In the Name of God

Advanced Engineering Mathematics

Linear Algebra

Matrices, Vectors, Determinants. Linear Systems

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Outlines

Linear Algebra: Matrices, Vectors, Determinants.Linear Systems

1 matrices, vectors: addition and scalar multiplication
2 matrix multiplication
3 linear systems of equations. Gauss elimination
4 linear independence. Rank of a matrix. Vector space
5 solutions of linear systems: existence, uniqueness
6 for reference: second- and third-order determinants
7 determinants. Cramer's rule
8 inverse of a matrix. Gauss–jordan elimination
9 vector spaces, inner product spaces. Linear transformations. *optional*

Ref: Erwin Kreyszig, "Advanced Engineering Mathematics," John Wiley & Sons, Inc., 10th Ed., 2011. (Chapter 7)

Matrices, Vectors: Addition and Scalar Multiplication

- A **matrix** is a rectangular array of numbers or functions which we will enclose in brackets.

- **Example1:** Linear Systems, a Major Application of Matrices

We are given a system of linear equations, briefly a **linear system**, such as

$$\begin{aligned} 4x_1 + 6x_2 + 9x_3 &= 6 \\ 6x_1 \qquad\quad - 2x_3 &= 20 \\ 5x_1 - 8x_2 + \ x_3 &= 10 \end{aligned}$$

Matrices, Vectors: Addition and Scalar Multiplication

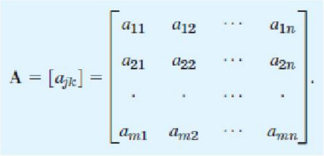
- Where $x1$, $x2$, $x3$ are the **unknowns**. We form the **coefficient matrix**, call it **A**, by listing the coefficients of the unknowns in the position in which they appear in the linear equations. In the second equation, there is no unknown $x2$, which means that the coefficient of $x2$ is 0 and hence in matrix **A**, $a22$ 0, Thus,

$$\mathbf{A} = \begin{bmatrix} 4 & 6 & 9 \\ 6 & 0 & -2 \\ 5 & -8 & 1 \end{bmatrix}. \quad \text{We form another matrix} \quad \tilde{\mathbf{A}} = \begin{bmatrix} 4 & 6 & 9 & 6 \\ 6 & 0 & -2 & 20 \\ 5 & -8 & 1 & 10 \end{bmatrix}$$

Matrices, Vectors: Addition and Scalar Multiplication

General Concepts and Notations

- Let us formalize what we just have discussed. We shall denote matrices by capital boldface letters **A**, **B**, **C**,… , or by writing the general entry in brackets; thus **A** [*ajk*], and so on. By an $m \times n$ **matrix** (read *m by n matrix*) we mean a matrix with m rows and n columns—rows always come first! is called the **size** of the matrix. Thus an matrix is of the form



$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Matrices, Vectors: Addition and Scalar Multiplication

- If $m = n$ we call A an $n \times n$ square matrix. Then its diagonal containing the entries $a_{11}, a_{22}, \dots, a_{nn}$ is called the main diagonal of A.

- **Vectors:** A **vector** is a matrix with only one row or column. Its entries are called the **components** of the vector.

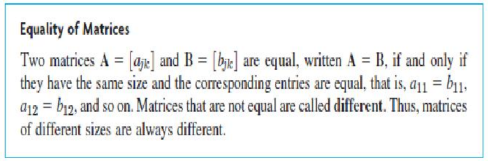
$$a = [a_1 \quad a_2 \quad \dots \quad a_n] \; for\ instance\ a = [-2 \quad 5 \quad 0.8 \quad 0 \quad 1]$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \; for\ instance\ b = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}$$

Matrices, Vectors: Addition and Scalar Multiplication

Addition and Scalar Multiplication of Matrices and Vectors

Definition



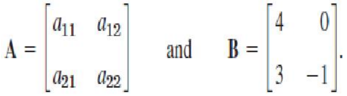
Equality of Matrices

Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are equal, written $\mathbf{A} = \mathbf{B}$, if and only if they have the same size and the corresponding entries are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on. Matrices that are not equal are called **different**. Thus, matrices of different sizes are always different.

Matrices, Vectors: Addition and Scalar Multiplication

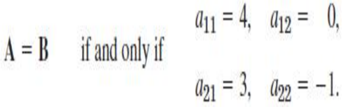
Example 3: **Equality of Matrices**

let



$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}.$$

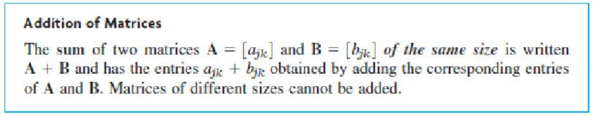
Then



$$\mathbf{A} = \mathbf{B} \quad \text{if and only if} \quad \begin{matrix} a_{11} = 4, & a_{12} = 0, \\ a_{21} = 3, & a_{22} = -1. \end{matrix}$$

Matrices, Vectors: Addition and Scalar Multiplication

Definition



Addition of Matrices

The sum of two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ *of the same size* is written $\mathbf{A} + \mathbf{B}$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$. Matrices of different sizes cannot be added.

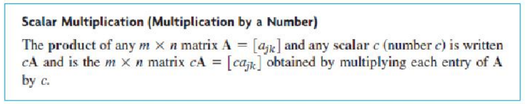
Example 4: **Addition of Matrices and Vectors**

$$\text{If}\quad \mathbf{A} = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix} \quad\text{and}\quad \mathbf{B} = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}, \quad\text{then}\quad \mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}.$$

A in Example 3 and our present **A** cannot be added.If a = [5 7 2] and b = [-6 2 0], then a + b = [-1 9 2].

Matrices, Vectors: Addition and Scalar Multiplication

Definition



Scalar Multiplication (Multiplication by a Number)

The product of any $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ and any scalar c (number c) is written $c\mathbf{A}$ and is the $m \times n$ matrix $c\mathbf{A} = [ca_{jk}]$ obtained by multiplying each entry of $\mathbf{A}$ by c.

Example 5: **Scalar Multiplication**

$$\text{If}\quad \mathbf{A} = \begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9.0 & -4.5 \end{bmatrix}, \quad\text{then}\quad -\mathbf{A} = \begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9.0 & 4.5 \end{bmatrix}, \quad \frac{10}{9}\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}, \quad 0\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Matrices, Vectors: Addition and Scalar Multiplication

- Rules for Matrix Addition and Scalar Multiplication.

(a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

(b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ (written $\mathbf{A} + \mathbf{B} + \mathbf{C}$)

(c) $\mathbf{A} + \mathbf{0} = \mathbf{A}$

(d) $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$.

Here **0** denotes the **zero matrix** (of size $m \times n$), that is, the m × n matrix with all entries zero. If $m = 1$ or $n = 1$, this is a vector, called a **zero vector**.

Matrices, Vectors: Addition and Scalar Multiplication

- for scalar multiplication we obtain the rules

(a) $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$

(b) $(c + k)\mathbf{A} = c\mathbf{A} + k\mathbf{A}$

(c) $c(k\mathbf{A}) = (ck)\mathbf{A}$ (written $ck\mathbf{A}$)

(d) $1\mathbf{A} = \mathbf{A}$.

Matrix Multiplication

Definition

Multiplication of a Matrix by a Matrix

The product $\mathbf{C} = \mathbf{AB}$ (in this order) of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ times an $r \times p$ matrix $\mathbf{B} = [b_{jk}]$ is defined if and only if $r = n$ and is then the $m \times p$ matrix $\mathbf{C} = [c_{jk}]$ with entries

$$(1) \qquad c_{jk} = \sum_{l=1}^{n} a_{jl}b_{lk} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk} \qquad \begin{matrix} j = 1, \cdots, m \\ k = 1, \cdots, p. \end{matrix}$$

Note that

$$\begin{matrix} \mathbf{A} & \mathbf{B} & = & \mathbf{C} \\ [m \times n] & [n \times p] & = & [m \times p]. \end{matrix}$$

Matrix Multiplication

- Example 1: **Matrix Multiplication**

$$\mathbf{AB} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

- Example 2**: Multiplication of a Matrix and a Vector**

$$\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \cdot 3 + 2 \cdot 5 \\ 1 \cdot 3 + 8 \cdot 5 \end{bmatrix} = \begin{bmatrix} 22 \\ 43 \end{bmatrix} \quad \text{whereas} \quad \begin{bmatrix} 3 \\ 5 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} \quad \text{is undefined.}$$

Matrix Multiplication

- Example 3: **Products of Row and Column Vectors**

$$\begin{bmatrix} 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = [19], \qquad \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 1 \\ 6 & 12 & 2 \\ 12 & 24 & 4 \end{bmatrix}.$$

- **CAUTION! Matrix Multiplication Is No Commutative, AB ^{1} BA**

Matrix Multiplication

Example 4:

$$\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{but} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} = \begin{bmatrix} 99 & 99 \\ -99 & -99 \end{bmatrix}.$$

- It is interesting that this also shows that AB = 0 does not necessarily imply BA = 0 or A = 0 or B = 0 .

Matrix Multiplication

$$
\begin{array}{lll}
\text{(a)} & (k\mathbf{A})\mathbf{B} = k(\mathbf{AB}) = \mathbf{A}(k\mathbf{B}) & \textit{written } k\mathbf{AB} \textit{ or } \mathbf{A}k\mathbf{B} \\
\text{(b)} & \mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C} & \textit{written } \mathbf{ABC} \\
\text{(c)} & (\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC} & \\
\text{(d)} & \mathbf{C}(\mathbf{A}+\mathbf{B}) = \mathbf{CA} + \mathbf{CB} &
\end{array}
$$

- here, k is any scalar.
- (b) is called the **associative law**
- (c) and (d) are called the **distributive laws**.

Matrix Multiplication

- Since matrix multiplication is a multiplication of rows into columns, we can write the defining formula (1) more compactly as

$$c_{jk} = \mathbf{a}_j\mathbf{b}_k, \qquad j = 1, \cdots, m; \quad k = 1, \cdots, p,$$

- Where a_j is the jth row vector of A and b_k is the kth column vector of B.

$$\mathbf{a}_j\mathbf{b}_k = [a_{j1} \;\; a_{j2} \;\; \cdots \;\; a_{jn}]\begin{bmatrix} b_{1k} \\ \vdots \\ b_{nk} \end{bmatrix} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk}.$$

Matrix Multiplication

Example 5: **Product in Terms of Row and Column Vectors**

If A = $[\boldsymbol{a_{jk}}]$ is of size 3× 3 and B = $[\boldsymbol{b_{jk}}]$ is of size 3×4 then

$$\mathbf{AB} = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_1\mathbf{b}_2 & \mathbf{a}_1\mathbf{b}_3 & \mathbf{a}_1\mathbf{b}_4 \\ \mathbf{a}_2\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 & \mathbf{a}_2\mathbf{b}_3 & \mathbf{a}_2\mathbf{b}_4 \\ \mathbf{a}_3\mathbf{b}_1 & \mathbf{a}_3\mathbf{b}_2 & \mathbf{a}_3\mathbf{b}_3 & \mathbf{a}_3\mathbf{b}_4 \end{bmatrix}.$$

Taking a_1=[3 5 -1], a_2=[4 0 2], etc.

Parallel processing of products on the computer is facilitated by a variant of (3) for computing C = AB , which is used by standard algorithms (such as in Lapack).

Matrix Multiplication

In this method, **A** is used as given, **B** is taken in terms of its column vectors, and the product is computed columnwise; thus,

$$\text{(5)} \quad \mathbf{AB} = \mathbf{A}[\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \cdots \;\; \mathbf{b}_p] = [\mathbf{Ab}_1 \;\; \mathbf{Ab}_2 \;\; \cdots \;\; \mathbf{Ab}_p].$$

Columns of **B** are then assigned to different processors (individually or several to each processor), which simultaneously compute the columns of the product matrix Ab_1,A$b_2$etc.

Matrix Multiplication

Example 6: **Computing Products Columnwise by (5)**

To obtain

$$\mathbf{AB} = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 & 7 \\ -1 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 11 & 4 & 34 \\ -17 & 8 & -23 \end{bmatrix}$$

from (5), calculate the columns

$$\begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ -17 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 34 \\ -23 \end{bmatrix}$$

of **AB** and then write them as a single matrix, as shown in the first formula on the right.

Matrix Multiplication

Motivation of Multiplication by Linear Transformations

- Let us now motivate the “unnatural” matrix multiplication by its use in **linear transformations**. For *n=2* variables these transformations are of the form

$$y_1 = a_{11}x_1 + a_{12}x_2$$

$$y_2 = a_{21}x_1 + a_{22}x_2$$

Matrix Multiplication

$$(6) \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ax} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}.$$

- Now suppose further that the x_1x_2-system is related to a w_1w_2-system by another linear transformation, say,

$$(7) \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{Bw} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} b_{11}w_1 + b_{12}w_2 \\ b_{21}w_1 + b_{22}w_2 \end{bmatrix}.$$

Matrix Multiplication

$$(7) \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{Bw} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} b_{11}w_1 + b_{12}w_2 \\ b_{21}w_1 + b_{22}w_2 \end{bmatrix}.$$

- Then the y_1y_2-system is related to the w_1w_2-system indirectly via the x_1x_2-system, and we wish to express this relation directly. Substitution will show that this direct relation is a linear transformation, too, say,

$$(8) \qquad \mathbf{y} = \mathbf{Cw} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} c_{11}w_1 + c_{12}w_2 \\ c_{21}w_1 + c_{22}w_2 \end{bmatrix}.$$

Matrix Multiplication

- Indeed, substituting (7) into (6), we obtain

$$\begin{aligned} y_1 &= a_{11}(b_{11}w_1 + b_{12}w_2) + a_{12}(b_{21}w_1 + b_{22}w_2) \\ &= (a_{11}b_{11} + a_{12}b_{21})w_1 + (a_{11}b_{12} + a_{12}b_{22})w_2 \\ y_2 &= a_{21}(b_{11}w_1 + b_{12}w_2) + a_{22}(b_{21}w_1 + b_{22}w_2) \\ &= (a_{21}b_{11} + a_{22}b_{21})w_1 + (a_{21}b_{12} + a_{22}b_{22})w_2. \end{aligned}$$

Matrix Multiplication

- Comparing this with (8), we see that

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21} & c_{12} &= a_{11}b_{12} + a_{12}b_{22} \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21} & c_{22} &= a_{21}b_{12} + a_{22}b_{22}. \end{aligned}$$

- This proves that C=AB with the product defined as in (1).

Matrix Multiplication

Transposition

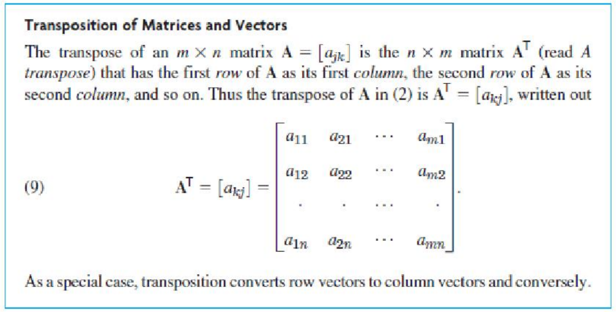
We obtain the transpose of a matrix by writing its rows as columns (or equivalently its columns as rows).

Example 7: **Transposition of Matrices and Vectors**

$$\mathbf{A} = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}, \quad \longrightarrow \quad \mathbf{A}^\mathsf{T} = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}.$$

Matrix Multiplication

Definition



Transposition of Matrices and Vectors

The transpose of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ is the $n \times m$ matrix $\mathbf{A}^\mathsf{T}$ (read *A transpose*) that has the first *row* of $\mathbf{A}$ as its first *column*, the second *row* of $\mathbf{A}$ as its second *column*, and so on. Thus the transpose of $\mathbf{A}$ in (2) is $\mathbf{A}^\mathsf{T} = [a_{kj}]$, written out

$$\mathbf{A}^\mathsf{T} = [a_{kj}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdot & \cdot & \cdots & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}. \tag{9}$$

As a special case, transposition converts row vectors to column vectors and conversely.

Matrix Multiplication

- Rules for transposition are

$$(10)\qquad \begin{aligned} &\text{(a)} & (\mathbf{A}^T)^T &= \mathbf{A} \\ &\text{(b)} & (\mathbf{A}+\mathbf{B})^T &= \mathbf{A}^T + \mathbf{B}^T \\ &\text{(c)} & (c\mathbf{A})^T &= c\mathbf{A}^T \\ &\text{(d)} & (\mathbf{A}\mathbf{B})^T &= \mathbf{B}^T\mathbf{A}^T. \end{aligned}$$

- **CAUTION!** Note that in (10d) the transposed matrices are ***in reversed order.***

Matrix Multiplication

Special Matrices

- **Symmetric** matrices are square matrices whose transpose equals the matrix itself.
- **Skew-symmetric** matrices are square matrices whose transpose equals ***minus*** the matrix.

$$(11)\quad \mathbf{A}^T = \mathbf{A} \quad (\text{thus } a_{kj} = a_{jk}), \qquad \mathbf{A}^T = -\mathbf{A} \quad (\text{thus } a_{kj} = -a_{jk}, \text{ hence } a_{jj} = 0).$$

Symmetric Matrix

Skew-Symmetric Matrix

Matrix Multiplication

- Symmetric and Skew-Symmetric Matrices

$$\mathbf{A} = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix} \text{ is symmetric, and } \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \text{ is skew-symmetric.}$$

Triangular Matrices.

- **Upper triangular matrices** are square matrices that can have nonzero entries only on and above the main diagonal, whereas any entry below the diagonal must be zero.

Matrix Multiplication

- **lower triangular matrices** can have nonzero entries only on and ***below*** the main diagonal. Any entry on the main diagonal of a triangular matrix may be zero or not.

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 \\ 8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 9 & 3 & 6 \end{bmatrix}.$$

Upper triangular

Lower triangular

Matrix Multiplication

Diagonal Matrices. These are square matrices that can have nonzero entries only onthe main diagonal. Any entry above or below the main diagonal must be zero.

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

Example 11: **Computer Production. Matrix Times Matrix**

Supercomp Ltd produces two computer models PC1086 and PC1186.

Matrix Multiplication

- The matrix **A** shows the cost per computer (in thousands of dollars) and **B** the production figures for the year 2010 (in multiples of 10,000 units.)
- Find a matrix **C** that shows the shareholders the cost per quarter (in millions of dollars) for raw material, labor, and miscellaneous.

$$\mathbf{A} = \begin{array}{c} \begin{array}{cc} \text{PC1086} & \text{PC1186} \end{array} \\ \begin{bmatrix} 1.2 & 1.6 \\ 0.3 & 0.4 \\ 0.5 & 0.6 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Raw Components} \\ \text{Labor} \\ \text{Miscellaneous} \end{array} \qquad \mathbf{B} = \begin{array}{c} \text{Quarter} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 3 & 8 & 6 & 9 \\ 6 & 2 & 4 & 3 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \text{PC1086} \\ \text{PC1186} \end{array}$$

Matrix Multiplication

- Solution:

$$\mathbf{C} = \mathbf{AB} = \begin{array}{c} \text{Quarter} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 13.2 & 12.8 & 13.6 & 15.6 \\ 3.3 & 3.2 & 3.4 & 3.9 \\ 5.1 & 5.2 & 5.4 & 6.3 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \text{Raw Components} \\ \text{Labor} \\ \text{Miscellaneous} \end{array}$$

- Since cost is given in multiples \$1000 of and production in multiples of 10,000 units, the entries of C are multiples of \$10 millions; thus means c_{11}=13.2 million, etc.
- Please study Examples 12,13.

Linear Systems of Equations. Gauss Elimination

- We now come to one of the most important use of matrices, that is, using matrices to solve systems of linear equations.
- Linear systems model many applications in engineering, economics, statistics, and many other areas.

Linear System, Coefficient Matrix, Augmented Matrix

- A **linear system of m equations in n unknowns** $x_1, x_2, \dots, x_n$ is a set of equations of the form

$$\text{(1)} \qquad \begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Linear Systems of Equations. Gauss Elimination

- The system is called *linear* because each variable x_j appears in the first power only, just as in the equation of a straight line.
- $a_{11}, \dots, a_{mn}$ are given numbers, called the **coefficients** of the system.
- $b_1, \dots, b_m$ on the right are also given numbers.

- If all $b_j$the are zero, then (1) is called a **homogeneous system**.

  If at least one b_j is not zero, then (1) is called a **nonhomogeneous system**.

Linear Systems of Equations. Gauss Elimination

- From the definition of matrix multiplication we see that the m equations of (1) may be written as a single vector equation

$$(2) \qquad \mathbf{Ax} = \mathbf{b}$$

Where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Linear Systems of Equations. Gauss Elimination

- The matrix

$$\tilde{\mathbf{A}} = \left[\begin{array}{cccc|c} a_{11} & \cdots & a_{1n} & & b_1 \\ \cdot & \cdots & \cdot & & \cdot \\ \cdot & \cdots & \cdot & & \cdot \\ a_{m1} & \cdots & a_{mn} & & b_m \end{array}\right]$$

is called the augmented matrix of the system (1).

- Note that the augmented matrix $\tilde{\boldsymbol{A}}$ determines the system (1) completely because it contains all the given numbers appearing in (1).

Linear Systems of Equations. Gauss Elimination

- Example 1: **Geometric Interpretation. Existence and Uniqueness of Solutions**

If $m = n = 2$, we have two equations in two unknowns x_1, x_2

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2.$$

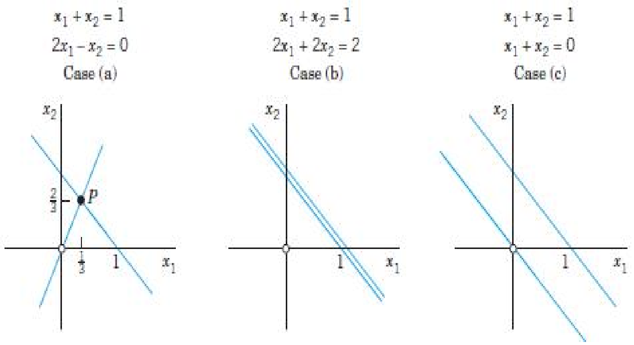
There are three possible cases

(*a*) Precisely one solution if the lines intersect

(*b*) Infinitely many solutions if the lines coincide

(*c*) No solution if the lines are parallel

Linear Systems of Equations. Gauss Elimination

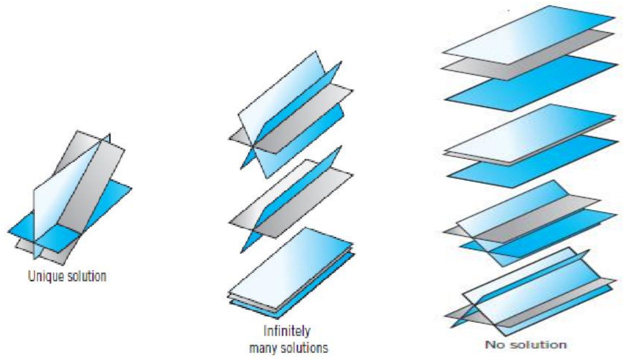


Linear Systems of Equations. Gauss Elimination

- If the system is homogenous, Case (c) cannot happen, because then those two straight lines pass through the origin, whose coordinates (0,0) constitute the trivial solution.

- Similarly, our present discussion can be extended from two equations in two unknowns to three equations in three unknowns.

Linear Systems of Equations. Gauss Elimination



Linear Systems of Equations. Gauss Elimination

Gauss Elimination and Back Substitution

Consider a linear system that is in ***triangular form*** (in full, *upper* triangular form) such as

$$\begin{aligned} 2x_1 + 5x_2 &= \quad 2 \\ 13x_2 &= -26 \end{aligned}$$

Then we can solve the system by **back substitution**, that is, we solve the last equation for the variable x_2= -26/13= -2, and then work backward, substituting x_2=-2 into the first equation and solving it for x_1 obtaining x_1= 0.5(2-5x_2)=6

Linear Systems of Equations. Gauss Elimination

let the given system be

$$\begin{aligned} 2x_1 + 5x_2 &= 2 \\ -4x_1 + 3x_2 &= -30. \end{aligned} \qquad \text{Its augmented matrix is} \qquad \left[\begin{array}{cc|c} 2 & 5 & 2 \\ -4 & 3 & -30 \end{array}\right].$$

We eliminate x_1 from the second equation, to get a triangular system. For this we add twice the first equation to the second, and we do the same operation on the ***rows*** of the augmented matrix.

Linear Systems of Equations. Gauss Elimination

$$\begin{aligned} 2x_1 + 5x_2 &= 2 \\ 13x_2 &= -26 \end{aligned} \qquad \text{Row 2} + 2\,\text{Row 1} \left[\begin{array}{cc|c} 2 & 5 & 2 \\ 0 & 13 & -26 \end{array}\right]$$

Where **Row 2 + 2 Row 1** means "**Add twice Row 1 to Row 2**" in the original matrix.

- Please study example 2,page 275

Linear Systems of Equations. Gauss Elimination

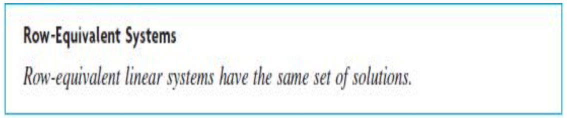
Elementary Row Operations. Row-Equivalent Systems

- **Elementary Row Operations for Matrices:**
  1. Interchange of two rows
  2. Addition of a constant multiple of one row to another row
  3. Multiplication of a row by a **nonzero** constant c
- **CAUTION!** These operations are for rows, ***not for columns***!

Linear Systems of Equations. Gauss Elimination

- **Elementary Operations for Equations:**
  1. Interchange of two equations
  2. Addition of a constant multiple of one equation to another equation
  3. Multiplication of an equation by a **nonzero** constant c

Theorem 1



Row-Equivalent Systems

Row-equivalent linear systems have the same set of solutions.

Linear Systems of Equations. Gauss Elimination

Because of this theorem, systems having the same solution sets are often called *equivalent systems.* But note well that we are dealing with ***row operations***. No column operations on the augmented matrix are permitted in this context because they would generally alter the solution set.

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}
\tag{1}
$$

Linear Systems of Equations. Gauss Elimination

- A linear system (1) is called **overdetermined** if it has more equations than unknowns.
- A linear system (1) is called **determined** if m=n
- A linear system (1) is called **underdetermined** if it has fewer equations than unknowns.

- ➢ a system (1) is called **consistent** if it has at least one solution (thus, one solution or infinitely many solutions),
- ➢ a system (1) is called **inconsistent** if it has no solutions at all.

Linear Systems of Equations. Gauss Elimination

Gauss Elimination: The Three Possible Cases of Systems

Example 3:**Gauss Elimination if Infinitely Many Solutions Exist**

Solve the following linear system of three equations in four unknowns whose augmented matrix is

$$
\left[\begin{array}{cccc|c}
3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\
0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\
1.2 & -0.3 & -0.3 & 2.4 & 2.1
\end{array}\right]. \tag{5}
$$

$$
\text{Thus,}\quad
\begin{aligned}
\enclose{circle}{3.0x_1} + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\
\boxed{0.6x_1} + 1.5x_2 + 1.5x_3 - 5.4x_4 &= 2.7 \\
\boxed{1.2x_1} - 0.3x_2 - 0.3x_3 + 2.4x_4 &= 2.1.
\end{aligned}
$$

Linear Systems of Equations. Gauss Elimination

Solution. As in the previous example, we circle pivots and box terms of equations and corresponding entries to be eliminated. We indicate the operations in terms of equations and operate on both equations and matrices.

Step 1. Elimination of x_1 from the second and third equations by adding

-0.6/3= -0.2 times the first equation to the second equation,

-1.2/3= -0.4 times the first equation to the third equation.

Linear Systems of Equations.
Gauss Elimination

This gives the following, in which the pivot of the next step is circled.

$$
(6)\quad \left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right] \begin{array}{l} \\ \text{Row 2} - 0.2 \text{ Row 1} \\ \text{Row 3} - 0.4 \text{ Row 1} \end{array} \qquad \begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ \boxed{1.1x_2} + 1.1x_3 - 4.4x_4 &= 1.1 \\ \boxed{-1.1x_2} - 1.1x_3 + 4.4x_4 &= -1.1. \end{aligned}
$$

Step 2. Elimination of x_2 from the third equation of (6) by adding

1.1/1.1= 1 times the second equation to the third equation.

Linear Systems of Equations.
Gauss Elimination

This gives

$$
(7)\quad \left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \text{Row 3} + \text{Row 2} \end{array} \qquad \begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ 0 &= 0. \end{aligned}
$$

Back Substitution. From the second equation, $x_2 = 1 - x_3 + 4x_4$. From this and the first equation, $x_1 = 2 - x_4$. Since x_3 and x_4 remain arbitrary, we have infinitely many solutions. If we choose a value of x_3 and a value of x_4, then the corresponding values of x_1 and x_2 are uniquely determined.

Linear Systems of Equations.
Gauss Elimination

Example 4: **Gauss Elimination if no Solution Exists**

Consider

$$
\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right] \qquad \begin{aligned} \boxed{3x_1} + 2x_2 + x_3 &= 3 \\ \boxed{2x_1} + x_2 + x_3 &= 0 \\ \boxed{6x_1} + 2x_2 + 4x_3 &= 6. \end{aligned}
$$

Step 1. Elimination of x_1 from the second and third equations by adding

-2/3 times the first equation to the second equation,

-6/3= -2 times the first equation to the third equation.

Linear Systems of Equations.
Gauss Elimination

This gives

$$
\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{array}\right] \begin{array}{l} \\ \text{Row 2} - \frac{2}{3} \text{ Row 1} \\ \text{Row 3} - 2 \text{ Row 1} \end{array} \qquad \begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ \boxed{-\tfrac{1}{3}x_2} + \tfrac{1}{3}x_3 &= -2 \\ \boxed{-2x_2} + 2x_3 &= 0. \end{aligned}
$$

Step 2. Elimination of x_2 from the third equation gives

$$
\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right] \begin{array}{l} \\ \\ \text{Row 3} - 6 \text{ Row 2} \end{array} \qquad \begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ -\tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 &= -2 \\ 0 &= 12. \end{aligned}
$$

The false statement 0=12 shows that the system has no solution.

Linear Systems of Equations. Gauss Elimination

Row Echelon Form and Information From It

At the end of the Gauss elimination the form of the coefficient matrix, the augmented matrix, and the system itself are called the **row echelon form**. In it, rows of zeros, if present, are the last rows, and, in each nonzero row, the leftmost nonzero entry is farther to the right than in the previous row. For instance, in Example 4 the coefficient matrix and its augmented in row echelon form are

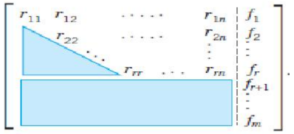
$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right].$$

Linear Systems of Equations. Gauss Elimination

The original system of m equations in n unknowns has augmented matrix [A|b] . This is to be row reduced to matrix [R|f] . The two systems Ax=b and Rx=f are equivalent:

if either one has a solution, so does the other, and the solutions are identical.

At the end of the Gauss elimination (before the back substitution), the row echelon form of the augmented matrix will be



Linear Systems of Equations. Gauss Elimination

Here is the method for determining whether has solutions and what they are:

- **a) No solution.** If r is less than m (meaning that **R** actually has at least one row of all 0s) *and* at least one of the numbers $f_{r+1}, f_{r+2}, \dots, f_m$ is not zero, then the system **Rx= f** is inconsistent: No solution is possible. Therefore the system Ax=b is inconsistent as well.
- **b) Unique solution.** If the system is consistent and r=n , there is exactly one solution, which can be found by back substitution.
- **c) Infinitely many solutions.** To obtain any of these solutions, choose values of $x_{r+1}, \dots, x_n$ arbitrarily. Then solve the rth equation for x_r (in terms of those arbitrary values), then the (r-1) st equation for x_{r-1}, and so on up the line.

Linear Independence. Rank of a Matrix. Vector Space

Linear Independence and Dependence of Vectors

Given any set of m vectors $a_{(1)}, \dots, a_m$ (with the same number of components), a **linear combination** of these vectors is an expression of the form

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)}$$

Where $c_1, c_2, \dots, c_m$ are any scalars. Now consider the equation

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)} = \mathbf{0}. \qquad (1)$$

Linear Independence. Rank of a Matrix. Vector Space

- our vectors $a_{(1)}, \dots, a_{(m)}$ are said to form a *linearly independent set* or, more briefly, we call them **linearly independent**.
- This means that we can express at least one of the vectors as a linear combination of the other vectors.
- For instance, if (1) holds with, say, $c_1 \neq 0$, we can solve (1) for $a_{(1)}$:

$$a_{(1)} = k_2 a_{(2)} + \cdots + k_m a_{(m)} \qquad \text{where } k_j = -c_j/c_1.$$

Linear Independence. Rank of a Matrix. Vector Space

Rank of a Matrix

Definition

The **rank** of a matrix **A** is the maximum number of linearly independent row vectors of **A**. It is denoted by rank **A**.

- Now the maximum number of linearly independent row vectors of a matrix does not change if we change the order of rows or multiply a row by a nonzero c or take a linear combination by adding a multiple of a row to another row. This shows that rank is **invariant** under elementary row operations.

Linear Independence. Rank of a Matrix. Vector Space

- Why is linear independence important?

Well, if a set of vectors is linearly dependent, then we can get rid of at least one or perhaps more of the vectors until we get a linearly independent set.

Example: Linear Independence and Dependence

The three vectors

$$\begin{aligned} \mathbf{a}_{(1)} &= [\ 3 \quad 0 \quad 2 \quad 2] \\ \mathbf{a}_{(2)} &= [-6 \quad 42 \quad 24 \quad 54] \\ \mathbf{a}_{(3)} &= [\ 21 \quad -21 \quad 0 \quad -15] \end{aligned}$$

are linearly dependent because

$$6\mathbf{a}_{(1)} - \tfrac{1}{2}\mathbf{a}_{(2)} - \mathbf{a}_{(3)} = \mathbf{0}.$$

Linear Independence. Rank of a Matrix. Vector Space

Theorem 1

Row-Equivalent Matrices

Row-equivalent matrices have the same rank.

Hence we can determine the rank of a matrix by reducing the matrix to row-echelon form

Example 3: **Determination of Rank**

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \quad \text{(given)}$$

Linear Independence. Rank of a Matrix. Vector Space

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \begin{matrix} \\ \text{Row 2} + 2\text{ Row 1} \\ \text{Row 3} - 7\text{ Row 1} \end{matrix} \qquad \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \text{Row 3} + \frac{1}{2}\text{Row 2.} \end{matrix}$$

The last matrix is in row-echelon form and has two nonzero rows. Hence rank A = 2.

Theorem 2

> **Linear Independence and Dependence of Vectors**
>
> *Consider p vectors that each have n components. Then these vectors are linearly independent if the matrix formed, with these vectors as row vectors, has rank p. However, these vectors are linearly dependent if that matrix has rank less than p.*

Linear Independence. Rank of a Matrix. Vector Space

Theorem 3

> **Rank in Terms of Column Vectors**
>
> *The rank r of a matrix* **A** *equals the maximum number of linearly independent column vectors of* **A**.
>
> *Hence* **A** *and its transpose* $\mathbf{A}^T$ *have the same rank.*

Please study the proof, page 283.

Example 4: Consider matrix A again,

Column 3 = (2/3) Column 1+ (2/3) Column 2

Column 4 = (2/3) Column 1+(29/21) Column 2

Linear Independence. Rank of a Matrix. Vector Space

Combining last two Theorems

Theorem 4

> **Linear Dependence of Vectors**
>
> *Consider p vectors each having n components. If $n < p$, then these vectors are linearly dependent.*

- Proof: The matrix **A** with those p vectors as row vectors has p rows and $n < p$ columns; hence by Theorem 3 it has rank A $\leq n < p$ which implies linear dependence by Theorem 2.

Linear Independence. Rank of a Matrix. Vector Space

Vector Space

- Consider a nonempty set V of vectors where each vector has the same number of components. If, for any two vectors **a** and **b** in V, we have that all their linear combinations $\alpha a + \beta b (\alpha, \beta$any real numbers) are also elements of V, and if, furthermore, **a** and **b** satisfy the laws (3a), (3c), (3d), and (4) in Sec. 7.1, as well as any vectors **a**, **b**, **c** in V satisfy (3b) then V is a vector space.

(a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

(b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ (written $\mathbf{A} + \mathbf{B} + \mathbf{C}$)

(c) $\mathbf{A} + \mathbf{0} = \mathbf{A}$

(a) $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$

(b) $(c + k)\mathbf{A} = c\mathbf{A} + k\mathbf{A}$

(c) $c(k\mathbf{A}) = (ck)\mathbf{A}$ (written $ck\mathbf{A}$)

(d) $1\mathbf{A} = \mathbf{A}$.

Linear Independence. Rank of a Matrix. Vector Space

- The maximum number of linearly independent vectors in V is called the **dimension** of V and is denoted by dim V.
- A linearly independent set in V consisting of a maximum possible number of vectors in V is called a **basis** for V.
- Thus, the number of vectors of a basis for V equals dim V.
- The set of all linear combinations of given vectors $a_{(1)}, \dots, a_{(p)}$ with the same number of components is called the **span** of these vectors.
- Obviously, a span is a vector space.

Linear Independence. Rank of a Matrix. Vector Space

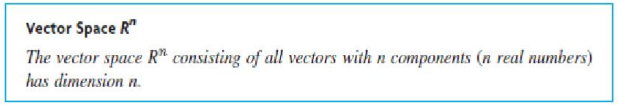
A set of vectors is a **basis** for a vector space V

1. if (1) the vectors in the set are linearly independent, and
2. if (2) any vector in V can be expressed as a linear combination of the vectors in the set.

- If (2) holds, we also say that the set of vectors **spans** the vector space V.
- By a **subspace** of a vector space V we mean a nonempty subset of V (including V itself)that forms a vector space with respect to the two algebraic operations (addition and scalar multiplication) defined for the vectors of V.

Linear Independence. Rank of a Matrix. Vector Space

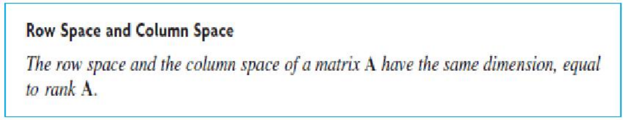
Theorem 5



Vector Space R^n

The vector space R^n consisting of all vectors with n components (n real numbers) has dimension n.

Theorem 6

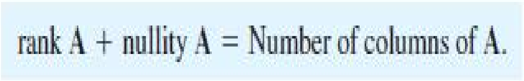


Row Space and Column Space

The row space and the column space of a matrix **A** *have the same dimension, equal to rank* **A**.

Linear Independence. Rank of a Matrix. Vector Space

- for a given matrix **A** the solution set of the homogeneous system Ax = 0 is a vector space, called the **null space** of **A**, and its dimension is called the **nullity** of **A**.



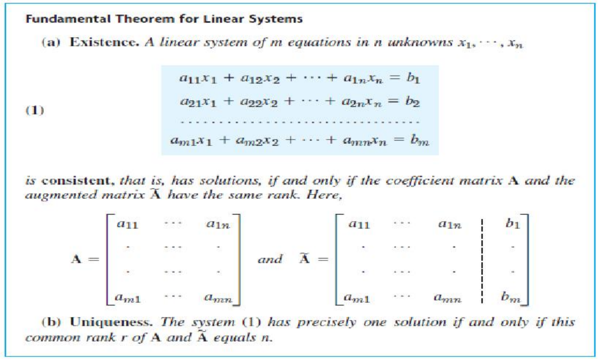
rank A + nullity A = Number of columns of A.

Solutions of Linear Systems: Existence, Uniqueness

1. A linear system of equations in *n* unknowns has a unique solution if the coefficient matrix and the augmented matrix have the same rank *n*.
2. The system has infinitely many solutions if that common rank is less than *n*.
3. The system has no solution if those two matrices have different rank.

Solutions of Linear Systems: Existence, Uniqueness

Theorem 1



Fundamental Theorem for Linear Systems

(a) Existence. *A linear system of m equations in n unknowns* $x_1, \cdots, x_n$

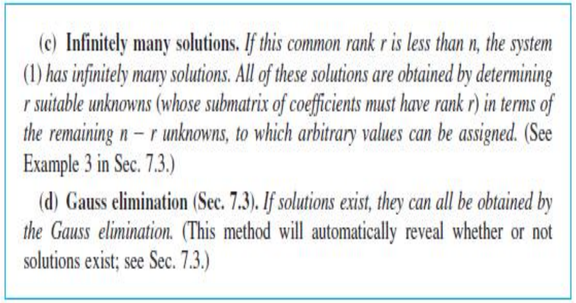
$$\begin{array}{ll} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ (1) & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}$$

is **consistent**, *that is, has solutions, if and only if the coefficient matrix* **A** *and the augmented matrix* $\tilde{\mathbf{A}}$ *have the same rank. Here,*

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

(b) Uniqueness. *The system* (1) *has precisely one solution if and only if this common rank r of* **A** *and* $\tilde{\mathbf{A}}$ *equals n.*

Solutions of Linear Systems: Existence, Uniqueness



(c) Infinitely many solutions. *If this common rank r is less than n, the system* (1) *has infinitely many solutions. All of these solutions are obtained by determining r suitable unknowns (whose submatrix of coefficients must have rank r) in terms of the remaining n − r unknowns, to which arbitrary values can be assigned.* (See Example 3 in Sec. 7.3.)

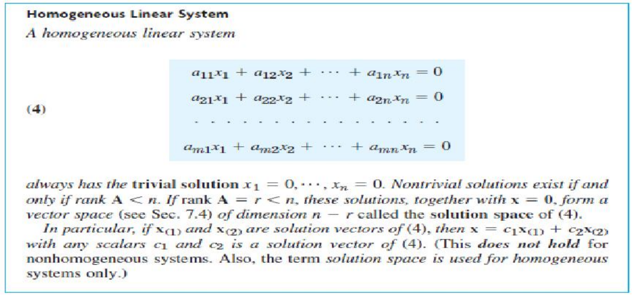
(d) Gauss elimination (Sec. 7.3). *If solutions exist, they can all be obtained by the Gauss elimination.* (This method will automatically reveal whether or not solutions exist; see Sec. 7.3.)

Please study the proof, page 289

Solutions of Linear Systems: Existence, Uniqueness

Homogeneous Linear System

Theorem 2



Homogeneous Linear System

A homogeneous linear system

$$\begin{array}{ll} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ (4) & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}$$

always has the **trivial solution** $x_1 = 0, \cdots, x_n = 0$. *Nontrivial solutions exist if and only if* rank $\mathbf{A} < n$. *If* rank $\mathbf{A} = r < n$, *these solutions, together with* $\mathbf{x} = \mathbf{0}$, *form a vector space* (see Sec. 7.4) *of dimension* $n - r$ called the **solution space** of (4).

In particular, if $\mathbf{x}_{(1)}$ *and* $\mathbf{x}_{(2)}$ *are solution vectors of* (4), *then* $\mathbf{x} = c_1\mathbf{x}_{(1)} + c_2\mathbf{x}_{(2)}$ *with any scalars* c_1 *and* c_2 *is a solution vector of* (4). (This ***does not hold*** for nonhomogeneous systems. Also, the term *solution space is used for homogeneous* systems only.)

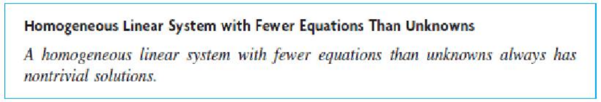
Solutions of Linear Systems: Existence, Uniqueness

The solution space of (4) is also called the **null space** of **A** because Ax = 0 for every **x** in the solution space of (4). Its dimension is called the **nullity** of **A**. Hence Theorem 2 states that

$$\text{rank A} + \text{nullity A} = n$$

Where *n* is the number of unknowns (number of columns of **A**).

Theorem 3



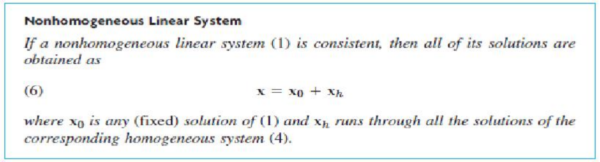
Homogeneous Linear System with Fewer Equations Than Unknowns

A homogeneous linear system with fewer equations than unknowns always has nontrivial solutions.

Solutions of Linear Systems: Existence, Uniqueness

Nonhomogeneous Linear Systems

Theorem 4



Nonhomogeneous Linear System

If a nonhomogeneous linear system (1) is consistent, then all of its solutions are obtained as

(6) $$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_h$$

where $\mathbf{x}_0$ *is any (fixed) solution of (1) and* $\mathbf{x}_h$ *runs through all the solutions of the corresponding homogeneous system (4).*

Proof: The difference $x_h = x - x_0$ of any two solutions of (1) is a solution of (4) because **A**x_h = **A**(**x** - x_0) = **Ax** - **A**x_0 =**b** – **b**= **0**.Since **x** is any solution of (1), we get all the solutions of (1) if in (6) we take any solution x_0 of (1) and let x_h vary throughout the solution space of (4).

For Reference: Second- and Third-Order Determinants

A **determinant of second order** is denoted and defined by

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Cramer's rule for solving linear systems of two equations in two unknowns

$$\begin{aligned} &\text{(a)} \quad a_{11}x_1 + a_{12}x_2 = b_1 \\ &\text{(b)} \quad a_{21}x_1 + a_{22}x_2 = b_2 \end{aligned}$$

For Reference: Second- and Third-Order Determinants

is

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} = \frac{b_1a_{22} - a_{12}b_2}{D},$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D} = \frac{a_{11}b_2 - b_1a_{21}}{D}$$

The value *D = 0* appears for homogeneous systems with nontrivial solutions.

For Reference:
Second- and Third-Order Determinants

Example 1: **Cramer’s Rule for Two Equations**

If $\begin{aligned} 4x_1 + 3x_2 &= 12 \\ 2x_1 + 5x_2 &= -8 \end{aligned}$ then $x_1 = \dfrac{\begin{vmatrix} 12 & 3 \\ -8 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \dfrac{84}{14} = 6,\quad x_2 = \dfrac{\begin{vmatrix} 4 & 12 \\ 2 & -8 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \dfrac{-56}{14} = -4.$

For Reference:
Second- and Third-Order Determinants

Third-Order Determinants

A **determinant of third order** can be defined by

$$(4)\quad D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

$$(4^*)\quad D = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{21}a_{13}a_{32} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22}.$$

For Reference:
Second- and Third-Order Determinants

Cramer’s Rule for Linear Systems of Three Equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \qquad x_1 = \frac{D_1}{D},\quad x_2 = \frac{D_2}{D},\quad x_3 = \frac{D_3}{D}$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix},\quad D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix},\quad D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}.$$

Determinants. Cramer’s Rule

A **determinant of order** n is a scalar associated with an $n \times n$ (hence ***square***!) matrix $A = [a_{jk}]$ and is denoted by

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

For $n = 1$ this determinant is defined by

$$D = a_{11}$$

Determinants. Cramer’s Rule

For n ≥ 2 by

$$D = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn} \qquad (j = 1, 2, \cdots, \text{or } n)$$

Or

$$D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \quad (k = 1, 2, \cdots, \text{or } n).$$

Here,

$$C_{jk} = (-1)^{j+k}M_{jk}$$

Determinants. Cramer’s Rule

- And M_{jk} is a determinant of order n – 1, namely, the determinant of the submatrix of **A** obtained from **A** by omitting the row and column of the entry a_{jk} , that is, the *j*th row and the *k*th column.
- M_{jk} is called the **minor** of a_{jk} in D, and C_{jk} the **cofactor** of a_{jk} in D.

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \qquad (j = 1, 2, \cdots, \text{or } n)$$

$$D = \sum_{j=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \qquad (k = 1, 2, \cdots, \text{or } n).$$

Determinants. Cramer’s Rule

Example: **Expansions of a Third-Order Determinant**

$$D = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix}$$

$$= 1(12 - 0) - 3(4 + 4) + 0(0 + 6) = -12.$$

This is the expansion by the first row. The expansion by the third column is

$$D = 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0 - 12 + 0 = -12.$$

Determinants. Cramer’s Rule

Example: **Determinant of a Triangular Matrix**

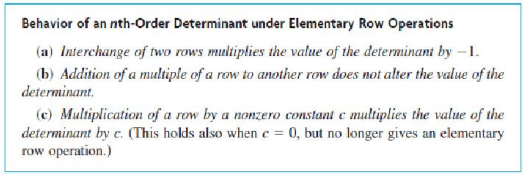
$$\begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix} = -3\begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix} = -3 \cdot 4 \cdot 5 = -60.$$

Inspired by this, can you formulate a little theorem on determinants of triangular matrices?

Determinants. Cramer’s Rule

General Properties of Determinants

Theorem 1



Behavior of an nth-Order Determinant under Elementary Row Operations

(a) *Interchange of two rows multiplies the value of the determinant by* -1.

(b) *Addition of a multiple of a row to another row does not alter the value of the determinant.*

(c) *Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c.* (This holds also when $c = 0$, but no longer gives an elementary row operation.)

CAUTION! det $(c\mathbf{A}) = c^n \det \mathbf{A}$ (not $c \det \mathbf{A}$).

Determinants. Cramer’s Rule

Example 4: **Evaluation of Determinants by Reduction to Triangular Form**

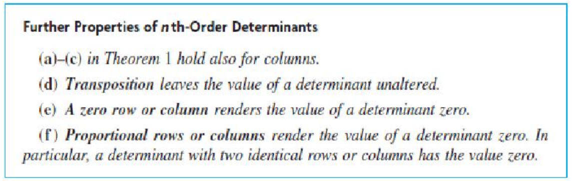
$$D = \begin{vmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{vmatrix} \qquad = \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 2 & 6 & -1 \\ 0 & 8 & 3 & 10 \end{vmatrix} \begin{matrix} \\ \text{Row 2} - 2\ \text{Row 1} \\ \\ \text{Row 4} + 1.5\ \text{Row 1} \end{matrix}$$

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & -11.4 & 29.2 \end{vmatrix} \begin{matrix} \\ \\ \text{Row 3} - 0.4\ \text{Row 2} \\ \text{Row 4} - 1.6\ \text{Row 2} \end{matrix} = \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & -0 & 47.25 \end{vmatrix} \begin{matrix} \\ \\ \\ \text{Row 4} + 4.75\ \text{Row 3} \end{matrix}$$

$$=2\times 5 \times 2.4 \times 47.25 = 1134$$

Determinants. Cramer’s Rule

Theorem 2



Further Properties of nth-Order Determinants

(a)–(c) *in Theorem 1 hold also for columns.*

(d) *Transposition leaves the value of a determinant unaltered.*

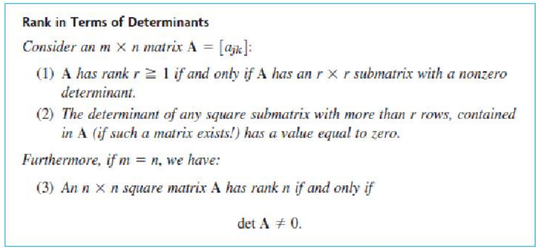
(e) *A zero row or column renders the value of a determinant zero.*

(f) *Proportional rows or columns render the value of a determinant zero. In particular, a determinant with two identical rows or columns has the value zero.*

It is quite remarkable that the important concept of the rank of a matrix **A**, which is the maximum number of linearly independent row or column vectors of **A** (see Sec. 7.4), can be related to determinants.

Determinants. Cramer’s Rule

Theorem 3



Rank in Terms of Determinants

Consider an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$:

(1) *$\mathbf{A}$ has rank $r \geqq 1$ if and only if $\mathbf{A}$ has an $r \times r$ submatrix with a nonzero determinant.*

(2) *The determinant of any square submatrix with more than r rows, contained in $\mathbf{A}$ (if such a matrix exists!) has a value equal to zero.*

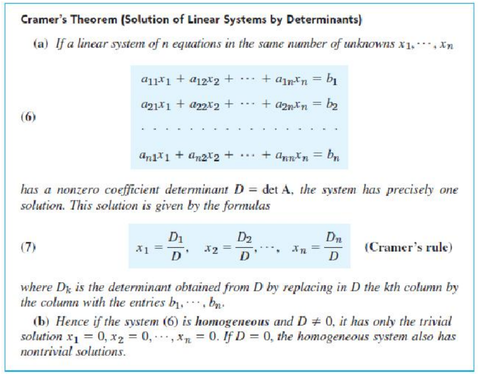
Furthermore, if $m = n$, we have:

(3) *An $n \times n$ square matrix $\mathbf{A}$ has rank n if and only if*

$$\det \mathbf{A} \neq 0.$$

Determinants. Cramer's Rule

Cramer's Rule



Cramer's Theorem (Solution of Linear Systems by Determinants)

(a) *If a linear system of n equations in the same number of unknowns* $x_1, \cdots, x_n$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{6}$$

has a nonzero coefficient determinant $D = \det \mathbf{A}$, *the system has precisely one solution. This solution is given by the formulas*

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \cdots, \quad x_n = \frac{D_n}{D} \qquad \textbf{(Cramer's rule)} \tag{7}$$

where D_k *is the determinant obtained from D by replacing in D the kth column by the column with the entries* $b_1, \cdots, b_n$.

(b) *Hence if the system (6) is* **homogeneous** *and* $D \neq 0$, *it has only the trivial solution* $x_1 = 0, x_2 = 0, \cdots, x_n = 0$. *If* $D = 0$, *the homogeneous system also has nontrivial solutions.*

Inverse of a Matrix.
Gauss–Jordan Elimination

- The **inverse** of an $n \times n$ matrix $A = [a_{jk}]$ is denoted by $A^{-1}$and is an $n \times n$ matrix such that

$$\mathrm{A}\ \mathrm{A}^{-1} = \mathrm{A}^{-1}\ \mathrm{A} = \mathrm{I}$$

where **I** is the $n \times n$ unit matrix.

- If **A** has an inverse, then **A** is called a **nonsingular matrix**.
- If **A** has no inverse, then **A** is called a **singular matrix**.
- If **A** has an inverse, the inverse is unique.

Inverse of a Matrix.
Gauss–Jordan Elimination

Indeed, if both **B** and **C** are inverses of **A**, then AB = I and CA = I, so that we obtain the uniqueness from

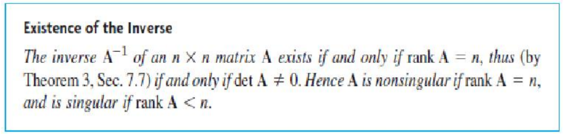
$$\mathbf{B = IB = (CA)B = C(AB) = CI = C}$$

- **A** has an inverse (is nonsingular) if and only if it has maximum

  possible rank *n.*
- Ax = b implies x = A^{-1}b provided A^{-1} exists, and will thus give a motivation for the inverse as well as a relation to linear systems.

Inverse of a Matrix.
Gauss–Jordan Elimination

Theorem 1



Existence of the Inverse

The inverse A^{-1} *of an* $n \times n$ *matrix* A *exists if and only if* rank A $= n$, *thus* (by Theorem 3, Sec. 7.7) *if and only if* det A $\neq 0$. *Hence* A *is nonsingular if* rank A $= n$, *and is singular if* rank A $< n$.

- **Proof:** Let **A** be a given $n \times n$ matrix and consider the linear system, **Ax = b.** If the inverse A^{-1} exists, then multiplication from the left on both sides and use of (1) gives, $\boldsymbol{A^{-1}}$**Ax = x =** $\boldsymbol{A^{-1}}$**b.** This shows that **Ax = b** has a solution **x**, which is unique because, for another solution **u**, we have Au = b , so that **u =** $\boldsymbol{A^{-1}}$**b = x** . Hence **A** must have rank *n* by the Fundamental Theorem in Sec. 7.5.

Inverse of a Matrix. Gauss–Jordan Elimination

- Conversely, let rank A = n . Then by the same theorem, the system Ax = b has a unique solution **x** for any **b**. Now the back substitution following the Gauss elimination (Sec. 7.3) shows that the components x_j of **x** are linear combinations of those of **b**. Hence we can write: x = Bb

Ax = A(Bb) (AB)b = Cb = b for any **b**.

Hence C = AB = I , the unit matrix.

x = Bb = B(Ax) = (BA)x

for any **x** (and **b = Ax**). Hence BA = I . Together, **B =** $\boldsymbol{A^{-1}}$ exists.

Inverse of a Matrix. Gauss–Jordan Elimination

Determination of the Inverse by the Gauss–Jordan Method

Using **A**, we form n linear systems

$$Ax_{(1)} = e_{(1)}, \dots, Ax_{(n)} = e_{(n)}$$

where the vectors $e_{(1)}$, …, $e_{(n)}$ are the columns of the $n \times n$ unit matrix **I;** thus, $e_{(1)} = [1 \quad 0 \quad \dots \quad 0]^T$, $e_{(2)}$ = $[0 \quad 1 \quad \dots \quad 0]^T$, etc.

We combine them into a single matrix equation AX = I, with the unknown matrix **X** having the columns x , …, $x_{(n)}$.

Correspondingly, we combine the n augmented matrices [A $e_{(1)}$],…, [A $e_{(n)}$] into one wide $n \times 2n$ “augmented matrix” $\tilde{A}$= [A I].

Inverse of a Matrix. Gauss–Jordan Elimination

Example 1: **Finding the Inverse of a Matrix by Gauss–Jordan Elimination**

Determine the inverse $\boldsymbol{A^{-1}}$ of

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

$$\tilde{A} = [\mathrm{A} \ \ \mathrm{I}] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Inverse of a Matrix. Gauss–Jordan Elimination

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \text{Row 2 + 3 Row 1} \\ \text{Row 3 − Row 1} \end{array}$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] \begin{array}{l} \\ \\ \text{Row 3 − Row 2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 3.5 & 1.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \begin{array}{l} \text{−Row 1} \\ \text{0.5 Row 2} \\ \text{−0.2 Row 3} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0.6 & 0.4 & -0.4 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \begin{array}{l} \text{Row 1 + 2 Row 3} \\ \text{Row 2 − 3.5 Row 3} \\ \\ \end{array}$$

Inverse of a Matrix.
Gauss–Jordan Elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \begin{array}{l} \text{Row 1 + Row 2} \\ \\ \\ \end{array}$$

The last three columns constitute A^{-1}. Check:

$$\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Inverse of a Matrix.
Gauss–Jordan Elimination

Example 2: Inverse of a 2 × 2 Matrix by Determinants

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}, \quad \mathbf{A}^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$$

Example 3: find the inverse of

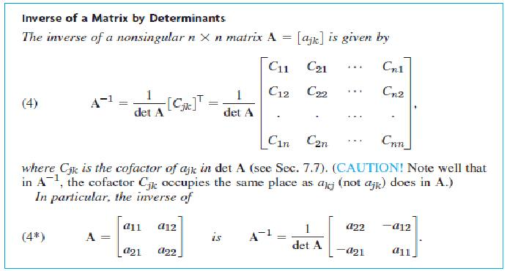
$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

Inverse of a Matrix.
Gauss–Jordan Elimination

Formulas for Inverses

Cramer’s rule was useful for theoretical study but not for computation.

Theorem 2



Inverse of a Matrix by Determinants

The inverse of a nonsingular $n \times n$ *matrix* $\mathbf{A} = [a_{jk}]$ *is given by*

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}[C_{jk}]^{\mathsf{T}} = \frac{1}{\det \mathbf{A}}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \cdot & \cdot & \cdots & \cdot \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}, \tag{4}$$

where C_{jk} *is the cofactor of* a_{jk} *in* det A (see Sec. 7.7). (CAUTION! Note well that in $\mathbf{A}^{-1}$, the cofactor C_{jk} occupies the same place as a_{kj} (not a_{jk}) does in A.)

In particular, the inverse of

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \textit{is} \quad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}. \tag{4*}$$

Inverse of a Matrix.
Gauss–Jordan Elimination

We obtain det A = -1(-7)-1(13)+2(8) = 10

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7, \quad C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2, \quad C_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13, \quad C_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2, \quad C_{32} = -\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = 7,$$

$$C_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8, \quad C_{23} = -\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2, \quad C_{33} = \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -2,$$

Then

$$\mathbf{A}^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}.$$

Inverse of a Matrix. Gauss–Jordan Elimination

Diagonal matrices A = $[a_{jk}]$, a_{jk} = 0 when j ≠ k have an inverse if and only if all $a_{jj} \neq 0$. Then A^{-1} is diagonal, too, with entries $1/a_{11}, \ldots, 1/a_{nn}$.

Proof: For a diagonal matrix we have in (4)

$$\frac{C_{11}}{D} = \frac{a_{22}\cdots a_{nn}}{a_{11}a_{22}\cdots a_{nn}} = \frac{1}{a_{11}}, \qquad \text{etc.}$$

Inverse of a Matrix. Gauss–Jordan Elimination

Example 4: **Inverse of a Diagonal Matrix**

$$\mathbf{A} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\mathbf{A}^{-1} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Inverse of a Matrix. Gauss–Jordan Elimination

Products can be inverted by taking the inverse of each factor and multiplying these inverses ***in reverse order***,

$$(\mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}.$$

Hence for more than two factors,

$$(\mathrm{AC}\cdots\mathrm{PQ})^{-1} = \mathrm{Q}^{-1}\mathrm{P}^{-1}\cdots\mathrm{C}^{-1}\mathrm{A}^{-1}.$$

We also note that the inverse of the inverse is the given matrix, as you may prove,

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}.$$

Inverse of a Matrix. Gauss–Jordan Elimination

Unusual Properties of Matrix Multiplication. Cancellation Laws

[1] Matrix multiplication is not commutative, that is, in general we have

$$\mathbf{AB} \neq \mathbf{BA}$$

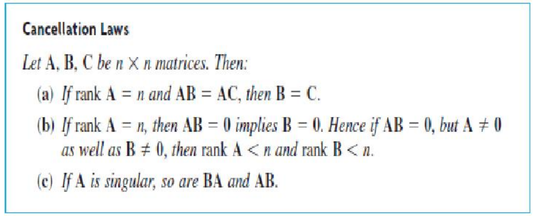
[2] B =0 does not generally imply A = 0 or B =0 (or BA = 0); for example,

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Inverse of a Matrix. Gauss–Jordan Elimination

[3] AC = AD does not generally imply C = D (even when A ≠ 0)

Theorem 3



Cancellation Laws

Let A, B, C *be* $n \times n$ *matrices. Then:*

(a) *If* rank A = n *and* AB = AC, *then* B = C.

(b) *If* rank A = n, *then* AB = 0 *implies* B = 0. *Hence if* AB = 0, *but* A ≠ 0 *as well as* B ≠ 0, *then* rank A < n *and* rank B < n.

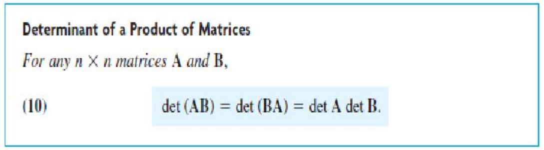
(c) *If* A *is singular, so are* BA *and* AB.

Inverse of a Matrix. Gauss–Jordan Elimination

Determinants of Matrix Products

It is interesting that det AB = det BA , although **AB ≠ BA** in general.

Theorem 4



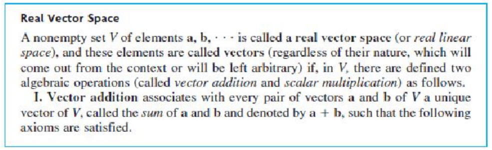
Determinant of a Product of Matrices

For any $n \times n$ *matrices* A *and* B,

(10) $\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det \mathbf{A} \det \mathbf{B}$.

Vector Spaces, Inner Product Spaces, Linear Transformations

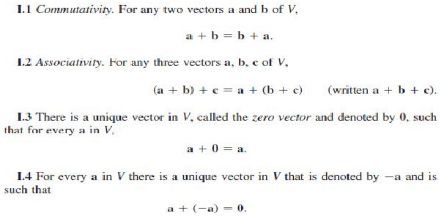
Definition



Real Vector Space

A nonempty set V of elements **a**, **b**, · · · is called a **real vector space** (or *real linear space*), and these elements are called **vectors** (regardless of their nature, which will come out from the context or will be left arbitrary) if, in V, there are defined two algebraic operations (called *vector addition* and *scalar multiplication*) as follows.

I. Vector addition associates with every pair of vectors **a** and **b** of V a unique vector of V, called the *sum* of **a** and **b** and denoted by **a** + **b**, such that the following axioms are satisfied.



I.1 *Commutativity.* For any two vectors **a** and **b** of V,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}.$$

I.2 *Associativity.* For any three vectors **a**, **b**, **c** of V,

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) \qquad (\text{written } \mathbf{a} + \mathbf{b} + \mathbf{c}).$$

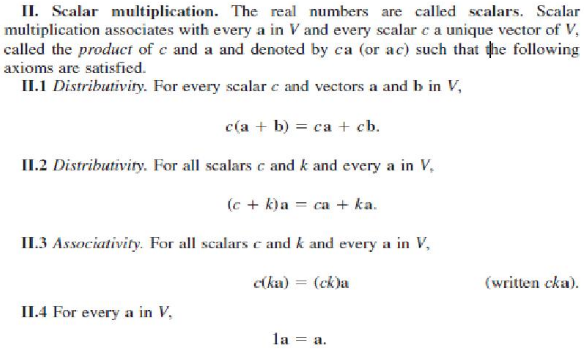
I.3 There is a unique vector in V, called the *zero vector* and denoted by **0**, such that for every **a** in V,

$$\mathbf{a} + \mathbf{0} = \mathbf{a}.$$

I.4 For every **a** in V there is a unique vector in V that is denoted by −**a** and is such that

$$\mathbf{a} + (-\mathbf{a}) = \mathbf{0}.$$

Vector Spaces, Inner Product Spaces, Linear Transformations



II. Scalar multiplication. The real numbers are called **scalars**. Scalar multiplication associates with every **a** in V and every scalar c a unique vector of V, called the *product* of c and **a** and denoted by $c\mathbf{a}$ (or $\mathbf{a}c$) such that the following axioms are satisfied.

II.1 *Distributivity.* For every scalar c and vectors **a** and **b** in V,

$$c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}.$$

II.2 *Distributivity.* For all scalars c and k and every **a** in V,

$$(c + k)\mathbf{a} = c\mathbf{a} + k\mathbf{a}.$$

II.3 *Associativity.* For all scalars c and k and every **a** in V,

$$c(k\mathbf{a}) = (ck)\mathbf{a} \qquad (\text{written } ck\mathbf{a}).$$

II.4 For every **a** in V,

$$1\mathbf{a} = \mathbf{a}.$$

Vector Spaces, Inner Product Spaces, Linear Transformations

A **linear combination** of vectors $a_{(1)}, \dots, a_{(m)}$ in a vector space V is an expression

$$c_1a_{(1)} + \cdots + c_m a_{(m)} \quad (c_1, \dots, c_m \text{ any scalars}).$$

These vectors form a **linearly independent set** (briefly, they are called **linearly independent**) if

$$(1)\, c_1a_{(1)} + \cdots + c_m a_{(m)} = 0$$

implies that $c_1 = 0, \dots, c_m = 0$.

Otherwise, if (1) also holds with scalars not all zero, the vectors are called linearly dependent.

Note that (1) with m = 1 is ca = 0 and shows that a single vector a is linearly independent if and only if a≠0.

Vector Spaces, Inner Product Spaces, Linear Transformations

1. V has ***dimension n***, or is ***n-dimensional***, if it contains a linearly independent set of n vectors, whereas any set of more than n vectors in V is linearly dependent.
2. That set of ***n*** linearly independent vectors is called a basis for V.
3. Then every vector in V can be written as a linear combination of the basis vectors. Furthermore, for a given basis, this representation is unique (see Prob. 2).

Example 1: **Vector Space of Matrices**

The 2× 2 real matrices form a four-dimensional real vector space. A basis is

Vector Spaces, Inner Product Spaces, Linear Transformations

$$B_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Because any 2× 2 matrix A = $[a_{jk}]$ has a unique representation

$$\mathbf{A} = a_{11}\boldsymbol{B}_{\mathbf{11}} + a_{12}\boldsymbol{B}_{\mathbf{12}} + a_{21}\boldsymbol{B}_{\mathbf{21}} + a_{22}\boldsymbol{B}_{\mathbf{22}}$$

Example 2:Vector Space of Polynomials

The set of all constant, linear, and quadratic polynomials in x together is a vector space of dimension 3 with basis$\{1, x, x^2\}$ under the usual addition and multiplication by real numbers because these two operations give polynomials not exceeding degree 2.

Vector Spaces, Inner Product Spaces, Linear Transformations

If a vector space V contains a linearly independent set of n vectors for every n, no matter how large, then V is called **infinite dimensional**, as opposed to a *finite dimensional* (n-dimensional) vector space just defined.

An example of an infinite dimensional vector space is the space of all continuous functions on some interval [a, b] of the x-axis.

Vector Spaces, Inner Product Spaces, Linear Transformations

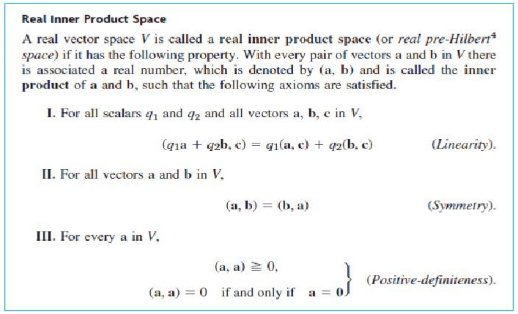
Inner Product Spaces

- If **a** and **b** are vectors in R^n , regarded as column vectors, we can form the product $\boldsymbol{a}^T\boldsymbol{b}$. This is a 1 × 1 matrix, which we can identify with its single entry, that is, with a number.
- This product is called the **inner product** or **dot product** of **a** and **b**. Other notations for it are (***a, b***) and $\boldsymbol{a} \cdot \boldsymbol{b}$. Thus

$$\mathbf{a}^T\mathbf{b} = (\mathbf{a}, \mathbf{b}) = \mathbf{a} \bullet \mathbf{b} = [a_1 \cdots a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + \cdots + a_n b_n.$$

Vector Spaces, Inner Product Spaces, Linear Transformations

Definition



Real Inner Product Space

A real vector space V is called a **real inner product space** (or *real pre-Hilbert*[4] *space*) if it has the following property. With every pair of vectors a and b in V there is associated a real number, which is denoted by (a, b) and is called the **inner product** of **a** and b, such that the following axioms are satisfied.

I. For all scalars q_1 and q_2 and all vectors **a, b, c** in V,

$$(q_1\mathbf{a} + q_2\mathbf{b}, \mathbf{c}) = q_1(\mathbf{a}, \mathbf{c}) + q_2(\mathbf{b}, \mathbf{c}) \qquad (\textit{Linearity}).$$

II. For all vectors a and **b** in V,

$$(\mathbf{a}, \mathbf{b}) = (\mathbf{b}, \mathbf{a}) \qquad (\textit{Symmetry}).$$

III. For every **a** in V,

$$\left.\begin{array}{r} (\mathbf{a}, \mathbf{a}) \geqq 0, \\ (\mathbf{a}, \mathbf{a}) = 0 \quad \text{if and only if} \quad \mathbf{a} = \mathbf{0} \end{array}\right\} \quad (\textit{Positive-definiteness}).$$

Vector Spaces, Inner Product Spaces, Linear Transformations

- Vectors whose inner product is zero are called **orthogonal**.
- The *length* or **norm** of a vector in V is defined by

$$\|\mathbf{a}\| = \sqrt{(\mathbf{a}, \mathbf{a})} \quad (\geqq 0).$$

- A vector

$$|(\mathbf{a}, \mathbf{b})| \leqq \|\mathbf{a}\| \, \|\mathbf{b}\| \qquad (\textit{Cauchy–Schwarz inequality}).$$

$$\|\mathbf{a} + \mathbf{b}\| \leqq \|\mathbf{a}\| + \|\mathbf{b}\| \qquad (\textit{Triangle inequality}).$$

Vector Spaces, Inner Product Spaces, Linear Transformations

$$\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 = 2(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2) \qquad (\textit{Parallelogram equality}).$$

Example 3: n-Dimensional Euclidean Space

R^n with the inner product

$$(\mathbf{a}, \mathbf{b}) = \mathbf{a}^T\mathbf{b} = a_1 b_1 + \cdots + a_n b_n$$

(where both **a** and **b** are *column* vectors) is called the ***n*** **dimensional Euclidean space** and is denoted by E^n or again simply by R^n . Axioms I–III hold, as direct calculation shows. Equation (2) gives the "**Euclidean norm**"

$$\|\mathbf{a}\| = \sqrt{(\mathbf{a}, \mathbf{a})} = \sqrt{\mathbf{a}^T\mathbf{a}} = \sqrt{a_1^2 + \cdots + a_n^2}.$$

Vector Spaces, Inner Product Spaces, Linear Transformations

Example 4: **An Inner Product for Functions. Function Space**

The set of all real-valued continuous functions $f(x), g(x), \ldots$ on a given interval $\alpha \le x \le \beta$ is a real vector space under the usual addition of functions and multiplication by scalars (real numbers). On this "***function space***" we can define an inner product by the integral

$$(f, g) = \int_{\alpha}^{\beta} f(x)\, g(x)\, dx.$$

Axioms I–III can be verified by direct calculation. Norm will be

$$\|f\| = \sqrt{(f, f)} = \sqrt{\int_{\alpha}^{\beta} f(x)^2\, dx}.$$

Vector Spaces, Inner Product Spaces, Linear Transformations

Linear Transformations

- Let X and Y be any vector spaces. To each vector **x** in X we assign a unique vector **y** in Y. Then we say that a **mapping** (or **transformation** or **operator**) of X into Y is given. Such a mapping is denoted by a capital letter, say F. The vector **y** in Y assigned to a vector **x** in X is called the **image** of **x** under F and is denoted by $F(x)$ [or F**x**, without parentheses].
- F is called a **linear mapping** or **linear transformation** if, for all vectors **v** and **x** in X and scalars c,

$$F(\mathbf{v} + \mathbf{x}) = F(\mathbf{v}) + F(\mathbf{x})$$

$$F(c\mathbf{x}) = cF(\mathbf{x}).$$

Vector Spaces, Inner Product Spaces, Linear Transformations

Linear Transformation of Space R^n into Space R^m

- From now on we let $\mathbf{X} = \boldsymbol{R^n}$ and $\mathbf{Y} = \boldsymbol{R^m}$. Then any real $\mathbf{m} \times \mathbf{n}$ matrix $\mathbf{A} = [a_{jk}]$ gives a transformation of $\boldsymbol{R^n}$ into $\boldsymbol{R^m}$,

$$\mathbf{y} = \mathbf{A}\mathbf{x}.$$

- Since **A(u+x)** = **Au** + **Au** and **A(cx)** = **cAx**, this transformation is linear.
- We show that, conversely, every linear transformation F of $\boldsymbol{R^n}$ into $\boldsymbol{R^m}$ can be given in terms of an **m** × **n** matrix **A**, after a basis for $\boldsymbol{R^n}$ and a basis for $\boldsymbol{R^m}$have been chosen.

Vector Spaces, Inner Product Spaces, Linear Transformations

- The purpose of a "**representation**" is the replacement of one object of study by another object whose properties are more readily apparent.
- In three-dimensional Euclidean space E^n the standard basis is usually writtene$_{(1)} = i, e_{(2)} = j, e_{(3)} = k$. Thus

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

- These are the three unit vectors in the positive directions of the axes of the **Cartesian coordinate system in space**, that is, the usual coordinate system with the same scale of measurement on the three mutually perpendicular coordinate axes.

Vector Spaces, Inner Product Spaces, Linear Transformations

Example 5: **Linear Transformations**

Interpreted as transformations of Cartesian coordinates in the plane, the matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

represent a reflection in the line $x_2 = x_1$, a reflection in the x_1-axis, a reflection in the origin, and a stretch (when a > 1 , or a contraction when 0 < a <1) in the x_1-direction, respectively.

Vector Spaces, Inner Product Spaces, Linear Transformations

Example 6: **Linear Transformations**

Our discussion preceding Example 5 is simpler than it may look at first sight. To see this, find **A** representing the linear transformation that maps (x_1, x_2) onto (2x_1-5x_2, 3x_1+4x_2)

Solution. Obviously, the transformation is

$$\begin{aligned} y_1 &= 2x_1 - 5x_2 \\ y_2 &= 3x_1 + 4x_2. \end{aligned}$$

$$A = \begin{bmatrix} 2 & -5 \\ 3 & 4 \end{bmatrix}. \qquad \text{Check:} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - 5x_2 \\ 3x_1 + 4x_2 \end{bmatrix}.$$

Vector Spaces, Inner Product Spaces, Linear Transformations

y = Ax, If **A** in *is square $\boldsymbol{n} \times \boldsymbol{n}$, , then *maps $\boldsymbol{R^n}$ into $\boldsymbol{R^m}$. If this **A** is nonsingular, so that $\boldsymbol{A^{-1}}$ exists (see Sec. 7.8), then multiplication of * by $\boldsymbol{A^{-1}}$ from the left and use of $\mathbf{A^{-1}A = I}$ gives the **inverse transformation**

$$\mathbf{x} = A^{-1}\mathbf{y}$$

It maps every y = $y_0$onto that **x**, which by *is mapped onto y_0 . *The inverse of a linear transformation is itself linear,* because it is given by a matrix, as $\mathbf{x} = \boldsymbol{A^{-1}}\mathbf{y}$ shows.

Vector Spaces, Inner Product Spaces, Linear Transformations

The last operation we want to discuss is composition of linear transformations. Let *X*, *Y*, *W* be general vector spaces. Let *F* be a linear transformation from *X* to *Y*. Let *G* be a linear transformation from *W* to *X*. Then we denote, by *H*, the **composition** of *F* and *G*, that is,

$$H = F \circ G = FG = F(G),$$

which means we take transformation G and then apply transformation F to it (***in that order!***, i.e. you go from left to right).

Vector Spaces, Inner Product Spaces, Linear Transformations

Now, to give this a more concrete meaning, if we let **w** be a vector in *W*, then ***G(w)*** is a vector in *X* and ***F(G(w))*** is a vector in *Y*. Thus, *H* maps *W* to *Y*, and we can write

$$H(\mathbf{w}) = (F \circ G)(\mathbf{w}) = (FG)(\mathbf{w}) = F(G(\mathbf{w})),$$

which completes the definition of composition in a general vector space setting. But is composition really linear?

Vector Spaces, Inner Product Spaces, Linear Transformations

Example 7: The Composition of Linear Transformations Is Linear

To show that *H* is indeed linear we must show that (10) holds. We have, for two vectors w_1, w_2 in *W*,

$$\begin{aligned} H(\mathbf{w}_1 + \mathbf{w}_2) &= (F \circ G)(\mathbf{w}_1 + \mathbf{w}_2) \\ &= F(G(\mathbf{w}_1 + \mathbf{w}_2)) \\ &= F(G(\mathbf{w}_1) + G(\mathbf{w}_2)) && \text{(by linearity of } G\text{)} \\ &= F(G(\mathbf{w}_1)) + F(G(\mathbf{w}_2)) && \text{(by linearity of } F\text{)} \\ &= (F \circ G)(\mathbf{w}_1) + (F \circ G)(\mathbf{w}_2) && \text{(by (15))} \\ &= H(\mathbf{w}_1) + H(\mathbf{w}_2) && \text{(by definition of } H\text{).} \end{aligned}$$

Similarly, $H(c\mathbf{w}_2) = (F \circ G)(c\mathbf{w}_2) = F(G(c\mathbf{w}_2)) = F(c(G(\mathbf{w}_2))$

$$= cF(G(\mathbf{w}_2)) = c(F \circ G)(\mathbf{w}_2) = cH(\mathbf{w}_2).$$

Vector Spaces, Inner Product Spaces, Linear Transformations

- We showed that the composition of linear transformations is indeed **linear**.
- Next we want to relate composition of linear transformations to matrix multiplication.
- To do so we let **X** = $\boldsymbol{R^n}$, **Y** = $\boldsymbol{R^m}$ and **W** = $\boldsymbol{R^p}$. This choice of particular vector spaces allows us to represent the linear transformations as matrices and form matrix equations.
- Thus F can be represented by a general real m × n matrix A = $[a_{jk}]$ and G by an n × p matrix B = $[b_{jk}]$. Then we can write for F, with column vectors x with n entries, and resulting vector y, with m entries

$$y = \mathbf{A}x$$

Vector Spaces, Inner Product Spaces, Linear Transformations

and similarly for *G*, with column vector **w** with *p* entries,

$$x = \mathbf{B}w$$

Then,

$$y = Ax = A(Bw) = Abw = Cw \quad \text{where } C = AB$$

we can define the composition of linear transformations in the Euclidean spaces as multiplication by matrices. Hence, m × p the real matrix **C** represents a linear transformation *H* which maps $\boldsymbol{R^p}$ to $\mathrm{R^n}$ with vector **w**, a column vector with *p* entries.

Vector Spaces, Inner Product Spaces, Linear Transformations

Example 8: **Linear Transformations. Composition**

In Example 5 of Sec. 7.9, let **A** be the first matrix and **B** be the fourth matrix with a > 1. Then, applying **B** to a vector w = $[w_1 w_2]^T$, stretches the element w_1 by a in the x_1 direction. Next, when we apply **A** to the “stretched” vector, we reflect the vector along the line $x_1 = x_2$, resulting in a vector y = $[w_1 a w_2]^T$. But this represents, precisely, a geometric description for the composition *H* of two linear transformations *F* and *G* represented by matrices **A** and **B**. We now show that, for this example, our result can be obtained by straightforward matrix multiplication, that is,

$$\mathbf{AB} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}$$

Vector Spaces, Inner Product Spaces, Linear Transformations

Then

$$\mathbf{ABw} = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_2 \\ aw_1 \end{bmatrix},$$

which is the same as before. This shows that indeed **AB = C** , and we see the composition of linear transformations can be represented by a linear transformation. It also shows that the order of matrix multiplication is important (!). You may want to try applying **A** first and then **B**, resulting in **BA**. What do you see? Does it make geometric sense? Is it the same result as **AB?**