Fundamentals of Engineering Exam

Spring 2006

Dynamics Review

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DYNAMICS

1. Rectilinear Motion of a Particle :-

$$v = ds/dt \quad ; \quad a = dv/dt = d^2s/dt^2 \qquad --(1)$$

(velocity) (accel.)

Also $a = v\, dv/ds$

Inverse Problem: (Important)

i) If $a = f(t)$, then

$$(1) \Longrightarrow \int_{v_0}^{v} dv = \int_0^t a\,dt = \int_0^t f(t)\,dt$$

or $$v - v_0 = \int_0^t f(t)\,dt$$

ii) $a = f(s)$, then

$$(1) \Longrightarrow \int_{v_0}^{v} v\,dv = \int_{s_0}^{s} a\,ds$$

or $$\frac{1}{2}\left[v^2 - v_0^2\right] = \int_{s_0}^{s} f(s)\,ds$$

iii) $a = f(v)$, then

$(1) \Longrightarrow \quad f(v) = dv/dt$, or

$$dt = dv/f(v) \qquad \text{--(a)}$$

Also,

$$f(v) = v\frac{dv}{ds} \quad \text{or} \quad ds = \frac{v\,dv}{f(v)} \qquad \text{---(b)}$$

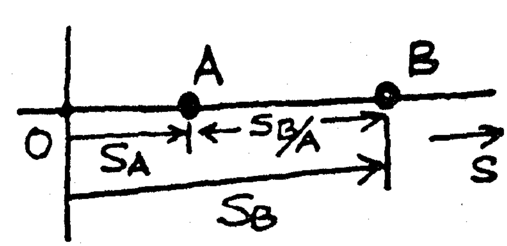
Note that the right hand sides of equations (a) and (b) can be integrated.

Special Cases: $v = \text{const.} \Longrightarrow \underline{a=0}$

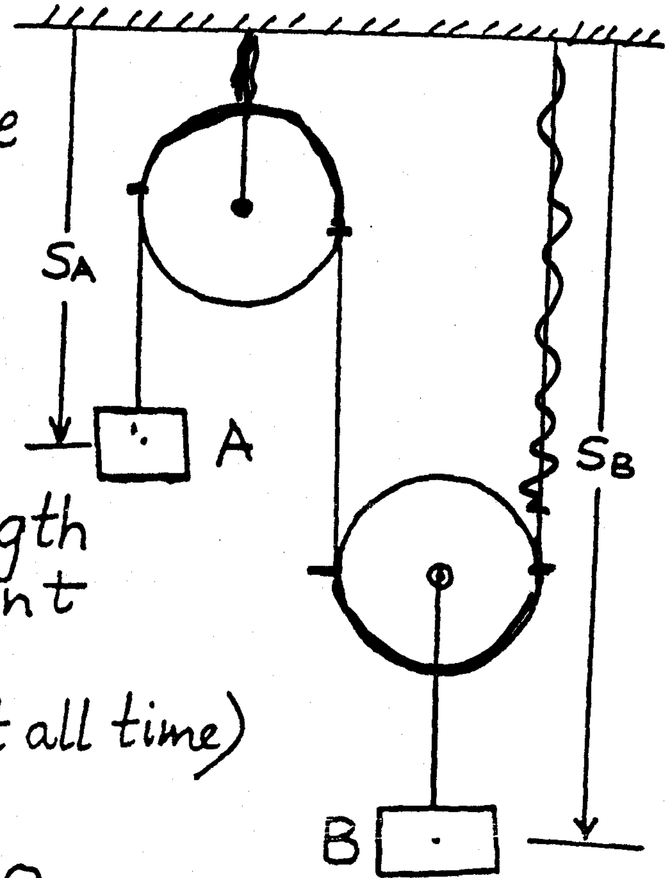
$$\frac{dv}{dt} = \underline{a = \text{const.}}$$

Relative Motion of Two Particles in Rectilinear Motion:

$$\vec{S}_B = \vec{S}_A + \vec{S}_{B/A}$$
$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$
$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$



Dependent Motion



Motion of one particle depends on the other.

→ Since the total length of rope is constant

$$S_A + 2S_B = \text{Const. (at all time)}$$

or $\quad \Delta S_A + 2\Delta S_B = 0$

Then $\quad \boxed{v_A + 2v_B = 0} \qquad a_A + 2a_B = 0$

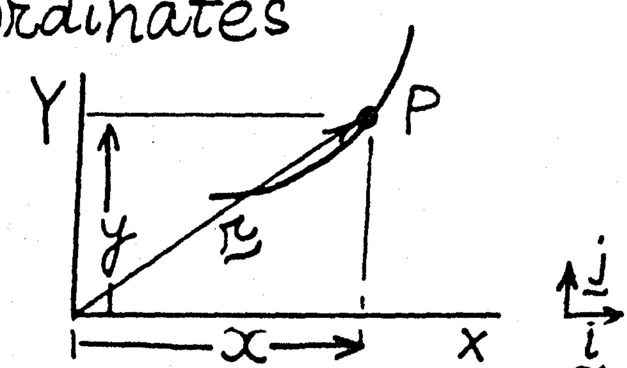
Curvilinear Motion

a) Rectangular Coordinates

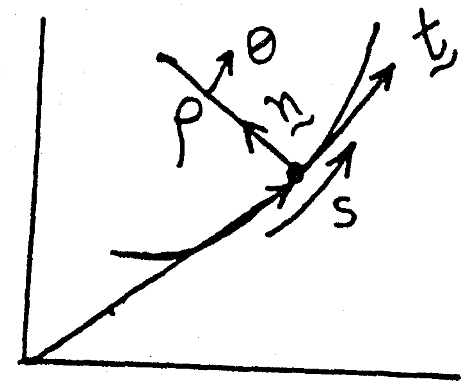
$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$

$\underset{\sim}{v} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}$

$\underset{\sim}{a} = \ddot{x}\underset{\sim}{i} + \ddot{y}\underset{\sim}{j}$



b) Normal ($\underset{\sim}{n}$) and Tangential ($\underset{\sim}{t}$) Coordinates :



$$\underset{\sim}{v} = \dot{s}\,\underset{\sim}{t} = \frac{ds}{dt}\,\underset{\sim}{t}$$

relating n & t to polar $\Rightarrow \rho \dfrac{d\theta}{dt}\,\underset{\sim}{t} = \rho\dot{\theta}\,\underset{\sim}{t}$

$$\rightarrow \underset{\sim}{a} = a_t\,\underset{\sim}{t} + a_n\,\underset{\sim}{n}$$

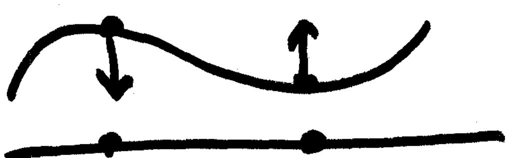
$$a_t = \frac{dv}{dt} = \underline{\underline{\ddot{s}}}$$

$$a_n = \underline{\underline{\frac{v^2}{\rho}}} = \underline{\underline{v\dot{\theta}}} = \rho\dot{\theta}^2$$

$\underset{\sim}{t}$ is the unit vector in the tangential direction.

$\underset{\sim}{n}$ is the unit vector in the normal direction.

Special Case : $\left.\begin{array}{l}\rho = \text{Const.} = r \\ \text{(Circular Motion)} \\ \dot{\theta} = \omega,\ \ddot{\theta} = \alpha\end{array}\right\}$



then

$$v = r\omega$$

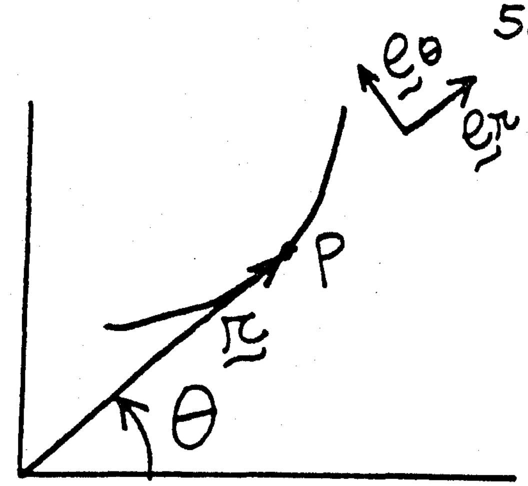
$$a_n = r\omega^2 = v^2/r = v\omega$$

$$a_t = \underline{r\dot{\omega}} = r\alpha$$

c) Polar Coordinates:

$$\vec{r} = r\,\hat{e}_r$$

$$\begin{cases} \underset{\sim}{v} = v_r\,\underset{\sim}{e}_r + v_\theta\,\underset{\sim}{e}_\theta \\ v_r = \dot{r}\,, \quad v_\theta = r\dot{\theta} \end{cases}$$



$$\begin{cases} \underset{\sim}{a} = a_r\,\underset{\sim}{e}_r + a_\theta\,\underset{\sim}{e}_\theta \\ a_r = \ddot{r} - r\dot{\theta}^2 \\ a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \end{cases}$$

$\underset{\sim}{e}_r$ is the unit vector in the direction of $\underset{\sim}{r}$.

$\underset{\sim}{e}_\theta$ is the unit vector $\perp$ to $\underset{\sim}{r}$.

Special Case: $r =$ Const. (Circular Motion)

$$\underset{\sim}{v} = v_\theta\,\underset{\sim}{e}_\theta = r\dot{\theta}\,\underset{\sim}{e}_\theta = r\omega\,\underset{\sim}{e}_\theta$$

$$\rightarrow \underset{\sim}{a} = -r\dot{\theta}^2\,\underset{\sim}{e}_r + r\ddot{\theta}\,\underset{\sim}{e}_\theta$$

$$= -\underline{r\omega^2}\,\underset{\sim}{e}_r + \underline{r\alpha}\,\underset{\sim}{e}_\theta$$

Note that radial component $r\omega^2$ is directed in $-\underset{\sim}{e}_r$ direction, i.e. towards the center of rotation.

<u>AND</u> $r\alpha$ in the $\underset{\sim}{e}_\theta$ direction.

Also

$$\int_{(v_y)_0}^{v_y} v\,dv = \int_{(y)_0}^{y} a\,dy$$

or $$v_y^2 = 90{,}000 - 64.4y \quad \text{--(iii)}$$

Horizontal :

$$(v_x)_0 = 600 \cos 30^\circ = 520 \text{ ft/s}$$

Note that the accel. in the x direction is zero.

Hence $\quad x = 520t. \qquad$ (IV)

When it hits the ground $\underline{y = -500'}$ --(V)

then (ii) & (V) $\Longrightarrow$ a quadratic in t

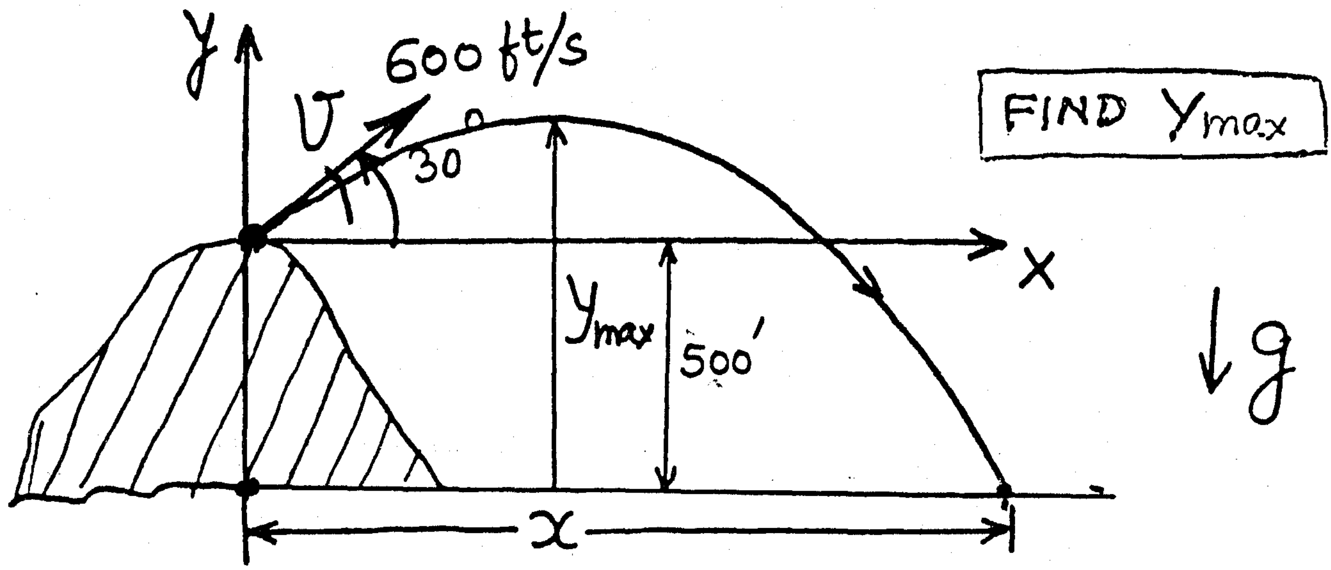
$\Longrightarrow \quad t = 20.1$ sec. $\leftarrow$

(IV) $\Longrightarrow \quad x = (520)(20.1) = \underline{10{,}450 \text{ ft.}}$

Now y_{max} can be obtained from (iii), because at y_{max}, $v_y = 0$

$\Longrightarrow \quad y = 1{,}398$ ft. $\therefore y_{max} = (500 + 1{,}398)$ ft

Ex. Projectile Problem:



Find x and y_{max}.

Consider the Vertical and the Horizontal Motion Separately:

Vertical: $(v_y)_0 = \underline{600 \sin 30^\circ} = 300$ ft/s

$$|\underset{\sim}{g}| = |\underset{\sim}{a}| = a_y = \underline{-32.2 \text{ ft/s}^2}$$

$$\int_{(v_y)_0}^{v_y} dv = \int_0^t a\,dt = \int_0^t (-32.2)\,dt$$

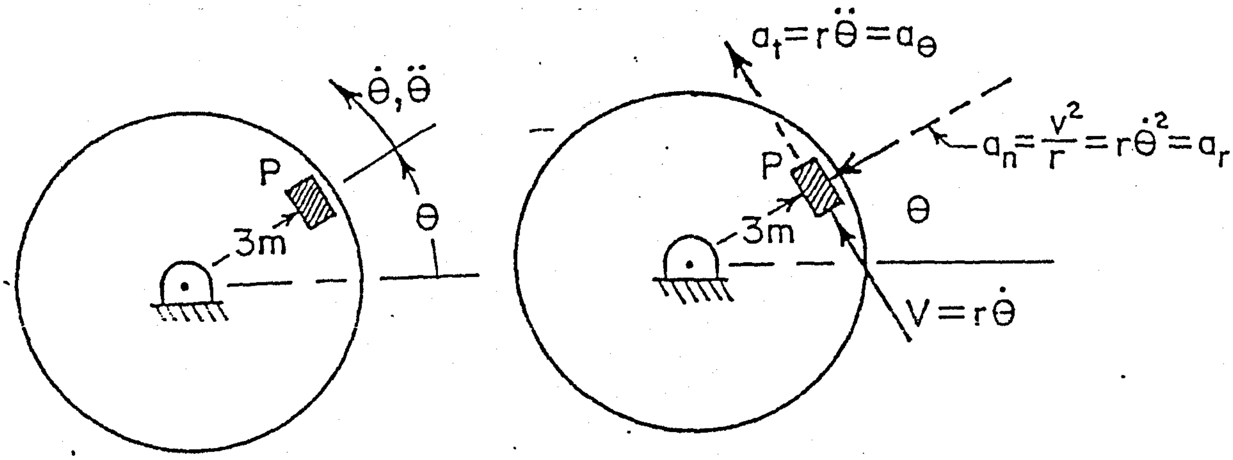
or $\quad v_y = 300 - 32.2t \qquad \text{---(i)}$

$\quad y = 300t - 16.1t^2 \qquad \text{---(ii)}$

-500

Example 2-8

Experiments show that package P will slide off the rotating table shown when the magnitude of its acceleration equals $30\ m/s^2$. The table starts from rest at $\theta = 0$ and accelerates counterclockwise at a constant angular acceleration of $\ddot{\theta} = 2\ rad/s^2$. Determine the angle θ at which the package slides.



angular rate of $\ddot{\theta} = 2\ rad/s^2$ (constant). Integration yields

$$\underline{\dot{\theta} = 2t} \quad \text{and} \quad \underline{\theta = t^2}$$

because r is const.

Since $a^2 = a_n^2 + a_t^2 = (r\dot{\theta}^2)^2 + (r\ddot{\theta})^2 = a_r^2 + a_\theta^2$

$$\therefore (30)^2 = \{(3)(2t)^2\}^2 + \{(3)(2)\}^2$$

Solving for t, $\underline{t = 1.565s}$ $\quad \therefore \boxed{\theta = (1.565)^2 = 2.45\ rad}$

2. Kinetics of Particles :—

Newton's Second Law: $\sum \underset{\sim}{F} = m\underset{\sim}{a}$

$\Longrightarrow$ Three scalar equations

$$\sum F_x = m a_x \ , \ \sum F_y = m a_y$$

and $\sum F_z = m a_z$

From the Previous Chapter we know how to calculate accel. in all the three coordinate Systems

$(\underset{\sim}{n}-\underset{\sim}{t})$ coordinates : $\sum F_n = m a_n$ $\nwarrow \underset{\sim}{n}$

& $\sum F_t = m a_t$ $\nearrow \underset{\sim}{t}$

Polar Coordinates (r, θ) :

$$\sum F_r = m a_r \qquad \nearrow \underset{\sim}{e_r}$$

& $$\sum F_\theta = m a_\theta \qquad \nwarrow \underset{\sim}{e_\theta}$$

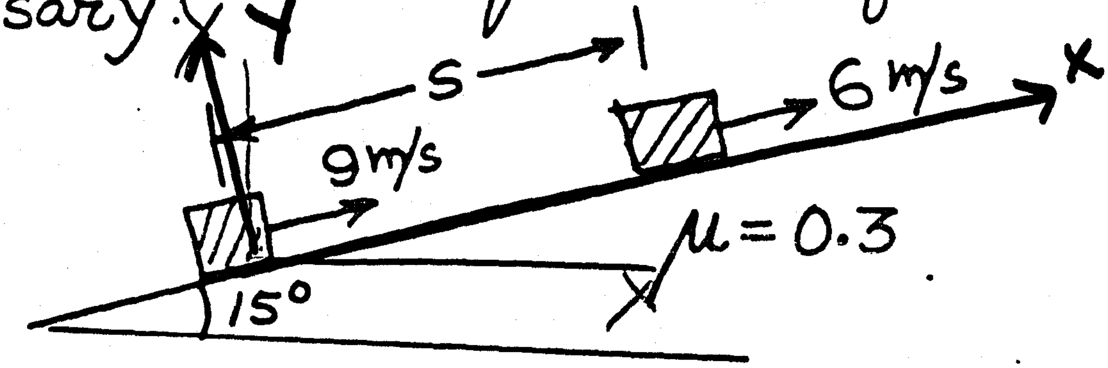
Procedure for Analysis :

i) Draw the <u>Free Body diagram</u> representing all the forces. (with directions)

ii) Apply $\sum \underset{\sim}{F} = m \underset{\sim}{a}$

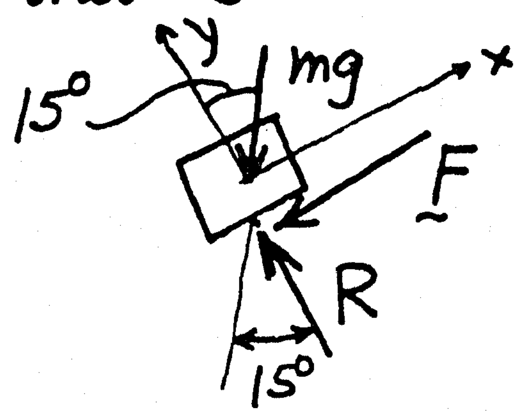
iii) Use kinematic equations if necessary.

<u>Ex.</u>



Find s

<u>F. B. D.</u>



$$\sum F_x = -mg\sin 15^\circ - F = m a_x \quad (a)$$

& $$\sum F_y = R - mg\cos 15^\circ = 0$$

$$\Longrightarrow \quad R = mg\cos 15^\circ \quad (b)$$

Also $F = \mu R = 0.3\, mg \cos 15^\circ$

then (a) $\Longrightarrow$

$$-mg \sin 15^\circ - (0.3)\, mg \cos 15^\circ = m a_x$$

$$\Longrightarrow \quad a_x = -5.38 \text{ m/s}^2$$

Now use Kinematics to get s :

We know : at $t=0,\ s_0=0,\ v_0 = 9$ m/s

" $t = t_1,\ s = s,\ v = 6$ m/s

$$v\,dv = a\,ds$$

or $$\int_9^6 v\,dv = \int_0^s (-5.38)\,dx$$

or $$\frac{1}{2}\left[6^2 - 9^2\right] = -5.38\,(s-0)$$

$$\Longrightarrow \quad s = 4.18 \text{ m.}$$

3. Work – Energy :

$$U_{1\to2} = T_2 - T_1$$

Where $U_{1\to2}$ = work done from position 1 to 2 by <u>all</u> the external forces

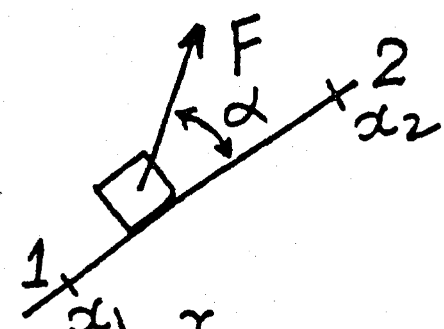
T_2 = Kinetic Energy at $2 = \frac{1}{2} m v_2^2$

T_1 = Kinetic Energy at $1 = \frac{1}{2} m v_1^2$

$$U = \int \underline{F} \cdot d\underline{r} = \int (F_x dx + F_y dy + F_z dz)$$

Various cases :

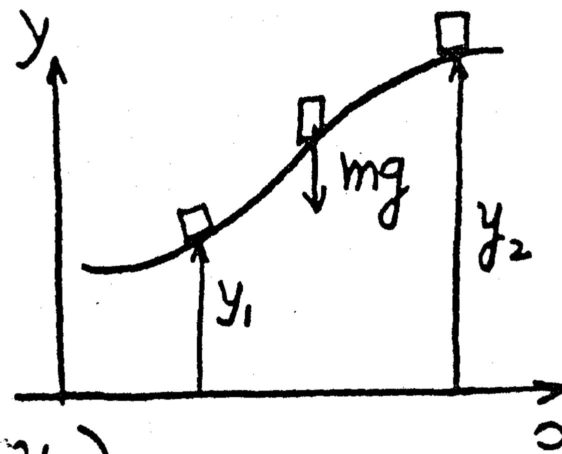
a) <u>Force at arb. dir.</u>



$$U_{1\to2} = \int_{x_1}^{x_2} F_x\, dx = \int_{x_1}^{x_2} (F\cos\alpha)\, dx$$

or $U_{1\to2} = (F\cos\alpha)(x_2 - x_1)$

b) Gravity Work

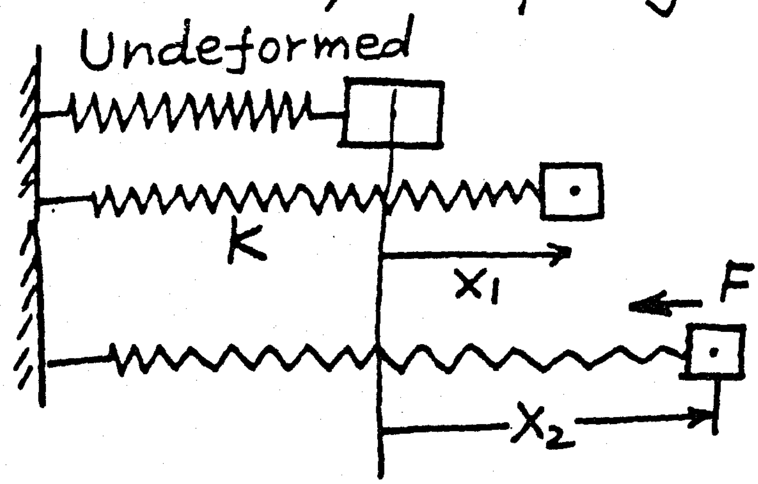


$$U_{1\to 2} = -\int_{y_1}^{y_2} mg\,dy$$

or $$U_{1\to 2} = -mg(y_2 - y_1)$$

Note that the work is +ve when the body moves down.

c) work of force exerted by a spring



Linear Spring:
i.e. $F = kx$

then

$$U_{1\to 2} = -\int_{x_1}^{x_2} kx\,dx = \frac{1}{2}k(x_1^2 - x_2^2)$$

i.e. Compression $\longrightarrow$ +ve work
Tension $\longrightarrow$ −ve work

Conservation of Energy: <u>NO friction</u>

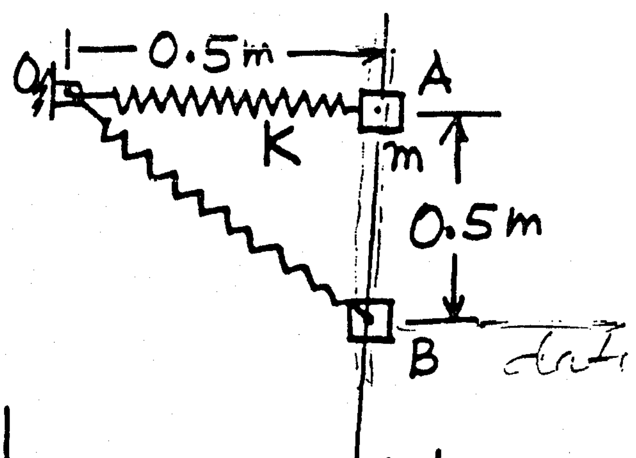
Then

$$T_1 + V_1 = T_2 + V_2 \qquad (*)$$

K.E. at 1 P.E. at 1 K.E. at 2 P.E. at 2

Because $U_{1\to 2} = T_2 - T_1 = -(V_2 - V_1)$

<u>Ex.</u>



Collar is at rest at A
Find V at B

$$T_A + V_A = T_B + V_B$$

or $0 + \frac{1}{2}K\left[0.5 - 0.25\right]^2$
$= \frac{1}{2}mV_2^2 + \frac{1}{2}K(\overline{OB} - 0.25)^2$
$- mg(0.5)$

Undeformed length of the spring $= 0.25$ m

$\Longrightarrow$ Can get V_2

4. Impulse-Momentum:

$$\sum \underset{\sim}{F} = m\underset{\sim}{a} = \frac{d}{dt}(m\underset{\sim}{v}) \quad ; \; m = \text{const.}$$

Hence
$$\int_{t_1}^{t_2} \sum \underset{\sim}{F}\, dt = \underset{\sim}{G}_2 - \underset{\sim}{G}_1 = m\underset{\sim}{v}_2 - m\underset{\sim}{v}_1$$

for the two positions

If the impulse $\int \underset{\sim}{F} dt = 0 \implies$

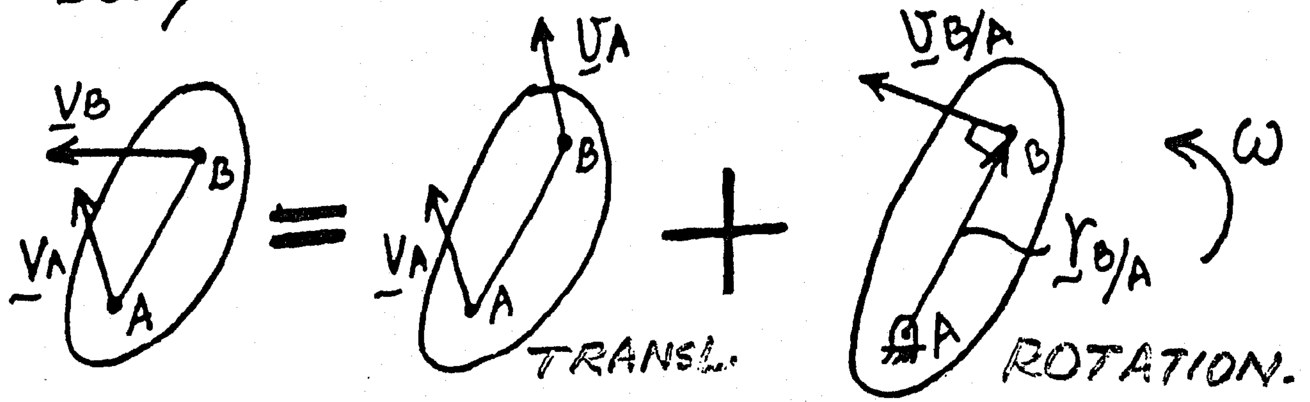
Conservation of Momentum $\implies$

$$\sum (mv)_2 = \sum (mv)_1$$

5. Plane Kinematics of Rigid Bodies:

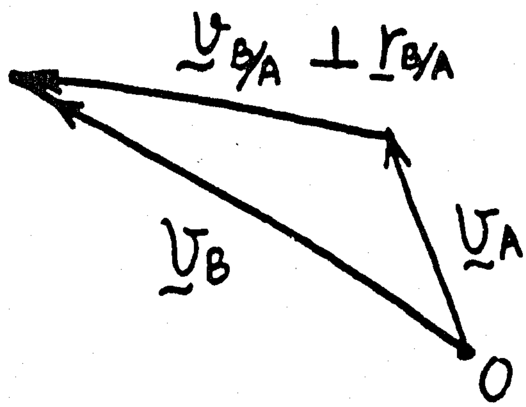
Velocity: $\underset{\sim}{v}_B = \underset{\sim}{v}_A + \underset{\sim}{v}_{B/A}$

A, B are the two points on the rigid body



$$\underset{\sim}{v}_B = \underset{\sim}{v}_A + \underset{\sim}{v}_{B/A} = \underset{\sim}{v}_A + \underset{\sim}{\omega} \times \underset{\sim}{r}_{B/A}$$

Graphically :



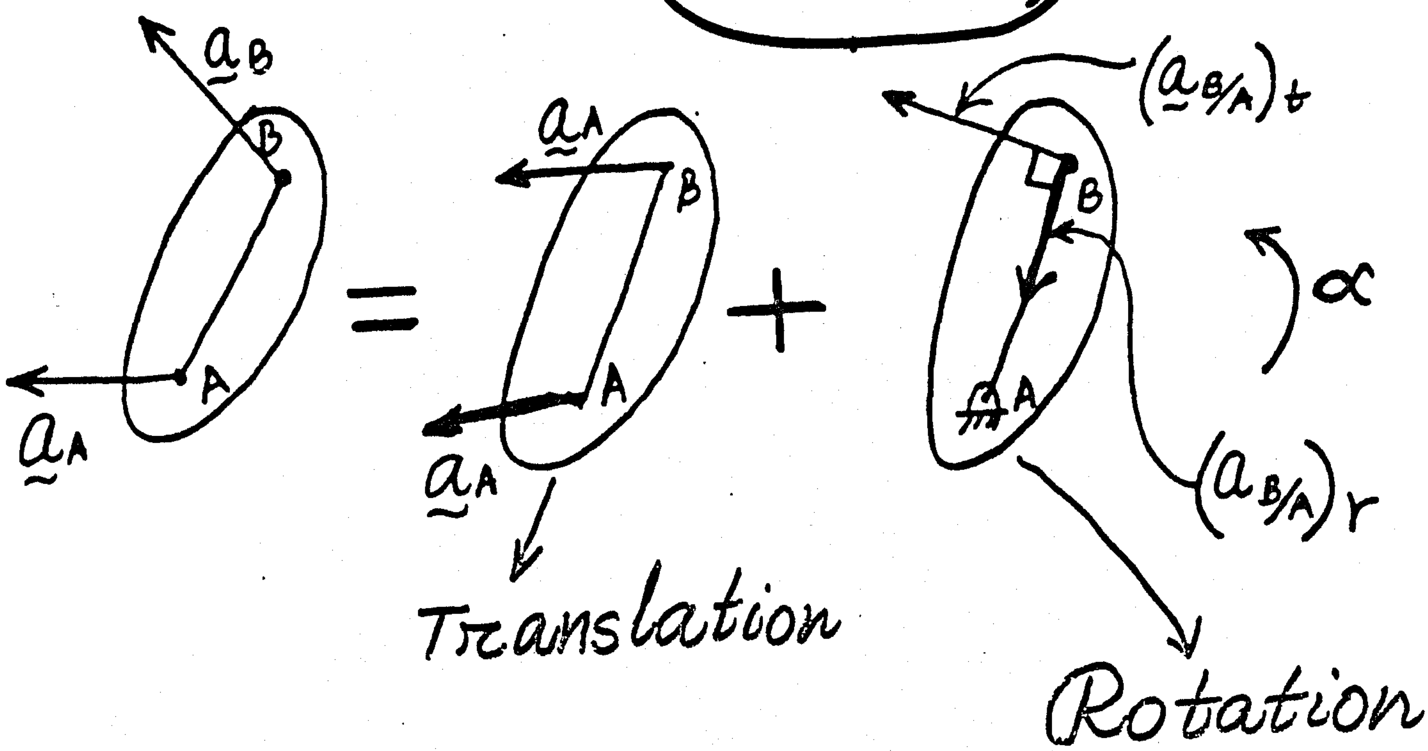
$$\boxed{V_{B/A} = \omega r_{B/A}}$$

Acceleration : Planar $-\omega \vec{r}_{B/A}$

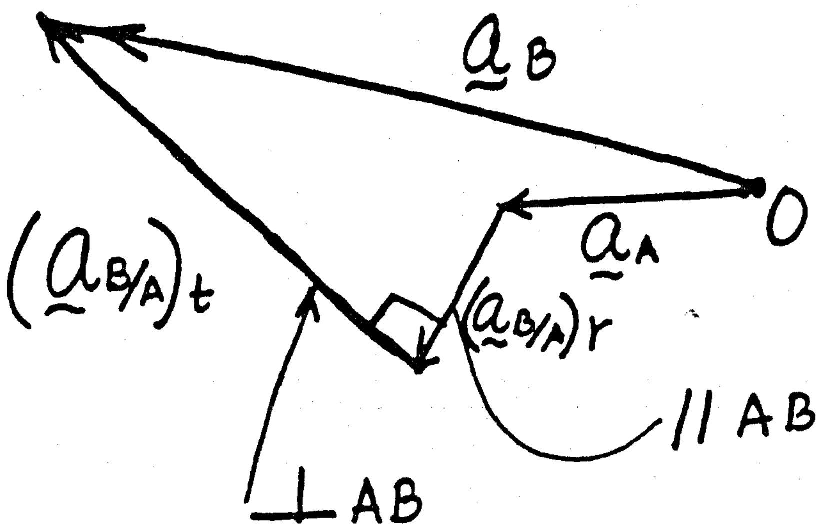
$$\underset{\sim}{a}_B = \underset{\sim}{a}_A + \underset{\sim}{a}_{B/A} = \underset{\sim}{a}_A + (\underset{\sim}{a}_{B/A})_r + (\underset{\sim}{a}_{B/A})_t$$

OR

$$\underset{\sim}{a}_B = \underset{\sim}{a}_A + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r}_{B/A}) + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r}_{B/A}$$



Graphically:

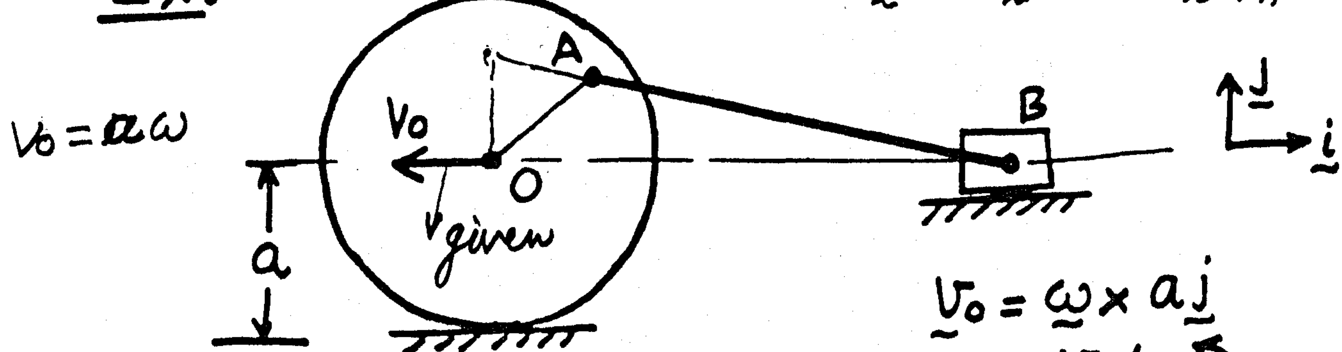


$(a_{B/A})_r = \omega^2 r_{B/A}$
$(a_{B/A})_t = \alpha r_{B/A}$

Note: You need to do the velocity analysis to get ω to do the accel. analysis.

Ex.

$\underset{\sim}{V}_B = \underset{\sim}{V}_A + \underset{\sim}{\omega} \times \underset{\sim}{r}_{B/A}$



$\underset{\sim}{V}_0 = \underset{\sim}{\omega} \times a\underset{\sim}{j}$

or $\omega = V_0/a$ ↺

Find $\underset{\sim}{V}_B$:

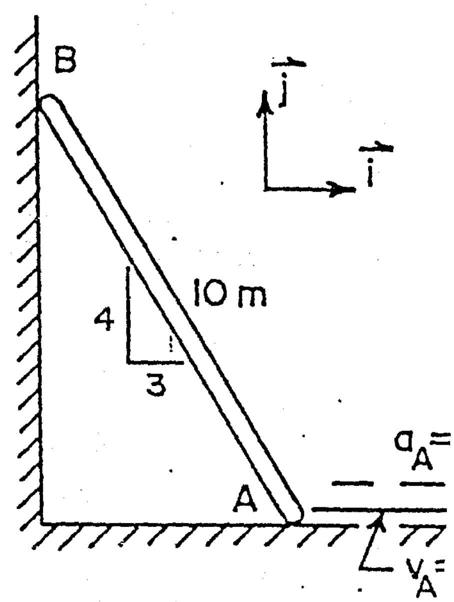
$\underset{\sim}{V}_A = \underset{\sim}{V}_0 + \underset{\sim}{\omega} \times \underset{\sim}{r}_{A/O} \Longrightarrow \underset{\sim}{V}_A$ known

Then $\underset{\sim}{V}_B = \underset{\sim}{V}_A + \underset{\sim}{V}_{B/A} = \underset{\sim}{V}_A + \underset{\sim}{\omega}_{AB} \times \underset{\sim}{r}_{B/A}$

GET $\underset{\sim}{\omega}_{AB}$ & $\underset{\sim}{V}_B$.

xample 7-6

he velocity of point A is 16 m/s nd its acceleration is 3 m/s^2, both o the right as shown. Determine the a) velocity and (b) acceleration of he top of the ladder B as it lides down the wall for the instant hown.



(a) $\vec{v}_B = \vec{v}_A + \vec{v}_{B/A} = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A}$

where: $\vec{v}_B = -v_B\vec{j}$ (assumed downward)

$\vec{v}_A = 16\vec{i}$ (given)

$\vec{\omega}_{AB} = \omega_{AB}\vec{k}$ (assumed counterclockwise)

$\vec{r}_{B/A} = -6\vec{i} + 8\vec{j}$

$\therefore \quad -v_B\vec{j} = 16\vec{i} + (\omega_{AB}\vec{k}) \times (-6\vec{i} + 8\vec{j}) = 16\vec{i} - 6\omega_{AB}\vec{j} - 8\omega_{AB}\vec{i}$

Equating $\vec{i}$ components: $0 = 16 - 8\omega_{AB}$ or $\omega_{AB} = +2$

Equating $\vec{j}$ components: $-v_B = -6(2)$ or $v_B = +12$

"Plus" answers simply indicate the directions were assumed c for both.

$$\therefore \quad \boxed{\vec{\omega}_{AB} = 2\vec{k}\ \text{(rad/s)}} \quad \text{and} \quad \boxed{\vec{v}_B = -12\vec{j}\ \text{(m/s)}}$$

(b) $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A} = \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{B/A} + \vec{\omega}_{AB} \times (\vec{\omega}_{AB} \times \vec{r}_{B/A})$

where: $\vec{a}_B = a_B\vec{j}$ (assumed upward)

$\vec{a}_A = 3\vec{i}$ (given)

$\vec{\alpha}_{AB} = \alpha\vec{k}$ (assumed counterclockwise)

$$\therefore \quad a_B\vec{j} = 3\vec{i} + (\alpha\,\vec{k}) \times (-6\vec{i} + 8\vec{j}) + (2\vec{k}) \times \{(2\vec{k}) \times (-6\vec{i} + 8\vec{j})\}$$

<u>or</u> $a_B\vec{j} = 3\vec{i} - 6\alpha\vec{j} - 8\alpha\vec{i} + 24\vec{i} - 32\vec{j}$

<u>Equating $\vec{i}$ components:</u> $0 = 27 - 8\alpha$ <u>or</u> $\alpha = +3.38$

<u>Equating $\vec{j}$ components:</u> $a_B = -6\alpha - 32$ <u>or</u> $a_B = -52.3$

$$\therefore \quad \boxed{\vec{\alpha}_{AB} = +3.38\vec{k}\ \text{(rad/s}^2\text{)}} \quad \text{and} \quad \boxed{\vec{a}_B = -52.3\vec{j}\ \text{(m/s}^2}$$

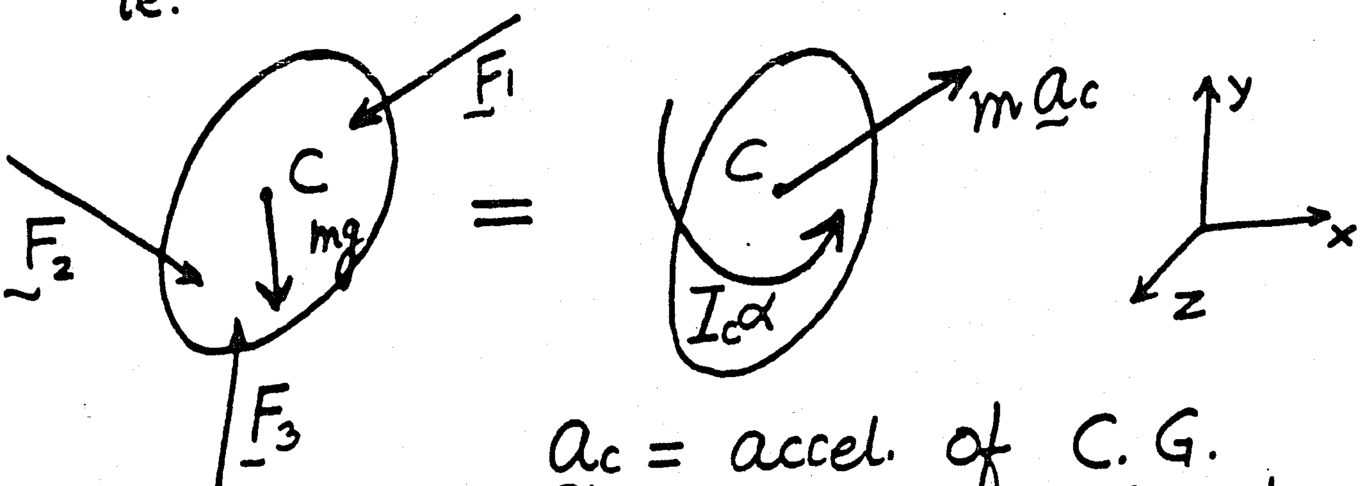
6. Kinetics of Rigid Bodies :

Procedure :

i) Draw F. B. D. and locate C. G. (This shows all the forces)

ii) Draw an equivalent diagram of the free rigid body representing $m\underset{\sim}{a}_c$ and $(I_c)\alpha$

ie.



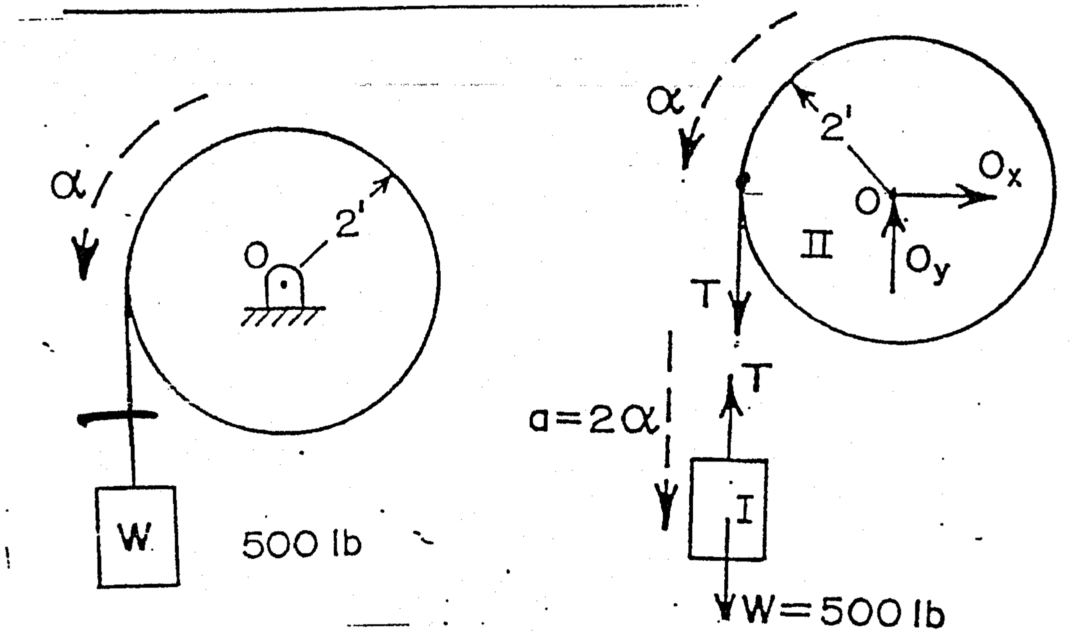
$\underset{\sim}{a}_c$ = accel. of C. G.
I_c = moment of Inertia about z axis at C.

iii) Use $\underline{\Sigma F_x = m(a_c)_x}$; $\underline{\Sigma F_y = m(a_c)_y}$ & $\underline{\Sigma M_c = I_c \alpha}$

To find $\underset{\sim}{a}_c$ and α, you may have to use kinematics.

Example 8-7

Cable unwinds from a 2 ft radius spool as the 500 lb weight is allowed to freely fall. The mass of the spool and cable is 50 slugs and its radius of gyration is $k_0 = 1.5$ ft. Neglecting friction, determine the angular acceleration of the spool as it unwinds and the tension in the cable.



$\Sigma F_y = ma(\downarrow)$ ∴ $500 - T = \dfrac{500}{32.2}\,\underbrace{(2\alpha)}_{a}$ (a)

where $a = 2\alpha$ (kinematics)

(2) From *FBD(II), $\Sigma M_0 = I_0\alpha$ $(+\circlearrowright)$

∴ $T(2) = (50)(1.5)^2\alpha = 112.5\alpha$ (b)

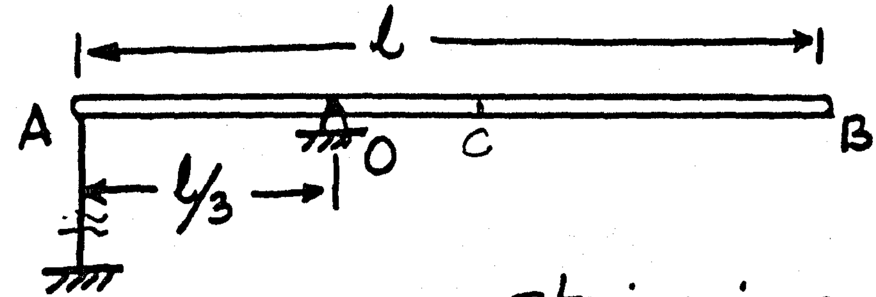
$\boxed{\text{where } I_0 = mk_0^2}$

Solving Eqs. (a) and (b) simultaneously yields:

{*FBD(I) means Free-Body Diagram (I).}

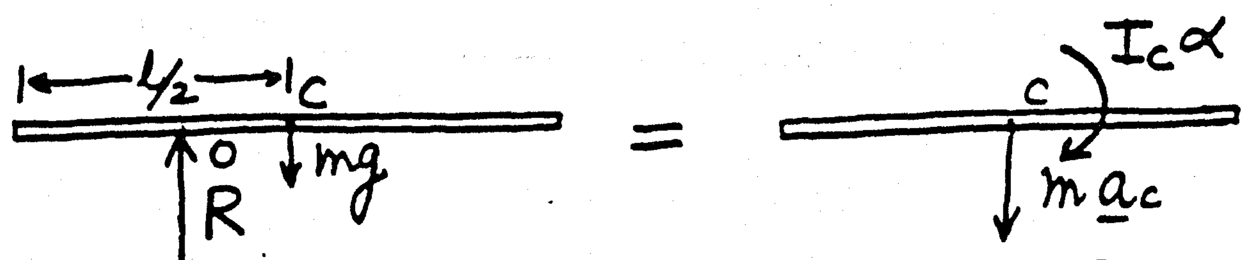
$$\boxed{\begin{array}{l}\alpha = 5.73 \text{ rad/s}^2 \circlearrowright \\ T = 322 \text{ lb}\end{array}}$$

Ex.



string is cut ; $\underline{\underline{\omega = 0}}$

Find a) Force at O
b) Accel. of pt. A.



Use $\Sigma M_c = I_c\alpha$; $I_c = \frac{1}{12}ml^2$

or $R\left(\frac{l}{2} - \frac{l}{3}\right) = \frac{1}{12}ml^2\alpha$ --- (1)

And $\Sigma F_y = R - mg = -ma_c$ -- (2)

In (1) & (2) R, α and a_c are unknowns

Kinematics :

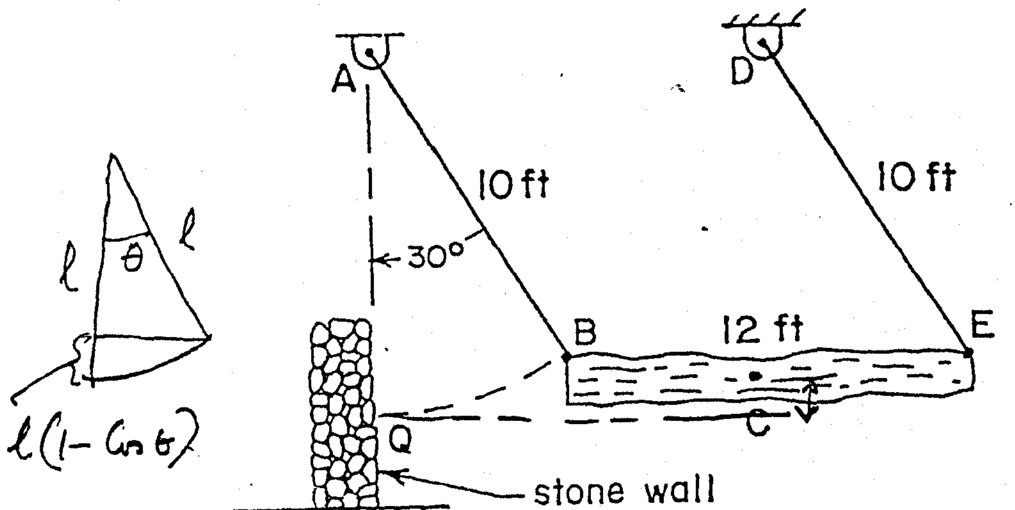
$$\underset{\sim}{a}_c = \underbrace{\cancel{\underset{\sim}{a}_o}}_{=0} + \underbrace{\cancel{\underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underline{r}_{c/o})}}_{=0} + \dot{\underset{\sim}{\omega}} \times \underline{r}_{c/o}$$

$$\therefore a_c = \alpha\left(\frac{l}{2} - \frac{l}{3}\right) \quad \text{--- (3)}$$

From (1), (2) & (3), we can solve for R and α.

Example 9-2

A long slender log weighing 1610 lb is used as a battering ram to knock down a stone wall. The log is suspended by two cables of length 10 ft attached to the ends of the log. Determine the speed at which the log strikes the wall at Q if it is released from rest at 30° as shown.



Pure Translation

$V_c = V_B = V_E$

$$\therefore\ U_{12} = T_2 - \cancelto{0}{T_1} = T_2\ .$$

where: $U_{12} = W(\Delta h_C) = 1610\{10(1 - \cos 30°)\} = 2157 (\text{ft}\cdot\text{lb})$

$$T_2 = \frac{1}{2} m v_C^2 = \frac{1}{2}\frac{1610}{32.2} v^2 = 25 v^2 (\text{ft}\cdot\text{lb})$$

$\therefore\ 2157 = 25v^2$ <u>or</u> $\boxed{v = 9.29 \text{ ft/s}}$

7. Work-Energy Methods for Rigid Bodies:

$$U_{1\to 2} = T_2 - T_1$$

where $U_{1\to 2}$ = Work done from position 1 to 2.

T_2 = K.E. at position 2.

T_1 = K.E. at position 1.

But Remember

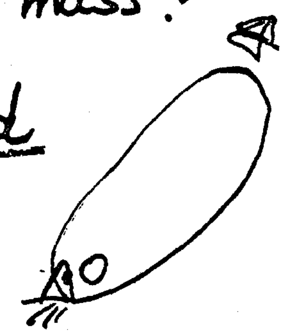
Translational Energy

$$\boxed{T = \frac{1}{2} m v_c^2 + \frac{1}{2} I_c \omega^2}$$

Rotation Energy

where v_c = velocity of center of mass.

For pure rotation about a fixed point O.



$$T = \frac{1}{2} I_o \omega^2 \text{ holds}$$

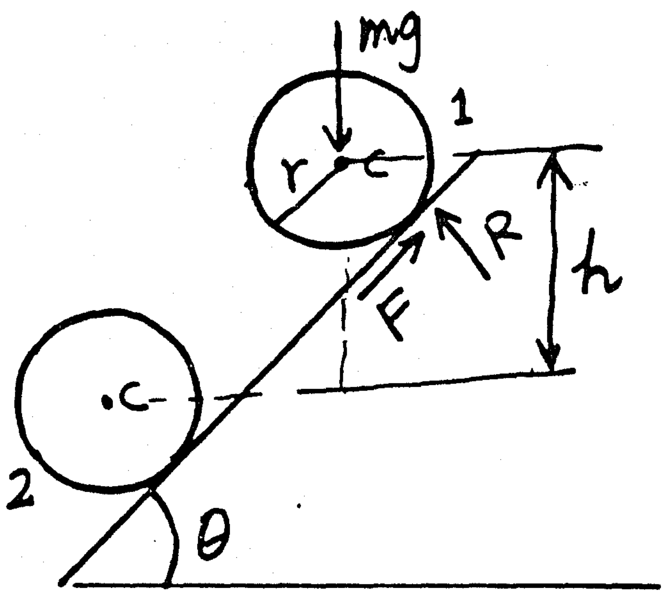
where I_o is the moment of Inertia about z axis at point O.

$U = \int \underline{F} \cdot d\underline{r}$: due to a force (as before)

$U = \int_{\theta_1}^{\theta_2} M \, d\theta = M(\theta_2 - \theta_1)$: due to a Moment.

Conservation of Energy

$T_1 + V_1 = T_2 + V_2$ holds if there is no friction.



Ex.

Cylinder released from rest. (1).

Find Velocity at 2

$U_{1\to2} = mgh$

$T_1 = 0$

$T_2 = \frac{1}{2} m v_c^2 + \frac{1}{2} I_c \omega^2$

$= \frac{1}{2} m v_c^2 + \frac{1}{2}\left(\frac{1}{2} m r^2\right)\left(\frac{v_c}{r}\right)^2$

then

$mgh = \frac{1}{2} m v_c^2 + \frac{1}{4} m v_c^2$

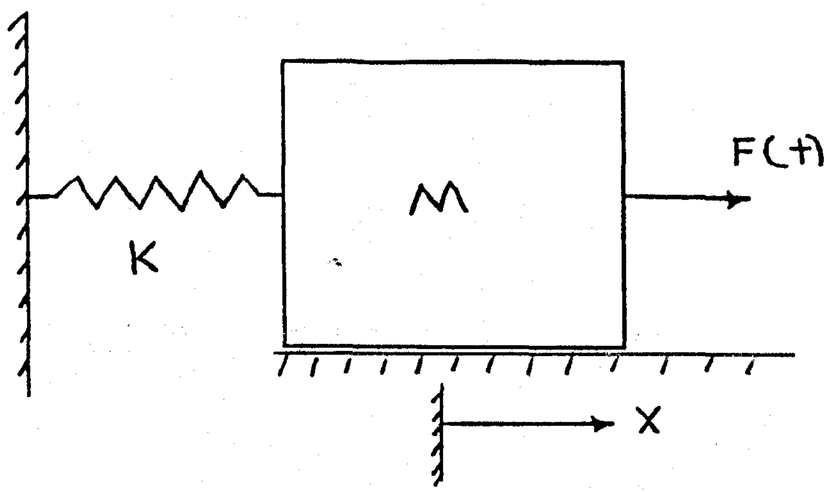
GET v_c

Note that $\underline{F}$ & $\underline{R}$ do no work

$I_c = \frac{1}{2} m r^2$

$\omega = v_c / r$

VIBRATION



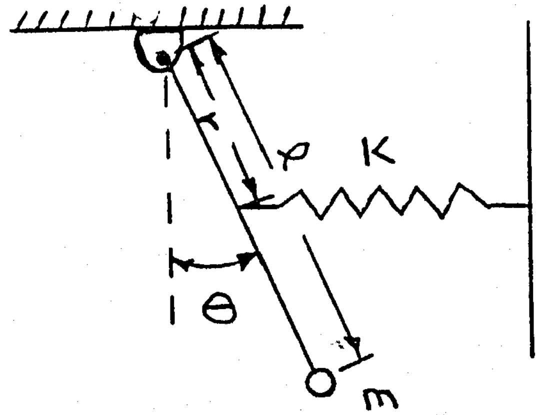
$$M\ddot{x} + Kx = F(t)$$

$$\ddot{x} + \omega_n^2 = \frac{F(t)}{M} \qquad (1)$$

$$\omega_n^2 = \frac{K}{M} \qquad (2)$$

For vibration problems, we are typically concerned with small amplitude motion and with free vibration. For the case of free vibration, $F(t) = 0$. The primary parameter of concern for such cases is typically the natural frequency of the system, ω_n .

EXAMPLE – SIMPLE PENDULUM



Assume θ is a small angle. Apply the relation,

$$\sum \text{(moments)} = \mathrm{I}\ddot{\theta}$$

Two moments are applied to the structure. The first is due to the spring force and can be described as

$$T_1 = -Kr\theta$$

The second is due to gravity and can be described as

$$T_2 = -mgl\sin(\theta) \approx -mgl\theta$$

Note: $\mathrm{I} = \mathrm{ml}^2$
Thus the equation of motion is

$$\mathrm{ml}^2\,\ddot{\theta} + \mathrm{Kr}\theta + \mathrm{mgl}\theta = 0$$

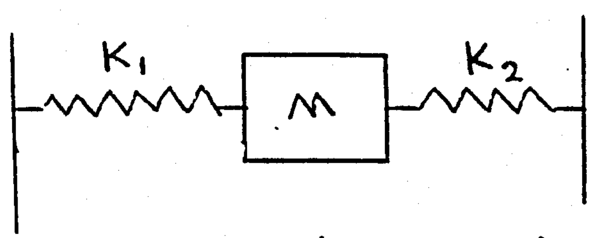
or

$$\ddot{\theta} + \left(\frac{Kr}{ml^2} + \frac{g}{l}\right)\theta = 0$$

Comparison of the above equation with equations 1 and 2 yields:

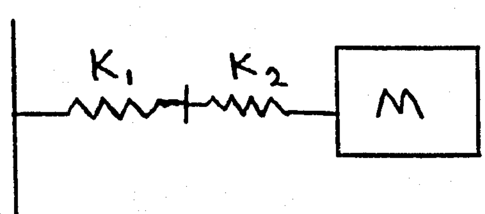
$$\omega_n^2 = \left(\frac{Kr}{ml^2} + \frac{g}{l}\right)$$

<u>Springs in Parallel</u>



$$\omega_n^2 = \left(\frac{(K_1 + K_2)}{M}\right)$$

<u>Springs in Series</u>



$$\omega_n^2 = \left(\frac{K_1 K_2}{(K_1 + K_2)M}\right)$$

DYNAMICS

KINEMATICS

Vector representation of motion in space: Let $\boldsymbol{r}(t)$ be the position vector of a particle. Then the velocity is

$\boldsymbol{v} = d\boldsymbol{r}/dt$, where

$\boldsymbol{v}$ = the instantaneous velocity of the particle, (length/time), and

t = time.

The acceleration is

$\boldsymbol{a} = d\boldsymbol{v}/dt = d^2\boldsymbol{r}/dt^2$, where

$\boldsymbol{a}$ = the instantaneous acceleration of the particle, (length/time/time).

Rectangular Coordinates

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

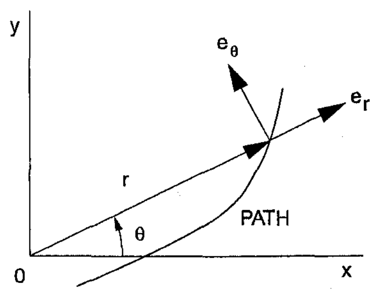
$$\boldsymbol{v} = d\boldsymbol{r}/dt = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}$$

$$\boldsymbol{a} = d^2\boldsymbol{r}/dt^2 = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}, \text{ where}$$

$\dot{x} = dx/dt = v_x$, etc.

$\ddot{x} = d^2x/dt^2 = a_x$, etc.

Transverse and Radial Components for Planar Problems



Unit vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are, respectively, colinear with and normal to the position vector.

$$\boldsymbol{r} = r\boldsymbol{e}_r$$

$$\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta$$

$$\boldsymbol{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\boldsymbol{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\boldsymbol{e}_\theta, \text{ where}$$

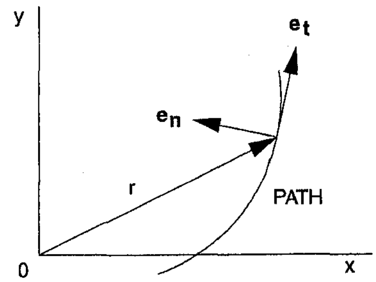
r = the radius,

θ = the angle between the x-axis and r,

$\dot{r} = dr/dt$, etc., and

$\ddot{r} = d^2r/dt^2$, etc.

Tangential and Normal Components



Unit vectors $\boldsymbol{e}_n$ and $\boldsymbol{e}_t$ are, respectively, normal and tangen to the path.

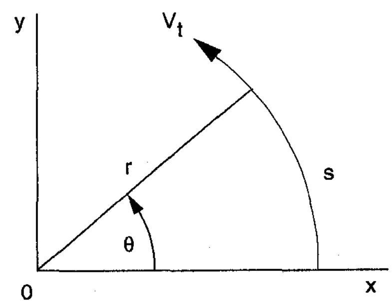
$$\boldsymbol{v} = v_t\boldsymbol{e}_t$$

$$\boldsymbol{a} = (dv_t/dt)\,\boldsymbol{e}_t + (v_t^2/\rho)\,\boldsymbol{e}_n, \text{ where}$$

ρ = instantaneous radius of curvature, and

v_t = tangential velocity.

Plane Circular Motion



Rotation about the origin with constant radius: The un vectors are $\boldsymbol{e}_t = \boldsymbol{e}_\theta$ and $\boldsymbol{e}_r = -\boldsymbol{e}_n$.

Angular velocity

$$\omega = \dot{\theta} = v_t/r$$

Angular acceleration

$$\alpha = \dot{\omega} = \ddot{\theta} = a_t/r$$

$$s = r\,\theta$$

$$v_t = r\,\omega$$

Tangential acceleration

$$a_t = r\,\alpha = dv_t/dt$$

Normal acceleration

$$a_n = v_t^2/r = r\,\omega^2$$

Straight Line Motion

Constant acceleration equations:

$$s = s_o + v_o t + (a_o t^2) / 2$$

$$v = v_o + a_o t$$

$$v^2 = v_o^2 + 2a_o(s - s_o), \text{ where}$$

s = distance along the line traveled,

s_o = an initial distance from origin (constant),

v_o = an initial velocity (constant),

a_o = a constant acceleration,

t = time, and

v = velocity at time t.

For a free-falling body, $a_o = g$ (downward)

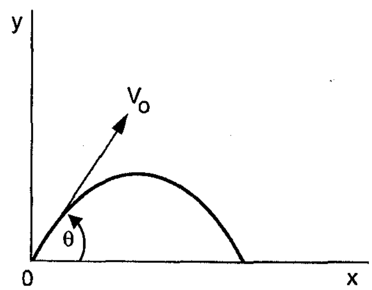
Using variable velocity, $v(t)$

$$s = s_o + \int_0^t v(t)\, dt$$

Using variable acceleration, $a(t)$

$$v = v_o + \int_0^t a(t)\, dt$$

PROJECTILE MOTION



$$a_x = 0; \quad a_y = -g$$

$$v_x = v_{xo} = v_o \cos \theta$$

$$v_y = v_{yo} - gt = v_o \sin \theta - gt$$

$$x = v_{xo} t = v_o t \cos \theta$$

$$y = v_{yo} t - gt^2 / 2 = v_o t \sin \theta - gt^2/2$$

CONCEPT OF WEIGHT

$$W = mg, \text{ where}$$

W = weight, N (lbf),

m = mass, kg (lbf-sec^2/ft), and

g = local acceleration of gravity, m/sec^2 (ft/sec^2).

KINETICS

Newton's second law for a particle

$$\Sigma \mathbf{F} = d(m\mathbf{v})/dt, \text{ where}$$

$\Sigma \mathbf{F}$ = the sum of the applied forces acting on the particle, N (lbf).

For a constant mass,

$$\Sigma \mathbf{F} = m d\mathbf{v}/dt = m\mathbf{a}$$

One-Dimensional Motion of Particle

When referring to motion in the x-direction,

$$a_x = F_x / m, \text{ where}$$

F_x = the resultant of the applied forces in the x-direction. F_x can depend on t, x and v_x in general.

If F_x depends only on t, then

$$v_x(t) = v_{xo} + \int_0^t [F_x(t')/m]\, dt'$$

$$x(t) = x_o + v_{xo} t + \int_0^t v_x(t')\, dt'$$

If the force is constant (independent of time, displacement, or velocity),

$$a_x = F_x / m$$

$$v_x = v_{x0} + (F_x / m)\, t = v_{x0} + a_x t$$

$$x = x_0 + v_{x0} t + F_x t^2/(2m)$$

$$= x_0 + v_{x0} t + a_x t^2/2$$

Tangential and Normal Kinetics for Planar Problems

Working with the tangential and normal directions,

$$\Sigma F_t = m a_t = m dv_t / dt \text{ and}$$

$$\Sigma F_n = m a_n = m\, (v_t^2/\rho)$$

Impulse and Momentum

Assuming the mass is constant, the equation of motion is

$$m dv_x / dt = F_x$$

$$m dv_x = F_x dt$$

$$m[v_x(t) - v_x(0)] = \int_0^t F_x(t') dt'$$

The left side of the equation represents the change in linear momentum of a body or particle. The right side is termed the impulse of the force $F_x(t')$ between $t' = 0$ and $t' = t$.

Work and Energy

Work W is defined as

$$W = \int \mathbf{F} \cdot d\mathbf{r}$$

(For particle flow, see **FLUID MECHANICS** section.)

<u>Kinetic Energy</u>

The kinetic energy of a particle is the work done by an external agent in accelerating the particle from rest to a velocity v.

$$T = mv^2 / 2$$

In changing the velocity from v_1 to v_2, the change in kinetic energy is

$$T_2 - T_1 = m v_2^2 / 2 - m v_1^2 / 2$$

<u>Potential Energy</u>

The work done by an external agent in the presence of a conservative field is termed the change in potential energy.

Potential Energy in Gravity Field

$U = mgh$, where

h = the elevation above a specified datum.

Elastic Potential Energy

For a linear elastic spring with modulus, stiffness, or spring constant k, the force is

$F_s = k\,x$, where

x = the change in length of the spring from the undeformed length of the spring.

The potential energy stored in the spring when compressed or extended by an amount x is

$$U = k\,x^2/2$$

The change of potential energy in deforming a spring from position x_1 to position x_2 is

$$U_2 - U_1 = k\,x_2^2/2 - k\,x_1^2/2$$

Principle of Work and Energy

If T_i and U_i are kinetic energy and potential energy at state i, then for conservative systems (no energy dissipation), the law of conservation of energy is

$$U_1 + T_1 = U_2 + T_2.$$

If nonconservative forces are present, then the work done by these forces must be accounted for.

$$U_1 + T_1 + W_{1\rightarrow 2} = U_2 + T_2$$

(Care must be exercised during computations to correctly compute the algebraic sign of the work term).

Impact

Momentum is conserved while energy may or may not be conserved. For direct central impact with no external forces

$m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 = m_1\boldsymbol{v}'_1 + m_2\boldsymbol{v}'_2$, where

m_1, m_2 = the masses of the two bodies,

$\boldsymbol{v}_1, \boldsymbol{v}_2$ = their velocities before impact, and

$\boldsymbol{v}'_1, \boldsymbol{v}'_2$ = their velocities after impact.

For impact with dissipation of energy, the relative velocity expression is

$$e = \frac{v'_{2_n} - v'_{1_n}}{v_{1_n} - v_{2_n}}, \text{ where}$$

e = the coefficient of restitution for the materials, and the subscript n denotes the components normal to the plane of impact.

Knowing e, the velocities after rebound are

$$v'_{1_n} = \frac{m_2 v_{2_n}(1+e) + (m_1 - e m_2) v_{1_n}}{m_1 + m_2}$$

$$v'_2 = \frac{m_1 v_{1_n}(1+e) - (e m_1 - m_2) v_{2_n}}{m_1 + m_2}$$

where $0 \le e \le 1$,

$e = 1$, perfectly elastic, and

$e = 0$, perfectly plastic (no rebound).

FRICTION

The Laws of Friction are

1. The total friction force F that can be developed is independent of the magnitude of the area of contact.
2. The total friction force F that can be developed is proportional to the normal force N.
3. For low velocities of sliding, the total friction force that can be developed is practically independent of the velocity, although experiments show that the force F necessary to start sliding is greater than that necessary to maintain sliding.

The formula expressing the laws of friction is

$F = \mu N$, where

μ = the coefficient of friction.

Static friction will be less than or equal to $\mu_s N$, where μ_s is the coefficient of static friction. At the point of impending motion,

$$F = \mu_s N$$

When motion is present

$F = \mu_k N$, where

μ_k = the coefficient of kinetic friction. The value of μ_k is often taken to be 75% of μ_s.

Belt friction is discussed in the **STATICS** section.

MASS MOMENT OF INERTIA

$$I_z = \int (x^2 + y^2)\, dm$$

A table listing moment of inertia formulas is available at the end of this section for some standard shapes.

Parallel Axis Theorem

$I_z = I_{zc} + md^2$, where

I_z = the mass moment of inertia about a specific axis (in this case, the z-axis),

I_{zc} = the mass moment of inertia about the body's mass center (in this case, parallel to the z-axis),

m = the mass of the body, and

d = the normal distance from the mass center to the specific axis desired (in this case, the z-axis).

Also,

$I_z = mr_z^2$, where

m = the total mass of the body, and

r_z = the radius of gyration (in this case, about the z-axis).

PLANE MOTION OF A RIGID BODY

For a rigid body in plane motion in the x-y plane

$$\Sigma F_x = ma_{xc}$$

$$\Sigma F_y = ma_{yc}$$

$$\Sigma M_{zc} = I_{zc}\alpha, \text{ where}$$

c = the center of gravity, and

α = angular acceleration of the body.

Rotation About a Fixed Axis

$$\Sigma M_O = I_O\alpha, \text{ where}$$

O denotes the axis about which rotation occurs.

For rotation about a fixed axis caused by a constant applied moment M

$$\alpha = M/I$$

$$\omega = \omega_O + (M/I)\,t$$

$$\theta = \theta_O + \omega_O t + (M/2I)\,t^2$$

The change in kinetic energy of rotation is the work done in accelerating the rigid body from ω_O to ω.

$$I_O\,\omega^2/2 - I_O\,\omega_O^2/2 = \int_{\theta_O}^{\theta} M d\theta$$

Kinetic Energy

The kinetic energy of a body in plane motion is

$$T = m\left(v_{xc}^2 + v_{yc}^2\right)/2 + I_c\,\omega^2/2$$

Instantaneous Center of Rotation

The instantaneous center of rotation for a body in plane motion is defined as that position about which all portions of that body are rotating.

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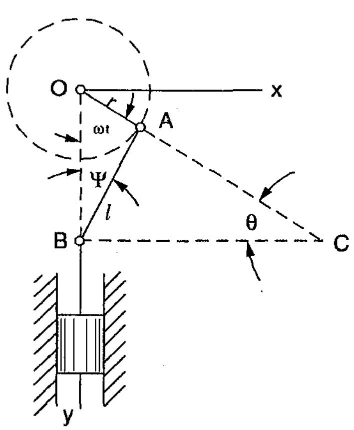


$$AC\dot{\theta} = r\omega, \text{ and}$$

$$v_b = BC\dot{\theta}, \text{ where}$$

C = the instantaneous center of rotation,

$\dot{\theta}$ = the rotational velocity about C, and

AC, BC = radii determined by the geometry of the situation.

CENTRIFUGAL FORCE

For a rigid body (of mass m) rotating about a fixed axis, the centrifugal force of the body at the point of rotation is

$$F_c = mr\omega^2 = mv^2/r, \text{ where}$$

r = the distance from the center of rotation to the center of the mass of the body.

BANKING OF CURVES (WITHOUT FRICTION)

$$\tan\theta = v^2/(gr), \text{ where}$$

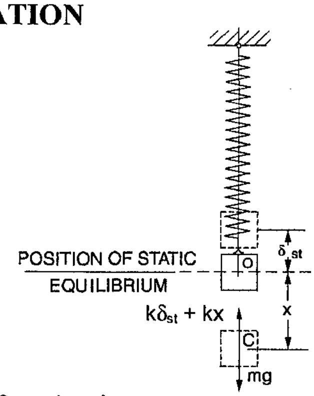
θ = the angle between the roadway surface and the horizontal,

v = the velocity of the vehicle, and

r = the radius of the curve.

FREE VIBRATION

•



The equation of motion is

$$m\ddot{x} = mg - k(x + \delta_{st})$$

From static equilibrium

$$mg = k\delta_{st},$$

where

k = the spring constant, and

δ_{st} = the static deflection of the spring supporting the weight (mg).

The above equation of motion may now be rewritten as

$$m\ddot{x} + kx = 0, \quad \text{or}$$

$$\ddot{x} + (k/m)x = 0.$$

The solution to this differential equation is

$$x(t) = C_1 \cos\sqrt{(k/m)}\,t + C_2 \sin\sqrt{(k/m)}\,t, \text{ where}$$

$x(t)$ = the displacement in the x-direction, and

C_1, C_2 = constants of integration whose values depend on the initial conditions of the problem.

The quantity $\sqrt{k/m}$ is called the undamped natural frequency ω_n or $\omega_n = \sqrt{k/m}$

• Timoshenko, S. and D.H. Young, *Engineering Mechanics*, Copyright © 1951 by McGraw-Hill Company, Inc. Diagrams reproduction permission pending.

From the static deflection relation

$$\omega_n = \sqrt{g/\delta_{st}}$$

The period of vibration is

$$\tau = 2\pi/\omega_n = 2\pi\sqrt{m/k} = 2\pi\sqrt{\delta_{st}/g}$$

If the initial conditions are $x(0) = x_0$ and $\dot{x}(0) = v_0$, then

$$x(t) = x_0 \cos \omega_n t + (v_0/\omega_n) \sin \omega_n t$$

If the initial conditions are $x(0) = x_0$ and $\dot{x}(0) = 0$, then

$$x(t) = x_0 \cos \omega_n t,$$

which is the equation for simple harmonic motion where the amplitude of vibration is x_0.

Torsional Free Vibration

$$\ddot{\theta} + \omega_n^2\theta = 0, \text{ where}$$

$\omega_n = \sqrt{k_t/I} = \sqrt{GJ/IL}$,

k_t = the torsional spring constant = GJ/L,

I = the mass moment of inertia of the body,

G = the shear modulus,

J = the area polar moment of inertia of the round shaft cross section, and

L = the length of the round shaft.

The solution to the equation of motion is

$$\theta = \theta_0 \cos\omega_n t + (\dot{\theta}_0/\omega_n)\sin\omega_n t, \text{ where}$$

θ_0 = the initial angle of rotation and

$\dot{\theta}_0$ = the initial angular velocity.

The undamped circular natural frequency of torsional vibration is

$$\omega_n = \sqrt{GJ/IL}$$

The period of torsional vibration is

$$\tau = 2\pi/\omega_n = 2\pi\sqrt{IL/GJ}$$