# Numerical Methods in Engineering

## 3- SET OF LINEAR EQUATIONS

DR. ETEMADI

## Linear Equations

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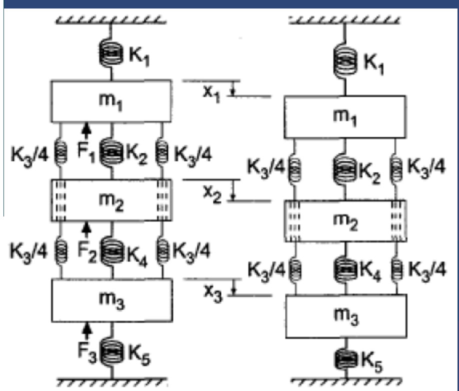
- Static deflection of springs in this example can be found by solving the following set of equations




$$(K_1+K_2+K_3)x_1-K_2x_2-K_3x_3=W_1$$
$$-K_2x_1+(K_2+K_4)x_2-K_4x_3=W_2$$
$$-K_3x_1-K_4x_2-(K_3+K_4+K_5)x_3=W_3$$

For these values, the set turns into

$$K_1=40N/cm,\ \ K_2=K_3=K_4=20N/cm$$
$$K_5=90N/cm \qquad W_1=W_2=W_3=20N$$

$$80x_1-20x_2-20x_3=20$$
$$-20x_1+40x_2-20x_3=20$$
$$-20x_1-20x_2-130x_3=20$$

## Set of Linear Equations (General Form)

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$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N &= b_2 \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N &= b_M
\end{aligned}
$$

- Matrix Form

$$A_{M\times N}x_{N\times 1} = b_{M\times 1}$$

$$A_{M\times N} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}$$



## Different Conditions

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- A set of equations like $A_{M\times N}x_{N\times 1} = b_{M\times 1}$ may have different conditions:
- $M=N \rightarrow$ Number of equations and variables are equal.
  - Straight forward (unique) solution
- $M<N \rightarrow$ Number of equations is less than variables.
  - Too many solution (choose the best by minimum norm)
- $M>N \rightarrow$ Number of equations is more than variables.
  - No solution (Use least squares method and find a solution based on minimum error.

## Set of Linear Equations

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If $M=N=n$ then:

- The solution can be obtained if:

  ***A*** is *Invertible* or is *Full Rank*.

  $\rightarrow$ Determinant of $\boldsymbol{A}=|\boldsymbol{A}|\neq 0$ or Rank($\boldsymbol{A}$)=n

- If $|\boldsymbol{A}|=0$ $\rightarrow$ Rank($\boldsymbol{A}$)<n or A is not full rank.

  then some equations are linearly dependent to others like:

$$2x+y=3$$
$$4x+2y=6$$



## *Determinant and Norm*

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- $|A|=\det(A)\approx 0$

Determinant is small if $|A|<<||A||$

- *Norms:*

  *Euclidian norm* $$\|\boldsymbol{A}\|=\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}a_{i,j}^{2}}$$

  *Maximum row sum* $$\|\boldsymbol{A}\|=\max_{1\le i\le n}\sum_{j=1}^{n}|a_{i,j}|$$

## Norm (Example)

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- Determine the norm of matrix A by maximum row sum

$$\boldsymbol{A} = \begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix}$$

$$\begin{aligned} \|\boldsymbol{A}\|_\infty &= \max_{1\le i\le 3} \sum_{j=1}^{3} |a_{i,j}| \\ &= \max\left[(|10|+|-7|+|0|),(|-3|+|2.099|+|6|),(|5|+|-1|+|5|)\right] \\ &= \max\left[(10+7+0),(3+2.099+6),(5+1+5)\right] \\ &= \max(17, 11.099, 11) \\ &= 17. \end{aligned}$$



## *ill - Conditioning*

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- Condition Number $cond(A) = \|\boldsymbol{A}\| \|\boldsymbol{A}^{-1}\|$

| | |
|---|---|
| Cond(A)>>1 | bad conditioned |
| Cond(A)≈1 | well conditioned |

- Well conditioned matrix can numerically lead to solution.

## Cramer's Rule

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- $A\,x = b$
- $A_j$ is constructed by replacing $j$th column of $A$ by $b$.
- Cramer's solution:

$$x_j = \frac{\det(A^j)}{\det(A)} \quad (j = 1, 2, \cdots, n)$$

Num. Methods: 3- Linear Equations

## Cramer's rule (Example)

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$$\begin{aligned} 80x_1 - 20x_2 - 20x_3 &= 20 \\ -20x_1 + 40x_2 - 20x_3 &= 20 \\ -20x_1 - 20x_2 + 130x_3 &= 20 \end{aligned}$$

$$\det(A) = \begin{vmatrix} 80 & -20 & -20 \\ -20 & 40 & -20 \\ -20 & -20 & 130 \end{vmatrix} = 300{,}000$$

$$\det(A^1) = \begin{vmatrix} 20 & -20 & -20 \\ 20 & 40 & -20 \\ 20 & -20 & 130 \end{vmatrix} = 180{,}000$$

$$x_1 = \frac{\det(A^1)}{\det(A)} = 0.60$$

$$x_2 = \frac{\det(A^2)}{\det(A)} = 1.00$$

$$x_3 = \frac{\det(A^3)}{\det(A)} = 0.40$$

Num. Methods: 3- Linear Equations

## Gauss-Elimination Rule

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- Added matrix

$$[A|\ b]=\left[\begin{array}{ccc|c} 80 & -20 & -20 & 20 \\ -20 & 40 & -20 & 20 \\ -20 & -20 & 130 & 20 \end{array}\right]$$

- Keep the first row
- Multiply the 1st row by proper coefficients
- Subtract it from other rows so that their first elements become zero



## Gauss-Elimination Rule

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$$\left[\begin{array}{ccc|c} 80 & -20 & -20 & 20 \\ -20 & 40 & -20 & 20 \\ -20 & -20 & 130 & 20 \end{array}\right] \begin{array}{l} \\ \leftarrow R_2-(-20/80)R_1 \\ \leftarrow R_3-(-20/80)R_1 \end{array}$$

- Now keep the 2nd row and do the same for other rows

$$\left[\begin{array}{ccc|c} 80 & -20 & -20 & 20 \\ 0 & 35 & -25 & 25 \\ 0 & -25 & 125 & 25 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow R_3-(-25/35)R_2 \end{array}$$

## Gauss-Elimination Rule

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$$\left[\begin{array}{ccc|c} 80 & -20 & -20 & 20 \\ 0 & 35 & -25 & 25 \\ 0 & 0 & 750/7 & 300/7 \end{array}\right]$$

- Back substitution yields: $\begin{cases} x_1 = 0.60 \\ x_2 = 1.00 \\ x_3 = 0.40 \end{cases}$



## Pivoting

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- If diagonal elements become zero, then the method fails. Use pivoting:
- Change rows of the added matrix to have biggest possible diagonal elements
- For example

$$[A \mid b] = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 2 & -1 & -1 & 0 \\ 1 & 1 & -1 & 1 \end{array}\right]$$

## Pivoting (Example)

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$$[A \mid b] = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 2 & -1 & -1 & 0 \\ 1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{Pivoting} \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & -1 & 1 \end{array}\right] \begin{array}{l} : r_1^{(1)} \\ : r_2^{(1)} \\ : r_3^{(1)} \end{array}$$

$$r_3^{(1)} - a_{31}^{(1)} / a_{11}^{(1)} \times r_1^{(1)} \rightarrow \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 3/2 & -1/2 & 1 \end{array}\right] \begin{array}{l} : r_1^{(2)} \\ : r_2^{(2)} \\ : r_3^{(2)} \end{array}$$

$$\xrightarrow{Pivoting} \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 3/2 & -1/2 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right] \begin{array}{l} : r_1^{(3)} \\ : r_2^{(3)} \\ : r_3^{(3)} \end{array}$$



## Pivoting (Example)

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$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 3/2 & -1/2 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right] \begin{array}{l} : r_1^{(3)} \\ : r_2^{(3)} \\ : r_3^{(3)} \end{array}$$

$$\begin{array}{l} r_1^{(3)} \\ r_2^{(3)} \\ r_3^{(3)} - a_{32}^{(3)} / a_{22}^{(3)} \times r_2^{(3)} \end{array} \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 4/3 & 4/3 \end{array}\right] \begin{array}{l} : r_1^{(4)} \\ : r_2^{(4)} \\ : r_3^{(4)} \end{array}$$

- Back substitution

$$x_3 = b_3^{(4)} / a_{33}^{(4)} = 1$$

$$x_2 = \left(b_2^{(4)} - a_{23}^{(4)} x_3\right) / a_{22}^{(4)} = 1$$

$$x_1 = \left(b_1^{(4)} - \sum_{n=2}^{3} a_{1n}^{(4)} x_n\right) \Big/ a_{11}^{(4)} = 1$$

## Scaling

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- If the pivoting element is too small then the error of rounding becomes very large
- Use scaling:

1. Before using gauss elimination, normalize the first column; divide elements of the first column by the biggest element of the first row.
2. Choose pivoting element and perform gauss elimination
3. Normalize the 2nd column for all elements except the 1st one; divide elements of the 2nd column (except the 1st one) by the biggest element of the 2nd row.
4. Choose pivoting element and perform gauss elimination
5. ……



## Gauss Elimination with Scaling (Example)

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$$[A|\ b] = \left[\begin{array}{ccc|c} 3 & 2 & 105 & 104 \\ 2 & -3 & 103 & 98 \\ 1 & 1 & 3 & 3 \end{array}\right] \qquad a_1 = \begin{bmatrix} 3/105 \\ 2/103 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0.0286 \\ 0.0194 \\ 0.333 \end{bmatrix}$$

- Normalization shows that the 1st and the 3rd columns must be replaced

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 2 & -3 & 103 & 98 \\ 3 & 2 & 105 & 104 \end{array}\right] \begin{array}{l} \\ R_2-(2/1)R_1 \\ R_3-(3/1)R_1 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & -5 & 97 & 92 \\ 0 & -1 & 96 & 95 \end{array}\right]$$

## Gauss Elimination with Scaling (Example)

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$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & -5 & 97 & 92 \\ 0 & -1 & 96 & 95 \end{array}\right] \qquad \boldsymbol{a_2} = \begin{bmatrix} - \\ -5/97 \\ -1/96 \end{bmatrix} = \begin{bmatrix} - \\ -0.0516 \\ -0.0104 \end{bmatrix}$$

- Normalization of the 2nd column shows that no replacement is needed.

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & -5 & 97 & 92 \\ 0 & -1 & 96 & 95 \end{array}\right] R_3 - (1/5)R_2 \qquad \left[\begin{array}{ccc|c} 1.0 & 1.0 & 3.0 & 3.0 \\ 0.0 & -5.0 & 9\ 70 & 9\ 20 \\ 0.0 & 0.0 & 7\ 66 & 7\ 66 \end{array}\right]$$

Num. Methods: 3- Linear Equations

## Gauss-Jordan Elimination

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- Similar to Gauss method, but elimination continues until all elements except diagonal elements vanish.
- Hence in added matrix, ***A*** converts to ***I*** and ***b*** converts to the answers of ***x***.

$$\left[\begin{array}{ccc|c} 80 & -20 & -20 & 20 \\ -20 & 40 & -20 & 20 \\ -20 & -20 & 130 & 20 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.60 \\ 0 & 1 & 0 & 1.00 \\ 0 & 0 & 1 & 0.40 \end{array}\right]$$

Num. Methods: 3- Linear Equations

## Gauss-Jordan Elimination (Example)



- Normalize the first row

$$\left[\begin{array}{ccc|c} 80 & -20 & -20 & 20 \\ -20 & 40 & -20 & 20 \\ -20 & -20 & 130 & 20 \end{array}\right] \begin{array}{l} R_1/80 \\ \\ \\ \end{array}$$

- Elimination of elemnts on the 1st column

$$\left[\begin{array}{ccc|c} 1 & -1/4 & -1/4 & 1/4 \\ -20 & 40 & -20 & 20 \\ -20 & -20 & 130 & 20 \end{array}\right] \begin{array}{l} \\ R_2-(-20)R_1 \\ R_3-(-20)R_1 \end{array}$$

- Normalize the second row

$$\left[\begin{array}{ccc|c} 1 & -1/4 & -1/4 & 1/4 \\ 0 & 35 & -25 & 25 \\ 0 & -25 & 125 & 25 \end{array}\right] \begin{array}{l} \\ R_2/35 \\ \\ \end{array}$$



## Gauss-Jordan Elimination (Example)



- Elimination of elements on 1st and 3rd row

$$\left[\begin{array}{ccc|c} 1 & -1/4 & -1/4 & 1/4 \\ 0 & 1 & -5/7 & 5/7 \\ 0 & -25 & 125 & 25 \end{array}\right] \begin{array}{l} R_1-(-1/4)R_2 \\ \\ R_3-(-25)R_2 \end{array}$$

- Normalize the 3rd row

$$\left[\begin{array}{ccc|c} 1 & 0 & -3/7 & 3/7 \\ 0 & 1 & -5/7 & 5/7 \\ 0 & 0 & 750/7 & 300/7 \end{array}\right] \begin{array}{l} \\ \\ R_3/(750/7) \end{array}$$

- Elimination of elements on the 1st and 2nd row

$$\left[\begin{array}{ccc|c} 1 & 0 & -3/7 & 3/7 \\ 0 & 1 & -5/7 & 5/7 \\ 0 & 0 & 1 & 215 \end{array}\right] \begin{array}{l} R_1-(-3/7)R_3 \\ R_2-(-5/7)R_3 \\ \\ \end{array}$$

## Gauss-Jordan Elimination (Example)

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- Final form of the added Matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.60 \\ 0 & 1 & 0 & 1.00 \\ 0 & 0 & 1 & 0.40 \end{array}\right]$$

- Answer

$$\boldsymbol{x}^T = \begin{bmatrix} 0.60 & 1.00 & 0.40 \end{bmatrix}$$



## Inverse of a Matrix by Gauss-Jordan

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- By definition

$$\boldsymbol{A} * \boldsymbol{A}\text{^-}1 = \boldsymbol{I} \qquad \boldsymbol{A}\, n \times n$$

$$\boldsymbol{A} * \boldsymbol{X} = \boldsymbol{I} \;\rightarrow\; \boldsymbol{X}\, n \times n = \boldsymbol{A}\text{^-}1$$

$$\left[\boldsymbol{A} \middle| \; \boldsymbol{I}\right] \longrightarrow \left[\boldsymbol{I} \middle| \; \boldsymbol{A}^{-1}\right]$$

## Inverse of a Matrix (Example)

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$$[A|\;I]=\left[\begin{array}{ccc|ccc}80 & -20 & -20 & 1 & 0 & 0\\ -20 & 40 & -20 & 0 & 1 & 0\\ -20 & -20 & 130 & 0 & 0 & 1\end{array}\right]$$

- From Gauss-Jordan:

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 2/125 & 1/100 & 1/250\\ 0 & 1 & 0 & 1/100 & 1/30 & 1/150\\ 0 & 0 & 1 & 1/250 & 1/150 & 7/750\end{array}\right]$$

$$A^{-1}=\begin{bmatrix}2/125 & 1/100 & 1/250\\ 1/100 & 1/30 & 1/150\\ 1/250 & 1/150 & 7/750\end{bmatrix}$$



## Jacobi Iterative Method

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$$\begin{aligned}4x_1 - x_2 + x_4 &= 100 \qquad & x_1 &= 1/4(100 + x_2 - x_4)\\ -x_1 + 4x_2 - x_3 + x_5 &= 100 & x_2 &= 1/4(100 + x_1 + x_3 - x_5)\\ -x_2 + 4x_3 - x_4 &= 100 & x_3 &= 1/4(100 + x_2 + x_4)\\ x_1 - x_3 + 4x_4 - x_5 &= 100 & x_4 &= 1/4(100 - x_1 + x_3 + x_5)\\ x_2 - x_4 + 4x_5 &= 100 & x_5 &= 1/4(100 - x_2 + x_4)\end{aligned}$$

$$\mathbf{x}^{(0)T} = [0.0 \quad 0.0 \quad 0.0 \quad 0.0 \quad 0.0]$$

$$\mathbf{x}^{(1)T} = [25.0 \quad 25.0 \quad 25.0 \quad 25.0 \quad 25.0]$$

## Gauss-Seidel Iterative Method

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$$x_1^{k+1} = 1/4\left(100 + x_2^k - x_4^k\right)$$

$$x_2^{k+1} = 1/4\left(100 + x_1^{k+1} + x_3^k - x_5^k\right)$$

$$x_3^{k+1} = 1/4\left(100 + x_2^{k+1} + x_4^k\right)$$

$$x_4^{k+1} = 1/4\left(100 - x_1^{k+1} + x_3^{k+1} + x_5^k\right)$$

$$x_5^{k+1} = 1/4\left(100 - x_2^{k+1} + x_4^{k+1}\right)$$

- Diagonal dominance is a necessity for convergence

$$\left|a_{i,i}\right| \geq \sum_{j=1, j\neq i}^{n} \left|a_{i,j}\right| \quad (i = 1, 2, \cdots, n)$$



## Iterative Methods

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- Jacobi

$$x_i^{(k+1)} = \frac{1}{a_{i,i}}\left(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{i,j} x_j^{(k)}\right)$$

- Gauss-Seidel

$$x_i^{(k+1)} = \frac{1}{a_{i,i}}\left(b_i - \sum_{j=1}^{i-1} a_{i,j} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{i,j} x_j^{(k)}\right)$$

Num. Methods: 3- Linear Equations

## Minimum Norm

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- If M<N: $A_{M\times N}x_{N\times 1} = b_{M\times 1}$

  Number of equations is less than the number of variables.
- Among many solutions, minimum norm gives:

$$x = A^T\left[AA^T\right]^{-1}b$$



## Minimum Norm (Example)

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$$\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = 3; \quad Ax = b; \quad A = \begin{bmatrix}1 & 2\end{bmatrix}; \quad b = 3$$

$$x = A^T\left[AA^T\right]^{-1}b = \begin{bmatrix}1\\ 2\end{bmatrix}\left(\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix}\right)^{-1} 3 = \frac{3}{5}\begin{bmatrix}1\\ 2\end{bmatrix} = \begin{bmatrix}0.6\\ 1.2\end{bmatrix}$$

- $A^T\left[AA^T\right]^{-1}$

  is called the right pseudo inverse of A

## Least squares error

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- If M>N: $A_{M\times N}x_{N\times 1} = b_{M\times 1}$

  Number of equations is more than the number of variables.
- No Exact solution exists An answer with the least error is:

$$e = Ax - b$$

$$J = \frac{1}{2}\|e\|^2 = \frac{1}{2}\|Ax-b\|^2 = \frac{1}{2}[Ax-b]^T[Ax-b]$$

$$\frac{\partial}{\partial x}J = A^T[Ax-b] = 0;$$

$$x = [A^TA]^{-1}A^Tb$$ Left pseudo inverse



## Least squares error (Example)

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$$\begin{bmatrix}1\\2\end{bmatrix}x = \begin{bmatrix}2.1\\3.9\end{bmatrix}; \qquad Ax = b, \quad A = \begin{bmatrix}1\\2\end{bmatrix}; \; b = \begin{bmatrix}2.1\\3.9\end{bmatrix}$$

$$x = [A^TA]^{-1}A^Tb = \left([1 \quad 2]\begin{bmatrix}1\\2\end{bmatrix}\right)^{-1}[1 \quad 2]\begin{bmatrix}2.1\\3.9\end{bmatrix} = 0.2[1 \quad 2]\begin{bmatrix}2.1\\3.9\end{bmatrix} = 1.9$$