فصل سوم

معادلات حرکت

- معادلات پيوستگي

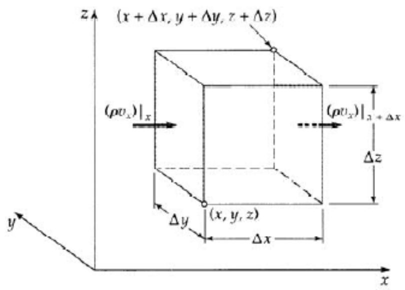


Fig. 3.1-1. Fixed volume element $\Delta x\, \Delta y\, \Delta z$ through which a fluid is flowing. The arrows indicate the mass flux in and out of the volume at the two shaded faces located at x and $x + \Delta x$.

$$\Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} = \Delta y \Delta z \left[(\rho V_x)\big|_x - (\rho V_x)\big|_{x+\Delta x} \right] + \Delta z \Delta x \left[(\rho V_y)\big|_y - (\rho V_y)\big|_{y+\Delta y} \right] + \Delta x \Delta y \left[(\rho V_z)\big|_z - (\rho V_z)\big|_{z+\Delta z} \right]$$

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial}{\partial x}\rho V_x + \frac{\partial}{\partial y}\rho V_y + \frac{\partial}{\partial z}\rho V_z \right)$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = -(\nabla.\rho V)$$

شدت افزایش جرم در واحد حجم

شدت افزایش جرم به واحد حجم بر اثر جابجایی

$$\nabla.V = 0$$

- معادله حرکت

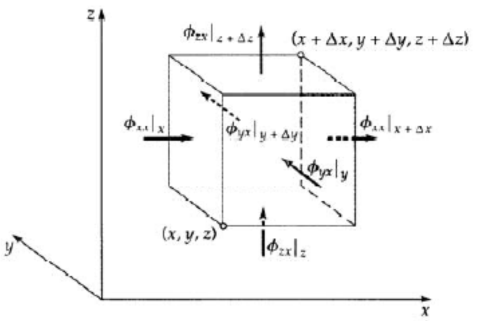
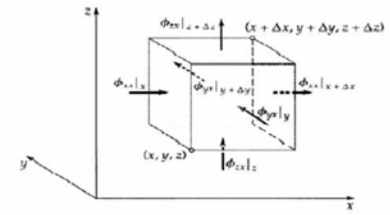


Fig. 3.2-1. Fixed volume element $\Delta x \Delta y \Delta z$, with six arrows indicating the flux of x-momentum through the surfaces by all mechanisms. The shaded faces are located at x and $x + \Delta x$.

{شدت افزایش اندازه حرکت} = {شدت ورود اندازه حرکت} − {شدت خروج اندازه حرکت} + {نیروی خارجی وارد بر سیال}



$$\Delta x \Delta y \Delta z \frac{\partial(\rho V_x)}{\partial t} = \Delta y \Delta z \left[\phi_{xx}\big|_x - \phi_{xx}\big|_{x+\Delta x}\right] + \Delta z \Delta x \left[\phi_{yx}\big|_y - \phi_{yx}\big|_{y+\Delta y}\right] + \Delta x \Delta y \left[\phi_{zx}\big|_z - \phi_{zx}\big|_{z+\Delta z}\right] + \rho g_x \Delta x \Delta y \Delta z$$

$$\Rightarrow \frac{\partial}{\partial t}\rho V_x = -\left(\frac{\partial}{\partial x}\phi_{xx} + \frac{\partial}{\partial y}\phi_{yx} + \frac{\partial}{\partial z}\phi_{zx}\right) + \rho g_x$$

همين رابطه براي y و z هم بدست مي آيد.

$$\Rightarrow \frac{\partial}{\partial t}\rho V_i = -[\nabla.\phi]_i + \rho g_i \qquad i = x, y, z$$

به صورت برداري $\Rightarrow \frac{\partial}{\partial t}\rho V = -[\nabla.\phi] + \rho g$

$$\phi = \rho VV + p\delta + \tau$$

$$\Rightarrow \frac{\partial}{\partial t}\rho V = -[\nabla.\rho VV] - \nabla p - [\nabla.\tau] + \rho g$$

- $\frac{\partial}{\partial t}\rho V$: شدت افزايش اندازه حركت بر واحد حجم
- $[\nabla.\rho VV]$: شدت افزايش اندازه حركت جابجايی در واحد حجم
- ∇p: شدت افزايش اندازه حركت از طريق كار فشار حجم
- $[\nabla.\tau]$: شدت افزايش اندازه حركت بخاطر نيروهای ويسكوز در واحد حجم
- ρg: نيروهای خارجی وارد بر سيال در واحد حجم

❖معادله انرژی مکانیکی

با معرفي انرژي پتانسيل به صورت $g = -\nabla\hat{\Phi}$ و استفاده از معادله پيوستگي، معادله تغيير انرژي مكانيكي كل به شكل زير در مي آيد:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho V^2 + \rho\hat{\Phi}\right) = -\left(\nabla.\left(\frac{1}{2}\rho V^2 + \rho\hat{\Phi}\right)V\right)$$
$$-(\nabla.pV) - p(-\nabla.V)$$
$$-(\nabla.[\tau.V]) - (-\tau:\nabla V)$$

معادله تغيير انرژي سينتيک از ضرب نقطه اي V در معادله حرکت بدست مي آيد:

- شدت افزايش انرژي سينتيک توسط جابجايي

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho V^2\right) = -\left(\nabla.\frac{1}{2}\rho V^2 V\right)$$

- شدت كار انجام شده توسط فشار از سوي محيط بر سيال

$$-(\nabla.pV)$$

- شدت تبديل برگشت پذير انرژي سينتيک به انرژي داخلي

$$-P(-\nabla.V)$$

- شدت کار انجام شده توسط نيروهاي ويسکوز بر سيال

$$-\left(\nabla.\left(\tau.V\right)\right)$$

- شدت تبديل برگشت ناپذير انرژي سينتيک به انرژي داخلي

$$-\left(-\tau:\nabla V\right)$$

- شدت کار انجام شده توسط نيروي خارجي بر سيال

$$+\rho\left(V.g\right)$$

با استفاده از تعريف مشتق کل معادلات ساده مي شوند:

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+V_x\frac{\partial}{\partial x}+V_y\frac{\partial}{\partial y}+V_z\frac{\partial}{\partial z}$$

- معادله پيوستگي

$$\frac{D\rho}{Dt}=-\rho\left(\nabla.V\right)$$

- معادله حرکت

$$\rho\frac{DV}{Dt}=-\nabla p-\left[\nabla.\tau\right]+\rho g$$

- نويه استوکس

$$=-\nabla p+\mu\nabla^2 V+\rho g$$

- **وقتي جريان شتاب نداشته باشد يعني** DV/Dt=0
جريان استوكس (جريان خزنده)

$$-\nabla p + \mu \nabla^2 V + \rho g = 0$$

- **وقتي نيروي ويسكوز صرف نظر شود يعني** $[\nabla . \tau] = 0$
معادله اويلر براي جريان غير ويسكوز

$$\rho \frac{DV}{Dt} = -\nabla p + \rho g$$

- معادله برنولي:

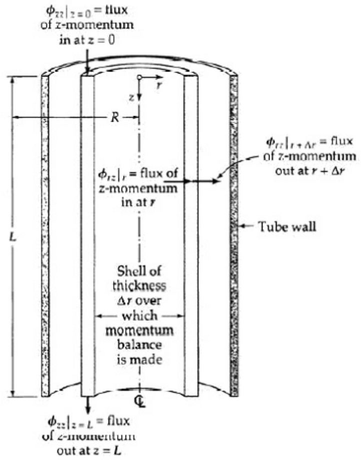
$$\rho V_z \frac{dV_z}{dz} = -\frac{dp}{dz} - \rho g$$

$$\Rightarrow \frac{1}{2}\rho\left(V_2^2 - V_1^2\right) = -\left(p_2 - p_1\right) - \rho g\left(z_2 - z_1\right)$$

- حل مثال جريان در لوله مدور

$$V=\delta_z\, V_z(r,z) \qquad ,V_r=V_\theta=0$$

V_z تابع z نيست.



- معادله پيوستگي:

$$\frac{\partial V_z}{\partial z}=0$$

- جهت r معادله حركت:

$$-\frac{\partial P}{\partial r}=0$$

- جهت θ معادله حركت:

$$-\frac{\partial P}{\partial \theta}=0$$

- جهت z معادله حركت:

$$-\frac{\partial P}{\partial z}+\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right)=0$$

فقط تابع z است. ⟶ $P=p+\rho g h$

$$\mu \frac{1}{r}\frac{d}{dr}\left(r\frac{dV_z}{dr}\right) = C_0 = \frac{dP}{dz}$$

$$\Rightarrow P = C_0 z + C_1$$

$$V_z = \frac{C_0}{4\mu} r^2 + C_2 \ln r + C_3$$

$$B.C.1: \quad z = 0 \ , \ P = P_0$$

$$B.C.2: \quad z = L \ , \ P = P_L$$

$$B.C.3: \quad r = R \ , \ V_z = 0$$

$$B.C.4: \quad r = 0 \ , \ V_z = finite$$

$$\frac{P_0 - P}{P_0 - P_L} = \frac{z}{L}$$

$$V_z = \frac{(P_0 - P_L)R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

- ویسکومتر کووت

$$V_\theta = V_\theta(r) \quad , V_z = V_r = 0$$

$$p = p(r, z)$$

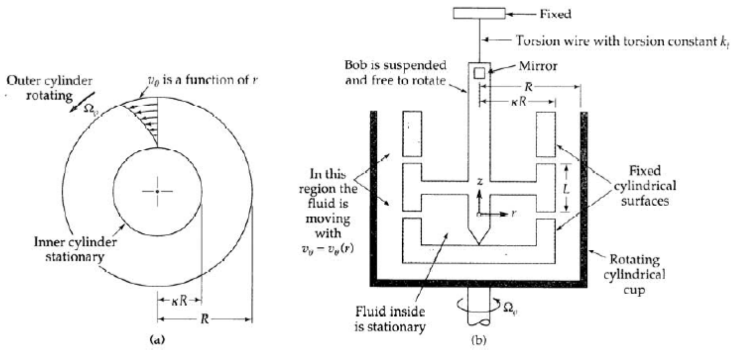


Fig. 3.6-1. (*a*)Tangential laminar flow of an incompressible fluid in the space between two cylinders; the outer one is moving with an angular velocity Ω_o. (*b*) A diagram of a Couette viscometer. One measures the angular velocity Ω_o of the cup and the deflection θ_B of the bob at steady-state operation. Equation 3.6-31 gives the viscosity μ in terms of Ω_o and the torque $T_z = k_t\theta_B$.

- جهت r

$$-\rho\frac{V_\theta^2}{r}=-\frac{\partial p}{\partial r}$$

- جهت θ

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(rV_\theta\right)\right)=0$$

– جهت z

$$-\frac{\partial p}{\partial z}-\rho g=0$$

انتگرال گيري در جهت θ توزيع سرعت را مي دهد:

$$V_\theta=\frac{1}{2}C_1 r+\frac{C_2}{r}$$

$$B.C.1: r=kR\ ,\ V_\theta=0$$

$$B.C.2: r=R\ ,\ V_\theta=\Omega_0 R$$

در نتيجه خواهيم داشت:

$$\Rightarrow V_\theta=\Omega_0 R\frac{\left(\frac{r}{kR}-\frac{kR}{r}\right)}{\left(\frac{1}{k}-k\right)}$$

$$\tau_{r\theta}=-\mu r\frac{d}{dr}\left(\frac{V_\theta}{r}\right)=-2\mu\Omega_0\left(\frac{R}{r}\right)^2\left(\frac{k^2}{1-k^2}\right)$$

$$T_z=\left(-\tau_{r\theta}\right)\Big|_{r=kR}.2\pi kRL.kR$$

$$=4\pi\mu\Omega_0 R^2 L\left(\frac{k^2}{1-k^2}\right)$$

همچنين:

$$T_z=k_t\theta_b$$

به اين ترتيب مي توان μ را محاسبه كرد.

اگر استوانه داخلي بچرخد و خارجي ثابت باشد:

$$V_\theta = \Omega_i kR \frac{\left(\frac{R}{r} - \frac{r}{R}\right)}{\left(\frac{1}{k} - k\right)}$$

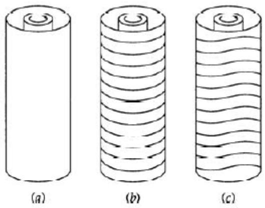
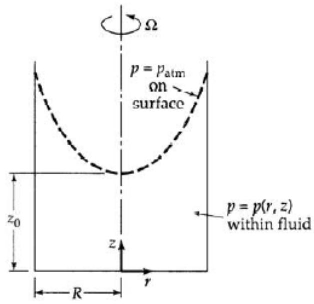


Fig. 3.6-4. Sketches showing the phenomena observed in the annular space between two cylinders: (a) purely tangential flow; (b) singly periodic flow (Taylor vortices); and (c) doubly periodic flow in which an undulatory motion is superposed on the Taylor vortices.

يعني فرضهايي که براي مسئله مي کنيم اهميت دارد.

- شکل سطح يک مايع در حال چرخش

Fig. 3.6-6. Rotating liquid with a free surface, the shape of which is a paraboloid of revolution.



$$V_r = V_z = 0$$

$$V_\theta = V_\theta(r)$$

$$p = p(z, r)$$

- جهت r

$$-\rho\frac{V_\theta^2}{r}=-\frac{\partial p}{\partial r}$$

- جهت θ

$$\mu\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rV_\theta)\right)=0$$

– جهت z

$$-\frac{\partial p}{\partial z}-\rho g=0$$

$$\Rightarrow V_\theta=\frac{1}{2}C_1r+\frac{C_2}{r}$$

$$B.C.1: r=0\ ,\ V_\theta=finit$$

$$B.C.2: r=R\ ,\ V_\theta=R\Omega$$

مثل چرخش یک جسم صلب

$$\Rightarrow V_\theta=\Omega r$$

$$\frac{\partial p}{\partial r}=\rho\Omega^2 r\quad ,\quad \frac{\partial p}{\partial z}=-\rho g$$

$$p=\frac{1}{2}\rho\Omega^2r^2+f(z)$$

$$\frac{\partial p}{\partial z}=-\rho g=\frac{df}{dz}$$

$$\Rightarrow f=-\rho gz+C$$

$$\Rightarrow p=\frac{1}{2}\rho\Omega^2r^2-\rho gz+C$$

$$\Rightarrow p = \frac{1}{2}\rho\Omega^2 r^2 - \rho g z + C$$

$$B.C. \quad : r = 0 \ , \ z = z_0 \ , \ p = p_{atm} \Rightarrow$$

$$p - p_{atm} = -\rho g\left(z - z_0\right) + \frac{1}{2}\rho\Omega^2 r^2$$

این معادله فشار براي همه نقاط درون مايع است.

در سطح: $p = p_{atm} = z - z_0 = \left(\frac{\Omega^2}{2g}\right) r^2$

- جریان در اطراف یک کره در حال چرخش

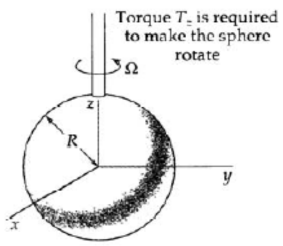


Fig. 3.6-7. A slowly rotating sphere in an infinite expanse of fluid. The primary flow is $v_\phi = \Omega R (R/r)^2 \sin\theta$.

$$V_\phi = V_\phi(r, \theta)$$

$$V_r = V_\theta = 0$$

$$P = P(r, \theta)$$

- جهت r

$$-\frac{\partial P}{\partial r}=0$$

- جهت θ

$$-\frac{1}{r}\frac{\partial P}{\partial \theta}=0$$

- جهت ϕ

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V_\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(V_\phi\sin\theta\right)\right)=0$$

$$B.C.1: r=R\ ,\ V_r=V_\theta=0\ ,\ V_\phi=R\Omega\ \sin\theta$$

$$B.C.2: r\to\infty\ ,\ V_r\to 0\ ,\ V_\theta\to 0\ ,\ V_\phi\to 0$$

$$B.C.3: r\to\infty\ ,\ P\to p_0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V_\phi}{\partial r}\right)+\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(V_\phi\sin\theta\right)\right)=0$$

$$V_\phi=R(r).T(\theta)$$

$$\Rightarrow\frac{1}{R}.\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{T}.\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\left(T\sin\theta\right)\right)=0$$

$$x=\cos\theta\Rightarrow$$

$$\frac{1}{R}.\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=-\frac{1}{T}.\frac{d}{dx}\left[\left(1-x^2\right)\frac{dT}{dx}\right]+\frac{1}{1-x^2}=n(n+1)$$

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dT}{dx}\right]+\left[n(n+1)-\frac{1}{1-x^2}\right]T=0$$

- معادله لژاندر وابسته

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{d\phi}{dx}\right]+\left[n(n+1)-\frac{m^2}{1-x^2}\right]\phi=0$$

$$\Rightarrow \phi = P_n^m(x)$$

$$P_n^m(x)=\left(1-x^2\right)^{m/2}\frac{d^m P_n(x)}{dx^m}$$

$$\Rightarrow T = P_n^1(x)$$

$$\Rightarrow r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-n(n+1)R=0$$

$$\Rightarrow r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-n(n+1)R=0$$

$$R=r^k \Rightarrow k^2+k-n(n+1)=0$$

$$\Rightarrow k=n\ ,\ \ k=-(n+1)$$

$$\Rightarrow R=C_1r^n+C_2r^{-(n+1)}$$

$$r\to\infty\ ,\ V_\phi\to 0\ \ \Rightarrow C_1=0$$

$$V_\phi=\sum A_n\frac{P_n^1(x)}{r^{(n+1)}}$$

$$r=R\ ,\ V_\phi=R\Omega\ \sin\theta \Rightarrow R\Omega\ \sin\theta=\sum A_n\frac{P_n^1(x)}{R^{(n+1)}}$$

$$\Rightarrow \frac{A_n}{R^{(n+2)}\Omega}=\frac{\int_{-1}^{1}\sin\theta\ P_n^1(x)dx}{\int_{-1}^{1}\left[P_n^1(x)\right]^2dx}$$

$$=\frac{\int_{-1}^{1}\left(1-x^2\right)^{\frac{1}{2}}P_n^1(x)dx}{\dfrac{2(n+1)!}{(2n+1)(n-1)!}}$$

$$\int_{-1}^{1}(1-x^2)^{\frac{1}{2}}P_n^1(x)dx=\int_{-1}^{1}(1-x^2)\left[\frac{d}{dx}P_n(x)\right]dx$$

$$(1-x^2)P_n'(x)=n\ P_{n-1}(x)-n\ xP_n(x)$$

$$\Rightarrow\int_{-1}^{1}\left[n\ P_{n-1}(x)-n\ xP_n(x)\right]dx$$

$$\Rightarrow n\int_{-1}^{1}P_{n-1}(x)dx-n\int_{-1}^{1}xP_n(x)dx$$

$$=n\int_{-1}^{1}P_0(x)P_{n-1}(x)dx-n\int_{-1}^{1}P_1(x)P_n(x)dx$$

$$\Rightarrow\int_{-1}^{1}1\times dx-\int_{-1}^{1}x^2dx=\frac{4}{3}$$

$$\Rightarrow\frac{A_n}{R^{(n+2)}\Omega}=1\Rightarrow A_1=R^3\Omega$$

$$A_n(n\neq 1)=0$$

$$\Rightarrow V_\phi=R^3\Omega\frac{P_1^1(x)}{r^2}$$

$$V_\phi=\frac{R^3\Omega}{r^2}(1-x^2)^{\frac{1}{2}}=R\Omega\left(\frac{R}{r}\right)^2\sin\theta$$

- محاسبه گشتاور وارد بر کره:

$$T_z = \int_0^{2\pi} \int_0^{\pi} \tau_{r\phi}\Big|_{r=R} (R\sin\theta)\ R^2 \sin\theta\ d\theta d\phi$$

$$= \int_0^{2\pi} \int_0^{\pi} (3\mu\Omega\ \sin\theta)\,(R\sin\theta)\ R^2 \sin\theta\, d\theta d\phi$$

$$= 6\pi\mu\Omega R^3 \int_0^{\pi} \sin^3\theta\, d\theta = 8\pi\mu\Omega R^3$$

Spherical coordinates (r, θ, ϕ):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right) = -\frac{\partial p}{\partial r}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial v_r}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}\right] + \rho g_r \qquad \text{(B.6-7)}^a$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2 \cot\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta \sin\theta)\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{2\cot\theta}{r^2 \sin\theta}\frac{\partial v_\phi}{\partial\phi}\right] + \rho g_\theta \qquad \text{(B.6-8)}$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi \cot\theta}{r}\right) = -\frac{1}{r \sin\theta}\frac{\partial p}{\partial \phi}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\phi \sin\theta)\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2} + \frac{2}{r^2 \sin\theta}\frac{\partial v_r}{\partial\phi} + \frac{2\cot\theta}{r^2 \sin\theta}\frac{\partial v_\theta}{\partial\phi}\right] + \rho g_\theta \qquad \text{(B.6-9)}$$

Cylindrical coordinates (r, θ, z):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r \quad \text{(B.6-4)}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta \quad \text{(B.6-5)}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \quad \text{(B.6-6)}$$

Cylindrical coordinates (r, θ, z):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r \quad \text{(B.6-4)}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta \quad \text{(B.6-5)}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \quad \text{(B.6-6)}$$

Cylindrical coordinates (r, θ, z):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r \qquad \text{(B.6-4)}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta \qquad \text{(B.6-5)}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \qquad \text{(B.6-6)}$$