INSTRUCTOR'S SOLUTIONS MANUAL

TO ACCOMPANY

A FIRST COURSE IN THE FINITE ELEMENT METHOD

FIFTH EDITION

DARYL L. LOGAN

Contents

Chapter 1

1.1. A finite element is a small body or unit interconnected to other units to model a larger structure or system.

1.2. Discretization means dividing the body (system) into an equivalent system of finite elements with associated nodes and elements.

1.3. The modern development of the finite element method began in 1941 with the work of Hrennikoff in the field of structural engineering.

1.4. The direct stiffness method was introduced in 1941 by Hrennikoff. However, it was not commonly known as the direct stiffness method until 1956.

1.5. A matrix is a rectangular array of quantities arranged in rows and columns that is often used to aid in expressing and solving a system of algebraic equations.

1.6. As computer developed it made possible to solve thousands of equations in a matter of minutes.

1.7. The following are the general steps of the finite element method.

Step 1

Divide the body into an equivalent system of finite elements with associated nodes and choose the most appropriate element type.

Step 2

Choose a displacement function within each element.

Step 3

Relate the stresses to the strains through the stress/strain law—generally called the constitutive law.

Step 4

Derive the element stiffness matrix and equations. Use the direct equilibrium method, a work or energy method, or a method of weighted residuals to relate the nodal forces to nodal displacements.

Step 5

Assemble the element equations to obtain the global or total equations and introduce boundary conditions.

Step 6

Solve for the unknown degrees of freedom (or generalized displacements).

Step 7

Solve for the element strains and stresses.

Step 8

Interpret and analyze the results for use in the design/analysis process.

1.8. The displacement method assumes displacements of the nodes as the unknowns of the problem. The problem is formulated such that a set of simultaneous equations is solved for nodal displacements.

1.9. Four common types of elements are: simple line elements, simple two-dimensional elements, simple three-dimensional elements, and simple axisymmetric elements.

1.10 Three common methods used to derive the element stiffness matrix and equations are

(1) direct equilibrium method

(2) work or energy methods

(3) methods of weighted residuals

1.11. The term 'degrees of freedom' refers to rotations and displacements that are associated with each node.

1.12. Five typical areas where the finite element is applied are as follows.

(1) Structural/stress analysis

(2) Heat transfer analysis

(3) Fluid flow analysis

(4) Electric or magnetic potential distribution analysis

(5) Biomechanical engineering

1.13. Five advantages of the finite element method are the ability to

(1) Model irregularly shaped bodies quite easily

(2) Handle general load conditions without difficulty

(3) Model bodies composed of several different materials because element equations are evaluated individually

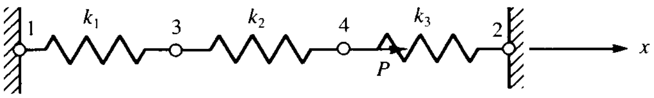
(4) Handle unlimited numbers and kinds of boundary conditions

(5) Vary the size of the elements to make it possible to use small elements where necessary

Chapter 2

2.1

(a)



$$[k^{(1)}] = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k^{(2)}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_2 & -k_2 \\ 0 & 0 & -k_2 & k_2 \end{bmatrix}$$

$$[k_3{}^{(3)}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_3 & 0 & -k_3 \\ 0 & 0 & 0 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix}$$

$$[K] = [k^{(1)}] + [k^{(2)}] + [k^{(3)}]$$

$$[K] = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix}$$

(b) Nodes 1 and 2 are fixed so $u_1 = 0$ and $u_2 = 0$ and $[K]$ becomes

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 0 \\ P \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$\{F\} = [K]\,\{d\} \Rightarrow [K^{-1}]\,\{F\} = [K^{-1}]\,[K]\,\{d\}$$

$$\Rightarrow [K^{-1}]\,\{F\} = \{d\}$$

Using the adjoint method to find $[K^{-1}]$

$C_{11} = k_2 + k_3$ $\qquad$ $C_{21} = (-1)^3\,(-k_2)$

$C_{12} = (-1)^{1+2}\,(-k_2) = k_2$ $\qquad$ $C_{22} = k_1 + k_2$

$$[C] = \begin{bmatrix} k_2+k_3 & k_2 \\ k_2 & k_1+k_2 \end{bmatrix} \text{ and } C^T = \begin{bmatrix} k_2+k_3 & k_2 \\ k_2 & k_1+k_2 \end{bmatrix}$$

$$\det [K] = |\,[K]\,| = (k_1 + k_2)\,(k_2 + k_3) - (-k_2)\,(-k_2)$$

$$\Rightarrow \quad |\,[K]\,| = (k_1 + k_2)\,(k_2 + k_3) - k_2^{\,2}$$

$$[K^{-1}] = \frac{[C^T]}{\det K}$$

$$[K^{-1}] = \frac{\begin{bmatrix} k_2+k_3 & k_2 \\ k_2 & k_1+k_2 \end{bmatrix}}{(k_1+k_2)(k_2+k_3)-k_2^{\,2}} = \frac{\begin{bmatrix} k_2+k_3 & k_2 \\ k_2 & k_1+k_2 \end{bmatrix}}{k_1\,k_2+k_1\,k_3+k_2\,k_3}$$

$$\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \frac{\begin{bmatrix} k_2+k_3 & k_2 \\ k_2 & k_1+k_2 \end{bmatrix}\begin{Bmatrix} 0 \\ P \end{Bmatrix}}{k_1\,k_2+k_1\,k_3+k_2\,k_3}$$

$$\Rightarrow \quad u_3 = \frac{k_2\,P}{k_1\,k_2+k_1\,k_3+k_2\,k_3}$$

$$\Rightarrow \quad u_4 = \frac{(k_1+k_2)\,P}{k_1\,k_2+k_1\,k_3+k_2\,k_3}$$

(c) In order to find the reaction forces we go back to the global matrix $F = [K]\,\{d\}$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1+k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2+k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

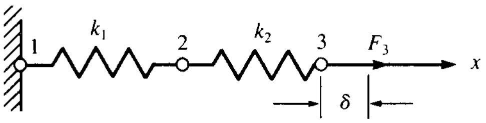
$$F_{1x} = -k_1\,u_3 = -k_1\,\frac{k_2\,P}{k_1\,k_2+k_1\,k_3+k_2\,k_3}$$

$$\Rightarrow F_{1x} = \frac{-k_1\,k_2\,P}{k_1\,k_2+k_1\,k_3+k_2\,k_3}$$

$$F_{2x} = -k_3\,u_4 = -k_3\,\frac{(k_1+k_2)\,P}{k_1\,k_2+k_1\,k_3+k_2\,k_3}$$

$$\Rightarrow F_{2x} = \frac{-k_3\,(k_1+k_2)\,P}{k_1\,k_2+k_1\,k_3+k_2\,k_3}$$

2.2



$$k_1 = k_2 = k_3 = 1000\ \frac{\text{lb}}{\text{in.}}$$

$$[k^{(1)}] = \begin{array}{c} \begin{array}{cc} (1) & (2) \end{array} \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{array} \begin{array}{c} \\ (1) \\ (2) \end{array} ; \quad [k^{(2)}] = \begin{array}{c} \begin{array}{cc} (2) & (3) \end{array} \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{array} \begin{array}{c} \\ (2) \\ (3) \end{array}$$

By the method of superposition the global stiffness matrix is constructed.

$$[K] = \begin{array}{c} \begin{array}{ccc} (1) & (2) & (3) \end{array} \\ \begin{bmatrix} k & -k & 0 \\ -k & k+k & -k \\ 0 & -k & k \end{bmatrix} \end{array} \begin{array}{c} \\ (1) \\ (2) \\ (3) \end{array} \Rightarrow [K] = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

Node 1 is fixed $\Rightarrow u_1 = 0$ and $u_3 = \delta$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = ? \end{Bmatrix} = \begin{bmatrix} k & k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = \delta \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 0 \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_2 \\ \delta \end{Bmatrix} \Rightarrow \begin{Bmatrix} 0 = 2k\,u_2 - k\delta \\ F_{3x} = -k\,u_2 + k\delta \end{Bmatrix}$$

$$\Rightarrow \quad u_2 = \frac{k\delta}{2\text{k}} = \frac{\delta}{2} = \frac{1 \text{ in.}}{2} \Rightarrow u_2 = 0.5''$$

$$F_{3x} = -k\,(0.5'') + k\,(1'')$$

$$F_{3x} = (-1000 \frac{\text{lb}}{\text{in.}})\,(0.5'') + (1000 \frac{\text{lb}}{\text{in.}})\,(1'')$$

$$F_{3x} = 500 \text{ lbs}$$

Internal forces

Element (1)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(2)} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = 0.5'' \end{Bmatrix}$$

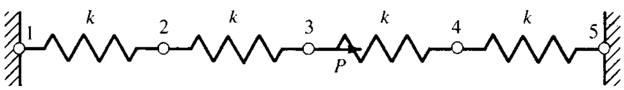
$$\Rightarrow \quad f_{1x}^{(1)} = (-1000 \frac{\text{lb}}{\text{in.}})\,(0.5'') \Rightarrow f_{1x}^{(1)} = -500 \text{ lb}$$

$$f_{2x}^{(1)} = (1000 \frac{\text{lb}}{\text{in.}})\,(0.5'') \Rightarrow f_{2x}^{(1)} = 500 \text{ lb}$$

Element (2)

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_2 = 0.5'' \\ u_3 = 1'' \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = -500 \text{ lb} \\ f_{3x}^{(2)} = 500 \text{ lb} \end{matrix}$$

2.3



(a) $[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$

By the method of superposition we construct the global $[K]$ and knowing $\{F\} = [K]\,\{d\}$ we have

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = P \\ F_{4x} = 0 \\ F_{5x} = ? \end{Bmatrix} = \begin{bmatrix} k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & 2k & -k & 0 \\ 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 \\ u_5 = 0 \end{Bmatrix}$$

(b) $\begin{Bmatrix} 0 \\ P \\ 0 \end{Bmatrix} = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} \Rightarrow \begin{matrix} 0 & = & 2ku_2 - ku_3 \\ P & = & -ku_2 + 2ku_3 - ku_4 \\ 0 & = & -ku_3 + 2ku_4 \end{matrix}$

$$\Rightarrow \quad u_2 = \frac{u_3}{2} \ ; u_4 = \frac{u_3}{2}$$

Substituting in the equation in the middle

$$P = -k\,u_2 + 2k\,u_3 - k\,u_4$$

$$\Rightarrow \quad P = -k\left(\frac{u_3}{2}\right) + 2k\,u_3 - k\left(\frac{u_3}{2}\right)$$

$$\Rightarrow \quad P = k\,u_3$$

$$\Rightarrow \quad u_3 = \frac{P}{k}$$

$$u_2 = \frac{P}{2k} \ ; u_4 = \frac{P}{2k}$$

(c) In order to find the reactions at the fixed nodes 1 and 5 we go back to the global equation $\{F\} = [K]\,\{d\}$

$$F_{1x} = -k\,u_2 = -k\,\frac{P}{2k} \Rightarrow F_{1x} = -\frac{P}{2}$$

$$F_{5x} = -k\,u_4 = -k\,\frac{P}{2k} \Rightarrow F_{5x} = -\frac{P}{2}$$

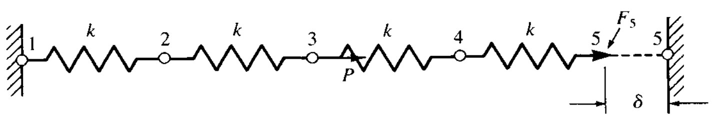
Check

$$\Sigma F_x = 0 \Rightarrow F_{1x} + F_{5x} + P = 0$$

$$\Rightarrow -\frac{P}{2} + \left(-\frac{P}{2}\right) + P = 0$$

$$\Rightarrow 0 = 0$$

2.4



(a) $[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$

By the method of superposition the global $[K]$ is constructed.

Also $\{F\} = [K]\,\{d\}$ and $u_1 = 0$ and $u_5 = \delta$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = 0 \\ F_{4x} = 0 \\ F_{5x} = ? \end{Bmatrix} = \begin{bmatrix} k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & 2k & -k & 0 \\ 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = ? \\ u_5 = \delta \end{Bmatrix}$$

(b) $0 = 2k\,u_2 - k\,u_3$ (1)

$0 = -\,ku_2 + 2k\,u_3 - k\,u_4$ (2)

$0 = -\,k\,u_3 + 2k\,u_4 - k\,\delta$ (3)

From (2)

$$u_3 = 2\,u_2$$

From (3)

$$u_4 = \frac{\delta + 2\,u_2}{2}$$

Substituting in Equation (2)

$$\Rightarrow -\,k\,(u_2) + 2k\,(2\,u_2) - k\left(\frac{\delta + 2\,\delta_{2x}}{2}\right)$$

$$\Rightarrow -\,u_2 + 4\,u_2 - u_2 - \frac{\delta}{2} = 0 \Rightarrow u_2 = \frac{\delta}{4}$$

$$\Rightarrow u_3 = 2\,\frac{\delta}{4} \Rightarrow u_3 = \frac{\delta}{2}$$

$$\Rightarrow u_4 = \frac{\delta + 2\left(\frac{\delta}{4}\right)}{2} \Rightarrow u_4 = \frac{3\,\delta}{4}$$

(c) Going back to the global equation

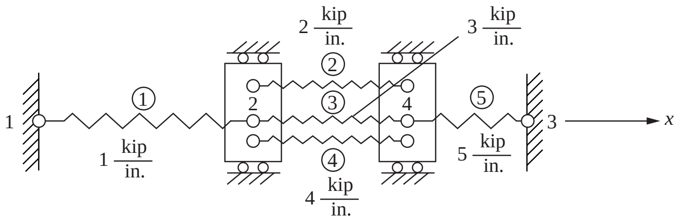
$$\{F\} = [K]\,\{d\}$$

$$F_{1x} = -\,k\;\,u_2 = -k\,\frac{\delta}{4} \Rightarrow F_{1x} = -\frac{k\,\delta}{4}$$

$$F_{5x} = -\,k\,u_4 + k\,\delta = -\,k\left(\frac{3\,\delta}{4}\right) + k\,\delta$$

$$\Rightarrow\; F_{5x} = \frac{k\,\delta}{4}$$

2.5



$$[k^{(1)}] = \begin{array}{cc} d_1 & d_2 \\ \end{array} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \quad [k^{(2)}] = \begin{array}{cc} d_2 & d_4 \\ \end{array} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$[k^{(3)}] = \begin{array}{cc} d_2 & d_4 \\ \end{array} \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}; \quad [k^{(4)}] = \begin{array}{cc} d_2 & d_4 \\ \end{array} \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$$

$$[k^{(5)}] = \begin{array}{cc} d_4 & d_3 \\ \end{array} \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}$$

Assembling global [K] using direct stiffness method

$$[K] = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+2+3+4 & 0 & -2-3-4 \\ 0 & 0 & 5 & -5 \\ 0 & -2-3-4 & -5 & 2+3+4+5 \end{bmatrix}$$

Simplifying

$$[K] = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 10 & 0 & -9 \\ 0 & 0 & 5 & -5 \\ 0 & -9 & -5 & 14 \end{bmatrix} \frac{\text{kip}}{\text{in.}}$$

2.6 Now apply + 2 kip at node 2 in spring assemblage of P 2.5.

$$\therefore \quad F_{2x} = 2 \text{ kip}$$

$$[K]\{d\} = \{F\}$$

[K] from P 2.5

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 10 & 0 & -9 \\ 0 & 0 & 5 & -5 \\ 0 & -9 & -5 & 14 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 2 \\ F_3 \\ 0 \end{Bmatrix} \qquad \text{(A)}$$

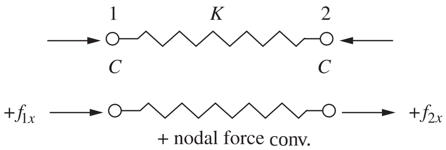
where $u_1 = 0$, $u_3 = 0$ as nodes 1 and 3 are fixed.
Using Equations (1) and (3) of (A)

$$\begin{bmatrix} 10 & -9 \\ -9 & 14 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}$$

Solving

$$u_2 = 0.475 \text{ in.}, \quad u_4 = 0.305 \text{ in.}$$

2.7



$$f_{1x} = C, \quad f_{2x} = -C$$

$$f = -k\delta = -k(u_2 - u_1)$$

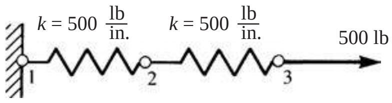
$$\therefore \quad f_{1x} = -k(u_2 - u_1)$$

$$f_{2x} = -(-k)(u_2 - u_1)$$

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{Bmatrix} k & -k \\ -k & k \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\therefore \quad [K] = \begin{Bmatrix} k & -k \\ -k & k \end{Bmatrix} \text{ same as for tensile element}$$

2.8



$$k_1 = 500\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix};\ k_2 = 500\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

So

$$[K] = 500\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\Rightarrow \begin{bmatrix} F_1 = ? \\ F_2 = 0 \\ F_3 = 1000 \end{bmatrix} = 500\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \end{Bmatrix}$$

$$\Rightarrow \quad 0 = 1000\,u_2 - 500\,u_3 \tag{1}$$

$$500 = -500\,u_2 + 500\,u_3 \tag{2}$$

From (1)

$$u_2 = \frac{500}{1000}\,u_3 \Rightarrow u_2 = 0.5\,u_3 \tag{3}$$

Substituting (3) into (2)

$$\Rightarrow \quad 500 = -500\,(0.5\,u_3) + 500\,u_3$$

$$\Rightarrow \quad 500 = 250\,u_3$$

$$\Rightarrow \quad u_3 = 2\text{ in.}$$

$$\Rightarrow \quad u_2 = (0.5)\,(2\text{ in.}) \Rightarrow u_2 = 1\text{ in.}$$

Element 1–2

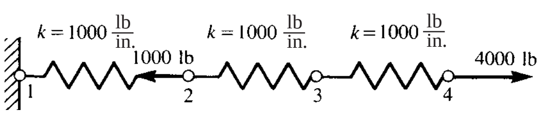
$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = 500\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 & \text{in.} \\ 1 & \text{in.} \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = -500\text{ lb} \\ f_{2x}^{(1)} = \ \ 500\text{ lb} \end{matrix}$$

Element 2–3

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = 500\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 1\text{ in.} \\ 2\text{ in.} \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = -500\text{ lb} \\ f_{3x}^{(2)} = \ \ 500\text{ lb} \end{matrix}$$

$$F_{1x} = 500\,[1\ \ -1\ \ 0]\begin{bmatrix} 0 \\ 1\text{ in.} \\ 2\text{ in.} \end{bmatrix} \Rightarrow F_{1x} = -500\text{ lb}$$

2.9



$$[k^{(1)}] = \begin{array}{c} \begin{array}{cc} (1) & (2) \end{array} \\ \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \end{array}$$

$$[k^{(2)}] = \begin{array}{c} \begin{array}{cc} (2) & (3) \end{array} \\ \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \end{array}$$

$$[k^{(3)}] = \begin{array}{c} \begin{array}{cc} (3) & (4) \end{array} \\ \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \end{array}$$

$$[K] = \begin{array}{c} \begin{array}{cccc} (1) & (2) & (3) & (4) \end{array} \\ \begin{bmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 2000 & -1000 & 0 \\ 0 & -1000 & 2000 & -1000 \\ 0 & 0 & -1000 & 1000 \end{bmatrix} \end{array}$$

$$\begin{Bmatrix} \cancel{F_{1x} = ?} \\ F_{2x} = -1000 \\ F_{3x} = 0 \\ F_{4x} = 4000 \end{Bmatrix} = \begin{bmatrix} \cancel{1000} & \cancel{-1000} & \cancel{0} & \cancel{0} \\ \cancel{-1000} & 2000 & -1000 & 0 \\ \cancel{0} & -1000 & 2000 & -1000 \\ \cancel{0} & 0 & -1000 & 1000 \end{bmatrix} \begin{Bmatrix} \cancel{u_1 = 0} \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\Rightarrow \quad \begin{aligned} u_1 &= 0 \text{ in.} \\ u_2 &= 3 \text{ in.} \\ u_3 &= 7 \text{ in.} \\ u_4 &= 11 \text{ in.} \end{aligned}$$

Reactions

$$F_{1x} = [1000 \quad -1000 \quad 0 \quad 0] \begin{Bmatrix} u_1 = & 0 \\ u_2 = & 3 \\ u_3 = & 7 \\ u_4 = & 11 \end{Bmatrix} \Rightarrow F_{1x} = -3000 \text{ lb}$$

Element forces

Element (1)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0 \\ 3 \end{Bmatrix} \Rightarrow \begin{aligned} f_{1x}^{(1)} &= -3000 \text{ lb} \\ f_{2x}^{(1)} &= 3000 \text{ lb} \end{aligned}$$

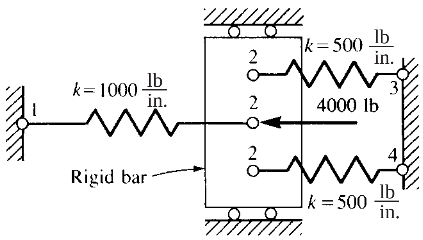
Element (2)

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 3 \\ 7 \end{Bmatrix} \Rightarrow \begin{aligned} f_{2x}^{(2)} &= -4000 \text{ lb} \\ f_{3x}^{(2)} &= 4000 \text{ lb} \end{aligned}$$

Element (3)

$$\begin{Bmatrix} f_{3x}^{(3)} \\ f_{4x}^{(3)} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 7 \\ 11 \end{Bmatrix} \Rightarrow \begin{aligned} f_{3x}^{(3)} &= -4000 \text{ lb} \\ f_{4x}^{(3)} &= 4000 \text{ lb} \end{aligned}$$

2.10



$$[k^{(1)}] = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix}$$

$$[k^{(2)}] = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix}$$

$$[k^{(3)}] = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix}$$

$$\{F\} = [K]\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = -4000 \\ F_{3x} = ? \\ F_{4x} = ? \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 2000 & -500 & -500 \\ 0 & -500 & 500 & 0 \\ 0 & -500 & 0 & 500 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = 0 \\ u_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad u_2 = \frac{-4000}{2000} = -2 \text{ in.}$$

Reactions

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 2000 & -500 & -500 \\ 0 & -500 & 500 & 0 \\ 0 & -500 & 0 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ -2 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{Bmatrix} 2000 \\ -4000 \\ 1000 \\ 1000 \end{Bmatrix} \text{ lb}$$

Element (1)

$$\begin{Bmatrix} f_{1x}{}^{(1)} \\ f_{2x}{}^{(1)} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0 \\ -2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{1x}{}^{(1)} \\ f_{2x}{}^{(1)} \end{Bmatrix} = \begin{Bmatrix} 2000 \\ -2000 \end{Bmatrix} \text{ lb}$$

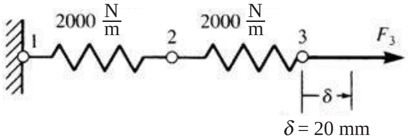
Element (2)

$$\begin{Bmatrix} f_{2x}{}^{(2)} \\ f_{3x}{}^{(2)} \end{Bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} \begin{Bmatrix} -2 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{2x}{}^{(2)} \\ f_{3x}{}^{(2)} \end{Bmatrix} = \begin{Bmatrix} -1000 \\ 1000 \end{Bmatrix} \text{ lb}$$

Element (3)

$$\begin{Bmatrix} f_{2x}{}^{(3)} \\ f_{4x}{}^{(3)} \end{Bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} \begin{Bmatrix} -2 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{2x}{}^{(3)} \\ f_{4x}{}^{(3)} \end{Bmatrix} \begin{Bmatrix} -1000 \\ 1000 \end{Bmatrix} \text{ lb}$$

2.11



$$[k^{(1)}] = \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix}; \quad [k^{(2)}] = \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = ? \end{Bmatrix} = \begin{bmatrix} 2000 & -2000 & 0 \\ -2000 & 4000 & -2000 \\ 0 & -2000 & 2000 \end{bmatrix} \begin{bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = 0.02\text{ m} \end{bmatrix}$$

$$\Rightarrow \quad u_2 = 0.01\text{ m}$$

Reactions

$$F_{1x} = (-2000)\,(0.01) \Rightarrow F_{1x} = -20\text{ N}$$

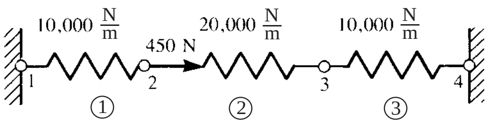
Element (1)

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.01 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{Bmatrix} -20 \\ 20 \end{Bmatrix}\text{ N}$$

Element (2)

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix} \begin{Bmatrix} 0.01 \\ 0.02 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{Bmatrix} -20 \\ 20 \end{Bmatrix}\text{ N}$$

2.12



$$[k^{(1)}] = [k^{(3)}] = 10000 \begin{Bmatrix} 1 & -1 \\ -1 & 1 \end{Bmatrix}$$

$$[k^{(2)}] = 10000 \begin{Bmatrix} 2 & -2 \\ -2 & 2 \end{Bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 4500\text{ N} \\ F_{3x} = 0 \\ F_{4x} = ? \end{Bmatrix} = 10000 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = 0 \end{Bmatrix}$$

$$0 = -2\,u_2 + 3\,u_3 \Rightarrow u_2 = \frac{3}{2}\,u_3 \Rightarrow u_2 = 1.5\,u_3$$

$$450\text{ N} = 30000\,(1.5\,u_3) - 20000\,u_3$$

$$\Rightarrow \quad 450\text{ N} = (25000\,\frac{\text{N}}{\text{m}})\,u_3 \Rightarrow u_3 = 1.8 \times 10^{-2}\text{ m}$$

$$\Rightarrow \quad u_2 = 1.5\,(1.8 \times 10^{-2}) \Rightarrow u_2 = 2.7 \times 10^{-2}\text{ m}$$

Element (1)

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = 10000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 2.7 \times 10^{-2} \end{Bmatrix} \Rightarrow \begin{matrix} \hat{f}_{1x}^{\ (1)} = -270\text{ N} \\ \hat{f}_{2x}^{\ (1)} = 270\text{ N} \end{matrix}$$

Element (2)

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = 20000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 2.7\times10^{-2} \\ 1.8\times10^{-2} \end{Bmatrix} \Rightarrow \begin{matrix} \hat{f}_{2x}{}^{(2)} = 180 \text{ N} \\ \hat{f}_{3x}{}^{(2)} = -180 \text{ N} \end{matrix}$$

Element (3)

$$\begin{Bmatrix} \hat{f}_{3x} \\ \hat{f}_{4x} \end{Bmatrix} = 10000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1.8\times10^{-2} \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} \hat{f}_{3x}{}^{(3)} = 180 \text{ N} \\ \hat{f}_{4x}{}^{(3)} = -180 \text{ N} \end{matrix}$$

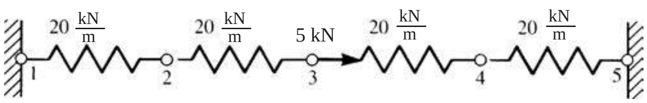
Reactions

$$\{F_{1x}\} = \left(10000\ \frac{\text{N}}{\text{m}}\right)[1 - 1] \begin{Bmatrix} 0 \\ 2.7\times10^{-2} \end{Bmatrix} \Rightarrow F_{1x} = -\,270 \text{ N}$$

$$\{F_{4x}\} = \left(10000\ \frac{\text{N}}{\text{m}}\right)[-1 \quad 1] \begin{Bmatrix} 1.8\times10^{-2} \\ 0 \end{Bmatrix}$$

$$\Rightarrow \quad F_{4x} = -\,180 \text{ N}$$

2.13



$$[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = 20 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = 10 \text{ kN} \\ F_{4x} = 0 \\ F_{5x} = ? \end{Bmatrix} = 20 \begin{Bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{Bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = ? \\ u_5 = 0 \end{Bmatrix}$$

$$\left.\begin{matrix} 0 = 2u_2 - u_3 \Rightarrow u_2 = 0.5\,u_3 \\ 0 = -\,u_3 + 2u_4 \Rightarrow u_4 = 0.5\,u_3 \end{matrix}\right\} \Rightarrow u_2 = u_4$$

$$\Rightarrow \quad 5 \text{ kN} = -\,20\,u_2 + 40\,(2\,u_2) - 20\,u_2$$

$$\Rightarrow \quad 5 = 40\,u_2 \Rightarrow u_2 = 0.125 \text{ m}$$

$$\Rightarrow \quad u_4 = 0.125 \text{ m}$$

$$\Rightarrow \quad u_3 = 2(0.125) \Rightarrow u_3 = 0.25 \text{ m}$$

Element (1)

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = 20 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.125 \end{Bmatrix} \Rightarrow \begin{matrix} \hat{f}_{1x}{}^{(1)} = -2.5 \text{ kN} \\ \hat{f}_{2x}{}^{(1)} = 2.5 \text{ kN} \end{matrix}$$

Element (2)

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = 20 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.125 \\ 0.25 \end{Bmatrix} \Rightarrow \begin{matrix} \hat{f}_{2x}{}^{(2)} = -2.5 \text{ kN} \\ \hat{f}_{3x}{}^{(2)} = 2.5 \text{ kN} \end{matrix}$$

Element (3)

$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = 20 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.25 \\ 0.125 \end{Bmatrix} \Rightarrow \begin{matrix} f_{3x}{}^{(3)} = 2.5 \text{ kN} \\ f_{4x}{}^{(3)} = -2.5 \text{ kN} \end{matrix}$$

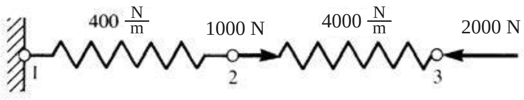
Element (4)

$$\begin{Bmatrix} \hat{f}_{4x} \\ \hat{f}_{5x} \end{Bmatrix} = 20 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.125 \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} \hat{f}_{4x}{}^{(4)} = 2.5 \text{ kN} \\ \hat{f}_{5x}{}^{(4)} = -2.5 \text{ kN} \end{matrix}$$

$$F_{1x} = 20 \, [1 \quad -1] \begin{Bmatrix} 0 \\ 0.125 \end{Bmatrix} \Rightarrow F_{1x} = -2.5 \text{ kN}$$

$$F_{5x} = 20 \, [-1 \quad 1] \begin{Bmatrix} 0.125 \\ 0 \end{Bmatrix} \Rightarrow F_{5x} = -2.5 \text{ kN}$$

2.14



$$[k^{(1)}] = [k^{(2)}] = 400 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{F\} = [K] \, \{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 100 \\ F_{3x} = -200 \end{Bmatrix} = 400 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \end{Bmatrix}$$

$$100 = 800 \, u_2 - 400 \, u_3$$

$$-200 = -400 \, u_2 + 400 \, u_3$$

$$-100 = 400 \, u_2 \Rightarrow u_2 = -0.25 \text{ m}$$

$$100 = 800 \, (-0.25) - 400 \, u_3 \Rightarrow u_3 = -0.75 \text{ m}$$

Element (1)

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = 400 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.25 \end{Bmatrix} \Rightarrow \begin{matrix} \hat{f}_{1x}{}^{(1)} = 100 \text{ N} \\ \hat{f}_{2x}{}^{(1)} = -100 \text{ N} \end{matrix}$$

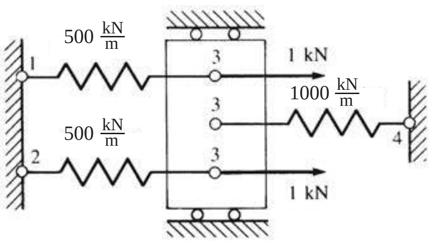
Element (2)

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = 400 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} -0.25 \\ -0.75 \end{Bmatrix} \Rightarrow \begin{matrix} \hat{f}_{2x}{}^{(2)} = 200 \text{ N} \\ \hat{f}_{3x}{}^{(2)} = -200 \text{ N} \end{matrix}$$

Reaction

$$\{F_{1x}\} = 400 \, [1 \quad -1] \begin{Bmatrix} 0 \\ -0.25 \end{Bmatrix} \Rightarrow F_{1x} = 100 \text{ N}$$

2.15



$$[k^{(1)}] = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix}; \ [k^{(2)}] = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix}; \ [k^{(3)}] = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = ? \\ F_{3x} = 2 \text{ kN} \\ F_{4x} = ? \end{Bmatrix} = \begin{bmatrix} 500 & 0 & -500 & 0 \\ 0 & 500 & -500 & 0 \\ -500 & -500 & 2000 & -1000 \\ 0 & 0 & -1000 & 1000 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = 0 \\ u_3 = ? \\ u_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad u_3 = 0.001 \text{ m}$$

Reactions

$$F_{1x} = (-500)(0.001) \Rightarrow F_{1x} = -0.5 \text{ kN}$$
$$F_{2x} = (-500)(0.001) \Rightarrow F_{2x} = -0.5 \text{ kN}$$
$$F_{4x} = (-1000)(0.001) \Rightarrow F_{4x} = -1.0 \text{ kN}$$

Element (1)

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.001 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{Bmatrix} -0.5 \text{ kN} \\ 0.5 \text{ kN} \end{Bmatrix}$$

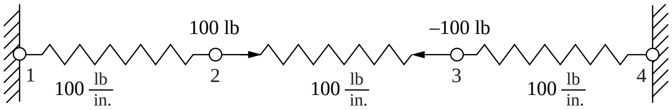
Element (2)

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.001 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{Bmatrix} -0.5 \text{ kN} \\ 0.5 \text{ kN} \end{Bmatrix}$$

Element (3)

$$\begin{Bmatrix} \hat{f}_{3x} \\ \hat{f}_{4x} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0.001 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \hat{f}_{3x} \\ \hat{f}_{4x} \end{Bmatrix} = \begin{Bmatrix} 1 \text{ kN} \\ -1 \text{ kN} \end{Bmatrix}$$

2.16



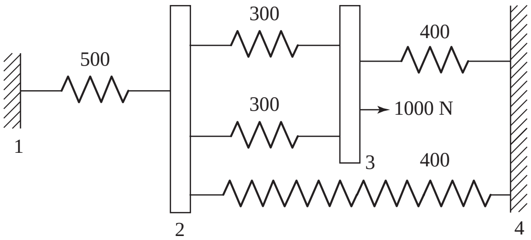
$$\begin{Bmatrix} F_{1x} \\ 100 \\ -100 \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 100+100 & -100 & 0 \\ 0 & -100 & 100+100 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} 100 \\ -100 \end{Bmatrix} = \begin{Bmatrix} 200 & -100 \\ -100 & 200 \end{Bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$u_2 = \frac{1}{3} \text{ in.}$$

$$u_3 = -\frac{1}{3} \text{ in.}$$

2.17



$$\begin{Bmatrix} F_{1x} = ? \\ 0 \\ 1000 \text{ N} \\ F_{4x} = ? \end{Bmatrix} = \begin{bmatrix} -500 & -500 & 0 & 0 \\ -500 & \begin{pmatrix} 400+300 \\ 500+300 \end{pmatrix} & -300-300 & -400 \\ 0 & -300-300 & (300+300+400) & -400 \\ 0 & -400 & -400 & 400+400 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 = 0 \end{Bmatrix}$$

$$0 = 1500\, u_2 - 600\, u_3$$

$$1000 = -600\, u_2 + 1000\, u_3$$

$$u_3 = \frac{15\cancel{0}\cancel{0}}{6\cancel{0}\cancel{0}}\; u_2 = 2.5\, u_2$$

$$1000 = -600\, u_2 + 1000\,(2.5\, u_2)$$

$$1000 = 1900\, u_2$$

$$u_2 = \frac{1000}{1900} = \frac{1}{1.9} \text{ mm} = 0.526 \text{ mm}$$

$$u_3 = 2.5 \left(\frac{1}{1.9}\right) \text{ mm} = 1.316 \text{ mm}$$

$$F_{1x} = -500 \left(\frac{1}{1.9}\right) = -263.16 \text{ N}$$

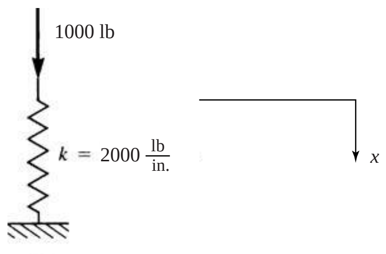
$$F_{4x} = -400 \left(\frac{1}{1.9}\right) - 400 \left(2.5\left(\frac{1}{1.9}\right)\right)$$

$$= -400 \left(\frac{1}{1.9} + \frac{2.5}{1.9}\right) = -736.84 \text{ N}$$

$$\Sigma F_x = -263.16 + 1000 - 736.84 = 0$$

2.18

(a)



As in Example 2.4

$$\pi_p = U + \Omega$$

$$U = \frac{1}{2}\, k\, x^2,\ \Omega = -Fx$$

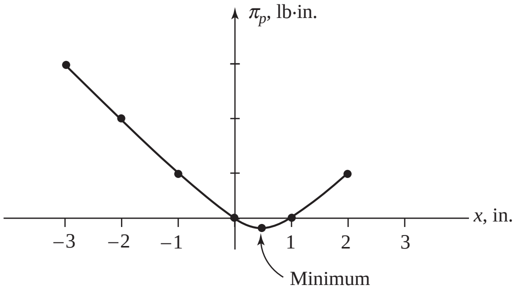
Set up table

$$\pi_p = \frac{1}{2}\,(2000)\, x^2 - 1000\, x = 1000\, x^2 - 1000\, x$$

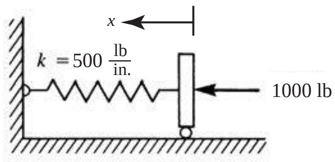
Deformation x, in.	π_p, lb·in.
– 3.0	6000
– 2.0	3000

– 1.0	1000
0.0	0
0.5	– 125
1.0	0
2.0	1000

$\dfrac{\partial \pi_p}{\partial x}$ = 2000 x – 1000 = 0 ⇒ x = 0.5 in. yields minimum π_p as table verifies.



(b)



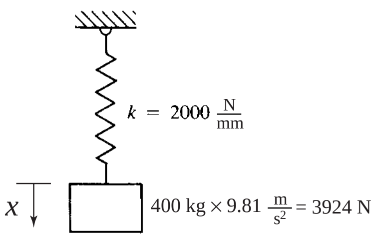
$\pi_p = \dfrac{1}{2}\,kx^2 - F_x = 250\,x^2 - 1000\,x$

x, in.	π_p, lb·in.
– 3.0	11250
– 2.0	3000
– 1.0	1250
0	0
1.0	– 750
2.0	– 1000
3.0	– 750

$\dfrac{\partial \pi_p}{\partial x}$ = 500 x – 1000 = 0

⇒ x = 2.0 in. yields π_p minimum

(c)



$$\pi_p = \frac{1}{2}\,(2000)\,x^2 - 3924\,x = 1000\,x^2 - 3924\,x$$

$$\frac{\partial \pi_p}{\partial x} = 2000\,x - 3924 = 0$$

$\Rightarrow \quad x = 1.962$ mm yields π_p minimum

$$\pi_{p\,\min} = \frac{1}{2}\,(2000)\,(1.962)^2 - 3924\,(1.962)$$

$\Rightarrow \; \pi_{p\,\min} = -\,3849.45$ N·mm

(d) $$\pi_p = \frac{1}{2}\,(400)\,x^2 - 981\,x$$

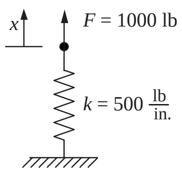
$$\frac{\partial \pi_p}{\partial x} = 400\,x - 981 = 0$$

$\Rightarrow \quad x = 2.4525$ mm yields π_p minimum

$$\pi_{p\,\min} = \frac{1}{2}\,(400)\,(2.4525)^2 - 981\,(2.4525)$$

$\Rightarrow \; \pi_{p\,\min} = -\,1202.95$ N·mm

2.19



Now let positive x be upward

$$\pi_p = \frac{1}{2}\,kx^2 - Fx$$

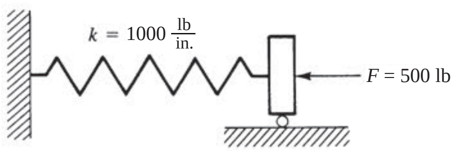
$$\pi_p = \frac{1}{2}\,(500)\,x^2 - 1000\,x$$

$$\pi_p = 250\,x^2 - 1000\,x$$

$$\frac{\partial \pi_p}{\partial x} = 500\,x - 1000 = 0$$

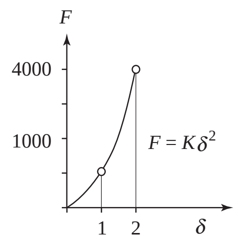
$\Rightarrow \; x = 2.0$ in. $\uparrow$

2.20



$F = k\delta^2 \qquad (x = \delta)$

$dU = F\,dx$



$$U = \int_0^x (kx^2)\,dx$$

$$U = \frac{kx^3}{3}$$

$$\Omega = -Fx$$

$$\pi_p = \frac{1}{3}\,kx^3 - 500\,x$$

$$\frac{\partial \pi_p}{\partial x} = 0 = kx^2 - 500$$

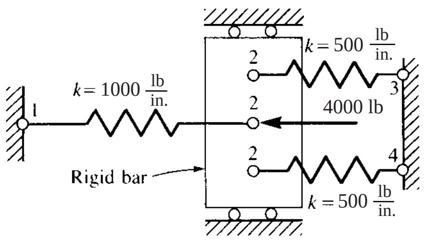
$$0 = 1000\,x^2 - 500$$

$$\Rightarrow \quad x = 0.707 \text{ in.} \quad \text{(equilibrium value of displacement)}$$

$$\pi_{p\,\min} = \frac{1}{3}\,(1000)\,(0.707)^3 - 500\,(0.707)$$

$$\pi_{p\,\min} = -235.7 \text{ lb}\cdot\text{in.}$$

2.21 Solve Problem 2.10 using P.E. approach



$$\pi_p = \sum_{e=1}^{3} \pi_p^{(e)} = \frac{1}{2}\,k_1\,(u_2 - u_1)^2 + \frac{1}{2}\,k_2\,(u_3 - u_2)^2 + \frac{1}{2}\,k_3\,(u_4 - u_2)^2$$

$$- f_{1x}^{(1)}\,u_1 - f_{2x}^{(1)}\,u_2 - f_{2x}^{(2)}\,u_2$$

$$- f_{3x}^{(2)}\,u_3 - f_{2x}^{(3)}\,u_2 - f_{4x}^{(3)}\,u_4$$

$$\frac{\partial \pi_p}{\partial u_1} = -k_1\,u_2 + k_1\,u_1 - f_{1x}^{(1)} = 0 \qquad (1)$$

$$\frac{\partial \pi_p}{\partial u_2} = k_1\,u_2 - k_1\,u_1 - k_2\,u_3 + k_2\,u_2 - k_3\,u_4$$

$$+ k_3\,u_2 - f_{2x}^{(1)} - f_{2x}^{(2)} - f_{2x}^{(3)} = 0 \qquad (2)$$

$$\frac{\partial \pi_p}{\partial u_3} = k_2\,u_3 - k_2\,u_2 - f_{3x}^{(2)} = 0 \qquad (3)$$

$$\frac{\partial \pi_p}{\partial u_4} = k_3\,u_4 - k_3\,u_2 - f_{4x}^{(3)} = 0 \qquad (4)$$

In matrix form (1) through (4) become

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_{1x}{}^{(1)} \\ f_{2x}{}^{(1)} + f_{2x}{}^{(2)} + f_{2x}{}^{(3)} \\ f_{3x}{}^{(2)} \\ f_{4x}{}^{(3)} \end{Bmatrix} \tag{5}$$

or using numerical values

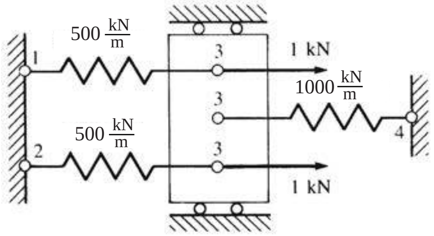
$$\begin{bmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 2000 & -500 & -500 \\ 0 & -500 & 500 & 0 \\ 0 & -500 & 0 & 500 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \\ u_4 = 0 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ -4000 \\ F_{3x} \\ F_{4x} \end{Bmatrix} \tag{6}$$

Solution now follows as in Problem 2.10

Solve 2nd of Equations (6) for $u_2 = -2$ in.

For reactions and element forces, see solution to Problem 2.10

2.22 Solve Problem 2.15 by P.E. approach



$$\pi_p = \sum_{e=1}^{3} \pi_p{}^{(e)} = \frac{1}{2}\, k_1 (u_3 - u_1)^2 + \frac{1}{2}\, k_2 (u_3 - u_2)^2$$

$$+ \frac{1}{2}\, k_3 (u_4 - u_3)^2 - f_{1x}{}^{(1)} u_1$$

$$- f_{3x}{}^{(1)} u_3 - f_{2x}{}^{(2)} u_2 - f_{3x}{}^{(2)} u_3$$

$$- f_{3x}{}^{(3)} u_3 - f_{3x}{}^{(4)} u_4$$

$$\frac{\partial \pi_p}{\partial u_1} = 0 = -k_1 u_3 + k_1 u_1 - f_{1x}{}^{(1)}$$

$$\frac{\partial \pi_p}{\partial u_2} = 0 = -k_2 u_3 + k_2 u_2 - f_{2x}{}^{(2)}$$

$$\frac{\partial \pi_p}{\partial u_3} = 0 = k_1 u_3 + k_2 u_3 - k_2 u_2 - k_3 u_4 + k_3 u_3 - f_{3x}{}^{(2)} - f_{3x}{}^{(3)} - f_{3x}{}^{(1)} - k_1 u_1$$

$$\frac{\partial \pi_p}{\partial u_4} = 0 = k_3 u_4 - k_3 u_3 - f_{3x}{}^{(4)}$$

In matrix form

$$\begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_2 & -k_2 & 0 \\ -k_1 & -k_2 & k_1+k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} = 2\text{ kN} \\ F_{4x} \end{Bmatrix}$$

For rest of solution, see solutions of Problem 2.15.

2.23

$$I = a_1 + a_2 x$$

$$I(0) = a_1 = I_1$$

$$I(L) = a_1 + a_2 L = I_2$$

$$a_2 = \frac{I_2 - I_1}{L}$$

$$\therefore \qquad I = I_1 + \frac{I_2 - I_1}{L} x$$

Now $V = IR$

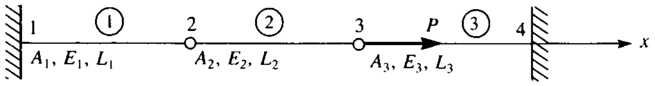
$$V = -V_1 = R(I_2 - I_1)$$

$$V = V_2 = R(I_2 - I_1)$$

$$\begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix} = R \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix}$$

Chapter 3

3.1



(a) $\left[\hat{k}^{(1)}\right] = \dfrac{A_1E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$\left[\hat{k}^{(2)}\right] = \dfrac{A_2E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$\left[\hat{k}^{(3)}\right] = \dfrac{A_3E_3}{L_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$[K] = \begin{bmatrix} \frac{A_1E_1}{L_1} & \frac{-A_1E_1}{L_1} & 0 & 0 \\ \frac{-A_1E_1}{L_1} & \frac{A_1E_1}{L_1} + \frac{A_2E_2}{L_2} & \frac{-A_2E_2}{L_2} & 0 \\ 0 & \frac{-A_2E_2}{L_2} & \frac{A_2E_2}{L_2} + \frac{A_3E_3}{L_3} & \frac{-A_3E_3}{L_3} \\ 0 & 0 & \frac{-A_3E_3}{L_3} & \frac{A_3E_3}{L_3} \end{bmatrix}$$

(b) $\dfrac{A_1E_1}{L_1} = \dfrac{A_2E_2}{L_2} = \dfrac{A_3E_3}{L_3} = \dfrac{AE}{L}$

$$[K] = \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

It is known that $\{F\} = [K]\,\{d\}$

$$\Rightarrow \begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = P \\ F_{4x} = ? \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad 0 = \frac{2AE}{L}\, u_2 - \frac{AE}{L}\, u_3 \quad \Rightarrow u_3 = 2\, u_2$$

$$P = \frac{-AE}{L}\, u_2 + \frac{2AE}{L}\, u_3$$

$$\Rightarrow \quad P = \frac{-AE}{L}\, u_2 + \frac{2AE}{L}\,(2\, u_2)$$

$$\Rightarrow \quad u_2 = \frac{1}{3}\,\frac{PL}{AE}$$

$$\Rightarrow \quad u_3 = 2 \cdot \frac{1}{3} \frac{PL}{AE} \quad \Rightarrow \quad u_3 = \frac{2}{3} \frac{PL}{AE}$$

(c) $A = 1$ in.^{2}; $E = 10 \times 10^6$ psi; $L = 10$ in.

$P = 1000$ lbs

(i) $u_2 = \frac{1}{3} \frac{PL}{AE} = \frac{1}{3} \frac{(1000)(10)}{(1)(10 \times 10^6)}$

$$\Rightarrow \quad u_2 = 3.33 \times 10^{-4} \text{ in.}$$

$$u_3 = \frac{2}{3} \frac{PL}{AE} = 2\, u_2$$

$$\Rightarrow \quad u_3 = 6.67 \times 10^{-4} \text{ in.}$$

(ii) Going back to $\{F\} = [K]\,\{d\}$

$$F_{1x} = \frac{-AE}{L}\, u_2 = \frac{-AE}{L}\left(\frac{1}{3} \frac{PL}{AE}\right) = \frac{-1}{3} P$$

$$\Rightarrow \quad F_{1x} = \frac{-1}{3}\,(1000) \quad \Rightarrow \quad F_{1x} = -\,333.3 \text{ lbs}$$

$$F_{4x} = \frac{-AE}{L}\, u_3 = \frac{-AE}{L}\left(\frac{2}{3} \frac{PL}{AE}\right) = -\frac{2}{3} P$$

$$\Rightarrow \quad F_{4x} = \frac{-2}{3}\,(1000) \quad \Rightarrow \quad F_{4x} = -\,666.7 \text{ lbs}$$

(iii) $f = \sigma A$, where f = force, σ = stress and A = area.

Going back to the local system and substituting

Element (1)

$$\begin{Bmatrix} \sigma_{1x} = \frac{f_{1x}}{A} \\ \sigma_{2x} = \frac{f_{2x}}{A} \end{Bmatrix} = \frac{AE}{AL} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = 3.33 \times 10^{-4} \end{Bmatrix}$$

$$\Rightarrow \quad \sigma_{1x} = -\frac{E}{L}\, u_2 = -\frac{10 \times 10^6}{10}\,(3.33 \times 10^{-4})$$

$$\Rightarrow \quad \sigma_{1x}^{(1)} = -\,333.33 \text{ psi (C)}$$

$$\Rightarrow \quad \sigma_{2x} = \frac{E}{L} u_2 = \frac{10 \times 10^6}{10}\,(3.33 \times 10^{-4})$$

$$\Rightarrow \quad \sigma_{2x}^{(2)} = 333.33 \text{ psi (T)}$$

Element (2)

$$\begin{Bmatrix} \sigma_{2x} = \frac{f_{2x}}{A} \\ \sigma_{3x} = \frac{f_{3x}}{A} \end{Bmatrix} = \frac{AE}{AL} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 = 3.33 \times 10^{-4} \\ u_3 = 6.67 \times 10^{-4} \end{Bmatrix}$$

$$\Rightarrow \quad \sigma_{2x} = \frac{E}{L}\,(u_2 - u_3) = \frac{10 \times 10^6}{10} \times 10^{-4}\,(3.33 - 6.67)$$

$$\Rightarrow \ \sigma_{2x}^{(2)} = -333.33 \text{ psi (C)}$$

$$\Rightarrow \ \sigma_{3x}^{(2)} = \frac{E}{L}(u_3 - u_2) = \frac{10\times10^6}{10} \times 10^{-4}\,(6.67 - 3.33)$$

$$\Rightarrow \ \sigma_{3x}^{(2)} = 333.33 \text{ psi (T)}$$

Element (3)

$$\begin{Bmatrix} \sigma_{3x} = \frac{f_{3x}}{A} \\ \sigma_{4x} = \frac{f_{4x}}{A} \end{Bmatrix} = \frac{AE}{AL}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_3 = 6.67\times10^{-4} \\ u_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \ \sigma_{3x} = \frac{E}{L}(u_3 - u_4) = \frac{10\times10^6}{10} \times 10^{-4}\,(6.67 - 0)$$

$$\Rightarrow \ \sigma_{3x}^{(3)} = 666.7 \text{ psi (T)}$$

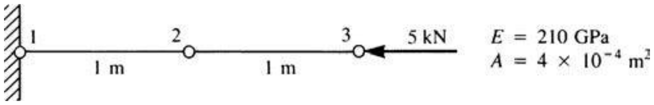
$$\Rightarrow \ \sigma_{4x}^{(3)} = \frac{E}{L}(u_4 - u_3) = \frac{10\times10^6}{10} \times 10^{-4}\,(-6.67)$$

$$\Rightarrow \ \sigma_{4x}^{(3)} = -666.7 \text{ psi (C)}$$

So $\quad \sigma^{(1)} = \sigma^{(2)} = -333.3$ psi (T) and

$\quad \sigma^{(3)} = 666.7$ psi (C)

3.2



Element 1–2

$$[k_{1-2}] = 84\times10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Element 2–3

$$[k_{2-3}] = 84\times10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\{F\} = [K]\,\{d\}$ and $u_1 = 0$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = -5000 \end{Bmatrix} = 84\times10^6 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \end{Bmatrix}$$

$$\Rightarrow \ 2u_2 - u_3 = 0 \ \Rightarrow \ u_3 = 2u_2 \qquad (1)$$

$$\Rightarrow \ -5000 = 84\times10^6\,[-u_2 + u_3] \qquad (2)$$

Substituting (1) in (2), we have

$$\frac{-5000}{84\times10^6} = -u_2 + 2u_2 \ \Rightarrow \ u_2 = -0.595\times10^{-4}\text{ m}$$

$$\Rightarrow \ u_3 = -1.19\times10^{-4}\text{ m}$$

Element 1–2

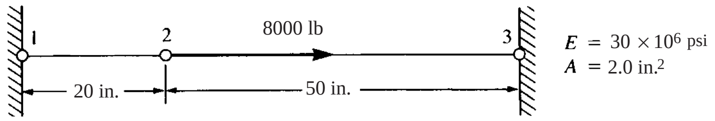
$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 84\times10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ -0.595\times10^{-4} \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = 5000\text{ N} \\ f_{2x}^{(1)} = -5000\text{ N} \end{matrix}$$

Element 2–3

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 84 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} -0.595 \times 10^{-4} \\ -1.19 \times 10^{-4} \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = 5000 \text{ N} \\ f_{3x}^{(2)} = -5000 \text{ N} \end{matrix}$$

$$F_{1x} = 84 \times 10^6 \; [1 \; -1 \;\; 0] \begin{Bmatrix} 0 \\ -0.595 \times 10^{-4} \\ -1.19 \times 10^{-4} \end{Bmatrix} \Rightarrow F_{1x} = 5000\text{N}$$

3.3



$$[k_{1-2}] = 3 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k_{2-3}] = 1.2 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K] = 10^6 \begin{bmatrix} 3 & -3 & 0 \\ -3 & 3+1.2 & -1.2 \\ 0 & -1.2 & 1.2 \end{bmatrix}$$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} = 8000 \\ F_{3x} \end{Bmatrix} = 10^6 \begin{bmatrix} 3 & -3 & 0 \\ -3 & 4.2 & -1.2 \\ 0 & -1.2 & 1.2 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \end{Bmatrix}$$

$$\Rightarrow \; u_2 = 1.905 \times 10^{-3} \text{ in.}$$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = 10^6 \begin{bmatrix} 3 & -3 & 0 \\ -3 & 4.2 & -1.2 \\ 0 & -1.2 & 1.2 \end{bmatrix} \begin{Bmatrix} 0 \\ 1.905 \times 10^{-3} \\ 0 \end{Bmatrix}$$

$$\Rightarrow \; F_{1x} = -5715 \text{ lb}$$

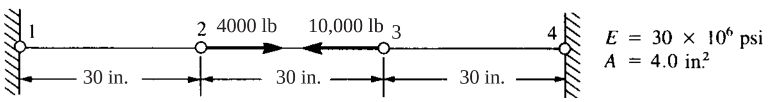
$$F_{2x} = 8000 \text{ lb}$$

$$F_{3x} = -2286 \text{ lb}$$

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 3 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 1.905 \times 10^{-3} \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = -5715 \text{ lb} \\ f_{2x}^{(1)} = 5715 \text{ lb} \end{matrix}$$

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 1.2 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1.905 \times 10^{-3} \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = 2286 \text{ lb} \\ f_{3x}^{(2)} = -2286 \text{ lb} \end{matrix}$$

3.4



$$[k_{1-2}] = 4 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k_{2-3}] = 4 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k_{3\text{–}4}] = 4\times10^6\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix}F_{1x}=0\\ F_{2x}=4000\\ F_{3x}=-10000\\ F_{4x}=0\end{Bmatrix} = 4\times10^6\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_1=0\\ u_2\\ u_3\\ u_4=0\end{Bmatrix}$$

$$\Rightarrow\ u_2 = -1.667\times10^{-4}\text{ in.}$$
$$u_3 = -1.333\times10^{-3}\text{ in.}$$

$$\begin{Bmatrix}F_{1x}\\ F_{2x}\\ F_{3x}\\ F_{4x}\end{Bmatrix} = 4\times10^6\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_1=0\\ u_2=-1.667\times10^{-4}\\ u_3=-1.333\times10^{-3}\\ u_4=0\end{Bmatrix}$$

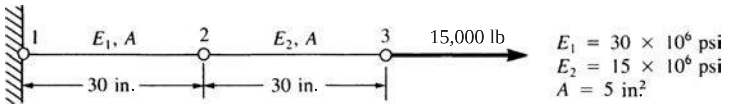
$$\Rightarrow\ F_{1x} = 666.7\text{ lb}$$
$$F_{2x} = 4000\text{ lb}$$
$$F_{3x} = -10000\text{ lb}$$
$$F_{4x} = 5333.3\text{ lb}$$

$$\begin{Bmatrix}f_{1x}\\ f_{2x}\end{Bmatrix} = 4\times10^6\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ -1.667\times10^{-4}\end{Bmatrix} \Rightarrow \begin{matrix}f_{1x}^{(1)} = 666.7\text{ lb}\\ f_{2x}^{(1)} = -666.7\text{ lb}\end{matrix}$$

$$\begin{Bmatrix}f_{2x}\\ f_{3x}\end{Bmatrix} = 4\times10^6\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}-1.667\times10^{-4}\\ -1.333\times10^{-3}\end{Bmatrix} \Rightarrow \begin{matrix}f_{2x}^{(2)} = 4666.67\text{ lb}\\ f_{3x}^{(2)} = -4666.7\text{ lb}\end{matrix}$$

$$\begin{Bmatrix}f_{3x}\\ f_{4x}\end{Bmatrix} = 4\times10^6\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}-1.333\times10^{-3}\\ 0\end{Bmatrix} \Rightarrow \begin{matrix}f_{3x}^{(3)} = -5333.3\text{ lb}\\ f_{4x}^{(3)} = 5333.3\text{ lb}\end{matrix}$$

3.5



Element 1–2

$$[k_{1\text{–}2}] = 5\times10^6\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Element 2–3

$$[k_{2\text{–}3}] = 5\times10^6\begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{-1}{2} & \frac{1}{2}\end{bmatrix}$$

$$\text{Global }[K] = 5\times10^6\begin{bmatrix}1 & -1 & 0\\ -1 & \frac{3}{2} & \frac{-1}{2}\\ 0 & -\frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

$$\{F\} = [K]\,\{d\}\text{ and }u_1 = 0$$

$$\begin{Bmatrix}F_{1x}=?\\ F_{2x}=0\\ F_{3x}=15000\end{Bmatrix} = 5\times10^6\begin{bmatrix}1 & -1 & 0\\ -1 & \frac{3}{2} & \frac{-1}{2}\\ 0 & \frac{-1}{2} & \frac{1}{2}\end{bmatrix}\begin{Bmatrix}u_1=0\\ u_2\\ u_3\end{Bmatrix}$$

$$\Rightarrow\ 0 = \frac{3}{2}\ u_2 - \frac{1}{2}\ u_3 \ \Rightarrow\ u_3 = 3\,u_2 \qquad (1)$$

$$\Rightarrow 15000 = 5 \times 10^6\,[\frac{1}{2}\ u_2 + \frac{1}{2}\ u_3] \qquad (2)$$

Substituting (1) in (2)

$$\frac{(2 \times 15000)}{5 \times 10^6} = -\,u_2 + 3\,u_2$$

$$\Rightarrow \qquad u_2 = 0.00075 \text{ in.}$$

$$\Rightarrow \qquad u_3 = 3\,(0.00075) \quad \Rightarrow \quad u_3 = 0.00225 \text{ in.}$$

Element 1–2

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 5 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.00075 \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x} = -15000 \text{ lb} \\ f_{2x} = 15000 \text{ lb} \end{matrix}$$

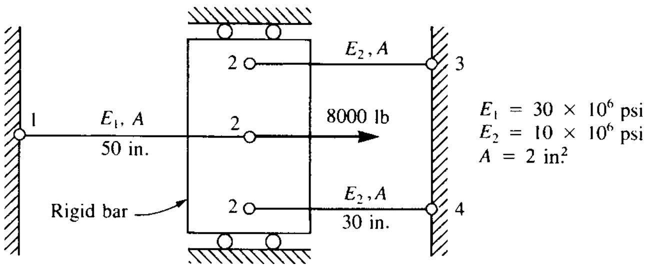
Element 2–3

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 5 \times 10^6 \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} 0.00075 \\ 0.00225 \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x} = -15000 \text{ lb} \\ f_{3x} = 15000 \text{ lb} \end{matrix}$$

$$F_{1x} = 5 \times 10^6\,[1 \quad -1 \quad 0] \begin{Bmatrix} 0 \\ 0.00075 \\ 0.00225 \end{Bmatrix}$$

$$\Rightarrow \qquad F_{1x} = -\,15000 \text{ lb}$$

3.6



$$[k_{1-2}] = 10^6 \begin{bmatrix} 1.2 & -1.2 \\ -1.2 & 1.2 \end{bmatrix}$$

$$[k_{2-3}] = [k_{2-4}] = 10^6 \begin{bmatrix} 0.667 & -0.667 \\ -0.667 & 0.667 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 8000 \\ F_{3x} = ? \\ F_{4x} = ? \end{Bmatrix} = 10^6 \begin{bmatrix} 1.2 & -1.2 & 0 & 0 \\ -1.2 & 2.533 & -0.667 & -0.667 \\ 0 & -0.667 & 0.667 & 0 \\ 0 & -0.667 & 0 & 0.667 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = 0 \\ u_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \qquad u_2 = 3.16 \times 10^{-3} \text{ in.}$$

Reactions

$$F_{1x} = (-\,1.2 \times 10^6)\,(u_2) \quad \Rightarrow \quad F_{1x} = -\,3789.5 \text{ lb}$$

$$F_{3x} = (-\,0.667 \times 10^6)\,(u_2) \quad \Rightarrow \quad F_{3x} = -\,2105.25 \text{ lb}$$

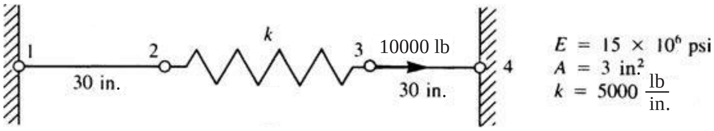
$$F_{4x} = (-0.667\times 10^6)\,(u_2) \;\Rightarrow\; F_{4x} = -2105.25 \text{ lb}$$

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 10^6 \begin{bmatrix} 1.2 & -1.2 \\ -1.2 & 1.2 \end{bmatrix} \begin{Bmatrix} 0 \\ 3.16\times 10^{-3} \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = -3789.5 \text{ lb} \\ f_{2x}^{(1)} = 3789.5 \text{ lb} \end{matrix}$$

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 10^6 \begin{bmatrix} 0.667 & -0.667 \\ -0.667 & 0.667 \end{bmatrix} \begin{Bmatrix} 3.16\times 10^{-3} \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = 2105.25 \text{ lb} \\ f_{3x}^{(2)} = -2105.25 \text{ lb} \end{matrix}$$

$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = 10^6 \begin{bmatrix} 0.667 & -0.667 \\ -0.667 & 0.667 \end{bmatrix} \begin{Bmatrix} 3.16\times 10^{-3} \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} f_{3x}^{(3)} = 2105.25 \text{ lb} \\ f_{4x}^{(2)} = -2105.25 \text{ lb} \end{matrix}$$

3.7



$$[k_{1-2}] = [k_{3-4}] = 1.5\times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k_{2-3}] = 5000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{2x} = 0 \\ F_{3x} = 10000 \\ F_{4x} = 0 \end{Bmatrix} = 10^3 \begin{bmatrix} 1500 & -1500 & 0 & 0 \\ -1500 & 1505 & -5 & 0 \\ 0 & -5 & 1505 & -1500 \\ 0 & 0 & -1500 & 1500 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = 0 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 10000 \end{Bmatrix} = 10^3 \begin{bmatrix} 1505 & -5 \\ -5 & 1505 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \Rightarrow \begin{matrix} u_2 = 2.21\times 10^{-5} \text{ in.} \\ u_3 = 6.65\times 10^{-3} \text{ in.} \end{matrix}$$

Reactions

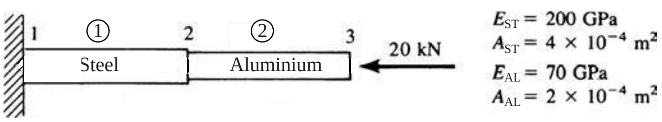
$$F_{1x} = (-1500\times 10^3)\,(u_2) \;\Rightarrow\; F_{1x} = -33.15 \text{ lb}$$
$$F_{4x} = (-1500\times 10^3)\,(u_3) \;\Rightarrow\; F_{4x} = -9975 \text{ lb}$$

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 1.5\times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 2.21\times 10^{-5} \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = -33.15 \text{ lb} \\ f_{2x}^{(1)} = 33.15 \text{ lb} \end{matrix}$$

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 5000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 2.21\times 10^{-5} \\ 6.65\times 10^{-3} \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = -33.15 \text{ lb} \\ f_{3x}^{(2)} = 33.15 \text{ lb} \end{matrix}$$

$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = 1.5\times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 6.65\times 10^{-3} \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} f_{3x}^{(3)} = 9975 \text{ lb} \\ f_{4x}^{(3)} = -9975 \text{ lb} \end{matrix}$$

3.8



$$[k^{(1)}] = \frac{(4\times10^{-4}\,\text{m}^2)(200\times10^6\,\frac{\text{kN}}{\text{m}^2})}{1\,\text{m}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$[k^{(1)}] = 800\times10^2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\frac{\text{kN}}{\text{m}}$$

$$[k^{(2)}] = \frac{(2\times10^{-4}\,\text{m}^2)(70\times10^6\,\frac{\text{kN}}{\text{m}^2})}{1\,\text{m}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$[k^{(2)}] = 140\times10^2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\frac{\text{kN}}{\text{m}}$$

$$\begin{Bmatrix}F_{1x}=0\\ F_{2x}\\ F_{3x}=-20\text{ kN}\end{Bmatrix} = 10^2\begin{bmatrix}800 & -800 & 0\\ -800 & 940 & -140\\ 0 & -140 & 140\end{bmatrix}\begin{Bmatrix}u_1=0\\ u_2\\ u_3\end{Bmatrix}$$

$$\Rightarrow\quad 0 = 10^2\,(940\,u_2 - 140\,u_3)\quad\Rightarrow\quad u_3 = 6.741\,u_2 \qquad (1)$$

$$\Rightarrow -20000 = 10^2\,(-140\,u_2 + 140\,u_3) \qquad (2)$$

Substituting (1) into (2)

$$\Rightarrow\quad -20000 = 10^2\,(-140\,u_2 + 140\,(6.714)\,u_2)$$

$$\Rightarrow\quad u_2 = -0.25\times10^{-3}\text{ m}$$

$$\Rightarrow\quad u_3 = -1.678\times10^{-3}\text{ m}$$

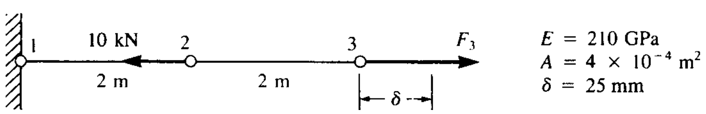
$$F_{1x} = 10^2\,(-800\times(-0.25\times10^{-3}))$$

$$\Rightarrow\quad F_{1x} = 20\text{ kN}$$

$$\begin{Bmatrix}f_{1x}\\ f_{2x}\end{Bmatrix} = 800\times10^2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ -0.25\times10^{-3}\end{Bmatrix}\Rightarrow \begin{matrix}f_{1x}^{(1)} = 20\text{ kN}\\ f_{2x}^{(1)} = -20\text{ kN}\end{matrix}$$

$$\begin{Bmatrix}f_{2x}\\ f_{3x}\end{Bmatrix} = 140\times10^2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}-0.25\times10^{-3}\\ -1.678\times10^{-3}\end{Bmatrix}\Rightarrow \begin{matrix}f_{2x}^{(2)} = 20\text{ kN}\\ f_{3x}^{(2)} = -20\text{ kN}\end{matrix}$$

3.9



$$[k_{1-2}] = [k_{2-3}] = 4.2\times10^4\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\frac{\text{kN}}{\text{m}}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{2x} = -10\text{ kN} \\ F_{3x} = ? \end{Bmatrix} = 4.2 \times 10^4 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = 0.025\text{ m} \end{Bmatrix}$$

$$\Rightarrow \frac{-10\text{ kN}}{4.2 \times 10^4} = 2\,u_2 - 1(0.025)$$

$$\Rightarrow \quad u_2 = 0.01238\text{ m}$$

$$F_{3x} = 4.2 \times 10^4\,[-1\ 1] \begin{Bmatrix} 0.01238 \\ 0.025 \end{Bmatrix} \Rightarrow F_{3x} = 530\text{ kN}$$

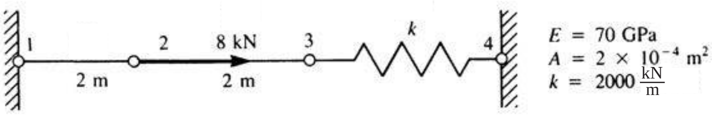
$$F_{1x} = 4.2 \times 10^4\,[1\ -1] \begin{Bmatrix} 0 \\ 0.01238 \end{Bmatrix} \Rightarrow F_{1x} = -520\text{ kN}$$

Element forces

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = 4.2 \times 10^4 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.01238 \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = -520\text{ kN} \\ f_{2x}^{(1)} = 520\text{ kN} \end{matrix}$$

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = 4.2 \times 10^4 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.01238 \\ 0.025 \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = -530\text{ kN} \\ f_{3x}^{(2)} = 530\text{ kN} \end{matrix}$$

3.10



$$[k^{(1)}] = [k^{(2)}] = 7000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(2)}] = 2000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 8\text{kN} \\ F_{3x} = 0 \\ F_{4x} = ? \end{Bmatrix} = 10^3 \begin{bmatrix} 7 & -7 & 0 & 0 \\ -7 & 14 & -7 & 0 \\ 0 & -7 & 9 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad 8 = 10^3\,[14\,u_2 - 7\,u_3] \qquad (1)$$

$$0 = 10^3\,[-7\,u_2 + 9\,u_3]$$

$$\Rightarrow \quad u_3 = \frac{7}{9}\,u_2 \qquad (2)$$

Substituting (2) into (1)

$$\Rightarrow \frac{8}{10^3} = 14\,u_2 - 7 \times \frac{7}{9}\,u_2$$

$$\Rightarrow \quad u_2 = 0.9351 \times 10^{-3}\text{ m}$$

$$\Rightarrow \quad u_3 = 0.7273 \times 10^{-3}\text{ m}$$

Element (1)

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 7 \times 10^3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.9351 \times 10^{-3} \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{Bmatrix} -6.546 \\ 6.546 \end{Bmatrix} \text{kN}$$

Element (2)

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 7 \times 10^3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.9351 \times 10^{-3} \\ 0.7273 \times 10^{-3} \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = \begin{Bmatrix} 1.455 \\ -1.455 \end{Bmatrix} \text{kN}$$

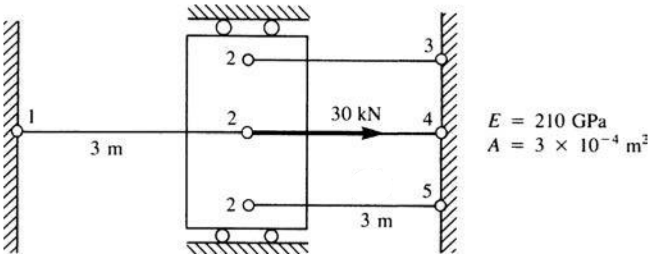
Element (3)

$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = 2 \times 10^3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.7273 \times 10^{-3} \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = \begin{Bmatrix} 1.455 \\ -1.455 \end{Bmatrix} \text{kN}$$

$$F_{1x} = 10^3 \, [7 \;\; -7] \begin{Bmatrix} 0 \\ 0.9351 \times 10^{-3} \end{Bmatrix} = F_{1x} = -6.546 \text{ kN}$$

$$F_{4x} = 10^3 \, [-2 \;\; 2] \begin{Bmatrix} 0.7273 \times 10^{-3} \\ 0 \end{Bmatrix} \Rightarrow F_{4x} = -1.455 \text{ kN}$$

3.11



$$[k_{1-2}] = [k_{2-3}] = [k_{2-4}] = [k_{2-5}] = 2.1 \times 10^7 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{2x} = 30 \text{ kN} \\ F_{3x} = 0 \\ F_{4x} = 0 \\ F_{5x} = 0 \end{Bmatrix} = 2.1 \times 10^7 \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \\ u_4 = 0 \\ u_5 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad u_2 = 3.572 \times 10^{-4} \text{ m}$$

Reactions

$$F_{1x} = (2.1 \times 10^7)(-1)(u_2) \Rightarrow F_{1x} = -7500 \text{ N}$$

$$F_{3x} = (2.1 \times 10^7)(-1)(u_2) \Rightarrow F_{3x} = -7500 \text{ N}$$

$$F_{4x} = (2.1 \times 10^7)(-1)(u_2) \Rightarrow F_{4x} = -7500 \text{ N}$$

$$F_{5x} = (2.1 \times 10^7)(-1)(u_2) \Rightarrow F_{5x} = -7500 \text{ N}$$

Element forces

$$f_{1-2} = -f_{2-3} = -f_{2-4} = -f_{2-5} = (2.1 \times 10^7)(u_2)$$

$$\Rightarrow \quad f_{1-2} = 7500 \text{ N}$$

$$f_{2-3} = -7500 \text{ N}$$
$$f_{2-4} = -7500 \text{ N}$$
$$f_{2-5} = -7500 \text{ N}$$

3.12

$$\frac{P}{A(x)} = E\frac{du}{dx}$$

$$u = \int \frac{P}{A(x)E}\ dx$$

$$u = \int \frac{P}{A_0\left(1+\frac{x}{L}\right)E}\,dx$$

$$= \int \frac{PL}{A_0 L\left(1+\frac{x}{L}\right)E}\,dx$$

$$= \int \frac{PL}{A_0\,(L+x)\,E}\,dx$$

$$= \int \frac{PL}{A_0 E\,u}\,du \quad \text{(Change variable } u = L + x \text{ and } du = dx\text{)}$$

$$= \frac{PL}{A_0 E}\int \frac{1}{u}\,du$$

$$= \frac{PL}{A_0 E}\ ln\ u$$

$$\Rightarrow \quad u = \frac{PL}{A_0 E}\ ln\ (L + x)$$

$$u = \frac{(-1000)\,(20)}{2\times 10\times 10^6}\ ln\ (20 + x)$$

$$u = -10^{-3}\ ln\ (20 + x)$$

$$u(x=0) = (-\,ln\ 20)\times 10^{-3}$$
$$= -\,2.996\times 10^{-3} \text{ in.}$$

$$u(x=10) = (-ln\ (20+10))\ (-10^{-3})$$
$$= -\,3.401\times 10^{-3} \text{ in.}$$

Two elements

$$A\left(\frac{L}{4}\right) = A_0\left(1+\frac{\frac{L}{4}}{L}\right) = A_0\left(1+\frac{1}{4}\right) = \frac{5}{4}A_0$$

$$A\left(\frac{3}{4}L\right) = A_0\left(1+\frac{\frac{3}{4}L}{L}\right) = A_0\ \left(1+\frac{3}{4}\right) = \frac{7}{4}A_0$$

$$[k^{(1)}] = \frac{5}{4}\,\frac{A_0 E}{\frac{L}{2}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(2)}] = \frac{7}{4}\,\frac{A_0 E}{\frac{L}{2}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{Bmatrix} -P \\ 0 \\ F_{3x} \end{Bmatrix} = \frac{A_0E}{\frac{L}{2}} \begin{bmatrix} \frac{5}{4} & \frac{-5}{4} & 0 \\ \frac{-5}{4} & \frac{5}{4}+\frac{7}{4} & \frac{-7}{4} \\ 0 & \frac{-7}{4} & \frac{7}{4} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_{3=0} \end{Bmatrix}$$

$$\Rightarrow \quad \frac{A_0E}{\frac{L}{2}} \left(\frac{5}{4}u_1 - \frac{5}{4}u_2\right) = -P \tag{1}$$

$$\Rightarrow \quad \frac{A_0E}{\frac{L}{2}} \left(\frac{-5}{4}u_1 + 3u_2\right) = 0$$

$$\Rightarrow \quad u_2 = \frac{5}{4\times 3}u_1 = \frac{5}{12}u_1 \tag{2}$$

Substituting (2) into (1)

$$\Rightarrow \quad \frac{A_0E}{\frac{L}{2}} \left(\frac{5}{4}u_1 - \frac{5}{4}\left(\frac{5}{12}u_1\right)\right) = -P$$

$$\Rightarrow \quad \left[\frac{12}{12}\left(\frac{5}{4}\right) - \frac{5}{4}\left(\frac{5}{12}\right)\right]u_1 = \frac{-PL}{2A_0E}$$

$$\Rightarrow \quad \left[\frac{60-25}{48}\right]u_1 = \frac{-PL}{2A_0E}$$

$$\Rightarrow \quad u_1 = \frac{-PL}{\cancel{2}A_0E}\;\frac{\overset{24}{\cancel{48}}}{35}$$

$$\Rightarrow \quad u_1 = \frac{-PL}{A_0E}\;\frac{24}{35}$$

$$\Rightarrow \quad u_2 = \frac{5}{12}\;\frac{24}{35}\left(\frac{-PL}{A_0E}\right)$$

$$\Rightarrow \quad u_2 = -\frac{2}{7}\;\frac{PL}{A_0E}$$

Now $\quad A_0 = 2 \text{ in.}^2, L = 20 \text{ in.}, E = 10\times 10^6 \text{ psi}$

$P = 1000 \text{ lb}$

$$u_1 = -\frac{(1000)(20)}{2(10\times 10^6)} \times \frac{24}{35}$$

$$\Rightarrow \quad u_1 = -0.6857\times 10^{-3} \text{ in.}$$

$$\Rightarrow \quad u_2 = \frac{5}{12}\left(-0.6857\times 10^{-3}\right)$$

$$\Rightarrow \quad u_2 = -0.2857\times 10^{-3} \text{ in.}$$

One element

$$A = A_0\left(1+\frac{\frac{L}{2}}{L}\right) = A_0\left(1+\frac{1}{2}\right) = \frac{3}{2}A_0$$

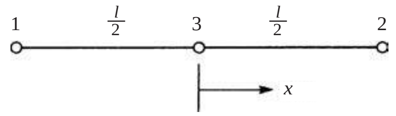
$$\begin{Bmatrix} -P \\ 0 \end{Bmatrix} = \frac{\frac{3}{2}A_0E}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad u_1 = \frac{-PL}{\frac{3}{2}A_0E}$$

$$\Rightarrow \quad u_1 = -\frac{2}{3}\frac{(1000)(20)}{(2)(10\times 10^6)}$$

$$\Rightarrow \quad u_1 = -0.667\times 10^{-3} \text{ in.}$$

3.13



$$u = a_1 + a_2x + a_3x^2 \tag{A}$$

$$u(0) = u_2 = a_1 \tag{1}$$

$$u(-\frac{l}{2}) = u_1 = u_2 + a_2(-\frac{l}{2}) + a_3(-\frac{l}{2})^2 \tag{2}$$

$$u(\frac{l}{2}) = u_3 = u_2 + a_2(\frac{l}{2}) + a_3(\frac{l}{2})^2 \tag{3}$$

Solving for a_2 and a_3 from (2) and (3)

$$a_2 = \frac{u_3 - u_1}{l},\ a_3 = \frac{2(u_1 + u_3 - 2u_2)}{l^2} \tag{4}$$

By (1) and (4) into (A)

$$u = u_2 + \left(\frac{u_3 - u_1}{l}\right)x + \frac{2(u_1 + u_3 - 2u_2)}{l^2}x^2 \tag{5}$$

$$u = [N]\{d\} \tag{6}$$

$$u = \begin{bmatrix} \frac{-x}{l} + \frac{2x^2}{l^2} & 1 - \frac{4x^2}{l^2} & \frac{x}{l} + \frac{2x^2}{l^2} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{7}$$

$$\{\in\} = \frac{\partial u}{\partial x} = [B]\{d\} = \frac{\partial N}{\partial x}\{d\} \tag{8}$$

Using (7) in (8)

$$\{\in\} = \begin{bmatrix} -\frac{1}{l} + \frac{4x}{l^2} & \frac{-8x}{l^2} & \frac{1}{l} + \frac{4x}{l^2} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{9}$$

$$\therefore\ [B] = \begin{bmatrix} -\frac{1}{l} + \frac{4x}{l^2} & \frac{-8x}{l^2} & \frac{1}{l} + \frac{4x}{l^2} \end{bmatrix} \tag{10}$$

$$[K] = A\int_{-l/2}^{l/2} [B^T]\,E\,[B]\,dx \tag{11}$$

A = cross sectional area of the bar

E = Young's Modulus of the bar

3.14 Given $u = a + bx^2$ for 2 noded bar

$$\in = \frac{du}{dx} = 2bx$$

$$u(0) = u_1 = a$$

$$u(L) = u_2 = u_1 + bL^2$$

$$\therefore \quad b = \frac{u_2 - u_1}{L^2}$$

$$u = u_1 + \left[\frac{u_2 - u_1}{L^2}\right]x^2$$

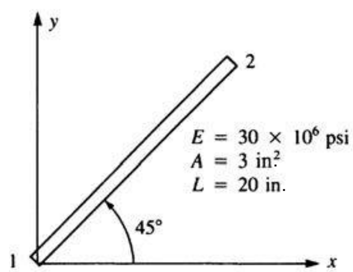
This displacement function allows for a rigid body displacement as the $a = u_1$ term does this. Also should allow for constant strain, but have $\in = 2bx$ or a linear strain. Therefore, not complete. Need to complete 2nd degree polynomial and 3rd node for compatible function.

Try $\quad u = a_1 + a_2\,x + a_3\,x^2$

$$\frac{du}{dx} = a_2 + 2a_3x$$

'a_2' allows for constant strain term.

3.15 (a)

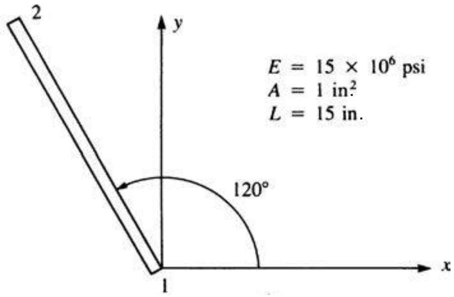


$$C = \frac{1}{\sqrt{2}}, S = \frac{1}{\sqrt{2}}$$

$$[K] = \frac{EA}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ & S^2 & -CS & -S^2 \\ & & C^2 & CS \\ & & & S^2 \end{bmatrix}$$

$$[K] = 2.25 \times 10^6 \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \frac{\text{lb}}{\text{in.}}$$

(b)

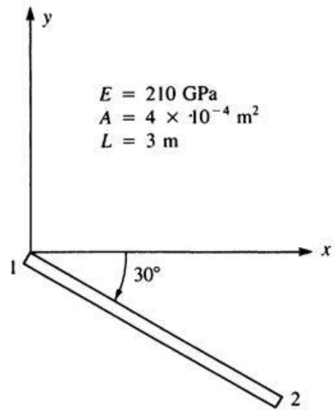


$$C = \frac{-1}{2},\ S = \frac{\sqrt{3}}{2}$$

$$[K] = \frac{15\times10^6\times1}{15}\begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}$$

$$[K] = \frac{10^6}{4}\begin{bmatrix} 1 & -\sqrt{3} & -1 & \sqrt{3} \\ -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & \sqrt{3} & 1 & -\sqrt{3} \\ \sqrt{3} & -3 & -\sqrt{3} & 3 \end{bmatrix}\frac{\text{lb}}{\text{in.}}$$

(c)

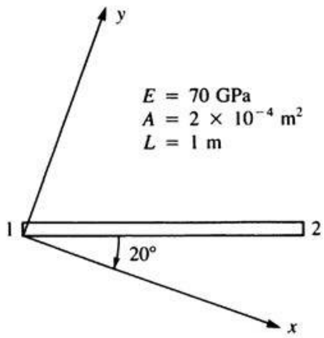


$$C = \frac{\sqrt{3}}{2},\quad S = -\frac{1}{2}$$

$$[K] = \frac{(210\times10^6)(4\times10^{-4})}{3}\begin{bmatrix} \frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$$

$$\underline{K} = 7000 \begin{bmatrix} 3 & \sqrt{3} & 3 & \sqrt{3} \\ \sqrt{3} & 1 & \sqrt{3} & 1 \\ 3 & \sqrt{3} & 3 & \sqrt{3} \\ \sqrt{3} & 1 & \sqrt{3} & 1 \end{bmatrix} \frac{\text{kN}}{\text{m}}$$

(d)



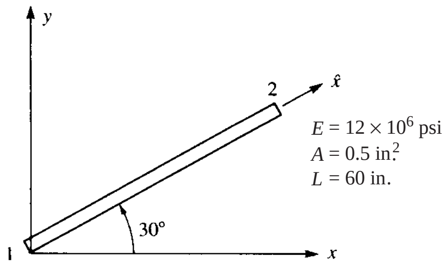
$C = 0.9397$ $\quad C^2 = 0.883$ $\quad CS = 0.321$

$S = 0.3420$ $\quad S^2 = 0.117$

$$[K] = \frac{(70\times10^4)(2\times10^{-4})}{1}\begin{bmatrix} 0.883 & 0.321 & -0.883 & -0.321 \\ 0.321 & 0.883 & -0.321 & -0.883 \\ -0.883 & -0.321 & 0.883 & 0.321 \\ -0.321 & -0.883 & 0.321 & 0.883 \end{bmatrix}$$

$$[K] = 1.4\times10^7 \begin{bmatrix} 0.883 & 0.321 & -0.883 & -0.321 \\ 0.321 & 0.883 & -0.321 & -0.883 \\ -0.883 & -0.321 & 0.883 & 0.321 \\ -0.321 & -0.883 & 0.321 & 0.883 \end{bmatrix} \frac{\text{N}}{\text{m}}$$

3.16 (a)



$C = 0.866$ $\quad u_1 = 0.5$ in. $\quad v_1 = 0.0$ in.

$S = 0.5$ $\quad u_2 = 0.25$ in. $\quad v_2 = 0.75$ in.

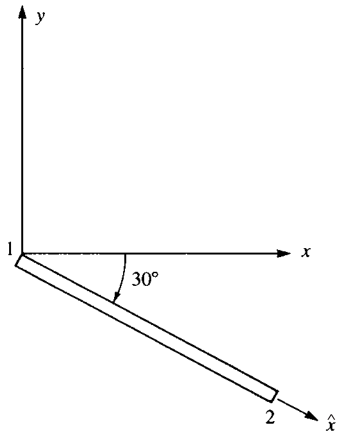
$$u_1' = u_1\,C + v_1\,S = 0.5\,(0.866) + (0.0)\,(0.5)$$

$$\Rightarrow \quad u_1' = 0.433 \text{ in.}$$

$$u_2' = u_2\,C + v_2\,S = (0.25)\,(0.866) + (0.75)\,(0.5)$$

$$u_2' = 0.592 \text{ in.}$$

(b)



$$C = \frac{\sqrt{3}}{2}, \quad S = -\frac{1}{2}$$

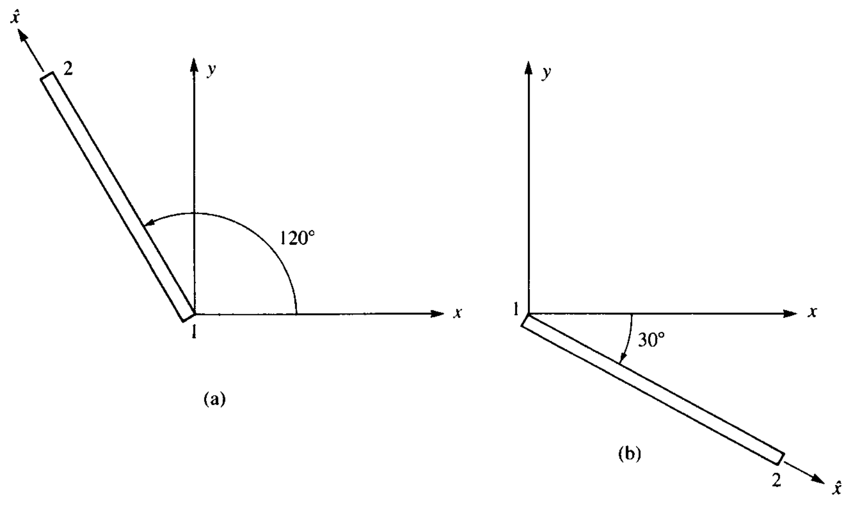
$$u_1' = u_1\,C + v_1\,S = \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + (0)\left(-\frac{1}{2}\right)$$

$$\Rightarrow \quad u_1' = 0.433 \text{ in.}$$

$$u_2' = u_2\,C + v_2\,S = \left(\frac{1}{4}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{3}{4}\right)\left(-\frac{1}{2}\right)$$

$$\Rightarrow \quad u_2' = -0.1585 \text{ in.}$$

3.17



$u_1 = 0.0$ $\quad u_2 = 5.0$ mm $\quad E = 210$ GPa

$v_1 = 2.5$ mm $\quad v_2 = 3.0$ mm $\quad A = 10 \times 10^{-4}$ m^2

$L = 3$ m

(a) We know that $\{d'\} = [T]\,\{d\}$

$$[T] = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix}$$

$C = \cos 120° = -0.5$, $S = \sin 120° = 0.866$

$$\begin{Bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{Bmatrix} = \begin{bmatrix} -0.5 & 0.866 & 0 & 0 \\ -0.866 & -0.5 & 0 & 0 \\ 0 & 0 & -0.5 & 0.866 \\ 0 & 0 & -0.866 & -0.5 \end{bmatrix} \begin{Bmatrix} 0.0 \\ 0.0025 \\ 0.005 \\ 0.003 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{Bmatrix} = \begin{Bmatrix} 0.002165 \\ -0.00125 \\ 0.000098 \\ -0.00583 \end{Bmatrix} \text{ m} = \begin{Bmatrix} 2.165 \\ -1.25 \\ 0.098 \\ -5.830 \end{Bmatrix} \text{ mm}$$

(b) $C = \cos(-30°) = 0.866$, $S = -0.5$

$$\begin{Bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{Bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 & 0 \\ 0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 0.866 & -0.5 \\ 0 & 0 & 0.5 & 0.866 \end{bmatrix} \begin{Bmatrix} 0.0 \\ 0.0025 \\ 0.005 \\ 0.003 \end{Bmatrix}$$

$$\begin{Bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{Bmatrix} = \begin{Bmatrix} -1.25 \\ 2.165 \\ 3.03 \\ 5.098 \end{Bmatrix} \text{ mm}$$

3.18

(a) $$\sigma = \frac{E}{L} [-C \quad -S \quad C \quad S] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$C = \frac{\sqrt{2}}{2}, \quad S = \frac{\sqrt{2}}{2}, \quad E = 30 \times 10^6 \text{ psi}, \quad L = 60 \text{ in.}$$

$$\sigma = \frac{30 \times 10^6}{60} \left[-\frac{\sqrt{2}}{2} \quad -\frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \right] \begin{Bmatrix} 0 \\ 0 \\ 0.01 \\ 0.02 \end{Bmatrix}$$

$\Rightarrow \sigma = 10600$ psi

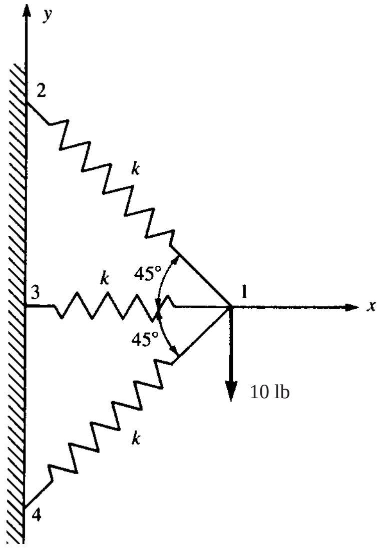
(b) $C = \frac{\sqrt{3}}{2}$, $S = \frac{1}{2}$, $E = 210$ GPa, $L = 3$ m

$$\sigma = \left[-\tfrac{\sqrt{3}}{2} \quad -\tfrac{1}{2} \quad \tfrac{\sqrt{3}}{2} \quad \tfrac{1}{2}\right]\begin{Bmatrix} 0.25 \\ 0 \\ 1.00 \\ 0 \end{Bmatrix} \times 10^{-3} \times \frac{210 \times 10^{6}}{3}$$

$$\Rightarrow \sigma = 45470 \ \frac{\text{kN}}{\text{m}^2}$$

$$\Rightarrow \sigma = 45.47 \text{ MPa}$$

3.19



$$\{f\} = \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ f_{4x} \\ f_{4y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10 \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ f_{4x} \\ f_{4y} \end{Bmatrix} \text{ and } \quad \{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

(a) For element 1–3; $\theta = 180°$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10 \\ f_{3x} \\ f_{3y} \end{Bmatrix} = K \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ 0 \\ 0 \end{Bmatrix}$$

For element 1–4; $\theta = 225°$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{4x} \\ f_{4y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10 \\ f_{4x} \\ f_{4y} \end{Bmatrix} = \frac{K}{2} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ 0 \\ 0 \end{Bmatrix}$$

For element 1–2; $\theta = 135°$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10 \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \frac{K}{2} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ 0 \\ 0 \end{Bmatrix}$$

Total K

$$[K] = K \begin{bmatrix} 2 & 0 & -\frac{1}{2} & \frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(b) Applying boundary conditions

$u_4 = v_4 = u_2 = v_2 = u_3 = v_3 = 0$

$[K]$ is reduced to

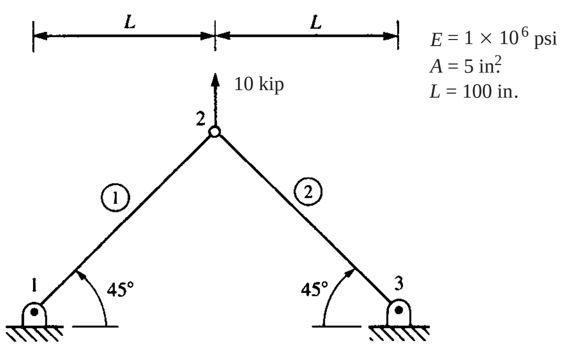
$$[K] = K \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \end{Bmatrix} = K \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} \Rightarrow \begin{Bmatrix} 0 \\ -10 \end{Bmatrix} = K \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$\Rightarrow u_1 = 0$

$$v_1 = \frac{-10}{K}$$

3.20



Element 1–2

$$C = \frac{\sqrt{2}}{2}; \quad S = \frac{\sqrt{3}}{2}; \quad L_{1-2} = \sqrt{2}\ L$$

$$[k_{1-2}] = \frac{A_1E_1}{L_{1-2}}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Element 2–3

$$C = \frac{\sqrt{2}}{2}; \quad S = -\frac{\sqrt{2}}{2}; \quad L_{2-3} = \sqrt{2}\ L$$

$$[k_{2-3}] = \frac{A_2E_2}{L_{2-3}}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Applying the boundary conditions

$$\begin{Bmatrix} 0 \\ 10 \end{Bmatrix} = \frac{1\times 5\times 10^3}{\sqrt{2}\times 100}\begin{bmatrix} \frac{1}{2}+\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \end{bmatrix}\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}$$

$$\Rightarrow u_2 = 0$$

$$v_2 = \frac{10\times\sqrt{2}\times 100}{5\times 10^3}$$

$$\Rightarrow v_2 = 0.283 \text{ in.}$$

$$\{f'\} = [k']\{d'\} = [k'][T^*]\{d\}$$

$$\begin{Bmatrix} f'_{1x} \\ f'_{2x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$= \frac{5\times 10^3}{\sqrt{2}\times 100}\begin{bmatrix} C & S & -C & -S \\ -C & -S & C & S \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.283 \end{Bmatrix}$$

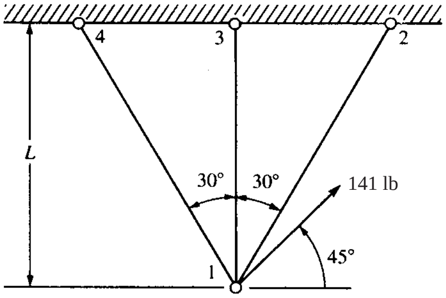
$$f'_{1x} = \frac{5\times 10^3}{\sqrt{2}\times 100}\left[-\frac{\sqrt{2}}{2}(0.283)\right] = -7.07 \text{ kips}$$

$$f'_{2x} = \frac{5\times 10^3}{\sqrt{2}\times 100}\left[\frac{\sqrt{2}}{2}(0.283)\right] = 7.07 \text{ kips}$$

$$\sigma_{1-2} = \frac{f'_{2x}}{A} = \frac{7.07 \text{ kips}}{5 \text{ in.}^2} \Rightarrow \sigma_{1-2} = 1414 \text{ psi (T)}$$

$$\sigma_{2-3} = \frac{f'_{3x}}{A} = \frac{7.07 \text{ kips}}{5 \text{ in.}^2} \Rightarrow \sigma_{2-3} = 1414 \text{ psi (T)}$$

3.21



Element 1–2

$$L_{1-2} = \frac{2}{\sqrt{3}}L;\ \theta = 60°$$

$$[k_{1-2}] = \frac{\sqrt{3}AE}{2L}\begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}$$

Element 1–3

$$L_{1-3} = L;\ \theta = 90°$$

$$[k_{1-2}] = \frac{AE}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Element 1–4

$$L_{1-4} = \frac{2}{\sqrt{3}}L;\ \theta = 120°$$

$$[k_{1-4}] = \frac{\sqrt{3}AE}{2L}\begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}$$

Applying the boundary conditions

$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$

$$[K] = \frac{AE}{L}\begin{bmatrix} \frac{\sqrt{3}}{2}(\frac{1}{4}) + 0 + \frac{\sqrt{3}}{2}(\frac{1}{4}) & \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{4}\right) + 0 + \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{4}\right) \\ \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{4}\right) + 0 + \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{4}\right) & \frac{\sqrt{3}}{2}(\frac{3}{4}) + 1 + \frac{\sqrt{3}}{2}(\frac{3}{4}) \end{bmatrix}$$

$$= \frac{AE}{L}\begin{bmatrix} \frac{\sqrt{3}}{4} & 0 \\ 0 & 1 + \frac{3\sqrt{3}}{4} \end{bmatrix}$$

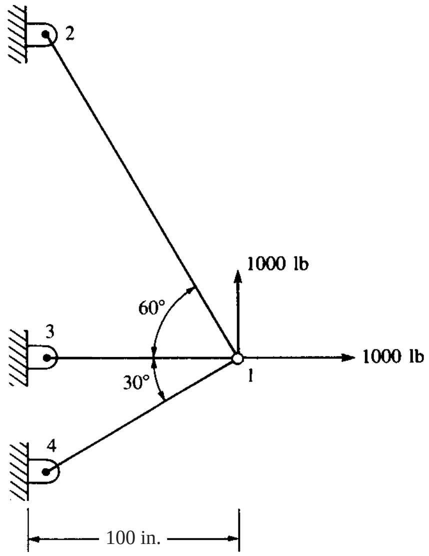
$$\begin{Bmatrix} F_{1x} \\ F_{1y} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} \frac{\sqrt{3}}{4} & 0 \\ 0 & 1 + \frac{3\sqrt{3}}{4} \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 100 \\ 100 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} \frac{\sqrt{3}}{4} & 0 \\ 0 & 1 + \frac{3\sqrt{3}}{4} \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$\Rightarrow u_1 = \frac{400L}{\sqrt{3}AE} \Rightarrow u_1 = \frac{231\,L}{AE}$$

$$v_1 = \frac{400\ L}{(3\sqrt{3}+4)AE} \Rightarrow v_1 = \frac{43.5\ L}{AE}$$

3.22



$$[K] = [T^T]\ [k']\ [T] = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

For element 1; $\theta = 120°$

$$[k^{(1)}] = \frac{AE}{2L}\begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}$$

For element 2; $\theta = 180°$

$$[k^{(2)}] = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For element 3; $\theta = 210°$

$$[k^{(3)}] = \frac{\sqrt{3}AE}{2\,L}\begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$$

Applying the boundary conditions

$$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$$

$$\begin{Bmatrix} 1000 \\ 1000 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} \frac{1}{8}+1+\frac{3\sqrt{3}}{8} & -\frac{\sqrt{3}}{8}+0+\frac{3}{8} \\ -\frac{\sqrt{3}}{8}+0+\frac{3}{8} & \frac{3}{8}+0+\frac{\sqrt{3}}{8} \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$\begin{Bmatrix} 1000 \\ 1000 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1.77 & 0.16 \\ 0.16 & 0.59 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$\Rightarrow u_1 = \frac{422\,(100)}{1\times 10\times 10^6} \Rightarrow u_1 = 0.00422 \text{ in.}$$

$$\Rightarrow v_1 = \frac{1570\,(100)}{1\times 10\times 10^6} \Rightarrow v_1 = 0.0157 \text{ in.}$$

Element (1)

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}\begin{Bmatrix} 422\,\frac{L}{AE} \\ 1570\,\frac{L}{AE} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow f_{2x} = 287 \text{ lb}$$

$$f_{2y} = -497 \text{ lb}$$

$$f^{(1)} = \sqrt{f_{2x}{}^2 + f_{2y}{}^2} \Rightarrow f^{(1)} = 5741 \text{ lb (C)}$$

$$\sigma^{(1)} = \frac{f^{(1)}}{A} = \frac{-5741}{\text{A}} \text{ psi}$$

$$\Rightarrow \sigma^{(1)} = -574 \text{ psi (C)}$$

Element (2)

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 422\frac{L}{AE} \\ 1570\frac{L}{AE} \\ 0 \\ 0 \end{Bmatrix}$$

$$f_{3x} = -422 \text{ lb}$$

$$f_{3y} = 0 \text{ lb}$$

$$f^{(2)} = \sqrt{f_{3x}{}^2 + f_{3y}{}^2} \Rightarrow f^{(2)} = 422 \text{ lb (T)}$$

$$\sigma^{(2)} = \frac{f^{(2)}}{A} = \frac{422}{A} \text{ psi}$$

$$\Rightarrow \sigma^{(2)} = 422 \text{ psi (T)}$$

Element (3)

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{4x} \\ f_{4y} \end{Bmatrix} = \frac{\sqrt{3}AE}{2\,L}\begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}\begin{Bmatrix} 422\,\frac{L}{AE} \\ 1570\,\frac{L}{AE} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow f_{4x} = -862.8 \text{ lb}$$

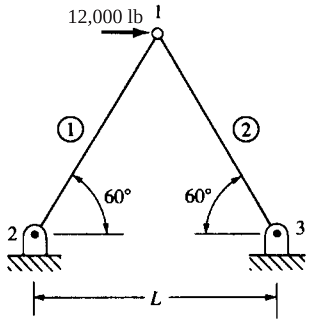
$$f_{4y} = -496 \text{ lb}$$

$$f^{(3)} = \sqrt{f_{4x}^{\ 2} + f_{4y}^{\ 2}} \Rightarrow f^{(3)} = 996 \text{ lb (T)}$$

$$\sigma^{(3)} = \frac{f^{(3)}}{A} = \frac{996}{A} \text{ psi (T)}$$

$$\sigma^{(3)} = 996 \text{ psi (T)}$$

3.23



$L = 100$ in.
$E = 10 \times 10^6$ psi
$A = 1$ in.^{2}

Element (1)

$$C = \frac{1}{2};\quad S = \frac{\sqrt{3}}{2}$$

$$[k^{(1)}] = \frac{AE}{L}\left[\begin{array}{cc|cc} \frac{1}{4} & \frac{\sqrt{3}}{4} & \multicolumn{2}{c}{-\lambda} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} & & \\ \hline & & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \multicolumn{2}{c|}{-\lambda} & \frac{\sqrt{3}}{4} & \frac{3}{4} \end{array}\right]$$

Element (2)

$$[k^{(2)}] = \frac{AE}{L}\left[\begin{array}{cc|cc} \frac{1}{4} & -\frac{\sqrt{3}}{4} & \multicolumn{2}{c}{-\lambda} \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & & \\ \hline & & \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \multicolumn{2}{c|}{-\lambda} & -\frac{\sqrt{3}}{4} & \frac{3}{4} \end{array}\right]$$

$$\{F\} = [K]\{d\}$$

$$\begin{Bmatrix} 12000 \\ 0 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$\Rightarrow 12000 = \frac{AE}{L}\,\frac{u_1}{2}$$

$$\Rightarrow u_1 = \frac{12000 \times 100 \times 2}{1 \times 10 \times 10^6}$$

$$\Rightarrow u_1 = 0.24 \text{ in.}$$

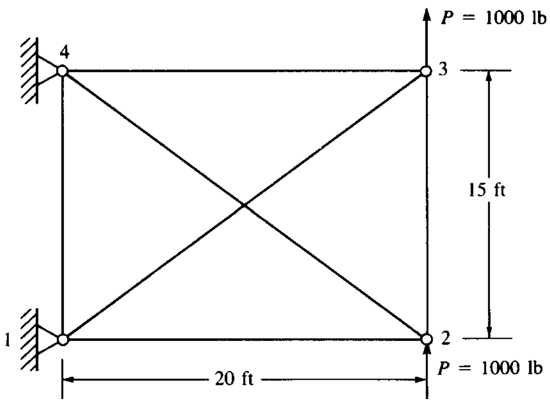
$$v_1 = 0$$

$$\sigma^{(1)} = [c']\{d\} = \frac{E}{L}\left[-\frac{1}{2} \quad -\frac{\sqrt{3}}{2} \quad \frac{1}{2} \quad \frac{\sqrt{3}}{2}\right]\begin{Bmatrix} u_2 = 0 \\ v_2 = 0 \\ u_1 = 0.24 \\ v_1 = 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(1)} = \frac{10\times 10^{6}}{10^{2}}\left[\frac{1}{2}(0.24)\right]$$

$$\Rightarrow \sigma^{(1)} = 12000 \text{ psi}$$

3.24



$L_{2-1} = 20'$	$L_{2-3} = 15'$	$L_{2-4} = 25'$
$\theta_{2-1} = 180°$	$\theta_{2-3} = 90°$	$\theta_{2-4} = 143.13°$
$\sin\theta_{2-1} = 0$	$\sin\theta_{2-3} = 1$	$\sin\theta_{2-4} = 1$
$\cos\theta_{2-1} = -1$	$\cos\theta_{2-3} = 0$	$\cos\theta_{2-4} = -0.8$

$L_{1-4} = 15'$	$L_{1-3} = 25'$	$L_{3-4} = 20'$
$\theta_{1-4} = 90°$	$\theta_{1-3} = 36.87°$	$\theta_{3-4} = 180°$
$\sin\theta_{1-4} = 1$	$\sin\theta_{1-3} = 0.6$	$\sin\theta_{3-4} = 0$
$\cos\theta_{1-4} = 0$	$\cos\theta_{1-3} = 0$	$\cos\theta_{3-4} = -1$

Boundary conditions $u_1 = v_1 = u_4 = v_4 = 0$

$$[k_{2-1}] = \frac{AE}{20}\begin{array}{c} \begin{array}{cccc} (2) & & (1) & \\ x & y & x & y \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}\begin{array}{c}(2)\\ \\(1)\end{array} ; \quad [k_{2-3}] = \frac{AE}{15}\begin{array}{c} \begin{array}{cccc} (2) & & (3) & \\ x & y & x & y \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}\begin{array}{c}(2)\\ \\(3)\end{array}$$

$$[k_{3-4}] = \frac{AE}{20}\begin{array}{c} \begin{array}{cccc} (3) & & (4) & \\ x & y & x & y \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}\begin{array}{c}(3)\\ \\(4)\end{array} ; \quad [k_{1-4}] = \frac{AE}{15}\begin{array}{c} \begin{array}{cccc} (1) & & (4) & \\ x & y & x & y \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}\begin{array}{c}(1)\\ \\(4)\end{array}$$

$$[k_{2-4}] = \frac{AE}{25}\begin{array}{c} \begin{array}{cccc} (2) & & (4) & \\ x & y & x & y \end{array} \\ \begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} \end{array}\begin{array}{c}(2)\\ \\(4)\end{array}$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} \multicolumn{2}{c}{(1)} & \multicolumn{2}{c}{(3)} \\ x & y & x & y \end{array} \\
[k_{1-3}] = \dfrac{AE}{25} & \begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix} \begin{array}{c} (1) \\ \\ (3) \\ \\ \end{array}
\end{array}
$$

$$\{F\} = [K]\,\{d\}$$

$$
\begin{Bmatrix} F_{1x} = ? \\ F_{1y} = ? \\ F_{2x} = 0 \\ F_{2y} = 1000 \\ F_{3x} = 0 \\ F_{3y} = 1000 \\ F_{4x} = ? \\ F_{4y} = ? \end{Bmatrix} = AE
\begin{bmatrix}
0.0756 & 0.0192 & -0.05 & 0 & 0.0256 & 0.0192 & 0 & 0 \\
0.0192 & 0.0811 & 0 & 0 & -0.0192 & -0.0144 & 0 & -0.0667 \\
-0.05 & 0 & 0.0756 & -0.0192 & 0 & 0 & -0.0256 & 0.0192 \\
0 & 0 & -0.0192 & 0.0811 & 0 & -0.0667 & 0.0192 & -0.0144 \\
-0.0256 & -0.0192 & 0 & 0 & 0.0756 & 0.0192 & -0.05 & 0 \\
-0.0192 & -0.0144 & 0 & -0.0667 & 0.0192 & 0.0811 & 0 & 0 \\
0 & 0 & -0.0256 & 0.0192 & -0.05 & 0 & 0.0756 & -0.0192 \\
0 & 0.0667 & 0.0172 & 0.0144 & 0 & 0 & -0.0192 & 0.0811
\end{bmatrix}
\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 = ? \\ v_2 = ? \\ u_3 = ? \\ v_3 = ? \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix}
$$

$$\Rightarrow 0 = [0.0756\, u_2 - 0.0192\, v_2 + 0\, u_3 + 0\, v_3]\, AE$$

$$\Rightarrow \quad u_2 = 0.254\, v_2 \qquad (1)$$

$$1000 = [-\,0.0192\, u_2 + 0.0811\, v_2 + 0\, u_3 - 0.0667\, v_3]\, AE$$

$$0 = [0\, u_2 + 0\, v_2 + 0.0756\, u_3 + 0.0192\, v_3]\, AE$$

$$\Rightarrow \quad u_3 = -\,0.254\, v_3 \qquad (2)$$

$$1000 = [0\, u_2 - 0.0667\, v_2 + 0.0192\, u_3 + 0.0811\, v_3]\, AE$$

$$1000 = [-0.0192\,(0.254\, v_2) + 0.0811\, v_2 - 0.0667\, v_3]\, AE$$

$$\Rightarrow \quad 1000 = [0.0762\, v_2 - 0.0667\, v_3]\, AE \qquad (3)$$

$$1000 = [\,-0.0667\, v_2 + 0.0192\,(-\,0.254\, v_3) + 0.0811\, v_3]\, AE$$

$$\Rightarrow \quad 1000 = [-\,0.0667\, v_2 + 0.0762\, v_3]\, AE \qquad (4)$$

Multiplying (4) by $\left(\frac{0.0762}{0.0667}\right)$

$$\Rightarrow 1142.4 = [-0.0762\, v_2 + 0.0870\, v_3]\, AE \qquad (5)$$

Adding (3) and (5)

$$2142.4 = [0\, v_2 + 0.204\, v_3]\, AE$$

$$\Rightarrow \quad v_3 = \frac{105021}{AE} \qquad (6)$$

Substituting (6) into (3)

$$1000 = \left[0.0762\, v_2 - 0.0667\left(\frac{105021}{AE}\right)\right] AE$$

$$\Rightarrow \quad v_2 = \frac{105021}{AE} \Rightarrow v_2 = v_3$$

Substituting in (1) and (2)

$$u_2 = 0.254\, v_2 = 0.254\left(\frac{105021}{AE}\right)$$

$$\Rightarrow \quad u_2 = \frac{26675}{AE}$$

$$u_3 = -\,0.254\, v_3 = -\,0.254\left(\frac{105021}{AE}\right)$$

$$\Rightarrow \quad u_3 = -\frac{26675}{AE}$$

Going back to the local stiffness matrices

$$\begin{Bmatrix} f_{2x} \\ f_{2y} \end{Bmatrix} = \frac{AE}{20}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_3 \\ v_2 \\ u_1 = 0 \\ v_1 = 0 \end{Bmatrix} \begin{matrix} \Rightarrow\ f_{2x} = 1\, u_2\ \dfrac{AE}{20} \\ \Rightarrow\ f_{2x} = \dfrac{26675}{\cancel{AE}}\ \dfrac{\cancel{AE}}{20} \end{matrix}$$

$$\Rightarrow \ f_{2x} = 1333 \text{ lb} ;\quad f_{2y} = 0 \text{ lb}$$

$$f_{1\text{–}2} = \sqrt{(f_{2x})^2 + (f_{2y})^2} \Rightarrow f_{1\text{–}2} = 1333 \text{ lb (T)}$$

Member 1–3

$$\begin{Bmatrix} f_{3x} \\ f_{3y} \end{Bmatrix} = \frac{AE}{25}\begin{bmatrix} -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\Rightarrow \ f_{3x} = \left[-0.64\left(\frac{-26675}{AE}\right) - 0.48\left(\frac{105021}{AE}\right)\right]\frac{AE}{25}$$

$$\Rightarrow \ f_{3x} = -\,1333 \text{ lb}$$

$$f_{3y} = \left[-0.48\left(\frac{-26675}{AE}\right) - 0.36\left(\frac{105021}{AE}\right)\right]\frac{AE}{25}$$

$$\Rightarrow \ f_{3y} = -\,1000 \text{ lb}$$

$$f_{1\text{–}3} = \sqrt{(f_{3x})^2 + (f_{3y})^2} \Rightarrow f_{1\text{–}3} = 1667 \text{ lb (T)}$$

Member 2–4

$$\begin{Bmatrix} f_{2x} \\ f_{2y} \end{Bmatrix} = \frac{AE}{25}\begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \ f_{2x} = 1333 \text{ lb (C)}$$

$$f_{2y} = 1000 \text{ lb (C)}$$

$$f_{2\text{–}4} = \sqrt{(f_{2x})^2 + (f_{2y})^2}$$

$$\Rightarrow \; f_{2\text{–}4} = 1667 \text{ lb (C)}$$

Member 2–3

$$\begin{Bmatrix} f_{2x} \\ f_{2y} \end{Bmatrix} = \frac{AE}{15}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x} = 0 \\ f_{2y} = 0 \end{matrix}$$

$$\Rightarrow \; f_{2\text{–}3} = 0 \text{ lb}$$

Member 3–4

$$\begin{Bmatrix} f_{3x} \\ f_{3y} \end{Bmatrix} = \frac{AE}{20}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_3 \\ v_3 \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix} \Rightarrow \begin{matrix} f_{3x} = -1333 \text{ lb} \\ f_{3y} = 0 \text{ lb} \end{matrix}$$

$$f_{3\text{–}4} = \sqrt{(f_{3x})^2 + (f_{3y})^2} \Rightarrow f_{3\text{–}4} = 1333 \text{ lb (C)}$$

Member 1-4

$$\begin{Bmatrix} f_{4x} \\ f_{4y} \end{Bmatrix} = \frac{AE}{15}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \; f_{1\text{–}4} = 0 \text{ lb}$$

3.25 The global stiffness matrix is changed since matrix $[k_{2\text{–}4}]$ is not incorporated in

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{1y} = 0 \\ F_{2x} = 0 \\ F_{2y} = 1000 \\ F_{3x} = 0 \\ F_{3y} = 1000 \\ F_{4x} = 0 \\ F_{4y} = 0 \end{Bmatrix} = AE \begin{array}{c} \begin{array}{cccc} \quad (1) & & \quad (2) & \end{array} \\ \left[\begin{array}{cccc} 0.0756 & 0.0192 & 0.05 & 0 \\ 0.0192 & 0.0811 & 0 & 0 \\ -0.05 & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 0.0667 \\ -0.0256 & -0.0192 & 0 & 0 \\ -0.0192 & -0.0144 & 0 & -0.0667 \\ 0 & 0 & 0 & 0 \\ 0 & -0.0667 & 0 & 0 \end{array}\right. \end{array}$$

$$\begin{array}{c} \begin{array}{cccc} \quad (3) & & \quad (4) & \end{array} \\ \left.\begin{array}{cccc} -0.0256 & -0.0192 & 0 & 0 \\ -0.0192 & -0.0144 & 0 & -0.0667 \\ 0 & 0 & 0 & 0 \\ 0 & -0.0667 & 0 & 0 \\ 0.0756 & 0.0192 & -0.05 & 0 \\ 0.0192 & 0.0811 & 0 & 0 \\ -0.05 & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 0.0667 \end{array}\right] \end{array} \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \; 0 = [0.05\, u_2 + 0\, v_2 + 0\, u_3 + 0\, v_3]\, AE$$

$$\Rightarrow \quad u_2 = 0 \tag{1}$$

$$1000 = [0\,u_2 + 0.0667\,v_2 + 0\,u_3 - 0.0667\,v_3]\,AE \tag{2}$$

$$0 = [0\,u_3 + 0\,v_2 + 0.0756\,u_3 + 0.0192\,v_3]\,AE \tag{3}$$

$$\Rightarrow \quad u_3 = -\,0.254\,v_3 \tag{3}$$

$$1000 = [0\,u_2 - 0.0667\,v_2 + 0.0192\,u_3 + 0.0811\,v_3]\,AE \tag{4}$$

Adding (2) and (4)

$$2000 = [0.0192\,u_3 + 0.0144\,v_3]\,AE \tag{5}$$

Substituting (3) in (5)

$$2000\,[0.0192\,(-\,0.254\,v_3) + 0.0144\,v_3]\,AE$$

$$\Rightarrow \quad v_3 = \frac{210000}{AE}$$

$$\Rightarrow \quad u_3 = (-\,0{\cdot}254)\,\frac{(210000)}{AE} \Rightarrow u_3 = \frac{-\,53340}{AE}$$

Substituting in (2)

$$\Rightarrow \quad 1000 = \left[0.0667\,v_2 - 0.0667\left(\frac{210000}{AE}\right)\right] AE$$

$$\Rightarrow \quad v_2 = \frac{224993}{AE}$$

Forces on members

Member 1–2

$$\begin{Bmatrix} f_{2x} \\ f_{2y} \end{Bmatrix} = \frac{AE}{20}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_2 = 0 \\ v_2 = \\ u_1 = 0 \\ v_1 = 0 \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x} = 0 \\ f_{2y} = 0 \end{matrix}$$

$$\Rightarrow \quad f_{1\text{–}2} = 0$$

Member 1–3

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \end{Bmatrix} = \frac{AE}{25}\begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\Rightarrow \quad f_{1x} = \frac{AE}{25}\left[-\,0.64\left(\frac{-\,53380}{AE}\right) - 0.48\left(\frac{210000}{AE}\right)\right]$$

$$\Rightarrow \quad f_{1x} = -\,2666.5 \text{ lb}$$

$$f_{1y} = \frac{AE}{25}\left[-\,0.48\left(\frac{-\,53380}{AE}\right) - 0.36\left(\frac{210000}{AE}\right)\right]$$

$$\Rightarrow \quad f_{1y} = -\,2000 \text{ lb}$$

$$f_{1\text{–}3} = \sqrt{(f_{1x})^2 + (f_{1y})^2} \qquad \Rightarrow f_{1\text{–}3} = 3333 \text{ lb (T)}$$

Member 2–3

$$\begin{Bmatrix} f_{2x} \\ f_{2y} \end{Bmatrix} = \frac{AE}{15}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}\begin{Bmatrix} u_2 = 0 \\ v_2 = 0 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\Rightarrow \quad f_{2x} = 0$$

$$\Rightarrow \quad f_{2y} = \frac{AE}{15}\left[1\frac{(224993)}{AE} - \frac{(210000)}{AE}\right]$$

$$\Rightarrow \quad f_{2y} = -1000 \text{ lb (C)}$$

$$f_{2-3} = \sqrt{(f_{2x})^2 + (f_{2y})^2}$$

$$\Rightarrow f_{2-3} = 1000 \text{ lb (C)}$$

Member 3–4

$$\begin{Bmatrix} f_{3x} \\ f_{3y} \end{Bmatrix} = \frac{AE}{20}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_3 \\ v_3 \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad f_{3x} = 1\ \frac{(-53340)}{AE}\ \frac{AE}{20} \Rightarrow f_{3x} = -2666.5 \text{ lb } (C)$$

$$\Rightarrow \quad f_{3y} = 0$$

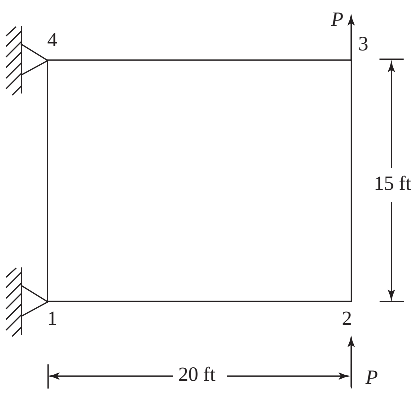
$$f_{3-4} = \sqrt{(f_{3x})^2 + (f_{3y})^2} \Rightarrow f_{3-4} = 2667 \text{ lb } (C)$$

Member 1–4

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \end{Bmatrix} = \frac{AE}{15}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x} = 0 \\ f_{1y} = 0 \end{matrix}$$

$$\Rightarrow \quad f_{1-4} = 0 \text{ lb}$$

3.26



Since both elements 2–4 and 1–3 are removed, the global stiffness matrix will change

$$\begin{Bmatrix} F_{1x}=? \\ F_{1y}=? \\ F_{2x}=0 \\ F_{2y}=1000 \\ F_{3x}=0 \\ F_{3y}=1000 \\ F_{4x}=? \\ F_{4y}=? \end{Bmatrix} = AE \begin{array}{c} \begin{array}{cccc} \quad(1)\quad\quad & \quad\quad(2)\quad\quad & \quad\quad(3)\quad\quad & \quad\quad(4)\quad \end{array} \\ \begin{bmatrix} 0.05 & 0 & -0.05 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0667 & 0 & 0 & 0 & 0 & 0 & -0.0667 \\ -0.05 & 0 & 0.05 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0667 & 0 & -0.0667 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.05 & 0 & -0.05 & 0 \\ 0 & 0 & 0 & -0.0667 & 0 & 0.0667 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.05 & 0 & 0.05 & 0 \\ 0 & 0.0667 & 0 & 0 & 0 & 0 & 0 & 0.0667 \end{bmatrix} \end{array} \begin{Bmatrix} u_1=0 \\ v_1=0 \\ u_2=? \\ v_2=? \\ u_3=? \\ v_3=? \\ u_4=0 \\ v_4=0 \end{Bmatrix}$$

$\Rightarrow\ 0 = 0.05\, u_2 \qquad \Rightarrow u_2 = 0$

$1000 = 0.0667\, v_2 - 0.0667\, v_3$ (1)

$0 = 0.05\, u_3 \qquad \Rightarrow u_3 = 0$

$1000 = -0.0067\, v_2 + 0.0667\, v_3$ (2)

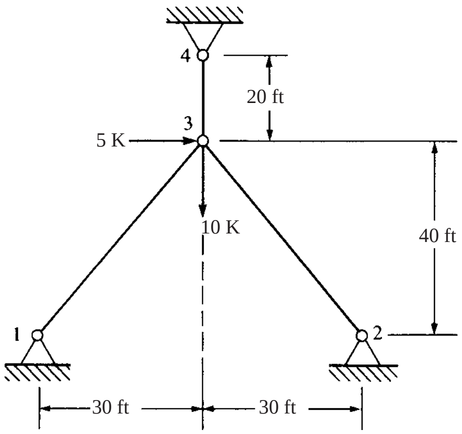
Adding (1) to (2)

$2000 = 0\, v_2 + 0\, v_3$

The matrix, therefore, is singular and we get an inconsistent equation.

Of course this should have been expected since the truss is unstable.

3.27



	(1)	(2)	(3)
	1–3	2–3	3–4
L	50 ft	50 ft	20 ft
θ	53.13°	126.87°	90°
$\cos\theta$	0.6	– 0.6	0
$\sin\theta$	0.8	0.8	1

$$[k^{(1)}] = \frac{AE}{50} \begin{array}{c} \begin{array}{cc} (1) & \quad\quad(3) \end{array} \\ \begin{bmatrix} 0.36 & 0.48 & -0.36 & 0.48 \\ 0.48 & 0.64 & 0.48 & -0.64 \\ & & & \\ & & & \end{bmatrix} \end{array} \begin{array}{c} (1) \\ \\ (3) \end{array}$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} \quad(2) & & \quad(3) & \end{array} \\
[k^{(2)}] = \dfrac{AE}{50} & \left[\begin{array}{cc|cc} 0.36 & -0.48 & -0.36 & 0.48 \\ -0.48 & 0.64 & 0.48 & -0.64 \\ \hline & & & \\ & & & \end{array}\right] \begin{array}{c} (2) \\ \\ (3) \end{array}
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} \quad(3) & & \quad(4) & \end{array} \\
[k^{(3)}] = \dfrac{AE}{20} & \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \hline & & & \\ & & & \end{array}\right] \begin{array}{c} (3) \\ \\ (4) \end{array}
\end{array}
$$

Invoking boundary conditions. Therefore, need only 3–3

$$[K] = \begin{bmatrix} \frac{AE}{50}(0.36+0.36) + \frac{AE}{20}(0) & \frac{AE}{50}(0.48-0.48) + \frac{AE}{20}(0) \\ \frac{AE}{50}(0.48-0.48) + \frac{AE}{20}(0) & \frac{AE}{50}(0.64+0.64) + \frac{AE}{20}(1) \end{bmatrix}$$

$$\Rightarrow \; K = \begin{bmatrix} (0.72)\frac{AE}{50} & 0 \\ 0 & (1.28)\frac{AE}{50} + \frac{AE}{20} \end{bmatrix}$$

$$\{F\} = [K]\{d\}$$

$$\begin{Bmatrix} F_{3x} = 5\text{ K} \\ F_{3y} = -10\text{ K} \end{Bmatrix} = AE \begin{bmatrix} \frac{0.72}{50} & 0 \\ 0 & \frac{1.28}{50} + \frac{1}{20} \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix}$$

$$\Rightarrow \qquad 5\text{ K} = \frac{(0.72)(AE)}{50}\, u_3$$

$$\Rightarrow \; u_3 = \frac{(5000)\times(50)\times(12)}{(0.72)(3)(30\times10^6)}$$

$$\Rightarrow \; u_3 = 0.0463 \text{ in.}$$

$$\Rightarrow \qquad -10\text{ K} = \left[\frac{1.28}{50} + \frac{1}{20}\right] AE\, v_3$$

$$\Rightarrow \; v_3 = -0.0176 \text{ in.}$$

Forces on the members

Member 1–3 (1)

$$\begin{Bmatrix} f'_{1x} \\ f'_{3x} \end{Bmatrix} = \frac{AE}{50} \begin{bmatrix} 0.6 & 0.8 & -0.6 & -0.8 \\ -0.6 & -0.8 & 0.6 & 0.8 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\Rightarrow \; f'^{(1)}_{1x} = -2.055 \text{ kips}$$

Member 2–3 (2)

$$\begin{Bmatrix} f'_{2x} \\ f'_{3x} \end{Bmatrix} = \frac{AE}{50} \begin{bmatrix} -0.6 & 0.8 & 0.6 & -0.8 \\ 0.6 & -0.8 & -0.6 & 0.8 \end{bmatrix} \begin{Bmatrix} u_2 = 0 \\ v_2 = 0 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\Rightarrow \quad f'^{(2)}_{2x} = 6.279 \text{ kips}$$

Member 3–4 (3)

$$\begin{Bmatrix} f'_{3x} \\ f'_{4x} \end{Bmatrix} = \frac{AE}{20} \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad f'_{3x} = \frac{AE}{20} (-0.0176)$$

$$\Rightarrow \quad f'_{3x} = -6.6 \text{ kips}$$

3.28

$$[T] = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix}$$

$$[T^T] = \begin{bmatrix} C & -S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & C & -S \\ 0 & 0 & S & C \end{bmatrix}$$

$$[T]\,[T^T] = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{bmatrix} C & -S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & C & -S \\ 0 & 0 & S & C \end{bmatrix}$$

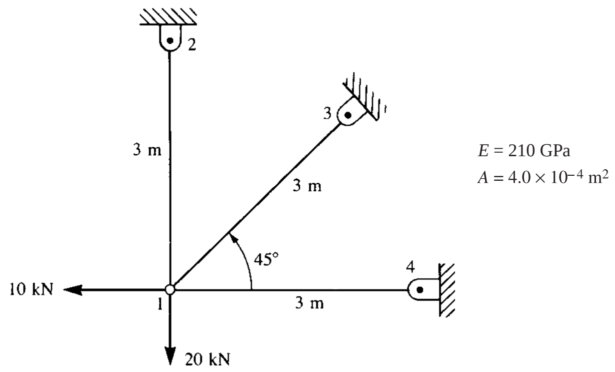
$$\Rightarrow \; T\,[T^T] = \begin{bmatrix} C^2 + S^2 & -SC + SC & 0 & 0 \\ -CS + CS & S^2 + C^2 & 0 & 0 \\ 0 & 0 & C^2 + S^2 & -SC + SC \\ 0 & 0 & -CS + CS & S^2 + C^2 \end{bmatrix}$$

$$[T]\,[T^T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \leftarrow \begin{matrix} \text{Identity} \\ \text{matrix} \end{matrix} = [I]$$

But $$[T]\,[T^T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I \Rightarrow$$

$$\Rightarrow \qquad [T^T] = [T^{-1}] = \begin{bmatrix} C & -S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & C & -S \\ 0 & 0 & S & C \end{bmatrix}$$

3.29



$$\frac{AE}{L} = \frac{(4\times 10^{-4})(210\times 10^{6})}{3} = 280\times 10^{2}\ \frac{\text{kN}}{\text{m}}$$

Element (1)

$$C = 0,\ S = 1$$

$$[k^{(1)}] = 280\times 10^{2}\ \begin{array}{c} \quad (1) \qquad\qquad (2) \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Element (2)

$$C = \frac{\sqrt{2}}{2},\ S = \frac{\sqrt{2}}{2}$$

$$[k^{(2)}] = 280\times 10^{2}\ \begin{array}{c} \quad (1) \qquad\qquad\qquad (3) \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

Element (3)

$$C = 1,\ S = 0$$

$$[k^{(3)}] = 280\times 10^{2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Boundary conditions

$$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$$

$$\begin{Bmatrix} F_{1x} = -10\text{ K} \\ F_{1y} = -20\text{ K} \end{Bmatrix} = 280\times 10^{2}\begin{bmatrix} 1\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1\frac{1}{2} \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$\Rightarrow\ u_1 = -0.893\times 10^{-4}\,\text{m}$$

$$v_1 = -4.46\times 10^{-4}\,\text{m}$$

Element stresses

$$\sigma^{(1)} = \frac{210 \times 10^6}{3}\ [0 \quad -1 \quad 0 \quad 1] \begin{Bmatrix} -0.893\times10^{-4} \\ -4.46\times10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow\ \sigma^{(1)} = 31.2 \text{ MPa (T)}$$

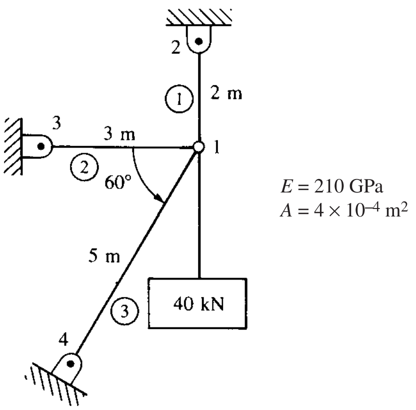
$$\sigma^{(2)} = 70 \times 10^6 \left[-\tfrac{\sqrt{2}}{2} \quad -\tfrac{\sqrt{2}}{2} \quad \tfrac{\sqrt{2}}{2} \quad \tfrac{\sqrt{2}}{2}\right] \begin{Bmatrix} -0.893\times10^{-4} \\ -4.46\times10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow\ \sigma^{(2)} = 26.5 \text{ MPa (T)}$$

$$\sigma^{(3)} = 70 \times 10^6 [-1 \quad 0 \quad 1 \quad 0] \begin{Bmatrix} -0.893\times10^{-4} \\ -4.46\times10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow\ \sigma^{(3)} = 6.25 \text{ MPa (T)}$$

3.30



Element 1–2

$C = 0, S = 1$

$$[k_{1-2}] = (4 \times 10^{-4})(210 \times 10^9) \begin{array}{c} \begin{array}{cc} (1) & (2) \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$

Element 1–3

$C = -1, S = 0$

$$[k_{1-3}] = 84 \times 10^6 \begin{array}{c} \begin{array}{cc} (1) & (3) \end{array} \\ \begin{bmatrix} \frac{1}{3} & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Element 1–4

$$C = -0.5,\ S = -\frac{\sqrt{3}}{2}$$

$$\begin{array}{cc} & \begin{array}{cccc} \quad & (1) & \quad & \quad (4) \end{array} \\ [k_{1-4}] = 84 \times 10^6 & \begin{bmatrix} 0.05 & 0.0866 & -0.05 & -0.0866 \\ 0.0866 & 0.15 & -0.0866 & -0.15 \\ -0.05 & -0.0866 & 0.05 & 0.0866 \\ -0.0866 & -0.15 & 0.0866 & 0.15 \end{bmatrix} \end{array}$$

$$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$$

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{1y} = -40 \end{Bmatrix} = 84 \times 10^6 \begin{bmatrix} 0.3883 & 0.0866 \\ 0.0866 & 0.65 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$\Rightarrow \quad u_1 = 1.71 \times 10^{-4}\ \text{m}$$

$$v_1 = -7.55 \times 10^{-4}\ \text{m}$$

Element stresses

$$\sigma^{(1)} = \frac{210 \times 10^9}{2} [0 \quad -1 \quad 0 \quad 1] \begin{Bmatrix} 1.71 \times 10^{-4} \\ -7.65 \times 10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(1)} = 79.28\ \text{MPa (T)}$$

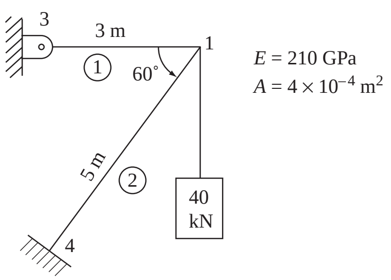
$$\sigma^{(2)} = \frac{210 \times 10^9}{3} [1 \quad 0 \quad -1 \quad 0] \begin{Bmatrix} 1.71 \times 10^{-4} \\ -7.55 \times 10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(2)} = 11.97\ \text{MPa (T)}$$

$$\sigma^{(3)} = \frac{210 \times 10^9}{5} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{Bmatrix} 1.71 \times 10^{-4} \\ -7.55 \times 10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(3)} = -23.87\ \text{MPa (C)}$$

3.31



$[k_{1-3}]$ and $[k_{1-4}]$ are the same as in Problem 3.30

$$u_3 = v_3 = u_4 = v_4 = 0$$

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{1y} = -40 \end{Bmatrix} = 8.4 \times 10^7 \begin{bmatrix} \frac{23}{60} & \frac{3\sqrt{3}}{60} \\ \frac{3\sqrt{3}}{60} & \frac{9}{60} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$\Rightarrow \quad u_1 = 8.248 \times 10^{-4}\,\text{m}$$

$$v_1 = -3.651 \times 10^{-3}\,\text{m}$$

Element stresses

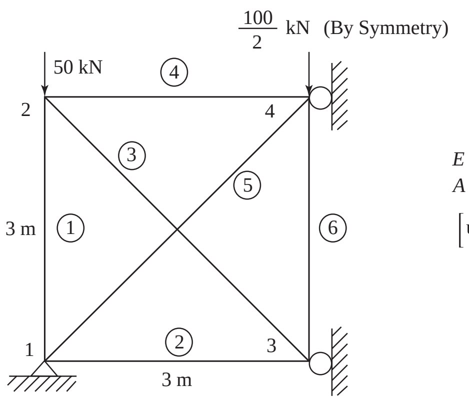
$$\sigma^{(1)} = \frac{210 \times 10^9}{3}\,[1 \quad 0 \quad -1 \quad 0] \begin{Bmatrix} 8.248 \times 10^{-4} \\ -3.651 \times 10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(1)} = 57.74 \text{ MPa (T)}$$

$$\sigma^{(2)} = \frac{210 \times 10^9}{5} \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{Bmatrix} 8.248 \times 10^{-4} \\ -3.657 \times 10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(2)} = -115.48 \text{ MPa (C)}$$

3.32



$E = 70$ GPa $= 70 \times 10^6 \frac{\text{kN}}{\text{m}^2}$
$A = 3.0 \times 10^{-4}\,\text{m}^2$
$\left[\text{use } A_{3-4} = \frac{3.0}{2} \times 10^{-4}\,\text{m}^2\right]$
(due to symmetry)

$$[k_{1-2}] = 7000 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}; \quad [k_{1-3}] = 7000 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k_{2-3}] = 2475 \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}; \quad [k_{2-4}] = 7000 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k_{1-4}] = 2475 \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}; \quad [k_{3-4}] = 3500 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$u_1 = v_1 = u_3 = v_4 = 0$$

$\therefore$ Global equations are

$$\begin{bmatrix} 9475 & -2475 & 2475 & 0 \\ & 9475 & -2475 & 0 \\ & & 5975 & -3500 \\ & & & 5975 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -50 \\ 0 \\ -50 \end{Bmatrix}$$

Solving simultaneously

$$u_3 = 0.135 \times 10^{-2}\,\text{m}$$

$$v_2 = -0.850 \times 10^{-2}\,\text{m}$$

$$v_3 = -0.137 \times 10^{-1}\,\text{m}$$

$$v_4 = -0.164 \times 10^{-1}\,\text{m}$$

$$\sigma_{1-2} = \sigma_{5-6} = \frac{70 \times 10^6}{3}\,[0 \;\; -1 \;\; 0 \;\; 1] \begin{Bmatrix} 0 \\ 0 \\ 0.135 \times 10^{-2} \\ -0.850 \times 10^{-2} \end{Bmatrix}$$

$$\Rightarrow \quad \sigma_{1-2} = \sigma_{5-6} = -198\text{ MPa (C)}$$

$$f'_{x_{1-2}} = f'_{x_{5-6}} = \sigma_{1-2} \times A_{1-2} = -198000 \times 3 \times 10^{-4}$$

$$\Rightarrow \; f'_{x_{1-2}} = f'_{x_{5-6}} = -59.5\text{ kN}$$

$$\sigma_{1-3} = \sigma_{5-3} = \frac{70 \times 10^6}{3}\,[-1 \;\; 0 \;\; 1 \;\; 0] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.137 \times 10^{-1} \end{Bmatrix}$$

$$\Rightarrow \quad \sigma_{1-3} = \sigma_{5-3} = 0$$

$$\Rightarrow \; f'_{x_{1-3}} = f'_{x_{5-3}} = 0$$

Similarly

$$\sigma_{2-3} = \sigma_{6-3} = \frac{70 \times 10^6}{3\sqrt{2}} \left[-\frac{\sqrt{2}}{2} \;\; \frac{\sqrt{2}}{2} \;\; \frac{\sqrt{2}}{2} \;\; \frac{\sqrt{2}}{2} \right] \begin{Bmatrix} 0.135 \times 10^{-2} \\ -0.850 \times 10^{-2} \\ 0 \\ -0.137 \times 10^{-1} \end{Bmatrix}$$

$$\Rightarrow \; \sigma_{2-3} = \sigma_{6-3} = 44.6\text{ MPa (T)}$$

$$f'_{x_{2-3}} = f'_{x_{6-3}} = 13.39\text{ kN}$$

$$\sigma_{2-4} = \sigma_{6-4} = -31.6 \text{ MPa (C)}$$

$$f'_{x_{2-4}} = f'_{x_{6-4}} = -9.47 \text{ kN}$$

$$\sigma_{1-4} = \sigma_{5-4} = -191 \text{ MPa (C)}$$

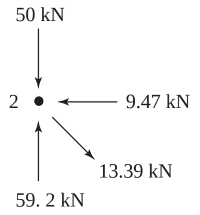
$$f'_{x_{1-4}} = f'_{x_{5-4}} = -57.32 \text{ kN}$$

$$\sigma_{3-4} = -63.1 \text{ MPa (C)}$$

$$f'_{x_{3-4}} = -18.93 \text{ kN}$$

Force equilibrium

Node 2

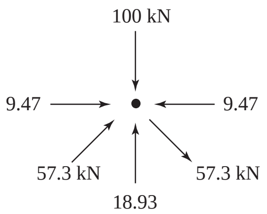


$$\Sigma F_y = -50 - 13.39 \sin 45° + 54.5$$

$$\Sigma F_y = 0$$

$$\Sigma F_x = -9.47 + 13.39 \cos 45° = -0.001 \text{ kN}$$

Node 4



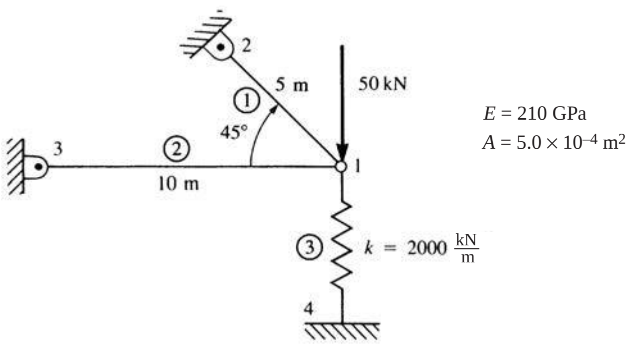
$$\Sigma F_y = -100 + 57.3 \sin 45° \times 2 + 18.93$$

$$\Sigma F_y = -0.003 \text{ kN}$$

$$\Sigma F_x = 9.47 + 57.3 \cos 45° - 9.47 - 57.3 \cos 45°$$

$$\Sigma F_x = 0$$

3.33 (a)



Element 1–2 ; $\theta = 135°$

$$C^2 = 0.5,\ CS = -0.5,\ S^2 = 0.5$$

$$[k_{1-2}] = \frac{(210\times10^9)(5.0\times10^{-4})}{5} \begin{array}{c} \begin{array}{cc} (1) & (2) \end{array} \\ \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \end{array}$$

$$\Rightarrow\ [k_{1-2}] = 105\times10^5 \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Element 1–3 ; $\theta = 180°$

$$C^2 = 1.0,\ CS = 0,\ S^2 = 0$$

$$[k_{1-3}] = \frac{(210\times10^9)(5\times10^{-4})}{10} \begin{array}{c} \begin{array}{cc} (1) & (3) \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\Rightarrow\ [k_{1-3}] = 105\times10^5 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 1–4 ; $\theta = 270°$

$$C^2 = 0,\ CS = 0,\ S^2 = 1.0$$

$$[k_{1-4}] = 20\times10^5 \begin{array}{c} \begin{array}{cc} (1) & (4) \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

$$\{F\} = [K]\{d\}$$

Boundary conditions are

$$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$$

The final matrix (assembled)

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{1y} = -50 \end{Bmatrix} = 10^5 \begin{bmatrix} 210 & -105 \\ -105 & 125 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$\Rightarrow\quad 0 = 210\,u_1 - 105\,v_1 \Rightarrow v_1 = 2\,u_1$$

$$-50000 = 10^5[-105\,u_1 + 125\,(2\,u_1)]$$

$$\Rightarrow\quad u_1 = -3.448\times10^{-3}\ \text{m}$$

$$\Rightarrow\quad v_1 = -6.896\times10^{-3}\ \text{m}$$

$$\sigma_{1-2} = \frac{210\times10^9}{5\text{ m}}\,[0.707 \quad -0.707 \quad -0.707 \quad 0.707]\begin{Bmatrix}-3.448\times10^{-3}\\ -6.896\times10^{-3}\\ 0\\ 0\end{Bmatrix}$$

$$\Rightarrow \sigma_{1-2} = 102.4\text{ MPa (T)}$$

$$\sigma_{1-3} = \frac{210\times10^9}{10}\,[1.0 \quad 0 \quad -1.0 \quad 0]\begin{Bmatrix}-3.448\times10^{-3}\\ -6.896\times10^{-3}\\ 0\\ 0\end{Bmatrix}$$

$$\Rightarrow \sigma_{1-3} = -72.4\text{ MPa (C)}$$

Note: Can show equilibrium at node 1

$$F_s = \left(2000\,\frac{\text{kN}}{\text{m}}\right)(6.896\times10^{-3}\text{ m})$$

$$= 13.792\text{ kN}$$

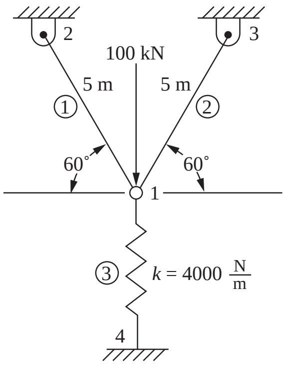
$$f_{1-3} = 35.6\text{ kN}$$

$$f_{1-2} = 51.2\text{ kN}$$

$$\Sigma F_y = 0$$

$$-50 + 13.79 + 36.198 = 0$$

(b)



$$A = 5\times10^{-4}\text{ m}^2,\; E = 210\times10^9\,\frac{\text{N}}{\text{m}^2}$$

$$L_1 = L_2 = 5\text{ m}$$

$$[k^{(1)}] = 2.1\times10^7 \begin{array}{c} \begin{array}{cc} u_1 & v_1 \end{array} \\ \left[\begin{array}{cc|c} \frac{1}{4} & -\frac{\sqrt{3}}{4} & \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & \\ \hline & & \end{array}\right] \end{array} \frac{\text{N}}{\text{m}}$$

$$[k^{(2)}] = 2.1\times10^7 \begin{array}{c} \begin{array}{cc} u_1 & v_1 \end{array} \\ \left[\begin{array}{cc|c} \frac{1}{4} & \frac{\sqrt{3}}{4} & \\ \frac{\sqrt{3}}{4} & \frac{3}{4} & \\ \hline & & \end{array}\right] \end{array} \frac{\text{N}}{\text{m}}$$

$$[k^{(3)}] = 4\times 10^{3}\begin{array}{c} \begin{array}{cc} u_1 & v_1 \end{array} \\ \left[\begin{array}{cc|c} 0 & 0 & \\ 0 & 1 & \\ \hline & & \\ & & \end{array}\right] \end{array}\frac{\text{N}}{\text{m}}$$

Boundary conditions

$$u_2 = v_2 = u_3 = v_3 = 0$$

$$u_4 = v_4 = 0$$

$$\{F\} = [K]\{d\}$$

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{1y} = -1\times 10^{5} \end{Bmatrix} = \begin{bmatrix} 5.25\times 10^{6} + 5.25\times 10^{6} & 0 \\ 0 & 1.58\times 10^{7} + 1.58\times 10^{7} + 4000 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

Solving

$$0 = 1.05\times 10^{7}\,u_1$$

$$\therefore\quad u_1 = 0$$

$$-1\times 10^{5} = 3.15\times 10^{7}\,v_1$$

$$v_1 = -0.00317\text{ m}$$

$$\sigma^{(1)} = \frac{E}{L}\,[-C \quad -S \quad C \quad S]\begin{Bmatrix} u_1 \\ v_1 \\ v_2 \\ v_2 \end{Bmatrix}$$

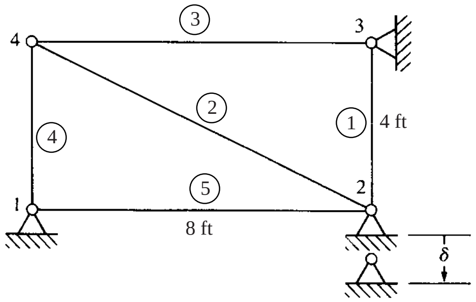
$$\sigma^{(1)} = \frac{210\times 10^{9}}{5}\left[\frac{1}{2} \quad -\frac{\sqrt{3}}{2} \quad -\frac{1}{2} \quad \frac{\sqrt{3}}{2}\right]\begin{Bmatrix} 0 \\ -0.00317 \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma^{(1)} = 1.155\times 10^{8}\,\frac{\text{N}}{\text{m}^2} = 115\text{ MPa}$$

$$\sigma^{(2)} = \frac{210\times 10^{9}}{5}\left[-\frac{1}{2} \quad -\frac{\sqrt{3}}{2} \quad \frac{1}{2} \quad \frac{\sqrt{3}}{2}\right]\begin{Bmatrix} 0 \\ -0.00317 \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma^{(2)} = 115\text{ MPa}$$

3.34



$$[k_{1-2}] = \frac{AE}{8}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};\quad [k_{1-4}] = \frac{AE}{4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$[k_{2-4}] = \frac{AE}{8.94}\begin{bmatrix} 0.80 & -0.40 & -0.80 & 0.40 \\ -0.40 & 0.20 & 0.40 & -0.20 \\ -0.80 & 0.40 & 0.80 & -0.40 \\ 0.40 & -0.20 & -0.40 & 0.20 \end{bmatrix}$$

$$[k_{2-3}] = \frac{AE}{4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix};\quad [k_{4-3}] = \frac{AE}{8}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Boundary conditions

$u_1 = v_1 = u_3 = v_3 = u_2 = 0$, $v_2 = -0.05$ in.

Applying the boundary conditions and superimposing the [k]s

$$AE\begin{bmatrix} 0.522 & 0.0447 & -0.0223 \\ 0.0447 & 0.214 & -0.0447 \\ -0.0223 & -0.0447 & 0.272 \end{bmatrix}\begin{Bmatrix} v_2 = -0.05 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Solving

$$u_4 = 9.93 \times 10^{-3} \text{ in.}$$

$$v_4 = -2.46 \times 10^{-3} \text{ in.}$$

Element stresses

$$\sigma^{(1)} = \frac{30 \times 10^3}{4 \times 12}\,[0 \;\; -1 \;\; 0 \;\; 1]\begin{Bmatrix} u_2 = 0 \\ v_2 = -0.05 \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix}$$

$$\Rightarrow\; \sigma^{(1)} = 31.25 \text{ ksi (T)}$$

$$\sigma^{(2)} = \frac{30 \times 10^3}{8.94 \times 12}\,[\,0.894 \;\; -0.447 \;\; -0.894 \;\; 0.447]\begin{Bmatrix} 0 \\ -0.05 \\ 9.93 \times 10^{-3} \\ -2.46 \times 10^{-3} \end{Bmatrix}$$

$$\Rightarrow\; \sigma^{(2)} = 3.459 \text{ ksi (T)}$$

$$\sigma^{(3)} = \frac{30\times10^3}{4\times12}\,[0 \quad -1 \quad 0 \quad 1]\begin{bmatrix} 0 \\ 0 \\ 9.93\times10^{-3} \\ -2.46\times10^{-3}\end{bmatrix}$$

$$\Rightarrow \quad \sigma^{(3)} = -\,1.538 \text{ ksi (C)}$$

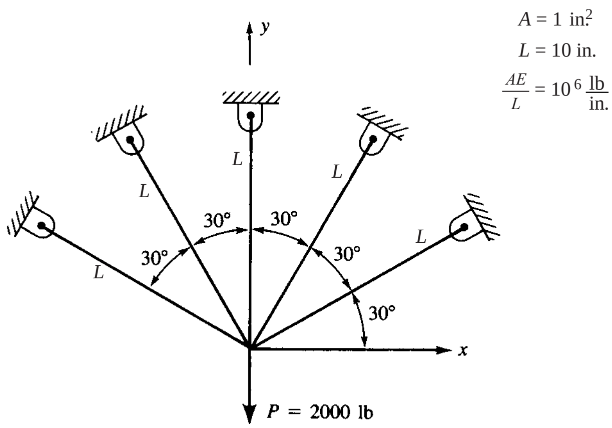
$$\sigma^{(4)} = \frac{30\times10^3}{8\times12}\,[-1 \quad 0 \quad 1 \quad 0]\begin{bmatrix} 9.93\times10^{-3} \\ -\,2.46\times10^{-3} \\ 0 \\ 0\end{bmatrix}$$

$$\Rightarrow \quad \sigma^{(4)} = -\,3.103 \text{ ksi (C)}$$

$$\sigma^{(5)} = \sigma_{1-2} = 0$$

Note: This solution was also verified by a computer program.

3.35



Using symmetry

Element (5)

$$\theta = 30°$$

$$[k^{(5)}] = \frac{AE}{L}\begin{array}{c} \begin{array}{cc} (1) & \qquad\qquad (5) \end{array} \\ \left[\begin{array}{cc|c} 0.75 & 0.433 & \\ 0.433 & 0.25 & \\ \hline & & \end{array}\right] \end{array}$$

Element (4)

$$\theta = 60°$$

$$[k^{(4)}] = \frac{AE}{L}\begin{array}{c} \begin{array}{cc} (1) & \qquad\qquad (5) \end{array} \\ \left[\begin{array}{cc|c} 0.25 & 0.433 & \\ & 0.75 & \\ \hline & & \end{array}\right] \end{array}$$

Element (5)

$$\theta = 90°$$

$$[k^{(3)}] = \left(\frac{2\,AE}{L}\right)\frac{1}{2}\left[\begin{array}{cc|c} 0 & 0 & \\ 0 & 1 & \\ \hline & & \end{array}\right]$$

Applying the boundary conditions and superimposing the K's

$$\frac{AE}{L}\begin{bmatrix} 1 & 0.866 \\ 0.866 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1000 \end{Bmatrix}$$

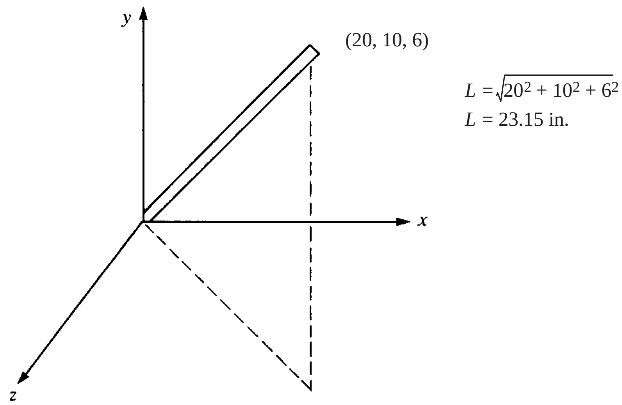
$$\Rightarrow \quad v_1 = \frac{-1000}{2 \times 10^6} \Rightarrow v_1 = -\,0.5 \times 10^{-3} \text{ in.}$$

$$u_1 = 0$$

$$\sigma^{(1)} = \sigma^{(5)} = \frac{E}{L}\,[0.866 \quad -0.5 \quad \ldots]\begin{Bmatrix} 0 \\ -0.0005 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(1)} = 250 \text{ psi (T)}$$

3.36



$$\{d'\} = [T^*]\,\{d\} \text{ and } [T^*] = [C_x \quad C_y \quad C_z]$$

$$C_x = \frac{20-0}{23.15} = 0.864 \qquad u_1 = 0.1 \text{ in.}$$

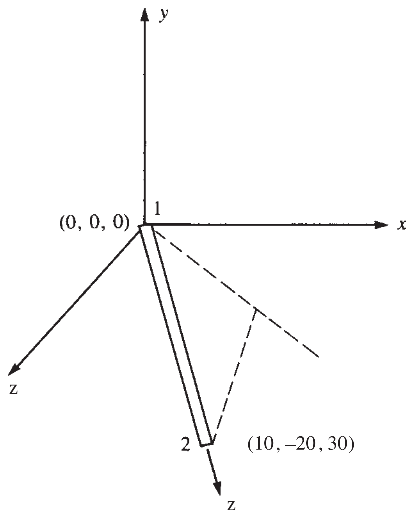
$$C_y = \frac{10-0}{23.15} = 0.432 \qquad v_1 = 0.2 \text{ in.}$$

$$C_z = \frac{6-0}{23.15} = 0.259 \qquad w_1 = 0.15 \text{ in.}$$

$$u_1' = 0.864\,(0.1) + 0.432\,(0.2) + 0.259\,(0.15)$$

$$\Rightarrow u_1' = 0.212 \text{ in.}$$

3.37



$L_{1-2} = 37.42$ in.

$$C_x = \frac{10}{37.42} = 0.267$$

$$C_y = \frac{-20}{37.42} = -0.534$$

$$C_z = \frac{30}{37.42} = 0.802$$

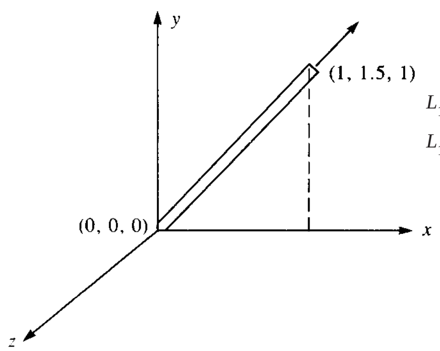
$$\{d'\} = [T]\{d\}$$

$$\{d'_{1x}\} = [C_x \; C_y \; C_z] \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \end{Bmatrix}$$

$$u'_1 = (0.267)(0.1) - (0.534)(0.2) + (0.802)(0.15)$$

$$\Rightarrow \quad u'_1 = 0.0397 \text{ in.}$$

3.38



$L_{1-2} = \sqrt{1^2 + 1.5^2 + 1^2}$

$L_{1-2} = 2.0616$ m

$$C_x = \frac{1}{2.0616} = 0.485$$

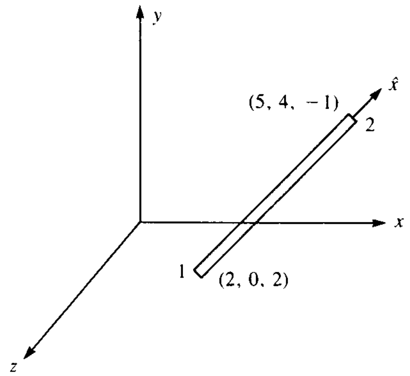
$$C_y = \frac{1.5}{2.0616} = 0.728$$

$$C_z = \frac{1}{2.0616} = 0.485$$

$$\{d'_{2x}\} = [0.485 \quad 0.728 \quad 0.485] \begin{Bmatrix} 5 \\ 10 \\ 15 \end{Bmatrix}$$

$$\Rightarrow \quad \hat{u}_2 = 16.98 \text{ mm}$$

3.39



$L_{1-2} = \sqrt{(5-2)^2 + (4-0)^2 + (-1-2)^2}$

$L_{1-2} = 5.831$ m

$$C_x = \frac{3}{5.831} = 0.515$$

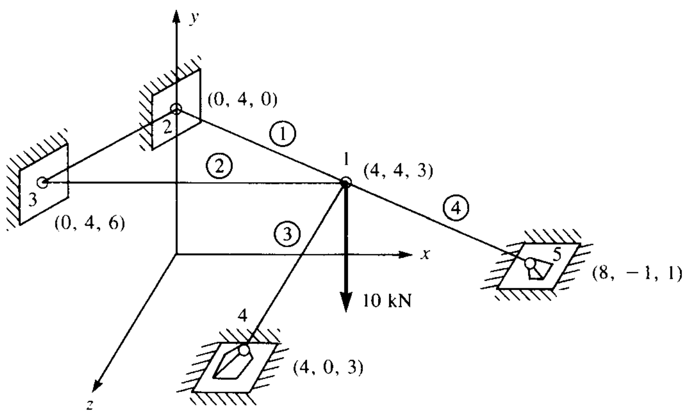
$$C_y = \frac{4}{5.831} = 0.686$$

$$C_z = -\frac{3}{5.831} = -0.515$$

$$\{d'_{2x}\} = [0.515 \quad 0.686 \quad -0.515] \begin{Bmatrix} 5 \\ 10 \\ 15 \end{Bmatrix}$$

$$\Rightarrow \quad \hat{u}_2 = 1.71 \text{ mm}$$

3.40 From Figure P. 3.40



$$L_{1-2} = \sqrt{(-4)^2 + 0 + (-3)^2} = 5 \text{ m}$$

$$C_x = \frac{0-4}{5} = -0.8;\ C_y = \frac{4-4}{5} = 0;\ C_2 = \frac{0-3}{5} = -0.6$$

$$L_{1-3} = \sqrt{(-4)^2 + 0 + (3)^2} = 5 \text{ m}$$

$$C_x = \frac{0-4}{5} = -0.8;\ C_y = \frac{4-4}{0} = 0;\ C_2 = \frac{6-3}{5} = 0.6$$

$$L_{1-4} = \sqrt{0 + (-4)^2 + 0} = 4 \text{ m}$$

$$C_x = 0;\ C_y = \frac{0-4}{4} = -1;\ C_z = 0$$

$$L_{1-5} = \sqrt{4^2 + (-5)^2 + (-2)^2} = 3\sqrt{5} \text{ m}$$

$$C_x = \frac{4}{3\sqrt{5}} = 0.596;\ C_y = \frac{-5}{3\sqrt{5}} = -0.745$$

$$C_z = \frac{-2}{3\sqrt{5}} = -0.298$$

Element 1–2

$$[\lambda] = \begin{bmatrix} 0.64 & 0 & 0.48 \\ 0 & 0 & 0 \\ 0.48 & 0 & 0.64 \end{bmatrix} \quad [k] = 42000 \begin{bmatrix} [\lambda] & [-\lambda] \\ [-\lambda] & [\lambda] \end{bmatrix}$$

Element 1–3

$$[\lambda] = \begin{bmatrix} 0.64 & 0 & -0.48 \\ 0 & 0 & 0 \\ -0.48 & 0 & 0.64 \end{bmatrix} \quad [k] = 42000 \begin{bmatrix} [\lambda] & [-\lambda] \\ [-\lambda] & [\lambda] \end{bmatrix}$$

Element 1–4

$$[\lambda] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [k] = 52500 \begin{bmatrix} [\lambda] & [-\lambda] \\ [-\lambda] & [\lambda] \end{bmatrix}$$

Element 1–5

$$[\lambda] = \begin{bmatrix} 0.356 & -0.444 & -0.178 \\ -0.444 & 0.556 & 0.222 \\ -0.178 & 0.222 & 0.0889 \end{bmatrix}$$

$$[k] = 31305 \begin{bmatrix} [\lambda] & [-\lambda] \\ [-\lambda] & [\lambda] \end{bmatrix}$$

Applying the boundary conditions where all deflections at node 2, 3, 4 and 5 are zero. The global equations are

$$\begin{Bmatrix} 0 \\ -10 \\ 0 \end{Bmatrix} = \begin{bmatrix} 64905 & -13899 & -5572 \\ -13899 & 69906 & 6950 \\ -5572 & 6950 & 33023 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \end{Bmatrix}$$

$$0 = 64905\, u_1 - 13899\, v_1 - 5572\, w_1 \qquad (1)$$

$$-10 = -13899\, u_1 + 69906\, v_1 + 6950\, w_1 \qquad (2)$$

$$0 = -5572\, u_1 + 6950\, v_1 + 33023\, w_1 \qquad (3)$$

From (1) and (3)

$$0 = 67058\, v_1 + 379094\, w_1 \qquad (4)$$

From (2) and (3)

$$-10 = 52570\, v_1 - 75424\, w_1 \qquad (5)$$

From (4) and (5), we get

$$w_1 = 2.68374 \times 10^{-5} \text{ m}$$

$$v_1 = -1.5171 \times 10^{-4} \text{ m}$$

Substituting in (1)

$$u_1 = -3.0183 \times 10^{-5} \text{ m}$$

Element stresses

$$\sigma_{1-2} = 42 \times 10^6\, [0.8 \quad 0 \quad -0.6 \quad -0.8 \quad 0 \quad 0.6] \begin{Bmatrix} -3.0183 \times 10^{-5} \\ -1.5171 \times 10^{-4} \\ 2.6837 \times 10^{-5} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma_{1-2} = -337.846 \ \frac{\text{kN}}{\text{m}^2} \ (\text{C})$$

$$\text{Force}_{\,1-2} = \frac{-337.846}{1000} = -0.337 \text{ kN (C)}$$

$$\sigma_{1-3} = 42 \times 10^6\, [0.8 \quad 0 \quad -0.6 \quad -0.8 \quad 0 \quad 0.6] \begin{Bmatrix} -3.0183 \times 10^{-5} \\ -1.5171 \times 10^{-4} \\ 2.6837 \times 10^{-5} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma_{1-3} = -1690 \ \frac{\text{kN}}{\text{m}^2} \ (\text{C})$$

$$\text{Force}_{\,1-3} = \frac{-1690}{1000} = -1.69 \text{ kN (C)}$$

$$\sigma_{1-4} = 52500 \times 10^3 \,[0 \quad 1 \quad 0 \quad 0 \quad -1 \quad 0] \begin{Bmatrix} -3.0183 \times 10^{-5} \\ -1.5171 \times 10^{-4} \\ 2.68374 \times 10^{-5} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma_{1-4} = -7965 \ \frac{\text{kN}}{\text{m}^2} \ \text{(C)}$$

$$\text{Force}_{\ 1-4} = -\frac{7965}{1000} = -7.695 \ \text{kN (C)}$$

$$\sigma_{1-5} = -2726 \ \frac{\text{kN}}{\text{m}^2}$$

$$\text{Force}_{\ 1-5} = -2.726 \ \text{kN}$$

Force equilibrium at node 1

x direction

$$0 = 0.388 \times 0.8 + 1.69 \times 0.8 - 2.726 \times 0.596$$
$$0 = -0.00307$$

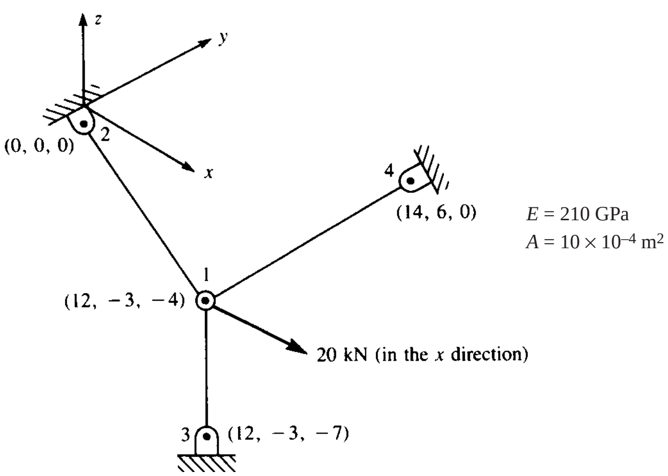
y direction

$$-10 = 0 + 0 + 7.965 \times 1 + 2.726 \times \frac{\sqrt{5}}{15}$$
$$-10 = 9.9968$$

z direction

$$0 = 0.338 \times 0.6 - 1.69 \times 0.6 + 0 + 2.726 \times 0.298$$
$$0 = 0.0015$$

3.41



$$L_{1-2} = \sqrt{(12-0)^2 + (-3-0)^2 + (-4-0)^2} \Rightarrow L_{1-2} = 13 \text{ m}$$

$$L_{1-3} = \sqrt{(12-12)^2 + (-3+3)^2 + (-7+4)^2} \Rightarrow L_{1-3} = 3 \text{ m}$$

$$L_{1-4} = \sqrt{(14-12)^2 + (6+3)^2 + (0+4)^2} \Rightarrow L_{1-4} = 10.05 \text{ m}$$

Element	$C_x = \dfrac{x_j - x_i}{L_{i-j}}$	$C_y = \dfrac{y_j - y_i}{L_{i-j}}$	$C_z = \dfrac{z_j - z_i}{L_{i-j}}$
$1-2$	$\frac{-12}{13}$	$\frac{3}{13}$	$\frac{4}{13}$
$1-3$	0	0	-1
$1-4$	$\frac{2}{10.05}$	$\frac{9}{10.05}$	$\frac{4}{10.05}$

Element	$C_x^{\ 2}$	C_xC_y	C_xC_z	$C_y^{\ 2}$	C_yC_z	$C_2^{\ 2}$
$1-2$	0.852	-0.213	-0.284	0.053	0.071	0.095
$1-3$	0	0	0	0	0	1
$1-4$	0.040	0.178	0.079	0.802	0.356	0.158

$$[\lambda] = \begin{bmatrix} C_x^{\ 2} & C_xC_y & C_xC_z \\ & C_y^2 & C_yC_z \\ & & C_z^2 \end{bmatrix}$$

$$[\lambda_{1-2}] = \begin{bmatrix} 0.852 & -0.213 & -0.284 \\ -0.213 & 0.053 & 0.071 \\ -0.284 & 0.071 & 0.095 \end{bmatrix}$$

$$[\lambda_{1-3}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\lambda_{1-4}] = \begin{bmatrix} 0.040 & 0.178 & 0.079 \\ 0.178 & 0.802 & 0.356 \\ 0.079 & 0.356 & 0.158 \end{bmatrix}$$

$$u_2 = v_2 = w_2 = u_3 = v_3 = w_3 = 0$$
$$u_4 = v_4 = w_4 = 0$$

$$\left[k_{1-2}^{(1)}\right] = \frac{AE}{L_{1-2}}\begin{bmatrix} [\lambda] & [-\lambda] \\ [-\lambda] & [\lambda] \end{bmatrix} = AE\begin{bmatrix} \frac{[\lambda]}{13} & -\frac{[\lambda]}{13} \\ -\frac{[\lambda]}{13} & \frac{[\lambda]}{13} \end{bmatrix}$$

$$\left[k_{1-3}^{(2)}\right] = \frac{AE}{L_{1-3}}\begin{bmatrix} [\lambda] & [-\lambda] \\ [-\lambda] & [\lambda] \end{bmatrix} = AE\begin{bmatrix} \frac{[\lambda]}{3} & -\frac{[\lambda]}{3} \\ -\frac{[\lambda]}{3} & \frac{[\lambda]}{3} \end{bmatrix}$$

$$\left[k_{1-4}^{(3)}\right] = \frac{AE}{L_{1-4}}\begin{bmatrix} [\lambda] & [-\lambda] \\ [-\lambda] & [\lambda] \end{bmatrix} = AE\begin{bmatrix} \frac{[\lambda]}{10.05} & -\frac{[\lambda]}{10.05} \\ -\frac{[\lambda]}{10.05} & \frac{[\lambda]}{10.05} \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = 20\,\text{kN} \\ F_{1y} = 0 \\ F_{1z} = 0 \end{Bmatrix} = 210 \times 10^3 \begin{bmatrix} 69.519 & 1.327 & -13.985 \\ 1.327 & 83.879 & 40.885 \\ -13.985 & 40.885 & 356.363 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \end{Bmatrix}$$

$$\Rightarrow \quad u_1 = 1.383 \times 10^{-3}\text{ m}$$
$$v_1 = -5.119 \times 10^{-5}\text{ m}$$
$$w_1 = 6.015 \times 10^{-5}\text{ m}$$

$$\sigma^{(1)} = \frac{E}{L^{(1)}} \begin{bmatrix} \frac{12}{13} & -\frac{3}{13} & -\frac{4}{13} & -\frac{12}{13} & \frac{3}{13} & \frac{4}{13} \end{bmatrix} \begin{Bmatrix} 1.383 \times 10^{-3} \\ -5.119 \times 10^{-5} \\ 6.015 \times 10^{-5} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(1)} = 20.51\text{ MPa (T)}$$

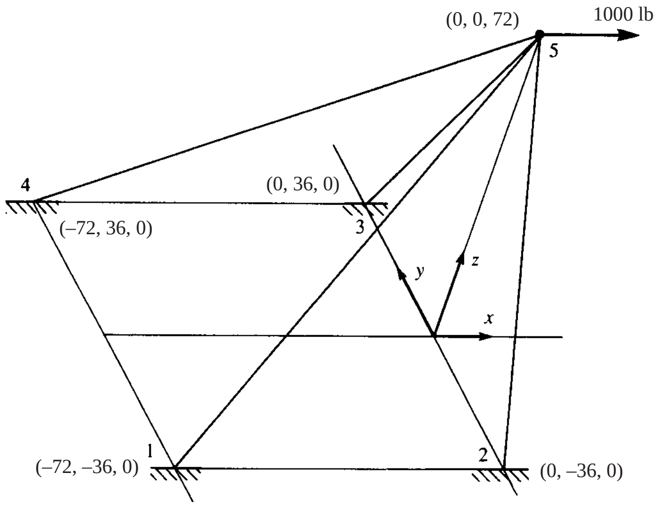
$$\sigma^{(2)} = \frac{E}{L^{(2)}} [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad -1] \begin{Bmatrix} 1.383 \times 10^{-3} \\ -5.119 \times 10^{-5} \\ 6.015 \times 10^{-5} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(2)} = 4.21\text{ MPa (T)}$$

$$\sigma^{(3)} = \frac{E}{L^{(3)}} \begin{bmatrix} \frac{-2}{10.05} & -\frac{9}{10.05} & -\frac{4}{10.05} & \frac{2}{10.05} & \frac{9}{10.05} & \frac{4}{10.05} \end{bmatrix} \begin{Bmatrix} 1.383 \times 10^{-3} \\ -5.119 \times 10^{-5} \\ 6.015 \times 10^{-5} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(3)} = -5.29\text{ MPa (C)}$$

3.42



Element 1–5

$$L_{1\text{–}5} = 108 \text{ in.}$$

$$C_x = \frac{x_5 - x_1}{L_{1-5}} = \frac{0-(-72)}{108} \Rightarrow C_x = 0.667$$

$$C_y = \frac{y_5 - y_1}{L_{1-5}} = \frac{0-(-36)}{108} \Rightarrow C_y = 0.333$$

$$C_z = \frac{z_5 - z_1}{L_{1-5}} = \frac{72-0}{108} \Rightarrow C_z = 0.667$$

$$[K] = \frac{4\times 30\times 10^6}{108}\begin{bmatrix} 0.444 & 0.222 & 0.444 & -0.444 & -0.222 & -0.444 \\ 0.222 & 0.111 & 0.222 & -0.222 & -0.111 & -0.222 \\ 0.444 & 0.222 & 0.444 & -0.444 & -0.222 & -0.444 \\ -0.444 & -0.222 & -0.444 & 0.444 & 0.222 & 0.444 \\ -0.222 & -0.111 & -0.222 & 0.222 & 0.111 & 0.222 \\ -0.444 & -0.222 & -0.444 & 0.444 & 0.222 & 0.444 \end{bmatrix}$$

Element 2–5

$$L_{2\text{–}5} = \sqrt{(0-0)^2 + (0-(-36)^2) + (72-0)^2}$$

$$\Rightarrow L_{2\text{–}5} = 80.5 \text{ in.}$$

$$C_x = \frac{0-0}{80.5} = 0$$

$$C_y = \frac{0-(-36)}{80.5} \Rightarrow C_y = 0.447$$

$$C_z = \frac{72-0}{80.5} \Rightarrow C_z = 0.894$$

$$[K] = \frac{4\times 30\times 10^6}{80.5}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.4 & 0 & -0.2 & -0.4 \\ 0 & 0.4 & 0.8 & 0 & -0.4 & -0.8 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.2 & -0.4 & 0 & 0.2 & 0.4 \\ 0 & -0.4 & -0.8 & 0 & 0.4 & 0.8 \end{bmatrix}$$

Since the structure is symmetric to the *x*-*z* plane then we can assume $v_5 = 0$ and a load of 500 lbs.

Disregarding all rows and columns of zero displacement we form the new global stiffness matrix comprised only of the non-zero displacements.

$$[K] = (4)\times 30\times 10^6\begin{bmatrix} 0.0041 & 0.0041 \\ 0.0041 & 0.014 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$F_{5x} = 500 = 492000\, u_5 + 492000\, w_5$$

$$F_{5z} = 0 = 492000\, u_5 + 1685880\, w_5$$

$$\Rightarrow \quad u_5 = -\frac{1685880}{492000}\, w_5$$

$$\Rightarrow \quad 500 = 492000\left[-\frac{1685880}{492000}\right] w_5 + 492000\, w_5$$

$$\Rightarrow \quad w_5 = -0.00042''$$
$$\Rightarrow \quad u_5 = 0.0014''$$

Element stresses

Element 1–5

$$\sigma_{1-5} = \frac{E}{L_{1-5}} [-C_x \quad -C_y \quad -C_z \quad C_x \quad C_y \quad C_z] \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_5 \\ v_5 \\ w_5 \end{Bmatrix}$$

$$\sigma_{1-5} = \frac{30 \times 10^6}{108} [-0.667 \quad -0.333 \quad -0.667 \quad 0.667 \; 0.333 \; 0.666] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.0014 \\ 0 \\ -0.00042 \end{Bmatrix}$$

$$\Rightarrow \quad \sigma_{1-5} = 180 \text{ psi (T)}$$

Element 2–5

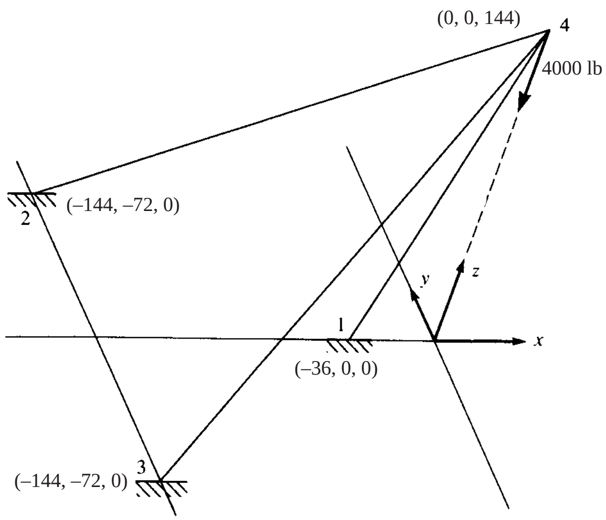
$$\sigma_{2-5} = \frac{30 \times 10^6}{80.5} [0 \quad -0.447 \quad -0.894 \quad 0 \quad 0.447 \quad 0.894] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.0014 \\ 0 \\ -0.00042 \end{Bmatrix}$$

$$\Rightarrow \quad \sigma_{2-5} = -140 \text{ psi (C)}$$

From symmetry

$$\sigma_{3-5} = 140 \text{ psi (C)}$$
$$\sigma_{4-1} = 180 \text{ psi (T)}$$

3.43



Element 1–4

$$L_{1\text{–}4} = \sqrt{(0-(-36)^2)+0+(144-0)^2} \Rightarrow L_{1\text{–}4} = 148.4 \text{ in.}$$

$$C_x = \frac{x_4 - x_1}{L_{1-4}} = \frac{36}{148.6}$$

$$\Rightarrow C_x = 0.2426$$

$$C_y = 0$$

$$C_z = 0.9704$$

$$[k_{1\text{–}4}] = \frac{AE}{L_{1-4}}\left[\begin{array}{ccc|ccc} & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & 0.05885 & 0 & 0\cdot 2354 \\ & & & 0 & 0 & 0 \\ & & & 0\cdot 2354 & 0 & 0\cdot 9417 \end{array}\right]$$

Element 2–4

$$L_{2\text{–}4} = \sqrt{(144)^2 + (-72)^2 + 144^2} = 216 \text{ in.}$$

$$C_x = \frac{144}{216} = 0.667,\ C_y = \frac{-72}{216} = -0.3333,\ C_z = \frac{144}{216} = 0.6667$$

$$[k_{2\text{–}4}] = \frac{AE}{L_{2-4}}\left[\begin{array}{ccc|ccc} & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & 0.4425 & -0.2222 & 0.4445 \\ & & & -0.2222 & 0.1111 & -0.2222 \\ & & & 0.4444 & -0.2222 & 0.4445 \end{array}\right]$$

Element 3–4

$$L_{3\text{–}4} = 216 \text{ in.},\ C_x = 0.6667,\ C_y = 0.3333,\ C_z = 0.6667$$

$$[k_{3\text{–}4}] = \frac{AE}{L_{3-4}}\left[\begin{array}{ccc|ccc} & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & 0.4425 & 0.2222 & 0.4425 \\ & & & 0.2222 & 0.1111 & 0.2222 \\ & & & 0.4445 & 0.2222 & 0.4445 \end{array}\right]$$

$$[K] = \frac{AE}{L_{1-4}}\begin{bmatrix} 0.05885 + 1.294 & 0 & 0.2354 + 1.294 \\ 0 & 0.3234 & 0 \\ 0.2354 + 1.294 & 0 & 0.9417 + 1.294 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} 0 \\ 0 \\ -4000 \end{Bmatrix} = \frac{AE}{148.4}\begin{bmatrix} 1.3529 & 0 & 1.5294 \\ 0 & 0.3234 & 0 \\ 1.5294 & 0 & 2.2357 \end{bmatrix}\begin{Bmatrix} u_4 \\ v_4 \\ w_4 \end{Bmatrix}$$

$$0 = \frac{AE}{148.4}\,[0\,u_4 + 0.3234\,v_4 + 0\,w_4]$$

$$\Rightarrow \quad v_4 = 0$$

$$0 = \frac{AE}{148.4}\,[1.3529\,u_4 + 1.5294\,w_4]$$

$$\Rightarrow \quad u_4 = -\,1.1305\,w_4$$

$$-\,4000 = \frac{AE}{148.4}\,[1.529\,u_4 + 2.2357\,w_4]$$

$$\Rightarrow \quad -\,4000 = \frac{AE}{148.4}\,[1.529\,(-\,1.1305\,w_4) + 2.2357\,w_4]$$

$$\Rightarrow \quad w_4 = \frac{1171501.87}{6 \times 30 \times 10^6}$$

$$\Rightarrow \quad w_4 = -\,0.00683 \text{ in.}$$

$$\Rightarrow \quad u_4 = 0.00863 \text{ in.}$$

Stresses

$$\sigma_{1-4} = \frac{E}{L_{1-4}}\,[-\,C_x \quad -\,C_y \quad -\,C_z \quad C_x \quad C_y \quad C_z] \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix}$$

$$\sigma_{1-4} = \frac{30 \times 10^6}{148.4}\,[-\,0.2426 \quad 0 \quad -\,0.9704 \quad 0.2426 \quad 0 \quad 0.9704] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.00863 \\ 0 \\ -\,0.00683 \end{bmatrix}$$

$$\Rightarrow \quad \sigma_{1-4} = -\,916 \text{ psi (C)}$$

3.44 Derive Equation (3.7.21)

$$\sigma = \frac{f'_{2x}}{A}$$

$$f'_{2x} = \frac{AE}{L}\,[1 \;\; -1] \begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix}$$

$$\sigma = \frac{E}{L}\,[-1 \quad 1] \begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix}$$

Now in 3–D

$$\{d'\} = [T^*]\,\{d\}$$

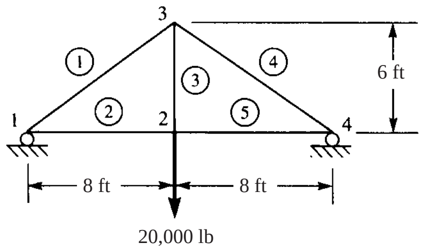
where by Equation (3.7.7)

$$[T^*] = \begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix}$$

$$\sigma = \frac{E}{L}\,[-1 \quad 1]\begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix}\{d\}$$

$$\sigma = \frac{E}{L}\,[-C_x \quad -C_y \quad -C_z \quad C_x \quad C_y \quad C_z]\begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}$$

3.46

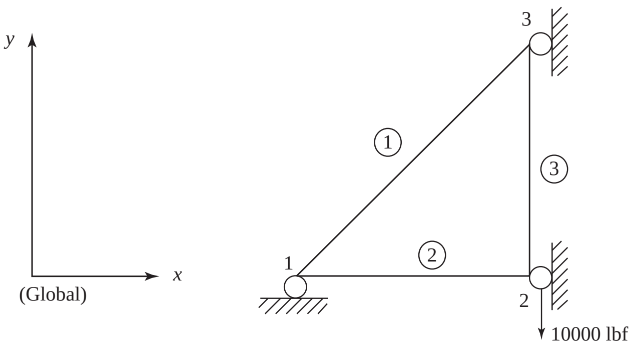


$$E = 30 \times 10^6 \text{ psi}$$

$$A^{(1)} = A^{(2)} = A^{(4)} = A^{(5)} = 10 \text{ in.}^2$$

$$A^{(2)} = 20 \text{ in.}^2$$

Reduce the given figure by symmetry.



$$A^{(1)} = A^{(2)} = A^{(3)} = 10 \text{ in.}^2$$

(Reducing given $A^{(3)}$ by half)

$$v_1 = 0,\ u_2 = u_3 = 0$$

find u_1, v_2, v_3

Data for reduced truss

Element	$\theta°$	C	S	C^2	S^2	CS
1	36.9°	0.8	0.6	0.64	0.36	0.48
2	0°	1.0	0	1.0	0	0
3	90°	0	1.0	0	1.0	0

$$[k^{(\alpha)}] = \left\{\frac{A_\alpha E_\alpha}{L_\alpha}\right\}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

$$\therefore \quad [k^{(1)}] = (10\text{ in.}^2)\left(\frac{30\times 10^6}{\text{in.}^2}\text{ lbf}\right)\left(\frac{1}{10\text{ft}}\right)\left(\frac{\text{ft}}{12\text{in.}}\right)\begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix}$$

So

$$[k^{(1)}] = 2.5\times 10^6 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.36 & -0.36 & 0.48 & 0.36 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_3 \\ v_3 \end{array}$$

Finally

$$[k^{(1)}] = 10^6 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1.6 & 1.2 & -1.6 & -1.2 \\ 1.2 & 0.9 & -1.2 & -0.9 \\ -1.6 & -1.2 & 1.6 & 1.2 \\ -1.2 & 0.9 & 1.2 & 0.9 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_3 \\ v_3 \end{array}$$

$$[k^{(2)}] = (10\text{ in.}^2)\left(\frac{30\times 10^6\text{ lbf}}{\text{in.}^2}\right)\left(\frac{1}{8\text{ ft}}\right)\left(\frac{1\text{ ft}}{12\text{ in.}}\right)\begin{bmatrix} 1.0 & 0 & -1.0 & 0 \\ 0 & 0 & 0 & 0 \\ -1.0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$[k^{(2)}] = 10^6 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 3.125 & 0 & -3.125 & 0 \\ 0 & 0 & 0 & 0 \\ -3.125 & 0 & 3.125 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_3 \\ v_3 \end{array}$$

and

$$[k^{(3)}] = (10\text{ in.}^2)\left(\frac{30\times 10^6\text{ lbf}}{\text{in.}^2}\right)\left(\frac{1}{6\text{ ft}}\right)\left(\frac{1\text{ ft}}{12\text{ in.}}\right)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & -1.0 \\ 0 & 0 & 0 & 0 \\ 0 & -1.0 & 0 & 1.0 \end{bmatrix}$$

$$[k^{(3)}] = 10^6 \begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4.167 & 0 & -4.167 \\ 0 & 0 & 0 & 0 \\ 0 & -4.167 & 0 & 4.167 \end{bmatrix} \end{array} \begin{array}{c} \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

$$[K] = 10^6 \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_3 & v_2 & u_3 & v_3 \end{array} \\ \left[\begin{array}{cccccc} 1.6+3.125 & 1.2 & -3.125 & & -1.6 & -1.2 \\ 1.2 & 0.9 & & & -1.2 & -0.9 \\ -3.125 & . & 3.125 & & & \\ & & & 4.167 & & -4.167 \\ -1.6 & -1.2 & & & 1.6 & 1.2 \\ -1.2 & -0.9 & & -4.167 & 1.2 & 4.1+4.167 \end{array}\right] \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

$[K]\,\{d\} = [F]$ gives

$$\begin{bmatrix} 4.725 & 1.2 & -3.125 & 0 & -1.6 & -1.2 \\ 1.2 & 0.9 & 0 & 0 & -1.2 & -0.9 \\ -3.125 & 0 & 3.125 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4.167 & 0 & -4.167 \\ -1.6 & -1.2 & 0 & 0 & 1.6 & 1.2 \\ -1.2 & -0.9 & 0 & -4.167 & 1.2 & 5.067 \end{bmatrix} \begin{Bmatrix} u_1 \\ 0 \\ 0 \\ v_2 \\ 0 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.01 \\ 0 \\ 0 \end{Bmatrix}$$

and

$$\begin{bmatrix} 4.725 & 0 & -1.200 \\ 0 & 4.167 & -4.167 \\ -1.2 & -4.167 & 5.067 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.01 \\ 0 \end{Bmatrix}$$

$$u_1 = \frac{\begin{vmatrix} 0 & 0 & -1.200 \\ -0.1 & 4.167 & -4.167 \\ 0 & -4.167 & 5.067 \end{vmatrix}}{\begin{vmatrix} 4.725 & 0 & -1.200 \\ 0 & 4.167 & -4.167 \\ -1.2 & -4.167 & 5.067 \end{vmatrix}}$$

$$u_1 = \frac{-1.2\begin{vmatrix} -0.01 & 4.167 \\ 0 & -4.167 \end{vmatrix}}{4.725\begin{vmatrix} 4.167 & -4.167 \\ -4.167 & 5.067 \end{vmatrix} - 1.2\begin{vmatrix} 0 & 4.167 \\ -1.2 & -4.167 \end{vmatrix}}$$

$$u_1 = \frac{-0.025}{17.72017 - 6.00048} = -0.00426 \text{ in.}$$

$$v_2 = \frac{\begin{vmatrix} 4.725 & 0 & -1.2 \\ 0 & -0.01 & -4.167 \\ -1.2 & 0 & 5.067 \end{vmatrix}}{11.71969}$$

$$= \frac{4.725\{-0.02534\} - 1.2\{-0.006\}}{11.71969}$$

$v_2 = -0.0192$ in.

$$v_3 = \frac{\begin{vmatrix} 4.725 & 0 & 0 \\ 0 & 4.167 & -0.01 \\ -1.2 & -4.167 & 0 \end{vmatrix}}{11.71969}$$

$$v_3 = \frac{4.725\{-(0.0)(4.167)\}}{11.71969} = -0.0168 \text{ in.}$$

now $\{\sigma\} = [C']\{d\}$ where $[C'] = \frac{E}{L}[-C \quad -S \quad C \quad S]$

So, for (1)

$$[C'] = \left(\frac{30 \times 10^6 \text{ lbf}}{120 \text{ in.}^2}\right)(-0.8 \;\; -0.6 \;\; 0.8 \;\; 0.6)\begin{Bmatrix} -0.00426 \\ 0 \\ 0 \\ -0.0168 \end{Bmatrix}$$

$$= \left(\frac{2.5 \times 10^5 \text{ lbf}}{\text{in.}^3}\right)(-0.0068 \text{ in.})$$

$\sigma^{(1)} = -1668$ psi (C)

$$\sigma^{(2)} = \left(\frac{30 \times 10^6 \text{ lbf}}{\text{in.}^2}\right)\left(\frac{1}{96 \text{ in.}}\right)[-1.0 \quad 0 \quad 1.0 \quad 0]\begin{Bmatrix} -0.00426 \\ 0 \\ 0 \\ -0.0192 \end{Bmatrix}$$

$$\sigma^{(2)} = \left(\frac{3.125 \times 10^5 \text{ lbf}}{\text{in.}^3}\right)(0.00426 \text{ in.})$$

$\sigma^{(2)} = 1332$ psi (T)

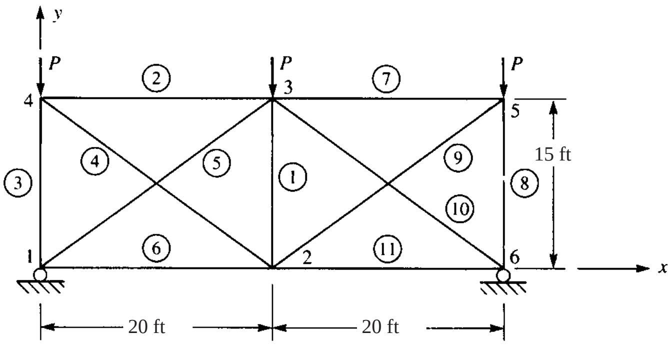
$$\sigma^{(3)} = \left(\frac{30 \times 10^6 \text{ lbf}}{\text{in.}^2}\right)\left(\frac{1}{72 \text{ in.}}\right)[0 \quad -1.0 \quad 0 \quad 1.0]\begin{Bmatrix} 0 \\ -0.0192 \\ 0 \\ -0.0168 \end{Bmatrix}$$

$$\sigma^{(3)} = \left(\frac{4.167 \times 10^5 \text{ lbf}}{\text{in.}^3}\right)(0.0096 - 0.0084)$$

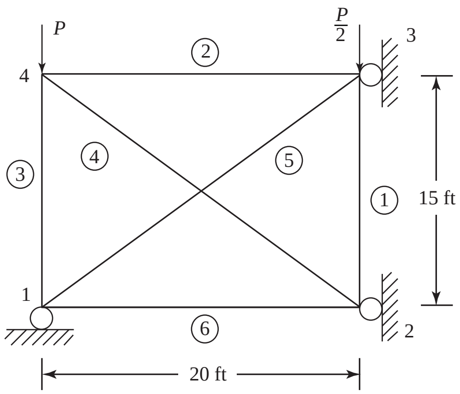
$= 1000$ psi

$\sigma^{(3)} = 1000$ psi (T)

3.47



Using symmetry



Boundary conditions

$$v_1 = u_2 = u_3 = 0$$

(see solution to Problem 3.24 for individual [k]'s for each element)
Global [K]

$$[K] = AE\begin{bmatrix} 0.0756 & 0.0192 & -0.05 & 0 & -0.0256 & -0.0192 & 0 & 0 \\ 0.0192 & 0.0811 & 0 & 0 & -0.0192 & -0.0144 & 0 & -0.0667 \\ -0.05 & 0.0 & 0.0756 & -0.0192 & 0 & 0 & -0.0256 & 0.0192 \\ 0 & 0 & -0.0192 & 0.0811 & 0 & -0.0667 & 0.0192 & -0.0144 \\ -0.0256 & -0.0192 & 0 & 0 & 0.0756 & 0.0192 & -0.05 & 0 \\ -0.0192 & -0.0144 & 0 & -0.0667 & 0.0192 & 0.0811 & 0 & 0 \\ 0 & 0 & -0.0256 & 0.0192 & -0.05 & 0 & 0.0756 & -0.0192 \\ 0 & -0.0667 & 0.0192 & -0.0144 & 0 & 0 & -0.0192 & 0.0811 \end{bmatrix}$$

Applying the boundary conditions, we obtain

$$AE\begin{bmatrix} 0.0756 & 0 & -0.0192 & 0 & 0 \\ 0 & 0.0811 & -0.0667 & 0.0192 & -0.0144 \\ -0.0192 & -0.0667 & 0.0811 & 0 & 0 \\ 0 & 0.0192 & 0 & 0.0756 & -0.0192 \\ 0 & -0.0144 & 0 & -0.0192 & 0.0811 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_2 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \frac{-P}{Z} \\ 0 \\ -P \end{Bmatrix}$$

Using the simultaneous equation solver, we obtain

$$u_1 = -110\ \frac{P}{AE},\ v_2 = -405\ \frac{P}{AE}$$

$$v_3 = -433\ \frac{P}{AE},\ u_4 = 50\ \frac{P}{AE}$$

$$v_4 = -208\ \frac{P}{AE}$$

where all displacements are now in units of inches.

(Computer program TRUSS was also used to verify the above displacements (setting $P = 1$, $A = 1$, $E = 1$))

Stresses

$$\sigma^{(2)} = \frac{E}{L}\ [-1\quad 0\quad 1\quad 0]\begin{Bmatrix} u_4 \\ v_4 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$= \frac{E}{L}\ [-1\quad 0\quad 1\quad 0]\begin{Bmatrix} 50 \\ -208 \\ 0 \\ -433 \end{Bmatrix}\ \frac{P}{AE}$$

$$= \frac{E}{240\text{ in.}}\left(-50\frac{P}{AE}\right)$$

$$\sigma^{(2)} = -0.208\ \frac{P}{A}$$

$$\sigma^{(3)} = \frac{E}{L}\ [0\ \ -1\ \ 0\ \ 1]\begin{Bmatrix} u_1 = -110 \\ v_1 = 0 \\ u_4 = 50 \\ v_4 = -208 \end{Bmatrix}\ \frac{P}{AE}$$

$$= \frac{-208}{15\times 12}\frac{P}{A}$$

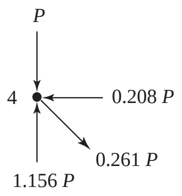
$$= -1.156\ \frac{P}{A}$$

$$\sigma^{(4)} = \frac{E}{L}\ [0.80\quad -0.60\quad -0.80\quad 0.60]\ \frac{P}{AE}\begin{Bmatrix} u_2 = 0 \\ v_2 = -405 \\ u_4 = 50 \\ v_4 = -208 \end{Bmatrix}$$

$$\sigma^{(4)} = \frac{(243 - 40 - 124.8)}{25'\times 12}\frac{P}{A}$$

$$\sigma^{(4)} = 0.261\frac{P}{A}$$

Verify equilibrium at node 4



$\Sigma F_x = -\,0.208\,P + 0.261\,P\,(0.8)$

$= (-0.208 + 0.208)P$

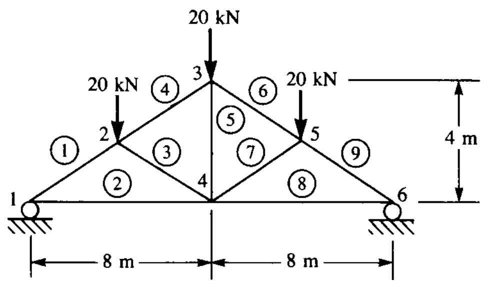
$= 0$

$\Sigma F_y = 1.156\,P - P + (0.261\,P)\,(0.6)$

$= (1.156 - 1 - 0.156)P$

$\cong 0$

3.48



Element	L	θ	C	S	C^2	S^2	CS
(1)	4.47	26.565°	0.89443	0.44721	0.8	0.2	0.4
(2)	8	0	1	0	1	0	0
(3)	4.47	153.435°	– 0.89443	0.44721	0.8	0.2	– 0.4
(4)	4.47	26.565°	0.89443	0.44721	0.8	0.2	0.4
(5)	4	90°	0	1	0	1	0

$$[k^{(1)}] = \frac{AE}{4.47}\begin{array}{c}\begin{bmatrix} 0.8 & 0.4 & -0.8 & -0.4 \\ & 0.2 & -0.4 & -0.2 \\ & & 0.8 & 0.4 \\ \text{Symmetry} & & & 0.2 \end{bmatrix}\end{array}\begin{array}{l} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}$$

$$[k^{(2)}] = \frac{AE}{8}\begin{array}{c}\begin{bmatrix} 1 & 0 & -1 & 0 \\ & 0 & 0 & 0 \\ & & 1 & 0 \\ \text{Symmetry} & & & 0 \end{bmatrix}\end{array}\begin{array}{l} u_1 \\ v_1 \\ u_4 \\ v_4 \end{array}$$

$$[k^{(3)}] = \frac{AE}{4.47}\begin{array}{c}\begin{bmatrix} 0.8 & -0.4 & -0.8 & 0.4 \\ & 0.2 & 0.4 & -0.2 \\ & & 0.8 & -0.4 \\ \text{Symmetry} & & & 0.2 \end{bmatrix}\end{array}\begin{array}{l} u_4 \\ v_4 \\ u_2 \\ v_2 \end{array}$$

$$[k^{(4)}] = \frac{AE}{4.47}\begin{array}{c}\begin{bmatrix} 0.8 & 0.4 & -0.8 & -0.4 \\ & 0.2 & -0.4 & -0.2 \\ & & 0.8 & 0.4 \\ \text{Symmetry} & & & 0.2 \end{bmatrix}\end{array}\begin{array}{l} u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

$$[k^{(5)}] = \frac{AE}{4}\begin{array}{c}\begin{bmatrix} 0 & 0 & 0 & 0 \\ & \frac{1}{2} & 0 & -\frac{1}{2} \\ & & 0 & 0 \\ \text{Symmetry} & & & \frac{1}{2} \end{bmatrix}\end{array}\begin{array}{l} u_4 \\ v_4 \\ u_3 \\ v_3 \end{array}$$

Boundary conditions

$$\because \quad v_1 = u_4 = u_3 = 0$$

$$[K] = 210 \times 10^6 \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} \frac{0.8}{4.47}+\frac{1}{8} & \frac{-0.8}{4.47} & \frac{-0.4}{4.47} & 0 & 0 \\ \frac{-0.8}{4.47} & \frac{2.4}{4.47} & \frac{0.4}{4.47} & \frac{-0.4}{4.47} & \frac{0.4}{4.47} \\ \frac{-0.4}{4.47} & \frac{0.4}{4.47} & \frac{0.6}{4.47} & \frac{-0.2}{4.47} & \frac{-0.2}{4.47} \\ 0 & \frac{-0.4}{4.47} & \frac{-0.2}{4.47} & \frac{0.2}{4.47}-\frac{0.5}{4} & \frac{-0.5}{4} \\ 0 & \frac{0.4}{4.47} & \frac{-0.2}{4.47} & \frac{-0.5}{4} & \frac{0.2}{4.47}+\frac{0.5}{4} \end{bmatrix} \end{array} \begin{array}{l} \\ u_1 \\ u_2 \\ v_2 \\ v_3 \\ v_4 \end{array}$$

$$\because \quad F_{2y} = -20 \text{ kN} \qquad F_{3y} = -10 \text{ kN}$$

$$F_{1x} = F_{1y} = F_{2x} = F_{3x} = F_{4x} = F_{4y} = ?$$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ -20\times10^3 \\ -10\times10^3 \\ F_{4y} \end{Bmatrix} = \begin{bmatrix} \left(\frac{2}{5\sqrt{5}}+\frac{1}{8}\right) & \frac{-2}{5\sqrt{5}} & \frac{-1}{5\sqrt{5}} & 0 & 0 \\ \frac{-2}{5\sqrt{5}} & \frac{6}{5\sqrt{5}} & \frac{1}{5\sqrt{5}} & \frac{-1}{5\sqrt{5}} & \frac{1}{5\sqrt{5}} \\ \frac{-1}{5\sqrt{5}} & \frac{1}{5\sqrt{5}} & \frac{3}{10\sqrt{5}} & \frac{-1}{10\sqrt{5}} & \frac{-1}{10\sqrt{5}} \\ 0 & \frac{-1}{5\sqrt{5}} & \frac{-1}{10\sqrt{5}} & \left(\frac{1}{10\sqrt{5}}+\frac{1}{8}\right) & \frac{-1}{8} \\ 0 & \frac{1}{5\sqrt{5}} & \frac{-1}{10\sqrt{5}} & \frac{-1}{8} & \left(\frac{1}{10\sqrt{5}}+\frac{1}{8}\right) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix} \times 210\times10^6$$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{4y} \\ -20\times10^3 \\ -10\times10^3 \end{Bmatrix} = \left[\begin{array}{ccc:cc} \left(\frac{2}{5\sqrt{5}}+\frac{1}{8}\right) & \frac{-2}{5\sqrt{5}} & 0 & \frac{-1}{5\sqrt{5}} & 0 \\ \frac{-2}{5\sqrt{5}} & \frac{6}{5\sqrt{5}} & \frac{-1}{10\sqrt{5}} & \frac{1}{5\sqrt{5}} & \frac{-1}{5\sqrt{5}} \\ 0 & \frac{1}{5\sqrt{5}} & \left(\frac{1}{10\sqrt{5}}+\frac{1}{8}\right) & \frac{-1}{10\sqrt{5}} & \frac{-1}{8} \\ \hdashline \frac{-1}{5\sqrt{5}} & \frac{1}{5\sqrt{5}} & \frac{1}{10\sqrt{5}} & \frac{3}{10\sqrt{5}} & \frac{-1}{10\sqrt{5}} \\ 0 & \frac{-1}{5\sqrt{5}} & \frac{-1}{8} & \frac{-1}{10\sqrt{5}} & \left(\frac{1}{10\sqrt{5}}+\frac{1}{5}\right) \end{array}\right] \begin{Bmatrix} u_1 \\ u_2 \\ v_4 \\ v_2 \\ v_3 \end{Bmatrix} \times 210\times10^6$$

$$\begin{Bmatrix} ? \\ P \end{Bmatrix} = 210 \times 10^6 \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$\left[k_{11}^{-1}\right] = \begin{bmatrix} 4.192 & 1.532 & -0.8074 \\ 1.532 & 2.602 & -1.3716 \\ 0.807 & -1.3716 & 6.6148 \end{bmatrix}$$

$$[k_c] = [k_{21}]\left[k_{11}^{-1}\right][k_{12}] = \begin{bmatrix} 0.04759 & 0.02923 \\ 0.02923 & 0.0935 \end{bmatrix}$$

$$[k_c] = [k_{22}] - [k_{21}]\left[k_{11}^{-1}\right][k_{12}] = \begin{bmatrix} 0.08657 & -0.073956 \\ -0.073956 & 0.076213 \end{bmatrix}$$

$$[k_c] = [N^1]^{-1} = \begin{bmatrix} 67.539 & 65.539 \\ 65.539 & 76.719 \end{bmatrix}$$

$$\begin{Bmatrix} v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 67.539 & 65.539 \\ 65.539 & 76.719 \end{bmatrix} \begin{Bmatrix} \frac{-2}{21} \times 10^{-3} \\ \frac{-1}{21} \times 10^{-3} \end{Bmatrix}$$

$$= \begin{Bmatrix} -9.5532489 \times 10^{-3} \\ -9.8951503 \times 10^{-3} \end{Bmatrix}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ v_4 \end{Bmatrix} = -\left[k_{11}{}^{-1}\right][k_{12}]\{d_2\} = -\begin{bmatrix} -0.201853644 & -0.03610867 \\ 0.157096846 & -0.06134 \\ -0.346288346 & -0.704175 \end{bmatrix} \begin{Bmatrix} -9.5532489 \times 10^{-3} \\ -9.8951503 \times 10^{-3} \end{Bmatrix}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} -2.285658 \times 10^{-3} \\ 8.93812942 \times 10^{-4} \\ -1.027609703 \times 10^{-2} \end{Bmatrix}$$

The stresses in each element

$$\sigma^{(1)} = \frac{E}{L}\,[-C \;\; -S \;\; C \;\; S] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$\sigma^{(1)} = \frac{210 \times 10^9}{2\sqrt{5}} \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{Bmatrix} -2.285658 \times 10^{-3} \\ 0 \\ 8.93812942 \times 10^{-4} \\ -9.5532489 \times 10^{-3} \end{Bmatrix}$$

$$\sigma^{(1)} = -67.08044733 \times 10^6 \, \frac{\text{N}}{\text{mm}^2} = -67.08 \text{ MPa (C)}$$

$$\sigma^{(2)} = \frac{210 \times 10^9}{8} \; [-1 \; 0 \; 1 \; 0] \begin{Bmatrix} -2.285658 \times 10^{-3} \\ 0 \\ 0 \\ -1.027609703 \times 10^{-2} \end{Bmatrix}$$

$$\sigma^{(2)} = 59.9985525 \times 10^6 \frac{\text{N}}{\text{mm}^2} = 60.0 \text{ MPa (T)}$$

$$\sigma^{(3)} = \frac{210 \times 10^9}{2\sqrt{5}} \left[\frac{2}{\sqrt{5}} \quad \frac{-1}{\sqrt{5}} \quad \frac{-2}{\sqrt{5}} \quad \frac{1}{\sqrt{5}} \right] \cdot \begin{Bmatrix} 0 \\ -1.027609703 \times 10^{-2} \\ 8.93812942 \times 10^{-4} \\ -9.5532489 \times 10^{-3} \end{Bmatrix}$$

$$= -22.36033275 \times 10^6 \frac{\text{N}}{\text{mm}^2} = -22.36 \text{ MPa (C)}$$

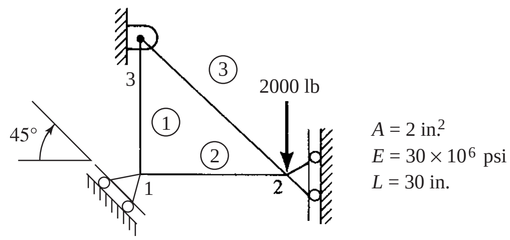
$$\sigma^{(4)} = \frac{210 \times 10^9}{2\sqrt{5}} \left[\frac{-2}{\sqrt{5}} \quad \frac{-1}{\sqrt{5}} \quad \frac{2}{\sqrt{5}} \quad \frac{1}{\sqrt{5}} \right] \begin{Bmatrix} 8.93812942 \times 10^{-4} \\ 9.5532489 \times 10^{-3} \\ 0 \\ -9.8951503 \times 10^{-3} \end{Bmatrix}$$

$$= -44.72007296 \times 10^6 \frac{\text{N}}{\text{mm}^2} = -44.72 \text{ MPa (C)}$$

$$\sigma^{(5)} = \frac{210 \times 10^9}{4} \; [0 \; -1 \; 0 \; 1] \begin{Bmatrix} 0 \\ -1.027609703 \times 10^{-2} \\ 0 \\ -9.8951503 \times 10^{-3} \end{Bmatrix}$$

$$= 19.99970332 \times 10^6 \frac{\text{N}}{\text{mm}^2} = 20.00 \text{ MPa (T)}$$

3.49



Element (1)

$\theta = 90°$

$$[k^{(1)}] = \frac{2 \times 30 \times 10^6}{30} \begin{array}{c} \begin{array}{cc} (1) & (3) \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Element (2)

$\theta = 0°$

$$[k^{(2)}] = \frac{2\times 30\times 10^6}{30}\begin{array}{c} \begin{array}{cc}(1) & (2)\end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\end{array}$$

Element (3)

$\theta = 135°$

$$[k^{(3)}] = \frac{2\times 30\times 10^6}{30\sqrt{2}}\begin{array}{c} \begin{array}{cc}(2) & (3)\end{array} \\ \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}\end{array}$$

Assembling the stiffness matrices

$$[K] = 2\times 10^6\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1+\frac{0.5}{\sqrt{2}} & -\frac{0.5}{\sqrt{2}} & -\frac{0.5}{\sqrt{2}} & \frac{0.5}{\sqrt{2}} \\ 0 & 0 & -\frac{0.5}{\sqrt{2}} & \frac{0.5}{\sqrt{2}} & \frac{0.5}{\sqrt{2}} & -\frac{0.5}{\sqrt{2}} \\ 0 & 0 & -\frac{0.5}{\sqrt{2}} & \frac{0.5}{\sqrt{2}} & \frac{0.5}{\sqrt{2}} & -\frac{0.5}{\sqrt{2}} \\ 0 & -1 & \frac{0.5}{\sqrt{2}} & -\frac{0.5}{\sqrt{2}} & -\frac{0.5}{\sqrt{2}} & 1+\frac{0.5}{\sqrt{2}} \end{bmatrix}$$

$$[K] = 10^6\begin{bmatrix} 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -2 \\ -2 & 0 & 2.707 & -0.707 & -0.707 & 0.707 \\ 0 & 0 & -0.707 & 0.707 & 0.707 & -0.707 \\ 0 & 0 & -0.707 & 0.707 & 0.707 & -0.707 \\ 0 & -2 & 0.707 & -0.707 & -0.707 & 2.707 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0.707 & -0.707 & 0 & 0 & 0 & 0 \\ 0.707 & 0.707 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T^T] = \begin{bmatrix} 0.707 & 0.707 & 0 & 0 & 0 & 0 \\ -0.707 & 0.707 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T]\,[K]\,[T^T] = 10^6 \begin{bmatrix} 2.0 & 0 & -1.4 & 0 & 0 & 1.4 \\ 0 & 2.0 & -1.4 & 0 & 0 & -1.4 \\ -1.4 & -1.4 & 2.707 & -0.707 & -0.707 & 0.707 \\ 0 & 0 & -0.707 & 0.707 & 0.707 & -0.707 \\ 0 & 0 & -0.707 & 0.707 & 0.707 & -0.707 \\ 1.4 & -1.4 & 0.707 & -0.707 & -0.7 & 2.707 \end{bmatrix}$$

Applying the boundary conditions

$$v_1 = u_2 = u_3 = v_3 = 0$$

$$\begin{Bmatrix} F'_{1x} = 0 \\ F_{2y} = -2000 \end{Bmatrix} = 10^6 \begin{bmatrix} 2.0 & 0 \\ 0 & 0.707 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_2 \end{Bmatrix}$$

$$\Rightarrow \quad u'_1 = 0$$

$$v_2 = -0.00283 \text{ in.}$$

$$\begin{Bmatrix} F'_{1x} = 0 \\ F'_{1y} = 0 \\ F_{2x} = 0 \\ F_{2y} = -2000 \\ F_{3x} = 0 \\ F_{3y} = 0 \end{Bmatrix} = 10^6 \begin{bmatrix} 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -2 \\ -2 & 0 & 2.707 & -0.707 & -0.707 & 0.707 \\ 0 & 0 & -0.707 & 0.707 & 0.707 & -0.707 \\ 0 & 0 & -0.707 & 0.707 & 0.707 & -0.707 \\ 0 & -2 & 0.707 & -0.707 & -0.707 & 0.707 \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00283 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow F_{1x} = 0$$

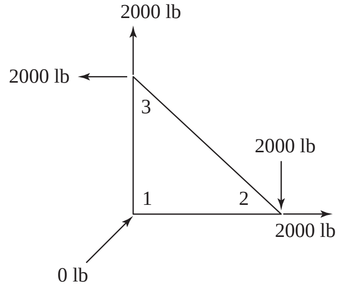
$$F_{1y} = 0$$

$$F_{2x} = 2000 \text{ lb}$$

$$F_{2y} = -2000 \text{ lb}$$

$$F_{3x} = -2000 \text{ lb}$$

$$F_{3y} = 2000 \text{ lb}$$



Element stresses

$$\{\sigma\} = [C']\,\{d\}$$

$$[C'] = \frac{E}{L}\,[-C \quad -S \quad C \quad S]$$

$$\sigma^{(1)} = \frac{30 \times 10^6}{30}\,[0 \quad -1 \quad 0 \quad 1] \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix}$$

$$\Rightarrow \ \sigma^{(1)} = 0$$

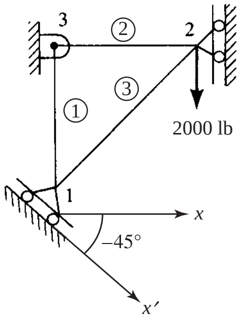
$$\sigma^{(2)} = \frac{30\times10^6}{30}\,[-1 \quad 0 \quad 1 \quad 0]\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 = 0 \\ v_2 = -0.00283 \end{Bmatrix}$$

$$\Rightarrow \ \sigma^{(2)} = 0$$

$$\sigma^{(3)} = \frac{30\times10^6}{30\sqrt{2}}\,[0.707 \quad -0.707 \quad -0.707 \quad 0.707]\begin{Bmatrix} 0 \\ -0.00283 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \ \sigma^{(3)} = 1414 \text{ psi (T)}$$

3.50



$$[k^{(1)}] = 2\times10^6\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$[k^{(2)}] = 2\times10^6\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k^{(3)}] = 1.414\times10^6\begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}$$

Assembling the stiffness matrices

$$[K] = 10^6\begin{bmatrix} 0.707 & 0.707 & -0.707 & -0.707 & 0 & 0 \\ 0.707 & 2.707 & -0.707 & -0.707 & 0 & -2 \\ -0.707 & -0.707 & 2.707 & 0.707 & -2.0 & 0 \\ -0.707 & -0.707 & 0.707 & 0.707 & 0 & 0 \\ 0 & 0 & -2.0 & 0 & -2.0 & 0 \\ 0 & -2.0 & 0 & 0 & 0 & 2.0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0.707 & -0.707 & 0 & 0 & 0 & 0 \\ 0.707 & 0.707 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T]\,[K]\,[T^T] \;=\; 10^6 \begin{bmatrix} 1.0 & -1.0 & 0 & 0 & 0 & 1.4 \\ -1.0 & 2.4 & -1.0 & -1.0 & 0 & -1.4 \\ 0 & -1.0 & 2.707 & 0.707 & -2 & 0 \\ 0 & -1.0 & 0.707 & 0.707 & 0 & 0 \\ 0 & 0 & 2.0 & 0 & -2.0 & 0 \\ 1.4 & -1.4 & 0 & 0 & 0 & 2.0 \end{bmatrix}$$

Applying the boundary conditions

$$v_1 = u_2 = u_3 = v_3 = 0$$

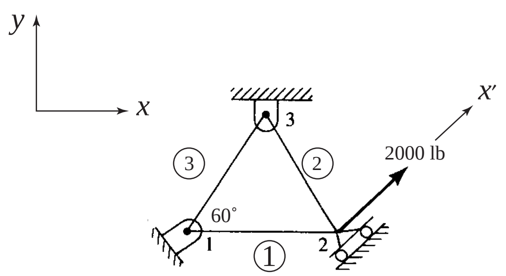
$$\begin{Bmatrix} F'_{1x} = 0 \\ F_{2y} = -2000 \end{Bmatrix} = \begin{bmatrix} 1000000 & 0 \\ 0 & 707106.75 \end{bmatrix} \begin{Bmatrix} u'_1 = 0 \\ v_2 \end{Bmatrix}$$

$$\Rightarrow \; u'_1 \;=\; 0 \qquad\qquad \sigma_{1-2} = -1414 \text{ psi}$$

$$v_2 = -\,0.00283 \text{ in.} \qquad\qquad \sigma_{1-3} = 0$$

$$\sigma_{3-2} = 0$$

3.51



$A = 2$ in.^{2}
$L = 30$ in.
$E = 30 \times 10^6$ psi

Element (1)

$$C = \; 1, S = 0, \; \theta = 0°$$

$$[k^{(1)}] = \; 2 \times 10^6 \begin{array}{c} \begin{array}{cc} (1) & \quad (2) \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Element (2)

$$C = -\frac{1}{2},\ S = \frac{\sqrt{3}}{2},\ \ \theta = 120°$$

$$[k^{(2)}] = 2 \times 10^6 \begin{array}{c} \begin{array}{cc} (2) & (3) \end{array} \\ \begin{bmatrix} 0.25 & \frac{-\sqrt{3}}{4} & -0.25 & \frac{\sqrt{3}}{4} \\ \frac{-\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{-3}{4} \\ -0.25 & \frac{\sqrt{3}}{4} & 0.25 & \frac{-\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{-3}{4} & \frac{-\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix} \end{array}$$

Element (3)

$$C = \frac{1}{2},\ \ S = \frac{\sqrt{3}}{2}$$

$$[k^{(3)}] = 2 \times 10^6 \begin{array}{c} \begin{array}{cc} (1) & (3) \end{array} \\ \begin{bmatrix} 0.25 & \frac{\sqrt{3}}{4} & -0.25 & \frac{-\sqrt{3}}{4} \\ & 0.75 & \frac{-\sqrt{3}}{4} & -0.75 \\ & & 0.25 & \frac{\sqrt{3}}{4} \\ \text{Symmetry} & & & 0.75 \end{bmatrix} \end{array}$$

Global $[K]$

$$[K] = 2 \times 10^6 \begin{bmatrix} 1.25 & \frac{\sqrt{3}}{4} & -1 & 0 & -0.25 & \frac{-\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & 0.75 & 0 & 0 & \frac{-\sqrt{3}}{4} & -0.75 \\ -1 & 0 & 1.25 & \frac{-\sqrt{3}}{4} & -0.25 & \frac{\sqrt{3}}{4} \\ 0 & 0 & \frac{-\sqrt{3}}{4} & 0.75 & \frac{\sqrt{3}}{4} & -0.75 \\ -0.25 & \frac{-\sqrt{3}}{4} & -0.75 & \frac{\sqrt{3}}{4} & 0.5 & 0 \\ \frac{-\sqrt{3}}{4} & -0.75 & \frac{\sqrt{3}}{4} & \frac{-3}{4} & 0 & 1.5 \end{bmatrix}$$

$$[T_1] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[K^*] = [T_1]\,[K]\,\left[T_1^T\right]$$

with boundary conditions

$$u_1 = v_1 = v_2 = u_3 = v_3 = 0$$

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F'_{2x} = 2000 \\ F'_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = [K^*] \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u'_2 = 0 \\ v'_2 = 0 \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix}$$

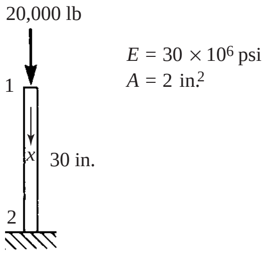
$$[T_1]\,[K] = \begin{bmatrix} 2.5\times10^6 & 8.66\times10^5 & -2\times10^6 & 0 & -5\times10^{-5} & -8.66\times10^5 \\ 8.66\times10^5 & 1.5\times10^6 & 0 & 0 & -8.66\times10^5 & -1.5\times10^6 \\ -1.414\times10^6 & 0 & 1.155\times10^6 & 4.483\times10^5 & 2.588\times10^5 & -4.483\times10^5 \\ 1.414\times10^6 & 0 & -2.38\times10^6 & 1.673\times10^6 & 9.659\times10^5 & -1.673\times10^6 \\ -5\times10^5 & -8.66\times10^5 & -1.5\times10^6 & 8.66\times10^5 & 1\times10^6 & 0 \\ -8.66\times10^5 & -1.5\times10^6 & 8.66\times10^5 & -1.5\times10^6 & 0 & 3\times10^6 \end{bmatrix}$$

$$[K^*] = [T_1]\,[K]\,\left[T_1^T\right] = \begin{bmatrix} 2.5\times10^6 & 8.66\times10^5 & -1.414\times10^6 & 1.414\times10^6 & -5\times10^5 & -8.66\times10^5 \\ 8.66\times10^5 & 1.5\times10^6 & 0 & 0 & -8.66\times10^5 & -1.5\times10^6 \\ -1.414\times10^6 & 0 & 1.134\times10^6 & -5\times10^5 & 2.588\times10^5 & -4.483\times10^5 \\ 1.414\times10^6 & 0 & -5\times10^5 & 2.866\times10^6 & 9.659\times10^5 & -1.673\times10^6 \\ -5\times10^{-5} & -8.66\times10^5 & -4.483\times10^5 & 1.673\times10^6 & 1\times10^6 & 0 \\ -8.66\times10^5 & -1.5\times10^6 & -4.483\times10^5 & -1.673\times10^6 & 0 & 3\times10^6 \end{bmatrix}$$

Solving the third equation of $[K^*]\,\{d'\} = \{F'\}$ yields

$$u'_2 = \frac{2000}{1.134\times10^6} = 1.764\times10^{-3}\text{ in.}$$

3.52 (a)



Using Equation (3.9.19), we get

$$\pi_p = \frac{AL}{2}\ \{d^T\}\ [B^T]\,[D]\,[B]\ \{d\} - \{d^T\}\{f\}$$

For the bar above, Equation 3.9.26 yields

$$\pi_p = \frac{AL}{2} u_1^2\ \frac{E}{L^2} - u_1 f$$

$$\pi_p = \frac{AE}{2L} u_1^2 - u_1 f$$

Putting in the numerical values

$$\pi_p = \frac{(2)(30 \times 10^6)}{(2)(30)}\ u_1^2\ - u_1\,(20000)$$

$$\pi_p = 10^6\ u_1^2\ - 2 \times 10^4\ u_1$$

u_1, in.	π_p lb·in.
–0.004	96
–0.002	44
0	0
0.002	–36
0.004	–54
0.006	–84
0.008	–96
0.010	–100 ←
0.012	96

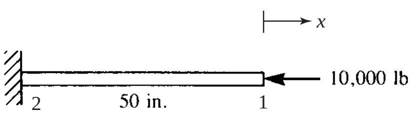
$\pi_{p\,\min} = -\,100$ lb·in.

Also by calculus

$$\frac{\partial \pi_p}{\partial u_1} = 2 \times 10^6\, u_1 - 2 \times 10^4 = 0$$

$\Rightarrow\ u_1 = 0.01$ in. for π_p minimum

(b)



$$\pi_p = \frac{AE}{2L} u_1^2 - u_1 f \text{ (see Problem 3.52 (a))}$$

$$\pi_p = \frac{2(30\times 10^6)}{2(50)} u_1^2 - u_1\,(10000)$$

$$\pi_p = 6\times 10^5 u_1^2 - 0.1\times 10^5 u_1$$

u_1 in.	π_p lb·in.
–0.004	49.6
–0.002	44.0
–0.001	10.6
0	0
0.002	4.0
0.004	–30.4
0.006	–38.4
0.008	–41.6 ←
0.010	–40.6

$\pi_{p\,\min} = -41.67$ lb·in.

By calculus

$$\frac{\partial \pi_p}{\partial u_1} = 12\times 10^5 u_1 - 0.1\times 10^5$$

$$\Rightarrow \quad u_1 = 0.00833 \text{ in. yields } \pi_{p\,\min}.$$

3.53

$$du = \sigma_x \; d\,\varepsilon_x \; du$$

$$U = \iint_v\!\int \left\{ \int_0^{\varepsilon_x} \sigma_x \, d\,\varepsilon_x \right\} dv$$

$$dv = A(x)\,dx \qquad A(x) = A_0\left(1+\frac{x}{L}\right)$$

$$\therefore \qquad U = \frac{1}{2}\int_0^L \sigma_x \; \varepsilon_x A_0 \left(1+\frac{x}{L}\right) dx$$

$$= \frac{1}{2}\int_0^L \{\hat{d}\}^T \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} [E] \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \{d\} \; A_0\left(1+\frac{x}{L}\right) dx$$

$$= \frac{1}{2}\int_0^L \{\hat{u}_1 \quad \hat{u}_2\} \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} [E] \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} A_0\left(1+\frac{x}{L}\right) dx$$

$$U = \frac{1}{2}\int_0^L \frac{EA_0}{L^2}(u_1^2 - 2u_1 u_2 + u_2^2)\left(1+\frac{x}{L}\right) dx$$

$$\frac{\partial U}{\partial u_1} = \frac{A_0 L}{2}\left[\frac{E}{L^2}(2u_1 - 2u_2)\right] + \frac{A_0}{2}\left[\frac{E}{L^2}(2u_1 - 2u_2)\right]\frac{1}{2}$$

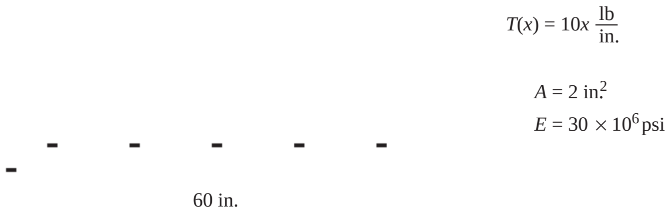
$$\frac{\partial U}{\partial u_1} = \frac{3A_0 L}{4}\left[\frac{E}{L^2}(2u_1 - 2u_2)\right] = f_1$$

Similarly

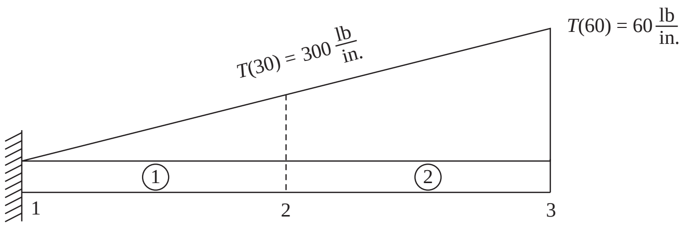
$$\frac{\partial U}{\partial u_2} = \frac{3AL}{4}\left[\frac{E}{L^2}(2u_2 - 2u_1)\right] = f_2$$

$$\therefore \quad K = \frac{3AE}{2L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

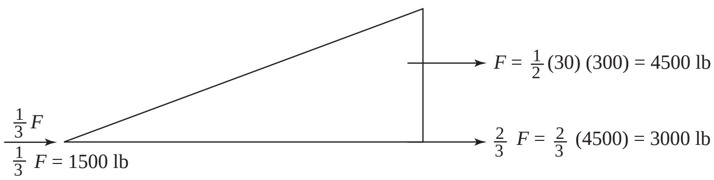
3.54



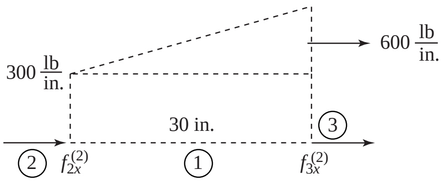
(a) Two element solution



For element (1) force matrix is from Example 3.9



$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1500 \\ 3000 \end{Bmatrix}$$



$$f_{2x}^{(2)} = \frac{1}{3}\,(4500) + \frac{1}{2}\,(30 \times 300)$$

$$f_{3x}^{(2)} = \frac{2}{3}\,(4500) + 30 \times 300 \times \frac{1}{2}$$

$$f_{2x}^{(2)} = 6000 \text{ lb} \qquad f_{3x}^{(2)} = 7500 \text{ lb}$$

$$\therefore \quad \begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{Bmatrix} 6000 \\ 7500 \end{Bmatrix}$$

$$\therefore \quad \{F\} = \begin{Bmatrix} F_{1x} + 1500 \\ 3000 + 6000 \\ 7500 \end{Bmatrix}$$

Stiffness matrices

$$[k^{(1)}] = \frac{AE}{30} \begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$[k^{(2)}] = \frac{AE}{30} \begin{array}{c} \begin{matrix} 2 & 3 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$[k^{(3)}] = \frac{AE}{30} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Global equations

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} + 1500 \\ 9000 \\ 7500 \end{Bmatrix} = \frac{(2)(30 \times 10^6)}{30} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \end{Bmatrix}$$

Solving equations 2 and 3 for u_2 and u_3, we obtain

$$u_2 = 0.00825 \text{ in.}$$

$$u_3 = 0.012 \text{ in.}$$

Element stresses

$$\sigma^{(1)} = [C']\,\{d\} = \frac{E}{L}\,[-C \;\; -S \;\; C \;\; S] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

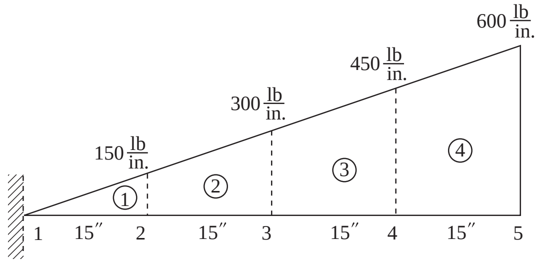
$$= \frac{30 \times 10^6}{30}\,[-1 \;\; 0 \;\; 1 \;\; 0] \begin{Bmatrix} 0 \\ 0 \\ 0.00825 \\ 0 \end{Bmatrix}$$

$$\sigma^{(1)} = 10^6\,(0.00825) = 8250 \text{ psi (T)}$$

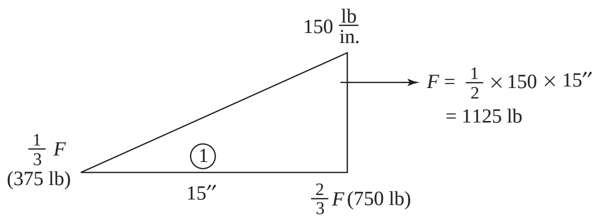
$$\sigma^{(2)} = \frac{30\times10^6}{30}\,[-1\ \ 0\ \ 1\ \ 0]\begin{Bmatrix} u_2 = 0.00825 \\ v_2 = 0 \\ u_3 = 0.012 \\ v_3 = 0 \end{Bmatrix}$$

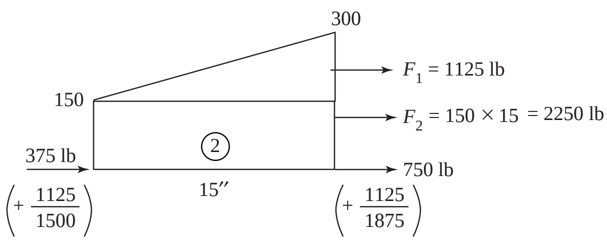
$$\sigma^{(2)} = 3750 \text{ psi (T)}$$

Four element solution

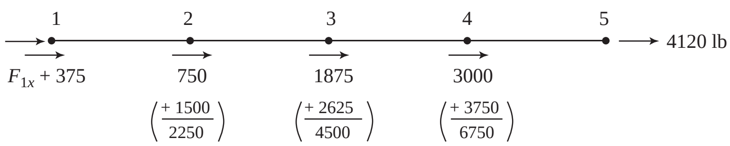


Basic triangular load





Total global forces at nodes



Global equations

$$\begin{Bmatrix} \cancel{F_{1x}+375} \\ 2250 \\ 4500 \\ 6750 \\ 4120 \end{Bmatrix} = \frac{AE}{15}\begin{bmatrix} \cancel{1} & \cancel{-1} & \cancel{0} & \cancel{0} & \cancel{0} \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} \cancel{u_1 = 0} \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}$$

Solve the last four equations for u_2 through u_5

$$2250 = \frac{AE}{15}\,(2\,u_2 - u_3) \qquad (1)$$

$$4500 = \frac{AE}{15}\,(-u_2 + 2u_3 - u_4) \tag{2}$$

$$6750 = \frac{AE}{15}\,(-u_3 + 2u_4 - u_5) \tag{3}$$

$$4120 = \frac{AE}{15}\,(-u_4 + u_5) \tag{4}$$

Using Gaussian Elimination, divide (1) by (2)

$$1125 = \left(u_2 - \frac{1}{2}u_5\right)\frac{AE}{15} \tag{5}$$

Add (5) to (2)

$$5625 = (1\tfrac{1}{2}\,u_3 - u_4)\,\frac{AE}{15} \tag{6}$$

$$6750 = (-u_3 + 2u_4 - u_5)\,\frac{AE}{15} \tag{7}$$

$$4120 = (-u_4 + u_5)\,\frac{AE}{15} \tag{8}$$

$$1125 = (u_2 - \tfrac{1}{2}\,u_3)\,\frac{AE}{15} \tag{9}$$

Divide (6) by 1.5

$$3750 = \left(0 + u_3 - \frac{1}{1.5}u_4\right)\frac{AE}{15} \tag{10}$$

Add (10) to (7)

$$10500 = [(2 - 0.067)\,u_4 - u_5)]\,\frac{AE}{15} \tag{11}$$

$$4120 = (-u_4 + u_5)\,\frac{AE}{15} \tag{12}$$

$$1125 = \left(u_2 - \frac{1}{2}u_3\right)\frac{AE}{15} \tag{13}$$

$$3750 = \left(-\frac{1}{1.5}u_4\right)\frac{AE}{15} \tag{14}$$

$$\frac{3}{4} \times (11)$$

$$\Rightarrow 7875 = \left(u_4 - \frac{3}{4}u_5\right)\frac{AE}{15} \tag{15}$$

Add (15) to (12)

$$11995 = \frac{1}{4}u_5\,\frac{AE}{15} \tag{16}$$

Solve (16) for u_5

$$u_5 = \frac{11995 \times 4 \times 15}{2 \times 30 \times 10^6} = 0.012 \text{ in.}$$

By (15)

$$u_4 = 7875 \times \frac{15}{2 \times 30 \times 10^6} + \frac{3}{4}\ (0.012 \text{ in.})$$

$$\Rightarrow \quad u_4 = 0.01097 \text{ in.}$$

By (10)

$$u_3 = \frac{3750\,(15'')}{2 \times 30 \times 10^6} + \frac{1}{1.5}\ (0.01097)$$

$$\Rightarrow \quad u_3 = 0.00825 \text{ in.}$$

By (9)

$$u_2 = \frac{1125 \times 15}{2 \times 30 \times 10^6} + \frac{1}{2}\ (0.00825)$$

$$\Rightarrow \quad u_2 = 0.00441 \text{ in.}$$

$$\sigma^{(1)} = \frac{30 \times 10^6}{15}\ [-1\ \ 1] \begin{Bmatrix} 0 \\ 0.00441 \end{Bmatrix}$$

$$\sigma^{(1)} = 8812 \text{ psi (T)}$$

$$\sigma^{(2)} = \frac{30 \times 10^6}{15}\ [-1\ \ 1] \begin{Bmatrix} 0.00441 \\ 0.00825 \end{Bmatrix}$$

$$\sigma^{(2)} = 7688 \text{ psi (T)}$$

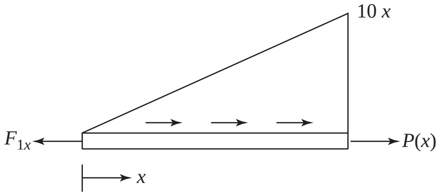
$$\sigma^{(3)} = \frac{30 \times 10^6}{15}\ [-1\ \ 1] \begin{Bmatrix} 0.00825 \\ 0.01097 \end{Bmatrix}$$

$$\sigma^{(3)} = 5440 \text{ psi (T)}$$

$$\sigma^{(4)} = \frac{30 \times 10^6}{15}\ [-1\ \ 1] \begin{Bmatrix} 0.01097 \\ 0.012 \end{Bmatrix}$$

$$\sigma^{(4)} = 2060 \text{ psi (T)}$$

Exact solution



$$F_{1x} = \frac{1}{2}\ (60'')\ (600\ \frac{\text{lb}}{\text{in.}})$$

$$F_{1x} = 18000 \text{ lb}$$

$$P(x) = 18000 - \frac{1}{2}\ (10\ x)\ x$$

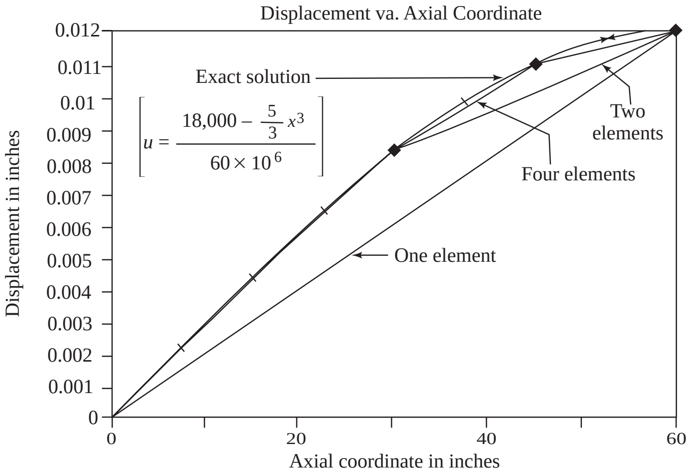
$$P(x) = 18000 - 5\ x^2$$

$$u(x) = \int_0^x \frac{(18000 - 5\ x^2)}{AE}\ dx$$

$$= \frac{1}{AE}\left[18000\ x - \frac{5}{3}\ x^3\right] + C$$

$$u(0) = 0 = C$$

$$u = \frac{18000 - \frac{5}{3}\ x^3}{60 \times 10^6}$$

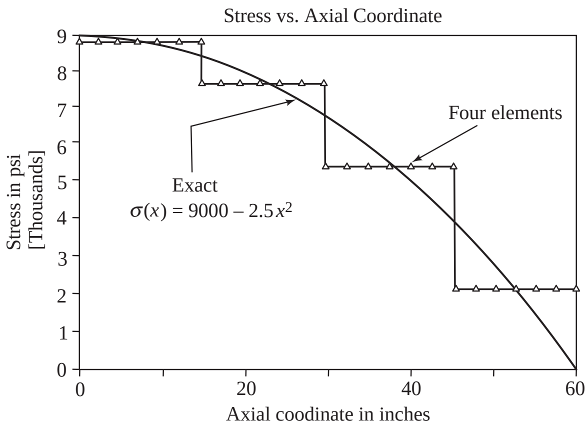


Analytical comparison with FEM

Element stress

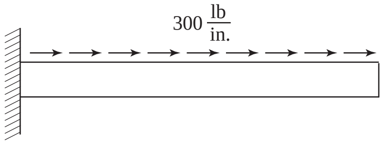
Exact $\sigma(x)$

$$\sigma(x) = \frac{P(x)}{A} = 9000 - 2.5\ x^2$$



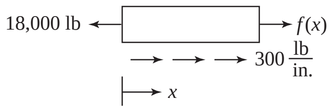
Analytical comparison with FEM

3.55



Analytical solution

$F(x)$ at wall = – 300 (60) = – 18000 lb



$$\therefore \quad f(x) = 18000 - 300\,x$$

$$\delta x = \text{displacement}$$

$$\sigma(x) = \frac{f(x)}{A} = \frac{18000 - 300\,x}{2}$$

$$\Rightarrow \quad \sigma(x) = 9000 - 150\,x \qquad (1)$$

$$\delta(x) = \int_0^x \frac{\sigma(x)}{E}\,dx$$

$$\Rightarrow \quad \delta(x) = \frac{1}{E}\left[\frac{-150\,x^2}{2} + 9000\,x\right] + C$$

Applying the boundary conditions

$$\delta(x = 0) = 0 = C$$

$$\delta(x) = -2.5 \times 10^{-6}\,x^2 + 3 \times 10^{-4}\,x \qquad (2)$$

Finite element solutions

(i) One element

Replace the distributed force with a concentrated force.

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 \end{Bmatrix}$$

$$F_{1x} = F_{2x} = \frac{1}{2}\,F = \frac{1}{2} \times 300 \times 60 = 9000 \text{ lb}$$

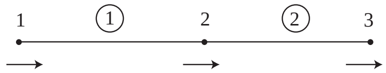
Solving for d_{2x}

$$u_2 = \frac{F_{2x}\,L}{A\,E} = \frac{9000 \times 60}{2 \times 30 \times 10^6}$$

$$u_2 = 0.009 \text{ in.}$$

$$\sigma = \frac{f_{2x}}{A} = \frac{9000}{2} = 4500 \text{ psi}$$

(2) Two elements



$$f_{1x} = \frac{300 \times 30}{2}$$

$$f_{2x} = \frac{300 \times 30}{2}(1+1)$$

$$f_{3x} = \frac{300 \times 30}{2}$$

$$f_{1x} = 4500 \text{ lb}$$

$$f_{2x} = 9000 \text{ lb}$$

$$f_{3x} = 4500 \text{ lb}$$

Global equation

$$\begin{Bmatrix} F_{1x} = 4500 \\ F_{2x} = 9000 \\ F_{3x} = 4500 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \end{Bmatrix}$$

Solving the two equations

$$u_2 = 6.75 \times 10^{-3} \text{ in.}$$

$$u_3 = 0.009 \text{ in.}$$

$$\sigma^{(1)} = \frac{E}{L}\,[-1 \;\; 1]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\sigma^{(1)} = \frac{30 \times 10^6}{30}\,[-1 \;\; 1]\begin{Bmatrix} 0 \\ 6.75 \times 10^{-3} \end{Bmatrix}$$

$$\Rightarrow \quad \sigma^{(1)} = 6750 \text{ psi (T)}$$

$$\sigma^{(2)} = \frac{30 \times 10^6}{30}\,[-1 \;\; 1]\begin{Bmatrix} 6.75 \times 10^{-2} \\ 0.009 \end{Bmatrix}$$

$$\Rightarrow \quad \sigma^{(2)} = 2250 \text{ psi (T)}$$

Computer solutions

One element

NUMBER OF ELEMENTS (NELE) = 1
NUMBER OF NODES (KNODE) = 2

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)
1	1 1 1	0.000000E+00	0.000000E+00	0.000000E+00
2	0 1 1	6.000000E+01	0.000000E+00	0.000000E+00

FORCE (1, K)	FORCE (2, K)	FORCE (3, K)
0.000000E+00	0.000000E+00	0.000000E+00
9.000000E+03	0.000000E+00	0.000000E+00

ELEMENTS

K	NODE (I, K)		E(K)	A(K)
1	1	2	3.0000E+07	2.0000E+00

NUMBER OF NONZERO UPPER CO-DIAGONALS (MUD) = 5

DISPLACEMENTS	X	Y	Z
NODE NUMBER 1	0.0000E+00	0.0000E+00	0.0000E+00
NODE NUMBER 2	0.9000E–02	0.0000E+00	0.0000E+00

STRESSES IN ELEMENTS (IN CURRENT UNITS)

ELEMENT NUMBER	STRESS
1 =	0.45000E+04

Two elements

NUMBER OF ELEMENTS (NELE) = 2
NUMBER OF MODES (KNODE) = 3

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)
1	1 1 1	0.000000E+00	0.000000E+00	0.000000E+00
2	0 1 1	3.000000E+01	0.000000E+00	0.000000E+00
3	0 1 1	6.000000E+01	0.000000E+00	0.000000E+00

FORCE (1, K)	FORCE (2, K)	FORCE (3, K)
0.000000E+00	0.000000E+00	0.000000E+00
9.000000E+03	0.000000E+00	0.000000E+00
4.500000E+03	0.000000E+00	0.000000E+00

ELEMENTS

K	MODE (I, K)		K(K)	A(K)
1	1	2	3.0000E+07	2.0000E+00
2	2	3	3.0000E+07	2.0000E+00

NUMBER OF NONZERO UPPER CO-DIAGONALS (MUD) = 5

DISPLACEMENTS	X	Y	Z
NODE NUMBER 1	0.0000E+00	0.0000E+00	0.0000E+00
NODE NUMBER 2	0.6750E–02	0.0000E+00	0.0000E+00
NODE NUMBER 3	0.9000E–02	0.0000E+00	0.0000E+00

STRESSES IN ELEMENT (IN CURRENT UNITS)

ELEMENT NUMBER	STRESS
1 =	0.67500E+04
2 =	0.22500E+04

Four elements

NUMBER OF ELEMENTS (NELE) = 4
NUMBER OF MODES (KNODE) = 5

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)
1	1 1 1	0.000000E+00	0.000000E+00	0.000000E+00
2	0 1 1	1.500000E+01	0.000000E+00	0.000000E+00
3	0 1 1	3.000000E+01	0.000000E+00	0.000000E+00
4	0 1 1	4.500000E+01	0.000000E+00	0.000000E+00
5	0 1 1	6.000000E+01	0.000000E+00	0.000000E+00

FORCE (1, K)	FORCE (2, K)	FORCE (3, K)
0.000000E+00	0.000000E+00	0.000000E+00
4.500000E+03	0.000000E+00	0.000000E+00
4.500000E+03	0.000000E+00	0.000000E+00
4.500000E+03	0.000000E+00	0.000000E+00
2.250000E+03	0.000000E+00	0.000000E+00

ELEMENTS

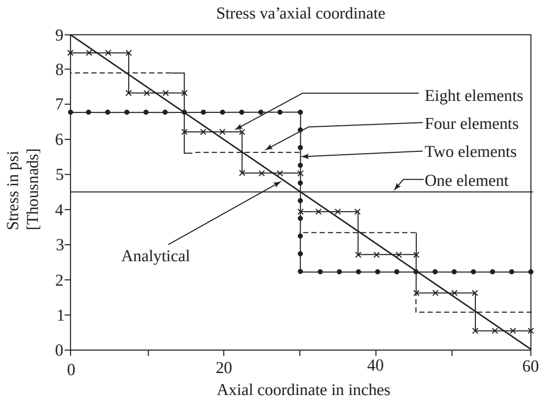
K	MODE (I, K)		K(K)	A(K)
1	1	2	3.0000E+07	2.0000E+00
2	2	3	3.0000E+07	2.0000E+00
3	3	4	3.0000E+07	2.0000E+00
4	4	5	3.0000E+07	2.0000E+00

NUMBER OF NONZERO UPPER CO-DIAGONALS (MUD) = 5

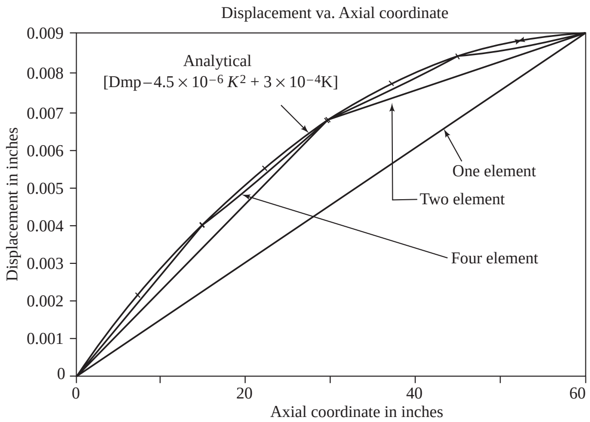
DISPLACEMENTS	X	Y	Z
NODE NUMBER 1	0.0000E+00	0.0000E+00	0.0000E+00
NODE NUMBER 2	0.3937E–02	0.0000E+00	0.0000E+00
NODE NUMBER 3	0.6750E–02	0.0000E+00	0.0000E+00
NODE NUMBER 4	0.8437E–02	0.0000E+00	0.0000E+00
NODE NUMBER 5	0.9000E–02	0.0000E+00	0.0000E+00

STRESSES IN ELEMENTS (IN CURRENT UNITS)

ELEMENT NUMBER	STRESS
1 =	0.78750E+04
2 =	0.56250E+04
3 =	0.33750E+04
4 =	0.11250E+04

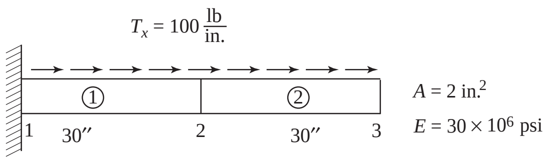


Analytical comparison with FEM

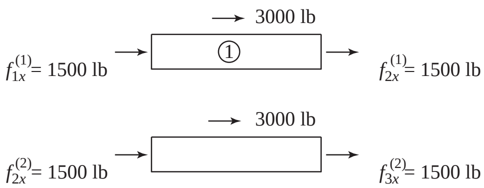


Analytical comparison with FEM

3.56



$F = (1000\,\frac{\text{lb}}{\text{in.}})\ (30\text{ in.}) = 3000\text{ lb}$



$$\{F\} = \begin{Bmatrix} F_{1x} + 1500\text{ lb} \\ 1500 + 1500 \\ F_{3x} + 1500 \end{Bmatrix}$$

$$[k^{(1)}] = \frac{AE}{30}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(2)}] = \frac{AE}{30}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Global equations

$$\frac{(2)(30\times10^6)}{30}\begin{bmatrix}1 & -1 & 0\\ -1 & 1+1 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_1=0\\ u_2\\ u_3=0\end{Bmatrix}=\begin{Bmatrix}F_{1x}+1500\\ 3000\\ F_{3x}+1500\end{Bmatrix}$$

Solving Equation (2)

$2\times10^6\,(2\,u_2) = 3000$

$$u_2 = 0.75\times10^{-3}\text{ in.}$$

Element stresses

$$\sigma^{(1)} = [C']\,\{d\} = \frac{E}{L}\,[-C \quad -S \quad C \quad S]\begin{Bmatrix}u_1\\ v_1\\ u_2\\ v_2\end{Bmatrix}$$

$$= \frac{30\times10^6}{30}\,[-1 \quad 0 \quad 1 \quad 0]\begin{Bmatrix}0\\ 0\\ 0.75\times10^{-3}\\ 0\end{Bmatrix}$$

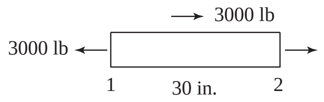
$$\Rightarrow\ \sigma^{(1)} = 750\text{ psi (T)}$$

$$F_{1x} + 1500 = 2\times10^6\,(-1)\,(0.75\times10^{-3})$$

$$\Rightarrow\ F_{1x} = -3000\text{ lb} \qquad (\leftarrow)$$

and $\qquad F_{3x} = -3000\text{ lb} \qquad (\leftarrow)$

Total applied force = $60\times100 = 6000$ lb $(\rightarrow)$

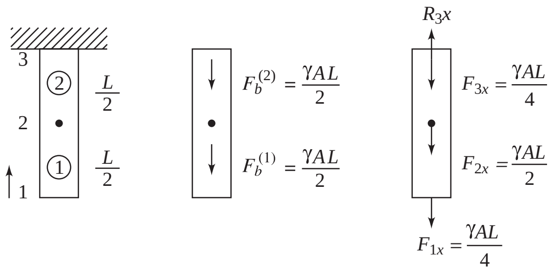


$$\sigma = 0\text{ (at node 2)}$$

$$\sigma\,(x = 15'') = \frac{3000-1500}{2} = 750\text{ psi}$$

3.57 Bar hanging under own weight

Two element solution



$$\frac{AE}{\frac{L}{2}}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3=0\end{Bmatrix}=\begin{Bmatrix}\frac{\gamma AL}{4}\\ \frac{\gamma AL}{2}\\ -R_{3x}+\frac{\gamma AL}{4}\end{Bmatrix}$$

$$u_1 - u_2 = \frac{\gamma AL^2}{2\,AE}\left(\frac{1}{4}\right) \qquad (1)$$

$$-u_1 + 2u_2 = \frac{\gamma AL^2}{2\,AE}\left(\frac{1}{2}\right) \qquad (2)$$

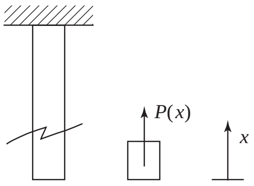
Adding (1) and (2)

$$u_2 = \frac{\gamma L^2}{2\,E}\left(\frac{3}{4}\right)$$

$$\Rightarrow \qquad u_2 = \frac{3\gamma L^2}{8\,E}(\downarrow)$$

$$u_1 = \frac{\gamma L^2}{8\,E} + \frac{3\gamma L^2}{8E} = \frac{\gamma L^2}{2\,E}\ (\downarrow)$$

Analytical solution



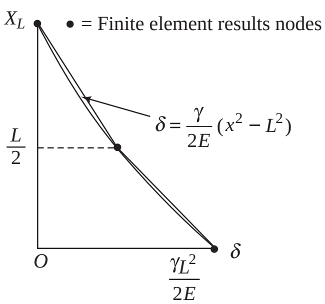
$$W_x = \gamma V(x) = \gamma Ax = P(x)$$

$$\delta = \int_0^x \frac{P(x)}{AE}\,dx = \int_0^x \frac{\gamma\,A\,x}{AE}\,dx$$

$$\delta = \frac{\gamma x^2}{2\,E} + C$$

$$\delta(L) = 0 = \frac{\gamma L^2}{2\,E} + C \Rightarrow C = -\frac{\gamma L^2}{2\,E}$$

$$\therefore \quad \delta = \frac{\gamma x^2}{2\,E} - \frac{\gamma L^2}{2\,E} = \frac{\gamma}{2\,E}(x^2 - L^2)$$



$$\sigma^{(1)} = \frac{E}{L}\,[1\ \ -1]\begin{Bmatrix} u_1 = \frac{\gamma L^2}{2\,E} \\ u_2 = \frac{3\gamma L^2}{8\,E} \end{Bmatrix}$$

$$= \gamma L = \left[\frac{1}{2} - \frac{3}{8}\right]$$

$$\sigma^{(1)} = \frac{\gamma L}{8} \text{ (T)}$$

$$\sigma^{(2)} = \frac{E}{L} [1 \ \ -1] \begin{Bmatrix} u_2 = \frac{3\gamma L^2}{8E} \\ u_3 = 0 \end{Bmatrix}$$

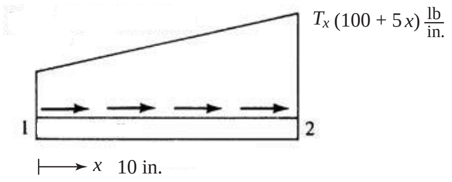
$$\sigma^{(2)} = \frac{3\gamma L}{8} \text{ (T)}$$

$$-R_{3x} = \frac{\gamma A L}{4} + \frac{2A\cancel{E}}{\cancel{L}}\left(-\frac{3\gamma L^{\cancel{2}}}{8\cancel{E}}\right)$$

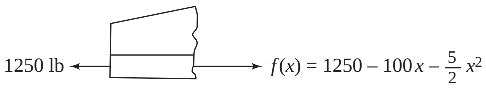
$$R_{3x} = \frac{\gamma A L}{4} + \frac{3}{4} \gamma A L$$

$$R_{3x} = \gamma A L$$

3.58



$$\text{Total } T_x = 100 \times 10 + \frac{1}{2} \times 5(10)^2 \text{ lb} = 1250 \text{ lb}$$



$$f_{1x} u_1 + f_{2x} u_2 = \int_0^{10} (100 + 5x) \left[\left(\frac{u_1 - u_2}{10}\right) x + u_1\right] dx$$

$$= \int_0^{10} \left[10(u_2 - u_1)x + 100u_1 + \frac{(u_2 - u_1)}{2}x^2 + 5u_1 x\right] dx$$

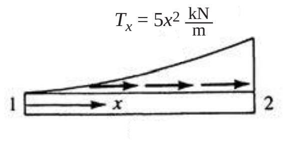
$$= 500(u_2 - u_1) + 1000u_1 + \frac{(u_2 - u_1)1000}{6} + 250u_1$$

Let $u_1 = 1$; $u_2 = 0$

$$\therefore \quad f_{1x} = -500 + 1000 - \frac{1000}{6} + 250 = 583.3$$

$$\therefore \quad f_{1x} = 583.33 \text{ lb}$$

(b)



$$f_{1x}\,u_1 + f_{2x}\,u_2 = \int_0^4 (5x^2)\left[\frac{u_2 - u_1}{4}\,x + u_1\right]$$

$$= \frac{5}{16}\,(u_2 - u_1)\,(4^4) + \frac{5\,u_1\,(4)^3}{3}$$

Let $u_1 = 0;\ u_2 = 1$

$$f_{2x} = (16)\,5 = 80 \text{ kN} = f_{2x}$$

$$u_2 = 0;\ u_1 = 1$$

$$f_{1x} = -\,(16)\,5 + \frac{5\,(4)^3}{3} + 26.67 \text{ kN} = f_{1x}$$

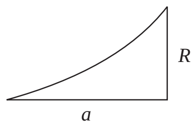
$$f_{1x} + f_{2x} = 106.67 \text{ kN (Total force)}$$

Check parabolic yields

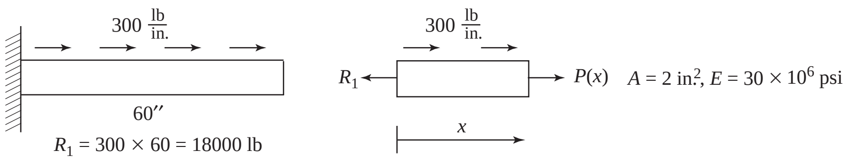
$$A = \frac{a\,R}{3}$$

$$= \frac{(4)\,\left[5\,(4^2)\right]}{3}$$

$$= 106.67 \text{ kN total force}$$



3.59



Exact solution

$$P(x) = 18000 - 300\,x$$

$$u = \frac{1}{AE}\int_0^x P(x)\,dx$$

$$= \frac{1}{AE}\int_0^x (18000 - 300\,x)\,dx$$

$$u = \frac{1}{AE}\left(18000\,x - 300\frac{x^2}{2}\right) + C_1$$

$$u\,(0) = 0 \Rightarrow C_1 = 0$$

$$u = \frac{1}{AE}(18000\,x - 150\,x^2)$$

$$u = \frac{1}{2 \times 30 \times 10^6}(18000\,x - 150\,x^2)$$

Now choose $u = C_0 + C_1x + C_2x^2$ (1)

$$\frac{du}{dx} = C_1 + 2C_2x$$

Differential equation $AE\frac{du}{dx} - P(x) = 0$ (2)

$$u(0) = 0 \Rightarrow C_0 = 0$$

Substituting (1) into (2) to yield R (error function)

$AE\,[C_1 + 2C_2\,x] - (18000 - 300\,x) = R$ (3)

By collocation:

2 unknowns so evaluate R at 2 points

a + $x = \frac{L}{2}$ and $x = L$

$$R\left(C, x = \frac{L}{2} = 0\right) = AE\left[C_1 + 2C_2\frac{L}{2}\right] - 18000 + 300\,\frac{L}{2} = 0$$

$R(C, x = L) = 0 = AE\,[C_1 + 2C_2\,L] - 18000 + 300\,L = 0$ (4)

Simplifying (4)

$-AE\,[C_1 + C_2\,L] + 18000 - 150\,L = 0$ (5.1)

$AE\,[C_1 + 2C_2\,L] - 18000 + 300\,L = 0$ (5.2)

Substituting (5.1) from (5.2)

$$AE\,C_2\,\cancel{L} + 0 + 150\,\cancel{L} = 0$$

$C_2 = \frac{-150}{AE}$ (6)

Substituting (6) into (5.1)

$$AE\left[C_1 + \frac{-150}{AE}L\right] - 18000 + 150\,L = 0$$

$$AEC_1 = 18000 - \cancel{150L} + \cancel{150L} = 0$$

$C_1 = \frac{18000}{AE}$ (7)

$u = \frac{18000}{AE}\,x + \frac{150}{AE}\,x^2 = \frac{1}{AE}(18000\,x - 150\,x^2)$ This gives the result which is same as the exact solution of differential equation.

Subdomain method:

Use 2 subintervals as 2 C's

$$\int_0^{L/2} R dx = 0 = \int_0^{L/2} \{AE\,[C_1 + 2C_2\,x] - 18000 + 300x\}dx = 0$$

$$\int_{L/2}^{L} Rdx = 0 = \int_{L/2}^{L} \{AE\,[C_1 + 2C_2x] - 18000 + 300x\}dx = 0 \quad (8)$$

$$\left\{AE\left[C_1x + 2C_2\,\frac{x^2}{2}\right] - 18000x + 300\,\frac{x^2}{2}\right\}\Bigg|_0^{L/2} = 0$$

$$\left\{AE\left[C_1x + 2C_2\,\frac{x^2}{2}\right] - 18000x + 150\,x^2\right\}\Bigg|_{L/2}^{L} = 0 \quad (9)$$

$$AE\left[C_1\frac{L}{2} + C_2\left(\frac{L}{2}\right)^2\right] - 18000\left(\frac{L}{2}\right) + 150\left(\frac{L}{2}\right)^2 = 0$$

and $$AE\left[C_1\left(L - \frac{L}{2}\right) + C_2\left(L^2 - \left(\frac{L}{2}\right)^2\right] - 18000\left(L - \frac{L}{2}\right) + 150\left[L^2 - \left(\frac{L}{2}\right)^2\right] = 0 \quad (10)$$

Simplifying (10)

$$-AE\left[C_1\frac{L}{2} + C_2\left(\frac{L}{2}\right)^2\right] - 9000\,L + 150\left(\frac{L}{2}\right)^2 = 0 \quad (11.1)$$

$$AE\left[C_1\frac{L}{2} + C_2\,\frac{3L^2}{4}\right] - 9000\,L + 150\left(\frac{3L^2}{4}\right) = 0 \quad (11.2)$$

Substituting (11.1) from (11.2)

$$AE\,C_2\,\cancel{\frac{L^2}{2}} + 0 + 150\,\cancel{\frac{L^2}{2}} = 0$$

$$C_2 = \frac{-150}{AE} \quad (12)$$

Substituting (12) into (11.1)

$$AE\left[C_1\,\frac{L}{2} + \cancel{\left(\frac{-150}{AE}\right)\left(\frac{L}{2}\right)^2}\right] - 9000\,\cancel{L} + 150\,\cancel{\left(\frac{L}{2}\right)^2} = 0$$

$$C_1 = \frac{18000}{AE} \quad (13)$$

Same values for C_1 and C_2 as previous solution. Same u as exact solution

Least square method:

2 C's need 2 integrals

$$\int_0^L R\,\frac{\partial R}{\partial C_1}\,dx = 0 \quad \text{by (3.13.10)} \quad (14)$$

$$\frac{\partial R}{\partial C_1}\ \underline{\underline{(3)}}\ AE\,(1) \text{ and } \frac{\partial R}{\partial C_2} = AE\,2x \quad (15)$$

$$\int_0^L \{AE\,[C_1 + 2C_2x] - 18000 + 300\,x\}\,\cancel{AE}\,dx = 0$$

$$\int_0^L [AE\,(C_1 + 2C_2x] - 18000 + 300\,x\}\,AE\,2x\,dx = 0 \quad (16)$$

Integrating and simplifying

$$-\frac{x}{2}\,AE\,[C_1x+\cancel{2}\,C_2\,\frac{x^2}{\cancel{2}}]-18000\,x+\frac{300\,x^2}{2}=0$$

$$AE\left[C_1\frac{x^2}{2}+2C_2\frac{x^3}{3}\right]-18000\,\frac{x^2}{2}+300\,\frac{x^3}{3}=0 \qquad (17)$$

$$AE\left[\frac{2}{3}x^3-\frac{x^3}{2}\right]C_2+0+300\left(\frac{x^3}{3}-\frac{x^3}{4}\right)=0$$

$$AE\;\cancel{x^3}\,\frac{1}{6}\,C_2+300\;\cancel{x^3}\left(\frac{1}{12}\right)=0$$

$$C_2=\frac{-300}{12}\,\frac{6}{AE}=\frac{-150}{AE} \qquad (18)$$

Substituting (18) into (17)

$$AE\left[C_1\cancel{x}+\cancel{\left(\frac{-150}{AE}\right)}\,x^2\right]-18{,}000\,\cancel{x}+150\,\cancel{x}^2=0$$

$$C_1=\frac{18000}{AE} \qquad (19)$$

Galerkin's method:

$$\int R\,W_i\,dx=0 \quad \text{by Equation (3.13.13)} \qquad (20)$$

Need 2 equations

Let $W_1 = x$ and $W_2 = x^2$ (21)

$$\int_0^L \{AE\,[C_1+2C_2x]-18000+300\,x]\,x\,dx=0$$

$$\int_0^L \{AE\,[C_1+2C_2x]-18000+300\,x\}\,x^2\,dx=0 \qquad (22)$$

$$\left\{AE\left[C_1\frac{x^2}{2}+2C_2\frac{x^3}{3}\right]-18000\,\frac{x^2}{2}+300\,\frac{x^3}{3}\right\}_0^L=0$$

$$\left\{AE\left[C_1\frac{x^3}{3}+2C_2\frac{x^4}{4}\right]-18000\,\frac{x^3}{3}+300\,\frac{x^4}{4}\right\}\Bigg|_0^L=0$$

$$\frac{-2L}{3}\;AE\left[C_1\frac{L^2}{2}+\frac{2}{3}C_2L^3\right]-9000\,L^2+100\,L^3=0$$

$$AE\left[C_1\frac{L^3}{3}+C_2\frac{L^4}{2}\right]-6000\,L^3+\frac{300}{4}\,L^4=0 \qquad (23)$$

$$AE\left[\frac{-4}{9}C_2\,\cancel{L^4}+C_2\,\frac{\cancel{L^4}}{2}\right]+0+\left[\frac{300}{4}-\frac{200}{3}\right]\cancel{L^4}=0$$

$$AE\left[\left(\frac{-8+9}{18}\right)C_2\right]+\frac{900-800}{12}=0$$

$$C_2 = -\left(\frac{100}{\cancel{12}}\right)\frac{\overset{1.5}{\cancel{18}}}{AE}$$

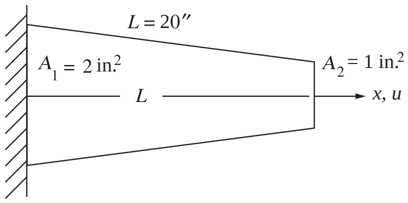
$$C_2 = \frac{-150}{AE} \tag{24}$$

(24) into (23)

$$AE\left[C_1\frac{L^2}{2}+\frac{2}{3}\left(\frac{-\cancel{150}}{\cancel{AE}}L^3\right)\right]-9000\,L^3+\cancel{100\,L^3}=0$$

$$C_1 = \frac{18000}{AE} \tag{25}$$

3.60



$E = 10 \times 10^6$ psi

$A(X) = A_1 + \left(\frac{A_2 - A_1}{L}\right)x$

Exact solution

$$P - EA(x)\,\frac{du}{dx} = 0$$

$$du = \frac{P\,dx}{EA\,(x)}$$

$$\int_0^x du = \int_0^x \frac{P\,dx}{EA\,(x)}\,dx$$

$$u(x) = \int_0^x \frac{P\,dx}{E\left[A_1+\left(\frac{A_2-A_1}{L}\right)x\right]}$$

$$\therefore\quad u(x) = \frac{PL}{E(A_2-A_1)}\left\{ln\left[A_1+\left(\frac{A_2-A_1}{L}\right)x\right]-ln\,A_1\right\}$$

x, in.	$u(x)$, in.
0	0
6.66	3.642×10^{-4}
13.32	8.099×10^{-4}
19.98	1.384×10^{-3}

Collocation method:

Let $u(x) = C_1x + C_2x^2 + C_3x^3$

$u(0) = 0$ satisfied

3 C_i's. So need error to go to zero at 3 points.

$$x = \frac{L}{3}, \frac{2L}{3}, L$$

$$A(x)\, E\, \frac{du}{dx} - P = 0$$

$$\therefore \qquad \left[A_1 + \left(\frac{A_2 - A_1}{L}\right)x\right]E(C_1 + 2C_2x + 3C_3x^2) - P = R \ (R = \text{the error})$$

$$\therefore \qquad R(C, x)\Big|_{x=\frac{L}{3}} = 0,\ R(C, x)\Big|_{x=\frac{2L}{3}} = 0,\ R(C, x)\Big|_{x=L} = 0$$

3 equations follow using Mathcad

Given

$$\left(2 - \frac{1\left(\frac{20}{3}\right)}{20}\right)E\left[C_1 + 2\,C_2\left(\frac{20}{3}\right) + 3\,C_3\left(\frac{20}{3}\right)^2\right] - 1000 = 0$$

$$\left[2 - \left(\frac{1}{20}\right)\left(\frac{40}{3}\right)\right]E\left[C_1 + 2\,C_2\left(\frac{40}{3}\right) + 3\,C_3\left(\frac{40}{3}\right)^2\right] - 1000 = 0$$

$$[2 - (1)]\, E\, (C_1 + 2\, C_2\ 20 + 3\, C_3\, 20^2) - 1000 = 0$$

$$\text{Find}\,(C_1, C_2, C_3) \rightarrow \begin{pmatrix} \frac{11}{200000} \\ 0 \\ \frac{3}{80000000} \end{pmatrix}$$

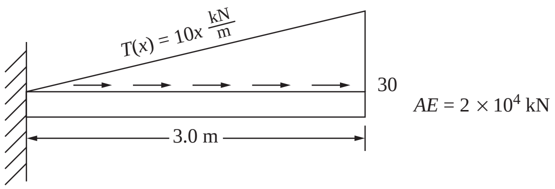
$$C_1 = \frac{11}{200000}; \qquad C_2 = 0; \qquad C_3 = \frac{3}{80000000}$$

$$uc(x) = C_1\, x + C_2\, x^2 + C_3\, x^3$$

$x =$	Collocation solution $uc(x) =$	Exact solution $u(x) =$
0	0	0
6.66	3.774×10^{-4}	3.642×10^{-4}
13.32	8.212×10^{-4}	8.099×10^{-4}
19.98	1.398×10^{-3}	1.384×10^{-3}

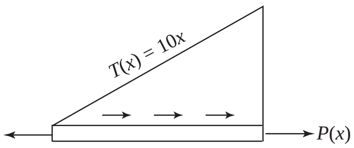
(For use of other methods see P. 3.59 and P. 3.61)

3.61



Exact solution

$$u = \frac{1}{AE}\int_0^x P(x)\, dx$$



$$R_1 = \frac{1}{2}\,(30 \times 3) = 45 \text{ kN}$$

$$P(x) = 45 - 10x\,\frac{x}{2} = 45 - 5x^2$$

$$u = \frac{1}{AE}\int_0^x (45 - 5x^2)\,dx$$

$$u = \frac{1}{AE}\left(45\,x - \frac{5x^3}{3}\right) + C$$

$$u(0) = 0 \Rightarrow C = 0$$

$$\therefore \quad u = \frac{1}{AE}\left(45\,x - \frac{5x^3}{3}\right) \qquad \text{(A)}$$

Collocation method:

$$\text{Let } u = C_1 x + C_2 x^2 + C_3 x^3 \qquad \text{(B)}$$

$$AE\,\frac{du}{dx} - P(x) = 0 \qquad (1)$$

$$\frac{du}{dx} = C_1 + 2C_2 x + 3C_3 x^2 \qquad (2)$$

$$R = AE\,[C_1 + 2C_2\,x + 3C_3 x^2] - (45 - 5x^2) = 0 \qquad (3)$$

3 C's, therefore, need 3 equations

$$R\left(C, x = \frac{L}{3}\right) = 0$$

$$R\left(C, x = \frac{2L}{3}\right) = 0 \qquad (4)$$

$$R\,(C, x = L) = 0$$

Substituting for R using Equation (3) into (4)

$$AE\left[C_1 + 2C_2\frac{L}{3} + 3C_3\left(\frac{L}{3}\right)^2\right] - 45 + 5\left(\frac{L}{3}\right)^2 = 0$$

$$AE\left[C_1 + 2C_2\left(\frac{2L}{3}\right) + 3C_3\left(\frac{2L}{3}\right)^2\right] - 45 + 5\left(\frac{2L}{3}\right)^2 = 0$$

$$AE\,[C_1 + 2\,C_2 L + 3\,C_3 L^2\,] - 45 + 5\,L^2 = 0 \qquad (5)$$

Solving for $C_1 - C_3$ in Mathcad

Given

$$AE\left(3C_3\,\frac{L^2}{9} + C_1 + 2\,C_2\,\frac{L}{3}\right) = 45 - 5\,\frac{L^2}{9}$$

$$AE\left(3\,C_3\,4\,\frac{L^2}{9}+C_1+2\,C_2\,2\,\frac{L}{3}\right) = 45 - 5\times 4\times\frac{L^2}{9}$$

$$AE\;(C_1 + 2\,C_2\;\;L + 3\,\mathrm{C}_3\;\;L^2) = 45 - 5\,L^2$$

Find $(C_1, C_2, C_3) \rightarrow \begin{pmatrix} \frac{45}{AE} \\ 0 \\ \frac{-5}{3\,AE} \end{pmatrix}$ (6)

Equation (6) into (B)

$$u = \frac{45\,x}{AE} + 0 + \frac{-5\,x^3}{3\,AE} \tag{7}$$

Equation (7) is identical to exact solution given by Equation (A).

Subdomain method:

3*C's*, 3 intervals needed

$$\int_0^{L/3} Rd_x = 0$$

$$0 = \int_0^{L/3} \{AE\,[C_1 + 2C_2x + 3C_3x^2] - (45 - 5x^2)\}dx$$

$$0 = \int_{L/3}^{2L/3} \{AE\,[C_1 + 2C_2x + 3C_3x^2] - (45 - 5x^2)\}dx$$

$$0 = \int_{2L/3}^{L} \{AE\,[C_1 + 2C_2\,x + 3C_3x^2] - (45 - 5x^2)\}dx \tag{8}$$

Integrate and simplify (8)

$$AE\left[C_1\,\frac{L}{3}+C_2\left(\frac{L}{3}\right)^2+C_3\left(\frac{L}{3}\right)^3\right]-45\,\frac{L}{3}+\frac{5}{3}\left(\frac{L}{3}\right)^3 = 0$$

$$AE\left[C_1\,\frac{L}{3}+C_2\left(\frac{1}{3}L^2\right)+C_3\,\frac{7}{27}\,L^3\right]-45\,\frac{L}{3}+\frac{5}{3}\left(\frac{7}{27}\right)L^3 = 0$$

$$AE\left[C_1\,\frac{L}{3}+C_2\,\frac{5}{9}L^2+C_3\left(\frac{19}{27}\right)L^3\right]-45\,\frac{L}{3}+\frac{5}{3}\left(\frac{19}{27}\right)L^3 = 0 \tag{9}$$

Solve using Mathcad for C_1-C_3

Given

$$AE\left[C_1\left(\frac{L}{3}\right)+C_2\left(\frac{L}{3}\right)^2+C_3\left(\frac{L}{3}\right)^3\right]-45\left(\frac{L}{3}\right)+\frac{5}{3}\left(\frac{L}{3}\right)^3 = 0$$

$$\frac{1}{3}\,AE\;\;C_1\,L + \frac{1}{3}\,AE\;C_2\,L^2 + \frac{7}{27}\,AE\;C_3\,L^3 - 15\,L + \frac{35}{81}\;\;L^3 = 0$$

$$\frac{1}{3}\,AE\;\;C_1\,L + \frac{5}{9}\,AE\;\;C_2\,L^2 + \frac{19}{27}\,AE\;C_3\,L^3 - 15\,L + \frac{95}{81}\,L^3 = 0$$

Find $(C_1, C_2, C_3) \rightarrow \begin{pmatrix} \frac{45}{AE} \\ 0 \\ \frac{-5}{3\,AE} \end{pmatrix}$ Same C's as in collocation method (10)

Least squares method:

3 C's need 3 integrals

$$\int_0^L R\,\frac{\partial R}{\partial C_i}\,dx = 0 \quad \text{by (3.13.10)} \tag{11}$$

$$\frac{\partial R}{\partial C_1} = AE, \qquad \frac{\partial R}{\partial C_2} = AE2x$$

$$\frac{\partial R}{\partial C_3} = AE3x^2 \tag{12}$$

$$\int_0^L \{AE\,[C_1 + 2C_2x + 3C_3x^2] - 45 + 5x^2]\,AE\,dx = 0$$

$$\int_0^L \{AE\,[C_1 + 2C_2x + 3C_3x^2] - 45 + 5x^2\}\,AE\,2x\,dx = 0$$

$$\int_0^L \{AE\,[C_1 + 2C_2x + 3C_3x^2] - 45 + 5x^2\}\,AE\,3x^2\,dx = 0 \tag{13}$$

Simplifying (13)

$$AE\,[C_1L + C_2L^2 + C_3L^3 - 45\,L + \frac{5}{3}\,L^3 = 0$$

$$AE\left[C_1\frac{L^2}{2} + \frac{2C_2L^3}{3} + \frac{3C_3L^4}{4}\right] - 45\frac{L^2}{2} + \frac{5\,L^4}{4} = 0$$

$$AE\left[C_1\frac{L^3}{3} + C_2\frac{L^4}{2} + \frac{3}{5}C_3\,L^5\right] - \frac{45}{3}L^3 + L^5 = 0 \tag{14}$$

Solving (14) for $C_1 - C_3$ using Mathcad.

Least squares method:

Given

$$AE\,(C_1\,L + C_2\,L^2 + C_3\,L^3) - 45\,L + \left(\frac{5}{3}\right)L^3 = 0$$

$$AE\left[C_1\left(\frac{L^2}{2}\right) + 2\,C_2\left(\frac{L^3}{3}\right) + 3\,C_3\left(\frac{L^4}{4}\right)\right] - 45\left(\frac{L^2}{2}\right) + 5\left(\frac{L^4}{4}\right) = 0$$

$$AE\left[C_1\left(\frac{L^3}{3}\right) + C_2\left(\frac{L^4}{2}\right) + \left(\frac{3}{5}\right)C_3\,L^5\right] - \frac{45}{3}L^3 + L^5 = 0$$

Find (C_1, C_2, C_3) $\begin{pmatrix} \frac{45}{AE} \\ 0 \\ \frac{-5}{3\,AE} \end{pmatrix}$ Same C's as in other methods

Galerkin's method:

$$\int_0^L R\,W_i\,dx = 0 \qquad (3.13.13)$$

Need 3 equations

Let $W_1 = x$, $W_2 = x^2$, $W_3 = x^3$

$$\int_0^L \{AE\,[C_1 + 2C_2x + 3C_3x^2] - 45 + 5x^2\}\, x\, dx = 0$$

$$\int_0^L \{AE\,[C_1 + 2C_2x + 3C_3x^2] - 45 + 5x^2\}\, x^2\, dx = 0$$

$$\int_0^L \{AE\,[C_1 + 2C_2x + 3C_3x^2] - 45 + 5x^2\}\, x^3\, dx = 0$$

$$AE\left[C_1\frac{x^2}{2} + \frac{2C_2x^3}{3} + \frac{3C_3x^4}{4}\right] - 45\frac{x^2}{2} + \frac{5x^4}{4}\Bigg|_0^L = 0$$

$$AE\left[C_1\frac{x^3}{3} + \frac{2C_2x^4}{4} + \frac{3C_3x^5}{5}\right] - 45\frac{x^3}{4} + \frac{5x^5}{5}\Bigg|_0^L = 0$$

$$AE\left[C_1\frac{x^4}{4} + \frac{2C_2x^5}{5} + \frac{3C_3x^6}{6}\right] - 45\frac{x^4}{4} + \frac{5x^6}{6}\Bigg|_0^L = 0$$

Simplifying

$$AE\left[C_1\frac{L^2}{2} + \frac{2C_2L^3}{3} + \frac{3C_3L^4}{4}\right] - 45\frac{L^2}{2} + \frac{5}{4}L^4 = 0$$

$$AE\left[C_1\left(\frac{L^3}{3}\right) + \frac{C_2L^4}{2} + \frac{3C_3L^5}{5}\right] - 15L^3 + L^5 = 0$$

$$AE\left[C_1\frac{L^4}{4} + \frac{2C_2L^5}{5} + \frac{C_3L^6}{2}\right] - \frac{45}{4}L^4 + \frac{5L^6}{6} = 0$$

Solve for $C_1 - C_3$ using Mathcad

Given

$$AE\left[C_1\left(\frac{L^2}{2}\right) + 2\,C_2\frac{L^3}{3} + 3\,C_3\frac{L^4}{4}\right] - 45\frac{L^2}{2} + \frac{5}{4}L^4 = 0$$

$$AE\left[C_1\left(\frac{L^3}{3}\right) + C_2\frac{L^4}{2} + 3\,C_3\frac{L^5}{5}\right] - 15\,L^3 + L^5 = 0$$

$$AE\left[C_1\frac{L^4}{4} + 2\,C_2\frac{L^5}{5} + C_3\frac{L^6}{2}\right] - 45\frac{L^4}{4} + 5\frac{L^6}{6} = 0$$

Find $(C_1, C_2, C_3) \rightarrow \begin{pmatrix} \frac{45}{AE} \\ 0 \\ \frac{-5}{3\,AE} \end{pmatrix}$

3.62

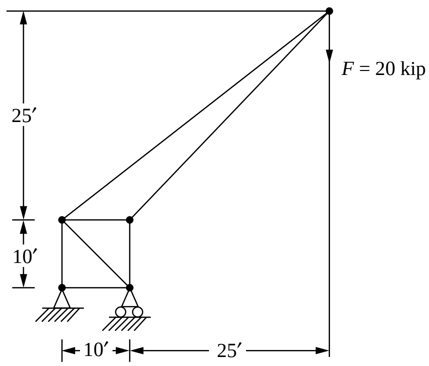


Figure P3–62 Derrick truss (FS = 4.0)

PRINT ELEMENT TABLE ITEMS PER ELEMENT
***** POST 1 ELEMENT TABLE LISTING ***
STAT CURRENT CURRENT

ELEM	SAXL	MFORX
1	$-3.3538E-12$	
2	50000	50000
3	$4.74E-12$	$4.74E-12$
4	-70000	-70000
5	-70000	-70000
6	86023	86023
7	-98995	-98995

MINIMUM	VALUES	
ELEM	7	7
VALUE	–98995	–98995
MAXIMUM	VALUES	
ELEM	6	6
VALUES	86023	86023

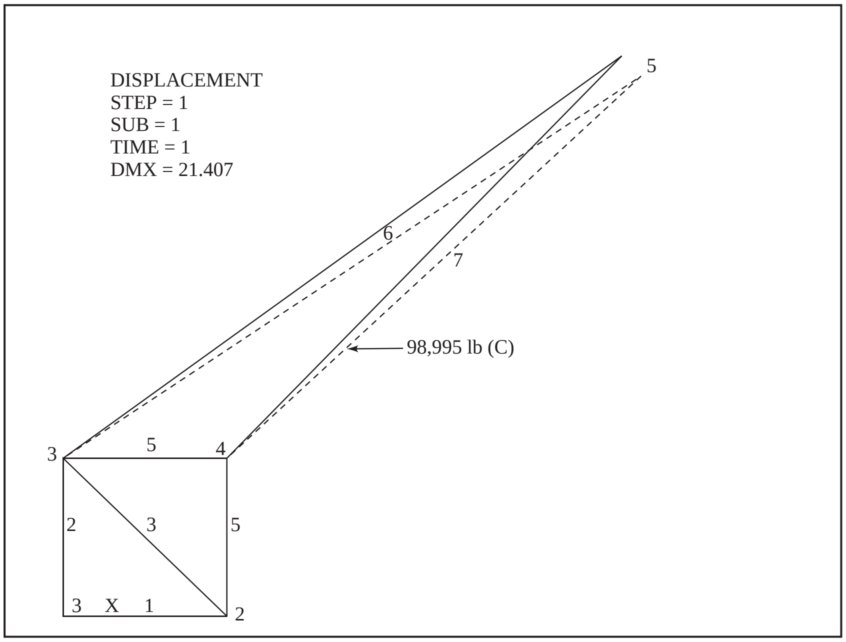


Figure 4 Free and deformed states

Try W 21 × 122 A36 steel based on critical buckling member 7. Assumed $L_e = 2.1\ L$ (conservative).

3.63

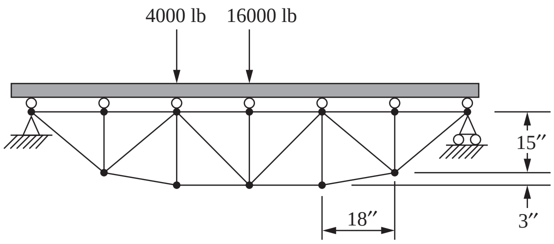
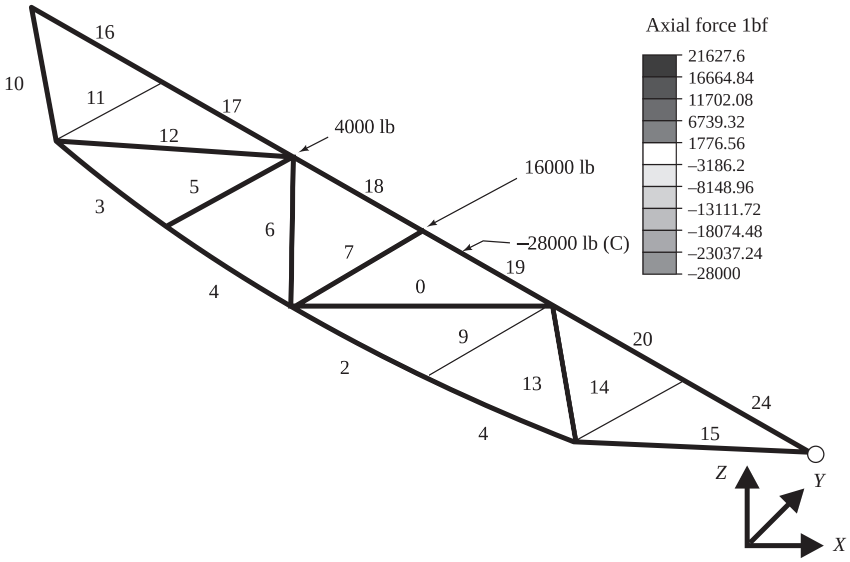


Figure P3–63 Truss bridge (FS = 3.0)



Load case: 1 of 1
Maximum value: 21627.6 lbf
Minimum value: –28000 lbf elements numbered
Try S 8 × 18.4 *A*36 steel
or 4.5″ × 4.5″ × $\frac{5}{16}$″ square tube
(assume pin-pin ends of members)

3.64

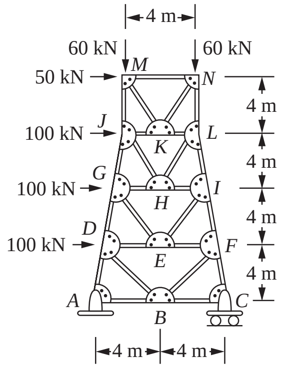


Figure P3–64 Tower (FS = 2.5)

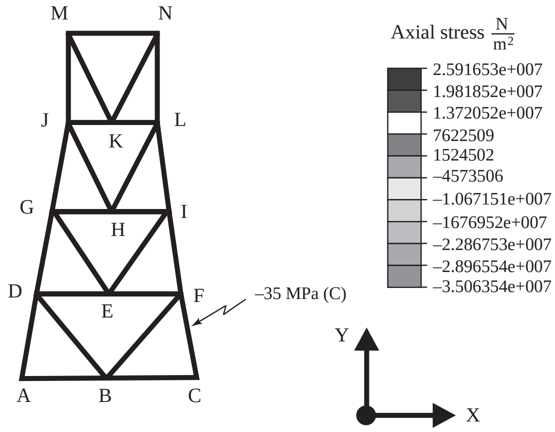


Figure 4 Stress analysis model

Try *S* 460 × 140 (mm × $\frac{\text{kg}}{\text{m}}$) *A*36 steel

$P_{\max}$ comp = – 466.35 kN in member CF

3.65

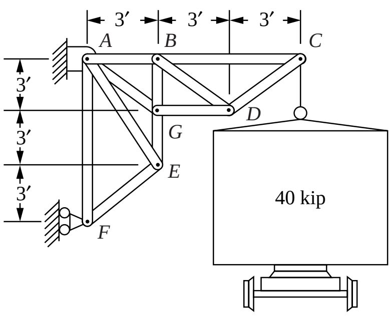


Figure P3–65 Boxcar lift (FS = 3.0)

Try one of these cross sections – 1) a square solid bar, 3.25 in. × 3.25 in., 2) a 6 × 6 × $\frac{1}{2}$ in. structural square tube, 3) a W 8 × 35 wide flange section. Any of these made of A 36 steel. The critical force of – 120,000 lb is in element *EG*. So Johnson buckling formula dictates the section selected.

3.66

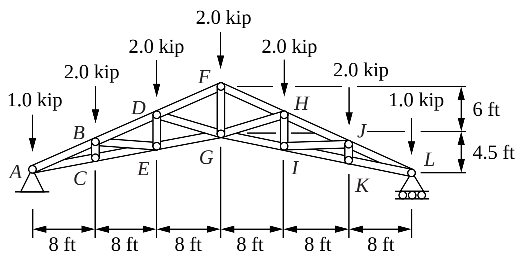


Figure P3–66 Howe scissors roof truss (FS = 2.0)

Try an *S* 6 × 12.5 or a 3 in. × 3 in. × $\frac{3}{6}$ in. structural square tube made of A 36 steel. The critical elements are *AB* and *JL* with force of –21,830 lb. Buckling dictates the cross section sizes recommended here.

3.67

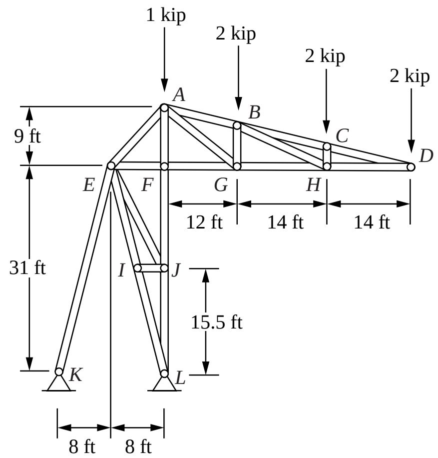
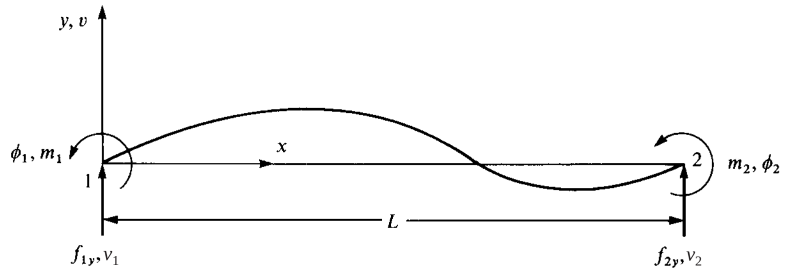


Figure P3–67 Stadium roof truss (FS = 3.0)

Try a W 6 × 25 of A 36 steel dictated by compressive force of –20,500 lb in elements *AF*, *FJ*, *JL*.

Chapter 4

4.1



Using Equation (4.1.7) plot N_1, $\dfrac{dN_2}{dx}$, N_3, $\dfrac{dN_4}{dx}$

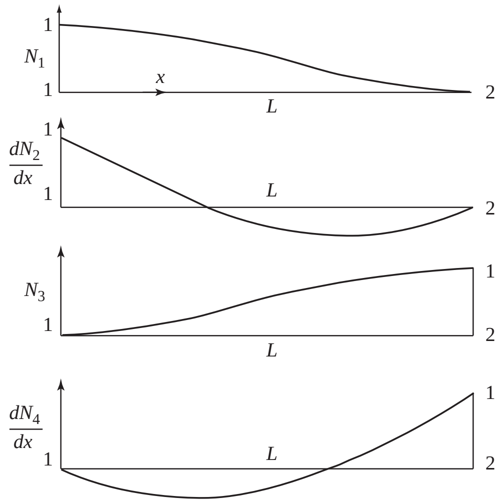
x	N_1	$\dfrac{dN_2}{dx}$	N_3	$\dfrac{dN_4}{dx}$
0	1	1	0	0
0.2L	0.896	0.32	0.104	–0.28
0.4L	0.648	–0.12	0.352	–0.32
0.6L	0.352	–0.32	0.648	–0.12
0.8L	0.104	–0.28	0.896	0.32
1.0L	0	0	1.00	1.00

where by Equation (4.1.7)

$$N_1 = \frac{1}{L^3}(2x^3 - 3x^2L + L^3)$$

$$N_2 = \frac{1}{L^3}(x^3L - 2x^2L^2 + xL^3) \text{ etc.}$$

Plots



4.2

Assumed nodal sign conventions

Note: moments opposite those in text.

$$v(0) = v_1 = a_4$$

$$\frac{dv(0)}{dx} = -\phi_1 = a_3$$

$$v(L) = v_2 = a_1L^3 + a_2L^2 + \overset{-\phi_1}{\underset{a_3L}{\Uparrow}} \;\; \overset{-v_1}{\underset{a_4}{\Uparrow}}$$

$$\frac{dv(L)}{dx} = -\phi_2 = 3a_1L^2 + 2a_2L + a_3$$

$$v_2 + \frac{1}{2}\phi_2(L) = a_1\left(L^3 - \frac{3}{2}L^3\right) + a_2(0) + a_3L - \frac{1}{2}a_3L + a_4$$

$$v_2 + \frac{\phi_2 L}{2} - \frac{1}{2}(-\phi_1)L - v_1 = a_1\left(\frac{-1}{2}L^3\right)$$

$$a_1 = \frac{2}{L^3}\left(-v_2 - \frac{\phi_2 L}{2} - \frac{\phi_1 L}{2} + v_1\right)$$

$$v_2 = \frac{2}{\cancel{L^3}}\left(-v_2 - \frac{\phi_2 L}{2} - \frac{\phi_1 L}{2} + v_1\right)\cancel{L^3} + a_2L^2 - \phi_1 L + v_1$$

$$a_2L^2 = v_2 + 2v_2 + \phi_2 L + \phi_1 L - 2v_1 + \phi_1 L - v_1$$

$$\therefore\ a_2 = \frac{3}{L^2}(v_2 - v_1) + \frac{(2\phi_1 + \phi_2)}{L^{\cancel{2}}}\cancel{L}$$

$$v = \left[\frac{2}{L^3}(v_1 - v_2) - \frac{1}{L^2}(\phi_1 + \phi_2)\right]x^3 + \left[\frac{-3}{L^2}(v_1 - v_2) + \frac{1}{L}(2\phi_1 + \phi_2)\right]x^2$$

$$- \phi_1 x + v_1$$

Note: Terms with $\phi_i s$ are opposite signs from v in Equation (4.1.4)

$$f_{1y} = V = \frac{EI}{L^3}\frac{d^3v(0)}{dx^3} = \frac{EI}{L^3}(12v_1 - 12v_2 - 6L\phi_1 - 6L\phi_2)$$

$$m_1 = m = EI\,\frac{d^2v(0)}{dx^2} = \frac{EI}{L^3}(-6Lv_1 + 6Lv_2 + 4L^2\phi_1 + 2L^2\phi_2)$$

$$f_{2y} = V = -EI\,\frac{d^3v(L)}{dx^3} = \frac{EI}{L^3}(-12v_1 + 12v_2 + 6\phi_1 L + 6\phi_2 L)$$

$$m_2 = -m = -EI\,\frac{d^2v(L)}{dx^2} = \frac{EI}{L^3}(-12Lv_1 + 12Lv_2 + 6\phi_1 L^2 +$$

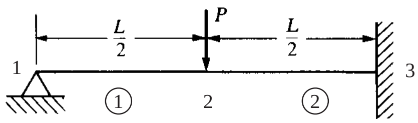
$$6\phi_2 L^2 + 6Lv_1 - 6Lv_2 - 4\phi_1 L^2 - 2\phi L^2)$$

$$m_2 = \frac{EI}{L^3}\;(-6Lv_1 + 6Lv_2 + 2L^2\phi_1 + 4L^2\phi_2)$$

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Note: All 6L terms have opposite signs from Equation (4.1.13), m_1 and m_2 are now negative of previous results.

4.3



Let $\quad \dfrac{L}{2} = l$

Element 1–2 $\qquad$ Element 2–3

$$[k_{1-2}] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}; \quad [k_{2-3}] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

$$\begin{Bmatrix} F_{1y} = ? \\ M_1 = 0 \\ F_{2y} = -P \\ M_2 = 0 \\ F_{3y} = ? \\ M_3 = ? \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ 6l & 4l^2 & -6l & 2l^2 & 0 & 0 \\ -12 & -6l & 24 & 0 & -12 & 6l \\ 6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\ 0 & 0 & -12 & -6l & 12 & -6l \\ 0 & 0 & 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} v_1 = 0 \\ \phi_1 = ? \\ v_2 = ? \\ \phi_2 = ? \\ v_3 = 0 \\ \phi_3 = 0 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 4l^2 & -6l & 2l^2 \\ -6l & 24 & 0 \\ 2l^2 & 0 & 8l^2 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Rearrange

$$\begin{array}{c} \qquad\qquad\qquad \alpha\alpha \qquad \alpha\beta \\ \begin{Bmatrix} 0 \\ 0 \\ \hline -P \end{Bmatrix} = \dfrac{EI}{l^3}\left[\begin{array}{cc:c} 4l^2 & 2l^2 & -6l \\ 2l^2 & 8l^2 & 0 \\ \hdashline -6l & 0 & 24 \end{array}\right]\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \hline v_2 \end{Bmatrix} \\ \qquad\qquad\qquad \beta\alpha \qquad \beta\beta \end{array}$$

Apply partition method

$$N = k_{\beta\beta} - k_{\beta\alpha}\ k_{\alpha\alpha}^{-1}\ k_{\alpha\beta}$$

$$= \frac{EI}{l^3}[24 - [-6l\ \ 0]\begin{bmatrix} 4l^2 & 2l^2 \\ 2l^2 & 8l^2 \end{bmatrix}^{-1}\begin{Bmatrix} -6l \\ 0 \end{Bmatrix}] = 13.7148\frac{EI}{l^3}$$

$$d_\beta = N^{-1}\ F \Rightarrow v_2 = \frac{l^3}{13.7148\ EI}\ (-P)$$

$$= \frac{-7Pl^3}{96EI} \Rightarrow \boxed{v_2 = \frac{-7PL^3}{768EI}}$$

$$\{d_\alpha\} = -[k_{\alpha\alpha}^{-1}]\ [k_{\alpha\beta}]\ \{d_\beta\}$$

$$\Rightarrow \{d_\alpha\} = \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = -\begin{bmatrix} 4l^2 & 2l^2 \\ 2l^2 & 8l^2 \end{bmatrix}\begin{Bmatrix} -6l \\ 0 \end{Bmatrix}\left[\frac{-7Pl^3}{96EI}\right]$$

$$= \frac{1}{l^2}\begin{bmatrix} 0.2857 & -0.0714 \\ -0.0714 & 0.1429 \end{bmatrix}\begin{Bmatrix} -6l \\ 0 \end{Bmatrix}\left[\frac{-7Pl^3}{96EI}\right]$$

$$\Rightarrow \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{bmatrix} \dfrac{-Pl^2}{8EI} \\ \dfrac{Pl^2}{32\ EI} \end{bmatrix} = \begin{bmatrix} \dfrac{-PL^2}{32\ EI} \\ \dfrac{PL^2}{128\ EI} \end{bmatrix}$$

Substituting back in the global matrix equation we have

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ 6l & 4l^2 & -6l & 2l^2 & 0 & 0 \\ -12 & -6l & 24 & 0 & -12 & 6l \\ 6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\ 0 & 0 & -12 & -6l & 12 & -6l \\ 0 & 0 & 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ \dfrac{-Pl^2}{8EI} \\ \dfrac{-7\ Pl^3}{96\ EI} \\ \dfrac{Pl^2}{32\ EI} \\ 0 \\ 0 \end{Bmatrix}$$

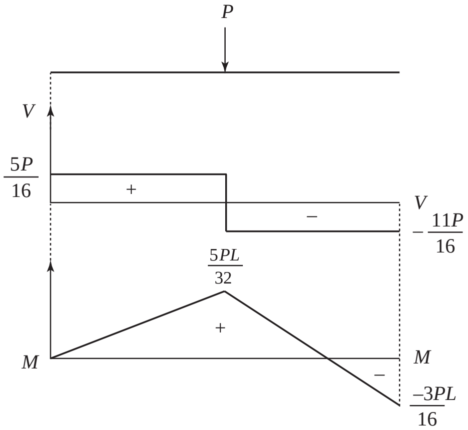
$$F_{1y} = \frac{EI}{l^3}\left[\frac{-6Pl^3}{8EI} + \frac{7Pl^3}{8EI} + \frac{6Pl^3}{32EI}\right] = \frac{EI}{l^3}\frac{10Pl^3}{32EI} \Rightarrow \boxed{F_{1y} = \frac{5P}{16}}$$

Similarly

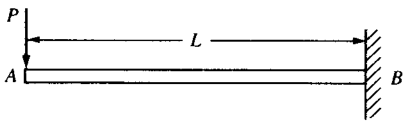
$$\boxed{M_1 = 0} \qquad \boxed{F_{3y} = \frac{11P}{16}}$$

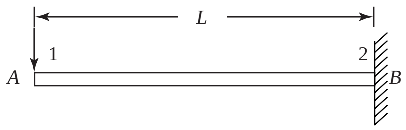
$$\boxed{F_{2y} = -P} \qquad \boxed{M_3 = \frac{-3PL}{16}}$$

$$\boxed{M_2 = 0}$$



4.4





$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & 6L & 2L^2 \\ & & 12 & -6L \\ \text{Symmetry} & & & 4L^2 \end{bmatrix}$$

Boundary conditions

$$v_2 = \phi_2 = 0$$

$$\begin{Bmatrix} -P \\ 0 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \end{Bmatrix} \Rightarrow \boxed{\phi_1 = \frac{PL^2}{2EI}}$$

$$\boxed{v_1 = \frac{-PL^3}{3EI}}$$

Matrix forces

$$\begin{Bmatrix} F_{1x} \\ M_1 \\ F_{2y} \\ M_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{Symmetry} & & & 4L^2 \end{bmatrix}\begin{Bmatrix} \frac{-PL^3}{3EI} \\ \frac{PL^2}{2EI} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \; F_{1y} = \frac{EI}{L^3}\left[-12\left(\frac{-PL^3}{3\,EI}\right) + 6L\left(\frac{-PL^2}{2\,EI}\right)\right]$$

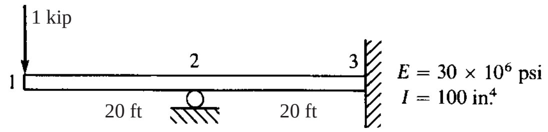
$$\Rightarrow \; F_{1y} = -P$$

Similarly $\quad M_1 = 0$

$$F_{2y} = P$$

$$M_2 = -PL$$

4.5



$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ & 4L^2 & -6L & 2L^2 & 0 & 0 \\ & & 24 & 0 & -12 & 6L \\ & & & 8L^2 & -6L & 2L^2 \\ & & & & 12 & -6L \\ \text{Symmetry} & & & & & 4L^2 \end{bmatrix}$$

$E = 30 \times 10^6$, $\quad I = 100$ in.^{4}, $\quad L = 20$ ft $= 240$ in.

$$\{F\} = [K]\{d\} \Rightarrow \begin{Bmatrix} F_{1y} = -10 \\ M_1 = 0 \\ F_{2y} = ? \\ M_2 = 0 \\ F_{3y} = ? \\ M_3 = ? \end{Bmatrix} = [K]\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} \text{ where } v_2 = v_3 = \phi_3 = 0$$

$$\begin{Bmatrix} -1000 \\ 0 \\ 0 \end{Bmatrix} = \frac{30 \times 10^6 (100)}{(240)^3}\begin{bmatrix} 12 & 6L & 6L \\ 6L & 4L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ \phi_2 \end{Bmatrix} \tag{1}$$

Solving for the displacements we have

$\phi_1 = 0.0144$ rad, $\phi_2 = 0.0048$ rad, $v_1 = -2.688$ in.

Substituting in the equation $\{F\} = [K]\{d\}$ we have

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{30 \times 10^6 (100)}{(240)}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ & 4L^2 & -6L & 2L^2 & 0 & 0 \\ & & 24 & 0 & -12 & 6L \\ & & & 8L^2 & -6L & 2L^2 \\ & & & & 12 & -6L \\ & \text{Symmetry} & & & & 4L^2 \end{bmatrix}\begin{Bmatrix} -2.688 \text{ in.} \\ 0.0144 \\ 0 \\ 0.0048 \\ 0 \\ 0 \end{Bmatrix}$$

$F_{1y} = -1000$, $M_1 = 0$,

$F_{2y} = 2500$ lb, $M_2 = 0$, $F_{3y} = -1500$ lb, $M_3 = 10$ ft $\cdot$ kip

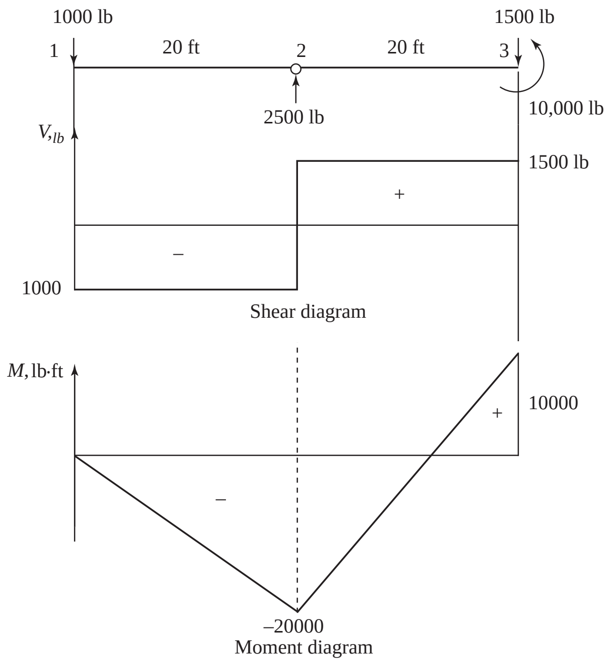
Element 1–2

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ & & & 4L^2 \end{bmatrix}\begin{Bmatrix} -2.688 \\ 0.0144 \\ 0 \\ 0.0048 \end{Bmatrix}$$

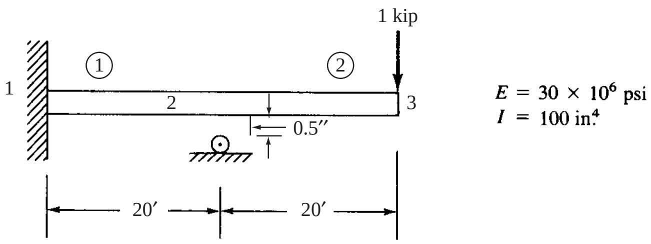
$$f_{1y} = -1000 \text{ lb}$$
$$m_1 = 0$$
$$f_{2y} = 1000 \text{ lb}$$
$$m_2 = -20000 \text{ ft}\cdot\text{lb}$$

Element 2–3

$$\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ & & & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.0048 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} f_{2y} = 1500 \text{ lb} \\ m_2 = 20000 \text{ ft}\cdot\text{lb} \\ f_{3y} = -1500 \text{ lb} \\ m_3 = 10000 \text{ ft}\cdot\text{lb} \end{matrix}$$



4.6



$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} \quad (1)$$

Boundary conditions $v_1 = 0$, $\phi_1 = 0$, $v_2 = -0.5$ in.

$$\begin{Bmatrix} F_{2y} \\ 0 \\ -1000 \\ 0 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 24 & 0 & -12 & 6L \\ 0 & 8L^3 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} -0.5\text{ in.} \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} \tag{2}$$

$$\frac{EI}{L^3} = \frac{(30\times 10^6\text{psi})(100\text{ in.}^4)}{(240\text{ in.})^3} = 217\ \frac{\text{lb}}{\text{in.}}$$

$$\begin{Bmatrix} 0 \\ -1000 \\ 0 \end{Bmatrix} = 217\begin{bmatrix} 0 & 8L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} -0.5\text{ in.} \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} \tag{3}$$

Solving (3)

$$v_3 = -\,3.938\text{ in.}$$
$$\phi_2 = -\,0.007925\text{ rad}$$
$$\phi_3 = -\,0.01753\text{ rad}$$

Back substituting into (1)

$$F_{1y} = -\,1174\text{ lb}$$
$$M_1 = -\,41{,}875\text{ lb}\cdot\text{in.}$$
$$F_{2y} = 2174\text{ lb}$$

Element 1

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 = 0 \\ \phi_1 = 0 \\ v_2 = -\,0.5\text{ in.} \\ \phi_2 = -\,0.00793 \end{Bmatrix}$$

$$= \begin{Bmatrix} -1174\text{ lb} \\ -41{,}875\text{ lb}\cdot\text{in.} \\ 1174\text{ lb} \\ -240{,}000\text{ lb}\cdot\text{in.} \end{Bmatrix}$$

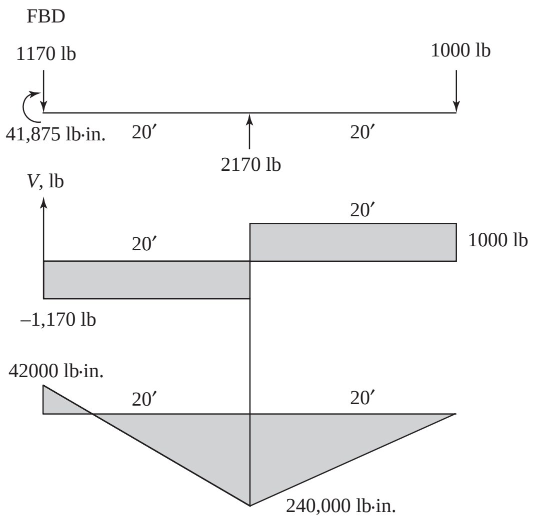
Element 2

$$\begin{Bmatrix} f_{2y}^{(2)} \\ m_2^{(2)} \\ f_{3y}^{(2)} \\ m_3^{(2)} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} -0.5\text{ in.} \\ -0.0079 \\ -3.938\text{ in.} \\ -0.01753 \end{Bmatrix}$$

$$f_{2y}^{(2)} = 1000\text{ lb} = -\,f_{3y}^{(2)}$$

$$m_2^{(2)} = 240{,}000\text{ lb}\cdot\text{in.}$$

$$m_3^{(2)} = 0$$



4.7

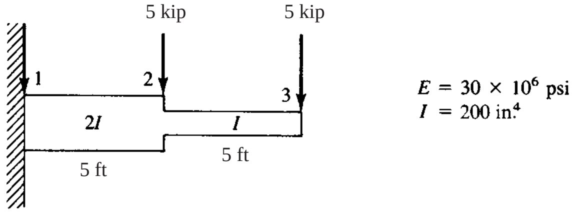
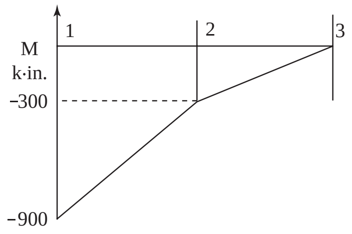


Figure P4-7



$$\begin{Bmatrix} F_{1y} \\ M_1 \\ -5000 \\ 0 \\ -5000 \\ 0 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 24 & 12L & -24 & 12L & 0 & 0 \\ 12L & 8L^2 & -12L & 4L^2 & 0 & 0 \\ -24 & -12L & 36 & -6L & -12 & 6L \\ 12L & 4L^2 & -6L & 12L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} \quad \text{(A)}$$

Solving the last four equations of (A)

$$\begin{aligned} v_2 &= -0.105 \text{ in.} \\ \phi_2 &= -0.003 \text{ rad} \\ v_3 &= -0.345 \text{ in.} \\ \phi_3 &= -0.0045 \text{ rad} \end{aligned} \quad \text{(B)}$$

(B) in (A)

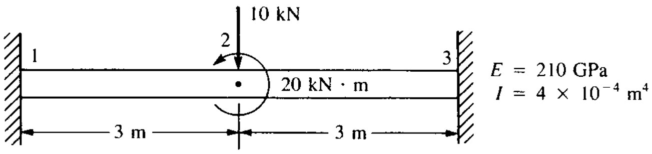
$$\Rightarrow F_{1y} = \frac{(30 \times 10^6)(200)}{60^3(1000)} [-24(-0.105) + 720(-0.003)]$$

$$= 10 \text{ kip}$$

$$M_1 = \frac{(30\times10^6)(200)}{60^3(1000\times12)}\ [-720\ (-0.105) + 14400\ (-0.003)]$$

$$= 75 \text{ kip}\cdot\text{ft}$$

4.8



$$[k_{1-2}] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{Symmetry} & & & 4L^2 \end{bmatrix}$$

$$[k_{2-3}] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{Symmetry} & & & 4L^2 \end{bmatrix}$$

Boundary conditions

$$v_1 = \phi_1 = v_3 = \phi_3 = 0$$

$$[K] = \frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix}$$

$$\begin{Bmatrix} F_{2y} = -1000 \text{ N} \\ M_2 = 20000 \text{ N}\cdot\text{m} \end{Bmatrix} = \frac{(210\times10^9)(4\times10^{-4})}{(3)^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix}$$

$$-0.0032142 = 24\ v_2 \Rightarrow v_2 = -1.34\times10^{-4} \text{ m}$$

$$0.0064285 = 72\ \phi_2 \Rightarrow \phi_2 = 8.93\times10^{-5} \text{ rad}$$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{(210\times10^9)(4\times10^{-4})}{3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -1.34\times10^{-4} \\ 8.93\times10^{-5} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow F_{1y} = 3.1\times10^6\ (-12\ (1.34\times10^{-4}) + 6(3)\ (8.93\times10^{-5})) \Rightarrow F_{1y} = 10000 \text{ N}$$

$$M_1 = 3.1\times10^6\ (-6(3)\ (1.34\times10^{-4}) + 2(3)^2\ (8.93\times10^{-5})) \Rightarrow M_1 = 12500 \text{ N}\cdot\text{m}$$

Similarly

$$F_{2y} = -10000 \text{ N}$$

$$M_2 = 20000 \text{ N}\cdot\text{m}$$

$$F_{3y} = 1.87 \text{ N}$$

$$M_3 = -2500 \text{ N}\cdot\text{m}$$

Element 1–2

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{(210\times10^9)(4\times10^{-4})}{(3)^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -1.34\times10^{-4} \\ 8.93\times10^{-5} \end{Bmatrix}$$

$$\Rightarrow f_{1y} = 10000 \text{ N}$$

$$m_1 = 12500 \text{ N}\cdot\text{m}$$

$$f_{2y} = -10000 \text{ N}$$

$$m_2 = 17500 \text{ N}\cdot\text{m}$$

Element 2–3

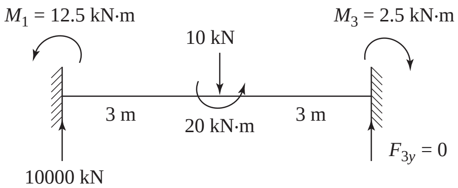
$$\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = \frac{(210\times10^9)(4\times10^{-4})}{(3)^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} -1.34\times10^{-4} \\ 8.93\times10^{-5} \\ 0 \\ 0 \end{Bmatrix}$$

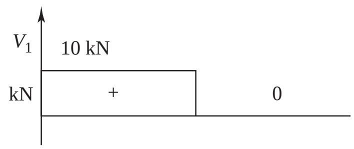
$$\Rightarrow f_{2y} = -1.87 \text{ N}$$

$$m_2 = 2500 \text{ N}\cdot\text{m}$$

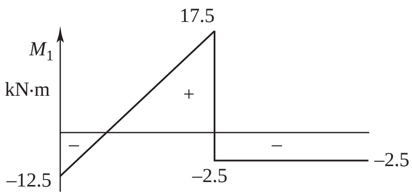
$$f_{3y} = 1.87 \text{ N}$$

$$m_3 = -2500 \text{ N}\cdot\text{m}$$



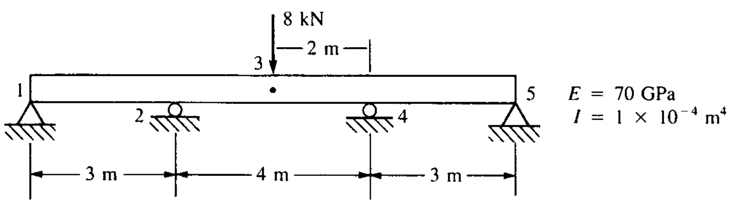


Shear diagram

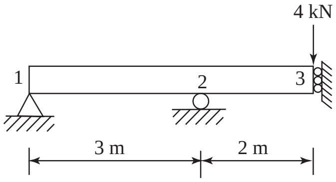


Moment diagram

4.9



Using symmetry



$$[k_{1-2}] = EI \begin{bmatrix} \frac{12}{27} & \frac{6}{9} & \frac{-12}{27} & \frac{6}{9} \\ & \frac{4}{3} & \frac{-6}{9} & \frac{2}{3} \\ & & \frac{12}{27} & \frac{-6}{9} \\ \text{Symmetry} & & & \frac{4}{3} \end{bmatrix}$$

$$[k_{2-3}] = EI \begin{bmatrix} \frac{12}{8} & \frac{6}{4} & \frac{-12}{8} & \frac{6}{4} \\ & \frac{4}{2} & \frac{-6}{4} & \frac{3}{2} \\ & & \frac{12}{8} & \frac{-6}{4} \\ & & & \frac{4}{2} \end{bmatrix}$$

Applying the boundary conditions

$v_1 = v_2 = \phi_3 = 0$ we have

$$\begin{Bmatrix} M_1 = 0 \\ M_2 = 0 \\ F_{3y} = -4000\text{ N} \end{Bmatrix} = EI \begin{bmatrix} \frac{4}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & \frac{10}{3} & \frac{-3}{2} \\ 0 & \frac{-3}{2} & \frac{3}{2} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ v_3 \end{Bmatrix}$$

which implies

$$0 = \frac{4}{3}\ \phi_1 + \frac{2}{3}\ \phi_2 \Rightarrow \phi_1 = -\frac{1}{2}\ \phi_2$$

$$0 = \frac{2}{3}\ \left(-\frac{1}{2}\phi_2\right) + \frac{10}{3}\ (\phi_2) - \frac{3}{2}\ v_3 \Rightarrow \phi_2 = \frac{1}{2}\ v_3$$

$$-4000\text{ N} = (70 \times 10^9)\ (1 \times 10^{-4}) \left[-\frac{3}{2}\left(\frac{1}{2}v_3\right) + \frac{3}{2}v_3\right]$$

$$v_3 = -7.619 \times 10^{-4}\text{ m}$$

$$\Rightarrow \quad \phi_2 = \frac{-1}{2}\ (-7.619 \times 10^{-4})\text{ m} \Rightarrow \phi_2 = -3.809 \times 10^{-4}\text{ rad}$$

$$\Rightarrow \quad \phi_1 = \frac{-1}{2}\ \phi_2$$

$$\Rightarrow \quad \phi_1 = 1.904 \times 10^{-4}\text{ rad}$$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = (70 \times 10^9)\ (1 \times 10^{-4}) \begin{bmatrix} \frac{4}{9} & \frac{2}{3} & \frac{-4}{9} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & \frac{4}{3} & \frac{-2}{3} & \frac{2}{3} & 0 & 0 \\ \frac{-4}{9} & \frac{-2}{3} & \frac{35}{18} & \frac{5}{6} & \frac{-3}{2} & \frac{3}{2} \\ \frac{2}{3} & \frac{2}{3} & \frac{5}{6} & \frac{4}{3} & \frac{-3}{2} & 1 \\ 0 & 0 & \frac{-3}{2} & \frac{-3}{2} & \frac{3}{2} & \frac{-3}{2} \\ 0 & 0 & \frac{3}{2} & 1 & \frac{-3}{2} & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ 1.904 \times 10^{-4} \\ 0 \\ -3.809 \times 10^{-4} \\ -7.619 \times 10^{-4} \\ 0 \end{Bmatrix}$$

$$F_{1y} = (7\times10^6)\left[\frac{2}{3}(1.904\times10^{-4}) + \frac{2}{3}\left(-3.809\times10^{-4}\right)\right]$$

$$\Rightarrow\ F_{1y} = -889\ \text{N}$$

Similarly

$$F_{2y} = 4889\ \text{N},\ M_3 = 5333\ \text{N}\cdot\text{m}$$

Element 1–2

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = (70\times10^9)\,(1\times10^{-4})\begin{bmatrix} \frac{4}{9} & \frac{2}{3} & \frac{-4}{9} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} & \frac{-2}{3} & \frac{2}{3} \\ \frac{-4}{9} & \frac{-2}{3} & \frac{4}{9} & \frac{-2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{-2}{3} & \frac{4}{3} \end{bmatrix}\begin{Bmatrix} 0 \\ 1.904\times10^{-4} \\ 0 \\ -3.809\times10^{-4} \end{Bmatrix}$$

$$\begin{aligned} \Rightarrow\ f_{1y} &= -889\ \text{N} \\ m_1 &= 0 \\ f_{2y} &= 889\ \text{N} \\ m_2 &= -2667\ \text{N}\cdot\text{m} \end{aligned}$$

Element 2–3

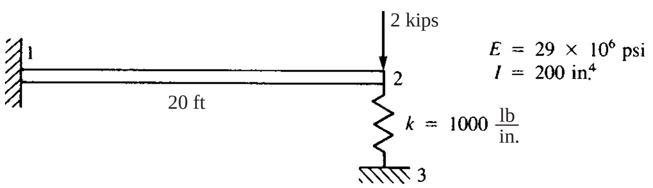
$$\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = (70\times10^9)\,(1\times10^{-4})\begin{bmatrix} \frac{3}{2} & \frac{3}{2} & \frac{-3}{2} & \frac{3}{2} \\ \frac{3}{2} & 2 & \frac{-3}{2} & 1 \\ \frac{-3}{2} & \frac{-3}{2} & \frac{3}{2} & \frac{-3}{2} \\ \frac{3}{2} & 1 & \frac{-3}{2} & 2 \end{bmatrix}\begin{Bmatrix} 0 \\ -3.809\times10^{-4} \\ -7.619\times10^{-4} \\ 0 \end{Bmatrix}$$

$$\Rightarrow\ f_{2y} = 4000\ \text{N} = -f_{3y}$$

$$m_2 = 2667\ \text{N}\cdot\text{m},\ m_3 = 5333\ \text{N}\cdot\text{m}$$

Elements 3–4 and 4–5 have same forces due to symmetry. Moments though will have opposite signs.

4.10



$$[K] = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12+\frac{KL^3}{EI} & -6L \\ \text{Symmetry} & & & 4L^2 \end{bmatrix} \end{array}\begin{array}{c} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{array}$$

Boundary conditions

$$v_1 = \phi_1 = 0$$

Applying the boundary conditions on equation $\{F\} = [K]\,\{d\}$

$$\begin{Bmatrix} F_2 = -2000\ \text{lb} \\ M_2 = 0 \end{Bmatrix} = \frac{(29\times10^6)(200)}{(20\times12)^3}\begin{bmatrix} 12+\frac{KL^3}{EI} & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix}$$

$$0 = -6(240)v_2 + 4(240)^2\,\phi_2$$

$$\Rightarrow 0 = -6v_2 + 4(240)\phi_2 \Rightarrow v_2 = 160\phi_2$$

$$-2000 = 419.56\,[(12 + (2.38)\,160\phi_2 - 6(240)\phi_2]$$

$$\Rightarrow \quad \phi_2 = -0.005538 \text{ rad}$$

$$\Rightarrow \quad v_2 = 160(-0.005538) \Rightarrow v_2 = -0.886 \text{ in.}$$

Beam element

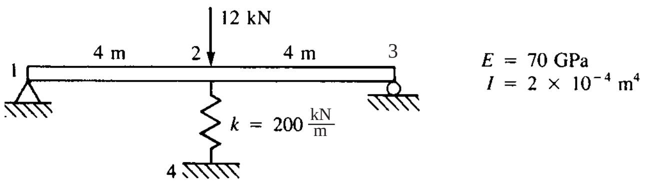
$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \end{Bmatrix} = \frac{(29\times10^6)(200)}{(20\times12)^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -0.886 \\ -0.005538 \end{Bmatrix}$$

$$F_{1y} = 1115 \text{ lbs}\uparrow,\ M_1 = -267 \text{ kip}\cdot\text{in.} \circlearrowright$$

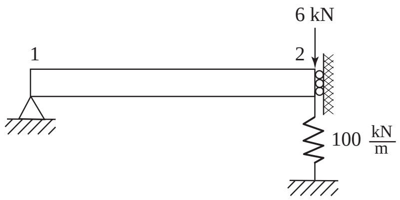
$$F_{2y} = -1115 \text{ lbs}\downarrow,\ M_2 = 0$$

The extra force at node 2 is resisted by the spring.

4.11



Applying symmetry



$$[K] = \frac{EI}{L_3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & \frac{12+KL^3}{EI} & -6L \\ \text{Symmetry} & & & 4L^2 \end{bmatrix}$$

Applying the boundary conditions $v_1 = 0$, $\phi_2 = 0$ we have

$$\begin{Bmatrix} M_1 = 0 \\ F_{2y} = -6000 \text{ N} \end{Bmatrix} = \frac{(70\times10^9)(2\times10^{-4})}{4^3}\begin{bmatrix} 4L^2 & -6L \\ -6L & \frac{12+KL^3}{EI} \end{bmatrix}\begin{Bmatrix} \phi_1 \\ v_2 \end{Bmatrix}$$

$$\Rightarrow \quad 0 = 4L^2\phi_1 - 6Lv_2 \Rightarrow \phi_1 = \frac{6}{4L}\,v_2 \Rightarrow \phi_1 = \frac{6}{16}\,v_2$$

$$-6000 = 218750\left[-24\left(\frac{6}{16}\right)v_2 + 12.457\,v_2\right]$$

$$\Rightarrow \quad v_2 = -7.9338\times10^{-3}\text{ m}$$

$$\phi_1 = \frac{6}{16}(-7.9338 \times 10^{-3}) \Rightarrow \phi_1 = -2.9752 \times 10^{-3}\ \text{rad}$$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \end{Bmatrix} = \frac{(70\times10^9)(2\times10^{-4})}{4^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12.457 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 2 \\ -2.9752\times10^{-3} \\ -7.9336\times10^{-3} \\ 0 \end{Bmatrix}$$

$F_{1y} = 5.208\ \text{kN} \uparrow,\ M_2 = 20.83\ \text{kN}\cdot\text{m}$ ↻

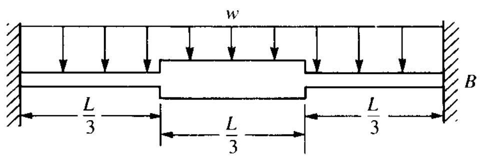
$F_{2y} = 0\ \text{kN} \downarrow$

$$F_{\text{spring}} = \left(200\ \frac{\text{kN}}{\text{m}}\right)(7.9338 \times 10^{-3}\ \text{m})$$

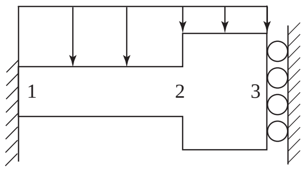
$F_{\text{spring}} = 1.587\ \text{kN}$

From symmetry $F_{3y} = 5.208\ \text{kN} \uparrow$

4.12

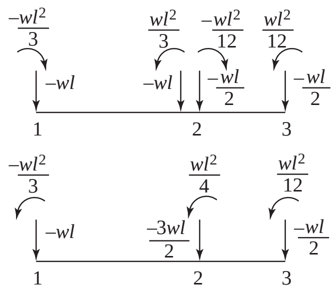


From symmetry



Let $\frac{L}{6} = l$

$\Rightarrow \frac{L}{3} = 2l$



Final nodal forces and moments

$$[k_{1-2}] = \frac{EI}{l^3}\begin{array}{c} \begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} \frac{3}{2} & \frac{3}{2}l & \frac{-3}{2} & \frac{3}{2}l \\ \frac{3}{2}l & 2l^2 & \frac{-3}{2}l & l^2 \\ \frac{-3}{2} & \frac{-3}{2}l & \frac{3}{2} & \frac{-3}{2}l \\ \frac{3}{2}l & l^2 & \frac{-3}{2}l & 2l^2 \end{bmatrix} \end{array}\begin{array}{c} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{array}$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\
[k_{2-3}] = & \begin{bmatrix} 24 & 12l & -24 & 12l \\ 12l & 8l^2 & -12l & 4l^2 \\ -24 & -12l & 24 & -12l \\ 12l & 4l^2 & -12l & 8l^2 \end{bmatrix}
\end{array}
\begin{array}{c} v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{array}
$$

$$
\begin{Bmatrix} -wl \\ \frac{-wl^2}{3} \\ \frac{-3wl}{2} \\ \frac{wl^2}{4} \\ \frac{-wl}{2} \\ \frac{wl^2}{12} \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 1.5 & 1.5l & 1.5 & 1.5l & 0 & 0 \\ 1.5l & 2l^2 & -1.5l & l^2 & 0 & 0 \\ -1.5 & -1.5l & 25.5 & 10.5l & -24 & 12l \\ 1.5l & l^2 & 10.5l & 10l^2 & -12l & 4l^2 \\ 0 & 0 & -24 & -12l & 24 & -12l \\ 0 & 0 & 12l & 4l^2 & 12l & 8l^2 \end{bmatrix}\begin{Bmatrix} v_1 = 0 \\ \phi_1 = 0 \\ v_2 = ? \\ \phi_2 = ? \\ v_3 = ? \\ \phi_3 = 0 \end{Bmatrix}
$$

Adding third row equation to fifth row equation we have

$$\frac{l^3}{EI}\left(\frac{-3wl}{2} - \frac{wl}{2}\right) = 25.5v_2 + 10.5l\phi_2 - 24v_3 - 24v_2 - 12l\phi_2 + 24v_3$$

$$\Rightarrow \frac{-2wl^4}{EI} = 1.5v_2 - 1.5l\phi_2$$

$$\Rightarrow \quad v_2 = \frac{-4wl^4}{3EI} + l\phi_2 \qquad \text{(A)}$$

Multiplying fourth row equation by –2 and third row equation by l and adding we have

$$\frac{l^3}{EI}\left(\frac{-3wl}{2} - \frac{wl^2}{2}\right) = 25.5lv_2 + 10.5l^2\phi_2 - 24lv_3 - 21lv_2 - 20l^2\phi_2 + 24lv_3$$

$$\Rightarrow \frac{-2wl^5}{EI} = 4.5lv_2 - 9.5l^2\phi_2 \qquad \text{(B)}$$

Substituting (A) into (B)

$$\frac{-2wl^5}{EI} = 4.5l\left[\frac{-4wl^4}{3EI} + l\phi_2\right] - 9.5l^2\phi_2$$

$$\Rightarrow \frac{-2wl^5}{EI} = \frac{-18wl^5}{3EI} + 4.5l^2\phi_2 - 9.5l^2\phi_2$$

$$\Rightarrow \frac{4wl^5}{EI} = -5l^2\phi_2 \Rightarrow \boxed{\phi_2 = \frac{-4wl^3}{5EI}}$$

$$v_2 = \frac{-4wl^4}{3EI} + l\left(\frac{-4wl^3}{5EI}\right) \quad \Rightarrow \quad \boxed{v_2 = \frac{-32wl^4}{15EI}}$$

Substituting in third row equation.

$$\frac{-3wl^4}{2EI} = 25.5\left(\frac{-32}{15}\frac{wl^4}{EI}\right) + 10.5l\left(\frac{-4}{5}\frac{wl^3}{EI}\right) - 24v_3$$

$$\frac{-3wl^4}{2EI}+\frac{816wl^4}{15EI}+\frac{42wl^4}{5EI} = -24v_3 \Rightarrow \boxed{v_3=\frac{-1839\,wl^4}{720\,EI}}$$

(Remember that $l = \frac{L}{6}$)

Now from symmetry

$$\boxed{v_4=v_2} \text{ and } \boxed{\phi_4=-\phi_2}$$

$$\begin{Bmatrix} F_{1y}^{(e)} \\ M_1^{(e)} \\ F_{2y}^{(e)} \\ M_2^{(e)} \\ F_{3y}^{(e)} \\ M_3^{(e)} \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 1.5 & 1.5l & -1.5 & 1.5l & 0 & 0 \\ 1.5l & 2l^2 & -1.5l & l^2 & 0 & 0 \\ -1.5 & -1.5l & 25.5 & 10.5l & -24 & 12l \\ 1.5l & l^2 & 10.5l & 10l^2 & -12l & 4l^2 \\ 0 & 0 & -24 & -12l & 24 & -12l \\ 0 & 0 & 12l & 4l^2 & -12l & 8l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \frac{-32wl^4}{15EI} \\ \frac{-4wl^3}{5EI} \\ \frac{-1839wl^4}{720\,EI} \\ 0 \end{Bmatrix}$$

$$F_{1y}^{(e)} = 2\,wl,\ M_1^{(e)} = 2.4\,wl^2,\ F_{2y}^{(e)} = -1.5\,wl$$

$$M_2^{(e)} = 0.25\,wl^2,\ F_{3y}^{(e)} = -0.5\,wl,\ M_3^{(e)} = 1.85\,wl^2$$

The equation

$\{F\} = [K]\,\{d\} - \{F_0\}$ is now used to find the global nodal concentrated forces.

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \begin{Bmatrix} 2wl \\ 2.4wl^2 \\ -1.5wl \\ 0.25wl \\ -0.5wl \\ 1.85wl^2 \end{Bmatrix} - \begin{Bmatrix} -wl \\ \frac{-wl^2}{3} \\ \frac{-3}{2wl} \\ \frac{wl^2}{4} \\ \frac{-wl}{2} \\ \frac{wl^2}{12} \end{Bmatrix} \Rightarrow$$

$$F_{1y} = 3wl = 3w\left(\frac{L}{6}\right) \Rightarrow \boxed{F_{1y}=\frac{wL}{2}}$$

$$M_1 = 2.73\,wl^2 = 3w\left(\frac{L}{6}\right)^2 \Rightarrow \boxed{M_1=\frac{wL^2}{12}}$$

$$F_{2y}=0,\ M_2=0,\ F_{3y}=0$$

$$M_3 = \frac{21.5\,wl^2}{12} = \frac{21.5}{12}w\left(\frac{L}{6}\right)^2 \Rightarrow M_3=\frac{wL^2}{24}$$

In our case M_3 is the maximum at midspan. From symmetry from elements 3–4 and 4–5

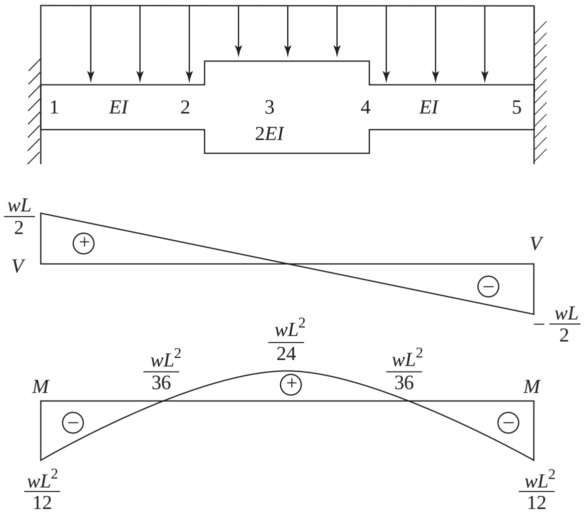
$$\Rightarrow M_3 = \frac{-wL^2}{24}$$

So $M_3 = 0,\ F_{4y} = 0,\ M_4 = 0$

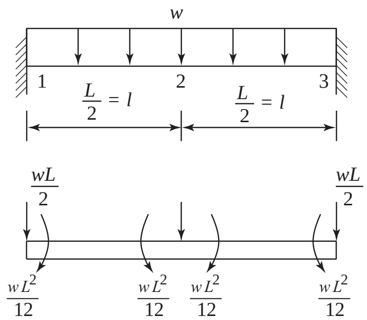
$$\boxed{F_{5y}=\frac{wL}{2}} \quad \boxed{M_5=\frac{-wL^2}{12}}$$

Going back to the deflections we have

$$v_3 = \frac{-1839\,wl^4}{720\,EI} = \frac{-1839}{720}\ \frac{w\left(\frac{L}{6}\right)^4}{EI} \Rightarrow \boxed{v_3=\frac{-wL^4}{507\,EI}}$$



4.13



Applying the boundary conditions

$$v_1 = \phi_1 = v_3 = \phi_3 = 0$$

$$\begin{Bmatrix} -wl \\ 0 \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 24 & 0 \\ 0 & 8l^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix} \Rightarrow 0 = \frac{EI}{l^3}\,(8l^2\phi_2) \Rightarrow \boxed{\phi_2 = 0}$$

$$-\,wl = \frac{EI}{l^3}\,(24v_2) \quad\Rightarrow\quad \boxed{v_2 = \frac{-wl^4}{24EI}}$$

$$f_{1y}^{(e)} = \frac{wl}{2},\ m_1^{(e)} = \frac{wl^2}{4},\ f_{2y}^{(e)} = -\,wl,\ m_2^{(e)} = 0,$$

$f_{3y}^{(e)} = \dfrac{wl}{2}$, $m_3 = \dfrac{-wl^2}{4}$. These are obtained from the following matrix equation.

$$\begin{Bmatrix} f_{1y}^{(e)} \\ m_1^{(e)} \\ f_{2y}^{(e)} \\ m_2^{(e)} \\ f_{3y}^{(e)} \\ m_3^{(e)} \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ & 4l^2 & -6l & 2l^2 & 0 & 0 \\ & & 24 & 0 & -12 & 6l \\ & & & 8l^2 & -6l & 2l^2 \\ & & & & 12 & -6l \\ & & & & & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \frac{-wl^4}{24EI} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{F\} = [K]\,\{d\} - \{F_0\}$$

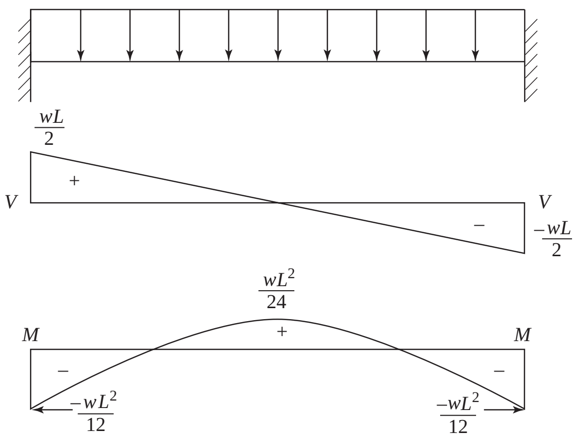
$$F_{1y} = \frac{wl}{2} - \left(\frac{-wl}{2}\right) = wl = \frac{wL}{2}$$

$$M_1 = \frac{wl^2}{4} - \left(\frac{-wl^2}{12}\right) = \frac{wl^2}{3} = \frac{wL^2}{12}$$

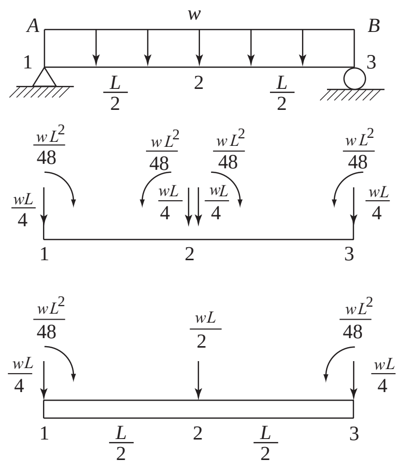
$$F_{2y} = -\,wl - (-\,wl) = 0,\ M_2 = 0 - 0 = 0$$

$$F_{3y} = \frac{wl}{2} - \left(\frac{-wl}{2}\right) = \frac{wL}{2},\ M_3 = \frac{wl^2}{4} - \frac{-wl^2}{12} = \frac{wL^2}{12}$$

$$v_2 = \frac{wl^4}{24EI} = \frac{w(\frac{l}{2})^4}{24EI} \Rightarrow \boxed{v_2 = \frac{-wL^4}{384EI}}$$



4.14



After applying the boundary conditions

$v_1 = v_3 = \phi_2 = 0$ we have

$$\begin{Bmatrix} \frac{-wL^2}{48} \\ \frac{wL}{2} \\ \frac{-wL^2}{48} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} L^2 & -3L & 0 \\ -3L & 24 & 3L \\ 0 & 3L & L^2 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix} \Rightarrow \begin{array}{c} \boxed{\phi_1 = \frac{-wL^3}{24EI}} \\ \boxed{v_2 = \frac{-5wL^4}{384EI}} \\ \boxed{\phi_3 = \frac{wL^3}{24EI}} \end{array}$$

Now $\{F^{(e)}\} = [K]\{d\}$

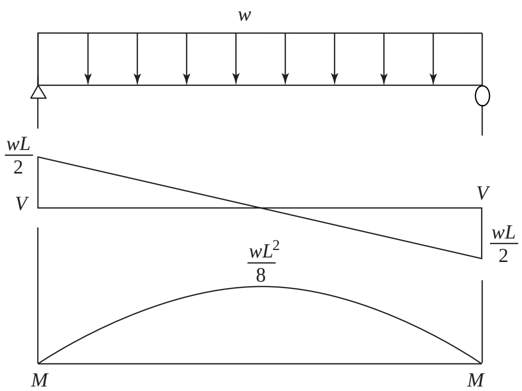
$$\begin{Bmatrix} F_{1y}^{(e)} \\ M_1^{(e)} \\ F_{2y}^{(e)} \\ M_2^{(e)} \\ F_{3y}^{(e)} \\ M_3^{(e)} \end{Bmatrix} = \frac{EI}{\left(\frac{L}{2}\right)^3}\begin{bmatrix} 12 & 3L & -12 & 3L & 0 & 0 \\ 3L & L^2 & -3L & \frac{L^2}{2} & 0 & 0 \\ -12 & -3L & 24 & 0 & -12 & 3L \\ 3L & \frac{L^2}{2} & 0 & 2L^2 & -3L & \frac{L^2}{2} \\ 0 & 0 & -12 & -3L & 12 & -3L \\ 0 & 0 & 3L & \frac{L^2}{2} & -3L & L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ \frac{-wL^3}{24EI} \\ \frac{-5wL^4}{384EI} \\ 0 \\ 0 \\ \frac{wL^3}{24EI} \end{Bmatrix}$$

$$F_{1y}^{(e)} = \frac{wL}{4},\ M_1^{(e)} = \frac{-wL^2}{48},\ F_{2y}^{(e)} = \frac{wL}{2},\ M_2^{(e)} = 0,$$

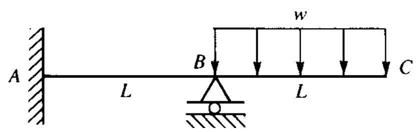
$$F_{3y}^{(e)} = \frac{wL}{4},\ M_3 = \frac{wL^2}{48}$$

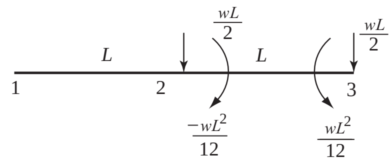
$$\{F\} = \{F^{(e)}\} - \{F_0\}$$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{3y} \\ M_3 \end{Bmatrix} = \begin{Bmatrix} \frac{wL}{4} \\ \frac{-wL^2}{48} \\ \frac{wL}{4} \\ \frac{wL^2}{48} \end{Bmatrix} - \begin{Bmatrix} \frac{-wL}{4} \\ \frac{-wL^2}{48} \\ \frac{-wL}{4} \\ \frac{wL^2}{48} \end{Bmatrix} \Rightarrow \begin{array}{c} \boxed{F_{1y} = \frac{wL}{2},\ M_1 = 0} \\ \boxed{F_{3y} = \frac{wL}{2},\ M_3 = 0} \end{array}$$



4.15





Total $[K]$ for the whole beam

$$[K] = \frac{EI}{L^4}\begin{bmatrix} 12L & 6L^2 & -12L & 6L^2 & 0 & 0 \\ 6L^2 & 4L^3 & -6L^2 & 2L^3 & 0 & 0 \\ -12L & -6L^2 & 24L & 0 & -12L & 6L^2 \\ 6L^2 & 2L^3 & 0 & 8L^3 & -6L^2 & 2L^3 \\ 0 & 0 & -12L & -6L^2 & 12L & -6L^2 \\ 0 & 0 & 6L^2 & 2L^3 & -6L^2 & 4L^3 \end{bmatrix}$$

$\{F\} = [K]\,\{d\}$

After applying boundary conditions

$$v_1 = 0,\ \phi_1 = 0,\ v_2 = 0$$

$$\begin{Bmatrix} M_2 = \frac{-wL^2}{12} \\ F_{3y} = \frac{-wL}{2} \\ M_3 = \frac{wL^2}{12} \end{Bmatrix} = \frac{EI}{L^4}\begin{bmatrix} 8L^3 & -6L^2 & 2L^3 \\ -6L^2 & 12L & -6L^2 \\ 2L^3 & -6L^2 & 4L^3 \end{bmatrix}\begin{Bmatrix} \phi_2 \\ v_2 \\ \phi_3 \end{Bmatrix}$$

Solving the 3 equations we get

$$\boxed{\phi_2 = \frac{-wL^3}{8EI} \text{ or } \curvearrowright} \quad \boxed{v_3 = \frac{-wL^4}{4EI}\downarrow} \quad \boxed{\phi_3 = \frac{-7wL^3}{24EI} \text{ or } \curvearrowright}$$

$$f_{1y}^{(1)} = \frac{EI}{L^4}\,(6L^2\phi_2) = \frac{3wL^5EI}{4EIL^4} = \frac{-3wL}{4} \text{ or } \downarrow$$

$$m_1^{(1)} = \frac{EI}{L^4}\,(2L^3\phi_2) = \frac{-2wL^6}{8EI}\,\frac{EI}{L^4} = \frac{-wL^2}{4} \text{ or } \curvearrowright$$

$$f_{2y}^{(1)} = \frac{EI}{L^4}\,(-6L^2\phi_2) = \frac{3wL}{4}\uparrow$$

Reactions

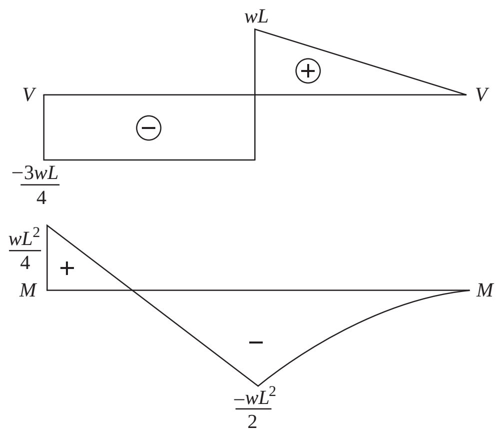
$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \end{Bmatrix} = \begin{Bmatrix} \frac{-3wL}{4} \\ \frac{-wL^2}{4} \\ \frac{3wL}{4} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ \frac{-wL}{2} \end{Bmatrix} = \begin{Bmatrix} \frac{-3wL}{4} \\ \frac{-wL^2}{4} \\ \frac{7wL}{4} \end{Bmatrix}$$

using

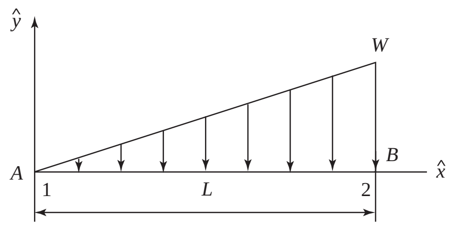
$$\{f\} = [k]\,\{d\} - \{f^{(0)}\}$$

$$\begin{bmatrix} f_{2y}^{(2)} \\ m_2^{(2)} \\ f_{3y}^{(2)} \\ m_3^{(3)} \end{bmatrix} = \frac{EI}{L^4}\begin{bmatrix} 12L & 6L^2 & -12L & 6L^2 \\ 6L^2 & 4L^3 & -6L^2 & 2L^3 \\ -12L & -6L^2 & 12L & -6L^2 \\ 6L^2 & 2L^3 & -6L^2 & 4L^3 \end{bmatrix}\begin{Bmatrix} 0 \\ \frac{-wL^3}{8EI} \\ \frac{-wL^4}{4EI} \\ \frac{-7wL^3}{24EI} \end{Bmatrix} - \begin{Bmatrix} \frac{-wL}{2} \\ \frac{-wL^2}{12} \\ \frac{-wL}{2} \\ \frac{wL^2}{12} \end{Bmatrix}$$

$$\Rightarrow \quad m_2^{(2)} = \frac{-wL^2}{2}$$



4.16



$$W_{\text{distributed}} = \int_0^L w(x)v(x)dx \ \text{ and}$$

$$W_{\text{discrete}} = m_1\phi_1 + m_2\phi_2 + f_{1y}v_1 + f_{2y}v_2$$

$$\text{and} \quad W_{\text{distributed}} = W_{\text{discrete}}$$

$$\int_0^L w(x)\,v(x)\,dx = m_1\phi_1 + m_2\phi_2 + f_{1y}v_1 + f_{2y}v_2$$

Now evaluting the left side of the equation by substituting where $w(x) = \frac{-wx}{L}$ (since the load is linear and increasing to the right), and in $V(x)$ we substitute with a's already evaluated, we get

$$\int_0^L w(x)v(x)dx = \int_0^L \frac{-wx}{L}\,[a_1x^3 + a_2x^2 + a_3x + a_4]$$

$$= \int_0^L \frac{-wx}{L}\left\{\left[\frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\phi_1 + \phi_2)\right]x^3 - \left[\frac{3}{L^2}(v_1 - v_2) + \frac{1}{L}(2\phi_1 + \phi_2)\right]x^2 + \phi_1 x + v_1\right\}dx$$

$$= \int_0^L \frac{-w}{L}\left[\frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\phi_1 + \phi_2)\right]x^4 -$$

$$\left[\frac{3}{L^2}(v_1 - v_2) + \frac{1}{L}(2\phi_1 + \phi_2)\right]x^3 + d_1x^2 + v_1\,x\Big]dx$$

$$= \frac{-w}{L}\left[\frac{2}{3}(v_1 - v_2) + \frac{1}{L^2}(\phi_1 + \phi_2)\right] + \frac{w}{L}\left[\frac{3}{L^2}(v_1 - v_2) + \frac{1}{L}(2\phi_1 + \phi_2)\right]$$

$$\left.\frac{x^4}{4}\right|_0^L - \frac{w}{L}\phi_1\left.\frac{x^3}{3}\right|_0^L - \frac{w}{L}v_1\left.\frac{x^2}{2}\right|_0^L$$

$$= \frac{-wL^4}{5}\left[\frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\phi_1 + \phi_2)\right] +$$

$$\frac{wL^3}{4}\left[\frac{3}{L^2}(v_1 - v_2) + \frac{1}{L}(2\phi_1 + \phi_2)\right] - \frac{wL^2}{3\phi_1} - \frac{wL}{2}v_1$$

$$= \frac{-2wL}{5}(v_1 - v_2) + \frac{wL^2}{5}(\phi_1 + \phi_2) + \frac{3wL}{4}(v_1 - v_2)$$

$$+ \frac{wL^2}{4}(2\phi_1 + \phi_2) - \frac{wL^2}{3}\phi_1 - \frac{wL}{2}v_1$$

$$= m_1\phi_1 + m_2\phi_2 + f_{1y}v_1 + f_{2y}v_2$$

Now if we take the last equation and set $\phi_1 = \phi_2 = v_1 = 0$ and $v_1 = 1$, we have

$$\frac{-2wL}{5} + \frac{3wL}{4} - \frac{wL}{2} = f_{1y}\,1$$

$$\Rightarrow \quad \frac{-8wL + 15wL - 10wL}{20} = f_{1y} \Rightarrow \boxed{f_{1y} = \frac{-3wL}{20}}$$

If $\quad \phi_2 = v_2 = v_1 = 0$ and $\phi_1 = 1$

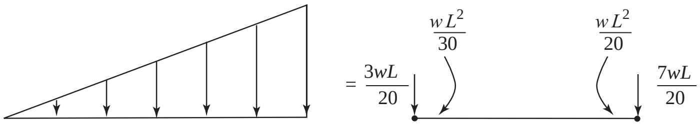
$$\Rightarrow \frac{-wL}{5} + \frac{2wL^2}{4} - \frac{wL^2}{3} = m_1 \Rightarrow \boxed{m_1 = \frac{-wL^2}{30}}$$

If $\quad \phi_2 = \phi_1 = v_1 = 0$ and $v_2 = 1$

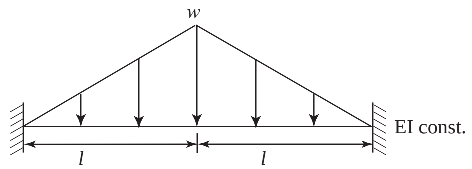
$$\Rightarrow \frac{2wL}{5} - \frac{3wL}{4} = f_{2y} \Rightarrow \boxed{f_{2y} = \frac{-7wL}{20}}$$

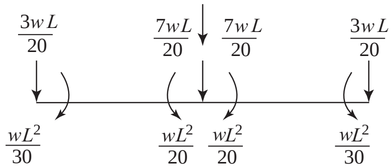
If $\phi_1 = v_1 = v_2 = 0$ and $\phi_2 = 1$

$$\Rightarrow \frac{-wL^5}{5} + \frac{wL^2}{4} = m_2\,1 \Rightarrow \boxed{m_2 = \frac{wL^2}{20}}$$



4.17





Work equivalent load system

$$\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ & 4l^2 & -6l & 2l^2 & 0 & 0 \\ & & 12+12 & -6l+6l & -12 & 6l \\ & & & 4l^2+4l^2 & -6l & 2l^2 \\ & & & & 12 & -6l \\ & & & & & 4l^2 \end{bmatrix}\begin{Bmatrix} v_1=0 \\ \phi_1=0 \\ v_2 \\ \phi_2 \\ v_3=0 \\ \phi_3=0 \end{Bmatrix} = \begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y}=-\frac{14wl}{20} \\ M_2=0 \\ F_{3y} \\ M_3 \end{Bmatrix} \begin{matrix} \\ \\ (3) \\ (4) \\ \\ \\ \end{matrix}$$

Boundary Conditions $v_1 = \phi_1 = v_3 = \phi_3 = 0$

Use Equations (3) and (4)

$$\begin{Bmatrix} \frac{-14wl}{20} \\ 0 \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 24 & 0 \\ 0 & 8l^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix}$$

$$v_2 = \frac{-7wl^4}{240EI} = \frac{-7L^4w}{3840EI} \quad (L = 2l)$$

$$\phi_2 = 0$$

Reactions

$$\{F\} = [K]\,\{d\} - \{F_0\}$$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ & 4l^2 & -6l & 2l^2 & 0 & 0 \\ & & 24 & 0 & -12 & 6l \\ & & & 8l^2 & -6l & 2l^2 \\ & & & & 12 & -6l \\ \text{Symmetry} & & & & & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \frac{-7wl^4}{240\,EI} \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} \frac{-3wl}{20} \\ \frac{-wl^2}{30} \\ \frac{-7wl}{10} \\ 0 \\ \frac{-3wl}{20} \\ \frac{wl^2}{30} \end{Bmatrix}$$

$$F_{1y} = \frac{EI}{l^3}\,(-12)\left(\frac{-7wl^4}{240EI}\right) - \left(\frac{-3wl}{20}\right)$$

$$F_{1y} = \frac{7}{20}wl + \left(\frac{3wl}{20}\right) = \frac{10}{20}wl = \frac{wl}{2} = \frac{wl}{4} \qquad \left(\frac{L}{2} = l\right)$$

$$M_1 = \frac{EI}{l^3}\,(-\,6l)\left(\frac{-7wl^4}{240EI}\right) - \left(\frac{-wl^2}{30}\right) = \left(\frac{7}{40} + \frac{1}{30}\right) wl^2$$

$$M_1 = \frac{5}{24}\,wl^2 = \frac{5wL^2}{96}$$

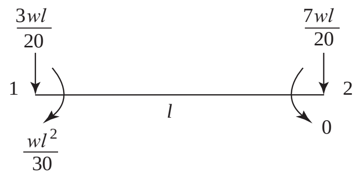
$$F_{2y} = \frac{EI}{l^3}\,(24)\left(\frac{-7wl^4}{240EI}\right) - \left(\frac{-7wl}{10}\right) = 0$$

$$M_2 = \frac{EI}{l^3}\,(0) - 0 = 0$$

$$F_{3y} = \frac{EI}{l^3}\,(-\,12)\left(\frac{-7wl^4}{240EI}\right) - \left(\frac{-3wl}{20}\right) = \frac{wl}{2} = \frac{wL}{4}$$

$$M_3 = \frac{EI}{l^3}\,(6l)\left(\frac{-7wl^4}{240EI}\right) - \left(\frac{wl^2}{30}\right) = \frac{-5}{24}\,wl^2 = \frac{-5}{96}\,wL^4$$

Note: Could use symmetry



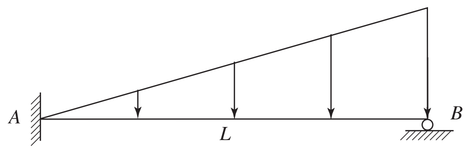
With $\quad v_1 = \phi_1 = \phi_2 = 0$

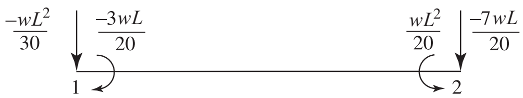
Get

$$\frac{12EI}{l^3}\,v_2 = \frac{-7wl}{20}$$

$$v_2 = \frac{-7wl^4}{240EI} \text{ as before}$$

4.18





$$\{F_0\} = [K]\,\{d\}$$

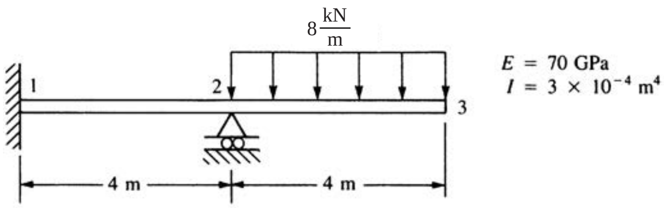
$$\begin{Bmatrix} F_{01y} = -\frac{3wL}{20} \\ M_{01} = -\frac{wL^2}{30} \\ F_{02y} = -\frac{7wL}{20} \\ M_{02} = \frac{wL^2}{20} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 = 0 \\ \phi_1 = 0 \\ v_2 = 0 \\ \phi_2 \end{Bmatrix}$$

$$\Rightarrow \frac{wL^2}{20} = \frac{EI}{L^3}\, 4L^2\phi_2 \Rightarrow \boxed{\phi_2 = \frac{wL^3}{80EI}}$$

$$\begin{Bmatrix} F_{1y}^{(e)} \\ M_1^{(e)} \\ F_{2y}^{(e)} \\ M_2^{(e)} \end{Bmatrix} = \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{wL^3}{80\,EI} \end{Bmatrix} \Rightarrow \begin{matrix} F_{1y}^{(e)} = \frac{3wL}{40} \\ M_1^{(e)} = \frac{wL^2}{40} \\ F_{2y}^{(e)} = -\frac{3wL}{40} \\ M_2^{(e)} = \frac{wL^2}{20} \end{matrix}$$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \end{Bmatrix} = \begin{Bmatrix} \frac{3wL}{40} \\ \frac{wL^2}{40} \\ \frac{-3wL}{40} \\ \frac{wL^2}{20} \end{Bmatrix} - \begin{Bmatrix} \frac{-3wL}{20} \\ \frac{-wL^2}{30} \\ \frac{-7wL}{20} \\ \frac{wL^2}{20} \end{Bmatrix} \Rightarrow \boxed{F_{1y} = \frac{9wL}{40}} \quad \boxed{M_1 = \frac{7wL^2}{120}} \quad \boxed{F_{2y} = \frac{11wL}{40}} \quad \boxed{M_2 = 0}$$

4.19



$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

After imposing the boundary conditions and using work equivalence

$v_1 = \phi_1 = v_2 = 0$, we have in $\{F\} = [K]\,\{d\}$

$$\begin{Bmatrix} \frac{-wL^2}{12} \\ \frac{-wL}{2} \\ \frac{wL^2}{12} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 8L^2 & -6L & 2L^2 \\ -6L & 12 & -6L \\ 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

From (1) and (3)

$$\frac{wL^2}{12} = \frac{EI}{L^3}\,[-8L^2\phi_2 + 6Lv_3 - 2L^2\phi_3]$$

$$\frac{wL^2}{12} = \frac{EI}{L^3}\,[2L^2\phi_2 - 6Lv_3 + 4L^2\phi_3]$$

$$\frac{wL^3}{6} = \frac{EI}{L^3}\,[-6L^2\phi_2 + 2L^2\phi_3] \qquad (4)$$

From (2) and (3)

$$\frac{-wL^2}{4} = \frac{EI}{L^3}\,[-\,3L^2\phi_2 + 6Lv_3 - 3L^2\phi_3]$$

$$\frac{wL^2}{12} = \frac{EI}{L^3}\,[2L^2\phi_2 - 6Lv_3 + 4L^2\phi_3]$$

$$\frac{-wL^2}{6} = \frac{EI}{L^3}\,[-\,L^2\phi_2 + L^2\phi_3] \qquad (5)$$

Adding (4) and (5) we have

$$\frac{wL^2}{6} = \frac{EI}{L^3}\,[-\,6L^2\phi_2 + 2L^2\phi_3]$$

$$\frac{2wL^2}{6} = \frac{EI}{L^3}\,[2L^2\phi_2 - 2L^2\phi_3]$$

$$\frac{wL^2}{6} = \frac{EI}{L^3}\,[-\,4L^2\phi_2] \;\Rightarrow\; \boxed{\phi_2 = \frac{-wL^3}{8EI} = -\,3.046\times 10^{-3}\,\text{rad}}$$

Substituting in (5) $\Rightarrow \dfrac{-wL^2}{6} = \dfrac{EI}{L^3}\left[-L^2\left(\dfrac{-wL^3}{8EI}\right) + L^2\phi_3\right]$

$$\Rightarrow \boxed{\phi_3 = \frac{-7wL^3}{24EI} = -\,0.00711\,\text{rad}}$$

Finally substituting in (1)

$$\Rightarrow \boxed{v_3 = \frac{-wL^4}{4EI} = -\,0.0244\text{ m}}$$

Reactions can be found from the global equation

$$\{F\} = [K]\,\{d\} - \{F_0\}$$

$$F_{1y} = \frac{EI}{L^3}[6L\phi_2] - 0 = \frac{EI}{L^3}\,6L\left(\frac{-wL^3}{8EI}\right) = \frac{-3wL}{4} = -\,24\text{ kN}$$

$$M_1 = \frac{EI}{L^3}\,[2L^2\phi_2] - 0 = \frac{EI}{L^3}2L^2\left(\frac{-wL^3}{8EI}\right) = \frac{-wL^2}{4} = -\,32\text{ kN}\cdot\text{m}$$

$$F_{2y} = \frac{EI}{L^3}\,[-12v_3 + 6L\phi_3] - \left[-\frac{wL}{2}\right] = \frac{7wL}{4} = 56\text{ kN}$$

and $M_2 = 0$

Element 1–2

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{-wL^3}{8EI} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$f_{1y} = \frac{-3}{4}wL = -24 \text{ kN}, \; m_1 = \frac{-wL^2}{4} = -32 \text{ kN}\cdot\text{m}$$

$$f_{2y} = \frac{3}{4}\, wL = 24 \text{ kN}, \; m_2 = -64 \text{ kN}\cdot\text{m}$$

Element 2–3

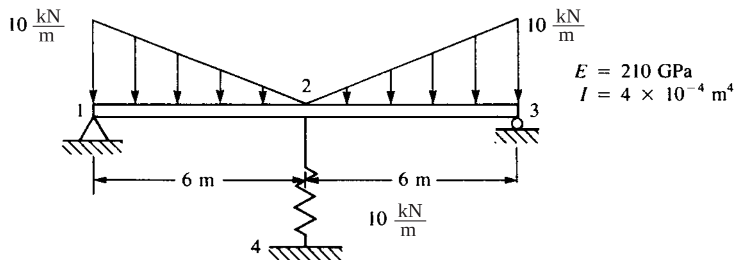
$$\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ \frac{-wL^3}{8EI} \\ \frac{-wL^4}{4EI} \\ \frac{-7wL^3}{24EI} \end{Bmatrix} - \begin{Bmatrix} \frac{-wL}{2} \\ \frac{-wL^2}{12} \\ \frac{-wL}{2} \\ \frac{wL^2}{12} \end{Bmatrix}$$

$$\Rightarrow f_{2y} = 32 \text{ kN}$$

$$m_2 = 64 \text{ kN}\cdot\text{m}$$

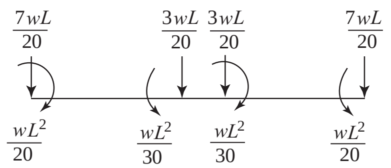
$$f_{3y} = m_3 = 0$$

4.20



Global stiffness matrix

$$[K] = \frac{EI}{L^3}\begin{array}{c} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ & 4L^2 & -6L & 2L^2 & 0 & 0 \\ & & \frac{24+KL^3}{EI} & 0 & -12 & 6L \\ & & & 8L^2 & -6L & 2L^2 \\ & & & & 12 & -6L \\ \text{Symmetry} & & & & & 4L^2 \end{bmatrix} \end{array}\begin{array}{l} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{array}$$



Boundary conditions

$$v_1 = v_3 = \phi_2 = 0 \text{ and } \phi_1 = -\phi_3$$

$$\{F\} = [K]\{d\}$$

$$\begin{Bmatrix} \frac{-wL^2}{20} \\ \frac{-6wL}{20} \\ \frac{wL^2}{20} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 4L^2 & -6L & 0 \\ -6L & \frac{24+KL^3}{EI} & 6L \\ 0 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ v_2 \\ \phi_3 \end{Bmatrix} \qquad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

since $\phi_1 = -\phi_3$ we can ignore Equation (3)

$\Rightarrow$ Multiplying (1) by 3 and (2) by L and adding we have

$$\frac{-3\,wL^2}{20} = \frac{EI}{L^3}\,[12L^2\phi_1 - 18Lv_2]$$

$$\frac{-6\,wL^2}{20} = \frac{EI}{L^3}\left[-12L^2\phi_1 + \left(24 + \frac{KL^4}{EI}\right)v_2\right]$$

$$-9\,\frac{wL^2}{20} = \left(\frac{6EI}{L^2} + KL\right)v_2$$

$$\Rightarrow\ -162000 = 140{,}060{,}000\ v_2 \Rightarrow \boxed{v_2 = -0.011522 \text{ m}}$$

Substituting in (1) we have

$$-18000 = 5.6 \times 10^7\ \phi_1 + 16.1309 \times 10^4$$

$$\Rightarrow\ \boxed{\phi_1 = -0.0032019 \text{ rad}}$$

$$\text{since } \phi_1 = -\phi_3 \Rightarrow \boxed{\phi_3 = 0.0032019 \text{ rad}}$$

The reactions can be found by the global matrix $\{F\} = [K]\{d\} - \{F_0\}$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & \frac{24+KL^3}{EI} & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ -0.0032019 \\ -0.0011522 \\ 0 \\ 0 \\ 0.0032019 \end{Bmatrix} - \begin{Bmatrix} \frac{-7wL}{20} \\ \frac{-wL^2}{20} \\ \frac{-6wL}{20} \\ 0 \\ \frac{-7wL}{20} \\ \frac{wL^2}{20} \end{Bmatrix}$$

$$\Rightarrow\ F_{1y} = 29.94 \text{ kN},\ M_1 = 0$$

$$F_{2y} = 0.11522 \text{ kN},\ M_2 = 0$$

$$F_{3y} = 29.94 \text{ kN},\ M_3 = 0$$

Element 1–2

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{Symmetry} & & & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ -0.00332019 \\ -0.011522 \\ 0 \end{Bmatrix} - \begin{Bmatrix} \frac{-7wL}{20} \\ \frac{-wL^2}{20} \\ \frac{-3wL}{20} \\ \frac{wL^2}{30} \end{Bmatrix}$$

$$\boxed{\Rightarrow\ f_{1y} = 29.44\text{kN},\ m_1 = 0,\ f_{2y} = 0.058\ \text{kN},\ m_2 = 59.65\ \text{kN}\cdot\text{m}}$$

Element 2–3

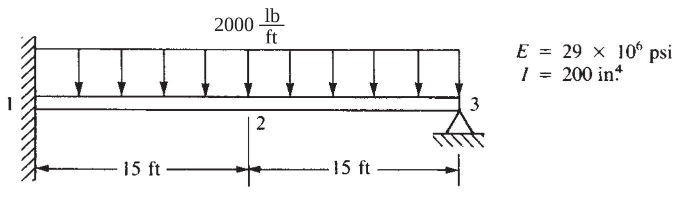
$$\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ & & & 4L^2 \end{bmatrix}\begin{Bmatrix} -0.011522 \\ 0 \\ 0 \\ -0.0032019 \end{Bmatrix} - \begin{Bmatrix} \frac{-3wL}{20} \\ \frac{-wL^2}{30} \\ \frac{-7wL}{20} \\ \frac{wL^2}{20} \end{Bmatrix}$$

$$\boxed{\Rightarrow\ f_{2y} = 0,\ m_3 = -59.65\ \text{kN}\cdot\text{m},\ f_{3y} = 29.94\ \text{kN},\ m_3 = 0}$$

Force in spring

$$F_S = \frac{10\,\text{kN}}{\cancel{m}} \times (0.011522)\cancel{m} = -0.1152\ \text{kN}$$

4.21



Global stiffness matrix of the beam

$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

After imposing the boundary conditions $v_1 = \phi_1 = v_3 = 0$ in $\{F\} = [K]\ \{d\}$

$$\begin{Bmatrix} -wL \\ 0 \\ \frac{wL^2}{12} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 24 & 0 & 6L \\ 0 & 8L^2 & 2L^2 \\ 6L & 2L^2 & 4L^2 \end{bmatrix}\begin{Bmatrix} v_3 \\ \phi_2 \\ \phi_3 \end{Bmatrix} \qquad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

Multiplying (1) by L and (3) by – 4 and adding them

$$-wL^2 = \frac{EI}{L^3}\,[24Lv_2 + 0\phi_2 + 6L^2\phi_3]$$

$$\frac{-wL^2}{3} = \frac{EI}{L^3}\,[-24Lv_2 - 8L^2\phi_2 - 16L^2\phi_3]$$

$$\frac{-4wL^2}{3} = \frac{EI}{L^3}\,[-8L^2\phi_2 - 10L^2\phi_3] \qquad (4)$$

Adding (2) and (4) we get

$$\frac{-4wL^2}{3} = \frac{EI}{L^3}\,[-8L^2\phi_3] \Rightarrow \boxed{\phi_3 = \frac{wL^3}{6EI}}$$

Substituting in (2)

$$0 = \frac{EI}{L^3}\left[8L^2\phi_2 + \frac{2L^2wL^3}{6EI}\right] \Rightarrow \boxed{\phi_2 = \frac{-wL^3}{24EI}}$$

Substituting in (1)

$$-wL = \frac{EI}{L^3}\left[24v_2 + \frac{6LwL^3}{6EI}\right] \Rightarrow -wL - wL = 24\,\frac{EI}{L^3}\,v_2$$

$$\Rightarrow \boxed{v_2 = \frac{-wL^4}{12EI}}$$

$$\Rightarrow \phi_3 = \frac{(\frac{2000}{12})(15\times 12)^3}{6\left(29\times 10^6\right)200} = 0.02793 \text{ rad}$$

$$\phi_2 = \frac{-(\frac{2000}{12})(15\times 12)^3}{24\left(29\times 10^6\right)200} = -0.0069827 \text{ rad}$$

$$v_2 = \frac{-1}{12}\,\frac{(\frac{2000}{12})(15\times 12)^4}{29\times 10^6\times 200} = -2.5138 \text{ in.}\downarrow$$

Substituting back in the global equation

$\{F\} = [K]\,\{d\} - \{F_0\}$ we can find the reactions

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -2.5138 \\ -0.0069827 \\ 0 \\ 0.027931 \end{Bmatrix}$$

$$- \begin{Bmatrix} \frac{-wL}{2} = -15000 \text{ lb} \\ \frac{-wL^2}{12} = -450000 \text{ lb}\cdot\text{in.} \\ -wL = -30000 \text{ lb} \\ 0 \\ \frac{-wL}{2} = -15000 \text{ lb} \\ \frac{-wL^2}{12} = 450000 \text{ lb}\cdot\text{in.} \end{Bmatrix}$$

$$F_{1y} = 37500 \text{ lbs}, \; M_1 = 225000 \text{ lb} \cdot \text{in.}$$

$$F_{2y} = 0, \; M_2 = 0$$

$$F_{1y} = 22500 \text{ lb}, \; M_3 = 0$$

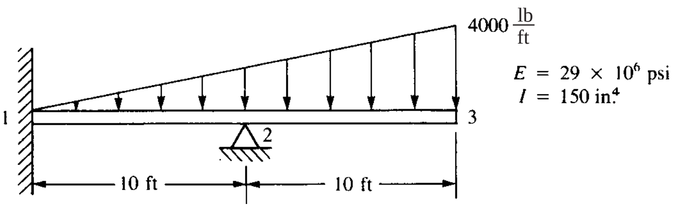
Element 1–2

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -2.5138 \\ -0.0069827 \end{Bmatrix} - \begin{Bmatrix} -15000 \\ -450000 \\ -15000 \\ 450000 \end{Bmatrix} \Rightarrow \begin{matrix} f_{1y} = 37500 \text{ lb} \\ m_1 = 225000 \text{ lb} \cdot \text{in.} \\ f_{2y} = -37500 \text{ lb} \\ m_2 = 112500 \text{ lb} \cdot \text{in.} \end{matrix}$$

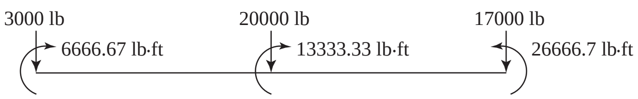
Element 2–3

$$\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -2.5138 \\ -0.0069827 \\ 0 \\ 0.027931 \end{Bmatrix} - \begin{Bmatrix} -15000 \\ -450000 \\ -15000 \\ 450000 \end{Bmatrix} \Rightarrow \begin{matrix} f_{2y} = 7500 \text{ lb} \\ m_2 = -112500 \text{ lb} \cdot \text{in.} \\ f_{3y} = 22500 \text{ lb} \\ m_3 = 0 \end{matrix}$$

4.22



$$[K] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ & 4L^2 & -6 & 2L^2 & 0 & 0 \\ & & 24 & 0 & -12 & 6L \\ & & & 8L^2 & -6L & 2L^2 \\ & & & & 12 & -6L \\ \text{Symmetry} & & & & & 4L^2 \end{bmatrix}$$



After applying the boundary conditions

$$v_1 = \phi_1 = v_2 = 0 \text{ in } \{F_0\} = [k]\,\{d\}$$

$$\begin{Bmatrix} -13333.33 \text{ ft} \cdot \text{lb} \\ -17000 \text{ lb} \\ 26666.67 \text{ ft} \cdot \text{lb} \end{Bmatrix} = 30208.33 \begin{bmatrix} 8L^2 & -6L & 2L^2 \\ -6L & 12 & -6L \\ 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

Rewriting equations (1) (2) and (3) we get

$$-0.441379 = 8L^2\phi_2 - 6Lv_3 + 2L^2\phi_3 \qquad (1)$$

$$-0.562759 = -6L\phi_2 + 12v_3 - 6L\phi_3 \qquad (2)$$

$$0.882759 = 2L^2\phi_2 - 6Lv_3 + 4L^2\phi_3 \qquad (3)$$

Adding (1) to $-4 \times (3)$ we get

$$-0.441379 = 8L^2\phi_2 - 6Lv_3 + 2L^2\phi_3$$

$$-3.53103 = -8L^2\phi_2 + 24Lv_3 - 16L^2\phi_3$$

$$-3.9724 = 18Lv_3 - 14L^2\phi_3 \qquad (4)$$

Adding $L \times (2)$ to $3 \times (3)$ we get (where L = 10 ft)

$$-5.62759 = -6L^2\phi_2 + 12Lv_3 - 6L^2\phi_3$$

$$2.64827 = 6L^2\phi_2 - 18Lv_3 + 12L^2\phi_3$$

$$-297931 = -6Lv_3 + 6L^2\phi_3 \qquad (5)$$

Adding Equation (4) to $3 \times (5)$ we have

$$-12.91034 = 4L^2\phi_3 \;\Rightarrow\; \boxed{\phi_3 = -3.22758 \times 10^{-2} \text{ rad}}$$

Substituting in (4)

$$\Rightarrow \; -3.9724 = 180v_3 - 1400\,(-3.22758 \times 10^{-2})$$

$$\Rightarrow \; \boxed{v_3 = -2.73103 \times 10^{-1} \text{ ft} = -3.27724 \text{ in.}}$$

Substituting in (1)

$$\Rightarrow -0.441379 = 8L^2\phi_2 - 6L\,(-2.73103 \times 10^{-1}) + 2L^2\,(-3.22758)$$

$$\Rightarrow \quad \boxed{\phi_2 = -1.29655 \times 10^{-2} \text{ rad}}$$

$$F_{1y}^{(e)} = \frac{6EI}{L^2}\phi_2 = \frac{6(29 \times 10^6)(150 \text{ in.}^4)}{(120)\text{ in.}^2}\;(-1.29655 \times 10^{-2}) = -23500 \text{ lb}$$

$$M_1^{(e)} = \frac{2EI}{L}\phi_2 = \frac{2(29 \times 10^6)(150 \text{ in.}^4)}{120 \times 12''} = (-1.29655 \times 10^{-2}) = 78333 \text{ lb}\cdot\text{ft}$$

$$F_{2y}^{(e)} = \frac{-12EI}{L^3}v_3 + \frac{6EI}{L^2}\phi_3$$

$$= \frac{-12(29 \times 10^6)(150)}{120 \times 120 \times 120}\;(-3.27724) + \frac{6(29 \times 10^6)}{(120)^2} \times 150$$

$$\times (-3.22758 \times 10^{-1}) = 40500 \text{ lb}$$

$$M_2^{(e)} = \frac{8L^2EI}{L^3}\phi_3 - \frac{6LEI}{L^3}v_2 + \frac{2L^2EI}{L^3}\phi_3 = -13333.33 \text{ ft}\cdot\text{lb}$$

$$F_{3y}^{(e)} = \frac{-6LEI}{L^3}\phi_2 + \frac{12LEI}{L^3}v_3 - \frac{6LEI}{L^3}\phi_3 = -17000 \text{ lb}$$

$$M_3^{(e)} = \frac{2L^2EI}{L^3}\phi_2 - \frac{6LEI}{L^3}v_3 + \frac{4L^2EI}{L^3}\phi_3 = 26{,}666.67 \text{ ft}\cdot\text{lb}$$

Global forces

$$F_{1y} = -23500 + 3000 = -20500 \text{ lb}$$

$$M_1 = -78333.33 + 6666.67 = -71{,}666.67 \text{ ft}\cdot\text{lb}$$

$$F_{2y} = 40{,}500 + 20{,}000 = 60500 \text{ lb}$$

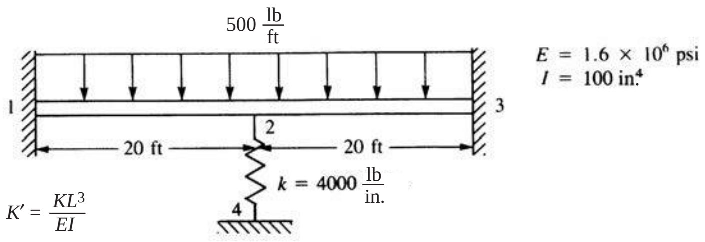
$$M_2 = -13333.33 + 13333.33 = 0$$

$F_{3y} = -17000 + 17000 = 0$

$M_3 = 26666.67 - 26666.67 = 0$

Element 1–2	Element 2–3
$f_{1y} = -20500$ lb	$f_{2y} = -30{,}000$ lb
$m_1 = -71666.67$ ft·lb	$m_2 = 2000$ kip·in.
$f_{2y} = 30{,}500$ lb	$f_{3y} = 0$
$m_2 = -2000$ kip·in.	$m_3 = 0$

4.23



After applying the boundary conditions on $\{F\} = [K]\{d\}$ we have

$v_1 = \phi_1 = v_3 = \phi_3 = \phi_2 = 0$

So

$$-wL = \frac{EI}{L^3}\left[24 + \frac{KL^3}{EI}\right] v_2$$

$$\Rightarrow -500 \times 20 = \frac{1.6\times10^6 \times 100}{(20\times12)^3}\left[24 + \frac{4000(20\times12)^3}{1.6\times10^6(100)}\right] v_2$$

$$\Rightarrow \quad -10000 = 277.78\, v_2 \quad \Rightarrow \quad v_2 = -2.338 \text{ in.}$$

Reactions

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = 11.574 \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24+k^1 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -2.338 \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -5000\text{ lb} \\ -200000\text{ in}\cdot\text{lb} \\ -10000 \\ 0 \\ -5000\text{ lb} \\ 200000\text{ in}\cdot\text{lb} \end{Bmatrix}$$

$\Rightarrow \; F_{1y} = 5325$ lb

$M_1 = 19{,}914$ lb·ft

$F_{2y} = 0$ lb

$M_2 = 0$

$F_{3y} = 5325$ lb

$M_3 = -19{,}914$ lb·ft

Element 1–2

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = 11.574 \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -2.338 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -5000 \\ -200000 \\ -5000 \\ 200000 \end{Bmatrix}$$

$$\Rightarrow f_{1y} = 5325 \text{ lb}, \; m_1 = 19914 \text{ lb}\cdot\text{ft}$$

$$f_{2y} = 4675 \text{ lb}, \; m_2 = -13419 \text{ lb}\cdot\text{ft}$$

Element 2–3

$$f_{2y} = 4675 \text{ lb}$$

$$m_2 = 13419 \text{ lb}\cdot\text{ft}$$

from symmetry

$$f_{3y} = 5325 \text{ lb}$$

$$m_3 = -19914 \text{ lb}\cdot\text{ft}$$

Note: Spring force is

$$F_s = \left(4000 \frac{\text{lb}}{\text{in.}}\right)(2.338 \text{ in.}) = 9352 \text{ lb}$$

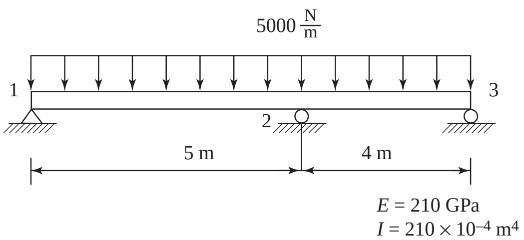
Equilibrium at node 2

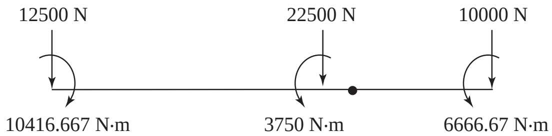
$\downarrow$ 4675 lb from element 1 $\qquad \Sigma F_y = 0$

$\downarrow$ 4675 lb from element 2

$\uparrow F_s = 9352$ lb

4.24





$$v_1 = 0 = v_2 = v_3$$

$$[k_{1-2}] = EI \begin{array}{c} \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix} \end{array} \begin{array}{c} \phi_1 \\ \phi_2 \end{array}$$

$$[k_{2-3}] = EI \begin{array}{c} \begin{array}{cc} \phi_2 & \phi_3 \end{array} \\ \begin{bmatrix} \frac{4}{4} & \frac{2}{4} \\ \frac{2}{4} & \frac{4}{4} \end{bmatrix} \end{array}$$

$$\{F_0\} = [K]\{d\}$$

$$\begin{Bmatrix} -10416.667 \\ 3750 \\ 6666.67 \end{Bmatrix} = (210 \times 10^9)\,(2 \times 10^{-4}) \begin{bmatrix} 0.8 & 0.4 & 0 \\ 0.4 & 1.8 & 0.5 \\ 0 & 0.5 & 1 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

$$-10416.667 = (210 \times 10^9)\,(2 \times 10^{-4})\,[0.8\phi_1 + 0.4\phi_2] \quad (1)$$

$$3750 = (210 \times 10^9)\,(2 \times 10^{-4})\,[0.4\phi_1 + 1.8\phi_2 + 0.5\phi_3] \quad (2)$$

$$6666.67 = (210 \times 10^9)\,(2 \times 10^{-4})\,[0.5\phi_2 + \phi_3] \quad (3)$$

Multiplying – 2 *x* (2) and adding it to (1) we have

$$-10416.667 = 4.2 \times 10^7\,[0.8\phi_1 + 0.4\phi_2]$$

$$-7500 = 4.2 \times 10^7\,[-0.8\phi_1 - 3.6\phi_2 - \phi_3]$$

$$-17916.667 = 4.2 \times 10^7\,[-3.2\phi_2 - \phi_3] \quad (4)$$

Adding (3) to (4) we have

$$-17916.667 = 4.2 \times 10^7\,[-3.2\phi_2 - \phi_3]$$

$$6666.667 = 4.2 \times 10^7\,[0.5\phi_2 + \phi_3]$$

$$-11250 = 4.2 \times 10^7\,(-2.7\phi_2)$$

$$\Rightarrow \boxed{\phi_2 = 9.92 \times 10^{-5}\ \text{rad}}$$

Substituting into (4) we have

$$-17916.667 = 4.2 \times 10^7\,[-3.2\,(9.92 \times 10^{-5}) - \phi_3]$$

$$\Rightarrow \boxed{\phi_3 = 1.091 \times 10^{-4}\ \text{rad}}$$

Substituting in (1)

$$\Rightarrow -10416.667 = 4.2 \times 10^7\,[0.8 + (u_0 + (-8)) + 0.4(9.92 \times 10^{-5})]$$

$$\Rightarrow \boxed{\phi_1 = -3.596 \times 10^{-4}\ \text{rad}}$$

Element 1–2

$$\begin{Bmatrix} f_{1y}^{(e)} \\ m_1^{(e)} \\ f_{2y}^{(e)} \\ m_2^{(2)} \end{Bmatrix} = \frac{(210 \times 10^9)(2 \times 10^{-4})}{5^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ -3.596 \times 10^{-4} \\ 0 \\ 9.92 \times 10^{-5} \end{Bmatrix}$$

$$\Rightarrow \hat{f}_{1y}^{(e)} = -2625\ \text{N}$$

$$m_1^{(e)} = -10416.67\ \text{N}\cdot\text{m}$$

$$f_{2y}^{(e)} = 2625\ \text{N}$$

$$m_2^{(e)} = -2708.33\ \text{N}\cdot\text{m}$$

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \begin{Bmatrix} -2625 \\ -10416.667 \\ 2625 \\ -2708.33 \end{Bmatrix} - \begin{Bmatrix} -12500 \\ -10416.667 \\ -12500 \\ -10416.667 \end{Bmatrix}$$

$$\Rightarrow f_{1y} = 9875\ \text{N},\ m_1 = 0$$

$$f_{2y} = 15125\ \text{N},\ m_2 = -13125\ \text{N}\cdot\text{m}$$

Element 2–3

$$\begin{Bmatrix} f_{2y}^{(e)} \\ m_2^{(e)} \\ f_{3y}^{(e)} \\ m_3^{(e)} \end{Bmatrix} = \frac{(210\times10^9)(2\times10^{-4})}{4^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 9.92\times10^{-5} \\ 0 \\ -1.091\times10^{-4} \end{Bmatrix}$$

$$\Rightarrow f_{2y} = 13281.25 \text{ N}$$

$$m_2 = 13125 \text{ N}\cdot\text{m}$$

$$f_3 = 6718.5 \text{ N}$$

$$m_3 = 0$$

Global

$$F_{1y} = f_{1y} = 9875 \text{ N}$$

$$M_1 = m_1 = 0$$

$$F_{2y} = 1525 + 13281.25 = 28406.25 \text{ N}$$

$$M_2 = -13125 + 13125 = 0$$

$$F_{3y} = f_{3y} = 6718.75 \text{ N}$$

$$M_3 = m_3 = 0$$

4.25

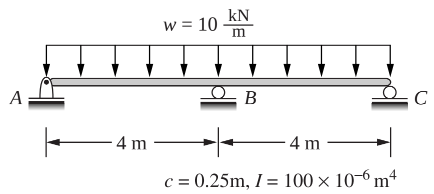
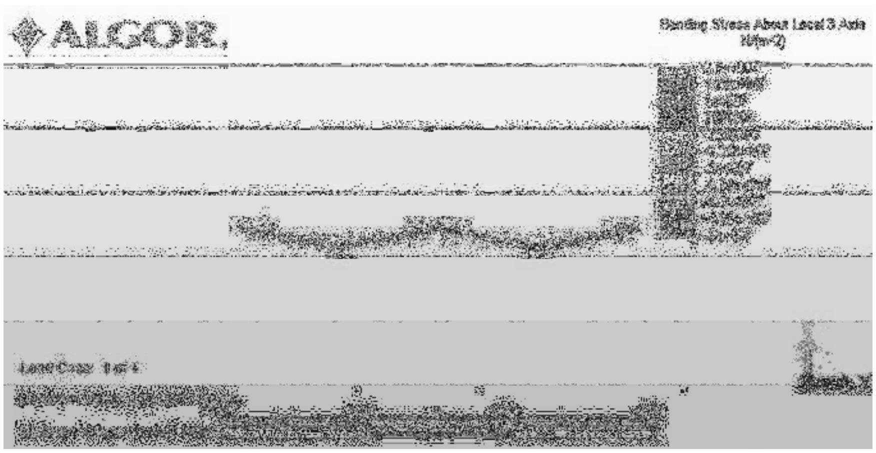
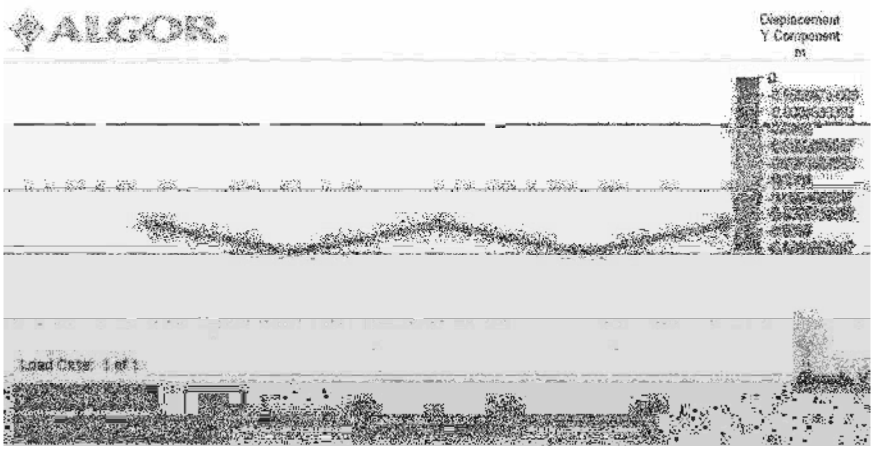


Figure P4–25





4.26

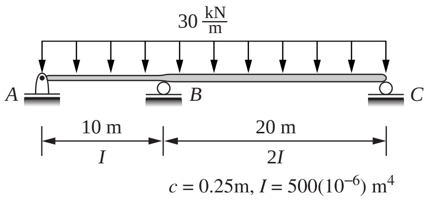
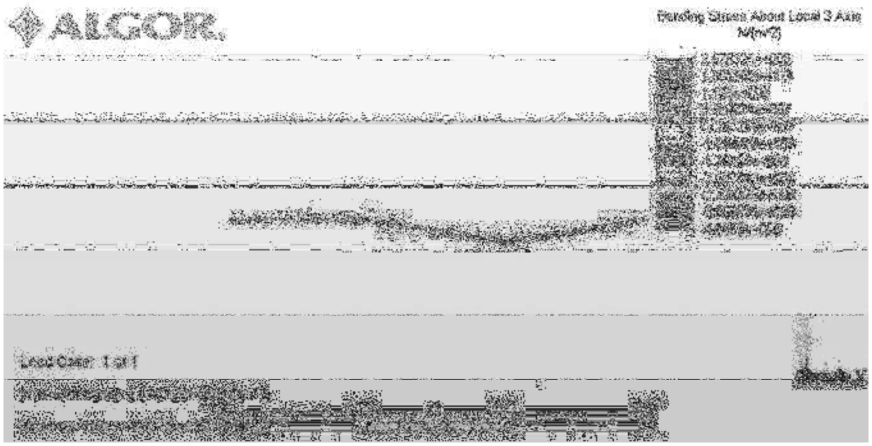
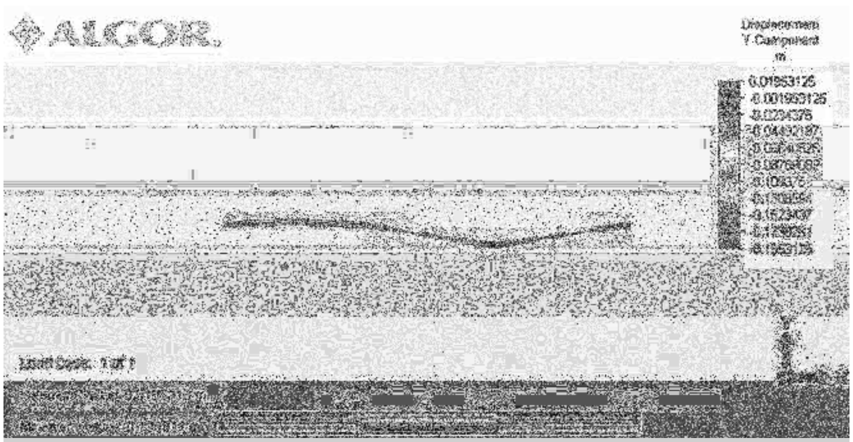


Figure P4–26





4.27

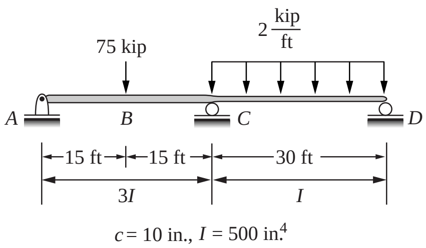
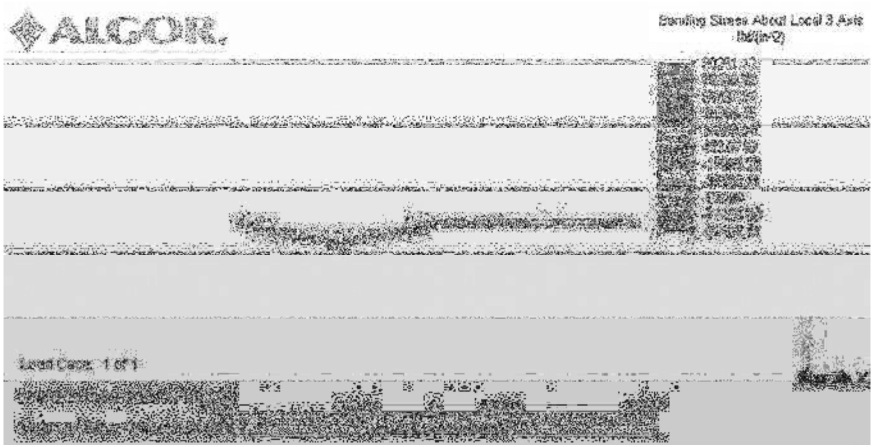
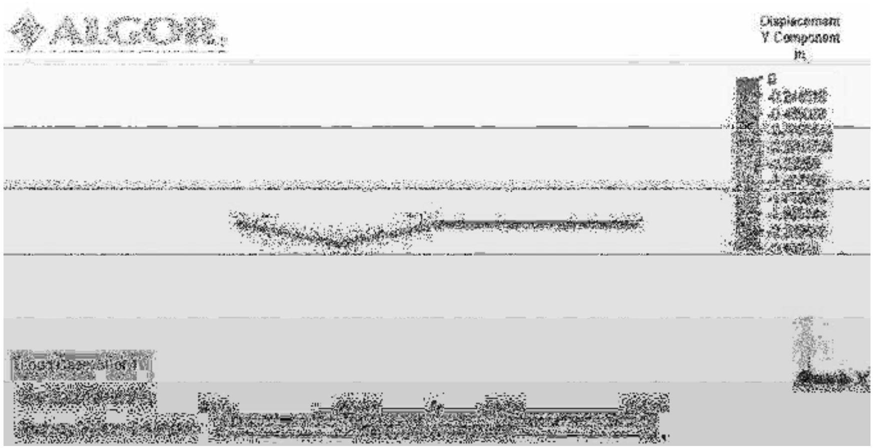


Figure P4–27





4.28

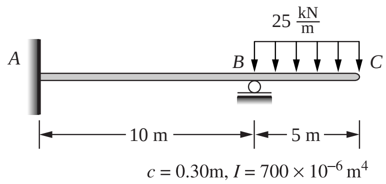


Figure P4–28

1 **** BEAM ELEMENTS

number of beam elements	= 2
number of area property sets	= 1
number of fixed end force sets	= 4

number of materials = 1
number of intermediate load sets = 4

1 **** BEAM ELEMENT FORCES AND MOMENTS

ELEMENT NO.	CASE (MODE)	AXIAL FORCE R1	SHEAR FORCE R2	SHEAR FORCE R3	TORSION MOMENT M1	BENDING MOMENT M2	BENDING MOMENT M3
1	1	0.000E+00	4.688E+04	0.000E+00	0.000E+00	0.000E+00	1.562E+05
		0.000E+00	4.688E+04	0.000E+00	0.000E+00	0.000E+00	–3.125E+05
2	1	0.000E+00	–1.250E+05	0.000E+00	0.000E+00	0.000E+00	–3.125E+05
		0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00

1 **** BEAM ELEMENT STRESSES

ELEMENT NO.	CASE (MODE)	P/A	P/A+M2/S2	P/A-M2/S2	P/A+M3/S3	P/A-M3/S3	WORST SUM
1	1	0.000E+00	0.000E+00	0.000E+00	6.6973+07	–6.697E+07	6.697E+07
		0.000E+00	0.000E+00	0.000E+00	–1.339E+08	1.339E+08	1.339E+08
2	1	0.000E+00	0.000E+00	0.000E+00	–1.339E+08	1.339E+08	1.339E+08
		0.000E+00	0.000E+00	0.000E+00	–1.192E–07	1.192E–07	1.192E–07

4.29

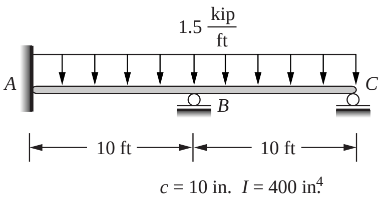
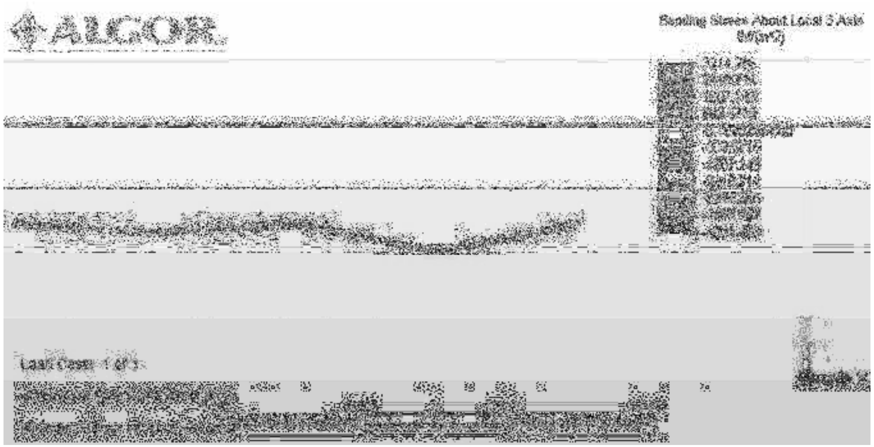
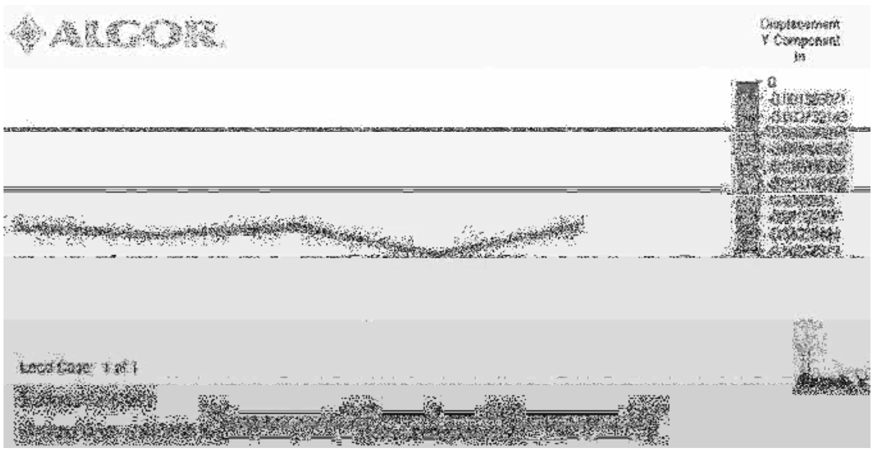
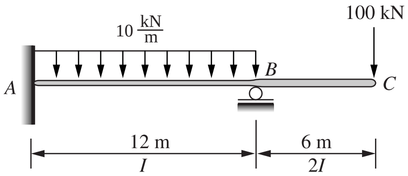


Figure P4–29



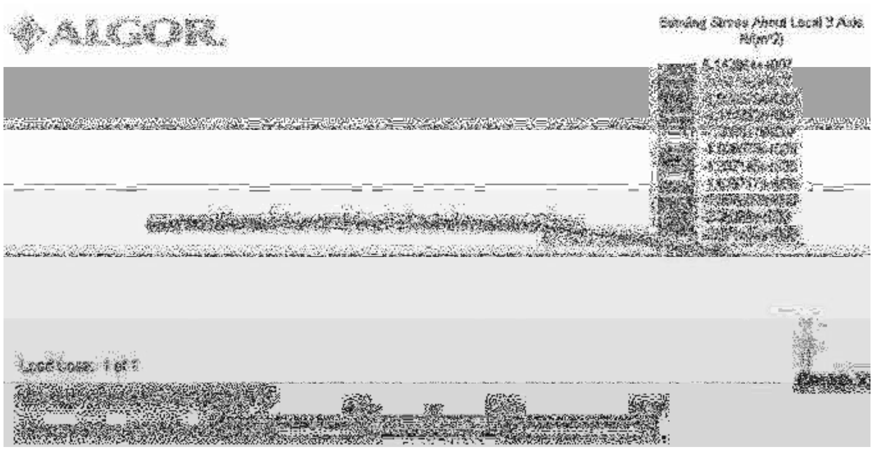


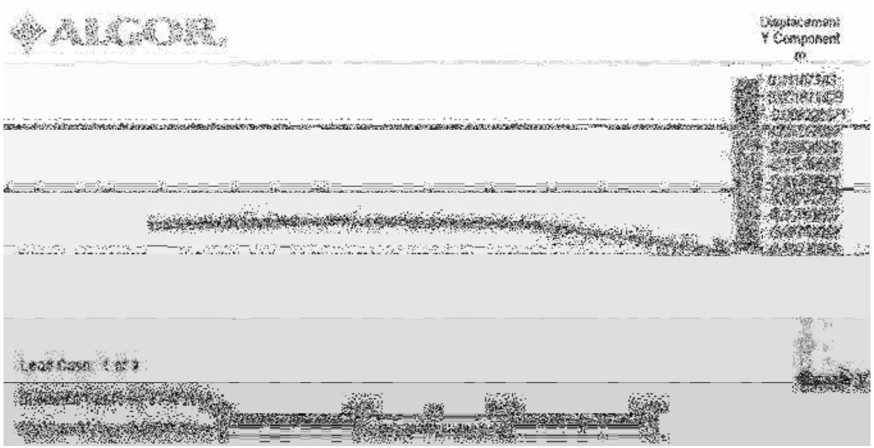
4.30



$c = 0.30\text{m}$, $I = 700 \times 10^{-6}\ \text{m}^4$

Figure P4–30





4.31 Design a beam of ASTM A36 steel with allowable bending stress of 160 MPa to support the load shown in Figure P4–31. Assume a standard wide flange beam from Appendix F or some other source can be used.

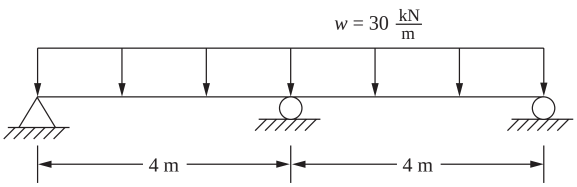


Figure P4–31

1**** BEAM ELEMENT FORCES AND MOMENTS

ELEMENT NO.	CASE (MODE)	AXIAL FORCE R1	SHEAR FORCE R2	SHEAR FORCE R3	TORSION MOMENT M1	BENDING MOMENT M2	BENDING MOMENT M3
1	1	0.000E+00	–4.500E+04	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		0.000E+00	1.500E+04	0.000E+00	0.000E+00	0.000E+00	3.000E+04
2	1	0.000E+00	1.500E+04	0.000E+00	0.000E+00	0.000E+00	3.000E+04
		0.000E+00	7.500E+04	0.000E+00	0.000E+00	0.000E+00	–6.000E+04
3	1	0.000E+00	–7.500E+04	0.000E+00	0.000E+00	0.000E+00	–6.000E+04
		0.000E+00	–1.500E+04	0.000E+00	0.000E+00	0.000E+00	3.000E+04
4	1	0.000E+00	–1.500E+04	0.000E+00	0.000E+00	0.000E+00	3.000E+04
		0.000E+00	4.500E+04	0.000E+00	0.000E+00	0.000E+00	0.000E+00

1 **** BEAM ELEMENT STRESSES

ELEMENT NO.	CASE (MODE)	P/A	P/A+M2/S2	P/A-M2/S2	P/A+M3/S3	P/A-M3/S3	WORST SUM
1	1	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		0.000E+00	0.000E+00	0.000E+00	4.658E+06	–4.658E+06	4.658E+06
2	1	0.000E+00	0.000E+00	0.000E+00	4.658E+06	–4.658E+06	4.658E+06
		0.000E+00	0.000E+00	0.000E+00	–9.317E+06	9.317E+06	9.317E+06
3	1	0.000E+00	0.000E+00	0.000E+00	–9.317E+06	9.317E+06	9.317E+06
		0.000E+00	0.000E+00	0.000E+00	4.658E+06	–4.658E+06	4.658E+06
4	1	0.000E+00	0.000E+00	0.000E+00	4.658E+06	-4.658E+06	4.658E+06
		0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00

S75 × 11.2	1430	76	64	6.6	8.9	1.20	31.6	29.0	0.254	7.72	13.1
8.5	1070	76	59	6.6	4.3	1.03	27.1	31.0	0.190	6.44	13.3

† It may be noted that an American Standard Beam is designated by the letter S followed by the nominal depth in millimeters and the mass in kilograms per meter. S75 × 8.5 acceptable for σ_{max} ≤ 160 MPa. But not for deflection. Try larger section. W 10 × 112 works.

4.32 Select a standard steel pipe from Appendix F to support the load shown. The allowable bending stress must not exceed 24 ksi, and the allowable deflection must note exceed $\frac{L}{360}$ of any span.

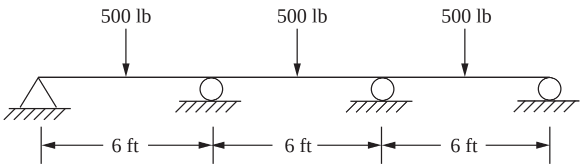
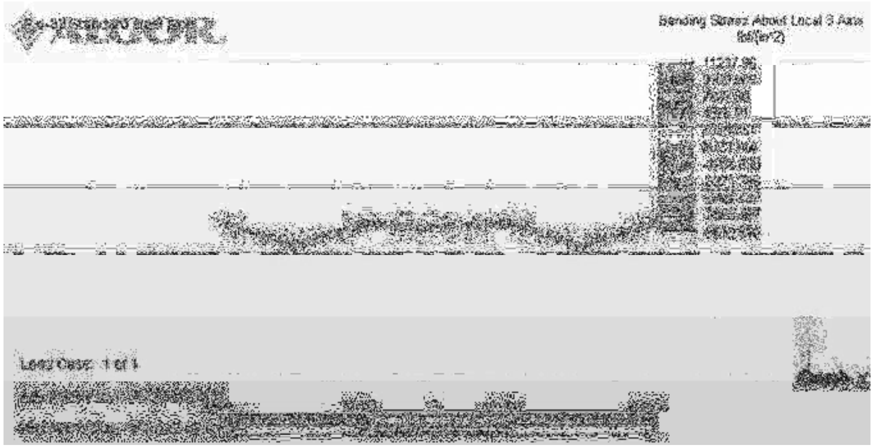
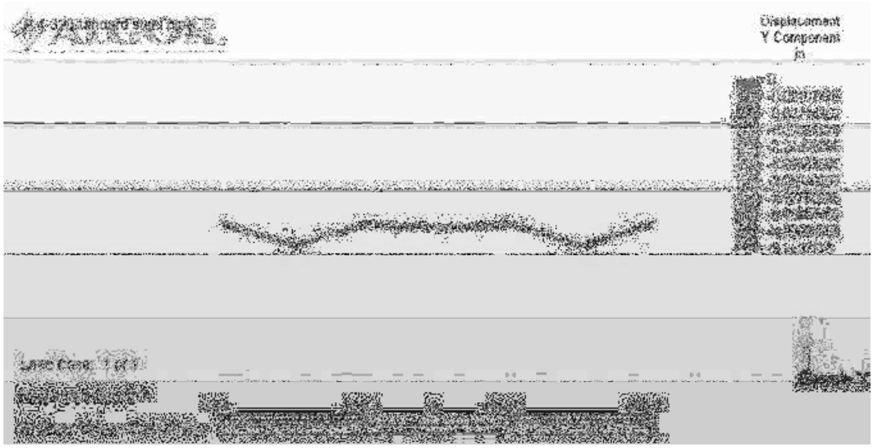


Figure P4–32





1 **** BEAM ELEMENTS

number of beam elements = 6
number of area property sets = 1
number of fixed end force sets = 4
number of materials = 1
number of intermediate load sets = 4

1**** BEAM ELEMENT FORCES AND MOMENTS

ELEMENT No.	CASE (MODE)	AXIAL FORCE R1	SHEAR FORCE R2	SHEAR FORCE R3	TORSION MOMENT M1	BENDING MOMENT M2	BENDING MOMENT M3
1	1	0.000E+00	–1.751E+02	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		0.000E+00	–1.751E+02	0.000E+00	0.000E+00	0.000E+00	6.303E+03
2	1	0.000E+00	3.249E+02	0.000E+00	0.000E+00	0.000E+00	6.303E+03
		0.000E+00	3.249E+02	0.000E+00	0.000E+00	0.000E+00	–5.394E+03
3	1	0.000E+00	–2.500E+02	0.000E+00	0.000E+00	0.000E+00	–5.394E+03
		0.000E+00	–2.500E+02	0.000E+00	0.000E+00	0.000E+00	3.606E+03
4	1	0.000E+00	2.500E+02	0.000E+00	0.000E+00	0.000E+00	3.606E+03
		0.000E+00	2.500E+02	0.000E+00	0.000E+00	0.000E+00	–5.394E+03
5	1	0.000E+00	–3.249E+02	0.000E+00	0.000E+00	0.000E+00	–5.394E+03
		0.000E+00	–3.249E+02	0.000E+00	0.000E+00	0.000E+00	6.303E+03
6	1	0.000E+00	1.751E+02	0.000E+00	0.000E+00	0.000E+00	6.303E+03
		0.000E+00	1.751E+02	0.000E+00	0.000E+00	0.000E+00	0.000E+00

1 **** BEAM ELEMENT STRESSES

ELEMENT No.	CASE (MODE)	P/A	P/A+M2/S2	P/A–M2/S2	P/A+M3/S3	P/A–M3/S3	WORST SUM
1	1	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		0.000E+00	0.000E+00	0.000E+00	1.124E+04	–1.124E+04	1.124E+04
2	1	0.000E+00	0.000E+00	0.000E+00	1.124E+04	–1.124E+04	1.124E+04
		0.000E+00	0.000E+00	0.000E+00	–9.621E+03	9.621E+03	9.621E+03
3	1	0.000E+00	0.000E+00	0.000E+00	–9.621E+03	9.621E+03	9.621E+03
		0.000E+00	0.000E+00	0.000E+00	6.432E+03	–6.432E+03	6.432E+03
4	1	0.000E+00	0.000E+00	0.000E+00	6.432E+03	–6.432E+03	6.432E+03
		0.000E+00	0.000E+00	0.000E+00	–9.621E+03	9.621E+03	9.621E+03
5	1	0.000E+00	0.000E+00	0.000E+00	–9.621E+03	9.621E+03	9.621E+03
		0.000E+00	0.000E+00	0.000E+00	1.124E+04	–1.124E+04	1.124E+04
6	1	0.000E+00	0.000E+00	0.000E+00	1.124E+04	–1.124E+04	1.124E+04
		0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00

$\sigma_{max} = 11.24$ ksi $< \sigma_{allow} = 24$ ksi (For 2 in. schedule 40 steel pipe, $I = 0.666$ in.^{4})

4.33 Select a rectangular structural tube from Appendix F to support the loads shown for the beam in Figure P4–33. The allowable bending stress should not exceed 24 ksi.

Rectangular tube $4'' \times 2\frac{1}{2}'' \times \frac{5}{16}''$
$I_2 = 2.89$ in.^{4}
$I_3 = 6.13$ in.^{4}

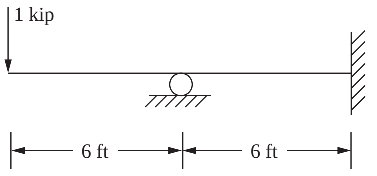


Figure P4–33

1 **** BEAM ELEMENT FORCES AND MOMENTS

ELEMENT No.	CASE (MODE)	AXIAL FORCE R1	SHEAR FORCE R2	SHEAR FORCE R3	TORSION MOMENT M1	BENDING MOMENT M2	BENDING MOMENT M3
1	1	0.000E+00	1.000E+03	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		0.000E+00	1.000E+03	0.000E+00	0.000E+00	0.000E+00	–3.600E+04
2	1	0.000E+00	1.000E+03	0.000E+00	0.000E+00	0.000E+00	–3.600E+04
		0.000E+00	1.000E+03	0.000E+00	0.000E+00	0.000E+00	–7.200E+04
3	1	0.000E+00	–1.500E+03	0.000E+00	0.000E+00	0.000E+00	–7.200E+04
		0.000E+00	–1.500E+03	0.000E+00	0.000E+00	0.000E+00	–1.800E+04
4	1	0.000E+00	–1.500E+03	0.000E+00	0.000E+00	0.000E+00	–1.800E+04
		0.000E+00	–1.500E+03	0.000E+00	0.000E+00	0.000E+00	3.600E+04

1 **** BEAM ELEMENT STRESSES

ELEMENT NO.	CASE (MODE)	P/A	P/A+M2/S2	P/A–M2/S2	P/A+M3/S3	P/A–M3/S3	WORST SUM
1	1	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		0.000E+00	0.000E+00	0.000E+00	–1.176E+04	1.176E+04	1.176E+04

2	1	0.000E+00	0.000E+00	0.000E+00	–1.176E+04	1.176E+04	1.176E+04
		0.000E+00	0.000E+00	0.000E+00	–2.353E+04	2.353E+04	2.353E+04
3	1	0.000E+00	0.000E+00	0.000E+00	–2.353E+04	2.353E+04	2.353E+04
		0.000E+00	0.000E+00	0.000E+00	–5.882E+03	5.882E+03	5.882E+03
4	1	0.000E+00	0.000E+00	0.000E+00	–5.882E+03	5.882E+03	5.882E+03
		0.000E+00	0.000E+00	0.000E+00	1.176E+04	–1.176E+04	1.176E+04

σ_{max} = 23,530 psi < σ_{allow} = 24,000 psi

4.34 Select a standard W section from Appendix F or some other source to support the loads shown for the beam in Figure P4–34. The bending stress must not exceed 160 MPa.

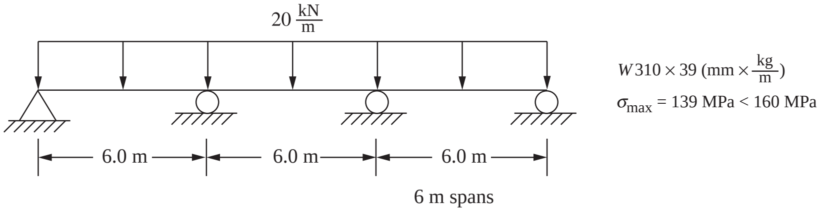


Figure P4 –34

Displacements/Rotations (degrees) of nodes

NODE number	X– translation	Y– translation	Z– translation	X– rotation	Y– rotation	Z– rotation
1	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	0.0000E+00	–4.2861E–03	0.0000E+00	0.0000E+00	0.0000E+00	–5.8470E–03
3	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	2.3388E–02
4	0.0000E+00	–2.4492E–03	0.0000E+00	0.0000E+00	0.0000E+00	1.7541E–02
5	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	–9.3553E–02
6	0.0000E+00	–9.7968E–03	0.0000E+00	0.0000E+00	0.0000E+00	–6.4317E–02

1 **** BEAM ELEMENT FORCE AND MOMENTS

ELEMENT NO.	CASE (MODE)	AXIAL FORCE R1	SHEAR FORCE R2	SHEAR FORCE R3	TORSION MOMENT M1	BENDING MOMENT M2	BENDING MOMENT M3
1	1	0.000E+00	–6.115E+04	0.000E+00	0.000E+00	0.000E+00	–6.231E+04
		0.000E+00	–1.154E+03	0.000E+00	0.000E+00	0.000E+00	3.115E+04
2	1	0.000E+00	–1.154E+03	0.000E+00	0.000E+00	0.000E+00	3.115E+04
		0.000E+00	5.885E+04	0.000E+00	0.000E+00	0.000E+00	–5.538E+04
3	1	0.000E+00	–5.654E+04	0.000E+00	0.000E+00	0.000E+00	–5.538E+04
		0.000E+00	3.462E+03	0.000E+00	0.000E+00	0.000E+00	2.423E+04
4	1	0.000E+00	3.462E+03	0.000E+00	0.000E+00	0.000E+00	2.423E+04
		0.000E+00	6.346E+04	0.000E+00	0.000E+00	0.000E+00	–7.615E+04
5	1	0.000E+00	–7.269E+04	0.000E+00	0.000E+00	0.000E+00	–7.615E+04
		0.000E+00	–1.269E+04	0.000E+00	0.000E+00	0.000E+00	5.192E+04
6	1	0.000E+00	–1.269E+04	0.000E+00	0.000E+00	0.000E+00	5.192E+04
		0.000E+00	4.731E+04	0.000E+00	0.000E+00	0.000E+00	0.000E+00

1**** BEAM ELEMENT STRESSES

ELEMENT NO.	CASE (MODE)	P/A	P/A + M2/S2	P/A–M2/S2	P/A+M3/S3	P/A-M3/S3	WORST SUM
1	1	0.000E+00	0.000E+00	0.000E+00	–1.139E+08	1.139E+08	1.139E+08
		0.000E+00	0.000E+00	0.000E+00	5.695E+07	– 5.695E+07	5.695E+07
2	1	0.000E+00	0.000E+00	0.000E+00	5.695E+07	– 5.569E+07	5.695E+07
		0.000E+00	0.000E+00	0.000E+00	–1.013E+08	1.013E+08	1.013E+08
3	1	0.000E+00	0.000E+00	0.000E+00	–1.013E+08	1.013E+08	1.013E+08
		0.000E+00	0.000E+00	0.000E+00	4.430E+07	– 4.430E+07	4.430E+07
4	1	0.000E+00	0.000E+00	0.000E+00	4.430E+07	–4.430E+07	4.430E+07
		0.000E+00	0.000E+00	0.000E+00	–1.392E+08	1.392E+08	1.392E+08
5	1	0.000E+00	0.000E+00	0.000E+00	–1.392E+08	1.392E+08	1.392E+08
		0.000E+00	0.000E+00	0.000E+00	9.492E+07	–9.492E+07	9.492E+07
6	1	0.000E+00	0.000E+00	0.000E+00	9.492E+07	–9.492E+07	9.492E+07
		0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00

4.35 For the beam shown in Figure P4–35, determine a suitable sized W section from Appendix F or from another suitable source such that the bending stress does not exceed 150 MPa and the maximum deflection does not exceed $\frac{L}{360}$ of any span.

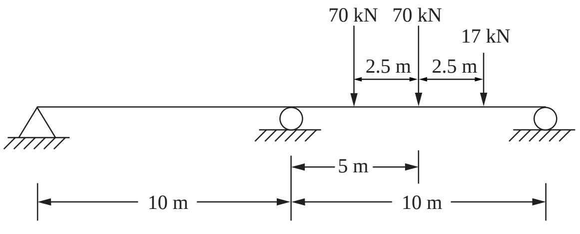


Figure P4–35

ASTM A36 steel

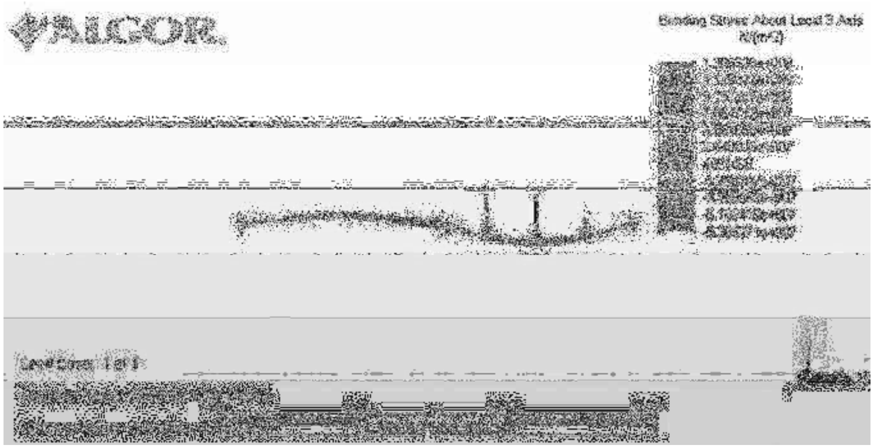
$$\frac{L}{360} = \frac{10\text{ m}}{360} = 0.0278\text{ m}$$

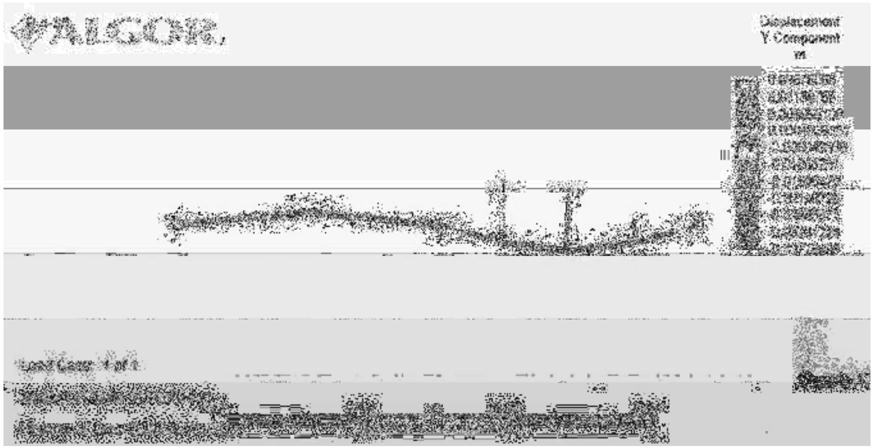
$$\Delta Y_{\max} = 0.0556\text{ m}$$

Bending stress max = 150 MPa = $1.50 \times 10^8\ \frac{\text{N}}{\text{m}^2}$

Beam	I_3 (m^4)	S_3(m^3)	A(m^2)	Bending stress ($\frac{\text{N}}{\text{m}^2}$)	$\Delta Y_{\max}$ (m)
W310 × 143	0.000348	0.002150	0.018200	1.010×10^8	– 0.0269
W460 × 158	0.000796	0.00340	0.0201	6.389×10^7	– 0.0118
W760 × 257	0.003420	0.008850	0.0326	2.455×10^7	– 0.00275
W310 × 44.5	0.0000992	0.000634	0.00569	3.426×10^8	– 0.0944
W310 × 74	0.000165	0.001060	0.009480	2.049×10^8	– 0.0568
W310 × 107	0.000248	0.001590	0.013600	1.366×10^8	– 0.03776

For the problem given a W310 × 107 beam was chosen made of ASTM A36 steel. This made for a maximum deflection of 0.0378 m (see above table) which is less than the 0.0556 m maximum restraint.





4.36 For the stepped shaft shown in Figure P4–36, determine a solid circular cross section for each section shown such that the bending stress does not exceed 160 MPa and the maximum deflection does not exceed $\frac{L}{360}$ of the span.

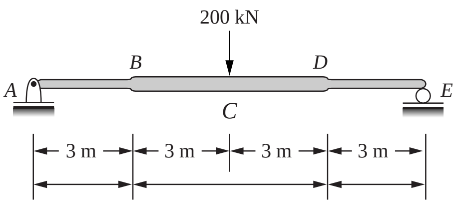


Figure P4–36

Try small radius of 140 mm

Large radius of 166 mm

Yields $\sigma_{max} = 166$ MPa close to

σ_{max} allow $= 160$ MPa.

Need to increase smaller diameter

$$\frac{L}{360} = \frac{12 \text{ m}}{360} = \frac{1}{30} = 0.0333 \text{ m}$$

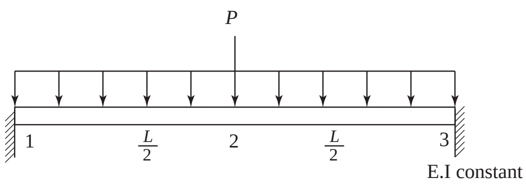
Other trials shown below

d_{AB} and d_{DE}	d_{BD}, mm	σ (MPa)	δ (m)	
279.6	332.5	166.3 MPa > 160	0.064 > 0.033	
290	340	155.5	0.058	δ Too Large
310	360	131	0.046	δ Too Large
340	390	103	0.0330	δ Finally at Limit of $\frac{L}{360}$

$\therefore$ Final $d_{AB} =$ 340 mm $= d_{DE}$

$d_{BD} =$ 390 mm

4.37



Applying the boundary conditions

$v_1 = \phi_1 = \phi_2 = v_3 = \phi_3 = 0$ in the global equation $\{F\} = [K]\{d\}$ we have

$$-\left(P + \frac{wL}{2}\right) = \frac{EI}{(\frac{L}{2})^3}\,[24v_2]$$

$$\Rightarrow \; -P - \frac{wL}{2} = \frac{8EI}{L^3}\,24v_2 \;\Rightarrow\; \boxed{v_2 = \frac{-PL^3}{192EI} - \frac{wL^4}{384EI}}$$

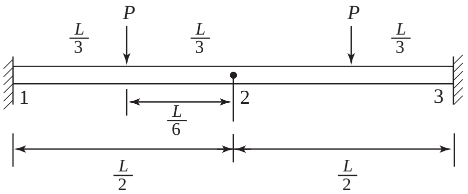
Reactions

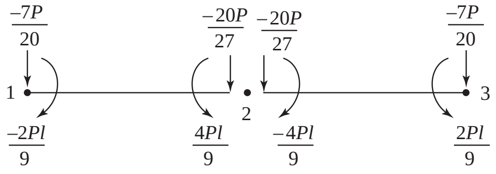
$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{8EI}{L^3}\begin{bmatrix} 12 & \frac{6L}{2} & -12 & \frac{6L}{2} & 0 & 0 \\ \frac{6L}{2} & 4\left(\frac{L}{2}\right)^2 & \frac{-6L}{2} & \frac{2L^2}{4} & 0 & 0 \\ -12 & \frac{-6L}{2} & 24 & 0 & -12 & \frac{6L}{2} \\ 6L & \frac{2L^2}{4} & 0 & \frac{8L^2}{4} & \frac{-6L}{2} & \frac{2L^2}{4} \\ 0 & 0 & -12 & \frac{-6L}{2} & 12 & \frac{-6L}{2} \\ 0 & 0 & \frac{6L}{2} & \frac{2L^2}{2} & \frac{-6L}{2} & \frac{4L^2}{4} \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \frac{-PL^3}{192\,EI} - \frac{wL^4}{384\,EI} \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} \frac{-wL}{4} \\ \frac{-wL^2}{48} \\ \frac{-wL}{2} \\ 0 \\ \frac{-wL}{4} \\ \frac{wL^2}{48} \end{Bmatrix}$$

$$\Rightarrow\; F_{1y} = \frac{P + wL}{2},\; M_1 = \frac{PL}{8} + \frac{wL^2}{12},\; F_{2y} = 0$$

$$M_2 = 0,\; F_{3y} = \frac{P + wL}{2},\; M_3 = \frac{-PL}{8} - \frac{wL^2}{12}$$

4.38





After applying the boundary conditions

$$v_1 = \phi_1 = \phi_2 = v_2 = \phi_3 = 0$$

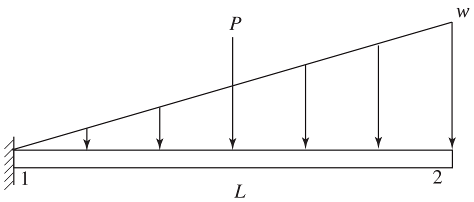
We have in the equation $\{F\} = [K]\{d\}$ the following

$$\frac{-40P}{27} = \frac{EI}{(3l)^3}[24v_2] \Rightarrow v_2 = \frac{-40Pl^3}{24EI}$$

$$\Rightarrow \quad v_2 = \frac{-5Pl^3}{3EI} \text{ since } l = \frac{L}{6}$$

$$\Rightarrow \quad v_2 = \frac{-5P(\frac{L}{6})^3}{3EI} \quad \Rightarrow \quad \boxed{v_2 = \frac{-5PL^3}{648\,EI}}$$

4.39



Applying the boundary conditions $v_1 = \phi_1 = 0$

$$\Rightarrow \begin{Bmatrix} -\dfrac{P}{2}-\dfrac{7wL}{20} \\ \dfrac{PL}{8}+\dfrac{wL^2}{20} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

Multiplying (1) by L and (2) by 2 and then adding

$$\frac{-PL}{2}-\frac{7wL^2}{20}=\frac{EI}{L^3}\,[12Lv_2-6L^2\phi_2]$$

$$\frac{PL}{4}+\frac{2wL^2}{20}=\frac{EI}{L^3}\,[-12Lv_2+8L^2\phi_2]$$

$$\frac{-PL}{4}-\frac{wL^2}{4}=\frac{EI}{L^3}\,(2L^2\phi_2)\Rightarrow \boxed{\phi_2=\frac{-(PL^2+wL^3)}{8EI}}$$

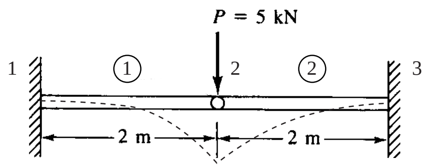
Substituting in (1) $\Rightarrow \dfrac{-P}{2}-\dfrac{7wL}{20}=\dfrac{EI}{L^3}\left[12v_2-6L\left[\dfrac{-PL^2-wL^3}{8EI}\right]\right]$

$$\Rightarrow \frac{-5P}{4}-\frac{22wL}{20}=\frac{12EI}{L^3}\,v_2\Rightarrow \boxed{v_2=\frac{-(25P+22wL)L^3}{240EI}}$$

$$\begin{Bmatrix}F_{1y}\\M_1\\F_{2y}\\M_2\end{Bmatrix}=\frac{EI}{L^3}\begin{bmatrix}12&6L&-12&6L\\6L&4L^2&-6L&2L^2\\-12&-6L&12&-6L\\6L&2L^2&-6L&4L^2\end{bmatrix}\begin{Bmatrix}0\\0\\\frac{-(25P+22wL)L^3}{240EI}\\\frac{-PL^2-wL^3}{EI}\end{Bmatrix}-\begin{Bmatrix}-\frac{P}{2}-\frac{3wL}{20}\\-\frac{PL}{8}-\frac{wL^2}{30}\\-\frac{P}{2}-\frac{7wL}{20}\\\frac{PL}{8}+\frac{wL^2}{20}\end{Bmatrix}$$

$$\Rightarrow \begin{aligned}&F_{1y}=P+\frac{wL}{2},\ M_1=\frac{PL}{2}+\frac{1}{3}wL^2\\&F_{2y}=0,\ M_2=0\end{aligned}$$

4.40



Assume the hinge as a part of the first element. Therefore, stiffness matrix for element 1 is

$$[k^{(1)}]=\frac{3EI}{8}\begin{array}{c}\begin{matrix}v_1&\phi_1&v_2&\phi_2\end{matrix}\\\begin{bmatrix}1&2&-1&0\\2&4&-2&0\\-1&2&1&0\\0&0&0&0\end{bmatrix}\end{array}$$

Stiffness matrix of element 2 is

$$[k^{(2)}]=\begin{array}{cc}\begin{bmatrix}12&12&-12&12\\12&16&-12&8\\-12&-12&12&-12\\12&8&-12&16\end{bmatrix}&\begin{matrix}v_2\\\phi_2\\v_3\\\phi_3\end{matrix}\end{array}$$

Adding the matrices by superposition

$$[K] = \begin{array}{c} \begin{array}{cccccc} v_1 & \phi_1 & v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ \begin{bmatrix} 3 & 6 & -3 & 0 & 0 & 0 \\ 6 & 12 & 6 & 0 & 0 & 0 \\ -3 & -6 & 15 & 12 & -12 & 12 \\ 0 & 0 & 12 & 16 & -12 & 8 \\ 0 & 0 & 12 & -12 & 12 & -12 \\ 0 & 0 & 12 & 8 & 12 & 16 \end{bmatrix} \end{array}$$

Applying the boundary conditions

$$v_1 = 0,\ \phi_1 = 0,\ v_3 = 0,\ \phi_3 = 0$$

$$\therefore\ \frac{EI}{8}\begin{bmatrix} 15 & 12 \\ 12 & 16 \end{bmatrix}\begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -5\text{ kN} \\ 0 \end{Bmatrix}$$

$$\Rightarrow \frac{EI}{8}[15v_2 + 12\phi_2] = -5000 \qquad (1)$$

$$12v_2 + 16\phi_2 = 0 \qquad (2)$$

From (2), $v_2 = -\dfrac{4}{3}\phi_2 \qquad (3)$

Substituting (3) in (1)

$$\overset{5}{\cancel{15}} \times \left(-\frac{4}{\cancel{3}}\right)\phi_2 + 12\phi_2 = \frac{-5 \times 10^3 \times 8}{210 \times 10^9 \times 2 \times 10^{-4}}$$

$$\Rightarrow -8\phi_2 = \frac{-5 \times 10^3 \times 8}{420 \times 10^5}$$

$$\Rightarrow \quad \phi_2 = \frac{5000}{420 \times 10^5}$$

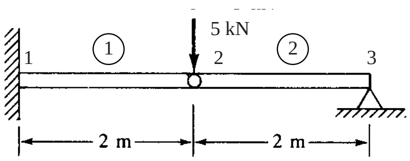
$$\Rightarrow \quad \boxed{\phi_2^{(2)} = 1.19 \times 10^{-4}\text{ rad}}$$

$$\therefore \quad v_2 = -\frac{4}{3}\phi_2$$

$$= -\frac{4}{3} \times 1.19 \times 10^{-4}$$

Hence $\boxed{v_2 = -1.57 \times 10^{-4}\text{ m}}$

4.41



$$[k^{(1)}] = \frac{3EI}{8}\begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k^{(2)}] = \frac{EI}{8}\begin{bmatrix} 12 & 12 & -12 & 12 \\ 12 & 16 & -12 & 8 \\ -12 & -12 & 12 & -12 \\ 12 & 8 & -12 & 16 \end{bmatrix}$$

By superposition

$$[K] = \frac{EI}{8}\begin{array}{c} \begin{array}{cccccc} v_1 & \phi_1 & v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ \begin{bmatrix} 3 & 6 & -3 & 0 & 0 & 0 \\ 6 & 12 & -6 & 0 & 0 & 0 \\ -3 & -6 & 15 & 12 & -12 & 12 \\ 0 & 0 & 12 & 16 & -12 & 8 \\ 0 & 0 & -12 & -12 & 12 & -12 \\ 0 & 0 & 12 & 8 & -12 & 16 \end{bmatrix} \end{array}$$

Boundary conditions

$$v_1 = 0,\ \phi_1 = 0,\ v_3 = 0$$

$$\frac{EI}{8}\begin{bmatrix} 15 & 12 & 12 \\ 12 & 16 & 8 \\ 12 & 8 & 16 \end{bmatrix}\begin{Bmatrix} v_3 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} -5000 \\ 0 \\ 0 \end{Bmatrix}$$

Solving by Gaussian Elimination we have

$$\left[\begin{array}{ccc|c} 15 & 12 & 12 & -9.5\times10^{-4} \\ 12 & 16 & 8 & 0 \\ 12 & 8 & 16 & 0 \end{array}\right]$$

Select $a_{11} = 15$ as the pivot

(a) Add the multiple $\frac{-a_{21}}{a_{11}} = \frac{-12}{15} = \frac{-4}{5}$ of the first row to the second row

(b) Add the multiple $\frac{-a_{31}}{a_{11}} = \frac{-4}{5}$ of the first row to the third row.

$$\left[\begin{array}{ccc|c} 15 & 12 & 12 & -9.5\times10^{-4} \\ 0 & 6.4 & -1.6 & 7.6\times10^{-4} \\ 0 & -1.6 & 6.4 & 7.6\times10^{-4} \end{array}\right]$$

Select $a_{22} = 6.4$ as the pivot

Repeating the same procedure

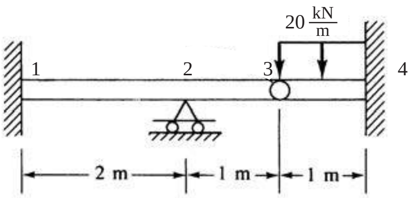
$$\begin{bmatrix} 15 & 12 & 12 & -9.5\times10^{-4} \\ 0 & 6.4 & -1.6 & 7.6\times10^{-4} \\ 0 & 0 & 6 & 9.5\times10^{-4} \end{bmatrix}$$

$$\Rightarrow \quad \underline{\underline{\phi_3 = 1.583 \times 10^{-4} \text{ rad}}}$$

$$\underline{\underline{\phi_2 = 1.583 \times 10^{-4} \text{ rad}}}$$

$$\underline{\underline{v_2 = -3.175 \times 10^{-4} \text{ m}}}$$

4.42



$$[K^{(1)}] = \frac{EI}{8} \begin{array}{c} \begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 12 & 12 & -12 & 12 \\ 12 & 16 & -12 & 8 \\ -12 & -12 & 12 & -12 \\ 12 & 8 & -12 & 16 \end{bmatrix} \end{array} = EI \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & \frac{-3}{2} & \frac{3}{2} \\ \frac{3}{2} & 2 & \frac{-3}{2} & 1 \\ \frac{-3}{2} & \frac{-3}{2} & \frac{3}{2} & \frac{-3}{2} \\ \frac{3}{2} & 1 & \frac{-3}{2} & 2 \end{bmatrix}$$

Assume the hinge as *a* + right end part of element (2)

$$[K^{(2)}] = \frac{3EI}{(1)^3} \begin{array}{c} \begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$[K^{(3)}] = \frac{EI}{(1)^3} \begin{array}{c} \begin{array}{cccc} v_3 & \phi_3 & v_4 & \phi_4 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \end{array}$$

By superposition

$$[K] = \begin{array}{c} \begin{array}{cccccccc} v_1 & \phi_1 & v_2 & \phi_2 & v_3 & \phi_3 & v_4 & \phi_4 \end{array} \\ \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & \frac{-3}{2} & \frac{3}{2} & 0 & 0 & 0 & 0 \\ \frac{3}{2} & 2 & \frac{-3}{2} & 1 & 0 & 0 & 0 & 0 \\ \frac{-3}{2} & \frac{-3}{2} & \frac{9}{2} & \frac{-3}{2} & -3 & 0 & 0 & 0 \\ \frac{3}{2} & 1 & \frac{-3}{2} & 5 & -3 & 0 & 0 & 0 \\ 0 & 0 & -3 & -3 & 15 & 6 & -12 & 6 \\ 0 & 0 & 0 & 0 & 6 & 4 & -6 & 2 \\ 0 & 0 & 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \end{array}$$

Applying the boundary conditions

$$v_1 = \phi_1 = v_2 = v_4 = \phi_4 = 0$$

$$\Rightarrow \quad EI \begin{bmatrix} 5 & -3 & 0 \\ -3 & 15 & 6 \\ 0 & 6 & 4 \end{bmatrix} \begin{Bmatrix} \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10 \text{ kN} \\ -1.67 \text{ kN}\cdot\text{m} \end{Bmatrix}$$

$$\Rightarrow \quad \phi_2 = \frac{3}{5} v_3 \qquad (1)$$

$$6v_3 + 4\phi_3 = \frac{1667}{EI}$$

$$\Rightarrow \quad \phi_3 = \left[\frac{-416.75}{EI} - 1.5v_3\right] \qquad (2)$$

$$EI(-3\phi_2 + 15v_3 + 6\phi_3) = -10000 \qquad (3)$$

Substituting (2) and (1) in (3)

$$-\frac{9}{5}v_3 + 15v_3 - \frac{2500}{EI} - 9v_3 = \frac{-10000}{EI}$$

$$\Rightarrow \quad \frac{21}{5}v_3 = -\frac{12500}{EI}$$

$$\underline{\underline{v_3 = -4.252 \times 10^{-5} \text{ m}}}$$

$$\underline{\underline{\phi_2 = -2.551 \times 10^{-5} \text{ rad}}}$$

$$\underline{\underline{\phi_3 = 5.386 \times 10^{-5} \text{ rad}}}$$

4.43 From Equation (4.7.15)

$$\pi_p = \int_0^L \frac{EI}{2}\{d\}[B]^T[B]\{d\}\,dx - \int_0^L w\{d\}^T[N]^T\,dx - \{d\}^T\{P\}$$

$$\{d\} = \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

EI $\rightarrow$ constant

$$\frac{\partial \pi_p}{\partial v_2} = \left(\frac{\cancel{2}EI}{\cancel{2}}\int_0^L [B]^T[B]dx\right)d_{1y} - \left(\int_0^L N_1 w dx\right) - f_{1y} = 0 \qquad (1)$$

$$\frac{\partial \pi_p}{\partial \phi_1} = \left(\frac{2EI}{2}\int_0^L [B]^T[B]dx\right)\phi_1 - \left(\int_0^L N_2 w dx\right)1_1 - m_1 = 0 \qquad (2)$$

$$\frac{\partial \pi_p}{\partial v_2} = \left(\frac{\cancel{2}EI}{\cancel{2}}\int_0^L [B]^T[B]dx\right)d_{2y} - \left(\int_0^L N_3 w dx\right)1_1 - f_{2y} = 0 \qquad (3)$$

$$\frac{\partial \pi_p}{\partial \phi_2} = \left(\frac{2EI}{2}\int_0^L [B]^T[B]dx\right)\phi_2 - \left(\int_0^L N_4 w dx\right)1_1 - m_2 = 0 \qquad (4)$$

Equations (1) – (4) in matrix form are

$$EI\int_0^L [B]^T[B]dx\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix} - \int_0^L \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} wdx - \begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = 0$$

Simplifying

$$EI\int_0^L [B]^T[B]dx\{d\} - \int_0^L [N]^T wdx - \{P\} = 0$$

4.44

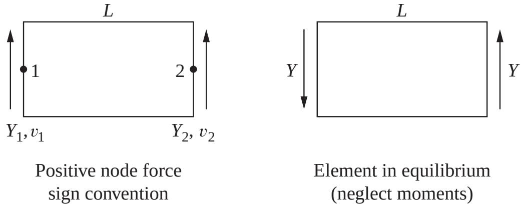


Figure P4–44

$$Y = A_W\,\tau = A_W\,G\gamma = A_w\,G\left(\frac{v_2 - v_1}{L}\right)$$

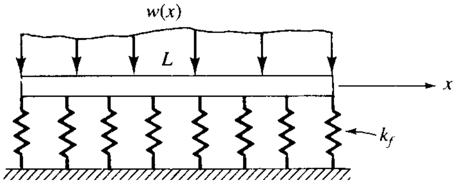
$$Y_1 = -Y = \frac{-A_W\,G}{L}\,(v_2 - v_1)$$

$$Y_2 = Y = \frac{A_W\,G}{L}\,(v_2 - v_1)$$

$$\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \frac{A_W\,G}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}$$

$$[k] = \frac{A_W\,G}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

4.47



$$\pi_p = \int_0^L \frac{1}{2}EI(v'')^2 dx + \int_0^L \frac{k_f v^2}{2}dx - \int_0^L wvdx$$

$$v = [N]\,\{d\} \quad \varepsilon_x = -y\frac{d^2v}{dx^2} = -yv''$$

$$\varepsilon_x = -\,\overline{y}\,[B]\,\{d\}$$

$[B]$ from Equation (4.7.10)

$$\pi_p = \int_0^L\int_A \frac{1}{2}\{\sigma_x^T\}\{\varepsilon_x\}dAdx - \int_0^L bT_y v dx\ \int_0^L \frac{1}{2}k_f\,\{d^T\}[N^T][N]\{d\}\,dx \qquad \underline{bT_y = w}$$

$$= \frac{EI}{2}\int_0^L \{d^T\}B^T[B]\{d\}\,dx - \int_0^L w\,\{d^T\}[N^T]dx + \int_0^L \frac{k_f}{2}\{d^T\}[N^T][N]\{d\}\,dx$$

$$\frac{\partial \pi_p}{\partial \underline{d}} = EI\int_0^L [B^T][B]dx\;\{d\} - \int_0^L [N^T]wd\,x + \int_0^L K_f[N^T][N]dx\;\{d\}$$

$$\therefore\; [k] = EI\int_0^L [B^T][B]dx + k_f\int_0^L [N^T][N]dx$$

Equation.(4.7.19)

New part similar to convection part of heat transfer stiffness matrix.

4.77 Find the deflection at the mid-span using four beam elements, making the shear area zero and then making the shear area equal to $\frac{5}{6}$ times the cross-sectional area (b times h). Then make the beam have decreasing spans of 200 mm, 100 mm, and 50 mm with zero shear area and then $\frac{5}{6}$ times the cross-sectional area. Compare the answers. Based on your program answers, can you conclude whether your program includes the effects of transverse shear deformation?

Beam Span (m)	Shear Area	Displacement at center (m)	% difference
0.400	0	1.28E-03	4.61.E+00
0.400	0.001042	1.34E-03	
0.200	0	1.60E-04	16.21
0.200	0.001042	1.91E-04	
0.100	0	2.00E-05	43.62
0.100	0.001042	3.55E-05	
0.0500	0	2.50E-06	75.58
0.0500	0.001042	1.02E-05	

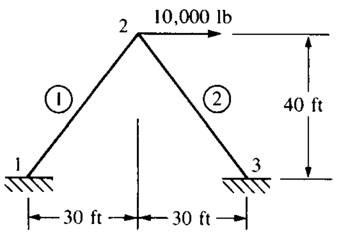
It would appear that the program **DOES** include the effects of transverse shear area which can be seen in the increasing per cent differences as the width of the beam approaches the span of the beam. As these width and span get closer and closer together the shear area becomes a larger factor, this would be the expected outcome if the program includes the effect of transverse shear area in the calculations.

Note: For all of the beams the element definitions remained the same except the beam spans were changed. The figure below shows one example of when the 400mm beam was run with shear area included.

Figure 1: 400mm beam deflection with shear area included

Chapter 5

5.1



$E = 30 \times 10^6$ psi
$A = 10$ in.^{2}
$I = 500$ in.^{4}

Element (1)

$$L^{(1)} = \sqrt{40^2 + 30^2} = 50 \text{ ft} = 600 \text{ in.}$$

$$\cos\theta = \frac{x_2 - x_1}{L^{(1)}} = \frac{30-0}{50} = 0.6$$

$$\sin\theta = \frac{y_2 - y_1}{L^{(1)}} = \frac{40-0}{50} = 0.8$$

$$\frac{E}{L} = 50000, \qquad \frac{12I}{L^2} = 0.0167, \qquad \frac{6I}{L} = 5.0$$

$$[k^{(1)}] = 50000\begin{bmatrix} 3.61 & 4.79 & -4 & -3.61 & -4.79 & -4 \\ 4.79 & 6.41 & 3 & -4.79 & -6.41 & 3 \\ -4 & 3 & 2000 & 4 & -3 & 1000 \\ -3.61 & -4.79 & 4 & 3.61 & 4.79 & 4 \\ -4.79 & -6.41 & -3 & 4.79 & 6.41 & -3 \\ -4 & 3 & 1000 & 4 & -3 & 2000 \end{bmatrix}$$

Element (2)

$$L^{(2)} = 50 \text{ ft} = 600 \text{ in.}$$

$$\cos\theta = \frac{-30-0}{50} = -0.6 \quad \sin\theta = \frac{40-0}{50} = 0.8$$

$$[k^{(2)}] = 50000\begin{bmatrix} 3.61 & -4.79 & -4 & -3.61 & 4.79 & -4 \\ -4.79 & 6.41 & -3 & 4.79 & -6.41 & -3 \\ -4 & -3 & 2000 & 4 & 3 & 1000 \\ -3.61 & 4.79 & 4 & 3.61 & -4.79 & 4 \\ 4.79 & -6.41 & 3 & -4.79 & 6.41 & 3 \\ -4 & -3 & 1000 & 4 & 3 & 2000 \end{bmatrix}$$

After imposing the boundary conditions on each element stiffness matrix and assembling, we have

$$\begin{Bmatrix} F_{2x} = 10000 \\ F_{2y} = 0 \\ M_2 = 0 \end{Bmatrix} = 50000 \begin{bmatrix} 7.22 & 0 & 8 \\ 0 & 12.82 & 0 \\ 0 & 0 & 4000 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Solving simultaneously, we obtain

$$u_2 = 0.0278 \text{ in.}, v_2 = 0, \phi_2 = -0.555 \times 10^{-4} \text{ rad}$$

The element forces are obtained using

$$\{f'\} = [k'][T]\{d\}$$

Element (1)

$$\{f'\} = [k'][T]\{d\} = 50000 \begin{bmatrix} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.0167 & 5 & 0 & -0.0167 & 5 \\ 0 & 5 & 2000 & 0 & -5 & 1000 \\ -10 & 0 & 0 & 10 & 0 & 0 \\ 0 & -0.0167 & -5 & 0 & 0.0167 & -5 \\ 0 & 5 & 0 & 0 & -5 & 2000 \end{bmatrix}$$

$$\times \begin{bmatrix} 0.6 & 0.8 & 0 & 0 & 0 & 0 \\ -0.8 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0.8 & 0 \\ 0 & 0 & 0 & -0.8 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ \phi_1 = 0 \\ 0.0278 \\ 0 \\ -0.555 \times 10^{-4} \end{Bmatrix} = \begin{Bmatrix} f'_{1x} = -8300 \text{ lb} \\ f'_{1y} = 4.6 \text{ lb} \\ m_1 = 2775 \text{ lb} \cdot \text{in.} \\ f'_{2x} = 8300 \text{ lb} \\ f'_{2y} = -4.6 \text{ lb} \\ m_2 = 0 \end{Bmatrix}$$

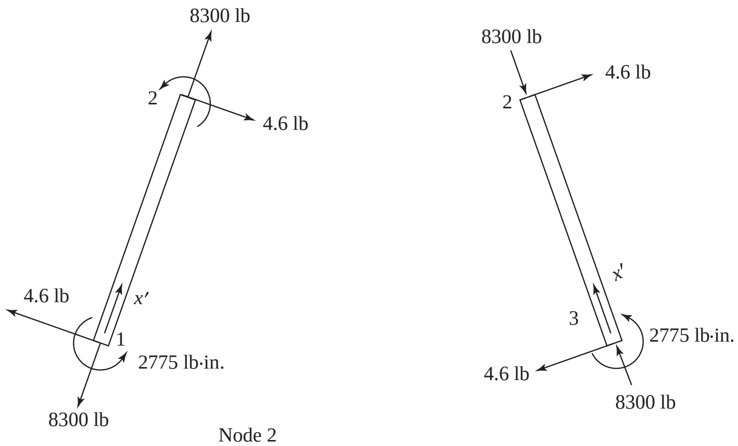
Element (2)

$$\{f'\} = \begin{bmatrix} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.0167 & 5 & 0 & -0.0167 & 5 \\ 0 & 5 & 2000 & 0 & -5 & 1000 \\ -10 & 0 & 0 & 10 & 0 & 0 \\ 0 & -0.0167 & -5 & 0 & 0.0167 & -0.5 \\ 0 & 5 & 1000 & 0 & -0.5 & 2000 \end{bmatrix}$$

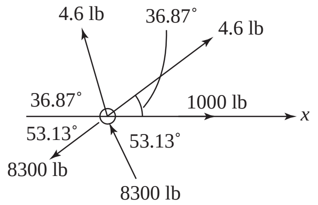
$$\times \begin{bmatrix} -0.6 & 0.8 & 0 & 0 & 0 & 0 \\ -0.8 & -0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.6 & 0.8 & 0 \\ 0 & 0 & 0 & -0.8 & -0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.0278 \\ 0 \\ -0.555 \times 10^{-4} \end{Bmatrix}$$

$$\begin{Bmatrix} f'_{3x} = 8300 \text{ lb} \\ f'_{3y} = 4.6 \text{ lb} \\ m_3 = 2775 \text{ lb} \cdot \text{in.} \\ f'_{2x} = -8300 \text{ lb} \\ f'_{2y} = -4.6 \text{ lb} \\ m_2 = 0 \end{Bmatrix}$$

Equilibrium check



Node 2



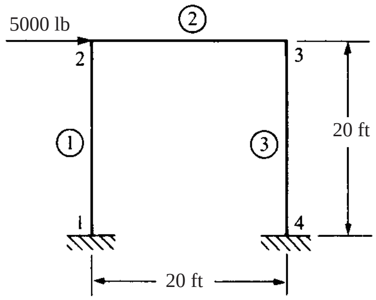
$\Sigma F_x = 0$

$1000 - 2(8300)\cos 53.13° - 2\,(4.6)\cos 36.87° = 32.6 \text{ lb} \cong 0$

$\Sigma F_y = 0$

$8300\,(\sin 53.13° - \sin 53.13°) + 4.6\,(\sin 36.87° - \sin 36.87°) = 0$

5.2



$$\frac{12I}{L^2} = 0.04167,\ \frac{6I}{L} = 5.0,\ \frac{E}{L} = 125{,}000$$

Element (1)

$C^{(1)} = 0, \quad S^{(1)} = 1$

$$[k^{(1)}] = 125{,}000\begin{bmatrix} 0.04167 & 0 & -5 & -0.04167 & 0 & -5 \\ & 10 & 0 & 0 & -10 & 0 \\ & & 800 & 5 & 0 & 400 \\ & & & 0.04167 & 0 & 5 \\ & & & & 10 & 0 \\ \text{Symmetry} & & & & & 800 \end{bmatrix}$$

Element (2)

$$C^{(2)} = 1, \quad S^{(2)} = 0$$

$$[k^{(2)}] = 125{,}000 \begin{bmatrix} 10 & 0 & 0 & -10 & 0 & 0 \\ & 0.04167 & 5 & 0 & 0.04167 & 5 \\ & & 800 & 0 & -5 & 400 \\ & & & 10 & 0 & 0 \\ & & & & 0.04169 & -5 \\ \text{Symmetry} & & & & & 800 \end{bmatrix}$$

Element (3)

$$C^{(3)} = 0, \quad S^{(3)} = -1$$

$$[k^{(3)}] = 125{,}000 \begin{bmatrix} 0.04167 & 0 & 10 & -0.04167 & 0 & 10 \\ & 10 & 0 & 0 & -10 & 0 \\ & & 800 & -10 & 0 & 400 \\ & & & 0.167 & 0 & -10 \\ & & & & 10 & 0 \\ \text{Symmetry} & & & & & 800 \end{bmatrix}$$

Boundary conditions are

$$u_1 = v_1 = \phi_1 = 0, u_4 = v_4 = \phi_4 = 0$$

Global equations $\{F\} = [K]\ \{d\}$ are

$$125000 \begin{bmatrix} 9.958 & 0 & -5 & -9.958 & 0 & 0 \\ & -9.958 & 5 & 0 & 10.0417 & 5 \\ & & 1200 & 10 & -5 & 1200 \\ & & & 0 & 0 & -10 \\ & & & & -9.958 & -5 \\ \text{Symmetry} & & & & & 1200 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \\ u_3 \\ v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 5000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Solving simultaneously

$$u_2 = 0.688 \text{ in.}, v_2 = 0.00171 \text{ in.}, \phi_2 = -0.00173 \text{ rad}$$

$$u_3 = 0.686 \text{ in.}, v_3 = -0.00171 \text{ in.}, \phi_3 = -0.00172 \text{ rad}$$

Local element forces

$$\{f'\} = [k']\{d'\} = [k']\ [T]\ \{d\} \text{ for each element}$$

Element (1)

$$[T]^{(1)}\ \{d\}^{(1)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.688 \\ 0.171 \times 10^{-2} \\ -0.173 \times 10^{-2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.171 \times 10^{-2} \\ -0.688 \\ -0.173 \times 10^{-2} \end{Bmatrix}$$

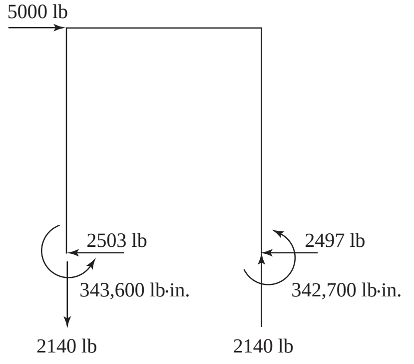
$$\{f'\}^{(1)} = [k']^{(1)}\,[T]^{(1)}\,\{d\}^{(1)} = \begin{Bmatrix} f'^{(1)}_{1x} = -2140\text{ lb} \\ f'^{(1)}_{1y} = 2503\text{ lb} \\ m'^{(1)}_{1} = 343{,}600\text{ lb}\cdot\text{in.} \\ f'^{(1)}_{2x} = 2140\text{ lb} \\ f'^{(1)}_{2y} = -2503\text{ lb} \\ m'^{(1)}_{2} = 257{,}000\text{ lb}\cdot\text{in.} \end{Bmatrix}$$

Similarly

$$\{f'\}^{(2)} = [k']^{(2)}\,[T]^{(2)}\,\{d\}^{(2)} = \begin{Bmatrix} f_{2x} = 2497\text{ lb} \\ f_{2y} = -2140\text{ lb} \\ m_{2} = -257{,}000\text{ lb}\cdot\text{in.} \\ f_{3x} = -2497\text{ lb} \\ f_{3y} = 2140\text{ lb} \\ m_{3} = -256{,}600\text{ lb}\cdot\text{in.} \end{Bmatrix}$$

$$\{f'\}^{(3)} = [k']^{(3)}\,[T]^{(3)}\,\{d\}^{(3)} = \begin{Bmatrix} f_{3x} = 2140\text{ lb} \\ f_{3y} = 2497\text{ lb} \\ m_{3} = 256{,}600\text{ lb}\cdot\text{in.} \\ f_{4x} = -2140\text{ lb} \\ f_{4y} = -2497\text{ lb} \\ m_{4} = 342.700\text{ lb}\cdot\text{in.} \end{Bmatrix}$$

Free body diagram of frame
(using local force results)



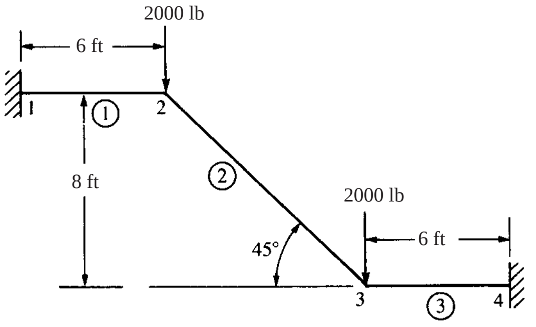
Check equation

$$\Sigma F_x = 0\text{: } 5000 - 2503 - 2497 \cong 0$$

$$\Sigma F_y = 0\text{: } -2140 + 2140 = 0$$

$$\Sigma M_1 = 343{,}600 + 342{,}700 + 2140\,(20')\left(12\ \frac{\text{in.}}{\text{ft}}\right) - 5000\,(20')\left(12\ \frac{\text{in.}}{\text{ft}}\right) \cong 0$$

5.3



$$\sigma_{\text{bend}} < \frac{2}{3}\ \sigma_y < 24000 \text{ psi}$$

$$\sigma_b = \frac{Mc}{I}$$

Assume channel section, $C6 \times 8.2$

$$I_x = 13.1 \text{ in.}^4, A = 2.40 \text{ in.}^2$$

Element (1)

$$C = 1 \qquad S = 0$$

$$\frac{12\,I}{L^2} = 2.3148 \times 10^{-3}\, I = 0.03032$$

$$\frac{6\,I}{L} = 8.33 \times 10^{-2}\, I = 1.092$$

$$\frac{E}{L} = \frac{29 \times 10^6}{(12)\,(6)} = 4.028 \times 10^5$$

$$[k^{(1)}] = (4.028 \times 10^5) \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & \phi_1 & u_2 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 2.4 & 0 & 0 & -2.4 & 0 & 0 \\ 0 & 0.0303 & 1.0917 & 0 & 0.303 & 1.0917 \\ 0 & 1.0917 & 52.4 & 0 & -1.0917 & 26.2 \\ -2.4 & 0 & 0 & 2.4 & 0 & 0 \\ 0 & -0.0303 & -1.0917 & 0 & 0.0303 & -1.0917 \\ 0 & 1.0917 & 26.2 & 0 & -1.0917 & 52.4 \end{bmatrix} \end{array}$$

Element (2)

$$C = \cos 45° = 0.707, S = \sin 45° = 0.707$$

$$\frac{12\,I}{L^2} = \frac{12\,I}{(8^2 + 8^2)^{\frac{1}{2}}\, 12} = 8.528 \times 10^{-3}$$

$$\frac{6\,I}{L} = 0.5789, \ \frac{E}{L} = 2.136 \times 10^6$$

$$[k^{(2)}] = 2.136 \times 10^5 \begin{array}{c} \begin{array}{cccccc} & (2) & & & (3) & \end{array} \\ \begin{bmatrix} 1.204 & 1.196 & -0.409 & -1.204 & -1.196 & -0.4094 \\ & 1.204 & 0.409 & -1.196 & -1.204 & 0.4094 \\ & & 52.4 & 0.4094 & -0.4094 & 26.2 \\ & & & 1.204 & 1.196 & 0.4094 \\ & & & & 1.204 & -0.4094 \\ \text{Symmetry} & & & & & 52.4 \end{bmatrix} \end{array}$$

Element (3)

$C = \cos 0° = 1, \quad S = \sin 0° = 0$

$$[k^{(3)}] = 4.028 \times 10^5 \begin{array}{c} \begin{array}{cccccc} & (3) & & & (4) & \end{array} \\ \begin{bmatrix} 2.4 & 0 & 0 & -2.4 & 0 & 0 \\ & 0.0303 & 1.092 & 0 & -0.0303 & 1.092 \\ & & 52.4 & 0 & -1.092 & 26.2 \\ & & & 2.4 & 0 & 0 \\ & & & & 0.0303 & -1.092 \\ \text{Symmetry} & & & & & 52.4 \end{bmatrix} \end{array}$$

Boundary conditions

$u_1 = v_1 = \phi_1 = 0 \qquad u_4 = v_4 = \phi_4 = 0$

Applying boundary conditions the reduced global equations become

$$10^5 \begin{bmatrix} 12.24 & 2.554 & -0.8745 & -2.573 & -2.554 & -0.8745 \\ & 2.695 & -3.523 & -2.554 & -2.573 & 0.8745 \\ & & 323 & 0.8745 & -0.8745 & 55.97 \\ & & & 12.24 & 2.55 & 0.8745 \\ & & & & 2.69 & 3.523 \\ \text{Symmetry} & & & & & 323 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_2 \\ \phi_2 \\ u_3 \\ v_3 \\ \phi_3 \end{Bmatrix}$$

$$= \begin{Bmatrix} F_{2x} = 0 \\ F_{2y} = -2000 \\ M_2 = 0 \\ F_{3x} = 0 \\ F_{3y} = -2000 \\ M_3 = 0 \end{Bmatrix}$$

Solving simultaneously, we obtain

$u_2 = -3.008 \times 10^{-9}$ in.

$v_2 = -0.402$ in.

$\phi_2 = -6.663 \times 10^{-3}$ rad

$u_3 = 3.30 \times 10^{-9}$ in.

$v_3 = -0.402$ in.

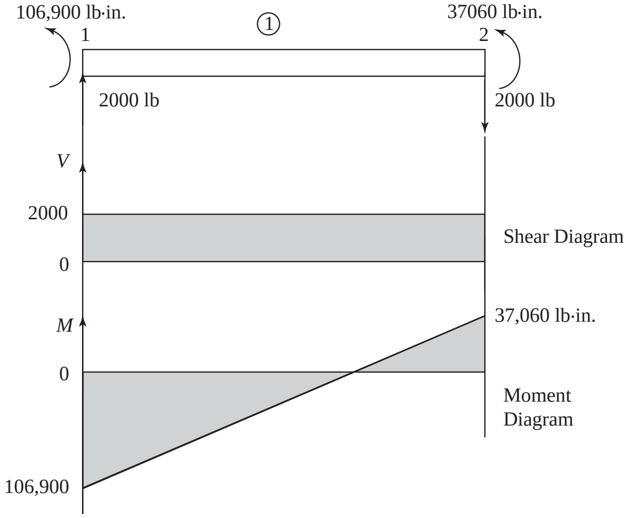
$\phi_3 = 6.663 \times 10^{-3}$ rad

Element forces

Element (1)

$$\begin{Bmatrix} f'^{(1)}_{1x} \\ f'^{(1)}_{1y} \\ m_1^{(1)} \\ f'^{(1)}_{2x} \\ f'^{(1)}_{2y} \\ m_2^{(1)} \end{Bmatrix} = 4.028 \times 10^5 \begin{bmatrix} 2.4 & 0 & 0 & -2.4 & 0 & 0 \\ & 0.0303 & 1.092 & 0 & -0.0303 & 1.092 \\ & & 52.4 & 0 & -1.092 & 26.2 \\ & & & 2.4 & 0 & 0 \\ & & & & 0.0303 & 0.1092 \\ \text{Symmetry} & & & & & 52.4 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -3.008 \times 10^{-9} \\ -0.402 \\ -6.663 \times 10^{-3} \end{Bmatrix}$$

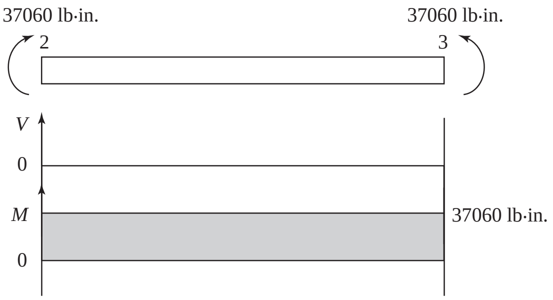
$$= \begin{Bmatrix} f'^{(1)}_{1x} = 0 \\ f'^{(1)}_{1y} = 2000 \text{ lb} \\ m_1^{(1)} = 106{,}900 \text{ lb}\cdot\text{in.} \\ f'^{(1)}_{2x} = 0 \\ f'^{(1)}_{2y} = -2000 \text{ lb} \\ m_2^{(1)} = 37060 \text{ lb}\cdot\text{in.} \end{Bmatrix}$$



Element (2)

$$2.316 \times 10^5 \begin{bmatrix} 1.204 & 1.196 & -0.409 & -1.204 & -1.196 & -0.409 \\ & 1.204 & 0.409 & -1.196 & -1.204 & 0.409 \\ & & 52.4 & 0.409 & -0.409 & 26.2 \\ & & & 1.204 & 1.196 & 0.409 \\ & & & & 1.204 & -0.409 \\ \text{Symmetry} & & & & & 52.4 \end{bmatrix} \begin{Bmatrix} 3.008 \times 10^{-9} \\ -0.402 \\ -6.66 \times 10^{-3} \\ 3.3 \times 10^{-9} \\ -0.402 \\ 6.66 \times 10^{-3} \end{Bmatrix}$$

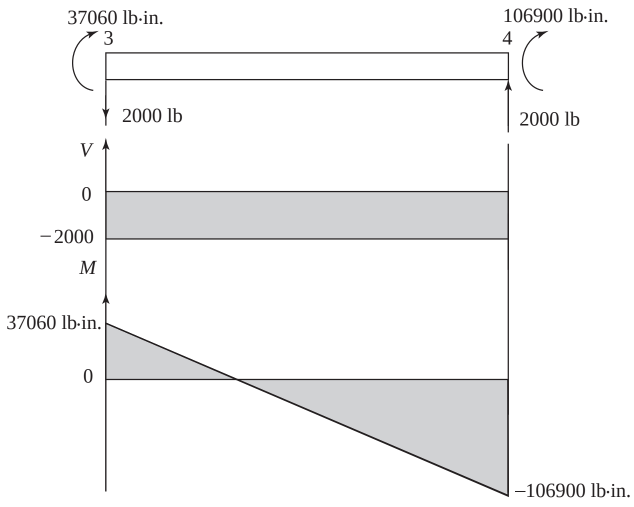
$$= \begin{Bmatrix} f'^{(2)}_{2x} \\ f'^{(2)}_{2y} \\ m_2^{(2)} \\ f'^{(2)}_{3x} \\ f'^{(2)}_{3y} \\ m_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -37060 \text{ lb}\cdot\text{in.} \\ 0 \\ 0 \\ 37060 \text{ lb}\cdot\text{in.} \end{Bmatrix}$$



Element (3)

$$4.028 \times 10^5 \begin{bmatrix} 2.4 & 0 & 0 & -2.4 & 0 & 0 \\ & 0.0303 & 1.092 & 0 & -0.0303 & 1.092 \\ & & 52.4 & 0 & -1.092 & 26.2 \\ & & & 2.4 & 0 & 0 \\ & & & & 0.0303 & -1.092 \\ \text{Symmetry} & & & & & 52.4 \end{bmatrix} \begin{Bmatrix} 3.3 \times 10^{-9} \\ -0.402 \\ 6.66 \times 10^{-3} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} f'^{(3)}_{3x} \\ f'^{(3)}_{3y} \\ m_3^{(3)} \\ f'^{(3)}_{4x} \\ f'^{(3)}_{4y} \\ m_4^{(3)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -2000 \text{ lb} \\ -37060 \text{ lb} \cdot \text{in.} \\ 0 \\ 2000 \text{ lb} \\ -106900 \text{ lb} \cdot \text{in.} \end{Bmatrix}$$



Reactions

$$F_{1x} = f'^{(1)}_{1x} = 0$$

$$F_{1y} = f'^{(1)}_{1y} = 2000 \text{ lb}$$

$$M_1 = m_1^{(1)} = 106900 \text{ lb} \cdot \text{in.}$$

$$F_{4x} = f'^{(3)}_{4x} = 0$$

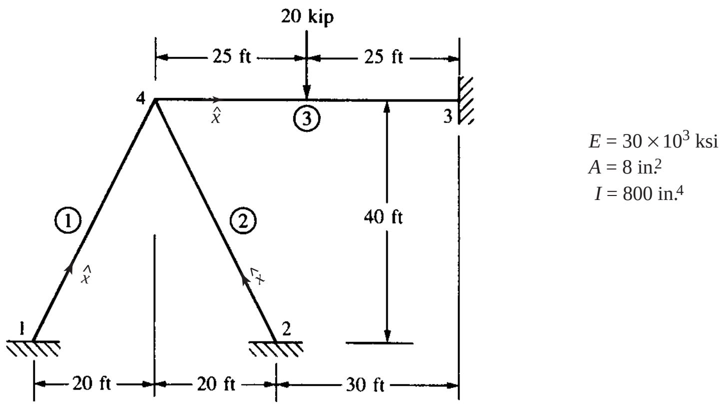
$$F_{4y} = f'^{(3)}_{4y} = 2000 \text{ lb}$$

$$M_4 = m_4^{(3)} = -106900 \text{ lb}\cdot\text{in.}$$

$$\sigma_b = \frac{Mc}{I} = \frac{106900\,(3\text{ in.})}{13.1\text{ in.}^4} = 24{,}480 \text{ psi}$$

Close to $\sigma_{allow} = 24{,}000$ psi

5.4



Element (1)

$$C = \frac{x_4 - x_1}{L^{\sim}} = \frac{20}{44.7} = 0.447,\ S = \frac{40}{44.7} = 0.895$$

$$\frac{12I}{L^2} = 0.0334,\ \frac{6I}{L} = 8.949,\ \frac{E}{L} = 55.93$$

Imposing boundary conditions $u_1 = v_1 = \phi_1 = 0,\ u_2 = v_2 = \phi_2 = 0,\ u_3 = v_3 = \phi_3 = 0$

$$\begin{array}{cc} & \begin{array}{ccc} u_4 & v_4 & \phi_4 \end{array} \\ [k^{(1)}] = & \begin{bmatrix} 90.87 & 178.2 & 447.9 \\ & 358.8 & -223.7 \\ \text{Symmetry} & & 179000 \end{bmatrix} \end{array}$$

Element (2)

$$[k^{(2)}] = \begin{bmatrix} 90.87 & -178.2 & 447.9 \\ & 358.8 & 223.7 \\ \text{Symmetry} & & 179000 \end{bmatrix}$$

$$[k^{(3)}] = \begin{bmatrix} 400 & 0 & 0 \\ & 1.334 & 400 \\ \text{Symmetry} & & 160000 \end{bmatrix}$$

Equivalent forces (element 3)

$$m_{04} = -\frac{PL}{8} = \frac{-(20\text{ kip})\,(50\text{ ft})\,(12\,\frac{\text{in.}}{\text{ft}})}{8} = -\,1500\text{ kip}\cdot\text{in.}$$

$$f_{04y} = -\,10\text{ kip}$$

Global equations

$$\begin{Bmatrix} 0 \\ -10\text{ kip} \\ -1500\text{ kip}\cdot\text{in.} \end{Bmatrix} = \begin{bmatrix} 582 & 0 & 896 \\ & 719 & 400 \\ \text{Symmetry} & & 517900 \end{bmatrix} \begin{Bmatrix} u_4 \\ v_4 \\ \phi_4 \end{Bmatrix}$$

Solving

$$u_4 = 0.445 \times 10^{-2}\text{ in.},\; v_4 = -\,0.123 \times 10^{-1}\text{ in.}$$

$$\phi_4 = -\,0.290 \times 10^{-2}\text{ rad}$$

Element forces

$$\{f'\} = [k']\,[T]\,\{d\} - \{f'_0\}$$

Element (1)

$$\begin{bmatrix} 447 & 0 & 0 & -477 & 0 & 0 \\ 0 & 1.868 & 500.5 & 0 & -1.868 & 500.5 \\ 0 & 500.5 & 179000 & 0 & -500.5 & 89490 \\ -447 & 0 & 0 & 447 & 0 & 0 \\ 0 & -1.868 & -500.5 & 0 & 1.868 & -500.5 \\ 0 & 500.5 & 89490 & 0 & -500.5 & 179000 \end{bmatrix} \begin{array}{c} [T]\,\{d\} \\ \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.90193\times10^{-2} \\ -0.94808\times10^{-2} \\ -0.2895\times10^{-2} \end{Bmatrix} \end{array} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ m_1 \\ f'_{4x} \\ f'_{4y} \\ m_4 \end{Bmatrix} = \begin{Bmatrix} 4.04\text{ kip} \\ -1.43\text{ kip} \\ -254\text{ kip}\cdot\text{in.} \\ -4.04\text{ kip} \\ 1.43\text{ kip} \\ -513\text{ kip}\cdot\text{in.} \end{Bmatrix}$$

Element (2)

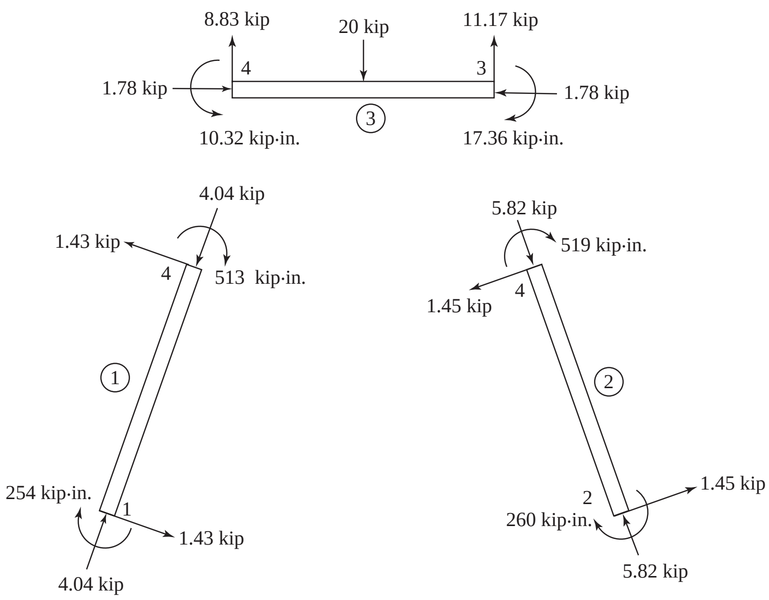
Similarly

$$\begin{Bmatrix} f'_{2x} \\ f'_{2y} \\ m_2 \\ f'_{4x} \\ f'_{4y} \\ m_4 \end{Bmatrix} = \begin{Bmatrix} 5.82\text{ kip} \\ -1.45\text{ kip} \\ -260\text{ kip}\cdot\text{in.} \\ -5.82\text{ kip} \\ 1.45\text{ kip} \\ -519\text{ kip}\cdot\text{in.} \end{Bmatrix}$$

Element (3)

$$\begin{Bmatrix} f'_{4x} \\ f'_{4y} \\ m_4 \\ f'_{3x} \\ f'_{3y} \\ m_3 \end{Bmatrix} = \begin{Bmatrix} -1.78\text{ kip} \\ 1.17\text{ kip} \\ -468\text{ kip}\cdot\text{in.} \\ 1.78\text{ kip} \\ -1.17\text{ kip} \\ -236\text{ kip}\cdot\text{in.} \end{Bmatrix} - \begin{array}{c} \{f'_0\} \\ \begin{Bmatrix} 0 \\ 10 \\ -1500 \\ 0 \\ 10 \\ 1500 \end{Bmatrix} \end{array} = \begin{Bmatrix} -1.78\text{ kip} \\ -8.83\text{ kip} \\ 1032\text{ kip}\cdot\text{in.} \\ 1.78\text{ kip} \\ -11.17\text{ kip} \\ -1736\text{ kip}\cdot\text{in.} \end{Bmatrix}$$

Free-body diagrams at each element

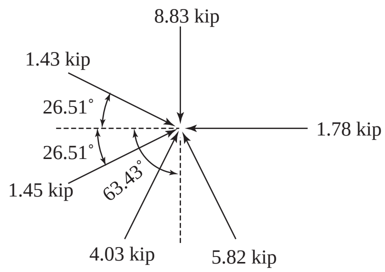


Equilibrium at node 4

$$M_4 = -513 \text{ kip}\cdot\text{in.} - 519 \text{ kip}\cdot\text{in.} + 1032 \text{ kip}\cdot\text{in.} \approx 0$$

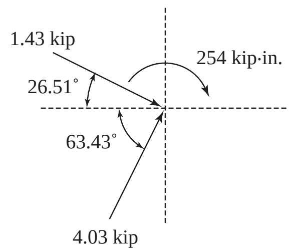
$$\Sigma F_x = 1.43 \cos 26.57° + 1.45 \cos 26.57° - 1.78 + 4.03 \cos 63.43° - 5.82 \cos 63.43° \approx 0$$

$$\Sigma F_y = \sin 26.57° \,(1.45 - 1.43) - 8.83 + \sin 63.43° \,(4.03 + 5.82) \approx 0$$



Reactions

Support node 1

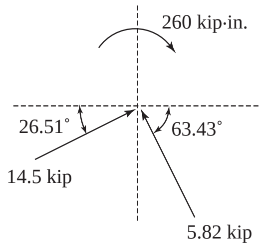


$$\Sigma F_y = 4.03 \sin 63.43° - 1.43 \sin 26.57° = 2.96 \text{ kip} \uparrow$$

$$\Sigma F_x = 4.03 \cos 63.43° + 1.43 \cos 26.57° = 31 \text{ kip} \rightarrow$$

$$M = 254 \text{ kip}\cdot\text{in.} \curvearrowright$$

Reactions support node 2

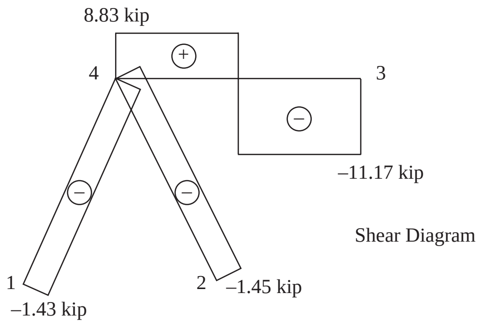


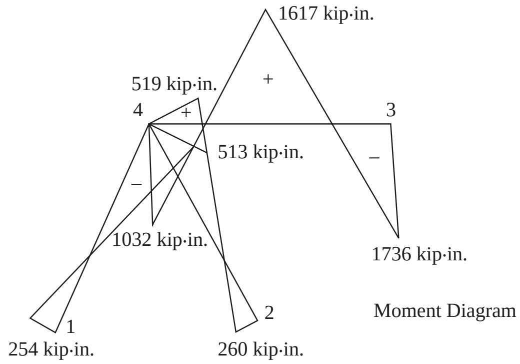
$$\Sigma F_x = 1.45 \cos 26.57° - 5.82 \cos 63.43° = -1.31 \text{ kip or} \leftarrow$$
$$\Sigma F_y = 5.82 \sin 63.43° + 1.45 \sin 26.57° = 58.86 \text{ kip} \uparrow$$
$$M_z = 260 \text{ kip} \cdot \text{in.}$$

Reactions support node 3

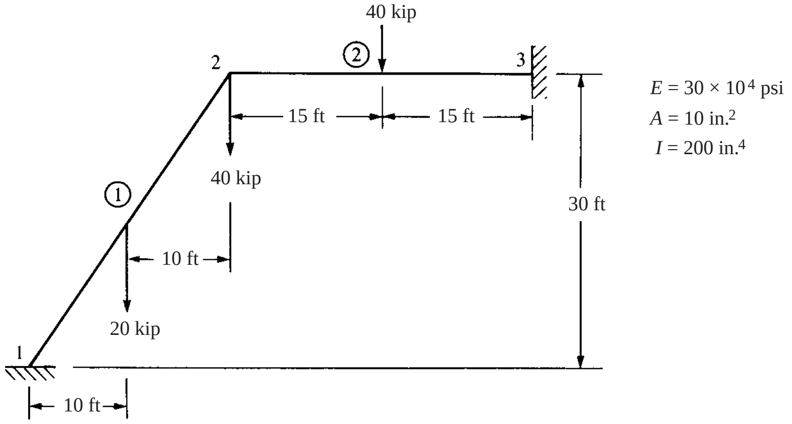
Already in global x-y directions

$$\Sigma F_y = 11.17 \text{ kip}, \Sigma F_x = 1.78 \text{ kip} \leftarrow, M = 1736 \text{ kip} \cdot \text{in.}$$





5.5



Element 1–2 (1)

$$\frac{12\,I}{L^2} = 0.0178,\ \frac{E}{L} = 69.338,\ \frac{6I}{L} = 2.7735,\ C = 0.555,\ S = 0.832$$

After imposing the boundary conditions $u_1 = v_1 = \phi_1$ and $u_3 = v_3 = \phi_3 =$ we have

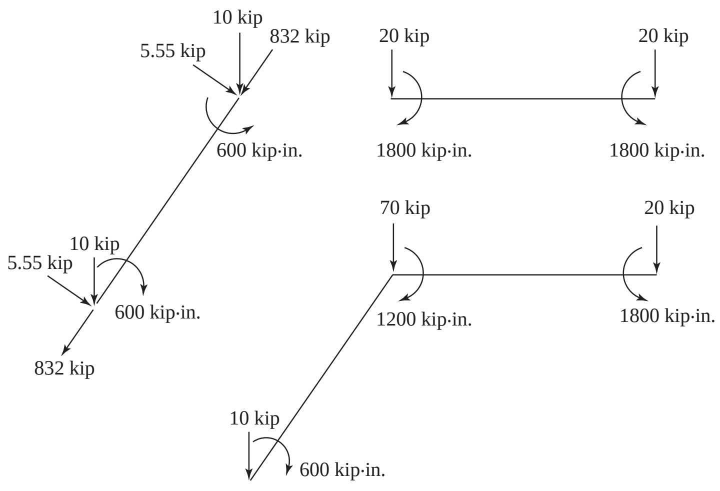
$$[k_{1-2}] = \begin{bmatrix} 214.2 & 319.8 & 160 \\ 319.8 & 480.2 & -106.7 \\ 160 & -106.7 & 55470 \end{bmatrix}$$

Element 2–3 (2)

$$\frac{12\,I}{L^2} = 0.0185,\ \frac{6I}{L} = 3.33,\ \frac{E}{L} = 85.53,\ C = 1;\ S = 0$$

$$[k_{2-3}] = \begin{bmatrix} -853.3 & 0 & 0 \\ 0 & 1.54 & 277.49 \\ 0 & 277.49 & 66664 \end{bmatrix}$$

Equivalent forces



Assembled global equations.

$$\begin{Bmatrix} 0 \\ -70 \\ -1200 \end{Bmatrix} = \begin{bmatrix} 1047.5 & 319.8 & 160 \\ & 481.74 & 170.8 \\ & & 122134.4 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Solving

$$u_2 = 0.0562 \text{ in.}$$

$$v_2 = -0.1792 \text{ in.}$$

$$\phi_2 = -0.00965 \text{ rad}$$

Element forces

$$\{f'\} = [k'] \{d'\} - \{f_0\} = [k'] [T] \{d\} - \{f'_0\}$$

Element (1)

$$[T]\{d\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.1179 \text{ in.} \\ -0.1462 \text{ in.} \\ -0.00965 \text{ rad} \end{Bmatrix}$$

$$[k'][T]\{d\} - \{f'_0\} =$$

$$\begin{bmatrix} 693.38 & 0 & 0 & -693.38 & 0 & 0 \\ & 0.8875 & 192 & 0 & -0.8875 & 192 \\ & & 55380 & 0 & -192 & 27690 \\ & & & 693.33 & 0 & 0 \\ & & & & 0.8875 & -192 \\ \text{Symmetry} & & & & & 55380 \end{bmatrix}$$

$$\times \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.1179 \\ -0.1462 \\ -0.00965 \end{Bmatrix} - \begin{Bmatrix} -8.32 \\ -5.55 \\ -600 \\ -8.32 \\ -5.55 \\ 600 \end{Bmatrix} = \begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ m_1 \\ f'_{2x} \\ f'_{2y} \\ m_2 \end{Bmatrix} = \begin{matrix} f'_{1x} = 90.07 \text{ kip} \\ f'_{1y} = 3.83 \text{ kip} \\ m_1 = 360.86 \text{ kip} \cdot \text{in.} \\ f'_{2x} = -73.43 \text{ kip} \\ f'_{2y} = 7.27 \\ m_2 = -110.635 \text{ kip} \cdot \text{in.} \end{matrix}$$

Reactions

From the free body diagram of equivalent forces gives

$$F_{1x} = f'_{1x}\ (0.555) - f'_{1y}\ (0.832) \Rightarrow F_{1x} = 46.8 \text{ kip}$$

$$F_{1y} = f'_{1x}\ (0.832) - f'_{1y}\ (0.555) \Rightarrow F_{1y} = 77.06 \text{ kip}$$

$$M_1 = 360.86 \text{ kip} \cdot \text{in.}$$

Element (2)

$$\begin{bmatrix} 833.3 & 0 & 0 & -833.3 & 0 & 0 \\ & 1.5416 & 277.486 & 0 & -1.5416 & 277.488 \\ & & 66591 & 0 & -277.488 & 33299 \\ & & & 833.3 & 0 & 0 \\ & & & & 1.5416 & -277.488 \\ \text{Symmetry} & & & & & 66597 \end{bmatrix} \begin{Bmatrix} 0.05618 \\ -0.1792 \\ -0.00965 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$- \begin{Bmatrix} 0 \\ -160 \\ -1800 \\ 0 \\ -20 \\ 1800 \end{Bmatrix} = \begin{Bmatrix} f'_{2x} \\ f'_{2y} \\ m_2 \\ f'_{3x} \\ f'_{3y} \\ m_3 \end{Bmatrix} = \begin{Bmatrix} 46.8 \text{ kip} \\ 17.1 \text{ kip} \\ 1108 \text{ kip} \cdot \text{in.} \\ -46.8 \text{ kip} \\ 22.95 \text{ kip} \\ -2171 \text{ kip} \cdot \text{in.} \end{Bmatrix}$$

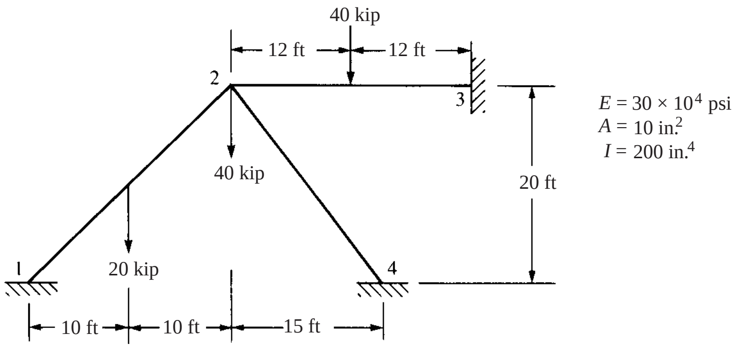
Reactions

$$F_{3x} = f'_{3x} = -46.8 \text{ kip} \leftarrow$$

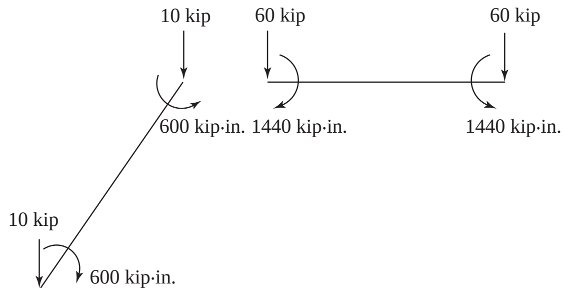
$$F_{3y} = f'_{3y} = 22.95 \text{ kip} \uparrow$$

$$M_3 = m_3 = -2171 \text{ kip} \cdot \text{in.} \circlearrowright$$

5.6



Replacement (Equivalent) force system



Element 1–2 (1)

$$C = S = 0.707,\ \frac{12I}{L^2} = 0.0208, \frac{E}{L} = 88.3881\ \frac{\text{kip}}{\text{in}^3}, \frac{6I}{L} = 3.536$$

Since $\qquad u_1 = v_1 = \phi_1 = 0$

$$\begin{array}{cc} & \begin{array}{ccc} u_2 & v_2 & \phi_2 \end{array} \\ [k_{1-2}] = & \begin{bmatrix} 442.82 & 441.06 & 220.97 \\ 441.06 & 442.82 & -220.97 \\ 220.97 & -220.97 & 70710 \end{bmatrix} \end{array}$$

Element 2–3 (2)

$$C = 1, S = 0,\ \frac{12I}{L^2} = 0.0289,\ \frac{6I}{L} = 4.166,\ \frac{E}{L} = 104.167$$

$$u_3 = v_3 = \phi_3 = 0$$

$$\begin{array}{cc} & \begin{array}{ccc} u_3 & v_3 & \phi_3 \end{array} \\ [k_{2-3}] = & \begin{bmatrix} 1041.67 & 0 & 0 \\ 0 & 3.01 & 434.03 \\ 0 & 434.03 & 83333.28 \end{bmatrix} \end{array}$$

Element 4–2 (3)

$$C = -0.60, S = 0.80,\ \frac{12I}{L^2} = 0.0267, \frac{6I}{L} = 4, \frac{E}{L} = 100$$

$$[k_{4-2}] = \begin{bmatrix} 361.71 & -478.72 & 320 \\ -478.72 & 641 & 240 \\ 320 & 240 & 80000 \end{bmatrix}$$

By direct stiffness method

$$\begin{Bmatrix} 0 \\ -70 \\ -840 \end{Bmatrix} = \begin{bmatrix} 1846.2 & -37.66 & 540.97 \\ -37.66 & 1086.83 & 453.06 \\ 540.97 & 453.06 & 234043.28 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Solving

$$\begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -0.000269 \text{ in.} \\ -0.0363 \text{ in.} \\ -0.00347 \text{ rad} \end{Bmatrix}$$

Element 1–2 (1)

$$[T]\,\{d\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.0447 \text{ in.} \\ -0.0426 \text{ in.} \\ -0.00347 \text{ rad} \end{Bmatrix}$$

$$\begin{array}{ccc} \{f'\} & [k'] & [T]\{d\} \\ \begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ m_1 \\ f'_{2x} \\ f'_{2y} \\ m_2 \end{Bmatrix} = (10^6) & \begin{bmatrix} 883.88 & 0 & 0 & -883.88 & 0 & 0 \\ & 1.838 & 311.92 & 0 & -1.838 & 311.92 \\ & & 70579.2 & 0 & -311.92 & 352.90 \\ & & & 883.88 & 0 & 0 \\ & & & & 1.838 & -311.92 \\ \text{Symmetry} & & & & & 70579.2 \end{bmatrix} & \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.0447 \\ -0.0426 \\ -0.00347 \end{Bmatrix} \end{array}$$

$$-\{f'_0\}$$

$$-\begin{Bmatrix} -7.07 \\ -7.07 \\ -600 \\ -7.07 \\ -7.07 \\ 600 \end{Bmatrix} = \begin{matrix} f'_{1x} = 46.6 \text{ kip} \\ f'_{1y} = 6.07 \text{ kip} \\ m_1 = 491.3 \text{ kip} \cdot \text{in.} \\ f'_{2x} = -32.4 \text{ kip} \\ f'_{2y} = 8.07 \text{ kip} \\ m_2 = -831.3 \text{ kip} \cdot \text{in.} \end{matrix}$$

From FBD of element (1)

$$F_{1x} = \frac{1}{2}\sqrt{2}\ (46.6 - 6.07) = 28.65 \text{ kip}$$

$$F_{1y} = \frac{1}{2}\sqrt{2}\ (46.6 + 6.07) = 37.24 \text{ kip}$$

$$M_1 = m_1 = 491.3 \text{ kip} \cdot \text{in.}$$

Similarly

Element 2–3 (2)

$$f'_{2x} = -0.28 \text{ kip}$$

$$f'_{2y} = 58.31 \text{ kip}$$

$$m_2 = 1123.9 \text{ kip} \cdot \text{in.}$$

$$f'_{3x} = 0.28 \text{ kip}$$

$$f'_{3y} = 21.69 \text{ kip} \cdot \text{in.}$$

$$m_3 = -1611 \text{ kN} \cdot \text{in.}$$

Element 4–2 (3)

$$f'_{4x} = 50.2 \text{ kip}$$

$$f'_{4y} = -1.49 \text{ kip}$$

$$m_4 = -154.2 \text{ kip} \cdot \text{in.}$$

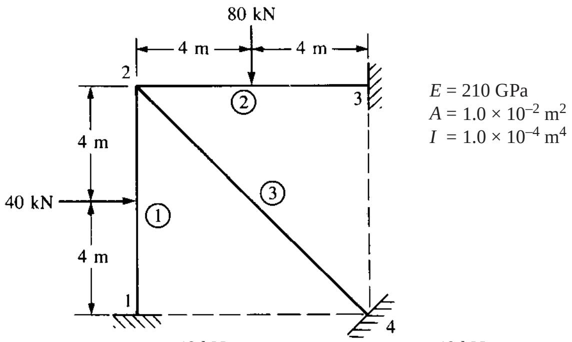
$$f'_{2x} = -50.2 \text{ kip}$$

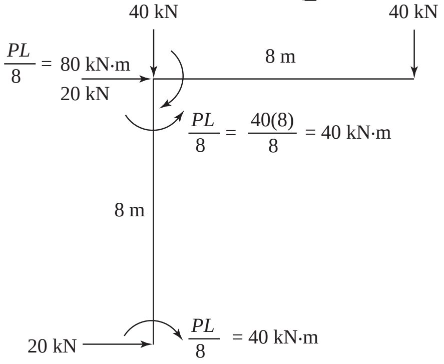
$$f'_{2y} = 1.49$$

$$m_2 = -293.2 \text{ kip} \cdot \text{in.}$$

$$\left.\begin{aligned} F_{2x} &= 0.28\text{ kip} \\ F_{3y} &= 21.69\text{ kip} \\ M_3 = m_3 &= -1611.8\text{ kip}\cdot\text{in.} \end{aligned}\right\} \leftarrow \text{Reaction} \rightarrow \left\{\begin{aligned} F_{4x} &= -28.93\text{ kip} \\ F_{4y} &= 41.05\text{ kip} \\ M_4 &= -154.2\text{ kip}\cdot\text{in.} \end{aligned}\right.$$

5.7





Equivalent force system

Boundary conditions $\quad u_1 = v_1 = \phi_1 = 0$

$u_3 = v_3 = \phi_3 = 0$

$u_4 = v_4 = \phi_4 = 0$

Element (1)

$$C = 0, \quad S = 1$$

$$\frac{12I}{L^3} = 2.344 \times 10^{-6}, \ \frac{4I}{L} = 5 \times 10^{-5}$$

$$\frac{6I}{L^2} = 9.375 \times 10^{-6}, \ \frac{A}{L} = 1.25 \times 10^{-3}$$

$$[k^{(1)}] = \frac{210 \times 10^6}{8}\begin{bmatrix} 1.875 \times 10^{-5} & 0 & 9.375 \times 10^{-6} \\ & 1.0 \times 10^{-2} & 0 \\ \text{Symmetry} & & 4.0 \times 10^{-4} \end{bmatrix}$$

Element (2)

$$C = 1, \quad S = 0$$

$$[k^{(2)}] = \frac{210\times10^6}{8}\begin{bmatrix} 1.0\times10^2 & 0 & 0 \\ & 1.875\times10^{-5} & 9.375\times10^{-6} \\ \text{Symmetry} & & 4.0\times10^{-4} \end{bmatrix}$$

Element (3)

$$C = 0.707, \quad S = -0.707$$

$$\frac{12\,I}{L^3} = 8.286\times10^{-7}, \quad \frac{4\,I}{L} = 3.536\times10^{-5}$$

$$\frac{6\,I}{L^2} = 4.6875\times10^{-6}, \quad \frac{A}{L} = 8.839\times10^{-4}$$

$$[k^{(3)}] = \frac{210\times10^6}{8\sqrt{2}}\begin{bmatrix} 5.005\times10^{-3} & -4.995\times10^{-3} & 3.75\times10^{-4} \\ & 5.005\times10^{-3} & 3.75\times10^{-5} \\ \text{Symmetry} & & 4.0\times10^{-4} \end{bmatrix}$$

Global equations

$$\begin{Bmatrix} 20 \\ -40 \\ -40 \end{Bmatrix} = \frac{210\times10^6}{8\sqrt{2}}\begin{bmatrix} 1.917\times10^{-2} & -4.995\times10^{-3} & 1.436\times10^{-4} \\ & 1.917\times10^{-2} & 1.436\times10^{-4} \\ \text{Symmetry} & & 1.531\times10^{-3} \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Solving simultaneously

$$u_2 = 0.4308\times10^{-4}\ \text{m}, \ v_2 = -0.9067\times10^{-4}\ \text{m}$$

$$\phi_2 = -0.1403\times10^{-2}\ \text{rad}$$

Local element forces

Element (1)

Effective forces = $[k'] \{d'\}$

$$f_{1x}^{(e)} = 23.8\ \text{kN}, \ f_{1y}^{(e)} = -2.74\ \text{kN}, \ m_1^{(e)} = -7.28\ \text{kN}\cdot\text{m}$$

$$f_{2x}^{(e)} = -23.8\ \text{kN}, \ f_{2y}^{(e)} = 2.74\ \text{kN}, \ m_2^{(e)} = -14.65\ \text{kN}\cdot\text{m}$$

Actual forces

$$\{f'\} = [k']\{d\} - \{f_0'\}$$

$$f'_{1x} = 23.8 - 0 = 23.8\ \text{kN} \uparrow$$

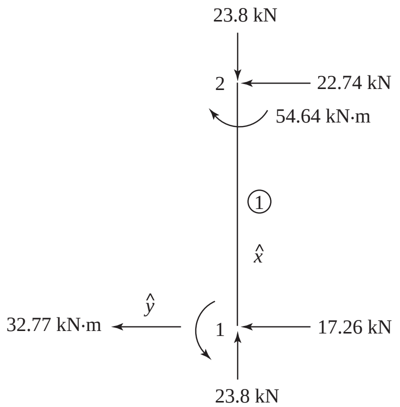
$$f'_{1y} = -2.74 + 20.0 = 17.26\ \text{kN} \leftarrow$$

$$m_1 = -7.28 + 40 = 32.77\ \text{kN}\cdot\text{m} \circlearrowleft$$

$$f'_{2x} = -23.8\ \text{kN} \downarrow$$

$$f'_{2y} = 2.74 + 20 = 22.74\ \text{kN} \leftarrow$$

$$m_2 = -14.65 - 40 = -54.64\ \text{kN}\cdot\text{m} \circlearrowright$$



FBD element (1)

Element (2)
Effective forces

$$f_{2x}^{(e)} = 11.31 \text{ kN}, \; f_{2y}^{(e)} = -2.81 \text{ kN}, \; m_2^{(e)} = -14.9 \text{ kN}\cdot\text{m}$$

$$f_{3x}^{(e)} = -11.31 \text{ kN}, \; f_{3y}^{(e)} = 2.81 \text{ kN}, \; m_3^{(e)} = -7.54 \text{ kN}\cdot\text{m}$$

Actual forces

$$\{f'\} = [k']\{d\} - \{f_0'\}$$

$$f'_{2x} = 11.31 - 0 = 11.31 \text{ kN} \rightarrow$$

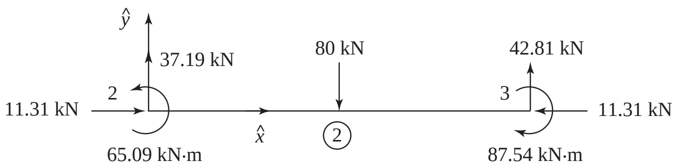
$$f'_{2y} = -2.81 - (-40) = 37.19 \text{ kN} \uparrow$$

$$m_2 = -14.91 - (-80) = 65.09 \text{ kN}\cdot\text{m}$$

$$f'_{3x} = -11.31 - 0 = -11.31 \text{ kN} \leftarrow$$

$$f'_{3y} = 2.81 - (-40) = 42.81 \text{ kN} \uparrow$$

$$m_3 = -7.54 - 80 = -87.54 \text{ kN}\cdot\text{m}$$



FBD of element 2

Element (3)

$$\{f_0'\} = 0$$

$$f'_{2x} = 17.55 \text{ kN}$$

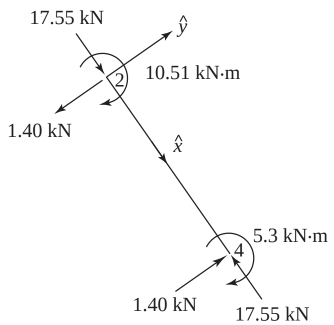
$$f'_{2y} = -1.40 \text{ kN}$$

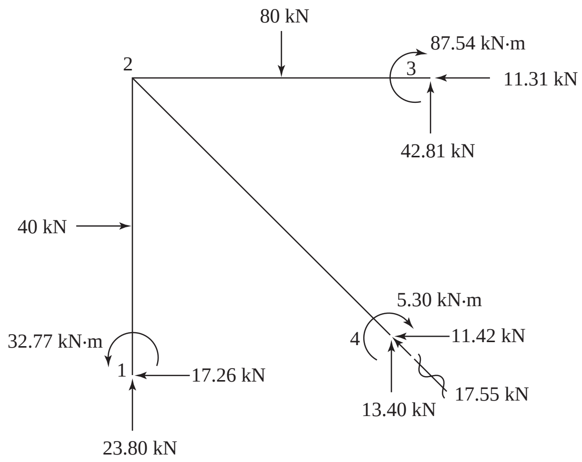
$$m_2 = -10.51 \text{ N}\cdot\text{m}$$

$$f'_{4x} = -17.55 \text{ kN}$$

$$f'_{4y} = 1.40 \text{ kN}$$

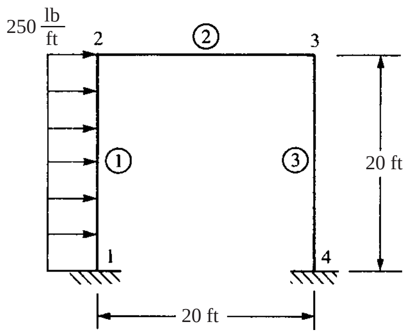
$$m_4 = -5.30 \text{ kN}\cdot\text{m}$$



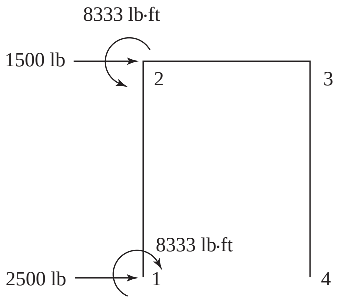


FBD of frame

5.8



$E = 30 \times 10^6$ psi
$A = 15$ in.^{2}
$I = 250$ in.^{4}



Equivalent force system

$$F = \frac{wL}{2} = \frac{250 \times 20}{2} = 2500 \text{ lb}$$

$$M = \frac{wL^2}{12} = \frac{250\,(20)^2}{12} = 8333 \text{ lb·ft}$$
$$= 100{,}000 \text{ lb·in.}$$

Calculate $[k]$'*s* based on node 2 and 3 contributions as

$$u_1 = v_1 = \phi_1 = 0,\ u_4 = v_4 = \phi_4 = 0$$

Element (1)

$$C = 0, \quad S = 1$$

$$[k^{(1)}] = \frac{E}{L}\begin{array}{c} (3) \\ \begin{bmatrix} \frac{12I}{L^2} & 0 & \frac{6I}{L} \\ & A & 0 \\ \text{Symmetry} & & 4I \end{bmatrix} \end{array}$$

Element (2)

$$C = 1, \quad S = 0$$

$$[k^{(2)}] = \frac{E}{L}\begin{array}{c} \begin{array}{cccccc} \quad & (2) & & & (3) & \end{array} \\ \begin{bmatrix} A & 0 & 0 & -A & 0 & 0 \\ & \frac{12I}{L^2} & \frac{16I}{L} & 0 & \frac{-12I}{L^2} & \frac{16I}{L} \\ & & 4I & 0 & \frac{-6I}{L} & 2I \\ & & & A & 0 & 0 \\ & & & & \frac{12I}{L^2} & \frac{-6I}{L} \\ \text{Symmetry} & & & & & 4I \end{bmatrix} \end{array}$$

Element (3)

$$C = 0, \quad S = -1$$

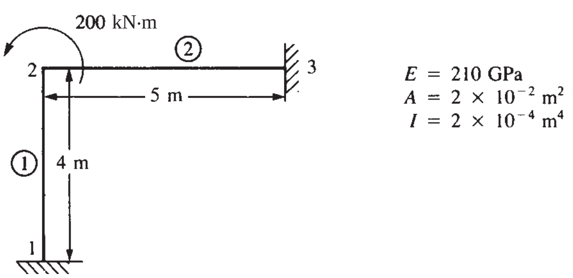
$$[k^{(3)}] = \frac{E}{L}\begin{array}{c} (3) \\ \begin{bmatrix} \frac{12I}{L^2} & 0 & \frac{6I}{L} \\ & A & 0 \\ \text{Symmetry} & & 4I \end{bmatrix} \end{array}$$

Assemble global $[K]$ and Equations $\{F\} = [K]\,\{d\}$ use numerical values for E, I, A, L

$$\begin{Bmatrix} 2500 \\ 0 \\ 100{,}000 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \frac{E}{L}\begin{bmatrix} 15.05 & 0 & 6.25 & -15 & 0 & 0 \\ & 15.05 & 6.25 & 0 & 0.0521 & 6.25 \\ & & 2000 & 0 & -6.25 & 500 \\ & & & 15.05 & 0 & 6.25 \\ & & & & 15.05 & -6.25 \\ \text{Symmetry} & & & & & 2000 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \\ u_3 \\ v_3 \\ \phi_3 \end{Bmatrix}$$

Solve simultaneously using an equation solver

5.9



Element $[k]$'s

Element (1)

$$C = 0, \quad S = 1$$

$$\frac{12I}{L^2} = \frac{12(2\times10^{-4})}{4^2} = 1.5\times10^{-4}\ \text{m}^2$$

$$\frac{6I}{L} = \frac{6(2\times10^{-4})}{4} = 3.0\times10^{-4}\ \text{m}^3$$

$$\frac{E}{L} = \frac{210\times10^{6}}{4} = 5.25\times10^{7}\ \frac{\text{kN}}{\text{m}}$$

$$[k^{(1)}] = 5.25\times10^{7}\begin{array}{c} \begin{array}{ccc} u_2 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 1.5\times10^{-4} & 0 & 3\times10^{-4} \\ & 2\times10^{-2} & 0 \\ & & 8\times10^{-4} \end{bmatrix} \end{array}$$

Element (2)

$$C = 1, \quad S = 0$$

$$\frac{12I}{L^2} = \frac{12(2\times10^{-4})}{5^2} = 9.6\times10^{-5}\ \text{m}^2$$

$$\frac{6I}{L} = \frac{6(2\times10^{-4})}{5} = 2.4\times10^{-4}\ \text{m}^3$$

$$\frac{E}{L} = \frac{210\times10^{6}}{5} = 4.2\times10^{7}\ \frac{\text{kN}}{\text{m}}$$

$$[k^{(2)}] = 4.2\times10^{7}\begin{array}{c} (2) \\ \begin{bmatrix} 2\times10^{-2} & 0 & 0 \\ & 9.6\times10^{-5} & 2.4\times10^{-4} \\ \text{Symmetry} & & 8\times10^{-4} \end{bmatrix} \end{array}$$

Assemble global equations for node 2

$$\begin{Bmatrix} 0 \\ 0 \\ 200 \end{Bmatrix} = 10^{7}\begin{bmatrix} 8.48\times10^{-2} & 0 & 1.58\times10^{-3} \\ & 1.05\times10^{-1} & 1.01\times10^{-3} \\ \text{Symmetry} & & 7.56\times10^{-3} \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Solving simultaneously

$$u_2 = -4.95\times10^{-5}\ \text{m},\ v_2 = -2.56\times10^{-5}\ \text{m}$$

$$\phi_2 = 2.66\times10^{-3}\ \text{rad}$$

Element forces $\{f'\} = [k']\ [T]\ \{d\}$

$$[T]\ \{d\} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -4.95\times10^{-5} \\ -2.56\times10^{-5} \\ 2.66\times10^{-3} \end{Bmatrix}$$

$$\{f'\} = [k']^{(1)} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -2.56\times10^{-5} \\ 4.95\times10^{-6} \\ 2.66\times10^{-3} \end{Bmatrix}$$

$$\{f'\} = 5.25\times10^{7} \begin{bmatrix} : & : & : & -2\times10^{-2} & 0 & 0 \\ . & . & . & 0 & -1.5\times10^{-4} & 3\times10^{-4} \\ . & . & . & 0 & -3\times10^{-4} & 4\times10^{-4} \\ . & . & . & 2\times10^{-2} & 0 & 0 \\ . & . & . & 0 & 1.5\times10^{-4} & -3\times10^{-4} \\ : & : & : & 0 & -3\times10^{-4} & 8\times10^{-4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -2.56\times10^{-5} \\ 4.95\times10^{-5} \\ 2.66\times10^{-3} \end{Bmatrix}$$

Multiplying matrices yields

$$f'^{(1)}_{1x} = 26.9 \text{ kN} = -f'^{(1)}_{2x}, \; f'^{(1)}_{1y} = 42 \text{ kN} = -f'^{(1)}_{2y}$$

$$m_1^{(1)} = 55.9 \text{ kN}\cdot\text{m}, \; m_2^{(1)} = 111.7 \text{ kN}\cdot\text{m}$$

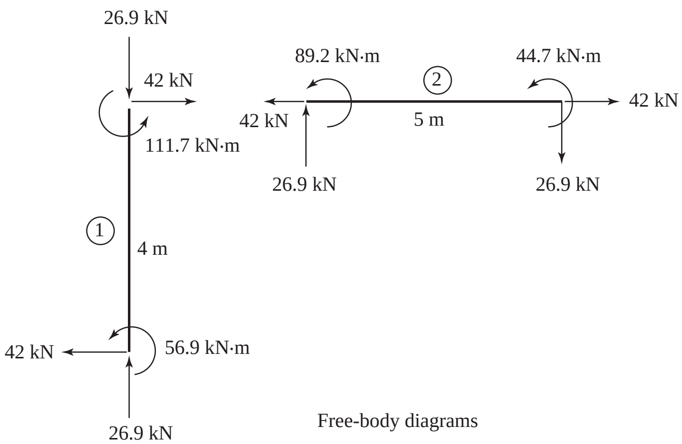
Similarly for element (2)

$$\{f'\}^{(2)} = [K']^{(2)}\,[T]^{(2)}\,\{d\}^{(2)} \text{ yields}$$

$$f'^{(2)}_{2x} = 42.0 \text{ kN} = -f'^{(2)}_{3x}$$

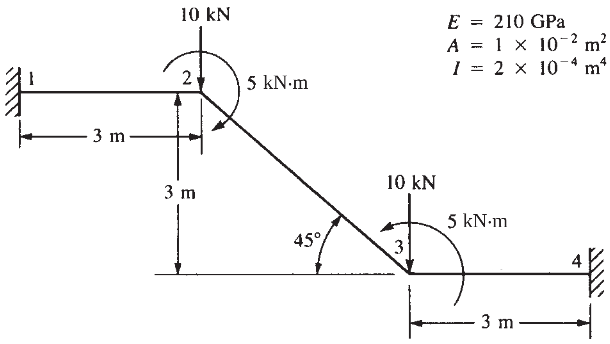
$$f'^{(2)}_{2y} = -26.9 \text{ kN} = -f'^{(2)}_{3y}$$

$$m_2^{(2)} = 89.2 \text{ kN}\cdot\text{m}, \; m_3^{(2)} = 44.7 \text{ kN}\cdot\text{m}$$



Free-body diagrams

5.10



Element (1)

$$\frac{E}{L} = \frac{210 \times 10^9}{3\text{m}} = 70 \times 10^9,\ \frac{12I}{L^2} = 2.67 \times 10^{-4}$$

$$\frac{6I}{L} = 4 \times 10^{-4},\ C = 1,\ S = 0$$

$$u_1 = v_1 = \phi_1 = 0 \text{ and } u_4 = v_4 = \phi_4 = 0$$

$$[k^{(1)}] = 70 \times 10^9 \begin{array}{c} \begin{array}{ccc} u_2 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 1\times10^{-2} & 0 & 0 \\ & 2.67\times10^{-4} & -4\times10^{-4} \\ \text{Symmetry} & & 8\times10^{-4} \end{bmatrix} \end{array}$$

Element (2)

$$\frac{E}{L} = \frac{210 \times 10^9}{3\sqrt{2}} = 49.5 \times 10^9,\quad \frac{12I}{L^2} = \frac{12(2\times10^{-4})}{18} = 1.33 \times 10^{-4}$$

$$\frac{6I}{L} = 2.83 \times 10^{-4},\ C = 0.707 = -S,\ 4I = 8 \times 10^{-4}$$

$$[k^{(2)}] = \begin{bmatrix} 3.58 & -3.48 & 0.1415 & 3.58 & 3.48 & 0.1415 \\ & 3.58 & 0.1415 & -3.48 & -3.58 & 0.1415 \\ & & 0.566 & -0.1415 & -0.1415 & 0.2828 \\ & & & 3.58 & -3.48 & -0.1415 \\ & & & & 3.58 & -0.1415 \\ \text{Symmetry} & & & & & 0.566 \end{bmatrix} 70 \times 10^6$$

Element (3)

$$\frac{E}{L} = 70 \times 10^9,\ \frac{12I}{L^2} = 2.67 \times 10^{-4},\ \frac{6I}{L} = 4 \times 10^{-4}$$

$$C = 1,\ S = 0$$

$$[k^{(3)}] = 70 \times 10^9 \begin{array}{c} \begin{array}{ccc} u_3 & v_3 & \phi_3 \end{array} \\ \begin{bmatrix} 1\times10^{-2} & 0 & 0 \\ 0 & 2.67\times10^{-4} & 4\times10^{-4} \\ \text{Symmetry} & & 8\times10^{-4} \end{bmatrix} \end{array}$$

Global equations

$$10 \times 10^{9}\begin{bmatrix} 0.0135 & -0.0035 & 0.00014 & 0.0036 & 0.0035 & 0.00014 \\ & 0.0038 & -0.0003 & -0.0035 & -0.0036 & 0.00014 \\ & & 0.0014 & -0.00014 & -0.00014 & 0.00028 \\ & & & 0.0135 & -0.0035 & -0.00014 \\ & & & & 0.0038 & 0.0003 \\ \text{Symmetry} & & & & & 0.0014 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \\ u_3 \\ v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10000 \\ -5000 \\ 0 \\ -10000 \\ 5000 \end{Bmatrix}$$

Using an equation solver

$$u_2 = 0.16 \times 10^{-10}\ \text{m} \qquad u_3 = 0.85 \times 10^{-11}$$

$$v_2 = -0.1423 \times 10^{-2}\ \text{m} \qquad v_3 = -0.1423 \times 10^{-2}\ \text{m}$$

$$\phi_2 = -0.5917 \times 10^{-3}\ \text{rad} \qquad \phi_3 = 0.5917 \times 10^{-3}\ \text{rad}$$

Element forces

Element (1)

$$[T]\,\{d\} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \sim 0 \\ 0 \\ -0.1423\times 10^{-2} \\ -0.5917\times 10^{-3} \end{Bmatrix}$$

$$\{f'\}^{(1)} = [k']^{(1)}\,[T]^{(1)}\,\{d\}^{(1)}$$

$$= 70 \times 10^{9}\begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ & 12C_2 & 6C_2L & 0 & -12C_2 & 6C_2L \\ & & 4C_2L^2 & 0 & -6C_2L & 2C_2L \\ & & & C_1 & 0 & 0 \\ & & & & 12C_2 & -6C_2L \\ \text{Symmetry} & & & & & 4C_2L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \overset{0}{\cancel{0.16\times 10^{-10}}} \\ -0.1923\times 10^{-2} \\ -0.5917\times 10^{-3} \end{Bmatrix}$$

$$f'^{(1)}_{1x} = 0$$

$$f'^{(1)}_{1y} = 10028\ \text{N}$$

$$m_1^{(1)} = 23276\ \text{N}\cdot\text{m}$$

$$f'^{(1)}_{2x} = 0$$

$$f'^{(1)}_{2y} = -10028\ \text{N}$$

$$m_2^{(1)} = 6709\ \text{N}\cdot\text{m}$$

Element (2)

$$[T] = \left[\begin{array}{ccc|ccc} 0.707 & -0.707 & 0 & 0 & 0 & 0 \\ 0.707 & 0.707 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline & & & & & \\ & & & & & \\ & & & & & \end{array}\right]$$

$$\{f'\}^{(2)} = [k']^{(2)}\,[T]^{(2)}\,\{d\}^{(2)} = \begin{Bmatrix} 0 \\ -0.387\times 10^{-4} \cong 0 \\ -11,720 \\ 0 \\ 0.387\times 10^{-4} \cong 0 \\ 11,720 \end{Bmatrix}$$

where $[k']^{(2)}$ from Equation (6.1.8) text

$$\{d\}^{(2)} = \begin{Bmatrix} 0.16\times 10^{-10} = 0 \\ -0.1423\times 10^{-2} \\ -0.5917\times 10^{-3} \\ 0.85\times 10^{-11} = 0 \\ -0.1423\times 10^{-2} \\ 0.5917\times 10^{-3} \end{Bmatrix}$$

Similarly for element (3)

$$f'^{(3)}_{3x} = 0$$

$$f'^{(3)}_{3y} = -10{,}028 \text{ N}$$

$$m_3^{(3)} = -6709 \text{ N}\cdot\text{m}$$

$$f'^{(3)}_{4x} = 0$$

$$f'^{(3)}_{4y} = 10{,}028 \text{ N}$$

$$m_4^{(3)} = -23276 \text{ N}\cdot\text{m}$$

5.11

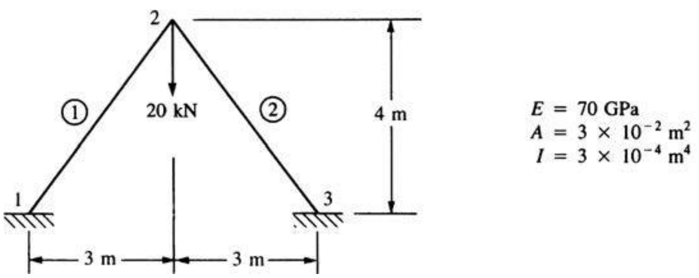


Figure P5–11

This problem is done using symmetry and Mathcad

$$E = 70 \times 10^9$$
$$A = 3 \times 10^{-2}$$
$$I = 3 \times 10^{-4}$$

Element 1

$$x_1 = 0 \qquad x_2 = 3$$
$$y_1 = 0 \qquad y_2 = 4$$
$$L_1 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$C = \frac{x_2 - x_1}{L_1} \qquad S = \frac{y_2 - y_1}{L_1}$$
$$C = 0.6 \qquad S = 0.8$$
$$N = \frac{12I}{L_1^{\,2}} \qquad M = \frac{6I}{L_1}$$

$$k_1 = \left(\frac{E}{L_1}\right)\begin{bmatrix} AC^2+NS^2 & (A-N)CS & -MS & -\left(AC^2+NS^2\right) & -(A-N)CS & -MS \\ (A-N)CS & AS^2+NC^2 & MC & -(A-N)CS & -\left(AS^2+NC^2\right) & MC \\ -MS & MC & 4I & MS & -MC & 2I \\ -\left(AC^2+NS^2\right) & -(A-N)CS & MS & AC^2+NS^2 & (A-N)CS & MS \\ -(A-N)CS & -\left(AS^2+NC^2\right) & -MC & (A-N)CS & AS^2+NC^2 & -MC \\ -MS & MC & 2I & MS & -MC & 4I \end{bmatrix}$$

Boundary conditions with symmetry

$$u_1 = 0 \qquad M_2 = 0 \qquad u_2 = 0$$
$$v_1 = 0 \qquad \phi_2 = 0$$
$$\phi_1 = 0$$

Reduced set of equations

Guess

$$v_2 = 1$$

Given

$$F_{2y} = -10 \times 10^3$$
$$(F_{2y}) = \left[\frac{E}{L_1}\left(AS^2 + NC^2\right)\right](v_2)$$
$$v_2 = \text{find}\,(v_2)$$

$v_2 = -3.7102 \times 10^{-5}$ Displacement of node 2

$$\begin{pmatrix} F_{1x} \\ F_{1y} \\ M_1 \\ F_{2x} \\ F_{2y} \\ M_2 \end{pmatrix} = k_1 \begin{pmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{pmatrix} \quad \begin{pmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -3.71\times10^{-5} \\ 0 \end{pmatrix} \text{ Displacements and rotations}$$

$$\begin{pmatrix} F_{1x} \\ F_{1y} \\ M_1 \end{pmatrix} = \begin{pmatrix} 7443.901 \\ 10000 \\ 112.197 \end{pmatrix} \text{ Reaction forces}$$

$$F_1 = \sqrt{F_{1x}{}^2 + F_{1y}{}^2}$$

$F_1 = 12466.422$ Reaction magnitude at node 1 same as node 3 by symmetry.

Forces in elements

$$T = \begin{pmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad d = \begin{pmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{pmatrix}$$

$$C_1 = \frac{AE}{L_1} \qquad C_2 = \frac{EI}{L_1{}^3}$$

$$[k'] = \begin{pmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6C_2 L_1 & 0 & -12C_2 & 6C_2 L_1 \\ 0 & 6C_2 L_1 & 4C_2 L_1^2 & 0 & -6C_2 L_1 & 2C_2 L_1^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6C_2 L_1 & 0 & 12C_2 & -6C_2 L_1 \\ 0 & 6C_2 L_1 & 2C_2 L_1^2 & 0 & -6C_2 L_1 & 4C_2 L_1^2 \end{pmatrix}$$

$$\begin{pmatrix} f_{1x} \\ f_{1y} \\ m_1 \\ f_{2x} \\ f_{2y} \\ m_2 \end{pmatrix} = [k']\,[T]\,\{d\} \qquad \begin{pmatrix} f_{1x} \\ f_{1y} \\ m_1 \\ f_{2x} \\ f_{2y} \\ m_2 \end{pmatrix} = \begin{pmatrix} 12466.341 \\ 44.879 \\ 112.197 \\ -12466.341 \\ -44.879 \\ 112.197 \end{pmatrix}$$

Forces in element 1 and 2 by symmetry.

5.12 Determine displacements and rotations of the nodes, element forces, and reactions.

Material properties & geometry

$EE = 210 \times 10^9$ Pa Modulus of elasticity

$AA = 80 \times 10^{-3}$ m^2 Area of cross section of all elements

$\text{II} = 1.2 \times 10^{-4}$ m^4 Area moment of inertia of all elements

$LL_1 = 3\ \text{m}$	Length of element 1
$LL_2 = 6\ \text{m}$	Length of element 2
$\theta_1 = 0\ \text{deg}$	Angle between local x and global x for element 1.
$\theta_2 = -90\ \text{deg}$	Angle between local x and global x for element 2.

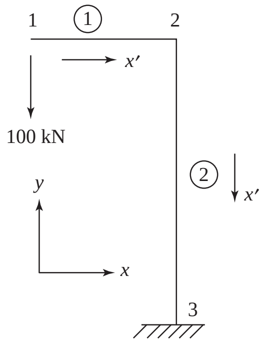


Figure P5.12

Applied loads

$F_{1y} = -100000\ \text{N}$	Applied load (down at node 1)

Boundary conditions

$d_{3x} = 0$	x-displacement 3 is zero.
$d_{3y} = 0$	y-displacement at node 3 is zero.
$\phi_3 = 0$	Angular displacement. at node 3 is zero.

Defining element properties in unitless format. (Mathcad does not allow elements with dissimilar units within the same matrix.)

$E = \dfrac{EE}{Pa}$	Modulus of elasticity (Pa).
$A_1 = \dfrac{AA}{m^2}$	Cross sectional area of element 1 (m^2)
$A_2 = \dfrac{AA}{m^2}$	Cross sectional area of element 2 (m^2)
$I_1 = \dfrac{II}{m^4}$	Area moment of element 1 (m^4)
$I_2 = \dfrac{II}{m^4}$	Area moment of element 2 (m^4)
$L_1 = \dfrac{LL_1}{m}$	Length of element 1(m)
$L_2 = \dfrac{LL_2}{m}$	Length of element 2 (m)
$C_1 = \cos(\theta_1)$	Cosine of angle between local x and global x for element 1.
$S_1 = \sin(\theta_1)$	Sine of angle between local x and global x for element 1.

$C_2 = \cos(\theta_2)$	Cosine of angle between local x and global x for element 2.
$S_2 = \sin(\theta_2)$	Sine of angle between local x and global x for element 2.

Functional equations for the global stiffness matrix for a 2D beam/frame element with axial effects.

Refer to text Equation (5.1.11)

Functional expressions for large repeated terms in global stiffness matrix.

$$M_1(A, C, S, I, L) = AC^2 + \frac{12I}{L^2}\ S^2$$

$$M_2(A, C, S, I, L) = \left(A - \frac{12I}{L^2}\right) CS$$

$$M_3(A, C, S, I, L) = AS^2 + \frac{12I}{L^2}\ C^2$$

Functional equation for global stiffness matrix of beam/frame element.

$$k(A, C, S, E, I, L) = \frac{E}{L}\left[\begin{matrix} M_1(A,C,S,I,L) & M_2(A,C,S,I,L) & \left(\frac{-6I}{L}S\right) \\ M_2(A,C,S,I,L) & M_3(A,C,S,I,L) & \frac{6I}{L}C \\ \frac{-6I}{L}S & \frac{6I}{L}C & 4I \\ -M_1(A,C,S,I,L) & -M_2(A,C,S,I,L) & \frac{6I}{L}S \\ -M_2(A,C,S,I,L) & -M_3(A,C,S,I,L) & \frac{-6I}{L}C \\ \frac{-6I}{L}S & \frac{6I}{L}C & 2I \end{matrix}\right.$$

$$\left.\begin{matrix} -M_1(A,C,S,I,L) & -M_2(A,C,S,I,L) & \frac{-6I}{L}S \\ -M_2(A,C,S,I,L) & -M_3(A,C,S,I,L) & \frac{-6I}{L}C \\ \frac{6I}{L}S & \frac{-6I}{L}C & 2I \\ M_1(A,C,S,I,L) & M_2(A,C,S,I,L) & \frac{6I}{L}S \\ M_2(A,C,S,I,L) & M_3(A,C,S,I,L) & -\left(\frac{6I}{L}C\right) \\ \frac{6I}{L}S & -\left(\frac{6I}{L}C\right) & 4I \end{matrix}\right]$$

Calculate global stiffness matrix for 1st element.

$i = 1$	Set $i = 1$ so that the properties of element 1 are used in the functional expression.

$$k_1 = k(A_i, C_i, S_i, E, I_i, L_i)$$

$$k_1 = \left(\begin{matrix} 5.6\times10^9 & 0 & 0 & -5.6\times10^9 \\ 0 & 1.12\times10^7 & 1.68\times10^7 & 0 \\ 0 & 1.68\times10^7 & 3.36\times10^7 & 0 \\ -5.6\times10^9 & 0 & 0 & 5.6\times10^9 \\ 0 & -1.12\times10^7 & -1.68\times10^7 & 0 \\ 0 & 1.68\times10^7 & 1.68\times10^7 & 0 \end{matrix}\right.$$

$$\left.\begin{matrix} 0 & 0 \\ -1.12\times10^{7} & 1.68\times10^{7} \\ -1.68\times10^{7} & 1.68\times10^{7} \\ 0 & 0 \\ 1.12\times10^{7} & -1.68\times10^{7} \\ -1.68\times10^{7} & 3.36\times10^{7} \end{matrix}\right)$$

Augment element global k matrix with rows and columns of zeros to facilitate the assembly of the total global stiffness matrix. The global k matrix for each element needs to have the same number of rows and columns as there is degrees of freedom in the model. In this case, it is 9 (3 nodes, $\frac{3\ \text{DOF}}{\text{node}}$).

$$\text{ZeroCol} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{ZeroRow} = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$$

$$kI_a = \text{augment}\ (k_1, \text{ZeroCol}, \text{ZeroCol}, \text{ZeroCol})$$

$$kI_b = \text{stack}(k_{1a}, \text{ZeroRow}, \text{ZeroRow}, \text{ZeroRow})$$

$k_{Ib} =$

0	1	2	3	4	5	6	7	8	9
1	5.6×10^{9}	0	0	-5.6×10^{9}	0	0	0	0	0
2	0	1.12×10^{7}	1.68×10^{7}	0	-1.12×10^{7}	1.68×10^{7}	0	0	0
3	0	1.68×10^{7}	3.36×10^{7}	0	-1.68×10^{7}	1.68×10^{7}	0	0	0
4	-5.6×10^{9}	0	0	5.6×10^{9}	0	0	0	0	0
5	0	-1.12×10^{7}	-1.68×10^{7}	0	1.12×10^{7}	-1.68×10^{7}	0	0	0
6	0	1.68×10^{7}	1.68×10^{7}	0	-1.68×10^{7}	3.36×10^{7}	0	0	0
7	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0

Calculate global stiffness matrix for 2nd element.

$i = 2$ Set $i = 2$ so that the properties of element 2 are used in the functional expression.

$$k_2 = k\ (A_i, C_i, S_i, E, I_i, L_i)$$

$$K_2 = \begin{pmatrix} 1.4\times10^{6} & \cancelto{0}{-1.714\times10^{-7}} & 4.2\times10^{6} & -1.4\times10^{6} & \cancelto{0}{1.714\times10^{-7}} & 4.2\times10^{6} \\ \cancelto{0}{-1.714\times10^{-7}} & 2.8\times10^{9} & \cancelto{0}{2.572\times10^{-10}} & \cancelto{0}{1.714\times10^{-7}} & -2.8\times10^{9} & \cancelto{0}{2.572\times10^{-10}} \\ 4.2\times10^{6} & \cancelto{0}{2.572\times10^{-10}} & 1.68\times10^{7} & -4.2\times10^{6} & -2.572\times10^{-10} & 8.4\times10^{6} \\ -1.4\times10^{6} & \cancelto{0}{1.714\times10^{-7}} & -4.2\times10^{6} & 1.4\times10^{6} & \cancelto{0}{-1.714\times10^{-7}} & -4.2\times10^{6} \\ \cancelto{0}{1.714\times10^{-7}} & -2.8\times10^{9} & \cancelto{0}{-2.572\times10^{-10}} & \cancelto{0}{-1.714\times10^{-7}} & 2.8\times10^{9} & \cancelto{0}{-2.572\times10^{-10}} \\ 4.2\times10^{6} & \cancelto{0}{2.572\times10^{-10}} & 8.4\times10^{6} & -4.2\times10^{6} & \cancelto{0}{-2.572\times10^{-10}} & 1.68\times10^{7} \end{pmatrix}$$

Augment element global k matrix with rows and columns of zeros to facilitate the assembly of the total global stiffness matrix. As before, we need 9 rows and columns.

$$k_{2a} = \text{augment (ZeroCol,ZeroCol, ZeroCol, } k_2)$$

$$k_{2b} = \text{stack(ZeroRow, ZeroRow, ZeroRow, } k_{2a})$$

$k_{2b} =$

0	1	2	3	4	5	6	7	8	9
1	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0
4	0	0	0	1.4×10^6	-1.714×10^{-7}	4.2×10^6	-1.4×10^6	1.714×10^{-7}	4.2×10^6
5	0	0	0	-1.714×10^{-7}	2.8×10^9	2.572×10^{-10}	1.714×10^{-7}	-2.8×10^9	2.572×10^{-10}
6	0	0	0	4.2×10^6	2.572×10^{-10}	1.68×10^7	-4.2×10^6	-2.572×10^{-10}	8.4×10^6
7	0	0	0	-1.4×10^6	1.714×10^{-7}	-4.2×10^6	1.4×10^6	-1.714×10^{-7}	-4.2×10^6
8	0	0	0	1.714×10^{-7}	-2.8×10^9	-2.572×10^{-10}	-1.714×10^{-7}	2.8×10^9	-2.572×10^{-10}
9	0	0	0	4.2×10^6	2.572×10^{-10}	8.4×10^6	-4.2×10^6	-2.572×10^{-10}	1.68×10^7

Calculate total global stiffness matrix by adding augmented matrices for each element.

$$K = k_{1b} + k_{2b}$$

$K =$

0	1	2	3	4	5	6	7	8	9
1	5.6×10^9	0	0	-5.6×10^9	0	0	0	0	0
2	0	1.12×10^7	1.68×10^7	0	-1.12×10^7	1.68×10^7	0	0	0
3	0	1.68×10^7	3.36×10^7	0	-1.68×10^7	1.68×10^7	0	0	0
4	-5.6×10^9	0	0	5.601×10^9	-1.714×10^{-7}	4.2×10^6	-1.4×10^6	1.714×10^{-7}	4.2×10^6
5	0	-1.12×10^7	-1.68×10^7	-1.714×10^{-7}	2.811×10^9	-1.68×10^7	1.714×10^{-7}	-2.8×10^9	2.572×10^{-10}
6	0	1.68×10^7	1.68×10^7	4.2×10^6	-1.68×10^7	5.04×10^7	-4.2×10^6	2.572×10^{-10}	8.4×10^6
7	0	0	0	-1.4×106	1.714×10^{-7}	-4.2×10^6	1.4×10^6	-1.714×10^{-7}	-4.2×10^6
8	0	0	0	1.714×10^{-7}	-2.8×10^9	2.572×10^{-10}	-1.714×10^{-7}	2.8×10^9	2.572×10^{-10}
9	0	0	0	4.2×10^6	2.572×10^{-10}	8.4×10^6	-4.2×10^6	2.572×10^{-10}	1.68×10^7

Solve for displacements and rotations at node 1 and 2.

First partition out rows and columns associated with homogenous boundary conditions (rows/columns 7, 8 and 9)

$$K_{\text{part}} = \text{submatrix } (K, 1, 6, 1, 6)$$

$$K_{\text{part}} = \begin{pmatrix} 5.6\times10^9 & 0 & 0 & -5.6\times10^9 \\ 0 & 1.12\times10^7 & 1.68\times10^7 & 0 \\ 0 & 1.68\times10^7 & 3.36\times10^7 & 0 \\ -5.6\times10^9 & 0 & 0 & 5.601\times10^9 \\ 0 & -1.12\times10^7 & -1.68\times10^7 & -1.714\times10^{-7} \\ 0 & 1.68\times10^7 & 1.68\times10^7 & 4.2\times10^6 \end{pmatrix}$$

$$\begin{matrix} 0 & 0 \\ -1.12\times10^{7} & 1.68\times10^{7} \\ -1.68\times10^{7} & 1.68\times10^{7} \\ -1.714\times10^{-7} & 4.2\times10^{6} \\ 2.811\times10^{9} & -1.68\times10^{7} \\ -1.68\times10^{7} & 5.04\times10^{7} \end{matrix}\Bigg)$$

Define partitioned vector of applied loads.

$$F_{\text{part}} = \begin{pmatrix} 0 & \dfrac{F_{1y}}{N} & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

Solve for displacements and rotations at node 1 and 2.

$$\begin{pmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{pmatrix} = K_{\text{part}}{}^{-1} F_{\text{part}}$$

$$\begin{pmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} -0.214 \\ -0.25 \\ 0.089 \\ -0.214 \\ -3.571\times10^{-5} \\ 0.071 \end{pmatrix} \begin{matrix} (\text{m}) \\ (\text{m}) \\ (\text{rad}) \\ (\text{m}) \\ (\text{m}) \\ (\text{rad}) \end{matrix}$$ [Displacements are in meters, rotations in radians.]

$$0.089 \text{ rad} = 5.099°$$

$$0.071 \text{ rad} = 4.068°$$

Solving for reactions.

$$\begin{pmatrix} F_{1x} \\ F_{1y} \\ M_1 \\ F_{2x} \\ F_{2y} \\ M_2 \\ F_{3x} \\ F_{3y} \\ M_3 \end{pmatrix} = K \begin{pmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \\ u_3 \\ v_3 \\ \phi_3 \end{pmatrix}$$ Multiplying global stiffness matrix by displacement vector.

$$\begin{pmatrix} F_{1x} \\ F_{1y} \\ M_1 \\ F_{2x} \\ F_{2y} \\ M_2 \\ F_{3x} \\ F_{3y} \\ M_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -10 \times 10^4 \\ 2.328 \times \cancel{10^{-10}}^{\,0} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \times 10^5 \\ -3 \times 10^5 \end{pmatrix}$$

Forces are in newtons, moment in N·m.

Values agree with what one would expect given the simple nature of the problem. Vertical reaction at node 3 should be equal and opposite the applied load F_{1y} and it is. Moment reaction at 3 (M_3) should equal the applied load (F_{1y}) times it's moment arm (L_1) and it does.

Solve for element forces.

Functional equations for local beam elements

Functional equation for local stiffness matrix for a 2D beam/frame element with axial affects. Refer to text Equation (5.1.8).

$$K_{\text{local}}\,(A, C, S, E, I, L) = \begin{pmatrix} \frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{6EI}{L^2} & \frac{4EI}{L} \end{pmatrix}$$

Functional equation for transformation matrix between local and global coordinates.

$$T(C, S) = \begin{pmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Solving for element forces in element 1

$$i = 1$$

$$k_{\text{local}}1 = k_{\text{local}}(A_i, C_i, S_i, E, I_i, L_i) \qquad \text{Local } k \text{ matrix for element 1}$$

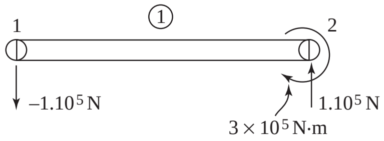
$$k_{\text{local}}1 = \begin{pmatrix} 5.6 \times 10^9 & 0 & 0 & -5.6 \times 10^9 \\ 0 & 1.12 \times 10^7 & 1.68 \times 10^7 & 0 \\ 0 & 1.68 \times 10^7 & 3.36 \times 10^7 & 0 \\ -5.6 \times 10^9 & 0 & 0 & 5.6 \times 10^9 \\ 0 & -1.12 \times 10^7 & -1.68 \times 10^7 & 0 \\ 0 & 1.68 \times 10^7 & 1.68 \times 10^7 & 0 \end{pmatrix}$$

$$\begin{matrix} 0 & 0 \\ -1.12\times10^{7} & 1.68\times10^{7} \\ -1.68\times10^{7} & 1.68\times10^{7} \\ 0 & 0 \\ 1.12\times10^{7} & -1.68\times10^{7} \\ -1.68\times10^{7} & 3.36\times10^{7} \end{matrix}\Bigg)$$

$T_1 = T(C_i, S_i)$ Transformation matrix for element 1.

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$f_1 = k_{\text{local}}\, T_1 \begin{pmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{pmatrix} \quad \text{Calculate local forces/moments in element 1.}$$



$$f_1 = \begin{pmatrix} 0 \\ -100000 \\ 0 \\ 0 \\ 100000 \\ -300,000 \end{pmatrix} \begin{pmatrix} f_{1x} \\ f_{1y} \\ m_1 \\ f_{2x} \\ f_{2y} \\ m_2 \end{pmatrix}$$

Forces are in Newtons, moments in $\text{N}\cdot\text{m}$.

f_{1y} comes back out of the equations as expected.

f_{2y} must be equal and opposite—and it is.

m_2 also checks (resists in CW direction the applied load of 1*10^5N over 3m moment arm)

Solving for element forces in element 2

$$i = 2$$

$$k_{\text{local}}2 = k_{\text{local}}(A_i, C_i, S_i, E, I_i, L_i) \quad \text{Local } k \text{ matrix for a element 2}$$

$$k_{\text{local}}2 = \begin{pmatrix} 2.8\times10^9 & 0 & 0 & -2.8\times10^9 & 0 & 0 \\ 0 & 1.4\times10^6 & 4.2\times10^6 & 0 & -1.4\times10^6 & 4.2\times10^6 \\ 0 & 4.2\times10^6 & 1.68\times10^7 & 0 & -4.2\times10^6 & 8.4\times10^6 \\ -2.8\times10^9 & 0 & 0 & 2.8\times10^9 & 0 & 0 \\ 0 & -1.4\times10^6 & -4.2\times10^6 & 0 & 1.4\times10^6 & -4.2\times10^6 \\ 0 & 4.2\times10^6 & 8.4\times10^6 & 0 & -4.2\times10^6 & 1.68\times10^7 \end{pmatrix}$$

$T_2 = T(C_i, S_i)$ Transformation matrix for element 2.

$$T_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

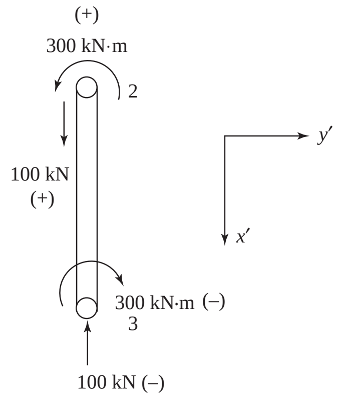
Calculate local forces/moments in element 1.

$$f_2 = k_{\text{local}}\,2\,T_2 \begin{pmatrix} u_2 \\ v_2 \\ \phi_2 \\ u_3 \\ v_3 \\ \phi_3 \end{pmatrix}$$

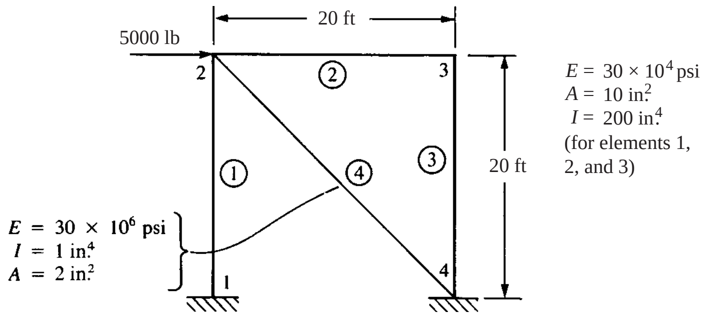
Forces are in Newtons, moments in N·m.
f_{2y} and f_{3y} again are of same magnitude as applied load at node 1 as expected.

m_2 and m_3 also have correct magnitude. m_3 must be equal and opposite of applied moment. m_2 must be opposite of m_3 for equilibrium.

$$f_2 = \begin{pmatrix} 100{,}000 \\ 0 \\ 300000 \\ -100000 \\ 0 \\ -300000 \end{pmatrix} \quad \begin{pmatrix} f_{2x} \\ f_{2y} \\ m_2 \\ f_{3x} \\ f_{3y} \\ m_3 \end{pmatrix}$$



5.13



Boundary conditions $\quad u_1 = v_1 = \phi_1 = 0, \quad u_4 = v_4 = \phi_4 = 0$

Element (1) by Equation (6.1.11)

$$C = 0; \quad S = 1$$

$$\begin{array}{cc} & \begin{array}{ccc} u_2 & v_2 & \phi_2 \end{array} \\ [k^{(1)}] = \dfrac{E}{L} & \begin{bmatrix} \frac{12I}{L^2} & 0 & \frac{6I}{L} \\ 0 & A & 0 \\ \frac{6I}{L} & 0 & 4I \end{bmatrix} \end{array}$$

Element (2)

$$C = 1, \; S = 0$$

$$\begin{array}{cc} & \begin{array}{cccccc} u_2 & v_2 & \phi_2 & u_3 & v_3 & \phi_3 \end{array} \\ [k^{(2)}] = \dfrac{E}{L} & \begin{bmatrix} A & 0 & 0 & -A & 0 & 0 \\ & \frac{12I}{L^2} & \frac{6I}{L} & 0 & \frac{-12I}{L^2} & \frac{6I}{L} \\ & & 4I & 0 & \frac{-6I}{L} & 2I \\ & & & A & 0 & 0 \\ & & & & \frac{12I}{L^2} & \frac{-6I}{L} \\ \text{Symmetry} & & & & & 4I \end{bmatrix} \end{array}$$

Element (3)

$$C = 0, \quad S = -1$$

$$\begin{array}{cc} & \begin{array}{ccc} u_3 & v_3 & \phi_3 \end{array} \\ [k^{(3)}] = \dfrac{E}{L} & \begin{bmatrix} \frac{12I}{L^2} & 0 & \frac{6I}{L} \\ & A & 0 \\ & & 4I \end{bmatrix} \end{array}$$

Element (4)

$$C = \frac{-\sqrt{2}}{2}, \quad S = \frac{-\sqrt{2}}{2}, \quad L = 20, \quad L^{(4)} = \sqrt{2}\ L$$

$$[k^{(4)}] = \frac{E}{L^{(4)}} \begin{bmatrix} \frac{A}{2} + \frac{6L}{L^2} & \frac{-A}{2} + \frac{6I}{L^2} & \sqrt{2}\,\frac{3I}{L} \\ & \frac{A}{2} + \frac{6I}{L^2} & \sqrt{2}\,\frac{3I}{L} \\ \text{Symmetry} & & 4I \end{bmatrix}$$

Assembling

$$\begin{Bmatrix} 5000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \frac{E}{L} \begin{bmatrix} 10.75 & -0.707 & 5.0 & -10 & 0 & 0 \\ & 10.75 & 5.0 & 0 & -0.0417 & 5 \\ & & 1603 & 0 & -5 & 400 \\ & & & 10.04 & 0 & 5 \\ & & & & 10.04 & -5 \\ & & & & & 1600 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \\ u_3 \\ v_3 \\ \phi_3 \end{Bmatrix}$$

$$\frac{E}{L} = \frac{30 \times 10^6}{20' \, (12 \, \frac{\text{in.}}{\text{ft}})} = 0.125 \times 10^6 \ \frac{\text{lb}}{\text{in.}^3}$$

Solving simultaneously

$$\begin{aligned} u_2 &= 0.055916 \text{ in.} & u_3 &= 0.05576 \text{ in.} \\ v_2 &= 0.003817 \text{ in.} & v_3 &= -0.000133 \text{ in.} \\ \phi_2 &= -0.00015 \text{ rad} & \phi_3 &= -0.000149 \text{ rad} \end{aligned}$$

PLANE FRAME PROBLEM 6.13
NUMBER OF ELEMENTS = 4
NUMBER OF NODES = 4

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)
1	1 1 1	0.000000	0.000000	0.000000	0.000000
2	0 0 0	0.000000	240.000000	0.000000	5000.000000
3	0 0 0	240.000000	240.000000	0.000000	0.000000
4	1 1 1	240.000000	0.000000	0.000000	0.000000

ELEMENTS

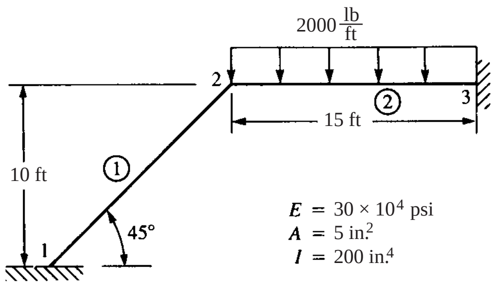
K	NODE(I, K)		E(K)	G(K)	A(K)	XI(K)
1	1	2	3.0000000E+07	0.0000000E+00	1.0000000E+01	2.0000000E+02
2	2	3	3.0000000E+07	0.0000000E+00	1.0000000E+01	2.0000000E+02
3	3	4	3.0000000E+07	0.0000000E+00	1.0000000E+01	2.0000000E+02
4	2	4	3.0000000E+07	0.0000000E+00	2.0000000E+00	1.0000000E+00

NODE	DISPLACEMENTS		Z-ROTATION
	X	Y	THETA
1	0.00000E+00	0.00000E+00	0.00000E+00
2	0.55918E–01	0.38170E–02	– 0.14987E–03
3	0.55760E–01	– 0.13304E–03	– 0.14913E–03
4	0.00000E+00	0.00000E+00	0.00000E+00

ELEMENTS

K	NODE(I, K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z–MOMENT
1	1	2	– 0.4771E+04	0.1976E+03	0.2746E+05	– 0.4771E+04	– 0.1976E+03	0.1996E+05
2	2	3	0.1972E+03	– 0.1663E+03	– 0.1997E+05	– 0.1972E+03	0.1663E+03	– 0.1994E+05
3	3	4	0.1663E+03	0.1972E+03	0.1994E+05	– 0.1663E+03	– 0.1972E+03	0.2739E+05
4	2	4	0.6513E+04	0.1547E+00	0.1301E+02	– 0.6513E+04	– 0.1547E+00	0.3950E+02

5.14



Element (1)

$$C = \frac{\sqrt{2}}{2}\ ,\ S = \frac{\sqrt{2}}{2}$$

Use only node 2 part of $[k^{(1)}]$

$$[k^{(1)}] = \frac{E}{L_1}\begin{bmatrix} AC^2 + \frac{12I}{L_1^2}S^2 & \left(A - \frac{12I}{L_1^2}\right)CS & \frac{6I}{L_1}S \\ & AS^2 + \frac{12I}{L_1^2}C^2 & \frac{-6I}{L_1}C \\ \text{Symmetry} & & 4I \end{bmatrix}$$

$$\begin{array}{c} \begin{array}{ccc} u_2 & v_2 & \phi_2 \end{array} \\ = \begin{bmatrix} \frac{61}{24} & \frac{59}{24} & 5\sqrt{2} \\ & \frac{61}{24} & -5\sqrt{2} \\ \text{Symmetry} & & 800 \end{bmatrix} \times 125\sqrt{2} \times 10^3 \end{array}$$

Element (2)

$$C = 1, \; S = 0$$

$$[k^{(2)}] = \frac{E}{L_2}\begin{bmatrix} AC^2 + \frac{12I}{L_2^2}S^2 & \left(A - \frac{12I}{L_2^2}\right)CS & \frac{-6I}{L_2}S \\ & AS^2 + \frac{12I}{L_2^2}C^2 & \frac{6I}{L_2}C \\ \text{Symmetry} & & 4I \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & \frac{2}{27} & \frac{20}{3} \\ & & 800 \end{bmatrix} \times \frac{5}{3} \times 10^5$$

$$[K] = [k^{(1)}] + [k^{(2)}]$$

Then

$$\{F\} = [K]\,\{d\}$$

Equivalent nodal forces

$$f_{2x} = 0 \qquad f_{2y} = \frac{-wL_2}{2} = -15000 \text{ lb}$$

$$m_2 = \frac{-wL_2^2}{12} = -45{,}000 \text{ lb} \cdot \text{in.}$$

$$\begin{Bmatrix} 0 \\ -15000 \\ -45000 \end{Bmatrix} = \begin{Bmatrix} 1282600 & 434580 & 125000 \\ & 461650 & -138900 \\ & & 27{,}475{,}000 \end{Bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Solving

$$u_2 = 0.0174 \text{ in.}$$

$$v_2 = -0.0481 \text{ in.}$$

$$\phi_2 = -0.00165 \text{ rad}$$

Element forces

$$\{f'\}^{(1)} = [K']^{(1)}\,[T]^{(1)}\,\{d\}^{(1)}$$

Element one (1)

$$\begin{Bmatrix} f'^{(1)}_{1x} \\ f'^{(1)}_{1y} \\ m_1^{(1)} \\ f'^{(1)}_{2x} \\ f'^{(1)}_{2y} \\ m_2^{(1)} \end{Bmatrix}^{(1)} = 10^6 \begin{bmatrix} \frac{5}{4\sqrt{2}} & 0 & 0 & \frac{-5}{4\sqrt{2}} & 0 & 0 \\ & \frac{1}{48\sqrt{2}} & \frac{5}{4} & 0 & \frac{-1}{48\sqrt{2}} & \frac{5}{4} \\ & & 100\sqrt{2} & 0 & \frac{-5}{4} & 50\sqrt{2} \\ & & & \frac{5}{4\sqrt{2}} & 0 & 0 \\ & & & & \frac{1}{48\sqrt{2}} & \frac{-5}{4} \\ \text{Symmetry} & & & & & 100\sqrt{2} \end{bmatrix} \overbrace{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.02167 \\ 0.0463 \\ -0.00165 \end{Bmatrix}}^{[T]\{d\}}$$

$$f'^{(1)}_{1x} = 19160 \text{ lb} = -f'^{(1)}_{2x}$$

$$f'^{(1)}_{2y} = -1385 \text{ lb} = -f'^{(1)}_{2y}$$

$$m_1^{(1)} = -59050 \text{ lb}\cdot\text{in.}$$

$$m_2^{(1)} = -176000 \text{ lb}\cdot\text{in.}$$

Element two (2)

$$\{f'\}^{(2)} = [K']^{(2)}[T]^{(2)}\{d\}^{(2)} - \{f'_0\}^{(2)}$$

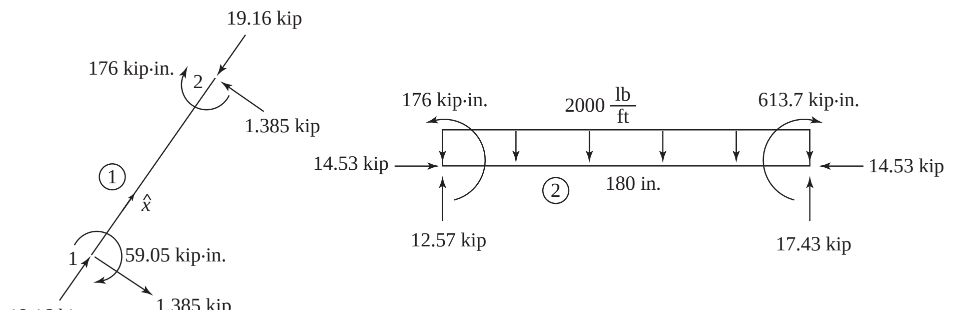
$$[T]^{(2)}\{d\}^{(2)} = \begin{bmatrix} 0.0174 \\ -0.0481 \\ -0.00165 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{Bmatrix} f'^{(2)}_{2xe} \\ f'^{(2)}_{2ye} \\ m_{2e}^{(2)} \\ f'^{(2)}_{3xe} \\ f'^{(2)}_{3ye} \\ m_{3e}^{(2)} \end{Bmatrix} = 10^6 \begin{bmatrix} \frac{5}{6} & 0 & 0 & \frac{-5}{6} & 0 & 0 \\ & \frac{1}{81} & \frac{10}{9} & 0 & \frac{-1}{81} & \frac{10}{9} \\ & & \frac{400}{3} & 0 & \frac{-10}{9} & \frac{200}{3} \\ & & & \frac{5}{6} & 0 & 0 \\ & & & & \frac{1}{81} & \frac{-10}{9} \\ & & & & & \frac{400}{3} \end{bmatrix} \begin{Bmatrix} 0.0174 \\ -0.0481 \\ -0.00165 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{f'_{(2)}\}^{(2)} = \begin{Bmatrix} 14530 \\ -2432 \\ -273{,}980 \\ -14530 \\ 2432 \\ -163{,}700 \end{Bmatrix}$$

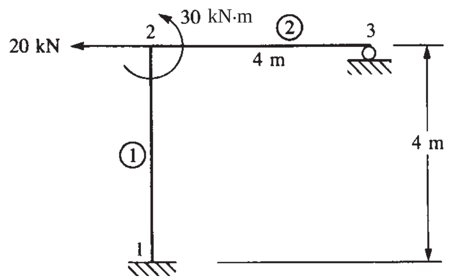
Finally $\{f'\} = \{f'_e\} - \{f'_0\}$

$$\begin{Bmatrix} f_{2x}'^{(2)} \\ f_{2y}'^{(2)} \\ m_2^{(2)} \\ f_{3x}'^{(2)} \\ f_{3y}'^{(2)} \\ m_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} 14530 \\ -2432 \\ -273{,}980 \\ -14530 \\ 2432 \\ -163{,}700 \end{Bmatrix} - \begin{Bmatrix} 0 \\ -15000 \\ -450{,}000 \\ 0 \\ -15000 \\ 450{,}000 \end{Bmatrix} = \begin{Bmatrix} 14530\text{ lb} \\ 12568\text{ lb} \\ 176020\text{ lb}\cdot\text{in.} \\ -14530\text{ lb} \\ 17432\text{ lb} \\ -613700\text{ lb}\cdot\text{in.} \end{Bmatrix}$$



Free-body diagrams

5.15



$E = 70$ GPa
$A = 4 \times 10^{-2}$ m^2
$I = 2 \times 10^{-4}$ m^4

Element (1)

$$C = 0, \quad S = 1$$

$$\frac{12\,I}{L^2} = \frac{12\,(2 \times 10^{-4})}{(4)^2} = 1.5 \times 10^{-4}\text{ m}^2$$

$$\frac{6\,I}{L} = 3.0 \times 10^{-4}\text{ m}^3$$

$$\frac{E}{L} = \frac{70 \times 10^6}{4} = 1.75 \times 10^7 \ \frac{\text{kN}}{\text{m}}$$

$$[k^{(1)}] = 1.75 \times 10^7 \begin{array}{c} \begin{array}{ccc} u_2 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 1.5 \times 10^{-4} & 0 & 3.0 \times 10^{-4} \\ & 4 \times 10^{-2} & 0 \\ \text{Symmetry} & & 8 \times 10^{-4} \end{bmatrix} \end{array}$$

Element (2)

$$S = 0, \quad C = 1$$

$$[k^{(2)}] = 1.75 \times 10^7 \begin{array}{c} \begin{array}{ccccc} u_2 & v_2 & \phi_2 & u_3 & \phi_3 \end{array} \\ \begin{bmatrix} 4\times10^{-2} & 0 & 0 & -4\times10^{-2} & 0 \\ & 1.5\times10^{-4} & 3\times10^{-4} & 0 & 3\times10^{-4} \\ & & 8\times10^{-4} & 0 & 4\times10^{-4} \\ & & & 4\times10^{-2} & 0 \\ \text{Symmetry} & & & & 8\times10^{-4} \end{bmatrix} \end{array}$$

where boundary conditions

$$u_1 = v_1 = \phi_1 = 0 \text{ and } v_3 = 0$$

have been used in $[k^{(1)}]$ and $[k^{(2)}]$

The global equations are

$$1.75 \times 10^7 \begin{bmatrix} 4.015\times10^{-2} & 0 & 3\times10^{-4} & -4\times10^{-2} & 0 \\ & 4.015\times10^{-2} & 3\times10^{-4} & 0 & 3\times10^{-4} \\ & & 1.6\times10^{-3} & 0 & 4\times10^{-4} \\ & & & 4\times10^{-2} & 0 \\ \text{Symmetry} & & & & 8\times10^{-4} \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_3 \\ u_3 \\ \phi_3 \end{Bmatrix}$$

$$= \begin{Bmatrix} F_{2x} = -20 \\ F_{2y} = 0 \\ M_2 = 30 \\ F_{3x} = 0 \\ M_3 = 0 \end{Bmatrix}$$

Solving simultaneously

$$u_2 = -1.76 \times 10^{-2} \text{ m}, \; v_2 = -1.87 \times 10^{-5} \text{ m}$$

$$\phi_2 = 5 \times 10^{-3} \text{ rad}$$

$$u_3 = -1.76 \times 10^{-2} \text{ m}, \quad \phi_3 = -2.49 \times 10^{-3} \text{ rad}$$

Element forces

Element (1)

$$\{f'\}^{(1)} = [k']^{(1)} \; [T]^{(1)} \, \{d\}^{(1)}$$

$$[T]^{(1)} \{d\}^{(1)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ \phi_1 = 0 \\ u_2 = -1.76\times10^{-2} \\ v_2 = -1.87\times10^{-5} \\ \phi_2 = 5.0\times10^{-3} \end{Bmatrix}$$

$$[T]^{(1)} \{d\}^{(1)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1.87\times10^{-5} \\ 1.76\times10^{-2} \\ 5.00\times10^{-3} \end{Bmatrix}$$

$\{f'\}^{(1)} = [k']^{(1)}\,[T]^{(1)}\,\{d\}^{(1)} =$

$$\begin{bmatrix} 7\times10^{-5} & 0 & 0 & -7\times10^{-5} & 0 & 0 \\ & 2.625\times10^{3} & 5.25\times10^{3} & 0 & -2.625\times10^{3} & 5.25\times10^{3} \\ & & 1.4\times10^{4} & 0 & -5.25\times10^{3} & 7\times10^{3} \\ & & & 7\times10^{5} & 0 & 0 \\ & & & & 2.625\times10^{3} & -5.25\times10^{3} \\ \text{Symmetry} & & & & & 1.4\times10^{4} \end{bmatrix}$$

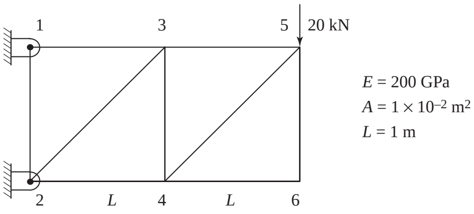
$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1.87\times10^{-5} \\ 1.76\times10^{-2} \\ 5.0\times10^{-3} \end{Bmatrix}$$

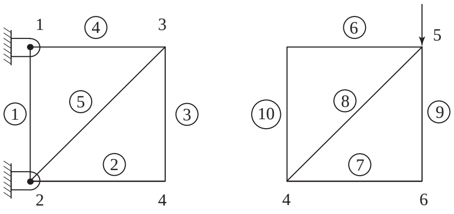
$$f'_{1x} = 13.1 \text{ kN}$$
$$f'_{1y} = -20.0 \text{ kN}$$
$$m_1 = -57.4 \text{ kN}\cdot\text{m}$$
$$f'_{2x} = -13.1 \text{ kN}$$
$$f'_{2y} = 20.0 \text{ kN}$$
$$m_2 = -22.4 \text{ kN}\cdot\text{m}$$

Similarly for element (2)

$$\{f'\}^{(2)} = [k']^{(2)}\,[T]^{(2)}\,\{d\}^{(2)}$$
$$f'_{2x} = f'_{3x} = 0$$
$$f'_{2y} = 13.1 \text{ kN},\ f'_{3y} = -13.1 \text{ kN}$$
$$m_2 = 52.47 \text{ kN}\cdot\text{m},\ m_3 = 0$$

5.16





Element (1)

$C = 0, \quad S = 1$

$$[k^{(1)}] = \frac{\overset{E}{\overset{|}{200 \times 10^9}}\ \overset{A}{\overset{|}{(1 \times 10^{-2})}}}{\underset{L}{\underset{|}{1}}} \begin{array}{c} \begin{array}{cccc} & (2) & & (1) \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Element (3)

$C = 0, \quad S = 1$

$$[k^{(3)}] = \frac{EA}{L} \begin{array}{c} \begin{array}{cccc} & (4) & & (3) \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Element (2)

$C = 1, \quad S = 0$

$$[k^{(2)}] = \frac{AE}{L} \begin{array}{c} \begin{array}{cccc} (2) & & (4) & \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Element (4)

$C = 1, \quad S = 0$

$$[k^{(4)}] = \frac{AE}{L} \begin{array}{c} \begin{array}{cccc} (1) & & (3) & \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Element (5)

$C = \frac{\sqrt{2}}{2}, \quad S = \frac{\sqrt{2}}{2}$

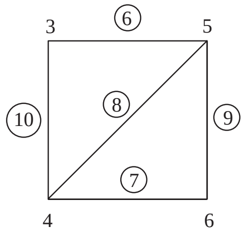
$$[k^{(5)}] = \frac{AE}{\sqrt{2}L} \begin{array}{c} \begin{array}{cccc} & (2) & & (3) \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \end{array}$$

Boundary conditions

$u_1 = v_1 = u_2 = v_2 = 0$

Assemble appropriate parts of $[k]$'s

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = 2 \times 10^9 \begin{bmatrix} 1.3535 & 0.3535 & 0 & 0 \\ & 1.3535 & 0 & -1 \\ & & 1 & 0 \\ \text{Symmetry} & & & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \qquad \text{(a)}$$



Similarly assembling $[k^{(6)}]$ through $[k^{(10)}]$ we obtain

$$\begin{array}{ccc} & \begin{array}{cccccccc} u_3 & v_3 & u_4 & v_4 & u_5 & v_5 & u_6 & v_6 \end{array} & \\ \begin{array}{r} [K_{ee}] = \\ \\ [K] = \\ \\ [K_{ie}] = \end{array} & \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.3535 & 0.3535 & -0.3535 & -0.3535 & -1 & 0 \\ 0 & -1 & 0.3535 & 1.3535 & -0.3535 & -0.3535 & 0 & 0 \\ \hline -1 & 0 & -0.3535 & -0.3535 & 1.3535 & 0.3535 & 0 & 0 \\ 0 & 0 & -0.3535 & -0.3535 & 0.3535 & 1.3535 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right] & \begin{array}{l} = [K_{ei}] \\ \\ \times 2 \times 10^9 \\ \\ = [K_{ii}] \end{array} \end{array}$$

Now $[K_{ii}] - [K_{ie}]\,[K_{ee}^{-1}]\;[K_{ei}]\;\{u_i\} = \{f_i\} - [K_{ie}]\,[K_{ee}^{-1}]\;\{f_e\}$

$$\left\{ \begin{bmatrix} 1.3535 & 0.3535 & 0 & 0 \\ 0.3535 & 1.3535 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 & -0.3535 & -0.3535 \\ 0 & 0 & -0.3535 & -0.3535 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right.$$

$$\left. \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1.3535 & 0.3535 \\ 0 & -1 & 0.3535 & 1.3535 \end{bmatrix}^{-1} - \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.3535 & -0.3535 & -1 & 0 \\ -0.3535 & -0.3535 & 0 & 0 \end{bmatrix} \right\} \begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{bmatrix} -1 & 0 & -0.3535 & -0.3535 \\ 0 & 0 & -0.3535 & -0.3535 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1.3535 & 0.3535 \\ 0 & -1 & 0.3535 & 1.3535 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ -20000 \\ 0 \\ 0 \end{Bmatrix}$$

$$\therefore \quad 2\times10^9 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} -20000 \\ -20000 \\ 20000 \\ 0 \end{Bmatrix} \qquad \text{(b)}$$

Adding two sections (a) and (b)

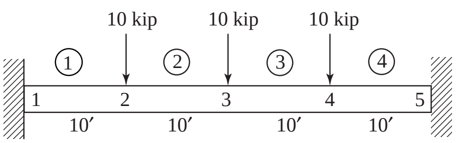
$$2\times10^9 \begin{bmatrix} 1.3535 & 0.3535 & 0 & 0 \\ 0.3535 & 2.3535 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} -20000 \\ -20000 \\ 20000 \\ 0 \end{Bmatrix}$$

Solving

$$u_3 = 2.832\times10^{-11}\ \text{m},\ v_3 = -2.828\times10^{-5}\ \text{m}$$

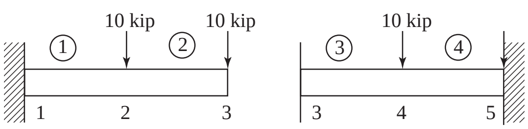
$$u_4 = 1.0\times10^{-5}\ \text{m},\ v_4 = -2.828\times10^{-5}\ \text{m}$$

5.17



Substructure 1 Substructure 2

Substructure 1 Substructure 2



$[k]$'s for each element are

$$[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = \frac{29\times10^3\times10^3}{(120)^3} \begin{array}{c} \begin{array}{cccc} & (1) & & (2) \end{array} \\ \begin{bmatrix} 12 & 720 & -12 & 720 \\ 720 & 57600 & -720 & -28800 \\ & & 12 & -720 \\ \text{Symmetry} & & & 57600 \end{bmatrix} \end{array}$$

Adding $[k]$'s of elements (1) and (2) for substructure (1) and apply boundary conditions

$$v_1 = \phi_1 = 0$$

$$16.78 \begin{bmatrix} 24 & 0 & -12 & 720 \\ & 115200 & -720 & 28800 \\ & & 12 & -720 \\ \text{Symmetry} & & & 57600 \end{bmatrix} \begin{Bmatrix} v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} -10 \\ 0 \\ -10 \\ 0 \end{Bmatrix}$$

Now rearrange the equations with interface displacement first

$$16.78\begin{bmatrix} 12 & -720 & -12 & -720 \\ -720 & 57600 & 720 & 28800 \\ -12 & 720 & 24 & 0 \\ -720 & 28800 & 0 & 115200 \end{bmatrix}\begin{Bmatrix} v_3 \\ \phi_3 \\ v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -10 \\ 0 \\ -10 \\ 0 \end{Bmatrix}$$

Using Equation (6.6.6) $[K_{ii}] - [K_{ie}]\ [K_{ee}^{-1}]\ [K_{ei}]\ \{u_i\} = \{f_i\} - [K_{ie}]\ [K_{ee}^{-1}]\ \{f_e\}$

$$\left\{16.78\begin{bmatrix} 12 & -720 \\ -720 & 57600 \end{bmatrix} - \begin{bmatrix} -12 & -720 \\ 720 & 28800 \end{bmatrix}\begin{bmatrix} 24 & 0 \\ 0 & 115200 \end{bmatrix}^{-1}\begin{bmatrix} -12 & 720 \\ -720 & 28800 \end{bmatrix}\right\}$$

$$\times\begin{Bmatrix} v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} -10 \\ 0 \end{Bmatrix} - \begin{bmatrix} -12 & -720 \\ 720 & 28800 \end{bmatrix}\begin{bmatrix} -24 & 0 \\ 0 & 115200 \end{bmatrix}^{-1}\begin{Bmatrix} -10 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 25.17 & -3020 \\ -3020 & 483260 \end{bmatrix}\begin{Bmatrix} v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} -15 \\ 300 \end{Bmatrix} \qquad (1)$$

Considering substructure (2) with boundary conditions $v_5 = \phi_5 = 0$

$$16.78\begin{bmatrix} 12 & 720 & -12 & 720 \\ & 57600 & -720 & 28800 \\ & & 24 & 0 \\ \text{Symmetry} & & & 115200 \end{bmatrix}\begin{Bmatrix} u_3 \\ \phi_3 \\ v_4 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -10 \\ 0 \end{Bmatrix}$$

Simplifying as per substructure (1)

$$\begin{bmatrix} 25.17 & 3020 \\ 3020 & 483264 \end{bmatrix}\begin{Bmatrix} v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} -5 \\ -300 \end{Bmatrix} \qquad (2)$$

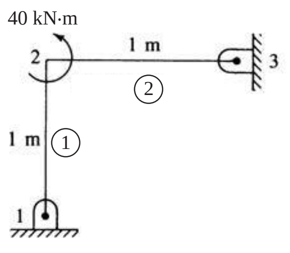
Adding (1) and (2)

$$\begin{bmatrix} 50.34 & 0 \\ 0 & 966528 \end{bmatrix}\begin{Bmatrix} v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} -20 \\ 0 \end{Bmatrix}$$

Solving

$$v_3 = -0.3973 \text{ in.} \quad \phi_3 = 0$$

5.18



$E = 200$ GPa
$I = 2 \times 10^{-4}$ m^4
$A = 1 \times 10^{-2}$ m^2

$$\frac{12\,I}{L^2} = \frac{12 \times 2 \times 10^{-4}}{1^2} = 2.4 \times 10^{-3}$$

$$\frac{6\,I}{L} = \frac{6 \times 2 \times 10^{-4}}{1} = 1.2 \times 10^{-3}$$

$$\frac{E}{L} = 200 \times 10^9 \ \frac{\text{N}}{\text{m}^3}$$

Considering substructure (1) and applying boundary conditions

$$u_1 = v_1 = 0$$

$$C = 0, S = 1$$

$$\begin{array}{ccc} & [k^{(1)}] & \{d\} \\ = 200 \times 10^9 & \begin{bmatrix} 0.0008 & 0.0012 & 0 & 0.0004 \\ & 0.0024 & 0 & 0.0012 \\ & & 0.01 & 0 \\ \text{Symmetry} & & & 0.0008 \end{bmatrix} & \begin{Bmatrix} \phi_1 \\ u_2 \\ v_2 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 40 \times 10^3 \end{Bmatrix} \end{array}$$

Rearranging equations (rows and columns) we place interface displacements first

$$200 \times 10^9 \left[\begin{array}{ccc|c} 0.0024 & 0 & 0.0012 & 0.0012 \\ & 0.01 & 0 & 0 \\ & & 0.0008 & 0.0004 \\ \hline \text{Symmetry} & & & 0.0008 \end{array}\right] \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \\ \phi_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 40 \times 10^3 \\ 0 \end{Bmatrix}$$

Using Equation (6.6.6)

$$200 \times 10^9 \left\{ \begin{bmatrix} 0.0024 & 0 & 0.0012 \\ & 0.01 & 0 \\ \text{Symmetry} & & 0.0008 \end{bmatrix} - \begin{Bmatrix} 0.0012 \\ 0 \\ 0.0004 \end{Bmatrix} [1250][0.0012 \quad 0 \quad 0.0004] \right\} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 20 \times 10^3 \end{Bmatrix} - \begin{Bmatrix} 0.0012 \\ 0 \\ 0.0004 \end{Bmatrix} [1250] \ [0]$$

Simplifying

$$1 \times 10^9 \begin{bmatrix} 0.12 & 0 & 0.12 \\ & 2.0 & 0 \\ & & 0.12 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 40 \times 10^3 \end{Bmatrix} \qquad (1)$$

Considering substructure (2) and applying boundary conditions

$$C = 1, S = 0$$

$$200 \times 10^9 \left[\begin{array}{ccc|c} 0.01 & 0 & 0 & -0.01 \\ 0 & 0.0024 & 0.0012 & 0 \\ 0 & 0.0012 & 0.0008 & 0 \\ \hline -0.01 & 0 & 0 & 0.01 \end{array}\right] \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Simplifying by applying Equation (6.6.6)

$$200 \times 10^9 \left\{ \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.0024 & 0.0012 \\ & & 0.0008 \end{bmatrix} - \begin{Bmatrix} -0.01 \\ 0 \\ 0 \end{Bmatrix} \left[\frac{1}{0.01} \right] [-0.01 \ 0 \ 0] \right\} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

$$= \{0\} - \{0\}$$

$$\therefore \quad 1\times10^9\begin{bmatrix}2.0 & 0 & 0\\ & 0.12 & 0.12\\ \text{Symmetry} & & 0.12\end{bmatrix}\begin{Bmatrix}u_2\\ v_2\\ \phi_2\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix} \qquad (2)$$

Adding (1) and (2)

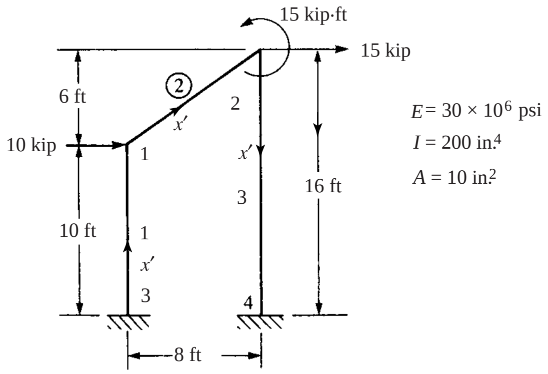
$$1\times10^9\begin{bmatrix}2.12 & 0 & 0.12\\ & 2.12 & 0.12\\ \text{Symmetry} & & 0.24\end{bmatrix}\begin{Bmatrix}u_2\\ v_2\\ \phi_2\end{Bmatrix}=\begin{Bmatrix}0\\0\\40\times10^3\end{Bmatrix}$$

Solving

$$u_2 = v_2 = -0.010\times10^{-3}\,\text{m}$$

$$\phi_2 = 17.67\times10^{-5}\,\text{rad}$$

5.19



NUMBER OF ELEMENTS = 3
NUMBER OF NODES = 4
NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1, K)
1	0 0 0	0.000000	120.000000	0.000000	10000.000000
2	0 0 0	96.000000	192.000000	0.000000	15000.000000
3	1 1 1	0.000000	0.000000	0.000000	0.000000
4	1 1 1	96.000000	0.000000	0.000000	0.000000

FORCE(2, K)	FORCE (3, K)
0.000000	0.000000
0.000000	180000.000000
0.000000	0.000000
0.000000	0.000000

ELEMENTS

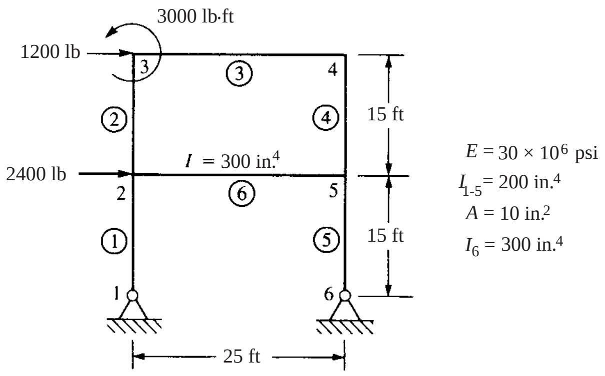
K	NODE(I, K)		E(K)	G(K)	A(K)	XI(K)
1	3	1	3.0000000E+07	1.0000000E+00	1. 0000000E+01	2.0000000E+02
2	1	2	3.0000000E+07	1.0000000E+00	1.0000000E+01	2.0000000E+02
3	2	4	3.0000000E+07	1.0000000E+00	1.0000000E+01	2.0000000E+02

DISPLACEMENT		Z-ROTATION
X	Y	THETA
0.70180E+00	0.79708E–02	– 0.44578E–02
0.72656E+00	– 0.12753E–01	– 0.49949E–03
0.00000E+00	0.00000E+00	0.00000E+00
0.00000E+00	0.00000E+00	0.00000E+00

ELEMENTS

K	NODE(I, K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	3	1	– 0.1993E+05	0.1810E+05	0.1309E+07	0.1993E+05	– 0.1810E+05	0.8629E+06
2	1	2	– 0.1843E+05	– 0.1108E+05	– 0.8629E+06	0.1843E+05	0.1108E+05	– 0.4671E+06
3	2	4	0.1993E+05	0.6903E+04	0.6471E+06	– 0.1993E+05	– 0.6903E+04	0.6783E+06

5.20



PLANE FRAME PROBLEM 5.20
NUMBER OF ELEMENTS = 6
NUMBER OF NODES = 6

NODE POINTS

K	IFIX			XC(K)	YC(K)	ZC(K)	FORCE(1, K)
1	1	1	0	0.000000	0.000000	0.000000	0.000000
2	0	0	0	0.000000	180.000000	0.000000	2400.000000
3	0	0	0	0.000000	360.000000	0.000000	1200.000000
4	0	0	0	300.000000	360.000000	0.000000	0.000000
5	0	0	0	300.000000	180.000000	0.000000	0.000000
6	1	1	0	300.000000	0.000000	0.000000	0.000000

FORCE(2, K)	FORCE(3, K)
0.000000	0.000000
0.000000	0.000000
0.000000	36000.000000
0.000000	0.000000
0.000000	0.000000
0.000000	0.000000

ELEMENTS

K	NODE(I, K)		E(K)	G(K)	A(K)	XI(K)
1	1	2	2.9000000E+07	0.0000000E+00	1.0000000E+01	2.0000000E+02
2	2	3	2.9000000E+07	0.0000000E+00	1.0000000E+01	2.0000000E+02
3	3	4	2.9000000E+07	0.0000000E+00	1.0000000E+01	2.0000000E+02
4	4	5	2.9000000E+07	0.0000000E+00	1.0000000E+01	2.0000000E+02
5	5	6	2.9000000E+07	0.0000000E+00	1.0000000E+01	2.0000000E+02
6	2	5	2.9000000E+07	0.0000000E+00	1.0000000E+01	3.0000000E+02

DISPLACEMENTS		Z-ROTATION
X	Y	THETA
0.00000E+00	0.00000E+00	– 0.69680E–02
0.95468E+00	0.17130E–02	– 0.19754E–02
0.12408E+01	0.20315E–02	– 0.55648E–03
0.12403E+01	0.20315E–02	– 0.79750E–03
0.95333E+00	– 0.17130E–02	– 0.19213E–02
0.00000E+00	0.00000E+00	– 0.69838E–02

ELEMENTS

K	NODE(I, K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	1	2	– 0.2760E+04	0.1787E+04	0.7438E–01	0.2760E+04	– 0.1787E+04	0.3217E+06
2	2	3	– 0.5131E+03	0.6953E+03	0.1686E+05	0.5131E+03	– 0.6953E+03	0.1083E+06
3	3	4	0.5045E+03	– 0.5131E+03	– 0.7230E+05	– 0.5045E+03	0.5131E+03	– 0.8162E+05
4	4	5	0.5131E+03	0.5046E+03	0.8162E+05	– 0.5131E+03	– 0.5046E+03	0.9200E+04
5	5	6	0.2760E+04	0.1813E+04	0.3263E+06	– 0.2760E+04	– 0.1813E+04	0.3091E–01
6	2	5	0.1308E+04	– 0.2247E+04	– 0.3386E+06	– 0.1308E+04	0.2247E+04	– 0.3355E+06

5.21 For the slant-legged rigid frame shown in Figure P5–21, size the structure for minimum weight based on a maximum bending stress of 20 ksi in the horizontal beam elements and a maximum compressive stress (due to bending and direct axial load) of 15 ksi in the slant-legged elements. Use the same element size for the two slant-legged elements and the same element size for the two 10-foot sections of the horizontal element. Assume A36 steel is used.

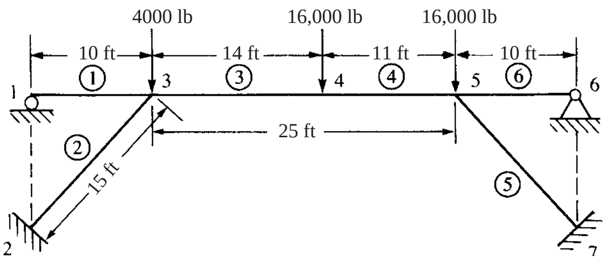


Figure P5–21

- I chose to use an ‘I’ beam for my cross sectional area because it is commonly used in bridges. My final design was chosen because the design meets the constraints of the problem while not being overly large.
- The center part of the cross member (25 foot section) was taken to be W14 × 26. This has a thickness of 0.420 inches, a depth of 13.91 inches and a width of 5.025 inches.
- The two angle members and the outside 10 foot members are also designed as ‘I’ beams. These members were taken to be W14 × 22. This size has a thickness of 0.335 inches, a depth of 13.74 inches and a width of 5.00 inches.
- The maximum bending stress (about the local axis 3) in this member is 19.51891 ksi. This is under to 20 ksi constraint put on the design in the problem. It is near the center of the cross member.
- The maximum stress in the angle member is – 11.51989 ksi and is well below the 15 ksi allowed in the problem.
- The sizes were determined by taking some commonly used sizes from my Mechanics of Materials book and using trial and error. When I got the lower members too small the bending stress went too high in the cross member. The final design was chosen because it minimized the size of the cross section while also minimizing the size of the angle members.

5.22 For the rigid building frame shown in Figure P5–22, determine the forces in each element and calculate the bending stresses. Assume all the vertical elements have $A = 10$ in.^{2} and $I = 100$ in^4 and all horizontal elements have $A = 15$ in.^{2} and $I = 150$ in.^{4}. Let $E = 29 \times 10^6$ psi for all elements. Let $c = 5$ in. for the vertical elements and $c = 6$ in. for the horizontal elements, where c denotes the distance from the neutral axis to the top or bottom of the beam cross section, as used in the bending stress formula $\sigma = \left(\frac{M_c}{I}\right)$.

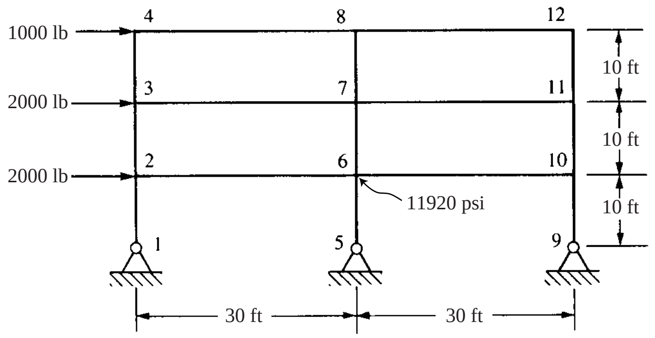
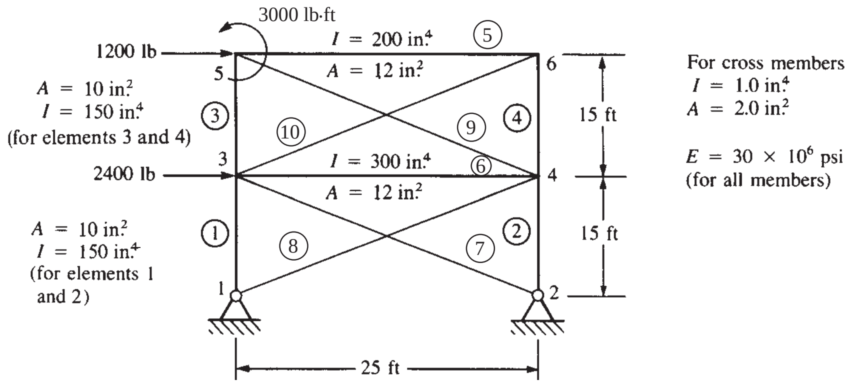


Figure P5–22

1**** BEAM ELEMENT STRESSES

ELEMENT NO.	CASE (MODE)	P/A	P/A+M2/S2	P/A–M2/S2	P/A+M3/S3	P/A–M3/S3	WORST SUM
1	1	1.500E+02	1.500E+02	1.500E+02	1.500E+02	1.500E+02	1.500E+02
		1.500E+02	1.500E+02	1.500E+02	4.676E+03	–4.376E+03	4.676E+03
2	1	1.500E+02	1.500E+02	1.500E+02	4.676E+03	–4.376E+03	4.676E+03
		1.500E+02	1.500E+02	1.500E+02	9.203E+03	–8.903E+03	9.203E+03
3	1	5.591E+01	5.591E+01	5.591E+01	1.605E+02	–4.866E+01	1.605E+02
		5.591E+01	5.591E+01	5.591E+01	2.054E+03	–1.942E+03	2.0543+03
4	1	5.591E+01	5.591E+01	5.591E+01	2.054E+03	–1.942E+03	2.054E+03
		5.591E+01	5.591E+01	5.591E+01	3.947E+03	–3.835E+03	3.947E+03
5	1	1.480E+01	1.480E+01	1.480E+01	1.329E+02	–1.033E+02	1.329E+02
		1.480E+01	1.480E+01	1.480E+01	7.855E+02	–7.559E+02	7.855E+02
6	1	1.480E+01	1.480E+01	1.480E+01	7.855E+02	–7.559E+02	7.855E+02
		1.480E+01	1.480E+01	1.480E+01	1.438E+03	–1.409E+03	1.438E+03
7	1	8.096E–02	8.096E–02	8.096E–02	8.096E–02	8.096E–02	8.096E–02
		8.096E–02	8.096E–02	8.096E–02	5.962E+03	–5.962E+03	5.962E+03
8	1	8.096E–02	8.096E–02	8.096E–02	5.962E+03	–5.962E+03	5.962E+03
		8.096E–02	8.096E–02	8.096E–02	1.192E+04	–1.192E+04	1.192E+04
9	1	1.920E–01	1.920E–01	1.920E–01	–4.031E+03	4.032E+03	4.032E+03
		1.920E–01	1.920E–01	1.920E–01	1.177E+03	–1.177E+03	1.177E+03
10	1	1.920E–01	1.920E–01	1.920E–01	1.177E+03	–1.177E+03	1.177E+03
		1.920E–01	1.920E–01	1.920E–01	6.386E+03	–6.386E+03	6.386E+03
11	1	1.098E–01	1.098E–01	1.098E–01	–8.724E+02	8.726E+02	8.726E+02
		1.098E–01	1.098E–01	1.098E–01	8.076E+02	–8.074E+02	8.076E+02
12	1	1.098E–01	1.098E–01	1.098E–01	8.076E+02	–8.074E+02	8.076E+02
		1.098E–01	1.098E–01	1.098E–01	2.488E+03	–2.487E+03	2.488E+03
13	1	–1.500E+02	–1.500E+02	–1.500E+02	–1.500E+02	–1.500E+02	–1.500E+02
		–1.500E+02	–1.500E+02	–1.500E+02	4.361E+03	–4.661E+03	–4.661E+03
14	1	–1.500E+02	–1.500E+02	–1.500E+02	4.361E+03	–4.661E+03	–4.661E+03
		–1.500E+02	–1.500E+02	–1.500E+02	8.873E+03	–9.173E+03	–9.173E+03
15	1	–5.610E+01	–5.610E+01	–5.610E+01	3.192E+01	–1.441E+02	–1.441E+02
		–5.610E+01	–5.610E+01	–5.610E+01	1.930E+03	–2.042E+03	–2.042E+03
16	1	–5.610E+01	–5.610E+01	–5.610E+01	1.930E+03	–2.042E+03	–2.042E+03
		–5.610E+01	–5.610E+01	–5.610E+01	3.828E+03	–3.940E+03	–3.940E+03
17	1	–1.491E+01	–1.491E+01	–1.491E+01	8.670E+01	–1.165E+02	–1.165E+02
		–1.491E+01	–1.491E+01	–1.491E+01	7.542E+02	–7.840E+02	–7.840E+02
18	1	–1.491E+01	–1.491E+01	–1.491E+01	7.542E+02	–7.840E+02	–7.840E+02
		–1.491E+01	–1.491E+01	–1.491E+01	1.422E+03	–1.451E+03	–1.451E+03

5.23



Problem 5-23

NUMBER OF ELEMENTS = 10

NUMBER OF NODES = 6

NODE POINTS

K	IFII	XC(K)	YC(K)	ZC(K)	FORCE(1, K)
1	1 1 0	0.000000	0.000000	0.000000	0.000000
2	1 1 0	300.000000	0.000000	0.000000	0.000000
3	0 0 0	0.000000	180.000000	0.000000	2400.000000
4	0 0 0	300.000000	180.000000	0.000000	0.000000
5	0 0 0	0.000000	360.000000	0.000000	1200.000000
6	0 0 0	300.000000	360.000000	0.000000	0.000000

ELEMENTS

K	NODE (I, K)		E(K)	G(K)	A(K)	XI(K)
1	1	3	3.0000000E+07	0.0000000E+00	1.0000000E+01	1.5000000E+02
2	2	4	3.0000000E+07	0.0000000E+00	1.0000000E+01	1.5000000E+02
3	3	5	3.0000000E+07	0.0000000E+00	1.0000000E+01	1.5000000E+02
4	4	6	3.0000000E+07	0.0000000E+00	1.0000000E+01	1.5000000E+02
5	5	6	3.0000000E+07	0.0000000E+00	1.2000000E+01	2.0000000E+02
6	3	4	3.0000000E+07	0.0000000E+00	1.2000000E+01	3.0000000E+02
7	3	2	3.0000000E+07	0.0000000E+00	2.0000000E+00	1.0000000E+00
8	1	4	3.0000000E+07	0.0000000E+00	2.0000000E+00	1.0000000E+00
9	5	4	3.0000000E+07	0.0000000E+00	2.0000000E+00	1.0000000E+00
10	3	6	3.0000000E+07	0.0000000E+00	2.0000000E+00	1.0000000E+00

FORCE(2, K)	FORCE(3, K)
0.000000	0.000000
0.000000	0.000000
0.000000	0.000000
0.000000	0.000000
0.000000	36000.000000
0.000000	0.000000

NODE	DISPLACEMENTS X	DISPLACEMENTS Y	Z-ROTATION THETA
1	0.00000E+00	0.00000E+00	– 0.90513E–04
2	0.00000E+00	0.00000E+00	– 0.11175E–03
3	0.15236E–01	0.10352E–02	– 0.72478E–04
4	0.14269E–01	– 0.99244E–03	– 0.13558E–04
5	0.20440E–01	0.12193E–02	0.20690E–03
6	0.20082E–01	– 0.11183E–02	– 0.74145E–04

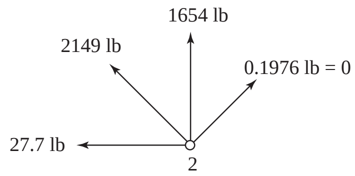
ELEMENTS

K	NODE(I, K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	1	3	– 0.1725E+04	0.5246E+01	0.2132E+02	0.1725E+04	– 0.5246E+01	0.9230E+03
2	2	4	0.1654E+04	0.2770E+02	0.3793E+02	0.1654E+04	– 0.2770E+02	0.4948E+04
3	3	5	– 0.3069E+03	0.1602E+03	0.7434E+04	0.3069E+03	– 0.1602E+03	0.2140E+05
4	4	6	0.2097E+03	– 0.1926E+02	– 0.2187E+03	– 0.2097E +03	0.1926E+02	– 0.3248E+04
5	5	6	0.4288E+03	0.5934E+02	0.1452E+05	– 0.4288E+03	– 0.5934E+02	0.3279E+04
6	3	4	0.1160E+04	– 0.4351E+02	– 0.8294E+04	– 0.1160E+04	0.4351E+02	– 0.4759E+04
7	3	2	0.2149E+04	– 0.1976E+00	– 0.3119E+02	– 0.2149E+04	0.1976E+00	– 0.3793E+02
8	1	4	– 0.2011E+04	– 0.8417E–01	– 0.2132E+02	0.2011E+04	0.8417E–01	– 0.8126E+01
10	3	6	– 0.5227E+03	0.1791E+00	– 0.3119E+02	0.5227+03	0.1791E+00	– 0.3148E+02

Reactions

Node 2

From x' and y'forces in elements (2) and (7)



$$F_{2x} = -27.7 - 1843 = 1871 \text{ lb} \leftarrow$$

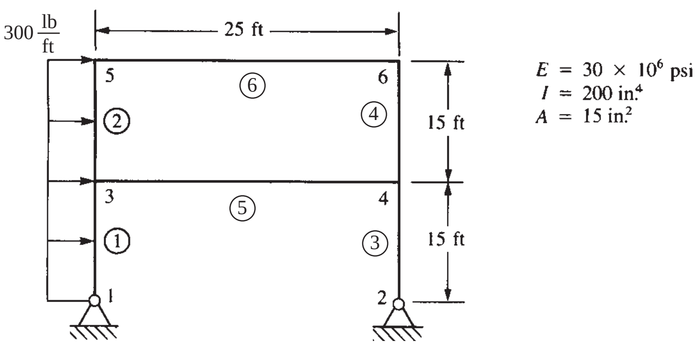
$$F_{2y} = 1654 + 1106 = 2760 \text{ lb} \uparrow$$

Similarly

$$F_{1x} = 1730 \text{ lb} \leftarrow$$

$$F_{1y} = 2760 \text{ lb} \downarrow$$

5.24



Problem 5.24

NUMBER OF ELEMENTS = 6

NUMBER OF NODES = 6

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1, K)	FORCE(2, K)	FORCE(3, K)
1	1 1 0	0.000000	0.000000	0.000000	0.000000	0.000000	–67500.000000
2	1 1 0	300.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0 0 0	0.000000	180.000000	0.000000	4500.000000	0.000000	0.000000
4	0 0 0	300.000000	180.000000	0.000000	0.000000	0.000000	0.000000
5	0 0 0	0.000000	360.000000	0.000000	2250.000000	0.000000	67500.000000
6	0 0 0	300.000000	360.000000	0.000000	0.000000	0.000000	0.000000

ELEMENTS

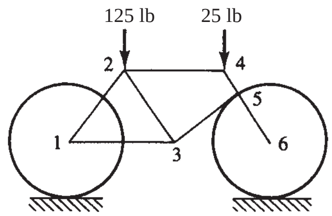
K	NODE(I, K)		E(K)	G(K)	A(K)	XI(K)
1	1	3	3.0000000E+07	0.0000000E+00	1.5000000E+01	2.0000000E+02
2	3	5	3.0000000E+07	0.0000000E+00	1.5000000E+01	2.0000000E+02
3	2	4	3.0000000E+07	0.0000000E+00	1.5000000E+01	2.0000000E+02
4	4	6	3.0000000E+07	0.0000000E+00	1.5000000E+01	2.0000000E+02
5	3	4	3.0000000E+07	0.0000000E+00	1.5000000E+01	2.0000000E+02
6	5	6	3.0000000E+07	0.0000000E+00	1.5000000E+01	2.0000000E+02

NODE	DISPLACEMENTS		Z-ROTATION
	X(in.)	Y(in.)	THETA
1	0.00000E+00	0.00000E+00	– 0.15591E–01
2	0.00000E+00	0.00000E+00	– 0.15054E–01
3	0.21212E+01	0.21601E–02	– 0.51828E–02
4	0.21193E+01	– 0.21601E–02	– 0.52135E–02
5	0.28221E+01	0.26614E–02	– 0.13916E–02
6	0.28215E+01	– 0.26614E–02	– 0.17770E–02

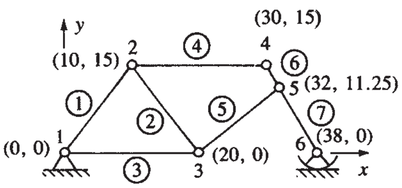
ELEMENTS

K	NODE (I, K)		X-FORCE (lb)	Y-FORCE (lb)	Z-MOMENT (lb · in.)	X-FORCE (lb)	Y-FORCE (lb)	Z-MOMENT (lb · in.)
1	1	3	– 0.5400E+04	0.3105E+04	– 0.6750E+05	0.5400E+04	– 0.3105E+04	0.6264E+06
2	3	5	– 0.1253E+04	0.1349E+04	– 0.4968E+04	0.1253E+04	– 0.1349E+04	0.2478E+06
3	2	4	0.5400E+04	0.3645E+04	0.8592E–01	– 0.5400E+04	– 0.3645E+04	0.6561E+06
4	4	6	0.1253E+04	0.9017E+03	– 0.3340E+05	– 0.1253E+04	– 0.9017E+03	0.1957E+06
5	3	4	0.2744E+04	– 0.4147E+04	– 0.6214E+06	– 0.2744E+04	0.4147E+04	– 0.6227E+06
6	5	6	0.9014E+03	– 0.1253E+04	– 0.1803E+06	– 0.9014E+03	0.1253E+04	– 0.1957E+06

5.25 (a)



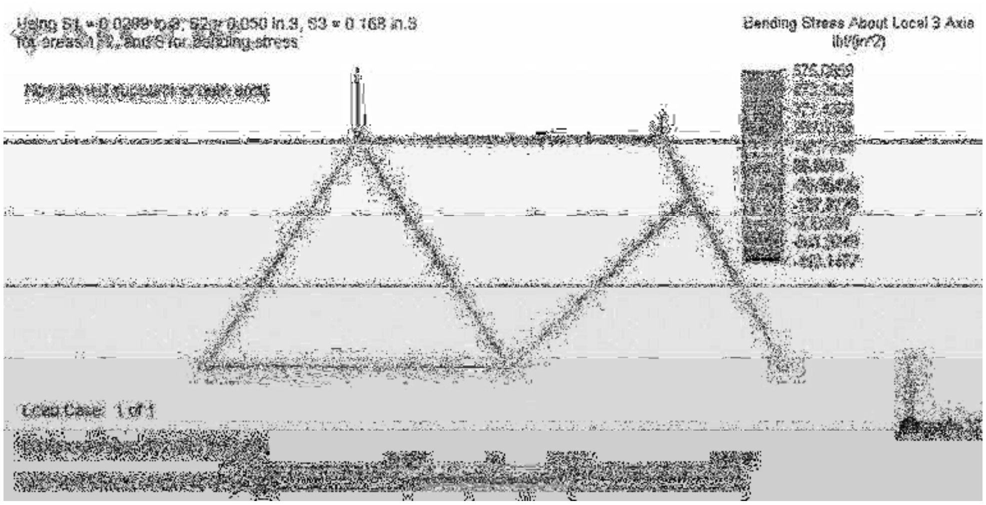
(a) Design 1

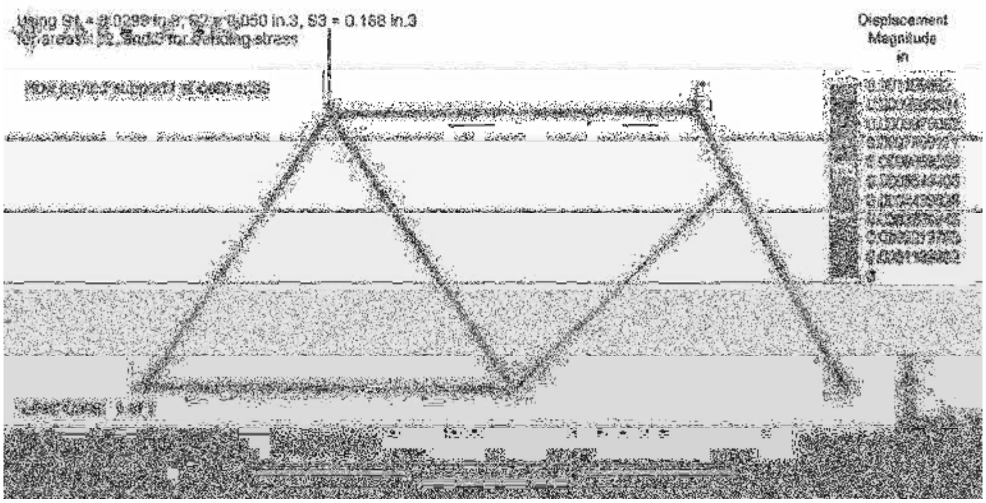


(a) Design 1

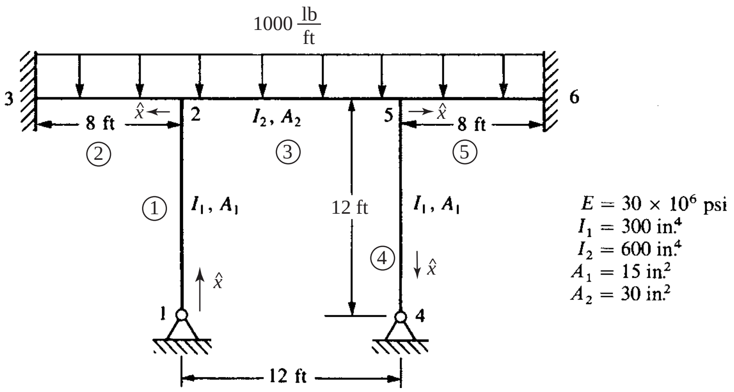
Case 1
$E = 30 \times 10^6$ psi
Case 2
$E = 10 \times 10^6$ psi

$A_1 = 0.1\text{ in}^2$
$A_2 = A_3 = A_4 = A_5 = 0.15\text{ in}^2$
$A_6 = A_7 = A_8 = 0.3\text{ in}^2$
$I_1 = 0.01\text{ in}^4$
$I_2 = I_3 = I_4 = I_5 = 0.02\text{ in}^4$
$I_6 = I_7 = I_8 = 0.1\text{ in}^4$

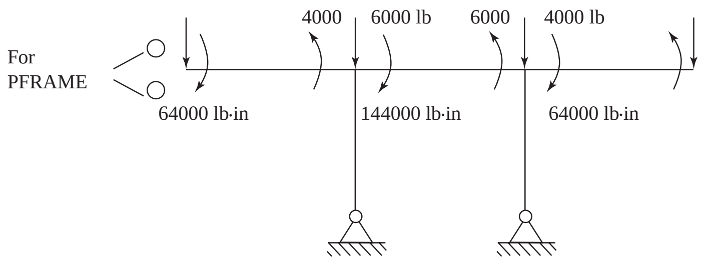




5.26



Solution: From appendix D for distributed load



NUMBER OF ELEMENTS = 5

NUMBER OF NODES = 6

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1, K)
1	1 1 0	–72.000000	0.000000	0.000000	0.000000
2	0 0 0	–72.000000	144.000000	0.000000	0.000000
3	1 1 1	–168.00000	144.000000	0.000000	0.000000
4	1 1 0	72.000000	0.000000	0.000000	0.000000
5	0 0 0	72.000000	144.000000	0.000000	0.000000
6	1 1 1	168.00000	144.000000	0.000000	0.000000

FORCE(2, K)	FORCE(3, K)
0.000000	0.000000
–10000.000000	–80000.000000
0.000000	0.000000
0.000000	0.000000
–10000.000000	80000.000000
0.0000000	0.000000

ELEMENTS

K	NODE(I, K)		E(K)	G(K)	A(K)	XI(K)
1	1	2	3.0000000E+07	0.0000000E+00	1.50000000E+01	3.0000000E+02
2	2	3	3.0000000E+07	0.0000000E+00	3.00000000E+01	6.0000000E+02
3	2	5	3.0000000E+07	0.0000000E+00	3.00000000E+01	6.0000000E+02
4	5	4	3.0000000E+07	0.0000000E+00	1.50000000E+01	3.0000000E+02
5	5	6	3.0000000E+07	0.0000000E+00	3.00000000E+01	6.0000000E+02

NODE	DISPLACEMENTS X	Y	Z-ROTATION THETA
1	0.00000E+00	0.00000E+00	0.49989E–04
2	0.59560E–05	– 0.33163E–02	– 0.10010E–03
3	0.00000E+00	0.00000E+00	0.00000E+00
4	0.00000E+00	0.00000E+00	– 0.49989E–04
5	– 0.59560E–05	– 0.33163E–02	0.10010E–03
6	0.00000E+00	0.00000E+00	0.00000E+00

ELEMENTS

K	NODE(I, K)		X-FORCE	Y- FORCE	Z-MOMENT	X-FORCE	Y-FORCE
1	1	2	0.1036E+05	– 0.1303E+03	– 0.7533E-04	– 0.1036E+05	0.1303E+03
2	2	3	– 0.5584E+02	– 0.3634E+03	– 0.3621E+05	0.5584E+02	0.3634E+03
3	2	5	0.7445E+02	0.1703E–04	– 0.2503E+05	– 0.7445E+02	– 0.1703E–04
4	5	4	0.1036E+05	0.1303E+03	0.1876E+05	– 0.1036E+05	– 0.1303E+03
5	5	6	– 0.5584E+02	0.3634E+03	0.3621E+05	0.5584E+02	– 0.3634E+03

		K	Z-MOMENT
1	1	2	– 1876E+05
2	2	3	.1325E+04
3	2	5	.2503E+05

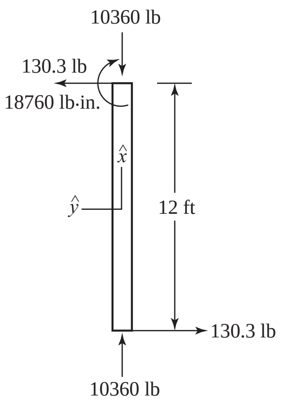
4	5	4	– 1137E+02
5	5	6	– 1325E+04

Refer to computer print out for displacement and rotations

Corrected elemental forces

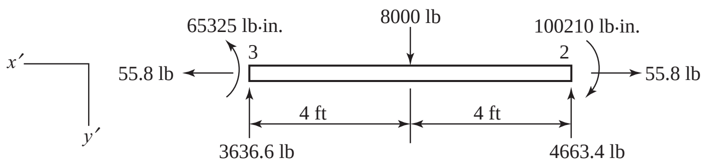
Element (1)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{1y}^{(1)} \\ m_1^{(1)} \\ f_{2x}^{(1)} \\ f_{2y}^{(1)} \\ m_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} 10360\text{ lb} \\ -130.3\text{ lb} \\ 0\text{ lb}\cdot\text{in.} \\ -10360\text{ lb} \\ 130.3\text{ lb} \\ -18760\text{ lb}\cdot\text{in.} \end{Bmatrix}$$



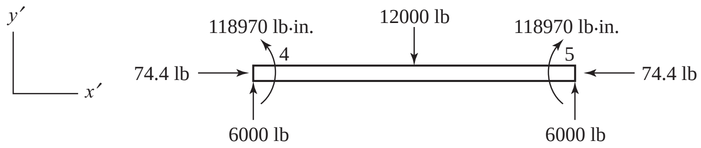
Element (2)

$$\begin{Bmatrix} f'_{2x} \\ f'_{2y} \\ m_2 \\ f'_{3x} \\ f'_{3y} \\ m_3 \end{Bmatrix} = \begin{Bmatrix} -55.8\text{ lb} \\ -363.4\text{ lb} \\ -36210\text{ lb}\cdot\text{in.} \\ 55.8\text{ lb} \\ 363.4\text{ lb} \\ 1325\text{ lb}\cdot\text{in.} \end{Bmatrix} - \begin{Bmatrix} 0 \\ +4000 \\ +64{,}000 \\ 0 \\ +4000 \\ -64{,}000 \end{Bmatrix} = \begin{Bmatrix} -55.8\text{ lb} \\ -4363.4\text{ lb} \\ -100{,}210\text{ lb}\cdot\text{in.} \\ 55.8\text{lb} \\ -3636.6\text{ lb} \\ 65325\text{ lb}\cdot\text{in.} \end{Bmatrix}$$



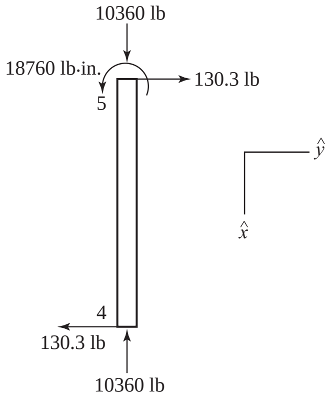
Element (3)

$$\begin{Bmatrix} f'^{(3)}_{2x} \\ f'^{(3)}_{2y} \\ m_2^{(3)} \\ f'^{(3)}_{5x} \\ f'^{(2)}_{5y} \\ m_5^{(2)} \end{Bmatrix} = \begin{Bmatrix} 74.4\text{ lb} \\ 0 \\ -25030\text{ lb} \\ -74.4\text{ lb} \\ 0 \\ 25030\text{ lb}\cdot\text{in.} \end{Bmatrix} - \begin{Bmatrix} 0 \\ -6000 \\ -144{,}000 \\ 0 \\ -6000 \\ 144{,}000 \end{Bmatrix} = \begin{Bmatrix} 74.4\text{ lb} \\ 6000\text{ lb} \\ -118870\text{ lb}\cdot\text{in.} \\ -74.4\text{ lb} \\ 6000\text{ lb} \\ -118870\text{ lb}\cdot\text{in.} \end{Bmatrix}$$



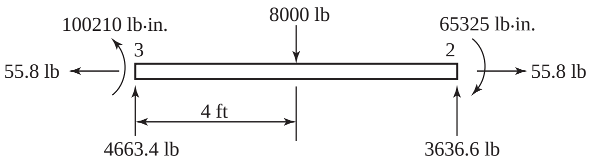
Element (4) No correction refer to computer print out

Element (4)



Element (5)

$$\begin{Bmatrix} f'^{(5)}_{5x} \\ f'^{(5)}_{5y} \\ m^{(5)}_5 \\ f'^{(5)}_{6x} \\ f'^{(5)}_{6y} \\ m^{(5)}_6 \end{Bmatrix} = \begin{Bmatrix} -55.81 \text{ lb} \\ -363.4 \text{ lb} \\ 36210 \text{ lb} \cdot \text{in.} \\ 55.8 \text{ lb} \\ -363.4 \text{ lb} \\ -1325 \text{ lb} \cdot \text{in.} \end{Bmatrix} - \begin{Bmatrix} 0 \\ -4000 \\ -64{,}000 \\ 0 \\ -4000 \\ 64{,}000 \end{Bmatrix} = \begin{Bmatrix} -55.81 \text{ lb} \\ 4363.4 \text{ lb} \\ 100{,}210 \text{ lb} \cdot \text{in.} \\ 55.8 \text{ lb} \\ 3636.6 \text{ lb} \\ -65325 \text{ lb} \cdot \text{in.} \end{Bmatrix}$$



Reactions

NODE 1

$$F_{1x} = f'^{(1)}_{1y} = +130.3 \text{ lb}$$

$$F_{1y} = f'^{(1)}_{1x} = +10360 \text{ lb}$$

$$M_1 = -m^{(1)}_1 = 0 \text{ lb} \cdot \text{in.}$$

NODE 3

$$F_{3x} = f'^{(2)}_{3x} = 55.8 \text{ lb}$$

$$F_{3y} = f'^{(2)}_{3y} = -3636.6 \text{ lb}$$

$$M_3 = -m^{(2)}_3 = -65235 \text{ lb} \cdot \text{in.}$$

NODE 4

$$F_{4x} = -\ f'^{(4)}_{4y} = 130.3 \text{ lb}$$

$$F_{4y} = \ f'^{(4)}_{4x} = -\ 10360 \text{ lb}$$

$$M_4 = -\ m_4^{(4)} = 0 \text{ lb} \cdot \text{in.}$$

NODE 6

$$F_{6x} = -\ f'^{(5)}_{6x} = -\ 55.8 \text{ lb}$$

$$F_{6y} = -\ f'^{(5)}_{6y} = -\ 3636.6 \text{ lb}$$

$$M_6 = -\ m_6^{(5)} = 65325 \text{ lb} \cdot \text{in.}$$

5.28

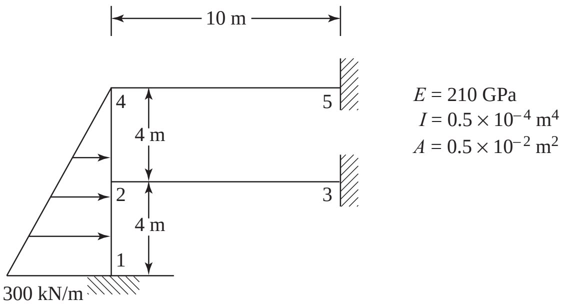
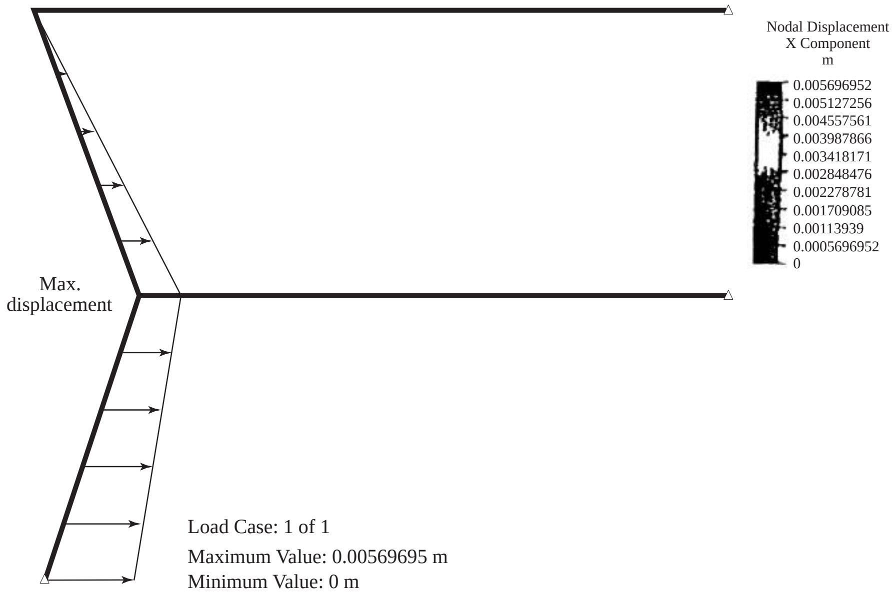


Figure P5-28



5.29

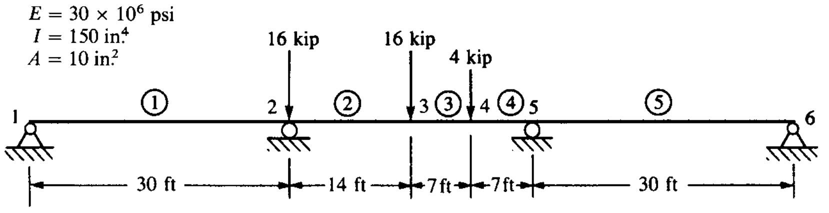
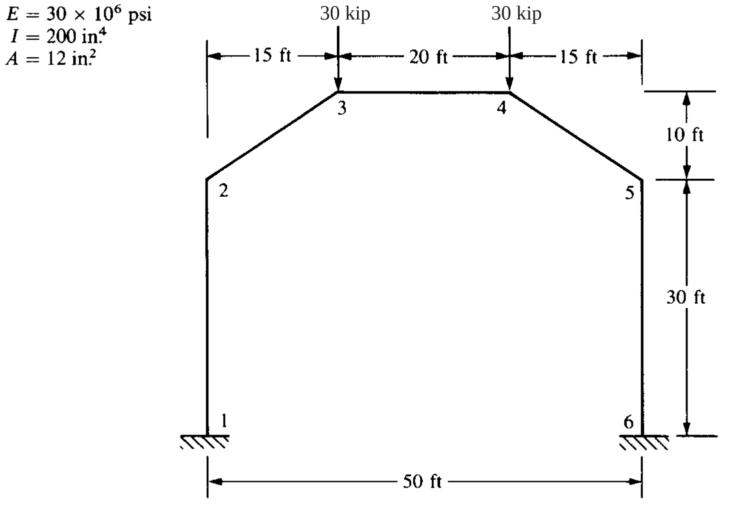


Figure P5–29

Displacements/Rotations (degrees) of nodes

Node number	X– translation	Y– translation	Z– translation	X– rotation	Y– rotation	Z– rotation
1	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	3.4030E–01
2	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	–6.8060E–01
3	0.0000E+00	–1.8330E+00	0.0000E+00	0.0000E+00	0.0000E+00	–3.7774E–02
4	0.0000E+00	–1.2242E+00	0.0000E+00	0.0000E+00	0.0000E+00	7.6168E–01
5	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	7.4186E–01
6	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	–3.7093E–01
7	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
8	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00

5.30



Plane Frame-Problem 5.30

NUMBER OF ELEMENTS = 5
NUMBER OF NODES = 6

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1, K)	FORCE(2, K)	FORCE(3, K)
1	1 1 1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0 0 0	0.000000	360.000000	0.000000	0.000000	0.000000	0.000000
3	0 0 0	180.000000	480.000000	0.000000	0.000000	–30000.000000	0.000000
4	0 0 0	420.000000	480.000000	0.000000	0.000000	–30000.000000	0.000000
5	0 0 0	600.000000	360.000000	0.000000	0.000000	0.000000	0.000000
6	1 1 1	600.000000	0.000000	0.000000	0.000000	0.000000	0.000000

Elements

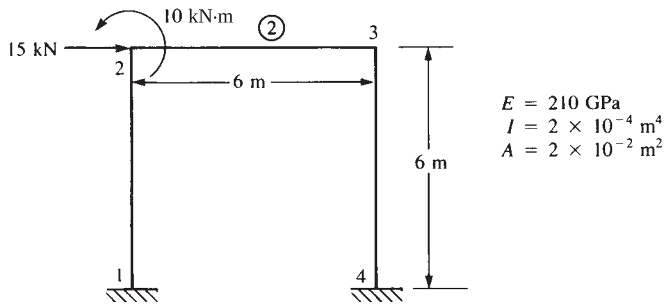
K	NODE (I, K)		E(K)	G(K)	A(K)	XI(K)	XJ(K)
1	1	2	3.0000000E+07	0.0000000E+00	1.2000000E+01	2.0000000E+02	0.0000000E+00
2	2	3	3.0000000E+07	0.0000000E+00	1.2000000E+01	2.0000000E+02	0.0000000E+00
3	3	4	3.0000000E+07	0.0000000E+00	1.2000000E+01	2.0000000E+02	0.0000000E+00
4	4	5	3.0000000E+07	0.0000000E+00	1.2000000E+01	2.0000000E+02	0.0000000E+00
5	5	6	3.0000000E+07	0.0000000E+00	1.2000000E+01	2.0000000E+02	0.0000000E+00

NODE	DISPLACEMENTS X	Y	Z-ROTATION THETA
1	0.00000E+00	0.00000E+00	0.00000E+0
2	– 0.44021E+01	– 0.30000E–01	– 0.17622E–0
3	0.33879E–02	– 0.66668E+01	– 0.32000E–0
4	– 0.44038E–02	– 0.66669E+01	0.32000E–0
5	0.44011E+01	– 0.30000E–01	0.17624E–0
6	0.00000E+00	0.00000E+00	0.00000E+0

ELEMENTS

K	NODE (I,K)		X–FORCE	Y–FORCE	Z– MOMENT	X–FORCE	Y–FORCE	Z–MOMENT
1	1	2	0.3000E+05	– 0.1169E+05	– 0.1810E+07	– 0.3000E+05	0.1169E+05	– 0.2398E+07
2	2	3	0.2637E+05	0.1848E+05	0.2398E+07	– 0.2637E+05	– 0.1848E+05	0.1600E+07
3	3	4	0.1169E+05	0.1738E+00	– 0.1600E+07	– 0.1169E+05	– 0.1738E+00	0.1600E+07
4	4	5	0.2637E+05	– 0.1848E+05	– 0.1600E+07	– 0.2637E+05	0.1848E+05	– 0.2397E+07
5	5	6	0.3000E+05	0.1169E+05	0.2397E+07	– 0.3000E+05	– 0.1169E+05	0.1810E+07

5.32



Problem 5.32

NUMBER OF ELEMENTS = 3

NUMBER OF NODES = 4

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
1	1 1 1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0 0 0	0.000000	6.000000	0.000000	15000.000000	0.000000	10000.000000
3	0 0 0	6.000000	6.000000	0.000000	0.000000	0.000000	0.000000
4	1 1 1	6.000000	0.000000	0.000000	0.000000	0.000000	0.000000

ELEMENTS

K	NODE(I,K)		E(K)	G(K)	A(K)	XI(K)
1	1	2	2.1000000E+11	1.0000000E+00	2.0000000E-02	2.0000000E-04
2	2	3	2.1000000E+11	1.0000000E+00	2.0000000E-02	2.0000000E-04
3	3	4	2.1000000E+11	1.0000000E+00	2.0000000E-02	2.0000000E-04

NODE	DISPLACEMENTS X	Y	Z-ROTATION THETA
1	0.00000E+00	0.00000E+00	0.00000E+00
2	0.42966E–02	0.71361E–05	– 0.24093E–03
3	0.42871E–02	– 0.71361E–05	– 0.47744E–03
4	0.00000E+00	0.00000E+00	0.00000E+00

ELEMENTS

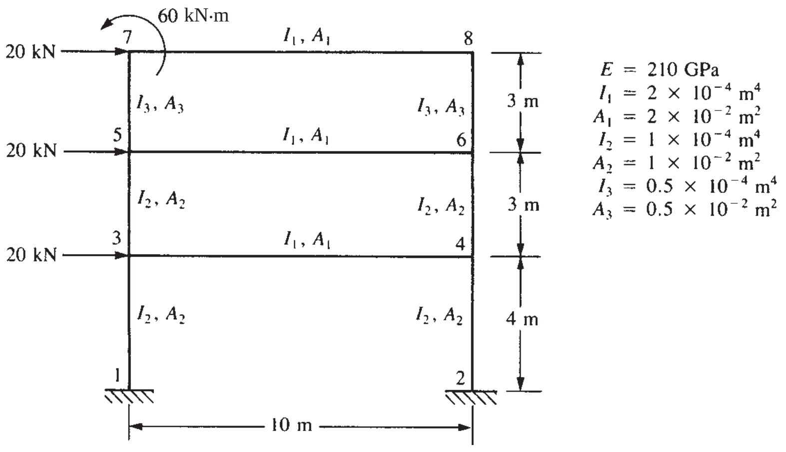
K	NODE (I,K)		X-FORCE	Y-FORCE	Z- MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	1	2	– 0.4995E+04	0.8339E+04	0.2620E+05	0.4995E+04	– 0.8339E+04	0.2333E+05
2	2	3	0.6661E+04	– 0.4995E+04	– 0.1333E+05	– 0.6661E+04	0.4995E+04	– 0.1664E+05
3	3	4	0.4995E+04	0.6661E+04	0.1664E+05	– 0.4999E+04	– 0.6661E+04	0.2333E+05

Reactions

$$F_{1x} = -8339 \text{ N}, F_{1y} = -4995 \text{ N}, M_1 = 26{,}700 \text{ N}\cdot\text{m}$$

$$F_{4x} = -6661 \text{ N}, F_{4y} = 4995 \text{ N}, M_4 = 23{,}330 \text{ N}\cdot\text{m}$$

5.33



***** Frame Problem 5.33 ******
NUMBER OF ELEMENTS = 9
NUMBER OF NODES = 8

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
1	1 1 1	–5.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	1 1 1	5.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0 0 0	–5.000000	4.000000	0.000000	20.000000	0.000000	0.000000
4	0 0 0	5.000000	4.000000	0.000000	0.000000	0.000000	0.000000
5	0 0 0	–5.000000	7.000000	0.000000	20.000000	0.000000	0.000000
6	0 0 0	5.000000	7.000000	0.000000	0.000000	0.000000	0.000000
7	0 0 0	–5.000000	10.000000	0.000000	20.000000	0.000000	60.000000
8	0 0 0	5.000000	10.000000	0.000000	0.0000000	0.000000	0.000000

ELEMENTS

K	NODE (I,K)		E(K)	G(K)	A(K)	XI(K)	XJ(K)
1	1	3	2.1000000E+06	0.0000000E+00	9.9999998E–03	9. 9999997E–03	0.0000000E+00
2	2	4	2.1000000E+06	0.0000000E+00	9.9999998E–03	9. 9999997E–03	0.0000000E+00
3	3	4	2.1000000E+06	0.0000000E+00	2.0000000E–02	1. 9999999E–04	0.0000000E+00
4	3	5	2.1000000E+06	0.0000000E+00	9.9999998E–03	9. 9999997E–03	0.0000000E+00

5	4	4	2.1000000E+06	0.0000000E+00	9.9999998E–03	9. 9999997E–03	0.0000000E+00
6	5	6	2.1000000E+06	0.0000000E+00	2.0000000E–02	1. 9999999E–04	0.0000000E+00
7	5	7	2.1000000E+06	0.0000000E+00	4.9999999E–03	4. 9999999E–03	0.0000000E+00
8	6	8	2.1000000E+06	0.0000000E+00	4.9999999E–03	4. 9999999E–03	0.0000000E+00
9	7	8	2.1000000E+06	0.0000000E+00	2.0000000E–02	1. 9999999E–04	0.0000000E+00

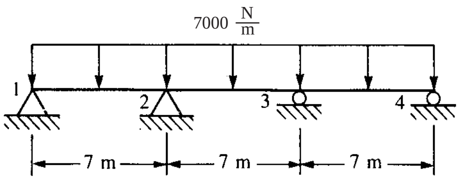
NODE	DISPLACEMENTS X	Y	Z-ROTATION THETA
1	0.00000E+00	0.00000E+00	0.00000E+00
2	0.00000E+00	0.00000E+00	0.00000E+00
3	0.13109E–01	0.40236E–04	–0.26944E–02
4	0.13092E–01	–0.40236E–04	–0.27869E–02
5	0.22043E–01	0.50737E–04	–0.19861E–02
6	0.21994E–01	–0.50737E–04	–0.15658E–02
7	0.26380E–01	0.46314E–04	0.17098E–02
8	0.26376E–01	–0.46314E–04	–0.11140E–02

ELEMENTS

K	NODE(I,K)		X-FORCE	Y-FORCE	Z-MOMENT
1	1	3	–0.2112E+02	0.3040E+02	0.7493E+02
2	2	4	0.2112E+02	0.2960E+02	0.7383E+02
3	3	4	0.7451E+01	–0.1377E+02	–0.4847E+02
4	3	5	–0.7351E+01	0.1785E+02	0.2182E+02
5	4	6	0.7351E+01	0.2215E+02	0.2468E+02
6	5	6	0.2046E+02	–0.8899E+01	–0.4626E+02
7	5	7	0.1548E+01	0.1831E+02	0.1453E+02
8	6	8	–0.1548E+01	0.1693E+01	0.9578E+00
9	7	8	0.1691E+01	0.1548E+01	0.1960E+02

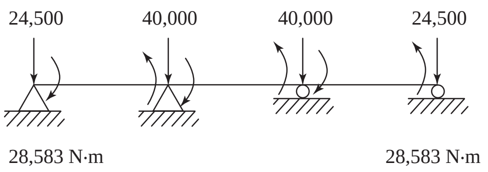
X-FORCE	Y-FORCE	Z-MOMENT
0.2112E+02	0.3040E+02	0.4665E+02
–0.2112E+02	–0.2960E+02	0.4457E+02
–0.7451E+01	0.1377E+02	–0.6925E+02
0.7351E+01	–0.1785E+02	0.3173E+02
–0.7351E+01	–0.2215E+02	0.4177E+02
–0.2046E+02	0.8899E+01	–0.4273E+02
–0.1548E+01	–0.1831E+02	0.4040E+02
0.1548E+01	–0.1693E+01	0.4120E+01
–0.1691E+01	–0.1548E+01	–0.4120E+01

5.34



$E = 210$ GPa
$I = 1 \times 10^{-4}$ m^4
$A = 1 \times 10^{-2}$ m^2

Solution: From appendix D for distributed load



NUMBER OF ELEMENTS = 3
NUMBER OF NODES = 4

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
1	1 1 0	0.000000	0.000000	0.000000	0.000000	0.000000	28583.000000
2	1 1 0	7.000000	0.000000	0.000000	0.000000	0.000000	0.000000

3	0 1 0	14.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4	0 1 0	21.000000	0.000000	0.000000	0.000000	0.000000	–28583.000000

ELEMENTS

K	NODE(I,K)		E(K)	G(K)	A(K)	XI(K)
1	1	2	2.1000000E+11	1.0000000E+00	1.0000000E–02	1.0000000E–04
2	2	3	2.1000000E+11	1.0000000E+00	1.0000000E–02	1.0000000E–04
3	3	4	2.1000000E+11	1.0000000E+00	1.0000000E–02	1.0000000E–04

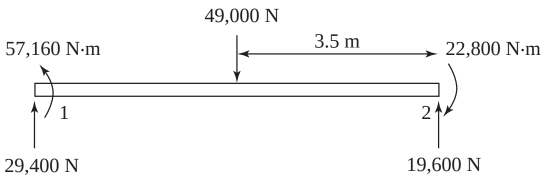
NODE	DISPLACEMENTS X	DISPLACEMENTS Y	Z-ROTATION THETA
1	0.00000E+00	0.00000E+00	0.28583E–02
2	0.00000E+00	0.00000E+00	– 0.95277E–03
3	0.00000E+00	0.00000E+00	0.95277E–03
4	0.00000E+00	0.00000E+00	– 0.28583E–02

ELEMENTS

K	NODE(I,K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	1	2	0.0000E+00	0.4900E+04	0.2858E+05	0.0000E+00	– 0.4900E+04	0.5717E+04
2	2	3	0.0000E+00	0.1201E–03	–0.5717E+04	0.0000E+00	– 0.1201E–03	0.5717E+04
3	3	4	0.0000E+00	– 0.4900E+04	–0.5717E+04	0.0000E+00	0.4900E+04	0.2858E+05

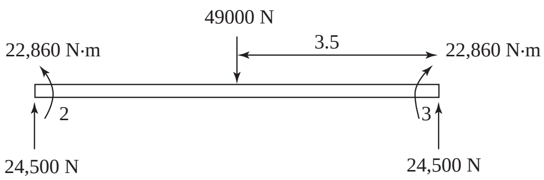
Element (1)

$$\begin{Bmatrix} f'^{(1)}_{1x} \\ f'^{(1)}_{1y} \\ m_1 \\ f'_{2x} \\ f'_{2y} \\ m_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 4900 \\ 28580 \\ 0 \\ -4900 \\ 5717 \end{Bmatrix} - \begin{Bmatrix} 0 \\ -24500 \\ -28580 \\ 0 \\ -24500 \\ 28580 \end{Bmatrix} = \begin{Bmatrix} 0\text{ N} \\ 29400\text{ N} \\ 57160\text{ N}\cdot\text{m} \\ 0\text{ N} \\ 19600\text{ N} \\ -22860\text{ N}\cdot\text{m} \end{Bmatrix}$$



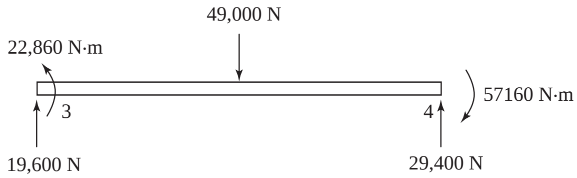
Element (2)

$$\begin{Bmatrix} f'^{(2)}_{2x} \\ f'^{(2)}_{2y} \\ m_2^{(2)} \\ f'^{(2)}_{3x} \\ f'^{(2)}_{3y} \\ m_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -5717 \\ 0 \\ 0 \\ 5717 \end{Bmatrix} - \begin{Bmatrix} 0 \\ -24500 \\ -28580 \\ 0 \\ -24500 \\ 28580 \end{Bmatrix} = \begin{Bmatrix} 0\text{ N} \\ 24500\text{ N} \\ 22{,}860\text{ N}\cdot\text{m} \\ 0 \\ 24500\text{ N} \\ -22860\text{ N}\cdot\text{m} \end{Bmatrix}$$



Element (3)

$$\begin{Bmatrix} f'^{(3)}_{3x} \\ f'^{(3)}_{3y} \\ m_3^{(3)} \\ f'^{(3)}_{4x} \\ f'^{(3)}_{4y} \\ m_4^{(3)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -4900 \\ -5717 \\ 0 \\ 4900 \\ -28580 \end{Bmatrix} - \begin{Bmatrix} 0 \\ -24500 \\ -28580 \\ 0 \\ -24500 \\ 28580 \end{Bmatrix} = \begin{Bmatrix} 0\text{ N} \\ 19600\text{ N} \\ 22860\text{ N}\cdot\text{m} \\ 0\text{ N} \\ 29400\text{ N} \\ -57160\text{ N}\cdot\text{m} \end{Bmatrix}$$



Reactions

Node 1

$$F_{1x} = -f'^{(1)}_{1x} = 0\text{ N}$$

$$F_{1y} = -f'^{(1)}_{1y} = -29400\text{ N}$$

$$M_1 = -m_1^{(1)} = 0\text{ N}\cdot\text{m}$$

Node 3

$$F_{3x} = -f'^{(2)}_{3x} = 0\text{ N}$$

$$F_{3y} = -\left(f'^{(2)}_{3y} + f'^{(1)}_{3y}\right) = -44100\text{ N}$$

$$M_3 = 0\text{ N}\cdot\text{m}$$

Node 2

$$F_{2x} = -f'^{(2)}_{2x} = 0\text{ N}$$

$$F_{2y} = -\left(f'^{(1)}_{2y} + f'^{(2)}_{2y}\right) = -44100\text{ N}$$

$$M_2 = 0\text{ N}\cdot\text{m}$$

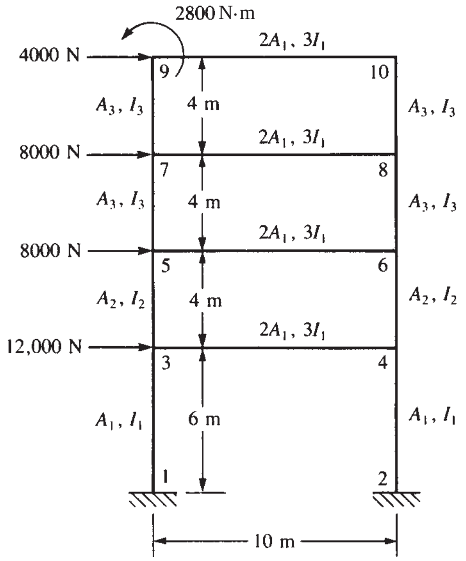
Node 4

$$F_{4x} = -f'^{(3)}_{3x} = 0\text{ N}$$

$$F_{4y} = -f'^{(3)}_{4y} = -29400\text{ N}$$

$$M_4 = -m_4^{(3)} = 0\text{ N}\cdot\text{m}$$

5.35



$E = 210$ GPa
$A_1 = 2.0 \times 10^{-2}$ m^2
$I_1 = 2.0 \times 10^{-4}$ m^4
$A_2 = 1.5 \times 10^{-2}$ m^2
$I_2 = 1.5 \times 10^{-4}$ m^4
$A_3 = 1.0 \times 10^{-2}$ m^2
$I_3 = 1.0 \times 10^{-4}$ m^4

NUMBER OF ELEMENTS = 12
NUMBER OF NODES = 10

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)
1	1 1 1	0.000000	0.000000	0.000000
2	1 1 1	10.000000	0.000000	0.000000
3	0 0 0	0.000000	6.000000	0.000000
4	0 0 0	10.000000	6.000000	0.000000
5	0 0 0	0.000000	10.000000	0.000000
6	0 0 0	10.000000	10.000000	0.000000
7	0 0 0	0.000000	14.000000	0.000000
8	0 0 0	10.000000	14.000000	0.000000
9	0 0 0	0.000000	18.000000	0.000000
10	0 0 0	10.000000	18.000000	0.000000

FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
0.000000	0.000000	0.000000
0.000000	0.000000	0.000000
12000.000000	0.000000	0.000000
0.000000	0.000000	0.000000
8000.000000	0.000000	0.000000
0.000000	0.000000	0.000000
8000.000000	0.000000	0.000000
0.000000	0.000000	0.000000
4000.000000	0.000000	2800.000000
0.000000	0.000000	0.000000

ELEMENTS

K	NODE(1,K)		E(K)	G(K)	A(K)	XI(K)
1	1	3	2.1000000E+11	0.00000000E+00	2.0000000E–02	2.0000000E–04
2	3	5	2.1000000E+11	0.00000000E+00	1.5000000E–02	1.5000000E–04
3	5	7	2.1000000E+11	0.00000000E+00	1.0000000E–02	1.0000000E–04
4	7	9	2.1000000E+11	0.00000000E+00	1.0000000E–02	1.0000000E–04
5	2	4	2.1000000E+11	0.00000000E+00	2.0000000E–02	2.0000000E–04
6	4	6	2.1000000E+11	0.00000000E+00	1.5000000E–02	1.5000000E–04
7	6	8	2.1000000E+11	0.00000000E+00	1.0000000E–02	1.0000000E–04
8	8	10	2.1000000E+11	0.00000000E+00	1.0000000E–02	1.0000000E–04
9	3	4	2.1000000E+11	0.00000000E+00	4.0000000E–02	6.0000000E–04
10	5	6	2.1000000E+11	0.00000000E+00	4.0000000E–02	6.0000000E–04
11	7	8	2.1000000E+11	0.00000000E+00	4.0000000E–02	6.0000000E–04
12	9	10	2.1000000E+11	0.00000000E+00	4.0000000E–02	6.0000000E–04

NODE	DISPLACEMENTS X	Y	Z-ROTATION THETA
1	0.00000E+00	0.00000E+00	0.00000E+00
2	0.00000E+00	0.00000E+00	0.00000E+00
3	0.92499E–02	0.32295E–04	– 0.79668E–03
4	0.92428E–02	– 0.32295E–04	– 0.79610E–03
5	0.13440E–01	0.45836E–04	– 0.45169E–03
6	0.13435E–01	– 0.45836E–04	– 0.45347E–03
7	0.16322E–01	0.53376E–04	– 0.23126E–03
8	0.16317E–01	– 0.53376E–04	– 0.22138E–03
9	0.17395E–01	0.54706E–04	– 0.25437E–04
10	0.17393E–01	– 0.54706E–04	– 0.88782E–04

ELEMENTS

K	NODE (I,K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	1	3	– 0.2261E+05	0.1601E+05	0.5360E+05	0.2261E+05	– 0.1601E+05	0.4244E+05
2	3	5	– 0.1066E+05	0.1000E+05	0.1728E+05	0.1066E+05	– 0.1000E+05	0.2272E+05
3	5	7	– 0.3959E+04	0.5969E+04	0.1078E+05	0.3959E+04	– 0.5969E+04	0.1310E+05
4	7	9	– 0.6981E+03	0.2206E+04	0.3331E+04	0.6981E+03	– 0.2206E+04	0.5492E+04
5	2	4	0.2261E+05	0.1599E+05	0.5355E+05	– 0.2261E+05	– 0.1599E+05	0.4241E+05
6	4	6	0.1066E+05	0.1000E+05	0.1730E+05	– 0.1066E+05	– 0.1000E+05	0.2270E+05
7	6	8	0.3959E+04	0.6032E+04	0.1085E+05	– 0.3959E+04	– 0.6032E+04	0.1328E+05
8	8	10	0.6981E+03	0.1796E+04	0.2896E+04	– 0.6981E+03	– 0.1796E+04	0.4288E+04
9	3	4	0.5993E+04	– 0.1194E+05	– 0.5973E+05	– 0.5993E+04	0.1194E+05	– 0.5971E+05
10	5	6	0.3968E+04	– 0.6704E+04	– 0.3350E+05	– 0.3968E+04	0.6704E+04	– 0.3354E+05
11	7	8	0.4237E+04	– 0.3261E+04	– 0.1643E+05	– 0.4237E+04	0.3261E+04	– 0.1618E+05
12	9	10	0.1799E+04	– 0.6981E+03	– 0.2692E+04	–0.1795E+04	0.6981E+03	– 0.4288E+04

5.36

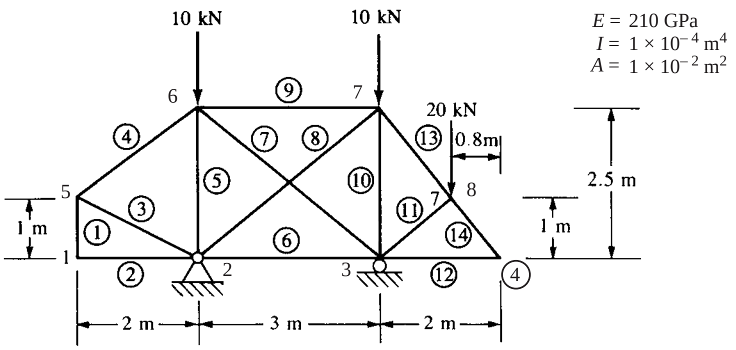
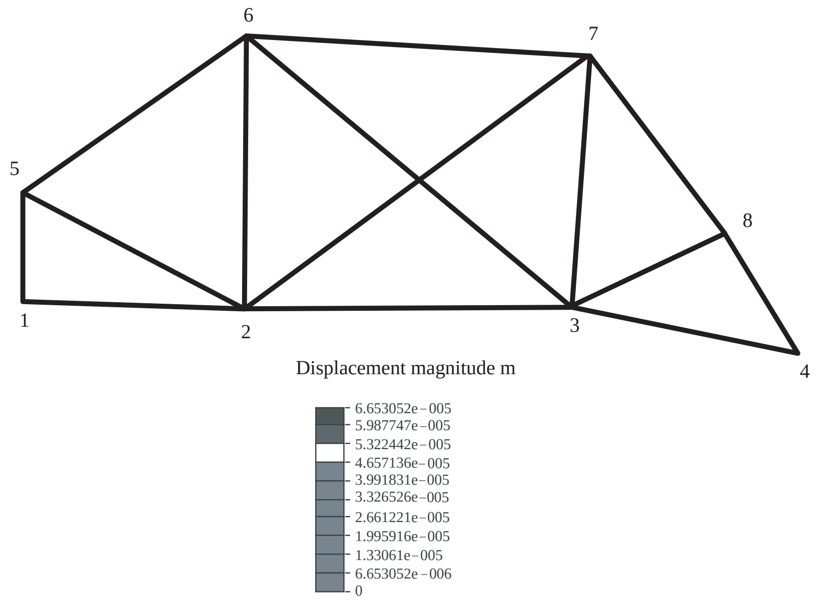


Figure P5–36

Displacements/Rotations (degrees) of nodes

NODE number	X-translation	Y-translation	Z-translation	X-rotation	Y-rotation	Z-rotation
1	–1.5416E–07	9.7204E–06	0.0000E+00	0.0000E+00	0.0000E+00	–3.2761E–04
2	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
3	–2.9451E–06	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	–1.2720E–03
4	–3.0149E–06	–6.6462E–05	0.0000E+00	0.0000E+00	0.0000E+00	–1.7954E–03
5	1.6304E–05	–4.5862E–06	0.0000E+00	0.0000E+00	0.0000E+00	–3.1306E–04
6	5.2760E–06	9.7805E–06	0.0000E+00	0.0000E+00	0.0000E+00	–3.6825E–04
7	2.7047E–05	–2.6136E–05	0.0000E+00	0.0000E+00	0.0000E+00	–3.4281E–04
8	2.2989E–05	–4.5872E–05	0.0000E+00	0.0000E+00	0.0000E+00	–1.4049E–03



5.37

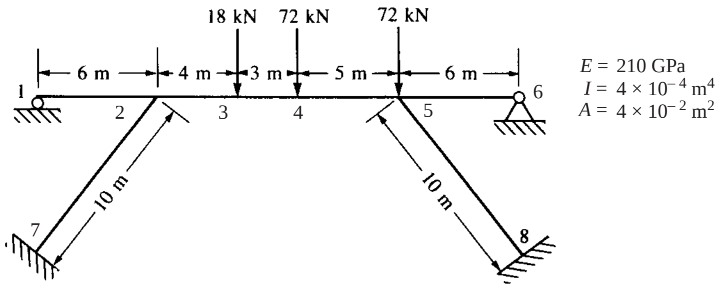
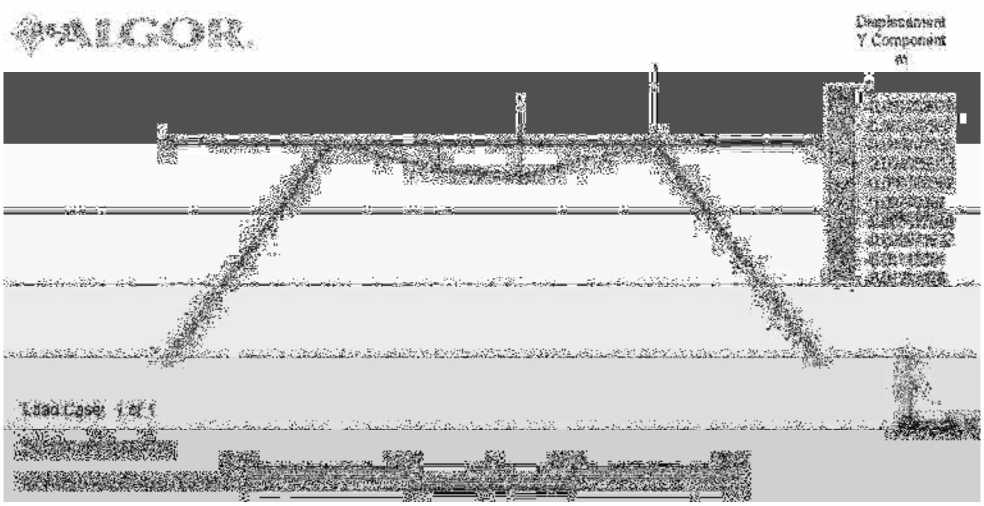


Figure P5–37

Displacements/Rotations (degrees) of nodes

NODE number	X-translation	Y-translation	Z-translation	X-rotation	Y-rotation	Z-rotation (deg)
1	1.1203E–05	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	4.0957E–02
2	1.1203E–05	–1.1000E–04	0.0000E+00	0.0000E+00	0.0000E+00	–8.5066E–02
3	–3.5606E–06	–1.0371E–02	0.0000E+00	0.0000E+00	0.0000E+00	–1.3515E–01
4	–1.4633E–05	–1.2879E–02	0.0000E+00	0.0000E+00	0.0000E+00	6.2412E–02
5	–3.3087E–05	–2.9915E–04	0.0000E+00	0.0000E+00	0.0000E+00	8.5353E–02
6	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
7	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
8	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00



5.38

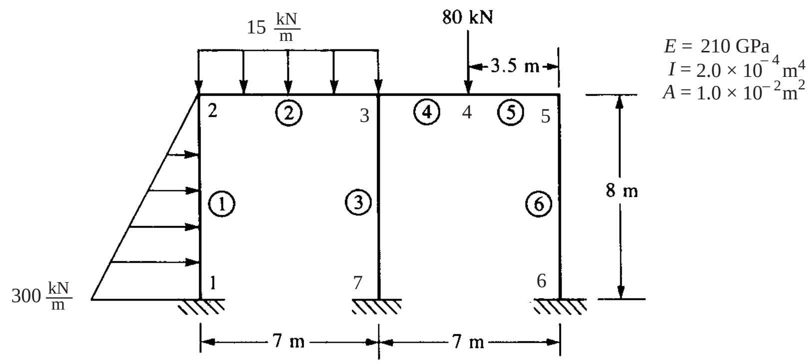


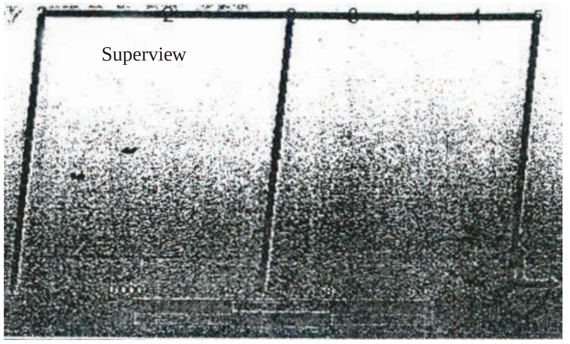
Figure P5–38

NODE number	X-translation	Y-translation	Z-translation	X-rotation	Y-rotation	Z-rotation (deg)
1	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	1.4372E–01	–1.1344E-04	0.0000E+00	0.0000E+00	0.0000E+00	1.4397E–01
3	1.4300E–01	–1.3696E–04	0.0000E+00	0.0000E+00	0.0000E+00	–4.1178E–01
4	1.4282E–01	–2.1948E–03	0.0000E+00	0.0000E+00	0.0000E+00	2.2790E–01
5	1.4265E–01	–4.7155E–04	0.0000E+00	0.0000E+00	0.0000E+00	–5.1623E–01
6	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
7	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00

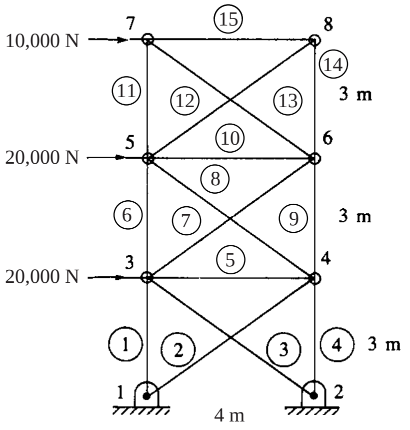
- BEAM ELEMENT FORCES AND MOMENTS

ELEMENT NO.	CASE (MODE)	AXIAL FORCE R_1	SHEAR FORCE R_2	SHEAR FORCE R_3	TORSION MOMENT M_1	BENDING MOMENT M_2	BENDING MOMENT M_3
1	1	–2.907E+04	–9.878E+05	0.000E+00	0.000E+00	0.000E+00	–1.538E+06
		–2.907E+04	2.122E+05	0.000E+00	0.000E+00	0.000E+00	–3.605E+04
2	1	–2.122E+05	–2.907E+04	0.000E+00	0.000E+00	0.000E+00	–3.605E+04
		–2.122E+05	7.593E+04	0.000E+00	0.000E+00	0.000E+00	–2.001E+05
3	1	–1.024E+05	4.084E+04	0.000E+00	0.000E+00	0.000E+00	2.022E+05
		–1.024E+05	4.084E+04	0.000E+00	0.000E+00	0.000E+00	5.932E+04
4	1	–1.024E+05	1.208E+05	0.000E+00	0.000E+00	0.000E+00	5.932E+04
		–1.024E+05	1.208E+05	0.000E+00	0.000E+00	0.000E+00	–3.636E+05
5	1	–1.208E+05	–1.024E+05	0.000E+00	0.000E+00	0.000E+00	–4.560E+05
		–1.208E+05	–1.024E+05	0.000E+00	0.000E+00	0.000E+00	3.636E+05

6	1	–3.510E+04	–1.098E+05	0.000E+00	0.000E+00	0.000E+00	–4.760E+05
		–3.510E+04	–1.098E+05	0.000E+00	0.000E+00	0.000E+00	4.023E+05



5.39



$E =$ 200 GPa
$I = 2.0 \times 10^{-4}$ m^2
$A = 4.0 \times 10^{-4}$ m^4

(a) Truss model
NUMBER OF ELEMENTS (NELE) = 15
NUMBER OF NODES (KNODE) = 8

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE (1,K)	FORCE (2,K)	FORCE (3,K)
1	1 1 1	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
2	1 1 1	4.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
3	0 0 1	0.000000E+00	3.000000E+00	0.000000E+00	2.000000E+04	0.000000E+00	0.000000E+00
4	0 0 1	4.000000E+00	3.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
5	0 0 1	0.000000E+00	6.000000E+00	0.000000E+00	2.000000E+04	0.000000E+00	0.000000E+00
6	0 0 1	4.000000E+00	6.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
7	0 0 1	0.000000E+00	9.000000E+00	0.000000E+00	1.000000E+04	0.000000E+00	0.000000E+00
8	0 0 1	4.000000E+00	9.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00

ELEMENTS

K	NODE(I,K)		E(K)	A(K)
1	1	3	2.0000E+11	2.0000E–04
2	1	4	2.0000E+11	2.0000E–04
3	2	3	2.0000E+11	2.0000E–04
4	2	4	2.0000E+11	2.0000E–04
5	3	4	2.0000E+11	2.0000E–04
6	3	5	2.0000E+11	2.0000E–04
7	3	6	2.0000E+11	2.0000E–04
8	4	5	2.0000E+11	2.0000E–04
9	4	6	2.0000E+11	2.0000E–04
10	5	6	2.0000E+11	2.0000E–04

11	5	7	2.0000E+11	2.0000E–04
12	5	8	2.0000E+11	2.0000E–04
13	6	7	2.0000E+11	2.0000E–04
14	6	8	2.0000E+11	2.0000E–04
15	7	8	2.0000E+11	2.0000E–04

NUMBER OF NONZERO UPPER CO-DIAGONALS (MUD)-11

DISPLACEMENTS	X	Y	Z
MODE NUMBER 1	0.0000E+00	0.0000E+00	0.0000E+00
MODE NUMBER 2	0.0000E+00	0.0000E+00	0.0000E+00
MODE NUMBER 3	0.7935E–02	0.3730E–02	0.0000E+00
MODE NUMBER 4	0.7315E–02	– 0.3583E–02	0.0000E+00
MODE NUMBER 5	0.1738E–01	0.5276E–01	0.0000E+00
MODE NUMBER 6	0.1681E–01	– 0.4849E–02	0.0000E+00
MODE NUMBER 7	0.2603E–01	0.5662E–02	0.0000E+00
MODE NUMBER 8	0.2572E–01	– 0.5026E–02	0.0000E+00

STRESSES IN ELEMENTS (IN CURRENT UNITS)

ELEMENT NUMBER	STRESS
1 =	0.24864E+09
2 =	0.14809E+09
3 =	– 0.16441E+09
4 =	– 0.23886E+09
5 =	– 0.30998E+08
6 =	0.10311E+09
7 =	0.78155E+08
8 =	– 0.10934E+09
9 =	– 0.84393E+08
10 =	– 0.28261E+08
11 =	0.25697E+08
12 =	0.19671E+08
13 =	– 0.42828E+08
14 =	– 0.11803E+08
15 =	– 0.15737E+08

(b) Rigid frame model

NUMBER OF ELEMENTS = 15

NUMBER OF NODES = 8

NODE POINTS

K	IFIK	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE (3,K)
1	1 1 0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	1 1 0	4.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0 0 0	0.000000	3.000000	0.000000	20000.000000	0.000000	0.000000
4	0 0 0	4.000000	3.000000	0.000000	0.000000	0.000000	0.000000
5	0 0 0	0.000000	6.000000	0.000000	20000.000000	0.000000	0.000000
6	0 0 0	4.000000	6.000000	0.000000	0.000000	0.000000	0.000000
7	0 0 0	0.000000	9.000000	0.000000	10000.000000	0.000000	0.000000
8	0 0 0	4.000000	9.000000	0.000000	0.000000	0.000000	0.000000

ELEMENTS

K	NODE(I, K)		E(K)	C(K)	A(K)	XI(K)
1	1	3	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
2	1	4	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
3	2	3	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
4	2	4	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
5	3	4	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
6	3	5	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
7	3	6	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
8	4	5	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04

9	4	6	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
10	5	6	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
11	5	7	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
12	5	8	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
13	6	7	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
14	6	8	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
15	7	8	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04

NODE	DISPLACEMENTS X	Y	K-ROTATION THETA
1	0.00000E+00	0.00000E+00	– 0.14936E–02
2	0.00000E+00	0.00000E+00	– 0.14200E–02
3	0.54772E–02	0.32294E–02	– 0.17329E–02
4	0.51685E–02	– 0.32554E–02	– 0.16980E–02
5	0.11556E–01	0.40230E–02	– 0.20908E–02
6	0.11199E–01	– 0.40706E–02	– 0.20924E–02
7	0.18021E–01	0.42395E–02	– 0.21639E–02
8	0.17667E–01	– 0.42509E–02	– 0.21228E–02

ELEMENTS

K	NODE (I,K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	1	3	– 0.4306E+05	0.2267E+05	0.4038E+05	0.4306E+05	– 0.2267E+05	0.2762E+05
2	1	4	– 0.1745E+05	– 0.1746E+05	– 0.4038E+05	0.1745E+05	0.1746E+05	– 0.4692E+05
3	2	3	0.1955E+05	– 0.1545E+05	– 0.3363E+05	– 0.1955E+05	0.1545E+05	– 0.4364E+05
4	2	4	0.4340E+05	0.1748E+05	0.3363E+05	– 0.4340E+05	– 0.1748E+05	0.1860E+05
5	3	4	0.3087E+04	– 0.5654E+04	– 0.1201E+05	– 0.3087E+04	0.5654E+04	– 0.1061E+05
6	3	5	– 0.1058E+05	0.1220E+05	0.2784E+05	0.1058E+05	– 0.1220E+05	0.8749E+04
7	3	6	– 0.1582E+04	– 0.2226E+04	0.1879E+03	0.1582E+04	0.2226E+04	– 0.1132E+05
8	4	5	0.5942E+04	0.1406E+04	0.9799E+04	– 0.5942E+04	– 0.1406E+04	– 0.2770E+04
9	4	6	0.1087E+05	0.1228E+05	0.2893E+05	– 0.1087E+05	– 0.1228E+05	0.7899E+04
10	5	6	0.3563E+04	– 0.4091E+04	– 0.8150E+04	– 0.3563E+04	0.4091E+04	– 0.8214E+04
11	5	7	– 0.2887E+04	0.2978E+04	0.6416E+04	0.2887E+04	– 0.2978E+04	0.2517E+04
12	5	8	0.6004E+03	– 0. 1903E+04	– 0.4245E+04	– 0.6004E+03	0.1903E+04	– 0.5270E+04
13	6	7	0.3772E+04	0.7733E+03	0.3077E+04	– 0.3772E+04	– 0.7733E+03	0.7894E+03
14	6	8	0.2405E+04	0.5163E+04	0.8555E+04	– 0.2405E+04	– 0.5163E+04	0.6933E+04
15	7	8	0.3541E+04	– 0.1243E+04	– 0.3307E+04	– 0.3541E+04	0.1243E+04	– 0.1664E+04

(c) Use program PFRAME to model a truss

(Use PFRAME to model a Truss, i.e., MAKE $I \cong 0$).

NUMBER OF ELEMENTS = 15

NUMBER OF NODES = 8

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)
1	1 1 0	0.000000	0.000000	0.000000	0.000000
2	1 1 0	4.000000	0.000000	0.000000	0.000000
3	0 0 0	0.000000	3.000000	0.000000	20000.000000
4	0 0 0	4.000000	3.000000	0.000000	0.000000
5	0 0 0	0.000000	6.000000	0.000000	20000.000000
6	0 0 0	4.000000	6.000000	0.000000	0.000000
7	0 0 0	0.000000	9.000000	0.000000	10000.000000
8	0 0 0	4.000000	9.000000	0.000000	0.000000

ELEMENTS

K	NODE(I,K)		E(K)	C(K)	A(K)	XI(K)
1	1	3	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
2	1	4	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
3	2	3	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
4	2	4	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06

5	3	4	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
6	3	5	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
7	3	6	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
8	4	5	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
9	4	6	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
10	5	6	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
11	5	7	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
12	5	8	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
13	6	7	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
14	6	8	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
15	7	8	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06

NODE	DISPLACEMENTS X	Y	Z-ROTATION THETA
1	0.00000E+00	0.00000E+00	– 0.21033E–02
2	0.00000E+00	0.00000E+00	– 0.19714E–02
3	0.79234E–02	0.37278E–02	– 0.23957E–02
4	0.73056E–02	– 0.35821E–02	– 0.23447E–02
5	0.17337E–01	0.52713E–02	– 0.28898E–02
6	0.16772E–01	– 0.48468E–02	– 0.29231E–02
7	0.25985E–01	0.56570E–02	– 0.27580E–02
8	0.25670E–01	–0.50248E–02	– 0.27544E–02

ELEMENTS

K	NODE (I,K)	X-FORCE	Y-FORCE at 1st node	Z-MOMENT	X-FORCE	Y-FORCE at 2nd node	Z-MOMENT
1	1 3	– 0.4970E+05	0.1044E+03	0.1762E+03	0.4970E+05	– 0.1044X+03	0.1372E+03
2	1 4	– 0.2956E+05	– 0.7432E+02	– 0.1762E+03	0.2956E+05	– 0.7432E+02	– 0.1955E+03
3	2 3	0.3282E+05	– 0.6108E+02	– 0.1357E+03	– 0.3282E+05	0.6108E+02	– 0.1697E+03
4	2 4	0.4776E+05	0.7391E+02	0.1357E+03	– 0.4776E+05	– 0.7391E+02	0.8597E+02
5	3 4	0.6178E+04	– 0.8141E+02	– 0.1654E+03	– 0.6178E+04	0.8141E+02	– 0.1603E+03
6	3 5	– 0.2058E+05	0.1320E+03	0.2309E+03	0.2058E+05	– 0.1320E+03	0.1650E+03
7	3 6	– 0.1548E+05	– 0.2166E+02	– 0.3305E+02	0.1548E+05	0.2166E+02	– 0.7524E+02
8	4 5	0.2170E+05	0.2894E+00	0.2253E+02	– 0.2170E+05	– 0.2894E+00	– 0.2108E+02
9	4 6	0.1686E+05	0.1391E+03	0.2472E+03	– 0.1686E+05	– 0.1391E+03	0.1701E+03
10	5 6	0.5641E+04	– 0.5654E+02	– 0.1114E+03	– 0.5641E+04	0.5654E+02	– 0.1148E+03
11	5 7	– 0.5143E+04	0.1572E+02	0.1480E+02	0.5143E+04	– 0.1572E+02	0.3237E+02
12	5 8	– 0.3912E+04	– 0.1677E+02	– 0.4735E+02	0.3912E+04	0.1677E+02	– 0.3652E+02
13	6 7	0.8543E+04	– 0.5227E+01	– 0.1967E+02	– 0.8543E+04	0.5227E+01	– 0.6461E+01
14	6 8	0.2373E+04	0.3387E+02	0.3956E+02	– 0.2373E+04	– 0.3387E+02	0.6206E+02
15	7 8	0.3153E+04	– 0.1286E+02	– 0.2591E+02	– 0.3153E+04	0.1286E+02	– 0.2554E+02

Comparison of TRUSS, PFRAME and modeling a truss using PFRAME

DISPLACEMENTS

	u_5	v_5	u_7	v_7
TRUSS	0.01738	0.005276	0.02603	0.005662
PFRAME	0.011556	0.004023	0.018021	0.0042395
Truss using PFRAME	0.017337	0.0052713	0.025985	0.005657

Note: Global displacements in meters

FORCES

	ELEMENT 1 f_{1x}	f_{1y}	ELEMENT 2 f_{1x}	f_{1y}	ELEMENT 3 f_{2x}	f_{2y}
Truss	–49728	0	–29618	0	32882	0
PFRAME	–43060	22670	–17450	–17460	19550	–15450
Truss using PFRAME	–49700	104.4	–29560	–74.32	32820	–61.00

Note 1: From equilibrium, only forces for one node of an element are shown
Note 2: All forces are in local element coordinates and in Newtons.

5.40

For the two-story, two-bay rigid frame shown, determine (1) the nodal displacement components and (2) the shear force and bending moments in each member. Let $E = 200$ GPa, $I = 2 \times 10^{-4}$ m^4 for each horizontal member and $I = 1.5 \times 10^{-4}$ m^4 for each vertical member.

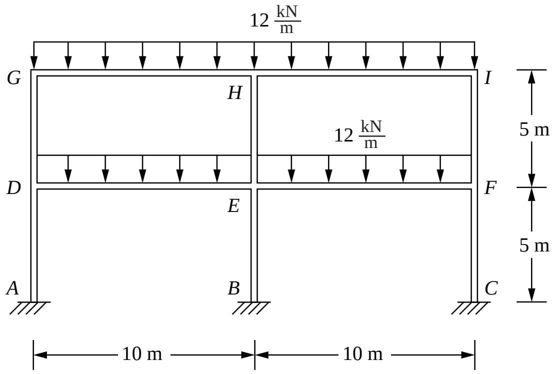


Figure P5–40

Displacements/Rotations (degrees) of nodes

NODE number	X– translation	Y– translation	Z– translation	X– rotation	Y– rotation	Z– rotation
1	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
3	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
4	–7.2482E-04	–1.4007E–06	0.0000E+00	0.0000E+00	0.0000E+00	1.6616E–02
5	0.0000E+00	–3.1986E–06	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
6	7.2482E-04	–1.4007E–06	0.0000E+00	0.0000E+00	0.0000E+00	–1.6616E–02
7	–7.7467E–07	–2.8014E–06	0.0000E+00	0.0000E+00	0.0000E+00	–6.6411E–02
8	–3.8733E–07	–9.2660E–03	0.0000E+00	0.0000E+00	0.0000E+00	1.6572E–02
9	0.0000E+00	–6.3972E–06	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
10	3.8733E–07	–9.2660E–03	0.0000E+00	0.0000E+00	0.0000E+00	–1.6572E–02
11	7.7467E–07	–2.8014E–06	0.0000E+00	0.0000E+00	0.0000E+00	6.6411E–02
12	–6.2096E–04	–3.4868E–06	0.0000E+00	0.0000E+00	0.0000E+00	4.7408E–02
13	0.0000E+00	–8.0263E–06	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
14	6.2096E–04	–3.4868E–06	0.0000E+00	0.0000E+00	0.0000E+00	–4.7408E–02
15	1.1921E–06	–4.1723E–06	0.0000E+00	0.0000E+00	0.0000E+00	–1.2336E–01
16	5.9603E–07	–1.0511E–02	0.0000E+00	0.0000E+00	0.0000E+00	3.0792E–02
17	0.0000E+00	–9.6555E–06	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
18	–5.9603E–07	–1.0511E–02	0.0000E+00	0.0000E+00	0.0000E+00	–3.0792E–02
19	–1.1921E–06	–4.1723E–06	0.0000E+00	0.0000E+00	0.0000E+00	1.2336E–01

1 **** BEAM ELEMENT FORCES AND MOMENTS

ELEMENT NO.	CASE (MODE)	AXIAL FORCE R_1	SHEAR FORCE R_2	SHEAR FORCE R_3	TORSION MOMENT M_1	BENDING MOMENT M_2	BENDING MOMENT M_3
1	1	–1.121E+05	8.348E+03	0.000E+00	0.000E+00	0.000E+00	1.391E+04
		–1.121E+05	8.348E+03	0.000E+00	0.000E+00	0.000E+00	–6.955E+03
2	1	–2.559E+05	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		–2.559E+05	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
3	1	–1.121E+05	–8.348E+03	0.000E+00	0.000E+00	0.000E+00	–1.391E+04
		–1.121E+05	–8.348E+03	0.000E+00	0.000E+00	0.000E+00	6.955E+03
4	1	–1.121E+05	8.348E+03	0.000E+00	0.000E+00	0.000E+00	–6.955E+03

		–1.121E+05	8.348E+03	0.000E+00	0.000E+00	0.000E+00	–2.782E+04
5	1	–2.559E+05	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		–2.559E+05	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
6	1	–1.121E+05	–8.348E+03	0.000E+00	0.000E+00	0.000E+00	6.955E+03
		–1.121E+05	–8.348E+03	0.000E+00	0.000E+00	0.000E+00	2.782E+04
7.	1	–5.484E+04	2.384E+04	0.000E+00	0.000E+00	0.000E+00	5.364E+04
		–5.484E+04	2.384E+04	0.000E+00	0.000E+00	0.000E+00	–5.963E+03
8	1	–1.303E+05	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		–1.303E+05	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
9	1	–5.484E+04	–2.384E+04	0.000E+00	0.000E+00	0.000E+00	–5.364E+04
		–5.484E+04	–2.384E+04	0.000E+00	0.000E+00	0.000E+00	5.963E+04
10	1	–5.484E+04	2.384E+04	0.000E+00	0.000E+00	0.000E+00	–5.963E+03
		–5.484E+04	2.384E+04	0.000E+00	0.000E+00	0.000E+00	–6.557E+04
11	1	–1.303E+05	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		–1.303E+05	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
12	1	–5.484E+04	–2.384E+04	0.000E+00	0.000E+00	0.000E+00	5.963E+03
		–5.484E+04	–2.384E+04	0.000E+00	0.000E+00	0.000E+00	6.557E+04

5.41 For the two-story, three-bay rigid frame shown, determine (1) the nodal displacements and (2) the member end shear forces and bending moments. (3) Draw the shear force and bending moment diagrams for each member. Let E = 200 GPa, $I = 1.29 \times 10^{-4}$ m^4 for the beams and $I = 0.462 \times 10^{-4}$ m^4 for the columns. The properties for I correspond to a W 610 × 155 and a W 410 × 114 wide-flange section, respectively, in metric units.

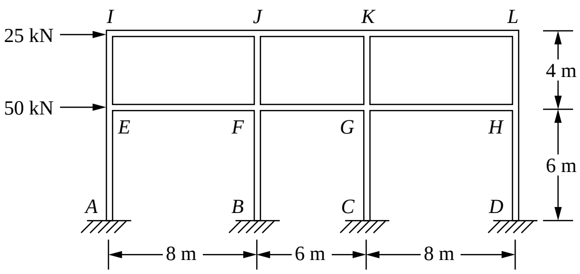


Figure P5–41

1**** BEAM ELEMENT FORCES AND MOMENTS

ELEMENT NO.	CASE (MODE)	AXIAL FORCE R_1	SHEAR FORCE R_2	SHEAR FORCE R_3	TORSION MOMENT M_1	BENDING MOMENT M_2	BENDING MOMENT M_3
1	1	1.347E+04	–1.750E+04	0.000E+00	0.000E+00	0.000E+00	–5.719E+04
		1.347E+04	–1.750E+04	0.000E+00	0.000E+00	0.000E+00	–4.685E+03
2	1	2.471E+03	–2.000E+04	0.000E+00	0.000E+00	0.000E+00	–6.219E+04
		2.471E+03	–2.000E+04	0.000E+00	0.000E+00	0.000E+00	–2.184E+03
3	1	–2.470E+03	–2.000E+04	0.000E+00	0.000E+00	0.000E+00	–6.218E+04
		–2.470E+03	–2.000E+04	0.000E+00	0.000E+00	0.000E+00	–2.184E+03
4	1	–1.347E+04	–1.750E+04	0.000E+00	0.000E+00	0.000E+00	–5.718E+04
		–1.347E+04	–1.750E+04	0.000E+00	0.000E+00	0.000E+00	–4.685E+03
5	1	1.347E+04	–1.750E+04	0.000E+00	0.000E+00	0.000E+00	–4.685E+03
		1.347E+04	–1.750E+04	0.000E+00	0.000E+00	0.000E+00	4.782E+04
6	1	2.471E+03	–2.000E+04	0.000E+00	0.000E+00	0.000E+00	–2.184E+03
		2.471E+03	–2.000E+04	0.000E+00	0.000E+00	0.000E+00	5.782E+04
7	1	–2.470E+03	–2.000E+04	0.000E+00	0.000E+00	0.000E+00	–2.184E+03
		–2.470E+03	–2.000E+04	0.000E+00	0.000E+00	0.000E+00	5.782E+04
8	1	–1.347E+04	–1.750E+04	0.000E+00	0.000E+00	0.000E+00	–4.685E+03
		–1.347E+04	–1.750E+04	0.000E+00	0.000E+00	0.000E+00	4.781E+04
9	1	2.686E+03	–3.022E+03	0.000E+00	0.000E+00	0.000E+00	–5.747E+02
		2.686E+03	–3.022E+03	0.000E+00	0.000E+00	0.000E+00	5.469E+03
10	1	1.063E+03	–9.477E+03	0.000E+00	0.000E+00	0.000E+00	–1.668E+04
		1.063E+03	–9.477E+03	0.000E+00	0.000E+00	0.000E+00	2.269E+03

11	1	–1.063E+03	–9.478E+03	0.000E+00	0.000E+00	0.000E+00	–1.669E+04
		–1.063E+03	–9.478E+03	0.000E+00	0.000E+00	0.000E+00	2.269E+03
12	1	–2.686E+03	–3.024E+03	0.000E+00	0.000E+00	0.000E+00	–5.786E+02
		–2.686E+03	–3.024E+03	0.000E+00	0.000E+00	0.000E+00	5.469E+03
13	1	2.686E+03	–3.022E+03	0.000E+00	0.000E+00	0.000E+00	5.469E+03
		2.686E+03	–3.022E+03	0.000E+00	0.000E+00	0.000E+00	1.151E+04
14	1	1.063E+03	–9.477E+03	0.000E+00	0.000E+00	0.000E+00	2.269E+03
		1.063E+03	–9.477E+03	0.000E+00	0.000E+00	0.000E+00	2.122E+04
15	1	–1.063E+03	–9.478E+03	0.000E+00	0.000E+00	0.000E+00	2.269E+03
		–1.063E+03	–9.478E+03	0.000E+00	0.000E+00	0.000E+00	2.122E+04
16	1	–2.686E+03	–3.024E+03	0.000E+00	0.000E+00	0.000E+00	5.469E+03
		–2.686E+03	–3.024E+03	0.000E+00	0.000E+00	0.000E+00	1.152E+04

5.42 For the rigid frame shown, determine (1) the nodal displacements and rotations and (2) the member shear forces and bending moments. Let E = 200 GPa, $I = 0.795 \times 10^{-4}$ m^4 for the horizontal members and $I = 0.316 \times 10^{-4}$ m^4 for the vertical members. These I values correspond to a W 460 × 158 and a W 410 × 85 wide-flange section, respectively.

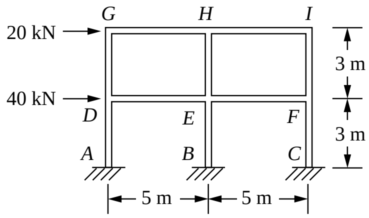


Figure P5–42

Displacements/Rotations (degrees) of nodes

NODE number	X– translation	Y– translation	Z– translation	X– rotation	Y– rotation	Z– rotation
1	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	7.8454E–04	8.2939E–06	0.0000E+00	0.0000E+00	0.0000E+00	–1.5312E–02
3	1.5075E–03	1.1468E–05	0.0000E+00	0.0000E+00	0.0000E+00	–9.6245E–03
4	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
5	7.3108E–04	2.5665E–08	0.0000E+00	0.0000E+00	0.0000E+00	–1.2692E–02
6	1.4698E–03	1.8869E–07	0.0000E+00	0.0000E+00	0.0000E+00	–7.8815E–03
7	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
8	7.0278E–04	–8.3195E–06	0.0000E+00	0.0000E+00	0.0000E+00	–1.4567E–02
9	1.4574E–03	–1.1657E–05	0.0000E+00	0.0000E+00	0.0000E+00	–1.0523E–02
10	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
11	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
12	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
13	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
14	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
15	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00

ELEMENT NO.	CASE (MODE)	AXIAL FORCE R_1	SHEAR FORCE R_2	SHEAR FORCE R_3	TORSION MOMENT M_1	BENDING MOMENT M_2	BENDING MOMENT M_3
			Columns				
1	1	1.106E+04	–2.085E+04	0.000E+00	0.000E+00	0.000E+00	–4.540E+04
		1.106E+04	–2.085E+04	0.000E+00	0.000E+00	0.000E+00	1.716E+04
2	1	4.232E+03	–3.811E+03	0.000E+00	0.000E+00	0.000E+00	–4.719E+02
		4.232E+03	–3.811E+03	0.000E+00	0.000E+00	0.000E+00	1.096E+04
3	1	3.422E+01	–2.168E+04	0.000E+00	0.000E+00	0.000E+00	–4.422E+04
		3.422E+01	–2.168E+04	0.000E+00	0.000E+00	0.000E+00	2.081E+04

4	1	2.174E+02	–1.088E+04	0.000E+00	0.000E+00	0.000E+00	–1.188E+04
		2.174E+02	–1.088E+04	0.000E+00	0.000E+00	0.000E+00	2.075E+04
5	1	–1.109E+04	–1.747E+04	0.000E+00	0.000E+00	0.000E+00	–3.964E+04
		–1.109E+04	–1.747E+04	0.000E+00	0.000E+00	0.000E+00	1.277E+04
6	1	–4.450E+03	–5.314E+03	0.000E+00	0.000E+00	0.000E+00	–4.242E+03
		–4.450E+03	–5.314E+03	0.000E+00	0.000E+00	0.000E+00	1.170E+04

ELEMENT NO.	CASE (MODE)	AXIAL FORCE R_1	SHEAR FORCE R_2	SHEAR FORCE R_3	TORSION MOMENT M_1	BENDING MOMENT M_2	BENDING MOMENT M_3
				Beams			
1	1	–1.619E+04	4.232E+03	0.000E+00	0.000E+00	0.000E+00	1.096E+04
		–1.619E+04	4.232E+03	0.000E+00	0.000E+00	0.000E+00	–1.020E+04
2	1	–5.314E+03	4.450E+03	0.000E+00	0.000E+00	0.000E+00	1.055E+04
		–5.314E+03	4.450E+03	0.000E+00	0.000E+00	0.000E+00	–1.170E+04
3	1	–2.296E+04	6.825E+03	0.000E+00	0.000E+00	0.000E+00	1.763E+04
		–2.296E+04	6.825E+03	0.000E+00	0.000E+00	0.000E+00	–1.649E+04
4	1	–1.216E+04	6.642E+03	0.000E+00	0.000E+00	0.000E+00	1.620E+04
		–1.216E+04	6.642E+03	0.000E+00	0.000E+00	0.000E+00	–1.701E+04

5.43 For the rigid frame shown, determine (1) the nodal displacements and rotations and (2) the shear force and bending moments in each member. Let $E = 29 \times 106$ psi, $I = 3100$ in.^{4} for the horizontal members and $I = 1110$ in.^{4} for the vertical members. The I values correspond to a W 24×104 and a W 16×77.

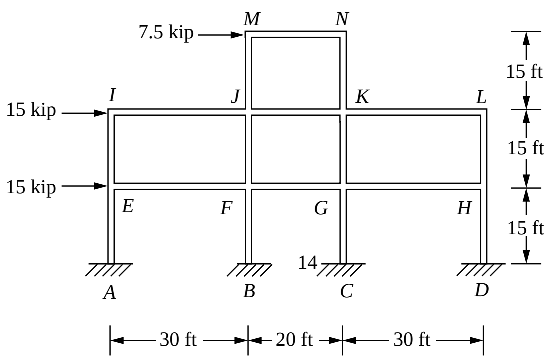


Figure P5–43

Displacements/Rotations (degrees) of nodes

NODE number	X– translation	Y– translation	Z– translation	X– rotation	Y– rotation	Z– rotation
1	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	1.8762E–03	1.4645E–04	0.0000E+00	0.0000E+00	0.0000E+00	–1.5869E–03
3	3.0993E-03	2.1717E–04	0.0000E+00	0.0000E+00	0.0000E+00	–1.4409E–03
4	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
5	1.5614E–03	3.5423E–05	0.0000E+00	0.0000E+00	0.0000E+00	–1.3154E–03
6	2.7811E–03	7.7093E–05	0.0000E+00	0.0000E+00	0.0000E+00	–1.4087E–03
7	3.6829E–03	1.2865E–04	0.0000E+00	0.0000E+00	0.0000E+00	–1.5467E–03
8	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
9	1.4331E–03	–4.4935E–05	0.0000E+00	0.0000E+00	0.0000E+00	–1.2573E–03
10	2.6228E–03	–8.8701E–05	0.0000E+00	0.0000E+00	0.0000E+00	–1.3909E–03
11	3.5926E-03	–1.4025E–04	0.0000E+00	0.0000E+00	0.0000E+00	–1.5614E–03
12	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
13	1.3394E–03	–1.3694E–04	0.0000E+00	0.0000E+00	0.0000E+00	–1.3297E–03
14	2.4458E–03	–2.0556E–04	0.0000E+00	0.0000E+00	0.0000E+00	–1.3014E–03
15	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
16	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
17	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
18	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00

19	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
20	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00

1 **** BEAM ELEMENT FORCES AND MOMENTS

ELEMENT NO.	CASE (MODE)	AXIAL FORCE R_1	SHEAR FORCE R_2	SHEAR FORCE R_3	TORSION MOMENT M_1	BENDING MOMENT M_2	BENDING MOMENT M_3
				Columns			
1	1	6.363E+03	–1.139E+04	0.000E+00	0.000E+00	0.000E+00	–1.444E+05
		6.363E+03	–1.139E+04	0.000E+00	0.000E+00	0.000E+00	2.653E+04
2	1	3.073E+03	–5.646E+03	0.000E+00	0.000E+00	0.000E+00	–3.692E+04
		3.073E+03	–5.646E+03	0.000E+00	0.000E+00	0.000E+00	4.777E+04
3	1	1.539E+03	–9.487E+03	0.000E+00	0.000E+00	0.000E+00	–1.200E+05
		1.539E+03	–9.487E+03	0.000E+00	0.000E+00	0.000E+00	2.231E+04
4	1	1.810E+03	–5.894E+03	0.000E+00	0.000E+00	0.000E+00	–4.766E+04
		1.810E+03	–5.894E+03	0.000E+00	0.000E+00	0.000E+00	4.074E+04
5	1	2.240E+03	–3.517E+03	0.000E+00	0.000E+00	0.000E+00	–3.150E+04
		2.240E+03	–3.517E+03	0.000E+00	0.000E+00	0.000E+00	2.125E+04
6	1	–1.952E+03	–8.662E+03	0.000E+00	0.000E+00	0.000E+00	–1.117E+05
		–1.952E+03	–8.662E+03	0.000E+00	0.000E+00	0.000E+00	1.828E+04
7	1	–1.902E+03	–5.757E+03	0.000E+00	0.000E+00	0.000E+00	– 4.814E+04
		–1.902E+03	–5.757E+03	0.000E+00	0.000E+00	0.000E+00	3.822E+04
8	1	–2.240E+03	–3.983E+03	0.000E+00	0.000E+00	0.000E+00	–3.620E+04
		–2.240E+03	–3.983E+03	0.000E+00	0.000E+00	0.000E+00	2.355E+04
9	1	–5.950E+03	–7.958E+03	0.000E+00	0.000E+00	0.000E+00	–1.091E+05
		–5.950E+03	–7.958E+03	0.000E+00	0.000E+00	0.000E+00	1.031E+04
10	1	–2.982E+03	–5.203E+03	0.000E+00	0.000E+00	0.000E+00	–3.797E+04
		–2.982E+03	–5.203E+03	0.000E+00	0.000E+00	0.000E+00	4.008E+04

5.44 A structure is fabricated by welding together three lengths of I-shaped members as shown in Figure P5-44. The yield strength of the members is 36 ksi, E = 29e6 psi, and Poisson's ratio is 0.3. The members all have cross-section properties corresponding to a W 18 × 76. That is, A = 22.3 in.^{2}, depth of section is d = 18.21 in., I_x = 1330 in.^{4}, S_x = 146 in.^{3}, I_y = 152 in.^{4}, and S_y = 27.6 in.^{3}. Determine whether a load of Q = 10,000 lb downward is safe against general yielding of the material. The factor of safety against general yielding is to be 2.0. Also, determine the maximum vertical and horizontal deflections of the structure.

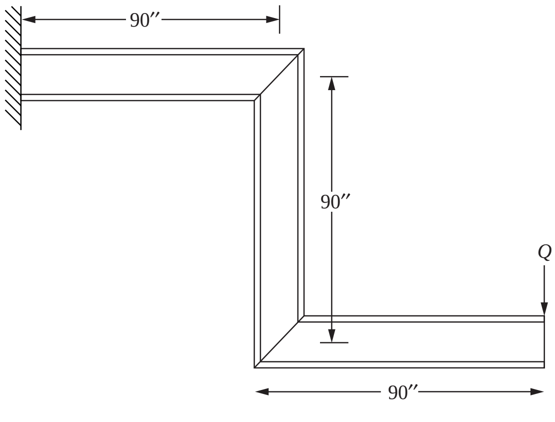
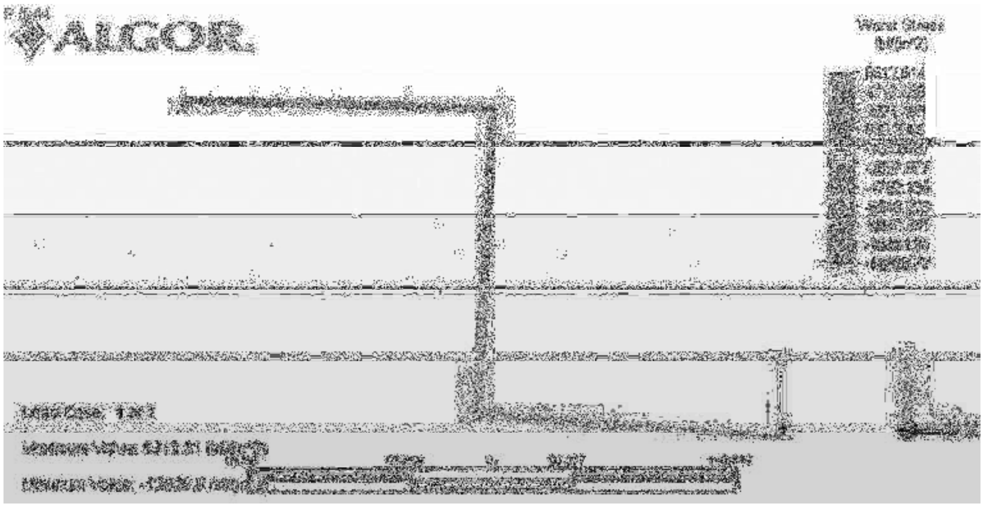


Figure P5-44

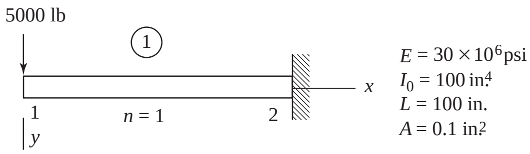


$$\sigma_{max} = 12{,}329 \text{ psi} < \frac{36000}{2} = 18{,}000 \text{ psi}$$

$\therefore$ Safe against yielding

5.45

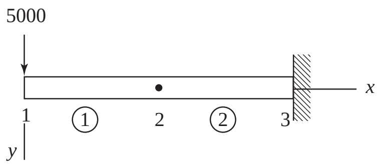
Tapered beam using 1 element



$$I(x) = I_0\left(1 + n\frac{x}{L}\right) = 100\left(1 + \frac{x}{100}\right) = 100 + x$$

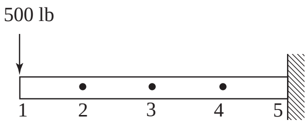
$$I\left(\frac{L}{2}\right) = 150 \text{ in.}^4$$

Tapered beam using 2 elements



$$I_{1\text{–}2}\left(\frac{L}{4}\right) = 125 \text{ in.}^4;\ I_{2\text{–}3} = \left(\frac{3L}{4}\right) = 175 \text{ in.}^4$$

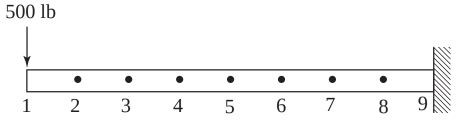
Tapered beam using 4 elements



$$I_{1-2}\left(\frac{L}{8}\right) = 112.5 \text{ in.}^4;\ I_{2-3}\left(\frac{3L}{8}\right) = 137.5 \text{ in.}^4;\ I_{3-4}\left(\frac{5L}{8}\right) = 162.5 \text{ in.}^4$$

$$I_{4-5} = \left(\frac{7L}{8}\right) = 187.5 \text{ in.}^4$$

Tapered beam using 8 elements



$$I_{1-2}\left(\frac{L}{16}\right) = 106.25 \qquad I_{2-3}\left(\frac{3L}{16}\right) = 118.75$$

$$I_{3-4}\left(\frac{5L}{16}\right) = 131.25 \qquad I_{4-5}\left(\frac{7L}{16}\right) = 143.75$$

$$I_{5-6}\left(\frac{9L}{16}\right) = 156.25 \qquad I_{6-7}\left(\frac{11L}{16}\right) = 168.75$$

$$I_{7-8}\left(\frac{13L}{16}\right) = 181.25 \qquad I_{8-9}\left(\frac{15L}{16}\right) = 193.75$$

Analytical solution

$$v = \frac{Pl^2}{n^2\, EI_0}\left[\left(\frac{n}{2l}x^2 + 2x + \frac{l}{n}\right) - \frac{1}{n}\left(1+\frac{n}{l}x\right) ln\left(1+\frac{n}{l}x\right)\right] + Ax + B$$

$$A = \frac{Pl^2}{n^2\, EI_0}\ [ln\ (1+n) - (1+n)]$$

$$B = \frac{Pl^3}{n^2\, EI_0}\left[\frac{1}{n} ln\ (1+n) + \frac{n}{2} - 1 - \frac{1}{n}\right]$$

$$x = 0,\ n = 1,\ P = 500,\ L = 100,\ E = 30 \times 10^6,\ I_0 = 100$$

$$A = \frac{(500)\,(100 \text{ in.})^2}{(1)^2\ (30\times 10^6)\,(100 \text{ in.}^4)}\ [ln(2) - 2]$$

$$= -2.1781 \times 10^{-3}$$

$$B = \frac{500\,(100 \text{ in.})^3}{(1)^2\ (30\times 10^6)\,(100 \text{ in.}^4)}\left[ln\ (2) + \frac{1}{2} - 1 - 1\right]$$

$$= -1.3448 \times 10^{-1}$$

$$v = \frac{500\,(100 \text{ in.})^2}{(1)^2\ (30\times 10^6)\,(100 \text{ in.})}\left[\left(\frac{100 \text{ in.}}{(1)}\right)\right] - 1.3448 \times 10^{-1}$$

$$= 0.032187 \text{ in.}$$

PROBLEM 2 USING 1 ELEMENT--
NUMBER OF ELEMENTS = 1
NUMBER OF NODES = 2

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
1	0 0 0	0.000000	0.000000	0.000000	0.000000	500.000000	0.000000
2	1 1 1	100.000000	0.000000	0.000000	0.000000	0.000000	0.000000

ELEMENTS

K	NODE(I,K)		E(K)	G(K)	A(K)	XI(K)
1	1	2	3,0000000E+07	1.0000000E+00	1.0000000E–01	1.5000000E+02

NODE	DISPLACEMENTS X	Y	Z-ROTATION THETA
1	0.00000E+00	0.37037E–01	– 0.55556E–03
2	0.00000E+00	0.00000E+00	0.00000E+00

ELEMENTS

K	NODE(I,K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	1	2	0.0000E+00	0.5000E+03	0.2434E–03	0.0000E+00	– 0.5000E+03	0.5000E+05

PROBLEM 2 USING 2 ELEMENTS--
NUMBER OF ELEMENTS = 2
NUMBER OF NODES = 3

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
1	0 0 0	0.000000	0.000000	0.000000	0.000000	500.000000	0.000000
2	0 0 0	50.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	1 1 1	100.000000	0.000000	0.000000	0.000000	0.000000	0.000000

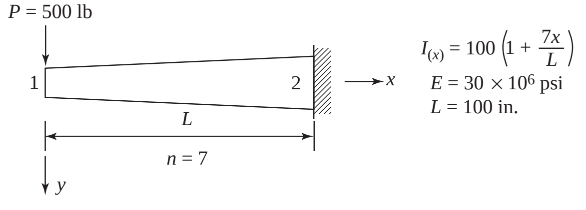
ELEMENTS

K	NODE(1,K)		E(K)	G(K)	A(K)	XI(K)
1	1	2	3.0000000E+07	1.0000000E+00	1.0000000E–01	1.2500000E+02
2	2	3	3.0000000E+07	1.0000000E+00	1.0000000E–01	1.7500000E+02

NODE	DISPLACEMENTS X	Y	Z-ROTATION THETA
1	0.00000E+00	0.33333E–01	– 0.52381E–03
2	0.00000E+00	0.99206E–02	– 0.35714E–03
3	0.00000E+00	0.00000E+00	0.000000E+00

ELEMENTS

K	NODE (I,K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	1	2	0.0000E+00	0.5000E+03	0.6515E–02	0.0000E+00	– 0.5000E+03	0.2500E+05
2	2	3	0.0000E+00	0.5000E+03	– 0.2500E+05	0.0000E+00	– 0.5000E+03	0.5000E+05



For 1 element

$$I_1 = 100\left(1 + \frac{7}{2}\right) = 450 \text{ in.}^4$$

For 2 elements

$$I_1 = 100\left(\frac{11}{4}\right) = 275 \text{ in.}^4$$

$$I_2 = 100\left(\frac{25}{4}\right) = 625 \text{ in.}^4$$

For 4 elements

$$I_1 = 100\left(\frac{15}{8}\right) = 187.5 \text{ in.}^4$$

$$I_2 = 100\left(\frac{29}{8}\right) = 362.5 \text{ in.}^4$$

$$I_3 = 100\left(\frac{43}{8}\right) = 537.5 \text{ in.}^4$$

$$I_4 = 100\left(\frac{57}{8}\right) = 712.5 \text{ in.}^4$$

For 8 elements

$$I_1 = 100\left(\frac{23}{16}\right) = 143.75 \text{ in.}^4$$

$$I_2 = 100\left(\frac{37}{16}\right) = 231.25 \text{ in.}^4$$

$$I_3 = 100\left(\frac{51}{16}\right) = 318.75 \text{ in}^4$$

$$I_4 = 100\left(\frac{65}{16}\right) = 406.25 \text{ in.}^4$$

$$I_5 = 100\left(\frac{79}{16}\right) = 493.75 \text{ in.}^4$$

$$I_6 = 100\left(\frac{93}{16}\right) = 581.25 \text{ in.}^4$$

$$I_7 = 100\left(\frac{107}{16}\right) = 668.75 \text{ in.}^4$$

$$I_8 = 100\left(\frac{121}{16}\right) = 756.25 \text{ in.}^4$$

The analytical solution is

$$v_{\max} = y_{\max} = \frac{1}{17.55}\,\frac{(500)\,(100)^3}{(30\times 10^6)\,(100)} = 0.0095 \text{ in.}$$

		$y_{\max}$
	Analytical	0.0095 in.
FEM	1 element	0.0123 in.
FEM	2 elements	0.0103 in.
FEM	4 elements	0.0097 in.
FEM	8 elements	0.0096 in.

NUMBER OF ELEMENTS = 1
NUMBER OF NODES = 2

NODE POINTS

K	IFIK	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
1	0 0 0	0.000000	0.000000	0.000000	0.000000	500.000000	0.000000
2	1 1 1	100.000000	0.000000	0.000000	0.000000	0.000000	0.000000

ELEMENTS

K	NODE(I,K)		E(K)	G(K)	A(K)	XI(K)
1	1	2	3.0000000E+07	0.0000000E+00	1.0000000E–01	4.5000000E+02

NODE	DISPLACEMENTS X	Y	Z-ROTATION THETA
1	0.00000E+00	0.12346E-01	–0.18519E-03
2	0.00000E+00	0.00000E+00	0.00000E+00

ELEMENTS

K	NODE(I,K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	1	2	0.0000E+00	0.5000E+03	–0.7706E–02	0.0000E+00	–0.5000E+03	0.5000E+05

NUMBER OF ELEMENTS = 4
NUMBER OF NODES = 5

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
1	0 0 0	0.000000	0.000000	0.000000	0.000000	500.000000	0.000000
2	0 0 0	25.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0 0 0	50.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4	0 0 0	75.000000	0.000000	0.000000	0.000000	0.000000	0.000000
5	1 1 1	100.000000	0.000000	0.000000	0.000000	0.000000	0.000000

ELEMENTS

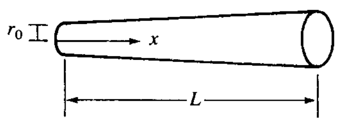
K	NODE(I,K)		E(K)	G(K)	A(K)	XI(K)
1	1	2	3.0000000E+07	0.0000000E+00	1.0000000E–01	1.8750000E+02
2	2	3	3.0000000E+07	0.0000000E+00	1.0000000E–01	3.6250000E+02
3	3	4	3.0000000E+07	0.0000000E+00	1.0000000E–01	5.3750000E+02
4	4	5	3.0000000E+07	0.0000000E+00	1.0000000E–01	7.1250000E+02

NODE	DISPLACEMENTS X	Y	Z-ROTATION THETA
1	0.00000E+00	0.97156E–02	– 0.17050E–03
2	0.00000E+00	0.56845E–02	– 0.14272E–03
3	0.00000E+00	0.25953E–02	– 0.99620E–04
4	0.00000E+00	0.67008E–03	– 0.51170E–04
5	0.00000E+00	0.00000E+00	0.00000E+00

ELEMENTS

K	NODE (1,K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	1	2	0.0000E+00	0.5000E+03	0.3812E –01	0.0000E+00	– 0.5000E+03	0.1250E+05
2	2	3	0.0000E+00	0.5000E+03	– 0.1250E+05	0.0000E+00	– 0.5000E+03	0.2500E+05
3	3	4	0.0000E+00	0.5000E+03	– 0.2500E+05	0.0000E+00	– 0.5000E+03	0.3750E+05
4	4	5	0.0000E+00	0.5000E+03	– 0.3750E+05	0.0000E+00	– 0.5000E+03	0.5000E+05

5.46



$$U = \frac{1}{2}\int_V \tau \;\gamma dV \qquad \tau = \frac{Tr}{J} = G\gamma$$

$$U = \frac{1}{2}\int_V \frac{T^2 r^2}{G\,J^2}\ dV \qquad \gamma = \frac{\gamma}{L}\ \phi$$

$$= \frac{1}{2}\int \frac{T^2}{GJ^2}\left(\int r^2\, dA\right)\ dx$$

$$= \int \frac{T^2}{2GJ(x)}\ dx$$

Now

$$\gamma = \frac{\gamma}{L}\ (\phi_{2x} - \phi_{1x})$$

$$= \gamma \underbrace{\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}}_{[B]} \underbrace{\begin{Bmatrix} \phi_{1x} \\ \phi_{2x} \end{Bmatrix}}_{\{\phi\}}$$

$$\tau_{\max} = G\gamma_{\max} = G\ [B]\ \{\phi\}$$

$$U = \frac{1}{2}\int_A \int \gamma\ \{\phi^T\}\ B^T\ \{G\}[B]\{\phi\}\,\gamma\ dx\ \ dA$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^R r^2\, r\, dr\, d\theta \int_0^L \{\phi^T\}[B^T]\,G\,[B]\{\phi\}\,dx \quad \text{(If J constant)}$$

or

$$U = \frac{1}{2}\int_0^{2\pi}\int_0^R r^3\ \ dr\, d\theta \int_0^L \{\phi^T\}[B^T]\,G\,[B]\{\phi\}\,dx$$

$$r = r_0\left(1 + \frac{x}{L}\right) \qquad dr = \frac{r_0}{L}\ dx$$

$$\frac{\partial U}{\partial \phi_{1x}} = \int_0^{2\pi}\int_0^L \left(r_0 + r_0\,\frac{x}{L}\right)^3\ \ \frac{r_0}{L}\,dx\,\frac{GL}{2}\,(2\,\phi_{1x} - 2\,\phi_{2x})$$

$$\frac{\partial U}{\partial \phi_{2x}} = 2\pi\,\frac{r_0^{\,4}}{L}\int_0^L \left(1 + \frac{x}{L}\right)^3 dx\,\frac{GL}{2L^2}\,(-\,2\phi_{1x} + 2\phi_{2x})$$

Let $u = 1 + \dfrac{x}{L},\ du = \dfrac{dx}{L}$

$$\int_1^2 u^3 L\ du = \left.\frac{u^4}{4}\,L\right|_1^2$$

$$\therefore\ \frac{\partial U}{\partial \phi_{1x}} = \frac{2\,\pi\,r_0^{\,4}}{4}\,\frac{L}{4}\ \left.u^4\right|_1^2\ \ \frac{G}{2L}\,(2\phi_{1x} - 2\phi_{2x})$$

$$= J_0(16 - 1)\ \frac{G}{L}\,(\phi_{2x} - \phi_{1x})$$

$$\frac{\partial U}{\partial \phi_{2x}} = J_0(16 - 1)\ \frac{G}{L}\,(\phi_{2x} - \phi_{1x})$$

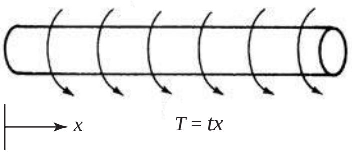
$$\therefore \quad [K] = \frac{15\,G\,J_0}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

5.47

$$U = \frac{1}{2}\int \frac{T^2}{2\,GJ^2}\left(\int r^2\,dA\right)\,dx$$

$$= \frac{1}{2}\int \frac{T^2}{2\,GJ}\,dx$$

$$T = t\,x \qquad t\left(\frac{\text{lb}\cdot\text{in.}}{\text{in.}}\right)$$

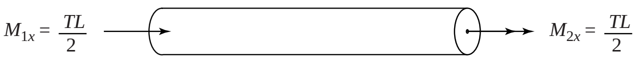


$$U = \frac{1}{2}\int_0^L \frac{(-\,tx)^2}{GJ}\,dx$$

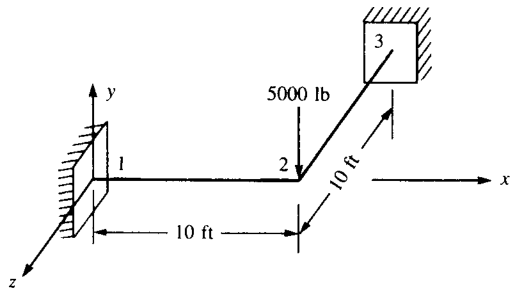
$$= \frac{1}{2\,GJ}\int_0^L (-\,tx)^2\,dx$$

$$= \frac{t^2}{2\,GJ}\,\frac{x^3}{3}\bigg|_0^L$$

$$U = \frac{t^2\,L^3}{6\,GJ} \text{ Total strain energy}$$



5.48



NUMBER OF ELEMENTS = 2
NUMBER OF NODES = 3

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
1	1 1 1	0.0000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0 0 0	120.000000	0.000000	0.000000	–5.000000	0.000000	120.000000
3	1 1 1	120.000000	0.000000	–120.000000	0.000000	0.000000	120.000000

ELEMENTS

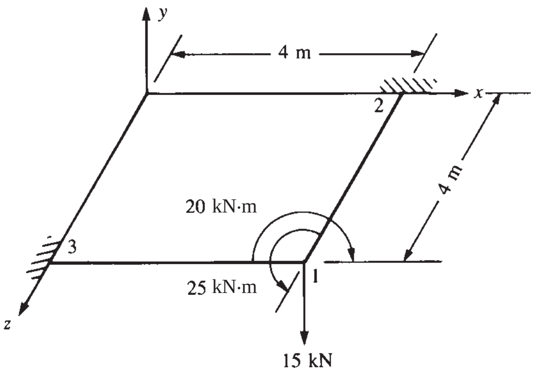
K	NODE(I,K)		E(K)	XI(K)	XJ(K)	G(K)
1	1	2	3.0000000E+04	2.0000000E+02	1.0000000E+02	1.0000000E+04
2	2	3	3.0000000E+04	2.0000000E+02	1.0000000E+02	1.0000000E+04

NODE	DISPLACEMENT	THETA-X	THETA-Z
1	0.00000E+00	0.00000E+00	0.00000E+00
2	–0.21429E+00	0.25714E–02	–0.25714E–02
3	0.00000E+00	0.00000E+00	0.00000E+00

ELEMENTS

K	NODE (I,K)		Y-FORCE	X-MOMENT	Z-MOMENT	Z-MOMENT	Y-FORCE	X-MOMENT
1	1	2	0.2500E+01	–0.2143E+02	0.2786E+03	0.2143E+02	–0.2500E+01	0.2143E+02
2	2	3	–0.2500E+01	0.2143E+02	–0.2143E+02	–0.2786E+03	0.2500E+01	–0.2143E+02

5.51



NUMBER OF ELEMENTS = 2
NUMBER OF NODES = 3

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
1	0 0 0	4.000000	0.000000	4.000000	–15.000000	25.000000	–20.000000
2	1 1 1	4.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	1 1 1	0.000000	0.000000	4.000000	0.000000	0.000000	0.000000

ELEMENTS

K	NODE(I,K)		E(K)	G(K)	A(K)	XI(K)	XJ(K)
1	1	2	2.1000000E+08	8.4000000E+07	9.9999998E–03	1.9999999E–04	9.9999997E–05
2	1	3	2.1000000E+08	8.4000000E+07	9.9999998E–03	1.9999999E–04	9.9999997E–05

NODE	DISPLACEMENT	THETA-X	THETA-Z
1	–0.69048E–02	0.30329E–02	–0.29195E–02
2	0.000000E+00	0.00000E+00	0.00000E+00
3	0.000000E+00	0.00000E+00	0.00000E+00

ELEMENTS

K	NODE (I,K)		Y-FORCE	X-MOMENT	Y-MOMENT	Y-FORCE	X-MOMENT	Z-MOMENT
1	1	2	–0.6607E+01	0.6131E+01	0.1863E+02	0.6607E+01	–0.6131E+01	–0.4506E+02
2	1	3	–0.8393E+01	–0.6369E+01	0.1387E+02	0.8393E+01	0.6369E+01	–0.4744E+02

5.52

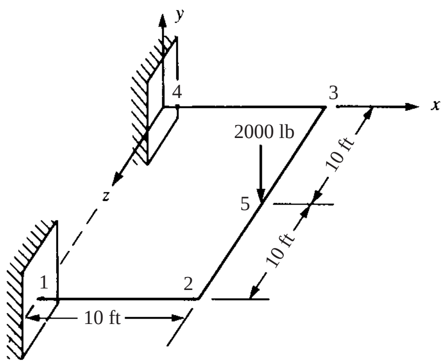
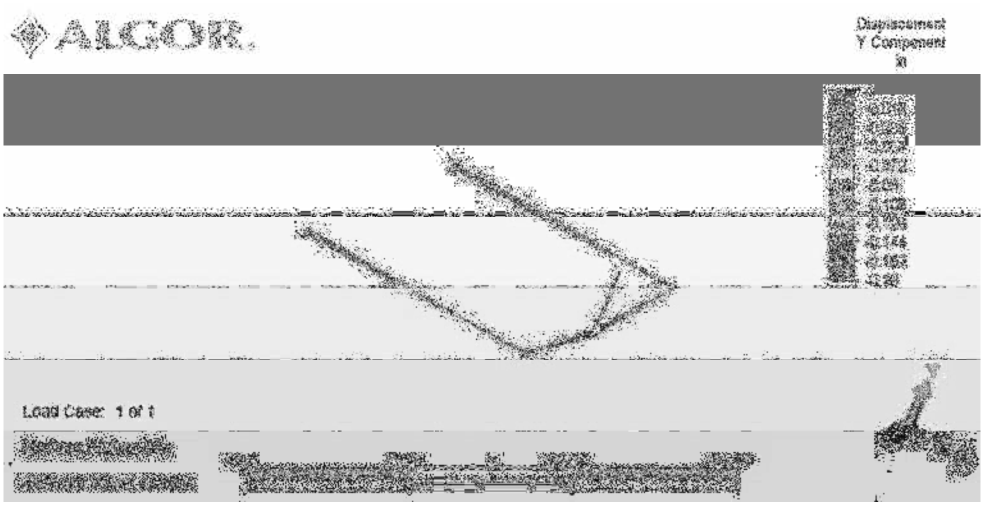


Figure P5–52



5.58-5.59 Determine the displacements and reactions for the space frames shown in Figures P5–58 and P5–59. Let I_x = 100 in.^{4}, I_y = 200 in.^{4}, I_z = 1000 in.^{4}, E = 30,000 ksi, G = 10,000 ksi, and A = 100 in.^{2} for both frames.

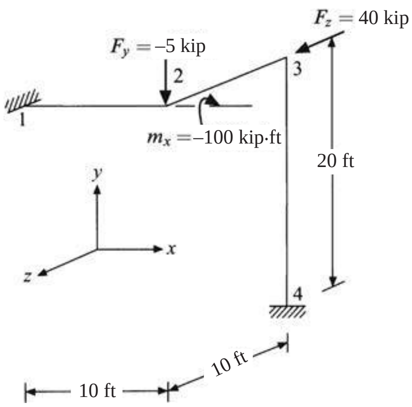


Figure P5–58

Displacements/Rotations (degrees) of nodes

NODE number	X– translation	Y– translation	Z– translation	X– rotation	Y– rotation	Z– rotation
1	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	6.8927E–01	–2.6320E–03	4.4137E–01	3.1601E–01	–7.5875E–01	–4.8913E–01

3	8.3682E–01	1.0091E+00	–5.8406E–02	–3.6119E–01	1.1418E–01	–4.8006E–01
4	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00

5.59

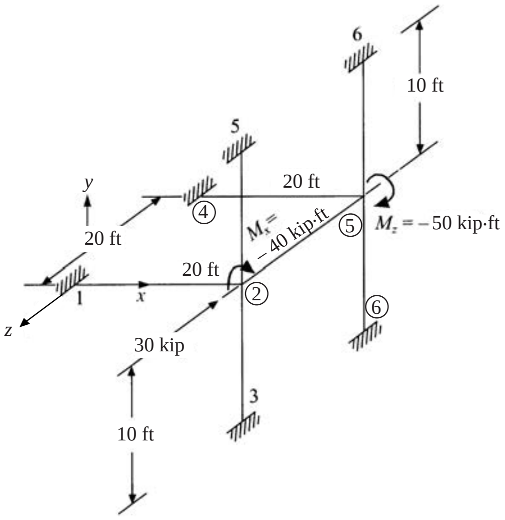
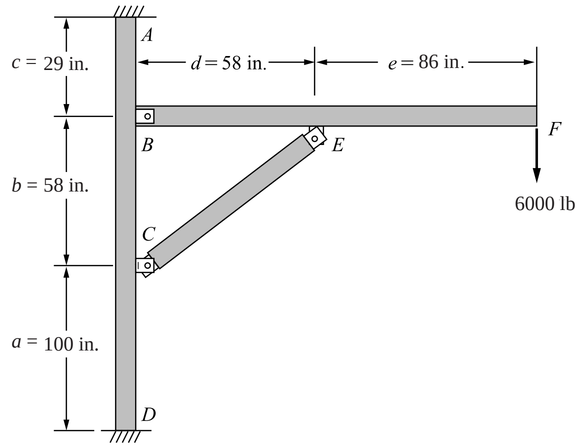


Figure P5–59

Displacements/Rotations (degrees) of nodes

NODE number	X– translation	Y– translation	Z– translation	X– rotation	Y– rotation	Z– rotation
1	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	3.1055E–04	2.5111E–01	1.0906E–05	–1.1112E–01	2.1045E–08	3.6115E–02
3	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
4	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
5	–3.1055E–04	2.4992E–01	–2.5905E–05	1.1109E–02	6.8756E–02	3.5829E–02
6	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
7	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
8	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00

5.60 Design a jib crane as shown in Figure P5–60 that will support a downward load of 6000 lb. Choose a common structural steel shape for all members. Use allowable stresses of 0.66 S_y (S_y is the yield strength of the material) in bending, and 0.60 S_y in. tension



FigureP5–60

Horizontal load beam	S12 × 50
Vertical support beam	S12 × 50
Cross brace	S6 × 12.5

All members A36 structural steel

- The required maximum deflection governed the selection of the material section size, as smaller sizing would be lighter and more than adequate to support the load of 6000 lb, but provide too much deflection than the required 0.400 in.
- The force in the cross brace (21066 lbf) does not yield to buckling as shown in the first set of calculations and with a *S*6 × 12.5 section the cross brace is designed above the imposed load of 6500 lb
- The horizontal load beam is designed to withstand above the imposed bending moment of 516000 lbf·in. The minimum required section of *S*10 × 25.4 was exceeded, as shown in the second set of calculations, to accommodate the required deflection constraint. Also, the excessive section will allow additional safety against failure from overloading.

5.61 Design the support members, *AB* and *CD*, for the platform lift shown in Figure P5-61. Select a mild steel and choose suitable cross-sectional shapes with no more than a 4 : 1 ratio of moments of inertia between the two principal directions of the cross section. You may choose two different cross sections to make up each arm to reduce weight. The actual structure has four support arms, but the loads shown are for one side of the platform with the two arms shown. The loads shown are under operating conditions. Use a factor of safety of 2 for human safety. In developing the finite element model, remove the platform and replace it with statically equivalent loads at the joints at *B* and *D*. Use truss elements or beam elements with low bending stiffness to model the arms from *B* to *D*, the intermediate connection, *E* to *F*, and the hydraulic actuator. The allowable stresses are $0.66S_y$ in bending and $0.60S_y$ in tension. Check buckling using either Euler's method or Johnson's method as appropriate. Also check maximum deflections. Any deflection greater than $\frac{1}{360}$ of the length of member *AB* is considered too large.

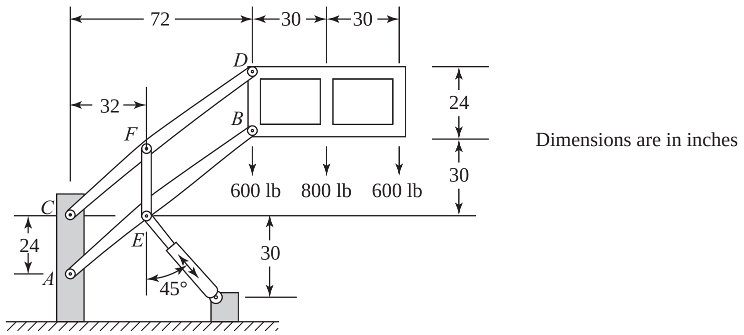


Figure P5-61

- Many viable solutions are possible.
- This design recommends 1020 steel with cross sections of 5 in. × 3 in. × $\frac{3}{8}$ in. rectangular tubing.
- The maximum deflection is then 0.244 in. which is less than the maximum allowable of 0.25 in.
- The bending stress is 11,418 psi which is less than the allowable of 31,600 psi.
- The axial stress is 2520 psi which is less than the allowable of 28,700 psi.

5.62 A two-story building frame is to be designed as shown in Figure P5-62. The members are all to be I-beams with rigid connections. We would like the floor joists beams to have a 15-in. depth and the columns to have a 10 in. width. The material is to be A36 structural steel. Two horizontal loads and vertical loads are shown. Select members such that the allowable bending in the beams is 24,000 psi. Check buckling in the columns using Euler's or Johnson's method as appropriate. The allowable deflection in the beams should not exceed $\frac{1}{360}$ of each beam span. The overall sway of the frame should not exceed 0.5 in.

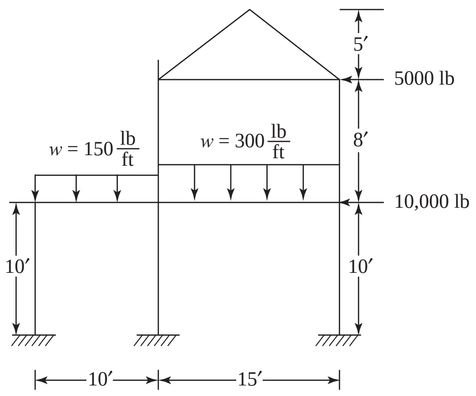


Figure P5-62

- Many viable solutions are possible.
- This design recommends A36 structural steel I-beams.
- W 16 × 26 beams are recommended for the horizontal and diagonal members with largest bending stress of 7000 psi which is less than the allowable of 24,000 psi.
- W 10 × 49 sections are recommended for the vertical members. Column buckling was verified to be satisfied.
- Maximum sway in the horizontal direction is 0.246 in. which is less than the allowable of 0.50 in.
- Another satisfactory solution is W 10 × 26 beams for horizontal and diagonal members and W 10 × 33 sections for the vertical members. The sway then becomes 0.417 in.

5.63 A pulpwood loader as shown in Figure P5–63 is to be designed to lift 2.5 kip. Select a steel and determine a suitable tubular cross section for the main upright member *BF* that has attachments for the hydraulic cylinder actuators *AE* and *DG*. Select a steel and determine a suitable box section for the horizontal load arm *AC*. The horizontal load arm may have two different cross sections *AB* and *BC* to reduce weight. The finite element model should use beam elements for all members except the hydraulic cylinders, which should be truss elements. The pinned joint at *B* between the upright and horizontal beam is best modeled with end release of the end node of the top element on the upright member. The allowable bending stress is 0.66 S_y in members *AB* and *BC*. Member *BF* should be checked for buckling. The allowable deflection at *C* should be less than $\frac{1}{360}$ of the length of *BC*. As a bonus, the client would like you to select the size of the hydraulic cylinders *AE* and *DG*.

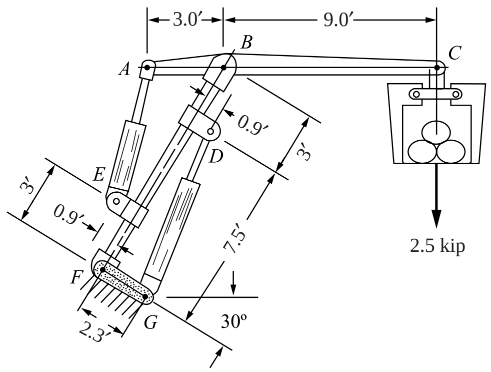


Figure P5–63

- Many viable solutions are possible.
- The design recommends AISI 1020 rolled steel.
- The horizontal beam, AC, is recommended to be a rectangular tube 4 in. by 16 in. with 0.25 in. thickness. The maximum bending stress in member AC is 7736 psi less than the allowable of 3100 psi. The maximum deflection is 0.299 in. less than the allowable of 0.300 in.
- The vertical member, BF, is recommended to be a square tube 10 in. by 10 in. with 0.5 in. thickness.

5.65 A small hydraulic floor crane as shown in Figure P5-65 carries a 5000 lb load. Determine the size of the beam and column needed. Select either a standard box section or a wide-flange section. Assume a rigid connection between the beam and column. The column is rigidly connected to the floor. The allowable bending stress in the beam is $0.60S_y$. The allowable deflection is $\frac{1}{360}$ of the beam length. Check the column for buckling.

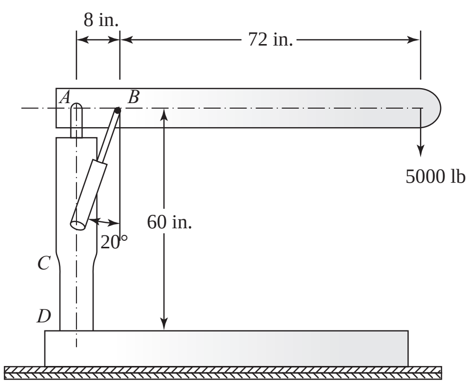


Figure P5-65

- Many viable solutions are possible.
- The design recommends A36 structural steel.
- The horizontal and vertical members are recommended to be W 10×68.
- The largest bending stress in the horizontal beam is 4756 psi less than the allowable of 21,600 psi. The maximum deflection is 0.215 in. less than the allowable of 0.222 in.
- The column, ACD, has a bending stress of 5284 psi.
- The column should be checked for buckling.

5.68 Design the gabled frame subjected to the external wind load shown (comparable to an 80 mph wind speed) for an industrial building. Assume this is one of a typical frame spaced every 20 feet. Select a wide flange section based on allowable bending stress of 20 ksi and an allowable compressive stress of 10 ksi in any member. Neglect the possibility of buckling in any members. Use ASTM A36 steel.

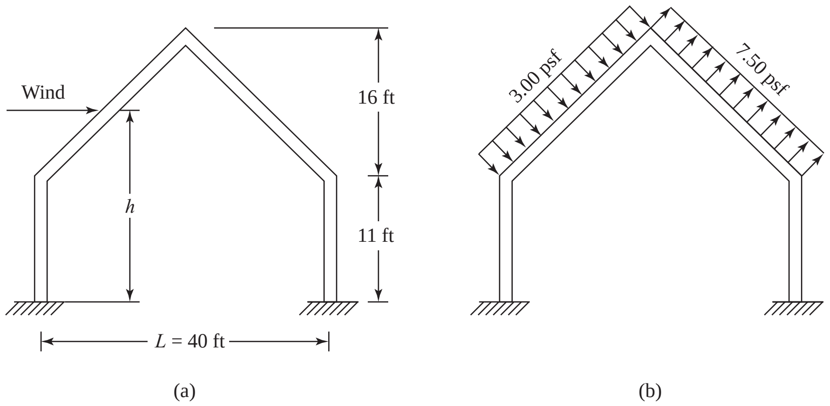


Figure P5-68

- Many viable solutions are possible.
- The design recommends A36 structural steel.
- The frame members are recommended to be W 10 × 12 wide flange shapes.
- The maximum worst stress (combined bending and compression) in any member is 19.5 ksi.
- The maximum displacement is 0.719 in.

5.69 Design the gabled frame shown for a balanced snow load shown (typical of the Mid-west) for an apartment building. Select a wide flange section for the frame. Assume the allowable bending stress not to exceed 140 MPa. Use ASTM A36 steel.

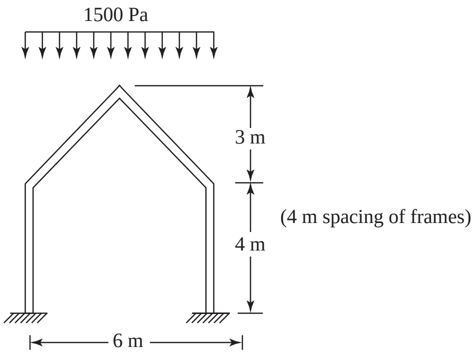


Figure P5-69

- Many viable solutions are possible.
- The table below lists some W sections that were considered.
- The recommended W 6 × 12 with bending stress of 120.4 MPa is less than the allowable value of 140 MPa.
- The maximum displacement is 0.0147 m.

Table 1: Beam Trial Runs

Beam Section	Bending Stress (Local 3)	Displacement
W30 × 173	1.6 MPa	6.26*10^–5
W12 × 45	15.2 MPa	0.0010 m
W8 × 13	88.7 MPa	0.0082 m
W6 × 12	120.4 MPa	0.0147 m

5.70 Design a gantry crane that must be able to lift 10 tons as it must lift compressors, motors, heat exchangers, and controls. This load should be placed at the center of one of the main 12-foot-long beams as shown in Figure P5-70 by the hoisting device location. Note that this beam is on one side of the crane. Assume you are using ASTM A36 structural steel.

- Many viable solutions are possible.
- The design is based on the load being applied to the center span of a 12 ft long beam.
- The design recommends W 10×100 for the horizontal beams.
- The bracing members are recommended to be W 4×13.
- The vertical columns are recommended to be 4 in. $\times$ 4 in. $\times \frac{1}{4}$ in. thick hollow square tubes.
- The table below is a summary of the final member sizes and deflection, stress, and buckling calculated and allowable results.

Table 1: This table shows all the members with corresponding material and size.

Member	Quantity	Material	Size (in. $\times \frac{\text{lb}}{\text{ft}}$)
loaded 12 ft beam	1	ASTM A36 St. Steel	W10 $\times$ 100
unloaded 12 ft beam	1	ASTM A36 St. Steel	W10 $\times$ 100
8 ft Beams	2	ASTM A36 St. Steel	W10 $\times$ 100
Corner Braces	8	ASTM A36 St. Steel	W4 $\times$ 13
Columns	4	ASTM A36 St. Steel	4×4 hollow $\times \frac{1}{4}$ thick (in.)

Table 2: This table shows that the maximum deflections are less than the allowable deflection, and that the calculated bending stresses are less than the allowable stresses in the beams.

	Calculations				
	Maximum Deflection (in.)		Allowable Deflection (in.)	Bending Stress (psi)	
Member	By Hand	Using Algor		Calculated	Allowable
Loaded 12 ft Beam	0.0722	0.0847	0.2667	6405	7200
Unloaded 12 ft Beam	–	0.0141	0.2667	23.145	7200
8 ft Beams	–	0.01977	0.2667	1.286	7200
Corner Braces	–	0.02921	0.1333	256.654	7200
Columns	–	0.2863	0.4000	–	–

Table 3: This table shows that the corner braces and the columns have loads smaller than the load that would cause buckling.

	Buckling Strength (lb)	
Member	Calculated load	Allowable load
Loaded 12 ft Beam	–	–
Unloaded 12 ft Beam	–	–
8 ft Beams	–	–
Corner Braces	24000	330000
Columns	24572	60000

5.71 Design the rigid highway bridge frame structure shown in Figure P5-71 for a moving truck load (shown below) simulating a truck moving across the bridge. Use the load shown and place it along the top girder at various locations. Use the allowable stresses in bending and compression and allowable deflection given in the *Standard Specification for Highway Bridges*, American Association of State Highway and Transportation Officials (AASHTO), Washington, D.C. or use some other reasonable values.

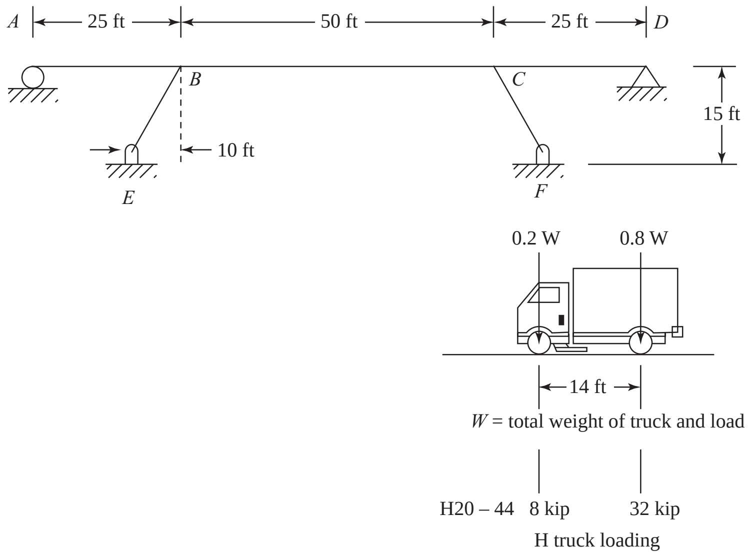


Figure P5-71

- Many viable solutions are possible.
- A36 structural steel is chosen in the design.
- After some iteration, W 24 × 94 wide flange sections were selected for all members.
- The largest bending stress of 12960 psi with the truck in the center span location is less than the allowable of 20,000 psi.
- The largest deflection of 0.731 in. is less than the allowable of 0.75 in. ($\frac{1}{800}$ of the span length).

5.73 The curved semi-circular frame shown in Figure P 5-73 is supported by a pin on the left end and a roller on the right end and is subjected to a load P = 1000 lb at its apex. The frame has a radius to centerline of cross section of R = 120 in. Select a structural steel W shape from Appendix F such that the maximum stress does not exceed 20 ksi. Perform a finite element analysis using 4, 8, and then 16 elements in your finite element model. Also determine the maximum deflection for each model. It is suggested that the finite element answers for deflection be compared to the solution obtained by classical methods, such as using Castigliano's theorem. The expression for deflection under the load is given by using Castigliano's theorem as

$$\delta_y = \frac{0.178\,PR^3}{EI} + \frac{0.393\,PR}{AE} + \frac{0.393\,PR}{A_v G}$$

where A is the cross sectional area of the W shape, A_v is the shear area of the W shape (use depth of web times thickness of web for the shear area), $E = 30 \times 10^6$ psi, and $G = 11.5 \times 10^6$ psi.

Now change the radius of the frame to 20 in. and repeat the problem. Run the finite element model with the shear area included in your computer program input and then without. Comment on the difference in results and compare to the predicted analytical deflection by using the equation above for δ_y.

For R = 20 in. A = 8.79 in. t_{web} = 0.260 in. depth = 12.34 in.

$SA2 = t_{\text{web}}$ depth $SA2 = 3.2084 \times 10^0$ in.

$$I_x = 238 \text{ in.}^4 \quad E = 29 \times 10^6 \text{ psi} \quad G = 11.6 \times 10^6 \text{ psi} \quad P = 1000 \text{ lb}$$

$$\delta_m = \frac{0.178\,PR^3}{EI_x} \qquad \delta_m = 2.06317 \times 10^{-4} \text{ in.}$$

$$\delta_n = \frac{0.393\,PR}{AE} \qquad \delta_n = 3.08344 \times 10^{-5} \text{ in.}$$

$$\delta_v = \frac{0.393\,PR}{SA2G} \qquad \delta_v = 2.11191 \times 10^{-4} \text{ in.}$$

$$\delta_{\max} = \delta_m + \delta_n + \delta_v$$

$$\delta_{\max} = 4.48343 \times 10^{-4} \text{ in.}$$

$$\delta_{\text{maxnoshear}} = \delta_m + \delta_n$$

$$\delta_{\text{maxnoshear}} = 2.37151 \times 10^{-4} \text{ in.}$$

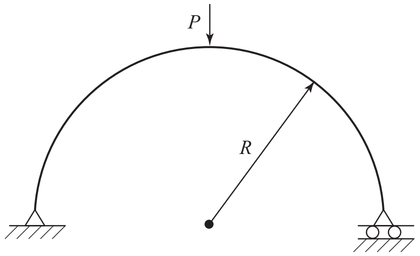


Figure P5-73

- The table below shows the results for the 2 – 16 element models without shear area and with shear area included in some cases for both radii.

Radius = 120 in.

No. of Elements	Max Def. (in.)	Max. Stress (psi)
2	5.92E-02	1595
4	4.69E-02	1576
8	4.52E-02	1566
16	4.48E-02	1560
8 (SA2 incl.)	4.65E-02	1566
16 (SA2 incl.)	4.62E-02	1560
Longhand	4.48E-02	
Longhand (SA2 incl.)	4.60E-02	

Radius = 20 in.

No. of Elements	Max Def. (in.)	Max. Stress (psi)
2	3.01E-04	299
4	2.47E-04	281
8	2.39E-04	270
16	2.39E-04	265
8 (SA2 incl.)	4.55E-04	270
16 (SA2 incl.)	4.55E-04	265
Longhand	2.37E-04	
Longhand (SA2 incl.)	4.48E-04	

Chapter 6

6.1 For sketch of N_i see Figure 6.8. Others follow similarly

By Equation (6.2.18)

$$N_i + N_j + N_m = \frac{1}{2A}\ [(\alpha_i + \alpha_j + \alpha_m) + (\beta_i + \beta_j + \beta_m)\, x + [\gamma_i + \gamma_j + \gamma_m)y] \tag{1}$$

By Equation (6.2.10)

$$\alpha_i + \alpha_j + \alpha_m = x_j\, y_m - y_j\, x_m + y_i\, x_m - x_i\, y_m + x_i\, y_j - y_i\, x_j$$

$$= 2A \text{ (by Equation (6.2.9))} \tag{2}$$

$$\beta_i + \beta_j + \beta_m = y_j - y_m + y_m - y_i + y_i - y_j = 0 \tag{3}$$

$$\gamma_i + \gamma_j + \gamma_m = x_m - x_j + x_i - x_m + x_j - x_i = 0 \tag{4}$$

By using (2)–(4) in (1), we obtain

$$N_i + N_j + N_m = 1 \text{ identically}$$

6.2 By Equation (6.2.47)

$$\pi_p = \frac{1}{2}\ \{d\}^T \iint_v\!\!\int\ [B]^T [D]\, [B]\, dV\{d\} - \{d\}^T\, \{f\}$$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix},\ \{d\} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

$$\therefore\quad \pi_p = \frac{1}{2}\ [u_i\, v_i\, u_j\, v_j\, u_m\, v_m] \iint_v\!\!\int \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \gamma_i \\ 0 & \gamma_i & \beta_i \\ \beta_j & 0 & \gamma_j \\ 0 & \gamma_j & \beta_j \\ \beta_m & 0 & \gamma_m \\ 0 & \gamma_m & \beta_m \end{bmatrix}$$

$$\frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_i & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix} dV \times \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

$$-\ [u_i\, v_i\, u_j\, v_j\, u_m\, v_m]\ \{f\}$$

$$\frac{\partial \pi}{\partial u_i} = \underbrace{2\frac{1}{2}\left(\frac{1}{2A}\right)\left(\frac{E}{1-v^2}\right)\frac{1}{2A}}_{\substack{\parallel \\ C}} \int_v [\beta_i \;\; 0 \;\; \gamma_i] \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{Bmatrix} \beta_i \\ 0 \\ \gamma_i \end{Bmatrix} dV \, u_i - f_{1x}$$

$$\frac{\partial \pi}{\partial v_i} = 2C \int_v [0 \;\; \gamma_i \;\; \beta_i] \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{Bmatrix} 0 \\ \gamma_i \\ \beta_i \end{Bmatrix} dV \, v_i - f_{1y}$$

$$\frac{\partial \pi}{\partial u_i} = 2C \int_v \left[\beta_j \;\; 0 \;\; \gamma_j\right] \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{Bmatrix} \beta_i \\ 0 \\ \gamma_j \end{Bmatrix} dV \, u_j - f_{2x}$$

$$\frac{\partial \pi}{\partial v_j} = 2C \int_v \left[0 \;\; \gamma_j \;\; \beta_j\right] \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{Bmatrix} 0 \\ \gamma_i \\ \beta_i \end{Bmatrix} dV \, v_j - f_{2y}$$

$$\frac{\partial \pi}{\partial u_m} = 2C \int_v [\beta_m \;\; 0 \;\; \gamma_m] \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{Bmatrix} \beta_m \\ 0 \\ \gamma_m \end{Bmatrix} dV \, u_m - f_{3x}$$

$$\frac{\partial \pi}{\partial v_m} = 2C \int_v [0 \;\; \gamma_m \;\; \beta_m] \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{Bmatrix} 0 \\ \gamma_m \\ \beta_m \end{Bmatrix} dV \, v_m - f_{3y}$$

$$\therefore \quad \frac{\partial \pi}{\partial \{d\}} = 2C \int_v \begin{bmatrix} \beta_i & 0 & \gamma_i \\ 0 & \gamma_i & \beta_i \\ \beta_i & 0 & \gamma_j \\ 0 & \gamma_j & \beta_j \\ \beta_m & 0 & \gamma_m \\ 0 & \gamma_m & \beta_m \end{bmatrix} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{bmatrix} \beta_i & 0 & \beta_i & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_m & \gamma_m & \beta_m \end{bmatrix}$$

$$\times dV \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} - \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{Bmatrix}$$

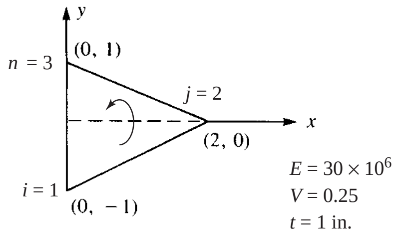
From Equation (3.10.27) or Equation (6.2.48)

$$\frac{\partial \pi}{\partial \{d\}} = 0$$

$$\therefore \quad \frac{\partial \pi}{\partial \{d\}} = \int_v [B]^T \, [D] \, [B] \, dV \{d\} - \{f\} = 0$$

6.3

(a)



(a)

$$[k] = t\,A\,[B]^T\,[D]\,[B]$$

$$x_i = 0,\ y_i = -1,\ x_j = 2,\ y_j = 0,\ x_m = 0,\ y_m = 1$$

$$A = \frac{1}{2}\,b\,h = \frac{1}{2}(2)(2) = 2 \text{ in.}^2$$

$$\beta_i = y_j - y_m = 0 - 1 = -1$$

$$\beta_j = y_m - y_i = 1 - (-1) = 2$$

$$\beta_m = y_i - y_j = -1 - 0 = -1$$

$$\gamma_i = x_m - x_j = 0 - 2 = -2$$

$$\gamma_j = x_i - x_m = 0 - 0 = 0$$

$$\gamma_m = x_j - x_i = 2 - 0 = 2$$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

Since it is plane stress $[D] = \dfrac{E}{(1-v^2)}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$

$$\text{So } [B]^T\,[D] = \frac{30\times10^6}{4(0.9375)}\begin{bmatrix} -1 & 0 & -2 \\ 0 & -2 & -1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -0.25 & -0.75 \\ -0.5 & -2 & -0.375 \\ 2 & 0.5 & 0 \\ 0 & 0 & 0.75 \\ -1 & -0.25 & 0.75 \\ 0.5 & 2 & -0.375 \end{bmatrix}\frac{30\times10^6}{4(0.9375)}$$

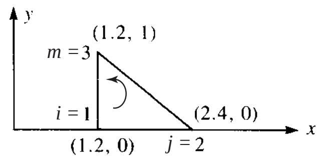
$$[k] = t\,A[B]^T\,[D]\,[B]$$

$$\Rightarrow \quad [k] = (1\,\text{in.})(2)\frac{30\times10^6}{4(0.9375)}\begin{bmatrix} -1 & -0.25 & -0.75 \\ -0.5 & -2 & -0.375 \\ 2 & 0.5 & 0 \\ 0 & 0 & 0.75 \\ -1 & -0.25 & 0.75 \\ 0.5 & 2 & -0.375 \end{bmatrix}$$

$$\frac{1}{4}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix}$$

$$\begin{array}{c} \qquad\qquad\quad i=1 \qquad\qquad\qquad j=2 \qquad\qquad\qquad m=3 \\ [k] = 4.0\times10^6 \begin{bmatrix} 2.5 & 1.25 & -2 & -1.5 & -0.5 & 0.25 \\ 1.25 & 4.375 & -1 & -0.75 & -0.25 & -3.625 \\ -2 & -1 & 4 & 0 & -2 & 1 \\ -1.5 & -0.75 & 0 & 1.5 & 1.5 & -0.75 \\ -0.5 & -0.25 & -2 & 1.5 & 2.5 & -1.25 \\ 0.25 & -3.625 & 1 & -0.75 & -1.25 & 4.375 \end{bmatrix} \end{array}$$

(b) $x_i = 1.2$, $y_i = 0$, $x_j = 2.4$, $y_j = 0$, $x_m = 1.2$, $y_m = 1$



(b)

$$\beta_i = y_j - y_m = 0 - 1 = -1$$

$$\beta_j = y_m - y_j = 1 - 0 = 1$$

$$\beta_m = y_i - y_j = 0 - 0 = 0$$

$$\gamma_i = x_m - x_j = 1.2 - 2.4 = -1.2$$

$$\gamma_j = x_i - x_m = 1.2 - 1.2 = 0$$

$$\gamma_m = x_j - x_i = 2.4 - 1.2 = 1.2$$

$$A = \frac{1}{2}(1.2)(1) = 0.6\ \text{in.}^2$$

$$\text{So } [B]^T[D] = \frac{30\times10^6}{(1.2)(0.9375)}\begin{bmatrix} -1 & 0 & -1.2 \\ 0 & -1.2 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1.2 \\ 0 & 1.2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$= \frac{25\times10^6}{0.9375}\begin{bmatrix} -1 & -0.25 & -0.45 \\ -0.3 & -1.2 & -0.375 \\ 1 & 0.25 & 0 \\ 0 & 0 & 0.375 \\ 0 & 0 & 0.45 \\ 0.3 & 1.2 & 0 \end{bmatrix}$$

$$[k] = t\,A[B]^T\,[D]\,[B]$$

$$[k] = \frac{(1\text{in.})(6\text{in.}^2)25\times10^6}{2(0.6)\,(0.9375)}\begin{bmatrix} -1 & -0.25 & -0.45 \\ -0.3 & -1.2 & -0.375 \\ 1 & 0.25 & 0 \\ 0 & 0 & 0.375 \\ 0 & 0 & 0.45 \\ 0.3 & 1.2 & 0 \end{bmatrix}$$

$$\times\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1.2 & 0 & 0 & 0 & 1.2 \\ -1.2 & -1 & 0 & 1 & 1.2 & 0 \end{bmatrix}$$

$$[k] = \frac{25\times10^6}{1.875}\begin{array}{c} \begin{array}{ccc} i=1 & j=2 & m=3 \end{array} \\ \begin{bmatrix} 1.54 & 0.75 & -1 & -0.45 & -0.54 & -0.3 \\ 0.75 & 1.815 & -0.3 & -0.375 & -0.45 & -1.44 \\ -1 & -0.3 & 1 & 0 & 0 & 0.3 \\ -0.45 & -0.375 & 0 & 0.375 & 0.45 & 0 \\ -0.54 & -0.45 & 0 & 0.45 & 0.54 & 0 \\ -0.3 & -1.44 & 0.3 & 0 & 0 & 1.44 \end{bmatrix} \end{array}$$

(c) $E = 30\times10^6 \qquad \nu = 0.25 \qquad t = 1$

Triangle coordinate definition

$i = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} x = 0 \\ y = 1 \end{matrix}$ This defines an array variable
x coordinate is the top
y coordinate is the bottom

$j = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ Area of triangle = $\frac{1}{2}$ base × height

$m = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad A = \frac{1}{2}\,(j_x - i_x)\,(m_y - i_y)$

$A = 1$

Develop stiffness matrix

$\beta_i = j_y - m_y \qquad \beta_i = -1 \qquad \gamma_i = m_x - j_x \qquad \gamma_i = -2$

$\beta_j = m_y - i_y \qquad \beta_j = 1 \qquad \gamma_j = i_x - m_x \qquad \gamma_j = 0$

$\beta_m = i_y - j_y \qquad \beta_m = 0 \qquad \gamma_m = j_x - i_x \qquad \gamma_m = 2$

$$[B_i] = \frac{1}{2A}\begin{pmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \gamma_i & \beta_i \end{pmatrix} \qquad [B_j] = \frac{1}{2A}\begin{pmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \gamma_i & \beta_j \end{pmatrix}$$

$$[B_m] = \frac{1}{2A}\begin{pmatrix} \beta_m & 0 \\ 0 & \gamma_m \\ \gamma_m & \beta_m \end{pmatrix}$$

Gradient matrix

$$[B] = \text{augment}\,(B_i, B_j, B_m)$$

$$[B] = \begin{pmatrix} -0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -0.5 & 0 & 0.5 & 1 & 0 \end{pmatrix}$$

Plane stress

Constitutive matrix

$$[D] = \frac{E}{1-v^2}\begin{pmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{pmatrix}$$

$$[D] = \begin{pmatrix} 3.2\times10^7 & 8\times10^6 & 0 \\ 8\times10^6 & 3.2\times10^7 & 0 \\ 0 & 0 & 1.2\times10^7 \end{pmatrix}$$

$[k] = t\,A\,[B]^T\,[D]\,[B]$ Constant-strain triangular element stiffness matrix

$$[k] = \begin{pmatrix} 2\times10^7 & 1\times10^7 & -8\times10^6 & -6\times10^6 & -1.2\times10^7 & -4\times10^6 \\ 1\times10^7 & 3.5\times10^7 & -4\times10^6 & -3\times10^6 & -6\times10^6 & -3.2\times10^7 \\ -8\times10^6 & -4\times10^6 & 8\times10^6 & 0 & 0 & 4\times10^6 \\ -6\times10^6 & -3\times10^6 & 0 & 3\times10^6 & 6\times10^6 & 0 \\ -1.2\times10^7 & -6\times10^6 & 0 & 6\times10^6 & 1.2\times10^7 & 0 \\ -4\times10^6 & -3.2\times10^7 & 4\times10^6 & 0 & 0 & 3.2\times10^7 \end{pmatrix}$$

6.4 In general we know that

$$\{\sigma\} = \frac{E}{(1-v^2)}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \times \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

(a) For the first element we have

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{30\times10^6}{(1-0.25^2)}\begin{bmatrix}1 & 0.25 & 0\\ 0.25 & 1 & 0\\ 0 & 0 & 0.375\end{bmatrix}\times\frac{1}{2(2)}\begin{bmatrix}-1 & 0 & 2 & 0 & -1 & 0\\ 0 & -2 & 0 & 0 & 0 & 2\\ -2 & -1 & 0 & 2 & 2 & -1\end{bmatrix}$$

$$\begin{bmatrix}0\\ 0.0025\\ 0.0012\\ 0\\ 0.0\\ 0.0025\end{bmatrix} = \begin{Bmatrix}19200\text{ psi}\\ 4800\text{ psi}\\ -15000\text{ psi}\end{Bmatrix}$$

The principal stresses are given by the equations

$$\sigma_{1,2} = \frac{\sigma_x+\sigma_y}{2}\pm\left[\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2\right]^{\frac{1}{2}}$$

and the plane that are acting upon is

$$\theta_p = \frac{1}{2}\tan^{-1}\frac{\tau_{xy}}{\left(\frac{\sigma_x-\sigma_y}{2}\right)}$$

$$\sigma_1 = \frac{19200+4800}{2}+\left[\left(\frac{19200-4800}{2}\right)^2+(-15000)^2\right]^{\frac{1}{2}}$$

$$= 28639\text{ psi}$$

$$\sigma_2 = \frac{19200+4800}{2}-\left[\left(\frac{19200-4800}{2}\right)^2+(-15000)^2\right]^{\frac{1}{2}}$$

$$= -4639\text{ psi}$$

$$\theta_p = \frac{1}{2}\tan^{-1}\frac{-15000}{\left(\frac{19200-4800}{2}\right)} = -32.2°$$

(b) For the second element we have

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{30\times10^6}{(1-0.25)^2}\begin{bmatrix}1 & 0.25 & 0\\ 0.25 & 1 & 0\\ 0 & 0 & 0.375\end{bmatrix}\times\frac{1}{2(0.6)}\begin{bmatrix}-1 & 0 & 1 & 0 & 0 & 0\\ 0 & -1.2 & 0 & 0 & 0 & 1.2\\ -1.2 & -1 & 0 & 1 & 1.2 & 0\end{bmatrix}$$

$$\times\begin{bmatrix}0\\ 0.0025\\ 0.0012\\ 0\\ 0.0\\ 0.0025\end{bmatrix}$$

$$= \begin{Bmatrix}\sigma_x = 32000\text{ psi}\\ \sigma_y = 8000\text{ psi}\\ \tau_{xy} = -25000\text{ psi}\end{Bmatrix}$$

$$\sigma_1 = \frac{32000+8000}{2} + \left[\left(\frac{32000-8000}{2}\right)^2 + (-25000)^2\right]^{\frac{1}{2}}$$

$$\sigma_1 = 47731 \text{ psi}$$

$$\sigma_2 = \frac{32000+8000}{2} - \left[\left(\frac{32000-8000}{2}\right)^2 + (-25000)^2\right]^{\frac{1}{2}}$$

$$\sigma_2 = -7731 \text{ psi}$$

$$\theta_p = \frac{1}{2} \tan^{-1} \frac{-25000}{\left(\frac{32000-8000}{2}\right)}$$

$$\theta_p = -32.2°$$

(c) For third element we have

$$[B] = \begin{pmatrix} -0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -0.5 & 0 & 0.5 & 1 & 0 \end{pmatrix} \frac{1}{\text{in.}}$$

$$[D] = \begin{pmatrix} 3.2\times10^7 & 8\times10^6 & 0 \\ 8\times10^6 & 3.2\times10^7 & 0 \\ 0 & 0 & 1.2\times10^7 \end{pmatrix} \frac{\text{lb}}{\text{in.}^2}$$

$u_1 = 0.0$ in. $\quad v_1 = 0.0025$ in.

$u_2 = 0.0012$ in. $\quad v_2 = 0.0$ in.

$u_3 = 0.0$ in. $\quad v_3 = 0.0025$ in.

$$\{d\} = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} \text{ Displacement matrix}$$

Stress evaluation

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = [D]\,[B]\,\{d\}$$

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} 1.92\times10^4 \\ 4.8\times10^3 \\ -1.5\times10^4 \end{pmatrix} \frac{\text{lb}}{\text{in.}^2}$$

Principal stresses

$$\sigma_1 = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_2 = \frac{\sigma_x+\sigma_y}{2} - \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_{\max} = 2.864 \times 10^4 \ \frac{\text{lb}}{\text{in.}^2} = \sigma_1$$

$$\sigma_{\min} = -4.639 \times 10^3 \ \frac{\text{lb}}{\text{in.}^2} = \sigma_2$$

Principal angle

$$\theta_p = \frac{\operatorname{atan}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$$\theta_p = -32.179 \text{ deg.}$$

6.5 Von Mises stress for biaxial stress state

(a) $\sigma_1 = 28639$ $\qquad \sigma_2 = -4639$

$$\sigma_e = \sqrt{\sigma_1{}^2 + \sigma_2{}^2 - \sigma_1\sigma_2} \qquad \sigma_e = 3.122 \times 10^4 \text{ psi}$$

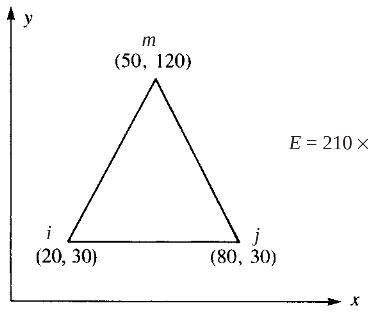
(b) $\sigma_1 = 47731$ $\qquad \sigma_2 = -7731$

$$\sigma_e = \sqrt{\sigma_1{}^2 + \sigma_2{}^2 - \sigma_1\sigma_2} \qquad \sigma_e = 5.203 \times 10^4 \text{ psi}$$

(c) $$\sigma_e = \sqrt{\sigma_1{}^2 + \sigma_2{}^2 - \sigma_1\sigma_2}$$

$$\sigma_e = 3.122 \times 10^4 \ \frac{\text{lb}}{\text{in.}^2}$$

6.6



$E = 210 \times 10^9$ Pa, $\nu = 0.25$, $t = 0.010$ m

(a)

$$\beta_i = y_j - y_m = 30 - 120 = -90 \qquad \gamma_i = x_m - x_i = 50 - 80 = -30$$

$$\beta_j = y_m - y_i = 120 - 30 = 90 \qquad \gamma_i = x_i - x_m = 20 - 50 = -30$$

$$\beta_m = y_i - y_j = 30 - 30 = 0 \qquad \gamma_m = x_j - x_i = 80 - 20 = 60$$

$$2A = x_i(y_j - y_m) + x_j(y_m - y_i) + x_m(y_i - y_j)$$

$$= 20(-90) + 80(90) + 50(0) = 5400 \text{ mm}^2$$

$$[B] = \frac{1}{5400}\begin{bmatrix} -90 & 0 & 90 & 0 & 0 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -90 & -30 & 90 & 60 & 0 \end{bmatrix}$$

$$[D] = \frac{210 \times 10^9}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} = 2.24 \times 10^{11}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$[k] = t\,A\,[B]^T\,[D]\,[B]$$

$$[k] = (0.01)\left(\frac{5.4\times10^{-3}}{2}\right)\left(\frac{1}{5.4\times10^{-3}}\right)\begin{bmatrix} -90 & 0 & -30 \\ 0 & -30 & -90 \\ 90 & 0 & -30 \\ 0 & -30 & 90 \\ 0 & 0 & 60 \\ 0 & 60 & 0 \end{bmatrix}(2.24\times10^{11})\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}[B]$$

$$[k] = 1.12\times10^{9}\begin{bmatrix} -90 & -22.5 & -11.25 \\ -7.5 & -30 & -33.75 \\ 90 & 22.5 & -11.25 \\ -7.5 & -30 & 33.75 \\ 0 & 0 & 22.5 \\ 15 & 60 & 0 \end{bmatrix}\frac{1}{5.4\times10^{-3}}\begin{bmatrix} -90 & 0 & 90 & 0 & 0 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -90 & -30 & 90 & 60 & 0 \end{bmatrix}$$

$$[k] = 2.074\times10^{5}\begin{bmatrix} 8437.5 & 1687.5 & -7762.5 & -337.5 & -675 & -1350 \\ 1687.5 & 3937.5 & 337.5 & -2137.5 & -2025 & -1800 \\ -7762.5 & 337.5 & 8437.5 & -1687.5 & -675 & 1350 \\ -337.5 & -2137.5 & -1687.5 & 3937.5 & 2025 & -1800 \\ -675 & -2025 & -675 & 2025 & 1350 & 0 \\ -1350 & -1800 & 1350 & -1800 & 0 & 3600 \end{bmatrix}$$

(b) Similarly

$\beta_i = -5 \qquad \gamma_i = 0$

$\beta_j = 2.5 \qquad \gamma_j = -5$

$\beta_m = 2.5 \qquad \gamma_m = 5$

$$[k] = 4.48\times10^{7}\begin{bmatrix} 25.0 & 0 & -12.5 & 6.25 & -12.5 & -6.25 \\ & 9.375 & 9.375 & -4.6875 & -9.375 & -4.6875 \\ & & 15.625 & -7.8125 & -3.125 & -1.5625 \\ & & & 27.343 & 1.5625 & -3.125 \\ & & & & 15.625 & 7.8125 \\ \text{Symmetry} & & & & & 27.343 \end{bmatrix}$$

Now solve P6-6c for stiffness matrix

$t = 0.01$

$A = \dfrac{tA}{2} \qquad A = 5\times10^{-5}$

$[k] = t\,A\,[B]^T\,[D]\,[B]$

$$[k] = \begin{pmatrix} 1.225\times10^9 & 3.5\times10^8 & -1.015\times10^9 & -7\times10^7 & -2.1\times10^8 & -2.8\times10^8 \\ 3.5\times10^8 & 7\times10^8 & 7\times10^7 & -1.4\times10^8 & -4.2\times10^8 & -5.6\times10^8 \\ -1.015\times10^9 & 7\times10^7 & 1.225\times10^9 & -3.5\times10^8 & -2.1\times10^8 & 2.8\times10^8 \\ -7\times10^7 & -1.4\times10^8 & -3.5\times10^8 & 7\times10^8 & 4.2\times10^8 & -5.6\times10^8 \\ -2.1\times10^8 & -4.2\times10^8 & -2.1\times10^8 & 4.2\times10^8 & 4.2\times10^8 & 0 \\ -2.8\times10^8 & -5.6\times10^8 & 2.8\times10^8 & -5.6\times10^8 & 0 & 1.12\times10^9 \end{pmatrix}$$

6.7 (a) By Equation (6.2.36)

$$\{\sigma\} = [D]\,[B]\,\{d\}$$

Using results of Problem 6.5 (a)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{2.24\times10^{11}}{5400\times10^{-3}} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \begin{bmatrix} -90 & 0 & 90 & 0 & 0 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -90 & -30 & 90 & 60 & 0 \end{bmatrix}$$

$$\times\, 10^{-3}\ \frac{\text{m}}{\text{mm}} \begin{Bmatrix} 0.002 \\ 0.001 \\ 0.0005 \\ 0 \\ 0.003 \\ 0.001 \end{Bmatrix}$$

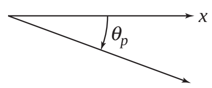
$$= \begin{Bmatrix} -5.29\ \text{GPa} \\ -0.766\ \text{GPa} \\ 0.233\ \text{GPa} \end{Bmatrix}$$

$$\sigma_{1,2} = \frac{-5.29 + (-\,0.156)}{2} \pm \sqrt{\left(\frac{-5.29 + 0.156}{2}\right)^2 + 0.233^2}$$

$$= -\,2.72 \pm 2.58$$

$$\sigma_1 = -\,0.14\ \text{GPa} \qquad \sigma_2 = -\,5.30\ \text{GPa}$$

$$\tan 2\theta_{p_1} = \frac{2(0.233)}{-5.29 + 0.156} = -\,0.091$$



$$\theta_p = -\,2.59°$$

(b) From Problem 6.5 (b) β's and γ's given

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{2.24\times10^{11}}{25\times10^{-6}} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \begin{bmatrix} -5 & 0 & 2.5 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 \\ 0 & -5 & -5 & 2.5 & 5 & 2.5 \end{bmatrix}$$

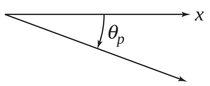
$$\times\, 10^{-3}\ \frac{\text{m}}{\text{mm}} \begin{Bmatrix} 0.002 \\ 0.001 \\ 0.0005 \\ 0 \\ 0.003 \\ 0.001 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 42.0\,\text{GPa} \\ 33.6\,\text{GPa} \end{Bmatrix}$$

$$\sigma_{1,2} = \frac{0+42.0}{2} \pm \sqrt{\left(\frac{-42}{2}\right)^2 + 33.6^2}$$

$$= 21 \pm 39.6$$

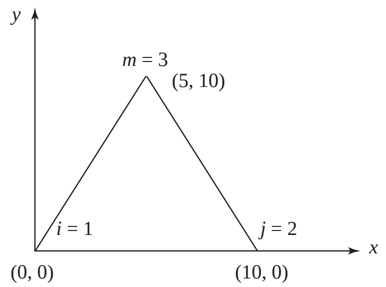
$$\sigma_1 = 60.6 \text{ GPa} \qquad \sigma_2 = -18.6 \text{ GPa}$$

$$\tan 2\theta_p = \frac{2(33.6)}{0-42.0}$$



$$\theta_p = -29°$$

(c)



Plane stress also find von Mises stress

$$\beta_i = -10 \text{ mm} \qquad \gamma_i = -5 \text{ mm}$$

$$\beta_j = 10 \text{ mm} \qquad \gamma_j = -5 \text{ mm}$$

$$\beta_m = 0 \qquad \gamma_m = 10 \text{ mm}$$

$$A = 2.5 \times 10^{-5} \text{ m}^2$$

$$\{\sigma\} = [D]\,[B]\,\{d\}$$

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$

$$[D] = \frac{(210\times 10^9 \frac{\text{N}}{\text{m}^2})}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$[D] = 224 \times 10^9 \ \frac{\text{N}}{\text{m}^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$\{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Determine the stresses in the element with nodal displacements listed

$u_1 = 0.002$ $v_1 = 0.001$ $u_2 = 0.0005$ $v_2 = 0$ $u_3 = 0.003$ $v_3 = 0.001$

Here are the coordinates for the element

$x_1 = 0$ $y_1 = 0$ $x_2 = 0.01$ $y_2 = 0$ $x_3 = 0.005$ $y_3 = 0.01$

$$E = 210 \times 10^9 \quad v = 0.25$$

$$\{d\} = \begin{pmatrix} u_1 \\ v_2 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} \qquad [D] = \begin{pmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{pmatrix} \qquad [D] = [D]\frac{E}{1-v^2}$$

Equation (6.1.8) $$[D] = \begin{pmatrix} 2.24 \times 10^{11} & 5.6 \times 10^{10} & 0 \\ 5.6 \times 10^{10} & 2.24 \times 10^{11} & 0 \\ 0 & 0 & 8.4 \times 10^{10} \end{pmatrix}$$

Equation (6.2.10)

$\beta_1 = y_2 - y_3$ $\beta_1 = -0.01$ $\gamma_1 = x_3 - x_2$ $\gamma_1 = -5 \times 10^{-3}$

$\beta_2 = y_3 - y_1$ $\beta_2 = 0.01$ $\gamma_2 = x_1 - x_3$ $\gamma_2 = -5 \times 10^{-3}$

$\beta_3 = y_1 - y_2$ $\beta_3 = 0$ $\gamma_3 = x_2 - x_1$ $\gamma_3 = 0.01$

$TA = x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)$

$TA = 1 \times 10^{-4}$ twice the area, Equation (6.2.9)

Equation (6.2.32) combined

$$[B] = \frac{1}{TA}\begin{pmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_2 & \beta_3 \end{pmatrix}$$

$$[B] = \begin{pmatrix} -100 & 0 & 100 & 0 & 0 & 0 \\ 0 & -50 & 0 & -50 & 0 & 100 \\ -50 & -100 & -50 & 100 & 100 & 0 \end{pmatrix}$$

In-plane stresses

$$\{\sigma\} = [D]\,[B]\,\{d\}$$

$$\{\sigma\} = \begin{pmatrix} -3.08 \times 10^{10} \\ 2.8 \times 10^9 \\ 6.3 \times 10^9 \end{pmatrix} \qquad \begin{matrix} \sigma_x = \sigma_0 \\ \sigma_y = \sigma_1 \\ \tau_{xy} = \sigma_2 \end{matrix} \qquad \begin{matrix} \sigma_x = -3.08 \times 10^{10} \\ \sigma_y = 2.8 \times 10^9 \\ \tau_{xy} = 6.3 \times 10^9 \end{matrix}$$

Note: use the left bracket after the sigma then the 0, or 1 or 2 for the values in the sigma matrix

Principal stresses

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \left[\left[\left[\frac{(\sigma_x - \sigma_y)}{2}\right]^2 + \tau_{xy}{}^2\right]\right]^{\frac{1}{2}}$$

$\sigma_1 = 3.942 \times 10^9 \dfrac{\text{N}}{\text{m}^2}$

$\text{sqrt} = \left[\left[\left[\dfrac{(\sigma_x - \sigma_y)}{2}\right]^2 + \tau_{xy}{}^2\right]\right]^{\frac{1}{2}}$

$\sigma_{\text{ave}} = \dfrac{\sigma_x + \sigma_y}{2}$

$\sigma_2 = \sigma_{\text{ave}} - \text{sqrt}$

$\theta_p = \dfrac{1}{2} \text{ a tan}\left(2 \dfrac{\tau_{xy}}{\sigma_x - \sigma_y}\right)$

$\sigma_2 = -3.194 \times 10^{10} \dfrac{\text{N}}{\text{m}^2}$

$\theta_p = -0.179$

$\theta_p = -10.278°$ Principal angle

6.8 Von Mises stress

(a) $\sigma_1 = -0.14$ $\sigma_2 = -5.30$

$\sigma_e = \sqrt{\sigma_1{}^2 + \sigma_2{}^2 - \sigma_1\sigma_2}$ $\sigma_e = 5.231$ GPa

(b) $\sigma_1 = 60.6$ $\sigma_2 = -18.6$

$\sigma_e = \sqrt{\sigma_1{}^2 + \sigma_2{}^2 - \sigma_1\sigma_2}$ $\sigma_e = 71.732$ GPa

(c) $\sigma_1 = 3.94$ GPa $\sigma_2 = -31.9$ GPa

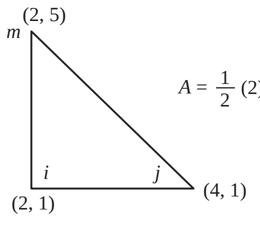
P6.8c von Mises stress, Equation (6.5.37a)

$$\sigma_{vm} = \frac{1}{\sqrt{2}}\left[\left[(\sigma_1 - \sigma_2)^2\right] + \sigma_2{}^2 + \sigma_1{}^2\right]^{\frac{1}{2}}$$

$$\sigma_{vm} = 3.409 \times 10^{10} \frac{\text{N}}{\text{m}^2}$$

6.9

(a)



Plane strain

$\beta_i = y_j - y_m = 1 – 5 = -4$ $\gamma_i = x_m - x_j = 2 - 4 = -2$

$\beta_j = y_m - y_i = 5 - 1 = 4$ $\gamma_j = x_i - x_m = 2 - 2 = 0$

$\beta_m = y_i - y_j = 1 - 1 = 0$ $\gamma_m = x_j - x_i = 4 - 2 = 2$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{30 \times 10^6}{(1+0.25)[1-2(0.25)]} \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.75 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$$

$$\times \left(\frac{1}{8}\right)\begin{bmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -4 & 0 & 4 & 2 & 0 \end{bmatrix}\begin{Bmatrix} 0.001 \\ 0.005 \\ 0.001 \\ 0.0025 \\ 0 \\ 0 \end{Bmatrix}$$

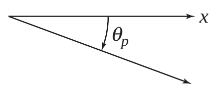
$$= \begin{Bmatrix} -15000 \\ -45000 \\ -18000 \end{Bmatrix} \text{ psi}$$

$$\sigma_{1,2} = \frac{-15000+(-45000)}{2} \pm \sqrt{\left(\frac{-15000-(-45000)}{2}\right)^2 (-18000)^2}$$

$$= -30000 \pm 23430$$

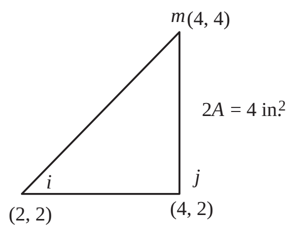
$$\sigma_1 = -6570 \text{ psi} \qquad \sigma_2 = -53{,}430 \text{ psi}$$

$$\tan 2\theta_p = \frac{2(-18000)}{-15000-(-45000)}$$



$$\theta_p = -25.1°$$

(b)



$$\beta_i = 2-4 = -2 \qquad \gamma_i = 4-4 = 0$$

$$\beta_j = 4-2 = 2 \qquad \gamma_j = 2-4 = -2$$

$$\beta_m = 2-2 = 0 \qquad \gamma_m = 4-2 = 2$$

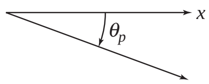
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 48 \times 10^6 \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.75 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}\left(\frac{1}{4}\right)\begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -2 & -2 & 2 & 2 & 0 \end{bmatrix}$$

$$\times \begin{Bmatrix} 0.001 \\ 0.005 \\ 0.001 \\ 0.0025 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -15{,}000 \\ -45{,}000 \\ -21{,}000 \end{Bmatrix} \text{ psi}$$

$$\sigma_{1,2} = \frac{-15000-45000}{2} \pm \sqrt{\left[\frac{-15000-(-45000)}{2}\right]^2 + (-21000)^2}$$

$$\sigma_1 = -4193 \text{ psi} \qquad \sigma_2 = -55{,}800 \text{ psi}$$

$$\tan 2\theta_p = \frac{2(-21000)}{-15000-(-45000)}$$

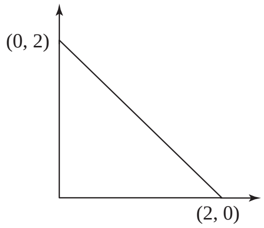


$$\theta_p = -27.2°$$

(c) Given displacements (in.)

$$u_1 = 0.001 \qquad v_1 = 0.005$$
$$u_2 = 0.001 \qquad v_2 = 0.0025$$
$$u_3 = 0 \qquad v_3 = 0$$

$$\{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$



Material definition

$$E = 30 \times 10^6 \text{ psi} \qquad v = 25$$

Geometry description

$$\beta_i = 0-2 \quad (y_j - y_m)$$
$$\beta_j = 2-0 \quad (y_m - y_i)$$
$$\beta_m = 0-0 \quad (y_i - y_j)$$
$$\gamma_i = 0-2 \quad (x_m - x_j)$$
$$\gamma_j = 0-0 \quad (x_i - x_m)$$
$$\gamma_m = 2-0 \quad (x_j - x_i)$$

$$A = \frac{1}{2}\times 2\times 2 \qquad A = 2$$

$$[B] = \frac{1}{2A}\begin{pmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{pmatrix}$$

Plane strain constitutive matrix

$$[D] = \frac{E}{(1+v)(1-2v)}\begin{pmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{pmatrix}$$

Stress matrix

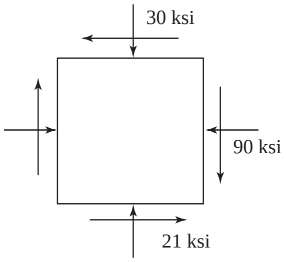
$$\{\sigma\} = [D]\,[B]\,\{d\}$$

$$\sigma = \begin{pmatrix} -3\times 10^4 \\ -9\times 10^4 \\ -2.1\times 10^4 \end{pmatrix}$$

$$\sigma_x = -3 \times 10^4 \text{ (psi)}$$

$$\sigma_y = -9 \times 10^4 \text{ (psi)}$$

$$\tau_{xy} = -2.1 \times 10^4 \text{ (psi)}$$



$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_1 = -2.338 \times 10^4 \text{ psi}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_2 = -9.662 \times 10^4 \text{ psi}$$

$$\theta_p = \frac{1}{2}\operatorname{atan}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$$

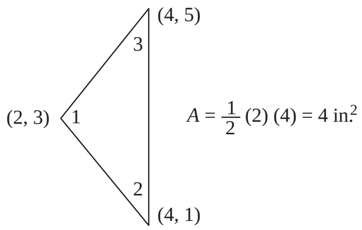
$$\theta_p = -0.305 \text{ (rad)}$$

$$\sigma_3 = 0$$

$$\sigma_{vm} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

$$\sigma_{vm} = 8.731 \times 10^4 \text{ (psi)}$$

(d)



$\beta_1 = -4 \qquad \gamma_1 = 0$

$\beta_2 = 2 \qquad \gamma_2 = -2$

$\beta_3 = 2 \qquad \gamma_3 = 2$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 48 \times 10^6 \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.75 & 0 \\ 0 & 0 & 0.25 \end{bmatrix} \left(\frac{1}{8}\right) \begin{bmatrix} -4 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -4 & -2 & 2 & 2 & 2 \end{bmatrix}$$

$$\begin{Bmatrix} 0.001 \\ 0.005 \\ 0.001 \\ 0.0025 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -16.5 \\ -25.5 \\ -25.5 \end{Bmatrix} \text{ ksi}$$

$$\sigma_{1,2} = \frac{-16.5 - 25.5}{2} \pm \sqrt{\left(\frac{-16.5 + 25.5}{2}\right)^2 + (-25.5)^2}$$

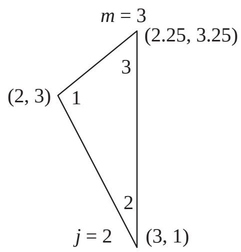
$$\sigma_1 = 4.89 \text{ ksi} \qquad \sigma_2 = -46.9 \text{ ksi}$$

$$\tan 2\theta_p = \frac{2\,(-25.5)}{-16.5 + 25.5}$$

$$\theta_p = -40.0°$$

(e)

$$A = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2.25 \\ 3 & 1 & 3.25 \end{vmatrix}$$



$$= 0.375 \text{ in.}^2$$

$$\beta_i = 1 - 3.25 = -2.25 \qquad \gamma_i = 2.25 - 3 = -0.75$$

$$\beta_j = 3.25 - 3 = 0.25 \qquad \gamma_j = 2 - 2.25 = -0.25$$

$$\beta_m = 3 - 1 = 2 \qquad \gamma_m = 3 - 2 = 1$$

$$\{\sigma\} = [D]\,[B]\,\{d\}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 4.8 \times 10^7 \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.75 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$$

$$\times \frac{1}{0.75}\begin{bmatrix} -2.25 & 0 & 0.25 & 0 & 2 & 0 \\ 0 & -0.75 & 0 & -0.25 & 0 & 1 \\ -0.75 & -2.25 & -0.25 & 0.25 & 1 & 2 \end{bmatrix}\begin{Bmatrix} 0.001 \\ 0.005 \\ 0.001 \\ 0.0025 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -166 \\ -242 \\ -186 \end{Bmatrix} \text{ ksi}$$

$$\sigma_{1,2} = \frac{-166 + (-242)}{2} \pm \sqrt{\left(\frac{-166 + 242}{2}\right)^2 + (-186)^2}$$

$$\sigma_1 = -14.2 \text{ ksi} \qquad \sigma_2 = -394 \text{ ksi}$$