# MODERN PHYSICS

## THIRD EDITION

## KENNETH KRANE

Instructor's Manual

to accompany

Modern Physics, 3$^{rd}$ Edition

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# Preface

This Instructor's Manual accompanies the 3$^{rd}$ edition of the textbook *Modern Physics* (John Wiley & Sons, 2012). It includes (1) explanatory material for each chapter; (2) suggested outside readings for instructor or student; (3) references to web sites or other generally available simulations of phenomena; (4) exercises that can be used in various active-engagement classroom strategies; (5) sample exam questions; and (6) complete solutions to the end-of-chapter problems in the text.

Perhaps the greatest influence on my teaching in the time since the publication of the 2$^{nd}$ edition of this textbook (1996) has been the growth into maturity of the field of physics education research (PER). Rather than indicating specific areas of misunderstanding, PER has demonstrated that student comprehension is enhanced by any of a number of interactive techniques that are designed to engage the students and make them active participants in the learning process. The demonstrated learning improvements are robust and replicable, and they transcend differences among instructors and institutional types. In my own trajectory in this process, I have been especially influenced by the work of Lillian McDermott and her group at the University of Washington[1] and Eric Mazur at Harvard University.[2] I am grateful to them not only for their contributions to PER but also for their friendship over the years.

With the support of a Course, Curriculum, and Laboratory Improvement grant from the National Science Foundation[3], I have developed and tested a set of exercises that can be used either in class as group activities or outside of class (for example, in a Peer Instruction mode following Mazur's format or in a Just-In-Time Teaching[4] mode). These exercises are included in this Instructor's Manual. I am grateful for the support of the National Science Foundation in enabling this project to be carried out. Two Oregon State University graduate students assisted in the implementations of these reformed teaching methods: K. C. Walsh helped with producing several simulations and illustrative materials, with implementing an interactive web site, and with corresponding developments in the laboratory that accompanies our course, and Pornrat Wattanakasiwich undertook a PER project[5] for her Ph.D. that involved the observation of student reasoning about probability, which lies at the heart of most topics in modern physics.

One of the major themes that has emerged from PER in the past two decades is that students can often learn successful algorithms for solving problems while lacking a fundamental understanding of the underlying concepts. The importance of the in-class or pre-class exercises is to force students to consider these concepts and to apply them to diverse situations that often cannot be analyzed with an equation. It is absolutely essential to devote class time to these exercises and to follow through with exam questions that require similar analysis and a similar articulation of the conceptual reasoning. I strongly believe that conceptual understanding is a necessary prerequisite to successful problem solving. In my own classes at Oregon State University I have repeatedly observed that improved conceptual understanding leads directly to improved problem-solving skills.

In training students to reason conceptually, it is necessary to force them to verbalize their reasons for selecting a particular answer to a conceptual or qualitative question, and you will learn much from listening to or reading their arguments. A simple

multiple-choice conceptual question, either as a class exercise or a test problem, gives you insufficient insight into the students' reasoning patterns unless you also ask them to justify their choice. Even when I have teaching assistants grade the exams in my class, I always grade the conceptual questions myself, if only to gather insight into how students reason. To save time I generally grade such questions with either full credit (correct choice of answer and more-or-less correct reasoning) or no credit (wrong choice or correct choice with incorrect reasoning).

Here's an example of why it is necessary to require students to provide conceptual arguments. After a unit on the Schrödinger equation, I gave the following conceptual test question: Consider a particle in the first excited state of a one-dimensional infinite potential energy well that extends from $x = 0$ to $x = L$. At what locations is the particle most likely to be found? The students were required to state an answer and to give their reasoning. One student drew a nice sketch of the probability density in the first excited state, correctly showing maxima at $x = L/4$ and $x = 3L/4$, and stated that those locations were the most likely ones at which to find the particle. Had I not required the reasoning, the student would have received full credit, and I would have been satisfied with the student's understanding of the material. However, in stating the reasoning, the student demonstrated what turned out to be a surprisingly common incorrect mode of reasoning. The student apparently confused the graph of probability density with a similar sort of roller-coaster potential energy diagram from introductory physics and reasoned as follows: The particle is moving more slowly at the peaks of the distribution, so it spends more time at those locations and is thus more likely to be found there. PER follow-up work indicated that the confusion was caused in part by combining probability distributions with energy level diagrams – students were unsure of what the ordinate represented. As a result, I adopted a policy in class (and in this edition of the textbook) of never showing the wave functions or probability distributions on the same plot as the energy levels.

The overwhelming majority of PER work has concerned the introductory course, but the effective pedagogic techniques revealed by that research carry over directly into the modern physics course. The collection of research directly linked to topics in modern physics is much smaller but no less revealing. The University of Washington group has produced several papers impacting modern physics, including the understanding of interference and diffraction of particles[,] time and simultaneity in special relativity, and the photoelectric effect (see the papers listed on their web site, ref. 1). The PER group of Edward F. Redish at the University of Maryland has also been involved in studying the learning of quantum concepts, including the student's prejudices from classical physics, probability, and conductivity.[6] (Further work on the learning of quantum concepts has been carried out by the research groups of two of Redish's Ph.D. students, Lei Bao at Ohio State University[7] and Michael Wittmann at the University of Maine.[8]) Dean Zollman's group at Kansas State University has developed tutorials and visualizations to enhance the teaching of quantum concepts at many levels (from pre-college through advanced undergraduate).[9] The physics education group at the University of Colorado, led by Noah Finkelstein and Carl Wieman, is actively pursuing several research areas involving modern physics and has produced numerous research papers as well as simulations on topics in modern physics.[10] Others who have conducted research on the teaching of quantum mechanics and developed interactive or evaluative materials include

Chandralekha Singh at the University of Pittsburgh[11] and Richard Robinett at Pennsylvania State University.[12]

## Classroom Materials for Active Engagement

### 1.  Reading Quizzes

I started developing the interactive classroom materials for modern physics after successfully introducing Eric Mazur's Peer Instruction techniques into my calculus-based introductory course.  Daily reading quizzes were a part of Mazur's original classroom strategy, but recently he has adopted a system that is more like Just-in-Time Teaching.  Nevertheless, I have found the reading quizzes to work effectively in both my introductory and modern physics classes, and I have continued using them.  We use electronic classroom communication devices ("clickers") to collect the responses, but in a small class paper quizzes work just as well.  Originally the quizzes were intended to get students to read the textbook before coming to class, and I have over the years collected evidence that the quizzes in fact accomplish that goal.  The quizzes are given just at the start of class, and I have found that they have two other salutary effects: (1) In the few minutes before the bell rings at the start of class, the students are not reading the campus newspaper or discussing last week's football game – they are reading their physics books. (2) It takes no time at the start of class for me to focus the students' attention or put them "in the mood" for physics; the quiz gets them settled into class and thinking about physics.  The multiple-choice quizzes must be very straightforward – no complex thinking or reasoning should be required, and if a student has done the assigned reading the quiz should be automatic and should take no more than a minute or so to read and answer.  Nearly all students get at least 80% of the quizzes correct, so ultimately they have little impact on the grade distribution.  The quizzes count only a few percent toward the student's total grade, so even if they miss a few their grade is not affected.

### 2.  Conceptual Questions

I spend relatively little class time "lecturing" in the traditional sense.  I prefer an approach in which I prod and coach the students into learning and understanding the material.  The students' reading of the textbook is an important component of this process – I do not see the need to repeat orally everything that is already written in the textbook. (Of course, there are some topics in any course that can be elucidated only by a well constructed and delivered lecture.  Separating those topics from those that the students can mostly grasp from reading the text and associated in-class follow-ups comes only from experience.  Feedback obtained from the results of the conceptual exercises and from student surveys is invaluable in this process.)  I usually take about 10 minutes at the beginning of class to summarize the important elements from that day's reading.  In the process I list on the board new or unfamiliar words and important formulas.  These remain visible during the entire class so I can refer back to them as often as necessary.  I explain any special or restrictive circumstances that accompany the use of any equation.  I do not do formal mathematical derivations in class – they cause a rapid drop-off in student attention.  However, I do discuss or explain mathematical processes or techniques

that might be unfamiliar to students. I encourage students to e-mail me with questions about the reading before class, and at this point I answer those questions and any new questions that may puzzle the students.

The remainder of the class period consists of conceptual questions and worked examples. I follow the Peer Instruction model for the conceptual questions: an individual answer with no discussion, then small group discussions, and finally a second individual answer. On my computer I can see the histograms of the responses using the clickers, and if there are fewer than 30% or more than 70% correct answers on the first response, the group discussions normally don't provide much benefit so I abandon the question and move on to another. During the group discussion time, I wander throughout the class listening to the comments and occasionally asking questions or giving a small nudge if I feel a particular group is moving in the wrong direction. After the second response I ask a member of the class to give the answer and an explanation, and I will supplement the student's explanation as necessary. I generally do not show the histograms of the clicker responses to the class, neither upon the first response nor the second. The daily quiz, summary, two conceptual questions or small group projects, and one or two worked examples will normally fill a 50-minute class period, with a few minutes at the end for recapitulation or additional questions. I try to end each class period with a brief teaser regarding the next class.

Some conceptual questions listed for class discussion may appear similar to those given on exams. I never use the same question for both class discussion and examination during any single term. However, conceptual questions used during one term for examinations may find use for in-class discussions during a subsequent term.

## 3. JITT Warm-up Exercises

Just-in-Time Teaching uses web-based "warm-up" exercises to assess the student's prior knowledge and misconceptions. The instructor can use the responses to the warm-up exercises to plan the content of the next class. The reading quizzes and conceptual questions intended for in-class activities can in many cases be used equally well for JiTT warm-up exercises.

### Lecture Demonstrations

Demonstrations are an important part of teaching introductory physics, and physics education research has shown that learning from the demos is enhanced if they are made interactive. (For example, you can ask students to predict the response of the apparatus, discuss the predictions with a neighbor, and then to reconcile an incorrect prediction with the observation.) Unfortunately, there are few demos that can be done in the modern physics classroom. Instead, we must rely on simulations and animations. There are many effective and interesting instructional software packages on the web that can be downloaded for your class, and you can make them available for the students to use outside of class. I have listed in this Manual some of the modern physics software that I have used in my classes. Of particular interest is the open-source collection of Physlets (physics applets) covering relativity and quantum physics produced by Mario Belloni, Wolfgang Christian, and Anne J. Cox.[13]

**Sample Test Questions**

This Instructor's Manual includes a selection of sample test questions. A typical midterm exam in my Modern Physics class might include 4 multiple-choice questions (no reasoning arguments required) worth 20 points, 2 conceptual questions (another 20 points) requiring the student to select an answer from among 2 or 3 possibilities and to give the reasons for that choice, and 3 numerical problems worth a total of 60 points. Students have 1 hour and 15 minutes to complete the exam. The final exam is about 1.5 times the length of a midterm exam.

One point worth considering is the use of formula sheets during exams. Over the years I have gone back and forth among many different exam systems: open book, closed book and notes, and closed book with a student-generated formula sheet. I have found that in the open book format students seem to spend a lot of time leafing through the book looking for an essential formula or constant. On the other hand, I have been amazed at how many equations a student can pack onto a single sheet of paper, and I often find myself wondering how much better such students would perform on exams if they spent as much study time working on practice problems as they do miniaturizing equations. (Students often have difficulty distinguishing important formulas, which represent a fundamental concept or relationship, from mere equations which might be intermediate steps in solving a problem or deriving a formula.) I have finally settled on a closed book format in which I supply the formula sheet with each exam. I feel this has a number of advantages: (1) It equalizes the playing field. (2) Students don't need to waste time copying equations. (3) The formula sheet, a copy of which I give to students at the beginning of the term, itself serves as a kind of study guide. (4) Students use the formula sheet when working homework problems and studying for the exams, so they know what formulas are on the sheet and where they are located. (5) I can be sure that the formulas that students need to work the exams are included on the formula sheet. A sample copy of my formula sheet is included in this Instructor's Manual.

This Instructor's Manual is always a work in progress. I would be grateful to receive corrections or suggestions from users.

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**References**

1. http://www.phys.washington.edu/groups/peg/

2. http://mazur.harvard.edu/education/educationmenu.php. Also see E. Mazur, *Peer Instruction: A User's Manual* (Prentice Hall, 1997).

3.  National Science Foundation grant DUE-0340818, "Materials for Active Engagement in the Modern Physics Course"

4.  http://jittdl.physics.iupui.edu/jitt/.  Also see G. Novak, A. Gavrin, W. Christian, and E. Patterson, *Just in Time Teaching: Blending Active Learning with Web Technology* (Benjamin Cummings, 1999).

5.  "Model of Student Understanding of Probability in Modern Physics," Pornrat Wattanakasiwich, Ph.D. dissertation, Oregon State University, 2005.

6.  http://www.physics.umd.edu/perg/

7.  http://www.physics.ohio-state.edu/~lbao/

8.  http://www.umaine.edu/per/

9.  http://web.phys.ksu.edu/

10.  http://www.colorado.edu/physics/EducationIssues/index.htm

11.  http://www.phyast.pitt.edu/~cls/

12.  http://www.phys.psu.edu/~rick/ROBINETT/robinett.html

13.  M. Belloni, W. Christian, and A. J. Cox, *Physlet Quantum Physics: An Interactive Introduction* (Pearson Prentice Hall, 2006).

# Table of Contents

# Chapter 1

This chapter presents a review of some topics from classical physics. I have often heard from instructors using the book that "my students have already studied a year of introductory classical physics, so they don't need the review." This review chapter gives the opportunity to present a number of concepts that I have found to cause difficulty for students and to collect those concepts where they are available for easy reference. For example, all students should know that kinetic energy is $\frac{1}{2}mv^2$, but few are readily familiar with kinetic energy as $p^2/2m$, which is used more often in the text. The expression connecting potential energy difference with potential difference for an electric charge $q$, $\Delta U = q\Delta V$, zips by in the blink of an eye in the introductory course and is rarely used there, while it is of fundamental importance to many experimental set-ups in modern physics and is used implicitly in almost every chapter. Many introductory courses do not cover thermodynamics or statistical mechanics, so it is useful to "review" them in this introductory chapter.

I have observed students in my modern course occasionally struggling with problems involving linear momentum conservation, another of those classical concepts that resides in the introductory course. Although we physicists regard momentum conservation as a fundamental law on the same plane as energy conservation, the latter is frequently invoked throughout the introductory course while former appears and virtually disappears after a brief analysis of 2-body collisions. Moreover, some introductory texts present the equations for the final velocities in a one-dimensional elastic collision, leaving the student with little to do except plus numbers into the equations. That is, students in the introductory course are rarely called upon to begin momentum conservation problems with $p_{\text{initial}} = p_{\text{final}}$. This puts them at a disadvantage in the application of momentum conservation to problems in modern physics, where many different forms of momentum may need to be treated in a single situation (for example, classical particles, relativistic particles, and photons). Chapter 1 therefore contains a brief review of momentum conservation, including worked sample problems and end-of-chapter exercises.

Placing classical statistical mechanics in Chapter 1 (as compared to its location in Chapter 10 in the 2$^{\text{nd}}$ edition) offers a number of advantages. It permits the useful expression $K_{\text{av}} = \frac{3}{2}kT$ to be used throughout the text without additional explanation. The failure of classical statistical mechanics to account for the heat capacities of diatomic gases (hydrogen in particular) lays the groundwork for quantum physics. It is especially helpful to introduce the Maxwell-Boltzmann distribution function early in the text, thus permitting applications such as the population of molecular rotational states in Chapter 9 and clarifying references to "population inversion" in the discussion of the laser in Chapter 8. Distribution functions in general are new topics for most students. They may look like ordinary mathematical functions, but they are handled and interpreted quite differently. Absent this introduction to a classical distribution function in Chapter 1, the students' first exposure to a distribution function will be $|\psi|^2$, which layers an additional level of confusion on top of the mathematical complications. It is better to have a chance to cover some of the mathematical details at an earlier stage with a distribution function that is easier to interpret.

## Suggestions for Additional Reading

Some descriptive, historical, philosophical, and nonmathematical texts which give good background material and are great fun to read:
A. Baker, *Modern Physics and Anti-Physics* (Addison-Wesley, 1970).
F. Capra, *The Tao of Physics* (Shambhala Publications, 1975).
K. Ford, *Quantum Physics for Everyone* (Harvard University Press, 2005).
G. Gamow, *Thirty Years that Shook Physics* (Doubleday, 1966).
R. March, *Physics for Poets* (McGraw-Hill, 1978).
E. Segre, *From X-Rays to Quarks: Modern Physicists and their Discoveries* (Freeman, 1980).
G. L. Trigg, *Landmark Experiments in Twentieth Century Physics* (Crane, Russak, 1975).
F. A. Wolf, *Taking the Quantum Leap* (Harper & Row, 1989).
G. Zukav, *The Dancing Wu Li Masters, An Overview of the New Physics* (Morrow, 1979).

Gamow, Segre, and Trigg contributed directly to the development of modern physics and their books are written from a perspective that only those who were part of that development can offer. The books by Capra, Wolf, and Zukav offer controversial interpretations of quantum mechanics as connected to eastern mysticism, spiritualism, or consciousness.


## Materials for Active Engagement in the Classroom

### A. Reading Quizzes

1. In an ideal gas at temperature *T,* the average speed of the molecules:
   (1) increases as the square of the temperature.
   (2) increases linearly with the temperature.
   (3) increases as the square root of the temperature.
   (4) is independent of the temperature.

2. The heat capacity of molecular hydrogen gas can take values of $3R/2$, $5R/2$, and $7R/2$ at different temperatures.  Which value is correct at low temperatures?
   (1) $3R/2$          (2) $5R/2$                (3) $7R/2$

**Answers**      1. 3     2. 1


### B. Conceptual and Discussion Questions

1. Equal numbers of molecules of hydrogen gas (molecular mass = 2 u) and helium gas (molecular mass = 4 u) are in equilibrium in a container.
   (a) What is the ratio of the average kinetic energy of a hydrogen molecule to the average kinetic energy of a helium molecule?
   $K_{\mathrm{H}} / K_{\mathrm{He}} =$   (1) 4     (2) 2     (3) $\sqrt{2}$     (4) 1     (5) $1/\sqrt{2}$     (6) 1/2     (7) 1/4

(b) What is the ratio of the average speed of a hydrogen molecule to the average speed of a helium molecule?

$v_H / v_{He} =$   (1) 4   (2) 2   (3) $\sqrt{2}$   (4) 1   (5) $1/\sqrt{2}$   (6) 1/2   (7) 1/4 (C)

(c) What is the ratio of the pressure exerted on the walls of the container by the hydrogen gas to the pressure exerted on the walls by the helium gas?

$P_H / P_{He} =$   (1) 4   (2) 2   (3) $\sqrt{2}$   (4) 1   (5) $1/\sqrt{2}$   (6) 1/2   (7) 1/4

2.  Containers 1 and 3 have volumes of 1 m$^3$ and container 2 has a volume of 2 m$^3$. Containers 1 and 2 contain helium gas, and container 3 contains neon gas. All three containers have a temperature of 300 K and a pressure of 1 atm.



|  | He | He | Ne |
|---|---|---|---|
|  | 1 | 2 | 3 |
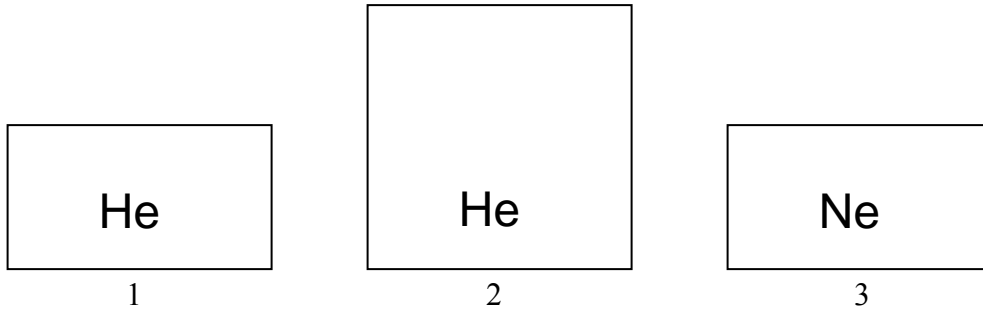
(a) Rank the average speeds of the molecules in the containers in order from largest to smallest.
    (1) $1 > 2 > 3$        (2) $1 = 2 > 3$        (3) $1 = 2 = 3$
    (4) $3 > 1 > 2$        (5) $3 > 1 = 2$        (6) $2 > 1 > 3$

(b) In which container is the average kinetic energy per molecule the largest?
    (1) 1            (2) 2            (3) 3
    (4) 1 and 2      (5) 1 and 3      (6) All the same

3.  (a) Consider diatomic nitrogen gas at room temperature, in which only the translational and rotational motions are possible. Suppose that 100 J of energy is transferred to the gas at constant volume. How much of this energy goes into the translational kinetic energy of the molecules?
    (1) 20 J         (2) 40 J        (3) 50 J
    (4) 60 J         (5) 80 J        (6) 100 J

(b) Now suppose that the gas is at a higher temperature, so that vibrational motion is also possible. Compared with the situation at room temperature, is the fraction of the added energy that goes into translational kinetic energy:
    (1) smaller?        (2) the same?       (3) greater?

**Answers**     1. (a) 4  (b) 3  (c) 4        2. (a) 2  (b) 6        3. (a) 4  (b) 1

## Sample Exam Questions

### A. Multiple Choice

1. A container holds gas molecules of mass $m$ at a temperature $T$. A small probe inserted into the container measures the value of the $x$ component of the velocity of the molecules. What is the average value of $\frac{1}{2}mv_x^2$ for these molecules?

    (a) $\frac{3}{2}kT$      (b) $\frac{1}{2}kT$      (c) $kT$      (d) $\sqrt{3}kT$

2. A container holds $N$ molecules of a diatomic gas at temperature $T$. At this temperature, rotational and vibrational motions of the gas molecules are allowed. A quantity of energy $E$ is transferred to the gas. What fraction of this added energy is responsible for increasing the temperature of the gas?

    (a) All of the added energy      (b) 3/5      (c) 2/5      (d) 2/7      (e) 3/7

3. Two identical containers with fixed volumes hold equal amounts of Ne gas and $N_2$ gas at the same temperature of 1000 K. Equal amounts of heat energy are then transferred to the two gases. How do the final temperatures of the two gases compare?

    (a) $T(\text{Ne}) = T(N_2)$      (b) $T(\text{Ne}) > T(N_2)$      (c) $T(\text{Ne}) < T(N_2)$

**Answers**      1. b      2. e      3. b

### B. Conceptual

1. A container of volume $V$ holds an equilibrium mixture of $N$ molecules of oxygen gas $O_2$ (molecular mass = 32.0 u) and also $2N$ molecules of He gas (mass = 4.00 u). Is the average molecular energy of $O_2$ *greater than, equal to, or less than* the average molecular energy of He? EXPLAIN YOUR ANSWER.

2. Consider two containers of identical volumes. Container 1 holds $N$ molecules of He at temperature $T$. Container 2 holds the same number $N$ molecules of $H_2$ at the same temperature $T$. Is the average energy per molecule of He *greater than, less than, or the same as* the average energy per molecule of $H_2$? EXPLAIN YOUR ANSWER.

**Answers**      1. equal to      2. the same as

### C. Problems

1. $N$ molecules of a gas are confined in a container at temperature $T$. A measuring device in the container can determine the number of molecules in a range of $0.002v$ at any speed $v$, that is, the number of molecules with speeds between $0.999v$ and $1.001v$. When the device is set for molecules at the speed $v_{\text{rms}}$, the result is $N_1$. When it is set for molecules at the speed $2v_{\text{rms}}$, the result is $N_2$. Find the value of $N_1/N_2$. Your answer should be a pure number, involving no symbols or variables.

2. A container holds 2.5 moles of helium (a gas with one atom per molecule; atomic mass = 4.00 u; molar mass = 0.00400 kg) at a temperature of 342 K. What fraction of the gas molecules has translational kinetic energies between 0.01480 eV and 0.01520 eV?

3. One mole of $N_2$ gas (molecular mass = 28 u) is confined to a container at a temperature of 387 K. At this temperature, you may assume that the molecules are free to both rotate and vibrate.
(a) What fraction of the molecules has translational kinetic energies within ±1% of the average translational kinetic energy?
(b) Find the total internal energy of the gas.

Answers:     1.  11.25       2.  0.0066      3.  (a) 0.0093   (b) 11.2 kJ

**Problem Solutions**

1. (a) Conservation of momentum gives $p_{x,\text{initial}} = p_{x,\text{final}}$, or

$$m_{\text{H}} v_{\text{H,initial}} + m_{\text{He}} v_{\text{He,initial}} = m_{\text{H}} v_{\text{H,final}} + m_{\text{He}} v_{\text{He,final}}$$

Solving for $v_{\text{He,final}}$ with $v_{\text{He,initial}} = 0$, we obtain

$$v_{\text{He,final}} = \frac{m_{\text{H}}(v_{\text{H,initial}} - v_{\text{H,final}})}{m_{\text{He}}}$$

$$= \frac{(1.674 \times 10^{-27} \text{ kg})[1.1250 \times 10^7 \text{ m/s} - (-6.724 \times 10^6 \text{ m/s})]}{6.646 \times 10^{-27} \text{ kg}} = 4.527 \times 10^6 \text{ m/s}$$

(b) Kinetic energy is the only form of energy we need to consider in this elastic collision. Conservation of energy then gives $K_{\text{initial}} = K_{\text{final}}$, or

$$\tfrac{1}{2} m_{\text{H}} v_{\text{H,initial}}^2 + \tfrac{1}{2} m_{\text{He}} v_{\text{He,initial}}^2 = \tfrac{1}{2} m_{\text{H}} v_{\text{H,final}}^2 + \tfrac{1}{2} m_{\text{He}} v_{\text{He,final}}^2$$

Solving for $v_{\text{He,final}}$ with $v_{\text{He,initial}} = 0$, we obtain

$$v_{\text{He,final}} = \sqrt{\frac{m_{\text{H}}(v_{\text{H,initial}}^2 - v_{\text{H,final}}^2)}{m_{\text{He}}}}$$

$$= \sqrt{\frac{(1.674 \times 10^{-27} \text{ kg})[(1.1250 \times 10^7 \text{ m/s})^2 - (-6.724 \times 10^6 \text{ m/s})^2]}{6.646 \times 10^{-27} \text{ kg}}} = 4.527 \times 10^6 \text{ m/s}$$

2. (a) Let the helium initially move in the $x$ direction. Then conservation of momentum gives:

$p_{x,\text{initial}} = p_{x,\text{final}}$ :    $m_{\text{He}} v_{\text{He,initial}} = m_{\text{He}} v_{\text{He,final}} \cos\theta_{\text{He}} + m_{\text{O}} v_{\text{O,final}} \cos\theta_{\text{O}}$

$p_{y,\text{initial}} = p_{y,\text{final}}$ :    $0 = m_{\text{He}} v_{\text{He,final}} \sin\theta_{\text{He}} + m_{\text{O}} v_{\text{O,final}} \sin\theta_{\text{O}}$

From the second equation,

$$v_{\text{O,final}} = -\frac{m_{\text{He}} v_{\text{He,final}} \sin\theta_{\text{He}}}{m_{\text{O}} \sin\theta_{\text{O}}} = -\frac{(6.6465 \times 10^{-27} \text{ kg})(6.636 \times 10^6 \text{ m/s})(\sin 84.7°)}{(2.6560 \times 10^{-26} \text{ kg})[\sin(-40.4°)]} = 2.551 \times 10^6 \text{ m/s}$$

(b) From the first momentum equation,

$$v_{He,initial} = \frac{m_{He} v_{He,final} \cos \theta_{He} + m_O v_{O,final} \cos \theta_O}{m_{He}}$$

$$= \frac{(6.6465 \times 10^{-27} \text{ kg})(6.636 \times 10^6 \text{ m/s})(\cos 84.7°) + (2.6560 \times 10^{-26} \text{ kg})(2.551 \times 10^6 \text{ m/s})[\cos(-40.4°)]}{6.6465 \times 10^{-27} \text{ kg}}$$

$$= 8.376 \times 10^6 \text{ m/s}$$

3.  (a) Using conservation of momentum for this one-dimensional situation, we have
$p_{x,initial} = p_{x,final}$, or

$$m_{He} v_{He} + m_N v_N = m_D v_D + m_O v_O$$

Solving for $v_O$ with $v_N = 0$, we obtain

$$v_O = \frac{m_{He} v_{He} - m_D v_D}{m_O} = \frac{(3.016 \text{ u})(6.346 \times 10^6 \text{ m/s}) - (2.014 \text{ u})(1.531 \times 10^7 \text{ m/s})}{15.003 \text{ u}} = -7.79 \times 10^5 \text{ m/s}$$

(b) The kinetic energies are:

$$K_{initial} = \tfrac{1}{2} m_{He} v_{He}^2 + \tfrac{1}{2} m_N v_N^2 = \tfrac{1}{2}(3.016 \text{ u})(1.6605 \times 10^{-27} \text{ kg/u})(6.346 \times 10^6 \text{ m/s})^2 = 1.008 \times 10^{-13} \text{ J}$$
$$K_{final} = \tfrac{1}{2} m_D v_D^2 + \tfrac{1}{2} m_O v_O^2 = \tfrac{1}{2}(2.014 \text{ u})(1.6605 \times 10^{-27} \text{ kg/u})(1.531 \times 10^7 \text{ m/s})^2$$
$$+ \tfrac{1}{2}(15.003 \text{ u})(1.6605 \times 10^{-27} \text{ kg/u})(7.79 \times 10^5 \text{ m/s})^2 = 3.995 \times 10^{-13} \text{ J}$$

As in Example 1.2, this is also a case in which nuclear energy turns into kinetic energy. The gain in kinetic energy is exactly equal to the loss in nuclear energy.

4.  Let the two helium atoms move in opposite directions along the $x$ axis with speeds $v_1$ and $v_2$. Conservation of momentum along the $x$ direction ($p_{x,initial} = p_{x,final}$) gives

$$0 = m_1 v_1 - m_2 v_2 \qquad \text{or} \qquad v_1 = v_2$$

The energy released is in the form of the total kinetic energy of the two helium atoms:

$$K_1 + K_2 = 92.2 \text{ keV}$$

Because $v_1 = v_2$, it follows that $K_1 = K_2 = 46.1 \text{ keV}$, so

$$v = \sqrt{\frac{2K_1}{m_1}} = \sqrt{\frac{2(46.1 \times 10^3 \text{ eV})(1.602 \times 10^{-19} \text{ J/eV})}{(4.00 \text{ u})(1.6605 \times 10^{-27} \text{ kg/u})}} = 1.49 \times 10^6 \text{ m/s}$$

$$v_2 = v_1 = 1.49 \times 10^6 \text{ m/s}$$

5. (a) The kinetic energy of the electrons is

$$K_i = \tfrac{1}{2}mv_i^2 = \tfrac{1}{2}(9.11\times10^{-31}\text{ kg})(1.76\times10^6\text{ m/s}) = 14.11\times10^{-19}\text{ J}$$

In passing through a potential difference of $\Delta V = V_f - V_i = +4.15$ volts, the potential energy of the electrons changes by

$$\Delta U = q\Delta V = (-1.602\times10^{-19}\text{ C})(+4.15\text{ V}) = -6.65\times10^{-19}\text{ J}$$

Conservation of energy gives $K_i + U_i = K_f + U_f$, so

$$K_f = K_i + (U_i - U_f) = K_i - \Delta U = 14.11\times10^{-19}\text{ J} + 6.65\times10^{-19}\text{ J} = 20.76\times10^{-19}\text{ J}$$

$$v_f = \sqrt{\frac{2K_f}{m}} = \sqrt{\frac{2(20.76\times10^{-19}\text{ J})}{9.11\times10^{-31}\text{ kg}}} = 2.13\times10^6\text{ m/s}$$

(b) In this case $\Delta V = -4.15$ volts, so $\Delta U = +6.65\times10^{-19}$ J and thus

$$K_f = K_i - \Delta U = 14.11\times10^{-19}\text{ J} - 6.65\times10^{-19}\text{ J} = 7.46\times10^{-19}\text{ J}$$

$$v_f = \sqrt{\frac{2K_f}{m}} = \sqrt{\frac{2(7.46\times10^{-19}\text{ J})}{9.11\times10^{-31}\text{ kg}}} = 1.28\times10^6\text{ m/s}$$

6. (a) $\Delta x_A = v\Delta t_A = (0.624)(2.997\times10^8\text{ m/s})(124\times10^{-9}\text{ s}) = 23.2\text{ m}$
   (b) $\Delta x_B = v\Delta t_B = (0.624)(2.997\times10^8\text{ m/s})(159\times10^{-9}\text{ s}) = 29.7\text{ m}$

7. With $T = 35°C = 308$ K and $P = 1.22$ atm $= 1.23\times10^5$ Pa,

$$\frac{N}{V} = \frac{P}{kT} = \frac{1.23\times10^5\text{ Pa}}{(1.38\times10^{-23}\text{ J/K})(308\text{ K})} = 2.89\times10^{25}\text{ atoms/m}^3$$

so the volume available to each atom is $(2.89\times10^{25}/\text{m}^3)^{-1} = 3.46\times10^{-26}\text{ m}^3$. For a spherical atom, the volume would be

$$\tfrac{4}{3}\pi R^3 = \tfrac{4}{3}\pi(0.710\times10^{-10}\text{ m})^3 = 1.50\times10^{-30}\text{ m}^3$$

The fraction is then

$$\frac{1.50\times10^{-30}}{3.46\times10^{-26}} = 4.34\times10^{-5}$$

8. Differentiating $N(E)$ from Equation 1.22, we obtain

$$\frac{dN}{dE} = \frac{2N}{\sqrt{\pi}} \frac{1}{(kT)^{3/2}} \left[ \tfrac{1}{2} E^{-1/2} e^{-E/kT} + E^{1/2} \left( -\frac{1}{kT} \right) e^{-E/kT} \right]$$

To find the maximum, we set this function equal to zero:

$$\frac{2N}{\sqrt{\pi}} \frac{1}{(kT)^{3/2}} E^{-1/2} e^{-E/kT} \left( \frac{1}{2} - \frac{E}{kT} \right) = 0$$

Solving, we find the maximum occurs at $E = \tfrac{1}{2} kT$. Note that $E = 0$ and $E = \infty$ also satisfy the equation, but these solutions give minima rather than maxima.

9. For this case $kT = (280\ \text{K})(8.617 \times 10^{-5}\ \text{eV/K}) = 0.0241\ \text{eV}$. We take $dE$ as the width of the interval (0.012 eV) and $E$ as its midpoint (0.306 eV). Then

$$dN = N(E)\,dE = \frac{2N}{\sqrt{\pi}} \frac{1}{(0.0241\ \text{eV})^{3/2}} (0.306\ \text{eV})^{1/2} e^{-(0.306\ \text{eV})/(0.0241\ \text{eV})} (0.012\ \text{eV}) = 6.1 \times 10^{-6}\,N$$

10. (a) From Eq. 1.31,

$$\Delta E_{\text{int}} = \tfrac{5}{2} nR\,\Delta T = \tfrac{5}{2} (2.37\ \text{moles})(8.315\ \text{J/mol} \cdot \text{K})(65.2\ \text{K}) = 3.21 \times 10^{3}\ \text{J}$$

(b) From Eq. 1.32,

$$\Delta E_{\text{int}} = \tfrac{5}{2} nR\,\Delta T = \tfrac{7}{2} (2.37\ \text{moles})(8.315\ \text{J/mol} \cdot \text{K})(65.2\ \text{K}) = 4.50 \times 10^{3}\ \text{J}$$

(c) For both cases, the change in the translational part of the kinetic energy is given by Eq. 1.29:

$$\Delta E_{\text{int}} = \tfrac{3}{2} nR\,\Delta T = \tfrac{5}{2} (2.37\ \text{moles})(8.315\ \text{J/mol} \cdot \text{K})(65.2\ \text{K}) = 1.93 \times 10^{3}\ \text{J}$$

11. After the collision, $m_1$ moves with speed $v_1'$ (in the $y$ direction) and $m_2$ with speed $v_2'$ (at an angle $\theta$ with the $x$ axis). Conservation of energy then gives $E_{\text{initial}} = E_{\text{final}}$:

$$\tfrac{1}{2} m_1 v_1^2 = \tfrac{1}{2} m_1 v_1'^2 + \tfrac{1}{2} m_2 v_2'^2 \qquad \text{or} \qquad v^2 = v_1'^2 + 3 v_2'^2$$

Conservation of momentum gives:

$$p_{x,\text{initial}} = p_{x,\text{final}} : \quad m_1 v_1 = m_2 v_2' \cos\theta \qquad \text{or} \qquad v = 3v_2' \cos\theta$$

$$p_{y,\text{initial}} = p_{y,\text{final}} : \quad 0 = m_1 v_1' - m_2 v_2' \sin\theta \quad \text{or} \qquad v_1' = 3v_2' \sin\theta$$

We first solve for the speeds by eliminating $\theta$ from these equations. Squaring the two momentum equations and adding them, we obtain $v^2 + v_1'^2 = 9v_2'^2$, and combining this result with the energy equation allows us to solve for the speeds:

$$v_1' = v/\sqrt{2} \qquad \text{and} \qquad v_2' = v/\sqrt{6}$$

By substituting this value of $v_2'$ into the first momentum equation, we obtain

$$\cos\theta = \sqrt{2/3} \qquad \text{or} \qquad \theta = 35.3°$$

12. The combined particle, with mass $m' = m_1 + m_2 = 3m$, moves with speed $v'$ at an angle $\theta$ with respect to the $x$ axis. Conservation of momentum then gives:

$$p_{x,\text{initial}} = p_{x,\text{final}} : \quad m_1 v_1 = m' v' \cos\theta \qquad \text{or} \qquad v = 3v' \cos\theta$$

$$p_{y,\text{initial}} = p_{y,\text{final}} : \quad m_2 v_2 = m' v' \sin\theta \qquad \text{or} \qquad \tfrac{4}{3} v = 3v' \sin\theta$$

We can first solve for $\theta$ by dividing these two equations to eliminate the unknown $v'$:

$$\tan\theta = \tfrac{4}{3} \qquad \text{or} \qquad \theta = 53.1°$$

Now we can substitute this result into either of the momentum equations to find

$$v' = 5v/9$$

The kinetic energy lost is the difference between the initial and final kinetic energies:

$$K_{\text{initial}} - K_{\text{final}} = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 - \tfrac{1}{2} m' v'^2 = \tfrac{1}{2} mv^2 + \tfrac{1}{2}(2m)(\tfrac{2}{3} v)^2 - \tfrac{1}{2}(3m)(\tfrac{5}{9} v)^2 = \tfrac{26}{27}(\tfrac{1}{2} mv^2)$$

The total initial kinetic energy is $\tfrac{1}{2} mv^2 + \tfrac{1}{2}(2m)(\tfrac{2}{3} v)^2 = \tfrac{17}{9}(\tfrac{1}{2} mv^2)$. The loss in kinetic energy is then $\tfrac{26}{51} = 51\%$ of the initial kinetic energy.

13. (a) Let $v_1$ represent the helium atom that moves in the $+x$ direction, and let $v_2$ represent the other helium atom (which might move either in the positive or negative $x$ direction). Then conservation of momentum ($p_{x,\text{initial}} = p_{x,\text{final}}$) gives

$$mv = m_1 v_1 + m_2 v_2 \qquad \text{or} \qquad 2v = v_1 + v_2$$

where $v_2$ may be positive or negative. The initial velocity $v$ is

$$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2(40.0 \times 10^3 \text{ eV})(1.602 \times 10^{-19} \text{ J/eV})}{(8.00 \text{ u})(1.6605 \times 10^{-27} \text{ kg/u})}} = 9.822 \times 10^5 \text{ m/s}$$

The energy available to the two helium atoms after the decay is the initial kinetic energy of the beryllium atom plus the energy released in its decay:

$$K + 92.2 \text{ keV} = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 (2v - v_1)^2$$

where the last substitution is made from the momentum equation. Solving this quadratic equation for $v_1$, we obtain $v_1 = 2.47 \times 10^6$ m/s or $-0.508 \times 10^6$ m/s. Because we identified $m_1$ as the helium moving in the positive $x$ direction, it is identified with the positive root and thus (because the two heliums are interchangeable in the equation) the second value represents the velocity of $m_2$:

$$v_1 = 2.47 \times 10^6 \text{ m/s}, \ v_2 = -0.508 \times 10^6 \text{ m/s}$$

(b) Suppose we were to travel in the positive $x$ direction at a speed of $v = 9.822 \times 10^5$ m/s, which is the original speed of the beryllium from part (a). If we travel at the same speed as the beryllium, it appears to be at rest, so its initial momentum is zero in this frame of reference. The two heliums then travel with equal speeds in opposite directions along the $x$ axis. Because they share the available energy equally, each helium has a kinetic energy of 46.1 keV and a speed of $\sqrt{2K/m} = 1.49 \times 10^6$ m/s, as we found in Problem 4. Let's represent these velocities in this frame of reference as $v_1' = +1.49 \times 10^6$ m/s and $v_2' = -1.49 \times 10^6$ m/s. Transforming back to the original frame, we find

$$v_1 = v_1' + v = 1.49 \times 10^6 \text{ m/s} + 9.822 \times 10^5 \text{ m/s} = 2.47 \times 10^6 \text{ m/s}$$

$$v_2 = v_2' + v = -1.49 \times 10^6 \text{ m/s} + 9.822 \times 10^5 \text{ m/s} = -0.508 \times 10^6 \text{ m/s}$$

14. (a) Let the second helium move in a direction at an angle $\theta$ with the $x$ axis. (We'll assume that the 30° angle for $m_1$ is measured above the $x$ axis, while the angle $\theta$ for $m_2$ is measured below the $x$ axis. Then conservation of momentum gives:

$$p_{x,\text{initial}} = p_{x,\text{final}}: \quad mv = m_1 v_1 \cos 30° + m_2 v_2 \cos \theta \qquad \text{or} \qquad 2v - \frac{\sqrt{3}}{2} v_1 = v_2 \cos \theta$$

$$p_{y,\text{initial}} = p_{y,\text{final}}: \quad 0 = m_1 v_1 \sin 30° - m_2 v_2 \sin \theta \qquad \text{or} \qquad \frac{1}{2} v_1 = v_2 \sin \theta$$

We can eliminate the angle $\theta$ by squaring and adding the two momentum equations:

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$$4v^2 + v_1^2 - 2\sqrt{3}vv_1 = v_2^2$$

The kinetic energy given to the two heliums is equal to the original kinetic energy $\frac{1}{2}mv^2$ of the beryllium plus the energy released in the decay:

$$\tfrac{1}{2}mv^2 + 92.2\ \text{keV} = \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2 = \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2(4v^2 + v_1^2 - 2\sqrt{3}vv_1)$$

$$\tfrac{1}{2}(m_1 + m_2)v_1^2 - \sqrt{3}m_2 vv_1 + (2m_2 v^2 - \tfrac{1}{2}mv^2 - 92.2\ \text{keV}) = 0$$

Solving this quadratic equation gives

$$v_1 = 2.405\times10^6\ \text{m/s},\ -0.321\times10^6\ \text{m/s}$$

Based on the directions assumed in writing the momentum equations, only the positive root is meaningful. We can substitute this value for $v_1$ into either the momentum or the energy equations to find $v_2$ and so our solution is:

$$v_1 = 2.41\times10^6\ \text{m/s},\ v_2 = 1.25\times10^6\ \text{m/s}$$

The angle $\theta$ can be found by substituting these values into either of the momentum equations, for example

$$\theta = \sin^{-1}\frac{v_1}{2v_2} = \sin^{-1}\frac{2.41\times10^6\ \text{m/s}}{2(1.25\times10^6\ \text{m/s})} = 74.9°$$

(b) The original speed of the beryllium atom is $v = \sqrt{2K/m} = 1.203\times10^6\ \text{m/s}$. If we were to view the experiment from a frame of reference moving at this velocity, the original beryllium atom would appear to be at rest. In this frame of reference, in which the initial momentum is zero, the two helium atoms are emitted in opposite directions with equal speeds. Each helium has a kinetic energy of 46.1 keV and a speed of $v_1' = v_2' = 1.49\times10^6\ \text{m/s}$. Let $\phi$ represent the angle that each of the helium atoms makes with the x axis in this frame of reference. Then the relationship between the x components of the velocity of $m_1$ in this frame of reference and the original frame of reference is

$$v_1 \cos 30° = v_1' \cos\phi + v$$

and similarly for the y components

$$v_1 \sin 30° = v_1' \sin\phi$$

We can divide these two equations to get

$$\cot 30° = \frac{v_1' \cos\phi + v}{v_1' \sin\phi}$$

which can be solved to give $\phi = 53.8°$. Using this value of $\phi$, we can then find $v_1 = 2.41 \times 10^6$ m/s. We can also write the velocity addition equations for $m_2$:

$$v_2 \cos\theta = -v_2' \cos\phi + v \qquad \text{and} \qquad v_2 \sin\theta = -v_2' \sin\phi$$

which describe respectively the $x$ and $y$ components. Solving as we did for $m_1$, we find $v_2 = 1.25 \times 10^6$ m/s and $\theta = 74.9°$.

15. (a) With $K = \frac{3}{2}kT$,

$$\Delta K = \frac{3}{2}k\Delta T = \frac{3}{2}(1.38 \times 10^{-23} \text{ J/K})(80 \text{ K}) = 1.66 \times 10^{-21} \text{ J} = 0.0104 \text{ eV}$$

(b) With $U = mgh$,

$$h = \frac{U}{mg} = \frac{1.66 \times 10^{-21} \text{ J}}{(40.0 \text{ u})(1.66 \times 10^{-27} \text{ kg/u})(9.80 \text{ m/s}^2)} = 2550 \text{ m}$$

16. We take $dE$ to be the width of this small interval: $dE = 0.04kT - 0.02kT = 0.02kT$, and we evaluate the distribution function at an energy equal to the midpoint of the interval ($E = 0.03kT$):

$$\frac{dN}{N} = \frac{N(E)\,dE}{N} = \frac{2}{\sqrt{\pi}} \frac{1}{(kT)^{3/2}} (0.03kT)^{1/2} e^{-(0.03kT)/kT} (0.02kT) = 3.79 \times 10^{-3}$$

17. If we represent the molecule as two atoms considered as point masses $m$ separated by a distance $2R$, the rotational inertia about one of the axes is $I_{x'} = mR^2 + mR^2 = 2mR^2$. On average, the rotational kinetic energy about any one axis is $\frac{1}{2}kT$, so $\frac{1}{2}I_{x'}\omega_{x'}^2 = \frac{1}{2}kT$ and

$$\omega_{x'} = \sqrt{\frac{kT}{I_{x'}}} = \sqrt{\frac{kT}{2mR^2}} = \sqrt{\frac{(1.38 \times 10^{-23} \text{ J/K})(300 \text{ K})}{2(15.995 \text{ u})(1.6605 \times 10^{-27} \text{ kg/u})(0.0605 \times 10^{-9} \text{ m})^2}} = 4.61 \times 10^{12} \text{ rad/s}$$

# Chapter 2

This chapter presents an introduction to the special theory of relativity. It is written assuming that students have not yet seen a full presentation of then topic, even though they might have seen selected bits in their introductory courses.

I have chosen not to introduce the speed parameter $\beta = v/c$ and the Lorentz factor $\gamma = (1 - \beta^2)^{-1/2}$, as I have found from practical experience in teaching the subject that while they may render equations more compact they also can make an intimidating subject more obscure. Students seem more comfortable with equations in which the velocities appear explicitly.

For similar reasons I have also chosen not to rely on spacetime (Minkowski) diagrams for the presentation of the space and time aspects of relativity, although in this edition I give a short introduction to their use in analyzing the twin paradox, where they do serve to enhance the presentation.

Any presentation of special relativity offers the instructor an opportunity to dwell on elucidating the new ways of thinking about space and time engendered by the theory. Some references to the resulting logical paradoxes are listed below. In terms of applicability, however, the remainder of the textbook relies more heavily on the more straightforward applications of relativistic dynamics. The Lorentz transformation, for example, does not reappear beyond this chapter, nor does reference to clock synchronization. Relativistic time dilation and Doppler shift do appear occasionally. Approximately 2/3 of this chapter deals with aspects of space and time, while only 1/3 deals with the more applicable issues of relativistic mass, momentum, and energy. In terms of division of class time, I try to divide the two topics more like 50/50, being especially careful to make sure that students understand how to apply momentum and energy conservation to situations involving high-speed motion.

## Supplemental Materials

Time dilation:
http://faraday.physics.utoronto.ca/PVB/Harrison/SpecRel/Flash/TimeDilation.html
http://galileoandeinstein.physics.virginia.edu/more_stuff/flashlets/lightclock.swf
*Physlet Quantum Physics*, Section 2.4

Length contraction:
http://faraday.physics.utoronto.ca/PVB/Harrison/SpecRel/Flash/LengthContract.html
http://science.sbcc.edu/physics/flash/relativity/LengthContraction.html

Simultaneity:
http://faraday.physics.utoronto.ca/GeneralInterest/Harrison/SpecRel/Flash/Simultaneity.html
http://science.sbcc.edu/physics/flash/relativity/Simultaniety.html

Twin paradox:
http://webphysics.davidson.edu/physletprob/ch10_modern/default.html
*Physlet Quantum Physics,* Sections 2.8 and 3.6

Other relativity paradoxes:
The best collection I know is that of Taylor and Wheeler's *Spacetime Physics* (2<sup>nd</sup> edition, 1992). See especially space war (pp. 79-80), the rising manhole (p. 116), the pole and barn paradox (p. 166), and the detonator paradox (pp. 185-186).

## Suggestions for Additional Reading

Special relativity has perhaps been the subject of more books for the nontechnical reader than any other area of science:
L. Barnett, *The Universe and Dr. Einstein* (Time Inc., 1962).
G. Gamow, *Mr. Tompkins in Paperback* (Cambridge University Press, 1967).
L. Marder, *Time and the Space Traveler* (University of Pennsylvania Press, 1971).
N. D. Mermin, *It's About Time: Understanding Einstein's Relativity* (Princeton
   University Press, 2009).
B. Russell, *The ABC of Relativity* (New American Library, 1958).
L. Sartori, *Understanding Relativity: A Simplified Approach to Einstein's Theories*
   (University of California Press, 1996).
J. T. Schwartz, *Relativity in Illustrations* (New York University Press, 1962).
R. Wolfson, *Simply Einstein: Relativity Demystified* (Norton, 2003).

Gamow's book takes us on a fanciful journey to a world where $c$ is so small that effects of special relativity are commonplace. Other introductions to relativity, more complete mathematically but not particularly more difficult than the present level, are the following:
P. French, *Special Relativity* (Norton, 1968).
H. C. Ohanian, *Special Relativity: A Modern Introduction* (Physics Curriculum and
   Instruction, 2001)
R. Resnick, *Introduction to Special Relativity* (Wiley, 1968).
R. Resnick and D. Halliday, *Basic Concepts in Relativity* (Macmillan, 1992).

For discussions of the appearance of objects traveling near the speed of light, see:
V. T. Weisskopf, "The Visual Appearance of Rapidly Moving Objects," *Physics Today,*
   September 1960.
I. Peterson, "Space-Time Odyssey," *Science News* **137**, 222 (April 14, 1990).

Some other useful works are:
L. B. Okun, "The Concept of Mass," *Physics Today,* June 1989, p. 31.
C. Swartz, "Reference Frames and Relativity," *The Physics Teacher,* September 1989, p. 437.
R. Baierlein, "Teaching $E = mc^2$," *The Physics Teacher,* March 1991, p. 170.

Okun's article explores the history of the "relativistic mass" concept and the connection between mass and rest energy. The article by Swartz gives some mostly classical descriptions of inertial and noninertial reference frames. Bayerlein's article discusses some of the common misconceptions about mass and energy in special relativity.

Finally, a unique and delightful exploration of special relativity; elegant and witty, with all of the relativity paradoxes you could want, carefully explained and diagrammed, with many worked examples:
E. F. Taylor and J. A. Wheeler, *Spacetime Physics,* 2nd ed. (Freeman, 1992).


## Materials for Active Engagement in the Classroom

### A. Reading Quizzes

1. Which of the following is <u>not</u> a consequence of the postulates of special relativity?
    (1) Clocks in motion appear to run slow.
    (2) Objects appear shortened in their direction of motion.
    (3) The velocity addition law allows relative velocities greater than the speed of
        light under certain circumstances.
    (4) The Doppler change in frequency does not distinguish between motion of the
        source and motion of the observer.

2. If two observers are in relative motion (one moves relative to the other) with constant relative velocity, in which of the following measurements would they obtain identical values?
    (1)  The velocity of an electron.　　(2)  The speed of a light beam.
    (3)  The ticking rate of a clock.　　(4)  The volume of a box.

3. Consider 2 observers moving toward each other at high speed.  One fires a light beam toward the other at speed $c$.  What speed $v$ does the second observer measure for the light beam?
    (1) $v > c$　　　　(2) $v = c$　　　　(3) $v < c$
    (4) Depends on magnitude and direction of relative speed of observers.

4. Observer O fires a particle at velocity $v$ in the positive $y$ direction.  Observer O′, who is moving relative to O with velocity $u$ in the $x$ direction, measures the $y$ component of the velocity of the same particle and obtains $v'$.  How does the $y$ component measured by O′ compare with the $y$ component measured by O?
    (1) $v' > v$　　(2) $v' = v$　　(3) $v' < v$　　(4) $v' = 0$

5. Two clocks in the reference frame of observer 1 are exactly synchronized.  For other observers in motion relative to observer 1, the clocks are:
    (1) synchronized for all observers.
    (2) not synchronized, but all observers will agree which of the two clocks is ahead.
    (3) not synchronized, but different observers may not agree which of the clocks is
        ahead.
    (4) either synchronized or not synchronized, depending on the locations of the
        observers.

6. In Tom's frame of reference, two events A and B take place at different locations along the *x* axis  but are observed by Tom to be simultaneous.  Which of the following statements is true?  (Consider observer motion along the *x* axis only.)
    (1) No observers moving relative to Tom will find A and B to be simultaneous, but some may see A before B and others B before A.
    (2) No observers moving relative to Tom will find A and B to be simultaneous, but they all will observe events A and B in the same order.
    (3) All observers moving relative to Tom will also perceive A and B to be simultaneous.
    (4) Some observers moving relative to Tom will find A and B to be simultaneous, while others will not.

7. The quantity $mc^2$ represents:
    (1) the kinetic energy of a particle moving at speed *c*.
    (2) the energy of a particle of mass *m* at rest.
    (3) the total relativistic energy of a particle of mass *m* moving at speed *c*.
    (4) the maximum possible energy of a moving particle of mass *m*.

8. Which one of the following statements is true?
    (1) The laws of conservation of energy and momentum are not valid in special relativity.
    (2) The laws of conservation of energy and momentum are valid in special relativity only if we use definitions of energy and momentum that differ from those of classical physics.
    (3) According to special relativity, particles have energy only if they are in motion.
    (4) $mc^2$ represents the energy of a  particle moving at speed *c*.

**Answers**      1. 3     2. 2     3. 2     4. 3     5. 3     6. 1     7. 2     8. 2

**B.  Conceptual and Discussion Questions**

1. Rockets A and C move with identical speeds in opposite directions relative to B, who is at rest in this frame of reference.  A, B, and C all carry identical clocks.



According to A:
    (1) B's clock and C's clock run at identical slow rates.
    (2) B's clock runs fast and C's clock runs slow.
    (3) B's clock runs slow and C's clock runs even slower.
    (4) B's clock runs fast and C's clock runs even faster.
    (5) B's clock runs slow and C's clock runs fast.

2. Rockets A and C move with identical speeds $v = 0.8c$ in opposite directions relative to B, who is at rest in this frame of reference. A stick of length $L_0$ carried by A has length $0.6L_0$ according to B. What is the length of the stick according to C?



    (1) $L_0$     (2) $0.6L_0$     (3) $0.36L_0$     (4) $0.22L_0$

3. Three identical triplets Larry, Moe, and Curly are testing the predictions of special relativity. Larry and Moe set out on round-trip journeys from Earth to distant stars. Larry's star is 12 light-years from Earth (as measured in the Earth reference frame), and he travels the round trip at a speed of $0.6c$. Moe's star is 16 light-years from Earth (also measured in the Earth frame), and he travels the round trip at a speed of $0.8c$. Both journeys thus take a total of 40 years, as measured by Curly who stays home on Earth. When Larry and Moe return, how do the ages of the triplets compare?
    (1) Larry = Moe > Curly     (2) Moe < Larry < Curly     (3) Larry < Moe < Curly
    (4) Larry = Moe = Curly     (5) Moe > Larry > Curly     (6) Larry = Moe < Curly

4. A star (assumed to be at rest relative to the Earth) is 100 light-years from Earth. (A light-year is the distance light travels in one year.) An astronaut sets out from Earth on a journey to the star at a constant speed of $0.98c$. (Note: At $v = 0.98c$, $\sqrt{1 - v^2/c^2} = 0.20$)
(a) How long does it take for a light signal from Earth to reach the star, according to an observer on Earth?
    (1) 100 y   (2) 98 y   (3) 102 y   (4) 20 y
(b) How long does it take for the astronaut to travel from Earth to the star, according to an observer on Earth?
    (1) 100 y   (2) 98 y   (3) 102 y   (4) 20 y
(c) According to the astronaut, what is the distance from Earth to the star?
    (1) 100 l.y.  (2) 102 l.y.  (3) 20 l.y.  (4) 98 l.y.
(d) According to the astronaut, how long does it take for the astronaut to travel from Earth to the star?
    (1) 100 y   (2) 102 y   (3) 20 y   (4) 20.4 y
(e) Light takes 100 years to travel from Earth to the star, but the astronaut makes the trip in 20.4 y. Does that mean that the astronaut travels faster than light?
    (1) Yes     (2) No     (3) Maybe

5. Two clocks, equidistant from $O$ and at rest in the reference frame of $O$, start running when they receive a flash of light from the light source midway between them. According to $O$, the two clocks are synchronized (they start at the same time). According to $O'$, who is moving with velocity $u$ relative to $O$, clock 2 starts ahead of clock 1 by an amount $\Delta t'$.
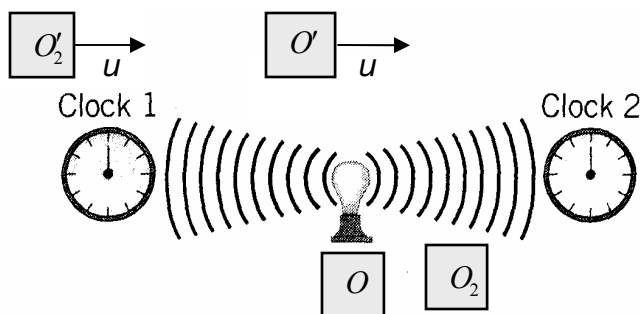
(a) Suppose there is a second observer $O_2$ at rest with respect to $O$ at a location midway between the light source and clock 2. Will $O_2$ conclude that the two clocks are synchronized?

    (1) Yes       (2) No       (3) Depends on location of $O_2$.

(b) Suppose there is a second observer $O_2'$ moving with the same speed $u$ as $O'$. At the instant shown, $O_2'$ is slightly to the left of clock 1. What will $O_2'$ conclude about the synchronization of the two clocks?

    (1) Clock 1 starts ahead of clock 2.
    (2) Clock 2 starts ahead of clock 1 by a time that is smaller than $\Delta t'$.
    (3) Clock 2 starts ahead of clock 1 by a time that is larger than $\Delta t'$.
    (4) Clock 2 starts ahead of clock 1 by a time that is equal to $\Delta t'$.

6. In a certain collision process, particles A and B collide, and after the collision particles C and D appear (C and D are different from A and B). Which quantities are conserved in this collision?

    (1) only linear momentum       (2) only total relativistic energy
    (3) only mass and linear momentum  (4) only linear momentum and kinetic energy
    (5) only mass and kinetic energy
    (6) only linear momentum and total relativistic energy
    (7) only linear momentum, kinetic energy, and total relativistic energy
    (8) linear momentum, kinetic energy, total relativistic energy, and mass

7. Two particles each of mass $m$ are moving at speed $v = 0.866c$ directly toward one another. After the head-on collision, all that remains is a new particle of mass $M$. What is the mass of this new particle? (Note: At $v = 0.866c$, $\sqrt{1 - v^2/c^2} = 0.50$)

    (1) $M = 2m$    (2) $M = 4m$    (3) $M = m$    (4) $M = 1.5m$    (5) None of these

**Answers**    1. 3   2. 4   3. 2   4. 1,3,3,4,2   5. 1,4   6. 6   7. 2

**Sample Exam Questions**

**A. Multiple Choice**

1. A certain particle at rest lives for 1.25 ns. When the particle moves through the laboratory at a speed of $0.91c$, what is its lifetime according to an observer in the laboratory?
   (a) 0.52 ns      (b) 3.01 ns      (c) 1.25 ns      (d) 7.27 ns

2. Two electrons, each with a kinetic energy of 2.52 MeV, collide head-on to produce a new particle. What is the rest energy of this new particle?
   (a) Zero      (b) 5.04 MeV      (c) 6.06 MeV      (d) 9.54 MeV

3. A newly created particle is moving through the laboratory at a speed of $0.765c$. It is observed to live for a time of 0.231 µs before decaying. What would be the lifetime of this particle according to someone who is moving along with the particle at a speed of $0.765c$?
   (a) 0.358 µs      (b) 0.149 µs      (c) 0.096 µs      (d) 0.557 µs

4. Tom fires a laser beam in the $y$ direction of his coordinate system. Mary is moving relative to Tom in the $x$ direction with a speed of $0.65c$. According to Mary, what is the $y$ component of the speed of Tom's laser beam?
   (a) $c$      (b) $0.89c$      (c) $0.76c$      (d) $0.35c$

5. In Albert's frame of reference, there is a stick of length $L_A$ at rest along the $x$ axis. Betty is traveling along the $x$ axis in either the positive or negative direction. In Betty's frame of reference, the length of the stick is:
   (a) always equal to $L_A$      (b) always greater than $L_A$      (c) always less than $L_A$
   (d) either greater than $L_A$ or less than $L_A$, depending on the direction of Betty's motion

6. Sitting in a chair in his laboratory, Albert observes a particle to be created at one instant moving at a speed of $0.65c$ and to decay after a time interval of 5.75 ns. Betty is moving along with the particle at a speed of $0.65c$. What is the time between the creation and decay of the particle according to Betty?
   (a) 2.43 ns      (b) 4.37 ns      (c) 5.75 ns      (d) 7.57 ns      (e) 9.64 ns

7. What is the momentum of a proton that has a kinetic energy of 750 MeV?
   (a) 750 MeV/$c$      (b) 1186 MeV/$c$      (c) 1404 MeV/$c$      (d) 1688 MeV/$c$

8. A certain particle at rest has a lifetime of 2.52 µs. What must be the speed of the particle for its lifetime to be observed to be 8.34 µs?
   (a) $0.953c$      (b) $0.302c$      (c) $0.913c$      (d) $0.985c$      (e) None of these

9. A particle moving through the laboratory at a speed of $v = 0.878c$ is observed to have a lifetime of 2.43 ns. If that particle had been produced at rest in the laboratory, what would its lifetime be?

(a) 2.43 ns    (b) 5.08 ns    (c) 1.16 ns    (d) 1.89 ns    (e) 3.79 ns

10. An unstable particle moving through the laboratory leaves a track of length 3.52 mm. The particle is moving at a speed of $0.943c$. How long would the particle's track appear to someone moving with the particle?
    (a) 1.17 mm    (b) 10.6 mm    (c) 3.52 mm    (d) 0.390 mm    (e) None of these

11. Tom observes a blinking light bulb that is at rest in his reference frame. Mary is moving relative to Tom at a speed of $0.735c$. According to Mary, the light blinks on for a time interval of 5.25 ms. What is the blinking interval according to Tom?
    (a) 7.74 ms    (b) 1.48 ms    (c) 3.56 ms    (d) 10.76 ms    (e) 5.25 ms

12. A distant star is 4.8 light-years (l.y.) from Earth, according to observers on Earth. An astronaut will travel to the star in a spaceship at a speed of $0.925c$. During the voyage, what distance does the astronaut measure between the Earth and the star?
    (a) 4.8 l.y.    (b) 12.6 l.y.    (c) 4.4 l.y.    (d) 1.8 l.y.    (e) 7.6 l.y.

13. An astronaut was told on Earth in 1993 that she had exactly 15 years to live. Starting in 1993 she made a journey at a speed of $0.80c$ to a distant star and back. What is the latest New Years Day she will be able to celebrate on Earth?
    (a) 2002   (b) 2008   (c) 2011   (d) 2018   (e) 2032

14. A particle of mass $m$ moving with speed $v$ collides with a particle of mass $2m$ at rest. The particles merge to form only a new particle of mass $M$ that moves with speed $V$. How is $M$ related to $m$?
    (a) $M < 3m$              (b) $M = 3m$              (c) $M > 3m$

15. A particle of mass $M$ at rest decays into two identical particles each of mass $m = 0.100M$ that travel in opposite directions. What is the speed of these particles?
    (a) $0.98c$        (b) $0.96c$        (c) $0.50c$        (d) $0.32c$

16. A certain particle has a proper lifetime of $1.00 \times 10^{-8}$ s. It is moving through the laboratory at a speed of $0.85c$. What distance does the particle travel in the laboratory?
    (a) 2.55 m        (b) 4.84 m        (c) 1.34 m        (d) 9.19 m

17. Two particles of the same mass $m$ and moving at the same speed $v$ collide head-on and combine to produce only a new particle of mass $M$. Which of the following is correct?
    (a) $M = 2m$              (b) $M < 2m$              (c) $M > 2m$

18. Two particles each of mass $m$ are each moving at a speed of $0.707c$ directly toward one another. After the head-on collision, all that remains is a new particle of mass $M$. What is the mass of this new particle?
    (a) $0.5m$    (b) $1.0m$    (c) $2.0m$    (d) $2.8m$    (e) $4.0m$

**Answers**    1. b    2. c    3. b    4. c    5. c    6. b    7. c    8. a    9. c
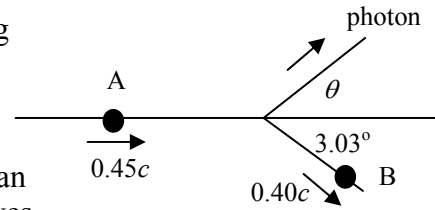         10. a    11. c    12. d    13. d    14. c    15. a    16. b    17. c    18. d


## B.  Conceptual

1.  A particle of mass $m$ is moving at a speed of $v = 0.80c$. It collides with and merges with another particle of the same mass $m$ that is initially at rest.  Is the mass of the resulting combined particle *greater than, less than, or equal to* $2m$?  EXPLAIN YOUR ANSWER.

2.  A particle of mass $M$ moving with velocity $v$ decays into two photons of energies $E_1$ and $E_2$.  Is the rest energy of the original particle *equal to $E_1 + E_2$, less than $E_1 + E_2$, or greater than $E_1 + E_2$*?  EXPLAIN YOUR ANSWER.

3.  Particle $X_1$ of mass $m_1$ is moving with speed $v_1 > 0.5c$ and kinetic energy $K_1$.  It collides with particle $X_2$ of mass $m_2$ that is initially at rest.  The collision produces ONLY a new particle $X_3$ of mass $m_3$ and kinetic energy $K_3$ (that is, $X_1 + X_2 \rightarrow X_3$).  Is $m_3$ *greater than, less than, or equal to* the sum of $m_1 + m_2$?  EXPLAIN YOUR ANSWER.

4.  Two spaceships A and B are approaching a space station from opposite directions. An observer on the station reports that both ships are approaching the station at the same speed $v$.  According to classical physics, each ship would see the other moving at a speed of $2v$.  According to special relativity, does each ship sees the other moving at a speed that is *greater than $2v$, less than $2v$, or equal to $2v$?*  EXPLAIN YOUR ANSWER.

**Answers**    1. greater than    2. less than    3. greater than    4. less than


## C.  Problems

1.  A photon of energy 1.52 MeV collides with and scatters from an electron that is initially at rest.  After the collision, the electron is observed to be moving with a speed of $0.937c$ at an angle of 64.1° relative to its original direction.
    (a) Find the energy of the scattered photon.
    (b) Find the direction of the scattered electron.

2.  A particle of rest energy 547 MeV is moving in the $x$ direction with a speed of $0.624c$. It decays into 2 new particles, each of rest energy 106 MeV.  One of the decay particles has a kinetic energy of 301 MeV and is moving at an angle of $38^\circ$ relative to the $x$ axis.
    (a) What is the kinetic energy of the second decay particle?
    (b) What is the direction of the second decay particle relative to the $x$ axis?

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3. Particle A has a rest energy of 1192 MeV and is moving through the laboratory in the positive $x$ direction with a speed of $0.45c$. It decays into particle B (rest energy = 1116 MeV) and a photon; particle A disappears in the decay process. Particle B moves at a speed of $0.40c$ at an angle of $3.03°$ with the positive $x$ axis. The photon moves in a direction at an angle $\theta$ with the positive $x$ axis.
   (a) Find the energy of the photon.
   (b) Find the angle $\theta$.

4. A star is at rest relative to the Earth and at a distance of 1500 light-years. An astronaut wishes to travel from Earth to the star and age no more than 30 years during the entire round-trip journey.
   (a) Assuming that the journey is made at constant speed and that the acceleration and deceleration intervals are very short compared with the rest of the journey, what speed is necessary for the trip?
   (b) According to the astronaut, what is the distance from Earth to the star?
   (c) According to someone on Earth, how long does it take the astronaut to make the round trip?
   (d) It takes light 1500 years to travel from Earth to the star, but the astronaut makes the trip in 15 years. Does this mean that the astronaut travels faster than light? Explain your answer.

5. A particle of mass $M$ is moving in the positive $x$ direction with speed $v$. It spontaneously decays into 2 photons, with the original particle disappearing in the process. One photon has energy 233 MeV and moves in the positive $x$ direction, and the other photon has energy 21 MeV and moves in the negative $x$ direction.
   (a) What is the total relativistic energy of the particle before its decay?
   (b) What is the momentum of the particle before its decay?
   (c) Find the mass $M$ of the particle, in units of MeV/$c^2$.
   (d) Find the original speed of the particle, expressed as a fraction of the speed of light.

6. In your laboratory, you observe particle A of mass 498 MeV/$c^2$ to be moving in the positive $x$ direction with a speed of $0.462c$. It decays into 2 particles B and C, each of mass 140 MeV/$c^2$. Particle B moves in the negative $x$ direction with a speed of $0.591c$.
   (a) Find the relativistic total energy of each of the three particles.
   (b) Find the velocity (magnitude and direction) of particle C.
   (c) Your laboratory supervisor is watching this experiment from a spaceship that is moving in the positive $x$ direction with a speed of $0.635c$. What values would your supervisor measure for the velocities of particles B and C?

7. The pi meson is a particle that has a rest energy of 135 MeV. It decays into two gamma-ray photons and no other particles. (The pi meson disappears after the decay.) Suppose a pi meson is moving through the laboratory in the positive x direction at a speed of $v = 0.90c$. One of the decay photons moves in the positive $x$ direction and

the other in the negative $x$ direction.

(a) What are the applicable conservation laws in this problem? Set up the equations for each applicable conservation law using the numerical values for this problem. THE ONLY UNKNOWNS IN YOUR EQUATIONS SHOULD BE THE ENERGIES OF THE TWO PHOTONS. Other than the unknown energies, all numerical factors in each equation should be evaluated. You don't need to solve the equations, just set them up.

(b) One of the photons has energy 15.5 MeV. Find the energy of the other photon, and show how all applicable conservation laws are satisfied.

(c) What are the speeds of the two photons in the rest frame of the pi meson and in the laboratory frame of reference? Explain your answer.

8. A pi meson (rest energy = 140 MeV) is moving through the laboratory with a kinetic energy of 405 MeV.

(a) Expressed as a fraction of the speed of light, what is the speed of the pi meson?

(b) At this speed, how long a track will the pi meson leave in the laboratory during its lifetime? The lifetime of a pi meson at rest in the laboratory is $1.0 \times 10^{-16}$ s.

9. A particle of rest energy 266.0 MeV is moving, according to a laboratory observer, in the $x$ direction with a speed of $0.720c$. It decays into 2 photons. Photon 1 has an energy of 260.6 MeV and travels at an angle of $26.2°$ with the direction of the motion of the original particle.

(a) Find the energy and direction of photon 2 in this frame of reference.

(b) A second observer is moving at a speed of $0.720c$, so that the original particle appears to be at rest. According to this observer, what are the energy and direction of travel of photon 1?

10. A pi meson ($m = 135$ MeV/$c^2$) moving through the laboratory at a speed of $v = 0.998c$ decays into two gamma-ray photons. The two photons have equal energies $E_\gamma$ and move at equal angles $\theta$ on opposite sides of the direction of motion of the original pi meson. Find $E_\gamma$ and $\theta$.

11. Space pilot Jim measures the length of his space ship to be 2450 m. The ship drifts at a constant velocity of $0.740c$ past a space platform from which Mary (at rest on the platform) observes its passage.

(a) What is the length of the space ship according to Mary?

(b) According to Mary, what is the time interval between the bow (front) of the ship passing her and the stern (rear) of the ship passing her?

(c) According to Jim, what is the time interval between the bow (front) of the ship passing Mary and the stern (rear) of the ship passing her?

**Answers**

1. (a) 0.568 MeV    (b) -21.9°
2. (a) 187 MeV   (b) 62.3°
3. (a)117 MeV   (b) 12.7°
4. (a) 0.99995$c$   (b) 15 l.y.   (c) 3000.15 y   (d) no
5. (a) 254 MeV    (b) 212 MeV/$c$   (c) 140 MeV/$c^2$   (d) 0.834$c$

6. (a) 561.5 MeV, 173.6 MeV, 388.0 MeV    (b) +0.933$c$    (c) -0.891$c$, +0.731$c$
7. (a) $E_1 + E_2 = 310$ MeV, $E_1 - E_2 = 279$ MeV    (b) 294.5 MeV    (c) $c$
8. (a) 0.966$c$    (b) 0.113 $\mu$m
9: (a) 122.7 MeV, -69.7°    (b) 133.0 MeV, 60.0°
10. 1069 MeV, 3.6°
11. (a) 1650 m    (b) 7.4 $\mu$s    (c) 11 $\mu$s

**Problem Solutions**

1.  Your air speed in still air is (750 km)/(3.14 h) = 238.8 km/h. With the nose of the plane pointed 22° west of north, you would be traveling at this speed in that direction if there were no wind. With the wind blowing, you are actually traveling due north at an effective speed of (750 km)/(4.32 h) = 173.6 km/h. The wind must therefore have a north-south component of (238.8 km/h)(cos 22°) − 173.6 km/h = 47.8 km/h (toward the south) and an east-west component of (238.8 km/h)(sin 22°) = 89.5 km/h (toward the east). The wind speed is thus

$$v = \sqrt{(47.8 \text{ km/h})^2 + (89.5 \text{ km/h})^2} = 101 \text{ km/h}$$

in a direction that makes an angle of

$$\theta = \tan^{-1}\frac{89.5 \text{ km/h}}{47.8 \text{ km/h}} = 62° \text{ east of south}$$

2.  (a)  $\dfrac{95 \text{ m}}{0.53 \text{ m/s}} = 179 \text{ s}$

    (b)  $\dfrac{95 \text{ m}}{1.24 \text{ m/s} + 0.53 \text{ m/s}} = 54 \text{ s}$

    (c)  $\dfrac{95 \text{ m}}{2.48 \text{ m/s} - 0.53 \text{ m/s}} = 49 \text{ s}$

3.  $$\Delta t = t_{\text{up}} + t_{\text{down}} - 2t_{\text{across}} = \frac{2L}{c}\left[\frac{1}{1-u^2/c^2} - \frac{1}{\sqrt{1-u^2/c^2}}\right]$$

    Assuming $u \ll c$,

    $$\frac{1}{1-u^2/c^2} \cong 1 + \frac{u^2}{c^2} \qquad \text{and} \qquad \frac{1}{\sqrt{1-u^2/c^2}} \cong 1 + \frac{1}{2}\frac{u^2}{c^2}$$

    $$\Delta t \cong \frac{2L}{c}\left[1 + \frac{u^2}{c^2} - \left(1 + \frac{1}{2}\frac{u^2}{c^2}\right)\right] = \frac{Lu^2}{c^3}$$

    $$u = \sqrt{\frac{c^3 \Delta t}{L}} = \sqrt{\frac{(3\times10^8 \text{ m/s})^3 (2\times10^{-15} \text{ s})}{11 \text{ m}}} = 7\times10^4 \text{ m/s}$$

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4.    (a)    $u = 100$ km/h $= 28$ m/s $\ll c$

$$\sqrt{1-u^2/c^2} \cong 1-\frac{1}{2}\frac{u^2}{c^2} = 1-\frac{1}{2}\frac{(28 \text{ m/s})^2}{(3\times10^8 \text{ m/s})^2} = 1-4.3\times10^{-15}$$

$$L = L_0\sqrt{1-u^2/c^2} = L_0(1-4.3\times10^{-15})$$

$$L_0 - L = (4.3\times10^{-15})L_0 = (4.3\times10^{-15})(5\times10^6 \text{ m}) = 2.1\times10^{-8} \text{ m}$$

This is less than the wavelength of light.

(b)    $$\Delta t = \frac{\Delta t_0}{\sqrt{1-u^2/c^2}} \cong \frac{\Delta t_0}{1-4.3\times10^{-15}} \cong \Delta t_0(1+4.3\times10^{-15})$$

$$\Delta t - \Delta t_0 = (4.3\times10^{-15})\Delta t_0 = (4.3\times10^{-15})(50 \text{ h})(3600 \text{ s/h}) = 7.7\times10^{-10} \text{ s}$$

5.    With $L = \frac{1}{2}L_0$, the length contraction formula gives $\frac{1}{2}L_0 = L_0\sqrt{1-u^2/c^2}$, so

$$u = \sqrt{3/4}\,c = 2.6\times10^8 \text{ m/s}$$

6.    The astronaut must travel 400 light-years at a speed close to the speed of light and must age only 10 years. To an Earth-bound observer, the trip takes about $\Delta t = 400$ years, but this is a dilated time interval; in the astronaut's frame of reference, the elapsed time is the proper time interval $\Delta t_0$ of 10 years. Thus, with $\Delta t = \Delta t_0/\sqrt{1-u^2/c^2}$,

$$400 \text{ years} = \frac{10 \text{ years}}{\sqrt{1-u^2/c^2}} \qquad \text{or} \qquad 1-\frac{u^2}{c^2} = \left(\frac{10}{400}\right)^2$$

$$u = \sqrt{1-(10/400)^2}\,c = 0.9997c$$

7.    (a)    $$\Delta t = \frac{\Delta t_0}{\sqrt{1-u^2/c^2}} = \frac{100.0 \text{ ns}}{\sqrt{1-(0.960)^2}} = 357.1 \text{ ns}$$

(b)    $$d = v\Delta t = 0.960(3.00\times10^8 \text{ m/s})(357.1\times10^{-9} \text{ s}) = 103 \text{ m}$$

(c)    $$d_0 = v\Delta t_0 = 0.960(3.00\times10^8 \text{ m/s})(100.0\times10^{-9} \text{ s}) = 28.8 \text{ m}$$

8.    In the laboratory reference frame, the lifetime is

$$\Delta t = \frac{d}{u} = \frac{1.25 \text{ mm}}{0.995(3.00 \times 10^8 \text{ m/s})} = 0.418 \times 10^{-11} \text{ s}$$

$$\Delta t_0 = \Delta t \sqrt{1 - u^2 / c^2} = (0.418 \times 10^{-11} \text{ s})\sqrt{1 - (0.995)^2} = 4.17 \times 10^{-13} \text{ s}$$

9.  From Equation 2.15, $\Delta t_1 = L /(v - u)$, and from Equation 2.16, $\Delta t_2 = L /(c + u)$.

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{L}{v - u} + \frac{L}{c + u} = \frac{\Delta t_0}{\sqrt{1 - u^2 / c^2}}$$

With $\Delta t_0 = L_0 / v' + L_0 / c$ and $L = L_0 \sqrt{1 - u^2 / c^2}$, this becomes

$$\frac{1}{v - u} + \frac{1}{c + u} = \frac{1}{1 - u^2 / c^2}\left(\frac{1}{v'} + \frac{1}{c}\right)$$

Solving for $v$, we obtain

$$v = \frac{v' + u}{1 + v'u / c^2}$$

10.  Let ship $A$ represent observer $O$, and let observer $O'$ be on Earth. Then $v' = 0.851c$ and $u = -0.753c$, and so

$$v = \frac{v' + u}{1 + v'u / c^2} = \frac{0.851c + 0.753c}{1 + (0.851)(0.753)} = 0.978c$$

If now ship $B$ represents observer $O$, then $v' = -0.753c$ and $u = -0.851c$.

$$v = \frac{v' + u}{1 + v'u / c^2} = \frac{-0.753c - 0.851c}{1 + (-0.753)(-0.851)} = -0.978c$$

11.  Let $O'$ be the observer on Earth, and let $O$ be the observer on ship $B$. Then $v' = 0.826c$ and $u = -0.635c$.

$$v = \frac{v' + u}{1 + v'u / c^2} = \frac{0.826c - 0.635c}{1 + (0.826)(-0.635)} = 0.402c$$

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12. (a) With $f' = f\sqrt{(1-u/c)/(1+u/c)}$ and $\lambda = c/f$, we obtain

$$\lambda' = \lambda\sqrt{\frac{1+u/c}{1-u/c}} \quad \text{or} \quad 366 \text{ nm} = 122 \text{ nm}\sqrt{\frac{1+u/c}{1-u/c}}$$

Solving, we get $u/c = 0.800$ or $u = 2.40 \times 10^8$ m/s.

(b) $\lambda' = \lambda\sqrt{\frac{1-u/c}{1+u/c}} = 122 \text{ nm}\sqrt{\frac{1-0.800}{1+0.800}} = 40.7$ nm

13. With $f' = f\sqrt{(1-u/c)/(1+u/c)}$ and $\lambda = c/f$, we obtain

$$\frac{1-u/c}{1+u/c} = \left(\frac{f'}{f}\right)^2 = \left(\frac{\lambda}{\lambda'}\right)^2 = \left(\frac{650 \text{ nm}}{550 \text{ nm}}\right)^2 = 1.397$$

Solving, $u/c = 0.166$ or $u = 5.0 \times 10^7$ m/s.

14.
$$dx' = \frac{dx - u\,dt}{\sqrt{1-u^2/c^2}} \quad \text{and} \quad dt' = \frac{dt - u\,dx/c^2}{\sqrt{1-u^2/c^2}}$$

$$v'_x = \frac{dx'}{dt'} = \frac{dx - u\,dt}{dt - u\,dx/c^2} = \frac{dx/dt - u}{1 - u(dx/dt)/c^2} = \frac{v_x - u}{1 - uv_x/c^2}$$

With $dz' = dz$, we obtain

$$v'_z = \frac{dz'}{dt'} = \frac{dz}{(dt - u\,dx/c^2)/\sqrt{1-u^2/c^2}} = \frac{(dz/dt)\sqrt{1-u^2/c^2}}{1-uv_x/c^2} = \frac{v_z\sqrt{1-u^2/c^2}}{1-uv_x/c^2}$$

15. For the light beam, observer $O$ measures $v_x = 0$, $v_y = c$. Observer $O'$ measures

$$v'_x = \frac{v_x - u}{1 - uv_x/c^2} = 0 - u = -u \quad \text{and} \quad v'_y = \frac{v_y\sqrt{1-u^2/c^2}}{1-uv_x/c^2} = c\sqrt{1-u^2/c^2}$$

According to $O'$, the speed of the light beam is

$$v' = \sqrt{(v'_x)^2 + (v'_y)^2} = \sqrt{u^2 + c^2(1-u^2/c^2)} = c$$

16.  $O$ measures times $t_1$ and $t_2$ for the beginning and end of the interval, while $O'$ measures $t_1'$ and $t_2'$. Using Equation 2.23d,

$$t_1' = \frac{t_1 - ux/c^2}{\sqrt{1 - u^2/c^2}} \quad \text{and} \quad t_2' = \frac{t_2 - ux/c^2}{\sqrt{1 - u^2/c^2}}$$

The same coordinate $x$ appears in both expressions, because the bulb is at rest according to $O$ (so $\Delta t$ is the proper time interval). Subtracting these two equations, we obtain

$$t_2' - t_1' = \frac{t_2 - t_1}{\sqrt{1 - u^2/c^2}} \quad \text{or} \quad \Delta t' = \frac{\Delta t}{\sqrt{1 - u^2/c^2}}$$

17.  Suppose observer $O$ is moving with the $K$ meson; to this observer, the $K$ meson appears to be at rest, and so $O$ measures $v_1 = +0.828c$ and $v_2 = -0.828c$ for the two $\pi$ mesons. Observer $O'$ is moving relative to $O$ with a velocity $u = -0.486c$; in the reference frame of $O'$, observer $O$ and the $K$ meson are moving in the positive $x$ direction with a velocity of $0.486c$. We can use the Lorentz velocity transformation (Equation 2.28a) to find the velocities of the two $\pi$ mesons according to $O'$:

$$v_1' = \frac{v_1 - u}{1 - v_1 u/c^2} = \frac{+0.828c - (-0.486c)}{1 - (0.828)(-0.486)} = +0.937c$$

$$v_2' = \frac{v_2 - u}{1 - v_2 u/c^2} = \frac{-0.828c - (-0.486c)}{1 - (-0.828)(-0.486)} = -0.572c$$

18.  Imagine the rod to be the hypotenuse of a right triangle having sides $L_x$ along the $x$ axis and $L_y$ in the $y$ direction. According to $O'$, the length $L_x$ is shortened by the length contraction, but the length $L_y$ is unaffected because it is perpendicular to the direction of motion. For $O$, $L_y = L_x \tan 31°$, while for $O'$, $L_y' = L_x' \tan 46°$ where

$L_x' = L_x \sqrt{1 - u^2/c^2}$.
Because $L_y = L_y'$, we have

$$L_x \tan 31° = L_x' \tan 46° = L_x \sqrt{1 - u^2/c^2}\, \tan 46°$$

or

$$u = c\sqrt{1 - (\tan 31°)/(\tan 46°)} = 0.648c$$

19.  (a)  Changing the coordinates in Equation 2.23d to intervals, we have

30

$$\Delta t' = \frac{\Delta t - u\,\Delta x/c^2}{\sqrt{1-u^2/c^2}} = \frac{0.465\ \mu s - (0.762)(53.4\ m)/(300\ m/\mu s)}{\sqrt{1-(0.762)^2}} = +0.508\ \mu s$$

(b) Changing coordinates to intervals in Equation 2.23a,

$$\Delta x' = \frac{\Delta x - u\,\Delta t}{\sqrt{1-u^2/c^2}} = \frac{53.4\ m - (0.762\times300\ m/\mu s)(0.465\ \mu s)}{\sqrt{1-(0.762)^2}} = -81.5\ m$$

The negative sign of $\Delta x'$ indicates that $O'$ finds the two events in inverted locations compared with $O$; for example, if $O$ finds that event 1 occurs at a smaller $x$ coordinate than event 2, then $O'$ finds that event 1 occurs at a *larger x'* coordinate than event 2. That is, $O$ sees event 1 to the left of event 2, while $O'$ sees event 1 to the right of event 2. Note than both observers find the time interval to be positive – event 2 occurs after event 1 to both observers.

20. From Equation 2.23d written in terms of intervals, for $O'$ to find $\Delta t' = 0$, it must be true that $\Delta t - (u/c^2)\Delta x = 0$. Thus

$$u = \frac{c^2\Delta t}{\Delta x} = c\frac{(300\ m/\mu s)(0.138\ \mu s - 0.124\ \mu s)}{(23.6\ m - 10.4\ m)} = +0.32c$$

21. (a) In your frame of reference on the plane, the distance between Los Angeles and Boston is contracted to $(3000\ mi)\sqrt{1-[(600\ mi/h)/(1000\ mi/h)]^2} = 2400\ mi$, and you measure the time for the trip to be $(2400\ mi)/(600\ mi/h) = 4\ h$. So your watch reads 2:00 pm.
(b) According to an observer on the ground, the trip takes $(3000\ mi)/(600\ mi/h) = 5\ h$. So the airport clock reads 3:00 pm.
(c) The return trip again takes 4 h as measured in your frame of reference and 5 h from the frame of reference of the ground. When you depart Boston, the airport clock reads 10:00 am but your watch reads 9:00 am, so when you land in Los Angeles, your watch reads 1:00 pm and the airport clock reads 3:00 pm.
   This analysis duplicates the result of the twin paradox: If your twin wearing an identical watch had remained at the Los Angeles airport, all observers would agree that you were 2 hours younger than your twin.

22. (a) On the outward journey at 0.60c, the rate at which signals are received is

$$f' = f\sqrt{\frac{1-u/c}{1+u/c}} = (1/\,year)\sqrt{\frac{1-0.60}{1+0.60}} = 0.5/\,year$$

(b) During the return journey,

$$f' = f\sqrt{\frac{1+u/c}{1-u/c}} = (1/\text{year})\sqrt{\frac{1+0.60}{1-0.60}} = 2/\text{year}$$

(c)  According to Casper, Amelia's outward journey lasts 16 years (it is 16 years before he sees her arrive at the planet), during which he receives 8 signals (0.5/year × 16 years).  Her total journey lasts 20 Earth years, so the return journey lasts 4 Earth years, during which he receives 8 signals (2/year × 4 years).  Thus Casper receives 16 signals (8 during the outward trip and 8 during the return) and concludes that his sister has aged 16 years.

23.  According to Amelia, the distance to the star is shortened to

$$L = L_0\sqrt{1-v^2/c^2} = (8.0\ \text{l-y})\sqrt{1-(0.80)^2} = 4.8\ \text{l-y}$$

and at a speed of $0.80c$ Amelia's travel time to the star is $(4.8\ \text{l-y})/(0.80c) = 6.0$ y.  The total round-trip time in Amelia's frame of reference is 12 years, so she is 8 years younger than her brother when she returns.

24.  (a)
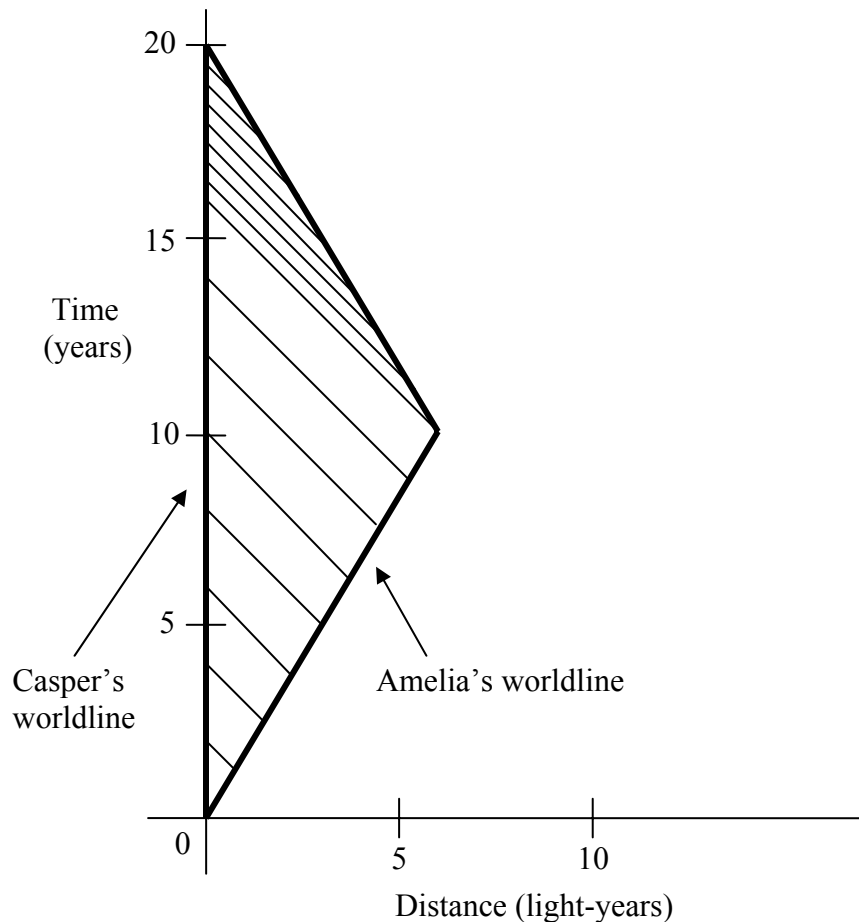
(b) 16 years

(c) 4 years

25. (a) $$K_i' = K_{1i}' + K_{2i}' = \frac{m_1 c^2}{\sqrt{1 - v_{1i}'^2 / c^2}} - m_1 c^2 + \frac{m_2 c^2}{\sqrt{1 - v_{2i}'^2 / c^2}} - m_2 c^2$$

$$= \frac{(2m)c^2}{\sqrt{1 - 0^2}} - (2m)c^2 + \frac{mc^2}{\sqrt{1 - (0.750)^2}} - mc^2 = 0.512mc^2$$

$$K_f' = K_{1f}' + K_{2f}' = \frac{m_1 c^2}{\sqrt{1 - v_{1f}'^2 / c^2}} - m_1 c^2 + \frac{m_2 c^2}{\sqrt{1 - v_{2f}'^2 / c^2}} - m_2 c^2$$

$$= \frac{(2m)c^2}{\sqrt{1 - (-0.585)^2}} - (2m)c^2 + \frac{mc^2}{\sqrt{1 - (0.294)^2}} - mc^2 = 0.512mc^2$$

(b) $$K_i = K_{1i} + K_{2i} = \frac{m_1 c^2}{\sqrt{1 - v_{1i}^2 / c^2}} - m_1 c^2 + \frac{m_2 c^2}{\sqrt{1 - v_{2i}^2 / c^2}} - m_2 c^2$$

$$= \frac{(2m)c^2}{\sqrt{1 - (0.550)^2}} - (2m)c^2 + \frac{mc^2}{\sqrt{1 - (-0.340)^2}} - mc^2 = 0.458mc^2$$

$$K_f = K_{1f} + K_{2f} = \frac{m_1 c^2}{\sqrt{1 - v_{1f}^2 / c^2}} - m_1 c^2 + \frac{m_2 c^2}{\sqrt{1 - v_{2f}^2 / c^2}} - m_2 c^2$$

$$= \frac{(2m)c^2}{\sqrt{1 - (-0.051)^2}} - (2m)c^2 + \frac{mc^2}{\sqrt{1 - (0.727)^2}} - mc^2 = 0.458mc^2$$

26. $$p = \frac{mv}{\sqrt{1 - v^2 / c^2}} = \frac{1}{c} \frac{(mc^2)(v/c)}{\sqrt{1 - v^2 / c^2}} = \frac{1}{c} \frac{(938.3 \text{ MeV})(0.756)}{\sqrt{1 - (0.756)^2}} = 1084 \text{ MeV}/c$$

$$K = \frac{mc^2}{\sqrt{1 - v^2 / c^2}} - mc^2 = \frac{938.3 \text{ MeV}}{\sqrt{1 - (0.756)^2}} - 938.3 \text{ MeV} = 495 \text{ MeV}$$

$$E = K + mc^2 = 495 \text{ MeV} + 938.3 \text{ MeV} = 1433 \text{ MeV}$$

27. $$E = K + mc^2 = 1.264 \text{ MeV} + 0.511 \text{ MeV} = 1.775 \text{ MeV}$$

Solving Equation 2.36 for $v$, we obtain

$$v = c\sqrt{1 - \left(\frac{mc^2}{E}\right)^2} = c\sqrt{1 - \left(\frac{0.511 \text{ MeV}}{1.775 \text{ MeV}}\right)^2} = 0.958c$$

28.
$$W = \int F \, dx = \int \frac{dp}{dt} \, dx = \int dp \, \frac{dx}{dt} = \int v \, dp$$

$$K = \int_0^v v \, dp = pv - \int_0^v p \, dv = \frac{mv^2}{\sqrt{1-v^2/c^2}} - \int_0^v \frac{mv}{\sqrt{1-v^2/c^2}} \, dv$$

$$= \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2\sqrt{1-v^2/c^2} - mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2$$

29. For what range of velocities is $K - \frac{1}{2}mv^2 \leq 0.01K$ ? At the upper limit of this range, where $K - \frac{1}{2}mv^2 = 0.01K$ , we have

$$0.99K = 0.99\left(\frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2\right) = \frac{1}{2}mv^2$$

With $x = v^2/c^2$, $\quad 0.99\left(\frac{1}{\sqrt{1-x}} - 1\right) = \frac{1}{2}x$ which gives $\quad \frac{1}{1-x} = \left(1 + \frac{0.5}{0.99}x\right)^2$

$$1 = (1-x)(1 + 1.0101x + 0.2551x^2) \qquad \text{or} \qquad 0.2551x^2 + 0.7550x - 0.0101 = 0$$

Solving using the quadratic formula, we find $x = 0.0133$ or -2.97. Only the positive solution is physically meaningful, so

$$v = \sqrt{0.0133}\,c = 0.115c$$

That is, for speeds smaller that $0.115c$, the classical kinetic energy is accurate to within 1%. For a different approach to that same type of calculation, see Problem 32.

30. As in Problem 29, let us now find the *lower* limit on the momentum such that

$$\sqrt{(pc)^2 + (mc^2)^2} - pc \leq 0.01\sqrt{(pc)^2 + (mc^2)^2}$$

From the lower limit, we obtain $0.99\sqrt{(pc)^2 + (mc^2)^2} = pc$ , which can be written as

$$(pc)^2 = \frac{m^2c^4}{1/(0.99)^2 - 1} \qquad \text{or} \qquad pc = 7.02mc^2$$

With $mvc/\sqrt{1-v^2/c^2} = 7.02mc^2$ , we obtain

$$\frac{v^2}{c^2} = 49.25\left(1 - \frac{v^2}{c^2}\right) \qquad \text{or} \qquad v/c = 0.990$$

Whenever $v/c \geq 0.990$, the expression $E = pc$ will be accurate to within 1%.

31. $$E^2 = \frac{(mc^2)^2}{1 - v^2/c^2} = (mc^2)^2 \left( \frac{1 - v^2/c^2 + v^2/c^2}{1 - v^2/c^2} \right) = (mc^2)^2 + \frac{m^2 c^2 v^2}{1 - v^2/c^2} = (mc^2)^2 + (pc)^2$$

$$E = \sqrt{(mc^2)^2 + (pc)^2}$$

32. With $\dfrac{1}{\sqrt{1 - v^2/c^2}} = 1 + \dfrac{1}{2}\dfrac{v^2}{c^2} + \dfrac{(-1/2)(-3/2)}{2}\left(\dfrac{v^2}{c^2}\right)^2 + \cdots$, we have

$$K = mc^2 \left( \frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right) = mc^2 \left( 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots - 1 \right) = \tfrac{1}{2} mv^2 \left( 1 + \frac{3}{4}\frac{v^2}{c^2} + \cdots \right)$$

so $K \cong \tfrac{1}{2} mv^2$ when $v \ll c$. The correction term is $3v^2/4c^2$, which has the value 0.01% when $3v^2/4c^2 = 0.0001$, or

$$v = \sqrt{0.0001(4/3)}\, c = 0.0115c$$

33. (a) With $E = 1125$ MeV and $pc = 817$ MeV, Equation 2.39 gives

$$m = \frac{1}{c^2}\sqrt{E^2 - (pc)^2} = \frac{1}{c^2}\sqrt{(1125\text{ MeV})^2 - (817\text{ MeV})^2} = 773 \text{ MeV}/c^2$$

(b) $E = \sqrt{(pc)^2 + (mc^2)^2} = \sqrt{(953\text{ MeV})^2 + (773\text{ MeV})^2} = 1227$ MeV

34.
$$K_f - K_i = (E_f - mc^2) - (E_i - mc^2) = E_f - E_i$$

$$= \frac{mc^2}{\sqrt{1 - v_f^2/c^2}} - \frac{mc^2}{\sqrt{1 - v_i^2/c^2}} = \frac{0.511\text{ MeV}}{\sqrt{1 - (0.91)^2}} - \frac{0.511\text{ MeV}}{\sqrt{1 - (0.81)^2}} = 0.361 \text{ MeV}$$

35. $$\Delta E = mc\,\Delta T = (1\text{ g})(0.40\text{ J/g}\cdot\text{K})(100\text{ K}) = 40 \text{ J}$$

$$\Delta m = \frac{\Delta E}{c^2} = \frac{40\text{ J}}{9 \times 10^{16}\text{ m}^2/\text{s}^2} = 4.4 \times 10^{-16} \text{ kg}$$

36. (a) At such low speed, the classical approximation is valid.

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}mc^2\left(\frac{v^2}{c^2}\right) = \tfrac{1}{2}(0.511\,\text{MeV})(1.00\times10^{-4})^2 = 2.56\times10^{-3}\,\text{eV}$$

(b) The relativistic expression gives

$$K = mc^2\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right) = 0.511\,\text{MeV}\left(\frac{1}{\sqrt{1-(0.01)^2}} - 1\right) = 25.6\,\text{eV}$$

For this speed, the classical expression $\tfrac{1}{2}mv^2$ also gives 25.6 MeV, so the two calculations agree to at least three significant figures. (Actually they agree to four significant figures, but not to five.)

(c) The relativistic expression gives

$$K = mc^2\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right) = 0.511\,\text{MeV}\left(\frac{1}{\sqrt{1-(0.3)^2}} - 1\right) = 24.7\,\text{keV}$$

For this speed, the classical expression gives 23.0 keV, which is incorrect by about 7%.

$$\text{(d)} \quad K = mc^2\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right) = 0.511\,\text{MeV}\left(\frac{1}{\sqrt{1-(0.999)^2}} - 1\right) = 10.9\,\text{MeV}$$

37. Because the electrons and the protons have charges of the same magnitude $e$, after acceleration through a potential difference of magnitude $\Delta V = 10.0$ million volts (a positive difference for the electron, a negative difference for the proton), each loses potential energy of $\Delta U = -e\,\Delta V = -10.0$ MeV and thus each acquires a kinetic energy of $K = +10.0$ MeV. For the electron, $E = K + mc^2 = 10.0$ MeV + 0.511 MeV = 10.5 MeV. The momentum is then

$$p = \frac{1}{c}\sqrt{E^2 - (mc^2)^2} = \frac{1}{c}\sqrt{(10.5\,\text{MeV})^2 - (0.511\,\text{MeV})^2} = 10.5\,\text{MeV}/c$$

The classical formula $K = p^2/2m$ gives

$$p = \sqrt{2mK} = \sqrt{2(0.511\,\text{MeV}/c^2)(10.0\,\text{MeV})} = 3.20\,\text{MeV}/c$$

which is far from the correct result (a discrepancy we would expect for such highly relativistic electrons). For the protons, $E = K + mc^2 = 10.0$ MeV + 938.3 MeV= 948.3 MeV, and the momentum is

$$p = \frac{1}{c}\sqrt{E^2 - (mc^2)^2} = \frac{1}{c}\sqrt{(948.3 \text{ MeV})^2 - (938.3 \text{ MeV})^2} = 137.4 \text{ MeV}/c$$

The classical formula gives

$$p = \sqrt{2mK} = \sqrt{2(938.3 \text{ MeV}/c^2)(10.0 \text{ MeV})} = 137.0 \text{ MeV}/c$$

The difference between the classical and relativistic formulas appears only in the fourth significant figure.

38. The mass of a uranium atom is about $(235 \text{ u})(1.66 \times 10^{-27} \text{ kg/u}) = 3.90 \times 10^{-25}$ kg, so 1.00 kg contains 1.00 kg/$3.90 \times 10^{-25}$ kg $= 2.56 \times 10^{24}$ atoms. The total energy released is

$$\Delta E = (200 \text{ MeV/atom})(2.56 \times 10^{24} \text{ atoms}) = 5.12 \times 10^{26} \text{ MeV}$$

and the change in mass is

$$\Delta m = \frac{\Delta E}{c^2} = \frac{(5.12 \times 10^{26} \text{ MeV})(1.602 \times 10^{-13} \text{ J/MeV})}{(2.998 \times 10^8 \text{ MeV})^2} = 9.14 \times 10^{-4} \text{ kg}$$

About one gram of matter vanishes for each kilogram that is fissioned!

39. (a)
$$E = E_\pi + E_p = \frac{m_\pi c^2}{\sqrt{1 - v^2/c^2}} + m_p c^2 = \frac{139.6 \text{ MeV}}{\sqrt{1 - (0.906)^2}} + 938.3 \text{ MeV} = 1268.1 \text{ MeV}$$

(b)
$$p = p_\pi + p_p = \frac{m_\pi v}{\sqrt{1 - v^2/c^2}} = \frac{1}{c}\frac{m_\pi c^2(v/c)}{\sqrt{1 - v^2/c^2}} = \frac{1}{c}\frac{(139.6 \text{ MeV})(0.906)}{\sqrt{1 - (0.906)^2}} = 298.8 \text{ MeV}/c$$

(c)
$$mc^2 = \sqrt{E^2 - (pc)^2} = \sqrt{(1268.1 \text{ MeV})^2 - (298.8 \text{ MeV})^2} = 1232 \text{ MeV}$$

40. Before the collision, the total relativistic energy of each electron is

$$E_e = \frac{m_e c^2}{\sqrt{1 - v^2/c^2}} = \frac{0.511 \, \text{MeV}}{\sqrt{1 - (0.99999)^2}} = 114.3 \, \text{MeV}$$

The total energy in the collision is therefore 2×114.3 MeV = 228.6 MeV. The total momentum is zero before the collision, because the two particles moves with equal and opposite velocities and have equal masses. After the collision, the total momentum is still zero, so we know that the two muons must move with equal speeds and thus have equal energies. The total energy of each muon is then 114.3 MeV and its kinetic energy is

$$K_\mu = E_\mu - m_\mu c^2 = 114.3 \, \text{MeV} - 105.7 \, \text{MeV} = 8.6 \, \text{MeV}$$

41. The two protons have equal (and opposite) momenta and thus equal energies $E_1$ and $E_2$. The new particle is created with zero momentum (at rest), so its total energy is equal to its rest energy $Mc^2 = 9700$ MeV. Conservation of energy then gives $E_1 + E_2 = Mc^2$, so $E_1 = E_2 = Mc^2/2$.

$$E_1 = \frac{m_p c^2}{\sqrt{1 - v^2/c^2}} = \frac{Mc^2}{2}$$

$$1 - \frac{v^2}{c^2} = \left[\frac{2(m_p c^2)}{Mc^2}\right]^2 = \left[\frac{2(938.3 \, \text{MeV})}{9700 \, \text{MeV}}\right]^2 = 0.0374 \qquad \text{so} \qquad v = 0.981c$$

42. For particle 1, moving in the positive $x$ direction,

$$E_1 = K_1 + mc^2 = 282 \, \text{MeV} + 140 \, \text{MeV} = 422 \, \text{MeV}$$
$$cp_1 = \sqrt{E_1^2 - (mc^2)^2} = \sqrt{(422 \, \text{MeV})^2 - (140 \, \text{MeV})^2} = +398 \, \text{MeV}$$

For particle 2, moving in the negative $x$ direction,

$$E_2 = K_2 + mc^2 = 25 \, \text{MeV} + 140 \, \text{MeV} = 165 \, \text{MeV}$$
$$cp_2 = -\sqrt{E_2^2 - (mc^2)^2} = -\sqrt{(165 \, \text{MeV})^2 - (140 \, \text{MeV})^2} = -87 \, \text{MeV}$$

The net final momentum is $p_f = p_1 + p_2 = 398 \, \text{MeV}/c - 87 \, \text{MeV}/c = 311 \, \text{MeV}/c$, and the net final energy is $E_f = E_1 + E_2 = 422 \, \text{MeV} + 165 \, \text{MeV} = 587 \, \text{MeV}$. Because of the conservation laws, these must be equal to the momentum and the energy of the initial particle, so that its rest energy is then

$$m_i c^2 = \sqrt{E_i^2 - (cp_i)^2} = \sqrt{(587 \, \text{MeV})^2 - (311 \, \text{MeV})^2} = 498 \, \text{MeV}$$

Solving Equation 2.36 for $v$, we obtain

$$v = c\sqrt{1 - \left(\frac{mc^2}{E}\right)^2} = c\sqrt{1 - \left(\frac{498 \text{ MeV}}{587 \text{ MeV}}\right)^2} = 0.529c$$

43.  $$pc = \frac{(mc^2)(v/c)}{\sqrt{1 - v^2/c^2}}$$

$$\frac{v}{c} = \sqrt{\frac{(pc/mc^2)^2}{1 + (pc/mc^2)^2}} = \sqrt{\frac{(3094 \text{ MeV}/105.7 \text{ MeV})^2}{1 + (3094 \text{ MeV}/105.7 \text{ MeV})^2}} = 0.99942$$

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} = \frac{2.198 \ \mu s}{\sqrt{1 - (0.99942)^2}} = 64.38 \ \mu s$$

44.  $$(pc)^2 = E^2 - (mc^2)^2 = (K + mc^2)^2 - (mc^2)^2 = K^2 + 2Kmc^2$$

$$p^2 = 2Km + \frac{K^2}{c^2} \qquad \text{so} \qquad \frac{p^2}{2K} = m + \frac{K}{2c^2}$$

45.  (a)  If the astronaut travels at speed $v$ for a distance $d = 200$ ly in the reference frame of Earth, then in the spacecraft reference frame the distance to the star is $d\sqrt{1 - v^2/c^2}$ and the time $T$ for the astronaut to reach the star (which must be 10 years in the spacecraft frame of reference) is

$$\frac{d\sqrt{1 - v^2/c^2}}{v} = T \qquad \text{or} \qquad \sqrt{1 - v^2/c^2} = v\frac{T}{d} = \frac{v}{c}\frac{cT}{d}$$

Solving, we find

$$v = c\sqrt{\frac{1}{1 + (Tc/d)^2}} = c\sqrt{\frac{1}{1 + (10/200)^2}} = 0.99875c$$

(b) According to an observer on Earth, the astronaut travels a total distance of 400 ly at a speed of $0.99875c$, so the total time for the round trip is (400 ly)/0.99875c 400.5 y.

46.  $$t_1' = \frac{t_1 - (u/c^2)x_1}{\sqrt{1 - u^2/c^2}} \qquad \text{and} \qquad t_2' = \frac{t_2 - (u/c^2)x_2}{\sqrt{1 - u^2/c^2}}$$

$$t_2' - t_1' = \frac{t_2 - t_1}{\sqrt{1 - u^2/c^2}} - \frac{(u/c^2)(x_2 - x_1)}{\sqrt{1 - u^2/c^2}}$$

In the $O$ frame of reference, the "cause" can travel to the "effect" with a speed that can be no greater than $c$; that is, $(x_2 - x_1)/(t_2 - t_1) \le c$, or $(x_2 - x_1) \le c(t_2 - t_1)$. Substituting for $(x_2 - x_1)$ in the second term, we obtain

$$t_2' - t_1' \ge \frac{t_2 - t_1}{\sqrt{1 - u^2/c^2}} - \frac{(u/c)(t_2 - t_1)}{\sqrt{1 - u^2/c^2}} = (t_2 - t_1)\frac{1 - u/c}{\sqrt{1 - u^2/c^2}} \ge 0$$

47. For the red flash, $x_1 = 0$ at $t_1 = 0$, so $x_1' = 0$ and $t_1' = 0$. For the blue flash, which occurs at

$x_2 = 3.26$ km and $t_2 = 7.63 \; \mu s$,

$$x_2' = \frac{x_2 - ut_2}{\sqrt{1 - u^2/c^2}} = \frac{3.26 \text{ km} - (0.625c)(7.63 \; \mu s)}{\sqrt{1 - (0.625)^2}} = 2.34 \text{ km}$$

$$t_2' = \frac{t_2 - (u/c^2)x_2}{\sqrt{1 - u^2/c^2}} = \frac{7.63 \; \mu s - (0.625/0.300 \text{ km}/\mu s)(3.26 \text{ km})}{\sqrt{1 - (0.625)^2}} = 1.07 \; \mu s$$

48. Let $O'$ be the observer on ship $A$, traveling at $u = 0.60c$ relative to the space station. Observer $O$ on the space station measures $v_{Bx} = 0$, $v_{By} = 0.50c$ for ship $B$. According to $O'$,

$$v_{Bx}' = \frac{v_{Bx} - u}{1 - v_{Bx}u/c^2} = \frac{0 - 0.60c}{1 - 0} = -0.60c$$

$$v_{By}' = \frac{v_{By}\sqrt{1 - u^2/c^2}}{1 - v_{Bx}u/c^2} = \frac{0.50c\sqrt{1 - (0.60)^2}}{1 - 0} = 0.40c$$

$$v_B' = \sqrt{(v_{Bx}')^2 + (v_{By}')^2} = \sqrt{(-0.60c)^2 + (0.40c)^2} = 0.72c$$

$$\theta_B = \tan^{-1}\frac{v_{By}'}{v_{Bx}'} = \tan^{-1}\frac{0.40c}{-0.60c} = 146°$$

For ship $C$, $O$ measures $v_{Cx} = -0.50c$, $v_{Cy} = 0$.

$$v_{Cx}' = \frac{-0.50c - 0.60c}{1 - (-0.50)(0.60)} = -0.85c, \quad v_{Cy}' = 0$$

$$v_C' = 0.85c \text{ at } \theta_C = 180° \text{ (negative } x \text{ direction)}$$

For ship $D$, $O$ measures $v_{Dx} = (-0.50c)\sin 45° = -0.35c$, $v_{Dy} = +0.35c$.

$$v'_{Dx} = \frac{-0.35c - 0.60c}{1-(-0.35)(0.60)} = -0.79c, \quad v'_{Dy} = \frac{0.35c\sqrt{1-(0.60)^2}}{1-(-0.35)(0.60)} = 0.23c$$

$$v'_D = \sqrt{(-0.79c)^2 + (0.23c)^2} = 0.82c \quad \text{at} \quad \theta_D = \tan^{-1}\frac{0.23c}{-0.79c} = 164°$$

49. (a) $t' = \dfrac{t-(u/c^2)x}{\sqrt{1-u^2/c^2}} = \dfrac{1.52\ \mu s - (0.563/300\ \text{m}/\mu s)(524\ \text{m})}{\sqrt{1-(0.563)^2}} = 0.648\ \mu s$

(b) $x' = \dfrac{x-ut}{\sqrt{1-u^2/c^2}} = \dfrac{524\ \text{m} - (0.563\times 300\ \text{m}/\mu s)(1.52\ \mu s)}{\sqrt{1-(0.563)^2}} = 335\ \text{m}$

50. (a) $O'$ measures $v' = (v-u)/(1-uv/c^2)$, and according to $O'$ the energy is

$$E' = \frac{mc^2}{\sqrt{1-v'^2/c^2}} = \frac{mc^2}{\sqrt{1-\left[(v-u)/(1-uv/c^2)\right]^2/c^2}}$$

and the momentum is

$$p' = \frac{mv'}{\sqrt{1-v'^2/c^2}} = \frac{m(v-u)/(1-uv/c^2)}{\sqrt{1-\left[(v-u)/(1-uv/c^2)\right]^2/c^2}}$$

(b) $E'^2 - (p'c)^2 = \dfrac{m^2c^4 - m^2\left[(v-u)/(1-uv/c^2)\right]^2}{1-\left[(v-u)/(1-uv/c^2)\right]^2/c^2} = m^2c^4$

The quantity $E'^2 - (p'c)^2$ is equal to $m^2c^4$ in *every* frame of reference, no matter what its relative speed $u$. In other words, every observer measures the same value for the rest energy or mass.

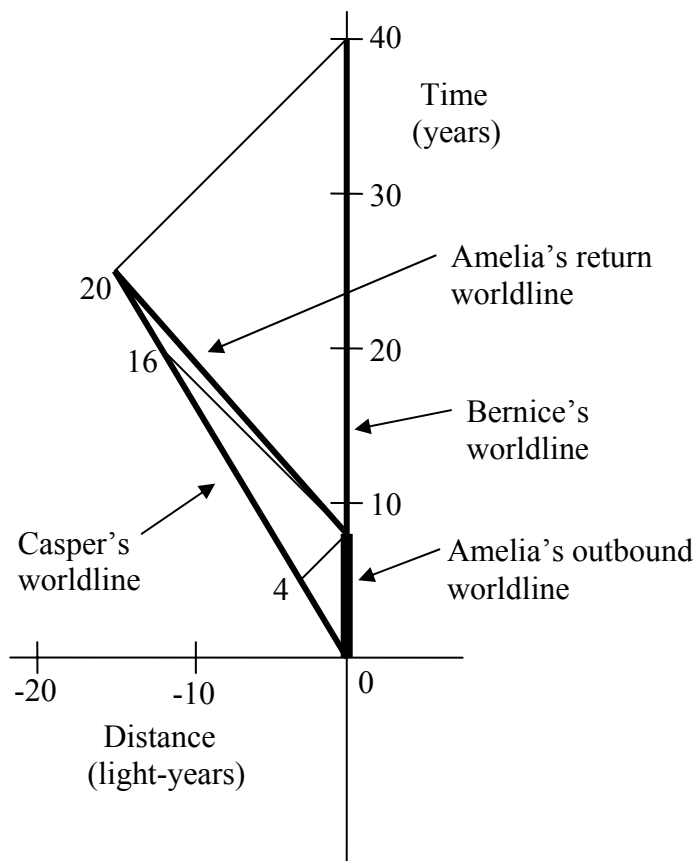51. (a) $O'$ measures $v'_x = -u$, $v'_y = v\sqrt{1 - u^2/c^2}$.

$$v' = \sqrt{(v'_x)^2 + (v'_y)^2} = \sqrt{u^2 + v^2 - u^2 v^2/c^2}$$

$$E' = \frac{mc^2}{\sqrt{1 - v'^2/c^2}} = \frac{mc^2}{\sqrt{1 - u^2/c^2 - v^2/c^2 + u^2 v^2/c^4}}$$

$$p' = \frac{mv'}{\sqrt{1 - v'^2/c^2}} = \frac{m\sqrt{u^2 + v^2 - u^2 v^2/c^2}}{\sqrt{1 - u^2/c^2 - v^2/c^2 + u^2 v^2/c^4}}$$

(b) $E'^2 - (p'c)^2 = \dfrac{m^2 c^4 - m^2 c^2 (u^2 + v^2 - u^2 v^2/c^2)}{1 - u^2/c^2 - v^2/c^2 + u^2 v^2/c^4} = m^2 c^4$

52.



(b) 40 years

42

(c) During Amelia's outbound journey, her worldline and Bernice's overlap. This part of her journey takes 8 years, as measured in the clocks in Bernice's frame of reference.

(d) Let $O$ = Casper and $O'$ = Bernice. Then $u = 0.60c$ and $v_x = -0.60c$.

$$v_x' = \frac{v_x - u}{1 - v_x u / c^2} = \frac{-0.60c - 0.60c}{1 - (-0.60)(0.60)} = -0.882c$$

(e) On his 4th birthday.

(f) On his 16th birthday.

53.  (a) Before the first acceleration, $E = E_0 = mc^2$. After the acceleration, the energy is

$$E_1 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \frac{0.511\,\text{MeV}}{\sqrt{1 - (0.99)^2}} = 3.6\,\text{MeV}$$

The change in energy is $\Delta E = E_1 - E_0 = 3.6\,\text{MeV} - 0.5\,\text{MeV} = 3.1\,\text{MeV}$, so the first stage adds 3.1 MeV to the energy of the electron.

(b)     $$E_2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \frac{0.511\,\text{MeV}}{\sqrt{1 - (0.999)^2}} = 11.4\,\text{MeV}$$

The change in energy is $\Delta E = E_2 - E_1 = 11.4\,\text{MeV} - 3.6\,\text{MeV} = 7.8\,\text{MeV}$, so the second stage adds about 2.5 times as much energy as the first stage, even though the second stage increases the velocity by only 0.9%.

54.     $$K = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 = \frac{0.511\,\text{MeV}}{\sqrt{1 - (0.732)^2}} - 0.511\,\text{MeV} = 0.239\,\text{MeV per particle}$$

$$E_{\text{beam}} = (0.239\,\text{MeV/particle})(1.35 \times 10^{11}\,\text{particles/s})(3600\,\text{s}) = 1.16 \times 10^{14}\,\text{MeV} = 18.6\,\text{J}$$

Copper has a density of $\rho = 8.92\,\text{g/cm}^3$ and a specific heat capacity of $c_p = 0.385$ J/g·K. The mass $M$ of the copper is $M = \rho V = (8.92\,\text{g/cm}^3)(2.54\,\text{cm})^3 = 146\,\text{g}$. Assuming all of the kinetic energy carried by the particles in the beam acts to produce a change in temperature of the copper ($E_{\text{beam}} = Mc_p \Delta T$), the temperature increase is

$$\Delta T = \frac{E_{\text{beam}}}{Mc_p} = \frac{18.6\,\text{J}}{(146\,\text{g})(0.385\,\text{J/g} \cdot \text{K})} = 0.33\,\text{K} = 0.33\,\text{C}°$$

55. (a) $$p_i = \frac{mv_i}{\sqrt{1-v_i^2/c^2}} = \frac{(0.511\,\text{MeV}/c^2)(0.960c)}{\sqrt{1-(0.960)^2}} = 1.752\,\text{MeV}/c$$

$$p_{1f} = \frac{mv_{1f}}{\sqrt{1-v_{1f}^2/c^2}} = \frac{(0.511\,\text{MeV}/c^2)(0.956c)}{\sqrt{1-(0.956)^2}} = 1.665\,\text{MeV}/c$$

$$p_{xf} = p_{1f}\cos\theta_1 + p_{2f}\cos\theta_2 = (1.665\,\text{MeV}/c)(\cos 9.7^\circ) + p_{2f}\cos\theta_2 = 1.642\,\text{MeV}/c + p_{2f}\cos\theta_2$$
$$p_{yf} = p_{1f}\sin\theta_1 + p_{2f}\sin\theta_2 = (1.665\,\text{MeV}/c)(\sin 9.7^\circ) + p_{2f}\sin\theta_2 = 0.281\,\text{MeV}/c + p_{2f}\sin\theta_2$$

Conservation of momentum gives $p_{xf} = p_i$ and $p_{yf} = 0$. Thus

$$p_{2f}\cos\theta_2 = 1.752\,\text{MeV}/c - 1.642\,\text{MeV}/c = 0.110\,\text{MeV}/c$$
$$p_{2f}\sin\theta_2 = 0 - 0.281\,\text{MeV}/c = -0.281\,\text{MeV}/c$$

Dividing the second result by the first gives

$$\tan\theta_2 = \frac{-0.281\,\text{MeV}/c}{0.110\,\text{MeV}/c} = -2.55 \qquad \text{or} \qquad \theta_2 = -68.6^\circ$$

With $p_{2f} = (0.110\,\text{MeV}/c)/[\cos(-68.6^\circ)] = 0.302\,\text{MeV}/c$, we have

$$p_{2f} = 0.302\,\text{MeV}/c = \frac{mv_{2f}}{\sqrt{1-v_{2f}^2/c^2}}$$

and solving, we find $v_{2f} = 0.508c$.

(b) $$E_i = \frac{mc^2}{\sqrt{1-v_i^2/c^2}} = \frac{0.511\,\text{MeV}}{\sqrt{1-(0.960)^2}} = 2.336\,\text{MeV}$$

$$E_{1f} = \frac{mc^2}{\sqrt{1-v_{1f}^2/c^2}} = \frac{0.511\,\text{MeV}}{\sqrt{1-(0.956)^2}} = 1.743\,\text{MeV}$$

Conservation of energy gives $E_i = E_{1f} + E_{2f}$, so $E_{2f} = 2.366\,\text{MeV} - 1.743\,\text{MeV} = 0.593\,\text{MeV}$ and

$$E_{2f} = 0.593\,\text{MeV} = \frac{mc^2}{\sqrt{1-v_{2f}^2/c^2}} = \frac{0.511\,\text{MeV}}{\sqrt{1-v_{2f}^2/c^2}}$$

and solving we find $v_{2f} = 0.508c$, in agreement with part (a).

56. The initial energy is

$$E_\pi = \frac{mc^2}{\sqrt{1-v^2/c^2}} = \frac{135 \text{ MeV}}{\sqrt{1-(0.98)^2}} = 678 \text{ MeV}$$

and the momentum is

$$p_\pi = \frac{1}{c}\sqrt{E_\pi^2 - (mc^2)^2} = \frac{1}{c}\sqrt{(678 \text{ MeV})^2 - (135 \text{ MeV})^2} = 664 \text{ MeV}/c$$

Because the two gamma ray photons have equal energies, each has an energy of $\frac{1}{2}(678 \text{ MeV})$, so $E_\gamma = 339$ MeV. Each gamma ray photon has a momentum of $p_\gamma = E_\gamma/c = 339$ MeV/$c$, which has a component $p_\gamma \cos\theta$ along the direction of the initial $\pi$ meson. Conservation of momentum then gives $p_\pi = 2p_\gamma \cos\theta$, so the angle is

$$\theta = \cos^{-1}\frac{p_\pi}{2p_\gamma} = \cos^{-1}\frac{664 \text{ MeV}/c}{2(339 \text{ MeV}/c)} = 11.7°$$

57.  The total energy of the kaon is $K_K + m_K c^2 = 77.0 \text{ MeV} + 497.7 \text{ MeV} = 574.7 \text{ MeV}$.
The speed of the kaon is found from $E_K = m_K c^2/\sqrt{1-v^2/c^2}$ :

$$574.7 \text{ MeV} = \frac{497.7 \text{ MeV}}{\sqrt{1-v^2/c^2}}$$

which gives $v = 0.500c$. This is the same as the transformation speed $u$ necessary to move from the laboratory frame to a frame in which the kaon is at rest.

When a kaon at rest decays into two pions, each pion has an energy of half the rest energy of the kaon: $E_\pi = \frac{1}{2}m_K c^2 = 248.9 \text{ MeV}$. In this frame, the speed of each pion is found from $E_\pi = m_\pi c^2/\sqrt{1-v'^2/c^2}$ :

$$248.9 \text{ MeV} = \frac{139.6 \text{ MeV}}{\sqrt{1-v'^2/c^2}}$$

Solving, we find $v' = 0.828c$, with the two pions moving in opposite directions with this speed $(v_1' = +0.828c, v_2' = -0.828c)$.

We can now transform back to the original laboratory frame:

$$v_1 = \frac{v_1' + u}{1 + v_1' u / c^2} = \frac{0.828c + 0.500c}{1 + (0.828)(0.500)} = 0.9392c$$

$$v_2 = \frac{v_2' + u}{1 + v_2' u / c^2} = \frac{-0.828c + 0.500c}{1 + (-0.828)(0.500)} = -0.5597c$$

$$K_1 = \frac{m_\pi c^2}{\sqrt{1 - v_1^2 / c^2}} - m_\pi c^2 = \frac{139.6 \text{ MeV}}{\sqrt{1 - (0.9392)^2}} - 139.6 \text{ MeV} = 267.0 \text{ MeV}$$

$$K_2 = \frac{m_\pi c^2}{\sqrt{1 - v_2^2 / c^2}} - m_\pi c^2 = \frac{139.6 \text{ MeV}}{\sqrt{1 - (0.5597)^2}} - 139.6 \text{ MeV} = 28.9 \text{ MeV}$$

# Chapter 3

This chapter presents the three experiments of the late 19th and early 20th centuries that pointed toward the particle nature of electromagnetic radiation: the photoelectric effect, thermal radiation, and Compton scattering.  I have chosen not to present these experiments in their historic order, because the barrier for understanding thermal radiation is significantly greater than that for the photoelectric effect.  The latter is clear and unambiguous, while the analysis of the former depends on a complex argument touching on statistical considerations, the full details of which are best left for a later time. For a first exposure to these ideas, the photoelectric effect presents fewer challenges. (However, see the discussion in Chapter 2 of the book by Greenstein and Zajonc cited in the reading list below; they summarize the argument that a classical radiation field and quantized levels in atoms can explain the photoelectric effect, without recourse to quantizing the radiation field, and that a semiclassical explanation can also be given for the Compton effect.  Nevertheless, for an introduction at this level I prefer to offer the traditional explanation of these experiments in terms of photons.)


**Supplemental Materials**

Double slit interference:
http://vsg.quasihome.com/interfer.htm
*Physlet Quantum Physics*, Section 5.4

Photoelectric effect:
*Physlet Quantum Physics*, Section 5.2

Thermal (blackbody radiation):
http://www.mhhe.com/physsci/astronomy/applets/Blackbody/frame.html

Compton scattering:
*Physlet Quantum Physics*, Section 5.2

Other processes:
http://www.upscale.utoronto.ca/GeneralInterest/Harrison/Flash/Nuclear/PairProduction/PairProduction.html

Wave-particle duality:
I have used in my classes a very nice interferometer simulation called "Quantum Eraser" developed at the University of Munich by Albert Huber.  [See also the discussion of the use of this simulation by Chandralekha Singh, *American Journal of Physics* **76**, 400 (2008).] The original version of this program, called "Polfilter", is available through the archives of the University of Munich physics education research group at:
http://www.didaktik.physik.uni-muenchen.de/archiv/inhalt_materialien/polfilter/index.html .

## Suggestions for Additional Reading

A collection of photographs illustrating the phenomena of classical optics can be found in
M. Cagnet, M. Francon, and J. C. Thrierr, *Atlas of Optical Phenomena* (Berlin, Springer-
Verlag, 1962).

For more complete discussions of blackbody radiation, including more detailed
derivations of the Rayleigh-Jeans law, see:
R. Eisberg and R. Resnick, *Quantum Physics of Atoms, Molecules, Solids, Nuclei,
   and Particles,* 2nd ed. (Wiley, 1985).
F. K. Richtmeyer, E. H. Kennard, and J. N. Cooper, *Introduction to Modern Physics,*
   6th ed. (McGraw-Hill, 1969).

The properties of X rays, including diffraction and scattering, are discussed in the
following:
L. Bragg, "X-Ray Crystallography," *Scientific American* **219**, 58 (July 1968).
G. L. Clark, *Applied X Rays* (McGraw-Hill, 1940).
A. H. Compton and S. K. Allison, *X Rays in Theory and Experiment* (Van Nostrand, 1935).
N. A. Dyson, *X Rays in Atomic and Nuclear Physics* (Cambridge University Press, 1990).

For more details of experiments discussed in this chapter and their importance in the
development of quantum theory, see:
W. H. Cropper, *The Quantum Physicists* (Oxford University Press, 1970), chapter 1.
G. Greenstein and A. Zajonc, *The Quantum Challenge* (Jones and Bartlett, 1997).
A. Shimony, "The Reality of the Quantum World," *Scientific American* **258**, 46 (Jan. 1988).

## Materials for Active Engagement in the Classroom

### A. Reading Quizzes

1. The brehmsstrahlung process occurs when
   (1) an isolated electron emits a photon.  (2) an electron encounters a positron.
   (3) an electron absorbs a photon.  (4) an electron decelerates near an atom.

2. The Compton effect is based on:
   (1) scattering of photons by the tightly bound inner electrons of an atom.
   (2) scattering of photons by the loosely bound, nearly free electrons of a material.
   (3) interference of light waves scattered from electrons.
   (4) the wave-like behavior of electrons.

3. In the Compton effect:
   (1) an electron emits an X-ray photon.
   (2) an X-ray photon is absorbed by a metal surface and knocks loose an electron.
   (3) an X-ray photon loses energy after colliding with an electron.
   (4) an X-ray photon gains energy after colliding with an electron.

4. The photoelectric effect:
   (1) verifies that electrons behave like waves.
   (2) involves the interference of light waves that reflect from the surface of a metal.
   (3) is consistent only with the wave theory of light.
   (4) verifies that light behaves as if it is composed of particles.

5. Which of the following processes is impossible under all circumstances?
   (1) photon $\rightarrow$ electron + positron     (2) electron + positron $\rightarrow$ photons
   (3) electron $\rightarrow$ electron + photon     (4) photon + electron $\rightarrow$ photon + electron
   (5) photon + electron $\rightarrow$ electron     (6) All of the above are possible.

**Answers**     1. 4    2. 2    3. 3    4. 4    5. 6


## B. Conceptual or Discussion Questions

1. Photons of wavelength $\lambda$ are Compton scattered from electrons, and the scattered photons of wavelength $\lambda'$ are observed at an angle of 90 degrees relative to the direction of the incident photons.
   (a) If the detector of the scattered photons is moved to an angle that is smaller than 90 degrees, the wavelength of the scattered photons will:
       (1) increase        (2) decrease        (3) remain the same
       (4) increase at some angles between 0 and 90 degrees and decrease at other angles
   (b) If the source of incident photons is replaced with one that produces photons of much larger wavelength, how does the change in wavelength $\Delta\lambda = \lambda' - \lambda$ measured at 90 degrees compare with that of the original source?
       (1) $\Delta\lambda$ increases     (2) $\Delta\lambda$ decreases     (3) $\Delta\lambda$ remains the same

2. Compton scattering suggests that the light scattered from objects should change wavelength when the objects are viewed from different angles. Why don't we observe objects changing color as we vary the viewing angle?

3. In a photoelectric experiment, changing one of the experimental conditions might produce one of the following results:
       (1) Increase the photoelectric current.        (2) Increase the stopping potential.
       (3) Decrease the photoelectric current.       (4) Decrease the stopping potential.
   Choose the outcome that would occur after the following changes:
   (a) Replace the light source with one of twice the frequency but emitting the same number of photons per second.
   (b) Replace the light source with one of twice the wavelength but emitting the same number of photons per second.
   (c) Replace the light source with one of the same frequency but emitting twice as many photons per second.
   (d) Without changing the light source, replace the emitter surface with a material having a smaller work function.

4. Possible interactions between electrons and photons can be represented symbolically by the 5 processes listed below. For each process, give the conventional name for the process, state whether it can occur for isolated particles, and describe how energy is conserved in the process.
    (a) photon $\rightarrow$ electron + positron      (b) electron + positron $\rightarrow$ photons
    (c) electron $\rightarrow$ electron + photon      (d) photon + electron $\rightarrow$ photon + electron
    (e) photon + electron $\rightarrow$ electron

**Answers**     1. (a) 2   (b) 3       2. $\Delta\lambda \sim 0.001$ nm     3. (a) 2   (b) 4   (c) 1   (d) 2

## Sample Exam Questions

### A. Multiple Choice

1. Some stars appear to have a color that is more blue than the color of the Sun. How would the surface temperature of a blue star compare with the surface temperature of the Sun?
    (a) $T_{\text{blue star}} > T_{\text{Sun}}$      (b) $T_{\text{blue star}} < T_{\text{Sun}}$

2. The most intense radiation emitted from a hot sample of metal has a wavelength of 60 μm. When the temperature of the sample is doubled, what will be the wavelength of the most intense radiation?
    (a) 30 μm           (b) 120 μm           (c) 960 μm           (d) 15 μm

3. A glowing object emits radiation with a spectrum in which there is one particular wavelength at which the maximum intensity occurs. If the temperature of the object is doubled, what happens to the wavelength of the intensity maximum?
    (a) Remains the same      (b) Becomes twice as large      (c) Becomes half as large
    (d) Becomes 4 times as large           (e) Becomes 16 times as large

4. The Sun's yellow color corresponds to a surface temperature of about 5500 K. What would be the color of a star with surface temperature of 7000 K?
    (a) red (longer wavelength)           (b) blue (shorter wavelength)
    (c) yellow (same wavelength)

5. Electrons are accelerated through a potential difference of 2000 volts and are incident on a metal surface, resulting in the emission of photons. Which of the following photon wavelengths would NOT be observed from this surface?
    (a) 0.24 nm           (b) 0.78 nm           (c) 1.25 nm           (d) 3.62 nm

6. Light of wavelength 477 nm is incident on the surfaces of several different metals. For which value of the work function will electrons be emitted from the surface?
    (a) 4.2 eV      (b) 3.7 eV      (c) 3.2 eV      (d) 2.3 eV

7.  When photons of wavelength 488 nm are incident on a metal surface, electrons of maximum kinetic energy 1.39 eV are emitted from the surface. What is the minimum energy needed to remove an electron from this metal?
    (a) 2.54 eV     (b) 3.93 eV     (c) 0.65 eV     (d) 1.15 ev

8.  For a certain metal surface, electrons are emitted when the surface is illuminated with light of wavelength below 435 nm but for no wavelengths above 435 nm. What is the work function of this surface?
    (a) 1.36 eV     (b) 2.85 eV     (c) 3.48 eV     (d) None of these values.

9.  At temperature $T$ a body emits its most intense radiation at a wavelength of 5.6 μm. What is the wavelength of the most intense radiation emitted by the same body at temperature 4$T$?
    (a) 22.4 μm     (b) 4.0 μm     (c) 1.4 μm     (d) 7.8 μm     (e) 5.6 μm

10. Electrons of maximum kinetic energy $K_1$ are released when X rays of wavelength $\lambda_1$ are incident on the surface of a metal. If the light source is replaced by another that emits photons at the same rate but of longer wavelength $\lambda_2$, how is the resulting maximum kinetic energy $K_2$ related to $K_1$?
    (a) $K_2 = K_1$     (b) $K_2 > K_1$     (c) $K_2 < K_1$

11. Electrons are emitted when an ultraviolet light source of wavelength $\lambda$ illuminates a certain metal surface. If you wanted to increase the number of electrons per unit time emitted from the surface, you should
    (a) increase the frequency of the light source
    (b) increase the wavelength of the light source
    (c) add a second light source identical to the first light source

12. A metal surface is illuminated with light of wavelength $\lambda$, and a resulting current $i$ is observed in an electric circuit connected to the surface. The source is replaced with a different one in which the photon emission rate is only half as large. What should be the wavelength of this second source in order that the current have the same value $i$?
    (a) 2$\lambda$    (b) $\lambda$/2   (c) > 2$\lambda$   (d) < $\lambda$/2
    (e) The experiment is impossible -- the second source can never give the same
        current $i$.

13. Which one of these processes involves a decrease in the kinetic energy of an electron?
    (a) bremsstrahlung     (b) photoelectric effect     (c) Compton scattering
    (d) pair production

**Answers**     1. a    2. a    3. c    4. b    5. a    6. d    7. d    8. b    9. c    10. c
                11. c   12. e   13. a

## B. Conceptual

1. Photons of wavelength $\lambda$ are incident on a metal target. Scattered photons are observed at an angle $\theta_1$ relative to the direction of the original photons. In this geometry, it is determined that the scattered electrons have kinetic energy $K_1$. If the scattered photons were observed at a larger angle, would the corresponding kinetic energy of the scattered electrons be *greater than $K_1$, equal to $K_1$, or less than $K_1$?* EXPLAIN YOUR ANSWER.

2. A beam of photons of energy $E$ is incident on a metal target. The photons scatter from the nearly free electrons in the target, and the scattered photons are observed at an angle $\theta$ relative to the direction of the original beam of photons. When the photons emerge at that angle, the scattered electrons have a certain kinetic energy. As the angle $\theta$ is made smaller, does the corresponding kinetic energy of the scattered electrons *increase, decrease, or remain the same?* EXPLAIN YOUR ANSWER.

3. Consider two monochromatic (single-wavelength) light sources emitting light of respective wavelengths $\lambda_1$ and $\lambda_2$, with $\lambda_2 > \lambda_1$. The two bulbs are otherwise identical and emit light with exactly the same intensity (in W/m$^2$). A detector placed a distance $d$ from bulb 1 (emitting at wavelength $\lambda_1$) records $N$ photons per second. When the same detector is placed at the same distance $d$ from bulb 2 (emitting at wavelength $\lambda_2$), is the number of photons per second recorded by the detector *greater than N, smaller than N, or equal to N?* EXPLAIN YOUR ANSWER.

4. A source of light of wavelength $\lambda$ is incident on a metal surface, and electrons of maximum kinetic energy $K$ are observed to be emitted from the surface. The source is replaced by a different source that emits the same power (in watts) but has a smaller wavelength. Does the rate at which electrons are emitted from the surface *increase, decrease, or remain the same,* and does their maximum kinetic energy *increase, decrease, or remain the same?* EXPLAIN YOUR ANSWERS.

5. A beam of photons of energy $E_1$ strikes a metal surface, and electrons are observed to be emitted at a rate that produces a current $i_1$ in an external circuit. If the photon energy is increased while the number of photons per second striking the surface is kept constant, does the current in the external circuit *increase, decrease, or remain the same?* EXPLAIN YOUR ANSWER.

6. When a certain light source illuminates a metal surface, electrons are emitted from the metal with kinetic energies up to the value $K$. The light source is replaced with one that has the same wavelength but less intensity. With this new light source, does the upper limit of the electron kinetic energies *increase, decrease, or remain the same*? EXPLAIN YOUR ANSWER.

7. Consider an experiment in which a beam of monoenergetic photons is incident on free electrons. The scattered electrons are observed with an energy $E_1$ when the scattered photons are observed at an angle $\theta_1$. For the same incident photons, when the scattered photons are observed at an angle that is smaller than $\theta_1$ do the scattered electrons have an energy that is *greater than $E_1$, less than $E_1$, or equal to $E_1$*? EXPLAIN YOUR ANSWER.

8. The walls of a hollow metal box are maintained at a temperature $T = 1000$ K. The box is filled with photons in equilibrium with the walls. A tiny hole in one of the walls allows a small number of the photons to escape. Your equipment measures the number of escaping photons in a small interval of wavelength $d\lambda$ at a wavelength of $10^{-5}$ m. If you raise the temperature of the box to 2000 K, would you expect the number of photons in the same interval at the same wavelength to *increase, decrease, or stay about the same*? EXPLAIN YOUR ANSWER.

9. A beam of ultraviolet light is incident on a metal surface. Electrons leave the surface with a range of kinetic energies from very small values up to some maximum value. If the light source is replaced by a different source that emits photons at the same rate but with smaller wavelength, does the range of electron kinetic energies *become larger, become smaller, or remain the same?* EXPLAIN YOUR ANSWER.

10. In Einstein's explanation of the photoelectric effect, for a fixed wavelength the number of emitted photoelectrons is [*directly proportional to, independent of*] the intensity of the incident radiation and the kinetic energy of the emitted photoelectrons is [*directly proportional to, independent of*] the intensity of the incident radiation. EXPLAIN YOUR ANSWERS.

11. X rays of wavelength $\lambda_1$ are incident on a material, and scattered X rays of wavelength $\lambda_1 + \Delta\lambda_1$ are observed at the scattering angle $\theta$. The source of X rays is now replaced with a source of wavelength $\lambda_2$ which is greater than $\lambda_1$, and at the same angle $\theta$ the scattered X rays now have wavelength $\lambda_2 + \Delta\lambda_2$. Is $\Delta\lambda_2$ *greater than, equal to, or less than* $\Delta\lambda_1$? EXPLAIN YOUR ANSWER.

12. The walls of a hollow metal box are maintained at a temperature $T = 1000$ K. The box is filled with photons in equilibrium with the walls. A tiny hole in one of the walls allows a small number of the photons to escape. Your equipment measures the number of escaping photons in a small interval of wavelength $d\lambda$ at a wavelength of $10^{-5}$ m. If you raise the temperature of the box to 2000 K, would you expect the number of photons in the same interval at the same wavelength to *increase, decrease, or stay about the same*? EXPLAIN YOUR ANSWER.

## C.  Problems

1.  Light of wavelength 435 nm is incident on a metal surface, and it is observed that
    electrons leave the surface with a maximum kinetic energy of 1.16 eV.
    (a) What is the work function of this metal?
    (b) What is the maximum kinetic energy of the electrons if light of wavelength
        560 nm is used?
    (c) What is the longest wavelength of light that will cause electrons to be emitted
        from this surface?

2.  Light from a source with a variable wavelength is incident on a metal.  It is observed
    that electrons are emitted from the surface of the metal for all wavelengths less than
    525 nm but never for wavelengths above 525 nm.
    (a) What is the minimum energy necessary to remove an electron from the surface of
        the metal?
    (b) After leaving the surface, electrons must cross through a potential difference $V$ in
        order to contribute to the current in an external circuit.  When the wavelength of
        the light is changed to 345 nm, what potential difference is necessary to prevent
        the most energetic electrons from completing the circuit?

3.  Ultraviolet radiation of wavelength 176 nm is incident on the surface of a metal.
    Electrons are released from the metal with a maximum kinetic energy of 4.52 eV.
    (a)  What is the maximum wavelength of the incident radiation that could cause
        electrons to be released from the metal?
    (b) If the wavelength of the incident radiation were changed to 288 nm, what would
        be the maximum kinetic energy of the emitted electrons?

4.  A beam of photons of energy 6250 eV is incident on an aluminum target.  Scattered
    photons are observed at an angle of $60^{\circ}$ relative to the direction of the incident beam.
    (a) What is the wavelength of the scattered photons?
    (b) Assuming that the photons scatter from nearly free electrons, what is the energy
        given to the scattered electrons?
    (c) Find the electron's momentum component in the direction of the original incident
        photons.

5.  X-ray photons of wavelength 0.01575 nm are incident on free electrons at rest.  After the interaction, photons of wavelength 0.01772 nm are observed.
    (a) Relative to the direction of the original X-rays, at what angle would we observe these photons?
    (b) What is the kinetic energy given to the electrons by this interaction?

6.  A beam of photons of energy 184 keV is incident on a target.  Scattered photons are observed at an angle of 60° relative to the direction of the incident beam.  Assume that the photons scatter from free electrons in the target.
    (a) What is the energy of the scattered photons?
    (b) Find the kinetic energy that is acquired by the scattered electrons.
    (c) Find the magnitude of the electron's momentum <u>and</u> the component of the electron's momentum perpendicular to the direction of the incident beam.

7.  The stopping potential for a certain surface is 1.25 V when it is illuminated with light of wavelength 471 nm.  When the wavelength is changed to a new value, the stopping potential becomes 1.68 V.
    (a) What is this new wavelength?
    (b) What is the work function of this surface?

8.  An X-ray photon of wavelength 0.00375 nm is scattered from an electron.
    (a) At what angle would we observe scattered photons of wavelength 0.00860 nm?
    (b) For these scattered photons, what is the kinetic energy of the scattered electrons?

9.  An ultraviolet lamp of adjustable wavelength is shining on a metal surface.  It is observed that electrons begin to emerge from the surface when the wavelength is 255 nm.
    (a) What is the minimum energy necessary to remove an electron from the surface of this metal?
    (b) If the wavelength is reduced to 215 nm, what is the energy of the electrons that leave the surface?

10. A metal surface is illuminated with light of different wavelengths. It is observed that electrons are emitted from the metal for wavelengths of light up to 525 nm but for no wavelengths above 525 nm.  When light of 420 nm is used, what is the maximum kinetic energy of the electrons?

11. A metal surface is illuminated with ultraviolet light of adjustable wavelength.  It is observed that electrons are emitted from the surface when the wavelength of the ultraviolet light is below 325 nm but never when the wavelength is above 325 nm.  What is the kinetic energy of the emitted electrons when light of wavelength 243 nm is used?

12. The largest wavelength of light that will cause emission of photoelectrons from a certain surface is 486 nm.
   (a) What is the work function of the surface?
   (b) When light of wavelength 325 nm is used, what is the stopping potential?

**Answers**      1. (a) 1.69 eV   (b) 0.52 eV   (c) 734 nm
         2. (a) 2.36 eV   (b) 1.23 V
         3. (a) 491 nm   (b) 1.78 eV
         4. (a) 0.1996 nm   (b) 38 eV   (c) 3144 eV/$c$
         5. (a) 79°   (b) 8.75 keV
         6. (a) 156 keV   (b) 28 keV   (c) 171 keV/$c$, 135 keV/$c$
         7. (a) 405 nm   (b) 1.38 eV
         8. (a) 178°   (b) 186 keV
         9. (a) 4.86 eV   (b) 0.91 eV
         10. 0.59 eV
         11. 1.29 eV
         12. (a) 2.55 eV   (b) 1.27 V

# Problem Solutions

1. $\Delta y = y_{n+1} - y_n = \dfrac{\lambda D}{d} = \dfrac{(589.0 \text{ nm})(2.357 \text{ m})}{1.05 \text{ mm}} = 1.32 \text{ mm}$

2. $\sin\theta = \dfrac{n\lambda}{2d} = \dfrac{2(0.250 \text{ nm})}{2(0.282 \text{ nm})} = 0.8865$    so    $\theta = \sin^{-1} 0.8865 = 62.4°$

3. (a) $\lambda = \dfrac{2d\sin\theta}{n} = \dfrac{2(0.347 \text{ nm})(\sin 34.0°)}{1} = 0.388 \text{ nm}$

   (b) The spacing between planes is $(0.347 \text{ nm})\sin 45° = 0.245 \text{ nm}$. The Bragg condition then gives

   $$\sin\theta = \frac{\lambda}{2d} = \frac{0.388 \text{ nm}}{2(0.245 \text{ nm})} = 0.791 \quad \text{or} \quad \theta = 52.2°$$

   This is the angle that the reflected ray makes with the plane of atoms. Because this plae makes and angle of 45° with the surface, the beam emerges at an angle of $52.2° - 45° = 7.2°$ with the surface.

4. The Bragg formula for the first-order peak is $\lambda = 2d\sin\theta$. We need to know the relationship between a small angular range $d\theta$ and the corresponding wavelength range $d\lambda$, which is found by taking the differentials from the Bragg formula:

   $$d\lambda = 2d\cos\theta\, d\theta$$

   Eliminating the lattice spacing $d$ using the Bragg formula, we obtain (after converting $d\theta$ to radians)

   $$d\lambda = 2\left(\frac{\lambda}{2\sin\theta}\right)\cos\theta\, d\theta = \lambda\cot\theta\, d\theta$$
   $$= (0.149 \text{ nm})(\cot 15.15°)(2.6\times10^{-4} \text{ rad}) = 1.4\times10^{-4} \text{ nm}$$

5. (a) $E = 10.0 \text{ MeV} = 1.60\times10^{-12} \text{ J}$

   $$p = \frac{E}{c} = \frac{10.0 \text{ MeV}}{c} = 1.00\times10^7 \text{ eV}/c$$
   $$p = \frac{1.60\times10^{-12} \text{ J}}{3.00\times10^8 \text{ m/s}} = 5.33\times10^{-21} \text{ kg}\cdot\text{m/s}$$

(b) $E = 25 \text{ keV} = 4.0 \times 10^{-15}$ J

$$p = \frac{E}{c} = \frac{25 \text{ keV}}{c} = 2.5 \times 10^4 \text{ eV}/c$$

$$p = \frac{4.0 \times 10^{-15} \text{ J}}{3.00 \times 10^8 \text{ m/s}} = 1.3 \times 10^{-23} \text{ kg} \cdot \text{m/s}$$

(c) $\lambda = 1.0 \ \mu\text{m} = 1.0 \times 10^3$ nm

$$p = \frac{h}{\lambda} = \frac{1}{c}\frac{hc}{\lambda} = \frac{1}{c}\frac{1240 \text{ eV} \cdot \text{nm}}{1.0 \times 10^3 \text{ nm}} = 1.2 \text{ eV}/c$$

$$p = \frac{6.6 \times 10^{-34} \text{ J} \cdot \text{s}}{1.0 \times 10^{-6} \text{ m}} = 6.6 \times 10^{-28} \text{ kg} \cdot \text{m/s}$$

(d) $E = hf = (4.14 \times 10^{-15} \text{ eV} \cdot \text{s})(150 \times 10^6 \text{ Hz}) = 6.2 \times 10^{-7} \text{ eV} = 9.9 \times 10^{-26}$ J

$$p = \frac{E}{c} = \frac{6.2 \times 10^{-7} \text{ eV}}{c} = 6.2 \times 10^{-7} \text{ eV}/c$$

$$p = \frac{9.9 \times 10^{-26} \text{ J}}{3.00 \times 10^8 \text{ m/s}} = 3.3 \times 10^{-34} \text{ kg} \cdot \text{m/s}$$

6.  At $1 \text{ MHz} = 10^6$ Hz, $E = hf = (4.14 \times 10^{-15} \text{ eV} \cdot \text{s})(10^6 \text{ s}^{-1}) = 4 \times 10^{-9}$ eV

At $100 \text{ MHz} = 10^8$ Hz, $E = hf = (4.14 \times 10^{-15} \text{ eV} \cdot \text{s})(10^8 \text{ s}^{-1}) = 4 \times 10^{-7}$ eV

The range is from $4 \times 10^{-9}$ eV to $4 \times 10^{-7}$ eV.

7.  (a) $\lambda = \dfrac{hc}{E} = \dfrac{1240 \text{ eV} \cdot \text{nm}}{1.00 \times 10^4 \text{ eV}} = 0.124$ nm

(b) $\lambda = \dfrac{1240 \text{ eV} \cdot \text{nm}}{1.00 \times 10^6 \text{ eV}} = 1.24 \times 10^{-3}$ nm

(c) 350 nm:     $E = \dfrac{hc}{\lambda} = \dfrac{1240 \text{ eV} \cdot \text{nm}}{350 \text{ nm}} = 3.5$ eV

700 nm:     $E = \dfrac{1240 \text{ eV} \cdot \text{nm}}{700 \text{ nm}} = 1.8$ eV

The range is from 1.8 eV to 3.5 eV.

8.	With $\phi = 4.08$ eV  for aluminum,

$$\lambda_c = \frac{hc}{\phi} = \frac{1240 \text{ eV} \cdot \text{nm}}{4.08 \text{ eV}} = 304 \text{ nm}$$

9.	$$\phi = \frac{hc}{\lambda_c} = \frac{1239.853 \text{ eV} \cdot \text{nm}}{325.6 \text{ nm}} = 3.808 \text{ eV}$$

$$eV_s = \frac{hc}{\lambda} - \phi = \frac{1239.853 \text{ eV} \cdot \text{nm}}{259.8 \text{ nm}} - 3.808 \text{ eV} = 0.964 \text{ eV}$$

$$V_s = 0.964 \text{ V}$$

10.	$$eV_{s,\text{Cu}} = \frac{hc}{\lambda} - \phi_{\text{Cu}} \qquad \text{and} \qquad eV_{s,\text{Na}} = \frac{hc}{\lambda} - \phi_{\text{Na}}$$

Subtracting, we obtain

$$eV_{s,\text{Cu}} - eV_{s,\text{Na}} = \phi_{\text{Na}} - \phi_{\text{Cu}} = 2.28 \text{ eV} - 4.70 \text{ eV} = -2.42 \text{ eV}$$

$$V_{s,\text{Na}} = V_{s,\text{Cu}} + 2.42 \text{ eV} = V + 2.42 \text{ volts}$$

11.	(a)	$\phi = \dfrac{hc}{\lambda_c} = \dfrac{1240 \text{ eV} \cdot \text{nm}}{254 \text{ nm}} = 4.88 \text{ eV}$

(b)	$\lambda < 254$ nm

12.	(a)	With $\phi = 4.31$ eV, $\lambda_c = \dfrac{hc}{\phi} = \dfrac{1240 \text{ eV} \cdot \text{nm}}{4.31 \text{ eV}} = 288$ nm

(b)	$eV_s = \dfrac{hc}{\lambda} - \phi = \dfrac{1240 \text{ eV} \cdot \text{nm}}{220.0 \text{ nm}} - 4.31 \text{ eV} = 1.33 \text{ eV}$,  so  $V_s = 1.33$ volts

13.	(a) The total number of oscillators is

$$\int_0^\infty n(E)\,dE = \int_0^\infty \frac{N}{kT} e^{-E/kT}\,dE = \frac{N}{kT}(-kT)e^{-E/kT}\bigg|_0^\infty = N$$

(b) The average energy is (from Equation 3.33)

$$E_{\text{avg}} = \frac{1}{N}\int_0^\infty E\,n(E)\,dE = \frac{1}{kT}\int_0^\infty E\,e^{-E/kT}\,dE = kT\int_0^\infty x e^{-x}\,dx \qquad \text{with } x = E/kT$$

The definite integral is a standard form that is equal to 1, so $E_{\text{avg}} = kT$ .

14. (a) The total number of oscillators at all energies is

$$\sum_{i=0}^{\infty} N_n = \sum_{n=0}^{\infty} A e^{-E_n/kT} = A\sum_{n=0}^{\infty} e^{-n\varepsilon/kT} = A\frac{1}{1-e^{-\varepsilon/kT}}$$

Setting this result equal to $N$, we obtain $A = N(1-e^{-\varepsilon/kT})$.

(b) On the left side $\dfrac{d}{dx}\sum_{n=0}^{\infty} e^{nx} = \sum_{n=0}^{\infty} ne^{nx}$ and on the right side $\dfrac{d}{dx}\dfrac{1}{1-e^x} = \dfrac{e^x}{(1-e^x)^2}$.

Setting these equal to each other gives $\sum_{n=0}^{\infty} ne^{nx} = \dfrac{e^x}{(1-e^x)^2}$.

(c) $E_{avg} = \dfrac{1}{N}\sum_{n=0}^{\infty} N_n E_n = (1-e^{-\varepsilon/kT})\sum_{n=0}^{\infty}(n\varepsilon)e^{-n\varepsilon/kT} = (1-e^{-\varepsilon/kT})\varepsilon\dfrac{e^{-\varepsilon/kT}}{(1-e^{-\varepsilon/kT})^2} = \dfrac{\varepsilon}{e^{\varepsilon/kT}-1}$

(d) For large $\lambda$, $e^{hc/\lambda kT} \approx 1+hc/\lambda kT$ and thus $E_{avg} = \dfrac{hc/\lambda}{e^{hc/\lambda kT}-1} \approx \dfrac{hc/\lambda}{1+hc/\lambda kT-1} = kT$

As $\lambda$ goes to 0, $e^{hc/\lambda kT} \to \infty$ and $E_{avg} \to 0$.

15. 
$$I(\lambda) = \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}$$

$$\frac{dI}{d\lambda} = 2\pi hc^2\left[\left(\frac{-5}{\lambda^6}\right)\frac{1}{e^{hc/\lambda kT}-1} + \left(\frac{1}{\lambda^5}\right)\frac{(-e^{hc/\lambda kT})(-hc/\lambda^2 kT)}{(e^{hc/\lambda kT}-1)^2}\right]$$

Setting $dI d\lambda$ equal to zero gives

$$-\frac{5}{\lambda} + \frac{(e^{hc/\lambda kT})(hc/\lambda^2 kT)}{e^{hc/\lambda kT}-1} = 0$$

or, with $x = hc/\lambda kT$,
$$(x-5)e^x + 5 = 0$$

This equation does not have an exact solution, but an approximate solution can be found by trial and error: $x = 4.9651 = hc/\lambda kT$, so

$$\lambda T = \frac{hc}{4.9651k} = \frac{1239.853\ \text{eV}\cdot\text{nm}}{4.9651(8.6174\times10^{-5}\ \text{eV/K})} = 2.8978\times10^{-3}\ \text{m}\cdot\text{K}$$

16. $$\int_0^\infty I(\lambda)\,d\lambda = \int_0^\infty \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}\,d\lambda$$

With $x = hc/\lambda kT$ and $dx = (-hc/\lambda^2 kT)\,d\lambda$,

$$\int_0^\infty I(\lambda)\,d\lambda = 2\pi hc^2\left(\frac{kT}{hc}\right)^3\left(-\frac{kT}{hc}\right)\int_0^\infty \frac{x^3\,dx}{e^x-1}$$

$$= 2\pi hc^2\left(\frac{k}{hc}\right)^4 T^4\int_0^\infty \frac{x^3\,dx}{e^x-1} = \frac{2\pi k^4}{h^3 c^2}T^4\frac{\pi^4}{15} = \sigma T^4$$

with $\sigma = 2\pi^5 k^4/15h^3 c^2$

17. $$h = \left(\frac{2\pi^5 k^4}{15\sigma c^2}\right)^{1/3} = \left[\frac{2\pi^5(1.38066\times10^{-23}\ \text{J/K})^4}{15(5.6704\times10^{-8}\ \text{W/m}^2\cdot\text{K}^4)(2.9979\times10^8\ \text{m/s})^2}\right]^{1/3}$$

$$= 6.626\times10^{-34}\ \text{J}\cdot\text{s}$$

18. $$\lambda = \frac{2.898\times10^{-3}\ \text{m}\cdot\text{K}}{T} = \frac{2.898\times10^{-3}\ \text{m}\cdot\text{K}}{6000\ \text{K}} = 483\ \text{nm}$$

This is in the middle of the visible spectrum, close to the peak sensitivity of the eye.

19. $$\lambda = \frac{2.898\times10^{-3}\ \text{m}\cdot\text{K}}{T} = \frac{2.898\times10^{-3}\ \text{m}\cdot\text{K}}{2.7\ \text{K}} = 1.1\ \text{mm (microwave region)}$$

$$E = \frac{hc}{\lambda} = \frac{1240\ \text{eV}\cdot\text{nm}}{1.1\ \text{mm}} = 1.1\times10^{-3}\ \text{eV}$$

20. (a) $$\lambda_{\text{max}} = \frac{2.898\times10^{-3}\ \text{m}\cdot\text{K}}{T} = \frac{2.898\times10^{-3}\ \text{m}\cdot\text{K}}{307\ \text{K}} = 9.4\ \mu\text{m (infrared)}$$

(b) Assume a person can be represented as a cylinder, about 6 feet (1.83 m) tall and 1 foot (0.30 m) in diameter. The surface area is $2\pi rL = 2\pi(0.15\ \text{m})(1.83\ \text{m}) = 1.72\ \text{m}^2$.

$$I = \sigma T^4 = (5.67\times10^{-8}\ \text{W/m}^2\cdot\text{K}^4)(307\ \text{K})^4 = 504\ \text{W/m}^2$$

$$P = IA = (504\ \text{W/m}^2)(1.72\ \text{m}^2) = 870\ \text{W}$$

(c) For $T = 20°\text{C} = 293\ \text{K}$,

$$I = \sigma T^4 = (5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4)(293 \text{ K})^4 = 418 \text{ W/m}^2$$

$$P = IA = (418 \text{ W/m}^2)(1.72 \text{ m}^2) = 719 \text{ W}$$

Thus the net power radiated by a person is about 150 W.

21.  $$I = \sigma T^4 = (5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4)(1650 \text{ K}) = 4.20 \times 10^5 \text{ W/m}^2$$

$$P = IA = I(\pi r^2) = (4.20 \times 10^5 \text{ W/m}^2)\pi(0.50 \times 10^{-3} \text{ m})^2 = 0.33 \text{ W}$$

22.  This small wavelength interval can be treated as a differential $d\lambda$. At $T = 1675$ K, $kT = (8.6174 \times 10^{-5} \text{ eV/K})(1675 \text{ K}) = 0.1443$ eV. From Equation 3.41 we obtain

$$dI = I(\lambda)\,d\lambda = \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}\,d\lambda$$

$$= \frac{2\pi(6.626 \times 10^{-34} \text{ J}\cdot\text{s})(2.998 \times 10^8 \text{ m/s})^2(1.55 \times 10^{-9} \text{ m})}{(875 \times 10^{-9} \text{ m})^5(e^{(1240\text{eV}\cdot\text{nm})/(875 \text{ nm})(0.1443 \text{ eV})}-1)} = 61.4 \text{ W/m}^2$$

23.  (a)  We consider an interval of width $d\lambda = 2.0$ nm at a central wavelength of 551.0 nm. At $T = 6000$ K, $kT = (8.6174 \times 10^{-5} \text{ eV/K})(6000 \text{ K}) = 0.517$ eV. The intensity in this interval is

$$dI = I(\lambda)\,d\lambda = \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}\,d\lambda$$

$$= \frac{2\pi(6.626 \times 10^{-34} \text{ J}\cdot\text{s})(2.998 \times 10^8 \text{ m/s})(2.0 \text{ nm})}{(551.0 \times 10^{-9} \text{ m})^5(e^{(1240\text{eV}\cdot\text{nm})/(551.0 \text{ nm})(0.517 \text{ eV})}-1)} = 1.9 \times 10^5 \text{ W/m}^2$$

(b)  The total radiant intensity emitted by the Sun is

$$I = \sigma T^4 = (5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4)(6000 \text{ K})^4 = 7.35 \times 10^7 \text{ W/m}^2$$

The fraction is then

$$\frac{1.9 \times 10^5 \text{ W/m}^2}{7.35 \times 10^7 \text{ W/m}^2} = 0.0026 = 0.26\%$$

24.  $$(E + m_e c^2 - E')^2 = c^2(p^2 - 2pp'\cos\theta + p'^2) + m_e^2 c^4$$

$$E^2 + E'^2 + m_e^2 c^4 + 2Em_e c^2 - 2EE' - 2E'm_e c^2 = c^2 p^2 - 2c^2 pp'\cos\theta + c^2 p'^2 + m_e^2 c^4$$

With $E^2 = c^2 p^2$ and $E'^2 = c^2 p'^2$,

$$Em_ec^2 - EE' - E'm_ec^2 = -EE'\cos\theta$$

$$m_ec^2(E - E') = EE'(1 - \cos\theta)$$

$$\frac{E - E'}{EE'} = \frac{1}{E'} - \frac{1}{E} = \frac{1}{m_ec^2}(1 - \cos\theta)$$

25. (a) $\dfrac{1}{E'} = \dfrac{1}{E} + \dfrac{1 - \cos\theta}{m_ec^2} = \dfrac{1}{10.39 \text{ keV}} + \dfrac{1 - \sqrt{2}/2}{511.0 \text{ keV}} = 0.09682 \text{ keV}^{-1}$

so $E' = 1/0.09682 \text{ keV}^{-1} = 10.33 \text{ keV}$ .

(b) $K_e = E_e - m_ec^2 = E - E' = 10.39 \text{ keV} - 10.33 \text{ keV} = 0.06 \text{ keV}$

26. (a) $\lambda' = \lambda + (h/m_ec)(1 - \cos\theta)$

$= 0.02480 \text{ nm} + (0.002426 \text{ nm})(1 - \cos 90°) = 0.02723 \text{ nm}$

(b) The momenta of the incident and scattered photons are

$$p = \frac{h}{\lambda} = \frac{1}{c}\frac{hc}{\lambda} = \frac{1}{c}\frac{1240 \text{ eV}\cdot\text{nm}}{0.02480 \text{ nm}} = 4.999\times10^4 \text{ eV}/c \quad (x \text{ direction})$$

$$p' = \frac{h}{\lambda} = \frac{1}{c}\frac{hc}{\lambda} = \frac{1}{c}\frac{1240 \text{ eV}\cdot\text{nm}}{0.02723 \text{ nm}} = 4.553\times10^4 \text{ eV}/c \quad (y \text{ direction})$$

(c) $K_e = E - E' = cp - cp' = 4.999\times10^4 \text{ eV} - 4.553\times10^4 \text{ eV} = 4.46\times10^3 \text{ eV}$

(d) Because momentum must be conserved, the $x$ component of the electron's momentum must equal $p$, and the $y$ component must equal $-p'$ :

$$p_{ex} = p = 4.999\times10^4 \text{ eV}/c \qquad \text{and} \qquad p_{ey} = -p' = -4.553\times10^4 \text{ eV}/c$$

$$p_e = \sqrt{p_{ex}^2 + p_{ey}^2} = \sqrt{(4.999\times10^4 \text{ eV}/c)^2 + (4.553\times10^4 \text{ eV}/c)^2} = 6.762\times10^4 \text{ eV}/c$$

in the direction given by $\theta = \tan^{-1}\dfrac{p_{ey}}{p_{ex}} = \tan^{-1}\dfrac{-4.553\times10^4 \text{ eV}/c}{4.999\times10^4 \text{ eV}/c} = -42.3°$

27. When $\theta = 180°$, $\cos\theta = -1$ and $\dfrac{1}{E'} = \dfrac{1}{E} + \dfrac{2}{m_ec^2}$ , so

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$$E' = \frac{Em_ec^2}{2E + m_ec^2} = \frac{m_ec^2}{2 + m_ec^2/E} \cong \frac{m_ec^2}{2} = 0.25 \text{ MeV} \quad \text{when } E \gg m_ec^2$$

28. (a) $\dfrac{1}{E'} = \dfrac{1}{E} + \dfrac{1-\cos\theta}{m_ec^2} = \dfrac{1}{0.662 \text{ MeV}} + \dfrac{1-\cos 60°}{0.511 \text{ MeV}} = 2.489 \text{ MeV}^{-1}$

   so $E' = 1/2.489 \text{ MeV}^{-1} = 0.402 \text{ MeV}$.

   (b) $K_e = E - E' = 0.662 \text{ MeV} - 0.402 \text{ MeV} = 0.260 \text{ MeV}$

29. The initial momentum of the atom is zero, so conservation of momentum requires that the total final momentum of the atom and the photon also must equal zero. Thus the recoil momentum of the atom $p_{\text{atom}}$ and the momentum $p$ of the photon must be equal in magnitude and opposite in direction. The photon momentum is

$$p = \frac{E}{c} = \frac{6.4 \text{ keV}}{c} = 6.4 \times 10^3 \text{ eV}/c$$

The recoil momentum of the atom is therefore $p_{\text{atom}} = 6.4 \times 10^3 \text{ eV}/c$. The kinetic energy associated with this momentum is certainly going to be far smaller than the atom's rest energy (the mass of an iron atom is about 56 u, and thus its rest energy is in the range of 50,000 MeV). We are therefore safe in using nonrelativistic kinetic energy:

$$K_{\text{atom}} = \frac{p_{\text{atom}}^2}{2m} = \frac{p_{\text{atom}}^2 c^2}{2mc^2} = \frac{(6.4 \times 10^3 \text{ eV})^2}{2(56 \text{ u})(931.5 \text{ MeV/u})} = 3.9 \times 10^{-4} \text{ eV}$$

30. After acceleration through a potential difference of $\Delta V = 2.5 \times 10^4 \text{ V}$, the electrons lose a potential energy of $\Delta U = q\,\Delta V = 2.5 \times 10^4 \text{ eV}$ and thus gain a kinetic energy of the same amount. From Equation 3.55,

$$\lambda_{\min} = \frac{hc}{K} = \frac{1240 \text{ eV} \cdot \text{nm}}{2.5 \times 10^4 \text{ eV}} = 0.0496 \text{ nm}$$

31. The energy of the absorbed photon is

$$E = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{375 \text{ nm}} = 3.31 \text{ eV}$$

Neglecting the small recoil kinetic energy of the atom, this is the amount by which the internal energy of the atom increases. The energy of the emitted photon is

$$E = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{580 \text{ nm}} = 2.14 \text{ eV}$$

Again neglecting the recoil of the atom, this is the amount by which the internal energy decreases what the photon is emitted.  The net change in energy of the atom is

$$\Delta E = 3.31 \text{ eV} - 2.14 \text{ eV} = 1.17 \text{ eV}$$

32.  (a)  $E = \dfrac{hc}{\lambda} = \dfrac{1240 \text{ eV} \cdot \text{nm}}{550 \text{ nm}} = 2.25 \text{ eV/photon} = 3.61 \times 10^{-19} \text{ J/photon}$

radiated power $= 55 \text{ W} \times 0.75 = 41.25 \text{ W} = 41.25 \text{ J/s}$

$$\text{photon emission rate} = \frac{41.25 \text{ J/s}}{3.61 \times 10^{-19} \text{ J/photon}} = 1.14 \times 10^{20} \text{ photons/s} = 9.12 \times 10^{23} \text{ photons/h}$$

(b)  At a distance of $d = 1.0$ m, the protons are spread uniformly over a sphere of surface area $4\pi d^2$.  The fraction that strikes the paper depends on the ratio between the area of the paper and the area of the spherical surface over which they are spread:

$$\text{Rate} = \text{emission rate} \times \frac{\text{area of paper}}{\text{area of sphere}}$$

$$= (1.14 \times 10^{20} \text{ photons/s}) \frac{(0.10 \text{ m})(0.10 \text{ m})}{4\pi (1.0 \text{ m})^2} = 9.1 \times 10^{16} \text{ photons/s}$$

33.  Applying $eV_s = hc/\lambda - \phi$ to both of the data points, we have

$$0.65 \text{ eV} = \frac{hc}{420 \text{ nm}} - \phi \qquad \text{and} \qquad 1.69 \text{ eV} = \frac{hc}{310 \text{ nm}} - \phi$$

Subtracting these two equations, we find

$$1.04 \text{ eV} = hc \left( \frac{1}{310 \text{ nm}} - \frac{1}{420 \text{ nm}} \right)$$

$$h = \frac{1.04 \text{ eV}}{(3.00 \times 10^8 \text{ m/s}) \left( \dfrac{1}{310 \text{ nm}} - \dfrac{1}{420 \text{ nm}} \right)} = 4.10 \times 10^{-15} \text{eV/s}$$

Returning to the two original equations, we multiply each equation by its wavelength:

$(0.65 \text{ eV})(420 \text{ nm}) = hc - \phi(420 \text{ nm})$   and   $(1.69 \text{ eV})(310 \text{ nm}) = hc - \phi(310 \text{ nm})$

and solving for $\phi$ by subtracting, we obtain

$$\phi = \frac{(1.69 \text{ eV})(310 \text{ nm}) - (0.65 \text{ eV})(420 \text{ nm})}{420 \text{ nm} - 310 \text{ nm}} = 2.28 \text{ eV}$$

34.  The photon's momentum is $p = h/\lambda = (1240 \text{ eV} \cdot \text{nm})/c(192 \text{ nm}) = 6.46 \text{ eV}/c$ and its energy is 6.46 eV. The work function of aluminum is 4.08 eV, so the maximum kinetic energy of the photoelectron is 6.46 eV − 4.08 eV = 2.38 eV. The momentum of the electron is

$$p = \sqrt{2mK} = \frac{1}{c}\sqrt{2mc^2 K} = \frac{1}{c}\sqrt{2(511{,}000 \text{ eV})(2.38 \text{ eV})} = 1560 \text{ eV}/c$$

Conservation of momentum requires that $p_{\text{photon}} = p_{\text{atom}} - p_{\text{electron}}$, taking the initial direction of the photon as positive. Thus

$$p_{\text{atom}} = p_{\text{photon}} + p_{\text{electron}} = 6.46 \text{ eV}/c + 1560 \text{ eV}/c = 1566 \text{ eV}/c$$

$$K_{\text{atom}} = \frac{c^2 p_{\text{atom}}^2}{2m_{\text{atom}}c^2} = \frac{(1566 \text{ eV})^2}{2(26.98 \text{ u})(931.5 \times 10^6 \text{ eV/u})} = 4.88 \times 10^{-5} \text{ eV}$$

This energy is negligible compared with the energy of the electron. The relatively heavy atom can take the recoil momentum at very little cost in energy.

35.  (a)  At $T = 1150$ K,
$$\lambda_{\text{max}} = \frac{2.898 \times 10^{-3} \text{ m} \cdot \text{K}}{T} = \frac{2.898 \times 10^{-3} \text{ m} \cdot \text{K}}{1150 \text{ K}} = 2.52 \ \mu\text{m}$$

(b)  $\dfrac{hc}{\lambda_{\text{max}}kT} = \dfrac{1240 \text{ eV} \cdot \text{nm}}{(2.898 \times 10^{-3} \text{ m} \cdot \text{K})(8.617 \times 10^{-5} \text{ eV/K})} = 4.9655$

Comparing $I(2\lambda_{\text{max}})$ with $I(\lambda_{\text{max}})$, we obtain

$$\frac{I(2\lambda_{\text{max}})}{I(\lambda_{\text{max}})} = \frac{(\lambda_{\text{max}})^5}{(2\lambda_{\text{max}})^5}\left[\frac{e^{hc/\lambda_{\text{max}}kT} - 1}{e^{hc/2\lambda_{\text{max}}kT} - 1}\right] = \frac{1}{32}\left[\frac{e^{4.9655} - 1}{e^{4.9655/2} - 1}\right] = 0.405$$

36.  $K_e$ is largest when $E'$ is smallest (because $K_e = E - E'$) and thus when $1/E'$ is largest, which occurs when $\cos \theta = -1$ (that is, when $\theta = 180°$).

$$\frac{1}{E'} = \frac{1}{E} + \frac{2}{m_e c^2} = \frac{m_e c^2 + 2E}{E m_e c^2} \qquad \text{so} \qquad E' = \frac{E m_e c^2}{m_e c^2 + 2E}$$

$$K_e = E - E' = E - \frac{E m_e c^2}{m_e c^2 + 2E} = \frac{2E^2}{m_e c^2 + 2E}$$

37. With $dI = I(\lambda)\,d\lambda = \dfrac{2\pi h c^2}{\lambda^5}\dfrac{1}{e^{hc/\lambda kT}-1}\,d\lambda$, we solve to get $e^{hc/\lambda kT} = 1 + \dfrac{2\pi h c^2 d\lambda}{\lambda^5 dI}$ and solving for $T$ then gives

$$T = \frac{hc}{\lambda k \ln\left(1 + \dfrac{2\pi h c^2 d\lambda}{\lambda^5 dI}\right)}$$

$$= \frac{(1240\ \text{eV}\cdot\text{nm})(10^{-7}\ \text{cm/nm})}{(0.133\ \text{cm})(8.617\times10^{-5}\ \text{eV/K})\ln\left(1 + \dfrac{2\pi(6.626\times10^{-34}\ \text{J}\cdot\text{s})(2.997\times10^{8}\ \text{m/s})^2(0.00833\times10^{-2}\ \text{m})}{(0.133\times10^{-2}\ \text{m})^5(1.440\times10^{-7}\ \text{W/m}^2)}\right)}$$

$$= 2.724\ \text{K}$$

38. The radiation intensity per unit wavelength interval is $I(\lambda)$. The peak wavelength can be found from Wien's displacement law:

$$\lambda_{\text{max}} = \frac{2.8978\times10^{-3}\ \text{m}\cdot\text{K}}{2.7250\ \text{K}} = 1.0634\times10^{-3}\ \text{m}$$

It is convenient for this calculation to evaluate the quantity $hc/\lambda kT$, using Wien's law to substitute for the product $\lambda T$:

$$\frac{hc}{\lambda kT} = \frac{(1240\ \text{eV}\cdot\text{nm})(10^{-9}\ \text{m/nm})}{(8.617\times10^{-5}\ \text{eV/K})(2.8978\times10^{-3}\ \text{m}\cdot\text{K})} = 4.9659$$

and thus $e^{hc/\lambda kT} = 143.44$. The intensity at this temperature and wavelength is

$$I(\lambda) = \frac{2\pi h c^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1} = \frac{2\pi(6.626\times10^{-34}\ \text{J}\cdot\text{s})(2.997\times10^{8}\ \text{m/s})^2}{(1.0634\times10^{-3}\ \text{m})^5(143.44-1)} = 1.9306\times10^{-3}\ \text{W/m}^3$$

The change in intensity corresponding to a small temperature difference is

$$\Delta I = \left|\frac{dI}{dT}\right|\Delta T = \frac{2\pi h c^2}{\lambda^5}\frac{e^{hc/\lambda kT}}{(e^{hc/\lambda kT}-1)^2}\left(\frac{hc}{\lambda kT^2}\right)\Delta T = R(\lambda)\frac{e^{hc/\lambda kT}}{e^{hc/\lambda kT}-1}\left(\frac{hc}{\lambda kT^2}\right)\Delta T$$

$$= (1.9306\times10^{-3}\ \text{W/m}^3)\left(\frac{143.44}{142.44}\right)(4.9659)\frac{2\times10^{-5}}{2.7250} = 7.1\times10^{-8}\ \text{W/m}^3$$

Measuring this tiny difference in the radiation intensity is roughly like trying to tell the difference between a 60 W bulb and a 100 W bulb from a distance of 4 miles.

39. (a) According to Wien's law, the maximum radiation intensity is emitted at a wavelength of $\lambda_{max} = (2.8977 \times 10^{-3} \text{ m} \cdot \text{K})/(2.725 \text{ K}) = 1.063 \times 10^{-3} \text{ m} = 1.063 \text{ mm}$. This is within the desired wavelength range, so the spectrometer records the maximum intensity at this wavelength. With

$$\frac{hc}{\lambda kT} = \frac{(1240 \text{ eV} \cdot \text{nm})(10^{-9} \text{ m/nm})}{(1.063 \times 10^{-3} \text{ m})(8.617 \times 10^{-5} \text{ eV/K})(2.725 \text{ K})} = 4.968$$

we have

$$dI_{max} = I(\lambda_{max})d\lambda = \frac{2\pi hc^2}{\lambda_{max}^5} \frac{1}{e^{hc/\lambda_{max}kT} - 1} d\lambda$$

$$= \frac{2\pi(6.626 \times 10^{-34} \text{ J} \cdot \text{s})(2.997 \times 10^8 \text{ m/s})^2}{(1.063 \times 10^{-3} \text{ m})^5} \frac{1}{e^{4.968} - 1}(3.0 \times 10^{-7} \text{ m}) = 5.79 \times 10^{-10} \text{ W/m}^2$$

The intensity falls off from the maximum in both directions, so the minimum intensity must occur at one of the endpoints. For the endpoint at $\lambda_1 = 0.5 \text{ mm}$, we find as above $hc/\lambda_1 kT = 10.56$ and $dI_1 = I(\lambda_1)d\lambda = 9.31 \times 10^{-11} \text{ W/m}^2$, and for the endpoint at $\lambda_2 = 5.0 \text{ mm}$, $hc/\lambda_2 kT = 1.056$, and $dI_1 = I(\lambda_1)d\lambda = 1.91 \times 10^{-11} \text{ W/m}^2$. Clearly the minimum intensity would be recorded at $\lambda_2$.

(b) The area of the detector is $A = \pi r^2 = \pi(0.0043 \text{ m})^2 = 5.81 \times 10^{-5} \text{ m}^2$. At the maximum intensity, the photon energy is

$$E = hc/\lambda = (1240 \text{ eV} \cdot \text{nm})/(1.063 \times 10^6 \text{ nm}) = 1.167 \times 10^{-3} \text{ eV} = 1.869 \times 10^{-22} \text{ J}$$

The number of photons striking the detector per unit time is

$$N = (5.79 \times 10^{-10} \text{ W/m}^2)(5.81 \times 10^{-5} \text{ m}^2)/(1.869 \times 10^{-22} \text{ J/photon}) = 1.81 \times 10^8 \text{ photons/s}$$

At the wavelength corresponding to the minimum intensity, we have a photon energy of $E = 3.973 \times 10^{-23} \text{ J}$ and a rate of $N = 2.97 \times 10^7 \text{ photons/s}$.

40. $\lambda' = \lambda + \dfrac{h}{mc}(1 - \cos\theta) = 7.52 \text{ pm} + (2.426 \text{ pm})(1 - \cos 180°) = 12.37 \text{ pm}$

$$p = \frac{1}{c}\frac{hc}{\lambda} = \frac{1}{c}\frac{1240 \text{ eV}\cdot\text{nm}}{0.00752 \text{ nm}} = 1.65\times10^5 \text{ eV}/c = p_{\text{initial}}$$

$$p' = \frac{1}{c}\frac{hc}{\lambda'} = \frac{1}{c}\frac{1240 \text{ eV}\cdot\text{nm}}{0.01237 \text{ nm}} = 1.00\times10^5 \text{ eV}/c \text{ in negative direction}$$

With $p_{\text{final}} = p_e - p'$ and $p_{\text{initial}} = p_{\text{final}}$, we have

$$p_e = p + p' = 1.65\times10^5 \text{ eV}/c + 1.00\times10^5 \text{ eV}/c = 2.65\times10^5 \text{ eV}/c$$

41. The initial speed of the atom can be expressed as $v = c(125.0 \text{ m/s})/(2.997 \times 10^8 \text{ m/s}) = 4.171 \times 10^{-7}c$. The initial momentum (which is nonrelativistic at this low speed) is

$$p_i = mv = \frac{1}{c}mc^2\frac{v}{c} = \frac{1}{c}(1.007825 \text{ u})(931.5\times10^6 \text{ eV/u})(4.171\times10^{-7}) = 391.6 \text{ eV}/c$$

The photon momentum, which is in the opposite direction, has magnitude

$$p = \frac{1}{c}\frac{hc}{\lambda} = \frac{1}{c}\frac{1240 \text{ eV}\cdot\text{nm}}{97 \text{ nm}} = 12.8 \text{ eV}/c$$

The atom's final momentum is $p_f = p_i - p = 391.6 \text{ eV}/c - 12.8 \text{ eV}/c = 378.8 \text{ eV}/c$ and its speed is

$$v_f = \frac{p_f}{m} = c\frac{p_f c}{mc^2} = c\frac{378.8 \text{ eV}}{(1.007825 \text{ u})(931.5\times10^6 \text{ eV/u})} = 4.035\times10^{-7}c = 120.9 \text{ m/s}$$

So the change in the speed of the atom is 125.0 m/s – 120.9 m/s = 4.1 m/s.

42. (a) From the equation for the Doppler shift (Eq. 2.22 converted from frequency to energy and evaluated for small speeds), $E' = E(1 + v/c)$, we solve for $v$ to find

$$v = c\left(\frac{E' - E}{E}\right) = c\frac{2.41 \text{ keV}}{511 \text{ keV}} = 0.0047c$$

(b) The speed of the electron is so small that we may safely use nonrelativistic equations:

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}mc^2\left(\frac{v}{c}\right)^2 = \tfrac{1}{2}(511,000 \text{ eV})(0.0047)^2 = 5.6 \text{ eV}$$

As we learn in Chapter 10, this is a very typical value for the electrons in a solid.

43. Suppose the electron moves with velocity $v$ in the $x$ direction, and let the photon be emitted at an angle $\theta$ with momentum $p$ and energy $E$. After emission of the photon, the electron moves with velocity $v'$ ay an angle $\phi$. Then conservation of $x$ and $y$ components of momentum gives

$$\frac{m_e v}{\sqrt{1-v^2/c^2}} = \frac{m_e v' \cos\phi}{\sqrt{1-v'^2/c^2}} + p\cos\theta \qquad \text{and} \qquad 0 = \frac{m_e v' \sin\phi}{\sqrt{1-v'^2/c^2}} - p\sin\theta$$

Squaring and adding these two equations, we obtain

$$\frac{m_e^2 v^2}{1-v^2/c^2} = \frac{m_e^2 v'^2}{1-v'^2/c^2} + p^2 - \frac{2m_e v' p}{\sqrt{1-v'^2/c^2}}\cos(\theta+\phi)$$

Conservation of energy gives $E_e = E_e' + pc$, or

$$\frac{m_e c^2}{\sqrt{1-v^2/c^2}} = \frac{m_e c^2}{\sqrt{1-v'^2/c^2}} + pc$$

Cancelling the common factor of $c$ and squaring, we find

$$\frac{m_e^2 c^2}{1-v^2/c^2} = \frac{m_e^2 c^2}{1-v'^2/c^2} + p^2 + \frac{2pm_e c}{\sqrt{1-v'^2/c^2}}$$

Now we subtract the results of the energy and momentum equations:

$$\frac{m_e^2(c^2-v^2)}{1-v^2/c^2} = \frac{m_e^2(c^2-v'^2)}{1-v'^2/c^2} + \frac{2m_e p}{\sqrt{1-v'^2/c^2}}\left[c + v'\cos(\theta+\phi)\right]$$

$$0 = \frac{2m_e p}{\sqrt{1-v'^2/c^2}}\left[c + v'\cos(\theta+\phi)\right]$$

The quantity in the square brackets can never be zero (because $v' < c$), so the only way the right side of this equation can be zero is if $p = 0$ – that is, no photon is emitted!  Thus it is not possible to have a photon emitted by an electron and satisfy both momentum and energy conservation.

It is easier to analyze this problem if we switch to a frame of reference in which the electron is at rest.  The total initial energy is just the electron's rest energy, $m_e c^2$.  If the electron were to emit a photon of energy $E$, the final energy would be $K_e + m_e c^2 + E$, which is equal to the initial energy only if $K_e = 0$ and $E = 0$, which again shows that no photon can be emitted.

44. Let $E$ be the energy of the initial photon, and let $p_e$ and $K_e$ be the momentum and kinetic energy of each of the three electrons after the encounter. Then momentum conservation gives

$$\frac{E}{c} = 3p_e \qquad \text{or} \qquad E^2 = 9(p_e c)^2$$

(where $E/c$ is the momentum of the initial photon), and energy conservation gives

$$E + m_e c^2 = 3E_e = 3\sqrt{(p_e c)^2 + (m_e c^2)^2}$$

Squaring this expression, we obtain

$$E^2 + 2Em_e c^2 + m_e^2 c^4 = 9(p_e c)^2 + 9(m_e^2 c^4)$$

Combining the energy and momentum results, we find

$$2Em_e c^2 + m_e^2 c^4 = 9(m_e^2 c^4) \qquad \text{or} \qquad E = 4m_e c^2$$

The total initial energy of the electron + photon is then $E + m_e c^2 = 5m_e c^2$. The final energy is $3E_e = 3(m_e c^2 + K_e)$. Equating the initial and final energies, we obtain

$$5m_e c^2 = 3(m_e c^2 + K_e) \qquad \text{so} \qquad K_e = \tfrac{2}{3}m_e c^2$$

# Chapter 4

This chapter presents some of the experimental evidence for the wave nature of particles (including the results of several beautiful particle diffraction experiments).  The resulting considerations affecting indeterminacy are discussed, followed by the representation of the behavior in terms of wave packets.  Some students may find the latter material mathematically challenging, especially those who are not yet familiar with Fourier analysis.

I have chosen to sidestep issues related to the nature of quantum reality, not because I find them uninteresting but rather because such philosophical discussions are not appropriate in a course at this level.  I have given in the reading list below some references that deal with these issues.

Students who are still reasoning from a semiclassical perspective will often fall into a "hidden-variable" mode of thinking: the particle has a definite location somewhere in the wave packet, and by measurement we find out what that location is (or was).  It's a rather large leap to get to the view that the particle does not have a location but that it is the measurement that imparts a location to the particle. A full discussion of such issues would invoke the EPR (Einstein, Podolsky, Rosen) argument, Bell's theorem, and the experiments that showed the violations of Bell's inequality and thus agreement with the traditional Copenhagen interpretation.  Along with entanglement, teleportation, and quantum computing, those are topics for a more detailed or higher-level presentation, but interested students can be referred to one of the general presentations in the reading list.

## Supplemental Materials

Chapter 5 of *Physlet Quantum Physics* offers several simulations showing particle diffraction experiments, wave packet construction, and the uncertainty principle.

I often use the particle double-slit simulation "Doppelspalt" that can be found in the archives of the University of Munich physics education research group at: http://www.didaktik.physik.uni-muenchen.de/archiv/inhalt_materialien/doppelspalt/index.html. This program allows one to simulate diffraction experiments with different particles, adjust the particle energy and slit separation, and symbolically choose whether or not to detect the passage of the particles through the slits.

## Suggestions for Additional Reading

For a delightful account of a world in which Planck's constant is so large that quantum effects are ordinary, see G. Gamow, *Mr. Tompkins in Paperback* (Cambridge University Press, 1967). Another nonmathematical discussion of quantum theory is B. Hoffmann, *The Strange Story of the Quantum* (Dover, 1959). An imaginary dialogue, in which the protagonists of Galileo's dialogues are reunited to discuss quantum theory, measurement, and uncertainty, is in J. M. Jauch, *Are Quanta Real?* (Indiana University Press, 1973). The paradox of Schrodinger's cat is discussed in this last reference. Other references, in

which the philosophy of quantum theory is mixed with mathematics at about the same level as this text, are as follows:

R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, 1965). Chapters 1 to 3 of Volume 3 are particularly good introductions to quantum waves and the philosophy of measurement.

R. Resnick and D. Halliday, *Basic Concepts in Relativity and Early Quantum Theory* (Macmillan, 1992). Chapter 6 discusses the wave nature of particles and the uncertainty principle.

E. H. Wichmann, *Quantum Physics, Volume 4 of the Berkeley Physics Course* (McGraw-Hill, 1971).

For a discussion of the uncertainty principle, see:

G. Gamow, "The Principle of Uncertainty," *Scientific American* **198,** 51 (January 1958).

A translation of Claus Jonsson's 1959 article on the electron double-slit experiment is given in *American Journal of Physics* **42**, 4 (1974). This short, clearly written paper is very readable and is highly recommended as an example of the careful experimental technique that is necessary in doing interference experiments to illustrate the wave nature of particles.

More recent experiments, in which an electron microscope has been used to demonstrate beautiful interference and diffraction effects with electrons, can be found in:

P. G. Merli, G. F. Missiroli, and G. Pozzi, *American Journal of Physics* **44,**306 (1976).

G. Matteucci and G. Pozzi, *American Journal of Physics* **46,** 619 (1978).

A. Tonomura, J. Endo, T. Matsuda, T. Kawasaki, and H. Ezawa, *American Journal of Physics* **57**, 117 (1989).

G. Matteucci, *American Journal of Physics* **58**, 1143 (1990).

A summary of experiments demonstrating interference and diffraction effects with neutrons is:

R. Gahler and A. Zeilinger, *American Journal of Physics* **59**, 316 (1991).

Some readable and nontechnical presentations of quantum measurement issues and more recent considerations (such as Bell's theorem and its experimental tests):

J. Baggott, *The Quantum Story: A History in 40 Moments* (Oxford University Press, 2011).

P. C. W. Davies and J. R. Brown, eds., *The Ghost in the Atom* (Cambridge University Press, 1986).

G. Greenstein and A. Zajonc, *The Quantum Challenge* (Jones and Bartlett, 1997).

J. Gribben, *Schrodinger's Kittens and the Search for Reality* (Little, Brown & Co., 1995).

D. F. Styer, *The Strange World of Quantum Mechanics* (Cambridge University Press, 2000).

**Materials for Active Engagement in the Classroom**

## A. Reading Quizzes

1. De Broglie waves:
   - (1) are a form of electromagnetic radiation.
   - (2) are a basic property of all particles, whether at rest or in motion.
   - (3) travel at the speed of light.
   - (4) describe the wave-type behavior of moving particles.

2. The deBroglie wave of a particle can best be described as:
   - (1) a form of electromagnetic wave.
   - (2) a characteristic of the oscillation of the particle.
   - (3) a probability wave.
   - (4) none of the above.

3. Which of the following is NOT true about the deBroglie wavelength?
   - (1) It is larger for an electron than for a baseball moving at the same speed.
   - (2) It applies only to charged particles.
   - (3) It describes the wave properties of particles such as electrons.
   - (4) It is a property of waves of probability.

4. The uncertainty relationships:
   - (1) apply to all types of waves.
   - (2) apply only to de Broglie waves.
   - (3) apply only to classical waves.
   - (4) apply only to light waves.

**Answers**      1. 4     2. 3     3. 2     4. 1

## B. Conceptual or Discussion Questions

1. In the following situations, choose which particle has the <u>larger de Broglie wavelength</u>:
        (1) the electron                (2) the proton            (3) they are both the same
   - (a) An electron and a proton moving with the same momentum.
   - (b) An electron and a proton moving at the same speed.
   - (c) An electron and a proton with the same nonrelativistic kinetic energy.
   - (d) An electron and a proton with the same kinetic energy, in both cases much larger than the rest energy.

2. Why is it not possible to observe double-slit interference with baseballs?
   - (1) The de Broglie wavelength of a baseball is too large.
   - (2) The de Broglie wavelength of a baseball is too small.
   - (3) Baseballs are too large to fit through a double-slit apparatus.
   - (4) Baseballs can't be accelerated to the speed of light.

3. A beam of electrons moving with speed $v$ passes through a single slit and strikes a screen, where it forms a diffraction pattern with a bright central maximum and some less intense maxima on either side of center.
   (a) If the speed of the electrons is increased to $2v$, what happens to the width of the central maximum?
       (1) Increases         (2) Decreases         (3) Remains the same
   (b) If the beam of electrons is replaced with a beam of protons moving with speed $v$, what happens to the width of the central maximum compared with that of electrons moving with the same speed?
       (1) Increases         (2) Decreases         (3) Remains the same

4. Suppose an electron is moving at speed $v$. In terms of $v$, what would be the speed of a baseball with the same deBroglie wavelength as the electron?
       (1) $v$    (2) $10^{10}v$    (3) $10^{-10}v$    (4) $10^{-20}v$    (5) $10^{-30}v$

5. A beam of monoenergetic electrons is incident on a mask that contains a single narrow slit. A pattern of diffraction maxima and minima appears on the screen.
   (a) If the slit width is halved, the diffraction minima on the screen would then be:
       (1) closer together   (2) farther apart   (3) unchanged
   (b) If the kinetic energy of the electrons in the original experiment is halved, the diffraction minima on the screen would be:
       (1) closer together   (2) farther apart   (3) unchanged
   (c) Suppose the beam of electrons were replaced with a beam of particles of greater mass, such that the resultant diffraction pattern was exactly the same as that in the original experiment. To accomplish this, the kinetic energy of the new particles would be:
       (1) greater than that of the original electrons
       (2) less than that of the original electrons
       (3) equal to that of the original electrons

6. (a) A packet of water waves of width $\Delta x$ contains a range of wavelengths $\Delta\lambda$ about a central wavelength $\lambda$; that is, the range of wavelengths is from about $\lambda - \Delta\lambda/2$ to $\lambda + \Delta\lambda/2$. If the packet were made half as wide, what would be the new range of wavelengths?
       (1) $2\Delta\lambda$     (2) $\Delta\lambda/2$     (3) $\Delta\lambda$     (4) None of these
   (b) A whistle blast lasts for a time interval $\Delta t$. It consists of a central frequency $\nu$ with a range $\Delta\nu$. If the blast were made twice as long, what would be the new range of frequencies?
       (1) $2\Delta\nu$     (2) $\Delta\nu/2$     (3) $\Delta\nu$     (4) None of these
   (c) A beam of electrons of momentum $p_x$ moving in the $x$ direction passes through a slit of width $\Delta y = a$. The beam diffracts through the slit so that the range in its $y$ momentum is $\Delta p_y$, that is, from $-\Delta p_y/2$ to $+\Delta p_y/2$. What is the new range in the $y$ momentum if the slit is made half as wide?
       (1) $2\Delta p_y$     (2) $\Delta p_y/2$     (3) $\Delta p_y$     (4) None of these

7. Consider the following three experiments:
   (a) The $x$ component of the position of an electron is measured to within $\pm\Delta x$, and simultaneously the $x$ component of its momentum is measured to within $\pm\Delta p_x$.
   (b) The $x$ component of the position of an electron is measured to within $\pm\Delta x$, and then later the $x$ component of its momentum is measured to within $\pm\Delta p_x$.
   (c) The $x$ component of the position of an electron is measured to within $\pm\Delta x$, and simultaneously the $y$ component of its momentum is measured to within $\pm\Delta p_y$.
   In which of these cases does the uncertainty principle NOT impose a limitation on the outcome of the experiment?
   (1) a only          (2) b only          (3) c only
   (4) a and b only     (5) a and c only    (6) b and c only        (7) a, b, and c

**Answers**     1. (a) 3   (b) 1   (c) 1   (d) 3     2. 3   3. (a) 2   (b) 2     4. 5
                5. (a) 2   (b) 2   (c) 2     6. (a) 1   (b) 2   (c) 1        7. 6

[Note to instructors: The following worksheet can serve as a transition between the material covered in chapters 4 and 5.]

## DeBroglie Wave Worksheet

A particle of momentum $p$ moving in the $x$ direction is represented by a deBroglie wave of wavelength $\lambda = h/p$. Suppose the deBroglie wave has amplitude $A$. Then the probability to find the particle in any small interval of width $dx$ is proportional to the square of the amplitude: probability $\propto |A|^2 dx$.

1. First consider a free particle – no forces act on the particle anywhere in space. The particle (of mass $m$) is moving in the $x$ direction with speed $v$. Write an expression that could represent the deBroglie wave of this particle. Your wave equation should be a function of both $x$ and $t$.

Consider an interval of width $dx$ located somewhere on the $x$ axis. What is the probability to locate the particle in that interval?

How does that probability change as the interval is moved to different locations along the $x$ axis? Explain.

How does the probability to locate the particle in any interval depend on the time? Explain.

2. Suppose now there is a constant potential energy $U_0$ everywhere along the $x$ axis, from $x = -\infty$ to $x = +\infty$. Describe how your answers to the above questions would change. Assume the particle is moving with the same speed $v$.

3. Now suppose that the potential energy changes at $x = 0$:
$$U(x) = U_0 \quad x < 0$$
$$U(x) = U_1 \quad x > 0$$
Assume $U_1 > U_0$ (both are positive) and assume that the kinetic energy of the particle is greater than $U_1$. Let the particle be originally moving in the region $x < 0$ with speed $v_0$ toward the origin. How does $\lambda_0$, the wavelength of the deBroglie wave in the region $x < 0$, compare with $\lambda_1$, the wavelength of the deBroglie wave in the region $x > 0$?
(1) $\lambda_0 = \lambda_1$         (2) $\lambda_0 < \lambda_1$         (3) $\lambda_0 > \lambda_1$
Explain.

How would you expect the probability $P_0$ to locate the particle in a small interval $dx$ somewhere in the region $x < 0$ to compare with the probability $P_1$ to locate it in a similar interval in the region $x > 0$? (Think about the speed of the particle in the two regions.)
(1) $P_0 = P_1$         (2) $P_0 < P_1$         (3) $P_0 > P_1$
Explain.

Describe the deBroglie wave that would represent this particle and sketch the waveform. Be especially careful to think about how the wave changes when it passes through $x = 0$, and be sure your sketch is consistent with your answers to the above questions about how the wavelength and the probability differ in the two regions.

4. Suppose we have trapped a particle in a region of space of length $L$:
$$U(x) = +\infty \quad x < 0$$
$$U(x) = 0 \quad 0 < x < L$$
$$U(x) = +\infty \quad x > L$$
Describe the physical motion of the particle in this case.

How would you represent the deBroglie wave for this particle? To what classical wave phenomenon is this similar?

**Sample Exam Questions**

**A. Multiple Choice**

1. A particle has a lifetime of $3.2 \times 10^{-23}$ s. The rest energy of the particle is about 600 MeV. What range of values will most likely result from a measurement of its rest energy?
   (a) 599 to 601 MeV   (b) 590 to 610 MeV   (c) 500 to 700 MeV   (d) 0 to 1200 MeV

2. What is the kinetic energy of a proton whose deBroglie wavelength is 15.4 fm?
   (a) 80.5 MeV          (b) 3.45 MeV          (c) 6340 MeV          (d) 6.90 MeV

3. What is the de Broglie wavelength of an electron with a kinetic energy of 12.8 eV?
   (a) 96.9 nm     (b) 0.34 nm     (c) 16.8 nm     (d) 0.66 nm

4. Experiments to measure the rest energy of a highly unstable elementary particle give a distribution of values centered at 2500 MeV with a width of ±40 MeV. Assuming the width is *not* due to any defect in the measuring instrument, what is the best estimate for the lifetime of the particle?
   (a) $10^{-19}$ s     (b) $10^{-21}$ s     (c) $10^{-23}$ s     (d) $10^{-25}$ s

5. A sodium atom, a neutron, a proton, and an electron all have the same nonrelativistic kinetic energy. Which has the <u>smallest</u> de Broglie wavelength?
   (a) sodium atom        (b) neutron        (c) proton        (d) electron

6. Which of the following does NOT provide evidence for the wave nature of matter?
   (a) the photoelectric effect              (b) neutron diffraction
   (c) the Heisenberg relationships          (d) electron diffraction

**Answers**        1. b     2. b     3. b     4. c     5. a     6. a

**B. Conceptual**

1. A beam of electrons of speed *v* is incident on a double slit and then strikes a screen where it is made visible. If the speed of the electrons is increased, does the spacing between the bright regions on the screen *increase, decrease, or remain the same?* EXPLAIN YOUR ANSWER.

2. An electron is trapped in a region of space between two walls that are a distance *L* apart. Measurements of the kinetic energy of the electron give an average value of *E*. If the walls are moved closer together, will the average kinetic energy *increase, decrease, or remain the same?* EXPLAIN YOUR ANSWER.

3. A beam of electrons with kinetic energy $K_1$ is incident on a single slit of width $a$. After passing through the slit, the beam strikes a screen where it is made visible. At the center of the screen directly in front of the slit, the intensity of the pattern on the screen is $I_1$. If the kinetic energy of the beam is increased (keeping the slit width and the number of electrons per second constant), does the intensity of the pattern at the center of the screen *increase, decrease, or remain the same?* EXPLAIN YOUR ANSWER.

4. A beam of electrons of kinetic energy $K$ is incident on a tiny hole of diameter $D$ in a metal foil. Electrons that pass through the hole strike a screen where they produce a visible image. Most of the electrons appear concentrated in a bright spot at the center of the screen. If the kinetic energy of the electrons is increased, does the size of the central spot on the screen *increase, decrease, or remain the same*? EXPLAIN YOUR ANSWER.

5. Atomic masses are often given in mass units (u) to an accuracy of six decimal places. A certain atom is very unstable, with a lifetime of about $10^{-20}$ s. Does this short lifetime have *a large effect* or *a negligible effect* on our ability to determine the mass of this atom to the desired precision of $10^{-6}$ u? EXPLAIN YOUR ANSWER.

6. The rest energies of two particles are measured in the laboratory. The measurements on particle A show a distribution of values centered at 500 MeV but with a spread of about ±10 MeV. Measurements on particle B show a distribution centered at 200 MeV but with a spread of about ±20 MeV. The measurements are done with equipment of very great precision (typically 1 eV). When measurements are made of the lifetimes of the two particles, is the lifetime of particle A *greater than, less than, or equal to* the lifetime of particle B? EXPLAIN YOUR ANSWER.

7. A beam of electrons of speed $v$ is incident on a single slit and then strikes a screen where it is made visible. If the speed of the electrons is increased, does the width of the central bright region on the screen *increase, decrease, or remain the same?* EXPLAIN YOUR ANSWER.

8. An electron is represented by a wave packet of length $d$. The electron cannot be at rest, but must have a certain minimum kinetic energy $K$. If the size of the electron's wave packet is doubled to $2d$, does the minimum kinetic energy *increase, decrease, or remain the same?* EXPLAIN YOUR ANSWER.

9. A beam of electrons passes through a narrow hole and then strikes a screen. After the beam passes through the hole, it is found that the beam spreads because the electrons have acquired an average momentum of $p$ transverse (perpendicular) to the direction of their original motion. If the hole is made larger, does the average transverse momentum *become larger, become smaller, or stay the same?* EXPLAIN YOUR ANSWER.

10. Consider the following three experiments: (a) The position of an electron is measured to within $\Delta x$, and simultaneously the $x$ component of its momentum is measured to within $\Delta p_x$. (b) The position of an electron is measured to within $\Delta x$, and then later the $x$ component of its momentum is measured to within $\Delta p_x$. (c) The position of an electron is measured to within $\Delta x$, and simultaneously the $y$ component of its momentum is measured to within $\Delta p_y$.
In which of these cases (possibly more than one) does the uncertainty principle NOT impose a limitation on the outcome of the experiment? EXPLAIN YOUR ANSWER.

**Answers**      1. decrease      2. increase      3. increase      4. decrease      5. large
                 6. greater than 7. decrease      8. decrease      9. become smaller      10. b,c

## C. Problems

1.  A certain solid is composed of molecules of diameter 0.30 nm (1 nm = $10^{-9}$ m). You are assigned the task of forming an image of the molecules in this solid by using neutron scattering.
    (a) What value should you choose for the kinetic energy of the neutrons?
    (b) The molecule consists of several atoms. If you wanted instead to produce images of the individual atoms rather than the entire molecule, should you increase or decrease the kinetic energy of the neutrons? Explain your answer.

2.  In a certain experiment, the momentum of a beam of electrons is measured to have a value of 14.2 keV/c, with a spread in values of ±0.7 keV/c.
    (a) What is the minimum size of the apparatus containing the electrons that is necessary to make this measurement?
    (b) What is the deBroglie wavelength of these electrons?
    (c) If an experiment were done to measure the wavelength of the electrons (for example, a double-slit interference experiment), there will be a distribution of values about some central mean or average wavelength. How likely would it be to find a particular value for the wavelength that is 15% larger or smaller than the mean value? Very likely? Somewhat likely? Very unlikely? Explain your answer.

3.  (a) An electron is moving in the $x$ direction with a speed of $0.0045c$. What is its de Broglie wavelength?
    (b) The electron is confined to a region of space along the $x$ axis that is 0.236 nm in length. What range of values will most likely result from a measurement of its speed in the x direction?
    (Hint: You can work this problem either in SI units or in eV units.)

4. (a) Calculate the uncertainty in momentum for a proton confined to a nucleus of radius 6.0 fm.
   (b) What is the kinetic energy of a proton with that momentum?
   (c) Suppose a proton in that nucleus had a kinetic energy of 5.6 MeV. If the proton were represented by a deBroglie wave, how many wavelengths could fit across the diameter of that nucleus?

5. (a) Find the de Broglie wavelength of electrons moving with a speed of $1.63 \times 10^5$ m/s.
   (b) Suppose these electrons are described by a wave packet of width 2.65 nm. What range of values will most likely result from a measurement of the speed of the electrons?
   (c) What range of values would most likely result from a measurement of the wavelength of these electrons?

6. An electron moving at a speed of $0.0202c$ is trapped in an atomic-sized region of width 0.12 nm.
   (a) What range of values would likely result from a measurement of the speed of the electron?
   (b) Find the de Broglie wavelength of the electron and sketch its wave packet.

**Answers**    1. (a) 0.0091 eV  (b) decrease
2. (a) 0.28 nm  (b) 0.0873 nm  (c) Very unlikely
3. (a) 837 eV/$c$ or 0.54 nm  (b) 0.0029$c$ to 0.0061$c$
4. (a) 16.5 MeV/$c$  (b) 0.144 MeV  (c) 1
5. (a) 4.46 nm  (b) $1.19 \times 10^5$ m/s to $2.07 \times 10^5$ m/s  (c) 3.26 nm to 5.66 nm
6. (a) 0.0170$c$ to 0.0234$c$  (b) 0.12 nm

**Problem Solutions**

1. (a) At 5 MeV, $K \ll mc^2$, so we use nonrelativistic kinetic energy.

$$p = \sqrt{2mK} = \frac{1}{c}\sqrt{2mc^2 K} = \frac{1}{c}\sqrt{2(938.3 \text{ MeV})(5 \text{ MeV})} = 96.9 \text{ MeV}/c$$

$$\lambda = \frac{hc}{pc} = \frac{1240 \text{ MeV} \cdot \text{fm}}{96.9 \text{ MeV}} = 13 \text{ fm}$$

(b) In this case $K \gg mc^2$, so the extreme relativistic approximation $E = pc$ is valid.

$$\lambda = \frac{hc}{pc} = \frac{hc}{E} = \frac{1240 \text{ MeV} \cdot \text{fm}}{50 \times 10^3 \text{ MeV}} = 0.025 \text{ fm}$$

(c) The speed is small compared with $c$, so nonrelativistic formulas apply. With $v/c = (1.00 \times 10^6 \text{ m/s})/(3.00 \times 10^8 \text{ m/s}) = 0.00333$.

$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{hc}{(mc^2)(v/c)} = \frac{1240 \text{ eV} \cdot \text{nm}}{(511,000 \text{ eV})(0.00333)} = 0.73 \text{ nm}$$

2. (a) $K = \frac{3}{2}kT = \frac{3}{2}(8.6174 \times 10^{-5} \text{ eV/K})(293 \text{ K}) = 0.0379 \text{ eV}$

(b) The neutrons are nonrelativistic, so

$$p = \sqrt{2mK} = \frac{1}{c}\sqrt{2mc^2 K} = \frac{1}{c}\sqrt{2(939.6 \times 10^6 \text{ eV})(0.0379 \text{ eV})} = 8.44 \times 10^3 \text{ eV}/c$$

$$\lambda = \frac{hc}{pc} = \frac{1240 \text{ eV} \cdot \text{nm}}{8.44 \times 10^3 \text{ eV}} = 0.147 \text{ nm}$$

3. (a) $p = \dfrac{h}{\lambda} = \dfrac{1}{c}\dfrac{hc}{\lambda} = \dfrac{1}{c}\dfrac{1240 \text{ MeV} \cdot \text{fm}}{9.16 \text{ fm}} = 135.4 \text{ MeV}/c$

From $p = mv/\sqrt{1 - v^2/c^2} = (1/c)mc^2(v/c)/\sqrt{1 - v^2/c^2}$ we solve for $v$:

$$v = \frac{c}{\sqrt{1 + (mc^2/pc)^2}} = \frac{c}{\sqrt{1 + [(938.3 \text{ MeV})/(135.4 \text{ MeV})]^2}} = 0.143c$$

(b)
$$K = E - E_0 = \sqrt{(pc)^2 + (mc^2)^2} - mc^2$$
$$= \sqrt{(135.4 \text{ MeV})^2 + (938.3 \text{ MeV})^2} - 938.3 \text{ MeV} = 9.72 \text{ MeV}$$

This gain in kinetic energy requires a loss in potential energy of $\Delta U = -9.72$ MeV and thus a potential difference of $\Delta V = \Delta U / q = -9.72$ MeV/$e = -9.72$ MV .

4. With $\Delta U = q\Delta V = (+e)(-2.36 \times 10^5 \text{ V}) = -0.236$ MeV , we have $\Delta K = -\Delta U = +0.236$ MeV . Then

$$p = \sqrt{2mK} = \frac{1}{c}\sqrt{2mc^2 K} = \frac{1}{c}\sqrt{2(938.3 \text{ MeV})(0.236 \text{ MeV})} = 21.0 \text{ MeV}/c$$

$$\lambda = \frac{hc}{pc} = \frac{1240 \text{ MeV} \cdot \text{fm}}{21.0 \text{ MeV}} = 59.0 \text{ fm}$$

5. (a) For $\lambda = 12$ nm,

$$p = \frac{h}{\lambda} = \frac{1}{c}\frac{hc}{\lambda} = \frac{1}{c}\frac{1240 \text{ eV} \cdot \text{nm}}{12 \text{ nm}} = 100 \text{ eV}/c$$

$$K = \frac{p^2}{2m} = \frac{p^2 c^2}{2mc^2} = \frac{(100 \text{ eV})^2}{2(511,000 \text{ eV})} = 0.010 \text{ eV}$$

To increase the kinetic energy by $\Delta K = 0.010$ eV, its potential energy must decrease by $\Delta U = -\Delta K$, where $\Delta U = q \Delta V$ :

$$\Delta V = \frac{\Delta U}{q} = \frac{-\Delta K}{q} = \frac{-0.010 \text{ eV}}{-e} = +0.010 \text{ V}$$

(b) For $\lambda = 0.12$ nm,

$$p = \frac{h}{\lambda} = \frac{1}{c}\frac{hc}{\lambda} = \frac{1}{c}\frac{1240 \text{ eV} \cdot \text{nm}}{0.12 \text{ nm}} = 1.0 \times 10^4 \text{ eV}/c$$

$$K = \frac{p^2}{2m} = \frac{p^2 c^2}{2mc^2} = \frac{(1.0 \times 10^4 \text{ eV})^2}{2(511,000 \text{ eV})} = 100 \text{ eV}$$

$$\Delta V = -\Delta K / q = +100 \text{ V}$$

(c) For $\lambda = 1.2$ fm,

$$p = \frac{1}{c}\frac{hc}{\lambda} = \frac{1}{c}\frac{1240 \text{ MeV} \cdot \text{fm}}{1.2 \text{ fm}} = 1000 \text{ MeV}/c$$

Here $pc \gg mc^2$ , so we can use the extreme relativistic approximation:

$$K \cong E \cong pc = 1000 \text{ MeV} = 1.0 \times 10^9 \text{ eV}$$

$$\Delta V = -\Delta K / q = +1.0 \times 10^9 \text{ V}$$

Although it is possible to accelerate electrons to such high energies, it is not done by a single acceleration through such a large potential difference.

6. (a) The wavelength should be roughly the size of (or smaller than) the object we want to study, so $\lambda \le 0.10 \ \mu\text{m}$.
   (b) Corresponding to $\lambda \le 0.10 \ \mu\text{m}$,

$$p = \frac{h}{\lambda} = \frac{1}{c} \frac{hc}{\lambda} = \frac{1}{c} \frac{1240 \text{ eV} \cdot \text{nm}}{100 \text{ nm}} = 12.4 \text{ eV}/c$$

$$K = \frac{p^2}{2m} = \frac{p^2 c^2}{2mc^2} = \frac{(12.4 \text{ eV})^2}{2(511,000 \text{ eV})} = 1.5 \times 10^{-4} \text{ eV}$$

$$\Delta V = \Delta U / q = -\Delta K / q = +1.5 \times 10^{-4} \text{ V}$$

This is a lower limit on the accelerating voltage. If $\Delta V$ is smaller than this value, the wavelength is too large and details of the particles could not be seen because of diffraction effects. As $\Delta V$ is increased above this value, finer details would be observed.

7. (a)
$$p = \frac{1}{c} \frac{hc}{\lambda} = \frac{1}{c} \frac{1240 \text{ MeV} \cdot \text{fm}}{14 \text{ fm}} = 88.6 \text{ MeV}/c$$

For electrons $pc \gg mc^2$, so the extreme relativistic approximation is valid.

$$E \cong pc = 88.6 \text{ MeV}$$

$$K = E - mc^2 = 88.6 \text{ MeV} - 0.5 \text{ MeV} = 88 \text{ MeV}$$

(b) For neutrons, $pc \ll mc^2$ so

$$K = \frac{p^2}{2m} = \frac{p^2 c^2}{2mc^2} = \frac{(88.6 \text{ MeV})^2}{2(939.6 \text{ MeV})} = 4.2 \text{ MeV}$$

(c)
$$K = \frac{p^2}{2m} = \frac{p^2 c^2}{2mc^2} = \frac{(88.6 \text{ MeV})^2}{2(3727.4 \text{ MeV})} = 1.1 \text{ MeV}$$

8. (a) $p = \sqrt{2mK} = \frac{1}{c}\sqrt{2mc^2 K} = \frac{1}{c}\sqrt{2(3727 \times 10^6 \text{ eV})(0.020 \text{ eV})} = 1.22 \times 10^4 \text{ eV}/c$

$$\lambda = \frac{hc}{pc} = \frac{1240 \text{ eV} \cdot \text{nm}}{1.22 \times 10^4 \text{ eV}} = 0.10 \text{ nm}$$

(b) The fringes are separated by about 9 $\mu$m.

$$\lambda = \frac{d \, \Delta y}{D} = \frac{(8 \times 10^{-6} \text{ m})(9 \times 10^{-6} \text{ m})}{0.64 \text{ m}} = 0.11 \text{ nm}$$

9. For $m = 10^{-9}$ g and taking the density to be $\rho = 2$ g/cm$^3$, the volume of the particle is $V = m/\rho = (10^{-9} \text{ g})/(2 \text{ g/cm}^3) = 5 \times 10^{-10} \text{ cm}^3$, which corresponds to a diameter of about 0.001 cm $= 10^{-5}$ m. The spacing between the fringes is then

$$\Delta y = \frac{\lambda D}{d} = \frac{(6.6 \times 10^{-20} \text{ m})(5 \times 10^6 \text{ m})}{10^{-5} \text{ m}} = 3.3 \times 10^{-8} \text{ m} = 33 \text{ nm}$$

which is about the size of an atom!

10.
$$\lambda = \frac{d \sin \phi}{2} = \frac{(0.215 \text{ nm})(\sin 55°)}{2} = 0.0881 \text{ nm}$$

$$pc = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{0.0881 \text{ nm}} = 1.408 \times 10^4 \text{ eV}$$

$$K = \frac{p^2}{2m} = \frac{(pc)^2}{2mc^2} = \frac{(1.408 \times 10^4 \text{ eV})^2}{2(0.511 \times 10^6 \text{ eV})} = 194 \text{ eV}$$

To achieve this kinetic energy, the electrons must be accelerated through a potential difference of $\Delta V = +194$ V.

11.
$$p = \sqrt{2mK} = \frac{1}{c}\sqrt{2mc^2K} = \frac{1}{c}\sqrt{2(0.511 \times 10^6 \text{ eV})(175 \text{ eV})} = 1.337 \times 10^4 \text{ eV}/c$$

$$\lambda = \frac{h}{p} = \frac{hc}{pc} = \frac{1240 \text{ eV} \cdot \text{nm}}{1.337 \times 10^4 \text{ eV}} = 0.0927 \text{ nm}$$

For $n = 1$:
$$\phi = \sin^{-1}\frac{\lambda}{d} = \sin^{-1}\frac{0.0927 \text{ nm}}{0.352 \text{ nm}} = 15.2°$$

For $n = 2$:
$$\phi = \sin^{-1}\frac{2\lambda}{d} = \sin^{-1}\frac{2(0.0927 \text{ nm})}{0.352 \text{ nm}} = 31.8°$$

For $n = 3$:
$$\phi = \sin^{-1}\frac{3\lambda}{d} = \sin^{-1}\frac{3(0.0927 \text{ nm})}{0.352 \text{ nm}} = 52.2°$$

There is no diffracted beam for $n = 4$.

12. The number of wave crests passing the observation point in time $\Delta t$ is $N = f\,\Delta t$. The uncertainty in this number is

$$\Delta N = \Delta f\,\Delta t$$

With $f = v/\lambda$, we take differentials and obtain

$$df = -\frac{v}{\lambda^2}d\lambda \qquad \text{or} \qquad \Delta f = \frac{v}{\lambda^2}\Delta\lambda$$

Using $\Delta t = \Delta x/v$ (where $\Delta x$ is the distance traveled by the wave in the time $\Delta t$),

$$\Delta N = \Delta f\,\Delta t = \left(\frac{v}{\lambda^2}\Delta\lambda\right)\left(\frac{\Delta x}{v}\right) = \frac{\Delta x\,\Delta\lambda}{\lambda^2}$$

If the uncertainty in counting is a fraction $\varepsilon$ of one wave ($\Delta N \sim \varepsilon$), then these results are consistent with $\Delta x\,\Delta\lambda \sim \varepsilon\lambda^2$ and $\Delta f\,\Delta t \sim \varepsilon$.

13. (a)  $\qquad\qquad \Delta x = v\,\Delta t = (330\text{ m/s})(2.0\text{ s}) = 660\text{ m}$

(b)  $\qquad\qquad \lambda = \dfrac{v}{f} = \dfrac{330\text{ m/s}}{1.0\times10^3\text{ Hz}} = 0.33\text{ m}$

(c)  $\qquad\qquad \Delta\lambda \sim \dfrac{\varepsilon\lambda^2}{\Delta x} = \dfrac{(0.1)(0.33\text{ m})^2}{(660\text{ m})} = 0.017\text{ mm}$

(d)  $\qquad\qquad \Delta f \sim \dfrac{\varepsilon}{\Delta t} = \dfrac{0.1}{2.0\text{ s}} = 0.050\text{ Hz}$

14. (a)  $\qquad\qquad \Delta x = v\,\Delta t = (25\text{ cm/s})(4.0\text{ s}) = 1.0\times10^2\text{ cm}$

(b) With $\lambda = \Delta x/N = (1.0\times10^2\text{ cm})/12 = 8.3\text{ cm}$, we have

$$\Delta\lambda \sim \frac{\varepsilon\lambda^2}{\Delta x} = \frac{(0.1)(8.3\text{ cm})^2}{1.0\times10^2\text{ cm}} = 0.069\text{ cm}$$

15. The central frequency is $f = c/\lambda = (2.997\times10^8\text{ m/s})/(0.225\text{ m}) = 1.33\times10^9\text{ Hz}$. The frequency range is

$$\Delta f \sim \frac{\varepsilon}{\Delta t} = \frac{0.1}{1.17 \times 10^{-6} \text{ s}} = 8.4 \times 10^{4} \text{ Hz}$$

The receiver should accept signals in a range of $8.4 \times 10^{4}$ Hz about a frequency of $1.33 \times 10^{9}$ Hz.

16.  For $\Delta f = 10^{4}$ Hz,

$$\Delta t \sim \frac{\varepsilon}{\Delta f} = \frac{0.1}{10^{4} \text{ Hz}} = 10^{-5} \text{ s}$$

The signal processing time must be at least $10^{-5}$ s.

17.  With $\Delta v = 2.0 \times 10^{4}$ m/s,

$$\Delta x \sim \frac{\hbar}{\Delta p} = \frac{\hbar}{m \, \Delta v} = \frac{1.05 \times 10^{-34} \text{ J} \cdot \text{s}}{(9.11 \times 10^{-31} \text{ kg})(2.0 \times 10^{4} \text{ m/s})} = 5.8 \times 10^{-9} \text{ m} = 5.8 \text{ nm}$$

18.  (a)  $$\Delta p \sim \frac{\hbar}{\Delta x} = \frac{h}{2\pi \, \Delta x} = \frac{1}{c} \frac{hc}{2\pi \, \Delta x} = \frac{1}{c} \frac{1240 \text{ eV} \cdot \text{nm}}{2\pi (0.1 \text{ nm})} = 2000 \text{ eV}/c$$

(b)  $$K = \frac{(\Delta p)^{2}}{2m} = \frac{(c \, \Delta p)^{2}}{2mc^{2}} = \frac{(2000 \text{ eV})^{2}}{2(0.511 \times 10^{6} \text{ eV})} = 4 \text{ eV}$$

19.  $$\Delta E \sim \frac{\hbar}{\Delta t} = \frac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{2.0 \times 10^{-23} \text{ s}} = 33 \text{ MeV}$$

Measurements of the $\Sigma^{+}$ rest energy are likely to fall in the range 1385 MeV ± 33 MeV, or from 1352 MeV to 1418 MeV.

20.  $$\Delta t \sim \frac{\hbar}{\Delta E} = \frac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{120 \times 10^{6} \text{ eV}} = 5.5 \times 10^{-24} \text{ s}$$

21.  $$\Delta E \sim \frac{\hbar}{\Delta t} = \frac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{1.2 \times 10^{-9} \text{ s}} = 5.5 \times 10^{-7} \text{ eV}$$

22.  With $\Delta E / E = 10^{-15}$, we have

$$\Delta E = 10^{-15} E = 10^{-15} (50 \times 10^3 \text{ eV}) = 5.0 \times 10^{-11} \text{ eV}$$

$$\Delta t \sim \frac{\hbar}{\Delta E} = \frac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{5.0 \times 10^{-11} \text{ eV}} = 1.3 \times 10^{-5} \text{ s}$$

23. As we did for electrons in Example 4.7, let's find the kinetic energy of an alpha particle with a momentum of 19.7 MeV/$c$:

$$K = \frac{p^2}{2m} = \frac{(pc)^2}{2mc^2} = \frac{(19.7 \text{ MeV})^2}{2(3727 \text{ MeV})} = 0.052 \text{ MeV}$$

This is negligible compared with the typical kinetic energies of alpha particles emitted in radioactive decays. Therefore, the uncertainty principle does not limit the existence of these alpha particles inside the nucleus.

24.
$$y(x) = \int A(k) \cos kx \, dk = A_0 \int_{k_0 - \Delta k/2}^{k_0 + \Delta k/2} \cos kx \, dk = A_0 \left. \frac{\sin kx}{x} \right|_{k_0 - \Delta k/2}^{k_0 + \Delta k/2}$$

$$= \frac{A_0}{x} \left[ \sin x(k_0 + \Delta k/2) - \sin x(k_0 - \Delta k/2) \right]$$

$$= \frac{A_0}{x} \left[ \sin k_0 x \cos \frac{x \Delta k}{2} + \cos k_0 x \sin \frac{x \Delta k}{2} - \left( \sin k_0 x \cos \frac{x \Delta k}{2} - \cos k_0 x \sin \frac{x \Delta k}{2} \right) \right]$$

$$= \frac{2A_0}{x} \sin \left( \frac{\Delta k}{2} x \right) \cos k_0 x$$

25.
$$y(x) = \int_{-\infty}^{+\infty} A(k) \cos kx \, dk = A_0 \int_{-\infty}^{+\infty} e^{-(k-k_0)^2/2(\Delta k)^2} \cos kx \, dk$$

Let $k' = k - k_0$.

$$y(x) = A_0 \int_{-\infty}^{+\infty} e^{-k'^2/2(\Delta k)^2} \left[ \cos k'x \, \cos k_0 x - \sin k'x \, \sin k_0 x \right] dk'$$

The integral over the second term (involving the sines) vanishes because $\sin k'x$ is an odd function of $k'$ (the contribution of the integral from $-\infty$ to 0 cancels the part from 0 to $+\infty$). The remaining integral is

$$y(x) = 2A_0 \cos k_0 x \int_0^{\infty} e^{-k'^2/2(\Delta k)^2} \cos k'x \, dk'$$

The integral is a standard form that can be found in integral tables:

$$y(x) = 2A_0 \cos k_0 x \sqrt{\pi}\, \frac{e^{-x^2(\Delta k)^2/2}}{\sqrt{2}/\Delta k} = A_0 \Delta k \sqrt{2\pi}\, e^{-x^2(\Delta k)^2/2} \cos k_0 x$$

26.
$$y(x) = A\cos(2\pi x/\lambda_1) + A\cos(2\pi x/\lambda_2) = A[\cos(2\pi x/\lambda_1) + \cos(2\pi x/\lambda_2)]$$

Using the identity $\cos x + \cos y = 2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y)$, we get directly

$$y(x) = 2A\cos\left(\frac{\pi x}{\lambda_1} + \frac{\pi x}{\lambda_2}\right)\cos\left(\frac{\pi x}{\lambda_1} - \frac{\pi x}{\lambda_2}\right)$$

27. $$\omega_1 = 2\pi f_1 = 2\pi\frac{v_1}{\lambda_1} = 2\pi\frac{6}{9} = \frac{4\pi}{3} \qquad \text{and} \qquad k_1 = \frac{2\pi}{\lambda_1} = \frac{2\pi}{9}$$

$$\omega_2 = 2\pi f_2 = 2\pi\frac{v_2}{\lambda_2} = 2\pi\frac{4}{11} = \frac{8\pi}{11} \qquad \text{and} \qquad k_2 = \frac{2\pi}{\lambda_2} = \frac{2\pi}{11}$$

$$v_{\text{group}} = \frac{\Delta\omega}{\Delta k} = \frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{4\pi/3 - 8\pi/11}{2\pi/9 - 2\pi/11} = 15$$

28. (a) $$v_{\text{phase}} = \omega/k$$

$$v_{\text{group}} = \frac{d\omega}{dk} = \frac{d}{dk}(kv_{\text{phase}}) = v_{\text{phase}} + k\frac{dv_{\text{phase}}}{dk}$$

$$\frac{dv_{\text{phase}}}{dk} = \frac{dv_{\text{phase}}}{d\lambda}\frac{d\lambda}{dk} = \frac{dv_{\text{phase}}}{d\lambda}\frac{d}{dk}\left(\frac{2\pi}{k}\right) = \frac{dv_{\text{phase}}}{d\lambda}\left(-\frac{2\pi}{k^2}\right) = \frac{dv_{\text{phase}}}{d\lambda}\left(-\frac{\lambda}{k}\right)$$

$$v_{\text{group}} = v_{\text{phase}} - \lambda\frac{dv_{\text{phase}}}{d\lambda}$$

(b) The index of refraction $n$ for light in glass decreases as $\lambda$ increases (shorter wavelengths are refracted more than longer wavelengths); that is $dn/d\lambda < 0$. Because $n = c/v_{\text{phase}}$, $dn/d\lambda$ and $dv_{\text{phase}}/d\lambda$ have opposite signs and so $dv_{\text{phase}}/d\lambda > 0$. Thus $v_{\text{group}} > v_{\text{phase}}$.

29. $$v_{\text{phase}} = \sqrt{\frac{b}{\lambda}} = \sqrt{\frac{bk}{2\pi}} = \frac{\omega}{k} \qquad \text{or} \qquad \omega = \sqrt{\frac{b}{2\pi}}\,k^{3/2}$$

$$v_{\text{group}} = \frac{d\omega}{dk} = \sqrt{\frac{b}{2\pi}}\,\frac{3}{2}k^{1/2} = \frac{3}{2}\sqrt{\frac{bk}{2\pi}} = \frac{3}{2}v_{\text{phase}}$$

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30. 
$$K = E - mc^2 = \sqrt{p^2c^2 + m^2c^4} - mc^2$$

$$\frac{dK}{dp} = \frac{1}{2}(p^2c^2 + m^2c^4)^{-1/2}(2pc^2) = \frac{pc^2}{\sqrt{p^2c^2 + m^2c^4}} = \frac{pc^2}{E} = c^2 \frac{mv/\sqrt{1-v^2/c^2}}{mc^2/\sqrt{1-v^2/c^2}} = v$$

31. (a) With a node at each end (say, at $x = 0$ and $x = L$) and no other nodes, we must have one half-wave between the two nodes. Thus $L = \lambda_1/2$ or $\lambda_1 = 2L$. If there is an additional node at the midpoint ($x = L/2$), then there is a full wave between the two ends, and $L = \lambda_2$ or $\lambda_2 = 2L/2$. The next shorter wavelength has (in addition to the nodes at either end ) nodes at $x = L/3$ and $x = 2L/3$, so there are three half-waves between the ends: $L = 3\lambda_3/2$ or $\lambda_3 = 2L/3$. Continuing in this way, we see that in the $n^{th}$ case there are $n$ half-waves in the length $L$, so $L = n(\lambda_n/2)$ or $\lambda_n = 2L/n$.

(b) With $p_n = h/\lambda_n = nh/2L$, we see that $cp_n$ is of order keV, so nonrelativistic equations can safely be used:

$$K_n = \frac{p_n^2}{2m} = \frac{c^2 p_n^2}{2mc^2} = \frac{n^2 h^2 c^2}{8mc^2 L^2} = n^2 \frac{(1240\text{ eV}\cdot\text{nm})^2}{8(511,000\text{ eV})(0.50\text{ nm})^2} = n^2(1.50\text{ eV})$$

Thus $K_1 = 1.50$ eV, $K_2 = 6.00$ eV, $K_3 = 13.5$ eV.

32. 
$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{2mc^2 K}} = \frac{1240\text{ eV}\cdot\text{nm}}{\sqrt{2(940\times10^6\text{ eV})(0.0105\text{ eV})}} = 0.279\text{ nm}$$

From the Bragg scattering formula (Eq. 3.18), we have

$$\sin\theta = \frac{n\lambda}{2d} = \frac{(1)(0.279\text{ nm})}{2(0.247\text{ nm})} = 0.565 \quad\text{or}\quad \theta = 34.4°$$

For second-order ($n = 2$) scattering at that angle, $\lambda = (2d\sin\theta)/2 = 0.140$ nm. The wavelength is reduced by half, so the momentum is doubled and the kinetic energy increases by a factor of 4 to 0.0420 eV. For third-order scattering ($n = 3$), the kinetic energy is 9 times as great, or 0.0945 eV. The scattered beam at that angle will consist of all energies that are $n^2$ times the original energy ($n = 1,2,3,\ldots$).

33. (a) The mass of a nitrogen molecule is 14 u. The average molecular kinetic energy is $\frac{3}{2}kT$, so the de Broglie wavelength is

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{2mc^2 K}} = \frac{1240\text{ eV}\cdot\text{nm}}{\sqrt{2(14\text{ u})(931.5\times10^6\text{ eV/u})(1.5)(8.617\times10^{-5}\text{ eV/K})(293\text{ K})}} = 0.0279\text{ nm}$$

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(b)  The number of nitrogen molecules per unit volume is

$$n = \frac{\rho N_{\mathrm{A}}}{M} = \frac{(1.292 \text{ kg/m}^3)(6.02 \times 10^{23} \text{ molecules/mole})}{(0.028 \text{ kg/mole})} = 2.78 \times 10^{25} \text{ molecules/m}^3$$

and the average spacing between molecules is $n^{-1/3} = 3.30 \times 10^{-9}$ m $= 3.3$ nm. The de Broglie wavelength is 2 orders of magnitude smaller than the molecular spacing, so that quantum effects are unimportant in gases at room temperature.
(c)  Let's estimate that quantum effects would be significant if the de Broglie wavelength were about 1/10 of the molecular separation (0.33 nm):

$$p = \frac{h}{\lambda} = \frac{1}{c} \frac{hc}{\lambda} = \frac{1}{c} \frac{1240 \text{ eV} \cdot \text{nm}}{0.33 \text{ nm}} = 3760 \text{ eV}/c$$

$$K = \frac{p^2}{2m} = \frac{p^2 c^2}{2mc^2} = \frac{(3760 \text{ eV})^2}{2(28 \text{ u})(931.5 \times 10^6 \text{ eV/u})} = 2.71 \times 10^{-4} \text{ eV}$$

.

The molecules have this tiny amount of average kinetic energy at a temperature

$$T = \frac{2K}{3k} = \frac{2(2.71 \times 10^{-4} \text{ eV})}{3(8.617 \times 10^{-5} \text{ eV/K})} = 2.1 \text{ K}$$

Nitrogen is no longer a gas at this temperature, so our calculation using the formula for the mean molecular energy of gases is not correct.  However, it does suggest that if quantum effects are to become important in gases, they will occur only at low temperatures.  (Recall the discussion in Chapter 1 about how the equipartition of energy fails for the rotational and vibrational motions of some gases at even moderate temperatures, so other effects of quantum behavior may be observable at these temperatures.)

34.   For both the photon and the electron,

$$p = \frac{h}{\lambda} = \frac{1}{c} \frac{hc}{\lambda} = \frac{1}{c} \frac{1240 \text{ eV} \cdot \text{nm}}{0.281 \text{ nm}} = 4.41 \times 10^3 \text{ eV}/c$$

For the photon,

$$E = pc = 4.41 \times 10^3 \text{ eV}$$

For the electron,

$$K = \frac{p^2}{2m} = \frac{p^2 c^2}{2mc^2} = \frac{(4.41 \times 10^3 \text{ eV})^2}{2(511,000 \text{ eV})} = 19.1 \text{ eV}$$

35. (a) The initial nucleus is at rest, the final momenta of the helium and the neutron must sum to zero: $p_n + p_{He} = 0$, and so $p_n = -p_{He}$. The energy released in the decay appears as the kinetic energy of the final products: $K_n + K_{He} = 0.89$ MeV. Using nonrelativisitc kinetic energies, we have

$$K_n + K_{He} = \frac{p_n^2}{2m_n} + \frac{p_{He}^2}{2m_{He}} = \frac{p_n^2}{2m_n} + \frac{p_n^2}{2m_{He}} = \frac{p_n^2}{2m_n}\left(1 + \frac{m_n}{m_{He}}\right) = 0.89 \text{ MeV}$$

$$K_n = \frac{0.89 \text{ MeV}}{1 + m_n / m_{He}} = \frac{0.89 \text{ MeV}}{1 + 1/4} = 0.71 \text{ MeV}$$

(b) $$\Delta E \sim \frac{\hbar}{\Delta t} = \frac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{1.0 \times 10^{-21} \text{ s}} = 0.66 \text{ MeV}$$

The measured neutron energies will then be in the range $0.89$ MeV $\pm 0.66$ MeV. The neutron energy is not very well defined in this process.

36. (a) $$\Delta p_x \sim \frac{\hbar}{\Delta x} = \frac{1}{c}\frac{hc}{2\pi\Delta x} = \frac{1}{c}\frac{1240 \text{ eV} \cdot \text{nm}}{2\pi(1.0 \text{ cm})} = 2.0 \times 10^{-5} \text{ eV}/c$$

(b) $(p^2)_{av} = (p_x^2 + p_y^2 + p_z^2)_{av} = (\Delta p_x)^2 + (\Delta p_y)^2 + (\Delta p_z)^2 = 3(\Delta p_x)^2$

using Eq. 4.15, where the last step can be made because all three components of the momentum have the same uncertainty.

$$K_{av} = \frac{(p^2)_{av}}{2m} = \frac{3(c\Delta p_x)^2}{2mc^2} = \frac{3(2.0 \times 10^{-5} \text{ eV})^2}{2(511{,}000 \text{ eV})} = 1.2 \times 10^{-15} \text{ eV}$$

(c) The 1 cm³ piece of copper has a mass of (1 cm³)(8.95 g/cm³) = 8.95 g.  The molar mass of copper is 63.5 g, so the cube is (8.95 g)/(63.5 g/mole) = 0.141 mole.  As a rough estimate, let's assume that the heat capacity of copper remains constant for most of the region from T = 0 K to 300 K (the heat capacity in fact falls to zero at low temperature, but the assumption is good enough for this estimate).  The internal energy added to copper to raise its temperature from 0 K to 300 K is then roughly

$$\Delta E_{int} = \mu C \Delta T = (0.141 \text{ mole})(24.5 \text{ J/mole} \cdot \text{K})(300 \text{ K}) = 1000 \text{ J}$$

Assuming that each atom contributes one free electron (and thus $1.2 \times 10^{-15}$ eV) to the metal, the contribution to the internal energy of the metal is

$$(0.141 \text{ mole})(6.02 \times 10^{23} \text{ atoms/mole})(1.2 \times 10^{-15} \text{ eV/atom}) = 1.0 \times 10^8 \text{ eV} = 1.6 \times 10^{-11} \text{ J}$$

This is some 14 orders of magnitude smaller than the internal energy of the metal, so it is clear that these mobile electrons do not make a significant contribution to the internal energy of the metal.

37. (a) $\Delta E \sim m_\pi c^2 = 135$ MeV

(b) $\Delta t \sim \dfrac{\hbar}{\Delta E} = \dfrac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{135 \times 10^6 \text{ eV}} = 4.87 \times 10^{-24}$ s

(c) $\Delta x = c\,\Delta t = (3.00 \times 10^8 \text{ m/s})(4.87 \times 10^{-24} \text{ s}) = 1.46$ fm

38. (a) $\Delta p_x \sim \dfrac{\hbar}{\Delta x} = \dfrac{1}{c}\dfrac{hc}{2\pi \Delta x} = \dfrac{1}{c}\dfrac{1240 \text{ eV} \cdot \text{nm}}{2\pi(0.20 \text{ nm})} = 990$ eV/$c$

(b) $(p^2)_{av} = (p_x^2 + p_y^2 + p_z^2)_{av} = (\Delta p_x)^2 + (\Delta p_y)^2 + (\Delta p_z)^2 = 3(\Delta p_x)^2$

$$K_{av} = \dfrac{(p^2)_{av}}{2m} = \dfrac{c^2(p^2)_{av}}{2mc^2} = \dfrac{3(990 \text{ eV})^2}{2(65 \text{ u})(931.5 \times 10^6 \text{ eV/u})} = 2.4 \times 10^{-5} \text{ eV}$$

(c) In a cube of copper 1.0 cm on edge (0.141 mole), the energy is

$(0.141 \text{ mole})(6.02 \times 10^{23} \text{ atoms/mole})(2.4 \times 10^{-5} \text{ eV/atom}) = 2.04 \times 10^{18} \text{ eV} = 0.33$ J

This energy is small compared with the internal energy (roughly 1000 J), but it is not quite as negligibly small as the energy of the electronic motion (see Problem 36). This energy of 0.33 J is independent of temperature, so it becomes relatively more important as the temperature of the copper is reduced (thereby decreasing the internal energy). This is one example of the phenomenon of "zero-point motion," a certain minimum energy that a confined quantum system must have. There is no counterpart to this zero-point motion in classical physics.

39. When the beam passes through a hole of width $\Delta x = d$, there is a resulting uncertainty in the transverse momentum of order $\Delta p_x \sim \hbar/d$ and thus in the transverse velocity of $\Delta v_x \sim \hbar/md$. From Eq. 4.16, we have $(p_x^2)_{av} = (\Delta p_x)^2$ or $(v_x^2)_{av} = (\Delta v_x)^2$. The diameter of the beam grows larger than its original diameter by an amount $\Delta d = t\,\Delta v_x$, where $t$ is the time the beam has been traveling. The speed of the atoms as they leave the oven at a temperature $T$ is found from

$$K = \tfrac{1}{2}mv^2 = \tfrac{3}{2}kT \qquad \text{so} \qquad v = \sqrt{\frac{3kT}{m}}$$

The beam travels the distance $L$ at a speed $v$ in a time $t = L/v$, and thus

$$\Delta d = t\,\Delta v_x \sim \frac{L}{d}\frac{\hbar}{md} = \frac{L\hbar}{md\sqrt{3kT/m}} = \frac{L\hbar}{d\sqrt{3mkT}}$$

$$= \frac{(2\text{ m})(1.05\times10^{-34}\text{ J}\cdot\text{s})}{(0.003\text{ m})\sqrt{3(7\text{ u})(1.66\times10^{-27}\text{ kg/u})(1.38\times10^{-23}\text{ J/K})(1500\text{ K})}} = 3\times10^{-9}\text{ m}$$

The spreading of the beam due to the uncertainty principle is thus a negligible effect.

# Chapter 5

This chapter presents an introduction to the methods of quantum mechanics as exemplified by solving the Schrödinger equation. It is helpful but not necessary for students to have some prior experience with differential equations. Because the application of boundary conditions is especially important, the chapter begins with a discussion of continuity of the classical wave and its first derivative, as applied to a wave that crosses the interface between two regions in which it propagates differently. Particularly for students with no prior training in solving differential equations (and even for students who have had such training but for whom the boundary conditions may be more a mathematical than a physical manifestation), it is important to introduce these exercises. The notion of confined particles then allows a quick review of issues related to wave packets and uncertainty from Chapter 4, before we launch into the Schrödinger formalism. New features in this edition include an earlier introduction of the complex nature of the wave function, which allows plots of the real part, imaginary part, and squared magnitude for the solutions to the problem of scattering from a step or barrier later in the chapter. As mentioned in the preface, this edition avoids plotting the wave function or probability density on the same plot as the energy levels of bound states, which many students found to be confusing. Also new to this edition is a section on the finite potential energy well.

I have long been curious why so many of us say "potential" instead of "potential energy" when we discuss the Schrödinger equation. That's a mistake we would certainly correct for our students if they were discussing electrostatics. Yet many quantum mechanics textbooks say "Coulomb potential" when the presence of the $e^2$ shows what is meant is potential energy, and one even occasionally sees $\frac{1}{2}kx^2$ called the "harmonic oscillator potential." This confusion is compounded by notation, because quantum mechanics textbooks often use $V$ in the potential energy term in the Schrödinger equation, but students in introductory physics learn to use $U$ to represent potential energy and $V$ to represent potential. I have been careful to use the term "potential energy well" rather than "potential well" in this edition and to use $U$ for the potential energy.

## Supplemental Materials

Chapters 6-12 of *Physlet Quantum Physics* offer numerous simulations that cover material presented in this chapter. It is especially instructive to be able to illustrate the time dependence of the real and imaginary parts of the wave function of non-stationary states, such as in the case of scattering from a step or barrier. Static illustrations in the text (such as Figures 5.24 and 5.25) don't quite convey the nature of the time dependence or the "waviness." Physlet exercises allowing one to vary the particle energy relative to the step or barrier height are available for classroom display or student use. A very useful simulation for step and barrier problems is available from the University of Colorado PhET simulations at http://phet.colorado.edu/en/simulation/quantum-tunneling. Another excellent PhET simulation for bound states in various potential energy wells is at http://phet.colorado.edu/en/simulation/bound-states.

**Suggestions for Additional Reading**

An introduction to the formalism of quantum physics and its historical development is found in W. H. Cropper, *The Quantum Physicists and an Introduction to Their Physics* (Oxford University Press, 1970).

Other discussions of the Schrödinger equation and examples of its use are given in the following books, in approximately increasing order of difficulty beginning at the level of this book:

N. Ashby and S. C. Miller, *Principles of Modern Physics* (Holden-Day, 1970).

P. A. Lindsay, *Introduction to Quantum Mechanics for Electrical Engineers* (McGraw-Hill, 1967).

E. H. Wichmann, *Quantum Physics, Volume 4 of the Berkeley Physics Course* (McGraw-Hill, 1971).

R. Eisberg and R. Resnick, *Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles,*2nd ed., (Wiley, 1985).

R. B. Leighton, *Principles of Modern Physics* (McGraw-Hill, 1959).

Illustrations showing the scattering of wave packets from barriers and the behavior of wave packets inside square-well and harmonic-oscillator potential energy wells may be found in:

S. Brandt and H. D. Dahmen, *The Picture Book of Quantum Mechanics*, 3$^{rd}$ ed., (Springer, 2000).

For a description of the scanning tunneling microscope, see:

G. Binnig and H. Rohrer, *Scientific American* **253**, 50 (August 1985).


**Materials for Active Engagement in the Classroom**

**A. Reading Quizzes**

1. The probability density for a particle in the ground state of a one-dimensional infinite potential energy well:
    (1) has a single maximum at the center of the well.
    (2) has a minimum at the center of the well and maxima at the sides of the well.
    (3) has several maxima and minima in the well.
    (4) is constant throughout the well.

2. In the one-dimensional infinite well, how does the energy spacing between the excited states change as the energy of the states increases?
    (1) The spacing is constant.            (2) The spacing decreases.
    (3) The spacing increases.              (4) The spacing changes randomly.

3. A beam of particles is incident from the negative $x$ axis onto a positive potential energy step located at $x = 0$. The kinetic energy of the particles is less than the potential energy of the step. Which is the best description of the behavior of the particles?
    (1) All particles are reflected precisely at $x = 0$.
    (2) Some particles are reflected at the step and some are transmitted into the $x > 0$ region.
    (3) Some particles are reflected and some are absorbed.
    (4) All particles are reflected, but they can penetrate a short distance into the $x > 0$ region.
    (5) All particles are absorbed at the step.

4. The probability to find a particle at any specific location in space:
    (1)  is directly proportional to the amplitude of the wave function.
    (2)  can never be zero.
    (3)  depends on the squared amplitude of the wave function.
    (4)  can sometimes be infinite.

5. The Schrödinger equation is
    (a) a second-order differential equation.
    (b) an equation based on conservation of energy.
    (c) an equation whose solution gives the wave function that describes a particle.
  How many of the above statements are true?
    (1) Zero      (2) One      (3) Two      (4) All three

**Answers**     1. 1    2. 3    3. 4    4. 3    5. 4

## B. Conceptual or Discussion Questions

1. A particle of mass $m$ is in the ground state of an infinite potential energy well of width $L$. The energy of the particle is 2.0 eV
(a) How much energy must be <u>added</u> to the particle to cause it to jump to the first excited state?
    (1) 2.0 eV    (2) 4.0 eV    (3) 6.0 eV    (4) 8.0 eV
(b) Suppose the particle is in the second excited state ($n = 3$) from which it can jump to any lower state by emitting a photon whose energy is equal to the difference in the energies of the two states. Considering all possible jumps that lead eventually to the ground state, which photon energy would NOT be observed?
    (1) 6.0 eV    (2) 10.0 eV    (3) 12.0 eV    (4) 16.0 eV
(c) What would be the ground-state energy of this particle if the width of the well were changed to $2L$?
    (1) 0.5 eV    (2) 1.0 eV    (3) 2.0 eV    (4) 4.0 eV

2. Suppose a particle is in the ground state ($n = 1$) of an in finite potential energy well. The energy is $E_1 = \hbar^2 \pi^2 / 2mL^2 = K + U = K$ because $U = 0$.

Thus $\quad \dfrac{\hbar^2 \pi^2}{2mL^2} = \dfrac{p_x^2}{2m} \quad$ or $\quad p_x = \dfrac{\hbar \pi}{L}$

If $p_x$ is exactly known, then $\Delta p_x = 0$. For this problem, $\Delta x = L$ (the particle is known to be somewhere in that interval). We then obtain $\Delta x \, \Delta p_x = (L)(0) = 0$, in violation of the uncertainty principle.

<div align="center">WHAT'S WRONG WITH THIS CALCULATION???</div>

3. Particles are incident from the negative $x$ axis onto a potential energy step at $x = 0$. At the step the potential energy drops from the positive value $U_0$ for all $x < 0$ to the value 0 for all $x > 0$. The energy of the particles is greater than $U_0$.
   (a) Which statement best describes the behavior of the particles?
       (1) All particles are transmitted from $x < 0$ to $x > 0$.
       (2) All particles are reflected back to $x < 0$ at the step.
       (3) Some particles are reflected and some are transmitted.
       (4) Some particles are reflected and some are absorbed.
   (b) How would the wavelength of a particle change as it moves from the $x < 0$ region to the $x > 0$ region?
       (1) Increases       (2) Decreases       (3) Remains the same
   (c) What are the continuity conditions at $x = 0$?
       (1) Both $\psi$ and $d\psi/dx$ are continuous at $x = 0$.
       (2) $\psi$ is continuous at $x = 0$, but $d\psi/dx$ is not.
       (3) $d\psi/dx$ is continuous at $x = 0$, but $\psi$ is not.
       (4) Neither $\psi$ nor $d\psi/dx$ are continuous at $x = 0$.

4. The ground-state energy of a simple harmonic oscillator is 1.0 eV.
   (a) If the oscillator is in its ground state, how much energy must be added for it to reach the first excited state?
       (1) 0.5 eV   (2) 1.0 eV   (3) 2.0 eV   (4) 3.0 eV   (5) 4.0 eV
   (b) How much energy must be added to move the oscillator from the first excited state to the second excited state?
       (1) 0.5 eV   (2) 1.0 eV   (3) 2.0 eV   (4) 3.0 eV   (5) 4.0 eV
   (c) The oscillator is in the third excited state ($n = 3$). It can jump to any lower state, in the process emitting a photon whose energy is equal to the difference in energy between the states. How many different photon energies can be emitted if these oscillators can take any possible path from the excited state to the ground state?
       (1) 1   (2) 2   (3) 3   (4) 4   (5) 5   (6) 6

**Answers**     1. (a) 3   (b) 3   (c) 1          2. $p_x = \pm \hbar \pi / L$, so $\Delta p_x \sim 2\hbar \pi / L$
                3. (a) 3   (b) 2   (c) 1          4. (a) 3   (b) 3   (c) 3

[Note to instructors: Some students have trouble understanding energy diagrams, which impedes their efforts to learn quantum mechanics. This worksheet helps them make the transition from classical to quantum energy diagrams.]

## Energy Diagrams Worksheet

1. A small puck is gliding with initial speed $v$ across a frictionless horizontal surface. It glides up a small hill and then moves on a horizontal surface that is a distance $h$ above the first surface.
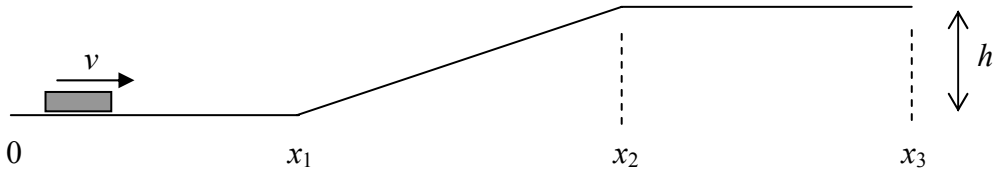


Sketch a plot that shows, on the same diagram, the gravitational potential energy $U$, kinetic energy $K$, and total energy $E = K + U$ for this system.
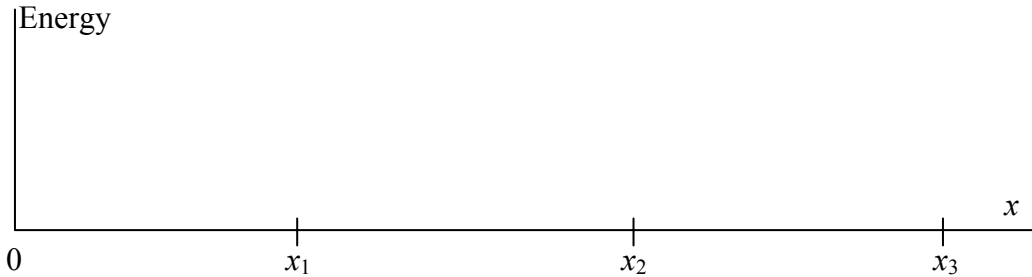


2. An electron is moving with initial speed $v$ inside a thin hollow metal tube. It emerges from the tube through a hole in a large metal plate and continues through a hole in a second plate into another thin tube. The two plates are connected across a battery of potential difference $V$.
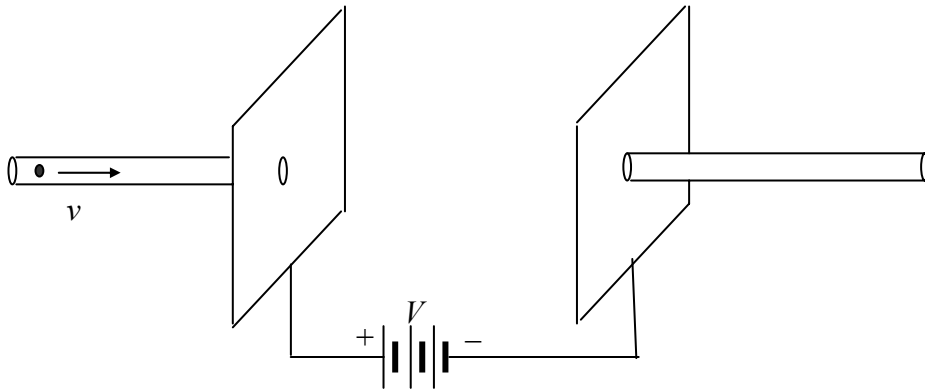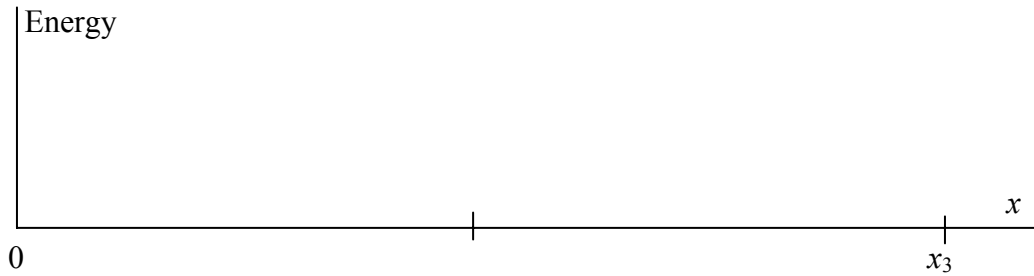


Sketch a plot that shows, on the same diagram, the electrical potential energy $U$, kinetic energy $K$, and total energy $E = K + U$ for this system.
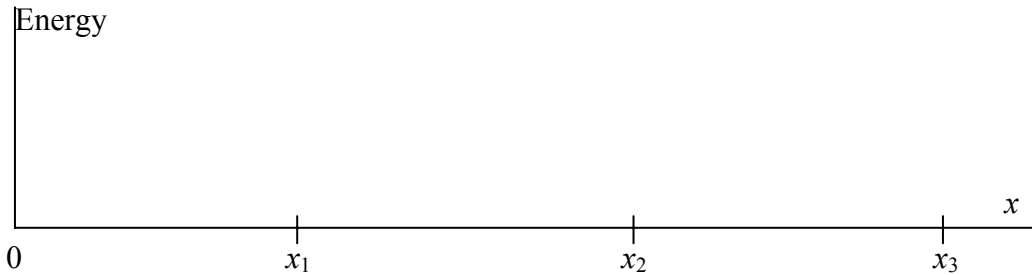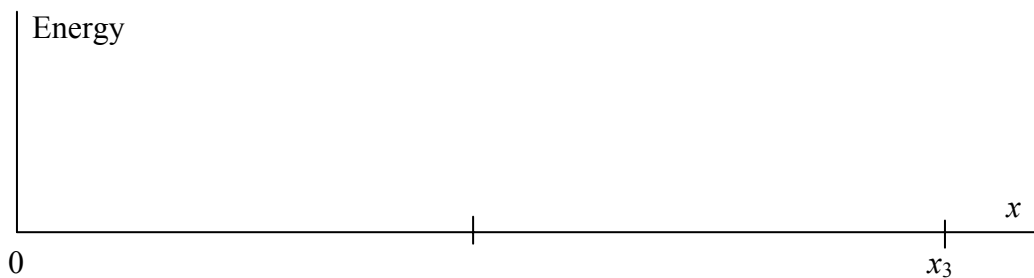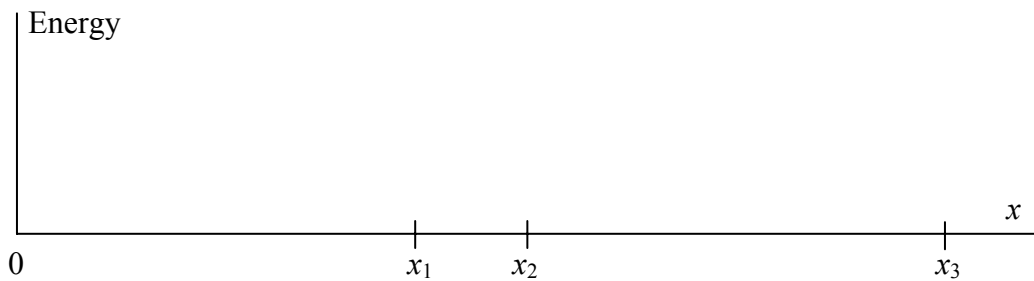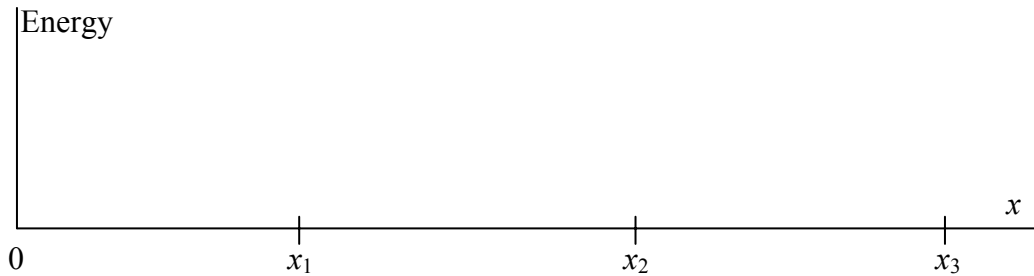
3.  Redraw the energy diagram for these situations if the distance $x_2 - x_1$ (over which the potential energy changes) becomes smaller.

Energy

$x$

$0$        $x_1$     $x_2$                    $x_3$
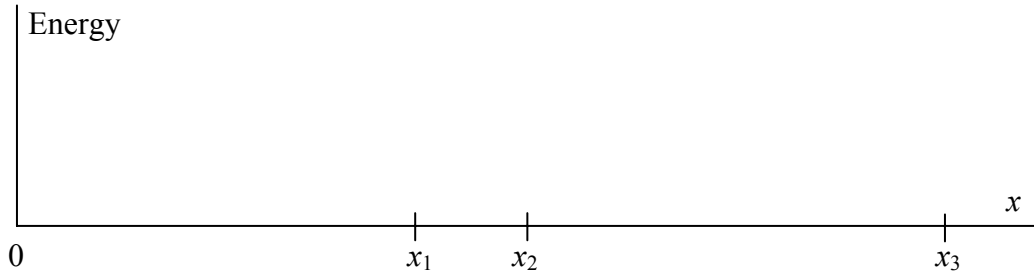
4.  Redraw the energy diagram for the limit in which $x_2 - x_1$ becomes zero.

Energy

$x$

$0$                              $x_3$

5.  Draw energy diagrams similar to those of questions 2, 3, and 4, but now assume that the initial kinetic energy is only half of the energy needed to climb the potential energy hill between the two plates.

Energy

$x$

$0$              $x_1$                $x_2$              $x_3$

Energy

$x$

$0$        $x_1$     $x_2$                    $x_3$

Energy

$x$

$0$                              $x_3$

[Note to instructors: This worksheet should be done by students working in groups, preferably in class.]

Schrödinger Worksheet

In this calculation you will analyze the quantum behavior of a particle moving in the following potential energy well:

Region 1:         $U(x) = \infty$          $x < 0$
Region 2:         $U(x) = 0$          $0 \leq x \leq L$
Region 3:         $U(x) = U_0$          $L \leq x \leq 2L$          $(U_0 > 0)$
Region 4:         $U(x) = \infty$          $x > 2L$

Begin by sketching this potential energy well.

**Part A.  First we will consider states of motion of the particle of energy $E < U_0$.**

1.  Write down solutions for the wave function in each of the 4 regions.

$\psi_1 =$
$\psi_2 =$                                       (use coefficients $A$ and $B$)
$\psi_3 =$                                       (use coefficients $C$ and $D$)
$\psi_4 =$

$\psi_2$ should be written in terms of the parameter $k = (2mE/\hbar^2)^{1/2}$.
$\psi_3$ should be written in terms of the parameter $k' = [2m(U_0 - E)/\hbar^2]^{1/2}$.

2.  Apply the <u>boundary condition</u> on $\psi$ at $x = 0$ [$\psi_1(x=0) = \psi_2(x=0)$] to eliminate one of the coefficients $A$ or $B$.  (Which one will be eliminated depends on how you wrote P$_2$.)

3.  Apply the boundary condition on $\psi$ at $x = 2L$ [$\psi_3(x=2L) = \psi_4(x=2L)$] to eliminate one of the coefficients $C$ or $D$.  (Solve for $C$ in terms of $D$ or for $D$ in terms of $C$.)

4. Apply the boundary condition on $\psi$ at $x = L$ [$\psi_2(x=L) = \psi_3(x=L)$] to express your remaining coefficient in $\psi_3$ (either $C$ or $D$) in terms of your remaining coefficient in $\psi_2$ (either $A$ or $B$).

5. Your entire wave function in the region $x = -\infty$ to $x = +\infty$ should now depend on only one coefficient (either $A$ or $B$). Explain how the normalization condition allows you to determine this remaining coefficient. Don't carry out the calculation, just show how it can be done.

6. There is still one boundary condition you haven't yet applied – the condition on $d\psi/dx$ at $x = L$. Explain how applying this boundary condition allows you to determine the energy $E$. Don't carry out the calculation, just show how it can be done.

**Part B. Now we will repeat the calculation for states of energy $E > U_0$.**

1. Write down solutions for the wave function in each of the 4 regions.

       $\psi_1 =$
       $\psi_2 =$                                     (use coefficients $A$ and $B$)
       $\psi_3 =$                                       (use coefficients $C$ and $D$)
       $\psi_4 =$

$\psi_2$ should be written in terms of the parameter $k = (2mE/\hbar^2)^{1/2}$.
$\psi_3$ should be written in terms of the parameter $k' = [2m(E - U_0)/\hbar^2]^{1/2}$.

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2. Apply the <u>boundary condition</u> on $\psi$ at $x = 0$ to eliminate one of the coefficients $A$ or $B$. (Which one will be eliminated depends on how you wrote P$_2$.)

3. Apply the boundary condition on $\psi$ at $x = 2L$ to eliminate one of the coefficients $C$ or $D$.

4. Apply the boundary condition on $\psi$ at $x = L$ to express your remaining coefficient in $\psi_3$ (either $C$ or $D$) in terms of your remaining coefficient in $\psi_2$ (either $A$ or $B$).

5. Your entire wave function in the region $x = -\infty$ to $x = +\infty$ should now depend on only one coefficient (either $A$ or $B$). Explain how the normalization condition allows you to determine this remaining coefficient. Don't carry out the calculation, just show how it can be done.

6. There is still one boundary condition you haven't yet applied – the condition on $d\psi/dx$ at $x = L$. Explain how applying this boundary condition allows you to determine the energy $E$. Don't carry out the calculation, just show how it can be done.

Part C.   Now consider the probability to locate the particle using the wave functions of Part B.

1.  What is the probability of finding the particle in each region (1, 2, 3, and 4)?  Don't evaluate any mathematical expressions, just write them down as completely as you can.

2.  How does the probability to find the particle in region 2 compare with the probability to find the particle in region 3?

          \_\_\_ probability (2) > probability (3)
          \_\_\_ probability (2) = probability (3)
          \_\_\_ probability (2) < probability (3)

Use the classical behavior of the particle to justify your answer.  What does this imply about the wave function?

**Part D.  Without solving the equations for the various coefficients that you found in Parts A and B, sketch $\psi(x)$ in the entire region $x = -\infty$ to $x = +\infty$ in the following cases.**

1.  The lowest energy state in Part A.
2.  The next higher state above the lowest one in Part A (the first excited state in Part A).
3.  The second excited state in Part A.
4.  The lowest energy state in Part B.
5.  The next higher state above the lowest one in Part B.

Your sketches need not be exact representations of the wave functions, but each sketch should clearly illustrate:
       (a)  the continuity conditions at each boundary;
       (b)  relative wavelengths associated with the particle motion;
       (c)  relative amplitudes of the wave function.

**Sample Exam Questions**

**A. Multiple Choice**

1. A particle in the first excited state of a one-dimensional infinite potential energy well (with $U = 0$ inside the well) has an energy of 6.0 eV. What is the energy of this particle in the ground state?
    (a) 1.0 eV             (b) 1.5 eV             (c) 2.0 eV             (d) 3.0 eV

2. The ground state of a particle in simple harmonic motion has energy 0.5 eV and the first excited state has energy 1.5 eV. What is the energy of the next excited state?
    (a) 2.5 eV             (b) 3.5 eV             (c) 3.0 eV             (d) 2.0 eV

3. The ground-state energy of a particle in an infinite one-dimensional potential energy well is 6.0 eV. What is the energy of the first excited state?
    (a) 12.0 eV           (b) 18.0 eV           (c) 7.5 eV             (d) 24.0 eV

4. In a certain infinite potential energy well, the particle has a ground-state energy of 2.0 eV. Which of the following is NOT a possible value for the energy of one of the excited states of this particle in the well?
    (a) 36 eV             (b) 50 eV             (c) 18 eV             (d) 8 eV

5. A particle is moving in an infinite potential energy well in one dimension. In the ground state, the energy of the particle is 5.0 eV. Which one of the following is a possible energy for an excited state?
    (a) 25.0 eV           (b) 40.0 eV           (c) 80.0 eV           (d) 100.0 eV

6. An electron in the ground state of an infinite potential energy well has an energy of 8.0 eV. How much <u>additional</u> energy must be supplied for the electron to jump from the ground state to the first excited state?
    (a) 8.0 eV            (b) 16.0 eV           (c) 24.0 eV           (d) 32.0 eV

7. Consider the following two possible solutions to the Schrödinger equation in the <u>entire</u> interval from $x = 0$ to $x = +\infty$:    (i) $\psi(x) = Ae^{-kx}/x$    (ii) $\psi(x) = Ae^{+kx}$
    ($A$ and $k$ are real positive constants.) Which are allowable wave functions?
    (a) Only $i$    (b) Only $ii$    (c) Both $i$ and $ii$    (d) Neither $i$ nor $ii$

8. A beam of particles is incident from the negative $x$ direction on a potential energy step at $x = 0$. When $x < 0$, the potential energy of the particles is zero, and for $x > 0$ the potential energy has the constant positive value $U_0$. In the region $x < 0$, the particles have a kinetic energy $K$ that is smaller than $U_0$. What is the form of the wave function in the region $x > 0$?
    (a) $Ae^{kx} + Be^{-kx}$    (b) $Ae^{ikx} + Be^{-ikx}$    (c) $Ae^{kx}$    (d) $Ae^{-kx}$    (e) $A\cos kx + B\sin kx$

**Answers**      1. b    2. a    3. d    4. a    5. c    6. c    7. d    8. d

## B. Conceptual

1.  A beam of particles is incident from the negative $x$ axis, where $U(x) = 0$, on a barrier of constant height $U_0$ which extends from $x = 0$ to $x = L$. Beyond the barrier (where $x > L$), the potential energy is again equal to zero. The particles have energy $E$ which is less than the height of the barrier. Let $B$ represent the amplitude of the wave function of particles that appear beyond the barrier (where $x > L$). Keeping the height of the barrier constant, we suddenly increase the thickness of the barrier from $L$ to $2L$ (that is, the barrier now extends from $x = 0$ to $x = 2L$). Is the amplitude of the wave function for particles beyond the new barrier (where $x > 2L$) *greater than B, equal to B, or less than B?* EXPLAIN YOUR ANSWER.

2.  The wave function shown is associated with the motion of an electron in one of the four energy levels of a finite potential energy well of width $L$ and depth $U$. Identify the number of the level with which this wave function is associated, and describe where in this region the electron would most likely be found. EXPLAIN YOUR ANSWERS.
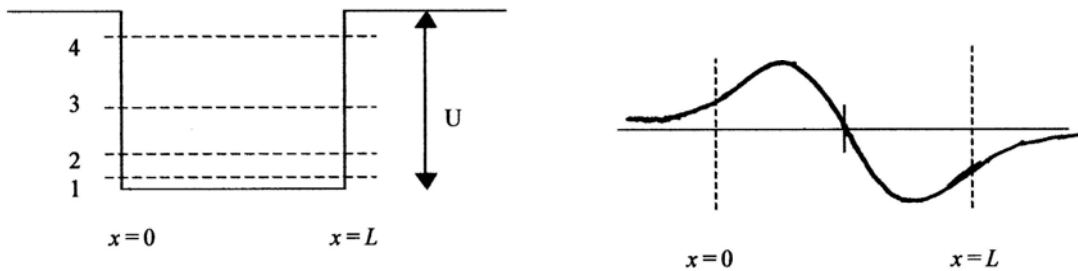


3.  A beam of particles of energy 3 eV is moving in the region $x < 0$ toward the origin. The potential energy is zero in this region. Starting at $x = 0$ and continuing until $x = L$, the potential energy is 10 eV, and beyond $x = L$ the potential energy is once again zero. Is the wavelength in the region $x > L$ *greater than, smaller than, or equal to* the wavelength in the region $x < 0$, and is the maximum probability to find particles in the region $x > L$ *greater than, smaller than, or equal to* the maximum probability to find particles in the region $x < 0$? EXPLAIN YOUR ANSWERS. (Note that this problem requires 2 answers.)

4.  A particle is trapped between $x = 0$ and $x = 0.120$ nm in a potential energy well of infinite height. The particle may be in either the **first excited state** or the **second excited state**. For which of these two states is there a greater probability of finding the particle in the region between $x = 0.039$ nm and $x = 0.041$ nm? EXPLAIN YOUR ANSWER.

5. A particle moves in a one-dimensional potential energy $U(x)$ specified by:

$U(x) = \infty$       $x < 0$       (region 1)
$U(x) = -30$ eV    $0 < x < L$     (region 2)
$U(x) = -15$ eV    $L < x < 3L$    (region 3)
$U(x) = 0$         $3L < x < 4L$   (region 4)
$U(x) = \infty$       $x > 4L$      (region 5)

The energy of the particle is -5 eV.
If we measure the position of the particle at a random time, in which region (1,2,3,4, or 5) are we most likely to find the particle? EXPLAIN YOUR ANSWER.
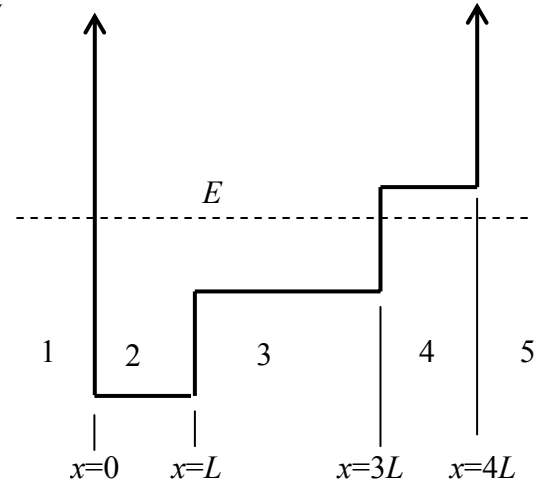


6. An electron is trapped in an infinite potential energy well with walls at $x = 0$ and $x = 0.100$ nm. Consider the probability to find the electron in a narrow region in the center of the well between $x = 0.049$ nm and $x = 0.051$ nm. Is the probability to find the electron in this region when the electron is in its ground state *greater than, less than, or equal to* the probability to find the electron in this same region when it is in the first excited state? EXPLAIN YOUR ANSWER.

7. A particle is trapped in the region between $x = 0$ and $x = L$ by two rigid walls. The particle is in its ground state. Is the probability to locate the particle in a small interval of width *dx* near $x = 0.5L$ *greater than, less than, or equal to* the probability to locate it in a small interval of width *dx* near $x = 0.1L$? EXPLAIN YOUR ANSWER.

8. An electron is trapped in a one-dimensional region of space between two rigid walls at $x = 0$ and $x = L$. In the first excited state, where would you expect that the electron is most likely found? EXPLAIN YOUR ANSWER.

**Answers**     1. less than     2. $n = 2$, near $x = L/4$ and $3L/4$     3. equal to, smaller than
                  4. first excited state         5. 3          6. greater than
                  7. greater than       8. $x = L/4$ and $3L/4$

## C. Problems

1. A particle of mass $m$ is moving in a one-dimensional region of space where there is a Coulomb-like potential energy $U(x) = -C/x$, where $C$ is a positive constant.
(a) Show that $\psi(x) = Axe^{-ax}$ (with $a = mC/\hbar^2$) is a solution to the one-dimensional Schrödinger equation, and find the corresponding energy $E$.
(b) Would you expect this wave function to be valid for the entire region $-\infty \leq x \leq +\infty$? Explain.
(c) Set up the equation that can be solved to find the constant $A$. Do not try to solve the equation, just set it up.

2.  A beam of electrons with energy $E$ is incident from the negative $x$ direction on a potential energy step located at $x = 0$.  The potential energy is zero for $x < 0$ and has the constant positive value $U_0$ for $x > 0$.  Assume $E$ is about $U_0/3$.
    (a) What are the wave functions that describe the particle in the region $x < 0$ and in the region $x > 0$?  You should use a total of 3 undetermined coefficients in your two wave functions.  Define all parameters that appear in your wave functions.
    (b) Show how to use the continuity conditions at $x = 0$ to find relationships among the coefficients.
    (c) Sketch the wave function.
    (d) Describe as carefully as you can how the wave functions in the regions $x < 0$ and $x > 0$ would change if $E$ were doubled (but still less than $U_0$).

3.  (a) A neutron ($mc^2 = 939.6$ MeV) is confined in a nucleus of diameter 11 fm.  Inside the nucleus, the neutron moves freely (no forces act on it), but at the edges of the nucleus a very strong force (which we can take to be infinitely strong) prevents the neutron from leaving the nucleus.  Treating this as a one-dimensional problem, find the energy difference between the ground state and the first excited state of the neutron.
    (b) In the ground state, what is the probability to find the neutron in a narrow region of width 0.10 fm located at the center of the nucleus?

4.  Consider the following potential energy:
    region 1:  $U(x) = U_0$          $x < 0$
    region 2:  $U(x) = 0$            $0 < x < L$
    region 3:  $U(x) = U_0$          $x > L$
    where $U_0 > 0$.  We want to consider a particle with energy $E$ such that $0 < E < U_0$.
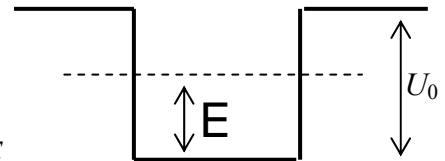
    There are two possible forms for the wave function that might be used to represent the particle:
    $$\psi(x) = A_i \sin k_i x + B_i \cos k_i x$$
    $$\psi(x) = A_i e^{k_i x} + B_i e^{-k_i x}$$
    where $i = 1,2,3$ indicates the three regions.
    (a) Write down wave functions that describe the behavior of the particle in region 1, region 2, and region 3.  Use appropriate subscripts to label all parameters.  If any of the coefficients $A_i$ or $B_i$ are zero, identify those coefficients and explain why they are equal to zero.
    (b) Sketch the probability distributions you would expect for the ground state and the first excited state.
    In region 2, how is the probability distribution of the ground state different from the probability distribution of the first excited state?
    In regions 1 and 3, how is the probability distribution of the ground state different from the probability distribution of the first excited state?
    (c) Use the continuity conditions at $x = 0$ to show how the coefficients of the wave function in region 2 are related to the coefficients of the wave function in region 1.

5. Consider the following one-dimensional potential energy barrier:

$$U(x) = 0 \qquad x < 0$$
$$U(x) = U_0 \qquad x > 0 \qquad\qquad (U_0 \text{ is a positive constant})$$

Particles of mass $m$ and energy $E$ are incident on this barrier from the negative $x$ axis.
(a) Suppose the particles have energy greater than $U_0$. Write down the wave function for the regions $x < 0$ and $x > 0$. Explain briefly how to determine all constants that appear in your solutions. Sketch a possible form for the wave function, paying careful attention to how the wavelength and amplitude change at $x = 0$.
(b) Now suppose the particles have energy less than $U_0$. Write down the wave function for the regions $x < 0$ and $x > 0$. Explain briefly how to determine all constants that appear in your solutions. Sketch a possible form for the wave function.

6. A beam of particles of mass $m$ is traveling from $x = -\infty$ toward $x = 0$. In that region the potential energy is zero. In the region $x > 0$ there is a constant negative potential energy $U_0$.
(a) By direct substitution into the Schrodinger equation, show that the particles in the region $x < 0$ can be represented by the wave function $\psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x}$. In the process of solving this equation, find the energy of the particles as a function of $k_1$.
(b) Describe the physical meaning of the constants $A$ and $B$.
(c) Write down the wave function $\psi_2$ that would describe the particles in the region $x > 0$. Define all constants that appear in your wave function. Show how you would apply the continuity conditions at $x = 0$ to determine these constants. It is not necessary to find the algebraic solutions to the continuity equations, just set them up in terms of the constants that need to be determined.

7. A beam of electrons is traveling along the $x$ axis starting at $x = -\infty$ in a region in which the potential energy can be taken to be zero. The electrons have a kinetic energy of $E = 4.0$ eV. At $x = 0$, the potential energy suddenly rises to $U_0 = 10.0$ eV and stays at that value to $x = +\infty$.
(a) Write down the wave function for this situation in the region $x < 0$. Explain the meaning of all terms and constants in your equation.
(b) Write down the wave function for this situation in the region $x > 0$. Explain the meaning of all terms and constants in your equation.
(c) Show how to apply the continuity conditions at $x = 0$.
(d) Sketch the probability density as a function of $x$ showing portions of both regions. Be sure your sketch clearly indicates the zero of the probability density axis.
(e) Sketch the probability density for the case in which the original kinetic energy is 8.0 eV, with all other parts of the problem remaining the same. Describe 2 ways in which this probability density differs from part (d).

8. An electron is trapped in a region between two perfectly rigid walls (which can be regarded as infinitely high energy barriers). In the region between the walls the potential energy of the electron is zero. The (normalized) wave function of the

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electron in the region between the walls is $\psi(x) = a \sin bx$ where $a = 0.50$ nm$^{-1/2}$ and $b = 1.18$ nm$^{-1}$.
(a) By substituting into the Schrödinger equation, find the energy of the electron in this state of motion.
(b) What is the probability to find the electron between $x = 0.99$ nm and $x = 1.01$ nm?

9. A particle of mass $m$ is trapped in a one-dimensional finite potential energy well. The potential energy is zero in the region between $x = 0$ and $x = L$, and it is equal to the constant positive value $U_0$ in the regions $x < 0$ and $x > L$. We want to examine bound-state solutions with $E < U_0$.
(a) Write down the wave functions that describe the particle in the regions $x < 0$, $0 < x < L$, and $x > L$. It is not necessary to apply the boundary conditions, but be sure all your wave functions remain finite in the regions in which they are defined. All unknown parameters that appear in your equations must be defined.
(b) Sketch the probability densities for the ground state and the first excited state. How do the ground-state and first excited-state wave functions differ from each other inside the well $(0 < x < L)$? How do they differ from each other outside the well $(x < 0$ or $x > L)$?.

10. In some situations, the behavior of an electron can be approximated as if the electron were bound to an equilibrium position by a spring force $(F = -kx, U = kx^2/2)$. Suppose such an electron were in the first excited state, with a wave function $\psi(x) = Axe^{-ax^2}$, where $A$ is a constant and $a = \sqrt{km}/2\hbar$. In this equation, $x$ represents the distance of the electron from its equilibrium position.
(a) Find the energy of the electron in terms of the spring constant $k$ and its mass $m$.
(b) If the electron behaved like a classical oscillating particle, the largest value of $x$ would be $x_m$. Find $x_m$ in terms of $k$ and $m$.
(c) Sketch a drawing that shows the probability to find the electron as a function of $x$ in the range $-\infty$ to $+\infty$. Indicate where $x_m$ would appear on your sketch.
(d) Explain the meaning the probability when $x > x_m$.

11. (a) For a particle of mass $m$ moving in the potential energy $U(x) = -A/x$, one solution to the Schrödinger equation for the region $0 < x < \infty$ is $\psi(x) = Bxe^{-bx}$, where $b = 4\pi^2 Am/h^2$. Show that this function is a solution, and find the corresponding energy $E$.
(b) Sketch the probability density for this wave function. Where would you expect the electron to be most likely to be found?

12. A particle moves in the following one-dimensional potential energy well:
$$\begin{array}{ll} U = \infty & x < 0 \\ U = 0 & 0 < x < L \\ U = U_0 & L < x < 2L \\ U = \infty & x > 2L \end{array}$$
where $U_0$ is a positive constant.
(a) Assume $E > U_0$. Without doing the complete mathematical solution, describe the wave function of the particle as completely as you can. Specify the behavior of the

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wave function in all 4 regions of the $x$ axis. Sketch wave functions that might represent the ground state and the first excited state. Pay special attention to the wave behavior of the particle. Describe clearly what happens to the wave function at each of the boundaries between the regions.
(b) Repeat part (a), but now assume $U_0 > E > 0$.

13. A beam of particles of mass $m$ and energy $E$ is traveling from $x = -\infty$ in a region in which the potential energy is zero. Between $x = 0$ and $x = L$ the potential energy has the negative value $-U_0$, and beyond $x = L$ the potential energy is again equal to 0.
(a) Find the wavelengths of the particles in the three regions $x < 0$, $0 < x < L$, and $x > L$.
(b) Using complex exponentials in the form $e^{\pm ikx}$, write down the wave functions that describe the particles in the three regions. Explain the meanings of any coefficients you use in your equations.
(c) Sketch the probability density, showing the regions from $x < 0$ to $x > L$.

**Answers**    2. (a) $\psi_1 = A\sin k_1 x + B\cos k_1 x$ with $k_1 = \sqrt{2mE/\hbar^2}$

$\psi_2 = Ce^{-k_2 x}$ with $k_2 = \sqrt{2m(U_0 - E)/\hbar^2}$

3. (a) 5.07 MeV   (b) 0.018

4. (a) $\psi_1 = A_1 e^{k_1 x}$, $\psi_2 = A_2\sin k_2 x + B_2\cos k_2 x$, $\psi_3 = B_3 e^{-k_3 x}$

5. (a) $\psi_1 = A\sin k_1 x + B\cos k_1 x$ with $k_1 = \sqrt{2mE/\hbar^2}$

$\psi_2 = C\sin k_2 x + D\cos k_2 x$ with $k_2 = \sqrt{2m(E - U_0)/\hbar^2}$

(b) $\psi_2 = De^{-k_2 x}$ with $k_2 = \sqrt{2m(U_0 - E)/\hbar^2}$

6. (a) $E = \hbar^2 k_1^2 / 2m$   (c) $\psi_2 = Ce^{ik_2 x}$ with $k_2 = \sqrt{2m(E + U_0)/\hbar^2}$

7. (a) $\psi_1 = A\sin k_1 x + B\cos k_1 x$ with $k_1 = \sqrt{2mE/\hbar^2}$

$\psi_2 = Ce^{-k_2 x}$ with $k_2 = \sqrt{2m(U_0 - E)/\hbar^2}$

8. (a) 0.053 eV   (b) 0.0043

9. (a) $\psi_1 = A_1 e^{k_1 x}$, $\psi_2 = A_2\sin k_2 x + B_2\cos k_2 x$, $\psi_3 = B_3 e^{-k_3 x}$

10. (a) $\frac{3}{2}\hbar\sqrt{k/m}$   (b) $(mk)^{-1/4}\sqrt{3\hbar}$

12. (a) $\psi_1 = 0$, $\psi_2 = A\sin kx + B\cos kx$ with $k = \sqrt{2mE/\hbar^2}$

$\psi_3 = A'\sin k'x + B'\cos k'x$ with $k' = \sqrt{2m(E - U_0)/\hbar^2}$, $\psi_4 = 0$

(b) $\psi_3 = A'e^{k'x} + B'e^{-k'x}$ with $k' = \sqrt{2m(U_0 - E)/\hbar^2}$

13. (a) $h/\sqrt{2mE}$, $h/\sqrt{2m(E + U_0)}$, $h/\sqrt{2mE}$

(b) $\psi_1 = Ae^{ikx} + Be^{-ikx}$ with $k = \sqrt{2mE/\hbar^2}$

$\psi_2 = Ce^{ik'x} + De^{-ik'x}$ with $k' = \sqrt{2m(E + U_0)/\hbar^2}$, $\psi_3 = Fe^{ikx}$

**Problem Solutions**

1.  (a) In air, the initial velocity is zero, so $v_1(t) = -gt$ and $y_1(t) = H - \frac{1}{2}gt^2$.

(b) The time $T$ at which the ball hits the water is determined by setting $y_1$ to zero:

$$y_1(T) = H - \frac{1}{2}gT^2 = 0 \qquad \text{so} \qquad T = \sqrt{2H/g}$$

At this time, we set $y_1(T) = y_2(T)$ and $v_1(T) = v_2(T)$:

$$H - \frac{1}{2}g\left(\sqrt{\frac{2H}{g}}\right)^2 = \frac{1}{2}\left(\frac{B-mg}{m}\right)\left(\sqrt{\frac{2H}{g}}\right)^2 + b\sqrt{\frac{2H}{g}} + c \quad \text{and} \quad -g\sqrt{\frac{2H}{g}} = \left(\frac{B-mg}{m}\right)\sqrt{\frac{2H}{g}} + b$$

The second equation can be solved directly to give

$$b = -\frac{B}{m}\sqrt{\frac{2H}{g}}$$

and inserting this value into the first equation gives

$$c = H\left(\frac{B}{mg} + 1\right)$$

2.  Let $y_1(x) = A\cos(2\pi x/\lambda + \pi/3)$ for $x < 0$. For $x > 0$, the wave is

$$y_2(x) = A_2 \cos(2\pi x/\lambda_2 + \phi)$$

with $\lambda_2 = \lambda/2$. The continuity conditions on $y$ and $dy/dx$ at $x = 0$ give

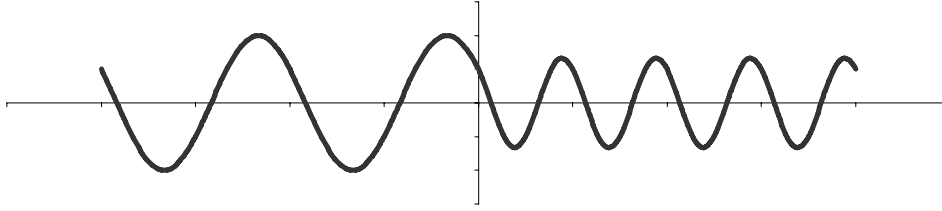$$y_1 = y_2 \qquad \text{or} \qquad A\cos(\pi/3) = A_2 \cos\phi$$

$$\frac{dy_1}{dx} = \frac{dy_2}{dx} \qquad \text{or} \qquad -\frac{2\pi A}{\lambda}\sin(\pi/3) = -\frac{4\pi A_2}{\lambda}\sin\phi$$

Dividing the second of these equations by the first, we obtain

$$\tan(\pi/3) = 2\tan\phi \qquad \text{or} \qquad \phi = \tan^{-1}\left[\tfrac{1}{2}\tan(\pi/3)\right] = 40.9°$$

and solving the first equation for $A_2$,

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$$A_2 = A\frac{\cos\pi/3}{\cos\phi} = 0.661A$$



3.                  $$E_1 = \frac{h^2}{8mL^2} = 4.4\text{ eV}$$

With $L' = 2L$,

$$E_1' = \frac{h^2}{8mL'^2} = \frac{h^2}{8m(2L)^2} = \frac{1}{4}\frac{h^2}{8mL^2} = \frac{1}{4}(4.4\text{ eV}) = 1.1\text{ eV}$$

4.  With $\lambda_n = 2L/n$,

$$\lambda_1 = \frac{2(0.120\text{ nm})}{1} = 0.240\text{ nm} \qquad \lambda_2 = \frac{0.240\text{ nm}}{2} = 0.120\text{ nm} \qquad \lambda_3 = \frac{0.240\text{ nm}}{3} = 0.080\text{ nm}$$

5.  The smallest energy is (using Equation 5.3)

$$E_1 = \frac{h^2}{8mL^2} = \frac{(hc)^2}{8(mc^2)L^2} = \frac{(1240\text{ eV}\cdot\text{nm})^2}{8(511,000\text{ eV})(0.050\text{ nm})^2} = 150\text{ eV}$$

Then $E_2 = 2^2 E_1 = 600\text{ eV}$ and $E_3 = 3^2 E_1 = 1350\text{ eV}$.

6.  With $L = 1.0 \times 10^{-14}$ m $= 10$ fm,

$$E_1 = \frac{h^2}{8mL^2} = \frac{(hc)^2}{8(mc^2)L^2} = \frac{(1240\text{ MeV}\cdot\text{fm})^2}{8(940\text{ MeV})(10\text{ fm})^2} = 2.0\text{ MeV}$$

7.  (a) At $x = a$, $\psi_1 = \psi_2$ and $d\psi_1/dx = d\psi_2/dx$ :

$$0 = (a-d)^2 - c \qquad \text{and} \qquad -2ab = 2(a-d)$$

From the second equation, $d = a(b+1)$. Inserting this into the first equation, we find $c = a^2 b^2$.

(b) With $\psi_2 = \psi_3$ at $x = w$, we get $(w-d)^2 - c = 0$, or

$$w = d + \sqrt{c} = a(b+1) + \sqrt{a^2 b^2} = a(2b+1)$$

The slope is discontinuous at $w$ suggesting an infinite discontinuity in the potential energy at that location.

8. (a) The regions with $x < a$ and $x > a$ do not contribute to the normalization. The normalization integral is

$$\int |\psi(x)|^2 \, dx = \int_{-a}^{+a} b^2 (a^2 - x^2)^2 \, dx = b^2 \int_{-a}^{+a} (a^4 - 2a^2 x^2 + x^4) \, dx = b^2 \left( a^4 x - 2a^2 \frac{x^3}{3} + \frac{x^5}{5} \right) \Bigg|_{-a/2}^{+a/2}$$

Evaluating the integral and setting it equal to 1, we find

$$b^2 \left( 2a^5 - \frac{4a^5}{3} + \frac{2a^5}{5} \right) = 1 \qquad \text{or} \qquad b = \sqrt{\frac{15}{16a^5}}$$

(b) $P(x) \, dx = |\psi(x)|^2 \, dx = b^2 (a^2 - x^2)^2 \, dx$, and with $x = +a/2$ and $dx = 0.010a$ we obtain

$$P(x) \, dx = \frac{15}{16a^5} \left( \frac{a^2}{4} - a^2 \right)^2 (0.010a) = 0.0053$$

(c) $P(a/2 : a) = \int_{a/2}^{a} |\psi(x)|^2 \, dx = \int_{a/2}^{a} b^2 (a^2 - x^2)^2 \, dx = \frac{15}{16a^5} \left( a^4 x - 2a^2 \frac{x^3}{3} + \frac{x^5}{5} \right) \Bigg|_{a/2}^{a}$

$$= \frac{15}{16a^5} \left[ a^4 \left( a - \frac{a}{2} \right) - \frac{2a^2}{3} \left( a^3 - \frac{a^3}{8} \right) + \frac{1}{5} \left( a^5 - \frac{a^5}{32} \right) \right] = 0.104$$

9. With $\psi(x) = Cxe^{-bx}$, we have $d\psi / dx = Ce^{-bx} - bCxe^{-bx}$ and

$$\frac{d^2\psi}{dx^2} = -2bCe^{-bx} + b^2 Cxe^{-bx}$$

We now substitute $\psi(x)$ and $d^2\psi / dx^2$ into the Schrödinger equation:

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$$-\frac{\hbar^2}{2m}(-2bCe^{-bx}+b^2Cxe^{-bx})+U(x)Cxe^{-bx}=ECxe^{-bx}$$

Canceling the common factor of $Ce^{-bx}$ and solving for $E$,

$$E=\frac{\hbar^2 b}{mx}-\frac{\hbar^2 b^2}{2m}+U(x)$$

The energy $E$ will be a constant only if the two terms that depend on $x$ cancel each other:

$$\frac{\hbar^2 b}{mx}+U(x)=0 \qquad \text{or} \qquad U(x)=-\frac{\hbar^2 b}{mx}$$

The cancellation of the two terms depending on $x$ leaves only the remaining term for the energy:

$$E=-\frac{\hbar^2 b^2}{2m}$$

10. (a) The regions with $x<-L/2$ and $x>+L/2$ do not contribute to the normalization integral. The remaining integral is:

$$\int|\psi(x)|^2\,dx=\int_{-L/2}^{0}C^2(2x/L+1)^2\,dx+\int_{0}^{L/2}C^2(-2x/L+1)^2\,dx$$

$$=C^2(4x^3/3L^2+2x^2/L+x)\Big|_{-L/2}^{0}+C^2(4x^3/3L^2-2x^2/L+x)\Big|_{0}^{L/2}=C^2L/3$$

Setting the integral equal to 1 gives $C=\sqrt{3/L}$.

(b) $P(x)\,dx=|\psi(x)|^2\,dx=C^2(4x^2/L^2-4x/L+1)\,dx$ and with $x=0.250L$ and $dx=0.010L$,

$$P(x)\,dx=\frac{3}{L}\left(\frac{4}{L^2}\frac{L^2}{16}-\frac{4}{L}\frac{L}{4}+1\right)(0.010L)=0.0075$$

(c)

$$P(0:L/4)=\int_{0}^{L/4}|\psi(x)|^2\,dx=C^2\int_{0}^{L/4}\left(\frac{4}{L^2}x^2-\frac{4}{L}x+1\right)dx=\frac{3}{L}\left(\frac{4}{L^2}\frac{x^3}{3}-\frac{4}{L}\frac{x^2}{2}+x\right)\Big|_{0}^{L/2}=\frac{1}{2}$$

(d) $\langle x\rangle=\int|\psi(x)|^2\,x\,dx=\frac{3}{L}\int_{-L/2}^{0}\left(\frac{4x^2}{L^2}+\frac{4x}{L}+1\right)x\,dx+\int_{0}^{L/2}\left(\frac{4x^2}{L^2}-\frac{4x}{L}+1\right)x\,dx$

$$= \frac{3}{L}\left(\frac{x^4}{L^2} + \frac{4x^3}{3L} + \frac{x^2}{2}\right)\Bigg|_{-L/2}^{0} + \frac{3}{L}\left(\frac{x^4}{L^2} - \frac{4x^3}{3L} + \frac{x^2}{2}\right)\Bigg|_{0}^{L/2} = 0$$

It is apparent from the shape of the wave function that the equal probability densities for positive and negative $x$ cancel to give an average of zero.

$$\langle x^2 \rangle = \int |\psi(x)|^2\, x^2\, dx = \frac{3}{L}\int_{-L/2}^{0}\left(\frac{4x^2}{L^2} + \frac{4x}{L} + 1\right) x^2\, dx + \int_{0}^{L/2}\left(\frac{4x^2}{L^2} - \frac{4x}{L} + 1\right) x^2\, dx$$

$$= \frac{3}{L}\left(\frac{4x^5}{5L^2} + \frac{x^4}{L} + \frac{x^3}{3}\right)\Bigg|_{-L/2}^{0} + \frac{3}{L}\left(\frac{4x^5}{5L^2} - \frac{x^4}{L} + \frac{x^3}{3}\right)\Bigg|_{0}^{L/2} = \frac{L^2}{40}$$

The rms value is then $x_{rms} = \sqrt{\langle x^2 \rangle} = \sqrt{L^2/40} = 0.158L$.

11.  With $E_1 = 1.26$ eV and $E_n = n^2 E_1$ we have

$$\Delta E_3 = E_3 - E_1 = 9E_1 - E_1 = 8E_1 = 8(1.26\text{ eV}) = 10.1\text{ eV}$$

$$\Delta E_4 = E_4 - E_1 = 16E_1 - E_1 = 15E_1 = 15(1.26\text{ eV}) = 18.9\text{ eV}$$

12.  (a)  $E_1 = \dfrac{h^2}{8mL^2} = \dfrac{(hc)^2}{8(mc^2)L^2} = \dfrac{(1240\text{ eV}\cdot\text{nm})^2}{8(511{,}000\text{ eV})(0.251\text{ nm})^2} = 5.97\text{ eV}$

$$4 \rightarrow 1: \quad \Delta E = E_4 - E_1 = 16E_1 - E_1 = 15E_1 = 15(5.97\text{ eV}) = 89.6\text{ eV}$$

(b)  $4 \rightarrow 3: \quad \Delta E = E_4 - E_3 = 16E_1 - 9E_1 = 7E_1 = 7(5.97\text{ eV}) = 41.8\text{ eV}$

  $4 \rightarrow 2: \quad \Delta E = E_4 - E_2 = 16E_1 - 4E_1 = 12E_1 = 12(5.97\text{ eV}) = 71.6\text{ eV}$

  $3 \rightarrow 2: \quad \Delta E = E_3 - E_2 = 9E_1 - 4E_1 = 5E_1 = 5(5.97\text{ eV}) = 29.9\text{ eV}$

  $3 \rightarrow 1: \quad \Delta E = E_3 - E_1 = 9E_1 - E_1 = 8E_1 = 8(5.97\text{ eV}) = 47.8\text{ eV}$

  $2 \rightarrow 1: \quad \Delta E = E_2 - E_1 = 4E_1 - E_1 = 3E_1 = 3(5.97\text{ eV}) = 17.9\text{ eV}$

13.  $\displaystyle\int_{0}^{L} A^2 \sin^2 \frac{n\pi x}{L}\, dx = A^2 \frac{L}{n\pi}\int_{0}^{n\pi} \sin^2 u\, du$ with $u = n\pi x/L$. The integral is a standard form that can be found in integral tables:

$$A^2 \frac{L}{n\pi} \int_0^{n\pi} \sin^2 u \; du = A^2 \frac{L}{n\pi} \left( \tfrac{1}{2}u - \tfrac{1}{4}\sin 2u \right)\Big|_0^{n\pi} = A^2 \frac{L}{2}$$

Setting the integral equal to 1 for normalization gives $A^2 L/2 = 1$ or $A = \sqrt{2/L}$.

14.  (a) $P(0:L/3) = \int_0^{L/3} |\psi_1(x)|^2 \; dx = \int_0^{L/3} \frac{2}{L}\sin^2 \frac{\pi x}{L} dx = \frac{2}{\pi}\int_0^{\pi/3} \sin^2 u \; du$

$$= \frac{2}{\pi}\left( \frac{u}{4} - \frac{\sin 2u}{4} \right)\Big|_0^{\pi/3} = 0.1955$$

(b) $P(L/3:2L/3) = \int_{L/3}^{2L/3} |\psi_1(x)|^2 \; dx = \int_{L/3}^{2L/3} \frac{2}{L}\sin^2 \frac{\pi x}{L} dx = \frac{2}{\pi}\int_{\pi/3}^{2\pi/3} \sin^2 u \; du$

$$= \frac{2}{\pi}\left( \frac{u}{4} - \frac{\sin 2u}{4} \right)\Big|_{\pi/3}^{2\pi/3} = 0.6090$$

(c) $P(2L/3:L) = \int_{2L/3}^{L} |\psi_1(x)|^2 \; dx = \int_{2L/3}^{L} \frac{2}{L}\sin^2 \frac{\pi x}{L} dx = \frac{2}{\pi}\int_{2\pi/3}^{\pi} \sin^2 u \; du$

$$= \frac{2}{\pi}\left( \frac{u}{4} - \frac{\sin 2u}{4} \right)\Big|_{2\pi/3}^{\pi} = 0.1955$$

15.  (a) $P(x)dx = |\psi_3(x)|^2 \; dx = \frac{2}{L}\sin^2 \frac{3\pi x}{L} dx = \frac{2}{0.189 \text{ nm}}\sin^2 \frac{3\pi(0.188 \text{ nm})}{0.189 \text{ nm}} 0.001 \text{ nm} = 2.63 \times 10^{-5}$

(b) $P(x)dx = \frac{2}{L}\sin^2 \frac{3\pi x}{L} dx = \frac{2}{0.189 \text{ nm}}\sin^2 \frac{3\pi(0.031 \text{ nm})}{0.189 \text{ nm}} 0.001 \text{ nm} = 0.0106$

(c) $P(x)dx = \frac{2}{L}\sin^2 \frac{3\pi x}{L} dx = \frac{2}{0.189 \text{ nm}}\sin^2 \frac{3\pi(0.079 \text{ nm})}{0.189 \text{ nm}} 0.001 \text{ nm} = 5.42 \times 10^{-3}$

A classical particle has a uniform probability to be found anywhere within the region, so $P(x) dx = (0.001 \text{ nm})/(0.189 \text{ nm}) = 5.29 \times 10^{-3}$.

16.  With $E = E_0(n_x^2 + n_y^2)$ the levels above $50E_0$ are as follows:

| $n_x$ | $n_y$ | $E$ | $n_x$ | $n_y$ | $E$ |
|-------|-------|-----|-------|-------|-----|
|       |       |     |       |       |     |

| 6 | 4 | $52E_0$ | 6 | 5 | $61E_0$ |
|---|---|---|---|---|---|
| 4 | 6 | $52E_0$ | 5 | 6 | $61E_0$ |
| 7 | 2 | $53E_0$ | 7 | 4 | $65E_0$ |
| 2 | 7 | $53E_0$ | 4 | 7 | $65E_0$ |
| 7 | 3 | $58E_0$ | 8 | 1 | $65E_0$ |
| 3 | 7 | $58E_0$ | 1 | 8 | $65E_0$ |

The level at $E = 65E_0$ is 4-fold degenerate.

17. With $E = E_0(n_x^2 + n_y^2/4)$ the levels are as follows:

| $n_x$ | $n_y$ | $E$ | $n_x$ | $n_y$ | $E$ |
|---|---|---|---|---|---|
| 1 | 1 | $1.25E_0$ | 2 | 3 | $6.25E_0$ |
| 1 | 2 | $2.00E_0$ | 1 | 5 | $7.25E_0$ |
| 2 | 1 | $2.25E_0$ | 2 | 4 | $8.00E_0$ |
| 1 | 3 | $3.25E_0$ | 3 | 1 | $9.25E_0$ |
| 2 | 2 | $5.00E_0$ | 1 | 6 | $10.00E_0$ |
| 1 | 4 | $5.00E_0$ | 3 | 2 | $10.00E_0$ |

The levels at $E = 5.00E_0$ and $E = 10.00E_0$ are both 2-fold degenerate.

18. Using Equations 5.39 and 5.40, we have

$$\frac{\partial \psi}{\partial x} = g(y)\frac{df}{dx} = g(y)(k_x A\cos k_x x - k_x B\sin k_x x)$$

$$\frac{\partial^2 \psi}{\partial x^2} = g(y)\frac{d^2 f}{dx^2} = g(y)(-k_x^2 A\sin k_x x - k_x^2 B\cos k_x x) = -k_x^2 g(y)f(x)$$

$$\frac{\partial \psi}{\partial y} = f(x)\frac{dg}{dy} = f(x)(k_y C\cos k_y y - k_y D\sin k_y y)$$

$$\frac{\partial^2 \psi}{\partial y^2} = f(x)\frac{d^2 g}{dy^2} = f(x)(-k_y^2 C\sin k_y y - k_y^2 D\cos k_y y) = -k_y^2 f(x)g(y)$$

With $U(x, y) = 0$ inside the well, Equation 5.37 gives

$$-\frac{\hbar^2}{2m}\left[-k_x^2 f(x)g(y) - k_y^2 f(x)g(y)\right] = Ef(x)g(y)$$

and so $E = \dfrac{\hbar^2}{2m}(k_x^2 + k_y^2)$.

19. Let $E_0 = \hbar^2 \pi^2/2mL^2$. The energy states are then

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| $n_x$ | $n_y$ | $n_z$ | $E$ | degeneracy | $n_x$ | $n_y$ | $n_z$ | $E$ | degeneracy | $n_x$ | $n_y$ | $n_z$ | $E$ | degeneracy |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | $3E_0$ | 1 | 2 | 2 | 2 | $12E_0$ | 1 | 1 | 1 | 4 | $18E_0$ | |
| | | | | | | | | | | 1 | 4 | 1 | $18E_0$ | 3 |
| 1 | 1 | 2 | $6E_0$ | | 1 | 2 | 3 | $14E_0$ | | 4 | 1 | 1 | $18E_0$ | |
| 1 | 2 | 1 | $6E_0$ | 3 | 1 | 3 | 2 | $14E_0$ | | | | | | |
| 2 | 1 | 1 | $6E_0$ | | 2 | 1 | 3 | $14E_0$ | 6 | 1 | 3 | 3 | $19E_0$ | |
| | | | | | 2 | 3 | 1 | $14E_0$ | | 3 | 1 | 3 | $19E_0$ | 3 |
| 1 | 2 | 2 | $9E_0$ | | 3 | 1 | 2 | $14E_0$ | | 3 | 3 | 1 | $19E_0$ | |
| 2 | 1 | 2 | $9E_0$ | 3 | 3 | 2 | 1 | $14E_0$ | | | | | | |
| 2 | 2 | 1 | $9E_0$ | | | | | | | 1 | 2 | 4 | $21E_0$ | |
| | | | | | 2 | 2 | 3 | $17E_0$ | | 1 | 4 | 2 | $21E_0$ | |
| 1 | 1 | 3 | $11E_0$ | | 2 | 3 | 2 | $17E_0$ | 3 | 2 | 1 | 4 | $21E_0$ | 6 |
| 1 | 3 | 1 | $11E_0$ | 3 | 3 | 2 | 2 | $17E_0$ | | 2 | 4 | 1 | $21E_0$ | |
| 3 | 1 | 1 | $11E_0$ | | | | | | | 4 | 1 | 2 | $21E_0$ | |
| | | | | | | | | | | 4 | 2 | 1 | $21E_0$ | |

$21E_0$ —————————————— 6  (1,2,4), (1,4,2), (2,1,4), (2,4,1), (4,1,2), (4,2,1)

$19E_0$ ——————————————— 3  (1,3,3), (3,1,3), (3,3,1)

$18E_0$ ——————————————— 3  (1,1,4), (1,4,1), (4,1,1)

$17E_0$ ——————————————— 3  (2,2,3), (2,3,2), (3,2,2)

$14E_0$ —————————————— 6  (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)

$12E_0$ ——————————————— 1  (2,2,2)

$11E_0$ ——————————————— 3  (1,1,3), (1,3,1), (3,1,1)

$9E_0$ —————————————— 3  (1,2,2), (2,1,2), (2,2,1)

$6E_0$ —————————————— 3  (1,1,2), (1,2,1), (2,1,1)

$3E_0$ —————————————— 1  (1,1,1)

Energy                     Degeneracy     $(n_x, n_y, n_z)$

20.   With $\psi(x) = Ae^{-ax^2}$, the normalization integral is

$$\int_{-\infty}^{+\infty} |\psi(x)|^2\, dx = A^2 \int_{-\infty}^{+\infty} e^{-2ax^2}\, dx = 2A^2 \int_0^\infty e^{-2ax^2}\, dx = A^2 \sqrt{\frac{\pi}{2a}}$$

The integral is a standard form that can be found in integral tables.  Setting this result equal to 1 for the normalization condition and using Equation 5.49 for $a$, we obtain $A^2 \sqrt{\pi/2a} = 1$ or

$$A = \left(\frac{2a}{\pi}\right)^{1/4} = \left(\frac{\sqrt{km}}{\pi\hbar}\right)^{1/4} = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}$$

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where the final result uses $\omega_0^2 = k/m$.

21.  (a)    $E_0 = \frac{1}{2}\hbar\omega_0 = \frac{1}{2}kx_0^2$    so    $x_0 = \sqrt{\hbar\omega_0/k}$

(b)    $E_1 = \frac{3}{2}\hbar\omega_0 = \frac{1}{2}kx_0^2$    so    $x_0 = \sqrt{3\hbar\omega_0/k}$

   $E_2 = \frac{5}{2}\hbar\omega_0 = \frac{1}{2}kx_0^2$    so    $x_0 = \sqrt{5\hbar\omega_0/k}$

22.    $$x_{av} = \int_{-\infty}^{\infty}|\psi(x)|^2\, x\, dx = A^2\int_{-\infty}^{\infty}e^{-2ax^2}x\, dx = 0$$

because the integrand is an odd function of $x$ (the integral from $-\infty$ to $0$ exactly cancels the integral from $0$ to $+\infty$).

$$(x^2)_{av} = \int_{-\infty}^{\infty}|\psi(x)|^2\, x^2 dx = A^2\int_{-\infty}^{\infty}e^{-2ax^2}x^2 dx = 2A^2\int_{0}^{\infty}e^{-2ax^2}x^2 dx = \frac{2A^2}{\sqrt{8a^3}}\int_{0}^{\infty}e^{-u^2}u^2 du$$

with the substitution $u = x\sqrt{2a}$. The integral is a standard form found in tables and is equal to $\sqrt{\pi}/4$. Substituting $A = (\omega_0 m/\pi\hbar)^{1/4}$ and $a = \sqrt{km}/2\hbar = \omega_0 m/2\hbar$, we find

$$(x^2)_{av} = 2\left(\frac{\omega_0 m}{\pi\hbar}\right)^{1/2}\frac{1}{2\sqrt{2}}\left(\frac{2\hbar}{\omega_0 m}\right)^{3/2}\frac{\sqrt{\pi}}{4} = \frac{\hbar}{2\omega_0 m}$$

$$\Delta x = \sqrt{(x^2)_{av}-(x_{av})^2} = \sqrt{\hbar/2m\omega_0}$$

23.  (a)  Because the oscillating particle moves with equal probability in the positive and negative $x$ directions, $p_{av} = 0$.

(b)    $$U_{av} = \frac{1}{2}k(x^2)_{av} = \frac{1}{2}k\frac{\hbar}{2\omega_0 m} = \frac{1}{2}\omega_0^2 m\frac{\hbar}{2\omega_0^2 m} = \frac{1}{4}\hbar\omega_0$$

   $$K_{av} = E - U_{av} = \frac{1}{2}\hbar\omega_0 - \frac{1}{4}\hbar\omega_0 = \frac{1}{4}\hbar\omega_0$$

   $$(p^2)_{av} = 2mK_{av} = 2m\left(\frac{1}{4}\hbar\omega_0\right) = \frac{\hbar\omega_0 m}{2}$$

(c)    $$\Delta p = \sqrt{(p^2)_{av}-(p_{av})^2} = \sqrt{\hbar\omega_0 m/2}$$

24.  $E_0 = 1.24\text{ eV} = \frac{1}{2}\hbar\omega_0$    so    $\hbar\omega_0 = 2.48\text{ eV}$

To $n = 2$ state:     $\Delta E = E_2 - E_0 = \frac{5}{2}\hbar\omega_0 - \frac{1}{2}\hbar\omega_0 = 2\hbar\omega_0 = 2(2.48\text{ eV}) = 4.96\text{ eV}$

To $n = 4$ state:     $\Delta E = E_4 - E_0 = \frac{9}{2}\hbar\omega_0 - \frac{1}{2}\hbar\omega_0 = 4\hbar\omega_0 = 4(2.48\text{ eV}) = 9.92\text{ eV}$

25.  $P(x)\,dx = |\psi(x)|^2\,dx = A^2 e^{-2ax^2}\,dx$   so at $x = 0$   $P(0)dx = A^2 dx$

At the classical turning points $x = \pm x_0$, $K = 0$ so $E = U$   or   $\frac{1}{2}\hbar\omega_0 = \frac{1}{2}kx_0^2$

$P(\pm x_0)dx = A^2 e^{-2(\sqrt{km}/2\hbar)(\hbar\omega_0/k)}dx = A^2 e^{-1}dx = e^{-1}P(0)dx = 0.368P(0)dx$

26.     $\dfrac{d(\Delta x)}{dK} = \dfrac{d}{dK}\left(\dfrac{1}{2}\sqrt{\dfrac{2K}{m}}\,\dfrac{\hbar}{U_0 - E + K}\right) = \dfrac{\hbar}{2}\sqrt{\dfrac{2}{m}}\left[\dfrac{\frac{1}{2}K^{-1/2}}{U_0 - E + K} - \dfrac{K^{1/2}}{(U_0 - E + K)^2}\right] = 0$

$$\frac{1}{2}K^{-1/2} = \frac{K^{1/2}}{U_0 - E + K} \qquad \text{or} \qquad K = U_0 - E$$

$$(\Delta x)_{max} = \frac{1}{2}\sqrt{\frac{2(U_0 - E)}{m}}\,\frac{\hbar}{2(U_0 - E)} = \frac{1}{2}\frac{\hbar}{\sqrt{2m(U_0 - E)}}$$

27.  $x < 0$:   $\psi_0 = A\sin k_0 x + B\cos k_0 x$     with   $k_0 = \sqrt{\dfrac{2mE}{\hbar^2}}$

$x > 0$:   $\psi_1(x) = Ce^{k_1 x} + De^{-k_1 x}$     with   $k_1 = \sqrt{\dfrac{2m(U_0 - E)}{\hbar^2}}$

We set $C = 0$ to keep $\psi_1$ finite as $x \to \infty$. We then apply the continuity conditions on $\psi$ and $d\psi/dx$ at $x = 0$:

$$\psi_0(0) = \psi_1(0): \qquad B = D$$
$$\left(\frac{d\psi_0}{dx}\right)_{x=0} = \left(\frac{d\psi_1}{dx}\right)_{x=0}: \qquad k_0 A = -k_1 D$$

Thus $D = B = -A(k_0/k_1) = -A\sqrt{E/(U_0 - E)}$.

28.  $x < 0$:   $\psi_0 = A'e^{ik_0 x} + B'e^{-ik_0 x}$     with   $k_0 = \sqrt{\dfrac{2mE}{\hbar^2}}$

$x > 0$:   $\psi_1(x) = C'e^{ik_1 x} + D'e^{-ik_1 x}$     with   $k_1 = \sqrt{\dfrac{2m(E - U_0)}{\hbar^2}}$

If the particles are incident from them negative $x$ direction, then $D'$ (the coefficient of the term that represents a wave in the region of positive $x$ traveling toward the origin) must be set to 0. We then apply the continuity conditions on $\psi$ and $d\psi/dx$ at $x = 0$:

$$\psi_0(0) = \psi_1(0): \qquad A' + B' = C'$$

$$\left(\frac{d\psi_0}{dx}\right)_{x=0} = \left(\frac{d\psi_1}{dx}\right)_{x=0}: \qquad k_0(A' - B') = k_1 C'$$

Solving these two equations, we obtain

$$C' = \frac{2A'}{1 + k_1/k_0} \qquad\qquad B' = \frac{1 - k_1/k_0}{1 + k_1/k_0}A'$$

The squares of the amplitude ratios give the relative probabilities for the particles to be reflected at $x = 0$ or transmitted into the $x > 0$ region:

Reflection probability: $\qquad \dfrac{|B'|^2}{|A'|^2} = \left(\dfrac{1 - k_1/k_0}{1 + k_1/k_0}\right)^2$

Transmission probability: $\qquad \dfrac{|C'|^2}{|A'|^2} = \dfrac{4}{(1 + k_1/k_0)^2}$

29. (a) $\quad x < 0: \quad \psi_0(x) = Ae^{ik_0 x} + Be^{-ik_0 x} \qquad$ with $\quad k_0 = \sqrt{2mE/\hbar^2}$

$\qquad 0 < x < L: \quad \psi_1(x) = Ce^{k_1 x} + De^{-k_1 x} \qquad$ with $\quad k_1 = \sqrt{2m(U_0 - E)/\hbar^2}$

$\qquad x > L: \quad \psi_2(x) = Fe^{ik_2 x} + Ge^{-ik_2 x} \qquad$ with $\quad k_2 = k_0$

(b) The continuity conditions on $\psi$ and $d\psi/dx$ at $x = 0$ are

$$\psi_0(0) = \psi_1(0): \quad A + B = C + D$$

$$\left(\frac{d\psi_0}{dx}\right)_{x=0} = \left(\frac{d\psi_1}{dx}\right)_{x=0}: \quad ik_0(A - B) = k_1(C - D)$$

and at $x = L$

$$\psi_1(L) = \psi_2(L): \quad Ce^{k_1 L} + De^{-k_1 L} = Fe^{ik_0 L} + Ge^{ik_0 L}$$

$$\left(\frac{d\psi_1}{dx}\right)_{x=L} = \left(\frac{d\psi_2}{dx}\right)_{x=L}: \quad k_1(Ce^{k_1 L} - De^{-k_1 L}) = ik_0(Fe^{ik_0 L} - Ge^{-ik_0 L})$$

(c) $G = 0$, because no wave can travel right to left in the region $x > L$ if the particles are incident from $x < 0$.

30. (a)  $x < 0$:  $\psi_0(x) = Ae^{ik_0x} + Be^{-ik_0x}$   with  $k_0 = \sqrt{2mE/\hbar^2}$

   $0 < x < L$:  $\psi_1(x) = Ce^{ik_1x} + De^{-ik_1x}$   with  $k_1 = \sqrt{2m(E - U_0)/\hbar^2}$

   $x > L$:  $\psi_2(x) = Fe^{ik_2x} + Ge^{-ik_2x}$   with  $k_2 = k_0$

(b) The continuity conditions on $\psi$ and $d\psi/dx$ at $x = 0$ are

   $\psi_0(0) = \psi_1(0)$:  $A + B = C + D$

   $\left(\dfrac{d\psi_0}{dx}\right)_{x=0} = \left(\dfrac{d\psi_1}{dx}\right)_{x=0}$:  $ik_0(A - B) = ik_1(C - D)$

and at $x = L$

   $\psi_1(L) = \psi_2(L)$:  $Ce^{ik_1L} + De^{-ik_1L} = Fe^{ik_0L} + Ge^{ik_0L}$

   $\left(\dfrac{d\psi_1}{dx}\right)_{x=L} = \left(\dfrac{d\psi_2}{dx}\right)_{x=L}$:  $ik_1(Ce^{ik_1L} - De^{-ik_1L}) = ik_0(Fe^{ik_0L} - Ge^{-ik_0L})$

(c) $G = 0$, because no wave can travel right to left in the region $x > L$ if the particles are incident from $x < 0$.
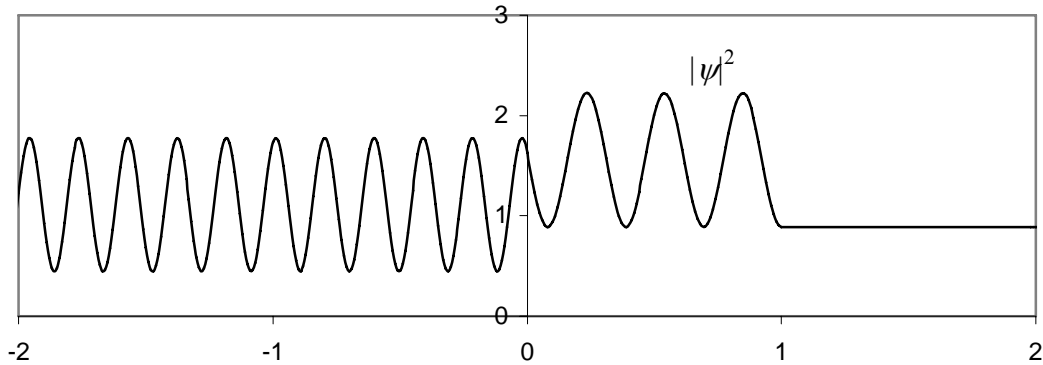


In this sketch the barrier extends from $x = 0$ to $x = 1$. To the left of the barrier, the incident and reflected waves combine to form a standing wave (but the minima do not go to zero, because the reflected wave has smaller amplitude than the indicent wave). Between $x = 0$ and $x = 1$, the particles have a smaller kinetic energy and thus: (1) a larger de Broglie wavelength, and (2) a slower speed, which means that the probability to find
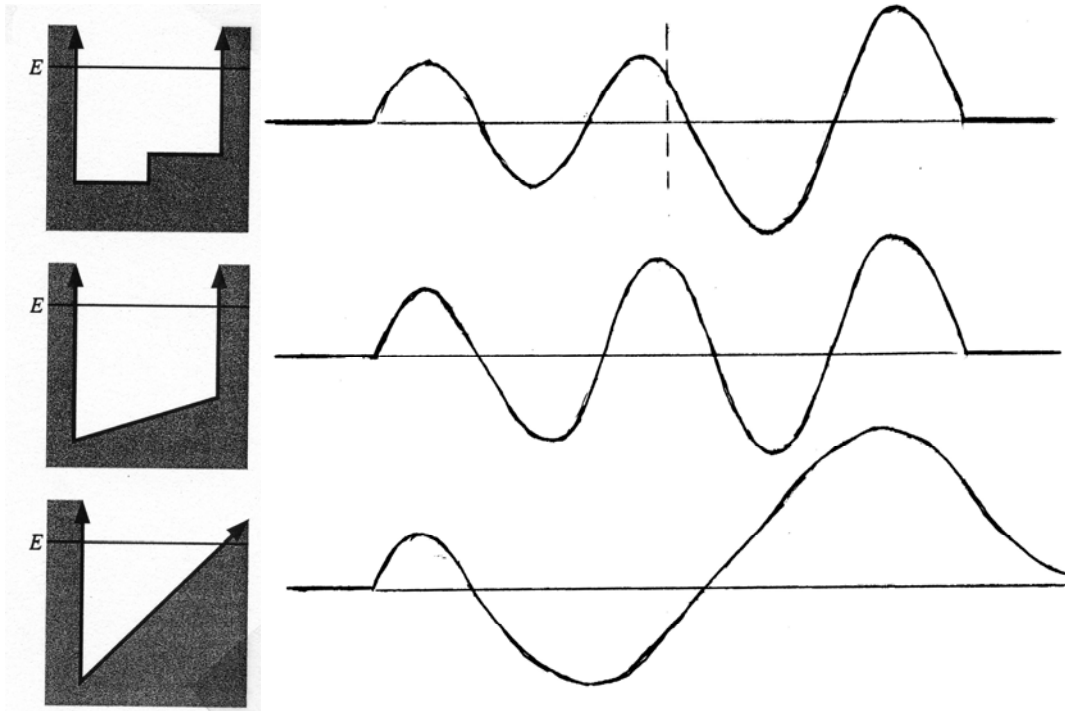
the particles in any interval must be larger. Beyond the barrier $(x > L)$ the probability density is flat.

31. (a) $E = n^2 E_1 = 10^2 \dfrac{h^2}{8mL^2} = \dfrac{100(hc)^2}{8mc^2 L^2} = \dfrac{100(1240 \text{ eV} \cdot \text{nm})^2}{8(511{,}000 \text{ eV})(0.132 \text{ nm})^2} = 2160 \text{ eV}$

(b) $\Delta p = \sqrt{\overline{p^2}} = \sqrt{2mE} = \dfrac{1}{c}\sqrt{2mc^2 E} = \dfrac{1}{c}\sqrt{2(511{,}000 \text{ eV})(2160 \text{ eV})} = 4.70 \times 10^4 \text{ eV}/c$

(c) $\Delta x \sim \dfrac{\hbar}{\Delta p} = \dfrac{1}{2\pi}\dfrac{hc}{c\Delta p} = \dfrac{1}{2\pi}\dfrac{1240 \text{ eV} \cdot \text{nm}}{4.70 \times 10^4 \text{ eV}} = 4.2 \times 10^{-3} \text{ nm}$

32.



33. $(x^2)_{av} = \displaystyle\int_0^L |\psi(x)|^2 \, x^2 dx = \dfrac{2}{L}\int_0^L x^2 \sin^2 \dfrac{n\pi x}{L} dx = \dfrac{2}{L}\left(\dfrac{L}{n\pi}\right)^3 \int_0^{n\pi} u^2 \sin u \, du \quad \text{with } u = \dfrac{n\pi x}{L}$

The integral is a standard form that can be found in integral tables.

$(x^2)_{av} = \dfrac{2L^2}{(n\pi)^3}\left[\dfrac{u^3}{6} - \left(\dfrac{u^2}{4} - \dfrac{1}{8}\right)\sin 2u - \dfrac{u \cos 2u}{4}\right]_0^{n\pi} = \dfrac{2L^2}{(n\pi)^3}\left[\dfrac{(n\pi)^3}{6} - \dfrac{n\pi}{4}\right] = L^2\left(\dfrac{1}{3} - \dfrac{1}{2n^2\pi^2}\right)$

34. With $x_{av} = L/2$ from Example 5.4, we have

$$\Delta x = \sqrt{(x^2)_{av} - (x_{av})^2} = \sqrt{L^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right) - \left(\frac{L}{2}\right)^2} = L\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}$$

35. (a) The particle has no preferred direction of motion, so it is equally likely to be moving in the positive and negative $x$ directions. We therefore expect that $p_{av} = 0$.

(b) Because the potential energy is zero inside the well, the kinetic energy is equal to the total energy:

$$K = E_n \quad \text{or} \quad \frac{p^2}{2m} = \frac{h^2 n^2}{8mL^2} \quad \text{so} \quad p^2 = \frac{h^2 n^2}{4L^2}$$

For a given level $n$, $p^2$ is constant so $(p^2)_{av}$ has that same value.

(c) $\quad \Delta p = \sqrt{(p^2)_{av} - (p_{av})^2} = \sqrt{\frac{h^2 n^2}{4L^2} - 0} = \frac{hn}{2L}$

36. $\quad \dfrac{d\psi}{dx} = A\dfrac{d}{dx}xe^{-ax^2} = A(e^{-ax^2} - 2ax^2 e^{-ax^2}) = Ae^{-ax^2}(1 - 2ax^2)$

$$\frac{d^2\psi}{dx^2} = A[(-4ax)e^{-ax^2} - (1 - 2ax^2)(-2ax)e^{-ax^2}] = Ae^{-ax^2}(-6ax + 4a^2 x^3)$$

Substituting the second derivative into the Schrödinger equation, we have

$$-\frac{\hbar^2}{2m}Ae^{-ax^2}(-6ax + 4a^2 x^3) + \frac{1}{2}kx^2 Axe^{-ax^2} = EAxe^{-ax^2}$$

After canceling common factors and combining terms,

$$x^2\left(\frac{k}{2} - \frac{2\hbar^2 a^2}{m}\right) + \left(\frac{3a\hbar^2}{m} - E\right) = 0$$

In order for this to be valid for all possible values of $x$, both of the quantities in parentheses must be zero:

$$\frac{k}{2} = \frac{2\hbar^2 a^2}{m} \quad \text{or} \quad a = \frac{\omega_0 m}{2\hbar} \quad \text{AND} \quad E = \frac{3a\hbar^2}{m} = \frac{3\hbar^2}{m}\frac{\omega_0 m}{2\hbar} = \frac{3}{2}\hbar\omega_0$$

$$\int_{-\infty}^{\infty}|\psi(x)|^2\,dx = A^2\int_{-\infty}^{\infty}x^2 e^{-2ax^2}\,dx = 2A^2\int_0^{\infty}x^2 e^{-2ax^2}\,dx = \frac{2A^2}{\sqrt{8a^3}}\int_0^{\infty}u^2 e^{-u^2}\,du = \frac{A^2}{\sqrt{2a^3}}\frac{\sqrt{\pi}}{4}$$

where we have made the substitution $u = x\sqrt{2a}$ to put the integral into a standard form that is found in integral tables. Setting the result equal to 1 gives

$$A^2 = \frac{4\sqrt{2a^3}}{\sqrt{\pi}} = 4\sqrt{\frac{2}{\pi}}\left(\frac{m\omega_0}{2\hbar}\right)^{3/2} \quad \text{or} \quad A = \frac{\sqrt{2}}{\pi^{1/4}}\left(\frac{m\omega_0}{\hbar}\right)^{3/4}$$

37.

$$P = \int_{-x_0}^{-\infty}|\psi(x)|^2\,dx + \int_{x_0}^{\infty}|\psi(x)|^2\,dx = 2\int_{x_0}^{\infty}|\psi(x)|^2\,dx = 2A^2\int_{x_0}^{\infty}e^{-2ax^2}\,dx$$

$$= 2\sqrt{\frac{m\omega_0}{\hbar\pi}}\frac{1}{\sqrt{2a}}\int_1^{\infty}e^{-u^2}\,du = \frac{2}{\sqrt{\pi}}\int_1^{\infty}e^{-u^2}\,du = 0.157$$

38. (a) The $x$ and $y$ motions are independent, and each contributes an energy of $\hbar\omega_0(n+\tfrac{1}{2})$, but the integer $n$ is not necessarily the same for the two independent motions. Thus the total energy is

$$E = \hbar\omega_0(n_x + \tfrac{1}{2}) + \hbar\omega_0(n_y + \tfrac{1}{2}) = \hbar\omega_0(n_x + n_y + 1)$$

(b)

| Energy | | Degeneracy | $(n_x, n_y)$ |
|---|---|---|---|
| $4\hbar\omega_0$ | ———————— | 4 | (0,3), (1,2), (2,1), (3,0) |
| $3\hbar\omega_0$ | ———————— | 3 | (0,2), (1,1), (2,0) |
| $2\hbar\omega_0$ | ———————— | 2 | (0,1), (1,0) |
| $\hbar\omega_0$ | ———————— | 1 | (0,0) |

(c) The level with energy $N\hbar\omega_0$ has $N$ different possible sets of quantum numbers $n_x, n_y$. Both $n_x$ and $n_y$ range from 0 to $N-1$ but with their sum fixed to $N$. The number of possible values of $n_x$ is then $N$ (the values are 0, 1, 2, …, N-2, N-1), and for each value of $n_x$ the value of $n_y$ is fixed. The total degeneracy of each level is thus $N = n_x + n_y + 1$.

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# Chapter 6

Why teach the Bohr model?  It lays the groundwork for the relationship between atomic spectroscopy and atomic structure, and as such displays for students a classic example of how theory is informed by experiment.  The model gives a useful (although ultimately incorrect) picture of how electrons move in atoms; even though the model is incorrect it offers a quick means to calculate or estimate atomic properties, for example magnetic dipole moments.  It allows some fundamental ideas (such as quantized energy levels in atoms, quantization of angular momentum, and the correspondence principle) to be introduced in a simple context.

The Bohr model is also important in its historical context.  Why then not introduce it in that context (that is, prior to de Broglie waves)?  The work of Rutherford and Bohr is more significant in elucidating the properties and structure of atoms than it is in leading to further developments in quantum physics. Hence it should be presented as a lead-in to atomic structure.

It is also important to minimize the time between the introduction of the Bohr model and its negation by the correct quantum theory calculation.  This avoids what the psychologists call "imprinting."  In this ordering of topics, the Bohr model is presented, its deficiencies are discussed (in the context of our previous discussion of quantum mechanics), and then we launch quickly into the correct quantum model.

Instructors will note that this text does not present the quantization of angular momentum based on a calculation in which standing de Broglie waves just fit around the circumference of a Bohr orbit.  In my view this piles misunderstanding on top of misunderstanding.  (Note that Bohr could not have derived quantization of angular momentum in this way, as his work occurred a decade before de Broglie's.)  Just what is it that students should regard those standing de Broglie waves as representing?  That is, is there some spatial variation of the electron probability density around the orbit?  And how do we reconcile this with the spherical symmetry of the ground state?  In a similar fashion, how would we draw the standing de Broglie waves for the $l = 0$ state?  This leads to no productive outcome, so it is better to leave it out.

## Supplemental Materials

The University of Colorado PhET collection offers a Rutherford scattering simulation at http://phet.colorado.edu/en/simulation/rutherford-scattering.  Chapter 4 of *Physlet Quantum Physics* includes simulations of Thomson and Rutherford scattering.

## Suggestions for Additional Reading

A discussion of many of the basic properties of atoms may be found in:
M. R. Wehr, J. A. Richards, and T. W. Adair, *Physics of the Atom* (Addison-Wesley, 1978).

For a historical perspective on the development of atomic theory, see:
H. A. Boorse and L. Motz, editors, *The World of the Atom* (Basic Books, 1966).
G. K. T. Conn and H. D. Turner, *The Evolution of the Nuclear Atom* (Iliffe Books, 1965).
F. Friedman and L. Sartori, *The Classical Atom* (Addison-Wesley, 1965).

For more details on the history of the Thomson and Bohr models, see:
J. L. Heilbron, "J. J. Thomson and the Bohr Atom," *Physics Today,* April 1977, p. 23.
J. L. Heilbron, "Bohr's First Theories of the Atom," *Physics Today,* October 1985, p. 28.

For a popular summary of Rutherford's work, see:
E. N. da C. Andrade, "The Birth of the Nuclear Atom," *Scientific American* **195**, 93 (November 1956).

The early papers on the Rutherford model and its experimental confirmation illustrate the difficulty of the experiments and the care and abilities of the experimenters. They are easily readable and require no mathematics beyond the present level.
E. Rutherford, *Philosophical Magazine* **21,** 669 (1911).
H. Geiger, *Proceedings of the Royal Society of London* **A83,** 492 (1910).
H. Geiger and E. Marsden, *Philosophical Magazine* **25**, 604 (1913).

## Materials for Active Engagement in the Classroom

### A. Reading Quizzes

1. Which of the following is NOT a characteristic of the Bohr model of the structure of the atom?
    (1) Electrons move in circular orbits about the nucleus.
    (2) Photons are emitted when an electron jumps from one circular orbit to a lower-energy orbit.
    (3) The circular motion of the electron is consistent with the uncertainty principle.
    (4) The angular momentum of each orbit can take only values that are integer multiples of the smallest value.

2. In the Bohr model:
    (1) Electrons move in circular orbits of definite radius.
    (2) Electrons move in elliptical orbits.
    (3) Electrons moving in the same orbit can have different energies.
    (4) Electrons can never jump from one orbit to another.

3. Which of the following is NOT used in the Bohr model of the atom?
    (1) Quantization of energy.
    (2) Relativistic energy and momentum.
    (3) Coulomb's law for electrostatic forces.
    (4) Quantization of angular momentum.

4.  In a Rutherford scattering experiment:
    (1) most of the particles are not scattered at all or scattered only at small angles, but a few are scattered at large angles.
    (2) the experimental results verify that the positive charge of the atom is spread throughout the volume of the atom.
    (3) most particles are scattered only once, but the ones that are scattered at large angles are scattered many times.
    (4) scattering by the negatively charged electrons can cancel scattering by the positively charged nucleus.

**Answers**      1. 3     2. 1     3. 2     4. 1


## B.  Conceptual or Discussion Questions

1.  An atom absorbs a photon, so that the electron's total energy increases by an amount equal to the photon energy.  In the Bohr model, the electron moves to an orbit of larger radius.  what happens to the orbital speed of the electron?
    (1) Increases          (2) Decreases          (3) Stays the same

2.  In the Bohr model of the hydrogen atom, the speed of the electron is
    (1) much smaller than the speed of light, so that nonrelativistic equations can safely be used.
    (2) large enough that the difference between the relativistic and nonrelativistic equations is important.
    (3) very close to the speed of light, so that the extreme relativistic approximation $E = pc$ should be used.

3.  Among the radiations emitted by atomic hydrogen we find one of energy 13.6 eV, the series limit of the Lyman series ($n = \infty$ to $n = 1$).  A photon of the same energy also exists as the limit of a certain series in singly ionized helium ($Z = 2$).  What is the final value of $n$ for this photon emission?
    (1) 1    (2) 2    (3) 3    (4) 4    (5) more than 4

4.  A Rutherford scattering experiment is set up with alpha particles incident on a silver foil, and the scattered alpha particles are observed by a detector at the scattering angle $\theta$.  How does the counting rate (number of counts per second) in the detector change in each of the following cases?  Possible answers are:
    (1) Increases   (2) Decreases    (3) Remains the same    (4) Needs more information
    (a) The detector moves to larger angles.
    (b) The kinetic energy of the incident alpha particles is increased.
    (c) The silver foil ($Z = 47$) is replaced with gold ($Z = 79$).
    (d) The alpha particles are replaced with protons.
    (e) The thickness of the foil is decreased.

**Answers**      1. 2     2. 1     3. 2     4. (a) 2  (b) 2  (c) 1  (d) 2  (e) 2

# Sample Exam Questions

## A. Multiple Choice

1. In Rutherford scattering we ignore the effect of the electrons because:
   (a) electrons do not exert any force on alpha particles.
   (b) electrons are uniformly distributed throughout the atom.
   (c) the electrons are in such rapid motion that the alpha particles do not collide with them.
   (d) the mass of an electron is much smaller than the mass of an alpha particle.

2. In a certain Rutherford scattering experiment with alpha particles, the distance of closest approach between the alpha particles and the nucleus is $r$. If the kinetic energy of the alpha particles is doubled, what is the new distance of closest approach?
   (a) $2r$　　　　　(b) $4r$　　　　(c) $r/2$　　　　(d) $r/4$

3. In Rutherford scattering, what is the approximate ratio of the number of particles scattered at 10° to the number scattered at 5°?
   (a) 1/4　　　　　(b) 1/8　　　　(c) 1/16　　　　(d) 1/64

4. When alpha particles of kinetic energy $E$ are incident on nuclei of gold atoms, the smallest distance between an alpha particle and a nucleus is $x$. What is the smallest separation distance when the alpha particle kinetic energy is doubled to $2E$?
   (a) $2x$　　　　　(b) $x/2$　　　　(c) $x/\sqrt{2}$　　　　(d) $x/4$　　　　(e) $x\sqrt{2}$

5. In Rutherford scattering of alpha particles by a gold foil, large deflections can occasionally be observed because:
   (a) a single alpha particle can be scattered many times in encounters with many atoms.
   (b) the alpha particle can be deflected by the many atomic electrons of the gold atoms.
   (c) the wave nature of the alpha particles causes interference effects.
   (d) none of the above.

6. Alpha particles are scattered from a certain nucleus and are observed at scattering angles of $45^o$ and $135^o$. For which of these two scattering angles does the nucleus exert a greater average force on the alpha particles?
   (a) $45^o$　　(b) $135^o$　　　　(c) The average force is the same for both angles.

7. In a scattering experiment, protons pass through a thin foil of silver ($Z = 47$). The probability to detect the scattered protons has the value $I$ at an angle of $10.0^o$ relative to the direction of the original beam of protons. At what angle is the scattering probability $0.1I$?
   (a) $40.0^o$　　　　(b) $5.6^o$　　　(c) $2.5^o$　　　(d) $17.8^o$　　　(e) None of these.

**Answers**　　　　　1. d　　2. c　　3. c　　4. b　　5. d　　6. b　　7. d

## B. Conceptual

1. Consider two different alpha-particle scattering experiments, one in which the particle is deflected by $45^o$ and another in which it is deflected from the same target by $135^o$. In the $45^o$ scattering experiment, is the maximum force experienced by the alpha particle *greater than, less than, or the same as* the maximum force experienced by the alpha particle in the $135^o$ scattering experiment? EXPLAIN YOUR ANSWER.

**Answers**      1. less than

## C. Problems

1. (a) The Balmer series of lines emitted by doubly ionized lithium ($Li^{++}$), which has atomic number 3, consists of electron transitions that end at the first excited state. Find the limiting wavelength of the Balmer series of doubly ionized lithium.
   (b) Find the longest wavelength at which doubly ionized lithium in its ground state can absorb a photon.

2. (a) Find the energies of the ground state and the first two excited states of the electron in doubly ionized lithium $Li^{++}$ (atomic number = 3). Sketch an energy-level diagram and label each level with its energy and principal quantum number.
   (b) If the electron is originally in the second excited state, find the wavelengths of all photons that can be emitted as the electron eventually reaches the ground state.

3. A neutral atom of Li has 3 electrons. Two of these are removed, leaving an ion of $Li^{++}$.
   (a) What is the longest wavelength at which the ground state of the $Li^{++}$ ion can absorb electromagnetic radiation?
   (b) How much energy is needed to remove the electron from the ground state of the ion?
   (c) When the electron is in its second excited state, it can return to the ground state through a variety of paths. Find the wavelengths of all possible radiations that can be emitted in these processes.

4. A neutral atom of the element boron (B) has 5 electrons. Four of the electrons are removed, forming an ion with the one remaining electron.
   (a) Sketch an energy level diagram for this ion, showing the ground state and the first two excited states. Label each state with the value of its energy.
   (b) Calculate the two longest wavelengths at which this ion can absorb radiation. Assume all absorption occurs from the ground state.
   (c) What is the minimum amount of energy needed to remove the electron from the ground state of this ion?

5. Consider an atom in which a single electron is attached to a positively charged nucleus. The ionization energy (the minimum energy necessary to completely remove the electron from the atom in its ground state) is 217.6 eV.
   (a) Find the energies of the ground state and the first and second excited states.

(b) How many positive charges are there in the nucleus?

(c) What are the longest and shortest wavelengths of the photons that can be absorbed by this atom in its ground state?

6. (a) Calculate the energies and draw an energy-level diagram showing the ground state and first 3 excited states of triply ionized $Be^{+++}$ ($Z = 4$). Label each state with its energy and principal quantum number.

(b) The Balmer series in this atom consists of electron transitions from higher states to the first excited state. Find the longest wavelength in the Balmer series for this atom, and draw this transition on your energy-level diagram.

(c) If the atom is in its ground state, what is the longest wavelength at which it can absorb a photon? Indicate this process on your energy-level diagram.

7. In a certain atom with a single electron, the longest wavelength at which absorption in the ground state will occur is 4.86 nm. For this atom, find:

(a) the energies of the ground state and the first two excited states (sketch an energy level diagram and label each state with its energy and its principal quantum number);

(b) the ionization energy (the minimum energy required for an electron in the ground state to be completely removed from the atom;

(c) the wavelengths of the photons emitted when the electron makes transitions to lower states beginning in the second excited state.

8. (a) In a certain atom with a single electron, the longest wavelength at which photons can be absorbed from the ground state is 4.863 nm. What is the atomic number $Z$ of this atom?

(b) What is the ionization energy (energy necessary to remove the electron from the ground state) of this atom?

9. In a certain single-electron ion in its ground state, the ionization energy (the energy needed to completely remove the electron from the ion) is 490 eV.

(a) How many positive charges are in the nucleus of this ion?

(b) What is the wavelength of the photon emitted when the electron jumps from the first excited state to the ground state?

10. (a) Draw an energy-level diagram showing the ground state and first 3 excited states of doubly ionized Li ($Z = 3$). Label each state with its energy.

(b) Calculate the longest-wavelength photon emitted in transitions from the third excited state.

(c) Calculate the ionization energy of the third excited state.

11. (a) Find the ionization energy (the energy needed to remove the electron) in an "atom" in which a single electron is attached to a nucleus of boron (atomic number = 5).

(b) Sketch an energy level diagram for this atom showing the ground state and the first 3 excited states. Label each state with its principal quantum number and its energy.

(c) Find the wavelength of the photon that is emitted when the electron makes a transition from the first excited state to the ground state.

12. (a) In the Balmer series of emitted transitions in a one-electron atom, the electron jumps from other states to the first excited state. In doubly ionized lithium (atomic number = 3), what are the wavelengths of the longest and shortest members of the Balmer series?
(b) The lifetime of the first excited state is about $10^{-8}$ s. Estimate the limit of precision with which it is possible to measure the energy of that state, and compare with the energy of the state.

13. (a) An atom of singly ionized helium (atomic number = 2) is initially in its second excited state. It ends up in the ground state after two photons are emitted from the atom. Find the energies of the two photons.
(b) How much energy is required to completely remove the electron when the He ion is in its ground state?

**Answers**
1. (a) 40.5 nm   (b) 13.5 nm
2. (a) -122.4 eV, -30.6 eV, -13.6 eV   (b) 72.9 nm, 11.4 nm, 13.4 nm
3. (a) 13.4 nm   (b) 122.4 eV   (c) 72.9 nm, 11.4 nm, 13.4 nm
4. (a) -340 eV, -85 eV, -37.8 eV   (b) 4.86 nm, 4.10 nm   (c) 340 eV
5. (a) -217.6 eV, -54.4 eV, -24.2 eV   (b) 4   (c) 7.50 nm, 5.70 nm
6. (a) -217.6 eV, -54.4 eV, -24.2 eV, -13.6 eV   (b) 41.1 nm   (c) 7.60 nm
7. (a) -340 eV, -85 eV, -37.8 eV   (b) 340 eV   (c) 26.3 nm, 4.10 nm
8. (a) 5   (b) 340 eV
9. (a) 6   (b) 3.37 nm
10. (a) -122.4 eV, -30.6 eV, -13.6 eV, -7.65 eV   (b) 208.4 nm   (c) 7.65 eV
11. (a) 340 eV   (b) -340 eV, -85 eV, -37.8 eV, -21.3 eV   (c) 4.86 nm
12. (a) 72.9 nm, 40.5 nm   (b) $6 \times 10^{-8}$ eV
13. (a) 7.6 eV, 40.8 eV   (b) 54.4 eV

# SOLUTIONS

1.
$$\Delta p \sim \frac{\hbar}{\Delta x} = \frac{1}{c}\frac{\hbar c}{\Delta x} = \frac{1}{c}\frac{197\,\text{eV}\cdot\text{nm}}{0.10\,\text{nm}} = 1970\,\text{eV}/c$$

$$K_{av} = \frac{p_{av}^2}{2m} = \frac{p_{av}^2 c^2}{2mc^2} = \frac{(1970\,\text{eV})^2}{2(0.511\times10^6\,\text{eV})} = 3.8\,\text{eV}$$

This energy is consistent with the observed energies of electrons in atoms.

2. (a)
$$4\pi r^2 E = \frac{q_{\text{enclosed}}}{\varepsilon_0} = \frac{Ze}{\varepsilon_0}\left(\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}\right) \qquad \text{or} \qquad E = \frac{1}{4\pi\varepsilon_0}\frac{Ze}{R^3}r$$

(b)
$$F = eE = \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{R^3}r$$

3. (a)
$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{Ze^2}{4\pi\varepsilon_0 R^3 m}} = \frac{1}{2\pi}\sqrt{\frac{Ze^2 c^2}{4\pi\varepsilon_0 R^3 mc^2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{(1.440\,\text{eV}\cdot\text{nm})(3.00\times10^8\,\text{m/s})^2(10^9\,\text{nm/m})^2}{(0.053\,\text{nm})^3(0.511\times10^6\,\text{eV})}} = 6.57\times10^{15}\,\text{Hz}$$

$$\lambda = \frac{c}{f} = \frac{3.00\times10^8\,\text{m/s}}{6.57\times10^{15}\,\text{Hz}} = 4.57\times10^{-8}\,\text{m} = 45.7\,\text{nm}$$

This is about a factor of 3 smaller than the observed wavelength.

(b) With $Z = 11$ and $R = 0.18$ nm,

$$f = \frac{1}{2\pi}\sqrt{\frac{(11)(1.440\,\text{eV}\cdot\text{nm})(3.00\times10^8\,\text{m/s})^2(10^9\,\text{nm/m})^2}{(0.18\,\text{nm})^3(0.511\times10^6\,\text{eV})}} = 3.48\times10^{15}\,\text{Hz}$$

$$\lambda = \frac{c}{f} = \frac{3.00\times10^8\,\text{m/s}}{3.48\times10^{15}\,\text{Hz}} = 8.62\times10^{-8}\,\text{m} = 86.2\,\text{nm}$$

This is nearly an order of magnitude smaller than the observed wavelength.

4. (a) The force on an electron at the radius $x$ due to the positive sphere is given by Equation 6.2 evaluated for $Z = 2$:

$$F_+ = \frac{1}{4\pi\varepsilon_0}\frac{2e^2x}{R^3}$$

The force exerted on one electron by the other electron (located a distance $2x$ away) is

$$F_- = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4x^2}$$

At equilibrium, $F_+ = F_-$, so

$$\frac{2e^2x}{R^3} = \frac{e^2}{4x^2} \qquad \text{or} \qquad x = \frac{R}{2}$$

5. (a) From Equation 6.8,

$$b = \frac{Zz}{2K}\frac{e^2}{4\pi\varepsilon_0}\cot\frac{\theta}{2} = \frac{(2)(79)}{2(5.00\,\text{MeV})}(1.440\,\text{MeV}\cdot\text{fm})\cot 45° = 22.8\,\text{fm}$$

(b) We can rewrite Equation 6.18 as

$$\tfrac{1}{2}mv^2 r_{\min}^2 - \frac{e^2}{4\pi\varepsilon_0}Zz r_{\min} - \tfrac{1}{2}mv^2 b^2 = 0$$

which can be analyzed using the quadratic formula to give

$$r_{\min} = \frac{e^2 Zz}{4\pi\varepsilon_0 mv^2} \pm \sqrt{\left(\frac{e^2 Zz}{4\pi\varepsilon_0 mv^2}\right)^2 + b^2}$$

$$= \frac{(1.440\,\text{MeV}\cdot\text{fm})(2)(79)}{10.00\,\text{MeV}} \pm \sqrt{\left(\frac{(1.440\,\text{MeV}\cdot\text{fm})(2)(79)}{10.00\,\text{MeV}}\right)^2 + (22.8\,\text{fm})^2} = 55.0\,\text{fm or } -9.5\,\text{fm}$$

Only the positive root is physically meaningful, so $r_{\min} = 55.0$ fm.

(c) 
$$U = \frac{e^2}{4\pi\varepsilon_0}\frac{Zz}{r_{\min}} = \frac{(1.440\,\text{MeV}\cdot\text{fm})(2)(79)}{55.0\,\text{fm}} = 4.14\,\text{MeV}$$

$$K = E - U = 5.00\,\text{MeV} - 4.14\,\text{MeV} = 0.86\,\text{MeV}$$

6. From Equation 6.19,

$$K = \frac{e^2}{4\pi\varepsilon_0}\frac{Zz}{d} = \frac{(1.440\,\text{MeV}\cdot\text{fm})(2)(79)}{7.0\,\text{fm}} = 33\,\text{MeV}$$

7.  $$d = \frac{e^2}{4\pi\varepsilon_0}\frac{Zz}{K} = \frac{(1.440\,\text{MeV}\cdot\text{fm})(2)(29)}{6.0\,\text{MeV}} = 14\,\text{fm}$$

8.  (a) The density $\rho$ of silver is 10.5 g/cm$^3$ and its molar mass $M$ is 107.9 g/mole.  Thus

$$n = \frac{N_A\rho}{M} = \frac{(6.02\times10^{23}\,\text{atoms/mole})(10.5\,\text{g/cm}^3)}{107.9\,\text{g/mole}} = 5.86\times10^{22}\,\text{atoms/cm}^3 = 5.86\times10^{28}\,\text{atoms/m}^3$$

At $\theta = 90°$, $b = \dfrac{Zz}{2K}\dfrac{e^2}{4\pi\varepsilon_0}\cot\dfrac{\theta}{2} = \dfrac{(1)(47)}{2(5.00\,\text{MeV})}\cot 45° = 6.77\,\text{fm}$

$f_{>90°} = nt\pi b^2 = (5.86\times10^{28}\,\text{m}^{-3})(4.0\times10^6\,\text{m})(\pi)(6.77\times10^{-15}\,\text{m})^2 = 3.37\times10^{-5}$

(b) At $\theta = 10°$, $b = \dfrac{Zz}{2K}\dfrac{e^2}{4\pi\varepsilon_0}\cot\dfrac{\theta}{2} = \dfrac{(1)(47)}{2(5.00\,\text{MeV})}\cot 5° = 77.4\,\text{fm}$

$f_{>10°} = nt\pi b^2 = (5.86\times10^{28}\,\text{m}^{-3})(4.0\times10^6\,\text{m})(\pi)(77.4\times10^{-15}\,\text{m})^2 = 4.41\times10^{-3}$

(c) At $\theta = 5°$, $b = \dfrac{Zz}{2K}\dfrac{e^2}{4\pi\varepsilon_0}\cot\dfrac{\theta}{2} = \dfrac{(1)(47)}{2(5.00\,\text{MeV})}\cot 2.5° = 155\,\text{fm}$

$f_{>5°} = nt\pi b^2 = (5.86\times10^{28}\,\text{m}^{-3})(4.0\times10^6\,\text{m})(\pi)(155\times10^{-15}\,\text{m}) = 1.77\times10^{-2}$

$f_{>5°} - f_{>10°} = 1.77\times10^{-2} - 4.43\times10^{-2} = 1.33\times10^{-2}$

(d) $f_{<5°} = 1 - f_{>5°} = 1 - 1.77\times10^{-2} = 0.982$

9.  (a) From Equation 6.19

$$K = \frac{e^2}{4\pi\varepsilon_0}\frac{Zz}{d} = \frac{(1.440\,\text{MeV}\cdot\text{fm})(1)(29)}{5.0\,\text{fm}} = 8.4\,\text{MeV}$$

(b)  $$b = \frac{Zz}{2K}\frac{e^2}{4\pi\varepsilon_0}\cot\frac{\theta}{2} = \frac{(1)(29)}{2(7.5\,\text{MeV})}(1.440\,\text{MeV}\cdot\text{fm})\cot 60° = 1.61\,\text{fm}$$

(c) We can rewrite Equation 6.18 as  $\frac{1}{2}mv^2 r_{min}^2 - \dfrac{e^2}{4\pi\varepsilon_0}Zz r_{min} - \frac{1}{2}mv^2 b^2 = 0$

and the quadratic formula then gives

$$r_{min} = \frac{e^2 Zz}{4\pi\varepsilon_0 mv^2} \pm \sqrt{\left(\frac{e^2 Zz}{4\pi\varepsilon_0 mv^2}\right)^2 + b^2}$$

$$= \frac{(1.440\,\text{MeV}\cdot\text{fm})(1)(29)}{15.0\,\text{MeV}} \pm \sqrt{\left(\frac{(1.440\,\text{MeV}\cdot\text{fm})(1)(29)}{15.0\,\text{MeV}}\right)^2 + (1.61\,\text{fm})^2} = 6.00\,\text{fm or } -0.43\,\text{fm}$$

Only the positive root is physically meaningful, so $r_{min} = 6.00$ fm.

(d) For copper, $\rho = 8.95$ g/cm$^3$ and $M = 63.5$ g/mole, so

$$n = \frac{N_A \rho}{M} = \frac{(6.02\times10^{23}\,\text{atoms/mole})(8.95\,\text{g/cm}^3)}{63.5\,\text{g/mole}} = 8.49\times10^{28}\,\text{atoms/m}^3$$

$$f_{>120°} = nt\pi b^2 = (8.49\times10^{28}\,\text{m}^{-3})(12\times10^{-6}\,\text{m})(\pi)(1.61\times10^{-15}\,\text{m})^2 = 8.3\times10^{-6}$$

10.  $$R = \frac{f_{>90°}(\text{Au})}{f_{>90°}(\text{Ag})} = \frac{n_{Au}t_{Au}\pi b_{Au}^2}{n_{Ag}t_{Ag}\pi b_{Ag}^2} = \frac{\rho_{Au}M_{Ag}Z_{Au}^2}{\rho_{Ag}M_{Au}Z_{Ag}^2} = \frac{(19.3\,\text{g/cm}^3)(107.9\,\text{g/mole})(79)^2}{(10.5\,\text{g/cm}^3)(197.0\,\text{g/mole})(47)^2} = 2.84$$

11.  As a rough first approximation, we can assume that the light alpha particle rebounds from the massive gold nucleus with a final momentum $p_f$ that is in magnitude nearly equal to its initial momentum $p_i$, so that the recoil momentum given to the gold nucleus is

$$p_{recoil} = p_i - p_f \cong p_i - (-p_i) = 2p_i$$

This is accurate to within about 2%.  The recoil kinetic energy of the gold nucleus is

$$K_{recoil} = \frac{p_{recoil}^2}{2M_{Au}} = \frac{4p_i^2}{2M_{Au}} = \frac{4(2m_\alpha K_\alpha)}{2M_{Au}} = 4(8.0\,\text{MeV})\frac{4}{197} = 0.63\,\text{MeV}$$

12.  Let $p_i$ and $p_f$ represent the initial and final momentum of the alpha particle, and let $K_i$ and $K_f$ be its initial and final kinetic energies.  Then conservation of momentum and energy give $p_i = p_f + p_e$ and $K_i = K_f + K_e$, or

$$\frac{p_i^2}{2m_\alpha} = \frac{p_f^2}{2m_\alpha} + \frac{p_e^2}{2m_e} = \frac{(p_i - p_e)^2}{2m_\alpha} + \frac{p_e^2}{2m_e} = \frac{p_i^2}{2m_\alpha} - \frac{2p_i p_e}{2m_\alpha} + \frac{p_e^2}{2m_\alpha} + \frac{p_e^2}{2m_e}$$

$$\frac{p_i p_e}{m_\alpha} = p_e^2 \left( \frac{1}{2m_\alpha} + \frac{1}{2m_e} \right) \qquad \text{or} \qquad p_e = \frac{2p_i}{1 + m_e/m_\alpha}$$

$$K_i - K_f = K_e = \frac{p_e^2}{2m_e} = \frac{4p_i^2}{2m_e(1 + m_e/m_\alpha)^2} = \frac{p_i^2}{2m_\alpha} \frac{4m_\alpha}{(1 + m_e/m_\alpha)^2}$$

$$= (8.0\,\text{MeV}) \frac{4(7294)}{(1 + 7294)^2} = 0.0044\,\text{MeV}$$

13. The potential energy at minimum separation is $U = E - K = 4.8$ MeV, so

$$r_{\min} = \frac{e^2}{4\pi\varepsilon_0} \frac{Zz}{U} = \frac{(1.440\,\text{MeV}\cdot\text{fm})(2)(47)}{4.8\,\text{MeV}} = 28.2\,\text{fm}$$

From conservation of angular momentum (Equation 6.17), we obtain

$$b = r_{\min} \frac{v_{\min}}{v} = r_{\min} \sqrt{\frac{K_{\min}}{K}} = (28.2\,\text{fm}) \sqrt{\frac{4.8\,\text{MeV}}{9.6\,\text{MeV}}} = 19.9\,\text{fm}$$

$$\cot\frac{\theta}{2} = \frac{8\pi\varepsilon_0 Kb}{Zze^2} = \frac{(2)(9.6\,\text{MeV})(19.9\,\text{fm})}{(2)(47)(1.440\,\text{MeV}\cdot\text{fm})} = 2.828 \quad \text{or} \quad \theta = 38.9°$$

14. $$n = \frac{N_A \rho}{M} = \frac{(6.02\times10^{23}\,\text{atoms/mole})(19.3\,\text{g/cm}^3)}{(197\,\text{g/mole})(10^{-2}\,\text{m/cm})^3} = 5.90\times10^{28}\,\text{atoms/m}^3$$

$$N(\theta) = \frac{nt}{4r^2} \left( \frac{Zz}{2K} \right)^2 \left( \frac{e^2}{4\pi\varepsilon_0} \right)^2 \frac{1}{\sin^4 \theta/2}$$

$$= \frac{(5.90\times10^{28}\,\text{m}^{-3})(3.0\times10^{-6}\,\text{m})}{4(0.12\,\text{m})^2} \left[ \frac{(2)(79)(1.440\,\text{MeV}\cdot\text{fm})}{2(6.0\,\text{MeV})} \right]^2 \frac{1}{\sin^4 15°} = 0.246\,\text{m}^{-2}$$

This gives the probability per unit area for an alpha particle to be scattered into the detector. The area of the detector is $\pi(0.50\,\text{cm})^2 = 7.96 \times 10^{-6}\,\text{m}^2$. The total probability for an alpha particle to strike the detector is $(0.246\,\text{m}^{-2})(7.96\times10^{-6}\,\text{m}^2) = 1.96\times10^{-6}$. That is, each alpha particle has a probability of $1.96 \times 10^{-6}$ to be scattered into the detector. If the rate of incident particles is $3.0 \times 10^7\,\text{s}^{-1}$, the rate at which they strike the detector is $(1.96\times10^{-6})(3.0\times10^7\,\text{s}^{-1}) = 59\,\text{s}^{-1}$.

15.  The shortest wavelength is the series limit.  For the Lyman series, $n_0 = 1$ and Equation 6.21 becomes

$$\lambda = (91.13 \text{ nm})\frac{n^2}{n^2 - 1} \qquad n = 2, 3, 4, \ldots$$

which gives $\lambda = 121.51$ nm ($n = 2$), 102.52 nm ($n = 3$), 97.21 nm ($n = 4$).

16.  For the Brackett series, $n_0 = 4$.  With $\lambda = 1944$ nm, Equation 6.21 gives

$$1944 \text{ nm} = (1458 \text{ nm})\frac{n^2}{n^2 - 4^2}$$

which can be solved to give $n = 8$.  The next higher ($n = 7$) and next lower ($n = 9$) lines are

$$\lambda = (1458 \text{ nm})\frac{7^2}{7^2 - 4^2} = 2165 \text{ nm} \qquad \lambda = (1458 \text{ nm})\frac{9^2}{9^2 - 4^2} = 1817 \text{ nm}$$

17.  For the Pfund series, $n_0 = 5$ and the longest wavelength corresponds to $n = 6$.  Solving Equation 6.21 for the series limit, we have

$$\lambda_{\text{limit}} = \lambda\frac{n^2 - n_0^2}{n^2} = (7459 \text{ nm})\frac{6^2 - 5^2}{6^2} = 2279 \text{ nm}$$

18.  $r_3 = 9a_0 = 9(0.0529 \text{ nm}) = 0.476 \text{ nm}$

$$v = \frac{n\hbar}{mr} = c\frac{n\hbar c}{mc^2 r} = c\frac{3(1240 \text{ eV} \cdot \text{nm})/2\pi}{(0.511 \times 10^6 \text{ eV})(0.476 \text{ nm})} = 2.43 \times 10^{-3} c = 7.30 \times 10^5 \text{ m/s}$$

$$U = -\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} = -\frac{1.440 \text{ eV} \cdot \text{nm}}{0.476 \text{ nm}} = -3.02 \text{ eV}$$

$$K = \frac{e^2}{8\pi\varepsilon_0}\frac{1}{r} = \frac{1.440 \text{ eV} \cdot \text{nm}}{2(0.476 \text{ nm})} = 1.51 \text{ eV}$$

19.  The Lyman series consists of transitions from higher levels to the $n_2 = 1$ level.  The series limit would be the transition with the highest energy, corresponding to a jump from $n_1 = \infty$ to $n_2 = 1$.  The wavelength is found from Equation 6.33:

$$\lambda = \frac{1}{R_\infty}\left(\frac{n_1^2 n_2^2}{n_1^2 - n_2^2}\right) = \frac{n_2^2}{R_\infty}\left(\frac{n_1^2}{n_1^2 - n_2^2}\right) = \frac{1}{1.09737 \times 10^7 \text{ m}^{-1}} = 91.13 \text{ nm}$$

For the Paschen series ($n_2 = 3$), the series limit is (with $n_1 = \infty$)

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$$\lambda_{\text{limit}} = \frac{n_2^2}{R_\infty} = \frac{9}{1.09737 \times 10^7 \text{ m}^{-1}} = 820.1 \text{ nm}$$

20.  (a) From Equation 6.26, $v = \dfrac{n\hbar}{mr} = \dfrac{n\hbar}{mn^2 a_0}$. Using Equation 6.29 for $a_0$, we obtain

$$v = \frac{\hbar}{nm(4\pi\varepsilon_0 \hbar^2 / me^2)} = \frac{e^2}{4\pi\varepsilon_0} \frac{1}{n\hbar} = \frac{\alpha c}{n}$$

(b)  When the nuclear charge is $Ze$, we must replace $e^2$ with $Ze^2$, so $v = Z\alpha c / n$.

21.  The energy of the initial $n = 5$ state is $E_5 = \dfrac{-13.6 \text{ eV}}{25} = -0.544 \text{ eV}$. An electron in this state can make transitions to any of the lower states with $n = 4$ ($E_4 = -0.850$ eV), $n = 3$ ($E_3 = -1.51$ eV), $n = 2$ ($E_2 = -3.40$ eV), and $n = 1$ ($E_1 = -13.6$ eV). The transition energies are:

$$5 \to 4: \quad \Delta E = E_5 - E_4 = -0.544 \text{ eV} - (-0.850 \text{ eV}) = 0.306 \text{ eV}$$
$$5 \to 3: \quad \Delta E = E_5 - E_3 = -0.544 \text{ eV} - (-1.51 \text{ eV}) = 0.97 \text{ eV}$$
$$5 \to 2: \quad \Delta E = E_5 - E_2 = -0.544 \text{ eV} - (-3.40 \text{ eV}) = 2.86 \text{ eV}$$
$$5 \to 1: \quad \Delta E = E_5 - E_1 = -0.544 \text{ eV} - (-13.6 \text{ eV}) = 13.1 \text{ eV}$$

22.  The Paschen series consists of transitions from higher levels that end in the $n = 3$ level. The energies and wavelengths are:
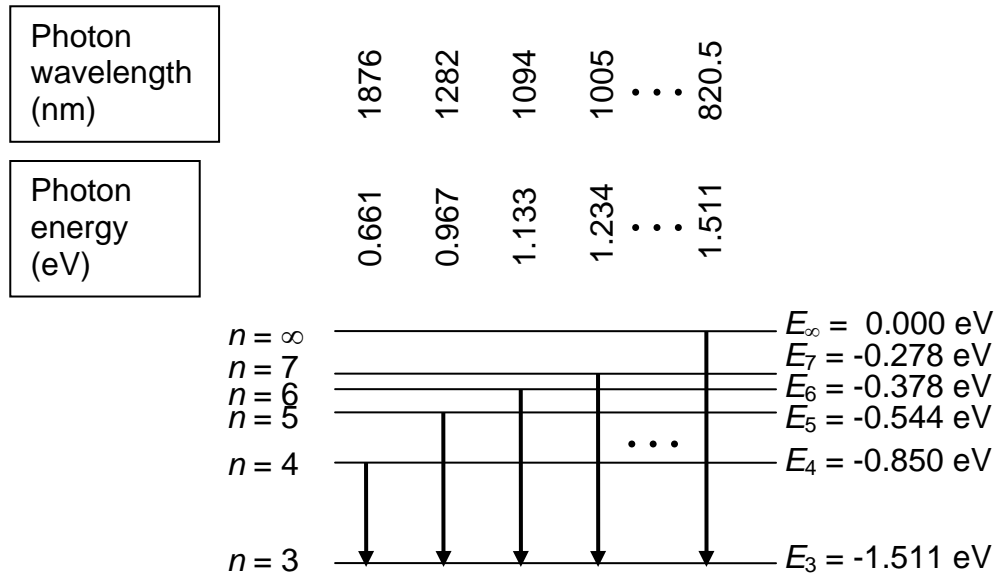
$$4 \to 3: \quad \Delta E = (-13.60 \text{ eV})\left(\frac{1}{4^2} - \frac{1}{3^2}\right) = 0.661 \text{ eV} \qquad \lambda = \frac{1240 \text{ eV} \cdot \text{nm}}{0.661 \text{ eV}} = 1876 \text{ nm}$$
$$5 \to 3: \quad \Delta E = (-13.60 \text{ eV})\left(\frac{1}{5^2} - \frac{1}{3^2}\right) = 0.967 \text{ eV} \qquad \lambda = \frac{1240 \text{ eV} \cdot \text{nm}}{0.967 \text{ eV}} = 1282 \text{ nm}$$
$$6 \to 3: \quad \Delta E = (-13.60 \text{ eV})\left(\frac{1}{6^2} - \frac{1}{3^2}\right) = 1.133 \text{ eV} \qquad \lambda = \frac{1240 \text{ eV} \cdot \text{nm}}{1.133 \text{ eV}} = 1094 \text{ nm}$$
$$7 \to 3: \quad \Delta E = (-13.60 \text{ eV})\left(\frac{1}{7^2} - \frac{1}{3^2}\right) = 1.234 \text{ eV} \qquad \lambda = \frac{1240 \text{ eV} \cdot \text{nm}}{1.234 \text{ eV}} = 1005 \text{ nm}$$

The series limit is 1.511 eV, corresponding to a wavelength of 820.5 nm.

Photon wavelength (nm): 1876   1282   1094   1005   ...   820.5

Photon energy (eV): 0.661   0.967   1.133   1.234   ...   1.511

$n = \infty$     $E_\infty = 0.000$ eV
$n = 7$     $E_7 = -0.278$ eV
$n = 6$     $E_6 = -0.378$ eV
$n = 5$     $E_5 = -0.544$ eV
$n = 4$     $E_4 = -0.850$ eV
$n = 3$     $E_3 = -1.511$ eV

23.  The photon energies of the incident light are

$$E = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{59.0 \text{ nm}} = 21.0 \text{ eV}$$

When an atom in the ground state absorbs a 21.0-eV photon, the atom is ionized (which takes 13.6 eV).  The excess energy, 21.0 eV − 13.6 eV = 7.4 eV, appears as the kinetic energy of the electron, which is now free of the atom.  Neglecting a small recoil kinetic energy given to the proton, the electrons have a kinetic energy of 7.4 eV.

24.  (a) The ionization energy is the magnitude of the energy of the electron.  For the $n = 3$ level of hydrogen

$$\left| E_3 \right| = \left| \frac{-13.6 \text{ eV}}{9} \right| = 1.51 \text{ eV}$$

(b) For singly ionized helium ($Z = 2$) we use Equation 6.38:

$$\left| E_n \right| = \left| \frac{(-13.6 \text{ eV})Z^2}{n^2} \right| = \left| \frac{(-13.6 \text{ eV})2^2}{2^2} \right| = 13.6 \text{ eV}$$

(c) $$\left| E_n \right| = \left| \frac{(-13.6 \text{ eV})Z^2}{n^2} \right| = \left| \frac{(-13.6 \text{ eV})3^2}{4^2} \right| = 7.65 \text{ eV}$$

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25. (a)    $E(4 \rightarrow 2) = E_4 - E_2 = (-13.6 \text{ eV}) \left( \dfrac{1}{4^2} - \dfrac{1}{2^2} \right) = 2.55 \text{ eV}$

$E(4 \rightarrow 3) = E_4 - E_3 = (-13.6 \text{ eV}) \left( \dfrac{1}{4^2} - \dfrac{1}{3^2} \right) = 0.661 \text{ eV}$

$E(3 \rightarrow 2) = E_3 - E_2 = (-13.6 \text{ eV}) \left( \dfrac{1}{3^2} - \dfrac{1}{2^2} \right) = 1.89 \text{ eV}$

$E(4 \rightarrow 3) + E(3 \rightarrow 2) = 0.661 \text{ eV} + 1.89 \text{ eV} = 2.55 \text{ eV} = E(4 \rightarrow 2)$

(b)    $E(4 \rightarrow 1) = E_4 - E_2 = (-13.6 \text{ eV}) \left( \dfrac{1}{4^2} - \dfrac{1}{1^2} \right) = 12.8 \text{ eV}$

$E(2 \rightarrow 1) = E_2 - E_1 = (-13.6 \text{ eV}) \left( \dfrac{1}{2^2} - \dfrac{1}{1^2} \right) = 10.2 \text{ eV}$

$E(4 \rightarrow 2) + E(2 \rightarrow 1) = 2.55 \text{ eV} + 10.2 \text{ eV} = 12.8 \text{ eV} = E(4 \rightarrow 1)$

26. The Lyman series consists of transitions that end in the $n = 1$ level. The smallest energy difference, corresponding to the longest wavelength, is $n = 2$ to $n = 1$.

$$\Delta E = E_2 - E_1 = (-13.6 \text{ eV}) 2^2 \left( \dfrac{1}{2^2} - \dfrac{1}{1^2} \right) = 40.8 \text{ eV}$$

$$\lambda = \dfrac{hc}{\Delta E} = \dfrac{1240 \text{ eV} \cdot \text{nm}}{40.8 \text{ eV}} = 30.4 \text{ nm}$$

The largest energy difference would correspond to transitions from $n = \infty$ to $n = 1$:

$$\Delta E = E_\infty - E_1 = (-13.6 \text{ eV}) 2^2 \left( 0 - \dfrac{1}{1^2} \right) = 54.4 \text{ eV}$$

$$\lambda = \dfrac{hc}{\Delta E} = \dfrac{1240 \text{ eV} \cdot \text{nm}}{54.4 \text{ eV}} = 22.8 \text{ nm}$$

27. Using Equation 6.38, we have $E_n = (-13.6 \text{ eV}) Z^2 / n^2 = (-54.4 \text{ eV}) / n^2$, so $E_1 = -54.40 \text{ eV}$, $E_2 = -13.60 \text{ eV}$, $E_3 = -6.04 \text{ eV}$, $E_4 = -3.40 \text{ eV}$. The possible transitions are:

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$$4 \rightarrow 1: \quad \Delta E = E_4 - E_1 = 51.00 \text{ eV} \qquad \lambda = hc / \Delta E = 24.31 \text{ nm}$$

$$4 \rightarrow 2: \quad \Delta E = E_4 - E_2 = 10.20 \text{ eV} \qquad \lambda = hc / \Delta E = 121.6 \text{ nm}$$

$$4 \rightarrow 3: \quad \Delta E = E_4 - E_3 = 2.64 \text{ eV} \qquad \lambda = hc / \Delta E = 469.7 \text{ nm}$$

$$3 \rightarrow 1: \quad \Delta E = E_3 - E_1 = 48.36 \text{ eV} \qquad \lambda = hc / \Delta E = 25.64 \text{ nm}$$

$$3 \rightarrow 2: \quad \Delta E = E_3 - E_2 = 7.56 \text{ eV} \qquad \lambda = hc / \Delta E = 164.0 \text{ nm}$$

$$2 \rightarrow 1: \quad \Delta E = E_2 - E_1 = 40.80 \text{ eV} \qquad \lambda = hc / \Delta E = 30.39 \text{ nm}$$
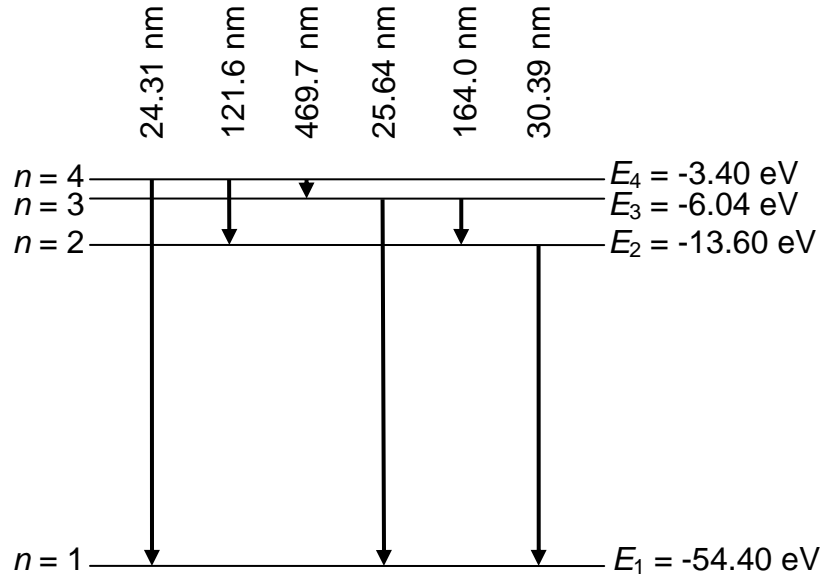


28.  The gravitational force law is $F = G m_e m_p / r^2$ instead of $F = e^2 / 4\pi\varepsilon_0 r^2$. The Bohr theory can thus be directly applied if we substitute $G m_e m_p$ for $e^2 / 4\pi\varepsilon_0$. Equation 6.29 becomes

$$a_0 = \frac{\hbar^2}{G m_e^2 m_p} = \frac{(1.05 \times 10^{-34} \text{ J} \cdot \text{s})^2}{(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2)(9.11 \times 10^{-31} \text{ kg})^2(1.67 \times 10^{-27} \text{ kg})} = 1.19 \times 10^{29} \text{ m}$$

$$E_2 - E_1 = -\frac{m_e}{2\hbar^2}(G m_e m_p)^2 \left( \frac{1}{2^2} - \frac{1}{1^2} \right) = \frac{3 G^2 m_e^3 m_p^2}{8\hbar^2}$$

$$= \frac{3(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2)^2(9.11 \times 10^{-31} \text{ kg})^3(1.67 \times 10^{-27} \text{ kg})^2}{8(1.05 \times 10^{-34} \text{ J} \cdot \text{s})^2} = 3.2 \times 10^{-97} \text{ J} = 2.0 \times 10^{-78} \text{ eV}$$

29. (a) If the circumference is an integral number of de Broglie wavelengths ($2\pi r = n\lambda$), then after each orbit the waves will align, peak to peak and valley to valley, to give standing waves.

(b) $2\pi r = n\lambda = n\dfrac{h}{p} = \dfrac{nh}{mv}$    so    $mvr = \dfrac{nh}{2\pi} = n\hbar$

30. Let $V_1 = 4$ V, $V_2 = 7$ V, and $V_3 = 9$ V. Then decreases in the current should be observed at the following voltages:

| | | |
|---|---|---|
| $V_1 = 4$ V | $3V_1 = 12$ V | $2V_1 + V_3 = 17$ V |
| $V_2 = 7$ V | $V_1 + V_3 = 13$ V | $2V_3 = 18$ V |
| $2V_1 = 8$ V | $2V_2 = 14$ V | $3V_1 + V_3 = 19$ V |
| $V_3 = 9$ V | $2V_1 + V_2 = 15$ V | $5V_1 = 20$ V |
| $V_1 + V_2 = 11$ V | $4V_1 = 16$ V | |

31. The energy difference between the ground state and the first excited state is

$$E = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{590 \text{ nm}} = 2.10 \text{ eV}$$

At $V = 2.10$ V, we expect to see a decrease in the current, as atoms are raised to the first excited state.

32. The energy uncertainty of a state with a lifetime of $10^{-8}$ s is

$$\Delta E \sim \frac{\hbar}{\Delta t} = \frac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{10^{-8} \text{ s}} = 6.58 \times 10^{-8} \text{ eV}$$

This energy uncertainty will be equal to the spacing $\Delta E = hf$ with $f$ given by Equation 6.43 when $n$ is large:

$$\Delta E = hf = \frac{me^4}{16\pi^2 \varepsilon_0^2 \hbar^2} \frac{1}{n^3} = \frac{mc^2}{(\hbar c)^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{1}{n^3}$$

$$n = \sqrt[3]{\frac{mc^2(e^2/4\pi\varepsilon_0)^2}{\Delta E(\hbar c)^2}} = \sqrt[3]{\frac{(0.511\times10^6 \text{ eV})(1.440 \text{ eV} \cdot \text{nm})^2}{(6.58\times10^{-8} \text{ eV})(197 \text{ eV} \cdot \text{nm})^2}} = 746$$

$$r = n^2 a_0 = (746)^2 (5.29\times10^{-11} \text{ m}) = 29 \ \mu\text{m}$$

33. (a) The frequency of revolution is given by Equation 6.41:

$$f_n = \frac{me^4}{32\pi^3\varepsilon_0^2\hbar^3}\frac{1}{n^3} = \frac{1}{\pi\hbar}\frac{me^4}{32\pi^2\varepsilon_0^2\hbar^2}\frac{1}{n^3} = \frac{13.6\text{ eV}}{\pi\hbar}\frac{1}{n^3} = \frac{6.58\times10^{15}\text{ Hz}}{n^3}$$

A similar calculation gives the radiation frequency from Equation 6.42:

$$f = \frac{me^4}{64\pi^3\varepsilon_0^2\hbar^3}\frac{2n-1}{n^2(n-1)^2} = \frac{13.6\text{ eV}}{2\pi\hbar}\frac{2n-1}{n^2(n-1)^2} = (6.58\times10^{15}\text{ Hz})\frac{2n-1}{2n^2(n-1)^2}$$

For $n = 10$, we get $f_n = 6.58\times10^{12}$ Hz and $f = 7.72\times10^{12}$ Hz.

(b) For $n = 100$, $f_n = 6.58\times10^{9}$ Hz and $f = 6.68\times10^{9}$ Hz.

(c) For $n = 1000$, $f_n = 6.58\times10^{6}$ Hz and $f = 6.59\times10^{6}$ Hz.

(d) For $n = 10{,}000$, $f_n = 6.58\times10^{3}$ Hz and $f = 6.58\times10^{3}$ Hz. Note how $f$ approaches $f_n$ as $n$ becomes large, in accordance with the correspondence principle.

34. The Rydberg constant in ordinary hydrogen is

$$R_H = R_\infty\left(1+\frac{m}{M_H}\right) = R_\infty\left(1+\frac{5.48580\times10^{-4}\text{ u}}{1.007825\text{ u}}\right) = R_\infty(1.000544)$$

and in "heavy" hydrogen or deuterium:

$$R_D = R_\infty\left(1+\frac{m}{M_D}\right) = R_\infty\left(1+\frac{5.48580\times10^{-4}\text{ u}}{2.104102\text{ u}}\right) = R_\infty(1.000272)$$

From Equation 6.33 the difference in wavelengths for the first line of the Balmer series ($n = 3$ to $n = 2$) is

$$\lambda_D - \lambda_H = \left(\frac{1}{R_D}-\frac{1}{R_H}\right)\left(\frac{3^2 2^2}{3^2-2^2}\right) = \frac{7.2}{1.09737\times10^7\text{ m}^{-1}}\left(\frac{1}{1.000272}-\frac{1}{1.000544}\right) = 0.178\text{ nm}$$

This small wavelength difference led to the discovery of deuterium in 1931.

35. (a) 15 different transitions are possible: $6 \to 5$, $6 \to 4$, $6 \to 3$, $6 \to 2$, $6 \to 1$, $5 \to 4$, $5 \to 3$, $5 \to 2$, $5 \to 1$, $4 \to 3$, $4 \to 2$, $4 \to 1$, $3 \to 2$, $3 \to 1$, $2 \to 1$.

(b) Only 5 of the transitions change $n$ by one unit.

(c) One.

36. (a) From the $n = 8$ level, downward transitions are possible to any level of smaller $n$. The transitions with the longest wavelengths are those with the smallest energy differences.

8→7:     $\Delta E = E_8 - E_7 = (-13.6 \text{ eV})Z^2 \left(\frac{1}{8^2} - \frac{1}{7^2}\right) = 0.260 \text{ eV}$

$\lambda = \dfrac{hc}{\Delta E} = \dfrac{1240 \text{ eV} \cdot \text{nm}}{0.260 \text{ eV}} = 4.77 \ \mu\text{m}$

8→6:     $\Delta E = E_8 - E_6 = (-13.6 \text{ eV})Z^2 \left(\frac{1}{8^2} - \frac{1}{6^2}\right) = 0.661 \text{ eV}$

$\lambda = \dfrac{hc}{\Delta E} = \dfrac{1240 \text{ eV} \cdot \text{nm}}{0.661 \text{ eV}} = 1.88 \ \mu\text{m}$

8→5:     $\Delta E = E_8 - E_5 = (-13.6 \text{ eV})Z^2 \left(\frac{1}{8^2} - \frac{1}{5^2}\right) = 1.33 \text{ eV}$

$\lambda = \dfrac{hc}{\Delta E} = \dfrac{1240 \text{ eV} \cdot \text{nm}}{1.33 \text{ eV}} = 0.935 \ \mu\text{m}$

(b) The transition with the shortest wavelength is the one with the largest energy difference.

8→1:     $\Delta E = E_8 - E_1 = (-13.6 \text{ eV})Z^2 \left(\frac{1}{8^2} - \frac{1}{1^2}\right) = 53.6 \text{ eV}$

$\lambda = \dfrac{hc}{\Delta E} = \dfrac{1240 \text{ eV} \cdot \text{nm}}{53.6 \text{ eV}} = 23.2 \text{ nm}$

(c) From the $n = 8$ level, the atom can absorb a photon and the electron will jump to a state of larger $n$. The longest absorption wavelengths correspond to the smallest energy differences.

8→9:     $\Delta E = E_9 - E_8 = (-13.6 \text{ eV})Z^2 \left(\frac{1}{9^2} - \frac{1}{8^2}\right) = 0.178 \text{ eV}$

$\lambda = \dfrac{hc}{\Delta E} = \dfrac{1240 \text{ eV} \cdot \text{nm}}{0.178 \text{ eV}} = 6.95 \ \mu\text{m}$

8→10:     $\Delta E = E_{10} - E_8 = (-13.6 \text{ eV})Z^2 \left(\frac{1}{10^2} - \frac{1}{8^2}\right) = 0.306 \text{ eV}$

$\lambda = \dfrac{hc}{\Delta E} = \dfrac{1240 \text{ eV} \cdot \text{nm}}{0.306 \text{ eV}} = 4.05 \ \mu\text{m}$

8→11:     $\Delta E = E_{11} - E_8 = (-13.6 \text{ eV})Z^2 \left(\frac{1}{11^2} - \frac{1}{8^2}\right) = 0.400 \text{ eV}$

$\lambda = \dfrac{hc}{\Delta E} = \dfrac{1240 \text{ eV} \cdot \text{nm}}{0.400 \text{ eV}} = 3.10 \ \mu\text{m}$

(d) The shortest absorption wavelength corresponds to the largest energy difference.

$8 \to \infty$:   $\Delta E = E_\infty - E_8 = (-13.6 \text{ eV})Z^2 \left(0 - \frac{1}{8^2}\right) = 0.850 \text{ eV}$

$$\lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV} \cdot \text{nm}}{0.850 \text{ eV}} = 1.46 \text{ } \mu\text{m}$$

37. The time $\Delta t$ in which we must measure the energy of the state is no greater than the lifetime of that state, and the energy measurement must be uncertain by an amount $\Delta E$ given by the uncertainty principle.

$$\Delta E \sim \frac{\hbar}{\Delta t} = \frac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{10^{-8} \text{ s}} = 7 \times 10^{-8} \text{ eV}$$

This energy uncertainty is negligible compared with the energy of the first excited state ($-3.4$ eV). There are very few measurements that are capable of determining energies to 1 part in $10^8$, and thus the energy uncertainty does not affect the observed photon energies in most experiments.

38. For a transition from $n$ to $n_0$,

$$\lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV} \cdot \text{nm}}{(-13.6 \text{ eV})Z^2(1/n^2 - 1/n_0^2)} = (22.78 \text{ nm})\frac{n^2 n_0^2}{n^2 - n_0^2}$$

Transitions ending on the state with $n_0 = 1$ (analogous to the Lyman series in hydrogen) have wavelengths ranging from 22.78 nm (for jumps from $n = \infty$) to 30.38 nm (for jumps from $n = 2$). Transitions ending with $n_0 = 2$ (similar to the Lyman series) have wavelengths from 91.13 nm ($n = \infty$) to 164.0 nm ($n = 3$). Those ending with $n_0 = 3$ range from 205.0 nm ($n = \infty$) to 468.6 nm ($n = 4$).

   For a given value of $n_0$, we can solve for $n$:

$$n = \sqrt{\frac{1}{1/n_0^2 - (22.78 \text{ nm})/\lambda}}$$

The first two given wavelengths (24.30 nm and 25.63 nm) fall in the range of transitions with $n_0 = 1$, so we find

$$n = \sqrt{\frac{1}{1 - (22.78 \text{ nm})/(24.30 \text{ nm})}} = 4 \quad \text{and} \quad n = \sqrt{\frac{1}{1 - (22.78 \text{ nm})/(25.63 \text{ nm})}} = 3$$

The transition with wavelength 102.5 nm belongs to the group with $n_0 = 2$:

$$n = \sqrt{\frac{1}{1/4 - (22.78 \text{ nm})/(102.5 \text{ nm})}} = 6$$

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and 320.4 nm belongs to the group with $n_0 = 3$:

$$n = \sqrt{\frac{1}{1/9 - (22.78 \text{ nm})/(320.4 \text{ nm})}} = 5$$

39. For a transition from $n$ to $n_0$ in lithium ($Z = 3$),

$$\lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV} \cdot \text{nm}}{(-13.6 \text{ eV})Z^2(1/n^2 - 1/n_0^2)} = (10.13 \text{ nm})\frac{n^2 n_0^2}{n^2 - n_0^2}$$

The range of wavelengths for each series is: 10.13 nm to 13.50 nm for $n_0 = 1$; 40.50 nm to 72.90 nm for $n_0 = 2$; 91.17 nm to 208.4 nm for $n_0 = 3$; and so forth. The wavelengths given for this problem clearly fall in the group with $n_0 = 2$, for which

$$n = \sqrt{\frac{1}{1/4 - (10.13 \text{ nm})/\lambda}}$$

This gives $n = 3$ for 72.90 nm and $n = 4$ for 54.00 nm. The next member of the series corresponds to $n = 5$, which has wavelength

$$\lambda = \frac{hc}{\Delta E} == (10.13 \text{ nm})\frac{5^2 2^2}{5^2 - 2^2} = 48.23 \text{ nm}$$

40. Let $p$ and $E$ be the momentum and energy of the photon, and assume the atom is at rest before the photon is emitted. Then conservation of momentum and energy give:

$$p_{\text{initial}} = p_{\text{final}}: \qquad 0 = p - p_{\text{R}}$$

$$E_{\text{initial}} = E_{\text{final}}: \qquad E_1 = E_2 + E + K_{\text{R}}$$

Using nonrelativistic kinetic energy for the recoil of the atom, $p_{\text{R}} = \sqrt{2MK_{\text{R}}}$, and substituting $E = cp = cp_{\text{R}}$, we obtain

$$E_1 - E_2 = cp_{\text{R}} + K_{\text{R}} = \sqrt{2Mc^2 K_{\text{R}}} + K_{\text{R}}$$

Because we expect $K_{\text{R}} \ll Mc^2$, the second term on the right can be neglected and we can solve for $K_{\text{R}}$:

$$K_{\text{R}} \cong \frac{(E_1 - E_2)^2}{2Mc^2}$$

For the $n = 2$ to $n = 1$ transition in hydrogen, we have

$$K_R = \frac{[-3.4\ \text{eV} - (-13.6\ \text{eV})]^2}{2(1.007825\ \text{u})(931.5 \times 10^6\ \text{eV/u})} = 5.5 \times 10^{-8}\ \text{eV}$$

This is indeed a very small recoil energy, which can be neglected when we consider the energies of the emitted photons.

41.  (a) Equation 6.30 shows that the energy levels are directly proportional to the mass of the electron.  If the electron is replaced by a particle with 207 times its mass, then the energies of the levels are increased by a factor of 207.  The ground-state energy is thus $E_1 = (-13.6\ \text{eV})(207) = -2.82\ \text{keV}$.  The shortest wavelength (largest energy) of the Lyman series represents a jump from $n = \infty$ ($E = 0$) to $n = 1$, so the energy of the photon is 2.82 keV and its wavelength is

$$\lambda = \frac{hc}{\Delta E} = \frac{1280\ \text{eV} \cdot \text{nm}}{2820\ \text{eV}} = 0.440\ \text{nm}$$

This is in the X-ray region of the electromagnetic spectrum.
(b) The correction for the finite nuclear mass is given by

$$R = R_\infty \left(1 + \frac{m}{M}\right)^{-1} = R_\infty \left(1 + \frac{207(5.49 \times 10^{-4}\ \text{u})}{1.007825\ \text{u}}\right)^{-1} = R_\infty / 1.113 = 0.899 R_\infty$$

The correction for muonic hydrogen is 11.3%, in contrast with the 0.055% correction for electronic hydrogen.  Because the wavelengths are inversely proportional to the Rydberg constant (Equation 6.33), the shortest wavelength in the Lyman series would be increased from 0.440 nm to (0.440 nm)/0.899 = 0.490 nm.

42.  We can use Equation 6.38, with the electron mass $m$ replaced by the muon mass which is 207 times the electron mass.  For $n = 1$, we obtain

$$r_1 = \frac{a_0}{(207)(82)} = 3.11 \times 10^{-15}\ \text{m} = 3.11\ \text{fm}$$

This is less than the nuclear radius of 7 fm, which suggests that the muon is inside the nucleus!  (However, we originally set up the calculation of the hydrogen-like atoms by assuming that the electron experienced the full nuclear charge of $Ze$.  If the muon spends at least part of its time inside the nucleus, it will not experience the full Coulomb force, so the calculation is no longer exact.)

# Chapter 7

This chapter presents the solutions to the Schrödinger equation for the Coulomb potential energy and thus obtains the wave functions for the hydrogen atom. A new feature of this edition is the opening section on the solutions for the one-dimensional Coulomb problem. This problem has no physical application, but it serves as a chance to review some aspects of solutions to the Schrödinger equation and their interpretation. It also addresses some of the deficiencies of the Bohr model discussed at the end of the previous chapter. Finally, it eases the transition into the full-blown formalism in spherical polar coordinates. References to papers on the one-dimensional hydrogen atom are given below.

## Supplemental Materials

Simulations of the wave functions for the one-dimensional and three-dimensional Coulomb potential energies are available in the University of Colorado PhET simulation of various bound state problems: http://phet.colorado.edu/en/simulation/bound-states. These simulations permit display of the real and imaginary parts of the wave function, which can be seen to oscillate even though the probability distribution remains constant in time. Chapter 14 of *Physlet Quantum Physics* gives plots of the radial wave functions and probability densities, and a Physlet for producing density plots of the angular wave functions is at http://webphysics.davidson.edu/physletprob/ch10_modern/angular.html. Another applet for displaying the complete hydrogenic wave functions can be found at http://www.falstad.com/qmatom/. A Maple worksheet that produces 3-dimensional displays of hydrogenic wave functions is at:
http://www.physics.oregonstate.edu/portfolioswiki/doku.php?id=activities:main&file=cfhydrogenvis.

## Suggestions for Additional Reading

For a more detailed treatment of the hydrogen atom, especially of the fine structure, see:
J. Norwood, *Twentieth Century Physics* (Prentice-Hall, 1976).

An excellent and detailed full-scale treatment of the solutions of the Schrödinger equation for the hydrogen atom is Chapter V of:
L. Pauling and E. B. Wilson, *Introduction to Quantum Mechanics* (McGraw-Hill, 1935).

Representing the three-dimensional probability distributions on a two-dimensional paper is a great challenge for illustrators, and the interpretation of such illustrations is often a similar challenge for students. It might be helpful to look at some other representations:
N. Ashby and S. C. Miller, *Principles of Modern Physics* (Holden-Day, 1970).
S. Brandt and H. D. Dahmen, *The Picture Book of Quantum Mechanics*, 3rd ed., (Springer, 2000).
R. Eisberg and R. Resnick, *Quantum Physics of Atoms, Molecules, Solids, Nuclei,*

*and Particles,* 2nd ed. (Wiley, 1985).

D. Kleppner, M. G. Littman, and M. L. Zimmerman, "Highly Excited Atoms," *Scientific American* **244,** 130 (May 1981).

R. B. Leighton, *Principles of Modern Physics* (McGraw-Hill, 1959).

References on the one-dimensional hydrogen atom are:

R. Loudon, "One-Dimensional Hydrogen Atom," *American Journal of Physics* **27**, 649 (1959).

L. K. Haines and D. H. Roberts, "One-Dimensional Hydrogen Atom," *American Journal of Physics* **37**, 1145 (1969).

## Materials for Active Engagement in the Classroom

### A. Reading Quizzes

1. How do the energy levels in a hydrogen atom depend on the orbital angular momentum quantum number?
    (1) The energy increases as the orbital angular momentum increases.
    (2) The energy decreases as the orbital angular momentum increases.
    (3) The energy does not depend on the orbital angular momentum.

2. In which of the following does the quantum mechanical picture of the hydrogen atom give the same result as the Bohr model?
    (1) The energy levels of the electron.
    (2) The orbital radius of the electron.
    (3) The angular momentum of the electron.
    (4) The probability distribution of the electron.

3. Which statement about electron spin is <u>not</u> true?
    (1) The spin can be measured only when the atom is in a magnetic field.
    (2) The $z$ component of the spin of the electron can never be zero.
    (3) The degeneracy of the hydrogen levels doubles when the electron spin is included.
    (4) The spin quantum number of the electron is always 1/2.

4. In which states of hydrogen is the probability to locate the electron independent of the direction in space?
    (1) None    (2) Only the $n = 1$ state    (3) Only $m_l$ states    (4) All $l = 0$ states

**Answers**    1. 3    2. 1    3. 1    4. 4

### B. Conceptual or Discussion Questions

1. How many of the following sets of quantum numbers $n,l,m_l$ are allowed for the hydrogen atom?   (i) 1,0,0    (ii) 1,0,1    (iii) 1,1,0    (iv) 1,1,1
    (1) One    (2) Two    (3) Three    (4) Four    (5) None

2. In the set of quantum numbers $n,l,m_l = 2,x,1$ what are the possible values of $x$?
   - (1) $x = 0$ only
   - (2) $x = 0$ or 1
   - (3) $x = 1$ or 2
   - (4) $x = 1$ only
   - (5) $x = \pm 1$

3. (a) How many sets of quantum numbers $n,l,m_l$ are possible for $n = 4$?
   (b) How many of the possible sets of quantum numbers $n,l,m_l$ for $n = 4$ have $l = 2$?
   (c) How many of the possible sets of quantum numbers $n,l,m_l$ for $n = 4$ have $m_l = 0$?

4. In a Stern-Gerlach type of experiment on an atom (such as boron) with a single $2p$ electron, into how many components would the beam be split?
   - (1) 2
   - (2) 3
   - (3) 5
   - (4) 6
   - (5) 8

**Answers**     1. 1     2. 4     3. (a) 16  (b) 5  (c) 4     4. 4

## Sample Exam Questions

### A. Multiple Choice

1. Which of the following can also be quantum numbers of an $l = 2$ electron in hydrogen?
   - (a) $m_l = 1/2$
   - (b) $n = 0$
   - (c) $n = 2$
   - (d) $m_l = 0$

2. Which of the following is an allowed set of quantum numbers $n,l,m_l,m_s$ for an electron in a hydrogen atom?
   - (a) 3,2,3,1/2
   - (b) 3,3,2,-1/2
   - (c) 3,1,0,-1/2
   - (d) 2,1,1,0

3. Which of the following sets of quantum numbers $n,l,m_l,m_s$ is <u>not</u> allowed for an electron in a hydrogen atom?
   - (a) 2,0,0,+1/2
   - (b) 3,2,-2,-1/2
   - (c) 3,1,1,-1/2
   - (d) 2,2,0,1/2

4. If an angular momentum vector has a maximum $z$ component of $+3\hbar$, how many different $z$ components can it have?
   - (a) 7
   - (b) 6
   - (c) 5
   - (d) 3

5. An electron is in an $n = 2$ state in a hydrogen atom. Which of the following can also be quantum numbers that describe that state of the electron?
   - (a) $l = 2$, $m_l = 0$
   - (b) $l = 1$, $m_l = +\frac{1}{2}$
   - (c) $l = -1$, $m_l = 0$
   - (d) $l = 1$, $m_l = -1$

6. The quantum number $m_l$ provides information about what property of a hydrogen atom?
   - (a) Energy
   - (b) Orbital radius
   - (c) Magnitude of the orbital angular momentum
   - (d) $z$ component of the orbital angular momentum

7. Which of the following sets of quantum numbers $n,l,m_l$ is <u>not</u> allowed for the electron in an atom of hydrogen?

(a) 3,0,0     (b) 3,2,-2     (c) 2,1,+1     (d) 2,2,-1

8.  Relative to the $z$ axis, how many possible directions are there in space for the orbital angular momentum vector that represents an electron in a 4$f$ state ($n = 4$, $l = 3$)?
    (a) 2          (b) 3          (c) 5          (d) 7          (e) 9

**Answers**     1. d     2. c     3. d     4. a     5. d     6. d     7. d     8. d


## B. Conceptual

1.  Excluding cases in which the angular momentum is zero, is the length of the angular momentum vector that describes an electron in an atom *always equal to, either greater than or equal to, or always greater than* the maximum possible $z$ component of the angular momentum?  EXPLAIN YOUR ANSWER.

2.  For a certain electronic state in hydrogen, the angular part of the wave function is $\Theta(\theta) = C \sin \theta \cos \theta$.  Is the electron described by this wave function most likely to be found *close to the z axis, close to the xy plane, or somewhere in between*?  EXPLAIN YOUR ANSWER.  ($\theta$ is the polar angle between $r$ and the $z$ axis.)

3.  The angular part of one of the 2p wave functions in atomic hydrogen is $A \sin \theta$, where $A$ is a constant.  ($\theta$ is the polar angle between the $z$ axis and the line connecting the electron to the origin.)  Does this electron have a greater probability to be found near the $z$ axis or near the $xy$ plane?  EXPLAIN YOUR ANSWER.

**Answers**     1. always greater than     2. somewhere in between     3. near the $xy$ plane


## C. Problems

1.  The radial part of the 2p wave function of atomic hydrogen is $Cre^{-r/2a_0}$ where $C = (24a_0^5)^{-1/2}$.  Consider two very thin spherical shells, each of thickness $dr$.  One shell has radius $2a_0$ and the other has radius $4a_0$.  Find the ratio of the probability to find the electron in the larger shell to the probability to find it in the smaller shell. (Here "in the shell" means between $r$ and $r + dr$.)

2.  The 2p ($l = 1$) radial wave function of an electron in atomic hydrogen is
    $$R(r) = A \frac{r}{a_0} e^{-r/2a_0}$$
    where $A$ is a constant.
    (a) Find the most probable value of $r$, that is, the most probable distance between the electron and the nucleus.
    (b) List all possible sets of quantum numbers that can describe an electron in this state

3. (a) The radial part of the wave function of the $n = 2$, $l = 1$ electron in a hydrogen atom is
$$R(r) = \frac{1}{\sqrt{3}(2a_0)^{3/2}} \frac{r}{a_0} e^{-r/2a_0}$$
Find all maxima and minima of the <u>radial probability density</u> and sketch the radial probability density as a function of $r$.
(b) List all possible $l$ values for an electron in hydrogen with $n = 2$. For each $l$ value, list all possible values of $m_l$.

4. (a) The radial part of the 2p hydrogen wave function is
$$R(r) = Are^{-r/2a_0}$$
where $A$ is a constant. Find the most probable distance of the electron from the nucleus.
(b) The complete 2p wave function for a particular value of $m_l$ is
$$\psi(r,\theta,\phi) = \frac{1}{(4\pi)^{1/2}(2a_0)^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \cos\theta$$
Describe the angular part of the electron probability density. Consider both the $\theta$ and $\phi$ dependences. Include in your discussion a sketch of the angular part of the probability density.

5. The radial part of the 3d wave function of atomic hydrogen is
$$R(r) = Ca_0^{-7/2} r^2 e^{-r/3a_0}$$
where $C = (2/81)\sqrt{2/15} = 9.016 \times 10^{-3}$.
(a) Find the locations of the maxima and minima of the radial probability density. Sketch the radial probability density as a function of $r$.
(b) Independent of the angular coordinates, what is the probability to find the electron in the region between $r = a_0$ and $r = 1.01a_0$?

6. (a) The radial part of the 2p ($l = 1$) wave function of hydrogen is
$$R(r) = Aa_0^{-5/2} re^{-r/2a_0}$$
where $A = 1/\sqrt{24} = 0.2041$. What is the probability to find the electron in the entire region of the thin spherical shell between $0.99a_0$ and $1.01a_0$?
(b) The angular part of this wave function is $\Theta(\theta) = B\sin\theta$, where $B = \frac{1}{2}\sqrt{3}$.
Where would you expect the probability to find the electron to be larger: closer to the z axis or closer to the xy plane? Explain your answer. ($\theta$ is the polar angle between the z axis and a line connecting the electron's volume element to the origin.)

**Answers**    1. 2.16        2. (a) $4a_0$  (b) $(2,1,\pm1,\pm1/2)$, $(2,1,0,\pm1/2)$
3. (a) min: $0,\infty$; max: $4a_0$  (b) $l = 0$ ($m_l = 0$), $l = 1$ ($m_l = 0,\pm1$)
4. (a) $4a_0$   (b) Independent of $\phi$; max for $\theta = 0$ and $\pi$, zero for $\theta = \pi/2$
5. (a) minima at $0,\infty$; maximum at $9a_0$  (b) $4.17 \times 10^{-7}$
6. (a) $3.07 \times 10^{-4}$  (b) near xy plane

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**Problem Solutions**

1. $\psi(x) = Axe^{-bx}$ gives $d\psi/dx = Ae^{-bx} - bAxe^{-bx}$ and $d^2\psi/dx^2 = -2bAe^{-bx} + b^2Axe^{-bx}$.
   Then substituting into Equation 7.2 we have

$$-\frac{\hbar^2}{2m}(-2Abe^{-bx} + b^2Axe^{-bx}) - \frac{e^2}{4\pi\varepsilon_0 x}Axe^{-bx} = EAxe^{-bx}$$

   Canceling common factors gives

$$\frac{\hbar^2 b}{m} - \frac{\hbar^2 b^2}{2m}x - \frac{e^2}{4\pi\varepsilon_0} = Ex \quad \text{or} \quad \left(\frac{\hbar^2 b}{m} - \frac{e^2}{4\pi\varepsilon_0}\right) + x\left(-\frac{\hbar^2 b^2}{2m} - E\right) = 0$$

   For this expression to equal zero for all $x$, both terms in parentheses must be zero. Thus

$$\frac{\hbar^2 b}{m} = \frac{e^2}{4\pi\varepsilon_0} \quad \text{or} \quad b = \frac{me^2}{4\pi\varepsilon_0\hbar^2} = \frac{1}{a_0} \qquad \text{and} \qquad E = -\frac{\hbar^2 b^2}{2m} = -\frac{me^4}{32\pi^2\varepsilon_0^2\hbar^2}$$

2. The probability density is $P(x) = |\psi(x)|^2 = A^2 x^2 e^{-2bx}$. To find the maximum, we set the first derivative equal to zero:

$$\frac{dP}{dx} = 2A^2 xe^{-2bx} - 2bA^2 x^2 e^{-2bx} = 0$$

   This has solutions at $x = 0$, $x = \infty$, and $x = 1/b = a_0$. The first two give minima and the third gives the maximum.

3. The probability to find the electron in a small interval is $P(x)dx = A^2 x^2 e^{-2bx}dx$.
   Substituting the values of $A$ and $b$, and evaluating the resulting expression for $x = a_0$ and $dx = 0.02a_0$ (appropriate to the interval from $x = 0.99a_0$ to $x = 1.01a_0$), we obtain

$$P(x)dx = \frac{4}{a_0^3}x^2 e^{-2x/a_0}dx = \frac{4}{a_0^3}a_0^2 e^{-2}(0.02a_0) = 0.0108$$

4. (a) $|\mathbf{L}| = \sqrt{l(l+1)}\hbar = \sqrt{(3)(4)}\hbar = \sqrt{12}\hbar$

   (b) There are $2l + 1 = 7$ possible $z$ components: $L_z = m_l\hbar = +3\hbar, +2\hbar, +\hbar, 0, -\hbar, -2\hbar, -3\hbar$.

   (c) $\cos\theta = m_l/\sqrt{l(l+1)} = m_l/\sqrt{12}$

$$m_l = +3 \qquad \theta = \cos^{-1} 3/\sqrt{12} = 30°$$

$$m_l = +2 \qquad \theta = \cos^{-1} 2/\sqrt{12} = 55°$$

$$m_l = +1 \qquad \theta = \cos^{-1} 1/\sqrt{12} = 73°$$

$$m_l = 0 \qquad \theta = \cos^{-1} 0 = 90°$$

$$m_l = -1 \qquad \theta = \cos^{-1}(-1/\sqrt{12}) = 107°$$

$$m_l = -2 \qquad \theta = \cos^{-1}(-2/\sqrt{12}) = 125°$$

$$m_l = -3 \qquad \theta = \cos^{-1}(-3/\sqrt{12}) = 150°$$

5.  For $l = 2$, $m_l = +2, +1, 0, -1, -2$. With $\cos\theta = m_l/\sqrt{l(l+1)} = m_l/\sqrt{6}$, we have

$$m_l = +2 \qquad \theta = \cos^{-1} 2/\sqrt{6} = 35°$$

$$m_l = +1 \qquad \theta = \cos^{-1} 1/\sqrt{6} = 66°$$

$$m_l = 0 \qquad \theta = \cos^{-1} 0 = 90°$$

$$m_l = -1 \qquad \theta = \cos^{-1}(-1/\sqrt{6}) = 114°$$

$$m_l = -2 \qquad \theta = \cos^{-1}(-2/\sqrt{6}) = 145°$$

6.  $l = 0$: (4, 0, 0)

$l = 1$: (4, 1, +1), (4, 1, 0), (4, 1, −1)

$l = 2$: (4, 2, +2), (4, 2, +1), (4, 2, 0), (4, 2, −1), (4, 2, −2)

$l = 3$: (4, 3, +3), (4, 3, +2), (4, 3, +1), (4, 3, 0), (4, 3, −1), (4, 3, −2), (4, 3, −3)

7.  (*a*) $l_{\max} = n - 1 = 5$ so $l = 0, 1, 2, 3, 4, 5$ for $n = 6$.

(*b*) $m_l = +6, +5, +4, +3, +2, +1, 0, -1, -2, -3, -4, -5, -6$

(*c*) $n \geq l + 1 = 5$ for $l = 4$, so the smallest possible $n$ is 5.

(*d*) For $m_l = 4$, $l \geq 4$ so the smallest possible $l$ is 4.

8.  The normalization integral for the (1, 0, 0) wave function is

$$\int_0^\infty r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \left| \psi_{1,0,0}(r,\theta,\phi) \right|^2 = \int_0^\infty 4a_0^{-3} e^{-2r/a_0} r^2 dr \int_0^\pi \tfrac{1}{2}\sin\theta \, d\theta \int_0^{2\pi} \tfrac{1}{2\pi} d\phi$$

$$= 4a_0^{-3} \int_0^\infty e^{-2r/a_0} r^2 dr = \frac{4}{a_0^3} \frac{2!}{(2/a_0)^3} = 1$$

The last integral is evaluated using the standard form $\int_0^\infty x^n e^{-ax} dx = n!/a^{n+1}$.

The normalization integral for the (2, 0, 0) wave function is

$$\int_0^\infty r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \left| \psi_{2,0,0}(r,\theta,\phi) \right|^2 = \int_0^\infty \tfrac{1}{8} a_0^{-3} (2 - r/a_0)^2 e^{-r/a_0} r^2 dr \int_0^\pi \tfrac{1}{2} \sin\theta \, d\theta \int_0^{2\pi} \tfrac{1}{2\pi} d\phi$$

$$= \tfrac{1}{8} a_0^{-3} \int_0^\infty e^{-r/a_0} \left( 4r^2 - 4\frac{r^3}{a_0} + \frac{r^4}{a_0^2} \right) dr = \tfrac{1}{8} a_0^{-3} \left[ 4\frac{2!}{(1/a_0)^3} - \frac{4}{a_0} \frac{3!}{(1/a_0)^4} + \frac{1}{a_0^2} \frac{4!}{(1/a_0)^5} \right] = \tfrac{1}{8}(8 - 24 + 24) = 1$$

9.  With $\psi_{2,0,0}(r,\theta,\phi) = \dfrac{1}{\sqrt{4\pi}} \dfrac{1}{\sqrt{8a_0^3}} \left( 2 - \dfrac{r}{a_0} \right) e^{-r/2a_0}$, we then have $\dfrac{\partial\psi}{\partial\theta} = 0$ and $\dfrac{\partial\psi}{\partial\phi} = 0$.

$$\frac{\partial\psi}{\partial r} = \frac{1}{\sqrt{32\pi a_0^3}} \left[ -\frac{1}{a_0} e^{-r/2a_0} - \frac{1}{2a_0} \left( 2 - \frac{r}{a_0} \right) e^{-r/2a_0} \right] = \frac{1}{\sqrt{32\pi a_0^3}} \left( -\frac{2}{a_0} + \frac{r}{2a_0^2} \right) e^{-r/2a_0}$$

$$\frac{\partial^2\psi}{\partial r^2} = \frac{1}{\sqrt{32\pi a_0^3}} \left[ \frac{1}{2a_0^2} e^{-r/2a_0} + \frac{1}{2a_0^2} e^{-r/2a_0} + \frac{1}{4a_0^2} \left( 2 - \frac{r}{a_0} \right) e^{-r/2a_0} \right] = \frac{1}{\sqrt{32\pi a_0^3}} \left( \frac{3}{2a_0^2} - \frac{r}{4a_0^3} \right) e^{-r/2a_0}$$

Substituting into the left side of Equation 7.10, we have

$$-\frac{\hbar^2}{2m} \left[ \frac{1}{\sqrt{32\pi a_0^3}} \left( \frac{3}{2a_0^2} - \frac{r}{4a_0^3} \right) e^{-r/2a_0} + \frac{2}{r} \frac{1}{\sqrt{32\pi a_0^3}} \left( -\frac{2}{a_0} + \frac{r}{2a_0^2} \right) e^{-r/2a_0} \right] - \frac{e^2}{4\pi\varepsilon_0 r} \frac{1}{\sqrt{32\pi a_0^3}} \left( 2 - \frac{r}{a_0} \right) e^{-r/2a_0}$$

$$= \frac{1}{\sqrt{32\pi a_0^3}} e^{-r/2a_0} \left[ -\frac{\hbar^2}{2m} \left( \frac{3}{2a_0^2} - \frac{r}{4a_0^3} - \frac{4}{ra_0} + \frac{1}{a_0^2} \right) - \frac{e^2}{2\pi\varepsilon_0 r} + \frac{e^2}{4\pi\varepsilon_0 a_0} \right]$$

$$= \frac{1}{\sqrt{32\pi a_0^3}} e^{-r/2a_0} \frac{e^2}{4\pi\varepsilon_0} \left( -\frac{5}{4a_0} + \frac{r}{8a_0^2} + \frac{2}{r} - \frac{2}{r} + \frac{1}{a_0} \right)$$

$$= \frac{1}{\sqrt{32\pi a_0^3}} e^{-r/2a_0} \frac{e^2}{4\pi\varepsilon_0} \left( -\frac{1}{4a_0} + \frac{r}{8a_0^2} \right) = \frac{e^2}{4\pi\varepsilon_0} \left( -\frac{1}{8a_0} \right) \psi_{2,0,0}(r,\theta,\phi) = E\psi_{2,0,0}(r,\theta,\phi)$$

with $E = \dfrac{e^2}{4\pi\varepsilon_0} \left( -\dfrac{1}{8a_0} \right) = \dfrac{e^2}{4\pi\varepsilon_0} \left( -\dfrac{me^2}{32\pi\varepsilon_0\hbar^2} \right) = \dfrac{1}{4} \left( -\dfrac{me^4}{32\pi^2\varepsilon_0^2\hbar^2} \right)$, which is the energy $E_2$ as defined in Equation 7.13.

Starting with $\psi_{2,1,0}(r,\theta,\phi) = \dfrac{1}{\sqrt{32\pi a_0^5}} r e^{-r/2a_0} \cos\theta$,

$$\frac{\partial \psi}{\partial r} = \frac{1}{\sqrt{32\pi a_0^5}}\left(e^{-r/2a_0} - \frac{r}{2a_0}e^{-r/2a_0}\right)\cos\theta$$

$$\frac{\partial^2 \psi}{\partial r^2} = \frac{1}{\sqrt{32\pi a_0^5}}\left(-\frac{1}{2a_0}e^{-r/2a_0} - \frac{1}{2a_0}e^{-r/2a_0} + \frac{r}{4a_0^2}e^{-r/2a_0}\right)\cos\theta$$

$$\frac{\partial \psi}{\partial \theta} = \frac{1}{\sqrt{32\pi a_0^5}}re^{-r/2a_0}(-\sin\theta)$$

$$\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \psi}{\partial \theta}\right) = -\frac{1}{\sqrt{32\pi a_0^5}}re^{-r/2a_0}(2\sin\theta\cos\theta)$$

Equation 7.10 then gives

$$\frac{\cos\theta}{\sqrt{32\pi a_0^5}}e^{-r/2a_0}\left(-\frac{\hbar^2}{2m}\left[-\frac{1}{a_0} + \frac{r}{4a_0^2} + \frac{2}{r}\left(1 - \frac{r}{2a_0}\right) - \frac{2}{r}\right] - \frac{e^2}{4\pi\varepsilon_0}\right)$$

$$= \frac{e^2}{4\pi\varepsilon_0}\psi_{2,1,0}(r,\theta,\phi)\left(\frac{1}{2r} - \frac{1}{8a_0} + \frac{1}{2r} - \frac{1}{r}\right) = \frac{e^2}{4\pi\varepsilon_0}\left(-\frac{1}{8a_0}\right)\psi_{2,1,0}(r,\theta,\phi) = E_2\psi_{2,1,0}(r,\theta,\phi)$$

10. With $\psi_{1,0,0}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}$, $\frac{\partial \psi}{\partial r} = \frac{1}{\sqrt{\pi a_0^3}}\left(-\frac{1}{a_0}\right)e^{-r/a_0}$ and $\frac{\partial^2 \psi}{\partial r^2} = \frac{1}{\sqrt{\pi a_0^3}}\left(\frac{1}{a_0^2}\right)e^{-r/a_0}$.

Substituting into Equation 7.10, we have

$$\frac{1}{\sqrt{\pi a_0^3}}\left[-\frac{\hbar^2}{2m}\left(\frac{1}{a_0^2}e^{-r/a_0} - \frac{2}{a_0 r}e^{-r/a_0}\right) - \frac{e^2}{4\pi\varepsilon_0 r}e^{-r/a_0}\right]$$

$$= \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}\frac{e^2}{4\pi\varepsilon_0}\left(-\frac{1}{2a_0} + \frac{1}{r} - \frac{1}{r}\right) = -\frac{1}{2a_0}\frac{e^2}{4\pi\varepsilon_0}\psi_{1,0,0}(r,\theta,\phi) = E\psi_{1,0,0}(r,\theta,\phi)$$

with $E = -\frac{1}{2a_0}\frac{e^2}{4\pi\varepsilon_0} = -\frac{1}{2}\frac{e^2}{4\pi\varepsilon_0}\frac{me^2}{4\pi\varepsilon_0\hbar^2} = -\frac{me^4}{32\pi^2\varepsilon_0^2\hbar^2}$ which is $E_1$ from Equation 7.13.

11. (a) For $n = 2$, $l = 1$, $m_l = 0$, the probability to find the electron in a volume element $dV$ is given by Equation 7.16:

$$\left|\psi_{2,1,0}(r,\theta,\phi)\right|^2 dV = \frac{1}{24a_0^3}\frac{r^2}{a_0^2}e^{-r/a_0}\left(\frac{3}{4\pi}\cos^2\theta\right)r^2\sin\theta\,dr\,d\theta\,d\phi$$

and for $m_l = \pm 1$,

$$\left|\psi_{2,1,\pm1}(r,\theta,\phi)\right|^2 dV = \frac{1}{24a_0^3}\frac{r^2}{a_0^2}e^{-r/a_0}\left(\frac{3}{8\pi}\sin^2\theta\right)r^2\sin\theta\,dr\,d\theta\,d\phi$$

For $\theta = 0$, both probabilities are zero due to the $\sin\theta$ terms. (b) For $\theta = 90°$, the 2,1,0 probability is zero due to the $\cos\theta$ term. With $dr = 0.02a_0$, $d\theta = 0.11° = 0.00192$ rad, and $d\phi = 0.25° = 0.00436$ rad, the 2,1,$\pm1$ probability is

$$\left|\psi_{2,1,\pm1}(r,\theta,\phi)\right|^2 dV = \frac{(0.50a_0)^2}{24a_0^5}e^{-0.5a_0/a_0}\left(\frac{3}{8\pi}\sin^2 90°\right)(0.50a_0)^2(\sin 90°)$$
$$\times(0.02a_0)(0.00192\,\text{rad})(0.00436\,\text{rad}) = 3.2\times10^{-11}$$

(c) Because the probability density associated with any particular state in hydrogen is always independent of $\phi$, the 2,1,0 probability is again zero and the 2,1,$\pm1$ probability is again $3.2 \times 10^{-11}$. (d) The only change in the 2,1,$\pm1$ probability is to replace $\sin 90°$ with $\sin 45°$ in three locations, so the new probability is $(3.2\times10^{-11})(\tfrac{1}{2}\sqrt{2})^3 = 1.1\times10^{-11}$. For the 2,1,0 probability, the angular factors are the same because $\cos 45° = \sin 45°$. The only change comes about because of the change from $3/8\pi$ to $3/4\pi$ in the $\Theta(\theta)$ term, so the 2,1,0 probability is $2.2 \times 10^{-11}$.

12. For $n = 1$, $l = 0$ we have $P(r) = r^2\left|R_{1,0}(r)\right|^2 = 4r^2e^{-2r/a_0}/a_0^3$. To find the maximum, we set $dP/dr$ to zero:

$$\frac{dP}{dr} = \frac{4}{a_0^3}\left[2re^{-2r/a_0} - r^2\left(\frac{2}{a_0}\right)e^{-2r/a_0}\right] = \frac{8r}{a_0^3}e^{-2r/a_0}\left(1-\frac{r}{a_0}\right) = 0$$

There are three solutions to this equation: $r = 0$, $r = \infty$, $r = a_0$. The first two solutions correspond to minima of $P(r)$; only the solution at $r = a_0$ gives a maximum.

13. For $n = 2$, $l = 0$, $P(r) = r^2\left|R_{2,0}(r)\right|^2 = r^2\frac{1}{8a_0^3}\left(2-\frac{r}{a_0}\right)^2 e^{-r/a_0} = \frac{1}{8a_0^3}\left(4r^2 - \frac{4r^3}{a_0} + \frac{r^4}{a_0^2}\right)e^{-r/a_0}$

Setting $dP/dr$ to zero, we have

$$\frac{1}{8a_0^3}re^{-r/a_0}\left(8 - \frac{16r}{a_0} + \frac{8r^2}{a_0^2} - \frac{r^3}{a_0^3}\right) = \frac{1}{8a_0^3}re^{-r/a_0}\left(2-\frac{r}{a_0}\right)\left(4 - \frac{6r}{a_0} + \frac{r^2}{a_0^2}\right) = 0$$

The five solutions are: $r = 0, r = \infty, r = 2a_0, r = (3\pm\sqrt{5})a_0$. The first three solutions give minima and the last two give maxima.

14. For $n = 2$, $l = 1$, we have $P(r) = r^2 |R_{2,1}(r)|^2 = r^2 \dfrac{1}{24a_0^3} \dfrac{r^2}{a_0^2} e^{-r/a_0}$. The total probability between $r = a_0$ and $r = 2a_0$ is

$$P(a_0 : 2a_0) = \int_{a_0}^{2a_0} P(r)\, dr = \frac{1}{24a_0^5} \int_{a_0}^{2a_0} r^4 e^{-r/a_0} dr$$

We can use Equation 7.4 to evaluate this integral. The result is

$$P(a_0 : 2a_0) = \frac{1}{24a_0^5} \left[ -a_0 e^{-r/a_0} (r^4 + 4a_0 r^3 + 12a_0^2 r^2 + 24a_0^3 r + 24) \right]_{a_0}^{2a_0} = 0.0490$$

15. $P(r)dr = r^2 |R_{1,0}(r)|^2 dr = (1.00a_0)^2 \dfrac{4}{a_0^3} e^{-2} (0.01a_0) = 0.0054$

16. The angular probability density is $P(\theta, \phi) = \frac{15}{8\pi} \sin^2 \theta \cos^2 \theta$. To find the locations of the maxima and minima, we set the derivative equal to zero:

$$\frac{dP}{d\theta} = \frac{15}{8\pi} (2\sin\theta\cos^3\theta - 2\sin^3\theta\cos\theta) = \frac{15}{4\pi}(\sin\theta)(\cos\theta)(\cos^2\theta - \sin^2\theta) = 0$$

The three angular terms in parentheses give three sets of solutions: $\theta = 0, \pi$; $\theta = \pi/2$; and $\theta = \pi/4, 3\pi/4$. By checking the second derivative, we find that the first two give minima (the second derivative is positive) and the third gives maxima (negative second derivative). The angular probability density thus starts at zero along the positive $z$ direction, rises to a maximum at $\theta = 45°$, falls again to zero in the $xy$ plane ($\theta = 90°$), rises again to a maximum at $\theta = 135°$, and finally falls again to zero on the negative $z$ axis.

17. The angular probability density is $P(\theta, \phi) = \frac{5}{16\pi}(3\cos^2\theta - 1)^2$. To find the locations of the maxima and minima, we set the derivative equal to zero:

$$\frac{dP}{d\theta} = \frac{5}{8\pi}(3\cos^2\theta - 1)(-6\sin\theta\cos\theta) = -\frac{15}{4\pi}(\sin\theta)(\cos\theta)(3\cos^2\theta - 1)$$

The three angular terms in parentheses give three sets of solutions: $\theta = 0, \pi$; $\theta = \pi/2$; and $\theta = \cos^{-1}(\pm 1/\sqrt{3}) = 0.955, 2.186$. The first two give maxima and the third gives minima. The angular probability density is a maximum on the positive $z$ axis, falls to zero at $\theta = 55°$, rises again to a maximum in the $xy$ plane ($\theta = 90°$), falls to zero at $\theta = 125°$, and rises to a maximum on the negative $z$ axis.

18. (a) degeneracy $= 2n^2 = 2(5)^2 = 50$

    (b) For each value of $l$, the degeneracy is $2(2l+1)$.

$$
\begin{aligned}
l = 0: &\quad 2(0+1) = \phantom{0}2 \\
l = 1: &\quad 2(2+1) = \phantom{0}6 \\
l = 2: &\quad 2(4+1) = 10 \\
l = 3: &\quad 2(6+1) = 14 \\
l = 4: &\quad \underline{2(8+1) = 18} \\
\text{total:} &\qquad\qquad\quad 50
\end{aligned}
$$

19.  $$\sum_{l=0}^{n-1} 2(2l+1) = 4\sum_{l=0}^{n-1} l + 2\sum_{l=0}^{n-1} 1 = 4\frac{n(n-1)}{2} + 2n = 2n^2$$

20. ($a$) $l$ exceeds the maximum permitted value $(n-1)$.

    ($b$) $m_l$ exceeds the maximum permitted value $(l)$

    ($c$) $m_s$ can be only +1/2 or -1/2

    ($d$) negative values of $l$ are not permitted

21.  The selection rule is $\Delta l = \pm 1$, so the $4p$ state can make transitions to any lower $s$ state ($\Delta l = -1$) or $d$ state ($\Delta l = +1$).  The possible transitions are then:

$$4p \to 3s,\ 4p \to 2s,\ 4p \to 1s,\ \text{and}\ 4p \to 3d$$

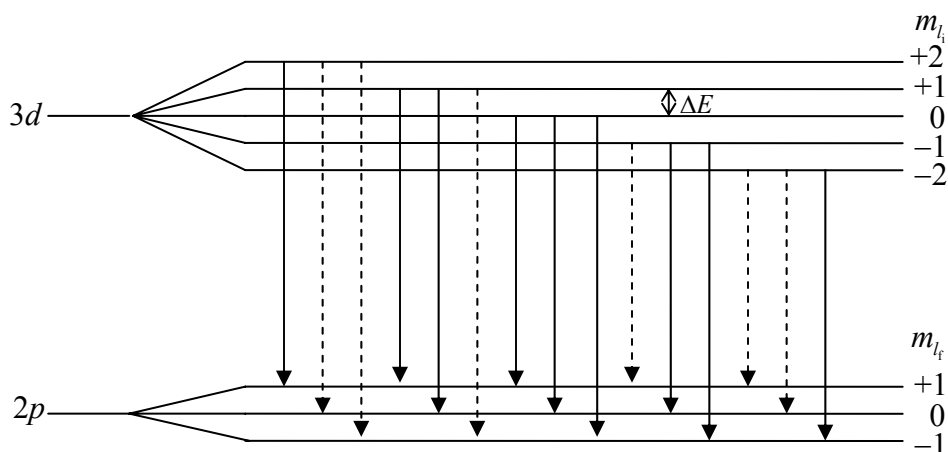22.  (a) The transitions that change $l$ by one unit are
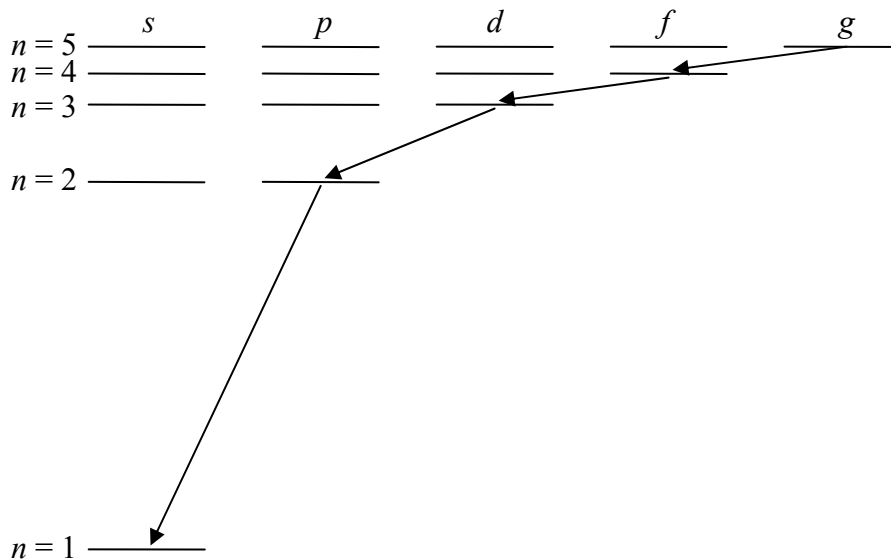


    (b) Starting instead with $5d$, the permitted transitions are

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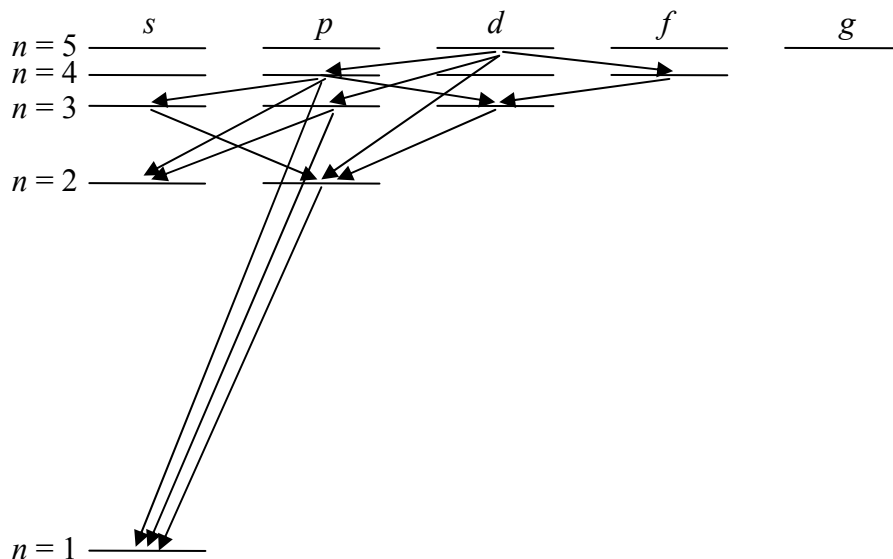
23.  (a) 7s, 7p, 7d, 7f, 7g, 7h, 7i  (b) 6p, 6f, 5p, 5f, 4p, 4f, 3p, 2p

24.  (a)



(b) Transitions shown with dashed lines violate the $\Delta m_l = \pm 1$ selection rule.

(c) The energy of the initial state is $E_i = E_{3d} + m_{l_i}\Delta E$ and the energy of the final state
is $E_f = E_{2p} + m_{l_f}\Delta E$ (where $\Delta E$ is the spacing between adjacent states). The transition
energies can be found from the energy difference:

$$E_i - E_f = (E_{3d} - E_{2p}) + (m_{l_i} - m_{l_f})\Delta E = (E_{3d} - E_{2p}) + \Delta m_l \Delta E$$

There are only three permitted values of $\Delta m_l$ $(0, \pm 1)$, so there are only three possible
values of the energy difference: $E_{3d} - E_{2p}, E_{3d} - E_{2p} + \Delta E, E_{3d} - E_{2p} + \Delta E$ .

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25. (a) In the absence of a magnetic field, the $3d$ to $2p$ energy difference is

$$E = (-13.6057 \text{ eV})\left(\frac{1}{3^2} - \frac{1}{2^2}\right) = 1.88968 \text{ eV}$$

and the wavelength is

$$\lambda = \frac{hc}{E} = \frac{1239.842 \text{ eV} \cdot \text{nm}}{1.88968 \text{ eV}} = 656.112 \text{ nm}$$

The magnetic field gives a change in wavelength of

$$\Delta\lambda = \frac{\lambda^2}{hc}\Delta E = \frac{(656.112 \text{ nm})^2}{1239.842 \text{ eV} \cdot \text{nm}}(5.79 \times 10^{-5} \text{ eV/T})(3.50 \text{ T}) = 0.0703 \text{ nm}$$

The wavelengths of the three normal Zeeman components are then 656.112 nm, 656.112 nm + 0.070 nm = 656.182 nm, and 656.112 nm − 0.070 nm = 656.042 nm.

26. The energy of the $2p$ to $1s$ Lyman transition is

$$E = (-13.60570 \text{ eV})\left(\frac{1}{2^2} - \frac{1}{1^2}\right) = 10.20428 \text{ eV}$$

and its wavelength (in the absence of fine structure) is

$$\lambda = \frac{hc}{E} = \frac{1239.842 \text{ eV} \cdot \text{nm}}{10.20428 \text{ eV}} = 121.5022 \text{ eV}$$

With the fine structure energy splitting of $4.5 \times 10^{-5}$ eV, the wavelength splitting is

$$\Delta\lambda = \frac{\lambda^2}{hc}\Delta E = \frac{(121.5 \text{ nm})^2}{1240 \text{ eV} \cdot \text{nm}}(4.5 \times 10^{-5} \text{ eV}) = 0.00054 \text{ nm}$$

The fine structure splits one level up by $0.5\Delta E$ and the other down by the same amount, so the wavelengths are

$$\lambda + \tfrac{1}{2}\Delta\lambda = 121.5024 \text{ nm} \qquad \text{and} \qquad \lambda - \tfrac{1}{2}\Delta\lambda = 121.5019 \text{ nm}$$

27. The $3d$ fine structure splitting is roughly

$$\Delta E = mc^2\alpha^4\tfrac{1}{3^5} = 6.0 \times 10^{-6} \text{ eV}$$

Assuming that the parallel and antiparallel states are affected equally by this splitting, the $3d$ fine structure levels are each shifted upward and downward by half this amount so their energies are $-1.51$ eV $\pm 3.0 \times 10^{-6}$ eV, while the $2p$ fine structure levels are each shifted upward and downward by half of the $2p$ fine structure splitting $(4.5 \times 10^{-5}$ eV$)$ so their energies are $-3.40$ eV $\pm 2.25 \times 10^{-5}$ eV. The possible energy differences are then

$$1.89 \text{ eV} \pm 2.55 \times 10^{-5} \text{ eV} \qquad \text{and} \qquad 1.89 \text{ eV} \pm 1.95 \times 10^{-5} \text{ eV}$$

The wavelength differences are

$$\Delta\lambda = \frac{\lambda^2}{hc}\Delta E = \frac{(656 \text{ nm})^2}{1240 \text{ eV} \cdot \text{nm}}(2.55 \times 10^{-5} \text{ eV}) = 0.00885 \text{ nm}$$

$$\Delta\lambda = \frac{\lambda^2}{hc}\Delta E = \frac{(656 \text{ nm})^2}{1240 \text{ eV} \cdot \text{nm}}(1.95 \times 10^{-5} \text{ eV}) = 0.00677 \text{ nm}$$

We might therefore observe as many as 4 component wavelengths: $656.1123$ nm $\pm$ $0.0089$ nm and $656.1123 \pm 0.0068$ nm.

28. With $\psi(x) = A(x + cx^2)e^{-bx}$, we find $d\psi/dx = A(1 + 2cx)e^{-bc} - bA(x + cx^2)e^{-bx}$ and $d^2\psi/dx^2 = 2cAe^{-bx} - 2bA(1 + 2cx)e^{-bx} + b^2 A(x + cx^2)e^{-bx}$. Substituting into Equation 7.2 and canceling the common factor $Ae^{-bx}$, we obtain

$$-\frac{\hbar^2}{2m}[2c - 2b(1 + 2cx) + b^2(x + cx^2)] - \frac{e^2}{4\pi\varepsilon_0}(1 + cx) = E(x + cx^2)$$

or

$$\left(-\frac{\hbar^2 c}{m} + \frac{\hbar^2 b}{m} - \frac{e^2}{4\pi\varepsilon_0}\right) + x\left(\frac{2\hbar^2 bc}{m} - \frac{\hbar^2 b^2}{2m} - \frac{e^2 c}{4\pi\varepsilon_0} - E\right) + x^2\left(-\frac{\hbar^2 b^2 c}{2m} - Ec\right) = 0$$

For this expression to be true for any $x$, all three expressions in parentheses must equal zero. Solving the third for $E$ and substituting into the second, we find $b = me^2/8\pi\varepsilon_0\hbar^2 = 1/2a_0$. Substituting this result into the first expression, we obtain $c = -me^2/8\pi\varepsilon_0 = -1/2a_0$. The third expression then gives $E = -\hbar^2 b^2/2m = -me^4/128\pi^2\varepsilon_0^2\hbar^2 = \frac{1}{4}(-me^4/32\pi^2\varepsilon_0^2\hbar^2)$, which is the energy of the first excited state in the Bohr model. To find the value of $A$, we must normalize the wave function:

$$\int_0^\infty |\psi(x)|^2 \, dx = A^2 \int_0^\infty (x - x^2/2a_0)^2 e^{-x/a_0} dx = A^2 \int_0^\infty (x^2 - x^3/a_0 + x^4/4a_0^2)e^{-x/a_0} dx = 1$$

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Evaluating the integrals with Equation 7.3, we find

$$A^2\left(\frac{2}{(1/a_0)^3}-\frac{1}{a_0}\frac{3!}{(1/a_0)^4}+\frac{1}{4a_0^2}\frac{4!}{(1/a_0)^5}\right)=1 \quad \text{or} \quad A=\sqrt{\frac{1}{2a_0^3}}$$

29.  For $n=2$, $l=0$, we have $P(r)=r^2\left|R_{2,0}(r)\right|^2=r^2\frac{1}{8a_0^3}\left(2-\frac{r}{a_0}\right)^2 e^{-r/a_0}$. The probability to find the electron beyond $r=5a_0$ is

$$P(5a_0:\infty)=\int_{5a_0}^\infty P(r)\,dr=\frac{1}{8a_0^3}\int_{5a_0}^\infty\left(4r^2-\frac{4r^3}{a_0}+\frac{r^4}{a_0^2}\right)e^{-r/a_0}dr=\frac{1}{8}\int_5^\infty(4x^2-4x^3+x^4)e^{-x}dx$$

with $x=r/a_0$. The integrals can be evaluated using Equation 7.4. The result of the integration is

$$P(5a_0:\infty)=\frac{1}{8}\left(4\times0.2493-4\times1.5902+10.5718\right)=0.651$$

For $n=2$, $l=1$, we have $P(r)=r^2\left|R_{2,1}(r)\right|^2=r^2\frac{1}{24a_0^3}\frac{r^2}{a_0^2}e^{-r/a_0}$. The probability is

$$P(5a_0:\infty)=\frac{1}{24a_0^5}\int_{5a_0}^\infty r^4 e^{-r/a_0}dr=\frac{1}{24}\int_5^\infty x^4 e^{-x}dx=0.440$$

Thus the $n=2$, $l=0$ electron is more likely to be found beyond $r=a_0$ than the $n=2$, $l=1$ electron.

30.

$$r_{\text{av}}=\int_0^\infty rP(r)\,dr=\int_0^\infty r^3\left|R_{1,0}(r)\right|^2 dr=\frac{4}{a_0^3}\int_0^\infty r^3 e^{-2r/a_0}dr=\frac{4}{a_0^3}\frac{3!}{(2/a_0)^4}=\frac{3}{2}a_0$$

31.  2s level:

$$r_{\text{av}}=\int_0^\infty rP(r)\,dr=\int_0^\infty r^3\left|R_{2,0}(r)\right|^2 dr=\frac{1}{8a_0^3}\int_0^\infty r^3\left(2-\frac{r}{a_0}\right)^2 e^{-r/a_0}dr$$

$$=\frac{1}{8a_0^3}\int_0^\infty\left(4r^3-4\frac{r^4}{a_0}+\frac{r^5}{a_0^2}\right)e^{-r/a_0}dr=\frac{1}{8a_0^3}\left(4\frac{3!}{(1/a_0)^4}-\frac{4}{a_0}\frac{4!}{(1/a_0)^5}+\frac{1}{a_0^2}\frac{5!}{(1/a_0)^6}\right)=6a_0$$

2p level:

$$r_{\text{av}}=\int_0^\infty rP(r)\,dr=\int_0^\infty r^3\left|R_{2,1}(r)\right|^2 dr=\frac{1}{24a_0^5}\int_0^\infty r^5 e^{-r/a_0}dr=\frac{1}{24a_0^5}\frac{5!}{(1/a_0)^6}=5a_0$$

32.
$$U_{av} = \int_0^\infty U(r)P(r)\,dr = \int_0^\infty \left(-\frac{e^2}{4\pi\varepsilon_0 r}\right) r^2 \left|R_{1,0}(r)\right|^2 \, dr = -\frac{e^2}{4\pi\varepsilon_0}\frac{4}{a_0^3}\int_0^\infty r e^{-2r/a_0}\,dr$$

$$= -\frac{e^2}{4\pi\varepsilon_0}\frac{4}{a_0^3}\frac{1}{(2/a_0)^2} = -\frac{e^2}{4\pi\varepsilon_0 a_0}$$

33. Assuming a beam of silver atoms ($m = 108$ u) and estimating the magnetic moment of the atom to be about one Bohr magneton, we have

$$F_z = \mu_z \frac{dB}{dz} = (9.27\times10^{-24}\text{ J/T})(10\text{ T/m}) = 9.3\times10^{-23}\text{ N}$$

The acceleration in the region of the field is

$$a_z = \frac{F_z}{m} = \frac{9.3\times10^{-23}\text{ N}}{(108\text{ u})(1.66\times10^{-27}\text{ kg/u})} = 518\text{ m/s}^2$$

For an oven temperature of 1000 K, the kinetic energy of the atoms is

$$K = \tfrac{3}{2}kT = \tfrac{3}{2}(1.38\times10^{-23}\text{ J/K})(1000\text{ K}) = 2.1\times10^{-20}\text{ J}$$

and the speed of these atoms is

$$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2(2.1\times10^{-20}\text{ J})}{(108\text{ u})(1.66\times10^{-27}\text{ kg/u})}} = 480\text{ kg/s}$$

The time for an atom to travel 1 meter through the magnetic field region is

$$t = \frac{1\text{ m}}{480\text{ m/s}} = 2.1\times10^{-3}\text{ s}$$

Let the atoms enter the field with $z = 0$ and $v_z = 0$. Then after passing through the 1-meter field region,

$$z = \tfrac{1}{2}a_z t^2 = \tfrac{1}{2}(518\text{ m/s}^2)(2.1\times10^{-3}\text{ s})^2 = 1.1\text{ mm}$$

$$v_z = a_z t = (518\text{ m/s}^2)(2.1\times10^{-3}\text{ s}) = 1.1\text{ m/s}$$

After leaving the region of the field there is no longer an acceleration, but the $z$ component of the velocity causes an additional displacement in the $z$ direction. The horizontal velocity is unchanged, to the atom takes $2.1 \times 10^{-3}$ s to pass through the field-free region, and the additional displacement is

$$z = v_z t = (1.1 \text{ m/s})(2.1 \times 10^{-3} \text{ s}) = 2.2 \text{ mm}$$

The total displacement is then 1.1 mm + 2.2 mm = 3.3 mm. We would thus expect to see images on the screen separated by a few mm.

34. For 1s: $(r^{-1})_{\text{av}} = \int_0^\infty r^{-1} P(r)\, dr = \int_0^\infty r \left| R_{1,0}(r) \right|^2 dr = \dfrac{4}{a_0^3} \int_0^\infty r e^{-2r/a_0}\, dr = \dfrac{4}{a_0^3} \dfrac{1}{(2/a_0)^2} = \dfrac{1}{a_0}$

For 2s: $(r^{-1})_{\text{av}} = \int_0^\infty r \left| R_{2,0}(r) \right|^2 dr = \dfrac{1}{8a_0^3} \int_0^\infty r \left( 2 - \dfrac{r}{a_0} \right)^2 e^{-r/a_0}\, dr$

$$= \dfrac{1}{8a_0^3} \int_0^\infty \left( 4r - 4\dfrac{r^2}{a_0} + \dfrac{r^3}{a_0^2} \right) e^{-r/a_0}\, dr = \dfrac{1}{8a_0^3} \left( 4\dfrac{1}{(1/a_0)^2} - \dfrac{4}{a_0}\dfrac{2}{(1/a_0)^3} + \dfrac{1}{a_0^2}\dfrac{3!}{(1/a_0)^4} \right) = \dfrac{1}{4a_0}$$

For 2p: $(r^{-1})_{\text{av}} = \int_0^\infty r \left| R_{2,1}(r) \right|^2 dr = \dfrac{1}{24a_0^3} \int_0^\infty r \dfrac{r^2}{a_0^2} e^{-r/a_0}\, dr = \dfrac{1}{24a_0^5} \dfrac{3!}{(1/a_0)^4} = \dfrac{1}{4a_0}$

# Chapter 8

The theme of this chapter is to extend our simple model for one-electron atoms to help us understand the properties of atoms with more than one electron. This attempt can meet with only limited success, but it is necessary to avoid delving into many-body theory or formal perturbation methods. As a result, this chapter is built mostly on conceptual rather than mathematical arguments.

## Supplemental Materials

Tabulations of atomic properties can be found in:
C. Kittel, *Introduction to Solid State Physics* 8$^{th}$ ed. (Wiley, 2004).
*Handbook of Chemistry and Physics* (CRC Press, published annually).
National Institute of Standards and Technology (http://www.nist.gov/pml/data/).
J. G. Speight, *Lange's Handbook of Chemistry,* 16$^{th}$ ed. (McGraw-Hill, 2004).

## Suggestions for Additional Reading

Some additional basic features of the vector representation of atomic states are in:
K. W. Ford, *Classical and Modern Physics* (Xerox College Publishing, 1974), Vol. 3,
   Chapt. 24.

More advanced and detailed works on atomic structure are:
C. J. Foot, *Atomic Physics* (Oxford University Press, 2005).
H. G. Kuhn, *Atomic Spectra* (Academic Press, 1969).
H. Semat and J. R. Albright, *Introduction to Atomic and Nuclear Physics* (Holt,
   Rinehart and Winston, 1972).
J. C. Willmott, *Atomic Physics* (Wiley, 1975).

Three classic works that include some introductory and advanced material covering almost all aspects of atomic structure:
A. C. Candler, *Atomic Spectra and the Vector Model* (Cambridge University Press, 1937).
G. Herzberg, *Atomic Spectra and Atomic Structure* (Prentice-Hall, 1937).
H. E. White, *Introduction to Atomic Spectra* (McGraw-Hill, 1934).

Many reference works, both popular and technical, are available about lasers. A good introductory work is:
B. A. Lengyel, *Lasers* (Wiley, 1971).

Two popular articles by the 1981 Nobel prize recipient (for his work with lasers):
A. W. Schawlow, "Optical Masers," *Scientific American* **204,** 52 (June 1961).
A. W. Schawlow, "Laser Light," *Scientific American* **219,** 120 (September 1968).

**Materials for Active Engagement in the Classroom**

## A. Reading Quizzes

1.  The Pauli principle states that
    (1) Atoms emit photons when electrons change from one quantum state to another.
    (2) No two electrons in an atom can have the same set of quantum numbers.
    (3) Electrons in an atom are forbidden from occupying certain orbits.
    (4) An atom with Z positive charges in its nucleus cannot hold more than Z electrons.

2.  Which statement best describes the emission of discrete X rays in atoms?
    (1) The energies of the X rays depend on the quantum numbers of the atomic states but not on the atomic number of the atom.
    (2) The energies of the X rays depend strongly on the orbital angular momentum quantum number.
    (3) X rays are emitted when one of the outer electrons returns to its original state after being promoted to a higher state.
    (4) X rays are emitted when one of the inner electrons jumps from one state to a state where a vacancy has occurred.

3.  X rays are emitted:
    (1) When one of the outer electrons jumps from an excited state to its original state.
    (2) Only from atoms of elements with small atomic number Z.
    (3) Only from atoms of elements with large atomic number Z.
    (4) After one of the inner electrons of an atom is removed.

4.  In an excited state, a certain atom has the configuration $3p^1 3d^1$. What range of values is possible for the total orbital angular momentum $L$ of the two electrons?
    (1) 0,1,2        (2) 1,2,3,4        (3) 1,2,3        (4) 1,2

**Answers**        1. 2        2. 4        3. 4        4. 3

## B. Conceptual or Discussion Questions

1.  (a) In what period (horizontal row) of the periodic table would we expect to find the element with $Z = 167$?
    (b) What would be the most probable electronic configuration of the element with $Z = 167$?
    (c) What would be the most likely atomic number of the next inert gas after $Z = 118$?
        (1) 136        (2) 150        (3) 168        (4) 180

2.  Comparing elements from Ne ($Z = 10$) to Ca ($Z = 20$), approximately how would you expect the energy necessary to remove a $1s$ electron to vary with $Z$?
        (1) Independent of $Z$        (2) Like $Z^{1/2}$        (3) Linearly with $Z$        (4) Like $Z^2$

**Answers**        1. (a) 8   (b) $8p^5$   (c) 3        2. 4

**Sample Exam Questions**

**A. Multiple Choice**

1. The $K_\alpha$ wavelength of an element with atomic number $Z = 17$ is 8. What is the atomic number of an element with a $K_\alpha$ wavelength of 48?
   (a) 3    (b) 7    (c) 9    (d) 33    (e) 65

2. In the ground state of a sodium atom, which has 11 electrons, the valence electron goes into the $3s$ state rather than the $3p$ state because:
   (a) the $3s$ state is lower in energy than the $3p$ state.
   (b) the $3p$ state is already filled with electrons.
   (c) the 3s orbit occupies less space than the $3p$ orbit.
   (d) the 3s orbit is always closer to the nucleus than the $3p$ orbit.

3. If electrons did not have spin (intrinsic angular momentum), how many elements would occupy the second row in the periodic table?
   (a) 3                                                  (b) 4                        (c) (

4. If the $K_\alpha$ ($2 \rightarrow 1$) X-ray energy in a certain element is $E$, what is the $K_\beta$ ($3 \rightarrow 1$) energy in that element?
   (a) $1.2E$        (b) $2.25E$        (c) $9E$        (d) $1.5E$

5. Put the energies of the $K_\alpha$, $K_\beta$, $L_\alpha$ X-rays from an element in order of increasing energy (from smallest to largest).
   (a) $L_\alpha$, $K_\alpha$, $K_\beta$        (b) $K_\alpha$, $K_\beta$, $L_\alpha$        (c) $K_\alpha$, $L_\alpha$, $K_\beta$        (d) $L_\alpha$, $K_\beta$, $K_\alpha$

6. Consider two elements with atomic numbers Z and 2Z. How do the wavelengths of the $K_\alpha$ X-rays compare in the two elements?
   (a) $\lambda(Z)/\lambda(2Z) = 4$        (b) $\lambda(Z)/\lambda(2Z) > 4$        (c) $\lambda(Z)/\lambda(2Z) < 4$

7. Oxygen has the electronic configuration $1s^2 2s^2 2p^4$. In the ground state, the total $m_s$ of all 8 of the electrons has the largest possible value consistent with the Pauli principle. What is this value?
   (a) 1            (b) 2            (c) 3            (d) 4

8. The sodium atom ($Z = 11$) has a single valence electron in the $3s$ subshell. The properties of the electron can be analyzed as if the electric force acting on that electron were due to an effective charge of about
   (a) $+e$          (b) $-e$          (c) $+5e$          (d) $+10e$

9. Consider an atom of oxygen, which has 4 electrons in the $2p$ ($l = 1$) subshell. If we add the $z$ components of the intrinsic spins of these 4 electrons, what is the maximum total $z$ component of the intrinsic spin that is consistent with the Pauli principle?
   (a) $+2\hbar$        (b) $+\frac{3}{2}\hbar$        (c) $+\hbar$        (d) $+\frac{1}{2}\hbar$        (e) 0

**Answers**    1. c    2. a    3. b    4. a    5. a    6. b    7. a    8. a    9. c


### B. Conceptual

1.  The lanthanide elements ($Z = 57$ to $Z = 70$) have the outer electron configuration
    $6s^2 4f^k$ where $k = 1$ to 14. Based on this configuration, explain why:
    (a) the atomic radius is very nearly constant over all 14 elements.
    (b) these elements have very similar chemical properties.

2.  Consider the ionization energy (the least energy needed to remove an electron) from
    the ground state of a neutral atom of Li ($Z = 3$). Compared with the ionization energy
    of the ground state of H, is the ionization energy of Li *much larger, much smaller, or
    about the same*? EXPLAIN YOUR ANSWER.

3.  The $L_\alpha$ x ray occurs when an electron in the $n = 3$ atomic level jumps to a vacancy in
    the $n = 2$ atomic level. Suppose you are doing an experiment to measure the $L_\alpha$ x ray
    frequencies in a variety of different elements with atomic numbers $Z$ greater than 20.
    You graph your data in the form of $(\text{frequency})^{1/2}$ on the $y$ axis against the atomic
    number $Z$ on the $x$ axis. You find that the data fall very nicely along a straight line.
    Would your straight line cross the $x$ axis at a value whose magnitude is *nearly equal
    to one, much greater than one, or much less than one*? EXPLAIN YOUR ANSWER.

4.  The Moseley formula for the frequency of the K x ray in an element of atomic
    number $Z$ depends not on $Z$ but on $Z - 1$. Explain in detail the reason for the 1. Base
    your discussion on the quantum mechanical wave functions of the atom, and specify
    whether the 1 is exact or an approximation.

5.  The electronic configuration of nitrogen ($Z = 7$) is $1s^2 2s^2 2p^3$. There is an empirical
    rule that requires the electrons in unfilled shells to have quantum numbers so that the
    sum of the $z$ components of their spins has the maximum possible value. Using this
    rule, give the complete set of quantum numbers $n, l, m_l, m_s$ for the three 2p electrons.
    EXPLAIN YOUR ANSWER.

**Answers**    1. all have "outer" $6s$ electrons that determine chemical properties
        2. much smaller        3. much greater then one
        4. approximate factor based on screening by remaining 1$s$ electron
        5. (2,1,+1,+1/2), (2,1,0,+1/2), (2,1,-1,+1/2)


### C. Problems

1.  (a) The $K_\alpha$ X ray in a certain element has an energy of 8585 eV. Identify the element
        by its atomic number.
    (b) The $K_\alpha$ X ray is emitted when an electron in the $n = 2$ shell makes a transition to
        fill a vacancy that has been created in the $n = 1$ shell. The $K_\beta$ X ray is emitted

when an electron in the $n = 3$ shell makes a transition to fill a vacancy that has been created in the $n = 1$ shell. Estimate the energy of the $K_\beta$ X ray in this element.

2. (a) In a certain one-electron atom, the longest wavelength at which the atom in its ground state can absorb a photon is 7.598 nm. What is the next longest wavelength at which a photon can be absorbed from the ground state of this atom?
   (b) A neutral atom of zinc has 30 electrons. What is the energy of the $K_\alpha$ X ray of zinc?

**Answers**     1. (a) 30  (b) 10,175 eV     2. (a) 6.411 nm  (b) 8.58 keV

**Problem Solutions**

1.  (a)  For a $2p$ electron, $n = 2$, $l = 1$, $m_l = 0, \pm 1$ and $m_s = \pm \frac{1}{2}$, so the possible sets of quantum numbers $(n, l, m_l, m_s)$ are:

$(2,1,+1,+\frac{1}{2})$, $(2,1,+1,-\frac{1}{2})$, $(2,1,0,+\frac{1}{2})$, $(2,1,0,-\frac{1}{2})$, $(2,1,-1,+\frac{1}{2})$, $(2,1,-1,-\frac{1}{2})$

(b) There are 6 possible sets of quantum numbers for each electron, so the total number of possibilities for 2 electrons is $6 \times 6 = 36$.

(c) The Pauli principle prevents the two sets from being identical. There will be 6 combinations in which the two sets are identical; eliminating these combinations leaves 30 allowed combinations.

(d) Because the $n$ values are different, the Pauli principle does not restrict the number of combinations, to there will be 36 possible combinations.

2.  (a) The two electrons in the $1s$ level have $m_s$ of $+1/2$ and $-1/2$, so they do not contribute to the total $m_s$, and the same is true for the two electrons in the $2s$ level. In the $2p$ level, there are three different possible values of $m_l$, and for each of those values we can assign a set of quantum numbers with $m_s = +1/2$, so the maximum possible value of the total $m_s$ is $+3/2$.

(b) $(n, l, m_l, m_s) = (2, 1, +1, +1/2), (2, 1, 0, +1/2), (2, 1, -1, +1/2)$

(c) There is only one possible value of the total $m_l$ in the states that maximize $m_s$, and from the states listed in (b) that value is $+1 + 0 + (-1) = 0$.

(d) We could maximize the total $m_l$ by giving the first $2p$ electron $m_l = +1$, and the second electron can also have $m_l = +1$ if we give these two electrons opposite values of $m_s$. The third electron cannot have $m_l = +1$, so we must assign it $m_l = 0$ and the maximum total $m_l$ is $+2$.

3.  (a) There are seven possible values of $m_l$ ($+3, +2, +1, 0, -1, -2, -3$). For each choice of $m_l$, the value of $m_s$ can be either $+1/2$ or $-1/2$, so there will be a total of 14 possible combinations of $m_l$ and $m_s$.

(b) By a suitable choice of $m_l$, it is possible to have as many as seven electrons in the $4f$ level with $m_s = +1/2$. So the three electrons can each be assigned $m_s = +1/2$ for a total of $+3/2$.

(c) Two electrons can be assigned $m_l = +3$ (one with $m_s = +1/2$ and one with $m_s = -1/2$). The next highest possible value of $m_l$ available for the third electron is $m_l = +2$, so the maximum value of the total is $+3 + 3 + 2 = +8$.

(d) We can assign $m_s = +1/2$ to no more than seven $4f$ electrons, and so the last three must have $m_s = -1/2$ which gives a total of $7(+1/2) + 3(-1/2) = +2$.

(e) We can maximize the total $m_l$ by assigning two electrons each to $m_l = +3, +2, +1, 0$, and $-1$, for a total of $+10$.

4.  (a) In beryllium ($1s^2 2s^2$) the smallest energy jump is from $2s$ to $2p$.

(b) In neon ($1s^2 2s^2 2p^6$) the smallest energy jump is from $2p$ to $3s$.

(c) From Figure 8.1 we see that the $2s \rightarrow 2p$ energy difference is smaller than the $2p \rightarrow 3s$ difference, so the minimum absorption energy would be smaller for beryllium.

5.  (a) From Figure 8.2 we find: N ($2p^3$), P ($3p^3$), As ($4p^3$), Sb ($5p^3$), Bi ($6p^3$).
    (b) Co ($3d^7$), Rh ($4d^7$), Ir ($5d^7$), Mt($6d^7$)

6.  (a) [Ne] $3s^2 3p^3$    (b) [Ar] $4s^2 3d^3$    (c) [Kr] $5s^2 4d^{10} 5p^3$    (d) [Xe] $6s^2 4f^{14} 5d^{10} 6p^2$

7.  (a) $[Ar]4s^2 3d^6$.
    (b) The $4s$ electrons and the Ar core have a total $m_s$ of zero. Of the six $3d$ electrons, at most five can have $m_s = +1/2$ without violating the Pauli principle (because there are only five different $m_l$ labels that can be used), and the sixth electron must then have $m_s = -1/2$. The total $m_s$ is then $5 \times (+1/2) + (-1/2) = 2$.
    (c) The five electrons with $m_s = +1/2$ use up all of the possible $m_l$ values (+2, +1, 0, −1, −2), which sum to zero. The sixth electron has a maximum $m_l$ of +2, so the maximum total $m_l$ is +2.
    (d) The next available level for one of the $3d$ electrons is $4p$. The remaining five $3d$ electrons can all have $m_s = +1/2$, and so can the single $4p$ electron. The maximum total $m_s$ is $5 \times (+1/2) + (+1/2) = 3$. The $m_l$ values of the five $3d$ electrons again sum to zero, and the $4p$ electron has a maximum $m_l$ of of +1, so the total $m_l$ is +1.

8.  Singly ionized lithium has two electrons. When one of those is excited to a higher level, it is screened by the one electron remaining in the $1s$ level so $Z_{\text{eff}} = 3 - 1 = 2$. The expected energy when the outer electron is excited to the $2p$ level is

$$E_n = (-13.6 \text{ eV})\frac{Z_{\text{eff}}^2}{n^2} = (-13.6 \text{ eV})\frac{2^2}{2^2} = -13.6 \text{ eV}$$

which agrees very well with the measured value of $-13.4$ eV. When the outer electron is in the $3d$ level, its expected energy is

$$E_n = (-13.6 \text{ eV})\frac{Z_{\text{eff}}^2}{n^2} = (-13.6 \text{ eV})\frac{2^2}{3^2} = -6.0 \text{ eV}$$

in excellent agreement with the measured value.

9.  The outer electron is screened by the three inner electrons, so $Z_{\text{eff}} = 4 - 3 = 1$. The expected energy of the $3p$ excitation is

$$E_n = (-13.6 \text{ eV})\frac{Z_{\text{eff}}^2}{n^2} = (-13.6 \text{ eV})\frac{1^2}{3^2} = -1.51 \text{ eV}$$

and for $4d$

$$E_n = (-13.6 \text{ eV})\frac{Z_{\text{eff}}^2}{n^2} = (-13.6 \text{ eV})\frac{1^2}{4^2} = -0.85 \text{ eV}$$

10. In lithium,

$$3d \to 3p \qquad \lambda = 610.4 \text{ nm} \qquad \Delta E = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{610.4 \text{ nm}} = 2.03 \text{ eV}$$

$$4d \to 3p \qquad \lambda = 460.3 \text{ nm} \qquad \Delta E = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{460.3 \text{ nm}} = 2.69 \text{ eV}$$

$$\Delta E(4d \to 3d) = \Delta E(4d \to 3p) - \Delta E(3d \to 3p) = 2.69 \text{ eV} - 2.03 \text{ eV} = 0.66 \text{ eV}$$

In sodium,

$$3d \to 3p \qquad \lambda = 819.1 \text{ nm} \qquad \Delta E = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{819.1 \text{ nm}} = 1.51 \text{ eV}$$

$$4d \to 3p \qquad \lambda = 568.6 \text{ nm} \qquad \Delta E = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{568.6 \text{ nm}} = 2.18 \text{ eV}$$

$$\Delta E(4d \to 3d) = \Delta E(4d \to 3p) - \Delta E(3d \to 3p) = 2.18 \text{ eV} - 1.51 \text{ eV} = 0.67 \text{ eV}$$

In hydrogen, $\Delta E(4 \to 3) = (-13.6 \text{ eV})\left( \frac{1}{4^2} - \frac{1}{3^2} \right) = 0.66 \text{ eV}$

The agreement is very good, because in lithium and sodium the outer electron is screened by the $Z-1$ inner electrons, and so the outer energy levels are well approximated by the hydrogenic levels with $Z = 1$.

11. (a) From Figure 8.4 we find the $3d \to 3p$ energy difference to be

$$\Delta E(3d \to 3p) = \Delta E(3d \to 2s) - \Delta E(3p \to 2s) = \Delta E(3d \to 2p) + \Delta E(2p \to 2s) - \Delta E(3d \to 2s)$$

$$= \frac{hc}{\lambda(3d \to 2p)} + \frac{hc}{\lambda(2p \to 2s)} - \frac{hc}{\lambda(3p \to 2s)}$$

$$= (1240 \text{ eV} \cdot \text{nm})\left( \frac{1}{610.4 \text{ nm}} + \frac{1}{670.8 \text{ nm}} - \frac{1}{323.3 \text{ nm}} \right) = 0.045 \text{ eV}$$

(b) $\Delta E(ns \to 2s) = \Delta E(ns \to 2p) + \Delta E(2p \to 2s) = \dfrac{hc}{\lambda(ns \to 2p)} + \dfrac{hc}{\lambda(2p \to 2s)}$

$$\Delta E(3s \to 2s) = (1240 \text{ eV} \cdot \text{nm})\left( \frac{1}{812.7 \text{ nm}} + \frac{1}{670.8 \text{ nm}} \right) = 3.374 \text{ eV}$$

$$\Delta E(4s \to 2s) = (1240 \text{ eV} \cdot \text{nm})\left( \frac{1}{491.2 \text{ nm}} + \frac{1}{670.8 \text{ nm}} \right) = 4.373 \text{ eV}$$

$$\Delta E(5s \to 2s) = (1240 \text{ eV} \cdot \text{nm})\left( \frac{1}{427.3 \text{ nm}} + \frac{1}{670.8 \text{ nm}} \right) = 4.750 \text{ eV}$$

(c)     $\Delta E(2p \rightarrow 2s) = E(2p) - E(2s)$

$$E(2p) = E(2s) + \Delta E(2p \rightarrow 2s) = E(2s) + \frac{hc}{\lambda(2p \rightarrow 2s)}$$

$$= -5.39 \text{ eV} + \frac{1240 \text{ eV} \cdot \text{nm}}{670.8 \text{ nm}} = -3.54 \text{ eV}$$

The 2p ionization energy is therefore 3.54 eV.

$$E(3s) = E(2s) + \Delta E(3s \rightarrow 2s) = -5.39 \text{ eV} + 3.37 \text{ eV} = -2.02 \text{ eV}$$

The 3s ionization energy is 2.02 eV.

12.   Solving Equation 8.4 for Z with $\Delta E = \dfrac{hc}{\lambda} = \dfrac{1240 \text{ eV} \cdot \text{nm}}{0.1940 \text{ nm}} = 6392 \text{ eV}$ , we obtain

$$Z = 1 + \sqrt{\frac{\Delta E}{10.2 \text{ eV}}} = 1 + \sqrt{\frac{6392 \text{ eV}}{10.2 \text{ eV}}} = 26$$

so the element is iron.

13.   Ca ($Z = 20$):    $\Delta E = (10.2 \text{ eV})(Z-1)^2 = (10.2 \text{ eV})(19)^2 = 3.68 \text{ keV}$
Zr ($Z = 40$):    $\Delta E = (10.2 \text{ eV})(Z-1)^2 = (10.2 \text{ eV})(39)^2 = 15.5 \text{ keV}$
Hg ($Z = 80$):   $\Delta E = (10.2 \text{ eV})(Z-1)^2 = (10.2 \text{ eV})(79)^2 = 63.7 \text{ keV}$

The values computed from Moseley's law are smaller than the measured values, and the discrepancy increases as Z increases.

14.   There is nothing that prevents all 6 electrons from having $m_s = +\frac{1}{2}$, so
$M_{S,\text{max}} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3$ and $S = 3$. According to the Pauli principle, the five
3d electrons must all have different $m_l$ values, which are +2, +1, 0, −1, −2. The 4s
electron has only $m_l = 0$. Thus $M_{L,\text{max}} = +2+1+0+(-1)+(-2)+0 = 0$ and so $L = 0$. The
configuration is $L = 0$, $S = 3$.

15.   (a) The two 6s electrons have opposite spins ($m_s = +\frac{1}{2}$ and $m_s = -\frac{1}{2}$) but each of the other
two electrons can have $m_s = +\frac{1}{2}$, so $S = M_{S,\text{max}} = +\frac{1}{2} + (-\frac{1}{2}) + \frac{1}{2} + \frac{1}{2} = 1$. The Pauli
principle does not restrict the $m_l$ values of the 4f and 5d electrons, so
$L = M_{L,\text{max}} = 0+0+3+2 = 5$. The ground state is $L = 5$, $S = 1$.
(b) Once again, the two 6s electrons have opposite $m_s$ values. The 4f shell can hold up to
14 electrons, so all 7 electrons can have $m_s = +\frac{1}{2}$, as can the single 5d electron. Thus
$M_{S,\text{max}} = 8(+\frac{1}{2}) = 4$ so $S = 4$. If all seven of the 4f electrons have the same $m_s$, they must
all have different $m_l$ values: +3, +2, +1, 0, −1, −2, −3, which sum to zero. The only
contribution to L comes from the 5d electron with $m_l = 2$. Thus $L = 2$.

(c) There is no contribution to $L$ or $S$ from the filled $4f$ subshell. The spin of the nine $5d$ electrons is maximized by giving five of them $m_s = +\frac{1}{2}$ while the other four must then have $m_s = -\frac{1}{2}$. The $6s$ electron also contributes $m_s = +\frac{1}{2}$, so $M_{S,\max} = 1$ and $S = 1$. The $6s$ electron has $m_l = 0$ and the five $5d$ electrons with $m_s = +\frac{1}{2}$ must have $m_l = +2, +1, 0, -1, -2$. To maximize $M_L$, the remaining four $5d$ electrons must have $m_l = +2, +1, 0, -1$, so $L = 2$.

16. (a) The configuration of fluorine is $[He]2s^2 2p^5$. Only the five $2p$ electrons contribute to $L$ or $S$. The total spin is maximized with three having $m_s = +\frac{1}{2}$ and two having $m_s = -\frac{1}{2}$, and thus $S = M_{S,\max} = \frac{1}{2}$. The three electrons with $m_s = +\frac{1}{2}$ have $m_l = +1, 0, -1$ for a total of zero. The two electrons with $m_s = -\frac{1}{2}$ maximize $M_L$ if they are assigned $m_l = +1$ and $0$, so $L = M_{L,\max} = 1$.

(b) The configuration of magnesium is $[Ne]2s^2$. The $2s$ electrons must have opposite spins, so $S = 0$, and they both have $l = 0$, so $L = 0$.

(c) The configuration of titanium is $[Ar]4s^2 3d^2$. The $4s$ electrons do not contribute to $L$ or $S$. The two $3d$ electrons can both have $m_s = +\frac{1}{2}$, so $S = M_{S,\max} = \frac{1}{2} + \frac{1}{2} = 1$. If they have the same $m_s$, they must have different $m_l$, and the maximum values are $+2$ and $+1$, so $L = M_{L,\max} = 2 + 1 = 3$.

(d) The configuration of iron is $[Ar]4s^2 3d^6$. The $4s$ electrons do not contribute to $L$ or $S$. Maximizing the spins of the six $3d$ electrons gives five with $m_s = +\frac{1}{2}$ and one with $m_s = -\frac{1}{2}$, so $S = M_{S,\max} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = 2$. The five $3d$ electrons with $m_s = +\frac{1}{2}$ must have $m_l = +2, +1, 0, -1, -2$, for a total of zero. The maximum $m_l$ of the remaining $3d$ electron is $+2$, so $L = M_{L,\max} = 2$.

17. Each electron has $s = \frac{1}{2}$, so the possible values of the total $S$ for the two electrons is $0$ or $1$. The two $d$ electrons, each with $l = 2$, can couple to give $L = 0, 1, 2, 3, 4$.

18. Figure 8.18 shows 6 states with different $L$ and $S$ values that make up the $2p^1 3p^1$ group. Each of these states has a degeneracy of $(2S + 1)(2L + 1)$:

|  | degeneracy |
|---|---|
| $L = 0, S = 0$ | 1 |
| $L = 1, S = 0$ | 3 |
| $L = 2, S = 0$ | 5 |
| $L = 0, S = 1$ | 3 |
| $L = 1, S = 1$ | 9 |
| $L = 2, S = 1$ | 15 |
| Total = 36 | |

The total number of individual states is 36, in agreement with the number found in Problem 1(d).

19.  (a) For $\lambda = 632.8$ nm,

$$E = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{632.8 \text{ nm}} = 1.96 \text{ eV} = 3.14 \times 10^{-19} \text{ J}$$

$$P = \frac{3.5 \times 10^{-3} \text{ J/s}}{3.14 \times 10^{-19} \text{ J/photon}} = 1.12 \times 10^{16} \text{ photons/s}$$

(b) From Equation 3.10,

$$E = \sqrt{\frac{2\mu_0 c P_{av}}{A}} = \sqrt{\frac{2(4\pi \times 10^{-7} \text{ T} \cdot \text{m/A})(3.00 \times 10^8 \text{ m/s})(3.5 \times 10^{-3} \text{ W})}{\pi (1.2 \times 10^{-3} \text{ m})^2}} = 763 \text{ V/m}$$

For the incandescent bulb,

$$E = \sqrt{\frac{2\mu_0 c P_{av}}{A}} = \sqrt{\frac{2(4\pi \times 10^{-7} \text{ T} \cdot \text{m/A})(3.00 \times 10^8 \text{ m/s})(100 \text{ W})}{\pi (1 \text{ m})^2}} = 77 \text{ V/m}$$

20.  (a) There are 6 possible sets of quantum numbers for an electron in the $2p$ level, corresponding to the three possible values of $m_l$ (+1, 0, −1) and the two possible values of $m_s$ (+1/2, −1/2).  The first electron can be assigned any one of these 6 sets, leaving 5 sets that could be assigned to the second electron, 4 for the third, and 3 for the fourth.  The total number of possibilities is then $6 \times 5 \times 4 \times 3 = 360$.  This would be the result if we could somehow tell the difference among the electrons.  Because all the electrons are identical, we must correct for the different permutations obtained by simply switching the assigned states among the four electrons; for each way of assigning the states, there are $4 \times 3 \times 2 = 24$ different permutations, so the actual number of different ways of assigning the quantum numbers is $360/24 = 15$.
(b) We cannot have all four electrons with $m_s = +1/2$ (nor all four with $m_s = -1/2$), because that would violate the Pauli principle (there are only three possible $m_l$ values that can be assigned to the electrons).  So the possible distributions of the $m_s$ values are three with +1/2 and one with −1/2, two each with +1/2 and −1/2, and one with +1/2 and three with −1/2, giving possible totals of +1, 0, and −1.
(c) We can assign at most two electrons with $m_l = +1$; if the two others both have $m_l = 0$, we get the largest possible total $m_l$ of +2.  Reversing the signs, we find that the smallest possible value is −2.  Other combinations can give all values in between, so the possible values of the total are +2, +1, 0, −1, and −2.
(d) We maximize $m_s$ by assigning three electrons with $m_s = +1/2$ and the fourth with $m_s = -1/2$.  The first three electrons must have $m_l = +1$. 0, and −1, which contributes a sum of 0 to the total $m_l$.  The fourth electron can have any allowed value of $m_l$, so the possible values of the total $m_l$ are +1, 0, or −1.
(e) The maximum total $m_l$ occurs when two electrons have $m_l = +1$ and two have $m_l = 0$.  In each case the two electrons must have $m_s = +1/2$ and −1/2, so the total $m_s$ is 0.

21. (a) For the $3s$ outer electron of sodium, inserting $E_3 = -5.14$ eV into Equation 8.1 gives

$$Z_{\text{eff}} = n\sqrt{\frac{E_n}{-13.6 \text{ eV}}} = 3\sqrt{\frac{-5.14 \text{ eV}}{-13.6 \text{ eV}}} = 1.84$$
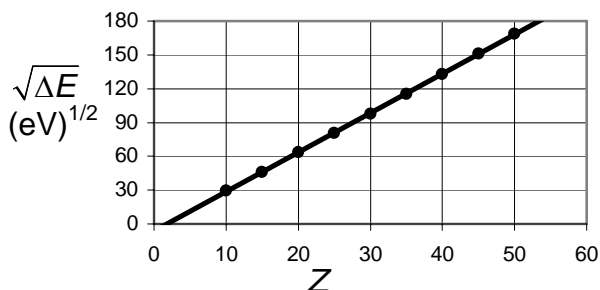
The simple screening model predicts $Z_{\text{eff}} = 1$, so clearly the $3s$ electron is slightly penetrating the inner orbits and so is less screened by the inner electrons.
(b) For the $4f$ state,

$$Z_{\text{eff}} = n\sqrt{\frac{E_n}{-13.6 \text{ eV}}} = 4\sqrt{\frac{-0.85 \text{ eV}}{-13.6 \text{ eV}}} = 1.00$$

so the screening is complete, with the 11 positive charges in the nucleus screened by the 10 electrons in the $n = 1$ and $n = 2$ shells.

22.



The slope of the line is 3.48 $\text{eV}^{1/2}$ and the intercept is 1.8.
The $K_\beta$ X rays originate from the $n = 3$ shell, so we must modify Equation 8.4 accordingly:

$$\Delta E = E_3 - E_1 = (-13.6 \text{ eV})(Z-1)^2 \left( \frac{1}{3^2} - \frac{1}{1^2} \right) = (12.1 \text{ eV})(Z-1)^2$$

The expected slope is then $(12.1 \text{ eV})^{1/2} = 3.48 \text{ eV}^{1/2}$, which agrees exactly with the slope of the graph. The screening model predicts an intercept of 1; the value from the graph is slightly larger, which suggests that our assumption that the screening is due to a single $1s$ electron is not quite correct.

23.
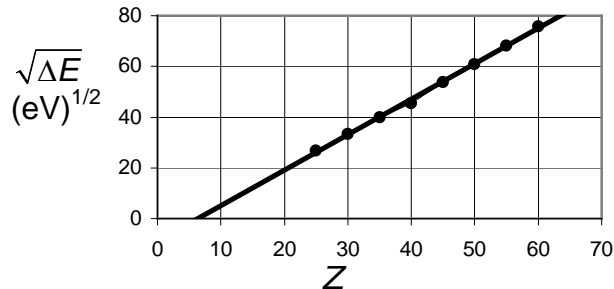


The slope is 1.40 eV$^{1/2}$ and the intercept is 6.4.

The $L_\alpha$ X rays originate from the $n = 3$ shell and lead to the $n = 2$ level. It is difficult to calculate the screening effect of the 1$s$, 2$s$, and 2$p$ electrons on the $n = 3$ electron that makes the transition, so we'll represent it by the value $k$.

$$\Delta E = E_3 - E_2 = (-13.6 \text{ eV})(Z - k)^2 \left( \frac{1}{3^2} - \frac{1}{2^2} \right) = (1.89 \text{ eV})(Z - k)^2$$

The expected slope is $(1.89 \text{ eV})^{1/2} = 1.38 \text{ eV}^{1/2}$, in very good agreement with the slope of the graph. If all of the $n = 1$ and $n = 2$ electrons contributed to the screening, we would expect $k = 9$, in contrast to the value 6.4 obtained from the intercept of the graph.

24. The wavelength difference is $\Delta\lambda = 0.59$ nm. By taking differentials of $E = hc/\lambda$, we can find the corresponding energy difference:

$$\Delta E = \frac{hc}{\lambda^2} \Delta\lambda = \frac{1240 \text{ eV} \cdot \text{nm}}{(590 \text{ nm})^2} (0.59 \text{ nm}) = 2.1 \times 10^{-3} \text{ eV}$$

This energy difference comes from the interaction of a magnetic field $B$ with a magnetic moment that we assume is of the order of 1 $\mu_B$. The energy difference between the cases with the magnetic moment parallel to $B$ and antiparallel to $B$ is (see Figure 7.25) $\Delta E = 2\mu_B B$, so

$$B = \frac{\Delta E}{2\mu_B} = \frac{2.1 \times 10^{-3} \text{ eV}}{2(5.8 \times 10^{-5} \text{ eV/T})} = 18 \text{ T}$$

This is quite a large magnetic field, of the order of the largest that can be produced in the laboratory with superconducting electromagnets.

25. (a) Figure 8.4 shows that the longest absorption wavelength from the ground state is 670.8 nm, in the visible region of the spectrum.
(b) From Figure 8.17, the longest ground-state absorption wavelength is 58.4 nm, in the ultraviolet region.

(c) The shortest absorption wavelengths (largest absorption energies) are those that ionize the atom. For lithium this is

$$\lambda = \frac{hc}{E} = \frac{1240 \text{ eV} \cdot \text{nm}}{5.39 \text{ eV}} = 230 \text{ nm}$$

in the near ultraviolet region. For helium,

$$\lambda = \frac{hc}{E} = \frac{1240 \text{ eV} \cdot \text{nm}}{24.5 \text{ eV}} = 50.6 \text{ nm}$$

in the ultraviolet region.

26.     $$\Delta E(4p \rightarrow 3p) = \Delta E(4p \rightarrow 2s) - \Delta E(3p \rightarrow 2s) = \frac{hc}{\lambda(4p \rightarrow 2s)} - \frac{hc}{\lambda(3p \rightarrow 2s)}$$

$$= \frac{1240 \text{ eV} \cdot \text{nm}}{396.5 \text{ nm}} - \frac{1240 \text{ eV} \cdot \text{nm}}{501.6 \text{ nm}} = 0.655 \text{ eV}$$

In hydrogen, $\Delta E(4 \rightarrow 3) = (-13.6 \text{ eV})(1/4^2 - 1/3^2) = 0.661 \text{ eV}$. The agreement is very good, because the $4p$ electron in helium is screened by the $1s$ electron, so that $Z_{\text{eff}} = 2 - 1 = 1$.

$$\Delta E(4d \rightarrow 3d) = \Delta E(4d \rightarrow 2p) - \Delta E(3d \rightarrow 2p) = \frac{hc}{\lambda(4d \rightarrow 2p)} - \frac{hc}{\lambda(3d \rightarrow 2p)}$$

$$= \frac{1240 \text{ eV} \cdot \text{nm}}{447.1 \text{ nm}} - \frac{1240 \text{ eV} \cdot \text{nm}}{587.6 \text{ nm}} = 0.663 \text{ eV}$$

Again, the agreement with hydrogen is very good.

# Chapter 9

This chapter provides an introduction to the structure and properties of molecules and how the spatial properties of the atomic wave functions described in Chapters 7 and 8 can be applied to understanding the arrangements of atoms in simple molecules. The mathematics of molecular orbitals is beyond the level of this text, so only very simple molecular wave functions are included: the hydrogen molecule ion $H_2^+$ and the hydrogen molecule $H_2$. The structures of some simple molecules containing $s$ and $p$ orbitals are illustrated. Both ionic and covalent bonds are briefly considered. Quantum states associated with rotation and vibration are reviewed, and then applied to the analysis of excited states in molecules. Electromagnetic transitions between states are discussed in connection with molecular spectroscopy. The major changes from the $2^{nd}$ edition involve the restructuring of the latter material on rotations, vibrations, and spectroscopy.

## Supplemental Materials

Information on the properties of molecules is tabulated in many locations; see, for example, the *Handbook of Chemistry and Physics* (Chemical Rubber Publishing Co.), the *American Institute of Physics Handbook* (American Institute of Physics), or the *Journal of Physical and Chemical Reference Data*. The National Institute of Standards and Technology (NIST) maintains a web site with chemistry reference data that includes properties of diatomic molecules (http://webbook.nist.gov/chemistry).

Chapter 14 of *Physlet Quantum Physics* has programs for displaying the $H_2^+$ and $H_2$ wave functions and also the representation of the potential energy for diatomic molecules by various functions such as Morse and Lennard-Jones. The PhET programs "Quantum Bound States" and "Double Wells and Covalent Bonds" have the option of displaying the one-dimensional wave functions and energy levels for an electron interacting with two positive charges separated by a distance that may be varied by the user.

More than 100 different types of molecules have been discovered in space, some with more than10 atoms. Two popular-level articles on the search for molecules in space:
B. E. Turner, "Interstellar Molecules," *Scientific American* **228,** 50 (March 1973).
B. Zuckerman, "Interstellar Molecules," *Nature* **268,** 491 (August 1977).
The web sites of NASA, the Jet Propulsion Laboratory, and the National Radio Astronomy Observatory provide links to lists of molecules discovered in space. Other good sites include http://astrochemistry.net/ and the University of Cologne's data base for molecular spectroscopy: http://www.astro.uni-koeln.de/site/vorhersagen/ .

## Suggestions for Additional Reading

Books on molecular spectroscopy that include introductory as well as advanced material:
G. M. Barrow, *Introduction to Molecular Spectroscopy* (McGraw-Hill, 1962).

M. Karplus and R. N. Porter, *Atoms and Molecules* (W. A. Benjamin, 1970).
P. S. C. Mathews, *Quantum Chemistry of Atoms and Molecules* (Cambridge University Press, 1986)
L. Pauling, *The Chemical Bond* (Cornell University Press, 1967).
W. G. Richards and P. R. Scott, *Structure and Spectra of Molecules* (Wiley, 1985).

An advanced but very detailed work:
G. Herzberg, *Molecular Spectra and Molecular Structure* (Van Nostrand, 1950). molecules.

## Materials for Active Engagement in the Classroom

### A. Reading Quizzes

1. Which is the correct procedure for combining the wave functions of the two electrons in the $H_2$ molecule to get the total electron probability density?
   (1) First add (or subtract) the two individual wave functions, then square the result; that is, $|\psi_1 \pm \psi_2|^2$.
   (2) First square the two wave functions, then add or subtract the results; that is, $|\psi_1|^2 \pm |\psi_2|^2$.

2. As the rotational energy of a molecule increases, what happens to the spacing between the rotational energy states?
   (1) The energy spacing increases.
   (2) The energy spacing decreases.
   (3) The energy spacing remains constant.

**Answers**    1. 1    2. 1

### B. Conceptual or Discussion Questions

1. Using the following atomic numbers for the $2p$ elements:

   |       | B | C | N | O | F | Ne |
   |-------|---|---|---|---|---|----|
   | $Z =$ | 5 | 6 | 7 | 8 | 9 | 10 |

   identify the molecule in each pair that has the larger dissociation energy.
   (a) CO or $O_2$        (b) $C_2$ or $B_2$        (c) BC or BO
   (d) CF or CN        (e) FO or $F_2$        (f) $N_2$ or NO

2. Molecules have three different types of excited states: electronic, vibrational, rotational. Put these in increasing order according to the amount of energy generally required for each type of excitation.
   (1) Vibrational, electronic, rotational        (2) Vibrational, rotational, electronic
   (3) Electronic, vibrational, rotational        (4) Electronic, rotational, vibrational
   (5) Rotational, electronic, vibrational        (6) Rotational, vibrational, electronic

3. As the angular momentum quantum number $L$ increases, the actual energy of the rotational states differs slightly from the predicted value $E_L = L(L+1)\hbar^2 / 2mR_{eq}^2$. How would you expect the actual energy to differ from the predicted energy?
   (1) The actual energy is a little larger than the predicted energy.
   (2) The actual energy is a little smaller than the predicted energy.

4. The figure shows the energy curves of two different molecules that have the same reduced mass. Which molecule has the larger (a) equilibrium radius? (b) dissociation energy? (c) rotational constant $B$? (d) rotational inertia? (e) vibrational energy spacing? (f) zero point energy?
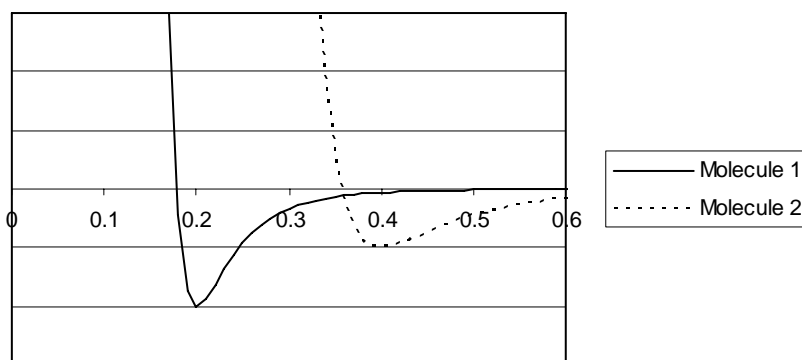


**Answers**    1. (a) CO  (b) $C_2$  (c) BO  (d) CN  (e) FO  (f) $N_2$      2. 6      3. 2
4. (a) 2  (b) 1  (c) 1  (d) 2  (e) 1  (f) 1


## Sample Exam Questions

### A. Multiple Choice

1. A certain diatomic molecule absorbs energy by rotating. The first excited rotational state is at an energy of 0.10 eV above the ground state. What is the energy of the second excited rotational state above the ground state?
   (a) 0.15 eV      (b) 0.20 eV      (c) 0.30 eV      (d) 0.40 eV

2. In a measurement of the absorption of microwaves by a diatomic gas, you find that the lowest absorption frequency is $f_0$. What is the next lowest absorption frequency?
   (a) $2f_0$          (b) $3f_0$          (c) $4f_0$          (d) $1.2f_0$

3. A certain diatomic molecule absorbs energy by vibrating. The first excited vibrational state is at an energy of 2.0 meV above the ground state. What is the energy of the second excited vibrational state above the ground state?
   (a) 2.5 meV    (b) 4.0 meV    (c) 6.0 meV    (d) 8.0 meV

4. Which has the smaller molecular binding energy, neutral $H_2$ or ionized $H_2^+$?

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(a) $H_2$          (b) $H_2^+$          (c) They are the same.

5. Photons emitted in transitions among the rotational or vibrational states of molecules generally are in what region of the electromagnetic spectrum?
   (a) X-ray          (b) Optical          (c) Ultraviolet          (d) Infrared          (e) Radio

6. The energy difference between the ground state and the first excited vibrational state of molecular $H_2$ is $E$. The energy difference between the first excited state and the second excited state is:
   (a) $E$          (b) $2E$          (c) $3E$          (d) $4E$

7. In a certain rotating diatomic molecule, the energy difference between the ground state and first excited state is 3.0 eV. What is the energy <u>difference</u> between the first and second excited states?
   (a) 1.5 eV          (b) 3.0 eV          (c) 6.0 eV          (d) 9.0 eV          (e) 12.0 eV

**Answers**          1. c          2. a          3. b          4. a          5. d          6. a          7. c

## B. Conceptual

1. Depending on whether their spins are parallel or antiparallel, two electrons (each with spin 1/2) can combine to give a total spin of 1 (parallel) or 0 (antiparallel). Only one of these two possible values is correct for the two electrons in a $H_2$ molecule. Which one is correct? EXPLAIN YOUR ANSWER.

2. Consider what happens to the rotational and vibrational spacings of KH ($m_K = 39$ u, $m_H = 1$ u) when the hydrogen is replaced by deuterium (heavy hydrogen, $m_D = 2$ u). Compared with KH, are the rotational spacings of KD *larger* or *smaller*? Are the vibrational spacings of KD *larger* or *smaller* than those of KH? Which changes by the larger amount, the *rotational* or the *vibrational* spacings? (Assume the equilibrium separations of KH and KD are the same.) EXPLAIN YOUR ANSWERS.

3. Put the following compounds in order of increasing dissociation energy of the carbon-carbon bond: $C_2H_2$, $C_2H_4$, $C_2H_6$. EXPLAIN YOUR ANSWER.

**Answers**          1. 0 (antiparallel)          2. smaller, smaller, rotational
                     3. $C_2H_6 < C_2H_4 < C_2H_2$

## C. Problems

1. In a certain diatomic molecule, the two atoms can vibrate relative to their common center of mass and also rotate about the center of mass. The masses of the two atoms

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are 10.0 u and 30.0 u. At equilibrium, the two atoms are separated by a distance of 0.236 nm. The vibrational frequency is found to be $2.42 \times 10^{13}$ s$^{-1}$.

(a) Considering only vibrational motion and assuming there to be no rotation, calculate the energies of the ground state and the first two excited states. Draw a diagram showing the vibrational energy levels.

(b) Considering only rotational motion and assuming there to be no vibration, calculate the energies of the ground state and the first two excited states. Draw a diagram showing the rotational energy levels.

2. (a) Calculate the energies of the first 3 rotational excited states of molecular CN. The equilibrium separation of CN is 0.117 nm and the masses of carbon and nitrogen are 12.00 u and 14.00 u.

(b) Draw an energy-level diagram showing the ground state and first 3 excited states. Label each state with its energy, angular momentum, and degeneracy.

(c) On your diagram show all allowed absorption transitions and calculate their wavelengths.

**Answers**   1. (a) 0.050 eV, 0.150 eV, 0.250 eV       (b) 0, 100 $\mu$eV, 300 $\mu$eV
          2. (a) 0.471 meV, 1.413 meV, 2.827 meV       (c) 2.632 mm, 1.316 mm, 0.877 mm

**Problem Solutions**

1.  The energy of $H_2^+$ is $-16.3$ eV, and the energy of $H_2$ is $-31.7$ eV. The ionization energy is the difference between these values:

    $$E_{ion}(H_2) = E(H_2^+) - E(H_2) = -16.3 \text{ eV} + 31.7 \text{ eV} = 15.4 \text{ eV}$$

2.  Let the negative charge on the sphere be $-q$. Then the potential energy between the sphere and either of the protons is

    $$U_e = -\frac{1}{4\pi\varepsilon_0}\frac{qe}{R_{eq}/2}$$

    The total energy of the system includes the repulsion of the protons and the attraction of the negative sphere for each of the protons:

    $$E = U_p + 2U_e = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{R_{eq}} - 2\frac{1}{4\pi\varepsilon_0}\frac{qe}{R_{eq}/2} = \frac{1}{4\pi\varepsilon_0}\frac{e}{R_{eq}}(e-4q)$$

    Inserting the value of $R_{eq} = 0.106$ nm and $E = -B = -2.7$ eV, we can solve to find

    $$q = 0.30e$$

    This quantity of charge is roughly consistent with the fraction of $\psi^2$ that appears between the two protons in Figure 9.3$a$.

3.  (a) Each mole contains $6.022 \times 10^{23}$ molecules, so

    $$E = \frac{410 \times 10^3 \text{ J/mole}}{6.022 \times 10^{23} \text{ molecules/mole}}\frac{1}{1.602 \times 10^{-19} \text{ J/eV}} = 4.25 \text{ eV/molecule}$$

    (b) $$E = \frac{106 \times 10^3 \text{ J/mole}}{6.022 \times 10^{23} \text{ molecules/mole}}\frac{1}{1.602 \times 10^{-19} \text{ J/eV}} = 1.10 \text{ eV/molecule}$$

    (c) $$E = \frac{945 \times 10^3 \text{ J/mole}}{6.022 \times 10^{23} \text{ molecules/mole}}\frac{1}{1.602 \times 10^{-19} \text{ J/eV}} = 9.80 \text{ eV/molecule}$$

4.  NO has 15 electrons, NF has 16, and OF has 17. A total of 8 electrons will occupy the 1$s$ and 2$s$ bonding and antibonding states, and 6 electrons can be placed in the three 2$p$ bonding states. The remaining electrons in each molecule must go into the 2$p$ antibonding states: one in NO, two in NF, and three in OF. So we would expect NO to

have the largest dissociation energy, followed by NF and then by OF. The measured bond strengths and corresponding dissociation energies are consistent with these predictions – NO: 631 kJ/mole and 6.54 eV; NF: 333 kJ/mole and 3.45 eV; OF: 222 kJ/mole and 2.30 eV.

5.  (a) $Be_2$ has four electrons, which fill the $1s$ bonding and antibonding states. $Li_2$ has six electrons, so in addition to filling the $1s$ bonding and antibonding states there are two electrons available for the $2s$ bonding state. So we expect $Li_2$ to have the greater dissociation energy. The measured values are 0.61 eV for $Be_2$ and 1.10 eV for $Li_2$.
(b) $B_2$ has 10 electrons, with 8 filling the $1s$ and $2s$ bonding and antibonding states so that there are 2 electrons left for the $2p$ bonding state. $C_2$ has 10 electrons, so 4 are available for the $2p$ bonding states. We thus expect $C_2$ to have the greater dissociation energy. The measured values are 3.08 eV for $B_2$ and 6.29 eV for $C_2$.
(c) There are 14 electrons in CO and 16 in $O_2$. In CO, 8 of the electrons fill the $1s$ and $2s$ bonding and antibonding states, and so there are 6 electrons available for the $2p$ bonding states. $O_2$ has 8 electrons available for the $2p$ states, of which 6 go into bonding states and 2 into antibonding states. We thus expect CO to have the greater dissociation energy. The measured values are 11.17 eV for CO and 5.16 eV for $O_2$.

6.  (a) $F_2$ has 18 electrons, 8 in the $1s$ and $2s$ bonding and antibonding states, 6 in the $2p$ bonding states, and 4 in the $2p$ antibonding states. $F_2^+$ has one less electron in the antibonding state, so it should have the greater dissociation energy. The measured values are: 1.6 eV for $F_2$ and 3.2 eV for $F_2^+$.
(b) $N_2$ (14 electrons) has 6 electrons in the $2p$ bonding state, while $N_2^+$ has only 5. Thus $N_2$ should have the greater dissociation energy. The measured values are: 9.8 eV for $N_2$ and 8.7 eV for $N_2^+$.
(c) NO (15 electrons) has 1 antibonding $2p$ electron while $NO^+$ has none, so $NO^+$ should have the greater dissociation energy. The measured values are: 6.5 eV for NO and 10.9 eV for $NO^+$.
(d) CN (13 electrons) has 5 bonding $2p$ electrons while $CN^+$ has only 4, so CN should have the greater dissociation energy. The measured values are: 7.9 eV for CN and 4.9 eV for $CN^+$.

7.  The equilibrium separation of KBr is 0.282 nm, so

$$U = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{R} = \frac{1.440\ eV \cdot nm}{0.282\ nm} = 5.11\ eV$$

8.  The difference between the ionization energy of K and the electgron affinity of I is

$$4.34\ eV - 3.06\ eV = 1.28\ eV$$

The potential energy will be 1.28 eV when the separation $R$ is

$$R = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{U} = \frac{1.440 \text{ eV} \cdot \text{nm}}{1.28 \text{ eV}} = 1.13 \text{ nm}$$

Whenever $R$ is less than 1.13 nm it is advantageous to have $K^+$ and $I^-$ ions.

9.  (a)    $p = qR_{eq} = (1.602 \times 10^{-19} \text{ C})(0.193 \times 10^{-9} \text{ m}) = 30.9 \times 10^{-30} \text{ C} \cdot \text{m}$

(b)    $\dfrac{p_{meas}}{qR_{eq}} = \dfrac{27.2 \times 10^{-30} \text{ C} \cdot \text{m}}{30.9 \times 10^{-30} \text{ C} \cdot \text{m}} = 0.88,$ so NaF is 88% ionic.

10.    $p = qR_{eq} = (1.602 \times 10^{-19} \text{ C})(0.160 \times 10^{-9} \text{ m}) = 25.6 \times 10^{-30} \text{ C} \cdot \text{m}$

$$\frac{p_{meas}}{qR_{eq}} = \frac{1.47 \times 10^{-30} \text{ C} \cdot \text{m}}{25.6 \times 10^{-30} \text{ C} \cdot \text{m}} = 0.057 = 5.7\% \text{ ionic}$$

11.    $p = qR_{eq} = 2(1.602 \times 10^{-19} \text{ C})(0.194 \times 10^{-9} \text{ m}) = 62.1 \times 10^{-30} \text{ C} \cdot \text{m}$

$$\frac{p_{meas}}{qR_{eq}} = \frac{26.5 \times 10^{-30} \text{ C} \cdot \text{m}}{62.1 \times 10^{-30} \text{ C} \cdot \text{m}} = 0.427 = 42.7\% \text{ ionic}$$

12.    (a) The reduced mass of the CN molecule is

$$m = \frac{m(\text{C})m(\text{N})}{m(\text{C}) + m(\text{N})} = \frac{(12.00 \text{ u})(14.00 \text{ u})}{12.00 \text{ u} + 14.00 \text{ u}} = 6.462 \text{ u}$$

The vibrational energy is

$$E = \hbar\omega = \hbar\sqrt{\frac{k}{m}} = \hbar c\sqrt{\frac{k}{mc^2}} = (197.3 \text{ eV} \cdot \text{nm})\sqrt{\frac{1.017 \times 10^4 \text{ eV/nm}^2}{(6.462 \text{ u})(931.5 \times 10^6 \text{ eV/u})}} = 0.2565 \text{ eV}$$

The first excited state would be at energy $E = 0.2565$ eV above the ground state and the second excited state would be at energy $2E = 0.5130$ eV above the ground state.
(b) The force constant is smaller by 3.42% or by a factor of 0.9658, so the energy, which is proportional to $k^{1/2}$, would be smaller by $0.9658^{1/2} = 0.9828$. The vibrational energy would then be $0.9828 \times 0.2565$ eV $= 0.2521$ eV, so the first excited state would be 0.2521 eV above the ground state and the second excited state would be 0.5042 eV above the ground state.

13.    The reduced mass of BeO is

$$m = \frac{m(\text{Be})m(\text{O})}{m(\text{Be}) + m(\text{O})} = \frac{(9.012 \text{ u})(15.995 \text{ u})}{9.012 \text{ u} + 15.995 \text{ u}} = 5.764 \text{ u}$$

The vibrational frequency is

$$f = \frac{c}{\lambda} = \frac{2.998 \times 10^8 \text{ m/s}}{6.724 \times 10^{-6} \text{ m}} = 4.459 \times 10^{13} \text{ Hz}$$

and the force constant is then

$$k = \omega^2 m = \frac{4\pi^2 f^2 mc^2}{c^2} = \frac{4\pi^2 (4.459 \times 10^{13} \text{ Hz})^2 (5.764 \text{ u})(931.5 \times 10^6 \text{ eV/u})}{(2.998 \times 10^8 \text{ m/s})^2 (10^9 \text{ nm/m})^2}$$

$$= 4.689 \times 10^3 \text{ eV/nm}^2$$

14. $$k = \frac{2(0.25 \text{ eV})}{(0.040 \text{ nm})^2} = 310 \text{ eV/nm}^2 = 310 \times 10^{18} \text{ eV/m}^2$$

$$m = \frac{m_1 m_2}{m_1 + m_2} = \frac{(23 \text{ u})(35.5 \text{ u})}{23 \text{ u} + 35.5 \text{ u}} = 13.96 \text{ u} \quad \text{using average mass of Cl}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{kc^2}{mc^2}} = \frac{1}{2\pi}\sqrt{\frac{(310 \times 10^{18} \text{ eV/m}^2)(3.0 \times 10^8 \text{ m/s})^2}{(13.96 \text{ u})(931.5 \times 10^6 \text{ eV/u})}} = 7.4 \times 10^{12} \text{ Hz}$$

$$\lambda = \frac{c}{f} = \frac{3.00 \times 10^8 \text{ m/s}}{7.4 \times 10^{12} \text{ Hz}} = 40 \mu\text{m} \quad \text{(infrared region)}$$

$$hf = 0.031 \text{ eV}$$

The approximation is valid for an energy range of about 0.25 eV, or up to about $N = 8$.

15. (a) The vibrational energy is $E = \hbar\omega = \hbar\sqrt{k/m}$, and so $E$ depends on the reduced mass as $m^{-1/2}$. The reduced mass of CO made with $^{12}\text{C}$ and $^{16}\text{O}$ is

$$m(^{12}\text{C}^{16}\text{O}) = \frac{m(^{12}\text{C})m(^{16}\text{O})}{m(^{12}\text{C}) + m(^{16}\text{O})} = \frac{(12.00 \text{ u})(16.00 \text{ u})}{12.00 \text{ u} + 16.00 \text{ u}} = 6.857 \text{ u}$$

Substituting $^{18}\text{O}$ for $^{16}\text{O}$ would give a reduced mass of

$$m(^{12}\text{C}^{18}\text{O}) = \frac{m(^{12}\text{C})m(^{18}\text{O})}{m(^{12}\text{C}) + m(^{18}\text{O})} = \frac{(12.00 \text{ u})(18.00 \text{ u})}{12.00 \text{ u} + 18.00 \text{ u}} = 7.200 \text{ u}$$

Because the vibrational energy depends on $m^{-1/2}$, the vibrational energy of $^{12}\text{C}^{18}\text{O}$ is

$$E(^{12}C^{18}O) = E(^{12}C^{16}O)\frac{m(^{12}C^{16}O)^{1/2}}{m(^{12}C^{18}O)^{1/2}} = (0.2691 \text{ eV})\sqrt{\frac{6.857 \text{ u}}{7.200 \text{ u}}} = 0.2626 \text{ eV}$$

(b) The reduced mass of $^{14}C^{16}O$ is

$$m(^{14}C^{16}O) = \frac{m(^{14}C)m(^{16}O)}{m(^{14}C)+m(^{16}O)} = \frac{(14.00 \text{ u})(16.00 \text{ u})}{14.00 \text{ u} + 16.00 \text{ u}} = 7.467 \text{ u}$$

The vibrational energy is then

$$E(^{14}C^{16}O) = E(^{12}C^{16}O)\frac{m(^{12}C^{16}O)^{1/2}}{m(^{14}C^{16}O)^{1/2}} = (0.2691 \text{ eV})\sqrt{\frac{6.857 \text{ u}}{7.467 \text{ u}}} = 0.2579 \text{ eV}$$

16.
$$I = m_1 x_1^2 + m_2 x_2^2 = (m_1 x_1^2 + m_2 x_2^2)\left(\frac{m_1+m_2}{m_1+m_2}\right) = \frac{m_1^2 x_1^2 + m_1 m_2 x_2^2 + m_1 m_2 x_1^2 + m_2^2 x_2^2}{m_1+m_2}$$

$$= \frac{m_1 m_2 x_1^2 + 2m_1 m_2 x_1 x_2 + m_1 m_2 x_2^2}{m_1+m_2} = \frac{m_1 m_2}{m_1+m_2}(x_1^2 + 2x_1 x_2 + x_2^2) = m$$

17. For NaCl (using the average mass of 35.5 u for Cl) and with $R_{eq} = 0.236$ nm,

$$m = \frac{m_1 m_2}{m_1+m_2} = \frac{(23.0 \text{ u})(35.5 \text{ u})}{23.0 \text{ u} + 35.5 \text{ u}} = 13.96 \text{ u}$$

$$B = \frac{\hbar^2}{2mR_{eq}^2} = \frac{\hbar^2 c^2}{2mc^2 R_{eq}^2} = \frac{(197 \text{ eV} \cdot \text{nm})^2}{2(13.96 \text{ u})(931.5 \times 10^6 \text{ eV/u})(0.236 \text{ nm})^2} = 2.68 \times 10^{-5} \text{ eV}$$

$L = 1$ to $L = 0$: $\Delta E = 2B = 5.36 \times 10^{-5}$ eV and $\lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV} \cdot \text{nm}}{5.36 \times 10^{-5} \text{ eV}} = 23.1 \text{ mm}$

$L = 2$ to $L = 1$: $\Delta E = 4B = 1.07 \times 10^{-4}$ eV and $\lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV} \cdot \text{nm}}{1.07 \times 10^{-4} \text{ eV}} = 11.6 \text{ mm}$

$L = 3$ to $L = 2$: $\Delta E = 6B = 1.61 \times 10^{-4}$ eV and $\lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV} \cdot \text{nm}}{1.61 \times 10^{-4} \text{ eV}} = 7.71 \text{ mm}$

18. For rotations about an axis that passes through the central carbon atom, the rotational inertia involves only the two oxygen atoms: $I = 2m_O R^2$, where $m_O$ is the mass of an oxygen atom and $R$ is the distance of the oxygens from the rotation axis. The rotational energy then depends on

$$\frac{\hbar^2}{2I} = \frac{\hbar^2}{4m_O R^2} = \frac{\hbar^2 c^2}{4m_O c^2 R^2} = \frac{(197 \ \text{eV} \cdot \text{nm})^2}{4(16 \ \text{u})(931.5 \times 10^6 \ \text{eV/u})(0.116 \ \text{nm})^2}$$

$$= 4.84 \times 10^{-5} \ \text{eV} = 48.4 \ \mu\text{eV}$$

and the energies of the rotational states are given by Equation 9.12, with
$E_L = (48.4 \ \mu\text{eV}) L(L+1)$:

$$E_0 = 0 \qquad E_1 = (48.4 \ \mu\text{eV})(1)(2) = 96.8 \ \mu\text{eV} \qquad E_2 = (48.4 \ \mu\text{eV})(2)(3) = 290.4 \ \mu\text{eV}$$

$$E_3 = (48.4 \ \mu\text{eV})(3)(4) = 580.8 \ \mu\text{eV} \qquad E_4 = (48.4 \ \mu\text{eV})(4)(5) = 968.0 \ \mu\text{eV}$$

19. The reduced masses are

$$m_{35} = \frac{m_1 m_2}{m_1 + m_2} = \frac{(1.007825 \ \text{u})(34.968853 \ \text{u})}{1.007825 \ \text{u} + 34.968853 \ \text{u}} = 0.979593 \ \text{u}$$

$$m_{37} = \frac{m_1 m_2}{m_1 + m_2} = \frac{(1.007825 \ \text{u})(36.965903 \ \text{u})}{1.007825 \ \text{u} + 36.965903 \ \text{u}} = 0.981077 \ \text{u}$$

$$\delta B = B(\text{H}^{35}\text{Cl}) - B(\text{H}^{37}\text{Cl}) = \frac{\hbar^2}{2R_{eq}^2}\left(\frac{1}{m_{35}} - \frac{1}{m_{37}}\right) = \frac{\hbar^2 c^2}{2R_{eq}^2}\left(\frac{1}{m_{35}c^2} - \frac{1}{m_{37}c^2}\right)$$

$$= \frac{1}{2}\left(\frac{197.3 \ \text{eV} \cdot \text{nm}}{0.127 \ \text{nm}}\right)^2 \left(\frac{1}{0.979593 \ \text{u}} - \frac{1}{0.981077 \ \text{u}}\right)\frac{1}{931.5 \times 10^6 \ \text{eV/u}} = 2.00 \times 10^{-6} \ \text{eV}$$

20. (a) Because there are no transitions with wavelengths between the two observed values, these wavelengths must correspond to emissions from $L + 1 \rightarrow L$ and $L + 2 \rightarrow L + 1$ (although we don't yet know the value of $L$). The transition energies are:

$$L + 2 \rightarrow L + 1: \qquad \Delta E = \frac{hc}{\lambda} = \frac{1239.8 \ \text{eV} \cdot \text{nm}}{526.6 \times 10^3 \ \text{nm}} = 2.354 \times 10^{-3} \ \text{eV}$$

$$L + 1 \rightarrow L: \qquad \Delta E = \frac{hc}{\lambda} = \frac{1239.8 \ \text{eV} \cdot \text{nm}}{658.3 \times 10^3 \ \text{nm}} = 1.883 \times 10^{-3} \ \text{eV}$$

From Equation 9.17, we see that successive values of $\Delta E$ increase by $2B$, and so the rotational spacing $2B$ is just the difference between the energies of successive emissions:

$$2B = 2.354 \times 10^{-3} \ \text{eV} - 1.883 \times 10^{-3} \ \text{eV} = 4.71 \times 10^{-4} \ \text{eV}$$

For the first transition, Equation 9.17 gives

$$L+1=\frac{\Delta E}{2B}=\frac{2.354\times10^{-3}\ \text{eV}}{4.71\times10^{-4}\ \text{eV}}=5$$

and thus the reported wavelengths must be from the $6\rightarrow5$ and $5\rightarrow4$ transitions.
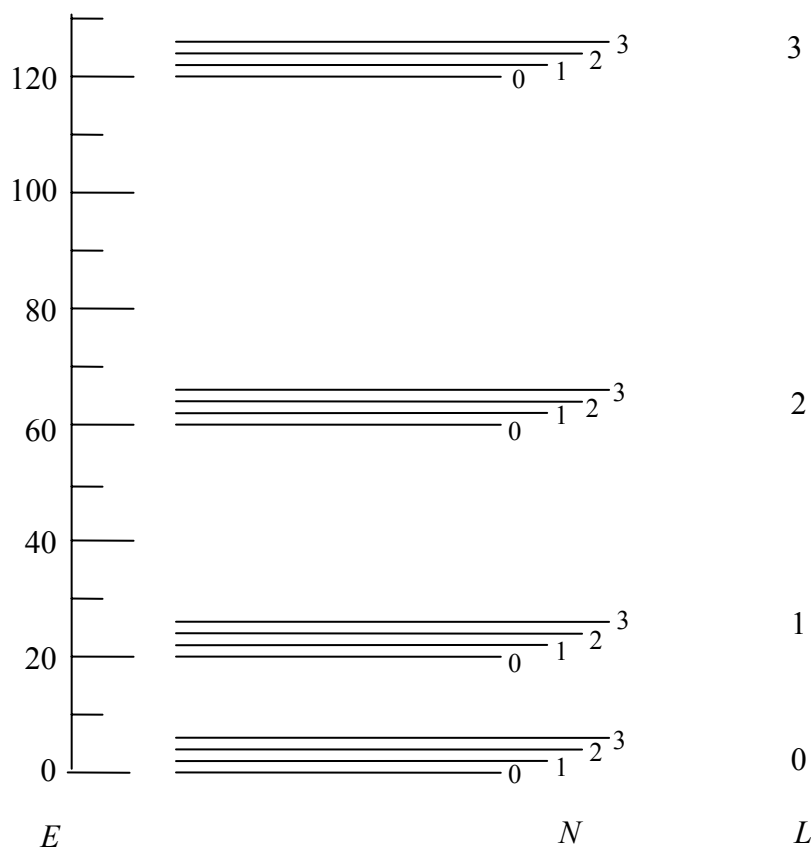(b) With $R_{eq}=0.1172$ nm, the reduced mass is

$$m=\frac{1}{c^2}\frac{(\hbar c)^2}{2BR_{eq}^2}=\frac{(197.3\ \text{eV}\cdot\text{nm})^2}{(931.5\times10^6\ \text{eV/u})(4.71\times10^{-4}\ \text{eV})(0.1172\ \text{nm})^2}=6.459\ \text{u}$$

With $m_1=12.00$ u for carbon, we solve Equation 9.10 for $m_2$ to give

$$m_2=\frac{m_1 m}{m_1-m}=\frac{(12.000\ \text{u})(6.459\ \text{u})}{12.000\ \text{u}-6.459\ \text{u}}=13.988\ \text{u}$$

The only stable atom with a mass near 14 u is nitrogen, so the molecule must be CN.

21. $E_{NL}=(N+1/2)hf+BL(L+1)=2(N+1/2)+10L(L+1)$



22. (a)
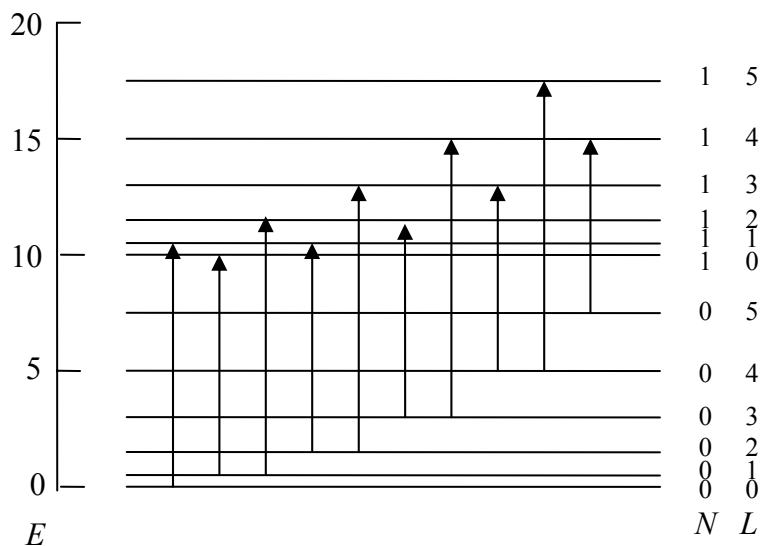
(b) $E_{NL} = (N + 1/2)hf + BL(L+1) = 10(N+1/2) + 0.25L(L+1)$

For $N = 0$ to $N = 1$, there are two sequences of absorption lines:

$$\Delta E = \left| E_{0L} - E_{1L\pm1} \right| = -E_{0L} + E_{1L\pm1}$$

$$L \to L-1: \quad \Delta E = 10 - L/2 \qquad\qquad L \to L+1: \quad \Delta E = 10 + (L+1)/2$$
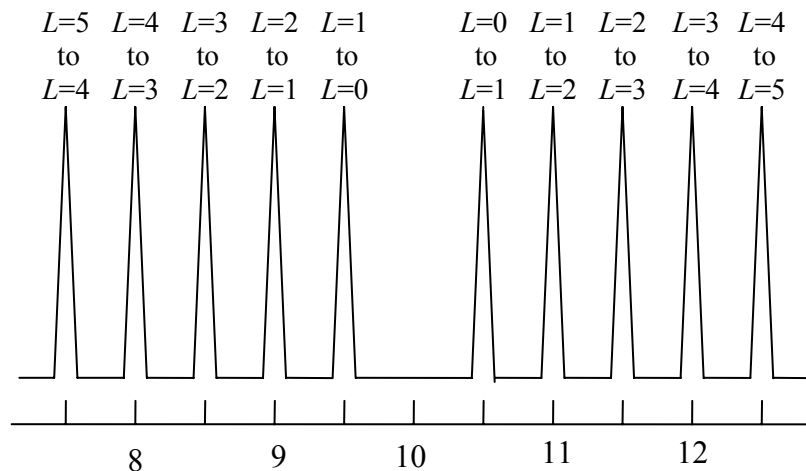


23. For emission transitions $N \to N - 1$ and $L \to L \pm 1$, so with $\Delta E = E_{NL} - E_{N-1L\pm1}$

$$L \to L+1: \quad \Delta E = hf + BL(L+1) - B(L+1)(L+2) = hf - 2B(L+1)$$

$$L \to L-1: \quad \Delta E = hf + BL(L+1) - B(L-1)(L) = hf + 2BL$$

24. (a) $\quad m = \dfrac{m_K m_{Cl}}{m_K + m_{Cl}} = \dfrac{(39.1\text{ u})(35.5\text{ u})}{39.1\text{ u} + 35.5\text{ u}} = 18.6\text{ u} = 1.73 \times 10^4 \text{ MeV}/c^2$

(b) $\quad B = \dfrac{\hbar^2}{2mR_{eq}^2} = \dfrac{\hbar^2 c^2}{2mc^2 R_{eq}^2} = \dfrac{(197.3 \text{ eV} \cdot \text{nm})^2}{2(1.73 \times 10^{10} \text{ eV})(0.267 \text{ nm})^2} = 1.58 \times 10^{-5} \text{ eV}$

The spacing of the transitions is $2B = 3.16 \times 10^{-5}$ eV.

25. $\quad p(E_{NL}) = (2L+1)e^{-E_{NL}/kT} = (2L+1)e^{-[(N+1/2)hf + BL(L+1)]/kT}$

$$\frac{dp}{dL} = 2e^{-[(N+1/2)hf + BL(L+1)]/kT} - (2L+1)e^{-[(N+1/2)hf + BL(L+1)]/kT}(2L+1)\frac{B}{kT} = 0$$

$$2L + 1 = \sqrt{\frac{2kT}{B}}$$

26. $\quad \dfrac{p(N=1)}{p(N=0)} = \dfrac{e^{-(3/2)hf/kT}}{e^{-(1/2)hf/kT}} = e^{-hf/kT}$

$$T = \frac{-hf}{k\ln[p(N=1)/p(N=0)]} = \frac{-0.358 \text{ eV}}{(8.617 \times 10^{-5} \text{ eV/K})\ln(1/3)} = 3780 \text{ K}$$

27. For CO, $m = \dfrac{m_C m_O}{m_C + m_O} = \dfrac{(12 \text{ u})(16 \text{ u})}{28 \text{ u}} = 6.86 \text{ u} = 6.39 \times 10^3 \text{ MeV}/c^2$

$$B = \frac{\hbar^2}{2mR_{eq}^2} = \frac{\hbar^2 c^2}{2mc^2 R_{eq}^2} = \frac{(197.3 \text{ eV} \cdot \text{nm})^2}{2(6.39 \times 10^9 \text{ eV})(0.113 \text{ nm})^2} = 2.39 \times 10^{-4} \text{ eV}$$

$$2L + 1 = \sqrt{\frac{2kT}{B}} = \sqrt{\frac{2(0.025 \text{ eV})}{2.39 \times 10^{-4} \text{ eV}}} = 14.5 \qquad \text{so} \qquad L = 7$$

28. At the equilibrium separation $R = R_{eq}$, $E = -E_0$ and $dE/dR = 0$:

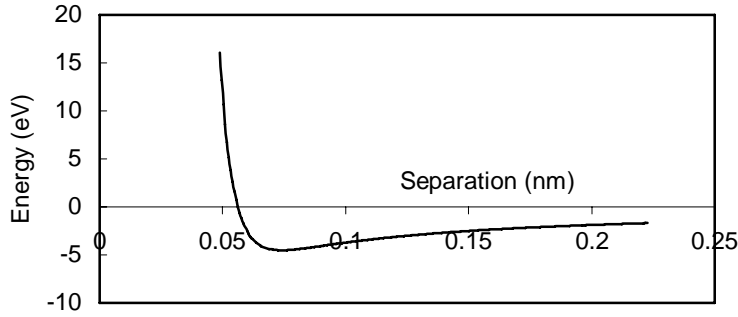$$-E_0 = \frac{A}{R_{eq}^9} - \frac{B}{R_{eq}} \qquad \text{and} \qquad \frac{dE}{dR} = \frac{-9A}{R_{eq}^{10}} + \frac{B}{R_{eq}^2} = 0$$

Solving these two equations simultaneously, we obtain

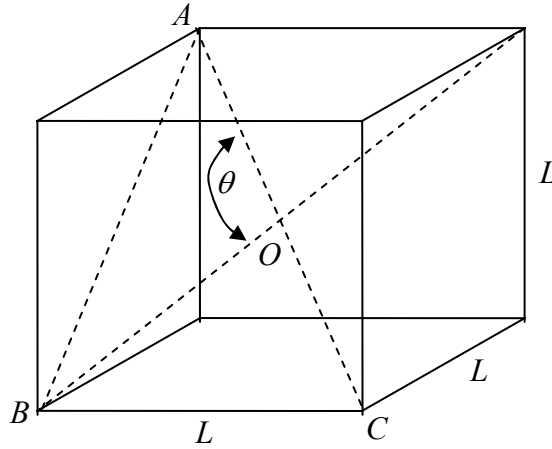$$A = \frac{E_0 R_{eq}^9}{8} \qquad \text{and} \qquad B = \frac{9A}{R_{eq}^8} = \frac{9}{8} E_0 R_{eq}$$

With these values we can write the energy equation as

$$E = \frac{E_0}{8}\frac{1}{(R/R_{eq})^9} - \frac{9E_0}{8}\frac{1}{R/R_{eq}}$$



29.



$$AB = \sqrt{2}L \quad \text{and} \quad AC = \sqrt{(AB)^2 + (BC)^2} = \sqrt{2L^2 + L^2} = \sqrt{3}L$$

Using the law of cosines on triangle $OAB$:

$$(AB)^2 = (AO)^2 + (OB)^2 - 2(AO)(OB)\cos\theta$$

but $AO = OB = \frac{1}{2}AC$, so

$$(AB)^2 = 2(\tfrac{1}{2}AC)^2 - 2(\tfrac{1}{2}AC)^2\cos\theta$$

$$\theta = \cos^{-1}\left[1 - \frac{(AB)^2}{\frac{1}{2}(AC)^2}\right] = \cos^{-1}\left[1 - \frac{2L^2}{\frac{1}{2}(3L^2)}\right] = \cos^{-1}\left(-\frac{1}{3}\right) = 109.5°$$

30. For $H_2$ 

$$m = \frac{m_1 m_2}{m_1 + m_2} = \frac{(m_H)(m_H)}{m_H + m_H} = \frac{1}{2} m_H$$

For HD 

$$m = \frac{m_1 m_2}{m_1 + m_2} = \frac{(m_H)(2m_H)}{m_H + 2m_H} = \frac{2}{3} m_H$$

For $D_2$ 

$$m = \frac{m_1 m_2}{m_1 + m_2} = \frac{(2m_H)(2m_H)}{2m_H + 2m_H} = m_H$$

With $f \propto \dfrac{1}{\sqrt{m}}$ we get $\dfrac{f_{HD}}{f_{H_2}} = \dfrac{\sqrt{m_H/2}}{\sqrt{2m_H/3}} = \sqrt{\dfrac{3}{4}} = 0.866$ and $\dfrac{f_{D_2}}{f_{H_2}} = \dfrac{\sqrt{m_H/2}}{\sqrt{m_H}} = \sqrt{\dfrac{1}{2}} = 0.707$

$$f_{HD} = 0.866(1.32 \times 10^{14} \text{ Hz}) = 1.14 \times 10^{14} \text{ Hz}$$

$$f_{D_2} = 0.707(1.32 \times 10^{14} \text{ Hz}) = 9.33 \times 10^{13} \text{ Hz}$$

$R_{eq}$ does not depend on the nuclear mass, so HD and $D_2$ should have the same $R_{eq}$ as $H_2$. The rotational parameter $B$ depends inversely on the mass, so the rotational parameters for HD and $D_2$ can be found from the rotational parameter for $H_2$ ( $B = 0.0076$ eV ) and the mass ratios:

For HD 

$$B = (0.0076 \text{ eV}) \frac{m_H/2}{2m_H/3} = 0.0057 \text{ eV}$$

For $D_2$ 

$$B = (0.0076 \text{ eV}) \frac{m_H/2}{m_H} = 0.0038 \text{ eV}$$

31. (a) The vibrational energies depend on the reduced mass like $m^{-1/2}$. With the deuterium mass $m_D = 2m_H$, where $m_H$ is the mass of the ordinary hydrogen atom, the reduced masses are $m(H_2) = m_H/2$, $m(HD) = 2\ m_H/3$, and $m(D_2) = m_H$. The vibrational energies are then

For HD 

$$E_{vib}(HD) = E_{vib}(H_2) \frac{m(H_2)^{1/2}}{m(HD)^{1/2}} = (0.54 \text{ eV}) \sqrt{\frac{0.5m_H}{0.667m_H}} = 0.47 \text{ eV}$$

For $D_2$ 

$$E_{vib}(D_2) = E_{vib}(H_2) \frac{m(H_2)^{1/2}}{m(D_2)^{1/2}} = (0.54 \text{ eV}) \sqrt{\frac{0.5m_H}{m_H}} = 0.38 \text{ eV}$$

The zero-point energy is half the vibrational energy, so the zero-point energies are 0.27 eV for $H_2$, 0.23 eV for HD and 0.19 eV for $D_2$.

(b) The depth of the minimum of the molecular energy curve is lower than the dissociation energy (which is the energy necessary to add to the vibrational ground state to reach $E = 0$) by the zero-point energy, so for $H_2$ the minimum would be at an energy of 4.52 eV + 0.27 eV = 4.79 eV below zero. Assuming the same value for the energy minimum for HD, the vibrational ground state of HD would then be 0.23 eV above the minimum, or 4.79 eV − 0.23 eV = 4.56 eV below zero. For $D_2$, the vibrational ground state would be 4.79 eV − 0.19 eV = 4.60 eV below zero. Thus the dissociation energies are 4.56 eV for HD and 4.60 eV for $D_2$.

32. (a) For $H_2$, $E_{vib} = 0.54$ eV and $B = 0.0076$ eV, so from Equation 9.16

$$L(L+1) = \frac{E_{vib}}{B} = \frac{0.54 \text{ eV}}{0.0076 \text{ eV}} = 71 \quad \text{and} \quad L \cong 8$$

so 9 rotational states ($L = 0$ to 8) are between the vibrational states.

(b) For HCl, $E_{vib} = 0.358$ eV and $B = 0.00132$ eV, so from Equation 9.16

$$L(L+1) = \frac{E_{vib}}{B} = \frac{0.358 \text{ eV}}{0.00132 \text{ eV}} = 271 \quad \text{and} \quad L \cong 16$$

so 17 rotational states ($L = 0$ to 16) are between the vibrational states.

(c) For NaCl, $E_{vib} = 0.30$ eV and $B = 2.68 \times 10^{-5}$ eV :

$$L(L+1) = \frac{E_{vib}}{B} = \frac{0.030 \text{ eV}}{2.68 \times 10^{-5} \text{ eV}} = 1119 \quad \text{and} \quad L \cong 33$$

so 34 rotational states are between the vibrational states.

33. (a) The "missing" transition is at an energy of about 0.317 eV.

(b) $\quad f = \frac{E}{h} = \frac{0.317 \text{ eV}}{4.14 \times 10^{-15} \text{ eV} \cdot \text{s}} = 7.66 \times 10^{13} \text{ Hz}$

$m = \frac{m_H m_{Br}}{m_H + m_{Br}} = \frac{(1 \text{ u})(80 \text{ u})}{81 \text{ u}} = 0.988 \text{ u} = 920 \text{ MeV}/c^2$

$k = \frac{mc^2 (2\pi f)^2}{c^2} = \frac{(920 \times 10^6 \text{ eV})(2\pi)^2 (7.66 \times 10^{13} \text{ Hz})^2}{9.00 \times 10^{16} \text{ m}^2/\text{s}^2} = 2.37 \times 10^{21} \text{ eV/m}^2$

(c) With $R_{eq} = 0.141$ nm,

$$2B = \frac{\hbar^2}{mR_{eq}^2} = \frac{\hbar^2 c^2}{mc^2 R_{eq}^2} = \frac{(197.3 \text{ eV} \cdot \text{nm})^2}{(920 \times 10^6 \text{ eV})(0.141 \text{ nm})^2} = 2.1 \times 10^{-3} \text{ eV}$$

This value agrees nicely with the spacing of the lines in the spectrum, estimated to be about 0.002 eV.

34. (a) The vibrational energy is $hf = 0.54$ eV. The vibrational states are non-degenerate, so

$$\frac{p(N=1)}{p(N=0)} = \frac{e^{-(3/2)hf/kT}}{e^{-(1/2)hf/kT}} = e^{-hf/kT}$$

At $T = 293$ K, $kT = 0.02525$ eV.

$$\frac{p(N=1)}{p(N=0)} = e^{-hf/kT} = e^{-0.54 \text{ eV}/0.02525 \text{ eV}} = 5.15 \times 10^{-10}$$

(b) In the $N = 2$ state,

$$\frac{p(N=2)}{p(N=0)} = \frac{e^{-(5/2)hf/kT}}{e^{-(1/2)hf/kT}} = e^{-2hf/kT} = e^{-2(0.54 \text{ eV})/0.02525 \text{ eV}} = 2.65 \times 10^{-19}$$

35. (a) For NaCl, $hf = 0.063$ eV.

$$\frac{p(N=1)}{p(N=0)} = \frac{e^{-(3/2)hf/kT}}{e^{-(1/2)hf/kT}} = e^{-hf/kT} = e^{-0.063 \text{ eV}/0.02525 \text{ eV}} = 8.2 \times 10^{-2}$$

(b) $$\frac{p(N=2)}{p(N=0)} = \frac{e^{-(5/2)hf/kT}}{e^{-(1/2)hf/kT}} = e^{-2hf/kT} = e^{-2(0.063 \text{ eV})/0.02525 \text{ eV}} = 6.8 \times 10^{-3}$$

36. (a) For H$_2$, $B = 0.0076$ eV. At $T = 293$ K, $kT = 0.02525$ eV.

$$\frac{p(L)}{p(0)} = (2L+1)e^{-BL(L+1)/kT} = (2L+1)e^{-L(L+1)(0.0076 \text{ eV})/(0.02525 \text{ eV})} = (2L+1)e^{-0.301L(L+1)}$$

$$\frac{p(1)}{p(0)} = 1.64, \qquad \frac{p(2)}{p(0)} = 0.822, \qquad \frac{p(3)}{p(0)} = 0.189$$

(b) At $T = 30$ K, $kT = 0.002585$ eV and $p(L)/p(0) = (2L+1)e^{-2.94L(L+1)}$.

$$\frac{p(1)}{p(0)} = 8.38 \times 10^{-3}, \qquad \frac{p(2)}{p(0)} = 1.09 \times 10^{-7}, \qquad \frac{p(3)}{p(0)} = 3.33 \times 10^{-15}$$

37. (a) For NaCl, $B = 2.68 \times 10^{-5}$ eV. At $T = 293$ K, $kT = 0.02525$ eV.

$$\frac{p(L)}{p(0)} = (2L+1)e^{-BL(L+1)\hbar^2/kT} = (2L+1)e^{-L(L+1)(2.68\times10^{-5}\ \text{eV})/(0.02525\ \text{eV})} = (2L+1)e^{-1.06\times10^{-3}L(L+1)}$$

$$\frac{p(1)}{p(0)} = 2.99, \qquad \frac{p(2)}{p(0)} = 4.97, \qquad \frac{p(3)}{p(0)} = 6.91$$

(b) At $T = 30$ K, $kT = 0.002585$ eV and $p(L)/p(0) = (2L+1)e^{-1.04\times10^{-2}L(L+1)}$.

$$\frac{p(1)}{p(0)} = 2.94, \qquad \frac{p(2)}{p(0)} = 4.70, \qquad \frac{p(3)}{p(0)} = 6.18$$

38. (a) The difference in electronegativity between K and Br is 2.96 – 0.82 = 2.14. Based on the values in Figure 9.21, we expect an ionic content in the range of 60-80%.
(b) The tabulated value of the electric dipole moment of KBr is $3.47 \times 10^{-29}$ C·m. The equilibrium separation is 0.282 nm so the expected dipole moment in a purely ionic molecule would be

$$p = qR_{\text{eq}} = (1.602\times10^{-19}\ \text{C})(0.282\times10^{-9}\ \text{m}) = 4.52\times10^{-29}\ \text{C}\cdot\text{m}$$

and thus the fractional ionic content is

$$\frac{3.47\times10^{-29}\ \text{C}\cdot\text{m}}{4.52\times10^{-29}\ \text{C}\cdot\text{m}} = 77\%$$

which agrees very well with the prediction based on the electronegativity values.

39. (a) The energies corresponding to the reported frequencies are 388.7 $\mu$eV, 583.1 $\mu$eV, and 971.7 $\mu$eV. The differences between successive values are 194.4 $\mu$eV and 388.6 $\mu$eV. For a sequence of rotational transitions, we know that the difference between *consecutive* emissions is $2B$. Noting that 388.6 $\mu$eV $\approx 2 \times 194.4$ $\mu$eV, these transitions would be consistent with a rotational sequence with $2B = 194.4$ $\mu$eV if the experiment had not been able to observe a transition at 583.1 $\mu$eV + 194.4 $\mu$eV = 777.5 $\mu$eV. (b) With 388.7 $\mu$eV/194.4 $\mu$eV = 2, the observed transitions must correspond to angular momentum quantum numbers $2 \rightarrow 1$, $3 \rightarrow 2$, and $5 \rightarrow 4$ (with $4 \rightarrow 3$ missing from the sequence). (c) The reduced mass of PN is

$$m = \frac{m(\text{P})m(\text{N})}{m(\text{P})+m(\text{N})} = \frac{(30.97\ \text{u})(14.00\ \text{u})}{30.97\ \text{u} + 14.00\ \text{u}} = 9.642\ \text{u}$$

The equilibrium separation of PN is 0.1491 nm, so the rotational spacing is expected to be

$$2B = \frac{\hbar^2}{mR_{eq}^2} = \frac{(\hbar c)^2}{mc^2 R_{eq}^2} = \frac{(197.3 \text{ eV} \cdot \text{nm})^2}{(9.642 \text{ u})(931.5 \times 10^6 \text{ eV/u})(0.1491 \text{ nm})^2} = 195.0 \ \mu\text{eV}$$

which agrees with the observed value of 194.4 $\mu$eV.

40. The energies corresponding to these two transitions are 663.05 $\mu$eV and 647.48 $\mu$eV. The reduced masses of AlCl for the two different stable isotopes of Cl (masses 34.969 u and 36.966 u) are:

$$m(\text{Al}^{35}\text{Cl}) = \frac{m(\text{Al})m(^{35}\text{Cl})}{m(\text{Al}) + m(^{35}\text{Cl})} = \frac{(26.982 \text{ u})(34.969 \text{ u})}{26.982 \text{ u} + 34.969 \text{ u}} = 15.230 \text{ u}$$

$$m(\text{Al}^{37}\text{Cl}) = \frac{m(\text{Al})m(^{37}\text{Cl})}{m(\text{Al}) + m(^{37}\text{Cl})} = \frac{(26.982 \text{ u})(36.966 \text{ u})}{26.982 \text{ u} + 36.966 \text{ u}} = 15.597 \text{ u}$$

The equilibrium separation of the atoms is 0.2130 nm, so the rotational constants are:

$$B(\text{Al}^{35}\text{Cl}) = \frac{\hbar^2}{2mR_{eq}^2} = \frac{(\hbar c)^2}{2mc^2 R_{eq}^2} = \frac{(197.3 \text{ eV} \cdot \text{nm})^2}{2(15.230 \text{ u})(931.50 \times 10^6 \text{ eV/u})(0.2130 \text{ nm})^2} = 30.240 \ \mu\text{eV}$$

$$B(\text{Al}^{37}\text{Cl}) = \frac{\hbar^2}{2mR_{eq}^2} = \frac{(\hbar c)^2}{2mc^2 R_{eq}^2} = \frac{(197.3 \text{ eV} \cdot \text{nm})^2}{2(15.597 \text{ u})(931.50 \times 10^6 \text{ eV/u})(0.2130 \text{ nm})^2} = 29.528 \ \mu\text{eV}$$

We assume that the larger of the observed close-lying emissions is associated with Al$^{35}$Cl, because it has the larger rotational constant. Equation 9.16 then gives $L+1 = \Delta E / 2B = 663.05 \ \mu\text{eV}/2(30.240 \ \mu\text{eV}) = 11$ and similarly for Al$^{37}$Cl the other emission gives $L+1 = \Delta E / 2B = 647.48 \ \mu\text{eV}/2(29.528 \ \mu\text{eV}) = 11$. Thus both emissions arise from the rotational transition with angular momentum quantum numbers $11 \rightarrow 10$. (b) The rotational state with $L = 11$ is at an energy given by Equation 9.16:

$$E_{11} = BL(L+1) = (30.240 \ \mu\text{eV})(11)(12) = 0.00399 \text{ eV}$$

To have a reasonable probability to reach an excited state at this energy, the environment of the molecules must be at a temperature of

$$T = \frac{E}{k} = \frac{0.00399 \text{ eV}}{8.617 \times 10^{-5} \text{ eV/K}} = 46 \text{ K}$$

(Here $k$ is the Boltzmann constant, not the force constant for vibrations.) Although this is cold by ordinary standards, it is far above the temperature of interstellar space (2.7 K), which suggests that the star is warming the dust cloud.

# Chapter 10

This chapter begins with a short exercise in discrete statistics that motivates the ensuing discussion of classical and quantum statistics. New to this edition is an explicit calculation of the density of states functions for particles and photons. The discussion of the classical statistics has been changed from the 2$^{nd}$ edition by starting with a discussion of the Maxwell-Boltzmann energy distribution and then deriving the speed and velocity distributions from the energy distribution. The discussion of Doppler broadening of spectral lines has been moved from a worked example to a full discussion. In the presentation of the applications of Bose-Einstein statistics, the discussion of the Einstein specific heat has been moved to Chapter 11 and a new section on Bose-Einstein condensation has been added. Applications of Fermi-Dirac statistics now include white dwarf stars and mixtures of $^3$He and $^4$He.

## Supplemental Materials

Chapter 15 of *Physlet Quantum Physics* has several animations and exercises that illustrate quantum statistical distributions. The statistical and thermal physics package in the ComPADRE digital library (http://www.compadre.org/stp/items/detail.cfm?ID=7308) also has animations that may be instructive for students.

## Suggestions for Additional Reading

An excellent introduction to statistical physics, including examples of arranging and sorting problems similar to the one in Section 10.2:
D. McLachlan Jr., *Statistical Mechanical Analogies* (Prentice-Hall, 1968).

More extensive treatments of statistical physics, at about the same mathematical level as this chapter:
F. Reif, *Statistical Physics,* Volume 5 of the Berkeley Physics Series (McGraw-Hill, 1967).
W. G. V. Rosser, *An Introduction to Statistical Physics* (Ellis Horwood, 1982).
D. V. Schroeder, *An Introduction to Thermal Physics* (Addison Wesley Longman, 2000).
M. D. Sturge, *Statistical and Thermal Physics: Fundamentals and Applications* (A. K. Peters, 2003).

Advanced texts, with many applications of classical and quantum statistics:
C. Kittel, *Thermal Physics* (Wiley, 1969).
P. M. Morse, *Thermal Physics* (Benjamin, 1965).
R. D. Reed and R. R. Roy, *Statistical Physics for Students of Science and Engineering* (Intext, 1971).

The properties of liquid helium are discussed in a very general and highly readable book:
K. Mendelssohn, *The Quest for Absolute Zero* (McGraw-Hill, 1966).

For additional work on the properties of $^3$He, see:

N. D. Mermin and D. M. Lee, "Superfluid Helium 3," *Scientific American* **235**, 56 (December 1976).

## Materials for Active Engagement in the Classroom

### A. Reading Quizzes

1.  Which distribution function provides the best description of the behavior of electrons in a metal?
    (1) Maxwell-Boltzmann  (2) Bose-Einstein  (3) Fermi-Dirac

2.  Which statistical distribution might describe the energy distribution of photons from the Sun?
    (1) Maxwell-Boltzmann  (2) Bose-Einstein  (3) Fermi-Dirac

3.  The density of states gives information about:
    (1) the number of filled states at a particular energy
    (2) the number of states available (either filled or empty) at a particular energy
    (3) the probability for a particular state to be occupied
    (4) the type of distribution function that describes the particles

4.  The unusual properties of liquid helium occur because:
    (1) there is a strong attractive force between the helium atoms
    (2) the liquid exists only at a particularly low temperature
    (3) two helium atoms in the liquid can combine to form a molecule
    (4) the helium atoms can occupy the same quantum state

5.  In a metal the Fermi energy describes
    (1) the highest occupied energy state of a free electron at zero temperature
    (2) the minimum energy necessary to remove an electron from the metal
    (3) the mean thermal energy of the atoms at temperature $T$
    (4) the energy necessary to break the bonds between the metal atoms

**Answers**      1. 3     2. 2     3. 2     4. 4     5. 1

### B. Conceptual or Discussion Questions

1.  Consider the states of the combined total spin of two particles, each of which has spin 5/2 with $m_s = +5/2, +3/2, +1/2, -1/2, -3/2,$ and $-5/2$.
    (a) How many macrostates are there corresponding to the different values of the total spin if the particles are distinguishable?
    (1) 2    (2) 5   (3) 6   (4) 12   (5) 36

(b) If the two particles are distinguishable, what is the total number of microstates for all the allowed macrostates?

    (1) 6   (2) 12   (3) 18   (4) 24   (5) 36

(c) If the two particles are indistinguishable, what is the total number of microstates for all the allowed macrostates?

    (1) 6   (2) 12   (3) 15   (4) 21   (5) 36

(d) What are the allowed values for the total spin if the two particles are indistinguishable?

    (1) 0,1,2,3,4,5     (2) 1,3,5        (3) 0,5        (4) 5        (5) 0

2.  Consider a collection of 4 identical atoms obeying the rules of quantum mechanics. The atoms can occupy a set of equally spaced energy levels:

$$\text{————————————— 8 eV}$$

$$\text{————————————— 6 eV}$$

$$\text{————————————— 4 eV}$$

$$\text{————————————— 2 eV}$$
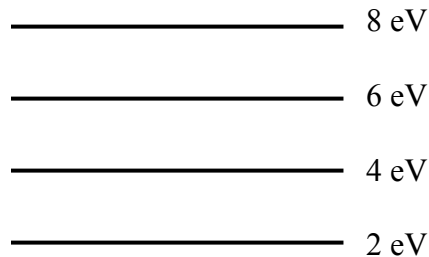
At a temperature of $T = 0$ K, what would be the average energy of these 4 atoms if they behaved like:

    (a) spin-1 particles

    (b) spin-1/2 particles

    (c) spin-1/2 particles in a strong magnetic field in which all the electron spins point in the same direction

Give your answer in units of eV rounded to the nearest integer.

**Answers**     1. (a) 3  (b) 5  (c) 4  (d) 2        2. (a) 2  (b) 3  (c) 5


## Sample Exam Questions

### A. Multiple Choice

1.  Protons and neutrons have spin ½, just like the electron. Which function would best describe the statistical energy distribution of protons and neutrons in a nucleus?

    (a) Fermi-Dirac          (b) Maxwell-Boltzmann    (c) Bose-Einstein

    (d) Protons and neutrons do not follow statistical rules in a nucleus.

2.  Which statistical distribution function allows all particles to occupy the same quantum state?

    (a) Maxwell-Boltzmann    (b) Fermi-Dirac    (c) Bose-Einstein

3.  In a "gas" of photons filling a container, the sharing of energy among the photons is best described by which statistical distribution?

(a) Maxwell-Boltzmann       (b) Fermi-Dirac       (c) Bose-Einstein

4.  A gas of atoms can be described with classical rather than quantum statistics only if
    (a) the atoms exert strong forces on one another
    (b) the atoms have no intrinsic spin
    (c) the temperature of the gas is very low
    (d) the average spacing between atoms is larger than their de Broglie wavelength

**Answers**      1. a      2. c      3. c      4. d

## B.  Conceptual

1.  Let $E$ represent the average energy of spin-1/2 electrons in a certain block of metal at a temperature of 0 K.  Now suppose the electrons have spin 1 instead of spin ½.  Would you expect the average energy of the spin $= 1$ electrons in an otherwise identical block of metal at 0 K to be *greater than, the same as, or less than* the energy $E$?  EXPLAIN YOUR ANSWER.

2.

**Answers**      1. less than

## C.  Problems

1.  (a) Consider a one-dimensional "lattice" of electrons in which a total of $N$ electrons occupy a length $L$.   Derive the expression $g(E)dE = \sqrt{2m/Eh^2}\,dE$ for the density of states for this system.
    (b) Assuming these particles to be described by the Fermi-Dirac distribution, find the total number of particles at $T = 0$ and derive an expression for the Fermi energy of the electrons.

2.  Consider an atom at the surface of the Sun, where the temperature is 6000 K.  The atom can exist in only 2 states.  The ground state is an $s$ state and the excited state at 1.25 eV is a $p$ state.  What is the probability to find the atom in the excited state?

**Answers**      1. (b) $E_F = (h^2/2m)(N/2L)^2$          2. 21%

**Problem Solutions**

1.  (a) There are 3 macrostates: *A,* in which one particle has 3 units of energy; *B,* in which one particle has 2 units of energy; and *C*, in which each particle has 1 unit.
    (b) In macrostate *A*, there are 3 different ways to choose which particle gets the 3 units. We can represent the distribution of energy to the three particles in these microstates as 300, 030, and 003. For macrostate *B*, the microstates are 210, 201, 021, 120, 012, and 102 for a total of 6. For macrostate *C*, there is only one microstate, 111.
    (c) There is a total of 10 microstates. Macrostate *A* has 3 of the 10 microstates, so there is a 30% probability of finding the system in macrostate *A*. None of the particles in macrostate *A* has 2 units of energy. There are 6 microstates in macrostate *B*, so there is a 60% probability of finding the system in *B*. Only one of the 3 particles in state *B* has 2 units of energy, so the probability is $1/3 \times 60\% = 20\%$. None of the particles in macrostate *C* has 2 units of energy, so the overall probability of finding a particle with 2 units of energy is 20%.
    There is a 2/3 chance of finding a particle with energy 0 in macrostate *A* and a 1/3 chance in macrostate *B*, so the net probability of finding a particle with energy 0 is $2/3 \times 30\% + 1/3 \times 60\% = 40\%$.

2.  (a) The possible macrostates (H = heads, T = tails) are 5H0T, 4H1T, 3H2T, 2H3T, 1H4T, and 0H5T, for a total of 6 macrostates.
    (b) Each toss has 2 possible outcomes, so the total number of possible outcomes is $2^5 = 32$ microstates.
    (c) 5H0T – 1 microstate        4H1T – 5 microstates        3H2T – 10 microstates
        0H5T – 1 microstate        1H4T – 5 microstates        2H3T – 10 microstates
    Total = 32 microstates

3.  (a) There are 3 different $z$ components ($+\hbar, 0, -\hbar$) for $s = 1$ and 2 different $z$ components ($+\hbar/2, -\hbar/2$) for $s = 1/2$, so the total number of possible combinations is $3 \times 2 = 6$.
    (b) Two particles of spins 1 and 1/2 can combine to give a total spin of either 3/2 or 1/2. So there are 2 macrostates of the total spin.
    (c) For total spin 3/2 there are 4 microstates ($z$ components $+3\hbar/2, +\hbar/2, -\hbar/2, -3\hbar/2$). For total spin 1/2 there are 2 microstates ($+\hbar/2, -\hbar/2$). The total number of microstates is $4 + 2 = 6$.

4.  $$p(0) = \frac{4\times5+3\times20+3\times20+2\times30+3\times10+2\times60+2\times10+1\times20+1\times30+0\times5}{5\times210} = 0.400$$

    $$p(3) = \frac{0\times5+0\times20+0\times20+0\times30+2\times10+1\times60+0\times10+1\times20+0\times30+0\times5}{5\times210} = 0.095$$

    $$p(5) = \frac{0\times5+1\times20+0\times20+0\times30+0\times10+0\times60+0\times10+0\times20+0\times30+0\times5}{5\times210} = 0.019$$

5. (a) Integral spin:

$$p(0) = \frac{4\times1+3\times1+3\times1+2\times1+2\times1+3\times1+2\times1+1\times1+1\times1+0\times1}{5\times10} = 0.420$$

$$p(3) = \frac{0\times1+0\times1+0\times1+0\times1+1\times1+2\times1+0\times1+1\times1+0\times1+0\times1}{5\times10} = 0.080$$

(b) Spin 1/2:

$$p(0) = \frac{2\times1+2\times1+1\times1}{5\times3} = 0.333$$

$$p(3) = \frac{0\times1+1\times1+0\times1}{5\times3} = 0.067$$

6. (a) The expected total number of microstates, using Equation 10.2, is 11!/8!3! = 165. Ignoring factors of 0! and 1!, which are equal to 1, we have

| Macrostate | Number of Microstates | | |
| --- | --- | --- | --- |
| | Distinguishable | Indistinguishable, integral spin | Indistinguishable, half-integral spin |
| 8000 | 4!/3! = 4 | 1 | 0 |
| 7100 | 4!/2! = 12 | 1 | 1 |
| 6200 | 4!/2! = 12 | 1 | 1 |
| 6110 | 4!/2! = 12 | 1 | 1 |
| 5300 | 4!/2! = 12 | 1 | 1 |
| 5210 | 4! = 24 | 1 | 1 |
| 5111 | 4!/3! = 4 | 1 | 0 |
| 4400 | 4!/2!2! = 6 | 1 | 1 |
| 4310 | 4! = 24 | 1 | 1 |
| 4220 | 4!/2! = 12 | 1 | 1 |
| 4211 | 4!/2! = 12 | 1 | 1 |
| 3320 | 4!/2! = 12 | 1 | 1 |
| 3311 | 4!/2!2! = 6 | 1 | 1 |
| 3221 | 4!/2! = 12 | 1 | 1 |
| 2222 | 4!/4! = 1 | 1 | 0 |
| Total | 165 | 15 | 12 |

(b) For distinguishable particles, including the 15 macrostates in order

$$p(2) = \frac{0+0+1\times12+0+0+1\times24+0+0+0+2\times12+1\times12+1\times12+0+2\times12+4\times1}{4\times165} = 0.170$$

For indistinguishable particles with integral spin,

$$p(2) = \frac{0+0+1\times1+0+0+1\times1+0+0+0+2\times1+1\times1+1\times1+0+2\times1+4\times1}{4\times15} = 0.200$$

and for indistinguishable particles with half-integral spin,

$$p(2) = \frac{0+1\times1+0+0+1\times1+0+0+2\times1+1\times1+1\times1+0+2\times1}{4\times12} = 0.167$$

7. (a) The $z$ components of the spin of each of the particles are $s_z = m_s\hbar$ with $m_s = +3/2$, $+1/2$, $-1/2$, $-3/2$. The $z$ component of the combined spin $S$ is $S_z = M_S\hbar$, with $M_S = m_{s1} + m_{s2}$. There are 4 different possible $m_s$ for each particle, so the total number of combinations is $4 \times 4 = 16$.

| $M_S$ | $m_{s1},m_{s2}$ | Multiplicity | |
|---|---|---|---|
| | | Distinguishable | Indistinguishable |
| +3 | +3/2,+3/2 | 1 | 1 |
| +2 | +3/2,+1/2; +1/2,+3/2 | 2 | 1 |
| +1 | +3/2,−1/2; +1/2,+1/2; −1/2,+3/2 | 3 | 2 |
| 0 | +3/2,−3/2; −3/2,+3/2; +1/2,−1/2; −1/2,+1/2 | 4 | 2 |
| −1 | −3/2,+1/2; −1/2,−1/2; +1/2,−3/2 | 3 | 2 |
| −2 | −3/2,−1/2; −1/2,−3/2 | 2 | 1 |
| −3 | −3/2,−3/2 | 1 | 1 |

Note that the total multiplicity is 16, as expected.
(b) The possible values of the total spin $S$ are 3, 2, 1, and 0. For $S = 3$, we can have $M_S = +3, +2, +1, 0, -1, -2, -3$ for a multiplicity of 7. For $S = 2$, $M_S = +2, +1, 0, -1, -2$ for a multiplicity of 5. For $S = 1$, $M_S = +1, 0, -1$ with multiplicity 3, and for $S = 0$, $M_S = 0$. The total multiplicity of these states is $7 + 5 + 3 + 1 = 16$.
(c) Because the particles are indistinguishable, we cannot count the arrangements obtained by interchanging $m_{s1}$ and $m_{s2}$ as separate microstates, for example +3/2,+1/2 and +1/2,+3/2. The multiplicities are then reduced to 1 for the states with $M_S = +3, +2, -2, -3$ and to 2 for states with $M_S = +1, 0, -1$. The total multiplicity is now 10.
(d) Because we have states with $M_S = \pm3$, there must be a state with total spin $S = 3$. This state must have $M_S = +3, +2, +1, 0, -1, -2, -3$. That uses up 7 of the 10 permitted states. The 3 remaining states have $M_S = +1, 0, -1$, which must be $S = 1$.

8. From Equation 10.15,

$$g(E)dE = \frac{8\pi}{(hc)^3}E^2\,dE = \frac{8\pi(10^9 \text{ nm/m})^3}{(1240 \text{ eV}\cdot\text{nm})^3}(2\times10^{-4} \text{ eV})^2(10^{-5} \text{ eV}) = 5\times10^6 \text{ m}^{-3} = 5 \text{ cm}^{-3}$$

This is an extremely small number of available states, which suggests that the actual density of these photons must be very small.

9.  In two dimensions, the energy of a particle in a square region is $E = (h^2 / 8mL^2)(n_x^2 + n_y^2)$, with $n^2 = n_x^2 + n_y^2$. If we imagine a two-dimensional $n_x n_y$ coordinate system, the points with a given value of $E$ lie on a circle of radius $n$, and the points with energies between $E$ and $E + dE$ lie on a circular ring of radius $n$ and thickness $dn$, and thus of area $2\pi n\, dn$. Because the segment of the ring with positive values of $n_x$ and $n_y$ occupies only ¼ of its area, the density of available states for particles of spin $s$ is

$$g(n)\, dn = \frac{1}{4}\frac{2s+1}{L^2}2\pi n\, dn$$

which gives the number of states per unit area. With $E = h^2 n^2 / 8mL^2$, we get $dE = (h^2 / 8mL^2)2n\, dn$ and so

$$g(E)\, dE = g(n)\, dn = \frac{1}{4}\frac{2s+1}{L^2}\frac{\pi\, dE}{h^2 / 8mL^2} = \frac{2\pi(2s+1)m}{h^2}dE$$

Note that $g(E)$ for this case is independent of $E$.

10.  The energy of the photons is $E = c\sqrt{p_x^2 + p_y^2} = (\hbar c\pi / L)\sqrt{n_x^2 + n_y^2}$. In two dimensions, we can represent the possible energies of the photons as points in a $n_x n_y$ coordinate system. The points with energy $E$ lie on a circle of radius $n = \sqrt{n_x^2 + n_y^2}$, and the number of photons with energies between $E$ and $E + dE$ is determined by the area of the ring between radii $n$ and $n + dn$, which is $2\pi n\, dn$. Because $n_x$ and $n_y$ must be positive, only ¼ of the area of the ring contributes to the density of states. Allowing for the 2 polarization states of the photons, the density of states is

$$g(n)\, dn = \frac{1}{4}\frac{2}{L^2}2\pi n\, dn$$

where the area occupied by the photons is $L^2$. With $E = \hbar cn / L$ and $dE = (\hbar c / L)dn$, we obtain

$$g(E)\, dE = g(n)\, dn = \frac{\pi}{L^2}\left(\frac{L}{\hbar c}\right)^2 E\, dE = \frac{4\pi^3}{(hc)^2}E\, dE$$

11.  With $s = 1/2$ and $mc^2$ 0.511 MeV for electrons, we obtain

$$g(E)dE = \frac{4\pi(2s+1)\sqrt{2}(mc^2)^{3/2}}{(hc)^3}E^{1/2}dE$$

$$= \frac{8\pi\sqrt{2}(0.511\times10^6 \text{ eV})^{3/2}}{(1240 \text{ eV}\cdot\text{nm})^3(1 \text{ m}/10^9 \text{ nm})^3}(0.0252 \text{ eV})^{1/2}(0.000252 \text{ eV}) = 2.7\times10^{23} \text{ m}^{-3}$$

The atomic density of copper metal is $8.5\times10^{28}$ atoms per $\text{m}^3$, so there seems to be a discrepancy between the density of states calculation and a model in which each atom contributes an electron. The error lies in assuming that the electrons behave as if they had thermal energies. In reality, the electrons that participate in electrical conduction have much greater average energies, of the order of a few eV, which is sufficient to provide rough agreement between the number of available energy states and the number of conduction electrons available to occupy those states.

12. With $kT = (8.617\times10^{-5} \text{ eV/K})(960 \text{ K}) = 0.0827 \text{ eV}$, the ratio of the number of particles $N_2$ in the excited state $E_2$ to the number $N_1$ in the ground state $E_1$ is

$$\frac{N_2}{N_1} = \frac{d_2 e^{-E_2/kT}}{d_1 e^{-E_1/kT}} = 3e^{-(E_2-E_1)/kT} = 3e^{-0.25 \text{ eV}/0.0827 \text{ eV}} = 0.146$$

The total number of particles is $N = N_1 + N_2$, so with $(N-N_1)/N_1 = 0.146$ we get

$$N_1 = \frac{1}{1+0.146}N = 0.87N \qquad \text{and} \qquad N_2 = N - N_1 = 0.13N$$

13. The levels are $E_1$, $E_2$, $E_3$ with numbers of particles $N_1$, $N_2$, $N_3$ such that the total number of particles is $N = N_1 + N_2 + N_3$, or $N/N_1 = 1 + N_2/N_1 + N_3/N_1$. With $kT = (8.617\times10^{-5} \text{ eV/K})(650 \text{ K}) = 0.0560 \text{ eV}$, the Maxwell-Boltzmann distribution for nondegenerate states gives

$$\frac{N_2}{N_1} = \frac{e^{-E_2/kT}}{e^{-E_1/kT}} = e^{-(E_2-E_1)/kT} = e^{-0.045 \text{ eV}/0.0560 \text{ eV}} = 0.448$$

$$\frac{N_3}{N_1} = \frac{e^{-E_3/kT}}{e^{-E_1/kT}} = e^{-(E_3-E_1)/kT} = e^{-0.135 \text{ eV}/0.0560 \text{ eV}} = 0.090$$

$$N/N_1 = 1 + N_2/N_1 + N_3/N_1 = 1 + 0.448 + 0.090 = 1.538 \qquad \text{so} \qquad N_1/N = 0.65$$
$$N_2/N = (N_2/N_1)(N_1/N) = (0.448)(0.65) = 0.29$$
$$N_3/N = 1 - N_1/N - N_2/N = 1 - 0.65 - 0.29 = 0.06$$

14. The most probable speed $v_p$ occurs where $N(v)$ has its maximum value, which we find by setting $dN/dv$ to zero:

$$\frac{dN}{dv} = 4\pi N\left(\frac{m}{2\pi kT}\right)^{3/2}\left[(2v)e^{-mv^2/2kT} + v^2(-2mv/2kT)e^{-mv^2/2kT}\right] = 0$$

Cancelling common factors we find $2v - mv^3/kT = 0$, so

$$v_p = \sqrt{\frac{2kT}{m}}$$

15. (a) With $kT = (8.617\times10^{-5}\text{ eV/K})(293\text{ K}) = 0.02525\text{ eV}$, then $E_m = \frac{3}{2}kT = 0.0379\text{ eV}$.

   (b) $dN = N(E_m)dE = \dfrac{2N_A}{\sqrt{\pi}(kT)^{3/2}}E_m^{1/2}e^{-E_m/kT}dE$

   $= \dfrac{2(6.02\times10^{23})}{\sqrt{\pi}(0.02525\text{ eV})^{3/2}}(0.0379\text{ eV})^{1/2}e^{-0.0379\text{ eV}/0.02525\text{ eV}}(0.000379\text{ eV}) = 2.78\times10^{21}$

16. (a) With $v = 505$ m/s and $dv = 10$ m/s, Equation 10.20 gives

$$dN = N(v)dv = N\sqrt{\frac{2}{\pi}}\left(\frac{m}{kT}\right)^{3/2}v^2e^{-mv^2/2kT}dv = (6.02\times10^{23})\sqrt{\frac{2}{\pi}}\left[\frac{(39.95\text{ u})(1.6605\times10^{-27}\text{ kg/u})}{(1.38\times10^{-23}\text{ J/K})(293\text{ K})}\right]^{3/2}$$

$$\times(505\text{ m/s})^2e^{-(39.95\text{ u})(1.6605\times10^{-27}\text{ kg/u})(505\text{ m/s})^2/2(1.38\times10^{-23}\text{ J/K})(293\text{ K})}(10\text{ m/s}) = 1.00\times10^{22}$$

   (b) Now using Equation 10.25 we get

$$dN = N(v_x)dv_x = N\left(\frac{m}{2\pi kT}\right)^{1/2}e^{-mv_x^2/2kT}dv_x = (6.02\times10^{23})\left[\frac{(39.95\text{ u})(1.6605\times10^{-27}\text{ kg/u})}{2\pi(1.38\times10^{-23}\text{ J/K})(293\text{ K})}\right]^{1/2}$$

$$\times e^{-(39.95\text{ u})(1.6605\times10^{-27}\text{ kg/u})(505\text{ m/s})^2/2(1.38\times10^{-23}\text{ J/K})(293\text{ K})}(10\text{ m/s}) = 1.20\times10^{21}$$

   Note that there are about an order of magnitude more atoms in the speed interval at 500 m/s than in the same velocity interval. This is because there are many ways to have a *speed* of 500 m/s that don't involve a *velocity* of 500 m/s in any particular direction. For example, it is just as likely to have an atom with 3 components of 289 m/s (which gives a speed of 500 m/s) as it is to have an atom with one component of 500 m/s and two components of 0.

17. (a) The energy of the $n = 2$ to $n = 1$ transition in hydrogen is 10.2 eV. Multiplying on both sides by Planck's constant $h$, we can turn Equation 10.30 into an equation for the energy linewidth, and with $kT = (8.617\times10^{-5}\text{ eV/K})(5800\text{ K}) = 0.500\text{ eV}$ we obtain

$$\Delta E = 2E_0\sqrt{(2\ln 2)kT/mc^2} = 2(10.2 \text{ eV})\sqrt{(2\ln 2)(0.500 \text{ eV})/(938.8\times 10^6 \text{ eV})} = 554 \ \mu eV$$

(b) The natural linewidth is determined by the uncertainty relationship:

$$\Delta E \sim \frac{\hbar}{\Delta t} = \frac{1.055\times 10^{-34} \text{ J}\cdot\text{s}}{(10^{-8} \text{ s})(1.602\times 10^{-19} \text{ J/eV})} = 0.066 \ \mu eV$$

The natural linewidth is negligible in comparison with the Doppler-broadened linewidth.

18.   (a) At a pressure of 1 atmosphere $= 1.0 \times 10^5$ N/m$^2$, the density of nitrogen gas is

$$\frac{N}{V} = \frac{P}{kT} = \frac{1.0\times 10^5 \text{ N/m}^2}{(1.38\times 10^{-23} \text{ J/K})(293 \text{ K})} = 2.5\times 10^{25} \text{ m}^{-3}$$

With $kT = (8.617\times 10^{-5} \text{ eV/K})(293 \text{ K}) = 0.0252 \text{ eV}$, the ratio in Equation 10.36 is

$$\frac{\lambda}{d} = \frac{hc(N/V)^{1/3}}{\sqrt{2mc^2 kT}} = \frac{(1240 \text{ eV}\cdot\text{nm})(2.5\times 10^{25} \text{ m}^{-3})^{1/3}(10^{-9} \text{ m/nm})}{\sqrt{2(28 \text{ u})(931.5\times 10^6 \text{ eV/u})(0.0252 \text{ eV})}} = 0.010$$

Thus Equation 10.36 is easily satisfied in this case.
(b) The density of water is $1.0 \times 10^3$ kg/m$^3$, which is equivalent to

$$\frac{N}{V} = (1.0\times 10^3 \text{ kg/m}^3)\left(\frac{1 \text{ molecule}}{18 \text{ u}}\right)\left(\frac{1 \text{ u}}{1.67\times 10^{-27} \text{ kg}}\right) = 3.3\times 10^{28} \text{ m}^{-3}$$

With $kT = (8.617\times 10^{-5} \text{ eV/K})(293 \text{ K}) = 0.0252 \text{ eV}$, the ratio in Equation 10.36 is

$$\frac{\lambda}{d} = \frac{hc(N/V)^{1/3}}{\sqrt{2mc^2 kT}} = \frac{(1240 \text{ eV}\cdot\text{nm})(3.3\times 10^{28} \text{ m}^{-3})^{1/3}(10^{-9} \text{ m/nm})}{\sqrt{2(18 \text{ u})(931.5\times 10^6 \text{ eV/u})(0.0252 \text{ eV})}} = 0.14$$

This ratio does not appear to be small enough to allow us to neglect quantum effects.
 (c) The density of liquid helium at 4 K is 0.125 kg/L $= 0.125 \times 10^3$ kg/m$^3$, so

$$\frac{N}{V} = (0.125\times 10^3 \text{ kg/m}^3)\left(\frac{1 \text{ atom}}{4 \text{ u}}\right)\left(\frac{1 \text{ u}}{1.67\times 10^{-27} \text{ kg}}\right) = 1.9\times 10^{28} \text{ m}^{-3}$$

With $kT = (8.617\times 10^{-5} \text{ eV/K})(4 \text{ K}) = 3.45\times 10^{-4} \text{ eV}$, the ratio in Equation 10.36 is

$$\frac{\lambda}{d} = \frac{hc(N/V)^{1/3}}{\sqrt{2mc^2kT}} = \frac{(1240 \text{ eV} \cdot \text{nm})(1.9 \times 10^{28} \text{ m}^{-3})^{1/3}(10^{-9} \text{ m/nm})}{\sqrt{2(4 \text{ u})(931.5 \times 10^6 \text{ eV/u})(3.45 \times 10^{-4} \text{ eV})}} = 2.1$$

Clearly it is not correct to use Maxwell-Boltzmann statistics in this case.
(d) The density of solid copper is 8.95 g/cm$^3$ and its molar mass is 63.5 g/mole, so

$$\frac{N}{V} = \frac{\rho N_A}{M} = \frac{(8.95 \text{ g/cm}^3)(6.02 \times 10^{23} \text{ atoms/mole})}{(63.5 \text{ g/mole})(10^{-6} \text{ m}^3/\text{cm}^3)} = 8.5 \times 10^{28} \text{ atoms/m}^3$$

With one conduction electron per atom, the density of conduction electrons has this same value. The ratio in Equation 10.36 is

$$\frac{\lambda}{d} = \frac{hc(N/V)^{1/3}}{\sqrt{2mc^2kT}} = \frac{(1240 \text{ eV} \cdot \text{nm})(8.5 \times 10^{28} \text{ m}^{-3})^{1/3}(10^{-9} \text{ m/nm})}{\sqrt{2(0.511 \times 10^6 \text{ eV})(0.0252 \text{ eV})}} = 34$$

It would not be appropriate to use Maxwell-Boltzmann statistics for this case, nor is it correct to estimate the kinetic energy of the conduction electrons as $kT$.

19. (a) Let's assume that Maxwell-Boltzmann statistics will fail when the ratio in Equation 10.36 becomes larger than 0.1. When thus occurs, $hc(N/V)^{1/3} = 0.1\sqrt{2mc^2kT}$, or with $kT = (8.617 \times 10^{-5} \text{ eV/K})(293 \text{ K}) = 0.0252 \text{ eV}$,

$$\frac{N}{V} = \left(\frac{0.1\sqrt{2mc^2kT}}{hc}\right)^3 = \left(\frac{0.1\sqrt{2(28 \text{ u})(931.5 \times 10^6 \text{ eV/u})(0.0252 \text{ eV})}}{(1240 \text{ eV} \cdot \text{nm})(10^{-9} \text{ m/nm})}\right)^3 = 2.5 \times 10^{28} \text{ m}^{-3}$$

Using the ideal gas law

$$P = (N/V)kT = (2.5 \times 10^{28} \text{ m}^{-3})(1.38 \times 10^{-23} \text{ J/K})(293 \text{ K}) = 1.01 \times 10^8 \text{ N/m}^2 = 10^3 \text{ atm}$$

At pressures below this extremely large value, we may safely use Maxwell-Boltzmann statistics for nitrogen gas.
(b) At a pressure of 1 atmosphere = $1.0 \times 10^5$ N/m$^2$, the density of nitrogen gas is

$$\frac{N}{V} = \frac{P}{kT} = \frac{1.0 \times 10^5 \text{ N/m}^2}{(1.38 \times 10^{-23} \text{ J/K})(293 \text{ K})} = 2.5 \times 10^{25} \text{ m}^{-3}$$

Again choosing 0.1 as the critical value of the ratio in Equation 10.36, we solve for $T$:

$$T = \frac{(hc)^2(N/V)^{2/3}}{0.01(2mc^2k)} = \frac{(1240 \text{ eV} \cdot \text{nm})^2(2.5 \times 10^{25} \text{ m}^{-3})^{2/3}(10^{-9} \text{ m/nm})^2}{(0.01)(2)(28 \text{ u})(931.5 \times 10^6 \text{ eV/u})(8.617 \times 10^{-5} \text{ eV/K})} = 2.9 \text{ K}$$

Nitrogen becomes a liquid at 77 K and a solid at 63 K, so clearly we are safe in using Maxwell-Boltzmann statistics over most of the range in which nitrogen is a gas.

20.  (a) Integrating Equation 10.38 over all photon energies, we obtain

$$N = \int dN = \int_0^\infty N(E)dE = \frac{8\pi V}{(hc)^3}\int_0^\infty \frac{E^2 dE}{e^{E/kT}-1}$$

With $x = E/kT$ and $dx = dE/kT$, we obtain $\dfrac{N}{V} = \dfrac{8\pi(kT)^3}{(hc)^3}\displaystyle\int_0^\infty \frac{x^2 dx}{e^x -1}$.

(b) At $T = 300$ K, $kT = (8.617\times10^{-5}\text{ eV})(300\text{ K}) = 0.02585$ eV, so

$$\frac{N}{V} = \frac{8\pi(0.02585\text{ eV})^3}{(1240\text{ eV}\cdot\text{nm})^3}(10^7\text{ nm/cm})^3(2.404) = 5.5\times10^8\text{ cm}^{-3}$$

At 3 K the number is reduced by $(100)^3 = 10^6$, so the number is 550 cm$^{-3}$.

21.  (a) The total energy density is given by Equation 10.41:

$$U = \frac{8\pi^5 k^4}{15(hc)^3}T^4 = \frac{8\pi^5(8.617\times10^{-5}\text{ eV/K})^4}{15(1240\text{ eV}\cdot\text{nm})^3(10^{-9}\text{ m/nm})^3}(2.50\times10^3\text{ K})^4 = 1.84\times10^7\text{ eV/m}^3$$

(b) At $T = 2.5 \times 10^3$ K, $kT = (8.617\times10^{-5}\text{ eV/K})(2.5\times10^3\text{ K}) = 0.215$ eV. The energy emitted in the interval $dE = 0.05$ eV at $E = 1.00$ eV is

$$u(E)\,dE = \frac{8\pi}{(hc)^3}\frac{E^3}{e^{E/kT}-1}dE = \frac{8\pi(10^9\text{ nm/m})^3}{(1240\text{ eV}\cdot\text{nm})^3}\frac{(1.00\text{ eV})^3}{e^{1.00\text{ eV}/0.215\text{ eV}}-1}(0.05\text{ eV}) = 6.36\times10^{15}\text{ eV/m}^3$$

The fraction of the total energy that this represents is

$$\frac{u(E)\,dE}{U} = \frac{6.36\times10^{15}\text{ eV/m}^3}{1.84\times10^{17}\text{ eV/m}^3} = 0.035$$

(c) At 10.00 eV,

$$u(E)\,dE = \frac{8\pi}{(hc)^3}\frac{E^3}{e^{E/kT}-1}dE = \frac{8\pi(10^9\text{ nm/m})^3}{(1240\text{ eV}\cdot\text{nm})^3}\frac{(10.00\text{ eV})^3}{e^{10.00\text{ eV}/0.215\text{ eV}}-1}(0.05\text{ eV}) = 4.16\text{ eV/m}^3$$

$$\frac{u(E)\,dE}{U} = \frac{4.16\text{ eV/m}^3}{1.84\times10^{17}\text{ eV/m}^3} = 2.3\times10^{-17}$$

22.
$$u(E) = \frac{8\pi E^3}{(hc)^3} \frac{1}{e^{E/kT} - 1}$$

$$\frac{du}{dE} = \frac{8\pi}{(hc)^3}\left[ \frac{3E^2}{e^{E/kT}-1} - \frac{E^3}{(e^{E/kT}-1)^2}\left(\frac{e^{E/kT}}{kT}\right)\right] = 0 \quad \text{for} \quad E = E_{max}$$

$$3(e^{E_{max}/kT} - 1) = (E_{max}/kT)e^{E_{max}/kT}$$

This equation cannot be solved exactly, but an approximate solution can be found numerically for $E_{max}/kT = 2.8214$ :

$$E_{max} = (2.8214)(8.617\times10^{-5} \text{ eV/K})T = (2.4313\times10^{-4} \text{ eV/K})T$$

Note that, using Equation 3.28,

$$\frac{hc}{\lambda_{max}} = \frac{1240 \text{ eV}\cdot\text{nm}}{(2.8978\times10^{-3} \text{ m}\cdot\text{K})T} = (4.2784\times10^{-4} \text{ eV/K})T$$

and thus $E_{max} \neq hc/\lambda_{max}$. This occurs because $u(E)$ and $u(\lambda)$ do not simultaneously reach their maximum values. That is, when $du/dE = 0$, $du/d\lambda \neq 0$.

23. We assume each copper atom contributes one free electron to the metal. The density of copper atoms (and therefore of the free electrons) is

$$\frac{N}{V} = \frac{\rho N_A}{M} = \frac{(8.95 \text{ g/cm}^3)(6.02\times10^{23} \text{ atoms/mole})}{(63.5 \text{ g/mole})(1 \text{ m}/100 \text{ cm})^3} = 8.48\times10^{28} \text{ m}^{-3}$$

$$E_F = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3} = \frac{(hc)^2}{2mc^2}\left(\frac{3}{8\pi}\frac{N}{V}\right)^{2/3} = \frac{(1240 \text{ eV}\cdot\text{nm})^2}{2(0.511\times10^6 \text{ eV})}\left(\frac{3(8.48\times10^{28} \text{ m}^{-3})}{8\pi(10^9 \text{ nm/m})^3}\right)^{2/3} = 7.04 \text{ eV}$$

$$E_m = \tfrac{3}{5}E_F = \tfrac{3}{5}(7.04 \text{ eV}) = 4.22 \text{ eV}$$

24.
$$\frac{N}{V} = (2 \text{ electrons/atom})\frac{\rho N_A}{M}$$

$$= (2/\text{atom})\frac{(1.74 \text{ g/cm}^3)(6.02\times10^{23} \text{ atoms/mole})}{(24.3 \text{ g/mole})(1 \text{ m}/100 \text{ cm})^3} = 8.62\times10^{28} \text{ n}$$

$$E_F = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3} = \frac{(hc)^2}{2mc^2}\left(\frac{3}{8\pi}\frac{N}{V}\right)^{2/3} = \frac{(1240 \text{ eV}\cdot\text{nm})^2}{2(0.511\times10^6 \text{ eV})}\left(\frac{3(8.62\times10^{28} \text{ m}^{-3})}{8\pi(10^9 \text{ nm/m})^3}\right)^{2/3} = 7.12 \text{ eV}$$

25. (a) At 295 K, $kT = (8.617 \times 10^{-5} \text{ eV/K})(295 \text{ K}) = 0.0254 \text{ eV}$.

$$\frac{N(E)dE}{V} = \frac{8\pi\sqrt{2}m^{3/2}}{h^3} E^{1/2} \frac{1}{e^{(E-E_F)/kT}+1} dE = \frac{8\pi\sqrt{2}(mc^2)^{3/2}}{(hc)^3} \frac{E^{1/2}dE}{e^{(E-E_F)/kT}+1}$$

$$= \frac{8\pi\sqrt{2}(0.511\times10^6 \text{ eV})^{3/2}}{(1240 \text{ eV}\cdot\text{nm})^3(10^{-9} \text{ m/nm})^3} \frac{(5.00 \text{ eV})^{1/2}(0.10 \text{ eV})}{e^{(5.00 \text{ eV}-3.00 \text{ eV})/(0.0254 \text{ eV})}+1} = 4.37\times10^{-36} \text{ m}^{-3}$$

(b) At 2500 K, $kT = (8.617 \times 10^{-5} \text{ eV/K})(2500 \text{ K}) = 0.215 \text{ eV}$.

$$\frac{N(E)dE}{V} = \frac{8\pi\sqrt{2}m^{3/2}}{h^3} E^{1/2} \frac{1}{e^{(E-E_F)/kT}+1} dE = \frac{8\pi\sqrt{2}(mc^2)^{3/2}}{(hc)^3} \frac{E^{1/2}dE}{e^{(E-E_F)/kT}+1}$$

$$= \frac{8\pi\sqrt{2}(0.511\times10^6 \text{ eV})^{3/2}}{(1240 \text{ eV}\cdot\text{nm})^3(10^{-9} \text{ m/nm})^3} \frac{(5.00 \text{ eV})^{1/2}(0.10 \text{ eV})}{e^{(5.00 \text{ eV}-3.00 \text{ eV})/(0.215 \text{ eV})}+1} = 6.00\times10^{-6} \text{ m}^{-3}$$

26. $E_m = \dfrac{1}{N}\displaystyle\int_0^\infty E\,N(E)\,dE = \dfrac{1}{N}V\dfrac{8\pi\sqrt{2}m^{3/2}}{h^3}\int_0^\infty E^{3/2}\dfrac{1}{e^{(E-E_F)/kT}+1}dE$

$$= \frac{1}{N}V\frac{8\pi\sqrt{2}m^{3/2}}{h^3}\int_0^{E_F} E^{3/2}dE = \frac{1}{N}V\frac{8\pi\sqrt{2}m^{3/2}}{h^3}\frac{2}{5}E_F^{5/2} = \frac{1}{N}V\frac{8\pi\sqrt{2}m^{3/2}}{h^3}\frac{2}{5}\left(\frac{h^2}{2m}\right)^{5/2}\left(\frac{3N}{8\pi V}\right)^{5/3}$$

$$= \frac{3}{5}\frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3} = \frac{3}{5}E_F$$

27. (a) Repeating the derivation of Section 10.7, we have, for a star of mass $M = Nm_n$ (where $m_n$ is the mass of a neutron)

$$E_{\text{grav}} = -\frac{3}{5}\frac{GM^2}{R} = -\frac{3}{5}\frac{GN^2m_n^2}{R}$$

and

$$E_{\text{neut}} = NE_m = \frac{3}{5}NE_F = \frac{3}{5}N\frac{h^2}{2m_n}\left(\frac{3N}{8\pi V}\right)^{2/3} = \frac{3}{5}N\frac{h^2}{2m_n}\left(\frac{3N}{8\pi\frac{4}{3}\pi R^3}\right)^{2/3} = \frac{3}{5}N\frac{h^2}{2m_n}\frac{1}{R^2}\left(\frac{9N}{32\pi^2}\right)^{2/3}$$

$$E = E_{\text{grav}} + E_{\text{neut}} = -\frac{3}{5}\frac{GN^2m_n^2}{R} + \frac{3}{5}N\frac{h^2}{2m_n}\frac{1}{R^2}\left(\frac{9N}{32\pi^2}\right)^{2/3}$$

The minimum radius is found by setting $dE/dR$ to zero:

$$\frac{dE}{dR} = \frac{3}{5}\frac{GN^2m_n^2}{R^2} + \frac{3}{5}N\frac{h^2}{2m_n}\left(\frac{-2}{R^3}\right)\left(\frac{9N}{32\pi^2}\right)^{2/3} = 0$$

Solving, we obtain

$$R = \frac{h^2}{GN^{1/3}m_n^3}\left(\frac{9}{32\pi^2}\right)^{2/3}$$

(b) For $M = 3M_{\text{Sun}} = 3(2.00\times10^{30}\text{ kg}) = 6.00\times10^{30}\text{ kg}$, we have
$N = M/m_n = 6.00\times10^{30}\text{kg}/1.67\times10^{-27}\text{ kg} = 3.6\times10^{57}$. Then

$$R = \frac{h^2}{GN^{1/3}m_n^3}\left(\frac{9}{32\pi^2}\right)^{2/3} = \frac{(6.626\times10^{-34}\text{ J}\cdot\text{s})^2}{(6.67\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(3.6\times10^{57})^{1/3}(1.67\times10^{-27}\text{ kg})^3}\left(\frac{9}{32\pi^2}\right)^{2/3}$$

which evaluates to 8.6 km.
(c) The density is

$$\rho = \frac{M}{V} = \frac{6.00\times10^{30}\text{ kg}}{\frac{4}{3}\pi(8600\text{ m})^3} = 2.2\times10^{18}\text{ kg/m}^3$$

This unimaginably large density is comparable to the central density of an atomic nucleus!

28. (a) For $M = 2M_{\text{Sun}} = 2(2.00\times10^{30}\text{ kg}) = 4.00\times10^{30}\text{ kg}$, the number of neutrons would be $N = M/m_n = 4.00\times10^{30}\text{kg}/1.67\times10^{-27}\text{ kg} = 2.4\times10^{57}$. The Fermi energy is

$$E_F = \frac{h^2}{2m_n}\left(\frac{3N}{8\pi V}\right)^{2/3} = \frac{6.626\times10^{-34}\text{ J}\cdot\text{s}}{2(1.67\times10^{-27}\text{ kg})}\left(\frac{3(2.4\times10^{57})}{8\pi\frac{4}{3}\pi(1\times10^4)^3}\right)^{2/3} = 140\text{ MeV}$$

This is smaller than the rest energy of the neutron (940 MeV), but not negligibly smaller. We probably don't make too large an error by using the nonrelativistic kinetic energy to compute the Fermi energy, but for larger stars relativistic effects should become important.
(b) The de Broglie wavelength is (again assuming nonrelativistic energy and momentum)

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} = \frac{hc}{\sqrt{2mc^2E}} = \frac{1240\text{ MeV}\cdot\text{fm}}{\sqrt{2(940\text{ MeV})(140\text{ MeV})}} = 2.4\text{ fm}$$

The radius of this neutron star would be

$$R = \frac{h^2}{GN^{1/3}m_n^3}\left(\frac{9}{32\pi^2}\right)^{2/3} = \frac{(6.626\times10^{-34}\ \text{J}\cdot\text{s})^2(9/32\pi^2)^{2/3}}{(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(2.4\times10^{57})^{1/3}(1.67\times10^{-27}\ \text{kg})^3} = 9.8\ \text{km}$$

giving a neutron density of $N/V = (2.4\times10^{57})/(\frac{4}{3}\pi)(9.8\times10^3\ \text{m})^3 = 6.1\times10^{44}\ \text{m}^{-3}$ and thus an average particle spacing of $(N/V)^{-1/3} = 1.18\ \text{fm}$. The de Broglie wavelength is much larger than the particle spacing, meaning that neighboring neutrons lie within each other's wave packets and so quantum effects are very important.

29.  The density of the $^3$He atoms is $(0.05)(2.2\times10^{28}\ \text{m}^{-3}) = 1.1\times10^{27}\ \text{m}^{-3}$. With $m = 3.02$ u and an effective mass 2.5 times as large, we have for the Fermi energy

$$E_F = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3} = \frac{h^2c^2}{2mc^2}\left(\frac{3N}{8\pi V}\right)^{2/3}$$

$$= \frac{(1240\ \text{eV}\cdot\text{nm})^2(10^{-9}\ \text{m/nm})^2}{(2)(2.5)(3.02\ \text{u})(931.5\times10^6\ \text{eV/u})}\left(\frac{3}{8\pi}1.1\times10^{27}\ \text{m}^{-3}\right)^{2/3} = 2.83\times10^{-5}\ \text{eV}$$

0.50 mole of $^4$He is $3.0 \times 10^{23}$ atoms, so a 5% solution would have $1.5 \times 10^{22}$ atoms of $^3$He. The heat capacity is then

$$C = \frac{\pi^2 k^2 NT}{2E_F} = \frac{\pi^2(1.38\times10^{-23}\ \text{J/K})^2(1.5\times10^{22})(0.025\ \text{K})}{2(2.83\times10^{-5}\ \text{eV})(1.6\times10^{-19}\ \text{J/eV})} = 0.078\ \text{J/K}$$

30.  The particles with $s = 2$ can have $m_s = +2, +1, 0, -1, -2$. The maximum value of the total $M_S = m_{s1} + m_{s2}$ is +4, for which we can have only $m_{s1}, m_{s2} = +2,+2$. With $M_S = +3$, we can have $m_{s1}, m_{s2} = +2,+1$ (but we cannot list +1,+2 as a separate microstate because the particles are indistinguishable). For $M_S = +2$ we have $m_{s1}, m_{s2} = +2,0$ or $+1,+1$. For $M_S = +1$, $m_{s1}, m_{s2} = +1,0$ or $+2,-1$, and for $M_S = 0$, $m_{s1}, m_{s2} = +2,-2$ or $+1,-1$ or $0,0$. For negative values of $M_S$ we get the same number of microstates as the positive values, with all plus signs becoming minus signs and minus signs becoming plus signs. So instead of the 25 microstates that would characterize distinguishable particles, we have 15 microstates. The 9 states with $M_S = +4,+3,+2,+1,0,-1,-2,-3,-4$ must go with $S = 4$. There are no addition microstates with $M_S = \pm 3$, so we cannot have a state with $S = 3$. The remaining 6 microstates have $M_S = +2,+1,0,0,-1,-2$. Five of them ($+2,+1,0,-1,-2$) can be assigned to go with $S = 2$, and the remaining $M_S = 0$ state must go with $S = 0$.

   This situation commonly arises in nuclear excited states, which can be treated as vibrational states in analogy with molecular vibrations. The vibrations behave as if they have two units of angular momentum. The first excited state has one of these vibrational quanta and thus has 2 units of angular momentum. At about twice the energy of the first excited state there is often a triplet of states with angular momentum 0,2,4.

31. (a) Let $E_1 = 0$, $E_2 = E$. The states are nondegenerate, so $d_1 = 1$ and $d_2 = 1$. If $N_1$ and $N_2$ are respectively the number of atoms with energies $E_1$ and $E_2$, then

$$\frac{N_2}{N_1} = \frac{d_2 f_{MB}(E_2)}{d_1 f_{MB}(E_1)} = e^{-(E_2 - E_1)/kT} = e^{-E/kT}$$

(b) $E_m = \dfrac{N_1 E_1 + N_2 E_2}{N_1 + N_2} = \dfrac{E_1 + (N_2 / N_1)E_2}{1 + N_2 / N_1} = \dfrac{0 + E e^{-E/kT}}{1 + e^{-E/kT}} = \dfrac{E}{e^{E/kT} + 1}$

(c) $E_{total} = N_1 E_2 + N_2 E_2 = N E_m = \dfrac{NE}{e^{E/kT} + 1}$

(d) Assume $N = N_A$ (corresponding to one mole of the atoms)

$$C = \frac{dE_{total}}{dT} = N_A E \frac{d}{dT}\left(\frac{1}{e^{E/kT}+1}\right) = N_A E \frac{-1}{(e^{E/kT}+1)^2}\left(-\frac{E}{kT^2}e^{E/kT}\right) = R\left(\frac{E}{kT}\right)^2 \frac{e^{E/kT}}{(e^{E/kT}+1)^2}$$

32. (a) At height $h$, $E = K + mgh$, so the ratios of the energy distributions would be

$$\frac{N(E_2)}{N(E_1)} = \frac{g(E_2)}{g(E_1)}\frac{e^{-E_2/kT}}{e^{-E_1/kT}} = \frac{e^{-(K+mgh)/kT}}{e^{-K/kT}} = e^{-mgh/kT}$$

(b) The density at height $h$ is proportional to the number of molecules at that height, so we expect

$$\frac{\rho(h)}{\rho(0)} = \frac{N(E_2)}{N(E_1)} \qquad \text{so} \qquad \rho(h) = \rho_0 \frac{N(E_2)}{N(E_1)} = \rho_0 e^{-mgh/kT}$$

(c) This simple model gives a surprisingly good account of the variation of density with altitude, even though our assumption that the gas at all altitudes is in thermal equilibrium at temperature $T$ is not an accurate description of the atmosphere.

33. (a) With $\mu = \mu_B$, we have $\Delta E = \mu_B B = (9.27 \times 10^{-27} \text{ J/T})(5.0 \text{ T}) = 4.64 \times 10^{-23}$ J for the energy splitting of the $m_l$ states. Because the degeneracies of the $m_l$ states are identical, the ratio of the number of atoms $N_{m_l}$ with different values of the energy $E_{m_l}$ is, for $m_l = 0$ and $m_l = -1$,

$$\frac{N_0}{N_{-1}} = \frac{d_0}{d_{-1}}\frac{e^{-E_0/kT}}{e^{-E_{-1}/kT}} = e^{-(E_0 - E_{-1})/kT} = e^{-\Delta E/kT} = e^{-(4.64\times10^{-23} \text{ J})/(1.38\times10^{-23} \text{ J/K})(293 \text{ K})} = 0.9886$$

Because the $m_l = +1$ and $m_l = 0$ states have the same energy splitting $\Delta E$, the ratio of their populations is the same: $N_{+1}/N_0 = 0.9886$. The fraction of the atoms in any particular state is the number in that state divided by the total number:

$$f_{m_l} = \frac{N_{m_l}}{N_{+1} + N_0 + N_{-1}}$$

so we have

$$f_{+1} = \frac{N_{+1}}{N_{+1} + N_0 + N_{-1}} = \frac{N_{+1}/N_0}{N_{+1}/N_0 + 1 + N_{-1}/N_0} = \frac{0.9886}{0.9886 + 1 + 1/0.9886} = 0.3295$$

$$f_0 = \frac{N_0}{N_{+1} + N_0 + N_{-1}} = \frac{1}{N_{+1}/N_0 + 1 + N_{-1}/N_0} = \frac{1}{0.9886 + 1 + 1/0.9886} = 0.3333$$

$$f_{-1} = 1 - f_{+1} - f_0 = 0.3372$$

(b) The three normal Zeeman components represent the $2p \rightarrow 1s$ transitions that begin in the initial levels with $m_l = +1$, 0, and $-1$. The relative intensities would then be proportional to the populations of the initial substates. The lowest energy substate ($m_l = -1$) has the largest population and so has the largest relative intensity (0.3372). The next substate ($m_l = 0$) has the next highest population (0.3333), and so it has a smaller relative intensity, and the transition from the n $m_l = +1$ substate, which lies highest in energy, has the smallest relative intensity (0.3295).

34.  Consider a molecule of the liquid at the location $x$ measured from the axis of rotation. The apparent "centrifugal" force on it is $F = mx\omega^2$. We assume that the forces in the liquid that give rise to this force have an associated potential energy $U(x)$, where $F = -dU/dx$:

$$U = -\int F\,dx = -\int_0^x mx\omega^2\,dx = -\tfrac{1}{2}m\omega^2 x^2$$

where we take $U = 0$ at $x = 0$. The Maxwell-Boltzmann distribution then gives

$$\frac{\rho(x)}{\rho(0)} = \frac{N(x)}{N(0)} = \frac{e^{m\omega^2 x^2/2kT}}{e^0} = e^{m\omega^2 x^2/2kT}$$

and thus $\rho(x) = \rho_0 e^{m\omega^2 x^2/2kT}$ with $\rho_0 = \rho(0)$.

35.  At room temperature (293 K), $kT = (8.617 \times 10^{-5}\text{ eV/K})(293\text{ K}) = 0.02525\text{ eV}$. With $E_F = 3.15\text{ eV}$ for sodium, the distribution function has the value 0.9 where

$$f_{\text{FD}}(E)=\frac{1}{e^{(E-E_F)/kT}+1}=0.9 \quad \text{or} \quad e^{(E-E_F)/kT}=1/9 \quad \text{and} \quad \frac{E-E_F}{kT}=-\ln 9$$

$$E = E_F - kT\ln 9 = 3.15 \text{ eV} - (0.02525 \text{ eV})(2.20) = 3.09 \text{ eV}$$

The distribution function has the value 0.1 where

$$f_{\text{FD}}(E)=\frac{1}{e^{(E-E_F)/kT}+1}=0.1 \quad \text{or} \quad e^{(E-E_F)/kT}=9 \quad \text{and} \quad \frac{E-E_F}{kT}=\ln 9$$

$$E = E_F + kT\ln 9 = 3.15 \text{ eV} + (0.02525 \text{ eV})(2.20) = 3.21 \text{ eV}$$

The energy difference is 3.21 eV – 3.09 eV = 0.12 eV, so the distribution is rather sharp. The distribution changes from 90% full to 10% full within ±2% of the Fermi energy.

36. For $E = 1.89$ eV and $kT = 0.02525$ eV (corresponding to $T = 293$ K), the Fermi-Dirac distribution function is

$$f_{\text{FD}}=\frac{1}{e^{(E-E_F)/kT}+1}=\frac{1}{e^{(1.89 \text{ eV} - 3.15 \text{ eV})/0.02525 \text{ eV}}+1}=1.000$$

$$\frac{dN}{V}=\frac{N(E_m)dE}{V}=\frac{8\pi\sqrt{2}m^{3/2}}{h^3}E_m^{1/2}f_{\text{FD}}(E_m)dE=\frac{8\pi\sqrt{2}(mc^2)^{3/2}}{(hc)^3}E_m^{1/2}f_{\text{FD}}(E_m)dE$$

$$=\frac{8\pi\sqrt{2}(0.511\times10^6 \text{ eV})^{3/2}}{(1240 \text{ eV}\cdot\text{nm})^3(10^{-9} \text{ m/nm})^3}(1.89 \text{ eV})^{1/2}(0.0315 \text{ eV})=2.95\times10^{28} \text{ m}^{-3}$$

37. The volume of the nucleus is $V=\frac{4}{3}\pi R^3=\frac{4}{3}\pi(7.4 \text{ fm})^3=1697 \text{ fm}^3$. The number of protons per unit volume is $N/V=92/1697 \text{ fm}^3=0.0542 \text{ fm}^{-3}$. The Fermi energy of the protons is

$$E_F=\frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}=\frac{(hc)^2}{2mc^2}\left(\frac{3}{8\pi}\frac{N}{V}\right)^{2/3}=\frac{(1240 \text{ MeV}\cdot\text{fm})^2}{2(938.3 \text{ MeV})}\left(\frac{3}{8\pi}0.0542 \text{ fm}^{-3}\right)^{2/3}=28.5 \text{ MeV}$$

$$E_m=\tfrac{3}{5}E_F=\tfrac{3}{5}(28.5 \text{ MeV})=17.1 \text{ MeV}$$

For the neutrons, $N/V=143/1697 \text{ fm}^3=0.0843 \text{ fm}^{-3}$

$$E_F=\frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}=\frac{(hc)^2}{2mc^2}\left(\frac{3}{8\pi}\frac{N}{V}\right)^{2/3}=\frac{(1240 \text{ MeV}\cdot\text{fm})^2}{2(939.6 \text{ MeV})}\left(\frac{3}{8\pi}0.0843 \text{ fm}^{-3}\right)^{2/3}=38.1 \text{ MeV}$$

$$E_{\text{m}} = \tfrac{3}{5} E_{\text{F}} = \tfrac{3}{5}(38.1 \text{ MeV}) = 22.9 \text{ MeV}$$

38. (a) The gravitational potential energy of two uniform spherical objects of masses $m_1$ and $m_2$ is $U = -Gm_1m_2/r$, where $r$ is the distance between their centers. When the separation is infinite, the initial potential energy is zero. When mass $dm$ is brought from an infinite separation to the surface of the spherical mass $m$, the final potential energy is

$$dU = -G\frac{m\,dm}{r} = -G\frac{(\rho\tfrac{4}{3}\pi r^3)dm}{r} = -\tfrac{4}{3}\pi G\rho r^2 dm$$

(b) The total mass brought in from infinity is in the spherical shell of radius $r$, surface area $4\pi r^2$, and volume $dV = 4\pi r^2 dr$. Its mass is $dm = \rho\,dV = 4\pi\rho r^2 dr$, and the corresponding change in potential energy is

$$dU = -\tfrac{4}{3}\pi G\rho r^2 (4\pi\rho r^2 dr) = -\tfrac{16}{3}\pi^2 G\rho^2 r^4 dr$$

(c) To assemble the entire sphere from $r = 0$ to $r = R$, we have

$$U = \int dU = -\frac{16}{3}\pi^2 G\rho^2 \int_0^R r^4 dr = -\frac{16}{3}\pi^2 G\rho^2 \frac{R^5}{5} = -\frac{16}{3}\pi^2 G\left(\frac{M}{\tfrac{4}{3}\pi R^3}\right)^2 \frac{R^5}{5} = -\frac{3}{5}\frac{GM^2}{R}$$

39. (a) When the Sun becomes a white dwarf star, the number of alpha particles in it will be $M_{\text{Sun}}/m_\alpha = 1.99 \times 10^{30} \text{ kg}/6.64 \times 10^{-27} \text{ kg} = 3.00 \times 10^{56}$. The number of electrons is twice this number, so $N = 6.00 \times 10^{56}$. Its radius will be

$$R = \frac{h^2}{GN^{1/3}m_e m_\alpha^2}\left(\frac{9}{4\pi^2}\right)^{2/3}$$

$$= \frac{(6.626 \times 10^{-34} \text{ J}\cdot\text{s})^2}{(6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2)(6.00 \times 10^{56})^{1/3}(9.11 \times 10^{-31} \text{ kg})(6.64 \times 10^{-27} \text{ kg})^2}\left(\frac{9}{4\pi^2}\right)^{2/3}$$

$$= 7.28 \times 10^6 \text{ m}$$

The Fermi energy is

$$E_{\text{F}} = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3} = \frac{(hc)^2}{2mc^2}\frac{1}{R^2}\left(\frac{9N}{32\pi^2}\right)^{2/3}$$

$$= \frac{(1240 \text{ eV}\cdot\text{nm})^2}{2(0.511 \times 10^6 \text{ eV})}\frac{(10^{-9} \text{ m/nm})^2}{(7.28 \times 10^6 \text{ m})^2}\left(\frac{9(6.00 \times 10^{56})}{32\pi^2}\right)^{2/3} = 1.88 \times 10^5 \text{ eV}$$

Using nonrelativistic mechanics, the de Broglie wavelength is

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$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} = \frac{hc}{\sqrt{2mc^2E}} = \frac{1240 \text{ eV}\cdot\text{nm}}{\sqrt{2(0.511\times10^6 \text{ eV})(1.88\times10^5 \text{ eV})}} = 2.83\times10^{-3} \text{ nm}$$

(b) The star contains $6.00 \times 10^{56}$ electrons in a volume $V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi(7.28\times10^6 \text{ m})^3$ $= 1.62 \times 10^{21} \text{ m}^3$, so the particle density is $N/V = 6.00 \times 10^{56}/1.62 \times 10^{21} \text{ m}^3 = 3.70 \times 10^{35}$ $\text{m}^{-3}$. The average distance between particles is $(N/V)^{-1/3} = 1.39\times10^{-3} \text{ nm}$. The de Broglie wavelength is larger than the spacing between the electrons, which means that quantum effects are indeed important.

40. (a) In a magnetic field, a state with spin 5 splits into 11 substates with $S_z = M_S\hbar$ ($M_S =$ $+5, +4, +3, \ldots, -4, -5$). The magnetic energy of the substate $M_S$ is $E_{M_S} = -\vec{\mu}\cdot\vec{B} = \gamma B\vec{S}\cdot\vec{k} = \gamma BS_z = \gamma B\hbar M_S$. The lowest state ($M_S = -5$) has energy $E_{-5} = -5\gamma B\hbar$ and the second lowest state ($M_S = -4$) has energy $E_{-4} = -4\gamma B\hbar$. The ratio of the populations $N_{M_S}$ is (with nondegenerate substates)

$$r = \frac{N_{-4}}{N_{-5}} = \frac{d_{-4}e^{-E_{-4}/kT}}{d_{-5}e^{-E_{-5}/kT}} = \frac{e^{4\gamma B\hbar/kT}}{e^{5\gamma B\hbar/kT}} = e^{-\gamma B\hbar/kT}$$

With $\ln r = -\gamma B\hbar/kT$, we have

$$T = -\frac{\gamma B\hbar}{k\ln r} = -\frac{(3.64\times10^7 \text{ T}^{-1}\text{s}^{-1})(29.0 \text{ T})(1.055\times10^{-34} \text{ J}\cdot\text{s})}{(1.38\times10^{-23} \text{ J/K})\ln(0.418)} = 0.00925 \text{ K}$$

(b) $$r = \frac{N_0}{N_{-5}} = \frac{d_0 e^{-E_0/kT}}{d_{-5}e^{-E_{-5}/kT}} = \frac{e^0}{e^{5\gamma B\hbar/kT}} = e^{-5\gamma B\hbar/kT}$$

$$= e^{-5(3.64\times10^7 \text{ T}^{-1}\text{s}^{-1})(29 \text{ T})(1.055\times10^{-34} \text{ J}\cdot\text{s})/(1.38\times10^{-23} \text{ J/K})(0.00925 \text{ K})} = 0.0128$$

41. (a) The degeneracies of the rotational states are $2L + 1$, so the ratio of the populations of the $L = 1$ and $L = 0$ states is

$$r_{10} = \frac{N_1}{N_0} = \frac{d_1 e^{-E_1/kT}}{d_0 e^{-E_0/kT}} = 3e^{-(E_1-E_0)/kT} = 3e^{-\Delta E/kT}$$

$$T = \frac{-\Delta E}{k\ln(r_{10}/3)} = \frac{-4.71\times10^{-4} \text{ eV}}{(8.617\times10^{-5} \text{ eV/K})\ln(0.421/3)} = 2.78 \text{ K}$$

The uncertainties can be analyzed by taking differentials of the first equation, which gives $dr_{10} = 3e^{-\Delta E/kT}(\Delta E/kT^2)dT = r_{10}(\Delta E/kT^2)dT$:

$$dT = \frac{dr_{10}}{r_{10}} \frac{kT^2}{\Delta E} = \frac{0.017}{0.421} \frac{(8.617 \times 10^{-5} \text{ eV/K})(2.78 \text{ K})^2}{4.71 \times 10^{-4} \text{ eV}} = 0.06 \text{ K}$$

so the deduced temperature is $2.78 \pm 0.06$ K.

(b) The second excited rotational state has $L = 2$. Because the energy of the rotational states is proportional to $L(L+1)$, this state has 3 times the energy of the first excited state:

$$r_{20} = \frac{N_2}{N_0} = \frac{d_2 e^{-E_2/kT}}{d_0 e^{-E_0/kT}} = 5e^{-(E_2-E_0)/kT} = 5e^{-3\Delta E/kT} = 5e^{-3(4.71\times10^{-4} \text{ eV})/(8.617\times10^{-5} \text{ eV/K})(2.78 \text{ K})} = 0.0137$$

The range of $\pm 0.06$ K in temperature corresponds to a range of $\pm 0.0017$ in $r_{20}$, so this ratio is in very good agreement with the observed value.

# Chapter 11

This chapter discusses several properties of solids in the context of applying quantum statistics and thus demonstrating the importance of quantum theory in studying condensed matter. The previous editions of this text included only applications to electrical conduction in solids, primarily that of semiconductors. In the present edition, the coverage of electrical conduction has been reduced somewhat in order to add discussions of the heat capacity of solids and paramagnetism. In both cases, theoretical formalisms have been coupled with analysis of actual experimental data, emphasizing the success of the interpretation based on quantum theory.

## Suggestions for Additional Reading

More detailed and comprehensive books on solid-state physics:

R. H. Bube, *Electrons in Solids: An Introductory Survey,* 3rd ed. (Academic Press, 1992).
J. R. Christman, *Fundamentals of Solid State Physics* (Wiley, 1988).
A. J. Dekker, *Solid State Physics* (Prentice-Hall, 1957).
A. Hart-Davis, *Solids: An Introduction* (McGraw-Hill, 1975).
C. Kittel, *Introduction to Solid State Physics,* 7th ed. (Wiley, 1996).
H. P. Myers, *Introductory Solid State Physics* (Taylor and Francis, 1990).
M. N. Rudden and J. Wilson, *Elements of Solid State Physics,* 2nd ed. (Wiley, 1993).
H. M. Rosenberg, *The Solid State*, 3$^{rd}$ ed. (Oxford, 1988).
R. T. Sanderson, *Chemical Bonds and Bond Energy* (Academic Press, 1976).
R. Turton, *The Physics of Solids* (Oxford, 2000).

High-temperature superconductors are discussed in R. J. Cava, "Superconductors beyond 1-2-3," *Scientific American* **263,** 42 (August 1990).

Bulk properties of solids are tabulated in many references, including the *Handbook of Chemistry and Physics* (Chemical Rubber Publishing Co.) and the *American Institute of Physics Handbook* (McGraw-Hill, 1963).

## Materials for Active Engagement in the Classroom

### A. Reading Quizzes

1. At the lowest temperatures, the molar heat capacity of metals can be explained primarily by the application of:
    (1) the equipartition theorem
    (2) Fermi-Dirac statistics of electrons
    (3) Bose-Einstein statistics of vibrating atoms

2. Which of the following processes can produce a substantial increase in the electrical conductivity of a semiconductor?
     (1) Lower the temperature.        (2) Add certain impurities.
     (3) Both 1 and 2.               (4) Neither 1 nor 2.

3. How does the electrical conductivity of a semiconductor depend on the temperature?
     (1) The conductivity increases as the temperature increases.
     (2) The conductivity decreases as the temperature increases.
     (3) The conductivity does not depend on the temperature.

**Answers**     1. 3    2. 2   3. 1


## B. Conceptual or Discussion Questions

1. Put these types of solids in order of increasing values of typical melting points:
     (1) ionic     (2) covalent   (3) metallic   (4) molecular

2. Classify the following materials as:
     (1) Conductor   (2) Insulator   (3) Semiconductor    (4) Conductor and insulator
     (5) Conductor and semiconductor     (6) Insulator and semiconductor
     (7) Conductor, insulator, and semiconductor       (8) None of these
    (a) Filled valence band, empty conduction band, energy gap = 8 eV.
    (b) Filled valence band, empty conduction band, energy gap = 1 eV.
    (c) Half-filled valence band, empty conduction band, energy gap = 1 eV.
    (d) The Fermi energy is located in the gap between the valence and conduction bands.
    (e) The electrical resistance decreases significantly as the temperature is lowered.

3. In an intrinsic semiconductor, the number of electrons in the conduction band is
    _____ the number of holes in the valence band.
     (1) exactly equal to        (2) approximately equal to
     (3) much greater than     (4) much less than

4. In an intrinsic semiconductor, the contribution of the electrons to the current is usually
    _____ the contribution of the holes to the current.
     (1) exactly equal to        (2) approximately equal to
     (3) much greater than     (4) much less than

5. In a p-type semiconductor, the number of electrons in the conduction band is _____
    the number of holes in the valence band.
     (1) exactly equal to        (2) approximately equal to
     (3) much greater than     (4) much less than

6. In a p-type semiconductor, the contribution of the electrons to the current is usually
    _____ the contribution of the holes to the current.
     (1) exactly equal to        (2) approximately equal to
     (3) much greater than     (4) much less than

## Sample Exam Questions

### A. Multiple Choice

1. In the range from 100 K to 200 K, the temperature variation of the molar heat capacity of metals is best explained in terms of
   (a) the equipartition theorem
   (b) Fermi-Dirac statistics of electrons
   (c) Bose-Einstein statistics of lattice vibrations

2. An intrinsic semiconductor at room temperature has:
   (a) a greater density of electrons in its conduction band than holes in its valence band.
   (b) a greater density of electrons in its conduction band than electrons it its valence band.
   (c) a greater density of electrons in its valence band than it would at a temperature of 0 K.
   (d) a greater density of holes in the valence band than it would at a temperature of 0 K.

3. In a certain intrinsic semiconductor, the probability for electron states at the bottom of the conduction band to be occupied at room temperature (300 K) is $1.0 \times 10^{-10}$. What would be the probability for these electron states to be occupied if the temperature were raised to 330 K?
   (a) $8.1 \times 10^{-10}$     (b) $1.2 \times 10^{-11}$     (c) $1.1 \times 10^{-10}$     (d) $2.4 \times 10^{-9}$   (e) $9.0 \times 10^{-11}$

4. In a certain intrinsic (undoped) semiconducting material, the occupation probability near the bottom of the conduction band at room temperature (300 K) is $10^{-9}$. If the temperature is raised to 390 K, what would you estimate to be the new occupation probability near the bottom of the conduction band?
   (a) $10^{-10}$    (b) $10^{-9}$         (c) $10^{-8}$        (d) $10^{-7}$         (e) $10^{-6}$

5. A piece of silicon contains a small amount of an impurity that forms states located in the energy gap very close to the edge of the valence band. What category would best describe this material?
   (a) insulator        (b) p-type semiconductor       (c) n-type semiconductor
   (d) intrinsic semiconductor         (e) conductor

6. A certain semiconductor has an energy gap of about 1 eV and a certain insulator has an energy gap of about 6 eV. Which of these would you expect to be transparent to all visible light?
   (a) semiconductor            (b) insulator      (c) both      (d) neither

7. A certain intrinsic (undoped) semiconducting material has an energy gap of 1.6 eV between the valence and conducting bands. What is the approximate probability to find electrons at the bottom of the conduction band at room temperature (300 K)?
   (a) $10^{-10}$    (b) $10^{-12}$         (c) $10^{-14}$        (d) $10^{-16}$        (e) $10^{-18}$

**Answers**     1. c     2. d     3. a     4. d     5. b     6. c     7. c

## B. Conceptual

1.  A certain insulator has an energy gap of 6.0 eV. When visible light (400 to 700 nm) shines on this material, would you expect the light to be *strongly absorbed* (so the material would be *opaque*) or *not absorbed* (so that the material would be *transparent*)? EXPLAIN YOUR ANSWER.

2.  Suppose we raise the temperature of a semiconductor from room temperature (20°C or about 300 K) to the temperature of hot tap water (40°C or about 320 K). Would you expect the electrical conductivity to *increase or decrease*, and would you expect the change in electrical conductivity to be *large or small*? EXPLAIN YOUR ANSWER.

3.  In a certain intrinsic (undoped) semiconductor, the Fermi energy at the absolute zero of temperature is at the middle of the gap between the valence band and the conduction band. The temperature is now raised to a value in the neighborhood of room temperature. At this temperature, it is found that there are $N_e$ empty states in the valence band. At this temperature, is the number of filled states in the conduction band *greater than $N_e$, less than $N_e$, or equal to $N_e$*? EXPLAIN YOUR ANSWER.

**Answers**     1. not absorbed     2. increase, large     3. equal to

## C. Problems

1.  In a certain conducting material at $T = 0$ K, all of the electron energy levels below 3.15 eV are filled, and all of the levels above 3.15 eV are empty. The material is now heated to a temperature of 120 K. Consider electrons with energies in a small interval $dE$ around $E_1 = 3.10$ eV and also electrons with energies in a small interval $dE$ around $E_2 = 3.20$ eV. Find the ratio of the probabilities to find electrons in these two intervals.

2.  In a certain semiconductor, the energy gap is 1.10 eV and the Fermi energy is at the center of the gap.
    (a) What is the probability that a state at the bottom of the conduction band is occupied at room temperature (300 K)?
    (b) What is the probability that a state at the top of the valence band is empty at room temperature?
    (c) What is the probability that a state at the bottom of the conduction band is occupied at 0 K?

**Answers**     1. 0.0079     2. (a) $5.76 \times 10^{-10}$  (b) $5.76 \times 10^{-10}$  (c) 0

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**Problem Solutions**

1.  (a) 1/8
    (b) If the spheres are in contact along each edge of the cube, then $a = 2r$.
    (c) With $V_{cube} = a^2 = (2r)^3$ and $V_{spheres} = 8 \times \frac{1}{8} \times \frac{4}{3} \pi r^3$, the packing fraction is

    $$\text{packing fraction} = \frac{V_{spheres}}{V_{cube}} = \frac{\frac{4}{3} \pi r^3}{8r^3} = \frac{\pi}{6} = 0.5236$$

2.  (a) In the fcc lattice, inside the basic cube there is 1/8 of a sphere at each of the 8 corners
    and 1/2 of a sphere at each of the 6 faces. Thus $V_{spheres} = \left(8 \times \frac{1}{8} + 6 \times \frac{1}{2}\right) \frac{4}{3} \pi r^3 = \frac{16}{3} \pi r^3$. The
    diagonal along any face of the cube has length $a\sqrt{2}$, where $a$ is the length of an edge of
    the cube. The spheres are in contact along one of these diagonals, so $a\sqrt{2} = 4r$,
    and $V_{cube} = a^3 = (4r / \sqrt{2})^3 = 16\sqrt{2}r^3$. The packing fraction is

    $$\text{packing fraction} = \frac{V_{spheres}}{V_{cube}} = \frac{\frac{16}{3} \pi r^3}{16\sqrt{2}r^3} = \frac{\pi}{3\sqrt{2}} = 0.7405$$

    (b) In the bcc lattice, inside the basic cube there is a full sphere at the center and 1/8 of a
    sphere at each of the 8 corners, so $V_{spheres} = \left(8 \times \frac{1}{8} + 1\right) \frac{4}{3} \pi r^3 = \frac{8}{3} \pi r^3$. The body diagonal has
    length $a\sqrt{3}$. The spheres are in contact along the body diagonal, so $a\sqrt{3} = 4r$, and
    $V_{cube} = a^3 = (4r / \sqrt{3})^3 = 64r^3 / 3\sqrt{3}$. The packing fraction is

    $$\text{packing fraction} = \frac{V_{spheres}}{V_{cube}} = \frac{\frac{8}{3} \pi r^3}{64r^3 / 3\sqrt{3}} = \frac{\pi\sqrt{3}}{8} = 0.6802$$

    The fcc arrangement allows for more dense packing of the spheres (and both fcc and bcc
    are more efficient packing schemes than the simple cubic).

3.  Starting with $E = -\alpha \dfrac{e^2}{4\pi\varepsilon_0} \dfrac{1}{R} + \dfrac{A}{R^n}$, we take the derivative to obtain:

    $$\frac{dE}{dr} = \alpha \frac{e^2}{4\pi\varepsilon_0} \frac{1}{R^2} - \frac{nA}{R^{n+1}} = 0 \quad \text{at} \quad R = R_0$$

    With $\alpha \dfrac{e^2}{4\pi\varepsilon_0} \dfrac{1}{R_0^2} = \dfrac{nA}{R_0^{n+1}}$ we get $A = \alpha \dfrac{e^2}{4\pi\varepsilon_0} \dfrac{R_0^{n-1}}{n}$ and so

$$E = -\alpha \frac{e^2}{4\pi\varepsilon_0}\frac{1}{R} + \alpha\frac{e^2 R_0^{n-1}}{4\pi\varepsilon_0 n}\frac{1}{R^n} = -\frac{\alpha e^2}{4\pi\varepsilon_0 R}\left[1 - \frac{1}{n}\left(\frac{R_0}{R}\right)^{n-1}\right]$$

The binding energy is

$$B = -E(R_0) = \frac{\alpha e^2}{4\pi\varepsilon_0 R_0}\left(1 - \frac{1}{n}\right)$$

4.  In the CsCl (bcc) structure, a $Cs^+$ ion at the center of a cube is surrounded by 8 $Cl^-$ ions at a distance $R$. With $a$ as the length of a cube edge, the distance $R$ is half the body diagonal $a\sqrt{3}$, so $a = 2R/\sqrt{3}$. At this distance from each $Cs^+$ ion, there are 6 $Cs^+$ ions, each at the center of an adjacent cube. At the 4 far corners of each of the 6 adjacent cubes there is a total of 24 $Cl^-$ ions at a distance of $\sqrt{(a\sqrt{2}/2)^2 + (3a/2)^2} = a\sqrt{11/4} = R\sqrt{11/3}$. The potential energy due to these three terms is

$$U = -8\frac{e^2}{4\pi\varepsilon_0}\frac{1}{R} + 6\frac{e^2}{4\pi\varepsilon_0}\frac{1}{2R\sqrt{3}} - 24\frac{e^2}{4\pi\varepsilon_0}\frac{1}{R\sqrt{11/3}} = -\frac{e^2}{4\pi\varepsilon_0 R}\left(8 - 3\sqrt{3} + \frac{24\sqrt{3}}{\sqrt{11}}\right)$$

5.  (a)    $B = \dfrac{E_{coh}}{N_A} = \dfrac{657\times10^3 \text{ J/mole}}{(6.02\times10^{23} \text{ ions/mole})(1.60\times10^{-19} \text{ J/eV})} = 6.82 \text{ eV}$

(b)    $B = \dfrac{\alpha e^2}{4\pi\varepsilon_0 R_0}\left(1 - \dfrac{1}{n}\right) = \dfrac{(1.7627)(1.440 \text{ eV}\cdot\text{nm})}{0.356 \text{ nm}}\left(1 - \dfrac{1}{10.5}\right) = 6.45 \text{ eV}$

(c) To find the binding energy per *atom* pair, we start with the binding energy per *ion* pair, add the electron affinity of Cl, and subtract the ionization energy of Cs: 6.82 eV + 3.61 eV − 3.89 eV = 6.54 eV. The binding energy per atom is half this value, or 3.27 eV.

6.  (a)    $B = \dfrac{E_{coh}}{N_A} = \dfrac{1030\times10^3 \text{ J/mole}}{(6.02\times10^{23} \text{ ions/mole})(1.60\times10^{-19} \text{ J/eV})} = 10.69 \text{ eV}$

(b)    $B = \dfrac{\alpha e^2}{4\pi\varepsilon_0 R_0}\left(1 - \dfrac{1}{n}\right) = \dfrac{(1.7476)(1.440 \text{ eV}\cdot\text{nm})}{0.201 \text{ nm}}\left(1 - \dfrac{1}{6}\right) = 10.43 \text{ eV}$

(c) To find the binding energy per *atom* pair, we start with the binding energy per *ion* pair, add the electron affinity of F, and subtract the ionization energy of Li: 10.69 eV + 3.45 eV − 5.39 eV = 8.75 eV. The binding energy per atom is half this value, or 4.38 eV.

7.    $U_C = -\alpha\dfrac{e^2}{4\pi\varepsilon_0}\dfrac{1}{R_0} = -1.7476\dfrac{1.440 \text{ eV}\cdot\text{nm}}{0.281 \text{ nm}} = -8.96 \text{ eV}$

$$U_R = \frac{A}{R_0^n} = \frac{\alpha e^2}{4\pi\varepsilon_0 R_0 n} = \frac{1}{n}(-U_C) = \frac{1}{9}(8.96 \text{ eV}) = 1.00 \text{ eV}$$

8. The density of atoms is

$$n = \frac{\rho N_A}{M} = \frac{(0.971 \text{ g/cm}^3)(6.02 \times 10^{23} \text{ atoms/mole})}{(23.0 \text{ g/mole})(1 \text{ m}/10^2 \text{ cm})^3} = 2.54 \times 10^{28} \text{ atoms/m}^3$$

In the bcc structure, each basic cell contains 2 atoms: 1/8 atom at each of the 8 corners and one at the center of the cube. If $a$ is the length of one edge of the cube, then $2 \text{ atoms}/a^3 = 2.54 \times 10^{28} \text{ atoms/m}^3$, or

$$a = \left(\frac{2 \text{ atoms}}{2.54 \times 10^{28} \text{ atoms/m}^3}\right)^{1/3} = 4.29 \times 10^{-10} \text{ m} = 0.429 \text{ nm}$$

The body diagonal has length $a\sqrt{3}$, so the atomic spacing is $\frac{1}{2}a\sqrt{3} = 0.371 \text{ nm}$.

9. The density of atoms is

$$n = \frac{\rho N_A}{M} = \frac{(8.96 \text{ g/cm}^3)(6.02 \times 10^{23} \text{ atoms/mole})}{(63.5 \text{ g/mole})(1 \text{ m}/10^2 \text{ cm})^3} = 8.49 \times 10^{28} \text{ atoms/m}^3$$

In the fcc structure, each basic cell contains 4 atoms: 1/8 atom at each of the 8 corners and 1/2 atom at the center of each of the 6 faces. If $a$ is the length of one edge of the cube, then $4 \text{ atoms}/a^3 = 8.49 \times 10^{28} \text{ atoms/m}^3$, or

$$a = \left(\frac{4 \text{ atoms}}{8.49 \times 10^{28} \text{ atoms/m}^3}\right)^{1/3} = 3.61 \times 10^{-10} \text{ m} = 0.361 \text{ nm}$$

The face diagonal has length $a\sqrt{2}$, so the atomic spacing is $\frac{1}{2}a\sqrt{2} = 0.255 \text{ nm}$.

10. Na: $B = \dfrac{E_{coh}}{N_A} = \dfrac{(107 \times 10^3 \text{ J/mole})(1 \text{ eV}/1.60 \times 10^{-19} \text{ J})}{6.02 \times 10^{23} \text{ atoms/mole}} = 1.11 \text{ eV}$

Cu: $B = \dfrac{E_{coh}}{N_A} = \dfrac{(337 \times 10^3 \text{ J/mole})(1 \text{ eV}/1.60 \times 10^{-19} \text{ J})}{6.02 \times 10^{23} \text{ atoms/mole}} = 3.50 \text{ eV}$

11. Equating the electronic (Equation 11.9) and lattice (Equation 11.12) contributions to the heat capacity gives

$$\frac{\pi^2}{2}\frac{kT}{E_F}R = \frac{12\pi^4}{5}R\left(\frac{T}{T_D}\right)^3$$

Solving for $T$, we obtain

$$T = \sqrt{\frac{5kT_D^3}{24\pi^2 E_F}} = \sqrt{\frac{5(8.617\times10^{-5}\ \text{eV/K})(343\ \text{K})^3}{24\pi^2(7.03\ \text{eV})}} = 3.23\ \text{K}$$

Above 3.23 K, the lattice contribution to the heat capacity is larger than the electronic contribution, but below 3.23 K the electronic contribution is larger.

12.  (a) From Equation 11.12,

$$T_D = T\left(\frac{12\pi^4 R}{5C}\right)^{1/3} = (2.00\ \text{K})\left[\frac{12\pi^4(8.31\ \text{J/mole}\cdot\text{K})}{5(0.0200\ \text{J/mole}\cdot\text{K})}\right]^{1/3} = 91.9\ \text{K}$$

(b) The heat capacity should increase as $T^3$, so the heat capacity at 3.00 K should be greater than the heat capacity at 2.00 K by the factor $(3.00\ \text{K}/2.00\ \text{K})^3 = 3.375$. The heat capacity at 3.00 K is therefore $(2.00\times10^{-2}\ \text{J/mole}\cdot\text{K})(3.375) = 6.75\times10^{-2}\ \text{J/mole}\cdot\text{K}$.

13.  From Equation 11.12, the slope $b$ is $12\pi^4 R/5T_D^3$, so

$$T_D = \left(\frac{12\pi^4 R}{5b}\right)^{1/3} = \left[\frac{12\pi^4(8.31\ \text{J/mole}\cdot\text{K})}{5(2.57\times10^{-3}\ \text{J/mole}\cdot\text{K}^4)}\right]^{1/3} = 91.1\ \text{K}$$

14.  (a) The lattice contribution to the heat capacity at 4 K is

$$C = \frac{12\pi^4 R}{5}\left(\frac{T}{T_D}\right)^3 = \frac{12\pi^4(8.31\ \text{J/mole}\cdot\text{K})}{5}\left(\frac{4\ \text{K}}{225\ \text{K}}\right)^3 = 0.0109\ \text{J/mole}\cdot\text{K}$$

The electronic contribution is then 0.0134 J/mole·K − 0.0109 J/mole·K = 0.0025 J/mole·K.
(b) The lattice contribution to the heat capacity is proportional to $T^3$, so at 2 K it is smaller by $(2\ \text{K}/4\ \text{K})^3 = 0.125$ and thus $0.125(0.0109\ \text{J/mole}\cdot\text{K}) = 0.00136\ \text{J/mole}\cdot\text{K}$. The electronic contribution is proportional to $T$ and so at 2 K its value is reduced by half, to 0.00125 J/mole·K. The total heat capacity at 2 K is 0.00136 J/mole·K + 0.00125 J/mole·K = 0.00261 J/mole·K.

15. (a)
$$f_{FD}(E) = \frac{1}{e^{(E-E_F)/kT}+1} = 0.1$$

$$e^{(E-E_F)/kT}+1 = 10 \qquad \text{so} \qquad \frac{E-E_F}{kT} = \ln 9$$

At room temperature (293 K), $kT = (8.617\times10^{-5}$ eV/K$)(293$ K$) = 0.0252$ eV , so

$$E = E_F + kT\ln 9 = 7.03 \text{ eV} + (0.0252 \text{ eV})(2.20) = 7.09 \text{ eV}$$

(b) $E - E_F$ is 2.20 times $kT$, indicating the sharpness of the Fermi-Dirac distribution – it drops from 0.9 to 0.1 within $4.40kT$ or about 0.11 eV.

16. An electron at the Fermi energy of 7.03 eV has momentum $p = \sqrt{2mE}$ and de Broglie wavelength

$$\lambda = \frac{h}{p} = \frac{hc}{pc} = \frac{hc}{\sqrt{2mc^2 E}} = \frac{1240 \text{ eV} \cdot \text{nm}}{\sqrt{2(0.511\times10^6 \text{ eV})(7.03 \text{ eV})}} = 0.463 \text{ nm}$$

This is comparable to the atomic spacing of copper (0.256 nm).

17. From Equation 10.47 for the Fermi-Dirac distribution, with $E_F = 3.15$ eV for sodium and $kT = (8.617\times10^{-5}$ eV/K$)(293$ K$) = 0.0252$ eV ,

$$\frac{dN}{V} = \frac{8\pi\sqrt{2}(mc^2)^{3/2}}{(hc)^3} \frac{E^{1/2}dE}{e^{(E-E_F)kT}+1}$$

$$= \frac{8\pi\sqrt{2}(0.511\times10^6 \text{ eV})^{3/2}}{(1240 \text{ eV}\cdot\text{nm})^3} \frac{(3.25 \text{ eV})^{1/2}(0.01 \text{ eV})}{e^{0.1 \text{ eV}/0.0252 \text{ eV}}+1} = 2.3\times10^{24} \text{ m}^{-3}$$

18. Solving Equation 11.19 for $\tau$, we obtain

$$\tau = \frac{\sigma m}{ne^2} = \frac{(5.96\times10^7 \ \Omega^{-1}\text{m}^{-1})(9.11\times10^{-31} \text{ kg})}{(8.49\times10^{28}\text{atoms/m}^3)(1.60\times10^{-19} \text{ C})^2} = 2.50\times10^{-14} \text{ s}$$

Using the Fermi velocity of $1.57 \times 10^6$ m/s, the mean free path is

$$l = \tau v_F = (2.50\times10^{-14} \text{ s})(1.57\times10^6 \text{ m/s}) = 39.2 \text{ nm}$$

In terms of the atomic spacing of Cu (0.256 nm), this amounts to more than 150 lattice spacings.

19.
$$\frac{E_F(T)}{E_F(0)} = 0.99 = 1 - \frac{\pi^2}{12}\left(\frac{kT}{E_F(0)}\right)^2$$

$$kT = E_F(0)\sqrt{\frac{12(1-0.99)}{\pi^2}} = (5.53\text{ eV})\sqrt{\frac{12(0.01)}{\pi^2}} = 0.610\text{ eV}$$

$$T = (0.610\text{ eV})/k = (0.610\text{ eV})/(8.617\times10^{-5}\text{ eV/K}) = 7080\text{ K}$$

20. The current density (current per unit area) is

$$j = i/A = 2.5\times10^{-3}\text{ A}/\pi(0.00025\text{ m})^2 = 1.27\times10^4\text{ A/m}^2$$

and the drift velocity is

$$v_d = \frac{j}{ne} = \frac{1.27\times10^4\text{ A/m}^2}{(8.49\times10^{28}\text{ m}^{-3})(1.60\times10^{-19}\text{ C})} = 9.37\times10^{-7}\text{ m/s}$$

The number of electrons in a narrow strip of width $dE$ is $dN = N(E)dE$. We'll assume that it is sufficient to evaluate $N(E)$ at the Fermi energy because the strip is very narrow. With $dE = (dE/dv)dv = mv\,dv$ (evaluating $dE/dv$ for $E = \frac{1}{2}mv^2$) we have

$$dN = N(E_F)dE = N(E_F)mv\,dv = N(E_F)mv_F\,v_d$$

using $dv = v_d$ for the width of the strip as in Figure 11.16. We can show that $N(E_F) = 1.5N/E_F$, and so

$$\frac{dN}{N} = \frac{1.5mv_Fv_d}{E_F} = \frac{1.5(9.11\times10^{-31}\text{ kg})(1.57\times10^6\text{ m/s})(9.37\times10^{-7}\text{ m/s})}{(7.03\text{ eV})(1.602\times10^{-19}\text{ J/eV})} = 1.8\times10^{-12}$$

Only a very small fraction of the electrons participates in the conduction.

21.
$$K = L\sigma T = (2.44\times10^{-8}\text{ W}\cdot\Omega/\text{K}^2)(5.96\times10^7\ \Omega^{-1}\cdot\text{m}^{-1})(293\text{ K}) = 426\text{ W/K}\cdot\text{m}$$

22. The concentration is dependent on the Fermi-Dirac distribution, which is to be evaluated at $kT = (8.617\times10^{-5}\text{ eV/K})(293\text{ K}) = 0.0252\text{ eV}$. For states at the bottom of the conduction band, $E - E_F = E_g/2$ and

$$\frac{1}{e^{(E-E_{\mathrm{F}})/kT}+1} = \frac{1}{e^{E_{\mathrm{g}}/2kT}+1} = \frac{1}{e^{5.5 \text{ eV}/2(0.0252 \text{ eV})}+1} = 4.0 \times 10^{-48} \text{ for C}$$

$$\frac{1}{e^{E_{\mathrm{g}}/2kT}+1} = \frac{1}{e^{1.1 \text{ eV}/2(0.0252 \text{ eV})}+1} = 3.3 \times 10^{-10} \text{ for Si}$$

The ratio of the electron concentrations of C and Si is about $1.2 \times 10^{-38}$.

23. At $kT = (8.617 \times 10^{-5} \text{ eV/K})(400 \text{ K}) = 0.0345 \text{ eV}$ the Fermi-Dirac distribution gives

$$\frac{1}{e^{(E-E_{\mathrm{F}})/kT}+1} = \frac{1}{e^{E_{\mathrm{g}}/2kT}+1} = \frac{1}{e^{0.66 \text{ eV}/2(0.0345 \text{ eV})}+1} = 7.0 \times 10^{-5} \text{ for Ge}$$

$$\frac{1}{e^{E_{\mathrm{g}}/2kT}+1} = \frac{1}{e^{1.12 \text{ eV}/2(0.0345 \text{ eV})}+1} = 8.9 \times 10^{-8} \text{ for Si}$$

The Ge/Si ratio is about 790.

24. (a) The cube has a volume of $(0.10 \text{ cm})^3 = 1.0 \times 10^{-3} \text{ cm}^3$ and thus a mass of $2.3 \times 10^{-3}$ g, so $N = (2.3 \times 10^{-3} \text{ g})/(28 \text{ g/mole})(6.02 \times 10^{23} \text{ atoms/mole}) = 5.0 \times 10^{19} \text{ atoms}$. The number of states in the valence band is $4N = 2.0 \times 10^{20}$.
(b) The valence band has a width of 12 eV, so the average spacing between the states is $12 \text{ eV} / 2.0 \times 10^{20} = 6.0 \times 10^{-20} \text{ eV}$.

25. (a) From Equation 11.25,

$$E_{\mathrm{g}} = 3.53kT_{\mathrm{c}} = 3.53(8.617 \times 10^{-5} \text{ eV/K})(0.65 \text{ K}) = 0.20 \text{ meV}$$

(b) $\quad \lambda = \dfrac{hc}{E} = \dfrac{1240 \text{ eV} \cdot \text{nm}}{0.20 \times 10^{-3} \text{ eV}} = 6.3 \times 10^6 \text{ nm} = 6.3 \text{ mm} \quad$ in the microwave region

26. The minimum energy of the photons that can destroy the superconductivity are

$$E = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{0.91 \times 10^6 \text{ nm}} = 1.36 \times 10^{-3} \text{ eV} = 1.36 \text{ meV}$$

This must be the same as the gap energy, so $E_{\mathrm{g}} = 1.36$ meV for Ta. According to the BCS theory, the critical temperature is

$$T_{\mathrm{c}} = \frac{E_{\mathrm{g}}}{3.53k} = \frac{1.36 \times 10^{-3} \text{ eV}}{3.53(8.617 \times 10^{-5} \text{ eV/K})} = 4.47 \text{ K}$$

27. The frequency $f = \omega/2\pi$ is

$$f = \frac{\omega}{2\pi} = \frac{2e\Delta V}{2\pi\hbar} = \frac{2(1.60\times10^{-19} \text{ C})(1.25\times10^{-6} \text{ V})}{6.63\times10^{-34} \text{ J}\cdot\text{s}} = 603 \text{ MHz}$$

Such frequencies in the radio region of the electromagnetic spectrum can be measured with very high precision.

28. Assuming the Fermi energy to be at the middle of the gap, the relative number of electrons excited from the valence band to the conduction band will be proportional to $e^{-(E-E_F)/kT} = e^{-E_g/2kT}$, so the ratio of the conductivities at two different temperatures should be roughly

$$\frac{e^{-E_g/2kT_1}}{e^{-E_g/2kT_2}} = \frac{e^{-(1.1 \text{ eV})/2(8.617\times10^{-5} \text{ eV/K})(393 \text{ K})}}{e^{-(1.1 \text{ eV})/2(8.617\times10^{-5} \text{ eV/K})(293 \text{ K})}} = 255$$

There are other temperature-dependent factors on which the conductivity depends, but their temperature variation is much weaker than the exponential factor. Note that a relatively small change in the temperature produces a huge change in the conductivity.

29. (a) With the Fermi energy at the middle of the gap, the number of excited electrons is proportional to $Ne^{-(E-E_F)/kT} = Ne^{-E_g/2kT}$ and so the fraction is approximately

$$e^{-E_g/2kT} = e^{-(1.1 \text{ eV})/2(8.617\times10^{-5} \text{ eV/K})(100 \text{ K})} = 1.9\times10^{-28} \quad \text{at 100 K}$$
$$e^{-E_g/2kT} = e^{-(1.1 \text{ eV})/2(8.617\times10^{-5} \text{ eV/K})(293 \text{ K})} = 3.5\times10^{-10} \quad \text{at 293 K}$$

(b) In a metal, the conductivity decreases linearly with the temperature, so it would drop by roughly a factor of 3 over this temperature range.

30. The occupation probability in the conduction band is roughly $e^{-(E-E_F)/kT} = e^{-E_g/2kT}$, assuming the Fermi energy is at the middle of the gap. If the occupation probabilities are equal, then the values of the exponents must be equal: $E_g(\text{Si})/2kT_{\text{Si}} = E_g(\text{Ge})/2kT_{\text{Ge}}$, or

$$T_{\text{Ge}} = T_{\text{Si}} \frac{E_g(\text{Ge})}{E_g(\text{Si})} = (293 \text{ K}) \frac{0.66 \text{ eV}}{1.12 \text{ eV}} = 173 \text{ K}$$

This is only a rough estimate, because we have ignored several factors that give an additional temperature dependence as well as a dependence on the different effective masses of electrons in Ge and Si. The correct value is closer to 200 K, but even this rough calculation gives a fairly good estimate of the temperature.

31. (a) The electric field in a dielectric is reduced by a factor of $1/\kappa$, where $\kappa$ is the dielectric constant. In electric field equations, that change can be accomplished by replacing $\varepsilon_0$ by

$\kappa\varepsilon_0$ everywhere it appears. In the Bohr theory (Equation 6.30), the energy levels include the factor $1/\varepsilon_0^2$, so the energy of the single phosphorous electron in Si is

$$E = \frac{-13.6 \text{ eV}}{\kappa^2} = \frac{-13.6 \text{ eV}}{12^2} = -0.094 \text{ eV}$$

(b) Again from the Bohr theory, the electron energy is directly proportional to the electron mass, so the effective mass changes the energy by the factor $m_{\text{eff}}/m = 0.43$. The energy would then be $(-0.094 \text{ eV})(0.43) = -0.040 \text{ eV}$. This is very close to the observed energy necessary to excite an electron for phosphorous impurities in Si (0.045 eV).

32. (a) For $kT = (8.617 \times 10^{-5} \text{ eV/K})(293 \text{ K}) = 0.0252 \text{ eV}$ and $E - E_F = E_g/2 = 0.55 \text{ eV}$, we have

$$f_{\text{FD}}(E) = \frac{1}{e^{(E-E_F)/kT}+1} = \frac{1}{e^{E_g/2kT}+1} = \frac{1}{e^{0.55 \text{ eV}/0.0252 \text{ eV}}+1} = 3.32 \times 10^{-10}$$

(b) The occupied states in the conduction band exactly correspond to the empty states in the valence band, so the occupation probability in the valence band is $1 - 3.32 \times 10^{-10}$.

33. The fraction of electrons in the conduction band is roughly $e^{-(E-E_F)/kT} = e^{-E_g/2kT}$, assuming the Fermi energy is at the center of the gap:

$$e^{-E_g/2kT} = e^{-0.66 \text{ eV}/2(8.617 \times 10^{-5} \text{ eV/K})(293 \text{ K})} = 2.1 \times 10^{-6}$$

To increase the population of the conduction band by a factor of 3, it is necessary to provide twice this number of donor atoms, so the relative population of donor atoms must be $4.2 \times 10^{-6}$.

34. (a) From Equation 11.31 with $kT = 0.0252 \text{ eV}$, we have

$$\frac{i(2 \text{ V})}{i(1 \text{ V})} = \frac{e^{2.00 \text{ eV}/0.0252 \text{ eV}}-1}{e^{1.00 \text{ eV}/0.0252 \text{ eV}}-1} = 1.7 \times 10^{17}$$

(b) At 400 K, $kT = (8.617 \times 10^{-5} \text{ eV/K})(400 \text{ K}) = 0.0345 \text{ eV}$ and

$$\frac{i(2 \text{ V})}{i(1 \text{ V})} = \frac{e^{2.00 \text{ eV}/0.0345 \text{ eV}}-1}{e^{1.00 \text{ eV}/0.0345 \text{ eV}}-1} = 3.9 \times 10^{12}$$

This sensitive dependence of current on temperature is in part what distinguishes semiconductors from ordinary metals.

35. From Equation 11.30, the coefficient $i_0$ is (with $kT = 0.025$ eV at room temperature)

$$i_0 = \frac{i}{e^{e\Delta V_{\text{ext}}/kT} - 1} = \frac{1.5 \text{ mA}}{e^{0.25 \text{ eV}/0.025 \text{ eV}} - 1} = 6.8 \times 10^{-5} \text{ mA}$$

Under reverse bias, the exponential factor in the denominator will be $e^{-10} = 4.5 \times 10^{-5}$, which is negligible compared with the 1 in the denominator. Thus under reverse biasing, $i = -i_0 = -6.8 \times 10^{-5}$ mA.

36.      GaP : $\lambda = \dfrac{hc}{\Delta E} = \dfrac{1240 \text{ eV} \cdot \text{nm}}{2.26 \text{ eV}} = 549$ nm (green)

   ZnSe:  $\lambda = \dfrac{hc}{\Delta E} = \dfrac{1240 \text{ eV} \cdot \text{nm}}{2.87 \text{ eV}} = 432$ nm (blue)

37. (a) For green light, the photon energy is $E = hc/\lambda = 1240 \text{ eV} \cdot \text{nm}/550 \text{ nm} = 2.25 \text{ eV}$. With a mixture of $In_xGa_{1-x}N$, the effective energy gap would be

$$x(0.7 \text{ eV}) + (1-x)(3.4 \text{ eV}) = 2.25 \text{ eV} \qquad \text{or} \qquad x = 0.43$$

so we would need 43% InN and 57% GaN.
(b) For violet light, the photon energy is $E = hc/\lambda = 1240 \text{ eV} \cdot \text{nm}/400 \text{ nm} = 3.10 \text{ eV}$ and

$$x(0.7 \text{ eV}) + (1-x)(3.4 \text{ eV}) = 3.10 \text{ eV} \qquad \text{or} \qquad x = 0.11$$

The mixture should be 11% InN and 89% GaN.

38. The number of electrons in the narrow strip is $dN = N(E)\,dE$, where the width of the strip is $dE = \mu_B B_{\text{app}}$. We'll assume that we are at a low enough temperature that we can approximate the Fermi-Dirac distribution by its $T = 0$ limit, and we'll also assume that the strip is so narrow that we can replace $N(E)$ by its value at $E = E_F$. Then using Equations 10.47 and 10.49 with $E = E_F$, we obtain for the fraction $dN/N$:

$$\frac{dN}{N} = \frac{E_F^{1/2}dE}{\frac{2}{3}E_F^{3/2}} = \frac{3\mu_B B_{\text{app}}}{2E_F} = \frac{3(9.27 \times 10^{-24} \text{ J/T})(1 \text{ T})}{2(3.15 \text{ eV})(1.6 \times 10^{-19} \text{ J/eV})} = 2.8 \times 10^{-5} = 0.0028\%$$

Even with a relatively large field of 1 T, only a small fraction of the electron spins will flip.

39. (a) The states have energies of $E = -g_J \mu_B m_J B_{\text{app}} = \mp \mu_B B_{\text{app}}$ with $g_J = 2$ and $m_J = \pm 1/2$.

$$E = \mp \mu_B B_{\text{app}} = \mp (9.27 \times 10^{-24} \text{ J/T})(0.25 \text{ T}) = \mp 2.32 \times 10^{-24} \text{ J} = \mp 1.45 \times 10^{-5} \text{ eV}$$

At room temperature, $kT = (8.617 \times 10^{-5} \text{ eV/K})(293 \text{ K}) = 0.0252 \text{ eV}$. The Boltzmann population factors are then

$$\frac{N_+}{N_+ + N_-} = \frac{e^{-E_+/kT}}{e^{-E_+/kT} + e^{-E_-/kT}} = \frac{e^{-1.45 \times 10^{-5} \text{ eV/0.0252 eV}}}{e^{-1.45 \times 10^{-5} \text{ eV/0.0252 eV}} + e^{+1.45 \times 10^{-5} \text{ eV/0.0252 eV}}} = 0.4997$$

$$\frac{N_-}{N_+ + N_-} = 1 - \frac{N_+}{N_+ + N_-} = 1 - 0.4997 = 0.5003$$

The difference between the relative spin-down and spin-up populations is 0.0006.
(b) At 4.2 K, $kT = (8.617 \times 10^{-5} \text{ eV/K})(4.2 \text{ K}) = 3.62 \times 10^{-4} \text{ eV}$ and so

$$\frac{N_+}{N_+ + N_-} = \frac{e^{-E_+/kT}}{e^{-E_+/kT} + e^{-E_-/kT}} = \frac{e^{-1.45 \times 10^{-5} \text{ eV/3.62} \times 10^{-4} \text{ eV}}}{e^{-1.45 \times 10^{-5} \text{ eV/3.62} \times 10^{-4} \text{ eV}} + e^{+1.45 \times 10^{-5} \text{ eV/3.62} \times 10^{-4} \text{ eV}}} = 0.480$$

$$\frac{N_-}{N_+ + N_-} = 1 - \frac{N_+}{N_+ + N_-} = 1 - 0.480 = 0.520$$

At the lower temperature, the population difference is about 4%.

40.  (a) For Li, with $\rho = 0.534$ g/cm$^3$ and $M = 6.94$ g/mole,

$$\frac{N}{V} = \frac{\rho N_A}{M} = \frac{(0.534 \times 10^3 \text{ kg/m}^3)(6.02 \times 10^{23} \text{ atoms/mole})}{6.94 \times 10^{-3} \text{ kg/mole}} = 4.63 \times 10^{28} \text{ m}^{-3}$$

$$\chi = \frac{3\mu_0 \mu_B^2}{2E_F} \frac{N}{V} = \frac{3(4\pi \times 10^{-7} \text{ T} \cdot \text{m/A})(9.27 \times 10^{-24} \text{ J/T})^2 (4.63 \times 10^{28} \text{ m}^{-3})}{2(4.70 \text{ eV})(1.60 \times 10^{-19} \text{ J/eV})} = 10.0 \times 10^{-6}$$

The experimental value for comparison is

$$\chi_{\text{expt}} = \frac{4\pi\rho}{M} \chi_{\text{molar}}^{\text{cgs}} = \frac{4\pi(0.534)}{6.94}(14.2 \times 10^{-6}) = 13.7 \times 10^{-6}$$

(b) For Ba, with $\rho = 3.50$ g/cm$^3$ and $M = 137.3$ g/mole,

$$\frac{N}{V} = \frac{\rho N_A}{M} = \frac{(3.504 \times 10^3 \text{ kg/m}^3)(6.02 \times 10^{23} \text{ atoms/mole})}{137.34 \times 10^{-3} \text{ kg/mole}} = 1.53 \times 10^{28} \text{ m}^{-3}$$

$$\chi = \frac{3\mu_0 \mu_B^2}{2E_F} \frac{N}{V} = \frac{3(4\pi \times 10^{-7} \text{ T} \cdot \text{m/A})(9.27 \times 10^{-24} \text{ J/T})^2 (2 \times 1.53 \times 10^{28} \text{ m}^{-3})}{2(3.65 \text{ eV})(1.60 \times 10^{-19} \text{ J/eV})} = 8.5 \times 10^{-6}$$

The experimental value for comparison is

$$\chi_{\text{expt}} = \frac{4\pi\rho}{M} \chi_{\text{molar}}^{\text{cgs}} = \frac{4\pi(3.50)}{137.3}(20.6\times10^{-6}) = 6.6\times10^{-6}$$

41. (a) For gold, the number of free electrons per unit volume is

$$\frac{N}{V} = \frac{\rho N_{\text{A}}}{M} = \frac{(19.3\times10^3 \text{ kg/m}^3)(6.02\times10^{23} \text{ atoms/mole})}{0.197 \text{ kg/mole}} = 5.90\times10^{28} \text{ m}^{-3}$$

The Pauli paramagnetic susceptibility is

$$\chi = \frac{3\mu_0\mu_{\text{B}}^2}{2E_{\text{F}}} \frac{N}{V} = \frac{3(4\pi\times10^{-7} \text{ T·m/A})(9.27\times10^{-24} \text{ J/T})^2(5.90\times10^8 \text{ m}^{-3})}{2(5.53 \text{ eV})(1.60\times10^{-19} \text{ J/eV})} = 10.8\times10^{-6}$$

(b) The experimental volume susceptibility in SI units is

$$\chi_{\text{volume}}^{\text{SI}} = 4\pi\frac{\rho}{M}\chi_{\text{molar}}^{\text{cgs}} = 4\pi\frac{19.3 \text{ g/cm}^3}{197 \text{ g/mole}}(-28.0\times10^{-6}) = -34.5\times10^{-6}$$

With $\chi_{\text{total}} = \chi_{\text{para}} + \chi_{\text{dia}}$, we have $\chi_{\text{dia}} = -45.3\times10^{-6}$.

42. (a) For $MnCl_2$, $\rho = 3.0$ g/cm$^3$ and $M = 126$ g/mole. The SI volume susceptibility is then

$$\chi_{\text{volume}}^{\text{SI}} = 4\pi\frac{\rho}{M}\chi_{\text{molar}}^{\text{cgs}} = 4\pi\frac{3.0 \text{ g/cm}^3}{126 \text{ g/mole}}(14350\times10^{-6}) = 4.29\times10^{-3}$$

$$\frac{N}{V} = \frac{\rho N_{\text{A}}}{M} = \frac{(3.0\times10^3 \text{ kg/m}^3)(6.02\times10^{23} \text{ atoms/mole})}{0.126 \text{ kg/mole}} = 1.43\times10^{28} \text{ m}^{-3}$$

$$g_J^2 J(J+1) = \frac{3kT\chi}{\mu_0(N/V)\mu_{\text{B}}^2} = \frac{3(1.38\times10^{-23} \text{ J/K})(293 \text{ K})(4.29\times10^{-3})}{(4\pi\times10^{-7} \text{ T·m/A})(1.43\times10^{28} \text{ m}^{-3})(9.27\times10^{-24} \text{ J/T})^2} = 33.7$$

(b) The neutral Mn atom has the electronic configuration $3d^5s^2$. If we remove the two outer $4s$ electrons, the $Mn^{++}$ ion will have the configuration $3d^5$ The $3d$ subshell can accommodate a total of 10 electrons with 5 different $m_l$ values. Without duplicating any $m_l$ values, each of the five $3d$ electrons in $Mn^{++}$ can have $m_s = +1/2$, so we have $S = 5/2$. The 5 electrons with $m_s = +1/2$ must use up all of the allowed $m_l$ values for $l = 2$ (+2, +1, 0, −1, −2), so the total $L$ is zero.
(c) With $J = L + S = 0 + 5/2 = 5/2$, we have

$$g_J = \sqrt{33.7/J(J+1)} = \sqrt{33.7/(2.5)(3.5)} = 1.96$$

This is in excellent agreement with the value 2.0 expected when the magnetic moment depends only on the spin and not on the orbital angular momentum.

43. Let $a$ represent the spacing between the ions. Any particular ion in the lattice will experience an attractive potential energy due to the two closest neighbors of the opposite charge:

$$U_1 = 2\frac{(+e)(-e)}{4\pi\varepsilon_0}\frac{1}{a} = -2\frac{e^2}{4\pi\varepsilon_0}\frac{1}{a}$$

At a distance of $2a$ from this ion, there will be two ions of the same charge, so the next term in the series is

$$U_2 = +2\frac{e^2}{4\pi\varepsilon_0}\frac{1}{2a}$$
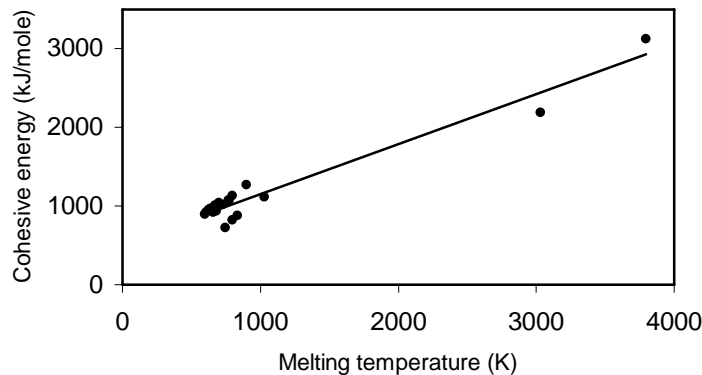
The series continues as follows:

$$U_3 = -2\frac{e^2}{4\pi\varepsilon_0}\frac{1}{3a} \qquad \text{and} \qquad U_4 = +2\frac{e^2}{4\pi\varepsilon_0}\frac{1}{4a}$$

and so forth. The total potential energy is $U = U_1 + U_2 + U_3 + U_4 + \cdots$, or

$$U = -2\frac{e^2}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{2a}+\frac{1}{3a}-\frac{1}{4a}+\cdots\right) = -2\frac{e^2}{4\pi\varepsilon_0 a}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = -2\frac{e^2}{4\pi\varepsilon_0 a}(\ln 2) = -\alpha\frac{e^2}{4\pi\varepsilon_0 a}$$
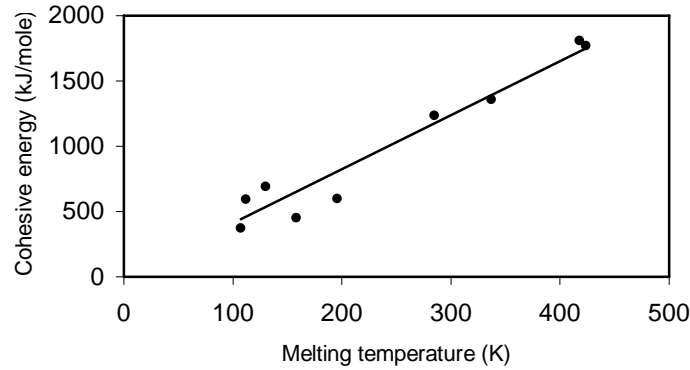
where $\alpha = 2\ln 2$.

44.



There does seem to be a rough correlation between cohesive energy and melting point.

45.



There seems to be a very good correlation between cohesive energy and melting point for the metallic solids – the larger the cohesive energy, the higher the melting point. The line shows that the relationship is very nearly linear.

46. (a) With $U(R) = -\alpha \dfrac{e^2}{4\pi\varepsilon_0} \dfrac{1}{R} + \dfrac{\alpha e^2 R_0^{n-1}}{4\pi\varepsilon_0 n} \dfrac{1}{R^n}$ , we obtain

$$F(R) = -\frac{dU}{dR} = -\frac{\alpha e^2}{4\pi\varepsilon_0}\left( \frac{1}{R^2} - \frac{R_0^{n-1}}{R^{n+1}} \right)$$

(b) $F(R_0 + x) = -\dfrac{dU}{dR} = -\dfrac{\alpha e^2}{4\pi\varepsilon_0}\left( \dfrac{1}{(R_0 + x)^2} - \dfrac{R_0^{n-1}}{(R_0 + x)^{n+1}} \right)$

$$= -\frac{\alpha e^2}{4\pi\varepsilon_0} \frac{1}{(R_0 + x)^2}\left( 1 - \frac{1}{(1 + x/R_0)^{n-1}} \right) \cong -\frac{\alpha e^2}{4\pi\varepsilon_0 R_0^2} \frac{1}{(1 + x/R_0)^2}\left[ (n-1)\frac{x}{R_0} \right]$$

$$= -\frac{\alpha e^2 (n-1)}{4\pi\varepsilon_0 R_0^3} x = -kx \qquad \text{with} \qquad k = \frac{\alpha e^2 (n-1)}{4\pi\varepsilon_0 R_0^3}$$

(c) $k = \dfrac{\alpha e^2 (n-1)}{4\pi\varepsilon_0 R_0^3} = \dfrac{(1.75)(1.44 \text{ eV}\cdot\text{nm})(8)}{(0.281 \text{ nm})^3} = 909 \text{ eV/nm}^2$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{kc^2}{mc^2}} = \frac{1}{2\pi}\sqrt{\frac{(909 \text{ eV/nm}^2)(9.00\times10^{16} \text{ m}^2/\text{s}^2)}{(23 \text{ u})(931.5\times10^6 \text{ eV/u})(10^{-9} \text{ m/nm})^2}} = 9.83\times10^{12} \text{ Hz}$$

(d) $\lambda = \dfrac{c}{f} = \dfrac{3.00\times10^8 \text{ m/s}}{9.83\times10^{12} \text{ Hz}} = 30.5\times10^{-6} \text{ m} = 30.6 \ \mu\text{m}$

which is in the infrared region of the electromagnetic spectrum.

47. The potential energy of electric dipole B in the electric field of dipole A is $U = -\vec{\mathbf{p}}_B \cdot \vec{\mathbf{E}}_A$.
With $p_B \propto E_A$ and $E_A \propto 1/r^3$, it follows that $p_B \propto 1/r^3$ and thus $U \propto 1/r^6$. With
$U = C/r^6$, we obtain

$$F = -\frac{dU}{dr} = -\frac{d}{dr}Cr^{-6} = 6Cr^{-7}$$

48.

| $T$ (K) | $C$ (J/mole·K) | | $T$ (K) | $C$ (J/mole·K) | | $T$ (K) | $C$ (J/mole·K) |
|---|---|---|---|---|---|---|---|
| 1 | 0.00138 | | 10 | 0.0378 | | 50 | 3.83 |
| 2 | 0.00292 | | 15 | 0.108 | | 60 | 5.78 |
| 3 | 0.00475 | | 20 | 0.240 | | 70 | 7.75 |
| 4 | 0.00705 | | 25 | 0.473 | | 80 | 9.64 |
| 6 | 0.0135 | | 30 | 0.851 | | 90 | 11.4 |
| 8 | 0.0238 | | 40 | 2.09 | | 100 | 13.0 |
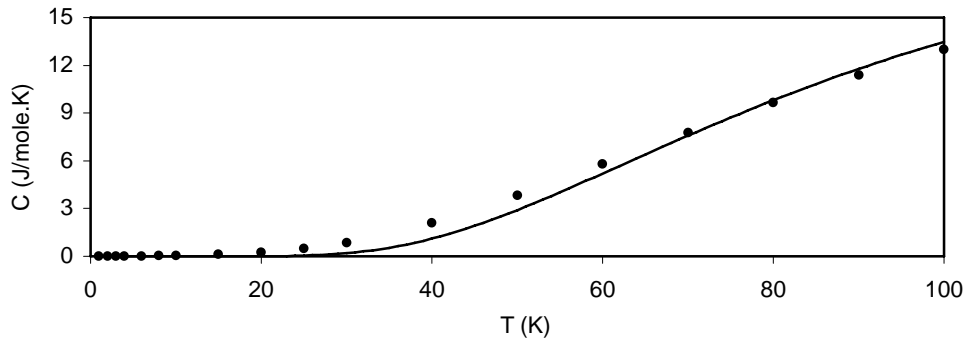


The best fit with the Einstein model is obtained for an Einstein temperature of about
280 K, but the lowest temperature data are not fit very well.
(b)



244
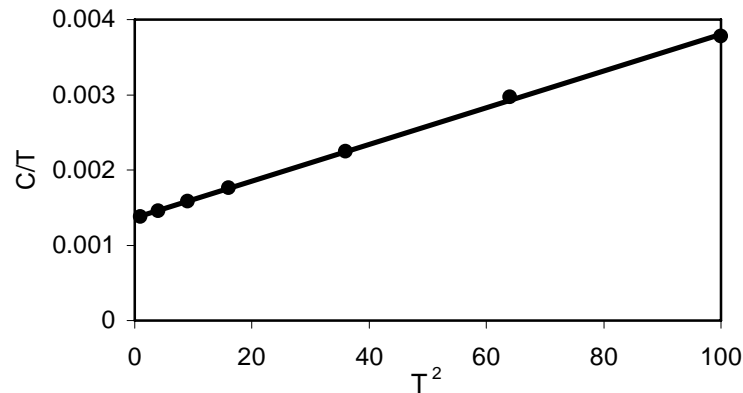
The slope of the line is $b = 2.55 \times 10^{-5}$ J/mole·K$^4$.  From Equation 11.12 we have

$$T_D = \left(\frac{12\pi^4 R}{5b}\right)^{1/3} = \left[\frac{12\pi^4 (8.31 \text{ J/mole} \cdot \text{K})}{5(2.44 \times 10^{-5} \text{ J/mole} \cdot \text{K}^4)}\right]^{1/3} = 430 \text{ K}$$

The intercept of the line is $1.37 \times 10^{-3}$ J/mole·K$^2$.  With $E_F = 11.7$ eV, Equation 11.9 gives the expected intercept as

$$a = \frac{3\pi^2 kR}{2E_F} = \frac{3\pi^2 (8.617 \times 10^{-5} \text{ eV/K})(8.31 \text{ J/mole} \cdot \text{K})}{2(11.7 \text{ eV})} = 9.06 \times 10^{-4} \text{ J/mole} \cdot \text{K}^2$$

The expected value can be forced into agreement with the value deduced from the graph if we modify the Fermi energy by reducing it by a factor of 1.51.  This reduction in the Fermi energy in turn comes about by using an effective mass for the electron that is 1.51 times its actual mass.

49.  (a)

| $T$ (K) | $C$ (J/mole·K) | | $T$ (K) | $C$ (J/mole·K) | | $T$ (K) | $C$ (J/mole·K) |
|---|---|---|---|---|---|---|---|
| 1 | 0.00118 | | 10 | 0.433 | | 50 | 14.3 |
| 2 | 0.00493 | | 15 | 1.46 | | 60 | 16.6 |
| 3 | 0.0138 | | 20 | 3.13 | | 70 | 18.3 |
| 4 | 0.0315 | | 25 | 5.18 | | 80 | 19.5 |
| 6 | 0.0985 | | 30 | 7.31 | | 90 | 20.5 |
| 8 | 0.236 | | 40 | 11.3 | | 100 | 21.3 |
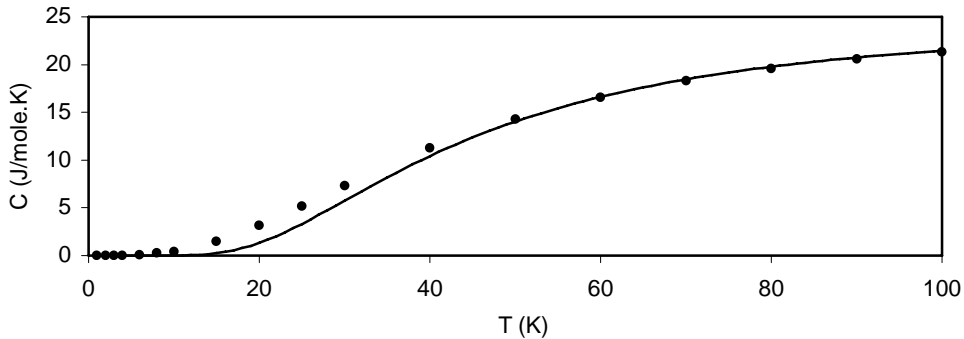


The best fit to the higher temperature data is for an Einstein temperature of about 135 K, although the Einstein model doesn't fit the lower temperature data very well.
(b)
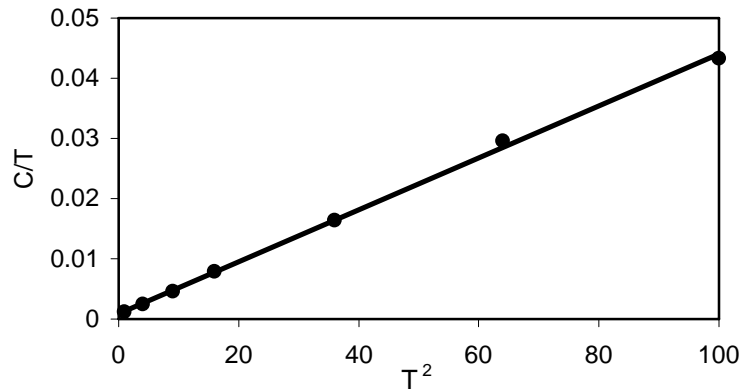
The slope of the line is $b = 4.31 \times 10^{-4}$ J/mole·K$^4$. From Equation 11.12 we have

$$T_{\text{D}} = \left( \frac{12\pi^4 R}{5b} \right)^{1/3} = \left[ \frac{12\pi^4 (8.31 \text{ J/mole} \cdot \text{K})}{5(4.31 \times 10^{-4} \text{ J/mole} \cdot \text{K}^4)} \right]^{1/3} = 165 \text{ K}$$

The intercept is $a = 8.95 \times 10^{-4}$ J/mole·K$^2$. From Equation 11.9 we have (with $E_{\text{F}} = 5.53$ eV)

$$a = \frac{\pi^2 kR}{2E_{\text{F}}} = \frac{\pi^2 (8.617 \times 10^{-5} \text{ eV/K})(8.31 \text{ J/mole} \cdot \text{K})}{2(5.53 \text{ eV})} = 6.39 \times 10^{-4} \text{ J/mole} \cdot \text{K}^2$$

If we reduce the Fermi energy by using an effective mass for the electron of 1.40 times its actual mass, the expected value of the intercept will agree with the value obtained from the graph.

50. (a) If we take $p = \sqrt{2mE}$, then $dp = \frac{1}{2}\sqrt{2m/E}\, dE$. With $E = E_{\text{F}}$, $dE = E_{\text{g}}$, and $m = 2m_{\text{e}}$, we have

$$\Delta x \sim \frac{\hbar}{\Delta p} = \frac{\hbar}{\sqrt{m_{\text{e}}/E_{\text{F}}}\, E_{\text{g}}} = \sqrt{\frac{E_{\text{F}}}{m_{\text{e}}c^2}} \frac{\hbar c}{E_{\text{g}}}$$

(b) For aluminum,

$$\Delta x \sim \sqrt{\frac{E_{\text{F}}}{m_{\text{e}}c^2}} \frac{\hbar c}{E_{\text{g}}} = \sqrt{\frac{11.7 \text{ eV}}{0.511 \times 10^6 \text{ eV}}} \frac{197 \text{ eV} \cdot \text{nm}}{0.34 \times 10^{-3} \text{ eV}} = 2.8 \times 10^3 \text{ nm}$$

This amounts to about $10^4$ times the atomic separation in aluminum, so the Cooper pair is indeed very large compared with the distance between atoms.

51. (a) $\quad N = \dfrac{662 \times 10^{3} \text{ eV}}{0.66 \text{ eV/electron}} = 1.00 \times 10^{6}$ electrons

(b) $\quad \delta N = \sqrt{N} = 1.00 \times 10^{3} \qquad \dfrac{\delta N}{N} = 1.00 \times 10^{-3}$

(c) $\quad \dfrac{\delta E}{E} = \dfrac{\delta N}{N} = 1.00 \times 10^{-3} \qquad \delta E = \dfrac{\delta E}{E} E = (1.00 \times 10^{-3})(662 \text{ keV}) = 0.662 \text{ keV}$

52. Grouping the constants in Equation 11.39 together, we can write the susceptibility as
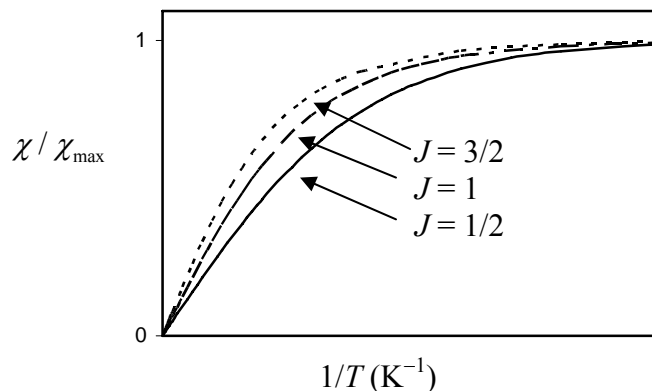
$$\chi = -C \frac{\displaystyle\sum_{m_J=-J}^{+J} m_J e^{-m_J c/T}}{\displaystyle\sum_{m_J=-J}^{+J} e^{-m_J c/T}} = C \frac{\displaystyle\sum_{m_J=+J}^{-J} m_J e^{m_J c/T}}{\displaystyle\sum_{m_J=-J}^{+J} e^{-m_J c/T}}$$

We then have:

$J = 1/2: \qquad \chi = C \dfrac{0.5 e^{0.5c/T} + (-0.5) e^{-0.5c/T}}{e^{-0.5c/T} + e^{0.5c/T}}$

$J = 1: \qquad \chi = C \dfrac{(1.0) e^{c/T} + 0 + (-1.0) e^{-c/T}}{e^{-c/T} + 1 + e^{c/T}}$

$J = 3/2: \qquad \chi = C \dfrac{(1.5) e^{1.5c/T} + (0.5) e^{0.5c/T} + (-0.5) e^{-0.5c/T} + (-1.5) e^{-1.5c/T}}{e^{-1.5c/T} + e^{-0.5c/T} + e^{0.5c/T} + e^{1.5c/T}}$



53. The interaction energy of two adjacent dipoles in iron separated by the atomic spacing (0.248 nm) is

$$E = \mu B = \frac{\mu_0 \mu^2}{2\pi r^3} = \frac{(4\pi \times 10^{-7} \text{ T} \cdot \text{m/A})(2.2 \times 9.27 \times 10^{-24} \text{ J/T})^2}{2\pi (0.248 \times 10^{-9} \text{ m})^3 (1.6 \times 10^{-19} \text{ J/eV})} = 3.4 \times 10^{-5} \text{ eV}$$

This would correspond to a thermal energy of $kT$ for $T = 0.4$ K. That is, above about 0.4 K the thermal energy is sufficient to destroy the tendency of the dipoles to align due to the interaction with neighboring dipoles. This indicates that the dipole-dipole interaction is far too weak to be responsible for ferromagnetism as observed at ordinary temperatures.

54. (a) In $N_2$, the six $2p$ electrons fill all of the bonding states, with 2 electrons in each state ($p_x$, $p_y$, $p_z$). These electrons are paired with spin up and spin down. When all electrons spins are paired, we expect the substance to be diamagnetic. $O_2$ has 2 additional electrons, which go into the first antibonding state. This state includes the degenerate $p_y$ and $p_z$ orbitals, so it is possible to have the 2 electron spins in the same direction without violating the Pauli principle. So for oxygen to be paramagnetic, we must indeed have the two spins in the antibonding states in the same direction. (b) NO would have a single electron in the antibonding state, and thus all electrons cannot be paired. We therefore expect NO to be paramagnetic. It is indeed observed to be paramagnetic, with a magnetic susceptibility of roughly half that of $O_2$.

# Chapter 12

This chapter presents a basic introduction to the structure of nuclei, including masses, binding energies, and radioactive decays. New to this edition are sections that improve the integration of nuclear properties with previous material in the text, thereby presenting a more coherent view of nuclear structure as a manifestation of phenomena that have already been discussed in quantum mechanics or applied to atoms or molecules. Examples of this material include proton and neutron separation energies (which are the counterpart of ionization energies in atoms), quantum states in nuclei (similar to states in potential energy wells discussed in Chapter 5 and filled in analogy to electron states in atoms as discussed in Chapter 8), and nuclear rotational and vibrational excited states (analogous to the molecular states considered in Chapter 9).

## Supplemental Materials

A complete tabulation of nuclear masses, decay properties, isotopic abundances, excited states is:
R. B. Firestone and V. S. Shirley, editors, *Table of Isotopes, 8th Edition* (Wiley, 1999).

A more compact listing of radioactive isotopes and their decay properties can be found in the *Table of Radioactive Isotopes*, which is available both in print format (E. Browne and R. B. Firestone, editors; Wiley, 1986) and electronic format (http://ie.lbl.gov/toi/).

## Suggestions for Additional Reading

The following are some intermediate-level, comprehensive nuclear physics texts.
B. L. Cohen, *Concepts of Nuclear Physics* (McGraw-Hill, 1971).
R. D. Evans, *The Atomic Nucleus* (McGraw-Hill, 1955).
I. Kaplan, *Nuclear Physics* (Addison-Wesley, 1962).
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## Materials for Active Engagement in the Classroom

### A. Reading Quizzes

1. The nuclear force
     (1) has infinite range, like the electromagnetic or gravitational force.
     (2) becomes infinite in strength as the distance between two particles approaches zero.
     (3) is exerted by each proton or neutron on only its nearest neighbors.

(4) is exerted by each proton or neutron on all other protons or neutrons in the nucleus.

2. The binding energy of a nucleus is:
    (1) the energy needed to remove one proton or neutron.
    (2) the energy needed to take apart a nucleus into its constituent protons and neutrons.
    (3) the energy with which the nucleus attracts the atomic electrons.
    (4) the energy equivalent of the mass of the nucleus.

3. The number of neutrons in a nucleus is:
    (1) always equal to the number of protons.
    (2) usually greater than the number of protons.
    (3) usually smaller than the number of protons.
    (4) always equal to the number of electrons in the atom.

4. In which type of decay process is the total number of protons before the decay <u>not</u> equal to the total number of protons after the decay?
    (1) Alpha decay        (2) Beta decay          (3) Gamma decay
    (4) All of the decay processes        (5) None of the decay processes

5. Which of the following is allowed to change in a radioactive decay process?
    (1) Total energy        (2) Total number of nucleons (protons plus neutrons)
    (3) Total electric charge        (4) Total number of electrons

**Answers**      1. 3      2. 2      3. 2      4. 2      5. 4


## B. Conceptual or Discussion Questions

1. Suppose we combine two nuclei of Ca to make a single nucleus of Zr.
    (a) Compared with one of the original Ca nuclei, the newly formed nucleus of Zr will have:
        (1) twice the radius      (2) twice the surface area        (3) twice the volume
    (b) If $B$ represents the <u>total</u> binding energy of a Ca nucleus, then the total binding energy of a Zr nucleus is approximately
        (1) $0.5B$         (2) $B$          (3) $2B$           (4) $4B$          (5) $8B$

2. For two protons separated by a distance of about 1 fm (a typical separation in a nucleus), the attractive nuclear (strong) force is stronger than the Coulomb repulsion force. Why then do nuclei need neutrons? Why don't we find nuclei with $Z$ protons and zero neutrons?

3. Light nuclei generally have $N \approx Z$, but more massive nuclei have $N \approx 1.5Z$. That is, as nuclei become more massive the number of neutrons increases more rapidly than the number of protons. Why?

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4. Choose from among the following decay processes:
    (1) Alpha decay    (2) All beta decays    (3) Negative beta decay
    (4) Positive beta decay    (5) Negative or positive beta decay
    (6) Electron capture decay    (7) Gamma decay    (8) Alpha and gamma decays
    (9) Alpha, electron capture, and gamma decays    (10) Alpha, beta, and gamma decays
    (a) In which type of decay might we expect to see bremsstrahlung?
    (b) Which type of decay is accompanied by monoenergetic X rays?
    (c) Which type of decay is accompanied by two 0.511-MeV photons?
    (d) In which type of decay is a monoenergetic particle emitted?
    (e) In which type of decay is new matter created?
    (f) Assuming the same Q value, in which type of decay would the residual nucleus
    have the largest kinetic energy?

**Answers**    1. (a) 3  (b) 3
    2. Neutrons are necessary to oppose the Coulomb force with its infinite range.
    3. The Coulomb force (with infinite range) affects all protons in the nucleus.
    4. (a) 5  (b) 6  (c) 4  (d) 9  (e) 2  (f) 1

## Sample Exam Questions

### A. Multiple Choice

1. A sample contains a large number of radioactive nuclei. At any instant of time, the
   rate of decay is:
    (a) directly proportional to the number of nuclei that have already decayed.
    (b) directly proportional to the number of nuclei that have not yet decayed.
    (c) constant in time.
    (d) directly proportional to the half-life of the decay.

2. The nuclear force:
    (a) has infinite range.
    (b) is generally stronger than the electromagnetic force.
    (c) becomes infinite as the distance between particles approaches zero.
    (d) acts on electrons that may be inside the nucleus.

3. Nucleus A has a half-life $T$ and nucleus B has a half-life $2T$. Initially the number of
   nuclei of type A equals the number of nuclei of type B. After a certain time, 10% of
   the nuclei of type B remain. At this same time, what fraction of the nuclei of type A
   remains?
    (a) 5%        (b) 1%        (c) 0.01%        (d) 20%        (e) 50%

4. The energy necessary to remove a proton or a neutron from a nucleus is typically
   about
    (a) 1 MeV        (b) 10 MeV        (c) 100 MeV        (d) 1000 MeV

5. Two nuclei of $^{40}$Ca (atomic number 20) undergo fusion to form a nucleus of $^{80}$Zr (atomic number 40). The total binding energy of $^{40}$Ca is $B$. What would be the best estimate for the total binding energy of $^{80}$Zr?
       (a) $4B$         (b) $2B$         (c) $B$         (d) $B/2$         (e) $B/4$

**Answers**     1. b    2. b    3. b    4. b    5. b

## B. Conceptual

1. A nucleus of $^4$He absorbs a photon of energy $E$ which causes it to split apart into two nuclei of $^2$H. The two $^2$H nuclei fly apart with kinetic energies $K_1$ and $K_2$. Is the total final kinetic energy $K$ (which is equal to $K_1 + K_2$) *greater than, less than, or equal to* the photon energy $E$? EXPLAIN YOUR ANSWER.

2. Suppose we can break apart a nucleus of $^{48}_{24}\text{Cr}_{24}$ in two different ways: 2 nuclei of $^{24}_{12}\text{Mg}_{12}$ or 3 nuclei of $^{16}_{8}\text{O}_8$. Is the amount of energy required to break it into two $^{24}_{12}\text{Mg}_{12}$ *greater than, less than, or equal to* the amount of energy required to break it into three $^{16}_{8}\text{O}_8$? EXPLAIN YOUR ANSWER.

3. You wish to obtain a supply of 20 free neutrons and 20 free protons. You have available either one nucleus of $^{40}$Ca ($Z = 20$, $N = 20$) or two nuclei of $^{20}$Ne ($Z = 10$, $N = 10$). Will the energy necessary to obtain the 20 neutrons and 20 protons from one $^{40}$Ca nucleus be *less than, greater than, or equal to* the energy necessary to obtain the same number of neutrons and protons from the two $^{20}$Ne nuclei? EXPLAIN YOUR ANSWER.

4. $^{224}$Ra can decay either by alpha emission or by $^{12}$C emission. The probability for alpha emission is about $10^9$ greater than the probability for $^{12}$C emission. How would you explain this great difference?

**Answers**     1. less than    2. less than    3. greater than
                      4. The Coulomb barrier for $^{12}$C is 3 times higher than for $^4$He.

## C. Problems

1. (a) Natural uranium today consists of about 0.7% of the isotope $^{235}$U (half life $= 7.1 \times 10^8$ y) and 99.3 % of $^{238}$U (half life $= 4.5 \times 10^9$ y). At some time in the past, natural uranium would have contained 3.0% of $^{235}$U, enough to make natural water-moderated fission reactors. How long before the present time did this occur?
   (b) Compute the binding energy per nucleon of $^{238}$U (atomic number $= 92$, atomic mass $= 238.050784$ u).

2. Consider the decay $^{226}_{88}\text{Ra} \rightarrow {}^{212}_{82}\text{Pb} + {}^{14}_{6}\text{C}$, which is similar to alpha decay.

   (a) The kinetic energy of the $^{14}_{6}\text{C}$ emitted in this decay process is 26.704 MeV. Calculate the kinetic energy of the "recoiling" $^{212}_{82}\text{Pb}$. Assume the original $^{226}_{88}\text{Ra}$ is at rest before the decay.

   (b) Given the atomic masses of $^{226}_{88}\text{Ra}$ (226.025410 u) and $^{14}_{6}\text{C}$ (14.003242 u), find the atomic mass of $^{212}_{82}\text{Pb}$.

   (c) Suppose the half-life for this decay is $T$. In terms of $T$, how long does it take for 95% of the nuclei in a sample of $^{226}_{88}\text{Ra}$ to decay?

3. In a process similar to alpha decay, a nucleus can emit a $^{12}\text{C}$ particle instead of an alpha particle ($^4\text{He}$). Consider the decay $^{224}_{88}\text{Ra} \rightarrow {}^{212}_{82}\text{Pb} + {}^{12}_{6}\text{C}$. The masses are:
   $m(^{224}\text{Ra}) = 224.020212$ u; $m(^{212}\text{Pb}) = 211.991898$ u; $m(^{12}\text{C}) = 12.000000$ u.
   (a) How much energy is released in this decay?
   (b) Suppose the original $^{224}\text{Ra}$ is at rest. Find the ratio of the kinetic energies $K(^{12}\text{C})/K(^{212}\text{Pb})$.

4. In the alpha decay of $^{226}\text{Ra}$ (originally at rest) to $^{222}\text{Rn}$ ($m = 222.017578$ u), an alpha particle ($m = 4.002603$ u) is emitted with a kinetic energy of 4.785 MeV. The masses given here are atomic masses.
   (a) Find the recoil kinetic energy of the $^{222}\text{Rn}$ and the total energy released in the decay.
   (b) From this information, find the atomic mass of $^{226}\text{Ra}$.
   (c) The half-life of $^{226}\text{Ra}$ is 1600 y. How long must you wait for 90% of the original Ra nuclei in a sample to decay?

5. (a) Find the total binding energy and the binding energy per nucleon for $^{61}_{28}\text{Ni}$ (atomic mass = 60.931056 u). (b) Would you expect $^{61}\text{Ni}$ to decay by beta decay? Neighboring nuclei are $^{61}_{29}\text{Cu}$ (mass = 60.933458 u) and $^{61}_{27}\text{Co}$ (mass = 60.932476 u). Explain your answer.

6. (a) Calculate the binding energy per nucleon of $^{8}_{4}\text{Be}$ (atomic mass = 8.005305 u).

   (b) $^{8}_{4}\text{Be}$ decays into two alpha particles, $^{4}_{2}\text{He}\,^{4}_{2}\text{He}_2$ (atomic mass = 4.002603 u). Calculate the kinetic energy of the two alphas if the original Be is at rest.

**Answers**
1. (a) $1.8 \times 10^9$ y   (b) 7.57 MeV
2. (a) 1.763 MeV   (b) 211.991608 u   (c) $4.32T$
3. (a) 26.4 MeV   (b) 17.7
4. (a) 0.086 MeV, 4.871 MeV   (b) 226.025410 u   (c) 5315 y
5. (a) 534.7 MeV, 8.77 MeV   (b) No, because $m_f > m_i$
6. (a) 7.06 MeV   (b) 0.0461

**Problem Solutions**

1. (a) Fluorine has $Z = 9$, and given $A = 19$, we have $N = A - Z = 19 - 9 = 10$. The symbol is $^{19}_{9}\text{F}_{10}$.

   (b) Gold has $Z = 79$, so $A = Z + N = 79 + 120 = 199$. The symbol is $^{199}_{79}\text{Au}_{120}$.

   (c) With $A = 107$ and $N = 60$, $Z = A - N = 107 - 60 = 47$. The element with $Z = 47$ is silver, and the symbol is $^{107}_{47}\text{Ag}_{60}$.

2. The atomic number of tin is $Z = 50$, and so the symbols are

$$^{114}_{50}\text{Sn}_{64}, \ ^{115}_{50}\text{Sn}_{65}, \ ^{116}_{50}\text{Sn}_{66}, \ ^{117}_{50}\text{Sn}_{67}, \ ^{118}_{50}\text{Sn}_{68}, \ ^{119}_{50}\text{Sn}_{69}, \ ^{120}_{50}\text{Sn}_{70}, \ ^{122}_{50}\text{Sn}_{72}, \ ^{124}_{50}\text{Sn}_{74}$$

3. (a) The radius of $^{16}\text{O}$ is $R = (1.2 \text{ fm})A^{1/3} = (1.2 \text{ fm})(16)^{1/3} = 3.0 \text{ fm}$. The Coulomb repulsion energy of two charges of $8e$ whose centers are separated by 6.0 fm is

$$U = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r} = \frac{e^2}{4\pi\varepsilon_0}\frac{Z^2}{r} = \frac{(1.440 \text{ MeV}\cdot\text{fm})(8^2)}{6.0 \text{ fm}} = 15 \text{ MeV}$$

   (b) For $^{238}\text{U}$, $R = (1.2 \text{ fm})A^{1/3} = (1.2 \text{ fm})(238)^{1/3} = 7.4 \text{ fm}$, and with $Z = 92$ the repulsion energy is

$$U = \frac{e^2}{4\pi\varepsilon_0}\frac{Z^2}{r} = \frac{(1.440 \text{ MeV}\cdot\text{fm})(92^2)}{14.8 \text{ fm}} = 824 \text{ MeV}$$

4. (a) $R = (1.2 \text{ fm})A^{1/3} = (1.2 \text{ fm})(197)^{1/3} = 7.0 \text{ fm}$

   (b) $R = (1.2 \text{ fm})A^{1/3} = (1.2 \text{ fm})(4)^{1/3} = 1.9 \text{ fm}$

   (c) $R = (1.2 \text{ fm})A^{1/3} = (1.2 \text{ fm})(20)^{1/3} = 3.3 \text{ fm}$

5. (a) $B = [Nm_n + Zm(^1\text{H}) - m(^A X)]c^2$
   $= [126(1.008665 \text{ u}) + 82(1.007825 \text{ u}) - 207.976652 \text{ u}](931.50 \text{ MeV/u}) = 1636.4 \text{ MeV}$
   $B/A = (1636.4 \text{ MeV})/208 = 7.868 \text{ MeV per nucleon}$

   (b) $B = [78(1.008665 \text{ u}) + 55(1.007825 \text{ u}) - 132.905452 \text{ u}](931.50 \text{ MeV/u}) = 1118.5 \text{ MeV}$
   $B/A = (1118.5 \text{ MeV})/133 = 8.410 \text{ MeV per nucleon}$

   (c) $B = [50(1.008665 \text{ u}) + 40(1.007825 \text{ u}) - 89.904704 \text{ u}](931.50 \text{ MeV/u}) = 783.9 \text{ MeV}$
   $B/A = (783.9 \text{ MeV})/90 = 8.710 \text{ MeV per nucleon}$

   (d) $B = [32(1.008665 \text{ u}) + 27(1.007825 \text{ u}) - 58.933195 \text{ u}](931.50 \text{ MeV/u}) = 517.3 \text{ MeV}$
   $B/A = (517.3 \text{ MeV})/59 = 8.768 \text{ MeV per nucleon}$

6. (a) $B = [Nm_n + Zm(^1H) - m(^A X)]c^2$

$= [2(1.008665 \text{ u}) + 2(1.007825 \text{ u}) - 4.002603 \text{ u}](931.50 \text{ MeV/u}) = 28.30 \text{ MeV}$

$B/A = (28.30 \text{ MeV})/4 = 7.074 \text{ MeV per nucleon}$

(b) $B = [10(1.008665 \text{ u}) + 10(1.007825 \text{ u}) - 19.992440 \text{ u}](931.50 \text{ MeV/u}) = 160.6 \text{ MeV}$

$B/A = (160.6 \text{ MeV})/20 = 8.032 \text{ MeV per nucleon}$

(c) $B = [20(1.008665 \text{ u}) + 20(1.007825 \text{ u}) - 39.962591 \text{ u}](931.50 \text{ MeV/u}) = 342.1 \text{ MeV}$

$B/A = (342.1 \text{ MeV})/40 = 8.551 \text{ MeV per nucleon}$

(d) $B = [30(1.008665 \text{ u}) + 25(1.007825 \text{ u}) - 54.938045 \text{ u}](931.50 \text{ MeV/u}) = 482.1 \text{ MeV}$

$B/A = (482.1 \text{ MeV})/55 = 8.765 \text{ MeV per nucleon}$

7. $^3$He: $B = [1(1.008665 \text{ u}) + 2(1.007825 \text{ u}) - 3.016029 \text{ u}](931.50 \text{ MeV/u}) = 7.718 \text{ MeV}$

$^3$H: $B = [2(1.008665 \text{ u}) + 1(1.007825 \text{ u}) - 3.016049 \text{ u}](931.50 \text{ MeV/u}) = 8.482 \text{ MeV}$

The radius of a nucleus with $A = 3$ is $R = 1.2 A^{1/3} = 1.2(3)^{1/3} = 1.7 \text{ fm}$ and the Coulomb repulsion energy of two protons separated by 1.7 fm is

$$U = \frac{e^2}{4\pi\varepsilon_0} \frac{1}{R} = \frac{1.440 \text{ eV} \cdot \text{nm}}{1.7 \text{ fm}} = 0.85 \text{ MeV}$$

This is very close to the difference in binding energy of 8.482 MeV – 7.718 MeV = 0.764 MeV, which suggests that the smaller binding energy of $^3$He arises primarily from the Coulomb repulsion of its two protons.

8. (a) $S_n(^{17}O) = [m_n + m(^{16}O) - m(^{17}O)]c^2$

$= [1.008665 \text{ u} + 15.994915 \text{ u} - 16.999132 \text{ u}](931.50 \text{ MeV/u}) = 4.143 \text{ MeV}$

(b) $S_n(^7Li) = [m_n + m(^6Li) - m(^7Li)]c^2$

$= [1.008665 \text{ u} + 6.015123 \text{ u} - 7.016005 \text{ u}](931.50 \text{ MeV/u}) = 7.250 \text{ MeV}$

(c) $S_n(^{57}Fe) = [m_n + m(^{56}Fe) - m(^{57}Fe)]c^2$

$= [1.008665 \text{ u} + 55.934937 \text{ u} - 56.935394 \text{ u}](931.50 \text{ MeV/u}) = 7.647 \text{ MeV}$

9. (a) $S_p(^4He) = [m(^1H) + m(^3H) - m(^4He)]c^2$

$= [1.007825 \text{ u} + 3.016049 \text{ u} - 4.002603 \text{ u}](931.50 \text{ MeV/u}) = 19.814 \text{ MeV}$

(b) $S_p(^{12}C) = [m(^1H) + m(^{11}B) - m(^{12}C)]c^2$

$= [1.007825 \text{ u} + 11.009305 \text{ u} - 12.000000 \text{ u}](931.50 \text{ MeV/u}) = 15.958 \text{ MeV}$

(c) $S_p(^{40}Ca) = [m(^1H) + m(^{39}K) - m(^{40}Ca)]c^2$

$= [1.007825 \text{ u} + 38.963707 \text{ u} - 39.962591 \text{ u}](931.50 \text{ MeV/u}) = 8.329 \text{ MeV}$

10. By analogy with Equation 12.8,

$$mc^2 = \frac{\hbar c}{x} = \frac{197 \text{ MeV} \cdot \text{fm}}{0.25 \text{ fm}} = 790 \text{ MeV}$$

11.  $x = \dfrac{\hbar c}{mc^2} = \dfrac{197 \text{ MeV} \cdot \text{fm}}{80 \times 10^3 \text{ MeV}} = 2.5 \times 10^{-3} \text{ fm}$

12.  (a) We first need the proton and neutron separation energies for $^{16}\text{O}$:

$$S_p(^{16}\text{O}) = [m(^1\text{H}) + m(^{15}\text{N}) - m(^{16}\text{O})]c^2$$
$$= [1.007825 \text{ u} + 15.000109 \text{ u} - 15.994915 \text{ u}](931.50 \text{ MeV/u}) = 12.127 \text{ MeV}$$
$$S_n(^{16}\text{O}) = [m_n + m(^{15}\text{O}) - m(^{16}\text{O})]c^2$$
$$= [1.008665 \text{ u} + 15.003066 \text{ u} - 15.994915 \text{ u}](931.50 \text{ MeV/u}) = 15.664 \text{ MeV}$$

The nuclear radius of $^{16}\text{O}$ is $R = R_0 A^{1/3} = (1.2 \text{ fm})16^{1/3} = 3.0 \text{ fm}$ so the volume of the nucleus is $V = \tfrac{4}{3}\pi R^3 = 116 \text{ fm}^3$.  The Fermi energy of the protons and neutrons is

$$E_{Fp} = \frac{(hc)^2}{2mc^2}\left(\frac{3Z}{8\pi V}\right)^{2/3} = \frac{(1240 \text{ eV}\cdot\text{nm})^2}{2(938.3 \text{ MeV})}\left(\frac{3 \cdot 8}{8\pi(116 \text{ fm}^3)}\right)^{2/3} = 33.$$
$$E_{Fn} = \frac{(hc)^2}{2mc^2}\left(\frac{3N}{8\pi V}\right)^{2/3} = \frac{(1240 \text{ eV}\cdot\text{nm})^2}{2(939.6 \text{ MeV})}\left(\frac{3 \cdot 8}{8\pi(116 \text{ fm}^3)}\right)^{2/3} = 33.$$

The well depths are then the sums of the separation energy and Fermi energy:

$$U_{0p} = S_p + E_{Fp} = 12.127 \text{ MeV} + 33.4 \text{ MeV} = 45.4 \text{ MeV}$$
$$U_{0n} = S_n + E_{Fn} = 15.664 \text{ MeV} + 33.4 \text{ MeV} = 49.1 \text{ MeV}$$

(b) For $^{235}\text{U}$ we have

$$S_p(^{235}\text{U}) = [m(^1\text{H}) + m(^{234}\text{Pa}) - m(^{235}\text{U})]c^2$$
$$= [1.007825 \text{ u} + 234.043308 \text{ u} - 235.043930 \text{ u}](931.50 \text{ MeV/u}) = 6.710 \text{ MeV}$$
$$S_n(^{235}\text{U}) = [m_n + m(^{234}\text{U}) - m(^{235}\text{U})]c^2$$
$$= [1.008665 \text{ u} + 234.040952 \text{ u} - 235.043930 \text{ u}](931.50 \text{ MeV/u}) = 5.297 \text{ MeV}$$

$$R = R_0 A^{1/3} = (1.2 \text{ fm})235^{1/3} = 7.4 \text{ fm} \qquad \text{and} \qquad V = \tfrac{4}{3}\pi R^3 = \tfrac{4}{3}.$$

$$E_{Fp} = \frac{(hc)^2}{2mc^2}\left(\frac{3Z}{8\pi V}\right)^{2/3} = \frac{(1240 \text{ eV}\cdot\text{nm})^2}{2(938.3 \text{ MeV})}\left(\frac{3 \cdot 92}{8\pi(1700 \text{ fm}^3)}\right)^{2/3} = 28.4 \text{ MeV}$$
$$E_{Fn} = \frac{(hc)^2}{2mc^2}\left(\frac{3N}{8\pi V}\right)^{2/3} = \frac{(1240 \text{ eV}\cdot\text{nm})^2}{2(939.6 \text{ MeV})}\left(\frac{3 \cdot 143}{8\pi(1700 \text{ fm}^3)}\right)^{2/3} = 38.1 \text{ MeV}$$

$$U_{0p} = S_p + E_{Fp} = 6.710 \text{ MeV} + 28.4 \text{ MeV} = 35.1 \text{ MeV}$$
$$U_{0n} = S_n + E_{Fn} = 5.297 \text{ MeV} + 38.1 \text{ MeV} = 43.4 \text{ MeV}$$

13. Starting with $^{160}$Dy, we first extract 2 neutrons and then extract 2 protons from the resulting $^{158}$Dy. The total energy cost is

$$E = S_{2n}(^{160}\text{Dy}) + S_{2p}(^{158}\text{Dy}) = 15.4 \text{ MeV} + 12.4 \text{ MeV} = 27.8 \text{ MeV}$$

The gain in energy if those 4 nucleons were formed into an alpha particle would be 28.3 MeV, and thus there is an excess energy of 0.5 MeV available and alpha decay is allowed. (However, the barrier penetration probability for such a low-energy alpha particle is extremely small, and as a result $^{160}$Dy is a stable nucleus.)

Starting instead with $^{164}$Dy, we first extract 2 neutrons and then 2 protons from the resulting $^{162}$Dy. The total energy cost is

$$E = S_{2n}(^{164}\text{Dy}) + S_{2p}(^{162}\text{Dy}) = 13.9 \text{ MeV} + 14.8 \text{ MeV} = 28.7 \text{ MeV}$$

The energy cost exceeds the gain (28.3 MeV) obtained by forming an alpha particle, so alpha decay is strictly forbidden for $^{164}$Dy.

14. $N/N_0 = e^{-\lambda t} = e^{-0.693t/t_{1/2}} = 0.5^{t/t_{1/2}}$

(a) $t/t_{1/2} = 2$ :    $N/N_0 = 0.5^2 = 0.25 = 1/4$

(b) $t/t_{1/2} = 4$ :    $N/N_0 = 0.5^4 = 0.0625 = 1/16$

(a) $t/t_{1/2} = 10$ :    $N/N_0 = 0.5^{10} = 1/1024 = 0.000977$

15. (a) With $a_0 = 548 \text{ s}^{-1}$ and $a = 213 \text{ s}^{-1}$ at $t = 48$ min, the radioactive decay equation $a = a_0 e^{-\lambda t}$ gives $213 \text{ s}^{-1} = (548 \text{ s}^{-1}) e^{-0.693(48 \text{ min})/t_{1/2}}$ or

$$\ln\frac{213 \text{ s}^{-1}}{548 \text{ s}^{-1}} = -\frac{0.693(48 \text{ min})}{t_{1/2}} \qquad \text{so} \qquad t_{1/2} = 35 \text{ min}$$

(b) $$\lambda = \frac{0.693}{t_{1/2}} = \frac{0.693}{35 \text{ min}} = 0.020 \text{ min}^{-1}$$

(c) $$a = a_0 e^{-\lambda t} = (548 \text{ s}^{-1}) e^{-(0.020 \text{ min}^{-1})(125 \text{ min})} = 46 \text{ s}^{-1}$$

16. With $t_{1/2} = 5.0 \text{ h} = 1.8 \times 10^4 \text{ s}$,    $\lambda = \dfrac{0.693}{t_{1/2}} = \dfrac{0.693}{1.8 \times 10^4 \text{ s}} = 3.9 \times 10^{-5} \text{ s}$

17. $$N/N_0 = e^{-0.693t/t_{1/2}} = e^{-0.693(50.0 \text{ y})/(12.3 \text{ y})} = 0.0598$$

18. (a)    $a_0 = (2.00 \text{ mCi})(3.7 \times 10^{10} \text{ s}^{-1}/\text{Ci}) = 7.40 \times 10^7 \text{ s}^{-1}$

(b)    $a = a_0 e^{-0.693t/t_{1/2}} = (7.40 \times 10^7 \text{ s}^{-1}) e^{-0.693(28 \text{ d})/(8.04 \text{ d})} = 6.62 \times 10^6 \text{ s}^{-1}$

19. (a) $$N = \frac{1000\text{ g}}{39.1\text{ g/mole}} 6.022 \times 10^{23}\text{ atoms/mole} = 1.54 \times 10^{25}\text{ atoms}$$

Of these, 0.012% or $1.85 \times 10^{21}$ are radioactive $^{40}$K. The activity is

$$a = \lambda N = \frac{0.693}{t_{1/2}} N = \frac{0.693}{(1.3 \times 10^9\text{ y})(3.156 \times 10^7\text{ s/y})} 1.85 \times 10^{21} = 3.1 \times 10^4\text{ s}^{-1} = 0.85\ \mu\text{Ci}$$

(b) In a sample of $N$ atoms, $N_r$ are radioactive and $N_{nr}$ are nonradioactive ($N = N_r + N_{nr}$). Presently, $N_r / N = 0.012\%$. With $N_r = N_{r0}e^{-\lambda t}$, we obtain for the original number of radioactive nuclei

$$N_{r0} = N_r e^{\lambda t} = N_r e^{0.693(4.5 \times 10^9\text{ y})/(1.3 \times 10^9\text{ y})} = 11.0 N_r$$

At that time ($4.5 \times 10^9$ years ago), the total number was $N_0 = N_{nr} + N_{r0}$ and the fraction of $^{40}$K was

$$\frac{N_{r0}}{N_0} = \frac{N_{r0}}{N_{nr} + N_{r0}} = \frac{11 N_r}{N_{nr} + 11 N_r} = \frac{11 N_r}{N + 10 N_r} = \frac{11(N_r / N)}{1 + 10(N_r / N)} = \frac{11(1.2 \times 10^{-4})}{1 + 10(1.2 \times 10^{-4})} = 0.13\%$$

20. Conservation of momentum gives $p_{X'} = p_\alpha$, if the original decaying nucleus is at rest.

$$Q = K_{X'} + K_\alpha = \frac{p_{X'}^2}{2m_{X'}} + \frac{p_\alpha^2}{2m_\alpha} = \frac{p_\alpha^2}{2m_{X'}} + \frac{p_\alpha^2}{2m_\alpha} = \frac{p_\alpha^2}{2m_\alpha}\left(1 + \frac{m_\alpha}{m_{X'}}\right) = K_\alpha\left(1 + \frac{m_\alpha}{m_{X'}}\right)$$

$$K_\alpha = \frac{Q}{1 + m_\alpha / m_{X'}} = Q\left(\frac{m_{X'}}{m_{X'} + m_\alpha}\right)$$

To a very good approximation,

$$m_{X'} \cong (A - 4)\text{ u} \qquad \text{and} \qquad m_{X'} + m_\alpha \cong (A - 4)\text{ u} + 4\text{ u} = A\text{ u}$$

so

$$K_\alpha \cong Q\left(\frac{A - 4}{A}\right)$$

21. (a) $^{210}$Bi $\rightarrow$ $^{206}$Tl $+ \alpha$: $\quad Q = [m(^{210}\text{Bi}) - m(^{206}\text{Tl}) - m(^4\text{He})]c^2$
$$Q = [209.984120\text{ u} - 205.976110\text{ u} - 4.002603\text{ u}](931.50\text{ MeV/u}) = +5.04\text{ MeV}$$

(b) $^{203}\text{Hg} \rightarrow {}^{199}\text{Pt} + \alpha$ :  $Q = [m(^{203}\text{Hg}) - m(^{199}\text{Pt}) - m(^4\text{He})]c^2$

$Q = [202.972872 \text{ u} - 198.970593 \text{ u} - 4.002603 \text{ u}](931.50 \text{ MeV/u}) = -0.30 \text{ MeV}$

(c) $^{211}\text{At} \rightarrow {}^{207}\text{Bi} + \alpha$ :  $Q = [m(^{211}\text{At}) - m(^{207}\text{Bi}) - m(^4\text{He})]c^2$

$Q = [210.987496 \text{ u} - 206.978471 \text{ u} - 4.002603 \text{ u}](931.50 \text{ MeV/u}) = +5.98 \text{ MeV}$

22.  $^{234}\text{U} \rightarrow {}^{230}\text{Th} + \alpha$ :  $Q = [m(^{234}\text{U}) - m(^{230}\text{Th}) - m(^4\text{He})]c^2$

$Q = [234.040952 \text{ u} - 230.033134 \text{ u} - 4.002603 \text{ u}](931.50 \text{ MeV/u}) = 4.859 \text{ MeV}$

$$K_\alpha = Q\left(\frac{A-4}{A}\right) = (4.859 \text{ MeV})\left(\frac{230}{234}\right) = 4.776 \text{ MeV}$$

23.  For negative beta decay, $^A_Z X \rightarrow {}^A_{Z+1}X' + \text{e}^- + \bar{\nu}$ , using nuclear masses $m_N$ and neglecting the mass of the neutrino,

$$Q = [m_N(^A_Z X) - m_N(^A_{Z+1}X') - m_e]c^2$$

Converting the nuclear masses to atomic masses and neglecting the atomic binding energy of the electrons, we then obtain

$$Q = \{[m(^A_Z X) - Zm_e] - [m(^A_{Z+1}X') - (Z+1)m_e] - m_e\}c^2 = [m(^A_Z X) - m(^A_{Z+1}X')]c^2$$

For positron beta decay, $^A_Z X \rightarrow {}^A_{Z-1}X' + \text{e}^+ + \nu$ ,

$$Q = [m_N(^A_Z X) - m_N(^A_{Z-1}X') - m_e]c^2$$
$$= \{[m(^A_Z X) - Zm_e] - [m(^A_{Z-1}X') - (Z-1)m_e] - m_e\}c^2 = [m(^A_Z X) - m(^A_{Z-1}X') - 2m_e]c$$

For electron capture, $^A_Z X + \text{e}^- \rightarrow {}^A_{Z-1}X' + \nu$ ,

$$Q = [m_N(^A_Z X) + m_e - m_N(^A_{Z-1}X')]c^2$$
$$= \{[m(^A_Z X) - Zm_e] + m_e - [m(^A_{Z-1}X') - (Z-1)m_e]\}c^2 = [m(^A_Z X) - m(^A_{Z-1}X')]c$$

24.  For $^{11}\text{Be} \rightarrow {}^{11}\text{B} + \text{e}^- + \bar{\nu}$ , assuming the antineutrino mass is negligible,

$$Q = [m(^{11}\text{Be}) - m(^{11}\text{B})] = [11.021658 \text{ u} - 11.009305 \text{ u}](931.50 \text{ MeV/u}) = 11.506 \text{ MeV}$$

With $Q = K_B + K_e + K_\nu$ , the electrons have their maximum kinetic energy when the antineutrino energy is zero, and assuming the kinetic energy of the $^{11}\text{B}$ is negligibly small, we conclude that the maximum electron kinetic energy is

$$K_{e,\text{max}} = Q = 11.506 \text{ MeV}$$

25. With $^{75}\text{Se} + e^- \rightarrow {}^{75}\text{As} + \nu$ and assuming the neutrino mass is negligible,

$$Q = [m(^{75}\text{Se}) - m(^{75}\text{As})]c^2 = [74.922523\ u - 74.921596\ u](931.5$$

Then $Q = K_{\text{As}} + E_\nu - K_e$ and neglecting the electron and As kinetic energies,

$$E_\nu = Q = 0.864\ \text{MeV}$$

26. (a) With $^{15}\text{O} \rightarrow {}^{15}\text{N} + e^+ + \nu$ and assuming a negligible neutrino mass,

$$Q = [m(^{15}\text{O}) - m(^{15}\text{N}) - 2m_e]c^2$$
$$= [15.003066\ u - 15.000109\ u](931.50\ \text{MeV/u}) - 2(0.511\ \text{MeV}) = 1.732\ \text{MeV}$$

(b) With $Q = K_N + K_e + K_\nu$, the kinetic energy of the $^{15}\text{N}$ is negligibly small and the electrons have their maximum kinetic energy when the neutrino energy is zero. Thus

$$K_{e,\text{max}} = Q = 1.732\ \text{MeV}$$

27. Let $E_1 = 0.000$ MeV represent the ground state, and $E_2 = 0.412$ MeV and $E_3 = 1.088$ MeV then represent the excited states. Neglecting the small recoil energy, the energies of the emitted gamma rays are equal to the energy differences between the levels:

$$\Delta E_{12} = E_2 - E_1 = 0.412\ \text{MeV} - 0.000\ \text{MeV} = 0.412\ \text{MeV}$$
$$\Delta E_{13} = E_3 - E_1 = 1.088\ \text{MeV} - 0.000\ \text{MeV} = 1.088\ \text{MeV}$$
$$\Delta E_{23} = E_3 - E_2 = 1.088\ \text{MeV} - 0.412\ \text{MeV} = 0.676\ \text{MeV}$$

28. (a) If we assume that 5 MeV is the kinetic energy of the alpha particle, then from Equation 12.23 we can find the $Q$ value to be $Q = K_\alpha A/(A-4)$, and combining this result with Equation 12.21 we can find the recoil energy of the nucleus $X'$:

$$K_{X'} = Q - K_\alpha = \frac{A}{A-4}K_\alpha - K_\alpha = \frac{4}{A-4}K_\alpha = \frac{4}{196}(5.0\ \text{MeV}) = 0.10\ \text{MeV}$$

(b) The gamma-ray recoil energy is given by Equation 12.38:

$$K_R = \frac{E_\gamma^2}{2Mc^2} = \frac{(5.0\ \text{MeV})^2}{2(200\ u)(931.5\ \text{MeV/u})} = 6.7 \times 10^{-5}\ \text{MeV} = 67\ \text{eV}$$

The recoil energy from the alpha decay is about three orders of magnitude larger than that from the gamma decay.

29.  $L = 1 \rightarrow L = 0$:  $\Delta E_{10} = E_1 - E_0 = 100.1 \text{ keV} - 0 = 100.1 \text{ keV}$
$L = 2 \rightarrow L = 0$:  $\Delta E_{20} = E_2 - E_0 = 300.9 \text{ keV} - 0 = 300.9 \text{ keV}$
$L = 2 \rightarrow L = 1$:  $\Delta E_{21} = E_2 - E_1 = 300.9 \text{ keV} - 100.1 \text{ keV} = 200.8 \text{ keV}$
$L = 3 \rightarrow L = 1$:  $\Delta E_{31} = E_3 - E_1 = 603.6 \text{ keV} - 100.1 \text{ keV} = 503.5 \text{ keV}$
$L = 3 \rightarrow L = 2$:  $\Delta E_{32} = E_3 - E_2 = 603.6 \text{ keV} - 300.9 \text{ keV} = 302.7 \text{ keV}$
$L = 4 \rightarrow L = 2$:  $\Delta E_{42} = E_4 - E_2 = 1010.0 \text{ keV} - 300.9 = 709.1 \text{ keV}$
$L = 4 \rightarrow L = 3$:  $\Delta E_{43} = E_4 - E_3 = 1010.0 \text{ keV} - 603.6 = 406.4 \text{ keV}$
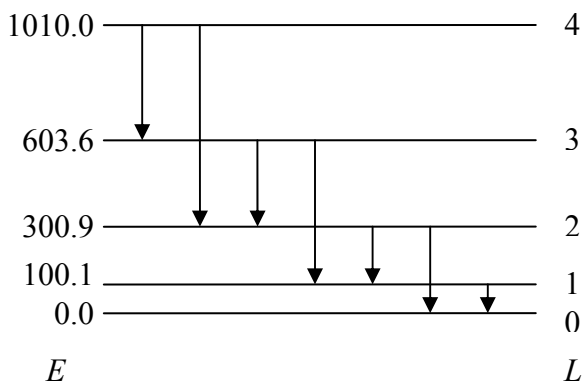


30.  The ratio of the number of Th atoms to the number of Pb atoms is

$$R = \frac{N_{\text{Th}}}{N_{\text{Pb}}} = \frac{m_{\text{Th}} N_A / M_{\text{Th}}}{m_{\text{Pb}} N_A / M_{\text{Pb}}} = \frac{m_{\text{Th}} M_{\text{Pb}}}{m_{\text{Pb}} M_{\text{Th}}} = \frac{(3.65 \text{ g})(208 \text{ g/mole})}{(0.75 \text{ g})(232 \text{ g/mole})} = 4.36$$

$$t = \frac{t_{1/2}}{0.693} \ln\left(\frac{1}{R} + 1\right) = \frac{1.41 \times 10^{10} \text{ y}}{0.693} \ln\left(\frac{1}{4.36} + 1\right) = 4.2 \times 10^9 \text{ y}$$

31.  $_{90}^{232}\text{Th}_{142} \rightarrow {}_{82}^{208}\text{Pb}_{126} + N_\alpha \, {}_2^4\text{He}_2 + N_e \, e^- + N_e \, \bar{\nu}$
(a) Balancing the mass numbers, we obtain

$$232 = 208 + 4N_\alpha \qquad \text{or} \qquad N_\alpha = 6$$

(b)  Balancing the electric charges, we obtain

$$90 = 82 + 2N_\alpha - N_e \qquad \text{or} \qquad N_e = 4$$

(c)  $Q = [m(^{232}\text{Th}) - m(^{208}\text{Pb}) - 6m(^4\text{He})]c^2$

$$= [232.038050 \text{ u} - 207.976636 \text{ u} - 6(4.002603 \text{ u})](931.50 \text{ }$$

(d) The number of atoms in the sample is

$$N = \frac{mN_A}{M} = \frac{(1000 \text{ g})(6.022 \times 10^{23} \text{ atoms/mole})}{232 \text{ g/mole}} = 2.60 \times 10^{24} \text{ atoms}$$

$$a = \lambda N = \frac{0.693}{(1.40 \times 10^{10} \text{ y})(3.156 \times 10^7 \text{ s/y})}(2.60 \times 10^{24}) = 4.07 \times 10^6 \text{ s}^{-1}$$

$$P = (42.659 \text{ MeV})(1.602 \times 10^{-13} \text{ J/MeV})(4.07 \times 10^6 \text{ s}^{-1}) = 27.8 \ \mu\text{W}$$

32.                     3.5 decays/min $= (12.4 \text{ decays/min})e^{-\lambda t}$

$$t = \frac{\ln(12.4 \text{ min}^{-1}/3.5 \text{ min}^{-1})}{\lambda} = \frac{1.26}{0.693/5730 \text{ y}} = 1.04 \times 10^4 \text{ y}$$

33. For 60° scattering with $K = 28$ MeV, Equation 6.8 gives

$$b = \frac{zZ}{2K}\frac{e^2}{4\pi\varepsilon_0}\cot\frac{\theta}{2} = \frac{(2)(82)}{2(28 \text{ MeV})}(1.440 \text{ MeV}\cdot\text{fm})(\cot 30°) = 7.3 \text{ fm}$$

The minimum distance between alpha particle and nucleus is found from Equation 6.18:

$$K = K\frac{b^2}{r_{min}^2} + \frac{e^2}{4\pi\varepsilon_0}\frac{zZ}{r_{min}}$$

$$r_{min}^2 - \frac{e^2}{4\pi\varepsilon_0}\frac{zZ}{K}r_{min} - b^2 = 0 \qquad \text{or} \qquad r_{min}^2 - 8.43 r_{min} - 53.35 = 0$$

with all distances expressed in fm. Using the quadratic formula, we find solutions for $r_{min} = 12.6$ fm or $-4.2$ fm. The negative root is physically unacceptable, so we conclude

$$r_{min} = 12.6 \text{ fm}$$

The nuclear radius of $^{208}$Pb is $R = (1.2 \text{ fm})A^{1/3} = (1.2 \text{ fm})(208)^{1/3} = 7.1$ fm. The minimum distance is thus greater than the nuclear radius of $^{208}$Pb, and even when we include the nuclear radius of the alpha particle (1.9 fm) we might question why the Rutherford formula fails even when the projectile and target are still "outside" of each other's nuclear charge distributions. Figure 12.1 shows that the nuclear charge actually extends 1-2 fm beyond the computed mean radius, so the alpha particle charge distribution extends to 3-4 fm and $^{208}$Pb to 8-9 fm. It is therefore not surprising that they begin to overlap in the range of 12-13 fm.

34. For diffraction by a circular disk, the first minimum occurs at $\theta = \sin^{-1}1.22\lambda/D$. For $^{12}$C at 420 MeV, $\theta = 51°$ and the diameter of the disk is

$$D = \frac{1.22\lambda}{\sin\theta} = \frac{1.22}{\sin\theta}\frac{hc}{E} = \frac{(1.22)(1240 \text{ MeV}\cdot\text{fm})}{(\sin 51°)(420 \text{ MeV})} = 4.64 \text{ fm} \qquad \text{so} \qquad R = 2.32 \text{ fm}$$

For $^{16}$O at 420 MeV, $\theta = 45°$ and

$$D = \frac{1.22}{\sin\theta}\frac{hc}{E} = \frac{(1.22)(1240 \text{ MeV}\cdot\text{fm})}{(\sin 45°)(420 \text{ MeV})} = 5.09 \text{ fm} \qquad \text{so} \qquad R = 2.54 \text{ fm}$$

For $^{16}$O at 360 MeV, $\theta = 53°$ and

$$D = \frac{1.22}{\sin\theta}\frac{hc}{E} = \frac{(1.22)(1240 \text{ MeV}\cdot\text{fm})}{(\sin 53°)(360 \text{ MeV})} = 5.26 \text{ fm} \qquad \text{so} \qquad R = 2.63 \text{ fm}$$

35.  We assume that the radiation is emitted by the source uniformly in all directions.  It is thus distributed over a sphere of radius $R = 25$ cm and area $4\pi R^2$, and the detector receives a fraction of the radiation equal to the fraction of the area that it occupies.  If $a_s$ represents the actual activity of the source and $a_d$ represents the activity measured by the detector of radius $r$, then $a_d = a_s(\pi r^2 / 4\pi R^2)$ or

$$a_s = a_d\frac{4\pi R^2}{\pi r^2} = (1250 \text{ s}^{-1})\frac{4\pi(25 \text{ cm})^2}{\pi(1.5 \text{ cm})^2} = 1.39\times 10^6 \text{ s}^{-1} = 37.5 \ \mu\text{Ci}$$

36.
$$n = \frac{PV}{RT} = \frac{(5.0\times 10^5 \text{ N/m}^2)(125\times 10^{-6} \text{ m}^3)}{(8.31 \text{ J/mole}\cdot\text{K})(300 \text{ K})} = 0.0251 \text{ mole}$$

$$N = (0.0251 \text{ mole})(6.022\times 10^{23} \text{ molecules/mole})(2 \text{ atoms/moleclule}) = 3.02\times 10^{22} \text{ atoms}$$

$$a = \lambda N = \frac{0.693N}{t_{1/2}} = \frac{0.693(3.02\times 10^{22})}{(12.3 \text{ y})(3.156\times 10^7 \text{ s/y})} = 5.39\times 10^{13} \text{ s}^{-1} = 1460 \text{ Ci}$$

37.  (a) Let $t_{n+1} = t_n + \Delta t$.  The number of decays $N_n$ between $t_n$ and $t_{n+1}$ is

$$N_n = N(t_n) - N(t_{n+1}) = N_0 e^{-\lambda t_n} - N_0 e^{-\lambda t_{n+1}} = N_0 e^{-\lambda t_n}(1 - e^{-\lambda\Delta t}) \cong N_0 e^{-\lambda t_n}(\lambda\,\Delta t)$$

for $\Delta t \ll 1/\lambda$.  The mean lifetime is then

$$\tau = \frac{1}{N_0}\sum_n N_n t_n = \sum_n e^{-\lambda t_n}t_n\lambda\,\Delta t = \lambda\sum_n e^{-\lambda t_n}t_n\,\Delta t$$

If the time interval $\Delta t$ becomes infinitesimally small, the sum becomes an integral:

$$\tau = \lambda\int_0^\infty e^{-\lambda t}t\,dt$$

(b) $\tau = \dfrac{1}{\lambda}\displaystyle\int_0^\infty e^{-\lambda t}(\lambda t)\,d(\lambda t) = \dfrac{1}{\lambda}\int_0^\infty e^{-x}x\,dx = \dfrac{1}{\lambda}[e^{-x}(-x-1)]_0^\infty = \dfrac{1}{\lambda}$

(c) $\tau = \dfrac{1}{\lambda} = \dfrac{t_{1/2}}{0.693} = 1.44\,t_{1/2}$   so   $\tau > t_{1/2}$

38. (a) $^{27}_{14}\text{Si}_{13} \rightarrow\ ^{27}_{13}\text{Al}_{14} + e^{+} + \nu$

(b) $^{74}_{33}\text{As}_{41} \rightarrow\ ^{74}_{34}\text{Se}_{40} + e^{-} + \overline{\nu}$

(c) $^{228}_{92}\text{U}_{136} \rightarrow\ ^{4}_{2}\text{He}_2 +\ ^{224}_{90}\text{Th}_{134}$

(d) $^{93}_{42}\text{Mo}_{51} + e^{-} \rightarrow\ ^{93}_{41}\text{Nb}_{52} + \nu$

(e) $^{131}_{53}\text{As}_{78} \rightarrow\ ^{131}_{54}\text{Xe}_{77} + e^{-} + \overline{\nu}$

39. 1.00 g of $^{239}$Pu is 1/239 mole or $(6.022 \times 10^{23})/239 = 2.52 \times 10^{21}$ atoms.  The activity is

$$a = \lambda N = \dfrac{0.693N}{t_{1/2}} = \dfrac{(0.693)(2.52 \times 10^{21})}{(2.41 \times 10^4\ \text{y})(3.156 \times 10^7\ \text{s/y})} = 2.30 \times 10^9\ \text{s}^{-1}$$

The energy output per $^{239}$Pu nucleus is

$$Q = [m(^{239}\text{Pu}) - m(^{235}\text{U}) - m(^4\text{He})]c^2$$
$$= [239.052163\ \text{u} - 235.043930\ \text{u} - 4.002603\ \text{u}](931.50\ \text{MeV/u}) = 5.245\ \text{MeV}$$

An energy output of 5.245 MeV $= 8.404 \times 10^{-13}$ J is released in each decay, and there are $2.30 \times 10^9$ decays per second, so the power output is

$$P = (8.404 \times 10^{-13}\ \text{J})(2.30 \times 10^9\ \text{s}^{-1}) = 1.93 \times 10^{-3}\ \text{W}$$

40. The mass of $^{224}$Ra in its excited state is $224.020212\ \text{u} + \dfrac{0.217\ \text{MeV}}{931.50\ \text{MeV/u}} = 224.020445\ \text{u}$.

The $Q$-value for the decay $^{228}\text{Th} \rightarrow\ ^{224}\text{Ra} + \alpha$ is
$$Q = [m(^{228}\text{Th}) - m(^{224}\text{Ra}) - m(^4\text{He})]c^2$$
$$= [228.028741\ \text{u} - 224.020445\ \text{u} - 4.002603\ \text{u}](931.50\ \text{MeV/u}) = 5.303\ \text{MeV}$$

$$K_\alpha = Q\left(\dfrac{A-4}{A}\right) = (5.303\ \text{MeV})\left(\dfrac{224}{228}\right) = 5.210\ \text{MeV}$$

41. For $^{232}$Th, $R = 1.2A^{1/3} = 1.2(232)^{1/3} = 7.37$ fm and

$$U_{\text{B}} = \frac{e^2}{4\pi\varepsilon_0}\frac{z(Z-z)}{R} = \frac{(1.440 \text{ MeV}\cdot\text{fm})(2)(88)}{7.37 \text{ fm}} = 34.37 \text{ MeV}$$

$$R' = \frac{e^2}{4\pi\varepsilon_0}\frac{z(Z-z)}{K_\alpha} = \frac{(1.440 \text{ MeV}\cdot\text{fm})(2)(88)}{4.01 \text{ MeV}} = 63.20 \text{ fm}$$

$$\frac{v}{c} = \sqrt{\frac{2K}{mc^2}} = \sqrt{\frac{2(34.01 \text{ MeV})}{3727 \text{ MeV}}} = 0.135$$

$$U_0 = \tfrac{1}{2}(U_{\text{B}} + K_\alpha) = 19.20 \text{ MeV} \quad \text{so} \quad U_0 - K_\alpha = 15.19 \text{ MeV}$$

$$k = \sqrt{\frac{2m}{\hbar^2}(U_0 - K_\alpha)} = \sqrt{\frac{2(3727 \text{ MeV})}{(197 \text{ MeV}\cdot\text{fm})^2}(15.19 \text{ MeV})} = 1.71 \text{ fm}^{-1}$$

$$L = \frac{R'-R}{2} = 27.92 \text{ fm} \quad \text{so} \quad kL = 47.74$$

$$\lambda = \frac{v}{2R}e^{-2kL} = \frac{(0.135)(3.00\times10^8 \text{ m/s})}{2(7.37\times10^{-15} \text{ m})}e^{-2(47.74)} = 1.04\times10^{-20}\,\text{s}^{-1}$$

For $^{218}$Th, $R = 1.2A^{1/3} = 1.2(218)^{1/3} = 7.22$ fm and

$$U_{\text{B}} = \frac{e^2}{4\pi\varepsilon_0}\frac{z(Z-z)}{R} = \frac{(1.440 \text{ MeV}\cdot\text{fm})(2)(88)}{7.22 \text{ fm}} = 35.10 \text{ MeV}$$

$$R' = \frac{e^2}{4\pi\varepsilon_0}\frac{z(Z-z)}{K_\alpha} = \frac{(1.440 \text{ MeV}\cdot\text{fm})(2)(88)}{9.85 \text{ MeV}} = 25.73 \text{ fm}$$

$$\frac{v}{c} = \sqrt{\frac{2K}{mc^2}} = \sqrt{\frac{2(39.85 \text{ MeV})}{3727 \text{ MeV}}} = 0.146$$

$$U_0 = \tfrac{1}{2}(U_{\text{B}} + K_\alpha) = 22.48 \text{ MeV} \quad \text{so} \quad U_0 - K_\alpha = 12.63 \text{ MeV}$$

$$k = \sqrt{\frac{2m}{\hbar^2}(U_0 - K_\alpha)} = \sqrt{\frac{2(3727 \text{ MeV})}{(197 \text{ MeV}\cdot\text{fm})^2}(12.63 \text{ MeV})} = 1.56 \text{ fm}^{-1}$$

$$L = \frac{R'-R}{2} = 9.26 \text{ fm} \quad \text{so} \quad kL = 14.45$$

$$\lambda = \frac{v}{2R}e^{-2kL} = \frac{(0.146)(3.00\times10^8 \text{ m/s})}{2(7.22\times10^{-15} \text{ m})}e^{-2(14.45)} = 9.0\times10^8\,\text{s}^{-1}$$

42. (a) For $^{226}$Ra, $R = 1.2A^{1/3} = 1.2(226)^{1/3} = 7.31$ fm and

$$U_B = \frac{e^2}{4\pi\varepsilon_0}\frac{z(Z-z)}{R} = \frac{(1.440 \text{ MeV}\cdot\text{fm})(2)(86)}{7.31 \text{ fm}} = 33.88 \text{ MeV}$$

$$R' = \frac{e^2}{4\pi\varepsilon_0}\frac{z(Z-z)}{K_\alpha} = \frac{(1.440 \text{ MeV}\cdot\text{fm})(2)(86)}{4.785 \text{ MeV}} = 51.76 \text{ fm}$$

$$\frac{v}{c} = \sqrt{\frac{2K}{mc^2}} = \sqrt{\frac{2(34.785 \text{ MeV})}{3727 \text{ MeV}}} = 0.1366$$

$$U_0 = \tfrac{1}{2}(U_B + K_\alpha) = 19.33 \text{ MeV} \quad \text{so} \quad U_0 - K_\alpha = 14.545 \text{ MeV}$$

$$k = \sqrt{\frac{2m}{\hbar^2}(U_0 - K_\alpha)} = \sqrt{\frac{2(3727 \text{ MeV})}{(197 \text{ MeV}\cdot\text{fm})^2}(14.545 \text{ MeV})} = 1.67 \text{ fm}^{-1}$$

$$L = \frac{R' - R}{2} = 22.23 \text{ fm} \quad \text{so} \quad kL = 37.12$$

$$\lambda = \frac{v}{2R}e^{-2kL} = \frac{(0.1366)(3.00\times10^8 \text{ m/s})}{2(7.31\times10^{-15} \text{ m})}e^{-2(37.12)} = 1.5\times10^{-11}\text{s}^{-1}$$

For $^{14}$C emission, $Q = 28.215$ MeV and $K_C = \dfrac{A-14}{A}Q = 26.47$ MeV .

$$U_B = \frac{e^2}{4\pi\varepsilon_0}\frac{z(Z-z)}{R} = \frac{(1.440 \text{ MeV}\cdot\text{fm})(6)(82)}{7.31 \text{ fm}} = 96.92 \text{ MeV}$$

$$R' = \frac{e^2}{4\pi\varepsilon_0}\frac{z(Z-z)}{K_\alpha} = \frac{(1.440 \text{ MeV}\cdot\text{fm})(6)(82)}{26.47 \text{ MeV}} = 26.77 \text{ fm}$$

$$\frac{v}{c} = \sqrt{\frac{2K}{mc^2}} = \sqrt{\frac{2(56.47 \text{ MeV})}{(14 \text{ u})(931.50 \text{ MeV/u})}} = 0.093$$

$$U_0 = \tfrac{1}{2}(U_B + K_C) = 61.70 \text{ MeV} \quad \text{so} \quad U_0 - K_C = 35.23 \text{ MeV}$$

$$k = \sqrt{\frac{2m}{\hbar^2}(U_0 - K_C)} = \sqrt{\frac{2(14 \text{ u})(931.50 \text{ MeV/u})}{(197 \text{ MeV}\cdot\text{fm})^2}(35.23 \text{ MeV})} = 4.87 \text{ fm}^{-1}$$

$$L = \frac{R' - R}{2} = 9.73 \text{ fm} \quad \text{so} \quad kL = 47.39$$

$$\lambda = \frac{v}{2R}e^{-2kL} = \frac{(0.093)(3.00\times10^8 \text{ m/s})}{2(7.31\times10^{-15} \text{ m})}e^{-2(47.39)} = 1.4\times10^{-20}\text{s}^{-1}$$

(b)  $\dfrac{\lambda_C}{\lambda_\alpha} = \dfrac{1.4\times10^{-20} \text{ s}^{-1}}{1.5\times10^{-11} \text{ s}^{-1}} = 1.1\times10^{-9}$

43. (a) In the decay $n \rightarrow p + e^- + \bar{\nu}$, assume the neutrino is of negligible mass and carries no energy. Then

$$E_n = E_p + E_e = m_p c^2 + K_p + E_e$$

$$K_p + E_e = m_n c^2 - m_p c^2 = (1.0086650 \text{ u} - 1.0072765 \text{ u})(931.50 \text{ MeV/u}) = 1.293 \text{ MeV}$$

Momentum conservation (with the initial neutron at rest) gives $p_p = p_e$. The proton can be treated nonrelativistically, but the electron is relativistic. With $E_e^2 = (p_e c)^2 + (m_e c^2)^2$,

$$(K_p - 1.293 \text{ MeV})^2 = E_e^2 = (p_e c)^2 + (m_e c^2)^2 = (p_p c)^2 + (m_e c^2)^2 = 2m_p K_p c^2 + (m_e c^2)^2$$

$$K_p^2 - (1879.15 \text{ MeV})K_p + 1.412 \text{ MeV}^2 = 0$$

Solving using the quadratic formula, we find two roots: $7.52 \times 10^{-4}$ MeV and 1879.15 MeV. Only the smaller root is physically meaningful, so

$$K_p = 7.52 \times 10^{-4} \text{ MeV}$$

(b) For the neutrino to have its maximum energy, the electron must have a kinetic energy of zero. In this case

$$E_n = m_n c^2 = K_p + m_p c^2 + m_e c^2 + E_\nu$$

$$E_\nu = (m_n c^2 - m_p c^2 - m_e c^2) - K_p = 0.782 \text{ MeV} - K_p$$

In analogy with the results of part (a), we know that $K_p$ will be small (of order $10^{-4}$ MeV). Thus to a good approximation $E_\nu = 0.782 \text{ MeV}$, and with $p_p = p_\nu = E_\nu / c$,

$$K_p = \frac{p_p^2}{2m_p} = \frac{p_\nu^2}{2m_p} = \frac{E_\nu^2}{2m_p c^2} = \frac{(0.782 \text{ MeV})^2}{2(938.3 \text{ MeV})} = 3.26 \times 10^{-4} \text{ MeV}$$

44. For $^{24}\text{Na} \rightarrow {}^{24}\text{Mg} + e^- + \bar{\nu}$,

$$Q = [m(^{24}\text{Na}) - m(^{24}\text{Mg})]c^2 = [23.990963 \text{ u} - 23.985042 \text{ u}](931.50 \text{ MeV/u}) = 5.515 \text{ MeV}$$

With $Q = K_{\text{Mg}} + K_e + E_\nu$ and neglecting $K_{\text{Mg}}$, we obtain

$$E_\nu = Q - K_e = 5.515 \text{ MeV} - 2.15 \text{ MeV} = 3.37 \text{ MeV}$$

45. (a) $\Delta E = \dfrac{\hbar}{\tau} = \dfrac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{141 \times 10^{-9} \text{ s}} = 4.67 \times 10^{-9} \text{ eV}$

(b) $K = \dfrac{E_\gamma^2}{2mc^2} = \dfrac{(14.4 \text{ keV})^2}{2(56.935 \text{ u})(931.50 \text{ MeV/u})} = 1.95 \times 10^{-3} \text{ eV}$

(c) $v = c\dfrac{\Delta E}{E} = c\dfrac{4.67 \times 10^{-9} \text{ eV}}{14.4 \text{ keV}} = 3.24 \times 10^{-13} c = 0.097 \text{ mm/s}$

46. The volume of $CO_2$ in one breath is $(0.0003)(0.5 \text{ L}) = 1.5 \times 10^{-4} \text{ L} = 1.5 \times 10^{-7} \text{ m}^3$. The number of $CO_2$ molecules is then

$$N = \frac{PV}{kT} = \frac{(1.01 \times 10^5 \text{ Pa})(1.5 \times 10^{-7} \text{ m}^3)}{(1.38 \times 10^{-23} \text{ J/K})(293 \text{ K})} = 3.75 \times 10^{18}$$

so the number of $^{14}C$ atoms in the lungs will be about $N_{14} = 10^{-12}N = 3.75 \times 10^6$.

Because the decay constant $\lambda$ gives the decay probability per nucleus per unit time, the total probability for decay in a time interval $\Delta t$ is

$$\text{probability} = \lambda N_{14}\Delta t = \frac{0.693}{(5730 \text{ y})(3.155 \times 10^7 \text{ s/y})}(3.75 \times 10^6)(3.5 \text{ s}) = 5.0 \times 10^{-5}$$

# Chapter 13

This chapter broadens the previous study of nuclear science from Chapter 12 by including nuclear reactions, including fission and fusion. To improve continuity, the section fission opens with a discussion of fission as a competing mode of radioactive decay for certain nuclei before moving on to neutron-induced fission. In this new edition there is less discussion of them technology of reactors compared with the $2^{nd}$ edition. The discussion of fusion has been updated to include reference to the current facilities attempting to exploit fusion by magnetic confinement and inertial confinement. Because much current nuclear research is oriented toward studying astrophysics, a section on nucleosynthesis has been added to this chapter.

## Supplemental Materials

The Nuclear Science Wall Chart (Contemporary Physics Education Project, 1997) contains much useful information about nuclear reactions. The accompanying on-line guide (http://www.lbl.gov/abc/wallchart/guide.html) provides supporting information as well as links to references and other sites. The content covers fission, fusion, nucleosynthesis, transuranic elements, and applications of nuclear science.

## Suggestions for Additional Reading

See the references listed in Chapter 12 for more detail on nuclear reactions. Other references at about the same level as this chapter are:
 H. Semat and J. R. Albright, *Introduction to Atomic and Nuclear Physics* (Holt, Rinehart and Winston, 1972).
 M. R. Wehr, J. A. Richards, and T. W. Adair, *Physics of the Atom* (Addison-Wesley, 1984). Particularly good discussion of fission reactors.

A detailed history of the development of fission is:
R. Rhodes, *The Making of the Atomic Bomb* (Simon and Schuster, 1986).

The natural fission reactor is discussed in:
G. A. Cowan, "A Natural Fission Reactor," *Scientific American* **235**, 36 (July 1976).
A. P. Meshik, "The Workings of an Ancient Nuclear Reactor," *Scientific American* **293**, 82 (November 2005).

For a comprehensive and nontechnical history of fusion, see:
R. Herman, *Fusion: The Search for Endless Energy* (Cambridge University Press, 1990).
T. K. Fowler, *The Fusion Quest* (Johns Hopkins University Press, 1997).

The history of progress in fusion power development can be traced through a series of articles in *Scientific American*, including the following: W. C. Gough and B. J. Eastland, "The Prospects of Fusion Power," (February 1971); M. J. Lubin and A. P. Fraas, "Fusion

by Laser," (June 1971); B. Coppi and J. Rem, "The Tokamak Approach in Fusion Research," (July 1972); J. L. Emmett, J. Nuckolls, and L. Wood, "Fusion Power by Laser Implosion," (June 1974); H. Furth, "Progress toward a Tokamak Fusion Reactor," (August 1979); R. W. Conn, V. A. Chuyanov, N. Inoue, and D. R. Sweetman, "The International Thermonuclear Experimental Reactor," (April 1992); M. Moyer, "Fusion's False Dawn," (March 2010).

For more information on medical applications of nuclear physics, see:
N. A. Dyson, *An Introduction to Nuclear Physics with Applications in Medicine and Biology* (Halsted Press, 1981).

More detail on the effects of radiation on living organisms can be found in:
S. C. Bushong, "Radiation Exposure in our Daily Lives," *The Physics Teacher,* March 1977, p. 135.

For more information on the stable elements beyond uranium, see:
G. T. Seaborg and W. D. Loveland, *The Elements Beyond Uranium* (Wiley, 1990)


## Materials for Active Engagement in the Classroom

### A. Reading Quizzes

1. Energy is released in the fission of $^{238}$U because:
    (1) the binding energy per nucleon increases with *A* for large *A*.
    (2) the binding energy per nucleon decreases with *A* for large *A*.
    (3) the binding energy per nucleon increases with *A* for small *A*.
    (4) the binding energy per nucleon decreases with *A* for small *A*.

2. In the radioactive decays of fission fragments, the most common emitted particles are:
    (1) electrons        (2) positrons        (3) neutrons        (4) alphas

3. Producing electricity from fusion reactors is currently not practical because:
    (1) the temperatures required to initiate the reaction in bulk matter are very high.
    (2) the energy release per nucleon is very small.
    (3) the nuclei needed for the reactions are found only in chemical elements that are very rare in nature.
    (4) fusion reactions occur only in the Sun and have never been achieved under laboratory conditions on Earth.

4. In a nuclear fusion reaction such as $^2$H + $^3$H → $^4$He + n, the binding energy per nucleon
    (1) Increases        (2) Decreases        (3) Does not change
    (4) Depends on the kinetic energy of the colliding particles

5. Thermonuclear fusion
      (1) requires isotopes that are very scarce.
      (2) requires extremely high temperatures.
      (3) requires that the particles move at speeds close to the speed of light.
      (4) presently is used to generate electricity in certain power plants.

**Answers**     1. 2   2. 1   3. 1   4. 1   5. 2


## B. Conceptual or Discussion Questions

1. Without doing any calculations, which reaction would you expect to have the higher $Q$ value?      (1) $^2H + {}^4He \rightarrow {}^6Li$   (2) $^3H + {}^3He \rightarrow {}^6Li$

2. Which of the following fusion reactions would you expect to release the most energy per reacting nucleon?
     (1) $^2H + {}^2H$          (2) $^{12}C + {}^{12}C$          (3) $^{40}Ca + {}^{40}Ca$     (4) $^{56}Fe + {}^{56}Fe$

3. Answer each of the following with:
     (1) Fission       (2) Fusion       (3) Both
   (a) Nuclei produced in the reaction are usually highly radioactive.
   (b) Energy release can be as large as several MeV per reacting nucleon.
   (c) It is usually necessary to overcome a Coulomb barrier for the reaction to occur.
   (d) Usually induced by the capture of a neutron.
   (e) Reacting nuclei come from commonly available chemical elements.
   (f) Electrical power could be generated by boiling water using heat obtained from kinetic energy of nuclei produced in the reaction.

**Answers**     1. 2   2. 1   3. (a) 1  (b) 2  (c) 2  (d) 1  (e) 2  (f) 3


## Sample Exam Questions

### A. Multiple Choice

1. Suppose a nucleus of $^{238}_{92}U$ fissions into two nuclei of $^{119}_{46}Pd$. Compared with the mass of $^{238}_{92}U$, the total mass of the two $^{119}_{46}Pd$ is
   (a) the same    (b) larger    (c) smaller

2. Fusion releases energy because the binding energy per nucleon:
   (a) increases with increasing mass number at high mass numbers.
   (b) increases with increasing mass number at low mass numbers.
   (c) decreases with increasing mass number at high mass numbers.
   (d) decreases with increasing mass number at low mass numbers.

3. The total nuclear binding energy of an alpha particle (a nucleus of $^4$He) is about 28 MeV. When 4 protons undergo fusion in the Sun to form an alpha particle, the total energy released is about
    (a) 7 MeV       (b) 14 MeV    (c) 28 MeV    (d) 56 MeV    (e) 112 MeV

**Answers**     1. c    2. b    3. c

## B. Conceptual

1. Would you expect the core of a fission reactor to emit *mostly neutrinos, mostly antineutrinos, or a roughly equal mixture of neutrinos and antineutrinos?* EXPLAIN YOUR ANSWER.

2. In the reactions $^2$H + $^2$H → $^3$He + n and $^2$H + $^3$H → $^4$He + n initiated in each case by $^2$H with kinetic energy $K$ incident on a target at rest, in which case would you expect the total kinetic energy of the reaction products to be greater? EXPLAIN YOUR ANSWER.
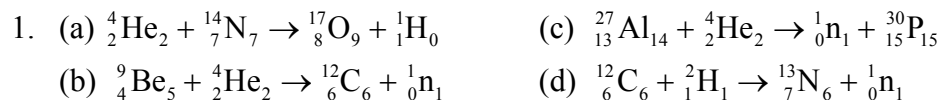
**Answers**     1. antineutrinos (because fission products have an excess of neutrons)
            2. the large binding energy of $^4$He gives the second reaction a much larger
                $Q$ value and thus a much larger total kinetic energy

## C. Problems

1. (a) The fusion reaction $^2$H + $^2$H → $^3$He + n is initiated by two $^2$H colliding head-on with equal kinetic energies of 0.124 MeV. Find the total kinetic energy of the products of this reaction. Atomic masses are: $^2$H – 2.014102 u, $^3$He – 3.016029 u.
    (b) Find the kinetic energy of the neutron.

2. (a) The nucleus $^{14}$C (halflife 5730 y) is used in radiocarbon dating to determine the age of previously living material. A certain sample of wood currently shows a decay rate of 25.0 decays/s of $^{14}$C. The wood is taken from a structure that was built from a tree that was cut 15,000 years ago. What would have been the $^{14}$C decay rate of this wood sample when the tree was cut?
    (b) Given that $^{14}$C has 6 protons and 8 neutrons and an atomic mass of 14.003242 u, find the total binding energy (in MeV) of $^{14}$C. Use 1.008665 u for the neutron mass and 1.007825 u for the mass of a hydrogen atom.
    (c) Suppose it were possible to form a nucleus of $^{14}$C by colliding two $^7_3$Li nuclei. Is energy *absorbed* or *released* in this process? (Choose which one.) Calculate the energy absorbed or released in MeV. The mass of $^7_3$Li is 7.016004 u.

**Answers**     1. (a) 3.52 MeV   (b) 2.64 MeV
            2. (a) 153 decays/s   (b) 105.3 MeV   (c) 26.8 MeV released

## Problem Solutions

1.  (a) $^4_2\text{He}_2 + ^{14}_7\text{N}_7 \rightarrow ^{17}_8\text{O}_9 + ^1_1\text{H}_0$       (c) $^{27}_{13}\text{Al}_{14} + ^4_2\text{He}_2 \rightarrow ^1_0\text{n}_1 + ^{30}_{15}\text{P}_{15}$

    (b) $^9_4\text{Be}_5 + ^4_2\text{He}_2 \rightarrow ^{12}_6\text{C}_6 + ^1_0\text{n}_1$       (d) $^{12}_6\text{C}_6 + ^2_1\text{H}_1 \rightarrow ^{13}_7\text{N}_6 + ^1_0\text{n}_1$
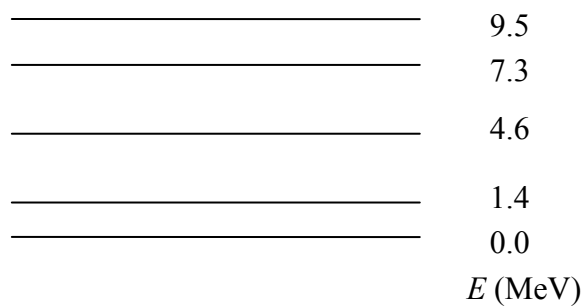
2.  Assuming the highest energy protons (16.2 MeV) lead to the ground state, the next highest to the first excited state, and so forth, the energies of the excited states are

    16.2 MeV – 14.8 MeV = 1.4 MeV             16.2 MeV – 8.9 MeV = 7.3 MeV

    16.2 MeV – 11.6 MeV = 4.6 MeV            16.2 MeV – 6.7 MeV = 9.5 MeV

<table>
<tr><td>─────────────────</td><td>9.5</td></tr>
<tr><td>─────────────────</td><td>7.3</td></tr>
<tr><td>─────────────────</td><td>4.6</td></tr>
<tr><td>─────────────────</td><td>1.4</td></tr>
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</table>

$$E \text{ (MeV)}$$

3.  Using the density of gold of 19.3 g/cm$^3$, the mass of the foil is

$$m = \rho V = (19.3 \text{ g/cm}^3)(\pi)(0.15 \text{ cm})^2 (1.81\times10^{-4} \text{ cm}) = 2.47\times10^{-4} \text{ g}$$

Solving Equation 13.2 for the cross section, we find

$$\sigma = \frac{RM}{\phi m N_A} = \frac{(5.37\times10^6 \text{ s}^{-1})(197 \text{ g/mole})}{(7.25\times10^{10} \text{ neutrons/cm}^2/\text{s})(2.47\times10^{-4} \text{ g})(6.02\times10^{23} \text{ mole}^{-1})} = 9.8\times10^{-23} \text{ cm}^2 = 98 \text{ b}$$

4. The mass of Co in the foil is (46 mg)(0.0044) = 0.202 mg, and the corresponding number of Co nuclei is

$$N = \frac{m}{M} N_A = \frac{0.202 \text{ mg}}{58.9 \text{ g}} 6.02\times10^{23} = 2.07\times10^{21}$$

and the neutron beam intensity is, from Equation 13.1,

$$I_0 = \frac{RS}{\sigma N} = \frac{(1.07\times10^{12} \text{ s}^{-1})(\pi)(0.50 \text{ cm})^2}{(37\times10^{-24} \text{ cm}^2)(2.07\times10^{21})} = 1.1\times10^{13} \text{ s}^{-1}$$

5. $$I_0 = \frac{20 \times 10^{-6} \text{ A}}{1.60 \times 10^{-19} \text{ C/proton}} = 1.25 \times 10^{14} \text{ protons/s}$$

Let $t$ represent the thickness of the target. Then

$$\frac{N}{S} = \frac{(N_A \rho / M)V}{S} = \frac{N_A \rho t}{M}$$

$$= \frac{(6.022 \times 10^{23} \text{ atoms/mole})(10.5 \text{ g/cm}^3)(4.5 \times 10^{-4} \text{ cm})}{107 \text{ g/mole}} = 2.65 \times 10^{19} \text{ atoms/cm}^2$$

Because 3 neutrons are produced in each reaction, the reaction rate $R$ is $\frac{1}{3}(8.5 \times 10^6 \text{ s}^{-1})$.

$$\sigma = \frac{R/I_0}{N/S} = \frac{\frac{1}{3}(8.5 \times 10^6 \text{ s}^{-1})/(1.25 \times 10^{14} \text{ s}^{-1})}{2.65 \times 10^{19} \text{ cm}^{-2}} = 8.6 \times 10^{-28} \text{ cm}^2 = 8.6 \times 10^{-4} \text{ b}$$

6. The irradiated target has a thickness of 2.5 μm and a diameter of 0.50 cm. Its mass is

$$m = \rho V = (8.96 \text{ g/cm}^3)(2.5 \times 10^{-4} \text{ cm})\pi(0.25 \text{ cm})^2 = 4.40 \times 10^{-4} \text{ g}$$

$$N = (6.022 \times 10^{23} \text{ atoms/mole})\frac{4.40 \times 10^{-4} \text{ g}}{63 \text{ g/mole}} = 4.20 \times 10^{18} \text{ atoms}$$

$$I_0 = \frac{7.5 \times 10^{-6} \text{ A}}{2(1.6 \times 10^{-19} \text{ C/particle})} = 2.34 \times 10^{13} \text{ particles/s}$$

$$R = \frac{I_0 \sigma N}{S} = \frac{(2.34 \times 10^{13} \text{ s}^{-1})(1.25 \times 10^{-24} \text{ cm}^2)(4.20 \times 10^{18})}{\pi(0.25 \text{ cm})^2} = 6.26 \times 10^8 \text{ neutrons/s}$$
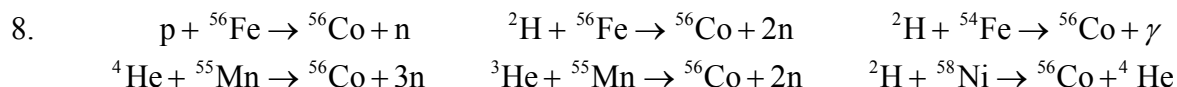
7. From Equation 13.6, the fraction of the maximum activity $f = a(t)/R$
   is $f = 1 - e^{-\lambda t} = 1 - e^{-0.693t/t_{1/2}} = 1 - (0.5)^{t/t_{1/2}}$
   (a) at $t = t_{1/2}$     $f = 1 - (0.5)^1 = 0.5$
   (b) at $t = 2t_{1/2}$     $f = 1 - (0.5)^2 = 0.75$
   (c) at $t = 4t_{1/2}$     $f = 1 - (0.5)^4 = 0.9325$

8. $$p + {}^{56}\text{Fe} \rightarrow {}^{56}\text{Co} + n \qquad {}^2\text{H} + {}^{56}\text{Fe} \rightarrow {}^{56}\text{Co} + 2n \qquad {}^2\text{H} + {}^{54}\text{Fe} \rightarrow {}^{56}\text{Co} + \gamma$$
$${}^4\text{He} + {}^{55}\text{Mn} \rightarrow {}^{56}\text{Co} + 3n \qquad {}^3\text{He} + {}^{55}\text{Mn} \rightarrow {}^{56}\text{Co} + 2n \qquad {}^2\text{H} + {}^{58}\text{Ni} \rightarrow {}^{56}\text{Co} + {}^4\text{He}$$

9. $$\frac{dN}{dt} + \lambda N = \frac{d}{dt}\left[\frac{R}{\lambda}(1 - e^{-\lambda t})\right] + R(1 - e^{-\lambda t}) = Re^{-\lambda t} + R(1 - e^{-\lambda t}) = R$$

10. (a) The number of nitrogen molecules in the cell is

$$N = \frac{PV}{kT} = \frac{(2.25 \text{ atm})(1.01 \times 10^5 \text{ Pa/atm})(0.0124 \text{ m})^3}{(1.38 \times 10^{-23} \text{ J/K})(293 \text{ K})} = 1.07 \times 10^{20}$$

and the number of nitrogen nuclei is twice that, or $2.14 \times 10^{20}$. The incident beam contains $I_0 = (2.05 \times 10^{-6} \text{ C/s})/(1.60 \times 10^{-19} \text{ C/deuteron}) = 1.28 \times 10^{13}$ deuterons/s. The reaction rate is then

$$R = \frac{\sigma N}{S} I_0 = \frac{(0.21 \times 10^{-28} \text{ m}^2)(2.14 \times 10^{20})}{(0.0124 \text{ m})^2}(1.28 \times 10^{13} \text{ s}^{-1}) = 4.03 \times 10^8 \text{ s}^{-1}$$

(b)   $a = R(1 - e^{-\lambda t}) = (4.03 \times 10^8 \text{ s}^{-1})(1 - e^{-(0.693)(60 \text{ s})/(122 \text{ s})}) = 1.16 \times 10^8 \text{ s}^{-1} = 3.14 \text{ mCi}$

11.   With $\phi\sigma = (2.5 \times 10^{13} \text{ neutrons/cm}^2 \cdot \text{s})(0.53 \times 10^{-24} \text{ cm}^2/\text{atom}) = 1.33 \times 10^{-11} \text{ s}^{-1}\text{atom}^{-1}$,

$$R = \phi\sigma \frac{m}{M} N_A = \left(1.33 \times 10^{-11} \text{s}^{-1}\text{atom}^{-1}\right)\left(\frac{1.0 \times 10^{-6} \text{ g}}{23.0 \text{ g/mole}}\right)\left(6.022 \times 10^{23} \frac{\text{atoms}}{\text{mole}}\right) = 3.47 \times 10^5 \text{ s}^{-1}$$

$$a = R(1 - e^{-\lambda t}) = (3.47 \times 10^5 \text{ s}^{-1})(1 - e^{-0.693(4 \text{ h})/(15 \text{ h})}) = 5.85 \times 10^4 \text{ s}^{-1} = 1.58 \ \mu\text{Ci}$$

12.
$$\tfrac{1}{2} m_x (v_x - v)^2 + \tfrac{1}{2} m_X (-v)^2 + m_x c^2 + m_X c^2 = m_y c^2 + m_Y c^2$$

Substituting $v = v_x m_x /(m_x + m_X)$, we obtain

$$\tfrac{1}{2} m_x \left( v_x - \frac{v_x m_x}{m_x + m_X} \right)^2 + \tfrac{1}{2} m_X \left( -\frac{v_x m_x}{m_x + m_X} \right)^2 = (m_y + m_Y - m_x - m_X)c^2 = -Q$$

$$\tfrac{1}{2} m_x v_x^2 \left[ \frac{m_X^2}{(m_x + m_X)^2} + \frac{m_x m_X}{(m_x + m_X)^2} \right] = -Q$$

$$K_{th} \left[ \frac{m_X}{m_x + m_X} \right] = -Q \qquad \text{or} \qquad K_{th} = -Q \left[ 1 + \frac{m_x}{m_X} \right]$$

13. (a)  $p + {}^{55}Mn \rightarrow {}^{54}Fe + 2n$

$Q = [m({}^{1}H) + m({}^{55}Mn) - m({}^{54}Fe) - 2m(n)]c^2$

$= [1.007825 \text{ u} + 54.938045 \text{ u} - 53.939611 \text{ u} - 2(1.008665 \text{ u})](931.50 \text{ MeV/u})$

$= -10.313 \text{ MeV}$

(b)  ${}^{3}He + {}^{40}Ar \rightarrow {}^{41}K + {}^{2}H$

$Q = [m({}^{3}He) + m({}^{40}Ar) - m({}^{41}K) - m({}^{2}H)]c^2$

$= (3.016029 \text{ u} + 39.962383 \text{ u} - 40.961826 \text{ u} - 2.014102 \text{ u})](931.50 \text{ MeV/u})$

$= 2.314 \text{ MeV}$

14. (a)  ${}^{6}Li + n \rightarrow {}^{3}H + {}^{4}He$

$Q = [m({}^{6}Li) + m(n) - m({}^{3}H) - m({}^{4}He)]c^2$

$= (6.015123 \text{ u} + 1.008665 \text{ u} - 3.016049 \text{ u} - 4.002603 \text{ u})(931.50 \text{ MeV/u})$

$= 4.784 \text{ MeV}$

(b)  $p + {}^{2}H \rightarrow 2p + n$

$Q = [m({}^{1}H) + m({}^{2}H) - 2m({}^{1}H) - m(n)]c^2$

$= [1.007825 \text{ u} + 2.014102 \text{ u} - 2(1.007825 \text{ u}) - 1.008665 \text{ u}](931.50 \text{ MeV/u})$

$= -2.224 \text{ MeV}$

(c)  ${}^{7}Li + {}^{2}H \rightarrow {}^{8}Be + n$

$Q = [m({}^{7}Li) + m({}^{2}H) - m({}^{8}Be) - m(n)]c^2$

$= (7.016005 \text{ u} + 2.014102 \text{ u} - 8.005305 \text{ u} - 1.008665 \text{ u})(931.50 \text{ MeV/u})$

$= 15.032 \text{ MeV}$

15.  ${}^{2}H + {}^{3}He \rightarrow p + {}^{4}He$

$Q = [m({}^{2}H) + m({}^{3}He) - m({}^{1}H) - m({}^{4}He)]c^2$

$= (2.014102 \text{ u} + 3.016029 \text{ u} - 1.007825 \text{ u} - 4.002603 \text{ u})(931.50 \text{ MeV/u})$

$= 18.353 \text{ MeV}$

Let $y = p$, $Y = {}^{4}He$, $x = {}^{2}H$.  Then $Q = K_y + K_Y - K_x$, so

$$K_y + K_Y = Q + K_x = 18.353 \text{ MeV} + 5.00 \text{ MeV} = 23.353 \text{ MeV}$$

$$\frac{p_y^2}{2m_y} + \frac{p_Y^2}{2m_Y} = 23.353 \text{ MeV}$$

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$$p_x = \sqrt{2m_x K_x} = \frac{1}{c}\sqrt{2m_x c^2 K_x} = \frac{1}{c}\sqrt{2(2.014102\ \text{u})(931.50\ \text{MeV/u})(5.00\ \text{MeV})} = 137.0\ \text{MeV}/c$$

Momentum conservation gives $p_y = p_x - p_Y = 137.0\ \text{MeV}/c - p_Y$. Substituting into the energy equation gives

$$\frac{(137.0\ \text{MeV}/c - p_Y)^2}{2m_y} + \frac{p_Y^2}{2m_Y} = 23.353\ \text{MeV}$$

$$c^2 p_Y^2 \left[\frac{1}{2m_y c^2} + \frac{1}{2m_Y c^2}\right] - \frac{137.0\ \text{MeV}}{m_y c^2} c p_Y + \frac{(137.0\ \text{MeV})^2}{2m_y c^2} - 23.353\ \text{MeV} = 0$$

Solving for $p_Y$ using the quadratic formula, we obtain

$$c p_Y = 288.4\ \text{MeV} \text{ or } -69.5\ \text{MeV}$$

The first solution corresponds to the $^4$He moving in the same direction as the original $^2$H, while the protons move in the opposite direction ($cp_y = 137.0\ \text{MeV} - cp_Y = -151.4\ \text{MeV}$). The kinetic energies are

$$K_Y = \frac{(cp_Y)^2}{2m_Y c^2} = \frac{(288.4\ \text{MeV})^2}{2(4.002603\ \text{u})(931.50\ \text{MeV/u})} = 11.152\ \text{MeV}$$

$$K_y = 23.353\ \text{MeV} - 11.152\ \text{MeV} = 12.201\ \text{MeV}$$

The second solution gives $^4$He moving opposite to the original direction of the $^2$H, with kinetic energy

$$K_Y = \frac{(69.5\ \text{MeV})^2}{2(4.002603\ \text{u})(931.50\ \text{MeV/u})} = 0.647\ \text{MeV}$$

$$K_y = 23.353\ \text{MeV} - 0.647\ \text{MeV} = 22.706\ \text{MeV}$$

16. (a)  $p + {}^4\text{He} \rightarrow {}^2\text{H} + {}^3\text{He}$

$$Q = [m({}^1\text{H}) + m({}^4\text{He}) - m({}^2\text{H}) - m({}^3\text{He})]c^2$$

$$= (1.007825\ \text{u} + 4.002603\ \text{u} - 2.014102\ \text{u} - 3.016029\ \text{u})(931.50\ \text{MeV/u})$$

$$= -18.353\ \text{MeV}$$

(b) For protons incident on $^4$He, $x = {}^1$H and $X = {}^4$He:

$$K_{th} = -Q\left(1 + \frac{m(^1H)}{m(^4He)}\right) = (18.353 \text{ MeV})\left(1 + \frac{1.007825 \text{ u}}{4.002603 \text{ u}}\right) = 22.974 \text{ MeV}$$

(c) For $^4He$ incident on protons, $x = {}^4He$ and $X = {}^1H$:

$$K_{th} = -Q\left(1 + \frac{m(^4He)}{m(^1H)}\right) = (18.353 \text{ MeV})\left(1 + \frac{4.002603 \text{ u}}{1.007825 \text{ u}}\right) = 91.242 \text{ MeV}$$

17. (a) The $Q$ value is

$$\begin{aligned} Q &= [m(^{254}Cf) - 2m(^{127}In)]c^2 \\ &= [254.087323 \text{ u} - 2(126.917353 \text{ u})](931.5 \text{ MeV/u}) = 235.3 \text{ MeV} \end{aligned}$$

(b) $Q = [m(^{254}Cf) - m(^{140}Xe) - m(^{110}Ru) - 4m(n)]c^2$

$$= (254.087323 \text{ u} - 139.921641 \text{ u} - 109.914136 \text{ u} - 4(1.00866$$
$$= 202.0 \text{ MeV}$$

18. The number of $^{235}U$ atoms in 1000 g of U (= 30 g of $^{235}U$) is

$$\frac{(30 \text{ g})(6.022 \times 10^{23} \text{ atoms/mole})}{235 \text{ g/mole}} = 5.9 \times 10^{22} \text{ atoms}$$

If each fission releases about 200 MeV, the total energy released is

$$E = (200 \text{ MeV/atom})(5.9 \times 10^{22} \text{ atoms}) = 1.2 \times 10^{31} \text{ eV} = 1.9 \times 10^{12} \text{ J}$$

19. (a) $\Delta E = [m(^{235}U) + m(n) - m(^{236}U)]c^2$
$$= (235.043930 \text{ u} + 1.008665 \text{ u} - 236.045568 \text{ u})(931.50 \text{ MeV/u}) = 6.546 \text{ MeV}$$

(b) $\Delta E = [m(^{238}U) + m(n) - m(^{239}U)]c^2$
$$= (238.050788 \text{ u} + 1.008665 \text{ u} - 239.054293 \text{ u})(931.50 \text{ MeV/u}) = 4.807 \text{ MeV}$$

(c) If $^{235}U$ + n gives enough energy of excitation to make the $^{236}U$ fission easily, then $^{238}U$ + n needs about 1.7 MeV of additional energy to reach the same state of excitation and therefore to have about the same probability to fission. This additional energy must come from the incident neutrons, so $^{238}U$ can be fissioned easily only by "fast" neutrons with 1-2 MeV of kinetic energy.

(d) $\Delta E = [m(^{239}Pu) + m(n) - m(^{240}Pu)]c^2$
$$= (239.052163 \text{ u} + 1.008665 \text{ u} - 240.053814 \text{ u})(931.50 \text{ MeV/u}) = 6.534 \text{ MeV}$$

Because $^{239}$Pu + n has about the same relative excitation as $^{235}$U + n, we expect that $^{239}$Pu (like $^{235}$U) can be easily fissioned by slow neutrons.

20.  $Q = [m(^{235}\text{U}) + m(\text{n}) - m(^{93}\text{Rb}) - m(^{141}\text{Cs}) - 2m(\text{n})]c^2$

$= (235.043930 \text{ u} - 92.922042 \text{ u} - 140.920046 \text{ u} - 1.008665 \text{ u})(931.50 \text{ MeV/u})$

$= 179.94 \text{ MeV}$

21.  (a)    $^{12}\text{C} + {}^{1}\text{H} \rightarrow {}^{13}\text{N} + \gamma$

$Q = [m(^{12}\text{C}) + m(^{1}\text{H}) - m(^{13}\text{N})]c^2$

$= (12.000000 \text{ u} + 1.007825 \text{ u} - 13.005739 \text{ u})(931.50 \text{ MeV/u}) = 1.943 \text{ MeV}$

$^{13}\text{N} \rightarrow {}^{13}\text{C} + e^+ + v$

$Q = [m(^{13}\text{N}) - m(^{13}\text{C}) - 2m(\text{e})]c^2$

$= (13.005739 \text{ u} - 13.003355 \text{ u} - 2 \times 0.0005486 \text{ u})(931.50 \text{ MeV/u}) = 1.199 \text{ MeV}$

$^{13}\text{C} + {}^{1}\text{H} \rightarrow {}^{14}\text{N} + \gamma$

$Q = [m(^{13}\text{C}) + m(^{1}\text{H}) - m(^{14}\text{N})]c^2$

$= (13.003355 \text{ u} + 1.007825 \text{ u} - 14.003074 \text{ u})(931.50 \text{ MeV/u}) = 7.551 \text{ MeV}$

$^{14}\text{N} + {}^{1}\text{H} \rightarrow {}^{15}\text{O} + \gamma$

$Q = [m(^{14}\text{N}) + m(^{1}\text{H}) - m(^{15}\text{O})]c^2$

$= (14.003074 \text{ u} + 1.007825 \text{ u} - 15.003066 \text{ u})(931.50 \text{ MeV/u}) = 7.296 \text{ MeV}$

$^{15}\text{O} \rightarrow {}^{15}\text{N} + e^+ + v$

$Q = [m(^{15}\text{O}) - m(^{15}\text{N}) - 2m(\text{e})]c^2$

$= (15.003066 \text{ u} - 15.000109 \text{ u} - 2 \times 0.0005486 \text{ u})(931.50 \text{ MeV/u}) = 1.732 \text{ MeV}$

$^{15}\text{N} + {}^{1}\text{H} \rightarrow {}^{12}\text{C} + {}^{4}\text{He}$

$Q = [m(^{15}\text{N}) + m(^{1}\text{H}) - m(^{12}\text{C}) - m(^{4}\text{He})]c^2$

$= (15.000109 \text{ u} + 1.007825 \text{ u} - 12.000000 \text{ u} - 4.002603 \text{ u})(931.50 \text{ MeV/u})$

$= 4.966 \text{ MeV}$

(b) When the 6 reactions or decays are combined, 4 electrons must be added to each side of the equations, as in the proton-proton cycle. (Two electrons are necessary to balance each of the two positron decays in the carbon cycle.) The net $Q$ value is then the sum of the $Q$ values for the 6 processes plus $4m_e c^2$, which gives 26.7 MeV.
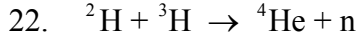
22. $^{2}H + {}^{3}H \rightarrow {}^{4}He + n$

$Q = [m(^{2}H) + m(^{3}H) - m(^{4}He) - m(n)]c^{2}$

$= (2.014102 \text{ u} + 3.016049 \text{ u} - 4.002603 \text{ u} - 1.008665 \text{ u})(931.50 \text{ MeV/u})$

$= 17.590 \text{ MeV}$

23. $^{2}H + {}^{3}H \rightarrow {}^{4}He + n$ with $Q = K_{He} + K_{n} = 17.6 \text{ MeV}$, neglecting the initial kinetic energies of the $^{2}H$ and $^{3}H$ (which are less then 0.01 MeV for temperatures up to $10^{8}$ K). Making also the approximation that the initial momentum is small, conservation of momentum then gives $p_{He} + p_{n} = 0$, or $p_{He} = -p_{n}$.

$$Q = K_{He} + K_{n} = \frac{p_{He}^{2}}{2m_{He}} + \frac{p_{n}^{2}}{2m_{n}} = \frac{p_{n}^{2}}{2m_{He}} + \frac{p_{n}^{2}}{2m_{n}} = \frac{p_{n}^{2}}{2m_{n}}\left(1 + \frac{m_{n}}{m_{He}}\right) = K_{n}\left(1 + \frac{m_{n}}{m_{He}}\right)$$

$$K_{n} = \frac{Q}{1 + m_{n}/m_{He}} = \frac{17.6 \text{ MeV}}{1 + 0.25} = 14.1 \text{ MeV}$$

24. (a) $n\tau \geq 10^{20} \text{ s} \cdot \text{m}^{-3}$ for $\tau = 0.60 \text{ s}$

$$n \geq \frac{10^{20} \text{ s} \cdot \text{m}^{-3}}{\tau} = \frac{10^{20} \text{ s} \cdot \text{m}^{-3}}{0.60 \text{ s}} = 1.67 \times 10^{20} \text{ m}^{-3}$$

(b) For $n = 1.67 \times 10^{21} \text{ m}^{-3}$, Figure 13.16 indicates $T > 1.5 \times 10^{8}$ K.

25. $^{4}He + {}^{4}He + {}^{4}He \rightarrow {}^{12}C$

$Q = [3m(^{4}He) - m(^{12}C)]c^{2}$

$= (3 \times 4.002603 \text{ u} - 12.000000 \text{ u})(931.50 \text{ MeV/u}) = 7.274 \text{ MeV}$

This energy is about 0.6 MeV per reacting nucleon, far smaller than the 6.7 MeV per reacting nucleon that is released in the fusion of 4 protons to form $^{4}He$.

26. We must first find the Coulomb barrier that keeps the helium nuclei from coming together. To find the potential energy of two helium nuclei that just touch at their surfaces, we need the nuclear radius, $R = 1.2A^{1/3} = 1.2(4)^{1/3} = 1.9 \text{ fm}$. When the charges of $+2e$ are separated by a distance of $2R$, the potential energy is

$$U = \frac{1}{4\pi\varepsilon_{0}}\frac{q_{1}q_{2}}{r} = \frac{1}{4\pi\varepsilon_{0}}\frac{(2e)^{2}}{2(1.9 \text{ fm})} = \frac{2(1.440 \text{ MeV} \cdot \text{fm})}{1.9 \text{ fm}} = 1.5 \text{ MeV}$$

We could overcome this barrier if each helium had a kinetic energy of 0.75 MeV. The corresponding temperature is determined from $K = \frac{3}{2}kT$:

$$T = \frac{K}{1.5k} = \frac{0.75 \text{ MeV}}{1.5(8.6 \times 10^{-5} \text{ eV/K})} = 5.8 \times 10^9 \text{ K}$$

27.  $^{63}\text{Cu} + \text{n} \rightarrow {}^{64}\text{Cu}$     $(t_{1/2} = 13 \text{ h})$         $^{69}\text{Ga} + \text{n} \rightarrow {}^{70}\text{Ga}$     $(t_{1/2} = 21 \text{ m})$

$^{64}\text{Cu} \rightarrow {}^{64}\text{Zn} + \text{e}^- + \bar{\nu}$                       $^{70}\text{Ga} \rightarrow {}^{70}\text{Ge} + \text{e}^- + \bar{\nu}$

$^{64}\text{Zn} + \text{n} \rightarrow {}^{65}\text{Zn}$    $(t_{1/2} = 244 \text{ d})$       $^{70}\text{Ge} + \text{n} \rightarrow {}^{71}\text{Ge}$     $(t_{1/2} = 11 \text{ d})$

$^{65}\text{Zn} \rightarrow {}^{65}\text{Cu} + \text{e}^+ + \nu$                       $^{71}\text{Ge} + \text{e}^- \rightarrow {}^{71}\text{Ga} + \bar{\nu}$

$^{65}\text{Cu} + \text{n} \rightarrow {}^{66}\text{Cu}$    $(t_{1/2} = 5 \text{ m})$         $^{71}\text{Ga} + \text{n} \rightarrow {}^{72}\text{Ga}$     $(t_{1/2} = 14 \text{ h})$

$^{66}\text{Cu} \rightarrow {}^{66}\text{Zn} + \text{e}^- + \bar{\nu}$                       $^{72}\text{Ga} \rightarrow {}^{72}\text{Ge} + \text{e}^- + \bar{\nu}$

$^{66}\text{Zn} + \text{n} \rightarrow {}^{67}\text{Zn}$    (stable)            $^{72}\text{Ge} + \text{n} \rightarrow {}^{73}\text{Ge}$     (stable)

$^{67}\text{Zn} + \text{n} \rightarrow {}^{68}\text{Zn}$    (stable)            $^{73}\text{Ge} + \text{n} \rightarrow {}^{74}\text{Ge}$     (stable)

$^{68}\text{Zn} + \text{n} \rightarrow {}^{69}\text{Zn}$    $(t_{1/2} = 56 \text{ m})$         $^{74}\text{Ge} + \text{n} \rightarrow {}^{75}\text{Ge}$     $(t_{1/2} = 83 \text{ m})$

$^{69}\text{Zn} \rightarrow {}^{69}\text{Ga} + \text{e}^- + \bar{\nu}$                       $^{75}\text{Ge} \rightarrow {}^{75}\text{As} + \text{e}^- + \bar{\nu}$

28.  $^{81}\text{Br} + \text{n} \rightarrow {}^{82}\text{Br}$    $(t_{1/2} = 35 \text{ h})$         $^{89}\text{Sr} \rightarrow {}^{89}\text{Y} + \text{e}^- + \bar{\nu}$

$^{82}\text{Br} \rightarrow {}^{82}\text{Kr} + \text{e}^- + \bar{\nu}$                       $^{89}\text{Y} + \text{n} \rightarrow {}^{90}\text{Y}$     $(t_{1/2} = 64 \text{ h})$

$^{82}\text{Kr} + \text{n} \rightarrow {}^{83}\text{Kr}$    (stable)            $^{90}\text{Y} \rightarrow {}^{90}\text{Zr} + \text{e}^- + \bar{\nu}$

$^{83}\text{Kr} + \text{n} \rightarrow {}^{84}\text{Kr}$    (stable)            $^{90}\text{Zr} + \text{n} \rightarrow {}^{91}\text{Zr}$     (stable)

$^{84}\text{Kr} + \text{n} \rightarrow {}^{85}\text{Kr}$    $(t_{1/2} = 11 \text{ y})$         $^{91}\text{Zr} + \text{n} \rightarrow {}^{92}\text{Zr}$     (stable)

$^{85}\text{Kr} + \text{n} \rightarrow {}^{86}\text{Kr}$    (stable)

        $^{92}\text{Zr} + \text{n} \rightarrow {}^{93}\text{Zr}$    $(t_{1/2} = 1.5 \times 10^6 \text{ y})$

$^{86}\text{Kr} + \text{n} \rightarrow {}^{87}\text{Kr}$    $(t_{1/2} = 76 \text{ m})$         $^{93}\text{Zr} + \text{n} \rightarrow {}^{94}\text{Zr}$     (stable)

$^{87}\text{Kr} \rightarrow {}^{87}\text{Rb} + \text{e}^- + \bar{\nu}$                       $^{94}\text{Zr} + \text{n} \rightarrow {}^{95}\text{Zr}$     $(t_{1/2} = 64 \text{ d})$

$^{87}\text{Rb} + \text{n} \rightarrow {}^{88}\text{Rb}$    $(t_{1/2} = 18 \text{ m})$

        $^{95}\text{Zr} \rightarrow {}^{95}\text{Nb} + \text{e}^- + \bar{\nu}$    $(t_{1/2} = 35 \text{ d})$

$^{88}\text{Rb} \rightarrow {}^{88}\text{Sr} + \text{e}^- + \bar{\nu}$                       $^{95}\text{Nb} \rightarrow {}^{95}\text{Mo} + \text{e}^- + \bar{\nu}$

$^{88}\text{Sr} + \text{n} \rightarrow {}^{89}\text{Sr}$    $(t_{1/2} = 51 \text{ d})(\quad)$

29.  Let the alpha particle move with velocity $v$ before the collision. After the collision, the alpha particle moves (in the reverse direction) with velocity $v'$ and kinetic energy $K'$, and the atom moves with recoil velocity $v_R$ and kinetic energy $K_R$. We assume the kinetic energies to be nonrelativistic.

Conservation of energy: $K = K' + K_R$    or    $\frac{1}{2}mv^2 = \frac{1}{2}mv'^2 + \frac{1}{2}Mv_R^2$

Conservation of momentum: $mv = Mv_R - mv'$    or    $v_R = \frac{m}{M}(v + v')$

Combining these two equations, we obtain $\frac{m}{M}(v^2 - v'^2) = v_R^2 = \left[\frac{m}{M}(v+v')\right]^2$ and so

$$v - v' = \frac{m}{M}(v+v') \qquad \text{or} \qquad v' = v\left(\frac{1 - m/M}{1 + m/M}\right)$$

$$\Delta K = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv'^2 = K\left(1 - \frac{v'^2}{v^2}\right) = K\left[1 - \left(\frac{1 - m/M}{1 + m/M}\right)^2\right] = K\left[\frac{4m/M}{(1+m/M)^2}\right]$$

30. (a)    $^{63}$Cu:    $\Delta K = K\left[\dfrac{4m/M}{(1+m/M)^2}\right] = (2.50 \text{ MeV})\left[\dfrac{4(4/63)}{(1+4/63)^2}\right] = 0.561 \text{ MeV}$
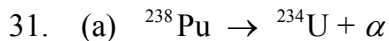
$^{107}$Ag:    $\Delta K = K\left[\dfrac{4m/M}{(1+m/M)^2}\right] = (2.50 \text{ MeV})\left[\dfrac{4(4/107)}{(1+4/107)^2}\right] = 0.347 \text{ MeV}$

$^{197}$Au:    $\Delta K = K\left[\dfrac{4m/M}{(1+m/M)^2}\right] = (2.50 \text{ MeV})\left[\dfrac{4(4/197)}{(1+4/197)^2}\right] = 0.195 \text{ MeV}$

(b)    $^{65}$Cu:    $\Delta K = K\left[\dfrac{4m/M}{(1+m/M)^2}\right] = (2.50 \text{ MeV})\left[\dfrac{4(4/65)}{(1+4/65)^2}\right] = 0.546 \text{ MeV}$

$^{109}$Ag:    $\Delta K = K\left[\dfrac{4m/M}{(1+m/M)^2}\right] = (2.50 \text{ MeV})\left[\dfrac{4(4/109)}{(1+4/109)^2}\right] = 0.342 \text{ MeV}$

The energy difference is 0.015 MeV for Cu and 0.005 MeV for Ag.

31. (a)    $^{238}$Pu $\rightarrow$ $^{234}$U $+ \alpha$
$Q = [m(^{238}\text{Pu}) - m(^{234}\text{U}) - m(^4\text{He})]c^2$

   $= (238.049560 \text{ u} - 234.040952 \text{ u} - 4.002603 \text{ u})(931.50 \text{ MeV/u}) = 5.594 \text{ MeV}$

(b)    $1.0 \text{ g} = \frac{1}{238} \text{ mole} = \frac{1}{238}(6.022 \times 10^{23} \text{ atoms}) = 2.53 \times 10^{21} \text{ atoms}$

$a = \lambda N = \dfrac{0.693}{(88 \text{ y})(3.16 \times 10^7 \text{ s/y})} 2.53 \times 10^{21} \text{ atoms} = 6.30 \times 10^{11} \text{ s}^{-1}$

$P = aQ = (6.30 \times 10^{11} \text{ decays/s})(5.594 \text{ MeV/decay})(1.60 \times 10^{-13}$

32. Titanium:    $R = a(1 - e^{-\lambda t})^{-1} = \dfrac{105 \text{ s}^{-1}}{1 - e^{-0.693(2.5 \text{ min})/(5.8 \text{ min})}} = 407 \text{ s}^{-1}$

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$$N = \frac{R}{\phi\sigma} = \frac{407 \text{ s}^{-1}}{(3.0 \times 10^{12} \text{ n/cm}^2\text{/s})(0.14 \times 10^{-24} \text{ cm}^2)} = 9.68 \times 10^{14} \text{ atoms of } {}^{50}\text{Ti}$$

Titanium is 5.25% ${}^{50}$Ti, so the total number $N$ of titanium atoms is $9.68 \times 10^{14} / 0.0525 = 1.84 \times 10^{16}$. With $N = N_A m / M$,

$$m = \frac{NM}{N_A} = \frac{(1.84 \times 10^{16} \text{ atoms})(47.9 \text{ g/mole})}{6.022 \times 10^{23} \text{ atoms/mole}} = 1.47 \ \mu\text{g}$$

Cobalt: $\quad R = a(1 - e^{-\lambda t})^{-1} = \dfrac{12 \text{ s}^{-1}}{1 - e^{-0.693(2.5 \text{ min})/[(5.27 \text{ y})(5.26 \times 10^5 \text{ min/y})]}} = 1.92 \times 10^7 \text{ s}^{-1}$

$$N = \frac{R}{\phi\sigma} = \frac{1.92 \times 10^7 \text{ s}^{-1}}{(3.0 \times 10^{12} \text{ n/cm}^2\text{/s})(19 \times 10^{-24} \text{ cm}^2)} = 3.36 \times 10^{17} \text{ atoms of } {}^{59}\text{Co}$$

$$m = \frac{NM}{N_A} = \frac{(3.36 \times 10^{17} \text{ atoms})(58.9 \text{ g/mole})}{6.022 \times 10^{23} \text{ atoms/mole}} = 33 \ \mu\text{g}$$

33. The total number of copper atoms in the target is

$$N = \frac{mN_A}{M} = \frac{(2.0 \times 10^{-3} \text{ g})(6.022 \times 10^{23} \text{ atoms/mole})}{63.5 \text{ g/mole}} = 1.90 \times 10^{19} \text{ atoms}$$

For ${}^{63}\text{Cu} + \text{n} \rightarrow {}^{64}\text{Cu}$, $N({}^{63}\text{Cu}) = 0.69 N(\text{Cu}) = 0.69(1.90 \times 10^{19}) = 1.31 \times 10^{19}$ atoms

$$R = \frac{a}{1 - e^{-\lambda t}} = \frac{(72 \times 10^{-6} \text{ Ci})(3.7 \times 10^{10} \text{ s}^{-1}/\text{Ci})}{1 - e^{-0.693(10 \text{ min})/[(12.7 \text{ h})(60 \text{ min/h})]}} = 2.94 \times 10^8 \text{ s}^{-1}$$

From Equation 13.2,

$$\sigma = \frac{RM}{\phi m N_A} = \frac{R}{\phi N} = \frac{2.94 \times 10^8 \text{ s}^{-1}}{(5.0 \times 10^{12} \text{ n/cm}^2\text{/s})(1.31 \times 10^{19})} = 4.49 \times 10^{-24} \text{ cm}^2 = 4.49 \text{ b}$$

For ${}^{65}\text{Cu} + \text{n} \rightarrow {}^{66}\text{Cu}$, $N({}^{65}\text{Cu}) = 0.31 N(\text{Cu}) = 0.31(1.90 \times 10^{19}) = 0.589 \times 10^{19}$ atoms

$$R = \frac{a}{1 - e^{-\lambda t}} = \frac{(1.30 \times 10^{-3} \text{ Ci})(3.7 \times 10^{10} \text{ s}^{-1}/\text{Ci})}{1 - e^{-0.693(10 \text{ min})/(5.1 \text{ min})}} = 6.47 \times 10^7 \text{ s}^{-1}$$

$$\sigma = \frac{RM}{\phi m N_{\rm A}} = \frac{R}{\phi N} = \frac{6.47 \times 10^7 \ {\rm s}^{-1}}{(5.0 \times 10^{12} \ {\rm n/cm^2/s})(0.589 \times 10^{19})} = 2.20 \times 10^{-24} \ {\rm cm}^2 = 2.20 \ {\rm b}$$

34.  (a) The number of atoms in the target is $N = m N_{\rm A} / M = \rho A N_{\rm A} \, dx / M = n A \, dx$
where $n = \rho N_{\rm A}/M$ is the number of target nuclei per unit volume. The rate at which reactions occur in the target is

$$R = \phi \sigma \frac{m}{M} N_{\rm A} = \phi \sigma n A \, dx = I \sigma n \, dx$$

where the neutron beam intensity $I$ (neutrons/s) is $\phi A$. The beam therefore loses intensity $dI$:

$$dI = -I \sigma n \, dx$$

(b) The total absorption after passing through a thickness $x$ is

$$\int_{I_0}^{I} \frac{dI}{I} = -\int_{0}^{x} \sigma n \, dx$$

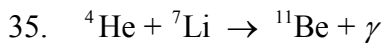$$\ln I - \ln I_0 = -\sigma n x \qquad \text{or} \qquad I = I_0 e^{-\sigma n x}$$

(c) $\quad n = \frac{\rho N_{\rm A}}{M} = \frac{(8.95 \ {\rm g/cm}^3)(6.022 \times 10^{23} \ {\rm atoms/mole})}{63.5 \ {\rm g/mole}} = 8.49 \times 10^{22} \ {\rm atoms/cm}^3$

$$n\sigma = (8.49 \times 10^{22} \ {\rm atoms/cm}^3)(5.0 \times 10^{-24} \ {\rm cm}^2) = 0.424 \ {\rm cm}^{-1}$$

$x = 1.0 \ {\rm mm} = 0.10 \ {\rm cm}: \quad I/I_0 = e^{-n\sigma x} = e^{-(0.424 \ {\rm cm}^{-1})(0.10 \ {\rm cm})} = 0.958 \quad$ so 4.2% is lost.

$x = 1.0 \ {\rm cm}: \quad I/I_0 = e^{-n\sigma x} = e^{-(0.424 \ {\rm cm}^{-1})(1.0 \ {\rm cm})} = 0.654 \quad$ so 34.6% is lost.

$x = 1.0 \ {\rm m} = 100 \ {\rm cm}: \quad I/I_0 = e^{-n\sigma x} = e^{-(0.424 \ {\rm cm}^{-1})(100 \ {\rm cm})} = 3.85 \times 10^{-19}$ so all is absorbed.

35.  $^4{\rm He} + {}^7{\rm Li} \rightarrow {}^{11}{\rm Be} + \gamma$
$\quad Q = [m(^4{\rm He}) + m(^7{\rm Li}) - m(^{11}{\rm B})]c^2$
$\quad\quad = (4.002603 \ {\rm u} + 7.016005 \ {\rm u} - 11.009305 \ {\rm u})(931.50 \ {\rm MeV/u}) = 8.666 \ {\rm MeV}$

Neglecting the energy of the incident particles,
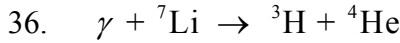
$$Q = K_{\rm B} + E_{\gamma} \qquad \text{and} \qquad p_{\rm B} = p_{\gamma}$$

$$K_B + E_\gamma = \frac{p_B^2}{2m_B} + E_\gamma = \frac{p_\gamma^2}{2m_B} + E_\gamma = \frac{E_\gamma^2}{2m_B c^2} + E_\gamma$$

$$\frac{E_\gamma^2}{2m_B c^2} + E_\gamma - Q = 0$$

$$E_\gamma = \left(-1 \pm \sqrt{1 + 4Q/2m_B c^2}\right) m_B c^2 = 8.662 \text{ MeV}$$

The kinetic energy of the boron is only 0.004 MeV, but it is not negligible at the precision of this calculation.

36.   $\gamma + {}^7\text{Li} \rightarrow {}^3\text{H} + {}^4\text{He}$

$$Q = [m({}^7\text{Li}) - m({}^3\text{H}) - m({}^4\text{He})]c^2$$

$$= (7.016005 \text{ u} - 3.016049 \text{ u} - 4.002603 \text{ u})(931.50 \text{ MeV/u}) = -2.4657 \text{ MeV}$$

With the ${}^7$Li at rest, energy conservation gives $Q = K_H + K_{He} - E_\gamma$. Similarly, conservation of momentum gives $p_\gamma = p_H + p_{He}$. If we supply only the minimum gamma-ray energy, the ${}^3$H and ${}^4$He will move together, as in a reaction at threshold. With $v_H = v_{He} = v$, we have

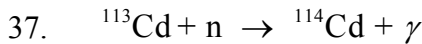$$p_\gamma = m_H v + m_{He} v \qquad \text{so} \qquad v = \frac{p_\gamma}{m_H + m_{He}} = \frac{E_\gamma}{c(m_H + m_{He})}$$

$$Q = -E_\gamma + K_H + K_{He} = -E_\gamma + \left(\tfrac{1}{2} m_H + \tfrac{1}{2} m_{He}\right) v^2 = -E_\gamma + \tfrac{1}{2}(m_H + m_{He}) \frac{E_\gamma^2}{c^2(m_H + m_{He})^2}$$

$$\frac{E_\gamma^2}{2(m_H + m_{He})c^2} - E_\gamma - Q = 0$$

$$E_\gamma = \left(1 \pm \sqrt{1 + 4Q/2(m_H + m_{He})c^2}\right)(m_H + m_{He})c^2 = 2.4661 \text{ MeV}$$

The gamma ray must supply the energy to break up the ${}^7$Li (2.4657 MeV) plus an additional 0.0004 MeV to give the kinetic energies to the products that is necessary to conserve momentum.

37.   ${}^{113}\text{Cd} + \text{n} \rightarrow {}^{114}\text{Cd} + \gamma$

$$Q = [m({}^{113}\text{Cd}) + m(\text{n}) - m({}^{114}\text{Cd})]c^2$$

$$= (112.904402 \text{ u} + 1.008665 \text{ u} - 113.903359 \text{ u})(931.50 \text{ MeV/u}) = 9.043 \text{ MeV}$$

Energy conservation gives $Q = K_{Cd} + E_\gamma - K_n$, where $K_{Cd}$ is the kinetic energy given to the product ${}^{114}$Cd (the original nucleus ${}^{113}$Cd being at rest). The neutron kinetic energy (0.025 eV) is negligible. We then have

$$K_{Cd} + E_\gamma = 9.043 \text{ MeV}$$

Momentum conservation gives $p_n = p_{Cd} + p_\gamma$, and with $p_n$ negligibly small, we get $p_{Cd} = -p_\gamma$. The heavy Cd nucleus can take this much momentum at a cost of very little energy, so we can neglect $K_{Cd}$ to obtain

$$E_\gamma = 9.043 \text{ MeV}$$

We can check these assumptions by computing the kinetic energy of the Cd nucleus. The result is $4 \times 10^{-4}$ MeV, which is indeed negligible within the precision of the numbers used in this problem.

38.     $$\frac{\Delta K}{K} = \frac{4m/M}{(1+m/M)^2} \qquad \text{with } m = 1 \text{ u (neutron)}$$

(a) $M = 1$ u (hydrogen): $\quad \dfrac{\Delta K}{K} = \dfrac{4}{2^2} = 1 \quad$ (all kinetic energy is lost)

$M = 2$ u (deuteron): $\quad \dfrac{\Delta K}{K} = \dfrac{4(1/2)}{(1+1/2)^2} = 0.89$

$M = 12$ u (carbon): $\quad \dfrac{\Delta K}{K} = \dfrac{4(1/12)}{(1+1/12)^2} = 0.28$

(b) After one scattering, the new kinetic energy is

$$K' = K - \Delta K = K - K\frac{4mM}{(1+m/M)^2} = K\left(\frac{1-m/M}{1+m/M}\right)^2$$

After $n$ scatterings, the kinetic energy is

$$K' = K\left(\frac{1-m/M}{1+m/M}\right)^{2n}$$

$$0.025 \text{ eV} = 2.0 \text{ MeV}\left(\frac{1-1/12}{1+1/12}\right)^{2n}$$

$$n = \frac{\log[(0.025 \text{ eV})/(2\times10^6 \text{ eV})]}{2\log(11/13)} = 54.5 \text{ or 55 scatterings}$$

39. (a) The number of molecules of water is

$$N = \frac{6.022\times10^{23} \text{ molecules/mole}}{18 \text{ g/mole}}(100 \text{ cm}^3)(1 \text{ g/cm}^3) = 3.3\times10^{24} \text{ molecules}$$

The number of $D_2O$ molecules is 0.015% of this, or $5.0 \times 10^{20}$. Each molecule includes 2 deuterons, but each fusion reaction takes 2 deuterons, so that consuming all of the $D_2O$ would result in $5.0 \times 10^{20}$ fusion reactions, each of which releases 4.0 MeV. The total energy release is

$$E = (5.0 \times 10^{20} \text{ reactions})(4.0 \text{ MeV/reaction}) = 2.0 \times 10^{21} \text{ MeV} = 3.2 \times 10^8 \text{ J}$$

(b) If only 2/3 of the deuterium participated in the first reaction, the energy released would be 2/3 of that found in part (a), or $2.1 \times 10^8$ J. Each reaction produces a single $^3H$, so the number of $^3H$ produced is $\frac{1}{3}(5.0 \times 10^{20})$. These $^3H$ then react with the remaining $\frac{1}{3}(5.0 \times 10^{20})$ deuterons, with each reaction releasing 17.6 MeV for an energy release of

$$E = \tfrac{1}{3}(5.0 \times 10^{20} \text{ reactions})(17.6 \text{ MeV/reaction}) = 2.9 \times 10^{21} \text{ MeV} = 4.7 \times 10^8 \text{ J}$$

The total energy released in the combined set of processes is

$$E = 2.1 \times 10^8 \text{ J} + 4.7 \times 10^8 \text{ J} = 6.8 \times 10^8 \text{ J}$$

More than twice as much energy is obtained from this second set of processes.

40. (a) $^4He + {}^4He \to {}^8Be$

$$Q = [2m(^4He) - m(^8Be)]c^2$$

$$= (2 \times 4.002603 \text{ u} - 8.005305 \text{ u})(931.50 \text{ MeV/u}) = -0.092 \text{ MeV}$$

(b) At a temperature $T$, the relative number of helium pairs with energy $\Delta E = 0.092$ MeV is approximately given by the Boltzmann factor

$$\tfrac{1}{2}e^{-\Delta E/kT} = \tfrac{1}{2}e^{-(0.092 \text{ MeV})/(8.62 \times 10^{-5} \text{ eV/K})(10^8 \text{ K})} = 1.12 \times 10^{-5}$$

The factor of ½ is necessary because we need to know the number of He pairs that have the necessary energy.

# Chapter 14

This chapter provides an introduction to particle physics, including classifications of particles, decays and reactions of particles, and a brief introduction to the explanation of particle structure in terms of quarks. The emphasis is on achieving a basic phenomenological understanding of the underlying structure of matter, rather than acquiring the mathematical skills necessary to analyze that structure.

## Supplemental Materials

Current listings of particle properties, quark contents, and decay modes can be found in the tables published by the Particle Data Group at the Lawrence Berkeley National Laboratory: http://pdg.lbl.gov/.

For an interactive and animated tour through the world of particle physics, see The Particle Adventure, also from the LBNL Particle Data Group: http://particleadventure.org/index.html.

A similar hands-on tutorial from CERN is at: http://hands-on-cern.physto.se/hoc_v21en/index.html.

And another from Fermilab: http://ed.fnal.gov/projects/labyrinth/games/index1.html.

For interactive questions interpreting bubble chamber tracks, see http://epweb2.ph.bham.ac.uk/user/watkins/seeweb_feb99/BubbleChamber.htm.

## Suggestions for Additional Reading

Advanced books on particle physics tend to be mathematically difficult, full of field theory and relativistic quantum mechanics. Fortunately there are many popular-level books and articles that can be read for background material. These are generally descriptive and nonmathematical. For example, see
G. Feinberg, *What is the World Made Of?* (Anchor Press, 1977).
J. C. Polkinghorne, *The Particle Play* (W. H. Freeman, 1981).
C. Schwarz, *A Tour of the Subatomic Zoo* (American Institute of Physics, 1992).
R. Gilmore, *The Wizard of Quarks* (Copernicus Books, 2001).
M. J. G. Veltman, *Facts and Mysteries in Elementary Particle Physics* (World
   Scientific, 2003).
The book by Gilmore is a fanciful tour through the world of particles written as a parody of *The Wizard of Oz*.

A little more challenging, but still containing lots of general introductory material:
G. D. Coughlan, J. E. Dodd and B. M. Gripaios, *The Ideas of Particle Physics,* 3rd ed.
   (Cambridge University Press, 2006).
D. Griffiths, *Introduction to Elementary Particles*, 2nd ed. (Wiley, 2008).

For histories of some discoveries in particle physics, see:
A. Pickering, *Constructing Quarks* (University of Chicago Press, 1984).
R. P. Crease and C. C. Mann, *The Second Creation,* (MacMillan, 1986).

A historical and lavishly illustrated introduction is:
F. Close, M. Marten, and C. Sutton, *The Particle Odyssey,* (Oxford University Press, 2002).

For speculations about unification by one of the developers of the electroweak theory, see:
S. Weinberg, *Dreams of a Final Theory* (Pantheon, 1992).

Good sources of information about particle physics at a general level are the occasional articles in the magazine *Scientific American*:
S. L. Glashow, "Quarks with Color and Flavor," (October 1975); Y. Nambu, "The Confinement of Quarks," (November 1976); C. Quigg, "Elementary Particles and Forces," (April 1985); J. M. LoSecco, F. Reines, and D. Sinclair, "The Search for Proton Decay," (June 1985); G. Kane, "The Dawn of Physics Beyond the Standard Model," (June 2003); C. Quigg, "The Coming Revolutions in Particle Physics," (February 2008).

Many articles on particle physics from *Scientific American* are collected in:
*Particles and Forces: At the Heart of the Matter,* edited by R. A. Carrigan, Jr. and W. P. Trower (Freeman, 1990).

### Materials for Active Engagement in the Classroom

#### A. Reading Quizzes

1. Which of the following particles is a composite (made up of other more fundamental particles)?
    (1) meson     (2) quark     (3) electron     (4) neutrino

2. Arrange the 4 basic forces in order of increasing values of the typical time over which the force acts.
    (1) electromagnetic, gravitational, weak, strong
    (2) weak, strong, electromagnetic, gravitational
    (3) gravitational, electromagnetic, weak, strong
    (4) strong, electromagnetic, weak, gravitational

3. If a new particle with spin 3/2 were discovered, it would be a
    (1) field particle    (2) lepton     (3) meson     (4) baryon

**Answers**    1. 1    2. 4    3. 4

## B. Conceptual or Discussion Questions

1. Give the quark content of mesons with the following properties:
   (a) charge = +1, charm = 0, strangeness = 0, topness = 0, bottomness = +1
   (b) charge = 0, charm = 0, strangeness = 0, topness = 0, bottomness = +1
   (c) charge = 0, charm = 0, strangeness = +1, topness = 0, bottomness = −1
   (d) charge = −1, charm = −1, strangeness = 0, topness = 0, bottomness = −1

2. The two dimensional (strangeness vs. charge) chart of the spin-0 mesons (Figure 14.11) became 3-dimensional (Figure 14.17) when a new axis was added for charm. Extend the 2 dimensional chart for the spin-3/2 baryons (Figure 14.16) to 3 dimensions by adding a charm axis and indicate the locations of all possible baryons that can be constructed from u, d, s, and c quarks.

3. In the multiplet of 10 spin-3/2 baryons (Figures 14.13 and 14.16), each row differs from the row below by replacing a u or d quark with an s quark. The average rest energy of the baryons in each row should therefore differ from the average rest energy of the next row by the difference in rest energy between the s quark and the u or d quark. Find the average rest energy differences between the rows and compare with the rest energy difference of the quarks.

4. The $\Omega^-$ baryon decays through several processes in sequence, ultimately resulting in only stable particles. After all decays have occurred, what are the remaining stable particles?

5. The rest energy of the $\Omega^-$ baryon (quark content sss) is 1672 MeV. Estimate the rest energy you would expect for the charmed $\Omega$ (ssc), the bottom $\Omega$ (ssb), and the charmed bottom $\Omega$ (scb). Find the measured values of the rest energies of these particles (if known) and compare with your estimates.

**Answers**    1. (a) $u\bar{b}$  (b) $d\bar{b}$  (c) $b\bar{s}$  (d) $b\bar{c}$
3. 153 MeV, 148 MeV, 139 MeV; 170 MeV
4. $p + 2e^- + 2\bar{\nu}_e$ (other paths might also include $2\nu_\mu + 2\bar{\nu}_\mu$)
5. 2672 MeV, 5872 MeV, 6872 MeV

### Sample Exam Questions

## A. Multiple Choice

1. In the reaction $p + p \rightarrow p + \bar{p} + X$ (where $\bar{p}$ represents an antiproton), what might X represent?
   (a) Two protons      (b) Two mesons      (c) Two positrons (positive electrons)
   (d) One proton                (e) One baryon with an electric charge of +2e

2. The decay $\Omega^- \rightarrow \Lambda^0 + K^-$ can be written in terms of quarks as $sss \rightarrow uds + s\bar{u}$.
   Which interaction is responsible for this decay?
   (a) Strong        (b) Weak     (c) Electromagnetic        (d) Gravitational

3. Other than gravity, two protons can interact through which forces?
   (a) only electromagnetic        (b) only electromagnetic and strong
   (c) only electromagnetic and weak       (d) strong, electromagnetic, and weak

4. In the reaction $p + p \rightarrow p + n + X$, the single particle represented by X could be a
   (a) proton        (b) positron, $e^+$      (c) quark     (d) meson     (e) photon

**Answers**     1. a    2. b   3. d   4. d

## B. Conceptual

1. The $\Delta$ baryon decays into 2 particles. To which families do these 2 particles belong?
   EXPLAIN YOUR ANSWER.

2. The $D^+$ meson (quark content $c\bar{d}$), which is the least massive charmed meson, decays
   with a lifetime of about $10^{-12}$ s. Some of its decay modes produce only 2 other
   mesons, for example 2 $\pi$ mesons or a $\pi$ and a K. Would you expect to find a decay
   mode for the $D^+$ meson in which its direct decay products are only leptons or
   antileptons? Either give an example of an allowed decay process and explain why it
   is allowed, or else explain why the decay process would not be allowed.

3. The $\Omega^-$ baryon (strangeness $= -3$) decays through a series of processes that eventually
   lead to a nucleon (proton or neutron). What is the minimum number of decay
   processes necessary for the $\Omega^-$ to transform into p or n? EXPLAIN YOUR
   ANSWER.

**Answers**      1. Baryon + meson (baryon + photon would be acceptable)
                2. $D^+ \rightarrow \mu^+ + \nu_\mu$, because the lifetime (and the change in charm) indicate
                     a decay by the weak interaction.
                3. Two (first a strangeness-changing weak decay including the emission
                 of a strange meson, followed by another strangeness-changing weak
                 decay. (Full credit for a well-explained answer arguing for a series of 3
                 weak decays each changing $S$ by 1.)

## C. Problems

1. An unknown particle X moving in the $x$ direction decays into a proton and a $\pi^-$.
   (a) To what family does X belong? Explain your answer.

(b) From the length of the track left by X, it is concluded that the decay lifetime is in the range of $10^{-10}$ s. What does this imply about the interaction that is responsible for this decay, and what can you conclude about the properties of particle X?

(c) You observe the proton to move with momentum 360 MeV/$c$ at 14° above the $x$ axis and the pion to move with momentum 155 MeV/$c$ at 34° below the $x$ axis. Find the momentum and energy of particle X. The proton rest energy is 938 MeV and the pion rest energy is 140 MeV.

(d) What is the rest energy of particle X?

2. (a) What is the minimum kinetic energy necessary to produce the reaction $\pi^0 + p \rightarrow p + p + \bar{p}$ with $\pi^0$ incident on protons at rest? (Rest energies are $m_\pi c^2 = 135\,\text{MeV}$, $m_p c^2 = 938\,\text{MeV}$.)

(b) Find the kinetic energy of each of the final particles if the pions are incident at the threshold energy.


**Answers**      1. (a) baryon   (b) weak interaction; a conservation law (such as strangeness) may be violated in the decay   (c) 478 MeV/$c$, 1214 MeV   (d) 1116 MeV
2. (a) 3607 MeV   (b) 622 MeV

# Problem Solutions

1. (a) Strong, because of the lifetime.    (b) Electromagnetic, because of the photons.
   (c) Weak, because of the leptons.    (d) Weak, because of the lifetime.
   (e) Strong, because of the lifetime.    (f) Weak, because of the lifetime.

2. We can estimate the range $\Delta x$ using the uncertainty principle in the form of $\Delta E \Delta t \sim \hbar$, with $\Delta E = mc^2$ and $\Delta t = \Delta x / c$ :

$$\Delta x \sim \frac{\hbar c}{mc^2} = \frac{197 \text{ MeV} \cdot \text{fm}}{80.2 \times 10^3 \text{ MeV}} = 2.46 \times 10^{-3} \text{ fm}$$

3. (a) $\pi^- \to \mu^- + \nu_\mu$         (b) $\rho^- \to \pi^- + \pi^0$
   (c) $D^- \to K^+ + \pi^- + \pi^-$         (d) $\overline{K}^0 \to \pi^- + \pi^+$

4. (a) $\overline{n} \to \overline{p} + e^+ + \nu_e$         (b) $\overline{\Lambda}^0 = \overline{p} + \pi^+$
   (c) $\overline{\Omega}^- \to \overline{\Lambda}^0 + K^+$         (d) $\overline{\Sigma}^0 \to \overline{\Lambda}^0 + \gamma$

5. (a) $K^0 \to \pi^- + e^+ + \nu_e$         (b) $K^0 \to \pi^+ + e^- + \overline{\nu}_e$
   (c) $K^0 \to \pi^- + \mu^+ + \nu_\mu$         (d) $K^0 \to \pi^+ + \mu^- + \overline{\nu}_\mu$

6. (a) Lepton number ($L_e = 0 \to L_e = -1$)
   (b) Energy ($m_\Lambda c^2 < m_p c^2 + m_K c^2$)
   (c) Strangeness ($S = -3 \to S = -1$; only $\Delta S = \pm 1$ or 0 is allowed for the weak interaction)
   (d) Baryon number ($B = 1 \to B = 0$)
   (e) Strangeness ($S = -1 \to S = 0$; only $\Delta S = 0$ is allowed for electromagnetic decays)
   (f) Energy ($m_\Omega c^2 < m_\Xi c^2 + m_K c^2$)
   (g) Energy ($m_\Xi c^2 < m_\Sigma c^2 + m_\pi c^2$)
   (h) Electron and muon lepton numbers ($L_e = 0 \to L_e = 1$ and $L_\mu = 1 \to L_\mu = 0$)

7. (a) Electron lepton number ($L_e = +1 \to L_e = -1$)
   (b) Strangeness ($S = 0 \to S = -1$)
   (c) Baryon number ($B = 2 \to B = 3$)
   (d) Strangeness ($S = 0 \to S = -2$)
   (e) Baryon number ($B = 1 \to B = 2$)

8. (a) $\nu_e$          (b) $\pi^0$          (c) $\pi^0$

9. (a) $K^+$     (b) $\overline{n}$     (c) $K^0$     (d) $\pi^-$     (e) $\mu^+$     (f) $\Xi^-$

10. (a)      $p_x = p_{1x} + p_{2x} = (528 \text{ MeV}/c)\cos 30° + (2166 \text{ MeV}/c)\cos 7° = 2607 \text{ MeV}/c$

$p_y = p_{1y} + p_{2y} = (528 \text{ MeV}/c)\sin 30° - (2166 \text{ MeV}/c)\sin 7° = 0$

$$E_1 = \sqrt{c^2 p_1^2 + m_\pi^2 c^4} = \sqrt{(528 \text{ MeV})^2 + (140 \text{ MeV})^2} = 546 \text{ MeV}$$

$$E_2 = \sqrt{c^2 p_2^2 + m_\pi^2 c^4} = \sqrt{(2166 \text{ MeV})^2 + (140 \text{ MeV})^2} = 2171 \text{ MeV}$$

$$mc^2 = \sqrt{(E_1 + E_2)^2 - c^2 p^2} = \sqrt{(546 \text{ MeV} + 2171 \text{ MeV})^2 - (2607 \text{ MeV})^2} = 764 \text{ MeV}$$

(b)      $p_x = p_{1x} + p_{2x} = (645 \text{ MeV}/c)\cos 20° + (4223 \text{ MeV}/c)\cos 3° = 4823 \text{ MeV}/c$

$p_y = p_{1y} + p_{2y} = (645 \text{ MeV}/c)\sin 20° - (4223 \text{ MeV}/c)\sin 3° = 0$

$$E_1 = \sqrt{c^2 p_1^2 + m_\pi^2 c^4} = \sqrt{(645 \text{ MeV})^2 + (140 \text{ MeV})^2} = 660 \text{ MeV}$$

$$E_2 = \sqrt{c^2 p_2^2 + m_\pi^2 c^4} = \sqrt{(4223 \text{ MeV})^2 + (140 \text{ MeV})^2} = 4225 \text{ MeV}$$

$$mc^2 = \sqrt{(E_1 + E_2)^2 - c^2 p^2} = \sqrt{(660 \text{ MeV} + 4225 \text{ MeV})^2 - (4823 \text{ MeV})^2} = 775 \text{ MeV}$$

(c)      $p_x = p_{1x} + p_{2x} = (119 \text{ MeV}/c)\cos 45° + (962 \text{ MeV}/c)\cos 5° = 1042 \text{ MeV}/c$

$p_y = p_{1y} + p_{2y} = (119 \text{ MeV}/c)\sin 45° - (962 \text{ MeV}/c)\sin 5° = 0$

$$E_1 = \sqrt{c^2 p_1^2 + m_\pi^2 c^4} = \sqrt{(119 \text{ MeV})^2 + (140 \text{ MeV})^2} = 183 \text{ MeV}$$

$$E_2 = \sqrt{c^2 p_2^2 + m_\pi^2 c^4} = \sqrt{(962 \text{ MeV})^2 + (140 \text{ MeV})^2} = 972 \text{ MeV}$$

$$mc^2 = \sqrt{(E_1 + E_2)^2 - c^2 p^2} = \sqrt{(183 \text{ MeV} + 972 \text{ MeV})^2 - (1042 \text{ MeV})^2} = 498 \text{ MeV}$$

11. (a) $\Delta E = \dfrac{\hbar}{\Delta t} = \dfrac{\hbar}{\tau} = \dfrac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{5.1 \times 10^{-19} \text{ s}} = 1.29 \text{ keV}$

(b) $\Delta E = \dfrac{\hbar}{\Delta t} = \dfrac{\hbar}{\tau} = \dfrac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{3.2 \times 10^{-21} \text{ s}} = 0.21 \text{ MeV}$

(c) $\Delta E = \dfrac{\hbar}{\Delta t} = \dfrac{\hbar}{\tau} = \dfrac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{7.4 \times 10^{-20} \text{ s}} = 8.9 \text{ keV}$

(d) $\Delta E = \dfrac{\hbar}{\Delta t} = \dfrac{\hbar}{\tau} = \dfrac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{5.6 \times 10^{-24} \text{ s}} = 118 \text{ MeV}$

12. For $\Sigma^-$ the proper lifetime is $\tau_0 = 1.5 \times 10^{-10} \text{ s}$.

$$E = K + mc^2 = 3642 \text{ MeV} + 1197 \text{ MeV} = 4839 \text{ MeV} = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

$$\frac{v}{c} = \sqrt{1 - \left(\frac{mc^2}{E}\right)^2} = \sqrt{1 - \left(\frac{1197 \text{ MeV}}{4839 \text{ MeV}}\right)^2} = 0.9689$$

$$\tau = \frac{\tau_0}{\sqrt{1 - v^2/c^2}} = \frac{1.5 \times 10^{-10} \text{ s}}{\sqrt{1 - (0.9689)^2}} = 6.07 \times 10^{-10} \text{ s}$$

$$D = v\tau = (0.9689)(3.00 \times 10^8 \text{ m/s})(6.07 \times 10^{-10} \text{ s}) = 0.18 \text{ m}$$

13. With $K_\pi = 0$, and assuming $m_\nu$ to be negligibly small,

$$Q = K_e + E_\nu = \sqrt{(p_e c)^2 + (m_e c^2)^2} - m_e c^2 - p_\nu c$$

Momentum conservation gives $p_e = p_\nu$, so

$$Q + m_e c^2 - p_e c = \sqrt{(p_e c)^2 + (m_e c^2)^2}$$

$$Q^2 + (m_e c^2)^2 + (p_e c)^2 + 2Q m_e c^2 - 2Q p_e c - 2m_e p_e c^3 = (p_e c)^2 + (m_e c^2)^2$$

$$cp_e = \frac{Q^2 + 2Q m_e c^2}{2Q + 2m_e c^2} = \frac{(358.2\ \text{MeV})^2 + 2(358.2\ \text{MeV})(0.511\ \text{MeV})}{2(358.2\ \text{MeV}) + 2(0.511\ \text{MeV})} = 179.4\ \text{MeV}$$

$$E_e = \sqrt{(p_e c)^2 + (m_e c^2)^2} = \sqrt{(179.4\ \text{MeV})^2 + (0.511\ \text{MeV})^2} = 179.4\ \text{MeV}$$

14. (a) $Q = m_i c^2 - m_f c^2 = m_\pi c^2 = 135\ \text{MeV}$

(b) $Q = m_i c^2 - m_f c^2 = m_\Sigma c^2 - m_p c^2 - m_\pi c^2$
$= 1189\ \text{MeV} - 938\ \text{MeV} - 135\ \text{MeV} = 116\ \text{MeV}$

(c) $Q = m_i c^2 - m_f c^2 = m_D c^2 - m_K c^2 - 2m_\pi c^2$
$= 1869\ \text{MeV} - 494\ \text{MeV} - 2(140\ \text{MeV}) = 1095\ \text{MeV}$

15. (a) $Q = m_i c^2 - m_f c^2 = m_\pi c^2 - m_\mu c^2 - m_\nu c^2 = 140\ \text{MeV} - 105.7\ \text{MeV} = 34\ \text{MeV}$

(b) $Q = m_i c^2 - m_f c^2 = m_K c^2 - 2m_\pi c^2 = 498\ \text{MeV} - 2(140\ \text{MeV}) = 218\ \text{MeV}$

(c) $Q = m_i c^2 - m_f c^2 = m_\Sigma c^2 - m_\Lambda c^2 = 1192\ \text{MeV} - 1116\ \text{MeV} = 76\ \text{MeV}$

16. (a) Momentum conservation: $p_{\pi^+} = p_{\pi^-}$ so $E_{\pi^+} = E_{\pi^-}$

Energy conservation: $E_K = E_{\pi^+} + E_{\pi^-}$

$$E_{\pi^+} = E_{\pi^-} = \tfrac{1}{2} E_K = \tfrac{1}{2}(498\ \text{MeV}) = 249\ \text{MeV}$$

$$K_{\pi^+} = K_{\pi^-} = E_{\pi^+} - m_\pi c^2 = 249\ \text{MeV} - 140\ \text{MeV} = 109\ \text{MeV}$$

(b) Momentum conservation: $p_n = p_\pi$

Energy conservation: $E_n + E_\pi = \sqrt{(p_n c)^2 + (m_n c^2)^2} + \sqrt{(p_\pi c)^2 + (m_\pi c^2)^2} = E_\Sigma$

$$\sqrt{(p_\pi c)^2 + (m_n c^2)^2} = E_\Sigma - \sqrt{(p_\pi c)^2 + (m_\pi c^2)^2} = m_\Sigma c^2 - \sqrt{(p_\pi c)^2 + (m_\pi c^2)^2}$$

$$(p_\pi c)^2 + (m_n c^2)^2 = (m_\Sigma c^2)^2 + (p_\pi c)^2 + (m_\pi c^2)^2 - 2m_\Sigma c^2 \sqrt{(p_\pi c)^2 + (m_\pi c^2)^2}$$

$$E_\pi = \sqrt{(p_\pi c)^2 + (m_\pi c^2)^2} = \frac{(m_\Sigma c^2) + (m_\pi c^2)^2 - (m_n c^2)^2}{2 m_\Sigma c^2}$$

$$= \frac{(1197 \text{ MeV})^2 + (140 \text{ MeV})^2 - (940 \text{ MeV})^2}{2(1197 \text{ MeV})} = 221 \text{ MeV}$$

$$K_\pi = E_\pi - m_\pi c^2 = 221 \text{ MeV} - 140 \text{ MeV} = 81 \text{ MeV}$$
$$E_n = E_\Sigma - E_\pi = 1197 \text{ MeV} - 221 \text{ MeV} = 976 \text{ MeV}$$
$$K_n = E_n - m_n c^2 = 976 \text{ MeV} - 940 \text{ MeV} = 36 \text{ MeV}$$

17. (a)    Momentum conservation: $p_\Lambda = p_K$

Energy conservation: $E_\Lambda + E_K = \sqrt{(p_\Lambda c)^2 + (m_\Lambda c^2)^2} + \sqrt{(p_K c)^2 + (m_K c^2)^2} = E_\Omega$

$$\sqrt{(p_K c)^2 + (m_\Lambda c^2)^2} = E_\Omega - \sqrt{(p_K c)^2 + (m_K c^2)^2} = m_\Omega c^2 - \sqrt{(p_K c)^2 + (m_K c^2)^2}$$

$$(p_K c)^2 + (m_\Lambda c^2)^2 = (m_\Omega c^2)^2 + (p_K c)^2 + (m_K c^2)^2 - 2 m_\Omega c^2 \sqrt{(p_K c)^2 + (m_K c^2)^2}$$

$$E_K = \sqrt{(p_K c)^2 + (m_K c^2)^2} = \frac{(m_\Omega c^2) + (m_K c^2)^2 - (m_\Lambda c^2)^2}{2 m_\Omega c^2}$$

$$= \frac{(1672 \text{ MeV})^2 + (494 \text{ MeV})^2 - (1116 \text{ MeV})^2}{2(1672 \text{ MeV})} = 537 \text{ MeV}$$

$$K_K = E_K - m_K c^2 = 537 \text{ MeV} - 494 \text{ MeV} = 43 \text{ MeV}$$
$$E_\Lambda = E_\Omega - E_K = 1672 \text{ MeV} - 537 \text{ MeV} = 1135 \text{ MeV}$$
$$K_\Lambda = E_\Lambda - m_\Lambda c^2 = 1135 \text{ MeV} - 1116 \text{ MeV} = 19 \text{ MeV}$$

(b)    Momentum conservation: $p_\mu = p_\nu$

Energy conservation: $E_\mu + E_\nu = \sqrt{(p_\mu c)^2 + (m_\mu c^2)^2} + p_\nu c = E_\pi = m_\pi c^2$

where we assume the $\nu$ has a negligible mass.  Then

$$(p_\nu c)^2 + (m_\mu c^2)^2 = (m_\pi c^2 - p_\nu c)^2 = (m_\pi c^2)^2 - 2 m_\pi p_\nu c^3 + (p_\nu c)^2$$

$$E_\nu = p_\nu c = \frac{(m_\pi c^2)^2 - (m_\mu c^2)^2}{2 m_\pi c^2} = \frac{(140 \text{ MeV})^2 - (105.7 \text{ MeV})^2}{2(140 \text{ MeV})} = 30 \text{ MeV}$$

$$E_\mu = E_\pi - E_\nu = 140 \text{ MeV} - 30 \text{ MeV} = 110 \text{ MeV}$$
$$K_\mu = E_\mu - m_\mu c^2 = 110 \text{ MeV} - 106 \text{ MeV} = 4 \text{ MeV}$$

18.
$$E_\Sigma = K_\Sigma + m_\Sigma c^2 = 250 \text{ MeV} + 1197 \text{ MeV} = 1447 \text{ MeV}$$
$$cp_\Sigma = \sqrt{E_\Sigma^2 - (m_\Sigma c^2)^2} = \sqrt{(1447 \text{ MeV})^2 - (1197 \text{ MeV})^2} = 813 \text{ MeV}$$

Let the neutron move at an angle $\theta$ with the positive $x$ axis, which is the original direction of motion of the $\Sigma$. Then conservation of momentum gives:

$x$ direction: $p_\Sigma = p_n \cos\theta$ \qquad\qquad $y$ direction: $0 = p_n \sin\theta - p_\pi$ or $p_\pi = p_n \sin\theta$

which combine to give $p_\Sigma^2 + p_\pi^2 = p_n^2$. Conservation of energy gives

$$E_\Sigma = E_n + E_\pi = \sqrt{(cp_n)^2 + (m_n c^2)^2} + \sqrt{(cp_\pi)^2 + (m_\pi c^2)^2}$$
$$(cp_\pi)^2 + (m_\pi c^2)^2 = E_\Sigma^2 - 2E_\Sigma \sqrt{(cp_n)^2 + (m_n c^2)^2} + (cp_n)^2 + (m_n c^2)^2$$
$$E_n = \sqrt{(cp_n)^2 + (m_n c^2)^2} = \frac{(m_n c^2)^2 + E_\Sigma^2 - (m_\pi c^2)^2 + (cp_n)^2 - (cp_\pi)^2}{2E_\Sigma}$$
$$= \frac{(m_n c^2)^2 + E_\Sigma^2 - (m_\pi c^2)^2 + (cp_K)^2}{2E_\Sigma}$$
$$= \frac{(940 \text{ MeV})^2 + (1447 \text{ MeV})^2 - (140 \text{ MeV})^2 + (813 \text{ MeV})^2}{2(1447 \text{ MeV})} = 1250 \text{ MeV}$$

$$K_n = E_n - m_n c^2 = 1250 \text{ MeV} - 940 \text{ MeV} = 310 \text{ MeV}$$
$$E_\pi = E_\Sigma - E_n = 1447 \text{ MeV} - 1250 \text{ MeV} = 197 \text{ MeV}$$
$$K_\pi = E_\pi - m_\pi c^2 = 197 \text{ MeV} - 140 \text{ MeV} = 57 \text{ MeV}$$
$$cp_n = \sqrt{E_n^2 - (m_n c^2)^2} = \sqrt{(1250 \text{ MeV})^2 - (940 \text{ MeV})^2} = 824 \text{ MeV}$$
$$\theta = \cos^{-1} \frac{p_\Sigma}{p_n} = \cos^{-1} \frac{813 \text{ MeV}/c}{824 \text{ MeV}/c} = 9.4°$$

19. Let $\theta$ be the angles that the directions of the two outgoing pi mesons make above and below the $x$ axis (which represents the direction of the original K meson). We'll use $p_+$ and $p_-$ to represent the momenta of the pions and $E_+$ and $E_-$ for their energies. Then conservation of momentum gives:

$x$ direction: $p_K = p_+ \cos\theta + p_- \cos\theta$ \qquad\qquad $y$ direction: $0 = p_+ \sin\theta - p_- \sin\theta$

From the $y$ equation, we get immediately $p_+ = p_-$, which in turn gives $E_+ = E_-$. The total energy of the K meson is $E_K = K_K + m_K c^2 = 276 \text{ MeV} + 498 \text{ MeV} = 774 \text{ MeV}$. Conservation of energy then gives:

$$E_K = E_+ + E_- \qquad \text{so} \qquad E_+ = E_- = \tfrac{1}{2} E_K = \tfrac{1}{2}(774 \text{ MeV}) = 387 \text{ MeV}$$

$$cp_K = \sqrt{E_K^2 - (m_K c^2)^2} = \sqrt{(774 \text{ MeV})^2 - (498 \text{ MeV})^2} = 592.5 \text{ MeV}$$
$$cp_+ = \sqrt{E_+^2 - (m_\pi c^2)^2} = \sqrt{(387 \text{ MeV})^2 - (140 \text{ MeV})^2} = 360.8 \text{ MeV}$$

From the $x$ component momentum equation with $p_+ = p_-$ we get $p_K = 2 p_+ \cos\theta$, so

$$\theta = \cos^{-1} \frac{p_K}{2 p_+} = \cos^{-1} \frac{592.5 \text{ MeV}}{2(360.8 \text{ MeV})} = 34.9°$$

20. In the nonrelativistic limit, all kinetic energies are very small compared with rest energies, so that the total rest energy in the reaction does not change:

$$m_1 + m_2 \cong m_3 + m_4 + m_5 + \cdots$$

The threshold kinetic energy is then

$$K_{th} = -Q \frac{m_1 + m_2 + m_3 + m_4 + m_5 + \cdots}{2m_2} = -Q \frac{2(m_1 + m_2)}{2m_2} = -Q\left(1 + \frac{m_1}{m_2}\right)$$

21. (a) $Q = m_i c^2 - m_f c^2 = m_K c^2 + m_p c^2 - m_\Lambda c^2 - m_\pi c^2$
    $= 494 \text{ MeV} + 938 \text{ MeV} - 1116 \text{ MeV} - 135 \text{ MeV} = 181 \text{ MeV}$

    (b) $Q = m_i c^2 - m_f c^2 = m_\pi c^2 + m_p c^2 - m_\Sigma c^2 - m_K c^2$
    $= 140 \text{ MeV} + 938 \text{ MeV} - 1189 \text{ MeV} - 494 \text{ MeV} = -605 \text{ MeV}$

    (c) $Q = m_i c^2 - m_f c^2 = m_p c^2 + m_p c^2 - m_p c^2 - m_\pi c^2 - m_\Lambda c^2 - m_K c^2$
    $= 938 \text{ MeV} - 140 \text{ MeV} - 1116 \text{ MeV} - 498 \text{ MeV} = -816 \text{ MeV}$

22. (a) $Q = m_i c^2 - m_f c^2 = m_n c^2 - m_\pi c^2 - m_p c^2$
    $= 940 \text{ MeV} - 140 \text{ MeV} - 938 \text{ MeV} = -138 \text{ MeV}$

    (b) $Q = m_i c^2 - m_f c^2 = m_K c^2 + m_p c^2 - m_\Omega c^2 - m_{K^+} c^2 - m_{K^0} c^2$
    $= 494 \text{ MeV} + 938 \text{ MeV} - 1672 \text{ MeV} - 494 \text{ MeV} - 498 \text{ MeV} = -1232 \text{ MeV}$

    (c) $Q = m_i c^2 - m_f c^2 = m_p c^2 + m_p c^2 - m_p c^2 - m_\Sigma c^2 - m_K c^2$
    $= 938 \text{ MeV} - 1189 \text{ MeV} - 498 \text{ MeV} = -749 \text{ MeV}$

23. (a) $Q = m_i c^2 - m_f c^2 = m_p c^2 + m_p c^2 - m_n c^2 - m_\Sigma c^2 - m_K c^2 - m_\pi c^2$
    $= 2(938 \text{ MeV}) - 940 \text{ MeV} - 1189 \text{ MeV} - 498 \text{ MeV} - 140 \text{ MeV} = -891 \text{ MeV}$

$$K_{th} = -Q\frac{2m_p + m_n + m_\Sigma + m_K + m_\pi}{2m_p}$$

$$= (891 \text{ MeV})\frac{2(938 \text{ MeV}) + 940 \text{ MeV} + 1189 \text{ MeV} + 498 \text{ MeV} + 140 \text{ MeV}}{2(938 \text{ MeV})}$$

$$= 2205 \text{ MeV}$$

(b)  $Q = m_i c^2 - m_f c^2 = m_\pi c^2 + m_p c^2 - m_\Sigma c^2 - m_K c^2$

$$= 140 \text{ MeV} + 938 \text{ MeV} - 1192 \text{ MeV} - 498 \text{ MeV} = -612 \text{ MeV}$$

$$K_{th} = -Q\frac{m_\pi + m_n + m_\Sigma + m_K}{2m_p}$$

$$= (612 \text{ MeV})\frac{140 \text{ MeV} + 938 \text{ MeV} + 1189 \text{ MeV} + 498 \text{ MeV}}{2(938 \text{ MeV})} = 903 \text{ MeV}$$

24.  (a)  $Q = m_i c^2 - m_f c^2 = m_p c^2 + m_n c^2 - m_p c^2 - m_\Sigma c^2 - m_K c^2$

$$= 940 \text{ MeV} - 1197 \text{ MeV} - 494 \text{ MeV} = -751 \text{ MeV}$$

$$K_{th} = -Q\frac{m_p + m_n + m_p + m_\Sigma + m_K}{2m_n}$$

$$= (751 \text{ MeV})\frac{2(938 \text{ MeV}) + 940 \text{ MeV} + 1197 \text{ MeV} + 494 \text{ MeV}}{2(940 \text{ MeV})} = 1800 \text{ MeV}$$

(b)  $Q = m_i c^2 - m_f c^2 = m_\pi c^2 + m_p c^2 - m_p c^2 - m_p c^2 - m_n c^2$

$$= 140 \text{ MeV} - 938 \text{ MeV} - 940 \text{ MeV} = -1738 \text{ MeV}$$

$$K_{th} = -Q\frac{m_\pi + 3m_p + m_n}{2m_p}$$

$$= (1738 \text{ MeV})\frac{140 \text{ MeV} + 3(938 \text{ MeV}) + 940 \text{ MeV}}{2(938 \text{ MeV})} = 3608 \text{ MeV}$$
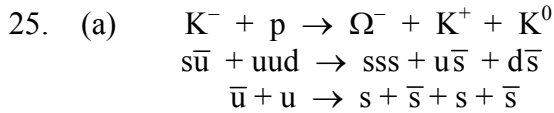
25.  (a)  $K^- + p \rightarrow \Omega^- + K^+ + K^0$
$s\bar{u} + uud \rightarrow sss + u\bar{s} + d\bar{s}$
$\bar{u} + u \rightarrow s + \bar{s} + s + \bar{s}$
The fundamental processes are annihilation of the $u\bar{u}$ pair followed by the creation of two $s\bar{s}$ pairs.
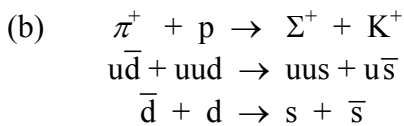
(b)  $\pi^+ + p \rightarrow \Sigma^+ + K^+$
$u\bar{d} + uud \rightarrow uus + u\bar{s}$
$\bar{d} + d \rightarrow s + \bar{s}$
The fundamental processes are annihilation of the $d\bar{d}$ pair followed by creation of the $s\bar{s}$ pair.

(c)  $\gamma + n \rightarrow \pi^- + p$
$\gamma + udd \rightarrow d\bar{u} + uud$
$\gamma \rightarrow u + \bar{u}$
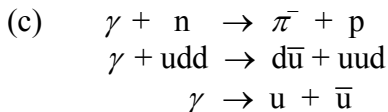
The incident photon creates a $u\bar{u}$ pair.

26. (a) $K^- + p \rightarrow \Lambda^0 + \pi^0$
$s\bar{u} + uud \rightarrow usd + u\bar{u}$
The quarks are merely rearranged among the particles; no quarks are created or destroyed.

(b) $p + p \rightarrow p + \pi^+ + \Lambda^0 + K^0$
$uud + uud \rightarrow uud + u\bar{d} + usd + d\bar{s}$
$energy \rightarrow d + \bar{d} + s + \bar{s}$
The incident proton energy creates $d\bar{d}$ and $u\bar{u}$ pairs.

(c) $\gamma + p \rightarrow D^+ + \bar{D}^0 + n$
$\gamma + uud \rightarrow c\bar{d} + u\bar{c} + udd$
$\gamma \rightarrow c + \bar{c} + d + \bar{d}$
The incident photon creates $c\bar{c}$ and $d\bar{d}$ pairs.

27. (a) $\Omega^- \rightarrow \Lambda^0 + K^-$
$sss \rightarrow usd + s\bar{u}$
$s \rightarrow u + d + \bar{u}$
The weak interaction processes are $s \rightarrow u + W^-$ followed by $W^- \rightarrow d + \bar{u}$.

(b) $n \rightarrow p + e^- + \bar{\nu}_e$
$udd \rightarrow uud + e^- + \nu_e$
$d \rightarrow u + e^- + \nu_e$
The weak interaction process is $d \rightarrow u + W^-$ followed by $W^- \rightarrow e^- + \nu_e$.

(c) $\pi^0 \rightarrow \gamma + \gamma$
$u\bar{u} \rightarrow \gamma + \gamma$
Photons are created from the annihilation of the $u\bar{u}$ pair.

(d) $D^+ \rightarrow K^- + \pi^+ + \pi^+$
$c\bar{d} \rightarrow s\bar{u} + u\bar{d} + u\bar{d}$
$c \rightarrow s + u + \bar{d} + u + \bar{u}$
The weak interaction processes are $c \rightarrow s + W^+$ followed by $W^+ \rightarrow u + \bar{d}$ and creation of a $u\bar{u}$ pair.
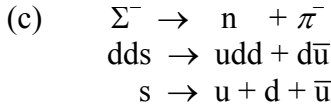
28. (a) $K^0 \rightarrow \pi^+ + \pi^-$
$d\bar{s} \rightarrow u\bar{d} + d\bar{u}$
$\bar{s} \rightarrow \bar{d} + u + \bar{u}$
The weak interaction processes are $\bar{s} \rightarrow \bar{u} + W^+$ followed by $W^+ \rightarrow u + \bar{d}$.
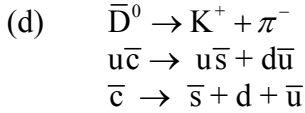
(b) $\Delta^{*++} \rightarrow p + \pi^+$
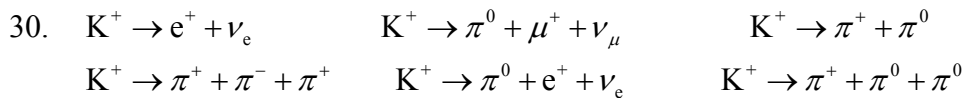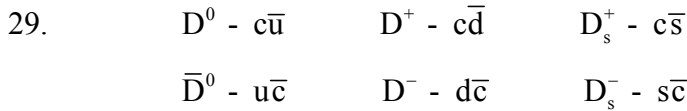$uuu \rightarrow uud + u\bar{d}$

energy $\rightarrow$ d + $\bar{\text{d}}$

The d$\bar{\text{d}}$ pair is created from the decay energy.

(c)     $\Sigma^- \rightarrow$ n $+ \pi^-$
        dds $\rightarrow$ udd $+ \text{d}\bar{\text{u}}$
          s $\rightarrow$ u + d $+ \bar{\text{u}}$

The weak interaction processes are s $\rightarrow$ u $+ \text{W}^-$ followed by $\text{W}^- \rightarrow$ d $+ \bar{\text{u}}$.

(d)     $\bar{\text{D}}^0 \rightarrow \text{K}^+ + \pi^-$
        $\text{u}\bar{\text{c}} \rightarrow$ u$\bar{\text{s}}$ + d$\bar{\text{u}}$
        $\bar{\text{c}} \rightarrow \bar{\text{s}}$ + d $+ \bar{\text{u}}$

The weak interaction processes are $\bar{\text{c}} \rightarrow \bar{\text{s}} + \text{W}^-$ followed by $\text{W}^- \rightarrow$ d $+ \bar{\text{u}}$.

29.     $\text{D}^0$ - c$\bar{\text{u}}$        $\text{D}^+$ - c$\bar{\text{d}}$        $\text{D}_s^+$ - c$\bar{\text{s}}$

        $\bar{\text{D}}^0$ - u$\bar{\text{c}}$        $\text{D}^-$ - d$\bar{\text{c}}$        $\text{D}_s^-$ - s$\bar{\text{c}}$

30.   $\text{K}^+ \rightarrow \text{e}^+ + \nu_e$          $\text{K}^+ \rightarrow \pi^0 + \mu^+ + \nu_\mu$          $\text{K}^+ \rightarrow \pi^+ + \pi^0$

      $\text{K}^+ \rightarrow \pi^+ + \pi^- + \pi^+$          $\text{K}^+ \rightarrow \pi^0 + \text{e}^+ + \nu_e$          $\text{K}^+ \rightarrow \pi^+ + \pi^0 + \pi^0$

31.   The mean lifetime in the laboratory $\tau$ is related to the halflife in the laboratory as $\tau = t_{1/2} / \ln 2$. The beam must travel a distance $d = 2.0$ m at speed $v$ within one halflife, so $t_{1/2} = d / v$. Starting with the time dilation formula for the particle/s mean lifetime $\tau_0$ we have

$$\tau = \frac{\tau_0}{\sqrt{1 - v^2/c^2}} = \frac{t_{1/2}}{\ln 2} = \frac{d/v}{\ln 2}$$

Solving for $v/c$,

$$\frac{v}{c} = \frac{1}{\sqrt{1 + (c\tau_0 \ln 2 / d)^2}} = 0.999635$$

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \frac{1116 \text{ MeV}}{\sqrt{1 - (0.999635)^2}} = 41.3 \text{ GeV}$$

$$K = E - mc^2 = 41.3 \text{ GeV} - 1.116 \text{ GeV} = 40.2 \text{ GeV}$$

32.   (a) $\Omega^- \rightarrow \Xi^- + \pi^0$        or        $\Xi^0 + \pi^-$
      (b) $\Lambda^0 \rightarrow \text{n} + \pi^0$        or        $\text{p} + \text{e}^- + \bar{\nu}_e$
      (c) $\Sigma^+ \rightarrow \Lambda^0 + \text{e}^+ + \nu_e$        or        $\text{n} + \pi^+$

33.   (a) The two protons have velocities of equal magnitude and thus equal kinetic energies. Conservation of energy then gives

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$$m_p c^2 + K_p + m_p c^2 + K_p = m_p c^2 + m_p c^2 + m_\pi c^2$$

so $K_p = \frac{1}{2} m_\pi c^2$ and $E_p = K_p + m_p c^2 = \frac{1}{2}(135 \text{ MeV}) + 938.3 \text{ MeV} = 1005.8 \text{ MeV}$. With $E = mc^2 / \sqrt{1 - v^2/c^2}$ we get
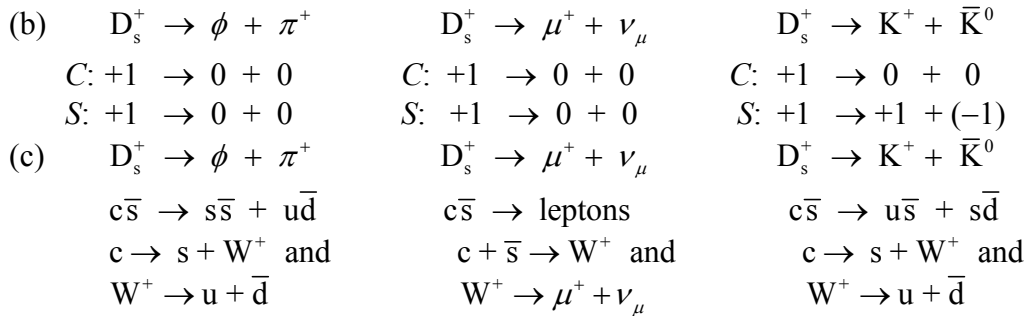
$$v = c\sqrt{1 - (mc^2/E)^2} = c\sqrt{1 - (938.3 \text{ MeV}/1005.8 \text{ MeV})^2} = 0.360c$$

(b) In this frame of reference the protons are moving with velocities $+v$ and $-v$. If we move at a transformation velocity $-v$, then the proton that was originally moving at velocity $-v$ will appear at rest, and the velocity of the proton originally moving at $+v$ can be found from the Lorentz velocity transformation:

$$v' = \frac{v - u}{1 - vu/c^2} = \frac{v + (-v)}{1 - v(-v)/c^2} = \frac{2v}{1 + v^2/c^2} = \frac{2(0.360c)}{1 + (0.360)^2} = 0.637c$$

(c) $K = E - mc^2 = \dfrac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 = \dfrac{938.3 \text{ MeV}}{\sqrt{1 - (0.637)^2}} - 938.3 \text{ MeV} = 279 \text{ MeV}$

34. (a) Weak interaction.

(b)  

| $D_s^+ \to \phi + \pi^+$ | $D_s^+ \to \mu^+ + \nu_\mu$ | $D_s^+ \to K^+ + \bar{K}^0$ |
|---|---|---|
| $C$: $+1 \to 0 + 0$ | $C$: $+1 \to 0 + 0$ | $C$: $+1 \to 0 + 0$ |
| $S$: $+1 \to 0 + 0$ | $S$: $+1 \to 0 + 0$ | $S$: $+1 \to +1 + (-1)$ |

(c)

| $D_s^+ \to \phi + \pi^+$ | $D_s^+ \to \mu^+ + \nu_\mu$ | $D_s^+ \to K^+ + \bar{K}^0$ |
|---|---|---|
| $c\bar{s} \to s\bar{s} + u\bar{d}$ | $c\bar{s} \to$ leptons | $c\bar{s} \to u\bar{s} + s\bar{d}$ |
| $c \to s + W^+$ and | $c + \bar{s} \to W^+$ and | $c \to s + W^+$ and |
| $W^+ \to u + \bar{d}$ | $W^+ \to \mu^+ + \nu_\mu$ | $W^+ \to u + \bar{d}$ |

(d) The weak interaction cannot change $S$ by 2 units.

35. One of the pions will have its maximum kinetic energy when the other 2 pions move in the opposite direction with equal momenta. In this way they have the least possible energy necessary for momentum conservation. To conserve momentum, if one pion has momentum $p$, then the other 2 pions each move in the opposite direction with momentum $p/2$. Conservation of energy then gives

$$E_K = E_{\pi 1} + E_{\pi 2} + E_{\pi 3} = \sqrt{(cp)^2 + (m_\pi c^2)^2} + 2\sqrt{(cp/2)^2 + (m_\pi c^2)^2}$$

$$(m_K c^2)^2 - 2m_K c^2 \sqrt{(cp)^2 + (m_\pi c^2)^2} + (cp)^2 + (m_\pi c^2)^2 = 4[(c^2 p^2/4 + (m_\pi c^2)^2]$$

$$E_{\pi 1} = \sqrt{(cp)^2 + (m_\pi c^2)^2} = \frac{(m_K c^2)^2 - 3(m_\pi c^2)^2}{2m_K c^2} = \frac{(494 \text{ MeV})^2 - 3(140 \text{ MeV})^2}{2(494 \text{ MeV})} = 187 \text{ MeV}$$

$$K_{\pi 1} = E_{\pi 1} - m_\pi c^2 = 187 \text{ MeV} - 140 \text{ MeV} = 47 \text{ MeV}$$

36. (a) The initial momentum of the pion is

$$p_\pi = \frac{mv}{\sqrt{1 - v^2/c^2}} = \frac{1}{c} \frac{(mc^2)(v/c)}{\sqrt{1 - v^2/c^2}} = \frac{1}{c} \frac{(139.6 \text{ MeV})(0.998)}{\sqrt{1 - (0.998)^2}} = 2204 \text{ MeV}/c$$

and the momentum components of the K meson are

$$p_{Kx} = p_K \cos\theta_K = (1560 \text{ MeV}/c)\cos 20.6° = 1460 \text{ MeV}/c$$
$$p_{Ky} = p_K \sin\theta_K = (1560 \text{ MeV}/c)\sin 20.6° = 549 \text{ MeV}/c$$

so that the momentum components of the unknown particle are

$$p_x = p_\pi - p_{Kx} = 2204 \text{ MeV}/c - 1460 \text{ MeV}/c = 744 \text{ MeV}/c$$
$$p_y = p_{Ky} = 549 \text{ MeV}/c$$

and its momentum and direction are thus

$$p = \sqrt{p_x^2 + p_y^2} = \sqrt{(744 \text{ MeV}/c)^2 + (549 \text{ MeV}/c)^2} = 925 \text{ MeV}/c$$
$$\theta = \tan^{-1}\frac{p_y}{p_x} = \tan^{-1}\frac{549 \text{ MeV}/c}{744 \text{ MeV}/c} = 36.4°$$

(b) The initial energy and the final energy of the K meson are

$$E_{\text{initial}} = E_\pi + E_p = \frac{m_\pi c^2}{\sqrt{1 - v^2/c^2}} + m_p c^2 = \frac{139.6 \text{ MeV}}{\sqrt{1 - (0.998)^2}} + 938.3 \text{ MeV} = 3147 \text{ MeV}$$
$$E_K = \sqrt{(cp_K)^2 + (m_K c^2)^2} = \sqrt{(1560 \text{ MeV})^2 + (498 \text{ MeV})^2} = 1638 \text{ MeV}$$

and so the energy of the second outgoing particle is

$$E = E_{\text{initial}} - E_K = 3147 \text{ MeV} - 1638 \text{ MeV} = 1509 \text{ MeV}$$

(c) The rest energy of the second outgoing particle is

$$mc^2 = \sqrt{E^2 - (cp)^2} = \sqrt{(1509 \text{ MeV})^2 - (925 \text{ MeV})^2} = 1192 \text{ MeV}$$

Because the reaction started with one baryon, this particle must also be a baryon, and a look at Table 14.6 shows that it must be a $\Sigma^0$.  This would be expected based also on the conservation of electric charge and strangeness.

# Chapter 15

**Supplemental Materials**

**Suggestions for Additional Reading**

There are many introductory, advanced, and popular-level books and articles that cover in more detail the subjects touched only briefly in this chapter. To pursue any of these subjects, you may first need some background material on astronomy and astrophysics:

B. W. Carroll and D. A. Ostlie, *An Introduction to Modern Astrophysics*, 2$^{nd}$ ed. (Benjamin Cummmings, 2006).
S. A. Gregory and M. Zeilik, *Introductory Astronomy and Astrophysics,* 4$^{th}$ ed. (Saunders, 1997).
M. Harwit, *Astrophysical Concepts*, 4$^{th}$ ed. (Springer, 2006).

Some nonmathematical introductory books on general relativity and cosmology are:

T. Ferris, *The Red Limit* (Wm. Morrow & Co., 1977).
S. W. Hawking, *A Brief History of Time* (Bantam, 1988).
L. Krauss, *Quintessence: The Mystery of Missing Mass in the Universe* (Basic Books, 2000).
K. Larsen, *Cosmology 101* (Greenwood Press, 2007).
A. Lightman, *Ancient Light* (Harvard University Press, 1991).
S. Weinberg, *The First Three Minutes* (Basic Books, 1977).

Some additional detail on general relativity, without the advanced math:

P. G. Bergmann, *The Riddle of Gravitation* (Scribner's, 1968).
M. Berry, *Principles of Cosmology and Gravitation* (Cambridge University Press, 1976).
P. C. W. Davies, *Space and Time in the Modern Universe* (Cambridge University Press, 1977).
W. Rindler, *Essential Relativity* (Van Nostrand, 1969).
R. M. Wald, *Space, Time, and Gravity,* 2nd ed. (University of Chicago Press, 1992).

Books on cosmology which include mostly calculus-level math (no tensors):

A. Liddle, *An Introduction to Modern Cosmology*, 2$^{nd}$ ed. (Wiley, 2003).
P. J. E. Peebles, *Principles of Physical Cosmology* (Princeton University Press, 1993).

M. Rowan-Robinson, *Cosmology,* 3$^{rd}$ ed. (Clarendon Press, 1996).
D. W. Sciama, *Modern Cosmology* (Cambridge University Press, 1971).
E. R. Harrison, *Cosmology,* 2$^{nd}$ ed. (Cambridge University Press, 2000).

A more challenging work, but still very readable and containing much information on observations:

J. V. Narlikar, *Introduction to Cosmology,* 3$^{rd}$ ed. (Cambridge University Press, 2002).

As with many of the subjects covered in this book, a good source for current, popular-level articles is the magazine *Scientific American*:

S. W. Hawking, "The Quantum Mechanics of Black Holes," *Scientific American* **236**, 34 (January 1977).
M. J. Rees, "Black Holes in Galactic Centers," *Scientific American* **263**, 56 (November 1990).
W. L. Freedman, "The Expansion Rate and Size of the Universe," *Scientific American* **267**, 54 (November 1992).
D. E. Osterbrock, J. A. Gwinn, and R. S. Brashear, "Edwin Hubble and the Expanding Universe," *Scientific American* **269**, 84 (July 1993).
C. H. Lineweaver and T. M. Davis, "Misconceptions About the Big Bang," *Scientific American* **292**, 36 (March 2005).
L. M. Krauss and R. J. Scherrer, "The End of Cosmology?", *Scientific American* **298**, 46 (March 2008).

## Materials for Active Engagement in the Classroom

### A. Reading Quizzes

### Answers

### B. Conceptual or Discussion Questions

### Answers

## Sample Exam Questions

### A. Multiple Choice

**Answers**

**B.  Conceptual**

**Answers**

**C.  Problems**

**Answers**

# Problem Solutions

1. (a)  $d = 1.0 \times 10^6$ light-years $= 0.307$ Mpc

$$v = Hd = (72 \text{ km/s/Mpc})(0.307 \text{ Mpc}) = 22.1 \text{ km/s} = 7.37 \times 10^{-5} c$$

$$\lambda' = \lambda \sqrt{\frac{1+v/c}{1-v/c}} = (590.0 \text{ nm}) \sqrt{\frac{1+7.67 \times 10^{-5}}{1-7.67 \times 10^{-5}}} = 590.0 \text{ nm}$$

   (b)  $d = 1.0 \times 10^9$ light-years $= 307$ Mpc

$$v = Hd = (72 \text{ km/s/Mpc})(307 \text{ Mpc}) = 22{,}100 \text{ km/s} = 7.37 \times 10^{-2} c$$

$$\lambda' = \lambda \sqrt{\frac{1+v/c}{1-v/c}} = (590.0 \text{ nm}) \sqrt{\frac{1+7.67 \times 10^{-2}}{1-7.67 \times 10^{-2}}} = 637.1 \text{ nm}$$

2. With $\lambda'/\lambda = 2$, Equation 15.1 gives

$$\frac{\lambda'}{\lambda} = 2 = \sqrt{\frac{1+v/c}{1-v/c}} \qquad \text{which gives} \qquad v/c = 0.60$$

   and from Hubble's law

$$d = \frac{v}{H_0} = \frac{(0.60)(3.00 \times 10^5 \text{ km/s})}{72 \text{ km/s/Mpc}} = 2.5 \times 10^3 \text{ Mpc} = 8.15 \times 10^3 \text{ light-years}$$

3. (a)  $$\frac{du}{dE} = \frac{d}{dE}\left( \frac{8\pi E^3}{(hc)^3} \frac{1}{e^{E/kT}-1} \right) = \frac{24\pi E^2}{(hc)^3} \frac{1}{e^{E/kT}-1} - \frac{8\pi E^3}{(hc)^3} \frac{e^{E/kT}(1/kT)}{(e^{E/kT}-1)^2} = 0$$

$$3 - \frac{E}{kT} \frac{e^{E/kT}}{e^{E/kT}-1} = 0 \qquad \text{or} \qquad e^{-x} = 1 - \frac{x}{3} \quad \text{with} \quad x = E/kT$$

   This equation must be solved numerically. Note that a good starting guess is $x = 3$. The numerical solution gives $x = 2.841$ and so

$$E = 2.821kT = (2.431 \times 10^{-4} \text{ eV/K})T$$

   (b)  $E = (2.431 \times 10^{-4} \text{ eV/K})(2.73 \text{ K}) = 6.64 \times 10^{-4} \text{ eV}$

4. From Equation 15.7 we have

$$\frac{N}{V} = \frac{8\pi k^3}{(hc)^3}(2.404)T^3 = \frac{8\pi(8.617\times10^{-5}\ \text{eV/K})^3(2.404)}{(1240\ \text{eV}\cdot\text{nm})^3(1\ \text{m}/10^9\ \text{nm})^3}T^3 = (2.03\times10^7\ \text{photons/m}^3\cdot\text{K}^3)T^3$$

Equation 10.41 gives

$$U = \frac{8\pi^5 k^4}{15(hc)^3}T^4 = \frac{8\pi^5(8.617\times10^{-5}\ \text{eV/K})^4}{15(1240\ \text{eV}\cdot\text{nm})^3(1\ \text{m}/10^9\ \text{nm})^3}T^4 = (4.72\times10^3\ \text{eV/m}^3\cdot\text{K}^4)T^4$$

5.  The tangential rotational speed of the Sun in our galaxy is 220 km/s. From Equation 15.1 we can find the Doppler shift when $v \ll c$, for $\lambda' = \lambda + \Delta\lambda$:

$$\lambda + \Delta\lambda \cong \lambda\left(1 + \frac{v}{c}\right) \qquad \text{so} \qquad \Delta\lambda = \lambda\frac{v}{c} = (121.5\ \text{nm})\frac{220\ \text{km/s}}{3.00\times10^5\ \text{km/s}} = 0.089\ \text{nm}$$

6.  (a) The tangential speed at the Sun's equator is

$$v = R\omega = (6.96\times10^8\ \text{m})\frac{2\pi}{(26\ \text{d})(86400\ \text{s/d})} = 1950\ \text{m/s}$$

For such a small speed, we can approximate the Doppler shift (see Equation 15.1) as

$$\Delta\lambda \cong \lambda\frac{v}{c} = (121.5\ \text{nm})\frac{1950\ \text{m/s}}{3.00\times10^8\ \text{m/s}} = 0.790\ \text{pm}$$

This is about 3 times larger than the gravitational shift.
(b) The width of the spectral line, due to the thermal Doppler broadening, can be found from Equation 10.30, with $T = 6000$ K (the surface temperature of the Sun) and thus $kT = (8.617\times10^{-5}\ \text{eV/K})(6000\ \text{K}) = 0.517\ \text{eV}$:

$$\Delta\lambda = 2\lambda\sqrt{(2\ln 2)kT/mc^2} = 2(121.5\ \text{nm})\sqrt{(2\ln 2)(0.517\ \text{eV})/(939\ \text{MeV})} = 6.7\ \text{pm}$$

It is difficult to measure the small gravitational effect for this relatively broad spectral line.

7.  $$\Delta f = f\frac{gH}{c^2} = (10^9\ \text{Hz})\frac{(9.8\ \text{m/s}^2)(150\times10^3\ \text{m})}{(3.00\times\ 10^8\ \text{m/s})^2} = 1.6\times10^{-2}\ \text{Hz}$$

8.  $$\Delta E = h\,\Delta f = h(2.5\times10^{-15}f) = 2.5\times10^{-15}E$$

$$\Delta t = \frac{\hbar}{\Delta E} = \frac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{(2.5 \times 10^{-15})(14.4 \times 10^3 \text{ eV})} = 1.8 \times 10^{-5} \text{ s}$$

9.  $$F_{\text{Coulomb}} = \frac{e^2}{4\pi\varepsilon_0} \frac{zZ}{r^2} \qquad \text{and} \qquad F_{\text{grav}} = \frac{GmM}{r^2}$$

so we can change expressions based on the Coulomb force into those based on the gravitational force by replacing $zZe^2/4\pi\varepsilon_0$ with $GmM$. Equation 6.8 then becomes

$$b = \frac{zZ}{2\left(\frac{1}{2}mv^2\right)} \frac{e^2}{4\pi\varepsilon_0} \cot \tfrac{1}{2}\theta \qquad \rightarrow \qquad b = \frac{GmM}{mv^2} \cot \tfrac{1}{2}\theta$$

For a photon that just grazes the Sun, $b = R$ and $v = c$, so $R = \dfrac{GM}{c^2} \cot \tfrac{1}{2}\theta$ and, assuming $\theta$ is a small angle,

$$\frac{GM}{Rc^2} = \tan \tfrac{1}{2}\theta \cong \tfrac{1}{2}\theta \qquad \text{so} \qquad \theta \cong \frac{2GM}{Rc^2}$$

10. The orbital period is 7.75 h, so with $(24)(365.25) = 8766$ hours per year that works out to 1131 orbits per year or 113,100 orbits per century. The mass of each star is about 1.4 solar masses, so the total mass $M$ is $(2.8)(2.0 \times 10^{30} \text{ kg}) = 5.6 \times 10^{30}$ kg. The eccentricity is 0.617, and the periastron distance $r_{\text{min}}$ is 746,600 km. From Equation 15.25 we then have

$$\Delta\phi = \frac{6\pi GM}{c^2 r_{\text{min}}(1+e)} = \frac{6\pi(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2)(5.6 \times 10^{30} \text{ kg})}{(3 \times 10^8 \text{ m/s})^2(7.5 \times 10^8 \text{ m})(1.617)} = 6.45 \times 10^{-5} \text{ rad}$$

Thus

$$N\Delta\phi = (113,100)(6.45 \times 10^{-5} \text{ rad})(57.3 \deg/\text{rad})(3600 \text{ arc sec/deg})$$
$$= 1.5 \times 10^6 \text{ arc seconds per century}$$

11. (a) If the star originally contains $N$ nucleons (protons and neutrons), then in comparison with the Sun we have $N = N_\odot(M/M_\odot)$. Also, $N_\odot = M_\odot/m_n = 1.188 \times 10^{57}$. Equation 10.58 is then

$$R = \frac{h^2 (9/32\pi^2)^{2/3}}{GN^{1/3}m_n^3} = \frac{h^2 (9/32\pi^2)^{2/3}}{GN_\odot^{1/3}(M/M_\odot)^{1/3}m_n^3}$$

$$= \frac{(6.626\times10^{-34} \text{ J}\cdot\text{s})^2 (9/32\pi^2)^{2/3}}{(6.673\times10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2)(1.188\times10^{57})^{1/3}(1.675\times10^{-27} \text{ kg})^3}\left(\frac{M}{M_\odot}\right)^{-1/3}$$

$$= (12.3 \text{ km})\left(\frac{M}{M_\odot}\right)^{-1/3}$$

(b) The radius of the neutron star is $R = (12.3 \text{ km})(1.5)^{-1/3} = 11$ km. The angular momentum $L = I\omega$ is conserved during the collapse, so $I_i\omega_i = I_f\omega_f$:

$$\omega_f = \omega_i \frac{I_i}{I_f} = \omega_i \frac{\frac{2}{5}MR_i^2}{\frac{2}{5}MR_f^2} = \omega_i \left(\frac{R_i}{R_f}\right)^2 = (1 \text{ rev/y})\left(\frac{7\times10^5 \text{ km}}{11 \text{ km}}\right)^2 = 4.0\times10^9 \text{ rev/y} = 128 \text{ rev/s}$$

12. (a) For the matter-dominated universe, $R = At^{2/3}$, so $dR/dt = \frac{2}{3}At^{-1/3}$ and $d^2R/dt^2 = -\frac{2}{9}At^{-4/3}$. The deceleration parameter is

$$q = -\frac{R(d^2R/dt^2)}{(dR/dt)^2} = -\frac{(At^{2/3})\left(-\frac{2}{9}At^{-4/3}\right)}{\left(\frac{2}{3}At^{-1/3}\right)^2} = 0.5$$

For the radiation-dominated universe, $R = A't^{1/2}$, so $dR/dt = \frac{1}{2}A't^{-1/2}$ and $d^2R/dt^2 = -\frac{1}{4}A't^{-3/2}$. The deceleration parameter is

$$q = -\frac{R(d^2R/dt^2)}{(dR/dt)^2} = -\frac{(A't^{1/2})\left(-\frac{1}{4}A't^{-3/2}\right)}{\left(\frac{1}{2}A't^{-1/2}\right)^2} = 1.0$$

(b) Starting with $(dR/dt)^2 = \frac{8\pi}{3}G\rho_m R^2$, we take the derivative with respect to time:

$$2\frac{dR}{dt}\frac{d^2R}{dt^2} = \frac{8\pi}{3}G\frac{d\rho_m}{dt}R^2 + \frac{8\pi}{3}G\rho_m\left(2R\frac{dR}{dt}\right)$$

With $\rho_m = CR^{-3}$ where $C$ is a constant, $d\rho_m/dt = -3CR^{-4}(dR/dt) = -3\rho_m R^{-1}(dR/dt)$ and

$$2\frac{dR}{dt}\frac{d^2R}{dt^2} = \frac{8\pi}{3}G\left(-3\rho_m R\frac{dR}{dt} + 2\rho_m R\frac{dR}{dt}\right) = -\frac{8\pi}{3}G\rho_m R\frac{dR}{dt}$$

$$q = -\frac{R(d^2R/dt^2)}{(dR/dt)^2} = \frac{(4\pi/3)G\rho_m R^2}{(dR/dt)^2} = \frac{4\pi G\rho_m}{H^2}$$

where the last step uses $H = R^{-1}(dR/dt)$.

13. With $U = (4.72\times10^3 \text{ eV/m}^3\cdot\text{K}^4)T^4 = (7.56\times10^{-16} \text{ J/m}^3\cdot\text{K}^4)T^4$, Equation 15.32 for the radiation-dominated universe gives

$$t = \sqrt{\frac{3}{32\pi G\rho_r}} = \sqrt{\frac{3c^2}{32\pi GU}} = \sqrt{\frac{3(3.00\times10^8 \text{ m/s})^2}{32\pi(6.67\times10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2)(7.56\times10^{-16} \text{ J/m}^3\cdot\text{K}^4)T^4}}$$

$$= (2.31\times10^{20} \text{ s}\cdot\text{K}^2)T^{-2}$$

$$T = \left(\frac{2.31\times10^{20} \text{ s}\cdot\text{K}^2}{t}\right)^{1/2} = \frac{1.5\times10^{10} \text{ s}^{1/2}\cdot\text{K}}{t^{1/2}}$$

14. (a)
$$T = \frac{mc^2}{k} = \frac{940\times10^6 \text{ eV}}{8.617\times10^{-5} \text{ eV/K}} = 1.09\times10^{13} \text{ K}$$

$$t = \left(\frac{1.5\times10^{10} \text{ s}^{1/2}\cdot\text{K}}{T}\right)^2 = \left(\frac{1.5\times10^{10} \text{ s}^{1/2}\cdot\text{K}}{1.09\times10^{13} \text{ K}}\right)^2 = 1.9\times10^{-6} \text{ s}$$

(b)
$$T = \frac{mc^2}{k} = \frac{140\times10^6 \text{ eV}}{8.617\times10^{-5} \text{ eV/K}} = 1.62\times10^{12} \text{ K}$$

$$t = \left(\frac{1.5\times10^{10} \text{ s}^{1/2}\cdot\text{K}}{T}\right)^2 = \left(\frac{1.5\times10^{10} \text{ s}^{1/2}\cdot\text{K}}{1.62\times10^{12} \text{ K}}\right)^2 = 8.6\times10^{-6} \text{ s}$$

15. (a)
$$T = \frac{mc^2}{k} = \frac{500\times10^6 \text{ eV}}{8.617\times10^{-5} \text{ eV/K}} = 5.8\times10^{12} \text{ K}$$

(b)
$$t = \left(\frac{1.5\times10^{10} \text{ s}^{1/2}\cdot\text{K}}{T}\right)^2 = \left(\frac{1.5\times10^{10} \text{ s}^{1/2}\cdot\text{K}}{5.8\times10^{12} \text{ K}}\right)^2 = 6.7\times10^{-6} \text{ s}$$

16.
$$\frac{N_{E>E_0}}{V} = V^{-1}\int_{E_0}^{\infty} N(E)dE = \frac{8\pi}{(hc)^3}\int_{E_0}^{\infty} E^2 e^{-E/kT}dE = \frac{8\pi}{(hc)^3}(kT)^3\int_{E_0/kT}^{\infty} x^2 e^{-x}dx$$

$$= \frac{8\pi}{(hc)^3}(kT)^3 \left[ -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} \right]_{E_0/kT}^{\infty} = \frac{8\pi}{(hc)^3}(kT)^3 e^{-E_0/kT} \left[ \left( \frac{E_0}{kT} \right)^2 + 2\left( \frac{E_0}{kT} \right) + 2 \right]$$

$$f = \frac{N_{E>E_0}/V}{N/V} = \frac{8\pi(kT)^3}{(hc)^3 (2.03\times10^7 \text{ m}^{-3}\cdot\text{K}^{-3})T^3} e^{-E_0/kT} \left[ \left( \frac{E_0}{kT} \right)^2 + 2\left( \frac{E_0}{kT} \right) + 2 \right]$$

$$= \frac{8\pi(8.617\times10^{-5} \text{ eV/K})^3 (10^9 \text{ nm/m})^3}{(1240 \text{ eV}\cdot\text{nm})^3 (2.03\times10^7 \text{ m}^{-3}\cdot\text{K}^{-3})} e^{-E_0/kT} \left[ \left( \frac{E_0}{kT} \right)^2 + 2\left( \frac{E_0}{kT} \right) + 2 \right]$$

$$= 0.42 e^{-E_0/kT} \left[ \left( \frac{E_0}{kT} \right)^2 + 2\left( \frac{E_0}{kT} \right) + 2 \right]$$

17. (a) $\qquad f = \frac{1}{6}\times10^{-8} = 0.42 e^{-E_0/kT} \left[ \left( \frac{E_0}{kT} \right)^2 + 2\left( \frac{E_0}{kT} \right) + 2 \right] \qquad$ gives $\qquad \dfrac{E_0}{kT} = 25$

$$T = \frac{E_0}{25k} = \frac{2.22\times10^6 \text{ eV}}{25(8.617\times10^{-5} \text{ eV/K})} = 1.0\times10^9 \text{ K}$$

(b) $\qquad t = \left( \dfrac{1.5\times10^{10} \text{ s}^{1/2}\cdot\text{K}}{T} \right)^2 = \left( \dfrac{1.5\times10^{10} \text{ s}^{1/2}\cdot\text{K}}{1.0\times10^9 \text{ K}} \right)^2 = 225 \text{ s}$

(c) $\qquad f = 10^{-8} = 0.42 e^{-E_0/kT} \left[ \left( \dfrac{E_0}{kT} \right)^2 + 2\left( \dfrac{E_0}{kT} \right) + 2 \right] \qquad$ gives $\qquad \dfrac{E_0}{kT} = 24$

$$T = \frac{E_0}{24k} = \frac{13.6 \text{ eV}}{24(8.617\times10^{-5} \text{ eV/K})} = 6600 \text{ K}$$

$$t = \left( \frac{1.5\times10^{10} \text{ s}^{1/2}\cdot\text{K}}{T} \right)^2 = \left( \frac{1.5\times10^{10} \text{ s}^{1/2}\cdot\text{K}}{6600 \text{ K}} \right)^2 = 5.2\times10^{12} \text{ s} = 1.6\times10^5 \text{ y}$$

These times are not very different from those computed using $10^{-9}$ for the matter-antimatter imbalance.

18. With $N_p/N_n = 0.60/0.40 = 1.5$, we have

$$\frac{N_n}{N_p} = e^{-\Delta E/kT} \qquad \text{or} \qquad \frac{\Delta E}{kT} = -\ln\left( \frac{N_n}{N_p} \right) = \ln(1.5) = 0.41$$

$$T = \frac{\Delta E}{0.41k} = \frac{1.3 \text{ MeV}}{0.41(8.62\times10^{-5} \text{ eV/K})} = 3.7\times10^{10} \text{ K}$$

$$t = \left(\frac{1.5\times10^{10} \text{ s}^{1/2}\cdot\text{K}}{T}\right)^2 = \left(\frac{1.5\times10^{10} \text{ s}^{1/2}\cdot\text{K}}{3.7\times10^{10} \text{ K}}\right)^2 = 0.17 \text{ s}$$

19. With $\rho = 0.046\rho_{cr} = 0.046(0.97\times10^{-26} \text{ kg/m}^3) = 4.5\times10^{-28} \text{ kg/m}^3$. This amounts to 0.27 nucleons per cubic meter, or about 1 nucleon in every 4 cubic meters.

20. (a) $\rho = 0.046\rho_{cr} = 0.046(0.97\times10^{-26} \text{ kg/m}^3)\dfrac{1 \text{ star}}{2.0\times10^{30} \text{ kg}} = 2.2\times10^{-58}$ stars/m$^3$ so the

    average spacing is $(2.2\times10^{-58})^{-1/3} = 1.65\times10^{19}$ m $= 1700$ light-years .

    (b) $\rho = 0.046\rho_{cr} = 0.046(0.97\times10^{-26} \text{ kg/m}^3)\dfrac{1 \text{ galaxy}}{1.2\times10^{42} \text{ kg}} = 3.7\times10^{-70}$ galaxy/m$^3$ so the

    average spacing is $(3.7\times10^{-70})^{-1/3} = 1.39\times10^{23}$ m $= 1.5\times10^{7}$ light-years .

21. From Equation 15.8, the present photon density (with $T = 2.73$ K) is

$$N/V = (2.07\times10^{7} \text{ photons/m}^3\cdot\text{K}^3)(2.73 \text{ K})^3 = 4.1\times10^{8} \text{ photons/m}^3$$

If the universe has its critical mass and if 23% is nonbaryonic dark matter, then the density of the nonbaryonic dark matter is $0.23(0.97\times10^{-26} \text{ kg/m}^3) = 2.2\times10^{-27} \text{ kg/m}^3$. If $4.1 \times 10^8$ neutrinos have a mass of $2.2 \times 10^{-27}$ kg, then one neutrino has a rest energy of

$$E = mc^2 = \frac{2.2\times10^{-27} \text{ kg}}{4.1\times10^{8}}(3.00\times10^{8} \text{ m/s})^2 \frac{1}{1.60\times10^{-19} \text{ J/eV}} = 3.0 \text{ eV}$$

This fairly close to the upper limit on the rest energy of the electron neutrino (see Table 14.4), but well within the limits on the rest energies of the muon and tau neutrinos.

22. (a) At $T = 2.73$ K, $kT = 2.352 \times 10^{-4}$ eV and $E/kT = 10630$ for $E = 2.5$ eV. Then from Equation 15.37 we have

$$\frac{N(E)dE}{V} = \frac{8\pi E^2}{(hc)^3}e^{-E/kT}dE = \frac{8\pi(2.5 \text{ eV})^2(1.0 \text{ eV})}{(1240 \text{ eV}\cdot\text{nm})^3(1 \text{ m}/10^9 \text{ nm})^3}e^{-10630} = 2.9\times10^{-4596} \text{ m}^{-3}$$

In doing this evaluation, it is helpful to replace $e^x$ with $10^{x \log e}$.
Such a small density is impossible to observe.
 (b) Again using Equation 15.37,

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$$e^{-E/kT} = \frac{N(E)dE}{V} \frac{(hc)^3}{8\pi E^2 dE} = \frac{100 \text{ cm}^{-3}}{(1 \text{ m}/100 \text{ cm})^3} \frac{(1240 \text{ eV}\cdot\text{nm})^3(1 \text{ m}/10^9 \text{ nm})^3}{8\pi(2.5 \text{ eV})^2(1.0 \text{ eV})} = 1.21\times10^{-12}$$

from which $E/kT = 27.4$, and so $kT = (2.5 \text{ eV})/27.4 = 0.091$ eV and $T = 1060$ K. This temperature occurred at time

$$t = \left(\frac{1.5\times10^{10} \text{ s}^{1/2}\cdot\text{K}}{T}\right)^2 = \left(\frac{1.5\times10^{10} \text{ s}^{1/2}\cdot\text{K}}{1060 \text{ K}}\right)^2 = 2.0\times10^{14} \text{ s} = 6.4\times10^6 \text{ y}$$

23. (a) $\quad t = \left(\dfrac{1.5\times10^{10} \text{ s}^{1/2}\cdot\text{K}}{T}\right)^2 = \left(\dfrac{1.5\times10^{10} \text{ s}^{1/2}\cdot\text{K}}{5000 \text{ K}}\right)^2 = 9.0\times10^{12} \text{ s} = 2.9\times10^5 \text{ y}$

This occurred during the time when atoms were forming.

(b) $\quad E_m = (2.33\times10^{-4} \text{ eV/K})T = (2.33\times10^{-4} \text{ eV/K})(5000 \text{ K}) = 1.17$ eV

(c) The nucleon rest energy is 940 MeV, so the ratio of the energy densities is

$$\frac{\rho_r}{\rho_m} = \frac{\rho_r c^2}{\rho_m c^2} = \frac{10^9(1.17 \text{ eV})}{940\times10^6 \text{ eV}} = 1.24$$

24. (a) We assume that $\rho_r = \rho_m$ occurred after antimatter annihilation. With $10^9$ photons per nucleon and an average photon energy of $E_m$, then

$$\frac{\rho_r}{\rho_m} = \frac{\rho_r c^2}{\rho_m c^2} = \frac{10^9 E_m}{m_{nucleon} c^2} = \frac{10^9(2.33\times10^{-4} \text{ eV/K})T}{940\times10^6 \text{ eV}} = 1 \quad \text{or} \quad T = 4000 \text{ K}$$

(b) $\quad t = \left(\dfrac{1.5\times10^{10} \text{ s}^{1/2}\cdot\text{K}}{T}\right)^2 = \left(\dfrac{1.5\times10^{10} \text{ s}^{1/2}\cdot\text{K}}{4000 \text{ K}}\right)^2 = 1.4\times10^{13} \text{ s} = 4.4\times10^5 \text{ y}$

25. (a) $\quad R = (12.3 \text{ km})(M/M_\odot)^{-1/3} = (12.3 \text{ km})(2.00)^{-1/3} = 9.79$ km

(b) $\quad K = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{2}{5}MR^2\right)\omega^2$

$\qquad = \frac{1}{5}(2.00)(1.99\times10^{30} \text{ kg})(9.79\times10^3 \text{ m})^2(2\pi \text{ rad/s})^2 = 3.01\times10^{39}$ J

(c) With $K = \frac{1}{2}I\omega^2$, then $dK = I\omega\, d\omega$ and $dK/K = 2d\omega/\omega$:

$$\Delta K = 2K\frac{\Delta\omega}{\omega} = 2(3.01\times10^{39} \text{ J})(10^{-9}) = 6.02\times10^{30} \text{ J}$$

(d) $\quad P = \dfrac{\Delta E}{\Delta t} = \dfrac{6.02\times10^{30} \text{ J}}{(1 \text{ d})(86400 \text{ s/d})} = 6.98\times10^{25} \text{ W}$

(e) If the radiation were distributed uniformly over a spherical surface at the Earth's distance $d = 10^4$ light years $= 9.46 \times 10^{19}$ m, the fraction of the power received by the antenna of area $A$ would be

$$P_{\text{received}} = P_{\text{transmitted}} \frac{A}{4\pi d^2} = (6.98 \times 10^{25} \text{ W}) \frac{10 \text{ m}^2}{4\pi (9.46 \times 10^{19} \text{ m})^2} = 6.21 \times 10^{-14} \text{ W}$$

26.  (a) With $t \propto h^i c^j G^k$, we have $[t] = [h]^i [c]^j [G]^k$ where $[\ ]$ indicates dimensions. Inserting the appropriate units, we have

$$s = (J \cdot s)^i (m \cdot s^{-1})^j (N \cdot m^2 \cdot kg^{-2})^k = (kg \cdot m^2 \cdot s^{-1})^i (m \cdot s^{-1})^j (m^3 \cdot s^{-2} \cdot kg^{-1})^k$$

$$= (kg)^{i-k} (m)^{2i+j+3k} (s)^{-i-j-2k}$$

Equating powers of the units on both sides of the equation, we obtain

$$i - k = 0 \qquad 2i + j + 3k = 0 \qquad -i - j - 2k = 1$$

which can be solved to give $i = 1/2$, $j = -5/2$, $k = 1/2$.

(b)  $$t = \sqrt{\frac{hG}{c^5}} = \sqrt{\frac{(6.626 \times 10^{-34} \text{ J} \cdot \text{s})(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2)}{(3.00 \times 10^8 \text{ m/s})^5}} = 1.3 \times 10^{-43} \text{ s}$$

(c)  $$R = ct = (3.00 \times 10^8 \text{ m/s})(1.3 \times 10^{-43} \text{ s}) = 3.9 \times 10^{-35} \text{ m}$$

27.  By taking differentials of Equations 2.23a and 2.23d, we obtain

$$dx' = \frac{dx - u\,dt}{\sqrt{1 - u^2/c^2}} \qquad \text{and} \qquad dt' = \frac{dt - (u/c^2)dx}{\sqrt{1 - u^2/c^2}}$$

Making the substitutions in the expression for $(ds')^2$,

$$(ds')^2 = (c\,dt')^2 - (dx')^2 = \left[ \frac{c\,dt - (u/c)dx}{\sqrt{1 - u^2/c^2}} \right]^2 - \left[ \frac{dx - u\,dt}{\sqrt{1 - u^2/c^2}} \right]^2$$

$$= \frac{c^2(dt)^2 - 2u\,dx\,dt + (u^2/c^2)(dx)^2 - (dx)^2 + 2u\,dx\,dt - u^2(dt)^2}{1 - u^2/c^2}$$

$$= \frac{c^2(1 - u^2/c^2)(dt)^2 - (1 - u^2/c^2)(dx)^2}{1 - u^2/c^2} = c^2(dt)^2 - (dx)^2 = (ds)^2$$

28. (a) For small angles, the distance between $I$ and the horizontal axis is $\theta d_S$, the distance between $S$ and the axis is $\beta d_S$, and the distance between $I$ and $S$ is $\alpha(d_S - d_L)$. Thus

$$\theta d_S = \beta d_S + \alpha(d_S - d_L)$$

From Equation 15.22, after inserting the extra factor of 2 to account for the difference between special and general relativity for the deflection angle, we have (again for small angles)

$$\alpha = \frac{4GM}{bc^2} = \frac{4GM}{\theta d_L c^2}$$

so

$$\theta d_S = \beta d_S + \frac{4GM}{\theta d_L c^2}(d_S - d_L)$$

(b) We can rewrite this equation as

$$\theta = \beta + \frac{4GM}{\theta d_L d_S c^2}(d_S - d_L) = \beta + \frac{\theta_E^2}{\theta}$$

or

$$\theta^2 - \beta\theta - \theta_E^2 = 0$$

with $\theta_E = \sqrt{4GM(d_S - d_L)/c^2 d_S d_L}$. Using the quadratic formula then gives the solutions

$$\theta_\pm = \tfrac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$$

and the difference between the angular positions of the two images is

$$\Delta\theta = \theta_+ - \theta_- = \sqrt{\beta^2 + 4\theta_E^2}$$

(c) For $\beta = 0$, we have $\theta = \theta_E$ and the problem has rotational symmetry about the horizontal axis. The image of the star is thus a circle.

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