of random variables is sometimes called the **delta method.** Equation (8.44) is often called the **transmission of error formula.**

EXAMPLE 8.11 A Product with Nonlinear Dimensions

Consider the simple DC circuit components shown in Fig. 8.21. Suppose that the voltage across the points (a, b) is required to be 100 ± 2 V. The specifications on the current and the resistance in the circuit are shown in Fig. 8.21. We assume that the component random variables I and R are normally and independently distributed with means equal to their nominal values.

From Ohm's law, we know that the voltage is

$$V = IR$$

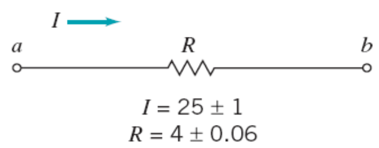


FIGURE 8.21 Electrical circuit for Example 8.11.

Since this involves a nonlinear combination, we expand V in a Taylor series about mean current μ_I and mean resistance μ_R, yielding

$$V \simeq \mu_I \mu_R + (I - \mu_I)\mu_R + (R - \mu_R)\mu_I$$

neglecting the terms of higher order. Now the mean and variance of voltage are

$$\mu_V \simeq \mu_I \mu_R$$

and

$$\sigma_V^2 \simeq \mu_R^2 \sigma_I^2 + \mu_I^2 \sigma_R^2$$

approximately, where σ_I^2 and σ_R^2 are the variances of I and R, respectively.

Now suppose that I and R are centered at their nominal values and that the natural tolerance limits are defined so that $\alpha = 0.0027$ is the fraction of values of each component falling outside these limits. Assume also that the specification limits are exactly equal to the natural tolerance limits. For the current I we have $I = 25 \pm 1$ A. That is, $24 \le I \le 26$ A correspond to the natural tolerance limits *and* the specifications. Since $I \sim N(25, \sigma_I^2)$, and since $Z_{\alpha/2} = Z_{0.00135} = 3.00$, we have

$$\frac{26 - 25}{\sigma_I} = 3.00$$

or $\sigma_I = 0.33$. For the resistance, we have $R = 4 \pm 0.06$ ohm as the specification limits *and* the natural tolerance limits. Thus,

$$\frac{4.06 - 4.00}{\sigma_R} = 3.00$$

and $\sigma_R = 0.02$. Note that σ_I and σ_R are the largest possible values of the component standard deviations consistent with the natural tolerance limits falling inside or equal to the specification limits.

Using these results, and if we assume that the voltage V is approximately normally distributed, then

$$\mu_V \simeq \mu_I \mu_R = (25)(4) = 100 \text{ V}$$

and

$$\sigma_V^2 \simeq \mu_R^2 \sigma_I^2 + \mu_I^2 \sigma_R^2 = (4)^2 (0.33)^2 + (25)^2 (0.02)^2 = 1.99$$

approximately. Thus $\sigma_V = \sqrt{1.99} = 1.41$. Therefore, the probability that the voltage will fall within the design specifications is

$$\begin{aligned} P\{98 \le V \le 102\} &= P\{V \le 102\} - P\{V \le 98\} \\ &= \Phi\left(\frac{102 - 100}{1.41}\right) - \Phi\left(\frac{98 - 100}{1.41}\right) \\ &= \Phi(1.42) - \Phi(-1.42) \\ &= 0.92219 - 0.07781 \\ &= 0.84438 \end{aligned}$$

That is, only 84% of the observed output voltages will fall within the design specifications. Note that the natural tolerance limits or process capability for the output voltage is

$$\mu_V \pm 3.00\sigma_V$$

or

$$100 \pm 4.23 \text{ V}$$

In this problem the process capability ratio is

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma} = \frac{102 - 98}{6(1.41)} = 0.47$$

Note that, although the individual current and resistance variations are not excessive relative to their specifications, because of tolerance stack-up problems, they interact to produce a circuit whose performance relative to the voltage specifications is very poor.

8.9 Estimating the Natural Tolerance Limits of a Process

For a normal distribution with unknown mean and variance:

$$\bar{x} \pm Z_{\alpha/2} s \qquad (8.45)$$

- Difference between tolerance limits and confidence limits
- Nonparametric tolerance limits can also be calculated

Important Terms and Concepts

ANOVA approach to a gauge R & R experiment
Components of gauge error
Components of measurement error
Confidence intervals for gauge R & R studies
Discrimination ratio (*DR*) for a gauge
Estimating variance components
Factorial experiment
Gauge R & R experiment
Graphical methods for process capability analysis
Measurement systems capability analysis
Natural tolerance limits for a normal distribution
Natural tolerance limits of a process
Nonparametric tolerance limits
Normal distribution and process capability ratios
One-sided process-capability ratios
PCR C_p
PCR C_{pk}
Confidence intervals on process capability ratios
Consumer's risk or missed fault for a gauge
Control charts and process capability analysis
Delta method
PCR C_{pm}
Precision and accuracy of a gauge
Precision-to-tolerence (P/T) ratio
Process capability
Process capability analysis
Process performance indices P_p and P_{pk}
Producer's risk or false failure for a gauge
Product characterization
Random effects model ANOVA
Signal-to-noise ratio (*SNR*) for a gauge
Tolerance stack-up problems
Transmission of error formula

Learning Objectives

1. Investigate and analyze process capability using control charts, histograms, and probability plots
2. Understand the difference between process capability and process potential
3. Calculate and properly interpret process capability ratios
4. Understand the role of the normal distribution in interpreting most process capability ratios
5. Calculate confidence intervals on process capability ratios
6. Know how to conduct and analyze a measurement systems capability (or gauge R & R) experiment
7. Know how to estimate the components of variability in a measurement system
8. Know how to set specifications on components in a system involving interaction components to ensure that overall system requirements are met
9. Estimate the natural limits of a process from a sample of data from that process