year month. day. 61 subject.

PDE $\begin{cases} \dfrac{\partial^2 w}{\partial x^2} = \dfrac{\partial w}{\partial t} \\ w(0,t) = 0 \\ w(1,t) = 0 \end{cases}$

$\begin{cases} w(x,t) = X_{(x)} T(t) \\ \dfrac{\partial^2 w}{\partial x^2} = X''T \quad , \dfrac{\partial w}{\partial t} = XT' \end{cases}$

$X''T = XT' \Rightarrow \dfrac{X''}{X} = \dfrac{T'}{T} = \sigma$

$w(0,t) = 0 \Rightarrow X_{(0)} T(t) = 0 \Rightarrow X(0) = 0$

$w(1,t) = 0 \Rightarrow X_{(1)} T(t) = 0 \Rightarrow X(1) = 0$

① $\sigma = 0 \Rightarrow \dfrac{X''}{X} = 0 \Rightarrow X = C_1 x + C_2$

$\begin{matrix} X(0) = 0 \\ X(1) = 0 \end{matrix} \Rightarrow \begin{matrix} C_2 = 0 \\ C_1 = 0 \end{matrix} \Rightarrow$

$X_{(x)} = 0$ is no good

② $\sigma = \delta^2$

$\dfrac{X''}{X} = +\delta^2 \Rightarrow X'' - \delta^2 X = 0 \Rightarrow X = C_1 \cosh \delta x + C_2 \sinh \delta x$

$X(0) = 0$

$X(1) = 0$

$\Rightarrow C_1(1) + C_2(0) = 0 \Rightarrow C_1 = 0$

$C_2 \sinh \delta(1) = 0 \Rightarrow C_2 = 0$

$\Rightarrow X_{(x)} = 0$ in no good

(3) $\sigma = -\delta^2$

$$\frac{X''}{X} = -\delta^2 \Rightarrow X'' + \delta^2 X = 0 \Rightarrow$$

$$X_{(x)} = C_1 \cos\delta x + C_2 \sin\delta x$$

$$X(0) = 0 \Rightarrow C_1(1) + C_2(0) = 0 \Rightarrow C_1 = 0$$

$$X(1) = 0 \qquad C_2 \sin\delta = 0 \Rightarrow \sin\delta = 0 \Rightarrow \delta_n = n\pi$$

$$X_n(x) = C_n \sin\delta_n x$$

$$\frac{T'}{T} = -\delta^2 \Rightarrow \ln(T) = -\delta^2 t + b$$

$$\Rightarrow T(t) = e^{-\delta^2 t + b} \Rightarrow T_n(t) = B_n e^{-\delta_n^2 t}$$

$$W_n(x,t) = X_n(x) T_n(t) \Rightarrow W_n(x,t) = A_n e^{-\delta_n^2 t} \sin\delta_n x$$

$$W(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t} \sin n\pi x$$

$$T^*(x,t) = V(x) + W(x,t) = 1 - x + \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t} \sin n\pi x$$

I.C $T^*(x,0) = 0$

$$\sum_{n=1}^{\infty} A_n \sin n\pi x = x - 1 \Rightarrow A_n = 2\int_0^1 (x-1)\sin n\pi x \, dx$$

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$$x-1 \qquad \sin n\pi x$$

$$1 \qquad -\frac{1}{n\pi}\cos n\pi x$$

$$0 \qquad -\frac{1}{n^2\pi^2}\sin n\pi x$$

$$A_n = 2\int_0^1 (x-1)\sin n\pi x = 2\left(-(x-1)\left(\frac{1}{n\pi}\cos n\pi x\right) + \frac{1}{n^2\pi^2}\sin n\pi x\right)\Big|_0^1$$

$$2\frac{1}{n\pi}(-1) = -\frac{2}{n\pi}$$

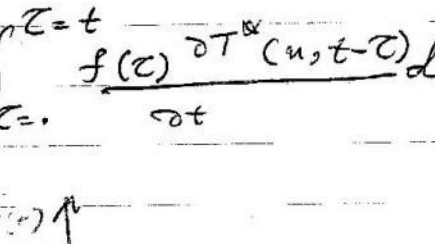
$$T^*(x,t) = 1-x+\sum_{n=1}^{\infty}\frac{-2}{n\pi}e^{-n^2\pi^2 t}\sin n\pi x$$

$$T^*(x,t-\tau) = 1-x+\sum_{n=1}^{\infty}\frac{-2}{n\pi}e^{-n^2\pi^2(t-\tau)}\sin n\pi x$$

$$\frac{\partial T^*(x,t-\tau)}{\partial t} = \sum_{n=1}^{\infty}\frac{-2}{n\pi}(-n^2\pi^2)e^{-n^2\pi^2(t-\tau)}\sin n\pi x$$

$$= \sum_{n=1}^{\infty} 2n\pi e^{-n^2\pi^2(t-\tau)}\sin n\pi x$$

$$T(x,t) = \int_{\tau=0}^{\tau=t} f(\tau)\frac{\partial T^*(x,t-\tau)}{\partial t}d\tau = \int_{\tau=0}^{\tau=t} f(\tau)\sum_{n=1}^{\infty}2n\pi e^{-n^2\pi^2(t-\tau)}\sin n\pi x \, d\tau$$



for $t < 25 \Rightarrow f(\tau) = 2\tau$

$$T(x,t) = \int_0^t 2\tau \sum_{n=1}^{\infty} 2n\pi e^{-n^2\pi^2(t-\tau)} \sin n\pi x \, d\tau$$

$$= \sum_{n=1}^{\infty} 4n\pi e^{-n^2\pi^2 t} \sin n\pi x \int_0^t \tau e^{n^2\pi^2\tau} d\tau$$

$$T(x,t) = \sum_{n=1}^{\infty} 4n\pi e^{-n^2\pi^2 t} \sin n\pi x \left[\frac{\tau}{n^2\pi^2} e^{n^2\pi^2\tau} - \frac{1}{n^4\pi^4} e^{n^2\pi^2\tau} \right]_0^t$$

$$T(x,t) = \sum_{n=1}^{\infty} 4n\pi e^{-n^2\pi^2 t} \sin n\pi x \left[\frac{t}{n^2\pi^2} e^{n^2\pi^2 t} - \frac{1}{n^4\pi^4} e^{n^2\pi^2 t} + \frac{1}{n^4\pi^4} \right]$$

$$T(x,t) = \sum_{n=1}^{\infty} 4\left(\frac{t}{n\pi} - \frac{1}{n^3\pi^3} + \frac{e^{-n^2\pi^2 t}}{n^3\pi^3} \right) \sin n\pi x \quad \text{for } t < 25$$

for $t > 25$

$$T(x,t) = \int_0^2 2\tau \sum_{n=1}^{\infty} 2n\pi e^{-n^2\pi^2(t-\tau)} \sin n\pi x \, d\tau =$$

$$\sum_{n=1}^{\infty} 4n\pi e^{-n^2\pi^2 t} \sin n\pi x \int_0^2 \tau e^{n^2\pi^2\tau} d\tau$$

$$= \sum_{n=1}^{\infty} 4n\pi e^{-n^2\pi^2 t} \sin n\pi x \left(\frac{\tau}{n^2\pi^2} e^{n^2\pi^2\tau} - \frac{1}{n^4\pi^4} e^{n^2\pi^2\tau} \right)_0^2$$

$$T(x,t) = \sum_{n=1}^{\infty} 4n\pi e^{-n^2\pi^2 t} \sin n\pi x \left(\frac{2}{n^2\pi^2} e^{2n^2\pi^2} - \frac{1}{n^4\pi^4} e^{2n^2\pi^2} + \frac{1}{n^4\pi^4} \right)$$

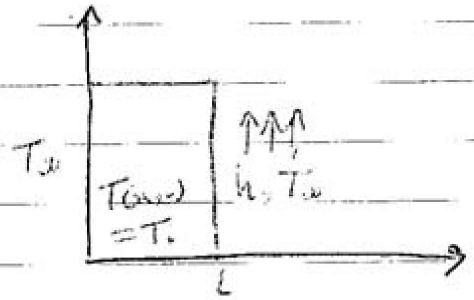
$$T(x,t) = \sum_{n=1}^{\infty} 4 \left(\frac{2}{n\pi} e^{(2-t)n^2\pi^2} - \frac{1}{n^3\pi^3} e^{(2-t)n^2\pi^2} + \frac{1}{n^3\pi^3} e^{-n^2\pi^2 t} \right) \sin n\pi x$$

for $t>2$ see

normalizing (بی‌بعدسازی)

یکی از مباحثی که در انتقال حرارت هدایت مطرح می‌شود (normalizing) نام دارد و باعث ساده شدن مسأله می‌شود. باید توجه داشت که normalizing تنها ظاهر مسأله را تغییر می‌دهد و در باطن مسأله هیچ تغییری ایجاد نمی‌کند. normalizing باعث می‌شود معادلات با شرایط مرزی آن تعریف شده و در نتیجه تعریف شود تا حد امکان مسائل به شکل‌های ساده‌تری در می‌آورد. لزوماً همه مسائل نرمالایز نمی‌شوند و برای حل هر مسأله لزومی ندارد که حتماً مسأله نرمالایز شود در حالی که حتماً باید به گونه‌ای تعریف شود تا بتوان آن را حل کرد

EX : Normalize this problem



$$\frac{\partial^2 T}{\partial x^2} = a^2 \frac{\partial T}{\partial T}$$

$$B.CS \begin{cases} T(0,t) = T_\infty \\ -k\dfrac{\partial T}{\partial x}\Big|_{x=L} = h(T_L - T_\infty) \end{cases}$$

$$T.C \ \{ T(x,0) = T_i$$

$\bar{T}$ = dimensionless temp $\quad \bar{t}$: dimensionless time

$$\bar{T} = \frac{T - T_\infty}{T_i - T_\infty} \qquad \bar{t} = \frac{t}{a^2 L^2} \qquad \bar{x} = \frac{x}{L}$$

$$\frac{1}{a} \quad \frac{m^2}{s} \quad a^2 = (\frac{1}{a})^2 = \frac{s}{m^2} \quad \bar{t} = \frac{t}{a^2 L^2}$$

$$\bar{T} = \frac{T - T_\infty}{T_i - T_\infty} \longrightarrow T = (T_i - T_\infty)\bar{T} + T_\infty$$

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left((T_i - T_\infty)\bar{T} + T_\infty\right) = (T_i - T_\infty)\frac{\partial \bar{T}}{\partial x} = (T_i - T_\infty)\frac{\partial \bar{T}}{(\partial \bar{x}}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}\left((T_i - T_\infty)\frac{\partial \bar{T}}{\partial x}\right) = (T_i - T_\infty)\frac{\partial^2 \bar{T}}{\partial x^2}$$

$$x = \bar{x} l \Rightarrow \frac{\partial^2 \bar{T}}{\partial x^2} = \frac{\partial^2 \bar{T}}{l^2 \partial \bar{x}^2} \Rightarrow$$

$$\frac{\partial^2 T}{\partial x^2} = (T_i - T_\infty) \frac{\partial^2 \bar{T}}{l^2 \partial \bar{x}^2}$$

$$\Rightarrow T = (T_i - T_\infty)\bar{T} + T_\infty \Rightarrow \frac{\partial T}{\partial t} = \frac{\partial}{\partial t}\left[(T_i - T_\infty)\bar{T} + T_\infty\right]$$

$$= (T_i - T_\infty)\frac{\partial \bar{T}}{\partial t}$$

$$t = \bar{t} a^2 l^2$$

$$\frac{\partial T}{\partial t} = (T_i - T_\infty)\frac{\partial \bar{T}}{\partial \bar{t}\, a^2 l^2} \Rightarrow$$ جایگذاری در معادله حاکم

$$\Rightarrow (T_i - T_\infty)\frac{\partial^2 \bar{T}}{l^2 \partial \bar{x}^2} = a^2 (T_i - T_\infty)\frac{\partial \bar{T}}{a^2 l^2 \partial \bar{t}} \Rightarrow$$

$$\frac{\partial^2 \bar{T}}{\partial \bar{x}^2} = \frac{\partial \bar{T}}{\partial \bar{t}}$$

شرایط مرزی
$$\begin{cases} T(0,t) = T_\infty \Rightarrow (T_i - T_\infty)\bar{T}(0,\bar{t}) + T_\infty = T_\infty \\ \bar{T}(0,\bar{t}) = 0 \end{cases}$$

$$\Rightarrow (T_i - T_\infty)\bar{T}(0,\bar{t}) = 0$$

$$-k\frac{\partial T}{\partial x}\Big|_{x=l} = h\left(T\big|_{l} - T_\infty\right) \Rightarrow -k(T_i - T_\infty)\frac{\partial \bar{T}}{l\partial \bar{x}} =$$

$$h\left((T_i - T_\infty)\bar{T} + T_\infty - T_\infty\right) \Rightarrow$$

$$\frac{\partial \bar{T}}{\partial \bar{x}}\Big|_{\bar{x}=1} = -\frac{hl}{k}\bar{T} \Rightarrow \frac{\partial \bar{T}}{\partial \bar{x}}\Big|_{\bar{x}=1} = -Bi\,\bar{T}\Big|_{\bar{x}=1} \qquad x = l \sim \underbrace{\bar{x} = 1}$$

$$T(x,0) = T_i \Rightarrow \underbrace{(T_i - T_\infty)\overset{\bar{T}(\bar{x},0)}{\bar{T}} + T_\infty}_{T(x,0)} = T_i$$

$$\bar{T}(\bar{x},0) = \frac{T_i - T_\infty}{T_i - T_\infty} = 1$$

$$\Rightarrow \bar{T}(\bar{x},0) = 1$$

$$\begin{cases} \dfrac{\partial^2 T}{\partial x^2} = a^2 \dfrac{\partial T}{\partial t} \\ T(0,t) = T_\infty \\ -k\dfrac{\partial T}{\partial x}\Big|_{x=l} = h\left(T\big|_{x=l} - T_\infty\right) \\ T(x,0) = T_i \end{cases} \Rightarrow \begin{cases} \dfrac{\partial^2 \bar{T}}{\partial \bar{x}^2} = \dfrac{\partial \bar{T}}{\partial \bar{t}} \\ \bar{T}(0,\bar{t}) = 0 \\ \dfrac{\partial \bar{T}}{\partial \bar{x}}\Big|_{\bar{x}=1} = -Bi\,\bar{T}\Big|_{\bar{x}=1} \\ \bar{T}(\bar{x},0) = 1 \end{cases}$$

$$\Rightarrow \bar{T}(\bar{x},\bar{t}) \qquad \bar{x} = \frac{x}{l} \qquad \bar{t} = \frac{t}{a^2 l^2}$$

$$\bar{T} = \frac{T - T_\infty}{T_i - T_\infty} \Rightarrow T \checkmark$$

General simplification for non homogeneous problem:

«شکل کلی ساده سازی مسائل غیر همگن»

«اصل سوپرپوزیشن»

$$\frac{\partial^2 u}{\partial x^2} + f(x,t) = \frac{\partial u}{\partial t}$$

$$u(0,t) = g(t)$$

$$u(L,t) = h(t)$$

$$u(x,0) = k(x)$$

حل:

$$u(x,t) = V(x,t) + w(x,t) + Z(x,t) + L(x,t)$$

این کار را انجام می دهیم که فقط یک عامل غیرهمگنی در هر کدام وجود داشته باشد

$$\frac{\partial u}{\partial x} = \frac{\partial V}{\partial x} + \frac{\partial w}{\partial x} + \frac{\partial Z}{\partial x} + \frac{\partial L}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 Z}{\partial^2 x} + \frac{\partial^2 L}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = \frac{\partial V}{\partial t} + \frac{\partial w}{\partial t} + \frac{\partial Z}{\partial t} + \frac{\partial L}{\partial t}$$

جانشین سازی

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 L}{\partial x^2}+f(x,t)=\frac{\partial v}{\partial t}+\frac{\partial z}{\partial t}+\frac{\partial L}{\partial t}+\frac{\partial w}{\partial t}$$

$$u(0,t)=g(t) \Rightarrow v(0,t)+w(0,t)+z(0,t)+l(0,t)=g(t)$$

$$u(L,t)=h(t) \Rightarrow v(L,t)+w(L,t)+z(L,t)+L(L,t)=h(t)$$

$$u(x,0)=k(x) \Rightarrow v(x,0)+w(x,0)+z(x,0)+l(x,0)=k(x)$$

$\Rightarrow$ ①
$$\begin{cases}\frac{\partial^2 v}{\partial x^2}+f(x,t)=\frac{\partial v}{\partial t}\\ v(0,t)=0\\ v(L,t)=0\\ v(x,0)=0\end{cases}$$
$v(x,t)=$ ✓

②
$$\begin{cases}\frac{\partial^2 w}{\partial x^2}=\frac{\partial w}{\partial t}\\ w(0,t)=g(t)\\ w(L,t)=0\\ w(x,0)=0\end{cases}$$
$\Rightarrow w(x,t)=$ ✓

③
$$\begin{cases}\frac{\partial^2 z}{\partial x^2}=\frac{\partial z}{\partial t}\\ z(0,t)=0\\ z(L,t)=h(t)\\ z(x,0)=0\end{cases}$$
$z(x,t)=$ ✓

④
$$\begin{cases}\frac{\partial^2 L}{\partial x^2}=\frac{\partial L}{\partial t}\\ L(0,t)=0\\ L(L,t)=0\\ L(x,0)=k(x)\end{cases}$$
$L(x,t)=$ ✓

Semi infinite solid problem:

بهترین روش برای حل مسائل نیمه بی نهایت تبدیل لاپلاس می باشد

Laplace transform method:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = f(s) \qquad s \geq 1$$

$$L[t] = \int_0^\infty t e^{-st} dt = -\frac{t}{s} e^{-st}\Big|_0^\infty + \int_0^\infty \frac{1}{s} e^{-st} dt$$

$$\left\{ \begin{array}{ll} u = t & dv = e^{-st} dt \\ du = dt & v = -\frac{1}{s} e^{-st} dt \end{array} \right\}$$

$$= 0 + \left(-\frac{1}{s^2} e^{-st}\right)_0^\infty = 1/s^2$$

$$L[c] = \frac{c}{s} \qquad L[t^n] = \frac{n!}{s^{n+1}}$$

$$L[\cos at] = \frac{s}{s^2 + a^2} \qquad L[e^{at}] = \frac{1}{s-a}$$

$$L[\sinh at] = \frac{a}{s^2 - a^2} \qquad L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$L[\cosh at] = \frac{s}{s^2 - a^2} \qquad \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

$$L[\sin at] = \frac{a}{s^2 + a^2} \qquad \Gamma(n+1) = n!$$

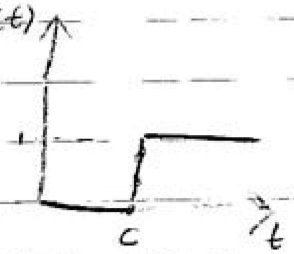
$$\Gamma(n+1) = n\,\Gamma(n)$$

$$\Gamma(1/2) = \sqrt{\pi}$$

$$\Gamma(1) = 1$$

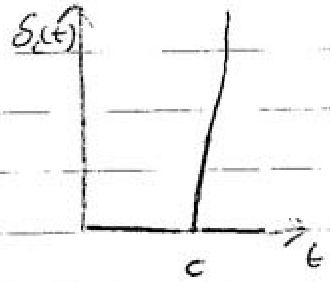
$$L[y'] = sL[y] - y(0)$$

$$L[y''] = s^2L[y] - sy(0) - y'(0)$$



$$u_c(t) = u(t-c) = \begin{cases} 1 & t \geqslant c \\ 0 & t < c \end{cases}$$

$$L[u_c(t)] = \frac{1}{s} e^{-cs}$$



$$\delta_c(t) = \delta(t-c) = \begin{cases} 0 & t \neq c \\ \infty & t = c \end{cases}$$

« مشتق تابع پله واحد تابع ضربه می‌باشد »

$$L[f(t)] = F(s) \Rightarrow L^{-1}[F(s)] = f(t)$$

$$L^{-1}\left[\frac{f(s)}{s^n}\right] = \underbrace{\int_0^t \int_0^t \cdots \int_0^t}_{n \text{ times}} f(t)\, \underbrace{dt\, dt \cdots dt}_{n \text{ times}}$$

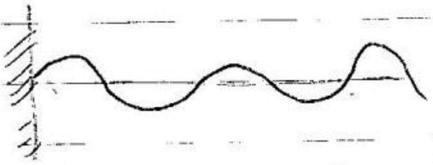
$$L[u_c(t) f(t-c)] = e^{-cs} f(s) \qquad L^{-1}[e^{-cs} f(s)] = u_c(t) f(t-c)$$

$$L[t^n f(t)] = (-1)^n F^{(n)}(s) \qquad L^{-1}[(-1)^n F^n(s)] = t^n f(t)$$

$$L[e^{at} f(t)] = f(s-a) \qquad L^{-1}[f(s-a)] = e^{at} f(t)$$

Ex) Deflection of a vibrting string

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{C^2}\frac{\partial^2 u}{\partial t^2}$$



$u(0,t) = f(t)$

$u(\infty, t) = \text{Bounded}$

$u(x,0) = 0$

$u_t(x,0) = 0$

«حل لاپلاس معکوس را از نرم افزار maple استفاده کنید»

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{C^2}\frac{\partial^2 u}{\partial t^2} \Rightarrow L\left[\frac{\partial^2 u}{\partial x^2}\right] = L\left[\frac{1}{C^2}\frac{\partial^2 u}{\partial t^2}\right]$$

$$\frac{\partial^2}{\partial x^2}[u] = \frac{1}{C^2} L\left[\frac{\partial^2 u}{\partial t^2}\right] \qquad L[u(x,t)] = U(x,s)$$

$$L\left[\frac{\partial^2 u}{\partial t}\right] = s^2 U(x,s) - s\,u(x,0) - u_t(x,0)$$

$$\frac{\partial^2}{\partial x^2} U(x,s) = \frac{1}{C^2}\left(s^2 U(x,s) - s\,\underbrace{\cancel{u(x,0)}}_{0} - \underbrace{\cancel{u_t(x,0)}}_{0}\right)$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{s^2}{C^2} U \Rightarrow \checkmark$$

$$\frac{\partial^2 U}{\partial x^2} - \frac{s^2}{C^2} U = 0 \Rightarrow U(x,s) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$

$$U(x,s) = C_1 e^{s/c\,x} + C_2 e^{-s/c\,x}$$

BC: $u(\infty, t) = \text{Bounded}$

$\Rightarrow L[u(\infty, t)] = \text{Bounded}$

$\Rightarrow \underset{x \to \infty}{U(x,s)} = \text{Bounded}$

$C_1 = 0 \Rightarrow U(x,s) = C_2 e^{-s/c\, x}$

$u(0,t) = f(t) \Rightarrow L[u(0,t)] = L[f(t)] \Rightarrow U(0,s) = f(s)$

$f(s) = C_2 \Rightarrow U(x,s) = f(s) e^{-s/c\, x}$

~~$u(0,t) = f(t) \Rightarrow L[u(0,t)] = L[f(t)] \Rightarrow U(0,s) = f(s)$~~

$u(x,t) = L^{-1}[u(x,s)] = L^{-1}[f(s) e^{-s/c\, x}] \Rightarrow$

$L^{-1}[e^{-as} F(s)] = u_a(t) f(t-a)$

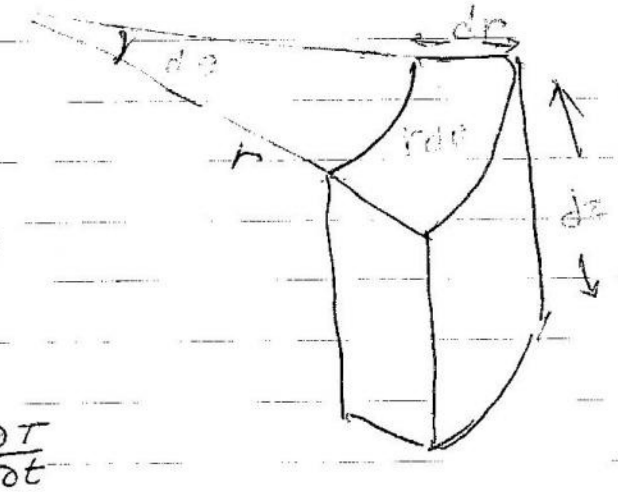
$u(x,t) = u_{x/c}(t)\, f(t - x/c) = u(t - x/c) f(t - x/c) \begin{cases} 1 \\ 0 \end{cases}$

$u(t - x/c) f(t - x/c) \begin{cases} 1 & t \geq x/c \Rightarrow u(x,t) = f(t - 1/c) \quad t \geq x/c \\ 0 & t < x/c \Rightarrow u(x,t) = 0 \quad t < x/c \end{cases}$

$u(x,t) = \begin{cases} f(t - 1/c) & t \geq t/c \\ 0 & t < x/c \end{cases}$

cylindrical cordinate $u = u(r, \theta, z)$ مختصات استوانه‌ای :

$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$ بالانس انرژی



$\dot{E}_{in} = q_r + q_\theta + q_z$

$\dot{E}_{out} = q_{r+dr} + q_{\theta+d\theta} + q_{z+dz}$

$\dot{E}_g = \dot{q}\, dv$

$\dot{E}_{st} = mc\frac{\partial T}{\partial t} = \rho dv c\frac{\partial T}{\partial t}$

$(q_r + q_\theta + q_z) - (q_{r+dr} + q_{\theta+d\theta} + q_{z+dz}) + \dot{q}_{gen} dv =$

$\rho dv c\frac{\partial T}{\partial t}$

$q_{r+dr} = q_r + \frac{\partial q_r}{\partial r} dr$

$q_{\theta+d\theta} = q_\theta + \frac{\partial q_\theta}{\partial \theta} d\theta$

$q_{z+dz} = q_z + \frac{\partial q_z}{\partial z} dz$

$(q_r + q_\theta + q_z) - (q_r + \frac{\partial q_r}{\partial r} dr + q_\theta + \frac{\partial q_\theta}{\partial \theta} d\theta + q_z + \frac{\partial q_z}{\partial z} dz)$

$+ \dot{q}_{gen} dv = (\rho dv) c\frac{\partial T}{\partial t} \Rightarrow$

$-\frac{\partial q_r}{\partial r} dr - \frac{\partial q_\theta}{\partial \theta} d\theta - \frac{\partial q_z}{\partial z} dz + \dot{q}_{gen} dv = \rho dv c\frac{\partial T}{\partial t}$

$$q_r = -k \underbrace{(r\,d\theta\,dz)}_{\text{سطح عمود}} \frac{\partial T}{\partial r}$$

$$q_\theta = -k (dr\,dz) \frac{\partial T}{r\,\partial \theta}$$

$$q_z = -k (r\,dr\,d\theta) \frac{\partial T}{\partial z}$$

$$-\frac{\partial}{\partial r}\left(-k r\,d\theta\,dz \frac{\partial T}{\partial r}\right) dr - \frac{\partial}{\partial \theta}\left(-k\,dr\,dz \frac{\partial T}{r\,\partial \theta}\right) d\theta -$$

$$\frac{\partial}{\partial z}\left(-k r\,dr\,d\theta \frac{\partial T}{\partial z}\right) dz + \dot{q}_{gen} (r\,dr\,d\theta\,dz) = \rho\, r\,dr\,d\theta\,dz\, c \frac{\partial T}{\partial t}$$

طرفین بر krz تقسیم

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right) + \frac{q_{gen}}{k} = \frac{\rho c}{k}\frac{\partial T}{\partial t}$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}_{gen}}{k} = \frac{\rho c}{k}\frac{\partial T}{\partial t}$$

$$\frac{1}{r}\left[\frac{\partial T}{\partial r} + r\frac{\partial^2 T}{\partial r^2}\right] + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}_{gen}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}_{gen}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

معادله توزیع دما در سیستم استوانه ای

Euler Caushy ODE:

$$x^n y^{(n)} + a_n x^{n-1} y^{(n-1)} + a_{n-1} x^{n-2} y^{(n-2)} + \dots + a_2 x y' + a_1 y = 0$$

$$n=2 \Rightarrow x^2 y'' + a_2 x y' + a_1 y = 0$$

فرض $z = \ln z$, $z = \ln x$ $\Longrightarrow$ $y' = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x} y'_z$

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{1}{x} y'_z\right) = -\frac{1}{x^2} y'_z + \frac{1}{x}\underbrace{\frac{d}{dx}(y'_z)}$$

$$\frac{dy'_z}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} y''_z$$

$$y'' = -\frac{1}{x^2} y'_z + \frac{1}{x^2} y''_z$$

$$n=2 : \quad x^2\left(-\frac{1}{x^2} y'_z + \frac{1}{x^2} y''_z\right) + a_n x\left(\frac{1}{x} y'_z\right) + a_1 y = 0$$

$$y''_z - y'_z + a_2 y'_z + a_1 y = 0 \Rightarrow$$

$$y''_z + (a_2 - 1) y'_z + a_1 y_z = 0$$

2 order linear hong with Const coefficie

Home word : Solve $x^3 y''' - 2x^2 y'' + x y' - 4y = 0$

Bessel's equation:

$$x^2y'' + xy' + (x^2 - \underset{\text{order}}{r^2})y = 0$$

$$y'' + \frac{1}{x}y' + \frac{1}{x^2}(x^2 - r^2)y = 0$$

$xp(x)$ $x^2Q(x)$

در $x=0$ تحلیلی نیست P, Q

اما $xp(x)$, $x^2Q(x)$ $\longrightarrow$ $\underset{\text{تحلیلی اند}}{x=0}$ $\Rightarrow x=0$ Reg-Singular point

معادله مشخصه $m^2 - r^2 = 0 \Rightarrow m = \pm r$

$$y_1 = x^r \sum_{n=0}^{\infty} a_n x^n$$

با توجه به تفاضل m ها

$$m_2 - m_1 = 2r \begin{cases} 2r \notin Z \Longrightarrow y_2 = x^{-r} \sum_{n=0} b_n x^n \\ 2r = 0 \Longrightarrow y_2 = y_1 \ln x + \sum_{n=0}^{\infty} b_n x^n \\ 2r \in Z \Longrightarrow y_2 = c_1 y_1 \ln x + x^{-r} \sum_{n=0}^{\infty} b_n x^n \end{cases}$$

جواب کلی $y = c_1y_1 + c_2y_2 = c_1 \sum_{n=0}^{\infty} a_n x^{n+1} + c_2 y_2$

شکل کلی تر $x^2y'' + xy' + ((\underset{\text{parameter}}{\lambda} x)^2 - \underset{\text{order}}{r^2})y = 0$

شکل کلی
Bessel

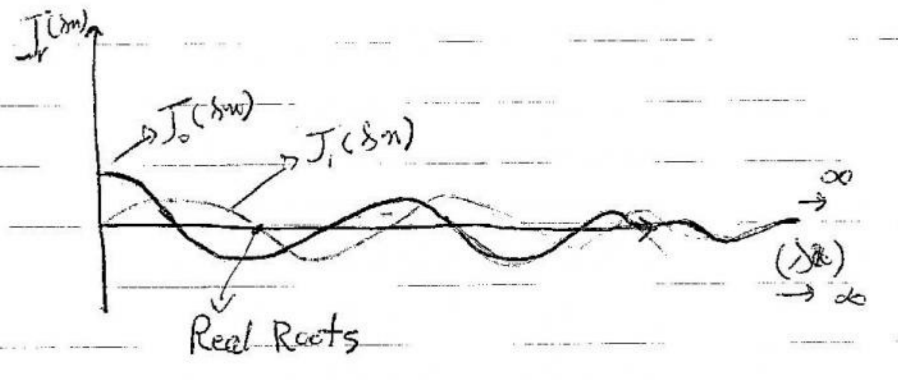
$$\underbrace{x^2y'' + xy' + ((\lambda x)^2 - r^2)y = 0}$$

جواب کلی

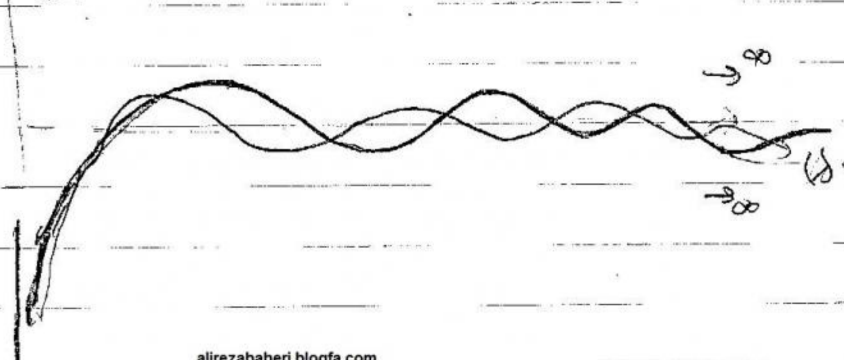
$$y = C_1 J_r(\lambda x) + C_2 \overbrace{Y_r(\lambda x)}^{\to J_{-r}(\lambda x)}$$

$$J_r(\lambda x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\underbrace{m!\,\Gamma(m+r+1)}_{(m+r)!}} \left(\frac{\lambda x}{2}\right)^{2m+r}$$

$$J_{-r}(\lambda x) = Y_r(\lambda x) = \sum_{n=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m-r+1)} \left(\frac{\lambda x}{2}\right)^{2m-r}$$



$Y_r(\lambda x) \neq J_r(\lambda x)$



modified bessel equation: بسل اصلاح شده

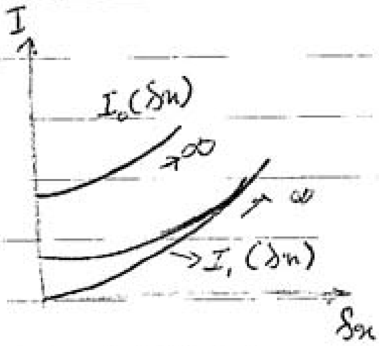
$$x^2y'' + xy' - ((\lambda x)^2 + r^2)y = 0$$

جواب عام : $y = C_1 I_r(\lambda x) + C_2 k_r(\lambda x)$

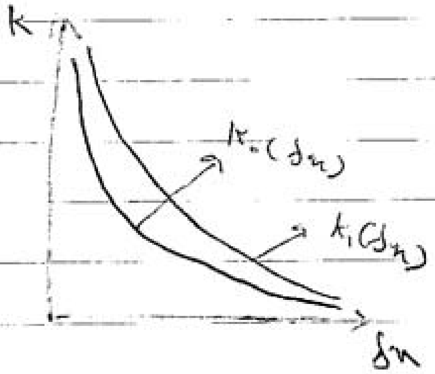
$$I_r(\lambda x) = \sum_{n=0}^{\infty} \frac{1}{m!\,\underbrace{\Gamma(x+r+1)}_{(m+r)!}} \left(\frac{\lambda x}{2}\right)^{2m+r}$$

$$K_r(\lambda x) = \frac{\pi}{2}\,\frac{I_{-r}(\lambda x) - I_r(\lambda x)}{\sin \pi r}$$

$$I_r(\lambda x)\Big|_{\lambda x = 0} = \text{Bounded}$$

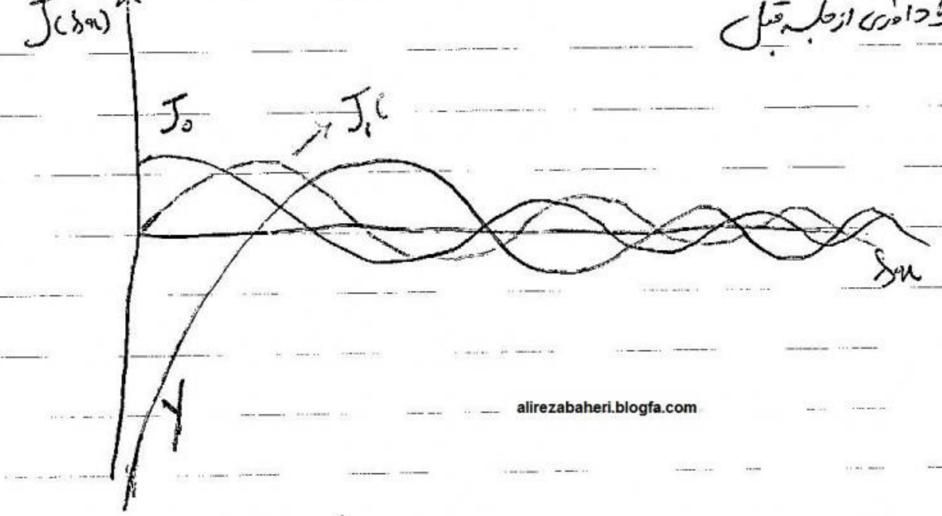


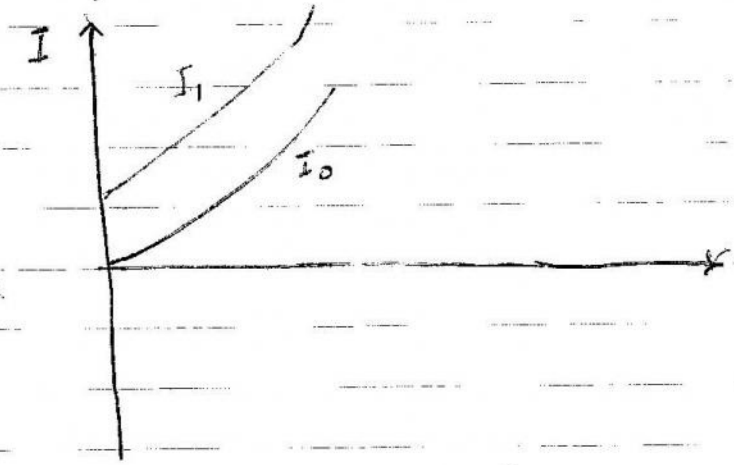
k ها همیشه نزولی است

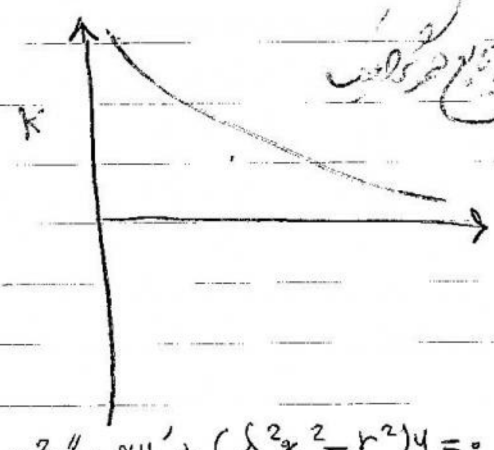


year month. day. 81 subject.

یادآوری از جلسه قبل







شبیه تابع نمایی کاهشی

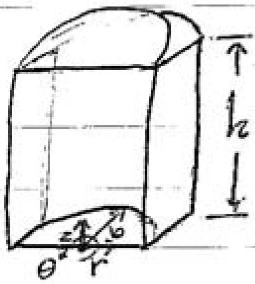
$$x^2y'' + xy' + (\lambda^2x^2 - r^2)y = 0$$ Bessel equation

$$y = C_1 J_r(\lambda x) + C_2 Y(\lambda x)$$

$$x^2y'' + xy' - (\lambda^2x^2 + r^2)y = 0$$ modified Bessel equation

$$y = C_1 I_r(\lambda x) + C_2 K_r(\lambda x)$$

Ex: a solid Consists of one-half of a right circular cylinder of radius b and height h.



the lower base, the curved surface, the vertical plane Faces are all maintened of the constant temperature of $T=0$ over the uper base the temperature is a known function of position $T=f(r,\theta)$

B.C.S

$T(r,\theta,0)=0 \rightarrow$ سطح زیرین lower base

$T(b,\theta,z)=0 \rightarrow$ curve

$T(r,0,z)=0$
$T(r,\pi,z)=0$ } the vertical plane

$T(r,\theta,h)=f(r,\theta)$ uper base

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}+\underbrace{\cancel{\frac{\dot q}{k}}}_{0}=\frac{1}{\alpha}\underbrace{\cancel{\frac{\partial T}{\partial t}}}_{0}$$

year month. day. 83 subject.

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}=0$$

$$T(r,\theta,z)=R(r)\,\theta(\theta)\,Z(z)$$

$$\frac{\partial^2 T}{\partial r^2}=R''\theta Z \quad , \quad \frac{\partial T}{\partial r}=R'\theta Z$$

$$\frac{\partial^2 T}{\partial \theta^2}=R\theta'' Z \quad , \quad \frac{\partial^2 T}{\partial z^2}=R\theta Z''$$

$$R''\theta Z+\frac{1}{r}R'\theta Z+\frac{1}{r^2}R\theta'' Z+R\theta Z''=0$$

$$\xrightarrow{\div R\theta Z}\quad \frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{\theta''}{\theta}+\frac{Z''}{Z}=0$$

$$\xrightarrow{\times r^2}\quad \frac{r^2R''}{R}+\frac{rR'}{R}+\frac{\theta''}{\theta}+r^2\frac{Z''}{Z}=0$$

$$\frac{r^2R''+rR'}{R}+r^2\frac{Z''}{Z}=-\frac{\theta''}{\theta}=\sigma$$

year month. day. 84 subject

$T(r,\theta,0)=0 \Rightarrow R_{(r)}\Theta_{(\theta)}Z_{(0)}=0 \Rightarrow Z(0)=0$

$T(b,\theta,z)=0 \Rightarrow R(b)\,\Theta(\theta)\,Z(z)=0 \Rightarrow R_{(b)}=0$

$T(r,0,z)=0 \Rightarrow R(r)\,\Theta(0)\,Z(z)=0 \Rightarrow \Theta(0)=0$

$T(r,\pi,z)=0 \Rightarrow R(r)\,\Theta(\pi)\,Z(z)=0 \Rightarrow \Theta(\pi)=0$

$T(r,\theta,h)=f(r,\theta)$ ~~⇒ R(r)~~

$\sigma=0 \Rightarrow$

$\Theta''=0 \Rightarrow \Theta_{(\theta)}=C_1\theta+C_2$

$\Theta(0)=0,\ \Theta(\pi)=0 \Rightarrow \begin{cases} C_1(0)+C_2=0 \Rightarrow C_2=0 \\ C_1\pi=0 \Rightarrow C_1=0 \end{cases} \Rightarrow \Theta_{(\theta)}=0$ is no good

$\sigma=-\delta^2 \Rightarrow$

$-\dfrac{\Theta''}{\Theta}=-\delta^2 \Rightarrow \Theta''-\delta^2\Theta=0 \Rightarrow$

$\Theta_{(\theta)}=C_1\cosh\delta\theta+C_2\sinh\delta\theta \xrightarrow{\Theta(0)=0} C_1(1)+C_2(0)=0 \Rightarrow C_1=0$

$\xrightarrow{\Theta(\pi)=0} C_2\sinh\delta\pi=0 \Rightarrow C_2=0$

$C_1=0,\ C_2=0 \Rightarrow \Theta_{(\theta)}=0 \Rightarrow T=0 =$ is no good

ANACO

year month day 85 subject

$\sigma = \lambda^2 \Rightarrow$

$-\dfrac{\theta''}{\theta} = \lambda^2 \Rightarrow \theta'' + \lambda^2\theta = 0$

$\Rightarrow \theta = C_1 \cos\lambda\theta + C_2 \sin\lambda\theta$

$\xrightarrow{\theta(0)=0} C_1(1) + C_2(0) = 0 \Rightarrow C_1 = 0$

$\xrightarrow{\theta(\pi)=0} C_2 \sin\lambda\pi = 0 \Rightarrow \sin\lambda\pi = 0 \Rightarrow \lambda\pi = n\pi \Rightarrow \lambda = n$

$$\theta_n(\theta) = C_n \sin\lambda_n\theta$$

$$T(r,\theta,z) = \sum_{n=1}^{\infty} R(r)\,\theta(\theta)\,Z(z)$$

$$T(r,\theta,z) = \sum_{n=1}^{\infty} C_n R_{(r)} \sin\lambda_n\theta . Z_{(z)}$$

$$\frac{r^2R'' + rR'}{R} + r^2\frac{Z''}{Z} = \lambda^2$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{Z''}{Z} = \frac{\lambda^2}{r^2} \Rightarrow \underbrace{\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{\lambda^2}{r^2} = -\frac{Z''}{Z} = \gamma}$$

$\gamma = 0 \Rightarrow$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{\delta^2}{r^2} = 0 \xrightarrow{\times r^2 R}$$

$$r^2R'' + rR' - \delta^2 R = 0 \Rightarrow$$

$$x^2y'' + axy' + by = 0$$
$$y''_z + (a-1)y'_z + by_z = 0$$

$$R''_z - \delta^2 R_z = 0 \Rightarrow$$

$$R(z) = C_1 e^{\delta z} + C_2 e^{-\delta z}$$

$$z = \ln r \Rightarrow e^z = r$$

$$R(r) = C_1 r^{\delta} + C_2 r^{-\delta}$$

at $r=0$ the temp is finite as other point, but limit $C_2 r^{-\delta} = \infty$

$$\Rightarrow C_2 = 0$$

$$R(r) = C_1 r^{\delta}$$

$$R(b) = 0 \Rightarrow C_1 b^{\delta} = 0 \Rightarrow C_1 = 0$$

$R(r) = 0 \Rightarrow T = 0$ is no good $(\gamma = 0)$

if $\gamma = \mu^2 \Rightarrow$

year month. day. 87 subject.

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}-\frac{\delta^2}{r^2}=\mu^2 \xrightarrow{r^2R}$$

$$r^2R''+rR'-\delta^2R=\mu^2r^2R \Rightarrow r^2R''+rR'-(\delta^2+\mu^2r^2)R=0$$

modify bessel
with order δ and paramter μ

Reminder
$$x^2y''+xy'-(x^2\mu^2+\delta^2)y=0$$
$$y=C_1I_\delta(\mu x)+C_2k_\delta(\mu x)$$

$$R(r)=C_1I_\delta(\mu r)+C_2k_\delta(\mu r)$$

as shown while $r\to 0$ $k_\delta(\mu r)\to\infty$
$R(r)\to\infty \Rightarrow$ then C_2 should be zero.

$$R(r)=C_1I_\delta(\mu r)$$

$R(b)=0 \Rightarrow C_1I_\delta(\mu b)=0$ نقطه‌ی صفر order I ندارد

as shown $I_\delta(\mu r)$ is not zero at all

(except at $r=0$, for zero-order) $\mu b\neq 0$

$\Rightarrow C_1=0$

$R(r)=0 \Rightarrow T=0$ (is no good) $(\upsilon=\mu^2)$

if $\nu = -\mu^2 \Rightarrow$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{\delta^2}{r^2} = -\mu^2 \xrightarrow{\times Rr^2}$$

$$r^2R'' + rR' - \delta^2 R = -\mu^2 r^2 R \Rightarrow$$

$$r^2R'' + rR' + (\mu^2 r^2 - \delta^2)R = 0$$

Reminder

$$x^2y'' + xy' + (\mu^2x^2 - \delta^2) = 0$$

$$y = c_1' J_\delta(\mu x) + c_2' Y_n(\mu x)$$

$$R(r) = c_1' J_\delta(\mu r) + c_2' Y_\delta(\mu r)$$

$$\downarrow$$

$$c_2' = 0 \quad r \to 0 \Rightarrow R \to \infty$$

$$\lim_{r \to 0} Y_\delta(\mu r) = \infty$$

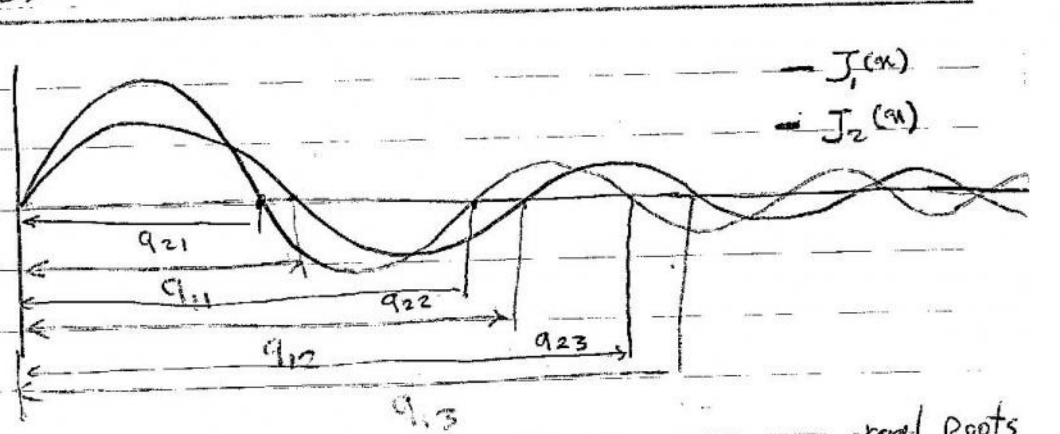
$R(r) = \infty$ then c_2' should be zero

$$R_n(r) = c_1' J_{\delta n}(\mu r)$$

$$R(b) = 0 \Rightarrow c_n' J_{\delta n}(\mu b) = 0$$

$$J_{\delta n}(\mu b) = 0$$

year month. day. 89 subject.



q_{nm} real Roots

order

(در جدول موجود است $J_1(x) = 0$ و $x =$ ✓)

$$J_{\delta n}(\mu b) = 0 \Rightarrow \mu b = q_{nm}$$

$$\mu_{nm} = \frac{q_{nm}}{b}$$

$$R_{nm}(r) = C_{nm} J_{\delta n}(\mu_{nm} r)$$

$$-\frac{Z''}{Z} = -\mu^2 \Rightarrow Z'' - \mu^2 Z = 0 \Rightarrow$$

$$Z_{(z)} = D_1 \cosh \mu z + D_2 \sinh \mu z$$

$$Z(0) = 0 \Rightarrow D_1(1) + D_2(0) = 0 \Rightarrow D_1 = 0$$

$$Z(z) = D_2 \sinh \mu z$$

$$T_{nm}(r,\theta,z) = R_{nm}(r)\,\Theta_n(\theta)\,Z_{nm}(z)$$

$$T_{nm}(r,\theta,z) = \underbrace{C'_{nm} J_{\lambda_n}(\mu_{nm} r) C_n \sin\lambda_n\theta\, D_{nm}}_{G_{mn}} \sinh\mu_{nm} z$$

$$T_{nm}(r,\theta,z) = G_{nm} J_n(\mu_{nm} r)\sin \overset{\lambda_n}{n}\theta \sinh(\mu_{nm} z)$$

$$T(r,\theta,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} G_{nm} J(\mu_{nm} r)\sin(n\theta)\sinh(\mu_{nm} z)$$

$$T(r,\theta,h) = f(r,\theta) \Rightarrow$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} G_{nm} J(\mu_{nm} r)\sin(n\theta)\sinh(\mu_{nm} h) = f(r,\theta)$$

$$\sum_{n=1}^{\infty}\sin n\theta \underbrace{\sum_{m=1}^{\infty} G_{nm} J_n(\mu_{nm} r)\sinh(\mu_{nm} h)}_{L_n(r)} = f(r,\theta)$$

$$\sum_{n=1}^{\infty} L_n(r)\sin n\theta = f(r,\theta)$$

$$\sum_{n=1}^{\infty} L_n(r)\sin m\theta \sin n\theta = f(r,\theta)\sin m\theta$$

$$\Rightarrow \int_0^{\pi}\sum_{n=1}^{\infty} L_n(r)\sin(m\theta)\sin(n\theta)\,d\theta = \int_0^{\pi} f(r,\theta)\sin(m\theta)\,d\theta$$

$$\sum_{n=1}^{\infty} L_n(r) \int_0^{\pi} \sin m\theta \sin n\theta \, d\theta = R.H.S \quad (\text{Right hand side})$$

$$L_1(r) = \int_0^{\pi} \sin m\theta \sin\theta \, d\theta + L_2(r) \int_0^{\pi} \sin m\theta \sin 2\theta \, d\theta +$$

$$L_3(r) \int_0^{\pi} \sin m\theta \sin 3\theta \, d\theta + \cdots + L_n(r) \int_0^{\pi} \sin m\theta \sin n\theta \, d\theta + \cdots$$

$$= R.H.S$$

Remider

if $m \neq n$ $\displaystyle\int_0^{\pi} \sin m\theta \sin n\theta = \int_0^{\pi} \frac{1}{2} \left(\cos(m-n)\theta - \cos(m+n)\theta \right) d\theta$

$$= \frac{1}{2} \left[\frac{1}{m-n} \sin(m-n)\theta - \frac{1}{m+n} \sin(m+n)\theta \right]_0^{\pi} = 0$$

if $m = \pi \Rightarrow$ $\displaystyle\int_0^{\pi} \sin m\theta \sin n\theta \, d\theta = \int_0^{\pi} \sin^2 n\theta \, d\theta$

$$= \int_0^{\pi} \frac{1 - \cos 2n\theta}{2} d\theta = \frac{1}{2} \left(\theta - \frac{1}{2} \sin 2n\theta \right)_0^{\pi} = \frac{\pi}{2}$$

$$\frac{\pi}{2} L_n(r) = R.H.S$$

$$\frac{\pi}{2} L_n(r) = \int_0^{\pi} f(r,\theta) \sin(n\theta) \, d\theta$$

$$L_n(r) = \frac{2}{\pi} \int_0^{\pi} f(r,\theta) \sin n\theta \, d\theta$$

year: month. day. 92 subject

$$\sum_{m=1}^{\infty} G_{nm} J_n(\mu_{nm} r) \sinh \mu_{nm} h = L_n(r)$$

r = weight function for Bessel function

orthogonality for Bessels function

$$\sum_{m=1}^{\infty} G_{nm} J_n(\mu_{nm} r) \underbrace{r J_n(\mu_{nm'} r) \sinh \mu_{nm} h = L_n(r) r}_{\text{weight function}} J_n(\mu_{nm'} r)$$

$$\int \sum_{m=1}^{\infty} G_{nm} J_n(\mu_{nm} r) r J_n(\mu_{nm'} r) \sinh(\mu_{nm} h) dr$$

$$= \int_0^b L_n(r) r J_n(\mu_{nm'} r) dr$$

$$\sum_{m=1}^{\infty} \sinh \mu_{nm} h G_{nm} \int_0^b \underbrace{r J_n(\mu_{nm} r) J_n(\mu_{nm'} r) dr}_{= 0 \text{ if } m \neq m'} =$$

$$\int_0^b L_n(r) r J_n(\mu_{nm'} r) dr$$

if $m \neq m'$ $\int_0^b r J_n(\mu_{nm} r) J_n(\mu_{nm'} r) dr = 0$

if $m = m'$ $\Rightarrow$

$$\sinh \mu_{nm} h G_{nm} \int_0^b r J^2(\mu_{nm} r) dr = \int_0^b L_n(r) J_n(\mu_{nm} r) dr$$

ANACO

year. month. day. 93 subject.

$$\int_0^b r J_n^2(\mu_{nm} r)\, dr = \tfrac{1}{2} b^2 J_{n+1}^2(\mu_{nm} b)$$

$$\Rightarrow \sinh(\mu_{nm} h)\, G_{nm} \left(\frac{b^2}{2} J_{n+1}^2(\mu_{nm} b)\right) =$$

$$\int_0^b L_n(r)\, r\, J_n(\mu_{nm} r)\, dr$$

$$G_{nm} = \frac{\int_0^b r L_n(r) J_n(\mu_{nm} r)\, dr}{\frac{b^2}{2} J_{n+1}^2(\mu_{nm} b) \sinh(\mu_{nm} h)}$$

$$T(r,\theta,z) = G_{11} J_1(\mu_{11} r) \sin\theta \sinh \mu_{11} z +$$

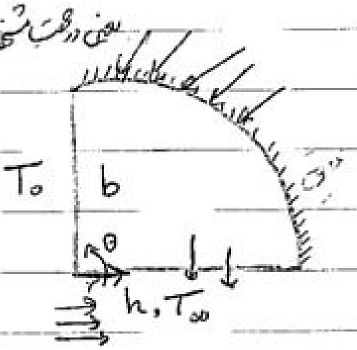
$$G_{12} J_1(\mu_{12} r) \sin\theta \sinh \mu_{12} z + G_{21} J_2(\mu_{21} r) \sin 2\theta \sinh \mu_{21} z$$

$$+ G_{22} J_2(\mu_{22} r) \sin 2\theta \sinh \mu_{22} z + \cdots$$

$T(5\,cm, 60^\circ, 40\,cm) = ?$ $b = 10$, $h = 50\,cm$

B.C'S

یعنی در جهت شعاعی شده عبور گرما نداریم



ربع استوانه با طول بی نهایت

(طول نامحدود مستقل از Z)

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\cancel{\frac{\partial^2 T}{\partial z^2}}+\cancel{\frac{\dot q}{k}}=\cancel{\frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

طول بی نهایت

$$\frac{\partial T}{\partial r}(b,\theta)=0 \qquad 0<\theta<\frac{\pi}{2}$$

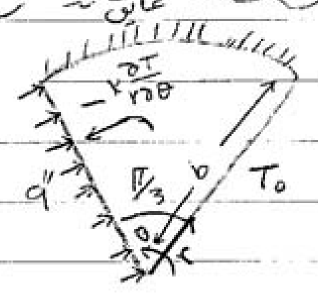
$$q''_r=-k\frac{\partial T}{\partial r}$$

$$q''_\theta=-k\frac{1}{r}\frac{\partial T}{\partial \theta}$$

$$T\left(r,\frac{\pi}{2}\right)=T_0$$

$$k\frac{\partial T}{r\partial \theta}(r,0)=h\left(T_{(r,0)}-T_\infty\right)$$

عایق



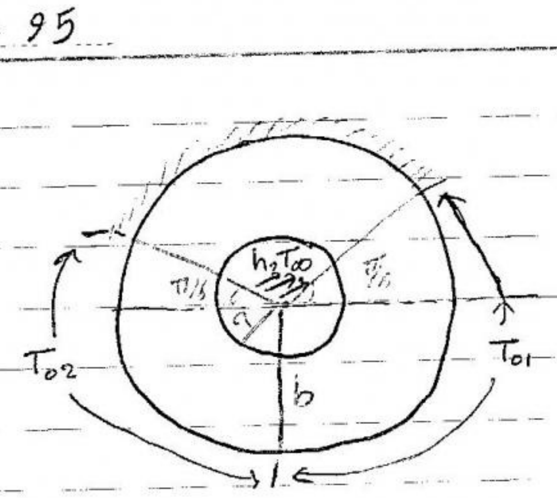
$$\dot E_{in}-\dot E_{out}+\dot E_g=\dot E_{st}$$

$$\left.\frac{\partial T}{\partial r}\right|_{(b,\theta)}=0 \qquad 0<\theta<\frac{\pi}{3}$$

$$T(r,0)=T_0$$

$$\left.-k\frac{\partial T}{r\partial \theta}\right|_{(r,\pi/3)}+q''=0$$

year. month. day. 95 subject.



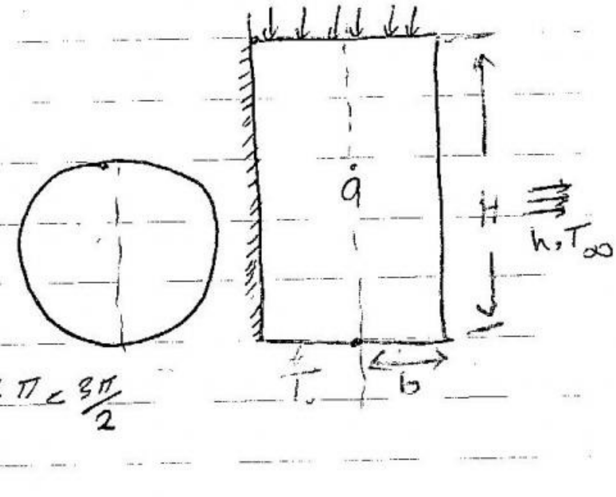
$$\frac{\partial T}{\partial r}\bigg|_{(b,\theta)} = 0 \qquad \pi/6 < \theta < 5\pi/6$$

$$T(b,\theta) = T_{o1} \qquad -\pi/2 < \theta < \pi/6$$

$$T(b,\theta) = T_{o2} \qquad 5\pi/6 < \theta < 3\pi/2$$

$$-k\frac{\partial T}{\partial r}\bigg|_{(a,\theta)} = h\left(T_{(a,\theta)} - T_\infty\right)$$

$$T(r,\theta) = T(r,\theta+2\pi)$$ از هندسه حل



$$T(r,\theta,0) = T_o$$

$$-k\frac{\partial T}{\partial z}\bigg|_{(r,\theta,H)} + q'' = 0$$

$$\frac{\partial T}{\partial r}(b,\theta,z) = 0 \qquad \pi/2 < \pi < 3\pi/2$$

$$-k\frac{\partial T}{\partial r}(b,\theta,z) = h\left(T - T_\infty\right)_{(b,\theta,z)}$$

year: month. day. 96 subject

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = 0$$

$$T(r,\theta,0) = 0$$

$$k\frac{\partial T}{\partial z}\Big|_{(r,\theta,H)} = q''$$

$\Rightarrow$ غیرهمگن

$$\frac{\partial T}{\partial r}(b,\theta,z) = 0 \qquad \pi/2 < \theta < 3\pi/2$$

$$-k\frac{\partial T}{\partial r}(b,\theta,z) = h(T - T_\infty)$$

$$T(r,\theta,z) = v(r,\theta,z) + w(r,\theta,z) + X(r,\theta,z)$$

$$\frac{\partial^2 T}{\partial r^2} = \frac{\partial^2 v}{\partial r^2} + \frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 X}{\partial r^2}$$

$$\frac{\partial T}{\partial r} = \frac{\partial v}{\partial r} + \frac{\partial w}{\partial r} + \frac{\partial X}{\partial r}$$

$$\frac{\partial^2 T}{\partial \theta^2} = \frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 X}{\partial \theta^2}$$

$$\frac{\partial^2 T}{\partial z^2} = \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 X}{\partial z^2}$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 X}{\partial r^2} + \frac{1}{r}\left(\frac{\partial v}{\partial r} + \frac{\partial w}{\partial r} + \frac{\partial X}{\partial r}\right) + \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 X}{\partial z^2}$$

$$+ \frac{\dot{q}}{k} = 0$$

$$T(r,\theta,0)=0 \Rightarrow V(r,\theta,0)+w(r,\theta,0)+X(r,\theta,0)=0$$

$$k\frac{\partial T}{\partial z}=q'' \Rightarrow \frac{\partial T}{\partial z}=\frac{q''}{k} \Rightarrow$$

$$\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}+\frac{\partial X}{\partial z}=\frac{q''}{k}$$

$$\frac{\partial T}{\partial r}(b,\theta,z)=0 \Rightarrow \frac{\partial T}{\partial r}=0 \Rightarrow \frac{\partial v}{\partial r}+\frac{\partial w}{\partial r}+\frac{\partial X}{\partial r}=0$$

$$\frac{\partial T}{\partial r}=-\frac{h}{k}(T-T_\infty) \Rightarrow$$

$$\frac{\partial v}{\partial r}+\frac{\partial w}{\partial r}+\frac{\partial X}{\partial r}=-\frac{h}{k}\left[v+w+X-T_\infty\right]$$

V

$$\begin{cases}\dfrac{\partial^2 v}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial v}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2}+\dfrac{\partial^2 v}{\partial z^2}+\dfrac{\dot q}{k}=0\\ v(r,\theta,0)=0\\ \dfrac{\partial v}{\partial z}=0\\ \dfrac{\partial v}{\partial r}=0\\ \dfrac{\partial v}{\partial r}=-\dfrac{h}{k}V_{(r,\theta,z)}\end{cases}$$

محدوده فرق می‌کند

W

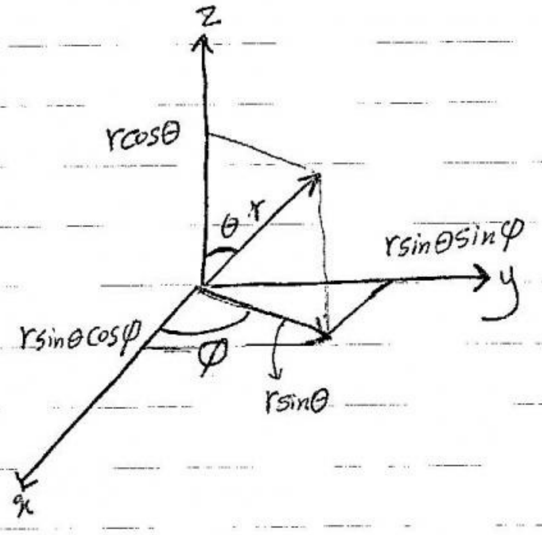
$$\begin{cases} \dfrac{\partial^2 w}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial w}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 w}{\partial \theta^2} + \dfrac{\partial^2 w}{\partial z^2} = 0 \\ w(r,\theta,0) = 0 \\ \dfrac{\partial w}{\partial z} = \dfrac{q''}{k} \\ \dfrac{\partial w}{\partial r} = 0 \\ \dfrac{\partial w}{\partial r} = -\dfrac{h}{k} w_{(r,\theta,z)} \end{cases}$$

X

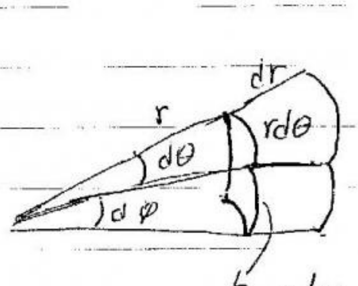
$$\begin{cases} \dfrac{\partial^2 X}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial X}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 X}{\partial \theta^2} + \dfrac{\partial^2 X}{\partial z^2} = 0 \\ X(r,\theta,0) = 0 \\ \dfrac{\partial X}{\partial z} = 0 \\ \dfrac{\partial X}{\partial r} = 0 \\ \dfrac{\partial X}{\partial r} = -\dfrac{h}{k}(X - T_\infty) \end{cases}$$

spherical Coordinates:

$T = T(r, \theta, \varphi)$



thermal equation in spherical Coordinates:



$$q_r = -k\, rd\theta\, (r\sin\theta d\varphi) \frac{\partial T}{\partial r}$$

$$q_\theta = -k\, dr\, (r\sin\theta d\varphi) \frac{\partial T}{r\partial\theta} \qquad 0 \le \theta \le \pi$$

$$q_\varphi = -k\, dr\, (rd\theta) \frac{\partial T}{r\sin\theta\, \partial\varphi} \qquad 0 \le \varphi \le 2\pi$$

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$

$$\dot{E}_{in} = q_r + q_\theta + q_\varphi$$

$$\dot{E}_{out} = q_{r+dr} + q_{\theta+d\theta} + q_{\varphi+d\varphi}$$

$$\dot{E}_g = \dot{q}\,dv = \dot{q}(dr)(r\sin\theta d\varphi)(rd\theta)$$

$$\dot{E}_{st} = \rho c \frac{\partial T}{\partial t} dv$$

$$\rho c \frac{\partial T}{\partial t}(dr)(rd\theta)(r\sin\theta d\varphi)$$

$$q_r + q_\theta + q_\varphi - (q_{r+dr} + q_{\theta+d\theta} + q_{\varphi+d\varphi}) + \dot{q}_g(r^2\sin\theta\, dr d\theta d\varphi)$$

$$q_{r+dr} = q_r + \frac{\partial q_r}{\partial r} dr$$

$$q_{\theta+d\theta} = q_\theta + \frac{\partial q_\theta}{\partial \theta} dr$$

$$q_{\varphi+d\varphi} = q_\varphi + \frac{\partial q_\varphi}{\partial \varphi} d\varphi$$

$$\Rightarrow \frac{\partial q_r}{\partial r} dr - \frac{\partial q_\theta}{\partial \theta} d\theta - \frac{\partial q_\varphi}{\partial \varphi} d\varphi + \dot{q}_g(r^2\sin\theta\, dr d\theta d\varphi) =$$

$$\rho c \frac{\partial T}{\partial t}(r^2\sin\theta\, dr d\theta d\varphi)$$

$$\frac{\partial}{\partial r}\left[-kr^2\sin\theta\, d\theta\, d\varphi\frac{\partial T}{\partial r}\right]dr-\frac{\partial}{\partial\theta}\left[-kr\sin\theta\, dr\, d\varphi\frac{\partial T}{r\partial\theta}\right]d\theta$$

$$-\frac{\partial}{\partial\varphi}\left[-kr\,dr\,d\theta\frac{\partial T}{r\sin\theta\,\partial\varphi}\right]d\varphi+\dot{q}_g(r^2\sin\theta\, dr\, d\theta\, d\varphi)$$

$$=\rho c\frac{\partial T}{\partial t}(r^2\sin\theta\, dr\, d\theta\, d\varphi)$$

$$\Leftrightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\varphi}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\alpha=\frac{k}{\rho c}$$

Legendre equation:

$$(1-x^2)y''-2xy'+\underset{\substack{\downarrow\\ \text{order}}}{n}(n+1)y=0$$

مرتبه معادله

$x=0$ نقطه عادی

$$y=P_n(x)$$

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

$(1-x^2)y''-2xy'=0$

$n=0$

$y=P_0(x)=1$

$(1-x^2)y''-2xy'+2y=0$

$n=1$

$y=P_1(x)=\frac{1}{2}\frac{d}{dx}(x^2-1)=x$

$$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = \begin{cases} 0 & n \neq m \\ \frac{2}{2n+1} & n = m \end{cases}$$

تعامد تابع لژاندر

شکل مثلثاتی معادله لژاندر :

$$(1-x^2)\frac{d^2M}{dx^2} - 2x\frac{dM}{dx} + n(n+1)M = 0 \quad *$$

$$M = \theta$$
$$x = \cos\theta$$

$$\frac{dM}{dx} = \frac{d\theta}{dx} = \frac{d\theta}{d\theta}\,\frac{d\theta}{dx}$$

$$x = \cos\theta \Rightarrow \frac{dx}{d\theta} = -\sin\theta \Rightarrow \frac{d\theta}{dx} = -\frac{1}{\sin\theta}$$

$$\frac{dM}{dx} = \frac{d\theta}{d\theta}\,\frac{-1}{\sin\theta}$$

$$\frac{d^2M}{dx^2} = \frac{d}{dx}\left(\frac{dM}{dx}\right) = \frac{d}{d\theta}\left(\frac{dM}{dx}\right)\frac{d\theta}{dx}$$

$$\frac{d^2M}{dx^2} = \frac{d}{d\theta}\left(-\frac{1}{\sin\theta}\,\frac{d\theta}{d\theta}\right)\frac{d\theta}{dx}$$

$$\Rightarrow \left[\frac{\cos\theta}{\sin^2\theta}\,\frac{d\theta}{d\theta} - \frac{1}{\sin\theta}\,\frac{d^2\theta}{d\theta^2}\right]\left(-\frac{1}{\sin\theta}\right)$$

by substit in * equation:

$$\overbrace{-\frac{1}{\sin\theta}\underbrace{(1-\cos^2\theta)}_{\sin^2\theta}}^{-\sin\theta}\left[-\frac{1}{\sin\theta}\,\frac{d^2\theta}{d\theta^2} + \frac{\cos\theta}{\sin^2\theta}\,\frac{d\theta}{d\theta}\right] - 2\cos\theta\left[-\frac{1}{\sin\theta}\,\frac{d\theta}{d\theta}\right] + n(n+1)\theta = 0$$

$$= \frac{d^2\Theta}{d\theta^2} - \frac{\cos\theta}{\sin\theta}\frac{d\Theta}{d\theta} + \frac{2\cos\theta}{\sin\theta}\frac{d\Theta}{d\theta} + n(n+1)\Theta = 0$$

$$\frac{\cos\theta}{\sin\theta}\frac{d\Theta}{d\theta}$$

$$\xrightarrow{\times \sin^2\theta} \sin^2\theta \frac{d^2\Theta}{d\theta^2} + \sin\theta\cos\theta\frac{d\Theta}{d\theta} + n(n+1)\Theta\sin^2\theta = 0$$

$\Downarrow$

Trignometric legendre equation

جواب $\Theta = A P_n(\cos\theta)$

$$P_n(\cos\theta) = \frac{1}{2^n n!}\frac{d^n}{d\theta^2}(\cos^2\theta - 1)^n$$

$$P_2(\cos\theta) = \frac{1}{2!2^2}\frac{d^2}{d\theta^2}\left[(\cos^2\theta-1)^2\right]$$

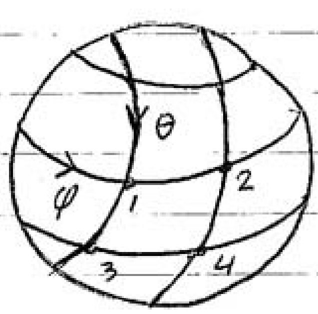
$$\int_0^\pi P_n(\cos\theta)P_m(\cos\theta)\underbrace{\sin\theta}_{\text{weight function}}\,d\theta = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases}$$

Ex) the known temperature distribution $T = f(\theta)$ is maintained over the entire surface of a sphere with a radius b, find the steady state temp at any point for the symmetrical.

$T_3 = T_4$

$T_1 = T_2$

$T_1 \neq T_4$



$T(r, \theta)$ چون که متقارن است
دماش از φ نیست

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \underbrace{\cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\varphi^2}}}_{0} + \underbrace{\cancel{\frac{\dot{q}}{k}}}_{0} = \underbrace{\cancel{\frac{1}{\alpha}\frac{\partial T}{\partial t}}}_{0}$$

تقارن

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) = 0$$

$$\xrightarrow{\times r^2} \frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) = 0$$

$$2r\frac{\partial T}{\partial r} + r^2\frac{\partial^2 T}{\partial r^2} + \frac{1}{\sin\theta}\left(\cos\theta\frac{\partial T}{\partial\theta} + \sin\theta\frac{\partial^2 T}{\partial\theta^2}\right) = 0$$

year month. day. 105 subject

$$\Rightarrow \quad r^2\frac{\partial^2 T}{\partial r^2}+2r\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial \theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial T}{\partial \theta}=0$$

$$T(r,\theta)=R_{(r)}\Theta_{(\theta)}$$

$$\frac{\partial^2 T}{\partial r^2}=R''\Theta$$

$$\frac{\partial T}{\partial r}=R'\Theta$$

$$\frac{\partial T}{\partial \theta}=R\Theta'$$

$$\frac{\partial^2 T}{\partial \theta^2}=R\Theta''$$

$$r^2R''\Theta+2rR'\Theta+R\Theta''+\frac{\cos\theta}{\sin\theta}R\Theta'=0$$

$$\frac{r^2R''}{R}+\frac{2rR'}{R}+\frac{\Theta''}{\Theta}+\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}=0$$

$$\frac{r^2R''+2rR'}{R}=-\frac{\Theta''}{\Theta}-\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}=\sigma=n(n+1)$$

$$\frac{r^2R''+2rR'}{R}=n(n+1)\Rightarrow$$

$$x^2y''+axy'+by=0$$
$$y_z''+(a-1)y_z'+by_z=0$$

$$r^2R''+2rR'-n(n+1)R=0$$

$$R_z''+rR_z'-n(n+1)R_z=0$$

characteristis equation :

$$m^2+m-n(n+1)=0 \Rightarrow \begin{array}{l} m_1 = n \\ m_2 = -1-n = -(1+n) \end{array}$$

$$R_z = C_1 e^{nz} + C_2 e^{-(1+n)z} \qquad z = \ln r$$

$$R_{(r)} = C_1 e^{n \ln r} + C_2 e^{-(1+n)\ln r}$$

$$= C_1 r^n + C_2 r^{-(1+n)}$$

$$\text{at } r=0 \quad C_2 r^{-(1+n)} \to \infty \Rightarrow C_2 = 0$$

$$R_{(r)} = C_n r^n$$

معمولاً در کره و استوانه از این شرط فیزیکی استفاده می‌کنیم ($r \to 0$ دما محدود است)

$$-\frac{\Theta''}{\Theta} - \frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta} = n(n+1)$$

$$\frac{\Theta''}{\Theta} + \frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta} + n(n+1) = 0 \xrightarrow{\times\ \Theta \sin^2\theta}$$

$$\sin^2\theta . \Theta'' + \sin\theta\cos\theta\ \Theta' + n(n+1)\sin^2\theta\ \Theta = 0$$

$$\Theta_{n(\theta)} = A_n P_n(\cos\theta)$$

year month. day. 107 subject.

$$T(r,\theta) = R_n(r)\,\Theta_n(\theta)$$

$$T_n(r,\theta) = D_n r^n P_n(\cos\theta)$$

$$T(r,\theta) = \sum_{n=0}^{\infty} T_n(r,\theta)$$

$$T(r,\theta) = \sum_{n=0}^{\infty} D_n r^n P_n(\cos\theta)$$

$$T(b,\theta) = f(\theta) \Rightarrow \sum_{n=0}^{\infty} D_n b^n P_n(\cos\theta) = f(\theta)$$

$$\sum_{n=0}^{\infty} D_n b^n P_n(\cos\theta) P_m(\cos\theta)\sin\theta = f(\theta) P_m(\cos\theta)\sin\theta$$

$$\Rightarrow \int_0^{\pi} \sum_{n=0}^{\infty} D_n b^n P_n(\cos\theta) P_m(\cos\theta)\sin\theta\, d\theta =$$

$$\int_0^{\pi} f(\theta) P_n(\cos\theta)(\sin\theta)\, d\theta$$

$$\Rightarrow \sum_{n=0}^{\infty} b^n D_n \int_0^{\pi} P_n(\cos\theta) P_m(\cos\theta)\sin\theta\, d\theta$$

$$\int_0^{\pi} f(\theta) P_m(\cos\theta)\sin\theta\, d\theta$$

$$n \neq m \quad \int_0^{\pi} P_n(\cos\theta) P_m(\cos\theta)\sin\theta\, d\theta = 0$$

$$n = m \quad \int P_n(\cos\theta) P_m(\cos\theta)\sin\theta = \int_0^{\pi} P_n^2(\cos\theta)\sin\theta\, d\theta$$

$$= \frac{2}{2n+1}$$

$$b^n D_n \frac{2}{2n+1} = \int_0^{\pi} f(\theta)\, P_n(\cos\theta) \sin\theta \, d\theta$$

$$\Rightarrow D_n = \frac{2n+1}{2b^n} \int_0^{\pi} f(\theta)\, P_n(\cos\theta) \sin\theta \, d\theta$$

پایان درس انتقال حرارت هدایت ساعت 16:30

91.9.8 کلاس 301

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« استاد دکتر علیرضا باهری »

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