Numerical Methods in Engineering

5- NUMERICAL SOLUTIONS FOR DIFFERENTIAL EQUATIONS

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Initial Value Differential Equations

- 1st order differential equation

$$y' = f(x,y), \quad y' = dy/dx, \quad f(x,y) \equiv given\ function$$

Initial value: $y(a) = \alpha$

- Order n: $y^{(n)} = f\left(x, y, y', \cdots, y^{(n-1)}\right)$

Initial value: $y^{(i)}(a) = \alpha_i, i = 0, \cdots, n-1$

State Space Representation

$$y^{(n)} = f\left(x, y, y', \cdots, y^{(n-1)}\right)$$

- Chose variables as following:

$$y_1 = y, \quad y_2 = y', \quad y_3 = y'', \cdots, \quad y_n = y^{(n-1)}$$

- So the eq. turns into *n* eqs.

$$y_1' = y_2, \quad y_2' = y_3, \quad y_3' = y_4, \cdots, \quad y_n' = f(x, y_1, y_2, \cdots, y_n)$$

$$\boldsymbol{y}' = \boldsymbol{F}(x, \boldsymbol{y}) \rightarrow \boldsymbol{F}(x, \boldsymbol{y}) = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ f(x, \boldsymbol{y}) \end{bmatrix} \qquad \boldsymbol{y}(a) = \boldsymbol{\alpha}$$

Example (Satellite Dynamics)

- Turn the diff. equ. Into state-space form:

$$J\ddot{\theta} = \tau(t)$$

- State variable: $\Rightarrow \begin{cases} y_1 = \theta \\ y_2 = \dot{\theta} \end{cases}$

- State-space form of the equation:

$$\Rightarrow \dot{\boldsymbol{y}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{y} + \frac{1}{J} \begin{bmatrix} 0 \\ \tau \end{bmatrix}$$

Example (Pendulum Dynamics)

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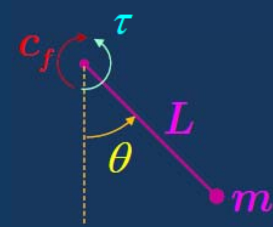
- Turn the diff. equ. Into state-space form:

$$mL^2\ddot{\theta}+c_f\dot{\theta}+mgL\sin\theta=\tau(t)$$



- State variable:

$$\begin{cases} x_1=\theta \\ x_2=\dot{\theta} \end{cases}$$

- State-space form of the equation:

$$\begin{cases} \dot{x}_1=x_2 \\ \dot{x}_2=-\dfrac{g}{L}\sin(x_1)-\dfrac{c_f}{mL^2}x_2+\dfrac{\tau(t)}{mL^2} \end{cases}$$

Taylor Series Method

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- Taylor series expansion

$$y(x+h)\approx$$

$$y(x)+y'(x)h+\frac{1}{2!}y''(x)h^2+\frac{1}{3!}y'''(x)h^3+\cdots+\frac{1}{m!}y^m(x)h^m$$

- Error:

$$E=\frac{1}{(m+1)!}y^{(m+1)}(\zeta)h^{m+1},\quad x<\zeta<x+h$$

$$y^{(m+1)}(\zeta)\approx\tfrac{1}{h}\left[y^{(m)}(x+h)-y^{(m)}(x)\right]$$

$$E\approx\frac{h^m}{(m+1)!}\left[y^{(m)}(x+h)-y^{(m)}(x)\right]$$

Taylor Series Method

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- At first construct the following matrix:

$$\boldsymbol{d} = \begin{bmatrix} (\boldsymbol{y}')^T \\ (\boldsymbol{y}'')^T \\ \vdots \\ (\boldsymbol{y}^m)^T \end{bmatrix} = \begin{bmatrix} y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^m & y_2^m & \cdots & y_n^m \end{bmatrix}$$

Taylor Series Method (Example 1)

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$y' = -4y + x^2, \qquad y(0) = 1$

- Choosing m=4, we have:

$$y(h) = y(0) + y'(0)h + \frac{1}{2!}y''(0)h^2 + \frac{1}{3!}y'''(0)h^3 + \frac{1}{4!}y^4(0)h^4$$

$$\begin{cases} y' = -4y + x^2 \\ y'' = -4y' + 2x = 16y - 4x^2 + 2x \\ y''' = 16y' - 8x + 2 = -64y + 16x^2 - 8x + 2 \\ y^4 = -64y' + 32x - 8 = 256y - 64x^2 + 32x - 8 \end{cases}$$

Taylor Series Method (Example 1)

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$$\begin{cases} y' = -4y + x^2 \\ y'' = -4y' + 2x = 16y - 4x^2 + 2x \\ y''' = 16y' - 8x + 2 = -64y + 16x^2 - 8x + 2 \\ y^4 = -64y' + 32x - 8 = 256y - 64x^2 + 32x - 8 \end{cases}$$

$$\begin{bmatrix} y' \\ y'' \\ y''' \\ y^4 \end{bmatrix} = \begin{bmatrix} -4y + x^2 \\ 16y - 4x^2 + 2x \\ -64y + 16x^2 - 8x + 2 \\ 256y - 64x^2 + 32x - 8 \end{bmatrix}$$

Num. Methods: 5-Differential Equations

Taylor Series Method (Example 1)

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- For $y(0)=1$

$$\begin{cases} y'(0) = -4(1) = -4 \\ y''(0) = 16y - 4x^2 + 2x = 16(1) = 16 \\ y'''(0) = -64y + 16x^2 - 8x + 2 = -64(1) + 2 = -62 \\ y^4(0) = 256y - 64x^2 + 32x - 8 = 256(1) - 8 = 248 \end{cases}$$

- For h=0.2 the first estimation of y can be obtained from Taylor expansion:

$$y(0.2) = 1 + (-4)(0.2) + \frac{1}{2!}(16)(0.2)^2 + \frac{1}{3!}(-62)(0.2)^3 + \frac{1}{4!}(248)(0.2)^4$$
$$= 0.4539$$

Num. Methods: 5-Differential Equations

Taylor Series Method (Example 1)

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- Rounding error:

$$E = \frac{h^4}{(4+1)!}\left[y^{(4)}(0.2) - y^{(4)}(0)\right] = -0.0018$$

- Analytical solution $y(x) = \frac{31}{32}e^{-4x} + \frac{1}{4}x^2 - \frac{1}{8}x + \frac{1}{32}$

$$y(0.2) = 0.4515$$

- Estimation error

$$y(0.2)\big|_{Exact} - y(0.2)\big|_{Estim} = 0.4515 - 0.4539 = -0.0024$$

Taylor Series Method (Example 1)

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- Estimation can be continued using $y(0.2)$ as initial value, to obtain $y(0.4)$, and also $y(0.6)$, $y(0.8)$, …, as well.
- $y(0.2) \rightarrow y(0.4)$
- $y(0.4) \rightarrow y(0.6)$
- $y(0.6) \rightarrow y(0.8)$
- $y(0.8) \rightarrow y(1.0)$
- Note: For more accurate estimation use smaller value for h.

Homework

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- Find $y(1)$ for h=0.5 and h=0.25 and then compare the results with the exact solution.

$$y' = -4y + x^2, \qquad y(0) = 1$$

- Class activity:
- Find $y(0.4)$ for h=0.2.

Taylor Series Method (Example 2)

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$$y'' = -0.1y' - x \qquad y(0) = 0, \quad y'(0) = 1$$

- State-space form:

$$y_1 = y, \; y_2 = y'$$

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ -0.1y_2 - x \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Taylor Series Method (Example 2)

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- Choose m=4

$$\boldsymbol{y}'' = \begin{bmatrix} y_2' \\ -0.1y_2' - 1 \end{bmatrix} = \begin{bmatrix} -0.1y_2 - x \\ 0.01y_2 + 0.1x - 1 \end{bmatrix}$$

$$\boldsymbol{y}''' = \begin{bmatrix} -0.1y_2' - 1 \\ 0.01y_2' + 0.1 \end{bmatrix} = \begin{bmatrix} 0.01y_2 + 0.1x - 1 \\ -0.001y_2 - 0.01x + 0.1 \end{bmatrix}$$

$$\boldsymbol{y}^{(4)} = \begin{bmatrix} 0.01y_2' + 0.1 \\ -0.001y_2' - 0.01 \end{bmatrix} = \begin{bmatrix} -0.0001y_2 - 0.01x + 0.1 \\ 0.0001y_2 + 0.001x - 0.01 \end{bmatrix}$$

Taylor Series Method (Example 2)

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$$\boldsymbol{d} = \begin{bmatrix} y_1' & y_2' \\ y_1'' & y_2'' \\ y_1''' & y_2''' \\ y_1^{(4)} & y_2^{(4)} \end{bmatrix} = \begin{bmatrix} y_2 & -0.1y_2 - x \\ -0.1y_2 - x & 0.01y_2 + 0.1x - 1 \\ 0.01y_2 + 0.1x - 1 & -0.001y_2 - 0.01x + 0.1 \\ -0.001y_2 - 0.01x + 0.1 & 0.0001y_2 + 0.001x - 0.01 \end{bmatrix}$$

- Class activity:
- For h=1; solve the differential equation. Use $\boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to find $y(1)$

2nd and 4th Order Runge-Kutta Methods

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- If F is the state-space form of the equation then:

1. RK2:

$$K_1 = hF(x,y) \qquad K_2 = hF\left(x+\frac{h}{2}, y+\frac{1}{2}K_1\right)$$

$$y(x+h) = y(x) + K_2$$

2. RK4:

$$K_1 = hF(x,y) \qquad K_2 = hF\left(x+\frac{h}{2}, y+\frac{1}{2}K_1\right)$$

$$K_3 = hF\left(x+\frac{h}{2}, y+\frac{1}{2}K_2\right) \qquad K_4 = hF(x+h, y+K_3)$$

$$y(x+h) = y(x) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

Runge-Kutta 2 (Example)

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$$y'' = -0.1y' - x \qquad y(0)=0, \quad y'(0)=1$$

$$y_1 = y$$

$$y_2 = y' = y_1' \qquad F(x,y) = y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ -0.1y_2 - x \end{bmatrix}$$

$$y_2' = -0.1y_2 - x$$

1. Class Activity: Solve by RK2 using h=0.5 to find y(1)

2. Homework: Solve by RK4 using h=0.5 to find y(1) and compare results with RK2

Computer Program 1

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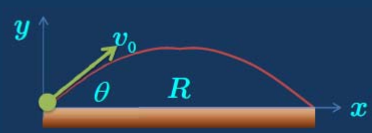
- Use Taylor and RK4 method with m=4 to find R. use similar value of *h* in both methods



$$\begin{cases} \ddot{x} = -\dfrac{C_D}{m}\dot{x}v^{0.5} \\ \ddot{y} = -\dfrac{C_D}{m}\dot{y}v^{0.5} - g \end{cases}$$

$v_0 = 50m/s$ $\quad m = 0.25kg$ $\quad v = \sqrt{\dot{x}^2 + \dot{y}^2}$

$\theta = 30^o$ $\quad g = 9.80665m/s^2$ $\quad C_D = 0.03$

Num. Methods: 5-Differential Equations

Computer Program 2

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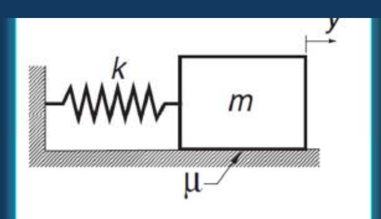
- Use RK2 and RK4 method with a proper value of h to find maximum deflection of the spring and final deflection of it.

  use similar value of *h* in both methods



$$\ddot{y} = -\frac{k}{m}y - \mu g\frac{\dot{y}}{|\dot{y}|} \qquad y(0) = 0.1$$

$m = 6kg$ $\quad k = 3000N/m$

$g = 9.80665m/s^2$ $\quad \mu = 0.5$

Num. Methods: 5-Differential Equations