In the Name of God

Advanced Engineering Mathematics

Linear Algebra

Matrix Eigenvalue Problems

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Fall 2014

Outlines

Linear Algebra: Matrix Eigenvalue Problems

Ref: Erwin Kreyszig, "Advanced Engineering Mathematics," John Wiley & Sons, Inc., 10th Ed., 2011. (Chapter 8)

Introduction

A matrix eigenvalue problem considers the vector equation

(1) Ax = lx

- Here **A** is a given square matrix, **l** an unknown scalar, and **x** an unknown vector.
- In a matrix eigenvalue problem, the task is to determine **l**'s and **x**'s that satisfy (1).
- Since **x** = **0** is always a solution for any **l** and thus not interesting, we only admit solutions with **x ¹ 0**.
- The **l**'s that satisfy (1) are called **eigenvalues of A** and the corresponding nonzero **x**'s that also satisfy (1) are called **eigenvectors of A**.

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The Matrix Eigenvalue Problem. Determining Eigenvalues and Eigenvectors

Consider multiplying nonzero vectors by a given square matrix, such as

$$\begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 33 \\ 27 \end{bmatrix}, \qquad \begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 30 \\ 40 \end{bmatrix}.$$

- In the first case, we get a totally new vector with a different direction and different length when compared to the original vector. This is what usually happens and is of no interest here.

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- In the second case something interesting happens. The multiplication produces a vector $[30\ 40]^T = 10[3\ 4]^T$, which means the new vector has the same direction as the original vector. The scale constant, which we denote by l is 10.
- We formalize our observation. Let A = $[a_{jk}]$ be a given nonzero square matrix of dimension n × n. Consider the following vector equation:

$$\mathbf{Ax} = \lambda \mathbf{x}. \quad (1)$$

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Remark:

- Geometrically, we are looking for vectors, **x**, for which the multiplication by **A** has the same effect as the multiplication by a scalar in other words, **Ax** should be proportional to **x**. Thus, the multiplication has the effect of producing, from the original vector **x**, a new vector that has the same or opposite (minus sign) direction as the original vector.
- A value of l for which (1) has a solution **x ¹ 0**, is called an **eigenvalue** or *characteristic value* of the matrix **A**. Another term for is a *latent* root. ("Eigen" is German and means "proper" or "characteristic.").

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- The set of all the eigenvalues of **A** is called the **spectrum** of **A**.
- We shall see that the spectrum consists of at least one eigenvalue and at most of *n* numerically different eigenvalues.
- The largest of the absolute values of the eigenvalues of **A** is called the *spectral radius* of **A**, a name to be motivated later.

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How to Find Eigenvalues and Eigenvectors (p.324)

Example1: Determination of Eigenvalues and Eigenvectors

We illustrate all the steps in terms of the matrix

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}.$$

Solution. **(a)** ***Eigenvalues.*** These must be determined ***first***. Equation (1) is

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$$\mathbf{Ax} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix};$$ in components,

$$-5x_1 + 2x_2 = \lambda x_1$$

$$2x_1 - 2x_2 = \lambda x_2.$$

Transferring the terms on the right to the left, we get

The Matrix Eigenvalue Problem. Determining Eigenvalues and Eigenvectors

(2*)
$$(-5 - \lambda)x_1 + 2x_2 = 0$$
$$2x_1 + (-2 - \lambda)x_2 = 0.$$

This can be written in matrix notation

(3*) $$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

We see that this is a ***homogeneous*** linear system. By Cramer's theorem in Sec. 7.7 it has a nontrivial solution (an eigenvector of **A** we are looking for) if and only if its coefficient determinant is zero, that is,

The Matrix Eigenvalue Problem. Determining Eigenvalues and Eigenvectors

(4*) $$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -5 - \lambda & 2 \\ 2 & -2 - \lambda \end{vmatrix} =$$

$$(-5 - \lambda)(-2 - \lambda) - 4 = \lambda^2 + 7\lambda + 6 = 0.$$

- We call *D(l)* the **characteristic determinant** or, if expanded, the **characteristic polynomial**, and *D(l) = 0* the **characteristic equation** of **A**.
- The solutions of this quadratic equation are $\lambda_1 = -1$ and $\lambda_2 = -6$. These are the eigenvalues of **A**.

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(b_1**)** ***Eigenvector of* A *corresponding to*** λ_1 . This vector is obtained from (2*) with, l = $\lambda_1 = -1$, that is

$$-4x_1 + 2x_2 = 0$$

$$2x_1 - x_2 = 0.$$

A solution is $x_2 = 2x_1$, as we see from either of the two equations, so that we need only one of them. This determines an eigenvector corresponding to $\lambda_1 = -1$ up to a scalar multiple. If we choose $x_1 = 1$, we obtain the eigenvector

The Matrix Eigenvalue Problem. Determining Eigenvalues and Eigenvectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ Check:}$$

$$\mathbf{A}\mathbf{x}_1 = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$= (-1)\mathbf{x}_1 = \lambda_1 \mathbf{x}_1.$$

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$$\mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \text{ Check:}$$

$$\mathbf{A}\mathbf{x}_2 = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -12 \\ 6 \end{bmatrix} =$$

$$(-6)\mathbf{x}_2 = \lambda_2 \mathbf{x}_2.$$

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(b_2) *Eigenvector of* A *corresponding to* λ_2 . For $\lambda_2 = -6$, equation (2*) becomes

$$x_1 + 2x_2 = 0$$

$$2x_1 + 4x_2 = 0$$

A solution is $x_2 = -x_1/2$ with arbitrary x_1. If we choose $x_1 = 2$, we get $x_2 = -1$. Thus an eigenvector of **A** corresponding to $\lambda_2 = -6$ is

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This example illustrates the general case as follows. Equation (1) written in components is

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= \lambda x_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= \lambda x_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= \lambda x_n. \end{aligned}$$

Transferring the terms on the right side to the left side, we have

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$$
(2)\quad
\begin{aligned}
(a_{11}-\lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0\\
a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + a_{2n}x_n &= 0\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\\
a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n &= 0.
\end{aligned}
$$

In matrix notation,

$$(3)\qquad (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}.$$

The Matrix Eigenvalue Problem. Determining Eigenvalues and Eigenvectors

By Cramer's theorem in Sec. 7.7, this homogeneous linear system of equations has a nontrivial solution if and only if the corresponding determinant of the coefficients is zero:

$$
(4)\quad D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) =
\begin{vmatrix}
a_{11}-\lambda & a_{12} & \cdots & a_{1n}\\
a_{21} & a_{22}-\lambda & \cdots & a_{2n}\\
\cdot & \cdot & \cdots & \cdot\\
a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda
\end{vmatrix} = 0.
$$

- Equation (4) is called the **characteristic equation** of **A**. By developing we obtain a polynomial of *n*th degree in λ.

The Matrix Eigenvalue Problem. Determining Eigenvalues and Eigenvectors

Theorem 1

Eigenvalues

The eigenvalues of a square matrix **A** *are the roots of the characteristic equation (4) of* **A**.

Hence an $n \times n$ *matrix has at least one eigenvalue and at most* n *numerically different eigenvalues.*

- For larger *n*, the actual computation of eigenvalues will, in general, require the use of Newton's method (Sec. 19.2) or another numeric approximation method in Secs.20.7–20.9.

The Matrix Eigenvalue Problem. Determining Eigenvalues and Eigenvectors

Theorem 2

Eigenvectors, Eigenspace

If **w** *and* **x** *are eigenvectors of a matrix* **A** *corresponding to* ***the same*** *eigenvalue* λ, *so are* $\mathbf{w} + \mathbf{x}$ (provided $\mathbf{x} \neq -\mathbf{w}$) *and* $k\mathbf{x}$ *for any* $k \neq 0$.

Hence the eigenvectors corresponding to one and the same eigenvalue λ *of* **A**, *together with* **0**, *form a vector space* (cf. Sec. 7.4), *called the* **eigenspace** *of* **A** *corresponding to that* λ.

Proof: Aw = lw and Ax = lx imply A(w+x) = Aw + Ax = lw + lx = l**(w +x) and A(kw) = k(Aw) = k(**l**w) =**l(kw); **hence A(kw+lx) = l(kw+lx)**

The Matrix Eigenvalue Problem. Determining Eigenvalues and Eigenvectors

- In particular, *an eigenvector* **x** *is determined only up to a constant factor.* Hence we can **normalize** **x**, that is, multiply it by a scalar to get a unit vector (see Sec. 7.9).

- For instance, $x_1 = [1\ 2]^T$ in Example 1 has the length $\|x_1\| = \sqrt{1+4} = \sqrt{5}$; hence $[\frac{1}{\sqrt{5}}\ \frac{2}{\sqrt{5}}]^T$ is a normalized eigenvector (a unit eigenvector).

The Matrix Eigenvalue Problem. Determining Eigenvalues and Eigenvectors

Example 2: **Multiple Eigenvalues**

Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}.$$

Solution. For our matrix, the characteristic determinant gives the characteristic equation

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0.$$

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The roots (eigenvalues of **A**) are $\lambda_1 = 5, \lambda_2 = \lambda_3 = -3$.
To find eigenvectors, we apply the Gauss elimination (Sec. 7.3) to the system (A - λI)x = 0 , first with λ = 5 and then with λ = -3 . For λ = 5 the characteristic matrix is

$$\mathbf{A} - \lambda\mathbf{I} = \mathbf{A} - 5\mathbf{I} = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}.$$ It row-reduces to

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -\frac{24}{7} & -\frac{48}{7} \\ 0 & 0 & 0 \end{bmatrix}.$$

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- Hence it has rank 2. Choosing $x_3 = -1$ we have $x_2 = 2$ and then $x_1 = 1$. Hence an eigenvector of **A** corresponding to λ = 5 is $x_1 = [1\ \ 2\ \ -1]^T$. For λ = -3 the characteristic matrix

$$\mathbf{A} - \lambda\mathbf{I} = \mathbf{A} + 3\mathbf{I} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$$ row-reduces to

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

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- Hence it has rank 1. From $x_1 + 2x_2 - 3x_3 = 0$ we have $x_1 = -2x_2 + 3x_3$. Choosing x_2=1, $x_3 = 0$ and x_2=0, $x_3 = 1$, we obtain two linearly independent eigenvectors of **A** corresponding to λ = -3

$$\mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

The Matrix Eigenvalue Problem. Determining Eigenvalues and Eigenvectors

- The order M_λ of an eigenvalue λ as a root of the characteristic polynomial is called the **algebraic multiplicity** of λ .
- The number m_λ of linearly independent eigenvectors corresponding to λ is called the **geometric multiplicity** of λ .Thus m_λ is the dimension of the eigenspace corresponding to this λ.
- In general, m_λ£ M_λ , as can be shown. The difference $\Delta_\lambda = M_\lambda - m_\lambda$ is called the **defect** of λ.

The Matrix Eigenvalue Problem. Determining Eigenvalues and Eigenvectors

Example 3: **Algebraic Multiplicity, Geometric Multiplicity. Positive Defect**

The characteristic equation of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{is} \quad \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0.$$

- Hence $\lambda = 0$ is an eigenvalue of algebraic multiplicity $M_0 = 2$. But its geometric multiplicity is only $m_0 = 2$, since eigenvectors result from $0x_1 + x_2 = 0$, hence$x_2 = 0$, in the form $[x_1 \;\; 0]^T$. Hence for $\lambda = 0$ the defect is $\Delta_0 = 1$.

The Matrix Eigenvalue Problem. Determining Eigenvalues and Eigenvectors

Similarly, the characteristic equation of the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{is} \quad \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = 0.$$

- Hence $\lambda = 3$ is an eigenvalue of algebraic multiplicity $M_3 = 2$, but its geometric multiplicity is only $m_3 = 1$, since eigenvectors result from $0x_1 + 2x_2 = 0$ in the form $[x_1 \;\; 0]^T$

The Matrix Eigenvalue Problem. Determining Eigenvalues and Eigenvectors

Example 4: **Real Matrices with Complex Eigenvalues and Eigenvectors**

Since real polynomials may have complex roots (which then occur in conjugate pairs), a real matrix may have complex eigenvalues and eigenvectors. For instance, the characteristic equation of the skew-symmetric matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{is} \quad \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0.$$

It gives the eigenvalues $\lambda_1 = \mathrm{i}(= \sqrt{-1})$, $\lambda_2 = -\mathrm{i}$. Eigenvectors are obtained from $-ix_1 + x_2 = 0$ and , $ix_1 + x_2 = 0$ respectively, and we can choose $x_1 = 1$ to get

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$$\begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

Theorem 3

Eigenvalues of the Transpose

The transpose $\mathbf{A}^{\mathsf{T}}$ of a square matrix $\mathbf{A}$ has the same eigenvalues as $\mathbf{A}$.

Some Applications of Eigenvalue Problems

Example 1: **Stretching of an Elastic Membrane**

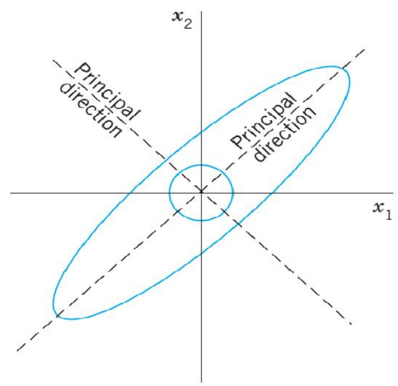
An elastic membrane in the x_1x_2-plane with boundary circle $x_1^2 + x_2^2 = 1$ is stretched so that a point P:(x_1,x_2) goes over into the point Q:(y_1,y_2) given by

$$(1) \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix};$$

in components,

$$y_1 = 5x_1 + 3x_2$$
$$y_2 = 3x_1 + 5x_2.$$

Some Applications of Eigenvalue Problems



Some Applications of Eigenvalue Problems

Find the **principal directions**, that is, the directions of the position vector **x** of *P* for which the direction of the position vector **y** of *Q* is the same or exactly opposite. ***Solution.***

We are looking for vectors **x** such that y = lx. Since y = lx , this gives Ax = lx , the equation of an eigenvalue problem. In components, Ax = lx is

$$(2) \quad \begin{aligned} 5x_1 + 3x_2 &= \lambda x_1 \\ 3x_1 + 5x_2 &= \lambda x_2 \end{aligned} \quad \text{or} \quad \begin{aligned} (5-\lambda)x_1 + \quad 3x_2 &= 0 \\ 3x_1 + (5-\lambda)x_2 &= 0. \end{aligned}$$

Some Applications of Eigenvalue Problems

For $\lambda_2 = 2$, our system (2) becomes

$$\begin{aligned} 3x_1 + 3x_2 &= 0, \\ 3x_1 + 3x_2 &= 0. \end{aligned} \quad \left| \quad \begin{aligned} &\text{Solution } x_2 = -x_1,\ x_1 \text{ arbitrary,} \\ &\text{for instance, } x_1 = 1, x_2 = -1. \end{aligned} \right.$$

- We thus obtain as eigenvectors of **A**, for instance, $[1 \quad 1]^T$ corresponding to λ_1 and $[1 \quad -1]^T$ corresponding to λ_2 (or a nonzero scalar multiple of these).
- These vectors make 45° and 135° angles with the positive $x1$-direction.

Some Applications of Eigenvalue Problems

The characteristic equation is

$$(3) \quad \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 9 = 0.$$

Its solutions are $\lambda_1 = 8$ and $\lambda_2 = 2$. These are the eigenvalues of our problem. For $\lambda_1 = 8$ our system (2) becomes

$$\begin{aligned} -3x_1 + 3x_2 &= 0, \\ 3x_1 - 3x_2 &= 0. \end{aligned} \quad \left| \quad \begin{aligned} &\text{Solution } x_2 = x_1,\ x_1 \text{ arbitrary,} \\ &\text{for instance, } x_1 = x_2 = 1. \end{aligned} \right.$$

Some Applications of Eigenvalue Problems

- The eigenvalues show that in the principal directions the membrane is stretched by factors 8 and 2, respectively;
- They give the principal directions, the answer to our problem. The eigenvalues show that in the principal directions the membrane is stretched by factors 8 and 2, respectively.
- if we choose the principal directions as directions of a new Cartesian u_1u_2-coordinate system, say, with the positive u_1 -semi-axis in the first quadrant and the positive u_2 -semi-axis in the second quadrant of the x_1x_2-system, and if we set $u_1 = r\sin\varphi,\ u_2 = r\cos\varphi$ then a boundary point of the unstretched circular membrane has coordinates $\cos\varphi, \sin\varphi$. Hence, after the stretch we have

Some Applications of Eigenvalue Problems

$$z_1 = 8\cos\phi, \qquad z_2 = 2\sin\phi.$$

Since $\cos^2\varphi + \sin^2\varphi = 1$, this shows that the deformed boundary is an ellipse

$$\frac{z_1^2}{8^2} + \frac{z_2^2}{2^2} = 1. \qquad (4)$$

Some Applications of Eigenvalue Problems

Example 2: **Eigenvalue Problems Arising from Markov Processes**

Markov processes as considered in Example 13 of Sec. 7.2 lead to eigenvalue problems if we ask for the limit state of the process in which the state vector **x** is reproduced under the multiplication by the stochastic matrix **A** governing the process, that is, **Ax** = **x** . Hence **A** should have the eigenvalue 1, and **x** should be a corresponding eigenvector. This is of practical interest because it shows the long-term tendency of the development modeled by the process.

In that example,

Some Applications of Eigenvalue Problems

$$\mathbf{A} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix}. \quad \text{For the transpose,}$$

$$\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.9 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Some Applications of Eigenvalue Problems

Hence A^T has the eigenvalue 1, and the same is true for **A** by Theorem 3 in Sec. 8.1. An eigenvector **x** of **A** for l = 1 is obtained from

$$\mathbf{A} - \mathbf{I} = \begin{bmatrix} -0.3 & 0.1 & 0 \\ 0.2 & -0.1 & 0.2 \\ 0.1 & 0 & -0.2 \end{bmatrix}, \quad \text{row-reduced to}$$

$$\begin{bmatrix} -\frac{3}{10} & \frac{1}{10} & 0 \\ 0 & -\frac{1}{30} & \frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix}.$$

Some Applications of Eigenvalue Problems

Taking $x_3 = 1$, we get $x_2 = 6$ from $-\frac{x_2}{6} + \frac{x_3}{5} = 0$ and then $x_1 = 2$ from $\frac{-3x_1}{10} + \frac{x_2}{10} = 0$.

This gives x = $[2 \;\; 6 \;\; 1]^T$. It means that in the long run, the ratio Commercial:Industrial:Residential will approach 2:6:1, provided that the probabilities given by **A** remain (about) the same. (We switched to ordinary fractions to avoid rounding errors.)

➢Please see example 3 on page 331.

Some Applications of Eigenvalue Problems

Example 4: **Vibrating System of Two Masses on Two Springs**

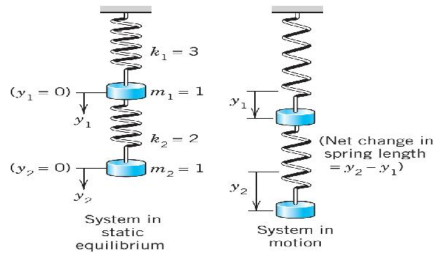
Mass–spring systems involving several masses and springs can be treated as eigenvalue problems. For instance, the mechanical system is governed by the system of ODEs

$$\text{(6)} \quad \begin{aligned} y_1'' &= -3y_1 - 2(y_1 - y_2) = -5y_1 + 2y_2 \\ y_2'' &= \qquad\qquad -2(y_2 - y_1) = \;\; 2y_1 - 2y_2 \end{aligned}$$

Where y_1 and y_2 are the displacements of the masses from rest, as shown in the figure, and primes denote derivatives with respect to time *t*. In vector form, this becomes

Some Applications of Eigenvalue Problems

$$\text{(7)} \quad y'' = \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \mathbf{Ay} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$



Some Applications of Eigenvalue Problems

We try a vector solution of the form

$$\text{(8)} \qquad \mathbf{y} = \mathbf{x}e^{\omega t}.$$

This is suggested by a mechanical system of a single mass on a spring (Sec. 2.4), whose motion is given by exponential functions (and sines and cosines). Substitution into (7) gives

$$\omega^2 \mathbf{x} e^{\omega t} = A\mathbf{x}e^{\omega t}.$$

Some Applications of Eigenvalue Problems

Dividing by $e^{\omega t}$ and writing $\omega^2 = \lambda$, we see that our mechanical system leads to the eigenvalue problem

$$\mathbf{Ax} = \lambda \mathbf{x} \tag{9}$$

From Example 1 in Sec. 8.1 we see that **A** has the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -6$. Consequently, $\omega = \pm\sqrt{-1} = \pm i$ and $\omega = \pm\sqrt{-6} = \pm i\sqrt{6}$ respectively. Corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}. \tag{10}$$

Some Applications of Eigenvalue Problems

From (8) we thus obtain the four complex solutions [see (10), Sec. 2.2]

$$\mathbf{x}_1 e^{\pm it} = \mathbf{x}_1(\cos t \pm i \sin t),$$

$$\mathbf{x}_2 e^{\pm i\sqrt{6}t} = \mathbf{x}_2(\cos \sqrt{6}\, t \pm i \sin \sqrt{6}\, t).$$

By addition and subtraction (see Sec. 2.2) we get the four real solutions

$$\mathbf{x}_1 \cos t, \quad \mathbf{x}_1 \sin t, \quad \mathbf{x}_2 \cos \sqrt{6}\, t, \quad \mathbf{x}_2 \sin \sqrt{6}\, t.$$

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A general solution is obtained by taking a linear combination of these,

$$\mathbf{y} = \mathbf{x}_1(a_1 \cos t + b_1 \sin t) + \mathbf{x}_2(a_2 \cos \sqrt{6}\, t + b_2 \sin \sqrt{6}\, t)$$

with arbitrary constants a_1, b_1, a_2, b_2 (to which values can be assigned by prescribing initial displacement and initial velocity of each of the two masses). By (10), the components of **y** are

$$y_1 = a_1 \cos t + b_1 \sin t + 2a_2 \cos \sqrt{6}\, t + 2b_2 \sin \sqrt{6}\, t$$

$$y_2 = 2a_1 \cos t + 2b_1 \sin t - a_2 \cos \sqrt{6}\, t - b_2 \sin \sqrt{6}\, t.$$

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Symmetric, Skew-Symmetric, and Orthogonal Matrices

Definition

Symmetric, Skew-Symmetric, and Orthogonal Matrices

A ***real*** square matrix $\mathbf{A} = [a_{jk}]$ is called

symmetric if transposition leaves it unchanged,

$$\mathbf{A}^{\mathsf{T}} = \mathbf{A}, \quad \text{thus} \quad a_{kj} = a_{jk}, \tag{1}$$

skew-symmetric if transposition gives the negative of **A**,

$$\mathbf{A}^{\mathsf{T}} = -\mathbf{A}, \quad \text{thus} \quad a_{kj} = -a_{jk}, \tag{2}$$

orthogonal if transposition gives the inverse of **A**,

$$\mathbf{A}^{\mathsf{T}} = \mathbf{A}^{-1}. \tag{3}$$

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Symmetric, Skew-Symmetric, and Orthogonal Matrices

Example 1: **Symmetric, Skew-Symmetric, and Orthogonal Matrices**

The matrices

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

are symmetric, skew-symmetric, and orthogonal.

Symmetric, Skew-Symmetric, and Orthogonal Matrices

➢ Any real square matrix **A** may be written as the sum of a symmetric matrix **R** and a skew-symmetric matrix **S**, where

$$(4) \quad \mathbf{R} = \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathsf{T}}) \quad \text{and} \quad \mathbf{S} = \tfrac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathsf{T}}).$$

Example 2: **Illustration of Formula (4)**

$$\mathbf{A} = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix} = \mathbf{R} + \mathbf{S} = \begin{bmatrix} 9.0 & 3.5 & 3.5 \\ 3.5 & 3.0 & -2.0 \\ 3.5 & -2.0 & 3.0 \end{bmatrix} + \begin{bmatrix} 0 & 1.5 & -1.5 \\ -1.5 & 0 & -6.0 \\ 1.5 & 6.0 & 0 \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

Theorem 1

Eigenvalues of Symmetric and Skew-Symmetric Matrices

(a) *The eigenvalues of a symmetric matrix are real.*

(b) *The eigenvalues of a skew-symmetric matrix are pure imaginary or zero.*

The proof is in Sec. 8.5.

Symmetric, Skew-Symmetric, and Orthogonal Matrices

Orthogonal Transformations and Orthogonal Matrices (p.336)

Orthogonal transformations are transformations

(5) $\mathbf{y} = \mathbf{A}\mathbf{x}$

With each vector **x** in R^n such a transformation assigns a vector **y** in R^n . For instance, the plane rotation through an angle θ

$$(6) \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

is an orthogonal transformation. It can be shown that any orthogonal transformation in the plane or in three-dimensional space is a **rotation**

Theorem 2

Invariance of Inner Product

An orthogonal transformation preserves the value of the **inner product** *of vectors* **a** *and* **b** *in* R^n, *defined by*

(7) $$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b} = [a_1 \ \cdots \ a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

That is, for any **a** *and* **b** *in* R^n, *orthogonal* $n \times n$ *matrix* **A**, *and* $\mathbf{u} = \mathbf{Aa}, \mathbf{v} = \mathbf{Ab}$ *we have* $\mathbf{u} \cdot \mathbf{v} = \mathbf{a} \cdot \mathbf{b}$.

Hence the transformation also preserves the **length** *or* **norm** *of any vector* **a** *in* R^n *given by*

(8) $$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{\mathbf{a}^T \mathbf{a}}.$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

Proof: Let u = Aa and v = Ab,

(9) $$\mathrm{u} \cdot \mathrm{v} = \mathrm{u}^T \mathrm{v} = (\mathrm{Aa})^T \mathrm{Ab} = \mathrm{a}^T \mathrm{A}^T \mathrm{Ab} = \mathrm{a}^T \mathrm{Ib} = \mathrm{a}^T \mathrm{b} = \mathrm{a} \cdot \mathrm{b}.$$

Orthonormality of Column and Row Vectors

A real square matrix is orthogonal if and only if its column vectors $\mathbf{a}_1, \cdots, \mathbf{a}_n$ *(and also its row vectors) form an* **orthonormal system**, *that is,*

(10) $$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j^T \mathbf{a}_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases}$$

Please see the proof on page 337

Symmetric, Skew-Symmetric, and Orthogonal Matrices

Theorem 4

Determinant of an Orthogonal Matrix

The determinant of an orthogonal matrix has the value $+1$ *or* -1.

Proof: (1) det (AA^{-1}) = det (A) det(A^{-1}) = 1(Sec. 7.8, Theorem 4)

(2) det (A) = det (A^T) = det (A^{-1}) (Sec. 7.7, Theorem 2d),

(1),(2): det² (A) = 1

Symmetric, Skew-Symmetric, and Orthogonal Matrices

Theorem 5

Eigenvalues of an Orthogonal Matrix

The eigenvalues of an orthogonal matrix **A** *are real or complex conjugates in pairs and have absolute value* 1.

Example 5: **Eigenvalues of an Orthogonal Matrix**

The orthogonal matrix in Example 1 has the characteristic equation

$$-\lambda^3 + \tfrac{2}{3}\lambda^2 + \tfrac{2}{3}\lambda - 1 = 0.$$

Eigenvalues: $-1, 5 + i\sqrt{\pi}, 5 - i\sqrt{\pi}$

Eigenbases. Diagonalization. Quadratic Forms

Eigenvectors of an $n \times n$ matrix **A** may (or may not!) form a basis for R^n. If we are interested in a transformation such an “**eigenbasis**” (basis of eigenvectors)—if it exists—is of great advantage because then we can represent any **x** in R^n uniquely as a linear combination of the eigenvectors $x_1, \dots, x_n$, say,

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n.$$

And, denoting the corresponding (not necessarily distinct) eigenvalues of the matrix **A** by $\lambda_1, \dots, \lambda_n$,we have $Ax_j = \lambda_j x_j$,so that we simply obtain

Eigenbases. Diagonalization. Quadratic Forms

$$\mathbf{y} = \mathbf{Ax} = \mathbf{A}(c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n)$$

(1) $$= c_1\mathbf{Ax}_1 + \cdots + c_n\mathbf{Ax}_n$$

$$= c_1\lambda_1\mathbf{x}_1 + \cdots + c_n\lambda_n\mathbf{x}_n.$$

This shows that we have decomposed the complicated action of **A** on an arbitrary vector **x** into a sum of simple actions (multiplication by scalars) on the eigenvectors of **A**. This is the point of an eigenbasis.

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Now if the n eigenvalues are all different, we do obtain a basis:

Theorem 1

Basis of Eigenvectors

*If an $n \times n$ matrix **A** has n **distinct** eigenvalues, then **A** has a basis of eigenvectors $\mathbf{x}_1, \cdots, \mathbf{x}_n$ for R^n.*

Please see the proof on page 339.

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Eigenbases. Diagonalization. Quadratic Forms

Example 1: **Eigenbasis. Nondistinct Eigenvalues. Nonexistenc**

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$ has a basis of eigenvectors $$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

On the other hand, **A** ***may not have enough linearly independent eigenvectors to make up a basis***. For instance, **A** in Example 3 of Sec. 8.1 is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$ one eigenvector $$\begin{bmatrix} k \\ 0 \end{bmatrix} \quad (k \neq 0, \text{ arbitrary}).$$

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Theorem 2

> **Symmetric Matrices**
>
> *A symmetric matrix has an orthonormal basis of eigenvectors for* R^n.

Example 2: **Orthonormal Basis of Eigenvectors**

The first matrix in Example 1 is symmetric, and an orthonormal basis of eigenvectors is

$$\left[\frac{1}{\sqrt{2}} \;\; \frac{1}{\sqrt{2}}\right]^T \qquad \left[\frac{1}{\sqrt{2}} \;\; -\frac{1}{\sqrt{2}}\right]^T$$

Eigenbases. Diagonalization. Quadratic Forms

Similarity of Matrices. Diagonalization (p.340)

Eigenbases also play a role in reducing a matrix **A** to a diagonal matrix whose entries are the eigenvalues of **A**. This is done by a "similarity transformation," which is defined as follows

Definition

> **Similar Matrices. Similarity Transformation**
>
> An $n \times n$ matrix $\hat{\mathbf{A}}$ is called **similar** to an $n \times n$ matrix **A** if
>
> (4) $$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$
>
> for some (nonsingular!) $n \times n$ matrix **P**. This transformation, which gives $\hat{\mathbf{A}}$ from **A**, is called a **similarity transformation**.

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Theorem 3

> **Eigenvalues and Eigenvectors of Similar Matrices**
>
> *If* $\hat{\mathbf{A}}$ *is similar to* **A**, *then* $\hat{\mathbf{A}}$ *has the same eigenvalues as* **A**.
>
> *Furthermore, if* **x** *is an eigenvector of* **A**, *then* $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ *is an eigenvector of* $\hat{\mathbf{A}}$ *corresponding to the same eigenvalue.*

Proof:

$$P^{-1}Ax = P^{-1}AIx = P^{-1}APP^{-1}x = (P^{-1}AP)P^{-1}x = \hat{A}(P^{-1}x) = \lambda P^{-1}x.$$

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Example 3: **Eigenvalues and Vectors of Similar Matrices**

Let,

$$\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{P} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}.$$

$$\hat{\mathbf{A}} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

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$\hat{A}$ has the eigenvalues $\lambda_1 = 3, \lambda_2 = 2$.

A also has the eigenvalues $\lambda_1 = 3, \lambda_2 = 2$.

Eigenvectors: $x_1 = [1 \ \ 1]^T$ and $x_2 = [3 \ \ 4]^T$

$$\mathbf{y}_1 = \mathbf{P}^{-1}\mathbf{x}_1 = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{y}_2 = \mathbf{P}^{-1}\mathbf{x}_2 = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- Indeed, these are eigenvectors of the diagonal matrix $\hat{A}$.
- Perhaps we see that x_1 and x_2 are the columns of **P**. This suggests the general method of transforming amatrix **A** to diagonal form **D** by using **P** = **X** , the matrix with eigenvectors as columns.

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Theorem 4

Diagonalization of a Matrix

If an $n \times n$ matrix **A** *has a basis of eigenvectors, then*

(5) $$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

is diagonal, with the eigenvalues of **A** *as the entries on the main diagonal. Here* **X** *is the matrix with these eigenvectors as column vectors. Also,*

(5*) $$\mathbf{D}^m = \mathbf{X}^{-1}\mathbf{A}^m\mathbf{X} \qquad (m = 2, 3, \cdots).$$

See the proof on page 342.

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Example 4: **Diagonalization**

Diagonalize

$$\mathbf{A} = \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1.0 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}.$$

Solution. characteristic equation:

$$-\lambda^3 - \lambda^2 + 12\lambda = 0 \Rightarrow \begin{cases} \lambda_1 = 3 \\ \lambda_2 = -4 \\ \lambda_3 = 0 \end{cases}$$

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$$\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \qquad \mathbf{X}^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

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$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

$$= \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} \begin{bmatrix} -3 & -4 & 0 \\ 9 & 4 & 0 \\ -3 & -12 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenbases. Diagonalization. Quadratic Forms

Quadratic Forms. Transformation to Principal Axes (p.343)

By definition, a **quadratic form** Q in the components $x_1, \dots, x_n$ of a n^2 vector **x** is a sum of terms, namely,

$$Q = \mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_j x_k$$

(7)

$$\begin{aligned} = \; & a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n \\ & + a_{21}x_2x_1 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n \\ & + \cdots\cdots\cdots\cdots\cdots\cdots \\ & + a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_n^2. \end{aligned}$$

Eigenbases. Diagonalization. Quadratic Forms

A = $[a_{jk}]$ is called the **coefficient matrix** of the form.

Example 5: **Quadratic Form. Symmetric Coefficient Matrix**

Let

$$\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1^2 + 4x_1x_2 + 6x_2x_1 + 2x_2^2 = 3x_1^2 + 10x_1x_2 + 2x_2^2.$$

Here 4+6 = 10 = 5+5 From the corresponding ***symmetric*** matrix C = $[c_{jk}]$, where $c_{jk} = \frac{1}{2}(a_{jk} + a_{kj})$

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Thus $c_{11} = 3, c_{12} = c_{21} = 5, c_{22} = 2$, we get the same result; indeed,

$$\mathbf{x}^{\mathsf{T}}\mathbf{C}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1^2 + 5x_1x_2 + 5x_2x_1 + 2x_2^2 = 3x_1^2 + 10x_1x_2 + 2x_2^2.$$

By Theorem 2, the *symmetric* coefficient matrix **A** of (7) has an orthonormal basis of eigenvectors. Hence if we take these as column vectors, we obtain a matrix **X** that is orthogonal, so that $X^{-1} = X^T$. From (5) we thus have $A = XDX^{-1} = XDX^T$. Substitution into (7) gives

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(8) $$Q = \mathbf{x}^{\mathsf{T}}\mathbf{X}\mathbf{D}\mathbf{X}^{\mathsf{T}}\mathbf{x}$$

If we set X^Tx = y , then, since $X^T = X^{-1}$, we have X^{-1}x = y and thus obtain

(9) $$\mathbf{x} = \mathbf{X}\mathbf{y}.$$

Furthermore, in (8) we have $x^T X = (X^T x)^T = y^T$ and $X^T x$ = y , so that Q becomes simply

Eigenbases. Diagonalization. Quadratic Forms

(10) $$Q = \mathbf{y}^{\mathsf{T}}\mathbf{D}\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2.$$

Theorem 5

Principal Axes Theorem

The substitution (9) transforms a quadratic form

$$Q = \mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_j x_k \qquad (a_{kj} = a_{jk})$$

to the principal axes form or **canonical form** *(10), where $\lambda_1, \cdots, \lambda_n$ are the (not necessarily distinct) eigenvalues of the (symmetric!) matrix* **A**, *and* **X** *is an orthogonal matrix with corresponding eigenvectors $\mathbf{x}_1, \cdots, \mathbf{x}_n$, respectively, as column vectors.*

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Example 6: **Transformation to Principal Axes. Conic Sections**

Find out what type of conic section the following quadratic form represents and transform it to principal axes:

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

Solution. We have $Q = x^T Ax$, where

$$\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

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$$(17 - \lambda)^2 - 15^2 = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 32$$

Hence (10) becomes

$$Q = 2y_1^2 + 32y_2^2.$$

We see $Q = 128$ that represents the ellipse $2y_1^2 + 32y_2^2 = 128,$ that is,

$$\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$$

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If we want to know the direction of the principal axes in the x_1x_2-coordinates, we have to determine normalizated eigenvectors from $(A - lI)x = 0$ with $l = l_1 = 2$ and $l = l_2 = 32$ and then use (9). We get

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Hence

$$\mathbf{x} = \mathbf{X}\mathbf{y} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad \begin{aligned} x_1 &= y_1/\sqrt{2} - y_2/\sqrt{2} \\ x_2 &= y_1/\sqrt{2} + y_2/\sqrt{2}. \end{aligned}$$

This is a rotation. Our results agree with those in Sec. 8.2, Example 1, except for the notations.

Complex Matrices and Forms

The three classes of matrices in Sec. 8.3 have complex counterparts which are of practical interest in certain applications, for instance, in quantum mechanics.

Notations

$\bar{A} = [\bar{a}_{jk}]$ is obtained from $A = [a_{jk}]$ by replacing each entry $a_{jk} = \alpha + i\beta$ (α, β real) with its complex conjugate $\bar{a}_{jk} = \alpha - i\beta$. Also, $\bar{A}^T = [\bar{a}_{kj}]$ is the transpose of $\bar{A}$, hence the conjugate transpose of A.

Example 1: **Notations**

If $A = \begin{bmatrix} 3+4i & 1-i \\ 6 & 2-5i \end{bmatrix}$, then $\bar{A} = \begin{bmatrix} 3-4i & 1+i \\ 6 & 2+5i \end{bmatrix}$ and $\bar{A}^T = \begin{bmatrix} 3-4i & 6 \\ 1-i & 2+5i \end{bmatrix}$

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Complex Matrices and Forms

Definition

Hermitian, Skew-Hermitian, and Unitary Matrices

A square matrix $A = [a_{kj}]$ is called

Hermitian	if	$\bar{A}^T = A$,	that is,	$\bar{a}_{kj} = a_{jk}$
skew-Hermitian	if	$\bar{A}^T = -A$,	that is,	$\bar{a}_{kj} = -a_{jk}$
unitary	if	$\bar{A}^T = A^{-1}$.		

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- If **A** is Hermitian, the entries on the main diagonal must satisfy $\bar{a}_{jj} = a_{jj}$ that is, they are real.
- Similarly, if **A** is skew-Hermitian, then $\bar{a}_{jj} = -a_{jj}$ If we set $a_{jj} = \alpha + i\beta$ this becomes $\alpha - i\beta = -(\alpha + i\beta)$. Hence so that a_{jj} must be pure imaginary or 0.

Example 2: **Hermitian, Skew-Hermitian, and Unitary Matrices**

$$A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix} \quad B = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix} \quad C = \begin{bmatrix} \frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}$$

are Hermitian, skew-Hermitian, and unitary matrices.

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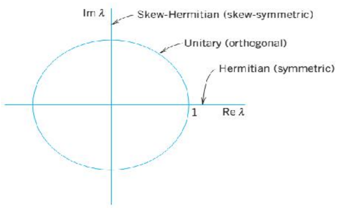
- If a Hermitian matrix is real, then $\bar{A}^T = A^T = A$. Hence a real Hermitian matrix is a symmetric matrix (Sec. 8.3).
- Similarly, if a skew-Hermitian matrix is real, then $\bar{A}^T = A^T = A$ Hence a real skew-Hermitian matrix is a skew-symmetric matrix.
- Finally, if a unitary matrix is real, then $\bar{A}^T = A^T = A^{-1}$ Hence a real unitary matrix is an orthogonal matrix.

- This shows that *Hermitian, skew-Hermitian, and unitary matrices generalize symmetric, skew-symmetric, and orthogonal matrices, respectively.*

Complex Matrices and Forms

Eigenvalues (p.347)

It is quite remarkable that the matrices under consideration have spectra (sets of eigenvalues; see Sec. 8.1) that can be characterized in a general way as follows



Location of the eigenvalues of Hermitian, skew-Hermitian,and unitary matrices in the complex l-plane

Complex Matrices and Forms

Theorem 1

Eigenvalues

(a) *The eigenvalues of a Hermitian matrix (and thus of a symmetric matrix) are real.*

(b) *The eigenvalues of a skew-Hermitian matrix (and thus of a skew-symmetric matrix) are pure imaginary or zero.*

(c) *The eigenvalues of a unitary matrix (and thus of an orthogonal matrix) have absolute value 1.*

Complex Matrices and Forms

Example 3: **Illustration of Theorem 1**

For the matrices in Example 2 we find by direct calculation

	Matrix	Characteristic Equation	Eigenvalues
A	Hermitian	$\lambda^2 - 11\lambda + 18 = 0$	$9,\ 2$
B	Skew-Hermitian	$\lambda^2 - 2i\lambda + 8 = 0$	$4i,\ -2i$
C	Unitary	$\lambda^2 - i\lambda - 1 = 0$	$\frac{1}{2}\sqrt{3} + \frac{1}{2}i,\ -\frac{1}{2}\sqrt{3} + \frac{1}{2}i$

and $|\pm\frac{1}{2}\sqrt{3} + \frac{1}{2}i|^2 = \frac{3}{4} + \frac{1}{4} = 1$

Please see the proof on page 348.

Complex Matrices and Forms

Theorem 2

Invariance of Inner Product

A **unitary transformation**, *that is,* $\mathbf{y} = \mathbf{Ax}$ *with a unitary matrix* $\mathbf{A}$, *preserves the value of the inner product (4), hence also the norm (5).*

Proof:

$$\mathbf{u} \bullet \mathbf{v} = \bar{\mathbf{u}}^T\mathbf{v} = (\overline{\mathbf{A}\mathbf{a}})^T\mathbf{A}\mathbf{b} = \bar{\mathbf{a}}^T\bar{\mathbf{A}}^T\mathbf{A}\mathbf{b} = \bar{\mathbf{a}}^T\mathbf{I}\mathbf{b} = \bar{\mathbf{a}}^T\mathbf{b} = \mathbf{a} \bullet \mathbf{b}$$

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Complex Matrices and Forms

Theorem 3

Unitary Systems of Column and Row Vectors

A complex square matrix is unitary if and only if its column vectors (and also its row vectors) form a unitary system.

Theorem 4

Determinant of a Unitary Matrix

Let $\mathbf{A}$ *be a unitary matrix. Then its determinant has absolute value one, that is,* $|\det \mathbf{A}| = 1$.

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Complex Matrices and Forms

Definition

Unitary System

A *unitary system* is a set of complex vectors satisfying the relationships

$$(6) \qquad \mathbf{a}_j \bullet \mathbf{a}_k = \bar{\mathbf{a}}_j^T\mathbf{a}_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases}$$

Theorem 3 in Sec. 8.3 extends to complex as follows.

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Example 4: **Unitary Matrix Illustrating Theorems 1c and 2–4**

$$\begin{cases} a^T = [2 \quad -i] \\ \bar{a}^T = [2 \quad i] \\ b^T = [1+i \quad 4i] \end{cases} \Rightarrow \bar{\mathbf{a}}^T\mathbf{b} = 2(1+i) - 4 = -2 + 2i$$

$$\mathbf{A} = \begin{bmatrix} 0.8i & 0.6 \\ 0.6 & 0.8i \end{bmatrix} \quad \text{also} \quad \mathbf{Aa} = \begin{bmatrix} i \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{Ab} = \begin{bmatrix} -0.8 + 3.2i \\ -2.6 + 0.6i \end{bmatrix}$$

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as one can readily verify. This gives $(\bar{A}\bar{a})^T Ab = -2 + 2i$, illustrating Theorem 2. The matrix is unitary. Its columns form a unitary system,

$$\bar{\mathbf{a}}_1^T\mathbf{a}_1 = -0.8i \cdot 0.8i + 0.6^2 = 1, \qquad \bar{\mathbf{a}}_1^T\mathbf{a}_2 = -0.8i \cdot 0.6 + 0.6 \cdot 0.8i = 0,$$

$$\bar{\mathbf{a}}_2^T\mathbf{a}_2 = 0.6^2 + (-0.8i)0.8i = 1$$

and so do its rows. Also, det A = -1. The eigenvalues are 0.6 + $0.8i$ and -0.6 + $0.8i$ with eigenvectors $[1 \quad 1]^T$ and $[1 \quad -1]^T$ respectively.

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Theorem 5

Basis of Eigenvectors

A Hermitian, skew-Hermitian, or unitary matrix has a basis of eigenvectors for C^n that is a unitary system.

For a proof see Ref. [B3], vol. 1, pp. 270–272 and p. 244 (Definition 2).

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Example 5: **Unitary Eigenbases**

The matrices **A**, **B**, **C** in Example 2 have the following unitary systems of eigenvectors, as you should verify.

A: $\frac{1}{\sqrt{35}}[1-3i \quad 5]^T \quad (\lambda = 9), \qquad \frac{1}{\sqrt{14}}[1-3i \quad -2]^T \quad (\lambda = 2)$

B: $\frac{1}{\sqrt{30}}[1-2i \quad -5]^T \quad (\lambda = -2i), \qquad \frac{1}{\sqrt{30}}[5 \quad 1+2i]^T \quad (\lambda = 4i)$

C: $\frac{1}{\sqrt{2}}[1 \quad 1]^T \quad (\lambda = \frac{1}{2}(i+\sqrt{3})), \qquad \frac{1}{\sqrt{2}}[1 \quad -1]^T \quad (\lambda = \frac{1}{2}(i-\sqrt{3}))$

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Hermitian and Skew-Hermitian Forms (p.351)

The concept of a quadratic form (Sec. 8.4) can be extended to complex. We call the numerator $\bar{x}^T Ax$ in (1) a **form** in the components $x_1, \dots, x_n$ of **x**, which may now be complex. This form is again a sum of n^2 terms

$$\begin{aligned} \bar{\mathbf{x}}^T\mathbf{A}\mathbf{x} &= \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}\bar{x}_j x_k \\ &= a_{11}\bar{x}_1x_1 + \cdots + a_{1n}\bar{x}_1x_n \\ &\quad + a_{21}\bar{x}_2x_1 + \cdots + a_{2n}\bar{x}_2x_n \\ &\quad + \cdots\cdots\cdots\cdots\cdots \\ &\quad + a_{n1}\bar{x}_nx_1 + \cdots + a_{nn}\bar{x}_nx_n \end{aligned} \tag{7}$$

A is called its **coefficient matrix**.

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Example 6: **Hermitian Form**

For **A** in Example 2 and, say, x = $[1+i \quad 5i]^T$ we get

$\bar{\mathbf{x}}^T \mathbf{A}\mathbf{x} =$

$$[1-i \quad -5i]\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}\begin{bmatrix} 1+i \\ 5i \end{bmatrix}$$

$$= [1-i \quad -5i]\begin{bmatrix} 4(1+i)+(1-3i)\cdot 5i \\ (1+3i)(1+i)+7\cdot 5i \end{bmatrix} = 223$$

Clearly, if **A** and **x** in (4) are real, then (7) reduces to a quadratic form, as discussed in the last section.