year month. day. جلسه اول

تعریف معادلات دیفرانسیل

معادلات دیفرانسیل معادلاتی هستند که رفتارهای فیزیکی، شیمیایی و حرارتی یک سیستم را با صورت مشتق تحت شرایطی که به آن شرایط مرزی و اولیه گفته می شود بیان می کنند

تعریف متغیر مستقل و وابسته

متغیر مستقل متغیرهایی هستند که مقادیر را از دامنه یک تابع برمی دارند و متغیرهای وابسته تابعی از متغیرهای مستقل می باشند

معادلات دیفرانسیل به طور کلی به دو نوع تقسیم می شوند

1- ODE (ordinary Differention Equation)

2- PDE (Partiah Differential Equation)

تفاوت معادلات معمولی و جزئی :

۱- ODE می تواند چند متغیر وابسته داشته باشد که البته PDE هم می تواند

۲- ODE تنها یک متغیر مستقل باید داشته باشد در حالی که PDE بیش از یک متغیر مستقل دارند
PDE حداقل دو متغیر مستقل باید داشته باشد

۳- ODE ها ایده آل اند به این معنا که مدلهای بسیار ایده آل و ساده شده را می توان به صورت یک ODE بیان کرد در حالی که PDE ها واقعی ترند به این مفهوم که سیستم های که در طبیعت وجود دارد رفتارشان به یک PDE مدل می شود

year: month: day: 2 subject:

۴ - کار با ODE ها راحت‌تر از PDE ها است و از آنها استفاده می‌شود

۵ - حل معادلات دیفرانسیل معمولی بسیار ساده است و برای جواب عمومی حالت خصوصی و حالت کلی است اما برای PDE ها حل به صورت کلی است و راه حل طولانی است

درجه و مرتبه معادله دیفرانسیل:

بیشترین مرتبه مشتق موجود در معادله را مرتبه معادله می‌نامند (order)

معادله دیفرانسیل خطی و غیر خطی: linear , non linear

معادله دیفرانسیل را خطی نامند هرگاه متغیر وابسته و مشتقات و یا مشتقات در هم ضرب نشده باشند و یا متغیر وابسته یا مشتقات توان غیر یک نداشته باشند

$uu_x + u_y = xy$ غیرخطی $u_x^2 + u_y = 1$ [illegible] غیرخطی

شرایط مرزی:

شرایط مرزی یعنی شرایط فیزیکی حاکم بر مسأله است که برای حل معادله حتماً به آن نیاز داریم. در حقیقت شرایط مرزی مقدار تابع را در نقاط معلوم بیان می‌کند. ممکن است شرایط مرزی به صورت معادله دیفرانسیل بیان شود. در این صورت بالاترین مرتبه مشتق آن را مرتبه شرایط مرزی می‌نامند. تعاریف خطی و غیرخطی که در مورد معادله بیان شد در مورد شرایط مرزی هم صادق است.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$ Governing Equation

$$\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$$ Boundary Conditions (B.C)

$$\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$ Initial Conditions (I.C)

معادلات همگن و غیر همگن : (homogeneus)

اگر u به cu تبدیل شود معادله تغییر نکند همگن است (هرگاه متغیر وابسته در عدد ثابتی مثل c ضرب شود معادله تغییر نکند همگن است)

$$\frac{\partial T}{\partial t} = T^2$$

$$\frac{\partial cT}{\partial t} = (cT)^2 \Rightarrow \frac{c\,\partial T}{\partial t} = c^2T^2$$ غیر همگن

مسائل خطی همگن

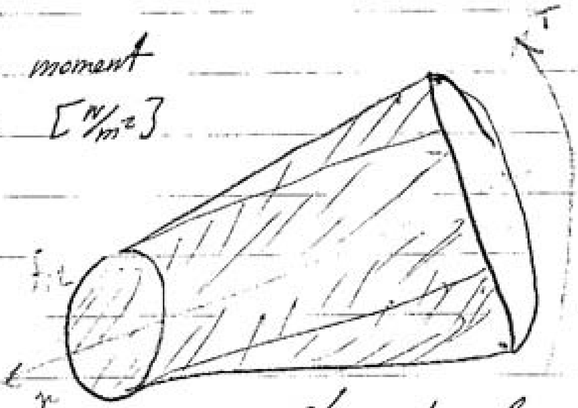
مسئله را همگن و خطی نامند هرگاه هم خود معادله و هم شرایط مرزی حاکم بر آن همگن و خطی باشد

year month. day. 4 subject

Ex: Consider a fixed shaft where affected by a moment at its end write Governing Eq. for it

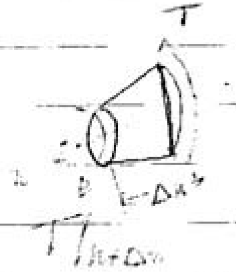
the important parameter are:

- γ: weight in volume unit $[N/m^3]$
- $A(x)$: sectional $[m^2]$
- $I(x)$: Inertia moment $[m^2 \cdot kg]$
- T: Torque $[N \cdot m]$
- M: mass $[kg]$
- θ: torsion angle $[rad]$
- $\dot{\theta}$: angular velocity $[rad/s]$
- $\ddot{\theta}$: angular ~~velo~~ accelration $[rad/s^2]$
- t: time $[s]$
- $J(x)$: $[m^4]$ Area moment
- G: shearing moduls $[N/m^2]$



$m(x,t)$: [illegible] (external moment for volume unit)

یک المان جدا می کنیم



$$\Sigma M = I\alpha \qquad \alpha = \frac{\partial^2\theta}{\partial t^2}$$

$$I = mk^2$$

↓ شعاع ژیراسیون

$$m = \frac{W}{g} = \frac{w A(x)\,\Delta x}{g}$$

$$J = A k^2 \Rightarrow k^2 = \frac{J(x)}{A(x)}$$

$$I = \frac{w A(x)\,\Delta x}{g}\,\frac{J(x)}{A(x)}$$

$$T\Big|_{x+\Delta x} - T\Big|_{x} + f(x,\theta,\dot\theta,t)\,A(x)\,\Delta(x)$$

$$= \frac{w A(x)\,\Delta x}{g}\,\frac{J(x)}{A(x)}\cdot\frac{\partial^2\theta}{\partial t^2}$$

طرفین را Δx تقسیم

$$\lim_{\Delta x\to 0}\frac{T\big|_{x+\Delta x} - T\big|_{x}}{\Delta x} + f(x,\theta,\dot\theta,t)\,(A(x)\,\Delta(x) = \frac{w}{g}\,J(x)\,\frac{\partial^2\theta}{\partial T^2}$$

$$\downarrow$$

$$\frac{\partial T}{\partial x}$$

$$\frac{\partial T}{\partial x} + f(x, \theta, \dot{\theta}, t)\, A(x) = \frac{w}{g} J(x) \frac{\partial^2 \theta}{\partial t^2}$$

$$\theta = \frac{Tl}{GJ} \qquad T = GJ\frac{\theta}{l} \quad \text{or} \quad T = GJ\frac{\partial \theta}{\partial x}$$

$$\frac{\partial}{\partial x}\left(GJ\frac{\partial \theta}{\partial x}\right) + f(x, \theta, \dot{\theta}, t)\, A(x) = \frac{w}{g} J(x) \frac{\partial^2 \theta}{\partial t^2}$$

$$\frac{\partial}{\partial x}\left(J(x)\frac{\partial \theta}{\partial x}\right) + f(x, \dot{\theta}, \theta, t)\, \frac{A(x)}{G} = \frac{w}{gG} J(x) \frac{\partial^2 \theta}{\partial t^2}$$

if no external moment and $J(x)$ be constant

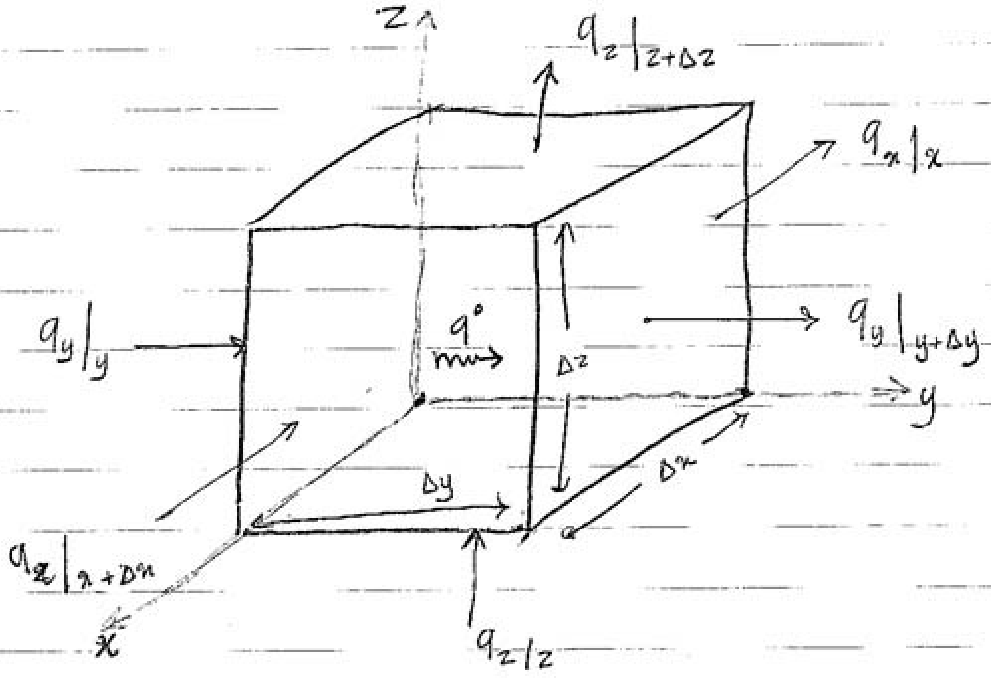
$$\frac{\partial^2 \theta}{\partial x^2} = \frac{w}{gG}\frac{\partial^2 \theta}{\partial t^2}$$

$$\frac{w}{gG} = a^2$$

$$\frac{\partial^2 \theta}{\partial x^2} = a^2\frac{\partial^2 \theta}{\partial t^2} \longrightarrow \text{T.V.S PDE}$$

torsion vibration shaft

Conduction heat transfer



Rate input energy − Rate output Energy + Rate genration Energy = rate storted Energy

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$

$$\underbrace{q_x|_x + q_y|_y + q_z|_z}_{\dot{E}_{in}} - \underbrace{\left(q_x|_{x+\Delta x} + q_y|_{y+\Delta y} + q_z|_{z+\Delta x}\right)}_{E_{out}}$$

$$+ \underbrace{q^\circ \Delta x \Delta y \Delta z}_{E_g} + \underbrace{\rho(\Delta x \Delta y \Delta z)}_{m} C \frac{\partial T}{\partial t}$$

year month. day. 8 subject.

$$f(x+\Delta x) = f(x) + f'(x)\Delta x$$

$$\left\{ q_x \big|_{x+\Delta x} = q_x \big|_x + \frac{\partial q_x}{\partial x}\Delta x \right.$$

$$\frac{\partial q_x}{\partial x}\Delta x + \frac{\partial q_y}{\partial y}\Delta y + \frac{\partial q_z}{\partial z}\Delta z + q'(\Delta x \Delta y \Delta z) = \rho(\Delta x \Delta y \Delta z) c \frac{\partial T}{\partial t}$$

$$\left\{ q_x = -k(\Delta z \Delta y)\frac{\partial T}{\partial x} \right.$$

$$q_y = -k(\Delta x \Delta z)\frac{\partial T}{\partial y}$$

$$q_z = -k(\Delta x \Delta y)\frac{\partial T}{\partial z}$$

$$-\frac{d}{\partial x}\left(-k\Delta z\Delta y \frac{\partial T}{\partial x}\Delta x\right) - \frac{\partial}{\partial y}\left(-k\Delta x\Delta z\frac{\partial T}{\partial y}\right)\Delta y - \frac{\partial}{\partial z}\left(-k\Delta x\Delta y\right)\frac{\partial T}{\partial z}\Delta z$$

$$+\dot{q}(\Delta x\Delta y\Delta z) = \rho(\Delta x\Delta y\Delta z)c\frac{\partial T}{\partial t}$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c\frac{\partial T}{\partial t}$$

$$\Rightarrow \frac{\partial^2 T}{\partial x} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{\rho c}{k}\frac{\partial T}{\partial t}$$

$$\frac{k}{\rho c} = \alpha$$ Diffusivity

$L \quad \nabla^2 T = 0$ laplace Temperture Eq

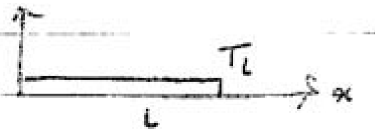
شرایط مرزی :

به طور کلی ۳ شرط مرزی وجود دارد

① شرط مرزی نوع اول (Dirichlet B.C)

دما به شکل صریح در این شرط بیان خواهد شد

states temperture as explcit



$$T(x,y,t)\Big|_{x=L} = T_L$$

② شرط مرزی نوع دوم (Neuman B.C)

دما در این شرط مرزی به شکل مشتق داده خواهد شد

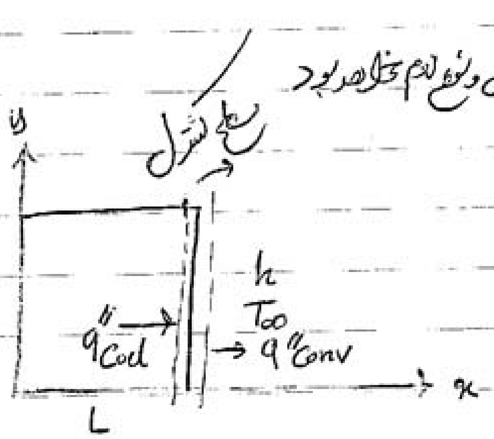
states temperature as differential

$$\frac{\partial T}{\partial x}\Big|_{x=0} = q''_0$$

* برای سطح عایق شرط مرزی مشتق صفر خواهد بود

for an adiabatic Surface $\frac{\partial T}{\partial x} = 0$

۳- شرط مرزی نوع سوم . ترکیبی از نوع اول و نوع دوم خواهد بود

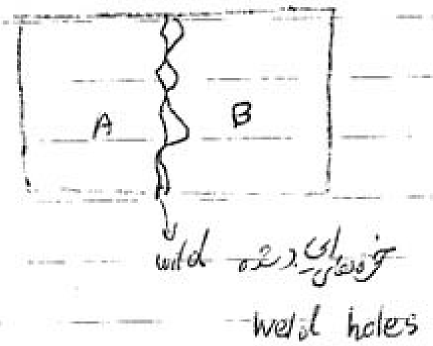


$$q''_{Cond} = q''_{Conv} \Rightarrow -k\frac{\partial T}{\partial x}\Big|_{x=l} = h\left(T_1\big|_{x=l} - T_\infty\right)$$

٤- شرط مرزی نوع چهارم . این شرط با سطوح تماس شده بکار خواهد رفت

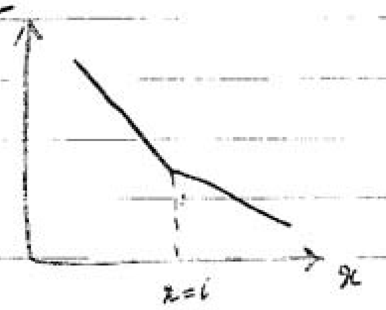
It used for interfaces it Comes up while weld

۱- تماس کامل (Perfect Contact)



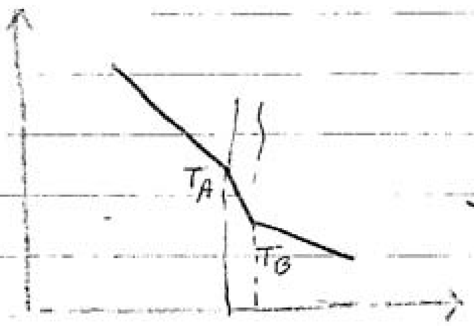
$$-k\frac{\partial T_1}{\partial x}\Big|_{x=L} = -k_L\frac{\partial T_2}{\partial n}\Big|_{x=l}$$

T_1 : منحنی تابع دما در نقطه A ، T_2 منحنی تابع دما در نقطه B



منحنی توزیع دما در حالت تماس کامل

۲- تماس ناقص (unperfect contact)



منحنی توزیع دما در حالت تماس ناقص

$$-k_1 \frac{\partial T_1}{\partial x} = h_c (T_A - T_B) = -k_2 \frac{\partial T_2}{\partial x}$$

$h_c \to \infty$

در حالت تماس کامل $T_A = T_B$

ظرفیت گرمایی گسترده (lumping method)

در این روش دما تابعی از زمان خواهد بود و تابعی از مکان نخواهد بود

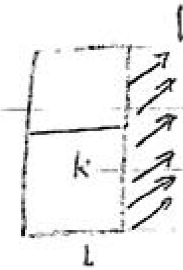
$T = T(x, y, z, t)$ non lumped

lumping method $\rightarrow$ $T = T(t)$

year month. day. 12 subject

شرط های اینکه ظرفیت گرمایی فشرده

1) Size smal
2) k باید بزرگ باشد
3) h باید بسیار کوچک باشه



$$R_{cond} = \frac{L}{kA}$$

$$R_{conv} = \frac{1}{hA}$$

$$R_{cond} \ll R_{conv}$$

$$\frac{R_{cond}}{R_{conv}} \ll$$

$$\text{Bojot numer} = \frac{R_{cond}}{R_{conv}} = \frac{L/kA}{1/hA} = \frac{hL}{k}$$

$$Bi = \frac{hL}{k} \qquad L = \frac{V}{A}$$

$Bi < 0.1$ شرط معتبر بودن ظرفیت گرمایی فشرده

year   month.   day. 13   subject.

lumping method
- total lump method
- Partial lump method

- حالت total برای حجم‌های بزرگ و کوچک معتبر خواهد بود

- حالت اول (total) : در این حالت از توزیع دما در جهت‌های مختلف چشم پوشی می‌کنیم. (اگر $Bi < 0.1$ باشد)

توزیع دما کمتر از 5% خواهد بود

- حالت دوم (Partial) :

It Can't be ignored temperature distribution in direction but it Can ignored in isothermal direction.

year month day 14 subject

: ex

the temperature of a gas stream is to be meaured by a thermocouple the junction may be approm/ted as a sphere of diamter ($D = 3/4$ mm) $K = 30\ W/mC$

$\rho = 8400\ kg/m^3$ $C = 0.4\ KJ/kg^{\circ}C$

if the heat transfer coefficient betwen the junction and the gas stram is $600\ W/m^2 {}^{\circ}C$

a) Can u total lumping formulation be applied?

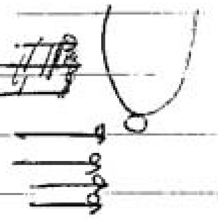
b) how long does it take for the themocouple to record 99% of the initial temperature of the thermocouple?

Solution :

$$Bi = \frac{hl}{K} \qquad l = \frac{V}{A} = \frac{4/3\,\pi r^3}{4\pi r^2} = 1/3\, r = 1/8 \qquad 1/3 \times 3/4 \times 1/2 = 1/8$$

$$L = 1/8 \times 10^{-3} = 1/8 \times 10^{-3}$$

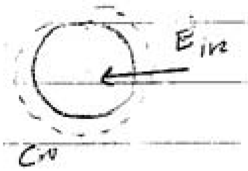
$$Bi = \frac{600 \times 1/8 \times 10^{-3}}{30} = 2.5 \times 10^{-3} < .1 \Rightarrow$$ lumped method is valid



Energy Conservation:

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st} \Rightarrow \dot{E}_{in} = \dot{E}_{st}$$

(with $\dot{E}_{out} = 0$ and $\dot{E}_g = 0$ crossed out)



$$hA(T_\infty - T) = \rho c v \frac{dT}{dt}$$

$$\Rightarrow \frac{dT}{T - T_\infty} = -\frac{hA}{\rho c v} dt$$

$$\int_{T_i}^{T} \frac{dT}{T - T_\infty} = \frac{-hA}{\rho c v} \int_0^t dt \Rightarrow$$

$$\ln (T - T_\infty) \Big|_{T_i}^{T} = -\frac{hA}{\rho c v} t$$

دمای اولیه

$$\ln \frac{T - T_\infty}{T_i - T_\infty} = -\frac{hA}{\rho v c} t \Rightarrow T = T_\infty + (T_i - T_\infty) e^{-\frac{hA}{\rho v c} t}$$

t: زمان T_i: دمای اولیه T_∞: دمای سیال

←

$$\overbrace{\qquad\qquad}^{.99\,(T_\infty - T_i)}\overbrace{\quad}^{.01\,(T_\infty - T_i)}$$

T_i ——————————— T_∞

$$T - T_i + T_\infty - T = T_\infty - T_i$$

$$= \quad .99\,(T_\infty - T_i) + T_\infty - T = T_\infty - T_i$$

$$\Rightarrow T_\infty - T = 0.01\,(T_\infty - T_i) \qquad \frac{T - T_\infty}{T_i - T_\infty} = e^{-\frac{hA}{\rho VC}t}$$

$$\Rightarrow 0.01 = e^{-\frac{hA}{\rho CV}t} \quad \Rightarrow \frac{-hA}{\rho CV}t = \ln 0.01$$

$$\Rightarrow t = \frac{\rho CV}{hA}\ln 0.01$$

$$\Rightarrow t = \frac{8400 \times 4 \times 10^{3}}{600}\left(\tfrac{1}{6} \times 10^{-3}\right)\ln 0.01 \Rightarrow t = 3.22\,s$$

تبدیل مختصات (Coordinates transformation)

در این مبحث به دنبال آن هستیم که روابطی را که در سیستم مختصات داریم به سیستم مختصات دیگری انتقال دهیم. مثلاً در سیستم کارتزین معادله توزیع دما معلوم است. به دنبال آن هستیم که این معادله را در سیستم استوانه‌ای بدست آوریم

Cartesian:
$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}=\alpha\frac{\partial T}{\partial t}$$

$$T=T(r,\theta,z,t)$$

از هم مستقل خواهند بود

قاعده زنجیری

$$\frac{\partial T}{\partial x}=\frac{\partial T}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial T}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}+\frac{\partial T}{\partial z}\cdot\frac{\partial z}{\partial x}+\frac{\partial T}{\partial t}\cdot\frac{\partial t}{\partial x}$$

($\frac{\partial z}{\partial x}=0$, $\frac{\partial t}{\partial x}=0$)

$$\frac{\partial T}{\partial x}=\frac{\partial T}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial T}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$x=r\cos\theta \Longrightarrow \frac{dx}{dx}=\frac{dr}{dx}\cos\theta+r\frac{\partial\cos\theta}{\partial x}$$

$$1=\cos\theta\frac{\partial r}{\partial x}-r\sin\theta\frac{\partial\theta}{\partial x} \quad (1)$$

$$y=r\sin\theta \Longrightarrow \frac{dy}{dx}=\frac{\partial r}{\partial x}\sin\theta+r\frac{\partial\sin\theta}{\partial x}$$

$$\frac{dy}{dx}=\frac{\partial r}{\partial x}\sin\theta+r\frac{\partial\sin\theta}{\partial\theta}\frac{\partial\theta}{\partial x}$$

year month. day. 18 subject.

$$\frac{dy}{dx} = \sin\theta\frac{\partial r}{\partial x} + r\cos\theta\frac{\partial\theta}{\partial x} \Rightarrow$$

$$0 = \sin\theta\frac{\partial r}{\partial x} + r\cos\theta\frac{\partial\theta}{\partial x} \quad (2)$$

$$1, 2 \Rightarrow \frac{\partial r}{\partial x} = \frac{\begin{vmatrix} 1 & -r\sin\theta \\ 0 & +r\cos\theta \end{vmatrix}}{\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial\theta}{\partial x} = \frac{\begin{vmatrix} \cos\theta & 1 \\ \sin\theta & 0 \end{vmatrix}}{\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}} = \frac{-\sin\theta}{r}$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial T}{\partial\theta}\frac{\partial\theta}{\partial x} \Rightarrow$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\cos\theta + \frac{\partial T}{\partial\theta}\left(-\frac{\sin\theta}{r}\right)$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial r}\left(\frac{\partial T}{\partial x}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial\theta}\left(\frac{\partial T}{\partial x}\right)\frac{\partial\theta}{\partial x}$$

$$+\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right)\frac{\partial z}{\partial x} + \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial x}\right)\frac{\partial t}{\partial x}$$

(last two terms crossed out, each marked 0)

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial r}\left[\frac{\partial T}{\partial r}\cos\theta + \frac{\partial T}{\partial\theta}\left(-\frac{\sin\theta}{r}\right)\right]\frac{\partial r}{\partial x} + \frac{\partial}{\partial\theta}\left[\frac{\partial T}{\partial r}\cos\theta + \frac{\partial T}{\partial\theta}\left(-\frac{\sin\theta}{r}\right)\right]\frac{\partial\theta}{\partial x}$$

$$\frac{\partial^2 T}{\partial x^2} = \left[\frac{\partial^2 T}{\partial r^2}\cos\theta + \frac{\partial^2 T}{\partial r\partial\theta}\left(-\frac{\sin\theta}{r}\right) + \frac{\sin\theta}{r^2}\frac{\partial T}{\partial\theta}\right]\frac{\partial r}{\partial x} +$$

$$\left[\frac{\partial^2 T}{\partial r\partial\theta}\cos\theta - \sin\theta\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial\theta^2}\left(-\frac{\sin\theta}{r}\right) - \frac{\cos\theta}{r}\frac{\partial T}{\partial\theta}\right]\frac{\partial\theta}{\partial x}$$

$$\Rightarrow \frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial r^2}\cos^2\theta + \frac{\partial^2 T}{\partial r\partial\theta}\left(-\frac{\sin\theta\cos\theta}{r}\right) + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial T}{\partial\theta}$$

$$+ \frac{\partial^2 T}{\partial r\partial\theta}\left(\frac{-\sin\theta\cos\theta}{r}\right) + \frac{\sin^2\theta}{r}\frac{\partial T}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 T}{\partial\theta^2} +$$

$$\frac{\sin\theta\cos\theta}{r^2}\frac{\partial T}{\partial\theta}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial r^2}\cos^2\theta - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 T}{\partial r\partial\theta} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial T}{\partial\theta} +$$

$$\frac{\sin^2\theta}{r}\frac{\partial T}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 T}{\partial\theta^2}$$

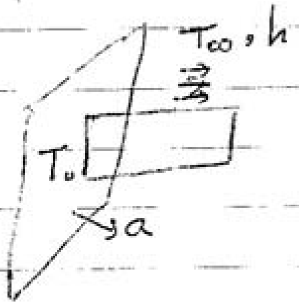
ex: obtain $\frac{\partial^2 T}{\partial y^2}$ and show that

$$T_{rr} + \frac{1}{r}T_r + \frac{1}{r^2}T_{\theta\theta} + T_{zz} + \frac{q}{\lambda} = \alpha\frac{\partial T}{\partial r}$$

in cylindrical system

فین‌ها :

از فین‌ها برای افزایش انتقال حرارت با استفاده از افزایش سطح موثر استفاده می‌شود



T_0 = base temperture

T_∞ = ambient temperture

h = Convection Coefficient

A_s = lateral Area

A_c = Cross sectional ares

k = Conductivity

P = Circumference

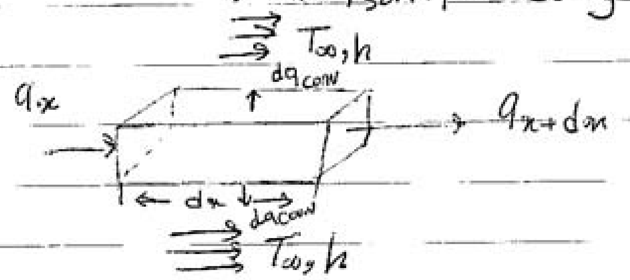
در فین‌ها ما به دنبال ۳ چیز هستیم:

- Govening Equation — معادله حاکم
- Tempeture distribution — توزیع دما
- Fin efficiency — راندمان فین

Ex) Consider an element of fin where is placed a surrounding of T_∞

assumption: 1) Constant cross sectional area $A_c = cte$

2) Conductivity is independent of temp and fin is an isotropic body $k = cte$



$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st} \Rightarrow \dot{E}_{in} = \dot{E}_{out}$$
($\dot{E}_g = 0$, $\dot{E}_{st} = 0$)

$$q_x = q_{x+dx} + dq_{conv} \quad , \quad q_{x+d} = q_x + \frac{dq_x}{dx} dx$$

$$dq_{conv} = h\, dA_s (T - T_\infty)$$

$$q_x = q_x + \frac{dq_x}{dx} dx + h\, dA_s (T - T_\infty)$$

$$q_x = -kA_c \frac{dT}{dx} \Rightarrow 0 = \frac{d}{dx}\left(-kA_c \frac{dT}{dx}\right) dx + hp\,\underbrace{dx}_{dA_s}(T - T_\infty)$$

$$\Rightarrow -kA_c \frac{d^2T}{dx^2} dx + hp\,dx (T - T_\infty) = 0$$

$$\Rightarrow \frac{d^2T}{dx^2} - \frac{hp}{kA_c}(T - T_\infty) = 0 \qquad \frac{hp}{kA_c} = m^2$$

$$\Rightarrow \frac{d^2T}{dx^2} - m^2 (T - T_\infty) = 0$$ Go Eq

$$\frac{d^2T}{dx^2} - m^2(T - T_\infty) = 0 \qquad m^2 = \frac{hP}{kA}$$

$$T - T_\infty = \theta \Rightarrow \frac{d^2T}{dx^2} = \frac{d^2\theta}{dx^2}$$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \Rightarrow \alpha^2 - m^2 = 0 \Rightarrow \alpha = \pm m$$

$$\theta = C_1 e^{mx} + C_2 e^{-mx}$$

$$T - T_\infty = C_1 e^{mx} + C_2 e^{-mx} \Rightarrow T = T_\infty + C_1 e^{mx} + C_2 e^{-mx}$$

for determine of C_1, C_2 we need two B.C's

1) the one is at $x = 0 \Rightarrow T = T_0$

2) the another one is deteminea by physical condition at other end of fin

there are for different case:

1) temperture is known in side end (at $x = L$ $T = T_L$)

2) the other end of fin is adiabatic (at $x = L$, $\frac{dT}{dx} = 0$)

3) the other end is placed at a surrounding at T_∞

(انتقال حرارت به بیرون داده خواهد شد)

at $x = L$ $\quad -k\frac{dT}{dx}\Big|_L = h(T_L - T_\infty)$

4) the fin is too long $(mL > 2.67)$ $T = T_\infty$

با این شرط یکی خواهد شد

assumption Case 2.1 :

at $x = 0 \Rightarrow T = T_0$

at $x = L \Rightarrow \frac{dT}{dx}$

$$\begin{cases} C_1 + C_2 = T_0 - T_\infty \\ C_1 m e^{mL} - C_2 m e^{-mL} = 0 \end{cases} \Rightarrow C_1 = C_2 e^{-2mL}$$

$$\Rightarrow C_2 e^{-2mL} + C_2 = T_0 - T_\infty \Rightarrow C_2 = \frac{T_0 - T_\infty}{1 + e^{-2mL}}$$

$$C_1 = \frac{T_0 - T_\infty}{1 + e^{-2mL}} e^{-2mL}$$

$$T = T_\infty + \frac{T_0 - T_\infty}{1 + e^{-2mL}} e^{-2mL} e^{mx} + \frac{T_0 - T_\infty}{1 + e^{-2mL}} e^{-mx}$$

$$\Rightarrow \frac{T - T_\infty}{T_0 - T_\infty} = \frac{e^{mx} e^{-2mL}}{1 + e^{-2mL}} + \frac{e^{-mx}}{1 + e^{-2mL}} = \frac{\left(e^{-mx} + e^{m(x-2L)}\right) e^{mL}}{\left(1 + e^{-2mL}\right) e^{mL}}$$

$$\frac{T - T_\infty}{T_0 - T_\infty} = \frac{e^{m(L-x)} + e^{m(x-L)}}{e^{mL} + e^{-mL}} = \frac{\dfrac{e^{m(L-x)} + e^{m(x-L)}}{2}}{\dfrac{e^{mL} + e^{-mL}}{2}}$$

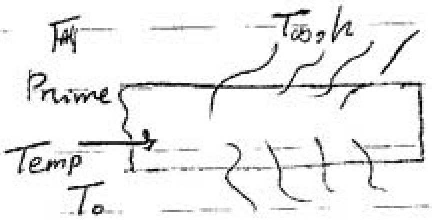
$$\frac{T - T_\infty}{T_0 - T_\infty} = \frac{\cosh m(L-x)}{\cosh mL}$$ Temp distribution

year: month. day. 24 subject .

Fin efficiency :

$$\eta = \frac{\text{actual heat transfer for the give geo ...}}{\text{ideal heat transfer for the given "}}$$

with prime temp (T_0) for all entire lenght



Case 2 $\quad \dfrac{T - T_\infty}{T_i - T_\infty} = \dfrac{\cosh m(L-x)}{\cosh mL}$

گرمای که از ریشه پره منتقل خواهد شد ← $q_{act} = -kA_c \dfrac{dT}{dx}\Big|_{x=0}$

مقدار گرمایی که از پره منتقل خواهد شد برابر با گرمایی است که پره به محیط می‌دهد

$$T = T_\infty + (T_i - T_\infty)\frac{\cosh m(L-x)}{\cosh mL} \Rightarrow$$

$$\frac{dT}{dx} = (T_i - T_\infty)\frac{-m \sinh m(L-x)}{\cosh mL}$$

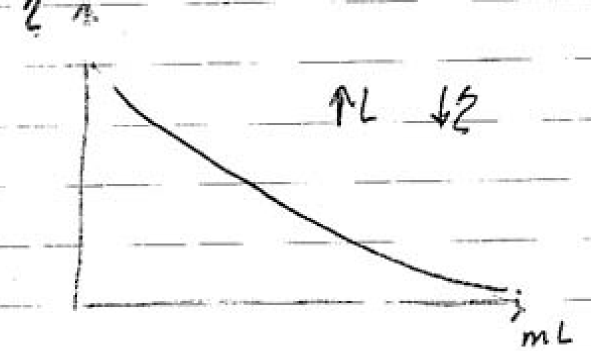
$$q_{act} = -kA_c \frac{dT}{dx}\Big|_{x=0} = -kA_c (T_i - T_\infty)\frac{-m\sinh mL}{\cosh mL} =$$

$$kmA_c (T_i - T_\infty)\,\text{tgh}\, mL$$

$$q_{ideal} = hA_s(T_i - T_\infty) = hpL(T_i - T_\infty)$$

$$\eta = \frac{kmA_c(T_i - T_\infty)\,tghmL}{hpL(T_i - T_\infty)} = \frac{k\sqrt{hp}\,A_c\,tghmL}{\sqrt{kA_c}\,hpL}$$

$$\eta = \frac{tghmL}{\frac{\sqrt{hp}\,L}{\sqrt{kA_c}}} \Rightarrow \eta = \frac{tghmL}{mL}$$



home work:

for $mL = .01, 0.1, 0.5, 1, 2, 3, 4, 5$

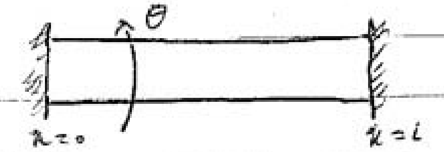
and $\frac{x}{L} = 0, 0.1, 0.2, \ldots, 1$ Plot $\frac{T - T_\infty}{T_i - T_\infty}$

v.s. $\frac{x}{L}$

and η v.s. mL

$$\frac{T - T_\infty}{T_i - T_\infty} = \frac{\cosh mL(1 - \frac{x}{L})}{\cosh mL} \qquad \eta = \frac{tghmL}{mL}$$

معادلات مشتقات جزئی (partial differential equation) :



$$\frac{\partial^2\theta}{\partial x^2} = a^2\frac{\partial^2\theta}{\partial t^2}$$

$$\theta = \theta(x,t)$$

$$B.C\begin{cases}\theta(0,t)=0\\ \theta(l,t)=0\end{cases}$$

$$IC\begin{cases}\theta(x,0)=f(x)\\ \theta_t(x,0)=g(x)\end{cases}$$

Solution seqration variables method:

$$\theta(x,t) = X_{(x)}\,T_{(t)}$$

$$\frac{\partial\theta}{\partial x} = \frac{\partial}{\partial x}(XT) = T\frac{dX}{dx} = TX'$$

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\theta}{\partial x}\right) = \frac{\partial}{\partial x}(TX') = T\frac{\partial X'}{\partial x} = T\frac{dX'}{dx} = TX''$$

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial t}(XT) = X\frac{\partial T}{\partial t} = X\frac{dT}{dt} = XT'$$

$$\frac{\partial^2\theta}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial\theta}{\partial t}\right) = \frac{\partial}{\partial t}(XT') = X\frac{\partial T'}{\partial t} = X\frac{dT'}{dt} = XT''$$

$$\frac{\partial^2\theta}{\partial x^2} = a^2\frac{\partial^2\theta}{\partial t^2} \Rightarrow X''T = a^2XT''$$

$\theta(x,t) = XT$ $\quad \theta=0$ although is acceptable mathematically but is not good phisycally assumption $\theta \neq 0$

$$X''T = a^2 X T'' \xrightarrow[\text{over } XT]{\text{Devide}} \frac{X''}{X} = a^2 \frac{T''}{T} = \sigma \quad \text{ثابت}$$

assume $\sigma = 0 \Rightarrow \frac{X''}{x} = 0 \Rightarrow X'' = 0 \Rightarrow X_{(x)} = C_1 x + C_2$

B.C $\begin{cases} \theta(0,t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow T(t) \neq 0 \\ \theta(L,t) = 0 \Rightarrow X(L)T(t) = 0 \Rightarrow T(t) \neq 0 \end{cases}$ چون θ صفر خواهد شد — در صورت صفر شدن T(t)

$$\Rightarrow \begin{cases} X(0) = 0 \\ X(L) = 0 \end{cases} \Rightarrow \begin{array}{l} X(0) = 0 \Rightarrow 0 = C_1 \times 0 + C_2 \Rightarrow C_2 = 0 \\ X(L) = 0 \Rightarrow 0 = C_1 L + 0 \Rightarrow C_1 = 0 \end{array}$$

$X(x) = 0 \Rightarrow \theta = 0$ (با فرض مغایر است) $\Rightarrow \sigma = 0$ is no good

assume $\sigma = \delta^2$ (Positive Value

$$\frac{X''}{x} = \delta^2 \Rightarrow X'' - \delta^2 X = 0 \Rightarrow \alpha^2 - \delta^2 = 0 \Rightarrow \alpha = \pm\delta$$

$$X_{(x)} = C_1 e^{\delta x} + C_2 e^{-\delta x}$$

or

$$X(x) = C_1 \cosh \delta x + C_2 \sinh \delta x$$

$$X(0) = 0 \Rightarrow 0 = C_1 \times 1 + C_2 \times 0 \Rightarrow C_1 = 0$$

$$X(L) = 0 \Rightarrow 0 = 0 + C_2 \sinh \delta L \Rightarrow C_2 \underbrace{\sinh \delta L}_{\neq 0} = 0 \Rightarrow C_2 = 0$$

$$\Rightarrow X_{(x)} = 0 \Rightarrow \theta = 0$$

$\sigma = \delta^2$ is no good

assume $\sigma = -\delta^2$ (Negative Value)

$$\frac{X''}{X} = -\delta^2 \Rightarrow X'' + \delta^2 X = 0 \Rightarrow \alpha^2 + \delta^2 = 0$$

$$\Rightarrow \alpha = \pm \delta i$$

$$X(x) = C_1 \cos\delta x + C_2 \sin\delta x$$

$$X(0) = 0 \Rightarrow 0 = C_1 \times 1 + C_2 \times 0 \Rightarrow C_1 = 0$$
$$X(L) = 0 \Rightarrow 0 = 0 + C_2 \sin\delta L \Rightarrow C_2 \sin\delta L = 0$$

$$\sin\delta L = 0 \Rightarrow \delta L = n\pi \Rightarrow \delta = \frac{n\pi}{L} \qquad n = 1, 2, \ldots N$$

$$n = 1 \Rightarrow \delta_1 = \frac{\pi}{L} \Rightarrow X_1 = k_1 \sin\delta_1 x$$

$$n = 2 \Rightarrow \delta_2 = \frac{2\pi}{L} \Rightarrow X_2 = k_2 \sin\delta_2 x$$

⋮

$$n = n \Rightarrow \delta_n = \frac{n\pi}{L} \Rightarrow X_n = k_n \sin\delta_n x$$ رابطه بازگشتی برای X

$$a^2 \frac{T''}{T} = -\delta^2 \Rightarrow$$

$$\frac{T''}{T} = -\frac{\delta^2}{a^2} \Rightarrow T'' + \frac{\delta^2}{a^2} T = 0 \Rightarrow$$

$$T(t) = C_1 \cos\frac{\delta}{a} t + C_2 \sin\frac{\delta}{a} t$$

$$T_n(t) = C_{1n} \cos\frac{\delta_n}{a} t + C_{2n} \sin\frac{\delta_n}{a} t$$

$$\theta_n(x,t) = X_n(x)\,T_n(t) \Longrightarrow$$

$$\theta_2(x,t) = \left(A_n \cos\frac{\delta_n}{a}t + B_n \sin\frac{\delta}{a}t\right)\sin\delta_n x$$

$$\theta(x,t) = \sum_{n=1}^{\infty}\theta_n(x,t) \quad \text{total solution}$$

$$\theta(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{\delta_n}{a}t + B_n\sin\frac{\delta_n}{a}t\right)\sin\delta_n x$$

$$IC \quad \theta(x,0) = f(x) \Rightarrow f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x \quad (I)$$

orthogonality:

طرفین I را در $\sin\frac{m\pi}{L}x$ ضرب و انتگرال می‌گیریم

$$\int_0^L f(x)\sin\frac{m\pi}{L}x\,dx = \sum_{n=1}^{\infty} A_n\int_0^L \sin\frac{n\pi}{L}x\,\sin\frac{m\pi}{L}x\,dx$$

$$\sin a\sin b = \tfrac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

$$L.H.S = \sum_{n=1}^{\infty}\frac{A_n}{2}\int_0^L\left[\cos(n-m)\frac{\pi}{L}x - \cos(n+m)\frac{\pi}{L}x\right]dx$$

$$L.H.S = \sum_{n=1}^{\infty}\frac{A_n}{2}\left[\frac{L}{\pi(n-m)}\sin(n-m)\frac{\pi}{L}x - \frac{L}{\pi(n+m)}\sin(n+m)\frac{\pi}{L}x\right]_0^L$$

$$L.H.S = 0 + A_n\int_0^L \sin^2\frac{n\pi}{L}x\,dx$$

$$L.H.S = A_n\int_0^L \frac{1-\cos\frac{2n\pi}{L}x}{2}\,dx \Rightarrow L.H.S = \tfrac{1}{2}A_n\left[x - \frac{L}{2n\pi}\sin\frac{2n\pi}{L}x\right]_0^L$$

$$L.H.S = \tfrac{1}{2}A_n L \Rightarrow A_n = \frac{2}{L}L.H.S \Rightarrow \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

$$\theta_t(x,0) = g(x)$$

$$\theta_t(x,t) = \sum_{n=1}^{\infty} \left(-A_n \frac{\lambda_n}{a} \sin\frac{\lambda_n}{a} t + B_n \frac{\lambda_n}{a} \cos\frac{\lambda_n}{a} t\right) \sin\lambda_n x$$

$$\theta_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{\lambda_n}{a} \sin\frac{n\pi}{L} x = g(x)$$

arthogonality or with Compuse with an

$$B_n \frac{\lambda_n}{a} = \frac{2}{L} \int_0^L g(x) \sin\frac{n\pi}{L} x \, dx$$

$$B_n = \frac{2a}{\lambda_n L} \int_0^L g(x) \sin\frac{n\pi}{L} x \, dx$$

$$\Rightarrow B_n = \frac{2a}{2\pi} \int_0^L g(x) \sin\frac{n\pi}{L} x \, dx$$

$$\Rightarrow B_n = \frac{2a}{2\pi}$$

a thin rad of lengh L has its lateral surface Perfectly isolated initialy the rad is at 100°C throughout at t=0 the left end at the rad is suddenly reduced to 50°C and it is remained thereafter at that value.

the right end of the rod is reamained at 100°C.

find the temperature at all points of rod at all subsequent times

i.C $\left\{ T(x,0) = 100°C \right.$

B.C $T(0,t) = 50°$

B.C $T(L,t) = 100°C$

$T(x,t) = ?$

$$\frac{\partial^2 T}{\partial x^2} + \cancel{\frac{\partial^2 T}{\partial y^2}} + \cancel{\frac{\partial^2 T}{\partial z^2}} + \cancel{\frac{\dot{q}}{k}} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \Rightarrow \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

0 0 0

$$\frac{\partial^2 T}{\partial x^2} = a^2 \frac{\partial T}{\partial t} \qquad \frac{1}{\alpha} = a^2$$

$$\begin{cases} T(0,t) = 50 \\ T(l,t) = 100 \end{cases}$$

$T(x,0) = 100$ non Homog problem

گذرا یا ناپایدار

$$T(x,t) = V(x) + W(x,t)$$

$V(x)$ → steady part, $W(x,t)$ → unsteady part

$$\frac{\partial T}{\partial x} = V' + \frac{\partial W}{\partial x}$$

$$\frac{\partial^2 T}{\partial x^2} = V'' + \frac{\partial^2 W}{\partial x^2}$$

$$\frac{\partial T}{\partial t} = \frac{\partial W}{\partial t}$$

$$\Rightarrow \quad V'' + \frac{\partial^2 W}{\partial x^2} = a^2 \frac{\partial W}{\partial t}$$

$$V'' = 0 \quad , \quad \frac{\partial^2 W}{\partial x^2} = a^2$$

$$T(0,t) = 50 \Rightarrow V(0) + W(0,t) = 50 \Rightarrow$$

$$V(0) = 50 \quad , \quad W(0,t) = 0$$

$$T(l,t) = 100 \Rightarrow V(l) + W(l,t) = 100 \Rightarrow$$

$$V(l) = 100 \quad , \quad W(l,t) = 0$$

PDE

$$\begin{cases} \frac{\partial^2 w}{\partial x^2} = a^2 \frac{\partial w}{\partial t} \\ w(0,t) = 0 \\ w(L,t) = 0 \end{cases}$$

ODE

$$\begin{cases} V'' = 0 \\ V(0) = 50 \\ V(L) = 100 \end{cases}$$

Solve PDE:

$$w(x,t) = X_{(x)}\,\Theta_{(t)}$$

$$\frac{\partial^2 w}{\partial x^2} = X''\Theta \quad , \quad \frac{\partial w}{\partial t} = X\,\Theta'$$

$$w(0,t) = 0 \Rightarrow X_{(0)}\,\Theta(t) = 0 \Rightarrow X(0) = 0$$

$$w(l,t) = 0 \Rightarrow X_{(l)}\,\Theta(t) = 0 \Rightarrow X(l) = 0$$

$$X''\Theta = a^2\Theta' \Rightarrow \frac{X''}{X} = a^2\frac{\Theta'}{\Theta} = \sigma$$

$$\sigma = 0 \Rightarrow \frac{X''}{X} = 0 \Rightarrow X'' = 0$$

$X_{(x)} = C_1 x + C_2$, $X(0) = 0 \Rightarrow C_1(0) + C_2 = 0 \Rightarrow C_2 = 0$

$X(l) = 0 \Rightarrow C_1(l) = 0 \Rightarrow C_1 = 0$

$\Rightarrow X_{(x)} = 0 \Rightarrow w(x,t) = 0$ ﮔﮕﮤ is no good

$\sigma = \delta^2 \Rightarrow \frac{X''}{X} = \delta^2 \Rightarrow X'' - \delta^2 X = 0 \Rightarrow$

$X_{(x)} = C_1 \cosh \delta x + C_2 \sinh \delta x$

$X(0) = 0 \Rightarrow C_1(1) + C_2(0) = 0 \Rightarrow C_1 = 0$

$X(L) = 0 \Rightarrow C_2 \sinh \delta L = 0 \xrightarrow{\delta L \neq 0} C_2 = 0 \Rightarrow w(x,t) = 0$
is no good

$\sigma = -\delta^2$

$\frac{X''}{X} = -\delta^2 \Rightarrow X'' + \delta^2 X = 0$

$X(x) = C_1 \cos \delta x + C_2 \sin \delta x$

$X(0) = 0 \Rightarrow C_1(1) + C_2(0) = 0 \Rightarrow C_1 = 0$

$X(L) = 0 \Rightarrow C_2 \sin \delta L = 0 \Rightarrow \sin \delta L = 0 \Rightarrow \delta L = n\pi$
$\delta = \frac{n\pi}{L}$

$X_n(x) = C_n \sin \frac{n\pi}{L} x$

$a^2 \frac{\theta'}{\theta} = -\delta^2 \Rightarrow \frac{\theta'}{\theta} = -\frac{\delta^2}{a^2}$

$\int \frac{\theta'}{\theta} dt = -\frac{\delta^2}{a^2} \int dt \Rightarrow \ln \theta_{(t)} = -\frac{\delta^2}{a^2} t + b$

$$\theta(t) = e^{-\frac{\delta^2}{a^2}t + b} = e^b e^{-\delta^2/a^2\, t}$$

$$\theta_n(t) = B_n e^{-\frac{\delta_n^2}{a^2} t}$$

$$w_n(x,t) = X_n(x)\,\theta_n(t) \Rightarrow w_n(x,t) = A_n e^{\frac{-\delta_n^2}{a^2} t} \sin\frac{n\pi}{L} x$$

$$\sum_{n=1}^{\infty} w_n(x,t) = W(x,t)$$

$$W(x,t) = \sum_{n=1}^{\infty} A_n e^{\frac{-n^2\pi^2}{L^2 a^2} t} \sin\frac{n\pi}{L} x$$

Solve ODE $\quad V'' = 0$

$V(0) = 50$

$V(L) = 100$

$V = C_1 x + C_2$

$50 = C_1(0) + C_2 \Rightarrow C_2 = 50$

$100 = C_1(L) + C_2 \Rightarrow C_1 = \frac{50}{L}$

$$\Rightarrow V_{(x)} = \frac{50}{L} x + 50$$

$$T(x,t) = V(x) + W(x,t)$$

$$T(x,t) = \frac{50}{L} x + 50 + \sum_{n=1}^{\infty} A_n e^{\frac{-n^2\pi^2}{L^2 a^2} t} \sin\frac{n\pi}{L} x$$

$T(x,0) = 100$

$$\frac{50}{L}x + 50 + \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x = 100$$

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x = 50 - \frac{50}{L}x$$

orthogonality $\Rightarrow$ $A_n = \frac{2}{L}\int_0^L \left(50 - \frac{50}{L}\right)x \sin\frac{n\pi}{L}x)\,dx$

$$A_n = \frac{2}{L}\left[-\frac{L}{n\pi}\left(50 - \frac{50x}{L}\right)\cos\frac{n\pi}{L}x - \frac{50L}{n^2\pi^2}\sin\frac{n\pi}{L}x\right]_0^L$$

$$= \frac{2}{L}\left[\frac{50L}{n\pi}\right] \Rightarrow A_n = \frac{100}{n\pi}$$

$$T(x,t) = \frac{50}{L}x + 50 + \sum_{n=1}^{\infty}\frac{100}{n\pi}e^{\frac{-n^2\pi^2}{L^2a^2}t}\sin\frac{n\pi}{L}x$$

$$T(x,t) = 50 + \frac{50x}{L} + \frac{100}{\pi}e^{\frac{-\pi^2 t}{L^2a^2}}\sin\frac{\pi}{L}x + \frac{100}{2\pi}e^{-\frac{4\pi^2 t}{L^2a^2}}\sin\frac{2\pi}{L}x$$

$+$ ———

Two Dimensional heat transfer :

2-D)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\dot{q}}{K} = a^2 \frac{\partial T}{\partial t} \qquad a^2 = \frac{1}{\alpha}$$

- heat genration
- non steady
- $T = (x, y, t)$

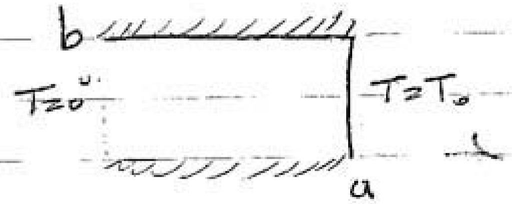
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = a^2 \frac{\partial T}{\partial t}$$

- without heat generation
- non steady
- $T = (x, y, t)$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad \text{or} \quad \nabla^2 T = 0$$

- without heat generation
- steady
- $T(x, y)$

Ex) Consider a rectangular plan where upper and lower side are isolated. the temperature at the left side is fixed at 0°C and the right side is fixed at T_0 find temperature distribution and heat rate and the center.



$$\begin{cases} \dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = 0 \\ T(0,y) = 0 \\ T(a,y) = T_0 \\ \left.\dfrac{\partial T}{\partial y}\right|_{(a,0)} = 0 \\ \left.\dfrac{\partial T}{\partial y}\right|_{(a,b)} = 0 \end{cases}$$

از معادله لاپلاس [illegible] --- حساب می‌شود

$$T(x,y) = X_{(x)}\,Y_{(y)}$$

$$\frac{\partial^2 T}{\partial x^2} = X''Y \quad , \quad \frac{\partial^2 T}{\partial y^2} = XY''$$

$$X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \sigma$$

$$T(0,y) = 0 \Rightarrow X_{(0)}Y_{(y)} = 0 \Rightarrow X_{(0)} = 0$$

$$\frac{\partial T}{\partial y}\Big|_{(x,0)} \Rightarrow XY'_{(0)} = 0 \Rightarrow Y'_{(0)} = 0$$

$$\frac{\partial T}{\partial y}\Big|_{(x,0)} = 0 \Rightarrow XY'_{(b)} = 0 \Rightarrow Y'_{(b)} = 0$$

$$\sigma = 0 \Rightarrow \frac{Y''}{Y} = 0 \Rightarrow Y'' = 0 \Rightarrow Y_{(y)} = C_1 y + C_2$$

$$Y'_{(y)} = C_1 \qquad Y'(0) = 0 \Rightarrow C_1 = 0$$

$$Y'(b) = 0 \Rightarrow C_1 = 0 \Rightarrow Y_{(y)} = C_2$$

$$\frac{X''}{X} = 0 \Rightarrow X'' = 0 \Rightarrow X_{(x)} = D_1 + D_2 x$$

$$X_{(0)} = 0 \Rightarrow D_1 + D_2(0) = 0 \Rightarrow D_1 = 0$$

$$\Rightarrow X_{(x)} = D_2 x$$

$$\sigma = 0 \Rightarrow T(x,y) = C_2 D_2 x = A_0 x$$

year. month. day. 40 subject.

if $\sigma = +\delta^2$

$$-\frac{Y''}{Y} = \delta^2 \Rightarrow Y'' + \delta^2 Y = 0$$

$$Y(y) = C_1 \cos \delta y + C_2 \sin \delta y$$

$$Y'(y) = -C_1 \delta \sin \delta y + C_2 \delta \cos \delta y$$

$$Y'(0) = 0 \Rightarrow -C_1 \delta (0) + C_2 \delta (1) = 0$$

$$\Rightarrow C_2 = 0$$

$$Y'(b) = 0 \Rightarrow -C_1 \delta \sin \delta b = 0 \Rightarrow \sin \delta b = 0 \Rightarrow \delta b = n\pi$$

$$\Rightarrow \delta_n = \frac{n\pi}{b}$$

$$Y_n(y) = C_n \cos \delta_n y$$

$$\frac{X''}{X} = \delta^2 \Rightarrow X'' - \delta^2 X = 0$$

~~Xlk~~ $X(x) = D_1 \cosh \delta x + D_2 \sinh \delta x$

$$X(0) = 0 \Rightarrow D_1(1) + D_2(0) = 0 \Rightarrow D_1 = 0$$

$$\Rightarrow X_n(x) = D_n \sinh \delta_n x$$

$$\sigma = \delta^2 \Rightarrow T_n(x, y) = X_n(x) Y_n(y)$$

$$T_n(x,y) = D_n C_n \sinh \delta_n x \cos \delta_n y$$

$\sigma = \delta^2$ $$T(x,y) = \sum A_n \sinh \frac{n\pi}{b} x \cos \frac{n\pi}{b} y$$

$\sigma = -\delta^2$

$$-\frac{Y''}{Y} = -\delta^2 \Rightarrow Y'' - \delta^2 Y = 0$$

$$\Rightarrow Y_{(y)} = C_1 \cosh \delta y + C_2 \sinh \delta y$$

$$Y'_{(y)} = C_1 \delta \sinh \delta y + C_2 \delta \cosh \delta y$$

$$Y'(0) = 0 \Rightarrow C_1 \delta (0) + C_2 \delta (1) = 0 \Rightarrow$$

$$C_2 = 0$$

$$Y'(b) = 0 \Rightarrow C_1 \delta \sinh \delta b = 0 \xrightarrow{\sinh \delta b \neq 0,\ \delta b \neq 0}$$

$$Y(y) = 0 \Rightarrow T(x,y) = 0$$

$\Rightarrow \sigma = -\delta^2$ is no good

final $$T(x,y) = A_0 x + \sum_{n=1}^{\infty} A_n \sinh \delta x \cos \delta y$$

(with $\delta = \frac{n\pi}{b}$ under each δ)

$$A_0 a + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi a}{b} \cos \frac{n\pi}{b} y = T_0$$

یادآوری

$$a_0 + \sum_{n=0}^{\infty} a_n \cos \frac{n\pi}{L} x = f(x)$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\, dx$$

$$a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi}{L} x\, dx$$

$$A_0 a = \frac{1}{b}\int_0^b T_0\, dy \Rightarrow \quad A_0 = \frac{T_0}{a}$$

$$A_n \sinh \frac{n\pi a}{b} = \frac{2}{b}\int_0^b T_0 \cos \frac{n\pi}{b} y\, dy \Rightarrow \quad A_n = 0$$

$$T(x,y) = \frac{T_0}{a} x$$

توزیع دما در این حالت
فقط تابع x می‌باشد

$$q'' = q''_x \hat{i} + q''_y \hat{j}$$

$$q''_x = -k \frac{\partial T}{\partial x}$$

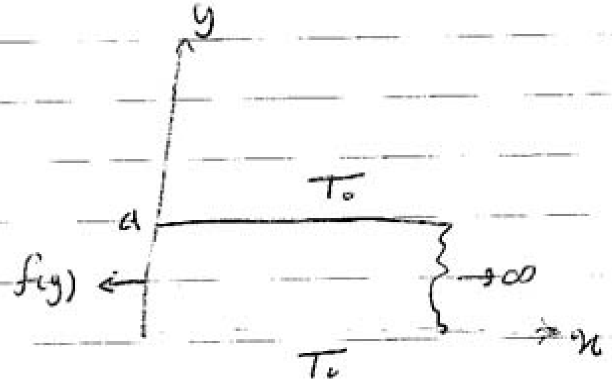
$$q''_y = -k \frac{\partial T}{\partial y}$$

$$q'' = -k \left(\frac{\partial T}{\partial x} \hat{i} + \frac{\partial T}{\partial y} \hat{j} \right)$$

در این مسئله $q'' = -k \frac{T_0}{a} \hat{i} + 0 \Rightarrow \quad q'' = -k \frac{T_0}{a}$

upper and lower side a rectangle are the same and temperature at T_0, the left side is as $f(y)$ and it has side to infinity as shown —

find temperature distribution



$$\begin{cases} T(0,y) = f(y) \\ T(x,0) = T_0 \\ T(x,a) = T_0 \\ \text{at } x \to \infty \quad T(\infty,y) = \text{finite} \end{cases}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$T(x,y) = V(y) + W(x,y)$$

V(y) → [illegible] y [illegible]:

$T(x,0) = T_0$

$T(x,a) = T_0$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 W}{\partial x^2}$$

$$\frac{\partial^2 T}{\partial y^2} = V'' + \frac{\partial^2 W}{\partial y^2} = 0 \Rightarrow$$

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = 0 \quad , V'' = 0$$

$$T(x,0)=T_0 \Rightarrow V(0)+W(x,0)=T_0 \Rightarrow \begin{cases} V(0)=T_0 \\ W(x,0)=0 \end{cases}$$

$$T(x,a)=T_0 \Rightarrow V(a)+W(x,a)=T_0 \Rightarrow \begin{cases} V(a)=T_0 \\ W(x,a)=0 \end{cases}$$

PDE : ODE

$$\begin{cases} \dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=0 \\ W(x,0)=0 \\ W(x,a)=0 \end{cases} \qquad \begin{cases} V''=0 \\ V=C_1 y+C_2 \\ V(0)=T_0, \ V(a)=T_0 \\ T_0=C_1(0)+C_2 \Rightarrow C_2=T_0 \\ T_0=C_1 a+T_0 \Rightarrow C_1=0 \end{cases}$$

$$V_{(y)}=T_0$$

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0$$

$$W(x,0)=0 \Rightarrow X_{(x)}Y_{(0)}=0 \Rightarrow Y_{(0)}=0$$

$$w(x,a)=0 \Rightarrow X_{(x)}Y_{(a)}=0 \Rightarrow Y_{(a)}=0$$

$$W(x,y)=X_{(x)}Y_{(y)}$$

$$\frac{\partial^2 w}{\partial x^2}=X''Y \quad , \quad \frac{\partial^2 w}{\partial y^2}=XY''$$

$$X''Y+XY''=0 \Rightarrow \frac{X''}{X}=-\frac{Y''}{Y}=\sigma$$

$\sigma = 0 \Rightarrow Y'' = 0 \Rightarrow Y = C_1 y + C_2$

$Y(0) = 0 \Rightarrow C_1(0) + C_2 = 0 \Rightarrow C_2 = 0$

$Y(a) = 0 \Rightarrow C_1(a) = 0 \Rightarrow C_1 = 0$

$Y = 0 \Rightarrow w(x,y) = 0$

$\Rightarrow \sigma = 0$ is no good

$\sigma = -\delta^2 \Rightarrow -\dfrac{Y''}{Y} = -\delta^2 \Rightarrow Y'' - \delta^2 Y = 0 \Rightarrow$

$Y(y) = C_1 \cosh \delta y + C_2 \sinh \delta y = 0$

$Y(0) = 0 \Rightarrow C_1(1) + C_2(0) = 0 \Rightarrow C_1 = 0$

$Y(a) = 0 \Rightarrow C_2 \sinh \delta a = 0 \Rightarrow C_2 = 0$

$Y(y) = 0 \Rightarrow w(x,y) = 0 \Rightarrow$ $\sigma = -\delta^2$ is no good

$\sigma = \delta^2 \Rightarrow -\dfrac{Y''}{Y} = \delta^2 \Rightarrow Y'' + \delta^2 Y = 0 \Rightarrow$

$Y(y) = C_2 \cos \delta y + C_2 \sin \delta y \Rightarrow Y(0) = 0 \Rightarrow$

$C_1(1) + C_2(0) = 0 \Rightarrow C_1 = 0$

$$y(a) = 0 \Rightarrow C_2 \sin \delta a = 0 \Rightarrow \sin \delta a = 0 \Rightarrow \delta a = n\pi \Rightarrow$$

$$\delta = \frac{n\pi}{a}$$

$$\Rightarrow Y_n(y) = C_n \sin \frac{n\pi}{a} y$$

$$\frac{X''}{X} = \delta^2 \Rightarrow X'' - \delta^2 X = 0 \Rightarrow X_{(x)} = D_1 e^{\delta x} + D_2 e^{-\delta x}$$

Since at $x \to \infty$ T is infite and

$\lim_{x \to \infty} e^{\delta x} = \infty$ then D_1 should be zero

$$\Rightarrow X_n(x) = D_n e^{-\delta x}$$

$$\Rightarrow W_n(x,y) = Y_n(y) X_n(x) = A_n e^{-\frac{n\pi}{a} x} \sin \frac{n\pi}{a} y$$

$$\Rightarrow W(x,y) = \sum_{n=1}^{\infty} A_n e^{-\frac{n\pi}{a} x} \sin \frac{n\pi}{a} y$$

$$T(x,y) = \underbrace{T_0}_{V(y)} + \underbrace{\sum_{n=1}^{\infty} A_n e^{-\frac{n\pi x}{a}} \sin \frac{n\pi}{a} y}_{W(x,y)}$$

$$T(0,y) = f(y) \Rightarrow T_0 + \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} y = f(y)$$

$$\Rightarrow \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} y = f(y) - T_0 \Rightarrow A_n = \frac{2}{a} \int_0^a (f(y) - T_0) \sin \frac{n\pi}{a} y \, dy$$

مسئله کامپیوتری

$$h = 100 \frac{W}{m^2 C} \qquad \Delta x = \Delta y = h = .05$$

$$T_\infty = 25°C$$

$$k = 401 \frac{W}{mK}$$

$$L = 1m$$

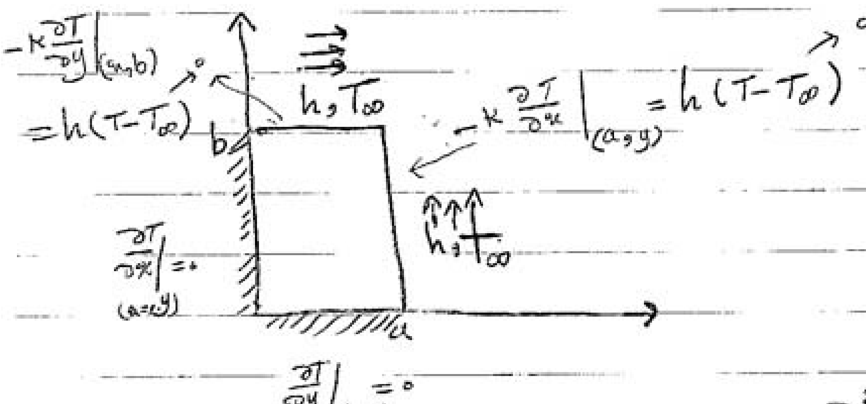
$$\frac{d^2T}{dx^2} - m^2(T - T_\infty) = 0$$

$$m^2 = \frac{hP}{kA_c}$$

$$\frac{d^2T}{dx^2} = \frac{1}{h^2}\left(T(x-h) - 2T(x) + T(x+h)\right]$$

$$\frac{dT}{dx} = \frac{1}{2h}\left[T(x-h) + T(x+h)\right]$$

EX: the plane wall shown having uniform energy generation $\dot{q}_{gen}$ due to a nuclear reaction the wall of length a and width b, has two of sides closest to origin isolated. the other side take place at a normolized zeror surrounding temperature and are exposed to convectiveit. find the steady state temp. distribution throughout the wall.



(دوبعدی) steady stay

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = a\frac{\partial T}{\partial t}$$

$$\begin{cases} \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\dot{q}}{k} = 0 \\ \frac{\partial T}{\partial x}\Big|_{(0,y)} = 0 \\ \frac{\partial T}{\partial y}\Big|_{(x,0)} = 0 \\ -k\frac{\partial T}{\partial x}\Big|_{(a,y)} = h(T-T_\infty), \quad -k\frac{\partial T}{\partial y}\Big|_{(x,b)} = h(T-T_\infty) \end{cases}$$

assumption:

$$T(x,y) = V(x) + w(x,y)$$

$$\frac{\partial^2 T}{\partial x^2} = V'' + \frac{\partial^2 w}{\partial x^2}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 w}{\partial y^2}$$

by substis in Govern-Eq:

$$V'' + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\dot{q}}{k} = 0$$

$$\begin{cases} \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = 0 \\ V'' + \dfrac{\dot{q}}{k} = 0 \end{cases}$$

$$\left.\frac{\partial T}{\partial x}\right|_{0,y} = 0 \Rightarrow V'(0) + \left.\frac{\partial w}{\partial x}\right|_{(0,y)} = 0 \quad \begin{cases} V'(0) = 0 \\ \left.\dfrac{\partial w}{\partial x}\right|_{(0,y)} = 0 \end{cases}$$

$$\left.\frac{\partial T}{\partial y}\right|_{(x,0)} = 0 \Rightarrow \left.\frac{\partial w}{\partial y}\right|_{(x,0)} = 0$$

$$-k\frac{\partial T}{\partial x}\Big|_{(a,y)} = hT \Rightarrow -k\left(v'(a) + \frac{\partial w}{\partial x}\Big|_{(a,y)}\right) = h\left(v(a) + w(a,y)\right)$$

$$\Rightarrow \quad v'(a) + \frac{\partial w}{\partial x}\Big|_{(a,y)} = -\frac{h}{k}\left(v(a) + w(a,y)\right)$$

$$v'(a) = -\frac{h}{k}v(a) \qquad , \qquad \frac{\partial w}{\partial x}\Big|_{(a,y)} = -\frac{h}{k}w(a,y)$$

$$-k\frac{\partial T}{\partial y}\Big|_{(x,b)} = hT \Rightarrow -k\left(\frac{\partial w}{\partial y}\Big|_{(x,b)}\right) = h\left(v(x) + w(x,b)\right)$$

$$\frac{\partial w}{\partial y}\Big|_{(x,b)} = -\frac{h}{k}\left(v(x) + w(x,b)\right)$$

یک B.C بیشتر از I.C دارد

$$v'' = -\frac{\dot q}{k}$$

alirezabaheri.blogfa.com

$$v'(0) = 0$$

$$v'(a) = -\frac{h}{k}v(a)$$

$$v(x) = -\frac{\dot q}{2k}x^2 + C_1 x + C_2$$

$$v'(x) = -\frac{\dot q}{k}x + C_1$$

$$v'(0) = 0 \Rightarrow C_1 = 0$$

$$v'(a) = -\frac{h}{k}v(a) \Rightarrow -\frac{\dot q a}{k} + C_1 = -\frac{h}{k}\left(-\frac{\dot q}{2k}a^2 + C_2\right)$$

$$C_2 = \frac{\dot q a}{h} + \frac{\dot q a^2}{2k}$$

$$v(x) = -\frac{\dot q}{2k}x^2 + \frac{\dot q a}{h} + \frac{\dot q a^2}{2k}$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

$$\left.\frac{\partial w}{\partial x}\right|_{(0,y)} = 0 \Rightarrow X'_{(0)} Y_{(y)} = 0 \Rightarrow X'_{(0)} = 0$$

$$\left.\frac{\partial w}{\partial y}\right|_{(x,0)} = 0 \Rightarrow X_{(x)} Y'_{(0)} = 0 \Rightarrow Y'_{(0)} = 0$$

$$\left.\frac{\partial w}{\partial x}\right|_{(a,y)} = -\frac{h}{K} w(a,y) \Rightarrow X'_{(a)} Y_{(y)} = -\frac{h}{K} X_{(a)} Y_{(y)}$$

$$\Rightarrow X'(a) = -\frac{h}{K} X(a)$$

seperation method.

$$W(x,y) = X_{(x)} Y_{(y)}$$

$$\frac{\partial^2 w}{\partial x^2} = X'' Y \quad , \quad \frac{\partial^2 w}{\partial y^2} = X_{(x)} Y''$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \Rightarrow X'' Y + X Y'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \sigma$$

if $\sigma = 0$

$$\frac{X''}{X} = 0 \Rightarrow X' = C_1 \quad , \quad X_{(x)} = C_1 x^2 + C_2$$

$$X'(0) = 0 \Rightarrow X'(0) = C_1 = 0 \Rightarrow C_1 = 0$$

$$X'(a) = -\frac{h}{k} X(a) \Rightarrow 0 = -\frac{h}{k} (\underset{X(a)}{C_2}) \Rightarrow C_2 = 0$$

$$\begin{matrix} C_1 = 0 \\ C_2 = 0 \end{matrix} \Rightarrow X_{(x)} = 0 \quad \text{is no good}$$

if $\sigma = \delta^2$

$$\frac{X''}{X} = \delta^2 \Rightarrow X'' - \delta^2 X = 0 \Rightarrow$$

$$X_{(x)} = C_1 \cosh \delta x + C_2 \sinh \delta x$$

$$X'(x) = C_1 \delta \sinh \delta x + C_2 \delta \cosh \delta x$$

$$X'(0) = C_1 \delta (0) + C_2 \delta (1) = 0 \Rightarrow C_2 = 0$$

$$X'(a) = -\frac{h}{k} X(a) \Rightarrow C_1 \delta \sinh \delta a = -\frac{h}{k} C_1 \cosh \delta a$$

$$\Rightarrow C_1 \left(\delta \sinh \delta a + \frac{h}{k} \cosh \delta a \right) = 0$$

$+\delta$, $+a$; همواره مثبت $\cosh$

علامت‌ها مثبت?

$$\Rightarrow C_1 = 0 \qquad X_{(x)} = 0 \quad \text{is no good}$$

if $\sigma^2 = -\delta^2$

$$\frac{X''}{X} = -\delta^2 \Rightarrow X'' + \delta^2 X = 0 \Rightarrow$$

$$X_{(x)} = C_1 \cos \delta x + C_2 \sin \delta x$$

$$X'(x) = -C_1 \sin \delta x + C_2 \delta \cos \delta x$$

$$X'_{(0)} = -C_1 \sin 0 + C_2 \delta \cos 0 = 0 \Rightarrow C_2 \delta (1) = 0 \Rightarrow$$

$$C_2 = 0$$

$$X'(a) = -\frac{h}{k} X(a)$$

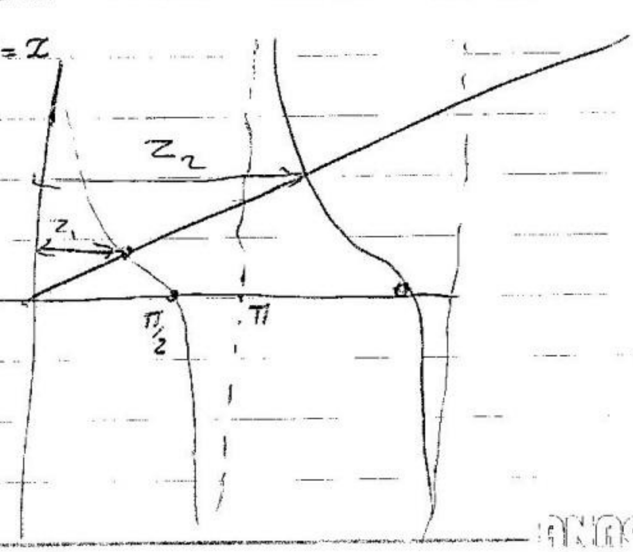
$$-C_1 \delta \sin \delta a = -\frac{h}{k} (C_1 \cos \delta a)$$

$$C_1 (\delta \sin \delta a - \frac{h}{k} \cos \delta a) = 0$$

$$\delta \sin \delta a = \frac{h}{k} \cos \delta a \Rightarrow \frac{\cos \delta a}{\sin \delta a} = \frac{\delta k}{h} \Rightarrow$$

$$\operatorname{Cotg} \delta a = \frac{\delta k}{h} \xrightarrow{\delta a = z} \operatorname{Cotg} z = \frac{z k}{a h}$$

assumption: $\frac{k}{ah} = 1 \Longrightarrow \operatorname{Cotg} z = z$



which produces no Particularity in the problem

$X_n(x) = C_n \cos \delta_n x$ Igen Value (مقادیر ویژه)

$\frac{Y''}{Y} = -\delta^2 \Rightarrow Y'' - \delta^2 Y = 0$

$Y(y) = D_1 \cosh \delta y + D_2 \sinh \delta y$

$Y'(y) = D_1 \delta \sinh \delta y + D_2 \delta \cosh \delta y$

$Y'(0) = D_1 \delta \sinh 0 + D_2 \delta \cosh 0 = 0 \Rightarrow D_2 \delta (1) = 0 \Rightarrow D_2 = 0$

$Y(y) = D_1 \cosh \delta y \Rightarrow Y_n(y) = D_2 \cosh \delta_n y$

$W_n(x,y) = X_n(x) Y_n(y) = A_n \cosh \delta_n y \cos \delta_n x$

$$w(x,y) = \sum_{n=1}^{\infty} A_n \cosh \delta_n y \cos \delta_n x$$

$$T(x,y) = -\frac{\dot{q}}{2k} x^2 + \frac{\dot{q}a}{h} + \frac{\dot{q}a^2}{2k} + \sum_{n=1}^{\infty} A_n \cosh \delta_n y \cos \delta_n x$$

$$\left.\frac{\partial w}{\partial y}\right|_{(x,b)} = -\frac{h}{k}\left(v(x) + w(x,b)\right)$$

$$\left.\frac{\partial w}{\partial y}\right|_{(x,b)} = \sum_{n=1}^{\infty} A_n \delta_n \sinh \delta_n y \cos \delta_n x$$

year month. day. 55 subject.

$$\frac{\partial w}{\partial y}\Big|_{(x,b)} = \sum_{n=1}^{\infty} A_n \delta_n \sinh \delta_n b \cos \delta_n x = -\frac{h}{k}\left(-\frac{\dot{q}}{2k}x^2 + \frac{\dot{q}a}{h} + \frac{\dot{q}a^2}{2k}\right)$$

$$+ \sum_{n=1}^{\infty} A_n \cosh \delta_n b \cos \delta_n x$$

$$\Rightarrow \sum_{n=1}^{\infty} \overbrace{A_n}^{A_n} \cos \delta_n x \overbrace{\left(\delta_n \sinh \delta_n b + \frac{h}{k}\cosh \delta_n b\right)}^{C_n}$$

$$= -\frac{h}{k}\left(-\frac{\dot{q}}{2k}x^2 + \frac{\dot{q}a}{h} + \frac{\dot{q}a^2}{2k}\right) \qquad A_n C_n = B_n$$

$$\sum_{n=1}^{\infty} B_n \cos \delta_n x = -\frac{h}{k}\left(-\frac{\dot{q}}{2k}x^2 + \frac{\dot{q}a}{h} + \frac{\dot{q}a^2}{2k}\right)$$

$$\Rightarrow \sum_{n=1}^{\infty} B_n \cos \delta_n x = f(x)$$

دوطرف را در $\cos\delta_m x$ ضرب
و انتگرال می‌گیریم

$$\sum_{n=1}^{\infty} B_n \int_0^a \cos \delta_n x \cos \delta_m x \, dx = \int_0^a f(x) \cos \delta_m x \, dx$$

$$\int_0^a \cos \delta_n x \cos \delta_m x = \begin{cases} 0 & n \neq m \\ \int_0^a \cos^2 \delta_n x \, dx & n = m \end{cases}$$

$$\text{for } n = m \quad \int_0^a \cos^2 \delta_n x \, dx = \int_0^a \frac{1 + \cos 2\delta_n x}{2} dx$$

$$= \frac{1}{2}\left(x + \frac{1}{2\delta_n}\sin 2\delta_n x\right)_0^a \Rightarrow \frac{1}{2}\left(a + \frac{1}{2\delta_n}\sin 2\delta_n a\right)$$

$$B_n = \frac{1}{2}\left(a + \frac{1}{2\delta_n}\sin 2\delta_n a\right) = \int_0^a f(x)\cos\delta_n x\, dx$$

$$B_1\int_0^a \cos\delta_1 x \cos\delta_m x\, dx + B_2\int \cos\delta_2 x \cos\delta_m x\, dx + \dots$$

$$B_m\int_0^a \cos\delta_m x \cos\delta_m x\, dx + \dots = \int_0^a f(x)\cos\delta_m x\, dx$$

$$B_n = \frac{2\int_0^a f(x)\cos\delta_n x\, dx}{a + \frac{1}{2}\delta_n \sin 2\delta_n a}$$

$$B_n = \frac{2\int_0^a -\frac{h}{k}\left(-\frac{\dot q}{2k}x^2 + \frac{\dot q a}{h} + \frac{\dot q a^2}{2k}\right)\cos\delta_n x\, dx}{a + \frac{1}{2}\delta_n \sin 2\delta_n a}$$

$$B_n = A_n C_n \Rightarrow A_n = \frac{B_n}{C_n}$$

Ex: Consider a plane wall of lenght l having large side perpendicular to the X-axes.

the wall is initially at uniform temperature T_i, the wall is suddenly reached to T_∞ throughout

obtain temperature distribution at the wall find temp at center rectangular if $T_i = 10°C$ and $T_\infty = 25°C$ for $L = 20$ Cm $t = 60$ s what is heat flux at that point if the wall is made of a) wood
b) iron

$$\frac{\partial^2 T}{\partial x^2} = a^2 \frac{\partial T}{\partial t}$$

I.C $\{ T(x,0) = T_i$ → the wall is initially at uniform temp

B.C $\begin{cases} T(0,t) = T_\infty \\ T(L,t) = T_\infty \end{cases}$ the wall suddenly reached to T_∞

در اینحالت پایداری شده توزیع دما ارتباطی به جنس دیواره نخواهد داشت

time dependent boundary Conditions: (شرایط مرزی وابسته به زمان)

Duhamel's theorem:

it can be used duhamel's theorem for problem with timedependent BCs

it is necessary that:

1. Governing equation be linear
2. initial condition be zero.
3. there is only a timedependent BCs and other be homogenes.

Unit step function method:

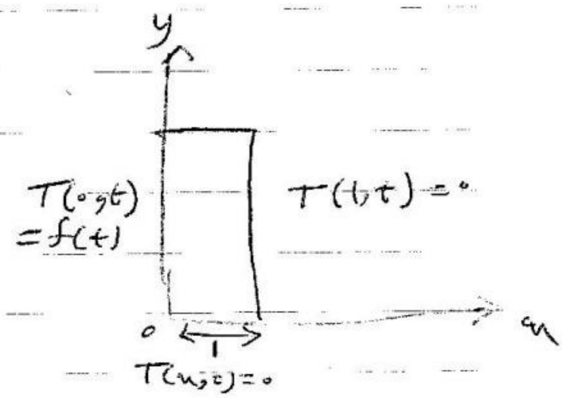
$$T(x,t) = \int_{\tau=0}^{\tau=t} f(\tau) \frac{\partial T^*(x,t-\tau)}{\partial t} d\tau$$

τ is a dummy variable

EX) Consider a plane wall of width 1, with a linear increase in temperature surface at $x=0$, given by $f(t)=2t$ until $t=2s$ and then a sudden drop back to zero. the wall is initial at $T=0$ throught the another side of the wall is kept at 0 normalized temperature find the temperature at any x and any y subsequent time.

$$G.E \quad \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \qquad C^2 = 1$$

$$T(x,0) = 0$$
$$T(0,t) = f(t)$$
$$T(1,t) = 0$$



unit function method : روش دوهامل از طریق روش تابع واحد
اعمال می شود

$$\frac{\partial^2 T^*}{\partial x^2} = \frac{\partial T^*}{\partial t}$$

$T^*(x,0) = 0$

$$BC's \begin{cases} T^*(0,t) = 1 \\ T^*(1,t) = 0 \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} T^*(x,t) = V(x) + W(x,t) \\ \dfrac{\partial^2 T^*}{\partial x^2} = V'' + \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial T}{\partial t} = \dfrac{\partial w}{\partial t} \end{cases}$$

$$\Rightarrow \quad V'' + \frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial t}$$

$BC's: \; T^*(0,t) = 1 \Rightarrow V(0) + w(0,t) = 1$

$T^*(1,t) = 0 \Rightarrow V(1) + w(1,t) = 0$

$$\Leftrightarrow \begin{cases} V(0) = 1 \\ w(0,t) = 0 \\ V(1) = 0 \\ w(1,t) = 0 \end{cases}$$

$$\begin{cases} V'' = 0 \\ \dfrac{\partial^2 w}{\partial x^2} = \dfrac{\partial w}{\partial t} \end{cases}$$

$$V'' = 0 \Rightarrow V' = C_1 \Rightarrow V = C_1 x + C_2$$

$$V(0) = C_1(0) + C_2 = 1 \Rightarrow C_2 = 1$$

$$V(1) = 0 \Rightarrow C_1(1) + 1 = 0 \Rightarrow C_1 = -1$$

$$V = -x + 1$$