Sharif University of Technology
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Computational Fluid Dynamics

Classification of PDEs

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Classification

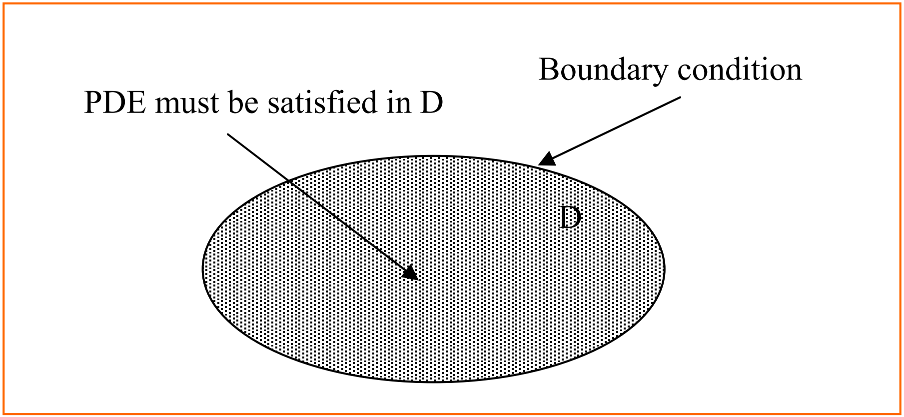
- **Classification of PDEs helps us to understand their behavior and choose suitable solution techniques.**
- **Classification of the governing equations can be performed in two ways:**
  - **Mathematical**
  - **physical**
- **PDEs are categorized in terms of mathematical types as follows:**
  - **Hyperbolic (wave)**
  - **Parabolic (transient heat-conduction)**
  - **Elliptic (diffusion)**
- **PDEs are categorize in terms of physical types as follows:**
  - **Equilibrium Problems (steady heat conduction)**
  - **Marching Problems (Unsteady heat conduction)**

Physical Classification

- **Equilibrium problems involve a closed domain and boundary conditions.**
- **They are in fact boundary value problems.**
- **Examples are Laplace and Poisson equations:**

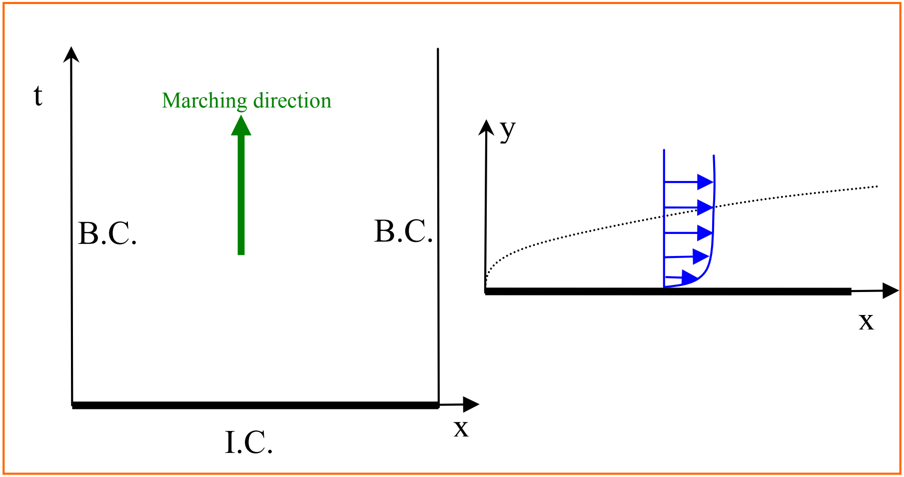
$$\nabla^2 \varphi = 0$$

$$\nabla^2 p = f(u, v, \nabla.\mathbf{V})$$



Physical Classification…

- **Marching problems involve an <u>open domain</u> (time or time-like) and <u>initial</u> conditions and/or <u>boundary</u> conditions.**
- **Example are <u>unsteady heat conduction</u> and <u>boundary layer</u> (without separation) problems.**



Physical Classification…

- **Boundary layer equations: (marching in x-direction)**

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$$

- **Pure initial value problems:**

$$I.C.\quad u(x,0)=f(x)\quad\rightarrow\begin{cases}\dfrac{\partial u}{\partial t}+a\dfrac{\partial u}{\partial x}=0 & \text{linear Berger Eq.}\\[2ex] \dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}=0 & \text{non}-\text{linear Berger Eq.}\end{cases}$$

$$I.C.\quad u(x,0)=f(x)\quad \& \quad u_t(x,0)=g(x)$$

$$\rightarrow\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}\quad\text{(wave Eq.)}$$

Mathematical Classification

- Consider the following general 2nd order PDE:

$$A\varphi_{xx} + B\varphi_{xy} + C\varphi_{yy} + D\varphi_x + E\varphi_y + F\varphi + G = 0$$

- The characteristic equation of this PDE is:

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) - C = 0$$

- Whose roots are:

$$\left(\frac{dy}{dx}\right) = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

- We can categorize the PDE according to its discriminant: $D = B^2 - 4AC$

Mathematical Classification...

- If $D > 0$ at $p(x_0, y_0)$, two real characteristics exist and the equation is Hyperbolic.
- If $D = 0$ at $p(x_0, y_0)$, only one real characteristic exists and the equation is Parabolic.
- If $D < 0$ at $p(x_0, y_0)$, no real characteristics exists and the equation is elliptic.
- If D changes sign then the PDE is of mixed type. For example, for 2D potential flow and Transonic flow, we have:

$$(1 - M^2)\varphi_{xx} + \varphi_{yy} = 0$$

$$D = -4(1 - M^2)$$

- Note that a PDE can be of different types in different regions.

Mathematical Classification...

- **Wave equation:**

$$u_{tt} = a^2 u_{xx} \quad ,$$
$$I.C.: \quad u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

- **The exact solution is**

$$u(x,t) = \frac{1}{2}\left[f(x+at) + f(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(z)dz$$

- **It is seen that**

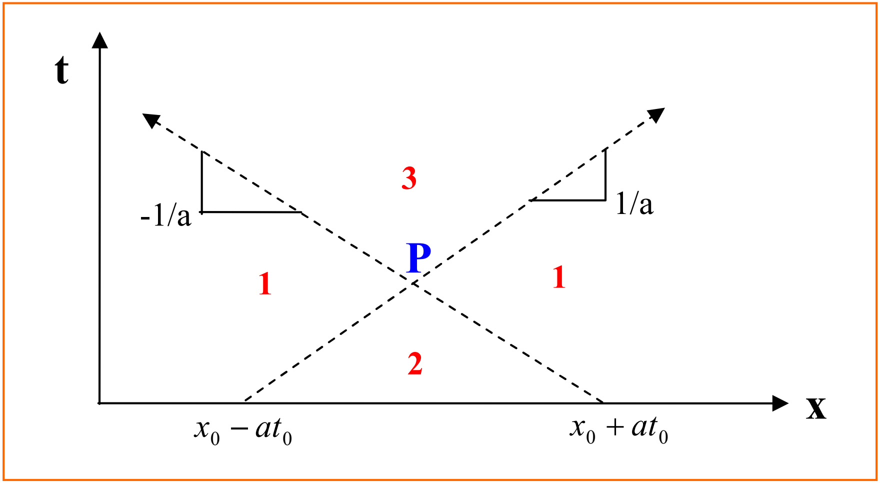
$$D = 4a^2 \quad \rightarrow \quad Hyperbolic$$

- **On the $x-t$ plane, we can show that two characteristic lines exist**

$$\frac{dt}{dx} = \pm\frac{1}{a} \quad \rightarrow \quad \begin{cases} x = at + c_1 \\ x = -at + c_2 \end{cases}$$

Mathematical Classification...

- Region 3 is called domain of influence for point P.
- Region 2 is called domain (zone) of dependence for point P.
- Region 1, is called domain of silence. This region does not feel the other two regions.



Model Equations

- **Laplace and Poisson:**

$$\varphi_{xx} + \varphi_{yy} = \begin{cases} 0 \\ f(x, y) \end{cases}$$

- **Unsteady Heat Conduction:**

$$\varphi_t = \alpha(\varphi_{xx} + \varphi_{yy})$$

- **First order linear wave equation (Linear Burger Eq.):**

$$\varphi_t + a\,\varphi_x = 0$$

- **First order nonlinear wave equation (Inviscid nonlinear burger):**

$$\varphi_t + \varphi\,\varphi_x = 0$$

- **Viscous Burger Equation:**

$$\varphi_t + \varphi\,\varphi_x = \nu\varphi_{xx}$$

- **Second order wave equation:**

$$\varphi_{tt} = a^2\varphi_{xx}$$

System of PDEs

- In general, any higher-order PDE can be converted to a system of first order PDEs.
- EXAMPLE: Let us consider the wave equation $\varphi_{tt} = a^2 \varphi_{xx}$:

$$\begin{cases} v = \dfrac{\partial u}{\partial t} \\ w = a\dfrac{\partial u}{\partial x} \end{cases} \rightarrow \begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial t^2} \\ \dfrac{\partial v}{\partial x} = \dfrac{\partial^2 u}{\partial t \partial x} \\ \dfrac{\partial w}{\partial x} = a\dfrac{\partial^2 u}{\partial x^2} \\ \dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 u}{\partial x \partial t} \end{cases} \rightarrow \frac{\partial}{\partial t}\begin{bmatrix} v \\ w \end{bmatrix} + \begin{bmatrix} 0 & -a \\ -a & 0 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} v \\ w \end{bmatrix} = 0$$

- Or equivalently

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} = \mathbf{0}$$

- Eigen-values of $\mathbf{A}$ are $\lambda = \pm a$.
- Therefore, we have two distinct real eigen-value.
- That is our system is Hyperbolic.

System of PDEs…

- **EXAMPLE: studying the Laplace equation,** $\varphi_{xx} + \varphi_{yy} = 0$ **, we find:**

$$\begin{cases} u = \dfrac{\partial \varphi}{\partial y} \\ v = \dfrac{\partial \varphi}{\partial x} \end{cases} \rightarrow \quad \frac{\partial}{\partial x}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial y}\begin{bmatrix} u \\ v \end{bmatrix} = 0$$

- **This have two distinct complex eigen-values,** $\lambda = \pm i$ **.**
- **Therefore, the equation is elliptic.**

Linear Unsteady System of Equations

- Consider

$$\frac{\partial \varphi}{\partial t} + A\frac{\partial \varphi}{\partial x} + B\frac{\partial \varphi}{\partial y} + \psi = 0$$

- Here, matrices A and B are functions of t, x and y.
- φ is a column vector and is the dependent variable.
- ψ is a column vector and is function of φ, x and y.
- The equation is hyperbolic at a point $p(x,t)$ if the eigen-values of A are all real and distinct.
- The equation is hyperbolic at a point $p(y,t)$ if the eigen-values of B are all real and distinct.
- The equation is parabolic at a point $p(x,t)$ if the eigen-values of A are all real but less than number of equations.
- The equation is parabolic at a point $p(y,t)$ if the eigen-values of B are all real but less than number of equations.

Linear Unsteady System of Equations

- The equation is elliptic at a point $p(x,t)$ if the eigen-values of A are all complex.
- The equation is elliptic at a point $p(y,t)$ if the eigen-values of B are all complex.
- The equation is mixed (Hyperbolic/Elliptic) at a point $p(x,t)$ if the eigen-values of A are mixed real and complex.
- The equation is mixed (Hyperbolic/Elliptic) at a point $p(y,t)$ if the eigen-values of B are mixed real and complex.

Linear Steady System of Equations

- **Consider**

$$A\frac{\partial \varphi}{\partial x} + B\frac{\partial \varphi}{\partial y} + \psi = 0$$

- **Method 1: define**

$$H = R^2 - 4PQ$$

where

$$P = |A| \qquad Q = |B| \qquad R = \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix} + \begin{vmatrix} a_3 & a_2 \\ b_3 & b_2 \end{vmatrix}$$

- **Then,**

$$\begin{array}{lcl} H > 0 & \rightarrow & Hyperbolic \\ H = 0 & \rightarrow & parabolic \\ H < 0 & \rightarrow & Elliptic \end{array}$$

- **Method 2: re-write the equation as**

$$(A\hat{i} + B\,\hat{j}) \bullet (\frac{\partial \varphi}{\partial x}\hat{i} + \frac{\partial \varphi}{\partial y}\hat{j}) + \psi = 0$$

Linear Steady System of Equations

- **In the characteristic direction $\vec{n}$, we can write**

$$T = (A\hat{i} + B\hat{j}) \bullet (n_x \hat{i} + n_y \hat{j}) = A n_x + B n_y$$

- **A wave-like solution will exist if**

$$|T| = 0 \quad or \quad |A n_x + B n_y| = 0$$

- **This gives**

$$Q\left(\frac{n_x}{n_y}\right)^2 + R\left(\frac{n_x}{n_y}\right) + P = 0$$

- **Or ($H = R^2 - 4PQ$)**

$$\frac{n_x}{n_y} = \frac{-R \pm \sqrt{H}}{2Q}$$

- **Therefore,**

$$H > 0 \rightarrow Hyperbolic$$
$$H = 0 \rightarrow Parabolic$$
$$H < 0 \rightarrow Elliptic$$

Linear Steady System of Equations

- **EXAMPLE: Steady, inviscid and incompressible flow**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = 0$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = 0$$

- **In vector form, we have**

$$A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = 0 \quad with \quad U = \begin{bmatrix} u \\ v \\ p \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 & 0 \\ u & 0 & 1 \\ 0 & u & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ v & 0 & 0 \\ 0 & v & 1 \end{bmatrix}$$

$$\Rightarrow |T| = -(u\,n_x + v\,n_y)\left(n_x^2 + n_y^2\right) = 0 \quad \rightarrow \frac{n_x}{n_y} = -\frac{u}{v} \quad and \quad \frac{n_x}{n_y} = \pm\sqrt{1}$$

- **Which means that the equations are of mixed hyperbolic/elliptic type.**

Linear Unsteady System of Equations

- **EXAMPLE: Unsteady, 1D, inviscid and compressible flow**

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial y}=0$$
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0$$
$$\frac{\partial p}{\partial t}+\rho a^{2}\frac{\partial u}{\partial x}+u\frac{\partial p}{\partial x}=0$$

- **In vector form,**

$$\frac{\partial \mathbf{U}}{\partial t}+\mathbf{A}\frac{\partial \mathbf{U}}{\partial x}=0 \quad with \quad \mathbf{U}=\begin{bmatrix}\rho\\ u\\ p\end{bmatrix} \quad \mathbf{A}=\begin{bmatrix}u & \rho & 0\\ 0 & u & 1/\rho\\ 0 & \rho u^{2} & u\end{bmatrix}$$

- **Whose eigen-values are**

$$\left|\mathbf{A}-\lambda I\right|=0 \quad \rightarrow \quad \lambda=u,\ u+a,\ u-a$$

- **System has 3 real and distinct eigen-values, therefore, it is hyperbolic.**

Second order PDEs

- Every second order PDE can be converted to two first order equations first and then dealt with as before.
- EXAMPLE: steady, 2D, viscous, incompressible flow

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{\partial p}{\partial x} + \dfrac{1}{\text{Re}}\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \\ u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -\dfrac{\partial p}{\partial y} + \dfrac{1}{\text{Re}}\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) \end{cases}$$

- Set,

$$\begin{cases} a = \dfrac{\partial v}{\partial x} \\ b = \dfrac{\partial v}{\partial y} \\ c = \dfrac{\partial u}{\partial y} \end{cases} \xrightarrow{\text{continuum}} \quad \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = -b$$

Second order PDEs

- **In vector form, we get**

$$\boxed{A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = C}$$

$$U = \begin{bmatrix} u \\ v \\ a \\ b \\ c \\ p \end{bmatrix} \quad A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\mathrm{Re}} & 0 & 1 \\ 0 & 0 & \frac{-1}{\mathrm{Re}} & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{\mathrm{Re}} & 0 \\ 0 & 0 & 0 & \frac{-1}{\mathrm{Re}} & 0 & 1 \end{bmatrix}$$

- **We get,**

$$T = \begin{bmatrix} n_y & 0 & 0 & 0 & 0 & 0 \\ n_x & n_y & 0 & 0 & 0 & 0 \\ 0 & 0 & -n_y & n_x & 0 & 0 \\ 0 & 0 & 0 & n_y & n_x & 0 \\ 0 & 0 & 0 & \frac{n_x}{\mathrm{Re}} & \frac{-n_y}{\mathrm{Re}} & n_x \\ 0 & 0 & \frac{-n_x}{\mathrm{Re}} & \frac{-n_y}{\mathrm{Re}} & 0 & n_y \end{bmatrix} \rightarrow |T| = \frac{1}{\mathrm{Re}} n_y^2 \left(n_x^2 + n_y^2\right)^2 = 0 \rightarrow \begin{cases} n_y = 0 \\ \left(\frac{n_y}{n_x}\right)^2 + 1 = 0 \end{cases}$$

- **Whose eigen-values are imaginary. Therefore, the system is mixed elliptic/parabolic.**