9 Cumulative Sum and Exponentially Weighted Moving Average Control Charts

Chapter Outline

Supplemental Material for Chapter 9

The supplemental material is on the textbook Website www.wiley.com/college/montgomery.

Learning Objectives

1. Set up and use cusum control charts for monitoring the process mean
2. Design a cusum control chart for the mean to obtain specific ARL performance
3. Incorporate a fast initial response feature into the cusum control chart
4. Use a combined Shewhart-cusum monitoring scheme
5. Set up and use EWMA control charts for monitoring the process mean
6. Design an EWMA control chart for the mean to obtain specific ARL performance
7. Understand why the EWMA control chart is robust to the assumption of normality
8. Understand the performance advantage of cusum and EWMA control charts relative to Shewhart control charts
9. Set up and use a control chart based on an ordinary (unweighted) moving average

9.1 The Cumulative Sum Control Chart

TABLE 9.1
Data for the Cusum Example

Sample, i	(a) x_i	(b) $x_i - 10$	(c) $C_i = (x_i - 10) + C_{i-1}$
1	9.45	−0.55	−0.55
2	7.99	−2.01	−2.56
3	9.29	−0.71	−3.27
4	11.66	1.66	−1.61
5	12.16	2.16	0.55
6	10.18	0.18	0.73
7	8.04	−1.96	−1.23
8	11.46	1.46	0.23
9	9.20	−0.80	−0.57
10	10.34	0.34	−0.23
11	9.03	−0.97	−1.20
12	11.47	1.47	0.27
13	10.51	0.51	0.78
14	9.40	−0.60	0.18
15	10.08	0.08	0.26
16	9.37	−0.63	−0.37
17	10.62	0.62	0.25
18	10.31	0.31	0.56
19	8.52	−1.48	−0.92
20	10.84	0.84	−0.08
21	10.90	0.90	0.82
22	9.33	−0.67	0.15
23	12.29	2.29	2.44
24	11.50	1.50	3.94
25	10.60	0.60	4.54
26	11.08	1.08	5.62
27	10.38	0.38	6.00
28	11.62	1.62	7.62
29	11.31	1.31	8.93
30	10.52	0.52	9.45

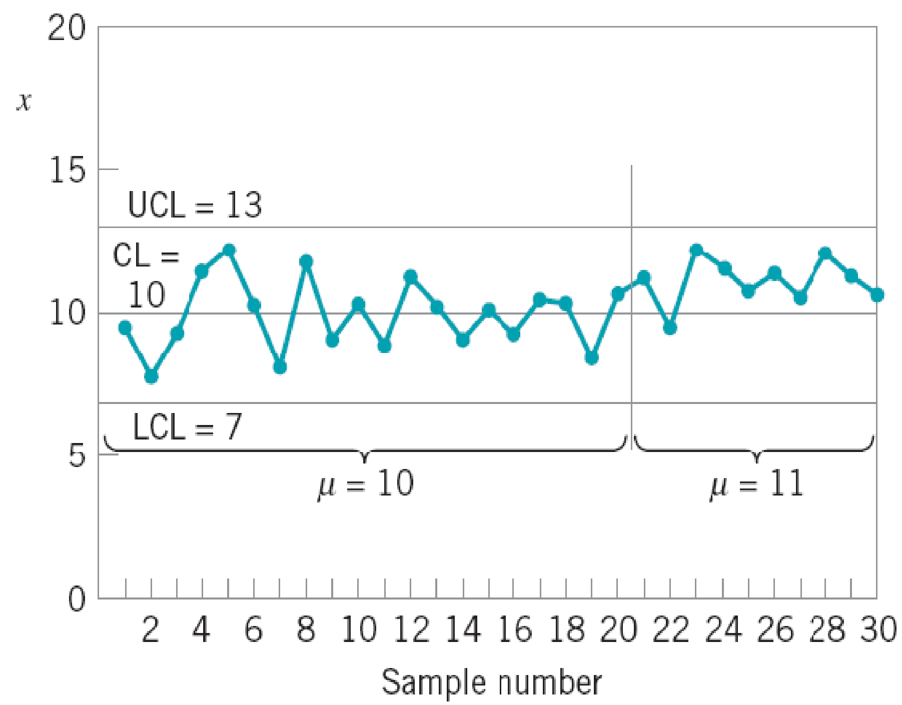


FIGURE 9.1 A Shewhart control chart for the data in Table 9.1.

The cusum chart directly incorporates all the information in the sequence of sample values by plotting the cumulative sums of the deviations of the sample values from a target value. For example, suppose that samples of size $n \geq 1$ are collected, and $\bar{x}_j$ is the average of the jth sample. Then if μ_0 is the target for the process mean, the cumulative sum control chart is formed by plotting the quantity

$$C_i = \sum_{j=1}^{i} \left(\bar{x}_j - \mu_0\right) \tag{9.1}$$

against the sample number i. C_i is called the cumulative sum up to and including the ith sample. Because they combine information from *several* samples, cumulative sum charts are more effective than Shewhart charts for detecting small process shifts. Furthermore, they are particularly effective with samples of size $n = 1$. This makes the cumulative sum control chart a good candidate for use in the chemical and process industries where rational subgroups are frequently of size 1, and in discrete parts manufacturing with automatic measurement of each part and on-line process monitoring directly at the work center.

The Cumulative Sum Control Chart

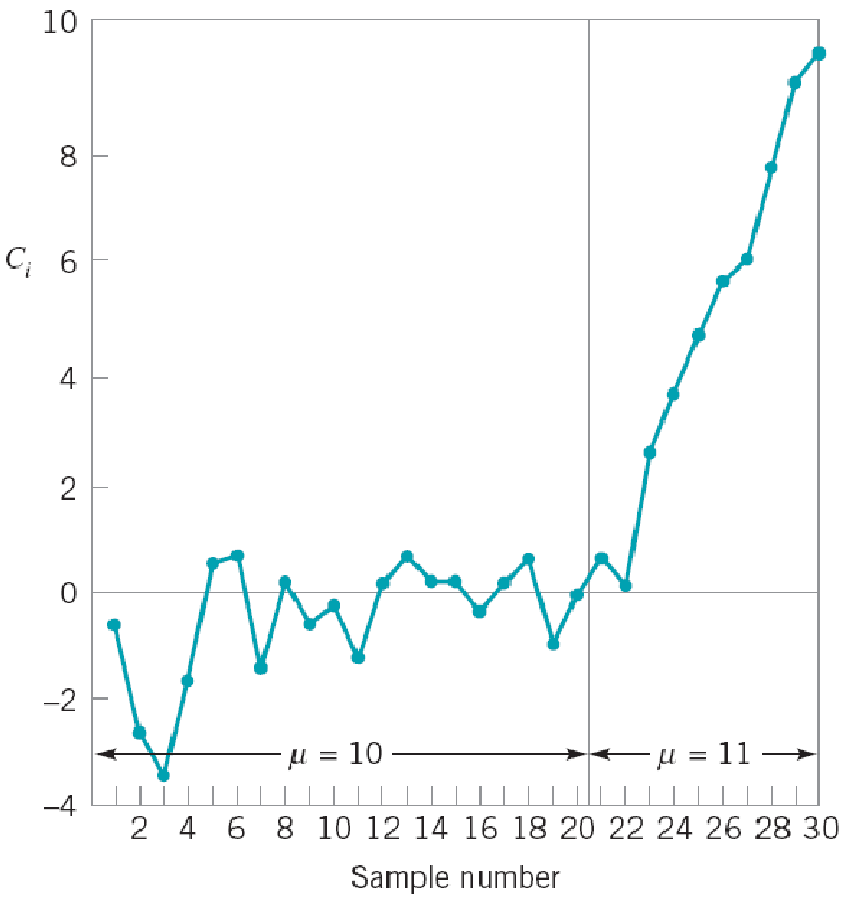


FIGURE 9.2 Plot of the cumulative sum from column (c) of Table 9.1.

$$C_i = \sum_{j=1}^{i} (x_j - 10)$$

$$= (x_i - 10) + \sum_{j=1}^{i-1} (x_j - 10)$$

$$= (x_i - 10) + C_{i-1}$$

The Tabular Cusum

The Tabular Cusum

$$C_i^+ = \max\left[0, x_i - \left(\mu_0 + K\right) + C_{i-1}^+\right] \quad (9.2)$$

$$C_i^- = \max\left[0, \left(\mu_0 - K\right) - x_i + C_{i-1}^-\right] \quad (9.3)$$

where the starting values are $C_0^+ = C_0^- = 0$.

$$K = \frac{\delta}{2}\sigma = \frac{\left|\mu_1 - \mu_0\right|}{2} \quad (9.4)$$

EXAMPLE 9.1 A Tabular Cusum

Set up the tabular cusum using the data from Table 9.1.

SOLUTION

Recall that the target value is $\mu_0 = 10$, the subgroup size is $n = 1$, the process standard deviation is $\sigma = 1$, and suppose that the magnitude of the shift we are interested in detecting is $1.0\sigma = 1.0(1.0) = 1.0$. Therefore, the out-of-control value of the process mean is $\mu_1 = 10 + 1 = 11$. We will use a tabular cusum with $K = \frac{1}{2}$ (because the shift size is 1.0σ and $\sigma = 1$) and $H = 5$

TABLE 9.2
The Tabular Cusum for Example 9.1

		(a)			(b)		
Period i	x_i	$x_i - 10.5$	C_i^+	N^+	$9.5 - x_i$	C_i^-	N^-
1	9.45	−1.05	0	0	0.05	0.05	1
2	7.99	−2.51	0	0	1.51	1.56	2
3	9.29	−1.21	0	0	0.21	1.77	3
4	11.66	1.16	1.16	1	−2.16	0	0
5	12.16	1.66	2.82	2	−2.66	0	0
6	10.18	−0.32	2.50	3	−0.68	0	0
7	8.04	−2.46	0.04	4	1.46	1.46	1
8	11.46	0.96	1.00	5	−1.96	0	0
9	9.20	−1.3	0	0	0.30	0.30	1
10	10.34	−0.16	0	0	−0.84	0	0
11	9.03	−1.47	0	0	0.47	0.47	1
12	11.47	0.97	0.97	1	−1.97	0	0
13	10.51	0.01	0.98	2	−1.01	0	0
14	9.40	−1.10	0	0	0.10	0.10	1
15	10.08	−0.42	0	0	−0.58	0	0
16	9.37	−1.13	0	0	0.13	0.13	1
17	10.62	0.12	0.12	1	−1.12	0	0
18	10.31	−0.19	0	0	−0.81	0	0
19	8.52	−1.98	0	0	0.98	0.98	1
20	10.84	0.34	0.34	1	−1.34	0	0
21	10.90	0.40	0.74	2	−1.40	0	0
22	9.33	−1.17	0	0	0.17	0.17	1
23	12.29	1.79	1.79	1	−2.79	0	0
24	11.50	1.00	2.79	2	−2.00	0	0
25	10.60	0.10	2.89	3	−1.10	0	0
26	11.08	0.58	3.47	4	−1.58	0	0
27	10.38	−0.12	3.35	5	−0.88	0	0
28	11.62	1.12	4.47	6	−2.12	0	0
29	11.31	0.81	5.28	7	−1.81	0	0
30	10.52	0.02	5.30	8	−1.02	0	0

(because the recommended value of the decision interval is $H = 5\sigma = 5(1) = 5$).

Table 9.2 presents the tabular cusum scheme. To illustrate the calculations, consider period 1. The equations for C_i^+ and C_i^- are

$$C_1^+ = \max\left[0, x_1 - 10.5 + C_0^+\right]$$

and

$$C_1^- = \max\left[0, 9.5 - x_1 + C_0^-\right]$$

since $K = 0.5$ and $\mu_0 = 10$. Now $x_1 = 9.45$, so since $C_0^+ = C_0^- = 0$,

$$C_1^+ = \max[0, 9.45 - 10.5 + 0] = 0$$

and

$$C_1^- = \max[0, 9.5 - 9.45 + 0] = 0.05$$

For period 2, we would use

$$C_2^+ = \max\left[0, x_2 - 10.5 + C_1^+\right] = \max\left[0, x_2 - 10.5 + 0\right]$$

and

$$C_2^- = \max\left[0, 9.5 - x_2 + C_1^-\right] = \max\left[0, 9.5 - x_2 + 0.05\right]$$

Since $x_2 = 7.99$, we obtain

$$C_2^+ = \max[0, 7.99 - 10.5 + 0] = 0$$

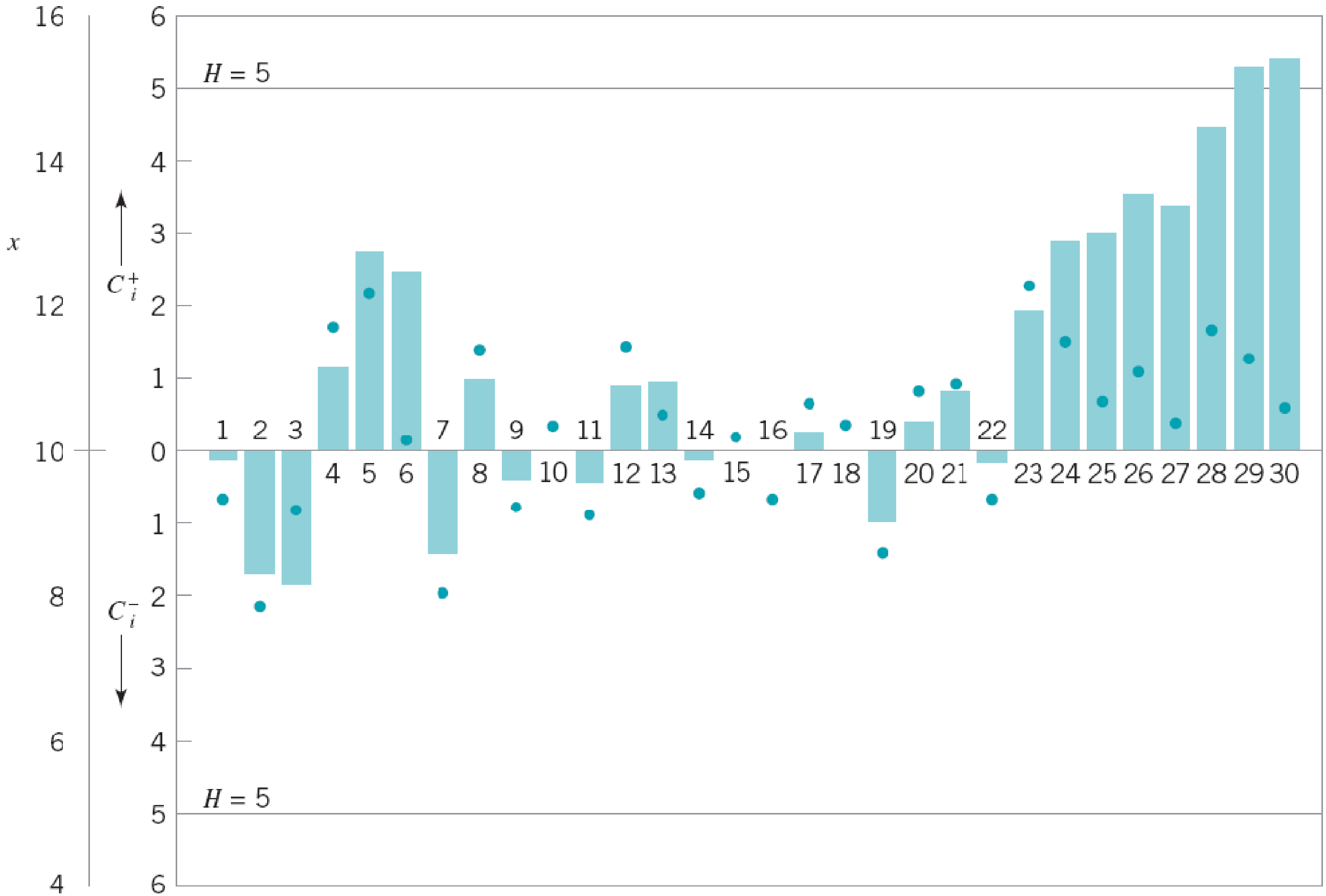
and

$$C_2^- = \max[0, 9.5 - 7.99 + 0.05] = 1.56$$

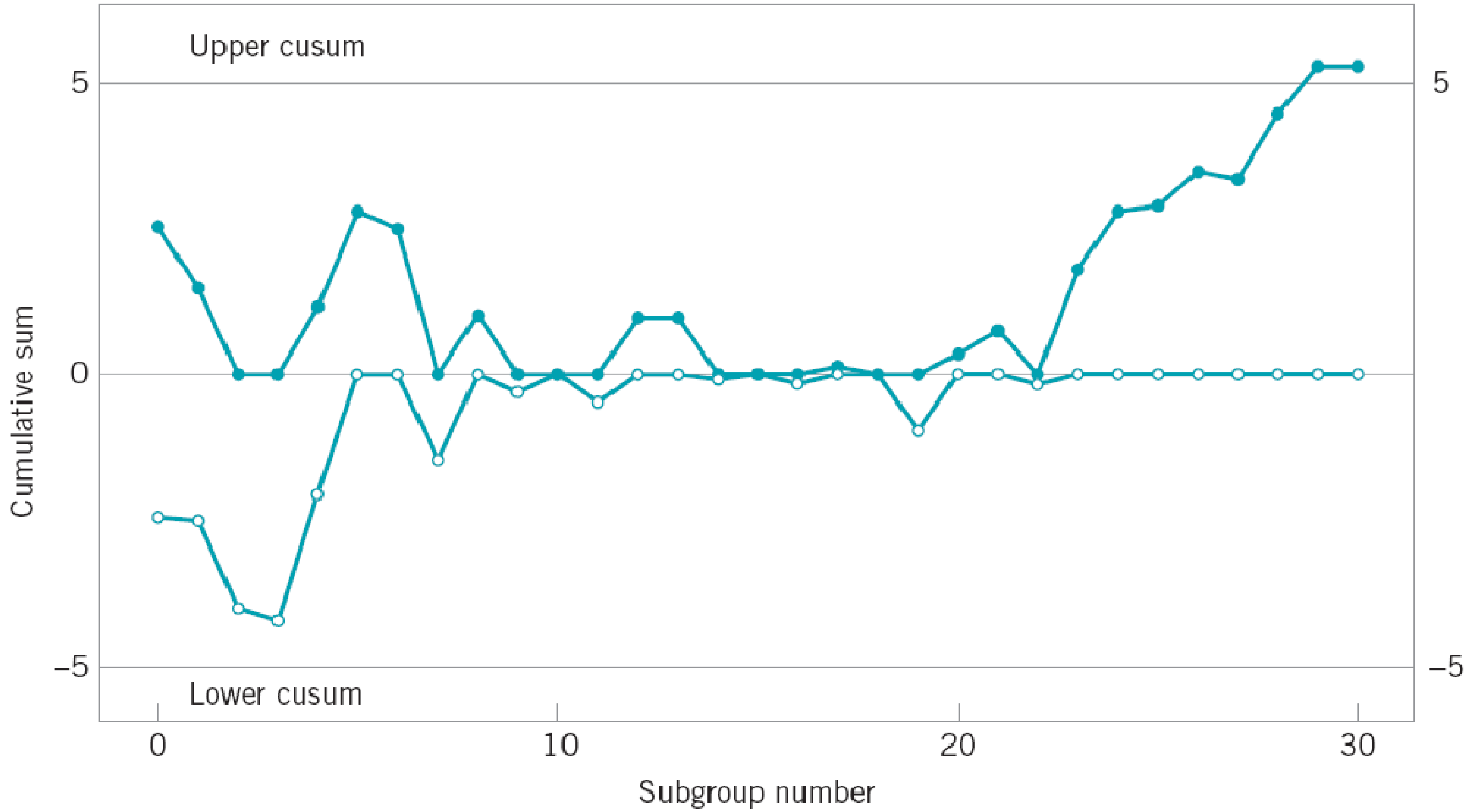
Panels (a) and (b) of Table 9.2 summarize the remaining calculations. The quantities N^+ and N^- in Table 9.2 indicate the number of consecutive periods that the cusums C_i^+ or C_i^- have been nonzero.

The cusum calculations in Table 9.2 show that the upper-side cusum at period 29 is $C_{29}^+ = 5.28$. Since this is the first period at which $C_i^+ > H = 5$, we would conclude that the process is out of control at that point. The tabular cusum also indicates when the shift probably occurred. The counter N^+ records the number of consecutive periods since the upper-side cusum C_i^+ rose above the value of zero. Since $N^+ = 7$ at period 29, we would conclude that the process was last in control at period $29 - 7 = 22$, so the shift likely occurred between periods 22 and 23.

Cusum Status Chart (Figure 9.3a)



MINITAB Version of Cusum Status Chart



$$C_i^- = \min\left(0, x_i - \mu_0 + k + C_{i-1}^-\right)$$ ← Minitab calculates the lower Cusum this way

In situations where an adjustment to some manipulatable variable is required in order to bring the process back to the target value μ_0, it may be helpful to have an estimate of the new process mean following the shift. This can be computed from

$$\hat{\mu} = \begin{cases} \mu_0 + K + \dfrac{C_i^+}{N^+}, & \text{if } C_i^+ > H \\ \mu_0 - K - \dfrac{C_i^-}{N^-}, & \text{if } C_i^- > H \end{cases} \quad (9.5)$$

To illustrate the use of equation (9.5), consider the cusum in period 29 with $C_{29}^+ = 5.28$. From equation (9.5), we would estimate the new process average as

$$\begin{aligned} \hat{\mu} &= \mu_0 + K + \frac{C_{29}^+}{N^+} \\ &= 10.0 + 0.5 + \frac{5.28}{7} \\ &= 11.25 \end{aligned}$$

Recommendations for Cusum Design

■ **TABLE 9.3**

ARL Performance of the Tabular Cusum with $k = \frac{1}{2}$ and $h = 4$ or $h = 5$

Shift in Mean (multiple of σ)	$h = 4$	$h = 5$
0	168	465
0.25	74.2	139
0.50	26.6	38.0
0.75	13.3	17.0
1.00	8.38	10.4
1.50	4.75	5.75
2.00	3.34	4.01
2.50	2.62	3.11
3.00	2.19	2.57
4.00	1.71	2.01

■ TABLE 9.4

Values of k and the Corresponding Values of h That Give $ARL_0 = 370$ for the Two-Sided Tabular Cusum [from Hawkins (1993a)]

k	0.25	0.5	0.75	1.0	1.25	1.5
h	8.01	4.77	3.34	2.52	1.99	1.61

The Standardized Cusum

Many users of the cusum prefer to standardize the variable x_i before performing the calculations. Let

$$y_i = \frac{x_i - \mu_0}{\sigma} \tag{9.8}$$

be the standardized value of x_i. Then the standardized cusums are defined as follows.

The Standardized Two-Sided Cusum

$$C_i^+ = \max\left[0, y_i - k + C_{i-1}^+\right] \tag{9.9}$$

$$C_i^- = \max\left[0, -k - y_i + C_{i-1}^-\right] \tag{9.10}$$

There are two advantages to standardizing the cusum. First, many cusum charts can now have the same values of k and h, and the choices of these parameters are not scale dependent (that is, they do not depend on σ). Second, a standardized cusum leads naturally to a cusum for controlling variability, as we will see in Section 9.1.8.

Improving Cusum Performance for Large Shifts: The Combined Shewhart-Cusum Scheme

■ **TABLE 9.5**

ARL Values for Some Modifications of the Basic Cusum with $k = \frac{1}{2}$ and $h = 5$ (If subgroups of size $n > 1$ are used, then $\sigma = \sigma_{\bar{x}} = \sigma/\sqrt{n}$)

Shift in Mean (multiple of σ)	(a) Basic Cusum	(b) Cusum-Shewhart (Shewhart limits at 3.5σ)	(c) Cusum with FIR	(d) FIR Cusum-Shewhart (Shewhart limits at 3.5σ)
0	465	391	430	360
0.25	139	130.9	122	113.9
0.50	38.0	37.20	28.7	28.1
0.75	17.0	16.80	11.2	11.2
1.00	10.4	10.20	6.35	6.32
1.50	5.75	5.58	3.37	3.37
2.00	4.01	3.77	2.36	2.36
2.50	3.11	2.77	1.86	1.86
3.00	2.57	2.10	1.54	1.54
4.00	2.01	1.34	1.16	1.16

The Fast Initial Response (FIR) Cusum

■ TABLE 9.6

A Cusum with a Headstart, Process Mean Equal to 100

		(a)			(b)		
Period i	x_i	$x_i - 103$	C_i^+	N^+	$97 - x_i$	C_i^-	N^-
1	102	−1	5	1	−5	1	1
2	97	−6	0	0	0	1	2
3	104	1	1	1	−7	0	0
4	93	−6	0	0	4	4	1
5	100	−3	0	0	−3	1	2
6	105	2	2	1	−8	0	0
7	96	−7	0	0	1	1	1
8	98	−5	0	0	−1	0	0
9	105	2	2	1	−8	0	0
10	99	−4	0	0	−2	0	0

$K = 3$, $H = 12$, headstart = $H/2 = 6$

$$C_i^+ = \max\left[0, x_1 - 103 + C_0^+\right] = \max[0, 102 - 103 + 6] = 5$$

$$C_i = \max\left[0, 97 - x_1 + C_0^-\right] = \max[0, 97 - 102 + 6] = 1$$

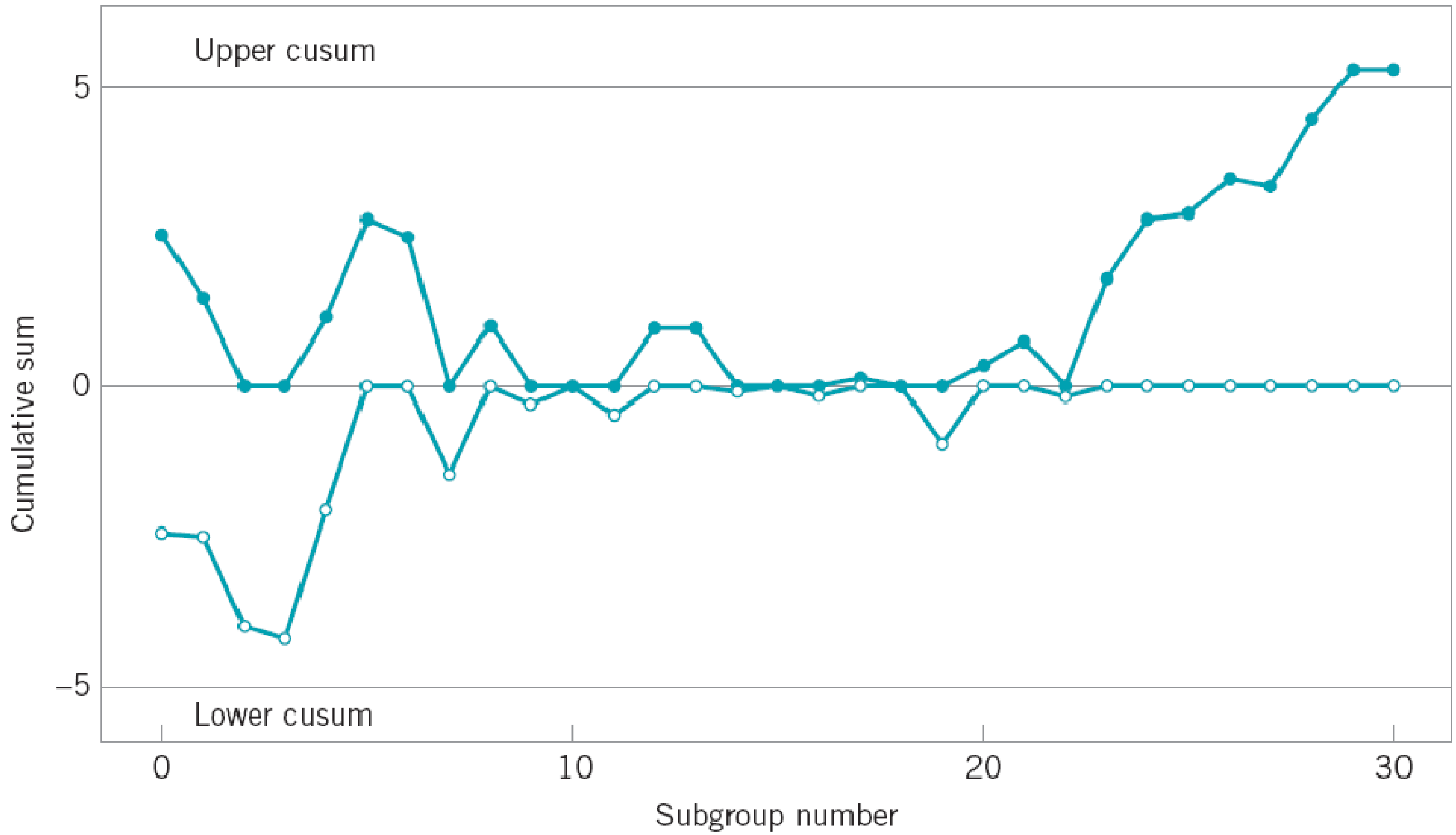
■ **TABLE 9.7**

A Cusum with a Headstart, Process Mean Equal to 105

		(a)			(b)		
Period i	x_i	$x_i - 103$	C_i^+	N^+	$97 - x_i$	C_i^-	N^-
1	107	4	10	1	−10	0	0
2	102	−1	9	2	−5	0	0
3	109	6	15	3	−12	0	0
4	98	−5	10	4	−1	0	0
5	105	2	12	5	−8	0	0
6	110	7	19	6	−13	0	0
7	101	−2	17	7	−4	0	0
8	103	0	17	8	−6	0	0
9	110	7	24	9	−13	0	0
10	104	1	25	10	−7	0	0

H = 12 implies that the cusum signals at sample 3

Without the headstart, it would not signal until sample 6



■ **FIGURE 9.4** A Minitab cusum status chart for the data in Table 9.1 illustrating the fast initial response or headstart feature.

More on Cusums

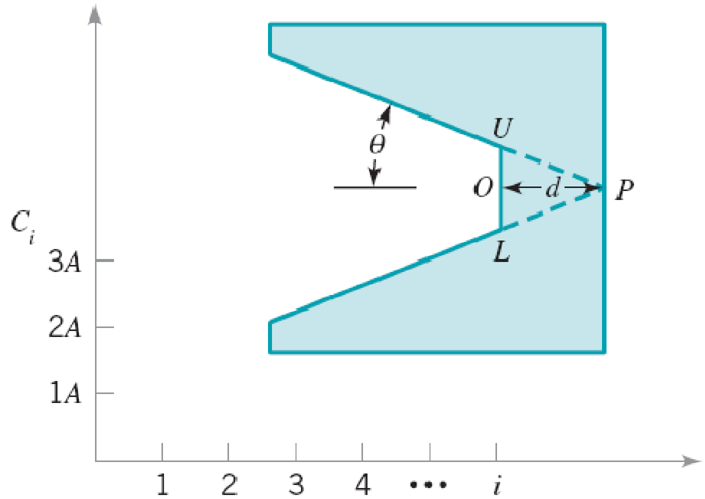
- Cusums are often used to determine if a process has shifted off a specified target because it is easy to calculate the required adjustment
- One-sided cusums are often useful
- Cusums can also be used to monitor variability
- Cusums are available for other sample statistics (ranges, standard deviations, counts, proportions)
- Rational subgroups and cusums

Although we have given the development of the tabular cusum for the case of individual observations ($n = 1$), it is easily extended to the case of averages of rational subgroups where the sample size $n > 1$. Simply replace x_i by $\bar{x}_i$ (the sample or subgroup average) in the above formulas, and replace σ with $\sigma_{\bar{x}} = \sigma/\sqrt{n}$.

With Shewhart charts, the use of averages of rational subgroups substantially improves control chart performance. However, this does not always happen with the cusum. If, for example, you have a choice of taking a sample of size $n = 1$ every half hour or a sample consisting of a rational subgroup of size $n = 5$ every 2.5 hours (not that both choices have the same sampling intensity), the cusum will often work best with the choice of $n = 1$ every half hour. For more discussion of this, see Hawkins and Olwell (1998). Only if there is some significant economy of scale or some other valid reason for taking samples of size greater than unity should one consider using $n > 1$ with the cusum.

One practical reason for using rational subgroups of size $n > 1$ is that we could now set up a cusum on the sample **variance** and use it to monitor **process variability.** Cusums for variances are discussed in detail by Hawkins and Olwell (1998); the paper by Chang and Gan (1995) is also recommended.

The Cusum V-Mask



■ FIGURE 9.5 A typical V-mask.

The tabular cusum and the V-mask scheme are equivalent if

$$k = A \tan \theta \tag{9.17}$$

and

$$h = A\ d \tan(\theta) = dk \tag{9.18}$$

We strongly advise **against using the V-mask procedure.** Some of the disadvantages and problems associated with this scheme are as follows:

1. The headstart feature, which is very useful in practice, cannot be implemented with the V-mask.
2. It is sometimes difficult to determine how far backward the arms of the V-mask should extend, thereby making interpretation difficult for the practitioner.
3. Perhaps the biggest problem with the V-mask is the ambiguity associated with α and β in the Johnson design procedure.

■ TABLE 9.8

Actual Values of ARL_0 for a V-Mask Scheme Designed Using Johnson's Method [Adapted from Table 2 in Woodall and Adams (1993)]

	Values of α [Desired Value of $ARL_0 = 1/(2\alpha)$]	
Shift to Be Detected, δ	**0.00135 (370)**	**0.001 (500)**
1.0	2350.6	3184.5
2.0	1804.5	2435.8
3.0	2194.8	2975.4

Adams, Lowry, and Woodall (1992) point out that defining 2α as the probability of a false alarm is incorrect. Essentially, 2α cannot be the probability of a false alarm on any single sample, because this probability changes over time on the cusum, nor can 2α be the probability of eventually obtaining a false alarm (this probability is, of course, 1). In fact, 2α must be the long-run proportion of observations resulting in false alarms. If this is so, then the in-control ARL should be $ARL_0 = 1/(2\alpha)$. However, Johnson's design method produces values of ARL_0 that are substantially larger than $1/(2\alpha)$.

9.1.12 The Self-Starting Cusum

The cusum is typically used as a phase II procedure; that is, it is applied to monitor a process that has already been through the phase I process and most of the large assignable causes have been removed. In phase II, we typically assume that the process parameters are reasonably well estimated. In practice, this turns out to be a fairly important assumption, as using estimates of the parameters instead of the true values has an effect on the average run length performance of the control chart [this was discussed in Chapter 4; also see the review paper by Jensen et al. (2006)]. Control charts that are designed to detect small shifts are particularly sensitive to this assumption, including the cusum. A Shewhart control chart with the Western Electric rules also would be very sensitive to the estimates of the process parameters. One solution to this is to use a large sample of phase I data to estimate the parameters.

An alternative approach for the cusum is to use a **self-starting cusum** procedure due to Hawkins (1987). The self-starting cusum for the mean of a normally distributed random variable is easy to implement. It can be applied immediately without any need for a phase I sample to estimate the process parameters, in this case the mean μ and the variance σ^2.

Let $\bar{x}_n$ be the average of the first n observations and let $w_n = \sum_{i=1}^{n} (x_i - \bar{x}_n)^2$ be the sum of squared deviations from the average of those observations. Convenient computing formulas to update these quantities after each new observation are

$$\bar{x}_n = \bar{x}_{n-1} + \frac{x_n - \bar{x}_{n-1}}{n}$$

$$w_n = w_{n-1} + \frac{(n-1)(x_n - \bar{x}_{n-1})^2}{n}$$

The sample variance of the first n observations is $s_n^2 = w_n/(n-1)$. Standardize each successive new process observation using

$$T_n = \frac{x_n - \bar{x}_{n-1}}{s_{n-1}}$$

for the case where n is greater than or equal to 3. If the observations are normally distributed, the distribution of $\sqrt{\frac{n-1}{n}}T_n$ is a t distribution with $n-1$ degrees of freedom. The cumulative distribution of T_n is

$$P(T_n \leq t) = F_{n-2}\left(t\sqrt{\frac{n-1}{n}}\right)$$

where F_n is the cumulative t distribution with $n-1$ degrees of freedom. It turns out that if the tail area for any continuous random variable is converted to a normal ordinate we obtain a new random variable that is distributed exactly as a standard normal random variable. That is, if Φ^{-1} is the inverse normal cumulative distribution, then the transformation

$$U_n = \Phi^{-1}[F_{n-2}(a_n T_n)] \quad \text{where } a_n = \sqrt{\frac{n-1}{n}}$$

converts the cusum quantity T_n into a standard normal random variable. It turns out that the values of U_n are statistically independent (this isn't obvious, because successive values of U_n share the same data points), so one can plot all values of U_n for $n \geq 3$ on a N(0, 1) cusum. This nicely avoids the problem of using a large sample of phase I data to estimate the process parameters for a conventional cusum.

TABLE 9.9

Calculations for a Self-Starting Cusum

n	x_n	$\bar{x}_n$	w_n	S_n	T_n	a_nT_n	$F_{n-2}(a_nT_n)$	U_n
1	9.45	9.45	0	—	—	—	—	—
2	7.99	8.72	1.07	1.03	—	—	—	—
3	9.29	8.91	1.25	0.80	0.55	0.45	0.6346	0.34406
4	11.66	9.60	6.92	1.52	3.44	2.98	0.9517	1.66157
5	12.16	10.11	12.16	1.75	1.68	1.50	0.8847	1.19881
6	10.18	10.12	12.16	1.56	0.04	0.04	0.5152	0.03811
7	8.04	9.82	15.87	1.63	−1.33	−1.23	0.1324	−1.11512
8	11.46	10.03	18.22	1.62	1.01	0.94	0.8107	0.88048

9.2 The Exponentially Weighted Moving Average Control Chart

The EWMA is

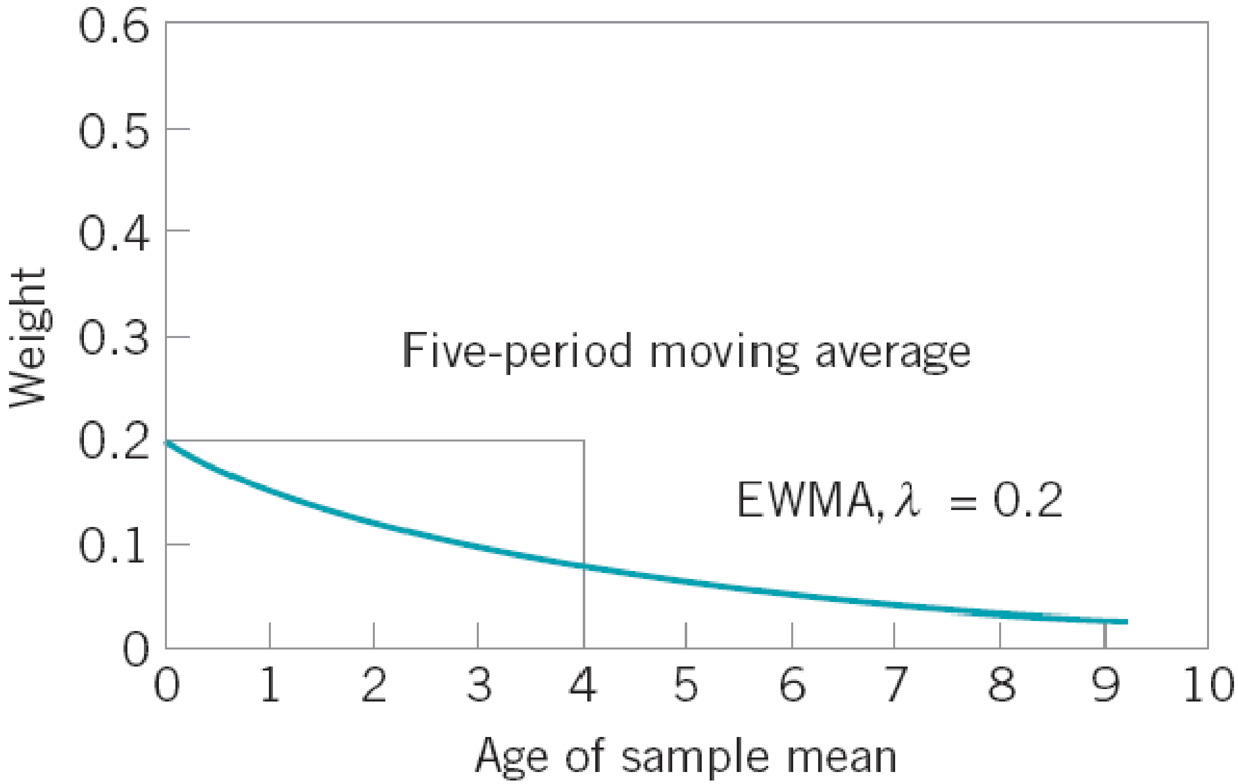
$$z_i = \lambda x_i + (1 - \lambda) z_{i-1} \qquad (9.22)$$

where $0 < \lambda \leq 1$ is a constant and the starting value (required with the first sample at $i = 1$) is the process target, so that

$$z_0 = \mu_0$$

Sometimes the average of preliminary data is used as the starting value of the EWMA, so that $z_0 = \bar{x}$.

$$z_i = \lambda \sum_{j=0}^{i-1} (1-\lambda)^j x_{i-j} + (1-\lambda)^i z_0$$



■ FIGURE 9.6 Weights of past sample means.

$$\sigma_{z_i}^2 = \sigma^2\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]$$

The EWMA Control Chart

$$\text{UCL} = \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2i}\right]} \tag{9.25}$$

$$\text{Center line} = \mu_0$$

$$\text{LCL} = \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2i}\right]} \tag{9.26}$$

$$\text{UCL} = \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}} \tag{9.27}$$

$$\text{LCL} = \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}} \tag{9.28}$$

Steady-state control limits

EXAMPLE 9.2 Constructing an EWMA Control Chart

Set up an EWMA control chart with $\lambda = 0.10$ and $L = 2.7$ to the data in Table 9.1.

SOLUTION

Recall that the target value of the mean is $\mu_0 = 10$ and the standard deviation is $\sigma = 1$. The calculations for the EWMA control chart are summarized in Table 9.10, and the control chart (from Minitab) is shown in Fig. 9.7.

To illustrate the calculations, consider the first observation, $x_1 = 9.45$. The first value of the EWMA is

$$\begin{aligned} z_1 &= \lambda x_1 + (1-\lambda) z_0 \\ &= 0.1(9.45) + 0.9(10) \\ &= 9.945 \end{aligned}$$

■ **TABLE 9.10**
EWMA Calculations for Example 9.2

Subgroup, i	* = Beyond Limits x_i	EWMA, z_i	Subgroup, i	* = Beyond Limits x_i	EWMA, z_i
1	9.45	9.945	16	9.37	9.98426
2	7.99	9.7495	17	10.62	10.0478
3	9.29	9.70355	18	10.31	10.074
4	11.66	9.8992	19	8.52	9.91864
5	12.16	10.1253	20	10.84	10.0108
6	10.18	10.1307	21	10.9	10.0997
7	8.04	9.92167	22	9.33	10.0227
8	11.46	10.0755	23	12.29	10.2495
9	9.2	9.98796	24	11.5	10.3745
10	10.34	10.0232	25	10.6	10.3971
11	9.03	9.92384	26	11.08	10.4654
12	11.47	10.0785	27	10.38	10.4568
13	10.51	10.1216	28	11.62	10.5731
14	9.4	10.0495	29	11.31	10.6468*
15	10.08	10.0525	30	10.52	10.6341*

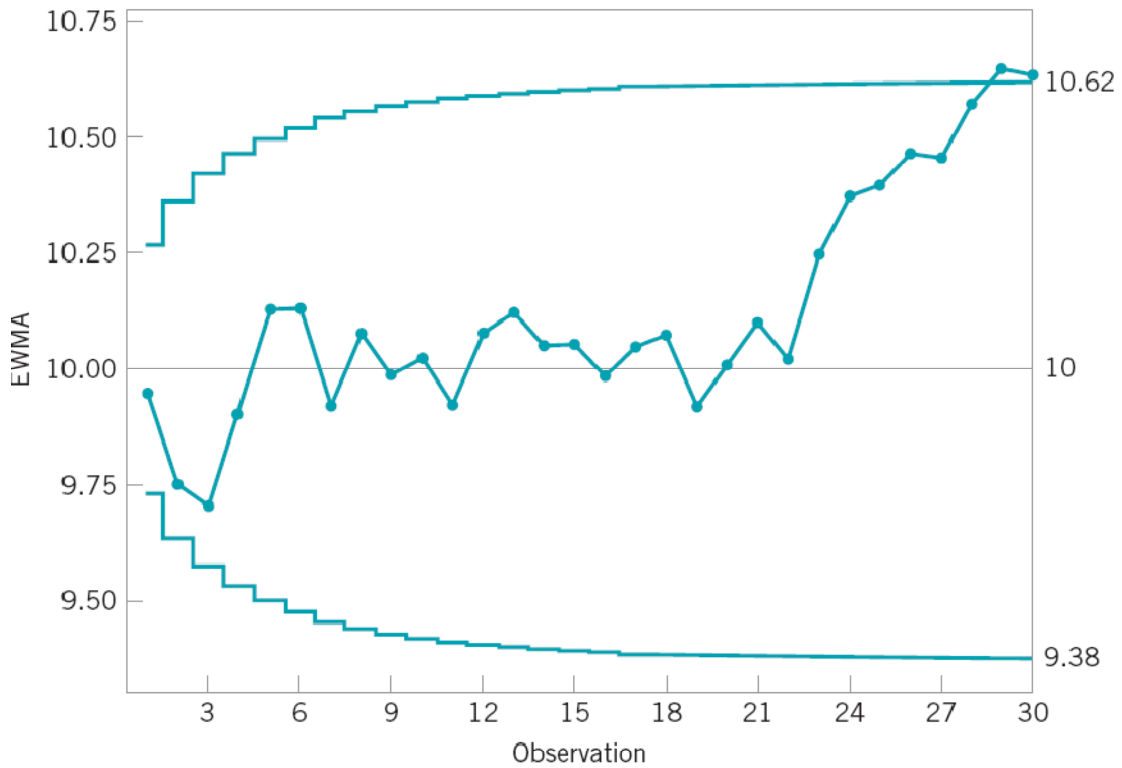


FIGURE 9.7 The EWMA control chart for Example 9.2.

Therefore, $z_1 = 9.945$ is the first value plotted on the control chart in Fig. 9.7. The second value of the EWMA is

$$\begin{aligned} z_2 &= \lambda x_2 + (1-\lambda) z_1 \\ &= 0.1(7.99) + 0.9(9.945) \\ &= 9.7495 \end{aligned}$$

The other values of the EWMA statistic are computed similarly.

The control limits in Fig. 9.7 are found using equations (9.25) and (9.26). For period $i = 1$,

$$\begin{aligned} \text{UCL} &= \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2i}\right]} \\ &= 10 + 2.7(1)\sqrt{\frac{0.1}{(2-0.1)}\left[1-(1-0.1)^{2(1)}\right]} \\ &= 10.27 \end{aligned}$$

and

$$\begin{aligned} \text{LCL} &= \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2i}\right]} \\ &= 10 - 2.7(1)\sqrt{\frac{0.1}{(2-0.1)}\left[1-(1-0.1)^{2(1)}\right]} \\ &= 9.73 \end{aligned}$$

For period 2, the limits are

$$\begin{aligned} \text{UCL} &= \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2i}\right]} \\ &= 10 + 2.7(1)\sqrt{\frac{0.1}{(2-0.1)}\left[1-(1-0.1)^{2(2)}\right]} \\ &= 10.36 \end{aligned}$$

and

$$\begin{aligned} \text{LCL} &= \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2i}\right]} \\ &= 10 - 2.7(1)\sqrt{\frac{0.1}{(2-0.1)}\left[1-(1-0.1)^{2(2)}\right]} \\ &= 9.64 \end{aligned}$$

Note from Fig. 9.7 that the control limits increase in width as i increases from $i = 1, 2, \ldots$, until they stabilize at the steady-state values given by equations (9.27) and (9.28)

$$\begin{aligned} \text{UCL} &= \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}} \\ &= 10 + 2.7(1)\sqrt{\frac{0.1}{(2-0.1)}} \\ &= 10.62 \end{aligned}$$

and

$$\begin{aligned} \text{LCL} &= \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}} \\ &= 10 - 2.7(1)\sqrt{\frac{0.1}{(2-0.1)}} \\ &= 9.38 \end{aligned}$$

The EWMA control chart in Fig. 9.7 signals at observation 28, so we would conclude that the process is out of control.

Design of the EWMA

■ **TABLE 9.11**

Average Run Lengths for Several EWMA Control Schemes [Adapted from Lucas and Saccucci (1990)]

Shift in Mean (multiple of σ)	$L = 3.054$ $\lambda = 0.40$	2.998 0.25	2.962 0.20	2.814 0.10	2.615 0.05
0	500	500	500	500	500
0.25	224	170	150	106	84.1
0.50	71.2	48.2	41.8	31.3	28.8
0.75	28.4	20.1	18.2	15.9	16.4
1.00	14.3	11.1	10.5	10.3	11.4
1.50	5.9	5.5	5.5	6.1	7.1
2.00	3.5	3.6	3.7	4.4	5.2
2.50	2.5	2.7	2.9	3.4	4.2
3.00	2.0	2.3	2.4	2.9	3.5
4.00	1.4	1.7	1.9	2.2	2.7

There is one potential concern about an EWMA with a small value of λ. If the value of the EWMA is on one side of the center line when a shift in the mean in the opposite direction occurs, it could take the EWMA several periods to react to the shift, because the small λ does not weight the new data very heavily. This is called the **inertia effect.** It can reduce the effectiveness of the EWMA in shift detection.

Woodall and Mahmoud (2005) have investigated the inertial properties of several different types of control charts. They define the **signal resistance of a control chart** to be the largest standardized deviation of the sample mean from the target or in-control value not leading to an immediate out-of-control signal. For a Shewhart $\bar{x}$ chart, the signal resistance is $SR(\bar{x}) = L$, the multiplier used to obtain the control limits. Thus the signal resistance is constant. For the EWMA control chart, the signal resistance is

$$SR(EWMA) = \frac{L\sqrt{\frac{\lambda}{2-\lambda}} - (1-\lambda)w}{\lambda}$$

where w is the value of the EWMA statistic. For the EWMA, the maximum value of the signal resistance averaged over all values of the EWMA statistic is $L\sqrt{(2-\lambda)/\lambda}$, if the chart has the asymptotic limits. These results apply for any sample size, as they are given in terms of shifts expressed as multiples of the standard error.

Clearly the signal resistance of the EWMA control chart depends on the value chosen for λ, with smaller values leading to larger values of the maximum signal resistance. This is in a sense unfortunate, because we almost always want to use the EWMA with a small value of λ as this results in good ARL performance in detecting small shifts. As we will see in Section 9.2.3, small values of λ are also desirable because they make the EWMA chart quite insensitive to normality of the process data. Woodall and Mahmoud (2005) recommend always using a Shewhart chart in conjunction with an EWMA (especially if λ is small) as one way to counteract the signal resistance.

Like the cusum, the EWMA performs well against small shifts but does not react to large shifts as quickly as the Shewhart chart. A good way to further improve the sensitivity of the procedure to large shifts without sacrificing the ability to detect small shifts quickly is to combine a Shewhart chart with the EWMA. These combined Shewhart-EWMA control procedures are effective against both large and small shifts. When using such schemes, we have found it helpful to use slightly wider than usual limits on the Shewhart chart (say, 3.25-sigma, or even 3.5-sigma). It is also possible to plot *both* x_i (or $\overline{x}_i$) and the EWMA statistic z_i on the same control chart along with both the Shewhart and EWMA limits. This produces one chart for the combined control procedure that operators quickly become adept at interpreting. When the plots are computer generated, different colors or plotting symbols can be used for the two sets of control limits and statistics.

Robustness of EWMA to Non-normal Process Data

■ TABLE 9.12

In-Control ARLs for the EWMA and the Individuals Control Charts for Various Gamma Distributions

	EWMA			Shewhart
λ	**0.05**	**0.1**	**0.2**	**1**
L	**2.492**	**2.703**	**2.86**	**3.00**
Normal	370.4	370.8	370.5	370.4
Gam(4, 1)	372	341	259	97
Gam(3, 1)	372	332	238	85
Gam(2, 1)	372	315	208	71
Gam(1, 1)	369	274	163	55
Gam(0.5, 1)	357	229	131	45

TABLE 9.13

In-Control ARLs for the EWMA and the Individuals Control Charts for Various t Distributions

	EWMA			Shewhart
λ	**0.05**	**0.1**	**0.2**	**1**
L	**2.492**	**2.703**	**2.86**	**3.00**
Normal	370.4	370.8	370.5	370.4
t_{50}	369	365	353	283
t_{40}	369	363	348	266
t_{30}	368	361	341	242
t_{20}	367	355	325	204
t_{15}	365	349	310	176
t_{10}	361	335	280	137
t_8	358	324	259	117
t_6	351	305	229	96
t_4	343	274	188	76

Rational Subgroups

The EWMA control chart is often used with individual measurements. However, if rational subgroups of size $n > 1$ are taken, then simply replace x_i with $\bar{x}_i$ and σ with $\sigma_{\bar{x}} = \sigma/\sqrt{n}$ in the previous equations.

Extensions of the EWMA

- Fast initial response feature
- Monitoring variability
- Monitoring count data
- The EWMA as a predictor of process level

9.3 The Moving Average Control Chart

Suppose that individual observations have been collected, and let $x_1, x_2, \ldots$ denote these observations. The moving average of span w at time i is defined as

$$M_i = \frac{x_i + x_{i-1} + \cdots + x_{i-w+1}}{w} \tag{9.37}$$

That is, at time period i, the oldest observation in the moving average set is dropped and the newest one added to the set. The variance of the moving average M_i is

$$V(M_i) = \frac{1}{w^2} \sum_{j=i-w+1}^{i} V(x_j) = \frac{1}{w^2} \sum_{j=i-w+1}^{i} \sigma^2 = \frac{\sigma^2}{w} \tag{9.38}$$

Therefore, if μ_0 denotes the target value of the mean used as the center line of the control chart, then the three-sigma control limits for M_i are

$$\text{UCL} = \mu_0 + \frac{3\sigma}{\sqrt{w}} \tag{9.39}$$

and

$$\text{LCL} = \mu_0 - \frac{3\sigma}{\sqrt{w}} \tag{9.40}$$

EXAMPLE 9.3 A Moving Average Control Chart

Set up a moving average control chart for the data in Table 9.1, using $w = 5$.

SOLUTION

The observations x_i for periods $1 \le i \le 30$ are shown in Table 9.14. The statistic plotted on the moving average control chart will be

$$M_i = \frac{x_i + x_{i-1} + \cdots + x_{i-4}}{5}$$

for periods $i \ge 5$. For time periods $i < 5$ the average of the observations for periods $1, 2, \ldots, i$ is plotted. The values of these moving averages are shown in Table 9.14.

■ TABLE 9.14

Moving Average Chart for Example 9.3

Observation, i	x_i	M_i	Observation, i	x_i	M_i
1	9.45	9.45	16	9.37	10.166
2	7.99	8.72	17	10.62	9.996
3	9.29	8.91	18	10.31	9.956
4	11.66	9.5975	19	8.52	9.78
5	12.16	10.11	20	10.84	9.932
6	10.18	10.256	21	10.9	10.238
7	8.04	10.266	22	9.33	9.98
8	11.46	10.7	23	12.29	10.376
9	9.2	10.208	24	11.5	10.972
10	10.34	9.844	25	10.6	10.924
11	9.03	9.614	26	11.08	10.96
12	11.47	10.3	27	10.38	11.17
13	10.51	10.11	28	11.62	11.036
14	9.4	10.15	29	11.31	10.998
15	10.08	10.098	30	10.52	10.982

The control limits for the moving average control chart may be easily obtained from equations (9.39) and (9.40). Since we know that $\sigma = 1.0$, then

$$\text{UCL} = \mu_0 + \frac{3\sigma}{\sqrt{w}} = 10 + \frac{3(1.0)}{\sqrt{5}} = 11.34$$

and

$$\text{LCL} = \mu_0 - \frac{3\sigma}{\sqrt{w}} = 10 - \frac{3(1.0)}{\sqrt{5}} = 8.66$$

The control limits for M_i apply for periods $i \geq 5$. For periods $0 < i < 5$, the control limits are given by $\mu_0 \pm 3\sigma/\sqrt{i}$. An alternative procedure that avoids using special control limits for periods $i < w$ is to use an ordinary Shewhart chart until at least w sample means have been obtained.

The moving average control chart is shown in Fig. 9.8. No points exceed the control limits. Note that for the initial periods $i < w$ the control limits are wider than their final steady-state value. Moving averages that are less than w periods apart are highly correlated, which often complicates interpreting patterns on the control chart. This is easily seen by examining Fig. 9.8.

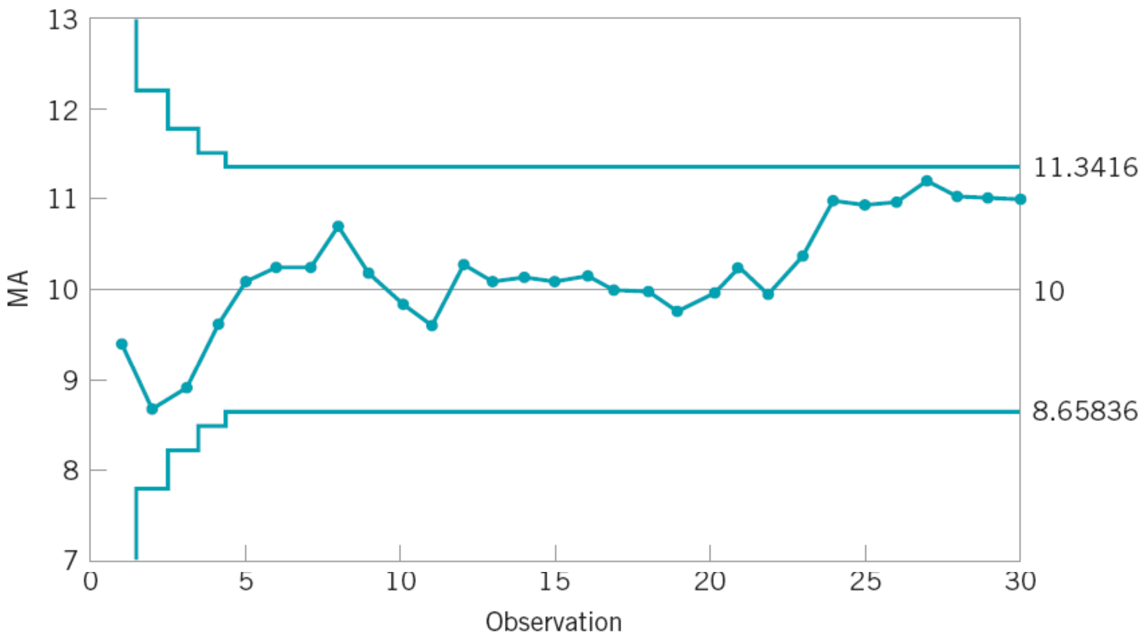


FIGURE 9.8 Moving average control chart with $w = 5$, Example 9.3.

Important Terms and Concepts

ARL calculations for the cusum
Average run length
Combined cusum-Shewhart procedures
Cusum control chart
Cusum status chart
Decision interval
Design of a cusum
Design of an EWMA control chart
EWMA control chart
Fast initial response (FIR) or headstart feature for a cusum
Fast initial response (FIR) or headstart feature for an EWMA
Moving average control chart
One-sided cusums
Poisson EWMA
Reference value
Robustness of the EWMA to normality
Scale cusum
Self-starting cusum
Signal resistance of a control chart
Standardized cusum
Tabular or algorithmic cusum
V-mask form of the cusum

Learning Objectives

1. Set up and use cusum control charts for monitoring the process mean
2. Design a cusum control chart for the mean to obtain specific ARL performance
3. Incorporate a fast initial response feature into the cusum control chart
4. Use a combined Shewhart-cusum monitoring scheme
5. Set up and use EWMA control charts for monitoring the process mean
6. Design an EWMA control chart for the mean to obtain specific ARL performance
7. Understand why the EWMA control chart is robust to the assumption of normality
8. Understand the performance advantage of cusum and EWMA control charts relative to Shewhart control charts
9. Set up and use a control chart based on an ordinary (unweighted) moving average