$$\theta_p = \frac{1}{2}\left(\tan^{-1}\frac{2(-186)}{-166+242}\right) = -39.2°$$

(f) Material properties

$E = 30 \times 10^6$ psi Modulus of elasticity.

$v = 0.25$ Poisson ratio

Nodal coordinates (coordinates defined CCW around element)

$x_1 = 0$ in.

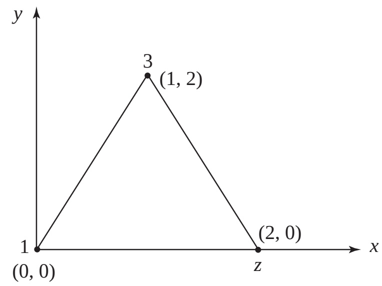
$y_1 = 0$ in.

$x_2 = 2$ in.

$y_2 = 0$ in.

$x_3 = 1$ in.

$y_3 = 2$ in.



Nodal displacements

$u_1 = 0.001$ in.

$v_1 = 0.005$ in.

$u_2 = 0.001$ in.

$v_2 = 0.0025$ in.

$u_3 = 0$ in.

$v_3 = 0$ in.

Set-up displacement vector

$$\{d\} = (u_1\ v_1\ u_2\ v_2\ u_3\ v_3)^T$$

$$\{d\}^T = (0.001 \quad 0.005 \quad 0.0001 \quad 0.0025 \quad 0 \quad 0)\text{ in.}$$

Area of triangular element ($\frac{1}{2} \times$ base $\times$ height)

$$A = \frac{1}{2}\,(x_2 - x_1)\,(y_3 - y_1)$$

$$A = 2\text{ in.}^2$$

Calculate gradient matrix, B, as given in text Equation (6.2.32)

Elements of B given by text Equation (6.2.10).

$$\beta_1 = y_2 - y_3 \qquad \beta_2 = y_3 - y_1 \qquad \beta_3 = y_1 - y_2$$

$$\gamma_1 = x_3 - x_2 \qquad \gamma_2 = x_1 - x_3 \qquad \gamma_3 = x_2 - x_1$$

$$[B] = \frac{1}{2A}\begin{pmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{pmatrix}$$

$$[B] = \begin{pmatrix} -0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & -0.25 & 0 & -0.25 & 0 & 0.5 \\ -0.25 & -0.5 & -0.25 & 0.5 & 0.5 & 0 \end{pmatrix}\frac{1}{\text{in.}}$$

Calculate constitutive matrix for plane strain

$$[D] = \frac{E}{(1+v)(1-2v)}\begin{pmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{pmatrix}$$

$$[D] = \begin{pmatrix} 3.6\times10^7 & 1.2\times10^7 & 0 \\ 1.2\times10^7 & 3.6\times10^7 & 0 \\ 0 & 0 & 1.2\times10^7 \end{pmatrix} \text{psi}$$

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = [D]\,[B]\,\{d\}$$

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} -22500.00 \\ -67500.00 \\ -21000.00 \end{pmatrix} \text{psi}$$

Principal stresses

$$\sigma_1 = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_1 = -\,1.422\times10^4 \text{ psi}$$

$$\sigma_2 = \frac{\sigma_x+\sigma_y}{2} - \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_2 = -\,75.777\times10^3 \text{ psi}$$

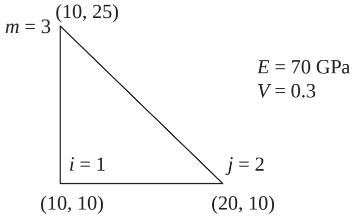
Angular location of principal stress plane

$$\theta_p = \frac{\operatorname{atan}\left(\frac{2\tau_{xy}}{\sigma_x-\sigma_y}\right)}{2}$$

$$\theta_p = -\,21.513°$$

6.10

(a)



$\beta_i = -\,15 \qquad \gamma_i = -\,10$

$\beta_j = 15 \qquad \gamma_j = 0$

$\beta_m = 0 \qquad \gamma_m = 10$

$2A = 150 \text{ mm}^2 = 150\times10^{-6} \text{ m}^2$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{70\times10^6}{(1+0.3)[1-2(0.3)]}\begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

$$\times \frac{1}{150}\begin{bmatrix} -15 & 0 & 15 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 & 0 & 10 \\ -10 & -15 & 0 & 15 & 10 & 0 \end{bmatrix}\begin{Bmatrix} 5.0 \\ 2.0 \\ 0 \\ 0 \\ 5.0 \\ 0 \end{Bmatrix} \times 10^{-3}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -52.5 \\ -32.8 \\ -5.38 \end{Bmatrix} \text{ MPa}$$

$$\sigma_{1,2} = \frac{-52.5+(-32.8)}{2} \pm \sqrt{\left(\frac{-52.5+32.8}{2}\right)^2 + (-5.38)^2}$$

$$= -42.65 \pm 11.22$$

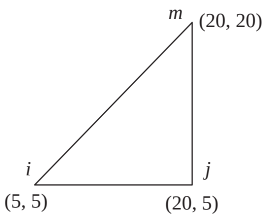
$$\sigma_1 = -31.4 \text{ MPa}$$

$$\sigma_2 = -53.9 \text{ MPa}$$

$$2\theta_p = \tan^{-1} \frac{2(-5.38)}{-52.5-(-32.8)} = 28.64°$$

$$\theta_p = 14.32°$$

(b)



$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 10^9 \begin{bmatrix} 94.2 & 40.4 & 0 \\ & 94.2 & 0 \\ & & 26.9 \end{bmatrix} \frac{1}{225\times10^{-6}}$$

$$\times \begin{bmatrix} -15 & 0 & 15 & 0 & 0 & 0 \\ 0 & 0 & 0 & -15 & 0 & 15 \\ 0 & -15 & -15 & 15 & 15 & 0 \end{bmatrix}\begin{Bmatrix} 0.005 \\ 0.002 \\ 0 \\ 0 \\ 0.005 \\ 0 \end{Bmatrix} \times 10^{-3}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -31.4 \\ -13.5 \\ 5.38 \end{Bmatrix} \text{ MPa}$$

$$\sigma_{1,2} = \frac{-31.4-13.5}{2} \pm \sqrt{\left(\frac{-31.4+13.5}{2}\right)^2 + (5.38)^2}$$

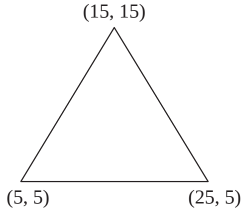
$$= -22.44 \pm 10.46$$

$\sigma_1 = -\,11.98$ MPa $\qquad \sigma_2 = -\,32.9$ MPa

$$\tan 2\theta_p = \frac{2(5.38)}{-31.4+13.46}$$

$$\theta_p = -\,15.5°$$

(c)



$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \left(\frac{70}{1.3\times 0.4}\right)\begin{bmatrix}0.7 & 0.3 & 0\\ 0.3 & 0.7 & 0\\ 0 & 0 & 0.2\end{bmatrix}\frac{1}{200}$$

$$\times\begin{bmatrix}-10 & 0 & 10 & 0 & 0 & 0\\ 0 & -10 & 0 & -10 & 0 & 20\\ -10 & -10 & -10 & 10 & 20 & 0\end{bmatrix}\begin{Bmatrix}5\\2\\0\\0\\5\\0\end{Bmatrix}\times 10^{-3}$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \begin{Bmatrix}-27.6\\ -19.5\\ 4.04\end{Bmatrix}\text{MPa}$$

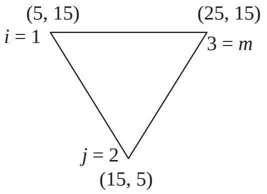
$$\sigma_{1,2} = \frac{-27.6-19.5}{2}\pm\sqrt{\left(\frac{-27.6+19.5}{2}\right)^2+4.04^2}$$

$\sigma_1 = -\,17.9$ MPa $\qquad \sigma_2 = -\,29.3$ MPa

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2\times 4.04}{-27.6+19.5}\right)$$

$$\theta_p = -\,22.5°$$

(d)



$E = 70\times 10^9$	$v = 3$	$t = 1$
$y_1 = 0.015$	$y_2 = 0.005$	$y_3 = 0.015$
$x_1 = 0.005$	$x_2 = 0.015$	$x_3 = 0.025$
$\beta_1 = y_2 - y_3$	$\beta_2 = y_3 - y_1$	$\beta_3 = y_1 - y_2$
$\gamma_1 = x_3 - x_2$	$\gamma_2 = x_1 - x_3$	$\gamma_3 = x_2 - x_1$

$$A = \frac{1}{2}\ [x_1\,(y_2 - y_3) + x_2\,(y_3 - y_1) + x_3\,(y_1 - y_2)]$$

$$[B] = \frac{1}{2A}\begin{pmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{pmatrix} \quad [D] = \frac{E}{(1+v)\,(1-2v)}\begin{pmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{pmatrix}$$

$$[k] = t\,A\,[B]^T\,[D]\,[B]$$

$u_1 = 0.000005$ $\qquad u_2 = 0$ $\qquad u_3 = 0.000005$

$v_1 = 0.000002$ $\qquad v_2 = 0$ $\qquad v_3 = 0$

$$\{d\} = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}$$

$$\{\sigma\} = [D]\,[B]\,\{d\}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \left(\frac{70}{1.3\times 0.4}\right)\begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}\frac{1}{200}$$

$$\times\begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -10 & 0 & 10 & 0 & 10 \\ -10 & 0 & 10 & 10 & 10 & -10 \end{bmatrix}\begin{Bmatrix} 5 \\ 2 \\ 0 \\ 0 \\ 5 \\ 0 \end{Bmatrix}\times 10^{-3}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -1.05 \\ 0.70 \\ 3.5 \end{Bmatrix}\text{ MPa}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + {\tau_{xy}}^2} \qquad \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + {\tau_{xy}}^2}$$

$$\sigma_{1,2} = \frac{-1.05 - 0.70}{2} \pm \sqrt{\left(\frac{-1.05 + 0.70}{2}\right)^2 + (-3.5)^2}$$

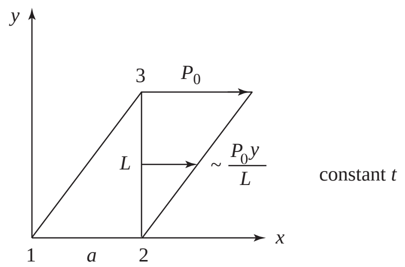
$$\sigma_1 = -3.43\text{ MPa} \qquad \sigma_2 = -3.78\text{ MPa}$$

$$\theta_p = \frac{1}{2}\ \tan^{-1}\left(\frac{2\,(-3.5)}{-1.05 - 0.70}\right)$$

$$\theta_p = 38°$$

6.11

(a)



Equation (6.3.7)

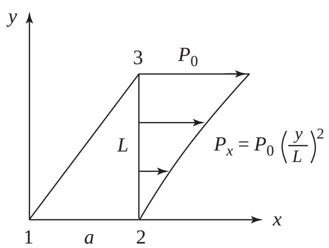
$$\{f_s\} = \int_0^t \int_0^L \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix}_{\text{evaluated of } \substack{x=a \\ y=y}} \begin{Bmatrix} \frac{P_0\, y}{L} \\ 0 \end{Bmatrix} dz\, dy = t \int_0^L \begin{Bmatrix} \frac{N_1\, P_0\, y}{L} \\ 0 \\ \frac{N_2\, P_0\, y}{L} \\ 0 \\ \frac{N_3\, P_0\, y}{L} \\ 0 \end{Bmatrix} dy$$

By Equations (6.3.12) – (6.3.17)

$$N_1 = \frac{L(a-x)}{2A},\ N_2 = \frac{Lx-ay}{2A},\ N_3 = \frac{ay}{2A}$$

$$\{f_s\} = \frac{t\, P_0}{2\left(\frac{aL}{2}\right)} \begin{Bmatrix} 0 \\ 0 \\ \frac{1}{6}aL^2 \\ 0 \\ \frac{1}{3}aL^2 \\ 0 \end{Bmatrix} \begin{Bmatrix} f_{s1x} \\ f_{s1y} \\ f_{s2x} \\ f_{s2y} \\ f_{s3x} \\ f_{s3y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \frac{1}{6}P_0 Lt \\ 0 \\ \frac{1}{3}P_0 Lt \\ 0 \end{Bmatrix}$$

(b)



By Equation (6.3.11)

$$\{f_s\} = t\int_0^L \begin{Bmatrix} N_1\, P_x \\ 0 \\ N_2\, P_x \\ 0 \\ N_3\, P_x \\ 0 \end{Bmatrix}_{\substack{x=a \\ y=y}} dy$$

Now $\quad N_1 = 0,\ N_2 = \dfrac{L_x - ay}{2A},\ N_3 = \dfrac{ay}{2A}$

$$\{f_s\} = t\int_0^L \left\{\begin{matrix} 0 \\ 0 \\ \left(\frac{Lx-ay}{2A}\right)P_0\left(\frac{y}{L}\right)^2 \\ 0 \\ \frac{ay}{2A}\left(\frac{y}{L}\right)^2 P_0 \\ 0 \end{matrix}\right\}_{\substack{x=a\\ y=y}} dy$$

Simplifying and integrating

$$\{f_s\} = \frac{P_0\,t}{2\left(\frac{1}{2}aL\right)}\left\{\begin{matrix} 0 \\ 0 \\ \left(\frac{Lay^3}{3L^2} - \frac{ay^4}{4L^2}\right)\Big|_0^L \\ 0 \\ \frac{ay^4}{4L^2}\Big|_0^L \\ 0 \end{matrix}\right\} = \left\{\begin{matrix} 0 \\ 0 \\ \frac{P_0\,tL}{12} \\ 0 \\ \frac{P_0\,tL}{4} \\ 0 \end{matrix}\right\}$$

or

$$f_{2x} = \frac{P_0\,tL}{12}$$

$$f_{3x} = \frac{P_0\,tL}{4}$$

6.12

(a)

$$P_y(x) = ax^2 + bx + c$$

Given

$$a\left(\frac{L}{2}\right)^2 + b\frac{L}{2} + p_1 = p_2$$

$$aL^2 + bL + p_1 = p_3$$

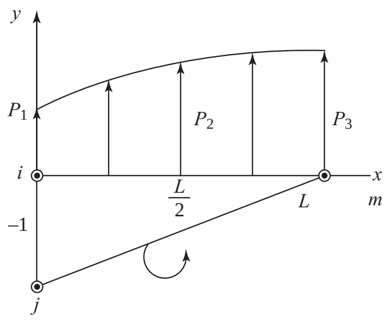
$$c = p_1$$

Find $\quad (\text{a, b, c}) \rightarrow \begin{bmatrix} 2\dfrac{(p_1 + p_3 - 2p_2)}{L^2} \\ \dfrac{(-3p_1 - p_3 + 4p_2)}{L} \\ p_1 \end{bmatrix}$

$$a(p_1, p_2, p_3) = 2\frac{(p_1 + p_3 - 2p_2)}{L^2}$$

$$b(p_1, p_2, p_3) = \frac{(-3p_1 - p_3 + 4p_2)}{L}$$

$$c(p_1, p_2, p_3) = p_1$$



Forces in y direction at nodes 1 and 3 are

$N_1 = N_i$

$$f_{s_{1y}} = \int_0^L \left(1-\frac{x}{L}\right)\left(a(p_1,\, p_2,\, p_3)x^2 + b(p_1,\, p_2,\, p_3)x + c\,(p_1,\, p_2,\, p_3)\right)dx \rightarrow \frac{1}{6}Lp_1 + \frac{1}{3}Lp_2$$

$$= \int_{x=0}^{L} N_1\, P_y(x)\, dx$$

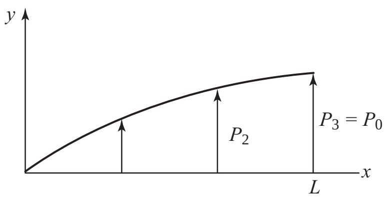
$N_3 = N_m$

$$f_{s_{3y}} = \int_0^L \frac{x}{L}\left(a(p_1,\, p_2,\, p_3)x^2 + b(p_1,\, p_2,\, p_3)x + c\,(p_1,\, p_2,\, p_3)\right)\, dx \rightarrow \frac{1}{6}Lp_3 + \frac{1}{3}Lp_2$$

$$= \int_0^L N_m\, P_y(x)$$

(Special case)

If $p_1 = 0 \qquad p_3 = p_0$



$$p(x) = ax^2 + bx + c$$

Want

$$p(x) = p_0 \left(\frac{x}{L}\right)^2 = ax^2 + \cancel{bx} + \cancel{c}$$

$$\therefore\; a = \frac{p_0}{L^2} = \frac{2}{L^2}\,(p_0 - 2p_2)$$

$$\frac{2p_0}{L^2} - \frac{p_0}{L^2} = \frac{4p_2}{L^2}$$

$$\frac{p_0}{\cancel{L^2}} = \frac{4p_2}{\cancel{L^2}}$$

$$p_2 = \frac{p_0}{4}$$

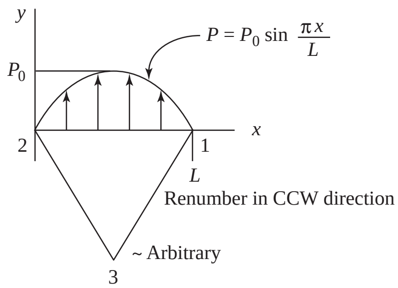
Then

$$f_{s_{14}} = \frac{1}{6}\,L\,\cancel{p_1}^{0} + \frac{1}{3}\,Lp_2 = \frac{1}{3}\,L\,\frac{p_0}{4} = \frac{p_0 L}{12} \rightarrow \text{Like 6.11 b with } f_{sx} = \frac{pL}{12}$$

$$f_{s_{34}} = \frac{1}{6}\,L\overset{p_0}{\overset{\|}{p_3}} + \frac{1}{3}\,L\overset{\frac{p_0}{4}}{\overset{\|}{p_2}} = \frac{3}{12}Lp_0 = \frac{p_0 L}{4} \quad \Rightarrow \quad f_{3x} \text{ in 6.11b}$$

(These answers match P6.11 for special case)

(b)



$$\{f_s\} = \int_s\!\int [N_s]^T \;[T_s]\; ds$$

$[T_s]$ = Surface tractions

$$= \begin{Bmatrix} T_{sx} \\ T_{sy} \end{Bmatrix} = \begin{Bmatrix} 0 \\ P_0 \sin\pi\frac{N}{L} \end{Bmatrix}$$

$[N_s]$ = Shape function matrix evaluated along edge 1-2

$$= \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix}$$

Let $i = 1$

$j = 2$

$m = 3$

$$N_i - N_1 = \frac{1}{2A}\;(\alpha_i + \beta_i x + \gamma_i y)$$

$$N_1(y = 0) = \frac{1}{2A}\;(\alpha_i + \beta_i x)$$

$$\alpha_i = x_j y_m - y_j x_m$$

$$= 0(ym) - 0(xm) = 0$$

$$\beta_i = y_j - y_m$$

$$= 0 - y_m = -y_m$$

$$N_1(y = 0) = \frac{1}{2A}\;(0 - y_m x) = \frac{-y_m x}{2A}$$

$$N_j = N_2 = \frac{1}{2A}\;(\alpha_j + \beta_j x + \gamma_j y)$$

$$N_2(y=0) = \frac{1}{2A}\ (\alpha_i + \beta_i x)$$

$$\alpha_j = y_j x_m - x_j y_m$$

$$= 0(x_m) - Ly_m = -Ly_m$$

$$\beta_j = y_m - y_i$$

$$= y_m - 0 = y_m$$

$$N_2(y=0) = \frac{1}{2A}\ [-Ly_m + y_m x] = \frac{y_m}{2A}\ [x-L]$$

$$N_m = N_3 = \frac{1}{2A}\ [\alpha_m + \beta_m x + \gamma_m y]$$

$$N_3(y=0) = \frac{1}{2A}\ [\alpha_m + \beta_m x]$$

$$\alpha_m = x_i y_j - y_i x_j = L(0) - 0(0) = 0$$

$$\beta_m = y_i - y_j = 0$$

$\therefore\ N_m(y=0) = 0$ As expected

$$\{f_s\} = \int_0^{x=L}\int_0^{z=t}\begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix}\begin{Bmatrix} 0 \\ P_0 \sin\frac{\pi x}{L} \end{Bmatrix}\ dz\ dx$$

$$= t\int_0^{x=L}\begin{bmatrix} 0 \\ N_1 P_0 \sin\frac{\pi x}{L} \\ 0 \\ N_2 P_0 \sin\frac{\pi x}{L} \\ 0 \\ N_3 P_0 \sin\frac{\pi x}{L} \end{bmatrix} dx = t\int_0^{x=L}\begin{bmatrix} 0 \\ \frac{-y_m x}{2A} P_0 \sin\left(\frac{\pi x}{L}\right) \\ 0 \\ \frac{y_m}{2A}(x-L)P_0 \sin\frac{\pi x}{L} \\ 0 \\ 0 \end{bmatrix} dx$$

2nd term in (A) $(y_m = y_3)$

$$f_{s1y} = \frac{-t\, y_3 P_0}{2A}\int_0^{x=L} x \sin\left(\frac{\pi x}{L}\right)\ dx$$

$$\int u\,dv = u - \int v\,du$$

$$u = x \qquad du = dx$$

$$dv = \sin\frac{\pi x}{L}\ dx \qquad v = -\cos\frac{\frac{\pi x}{L}}{\frac{\pi}{L}}$$

$$= \frac{-t\, y_3 P_0}{2A}\left[-\frac{xL}{\pi}\cos\pi\frac{x}{L} + \int_0^{x=L}\frac{L}{\pi}\cos\left(\pi\frac{x}{L}\right)dx\right]$$

$$= \frac{-t\, y_3 P_0}{2A}\left[-\frac{xL}{\pi}\cos\left(\pi\frac{x}{L}\right) + \frac{L^2}{\pi^2}\sin\left(\pi\frac{x}{L}\right)\right]\Bigg|_0^L$$

$$= \frac{-t\,y_3 P_0}{2A}\left[\frac{-L^2}{\pi}(-1) + 0 - 0 - 0\right]$$

$$f_{s1y} = \frac{-t\,y_3 P_0}{2A}\left(\frac{L^2}{\pi}\right) = \frac{t\,P_0 L}{\pi}$$

$$A = \frac{-1}{2}\;L y_3$$

4th term in (A)

$$f_{s2y} = t\int_0^L \left[\frac{y_3}{2A} P_0\; x \sin\frac{\pi x}{L} - \frac{y_3}{2A} P_0\, L \sin\left(\frac{x\pi}{L}\right)\right] dx$$

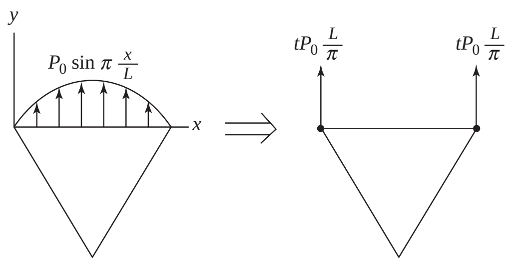
$$= \frac{t\,y_3}{2A} P_0 \underbrace{\int_0^L x \sin\frac{\pi x}{L}\, dx}_{\text{DONE IN 2nd TERM}} - \frac{t\,y_3}{2A} P_0 L \int_0^L \sin\frac{\pi x}{L}\, dx$$

$$= \frac{t\,y_3 P_0}{2A}\left[\frac{L^2}{\pi}\right] + \frac{t\,y_3 P_0 L}{2A}\left[\frac{L}{\pi}\cos\pi\frac{x}{L}\right]\Bigg|_0^L$$

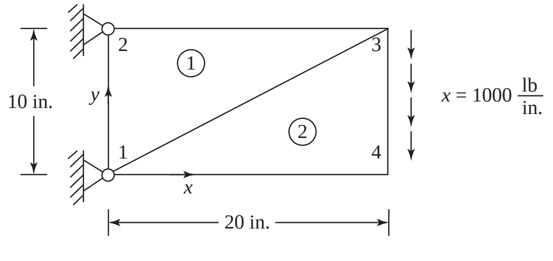
$$= \frac{t\,y_3 P_0}{2A}\,\frac{L^2}{\pi} + \frac{t\,y_3 P_0 L}{2A}\,\frac{L}{\pi}\;[-1 + 1]$$

$$f_{s2y} = \frac{t\,y_3 P_0\, L^2}{2A\,\pi} = \frac{t\,\cancel{y_3}\,L^{\cancel{2}} P_0}{\cancel{2}\left[\frac{1}{\cancel{2}}\cancel{L}\;\cancel{y_3}\right]\pi} = \frac{t\,L\,P_0}{\pi}$$

$$\therefore\quad \vec{f}_s = \begin{Bmatrix} f_{s1x} \\ f_{s1y} \\ f_{s2x} \\ f_{s2y} \\ f_{s3x} \\ f_{s3y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{t\,P_0\,L}{\pi} \\ 0 \\ \frac{t\,P_0\,L}{\pi} \\ 0 \\ 0 \end{Bmatrix}$$



6.13



Refer to Section 6.5 for $[K]$

Since $u_1 = v_1 = 0,\ u_2 = v_2 = 0$

$$\begin{Bmatrix} 0 \\ -5{,}000 \\ 0 \\ -5{,}000 \end{Bmatrix} = \frac{75000\times 5}{0.91}\begin{bmatrix} 48 & 0 & -28 & 14 \\ & 87 & 12 & -80 \\ & & 48 & -26 \\ \text{Symmetry} & & & 87 \end{bmatrix}\begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Solving

$$u_3 = 0.50\times 10^{-3} \text{ in.} \qquad v_3 = -0.275\times 10^{-2} \text{ in.}$$
$$u_4 = -0.609\times 10^{-3} \text{ in.} \qquad v_4 = -0.293\times 10^{-2} \text{ in.}$$

Using element (1)

By Equation (6.2.36), $\{\sigma\} = [D]\,[B]\,\{d\}$

$$\{\sigma\} = \frac{30\times 10^6}{(0.91)(200)}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -20 & 0 & 0 & 0 & 20 \\ -20 & 0 & 0 & 10 & 20 & -10 \end{bmatrix}$$

$$\times\begin{Bmatrix} 0 \\ 0 \\ 0.5\times 10^{-3} \\ -0.275\times 10^{-2} \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 824 \\ 247 \\ -1586 \end{Bmatrix} \text{ psi}$$

$$\sigma_{1,2} = \frac{824+247}{2} \pm \sqrt{\left(\frac{824-247}{2}\right)^2 + (-1586)^2}$$

$$\sigma_1 = 2149 \text{ psi} \qquad \sigma_2 = -1077 \text{ psi}$$

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{-2\times 1586}{824-247}\right)$$

$$\theta_p = -40^\circ$$

Using element (2)

$$\{\sigma\} = \frac{30\times 10^6}{0.91(200)}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -10 & -20 & 10 & 20 & 0 \end{bmatrix}$$

$$\times\begin{Bmatrix} 0 \\ 0 \\ -0.609\times 10^{-3} \\ -0.293\times 10^{-2} \\ 0.5\times 10^{-3} \\ -0.275\times 10^{-2} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -825 \\ 292 \\ -411 \end{Bmatrix} \text{ psi}$$

$$\sigma_{1,2} = \frac{-825+292}{2} \pm \sqrt{\left(\frac{-825-292}{2}\right)^2 + (-411)^2}$$

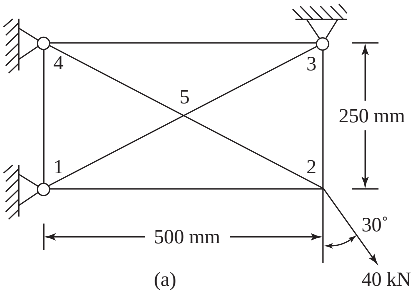
$$\sigma_1 = 426 \text{ psi} \qquad \sigma_2 = -960 \text{ psi}$$

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{-2\times 411}{-825-292}\right)$$

$$\theta_p = 18.15°$$

6.14

(a)



(a)

INPUT TABLE 1. BASIC PARAMETERS

NUMBER OF NODAL POINTS.	5
NUMBER OF ELEMENTS.	4
NUMBER OF DIFFERENT MATERIALS.	1
NUMBER OF SURFACE LOAD CARDS.	0
1 = PLANE STRAIN, 2 = PLANE STRESS	2
BODY FORCES (1 = IN – Y DIREC., 0 = NONE)	0

INPUT TABLE 2. MATERIAL PROPERTIES

MATERIAL NUMBER	MODULUS OF ELASTICITY	POISSON'S RATIO	MATERIAL DENSITY	MATERIAL THICKNESS
1	0.2100E+12	0.3000E+00	0.0000E+00	0.5000E–02

INPUT TABLE 3. NODAL POINT DATA

NODAL POINT	TYPE	X	Y
1	3	0.0000E+00	0.0000E+00
2	0	0.5000E+00	0.0000E+00
3	3	0.5000E+00	0.2500E+00
4	3	0.0000E+00	0.2500E+00
5	0	0.2500E+00	0.1250E+00

X-DISP. OR LOAD	Y-DISP. OR LOAD
0.0000E+00	0.0000E+00
0.2000E+05	– 0.3464E+05
0.0000E+00	0.0000E+00
0.0000E+00	0.0000E+00
0.0000E+00	0.0000E+00

INPUT TABLE 4. ELEMENT DATA

	GLOBAL INDICES OF ELEMENT NODES				
ELEMENT	1	2	3	4	MATERIAL
1	1	5	4	4	1
2	1	2	5	5	1
3	5	2	3	3	1
4	4	5	3	3	1

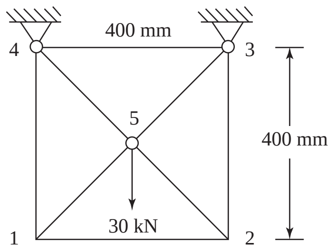
OUTPUT TABLE 1. NODAL DISPLACEMENTS, m

NODE	U = X-DISP.	V = Y-DISP.
1	0.00000000E+00	0.00000000E+00
2	0.28124740E–04	– 0.32985310E–04
3	0.00000000E+00	0.00000000E+00
4	0.00000000E+00	0.00000000E+00
5	0.11497950E–04	– 0.10347600E–04

OUTPUT TABLE 2. STRESSES AT ELEMENT CENTROIDS

ELEMENT	X	Y	SIGMA (X)	SIGMA (Y)	TAU (X, Y)
1	0.083	0.125	0.0613E+07	3.1840E+06	3.3431E+06
2	0.250	0.042	1.6384E+07	1.5239E+07	–6.9854E+06
3	0.417	0.125	1.1502E+07	3.1158E+07	–1.1072E+07
4	0.250	0.208	5.7310E+06	1.9103E+07	–7.4294E+06
			SIGMA (1)	SIGMA (2)	ANGLE
			1.1896E+07	1.9012E+06	2.0993E+01
			2.2820E+07	8.8026E+06	–4.2657E+01
			3.6135E+07	6.5251E+06	–6.5798E+01
			2.2412E+07	2.4221E+06	–6.5993E+01

(c)



(b)

INPUT TABLE 1. BASIC PARAMETERS

NUMBER OF NODAL POINTS.	5
NUMBER OF ELEMENTS.	4
NUMBER OF DIFFERENT MATERIALS.	1
NUMBER OF SURFACE LOAD CARDS.	0
1 = PLANE STRAIN, 2 = PLANE STRESS	2
BODY FORCES (1 = IN – Y DIREC., 0 = NONE)	0

INPUT TABLE 2. MATERAL PROPERTIES

MATERIAL NUMBER	MODULUS OF ELASTICITY	POISSON'S RATIO,	MATERIAL DENISTY	MATERIAL THICKNESS
1	0.2100E+12	0.3000E+00	0.0000E+00	0.5000E–02

INPUT TABLE 3. NODAL POINT DATA

POINT	TYPE	X	Y	X OR LOAD	Y OR LOAD
1	0	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	0	0.4000E+00	0.0000E+00	0.0000E+00	0.0000E+00
3	3	0.4000E+00	0.4000E+00	0.0000E+00	0.0000E+00
4	3	0.0000E+00	0.4000E+00	0.0000E+00	0.0000E+00
5	0	0.2000E+00	0.2000E+00	0.0000E+00	–0.3000E+05

INPUT TABLE 4. ELEMENT DATA

	GLOBAL INDICES OF ELEMENT NODES				
ELEMENT	1	2	3	4	MATERIAL
1	1	2	5	5	1
2	2	3	5	5	.1
3	3	4	5	5	1
4	4	1	5	5	–.1

OUTPUT TABLE 1. NODAL DISPLACEMENTS (m)

NODE	U = X–DISP.	V = Y–DISP.
1	–0.16515414E–05	–0.12504538E–04
2	0.16515442E–05	– 0.12504535E–04
3	0.00000000E+00	0.00000000E+00
4	0.00000000E+00	0.00000000E+00
5	0.27411561E–12	–0.16279491E–04

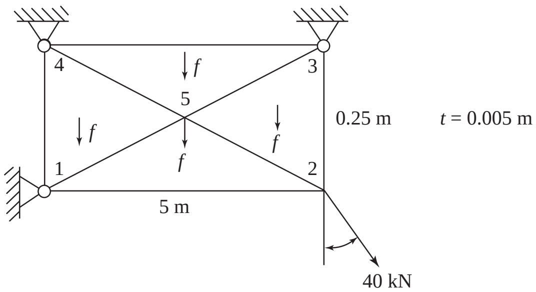
OUTPUT TABLE 2. STRESSES AT ELEMENT CENTROIDS $\left(\frac{\text{N}}{\text{m}^2}\right)$

ELEMENT	X	Y	SIGMA(X)	SIGMA(Y)	TAU(X, Y)
1	0.20	0.07	5.9891E+05	–3.7840E+06	4.0454E–01
2	0.33	0.20	3.1171E+06	7.5000E+06	3.7160E+06
3	0.20	0.33	5.6352E+06	1.8784E+07	–1.1070E–01
4	0.07	0.20	3.1171E+06	7.5000E+06	–3.7160E+06

SIGMA(1)	SIGMA(2)	ANGLE
5.9891E+05	–3.7840E+06	5.2883E–06
9.6226E+06	9.9449E+05	6.0265E+01
1.8784E+07	5.6352E+06	–9.0000E+01
9.6226E+06	9.9449E+05	–6.0265E+01

6.15

(a)



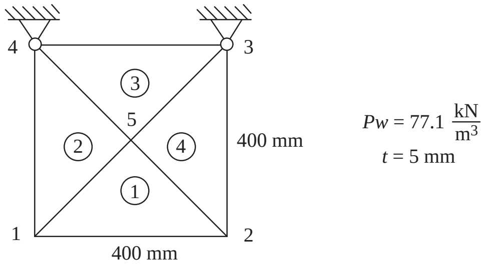
$$F = (0.5\ \cancel{\text{m}})\ (0.25\ \cancel{\text{m}})\ (0.005\ \cancel{\text{m}})\ \left(77.1\frac{\text{kN}}{\cancel{\text{m}^3}}\right) = 0.0482\ \text{kN}$$

$$f_1 = f_2 = f_3 = f_4 = \frac{-\frac{1}{2}(0.25)(0.25)(0.005)\left(77.1\ \frac{\text{kN}}{\text{m}^3}\right)\left[\frac{1000\ \text{N}}{1\text{kN}}\right](2)}{3} = -\ 8.031\ \text{N}$$

$$f_5 = (2) \times f_1 = 16.063\ \text{N} \downarrow$$

$$f_{5y} = -16.063\ \text{N} \downarrow$$

(c)



Equation (6.3.6)

$$\begin{Bmatrix} f_{B1x}^{(1)} \\ f_{B1y}^{(1)} \\ f_{B2x}^{(1)} \\ f_{B2y}^{(1)} \\ f_{B5x}^{(1)} \\ f_{B5y}^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 77.1 \\ 0 \\ 77.1 \\ 0 \\ 77.1 \end{Bmatrix} \frac{(0.4\,\text{m})(0.2\,\text{m})(0.005\,\text{m})}{(2)\,(3)} = \begin{Bmatrix} 0 \\ 5.14 \\ 0 \\ 5.14 \\ 0 \\ 5.14 \end{Bmatrix} 10^{-3}\ \text{kN}$$

All body force matrices for each element identical to above

Adding the 4 element body force matrices

$$\{F_B\} = \begin{Bmatrix} f_{B1x} \\ f_{B1y} \\ f_{B2x} \\ f_{B2y} \\ f_{B3x} \\ f_{B3y} \\ f_{B4x} \\ f_{B4y} \\ f_{B5x} \\ f_{B5y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10.28 \\ 0 \\ 10.28 \\ 0 \\ 10.28 \\ 0 \\ 10.28 \\ 0 \\ 20.56 \end{Bmatrix} \text{N}$$

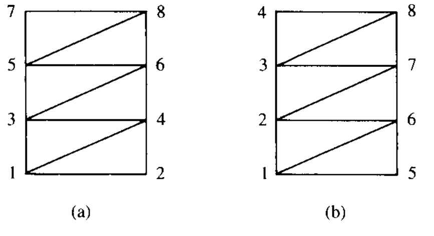
6.16 The triangular element is called a constant strain triangle (CST) because the strain is constant throughout the element.

6.17 The stresses are also constant as the strains are constant.

6.18
a. No, bending in the plane takes place
b. Yes, loads in-plane of the wall
c. Yes, a plane stress problem
d. Yes, if loads in-plane of the bar
e. Yes, a plane strain problem
f. Yes, a plane stress problem
g. No, loads out of the plane of the wrench
h. Yes, as loads in the plane
i. No, bending in the plane takes place

6.19 We must connect the beam element to two or more nodes of a plane stress element. The beam must be along the edge of the plane stress element.

6.20



$$n_b = n_0\,(m + 1)$$

(a) $n_b = 2(3 + 1) = 8$

(b) $n_b = 2(5 + 1) = 12$

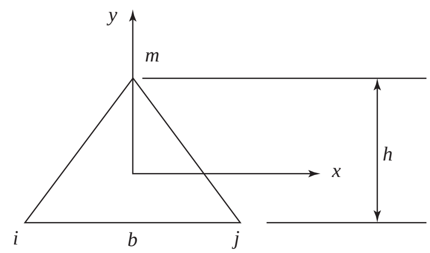
for model (a) model(b)

$$\underline{| \, n_b = 8 \, |} \qquad\qquad \underline{| \, n_b = 12 \, |}$$

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \quad \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$

$$\begin{bmatrix} \times & \times & \times & \times & & & & \\ & \times & 0 & \times & 0 & & & \\ & & \times & \times & \times & \times & & \\ & & & \times & 0 & \times & 0 & \\ & & & & \times & \times & \times & \times \\ & \text{Symmetry} & & & & \times & 0 & \times \\ & & & & & & \times & \times \\ & & & & & & & \times \end{bmatrix} \begin{bmatrix} \times & \times & 0 & 0 & \times & \times & & \\ & \times & \times & 0 & 0 & \times & \times & \\ & & \times & \times & 0 & 0 & \times & \times \\ & & & \times & 0 & 0 & 0 & \times \\ & & & & \times & \times & 0 & 0 \\ & & & & & \times & \times & 0 \\ & & & & & & \times & \times \\ \text{Symmetry} & & & & & & & \times \end{bmatrix}$$

6.21



By (6.2.10)

$$\alpha_i = x_j y_m - y_j x_m = \frac{b}{2}\left(\frac{2b}{3}\right) - \left(-\frac{h}{3}\right)(0) = \frac{bh}{3} = \frac{2A}{3}$$

$$\alpha_j = \left(-\frac{h}{3}\right)(0) - \left(-\frac{b}{2}\right)\left(\frac{2h}{3}\right) = \frac{bh}{3} = \frac{2A}{3}$$

$$\alpha_m = \left(-\frac{b}{2}\right)\left(-\frac{h}{3}\right) - \left(-\frac{h}{3}\right)\left(\frac{b}{2}\right) = \frac{bh}{3} = \frac{2A}{3}$$

$$\therefore \quad \{f_B\} = \int_v [N]^T \begin{Bmatrix} X_b \\ Y_b \end{Bmatrix} dV \qquad N_i = \frac{1}{2A}\left(\frac{2A}{3}\right) = \frac{1}{3} \text{ and } N_j = N_m = \frac{1}{3}$$

or

$$\{f_{b_i}\} = \int_v \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix} \begin{Bmatrix} X_b \\ Y_b \end{Bmatrix} dV$$

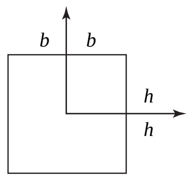
$$= \int_v \begin{bmatrix} \frac{1}{3} & \\ 0 & \frac{1}{3} \end{bmatrix} \begin{Bmatrix} X_b \\ Y_b \end{Bmatrix} tdA$$

$$\{f_{b_i}\} = \begin{Bmatrix} X_b \\ Y_b \end{Bmatrix} \frac{V}{3}$$

Similarly

$$\{f_{b_j}\} = \{f_{bm}\} = \begin{Bmatrix} X_b \\ Y_b \end{Bmatrix} \frac{V}{3} \qquad (6.3.6)$$

6.24



$$N_1 = \frac{(b-x)(h-y)}{4bh},\ N_2 = \frac{(b+x)(h-y)}{4bh}$$

$$N_3 = \frac{(b+x)(h+y)}{4bh},\ N_4 = \frac{(b-x)(h+y)}{4bh} \qquad (1)$$

at center $(x = 0, y = 0)$

$$N_1 = \frac{1}{4},\ N_2 = \frac{1}{4},\ N_3 = \frac{1}{4},\ N_4 = \frac{1}{4}$$

$N_1 + N_2 + N_3 + N_4 = 1$

at point $\left(x = \frac{b}{2}, y = \frac{h}{2}\right)$

$$N_1 = \frac{\left(b - \frac{b}{2}\right)\left(h - \frac{h}{2}\right)}{4bh} = \frac{1}{16}$$

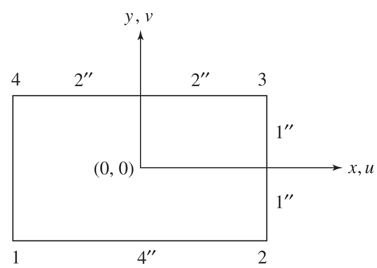
$$N_2 = \frac{3}{16},\ N_3 = \frac{9}{16},\ N_4 = \frac{3}{16}$$

$\therefore N_1 + N_2 + N_3 + N_4 = 1$

In general add the functions in Equation (1) and you get for all x and y on the element

$N_1 + N_2 + N_3 + N_4 = 1$

6.25



$$\{\sigma\} = [D]\,[B]\,\{d\}$$

$$[B] = \frac{1}{4bh}\begin{bmatrix} -(h-y) & 0 & h-y & 0 & h+y & 0 & -(h+y) & 0 \\ 0 & -(b-x) & 0 & -(b+x) & 0 & b+x & 0 & b-x \\ -(b-x) & -(h-y) & -(b+x) & h-y & b+x & h+y & b-x & -(h+y) \end{bmatrix}$$

At center $(x = 0, y = 0)$

$$[B] = \frac{1}{8}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}$$

$$\{\varepsilon\} = \frac{1}{8}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.005 \\ 0.0025 \\ 0.0025 \\ -0.0025 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} 0.0009375 \\ -0.00125 \\ -0.000625 \end{Bmatrix} \frac{\text{in.}}{\text{in.}}$$

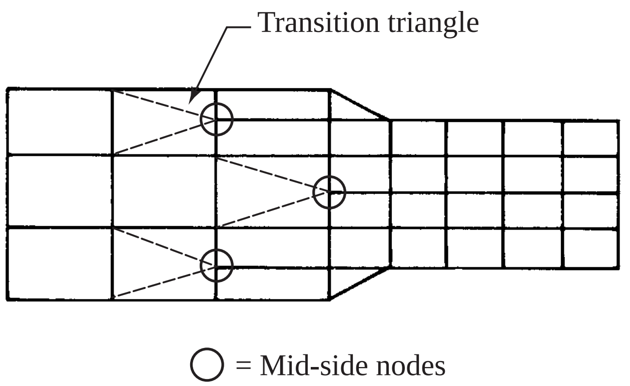
$$\{\sigma\} = [D]\,\{\varepsilon\}$$

$$= \frac{30\times 10^6}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{Bmatrix} 0.0009375 \\ -0.00125 \\ -0.000625 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 18.54 \\ -31.94 \\ -7.21 \end{Bmatrix} \text{ ksi}$$

Chapter 7

7.1 For the simple 4 noded elements it is a violation of displacement compatibility to have a mid-side node. Some of the elements have mid-side nodes in this model. Use 'transition' triangle to go from smaller to larger rectangular elements.



7.2 The mesh sizing is not fine enough in the reentrant corner region at *C*. We need smaller elements near point *C* and small radius at *C.*

7.3 Based on the formulation used here we can not have $\mu = 0.5$ for the plane strain case as the denominator in the material property matrices [*D*] (see Equation (6.1.10) and [*K*] (see Equation (6.4.3) becomes zero. A penalty formulation see Reference [7] can be used to avoid this problem.

7.4 The structure is plane strain if this section represents a cross section of a long structure in which the loads do not vary in the *z* direction.
The structure is a plane stress problem if this section is a thin plate type structure with loads in the plane of the structure only.

Also see Section 6.1 for descriptions of plane stress and plane strain and examples of each.

7.5 When abrupt changes in thickness at *E*'s occur from element to element.

7.6 Unit thickness/7.7 (a) best aspect ratio.

7.9 (a) No, as replacing a portion of the patch by a different material with different mechanical properties will in general produce non-uniform strain under constant state of applied stress. For rigid body mode tests, however, different mechanical property materials still result in rigid body displacement.

(b) Yes, the patch can be arbitrary in shape. If we apply a test displacement field of $u_x = 1$, $u_y = 0$ at the external nodes of a patch of say 4 elements and set the internal nodal force to zero, then solve for the displacement components at internal node *i*, these displacement components should agree with the value of the displacement function at that node. Also the strain function or field should vanish identically at any point over each element.

(c) Yes, we can mix triangular and quadrilateral elements in a 2-d patch test as long as the material properties are the same.

(d) No. Mixing bars with plane elements would alter the constant strain states as the plane element and bar are of different structural types.

(e) The patch test should be applied when developing new finite elements, to determine if the element can represent rigid body motion as well as states of constant strain when these conditions occur.

7.10 Using Mathcad

$A = 1 \times 10^{-4}$ $\quad E = 200 \times 10^{9}$ $\quad L_1 = 0.6$ $\quad [L_2] = 1.4$

$[k_1] = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad [k_1] = [k_1]\, A\, \dfrac{E}{L_1}$ $\quad [k_1] = \begin{pmatrix} 3.333 \times 10^{7} & -3.333 \times 10^{7} & 0 \\ -3.333 \times 10^{7} & 3.333 \times 10^{7} & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$[k_2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ $\quad [k_2] = [k_2]\, A\, \dfrac{E}{L_2}$ $\quad [k_2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1.429 \times 10^{7} & -1.429 \times 10^{7} \\ 0 & -1.429 \times 10^{7} & 1.429 \times 10^{7} \end{pmatrix}$

$[k] = [k_1] + [k_2]$ $\quad [k] = \begin{pmatrix} 3.333 \times 10^{7} & -3.333 \times 10^{7} & 0 \\ -3.333 \times 10^{7} & 4.762 \times 10^{7} & -1.429 \times 10^{7} \\ 0 & -1.429 \times 10^{7} & 1.429 \times 10^{7} \end{pmatrix}$

Set these 3 values to defined quantities of $u_1 = u_3 = 1$ for the rigid body patch test

$u_1 = 1$ $\quad u_3 = 1$ $\quad F_2 = 0$

Guess at F_1, F_3, and u_2 as shown below.

$F_1 = 1$ $\quad u_2 = 0$ $\quad F_3 = 1$

Given — Use the given command to create a solve block.

$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = [k] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ — Use control and equal sign here.

$\begin{pmatrix} F_1 \\ u_2 \\ F_3 \end{pmatrix} = \text{Find}\,(F_1, u_2, F_3)$ — Use the 'Find' command to find F_1, u_2, and F_3.

$F_1 = 0$ $\quad u_2 = 1$ $\quad F_3 = 0$

The rigid body motion patch test is satisfied as $u_2 = 1$.

Now check the constant strain test. Let $u(x) = x$ for the nodes at the boundaries, i.e., $u_1 = 0$ and $u_3 = 2$, Verify that $u_2\,(x = 0.6) = 0.6$.

$u_1 = 0$ $\quad u_3 = 2$ $\quad F_2 = 0$ — Initial these values

$F_1 = 1$ $\quad F_3 = 1$ $\quad u_2 = 0$ — Guesses for these values.

Given

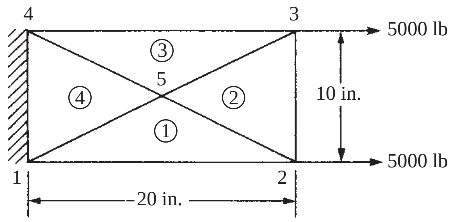
$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = [k] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$\begin{pmatrix} F_1 \\ u_2 \\ F_3 \end{pmatrix} = \text{Find}\,(F_1, u_2, F_3)$ — Use the 'Find' command to solve for F_1, u_2, and F_3.

$$F_1 = -2 \times 10^{7} \qquad F_3 = 2 \times 10^{7} \qquad u_2 = 0.6$$

Now upon solving the system of equations $u_2 = 0.6$ as it should to satisfy the patch test for constant strain.

7.12



INPUT TABLE 1.. BASIC PARAMETERS

NUMBER OF NODAL POINTS.	5
NUMBER OF ELEMENTS.	4
NUMBER OF DIFFERENT MATERIALS.	1
NUMBER OF SURFACE LOAD CARDS.	0
1 = PLANE STRAIN, 2 = PLANE STRESS	2
BODY FORCES (1 = IN – Y DIREC., 0 = NONE)	0

INPUT TABLE 2.. MATERIAL PROPERTIES

MATERIAL NUMBER	MODULUS OF ELASTICITY	POISSON'S RATIO,	MATERIAL DENSITY	MATERIAL THICKNESS
1	0.3000E+08	0.3000E+00	0.0000E+00	0.1000E+01

INPUT TABLE 3.. NODAL POINT DATA

NODAL POINT	TYPE	X	Y	X-DISP. OR LOAD	Y-DISP. OR LOAD
1	3	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	0	0.2000E+02	0.0000E+00	0.5000E+04	0.0000E+00
3	0	0.2000E+02	0.1000E+02	0.5000E+04	0.0000E+00
4	3	0.0000E+00	0.1000E+02	0.0000E+00	0.0000E+00
5	0	0.1000E+02	0.5000E+01	0.0000E+00	0.0000E+00

INPUT TABLE 4.. ELEMENT DATA

	GLOBAL	INDICES OF	ELEMENT	NODES	
ELEMENT	1	2	3	4	MATERIAL
1	1	2	5	5	1
2	2	3	5	5	1
3	5	3	4	4	1
4	1	5	4	4	1

OUTPUT TABLE 1.. NODAL DISPLACEMENTS

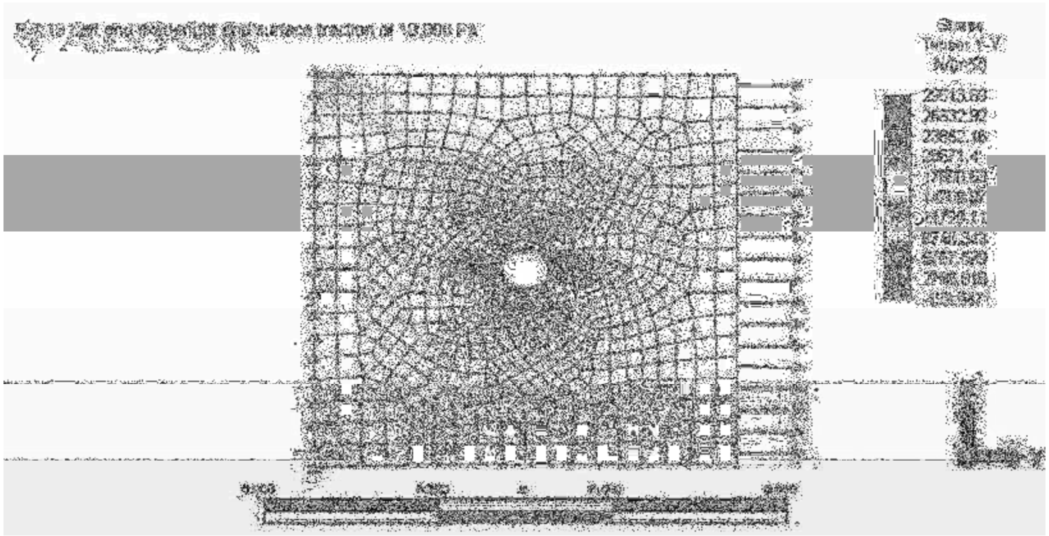
NODE	U = X-DISP.	V = Y-DISP.
1	0.00000000E+00	0.00000000E+00
2	0.64664544E–03	0.66631597E–04
3	0.61664509E–03	–0.66630528E–04
4	0.00000000E+00	0.000000000E+00
5	0.30527671E–03	0.24373945E–09

OUTPUT TABLE 2.. STRESSES AT ELEMENT CENTROIDS

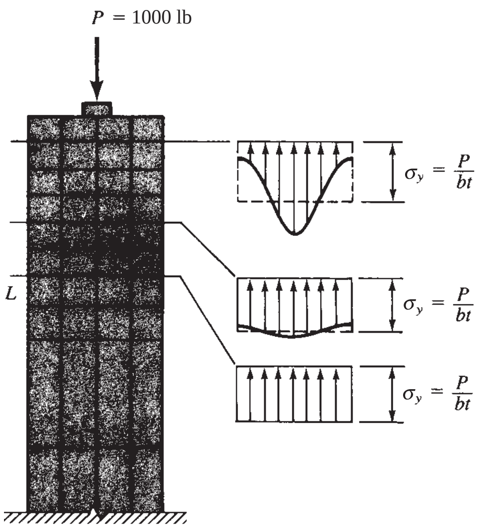
ELEMENT	X	Y	SIGMA(X)	SIGMA(Y)	TAU(X, Y)
1	10.00	1.67	1.0000E+03	1.0011E+02	–3.2032E+00
2	16.67	5.00	9.9359E+02	–1.0171E+02	1.2599E–05
3	10.00	8.33	1.0000E+03	1.0011E+02	3.2035E+00
4	3.33	5.00	1.0064E+03	3.0192E+02	2.8124E–04

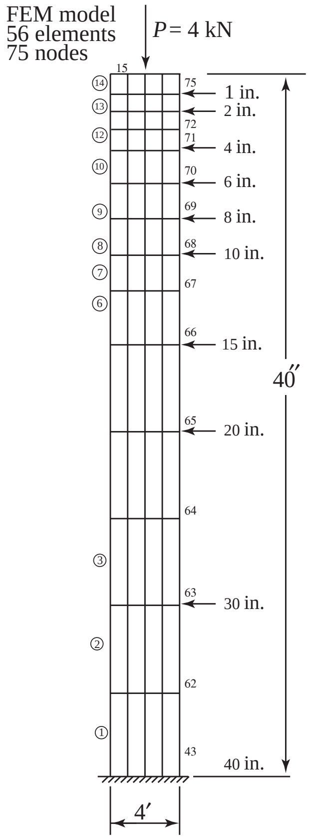
SIGMA(1)	SIGMA(2)	ANGLE
1.0000E+03	1.0010E+02	–2.0395E–01
9.9359E+02	–1.0171E+02	6.5906E–07
1.0000E+03	1.0010E+02	2.0396E–01
1.0064E+03	3.0192E+02	2.2873E–05

7.13



7.14

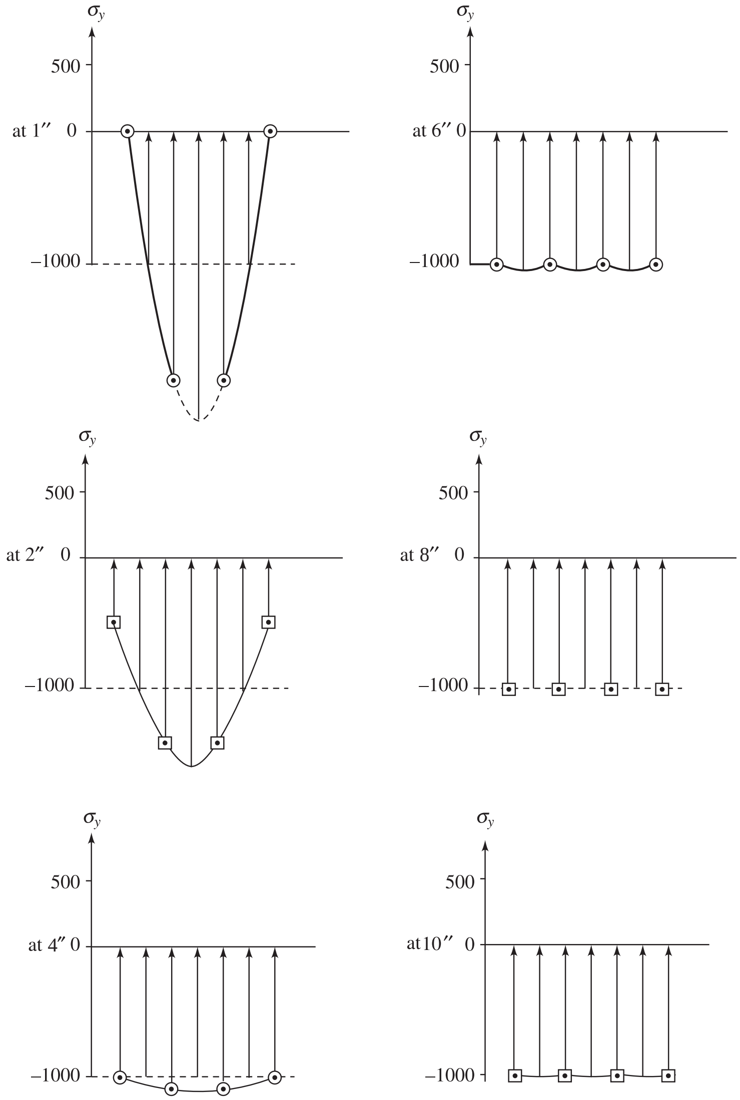




STRESS IN PSI AT VARIOUS DISTANCES
ALONG THE TENSILE PLATE WITH A 1000 # LOAD

ELEMENT #	14	28	42	56	
STRESS	26	–2026	–2026	26	(AT 1″)
ELEMENT #	13	27	41	55	
STRESS	–563	–1437	–1437	–563	(AT 2″)
ELEMENT #	11	25	39	53	
STRESS	–969	–1031	–1031	–969	(AT 4″)
ELEMENT #	10	24	38	52	
STRESS	–1002	–998	–998	–1002	(AT 6″)
ELEMENT #	9	23	37	51	
STRESS	–1002	–998	–998	–1002	(AT 8″)
ELEMENT #	8	22	36	50	
STRESS	–1001	–1000	–1000	–1001	(AT 10″)
ELEMENT #	6	20	34	48	
STRESS	–1000	–1000	–1000	–1000	(AT 15″)
ELEMENT #	5	21	33	47	
STRESS	–1000	–1000	–1000	–1000	(AT 20″)
ELEMENT #	3	17	31	45	
STRESS	–1002	–998	–998	–1002	(AT 30″)
ELEMENT #	1	15	29	43	
STRESS	–1005	–995	–995	–1005	(AT 40″)

Stress in plate at various depths (psi)



7.15

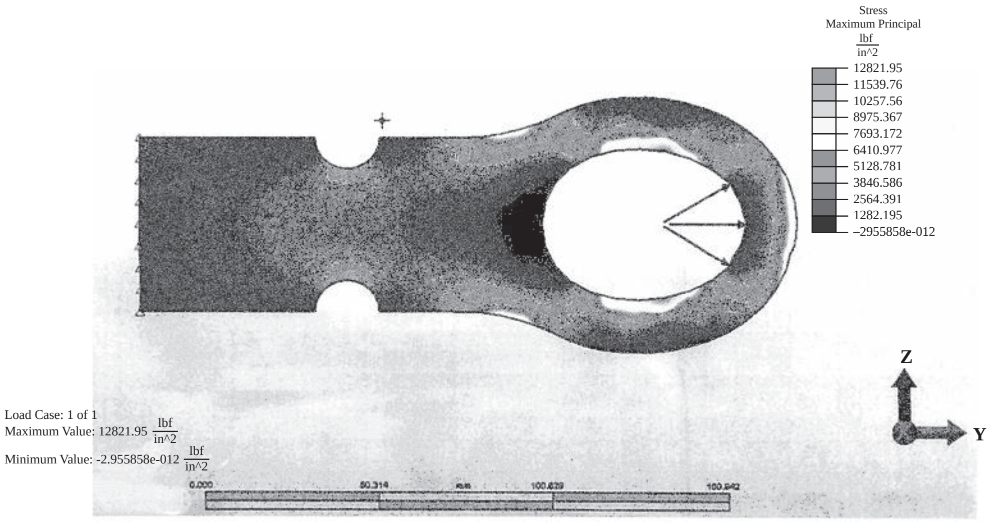
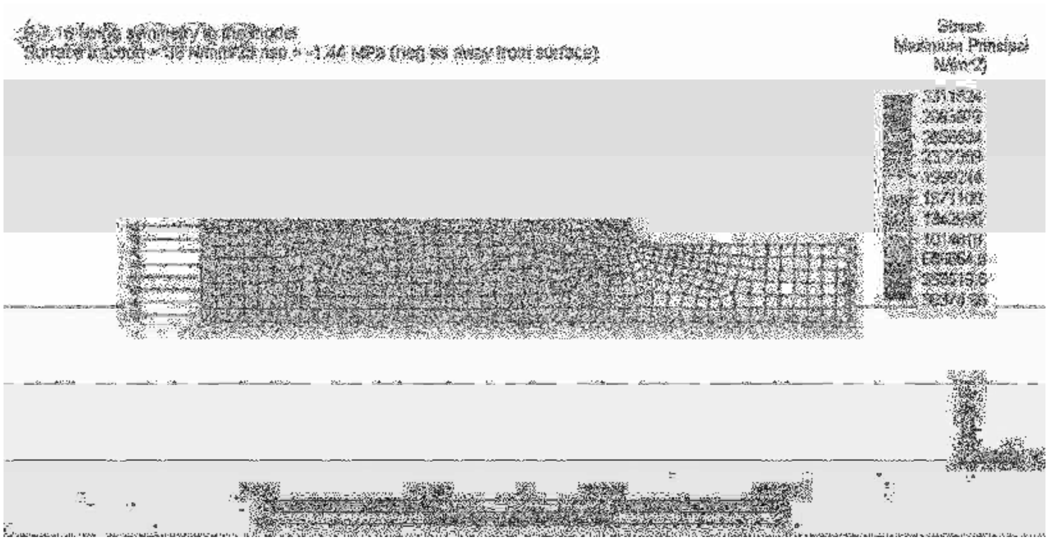


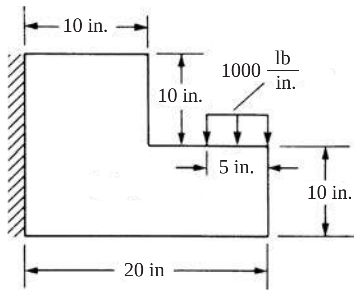
Figure. 8

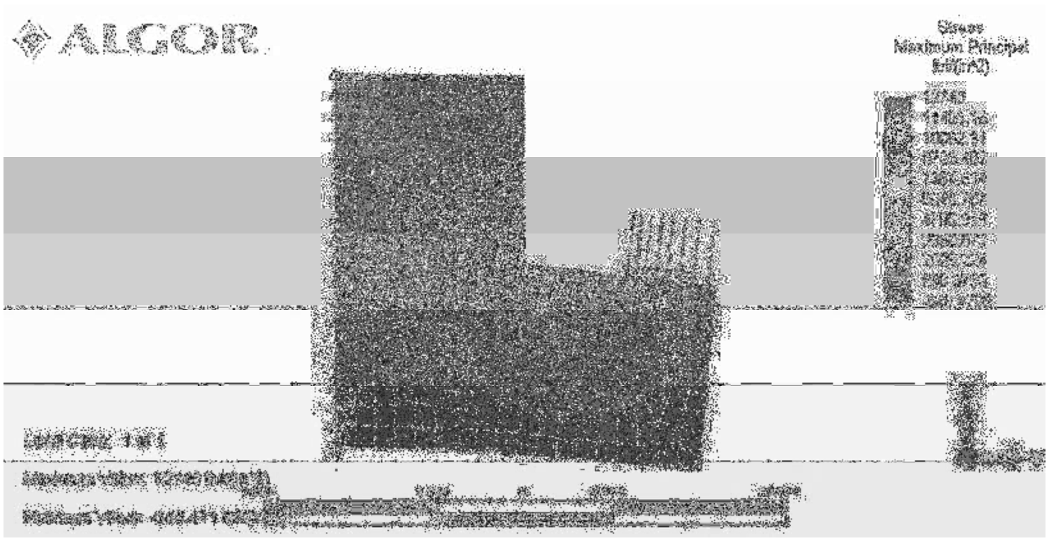
The maximum principal stresses are at the top and bottom of the circular opening of the connecting rod. The maximum stresses can be seen in Figure 8 above. Using a mesh density of only 400 (Figure 1- Figure 4) the precision in the area of interest (i. e., where the maximum stress occurs); see Figure 1 and Figure 4, the precision was about 0.28 further refinement was required. The mesh was refined to 800 and 1200 then finally to 1600. With the mesh density being 1600 the precision was less than 0.1 in. the place of interest and can be assumed correct. The maximum principal stress of about 12822 $\frac{\text{lb}}{\text{in.}^2}$ was determined.

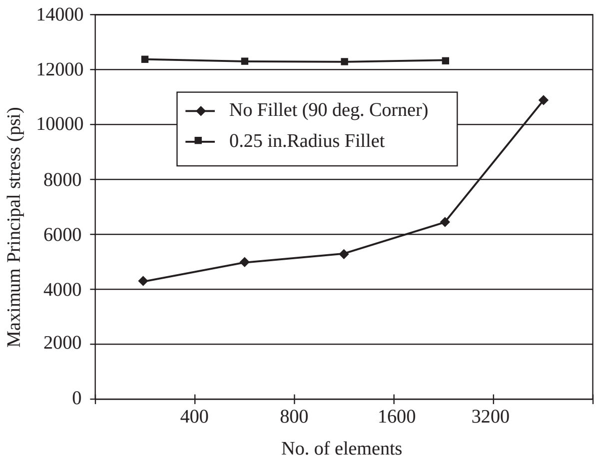
7.16



7.17

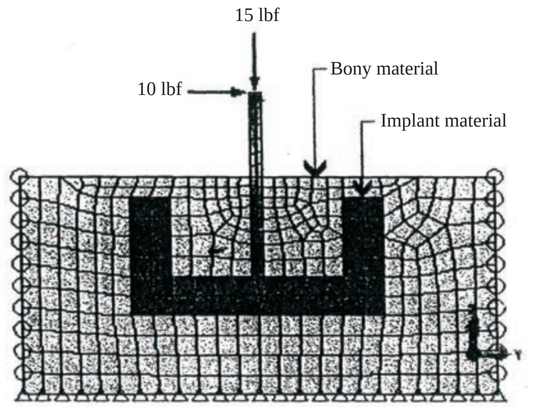






Maximum principal stress vs no. of elements

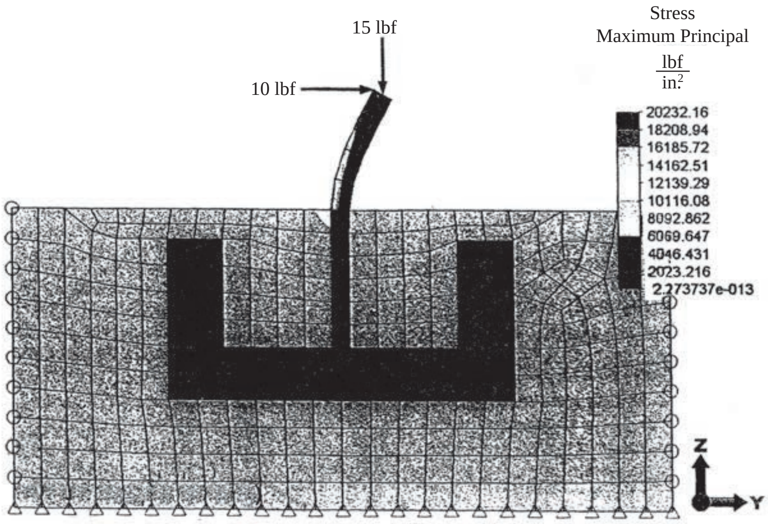
7.19



Model Parameters

Implant material modulus of elasticity 1.6×10^6 psi
Bony material modulus of elasticity 1.0×10^6 psi
Implant depth below bony material 0.100 in.

400 Mesh Density

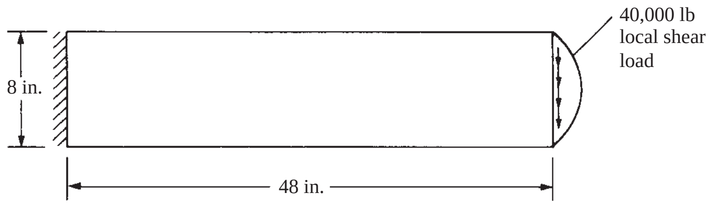


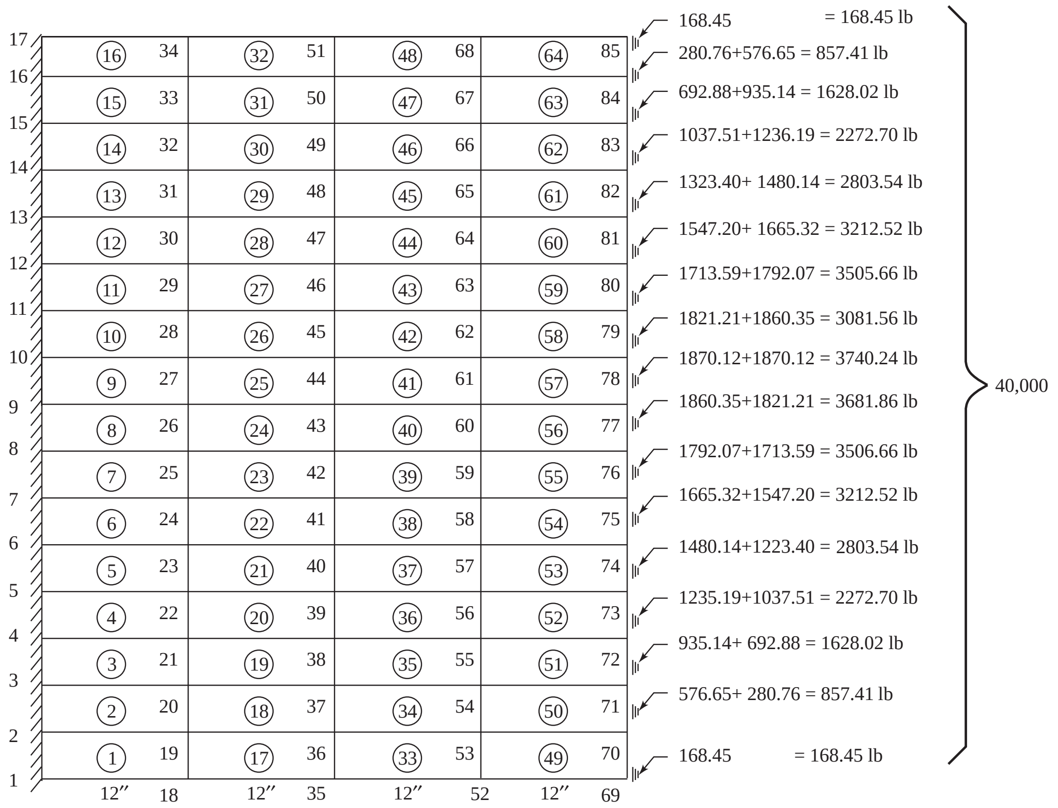
Load Case 1 of 1

Maximum Value: 20232.2 $\frac{\text{lbf}}{\text{in.}^2}$

Minimum Value: –2.27374e-013 $\frac{\text{lbf}}{\text{in.}^2}$

7.20





64 – 12 in. by $\frac{1}{2}$ in. elements

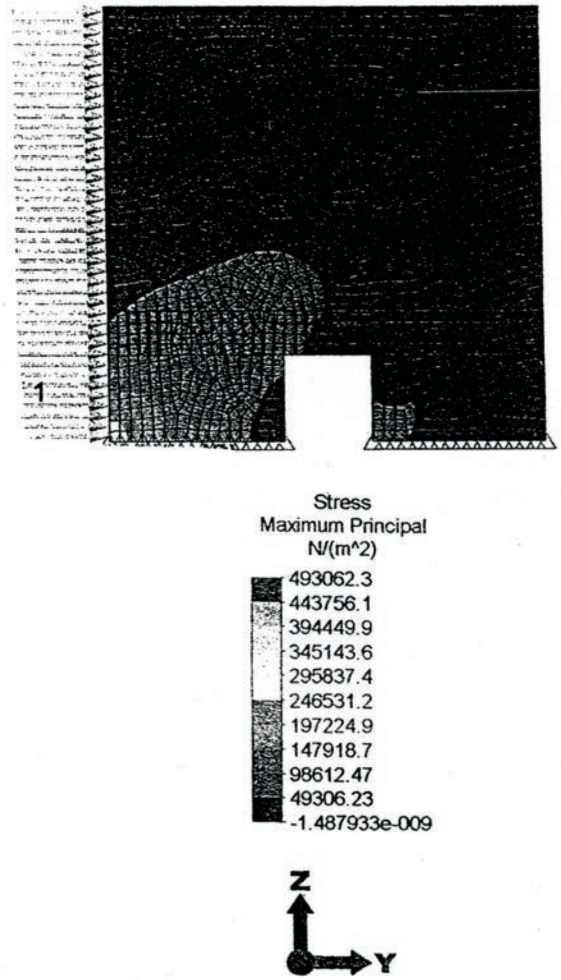
From computer program output

$$Y_{\text{maxA}} = -0.4993 \text{ in.} \quad Y_{\text{max exact}} = -1.152 \text{ in.} = \frac{PL^3}{3\,EI}$$

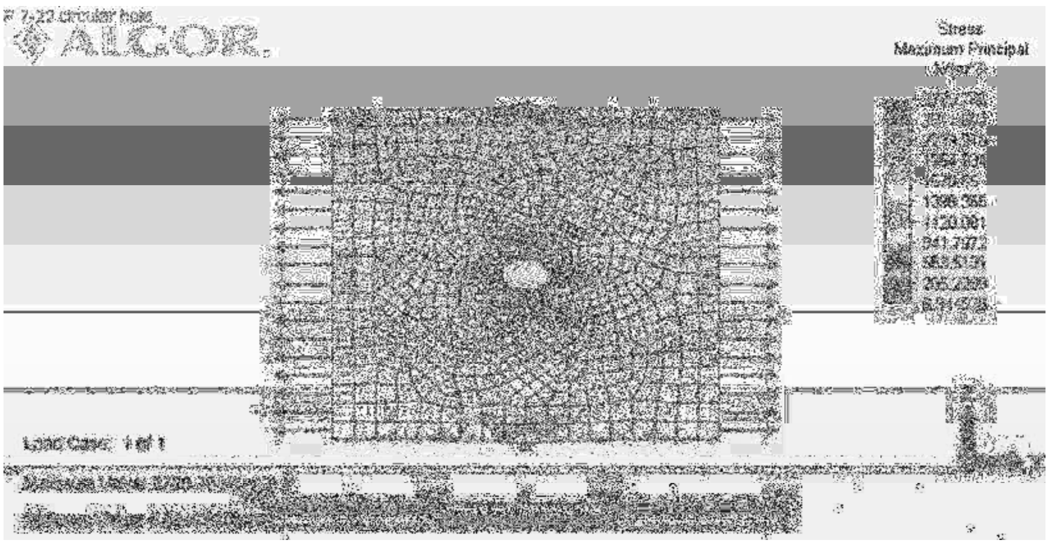
$$AR = \frac{12}{\frac{1}{2}} = 24 \text{ (56\% error due to large aspect ratio)}$$

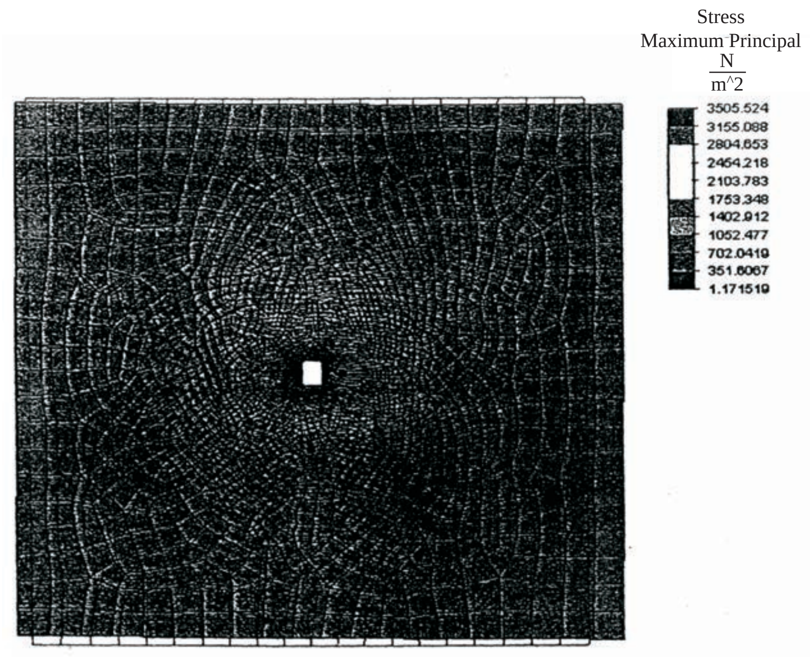
For other results see Example in Section 7.1, Table 7.1

7.21


1
Stress
Maximum Principal
N/(m^2)
493062.3
443756.1
394449.9
345143.6
295837.4
246531.2
197224.9
147918.7
98612.47
49306.23
-1.487933e-009
Z
Y

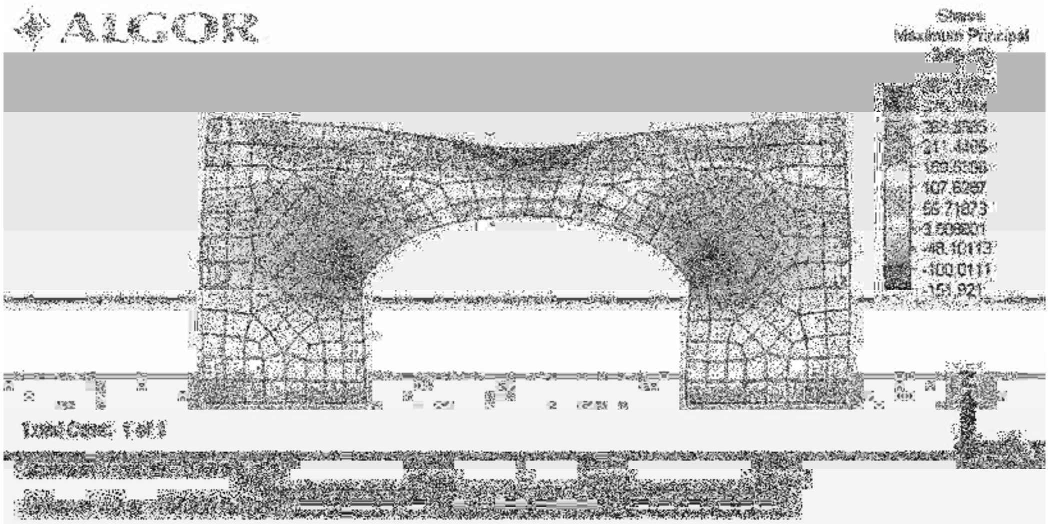
7.22


ALGOR
Stress
Maximum Principal
Load Case: 1 of 1

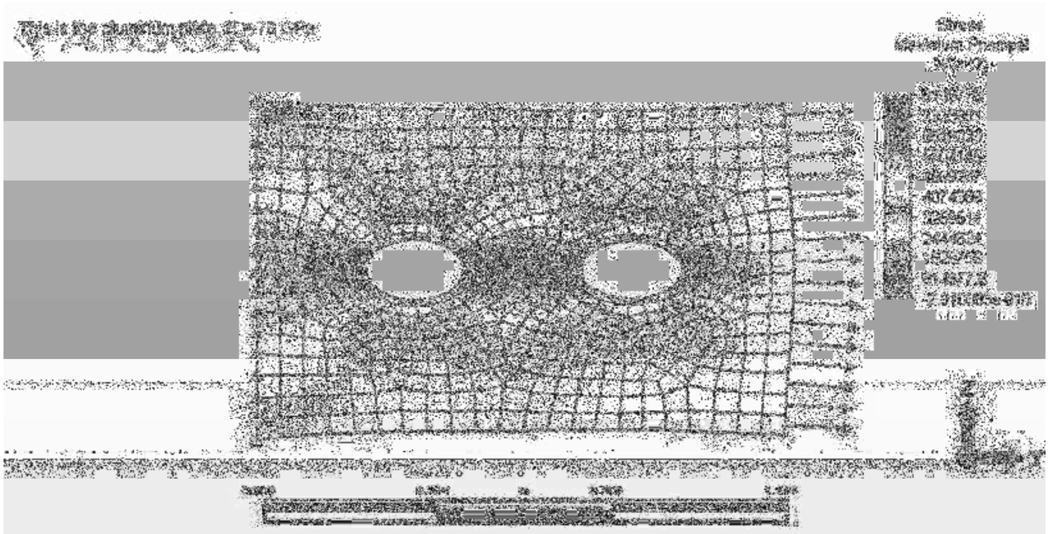


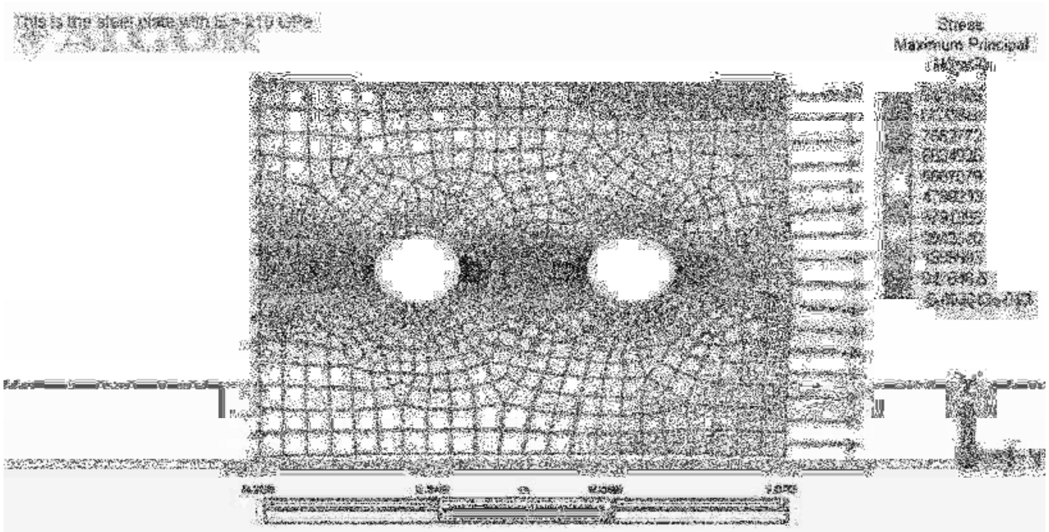
The figure above is the maximum principal stress. The maximum is 3505 $\frac{N}{m^{\wedge}2}$ The location of maximum stress occurs at the corners of the hole (with 1 mm radii)

7.23

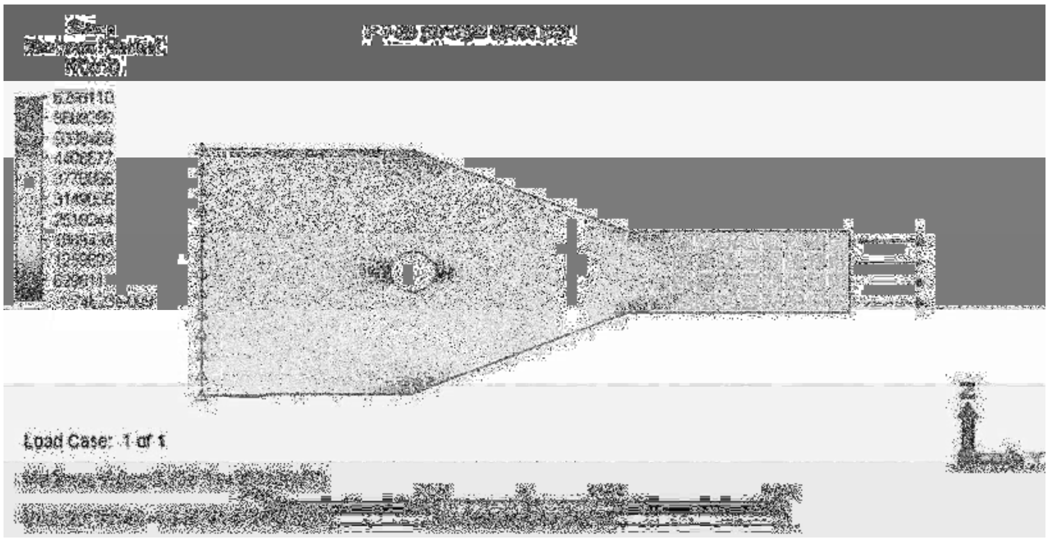


7.25



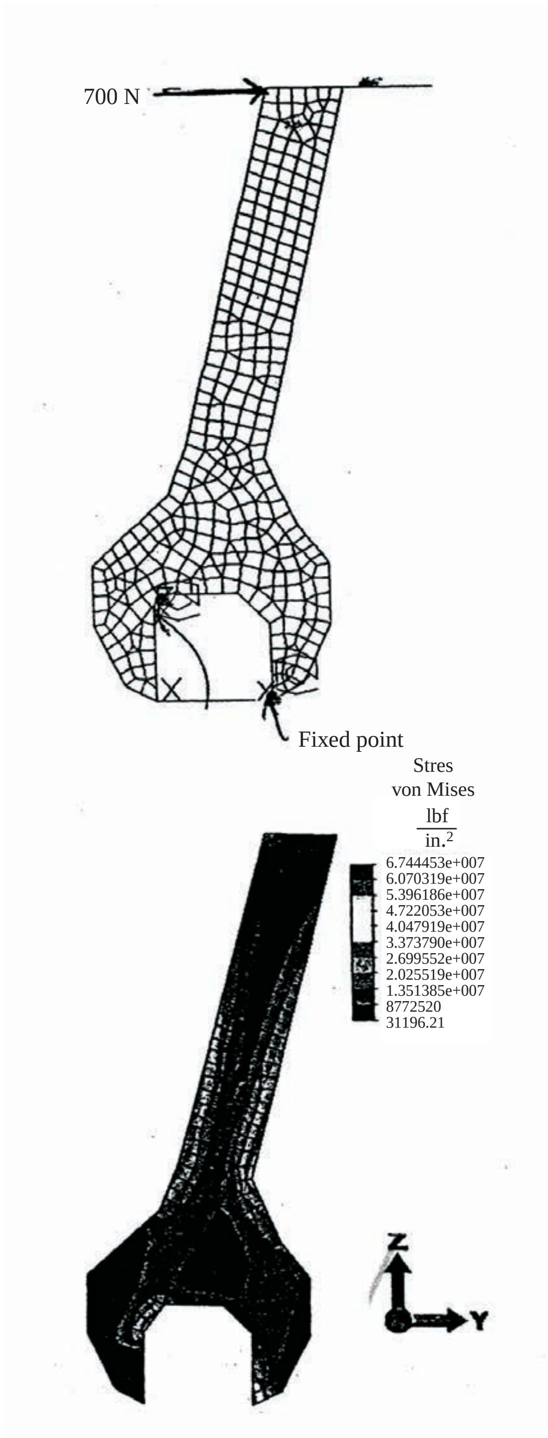

This is the steel plate with E = 210 GPa
Stress
Maximum Principal

7.26


Load Case: 1 of 1

The largest principal stress of 6.29 MPa occurs at the top and bottom inside edges of the hole. The second largest principal stress of 5.67 MPa occurs at the elbow between the smallest cross section and where the taper begins.

7.28



7.31

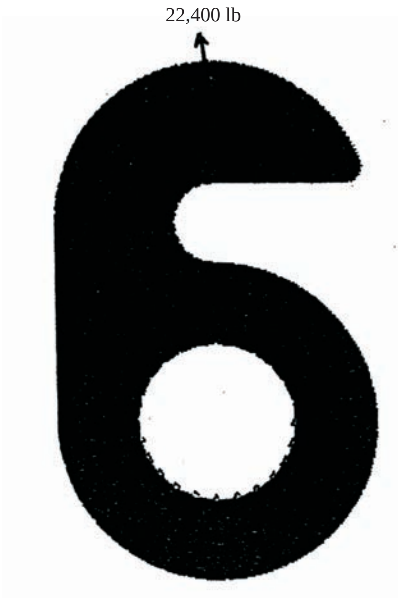


Figure 1: Mesh with Boundary Conditions and Nodal Force

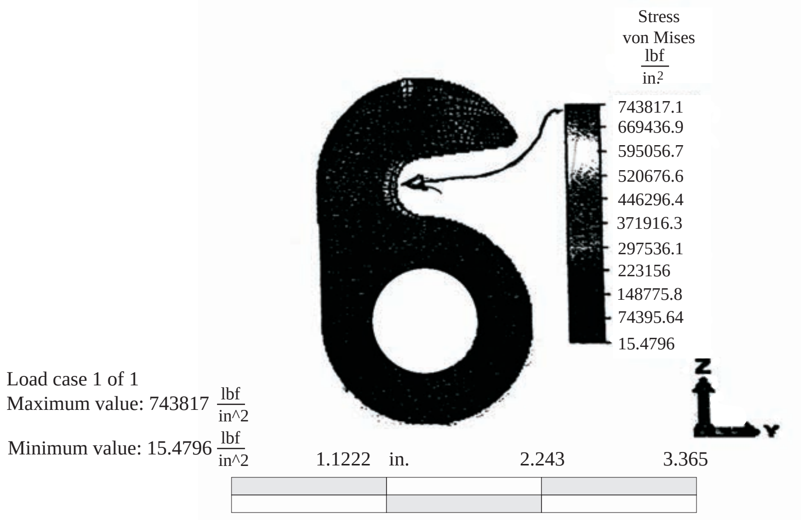
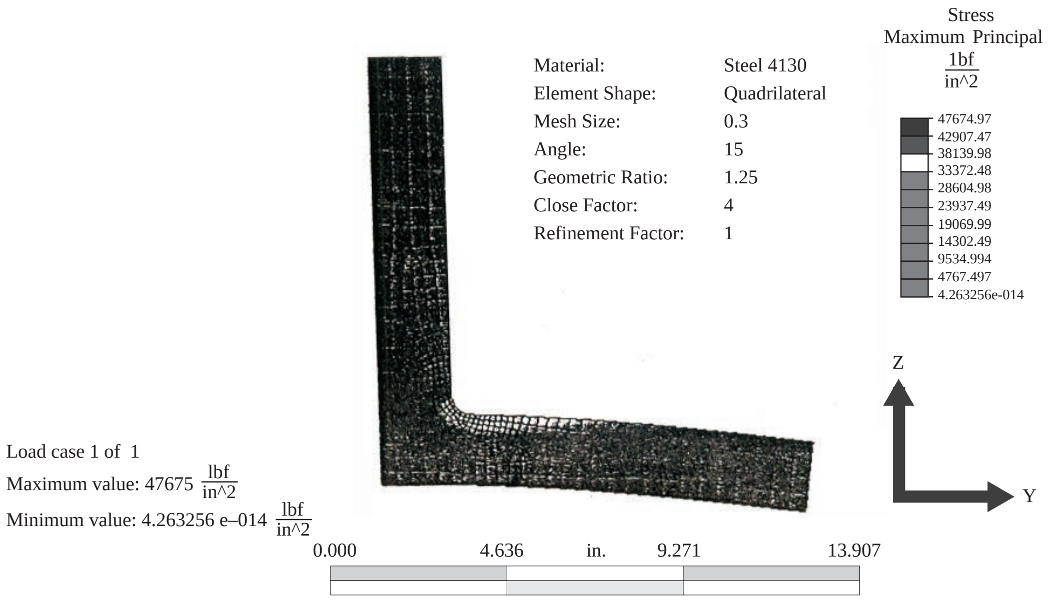


Figure 2: von Mises stresses (psi)

7.32

600 Mesh Density

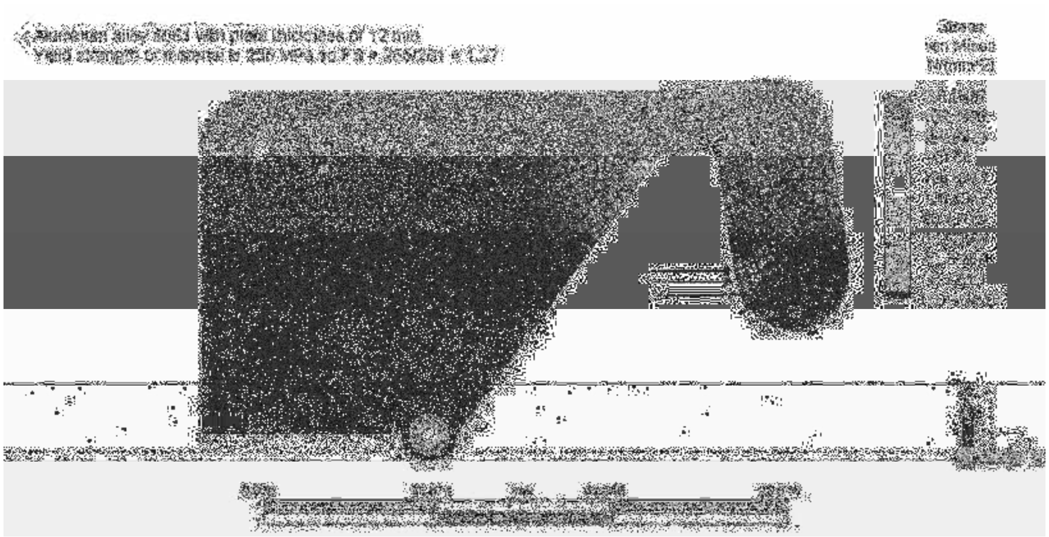


Mesh Density: 25-600

Bracket without Fillet	Maximum Principal Stress $\frac{lb}{in.^2}$
25 Mesh Density	22811.17
50	26114.15
100	27050.65
150	28179.32
200	28967.93
300	28800.52
400	35102.97
500	32852.23
600	33678.14
Bracket with Fillet	
25 Mesh Density	47481.11
50	47492.06
100	47502.16
150	47511.98
200	47521.59
300	47832.01
400	47688.08
500	47658.56
600	47674.97

In the FEA world re-entrant corners are a bad thing. These represent an infinite change in stiffness inside the part, which will result in an infinite stress concentration.

7.33



7.37

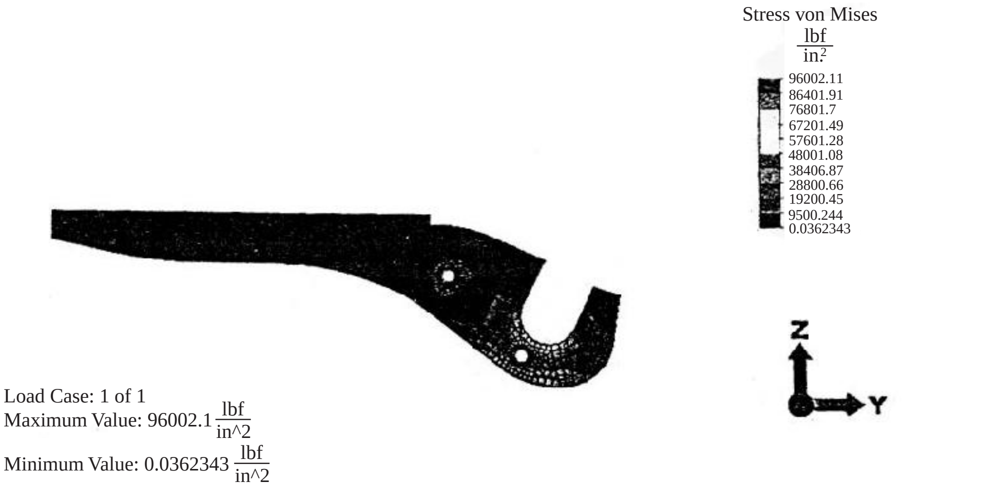
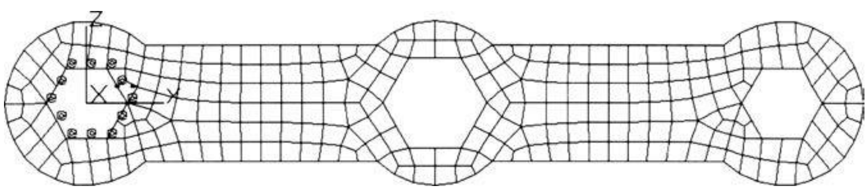


Figure 3: Maximum Stress Plot

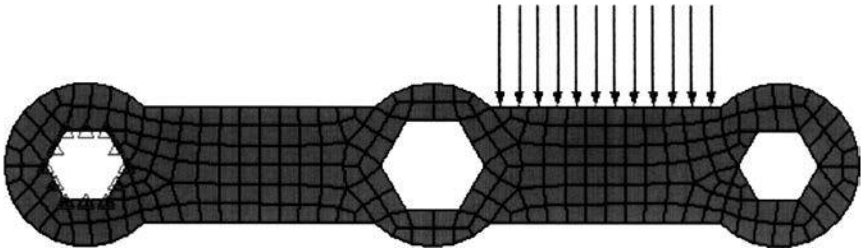
- Maximum von Mises stress occurred at the base of the notch in the crimper tool. The value was 96,002 psi.

7.38

The model is shown first with the boundary and loading conditions were then applied. The nodes of the far left hex were constrained from all movement. The red surface in the second figure below was selected and changed to surface 2. This allow the 100 $\frac{N}{cm^2}$ = 10,000,000 $\frac{dyn}{cm^2}$ force to be applied to the surface as shown.



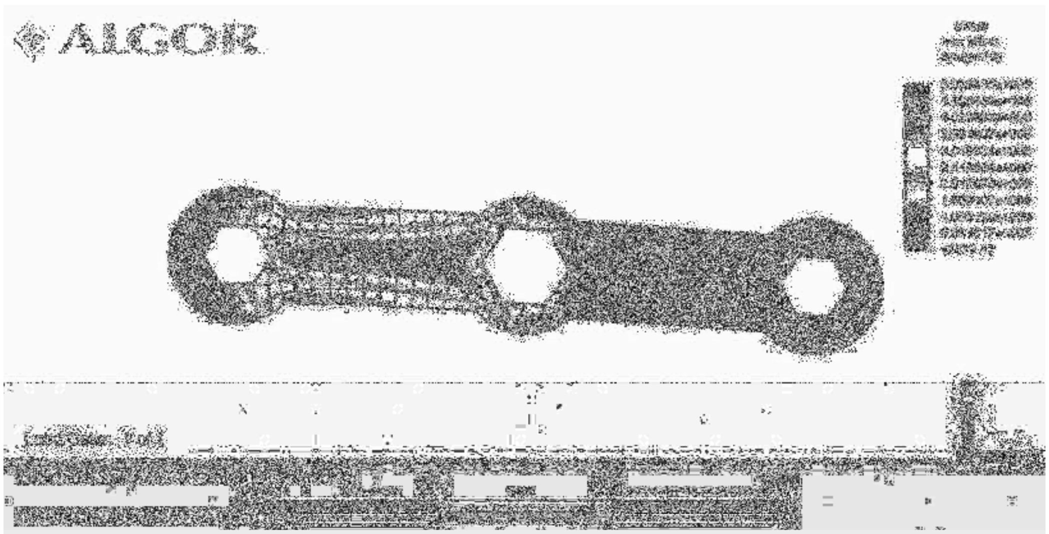
Next a material was chosen and an initial guess at the thickness *t* was made. ASTM A-514 was chosen, as this is a quenched and tempered steel with a high yield strength and will allow for the thickness to be minimized. A thickness of 1 cm was chosen as the initial guess, as this is an easy number to work with and it is compatible with the other wrench dimensions. A check was then performed to insure that the model properly reflected the problem. This check is shown below.



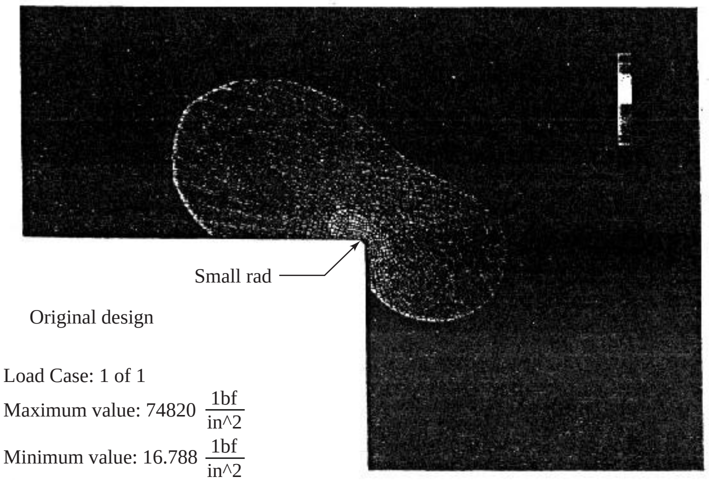
Material	Thickness (cm)	Pressure $\left(\frac{\text{N}}{\text{cm^2}}\right)$	Stress $\left(\frac{\text{N}}{\text{cm^2}}\right)$
ASTM A 514	1	100	5.03E+08
ASTM A 514	0.1	1000	5.03E+09
Al 3003-H16	0.3175	314.96	1.58E+09
Al 3003-H16	0.4	250	1.26E+09
Al 3003-H16	0.5	200	1.01E+09
Al 3003-H16	0.47625	209.97	1.06E+09

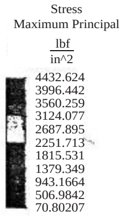
$$\frac{2}{3}S_y = \frac{2}{3} * 1.72\text{e}9 = 1.15\text{e}9 \frac{\text{dyn}}{\text{cm}^2}$$

With a thickness of 0.47625 cm, the stress was found to be $1.06 * 10^9 \ \frac{\text{dyn}}{\text{cm}^2}$.

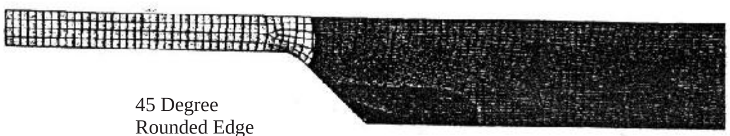


7.39 Zoomed in of the previous. To simulate a real cut, I inserted a very small radius at the point of concern.

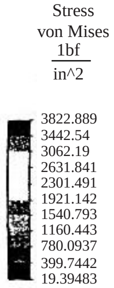


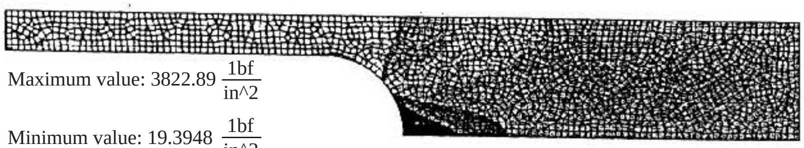


Maximum principal and von Mises look very similar



This geometry gave me the best result and had a better precision than the other configurations.



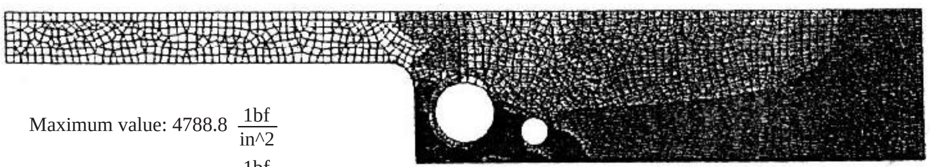


Maximum value: 3822.89 $\frac{1bf}{in^2}$

Minimum value: 19.3948 $\frac{1bf}{in^2}$

Radius Model

This geometry had some interesting results but the overall stress was still higher than the rounded off configuration



Maximum value: 4788.8 $\frac{1bf}{in^2}$

Minimum value: 62.2122 $\frac{1bf}{in^2}$

Relief Holes Model

Chapter 8

8.1 Triangular element

From Section 8.2

$$N_1 = 1 - \frac{3x}{b} - \frac{3y}{b} + \frac{2x^2}{b^2} + \frac{4xy}{bh} + \frac{2y^2}{h^2}$$

$$N_2 = \frac{-x}{b} + \frac{2x^2}{b^2}, \; N_3 = \frac{-y}{h} + \frac{2y^2}{h^2}$$

$$N_4 = \frac{4xy}{bh}, \; N_5 = \frac{4y}{h} - \frac{4xy}{bh} - \frac{4y^2}{h^2}$$

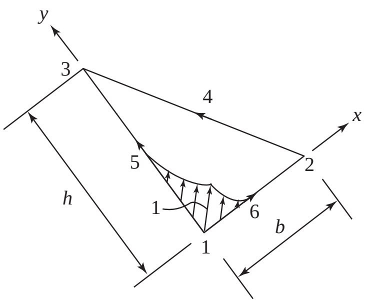
$$N_6 = \frac{4x}{b} - \frac{4x^2}{b^2} - \frac{4xy}{bh}$$

(a)

At $x = 0$ evaluate N's

$y = 0$

$N_1 = 1, N_2 = 0 = N_3 = N_4 = N_5 = N_6$



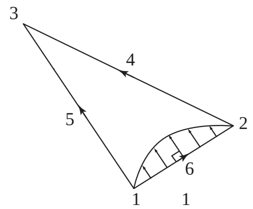
(b)

At $x = \dfrac{b}{2}$

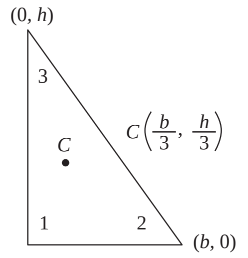
$y = 0$

$$N_1 = 1 - \frac{3(\frac{b}{2})}{b} - \frac{3(0)}{b} + \frac{2\left(\frac{b}{2}\right)^2}{b^2} + \frac{4\,(0)\,(0)}{bh} + \frac{2(0)^2}{h^2} = 1 - \frac{3}{2} + \frac{1}{2} = 0$$

and $N_2 = N_3 = N_4 = N_5 = 0, \quad N_6 = \dfrac{4\left(\frac{b}{2}\right)}{b} - \dfrac{4\left(\frac{b}{2}\right)^2}{b^2} = 1$



8.2



Strains

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 & \beta_4 & 0 & \beta_5 & 0 & \beta_6 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 & 0 & \gamma_4 & 0 & \gamma_5 & 0 & \gamma_6 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 & \gamma_4 & \beta_4 & \gamma_5 & \beta_5 & \gamma_6 & \beta_6 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ v_6 \end{Bmatrix} \quad (1)$$

Evaluate β's and γ's at centroid

$$\beta_1 = -3h + \frac{4hx}{b} + 4y = -3h + \frac{4h\left(\frac{b}{3}\right)}{b} + 4\left(\frac{h}{3}\right) = -\frac{h}{3}$$

$$\beta_2 = -h + \frac{4hx}{b} = -h + \frac{4h\left(\frac{b}{3}\right)}{b} = \frac{1}{3}h$$

$$\beta_3 = 0,\ \beta_4 = 4y = 4\left(\frac{h}{3}\right) = \frac{4h}{3}$$

$$\beta_5 = -4y = \frac{-4h}{3}$$

$$\beta_6 = 4h - \frac{-8hx}{b} - 4y = 4h - \frac{8h\left(\frac{b}{3}\right)}{b} - 4\left(\frac{h}{3}\right) = 0$$

$$\gamma_1 = -3b + 4x + \frac{4by}{h} = -3b + 4\left(\frac{b}{3}\right) + \frac{4b\left(\frac{h}{3}\right)}{h} = \frac{-b}{3}$$

$$\gamma_2 = 0,\ \gamma_3 = -b + \frac{4by}{h} = -b + \frac{4b\left(\frac{h}{3}\right)}{h} = \frac{b}{3}$$

$$\gamma_4 = 4x = \frac{4b}{3},\ \gamma_5 = 4b - 4x - \frac{8\,by}{h} = 4b - 4\left(\frac{b}{3}\right) - \frac{8b\left(\frac{h}{3}\right)}{h} = 0$$

$$\gamma_6 = \frac{-4b}{3} \quad (2)$$

Performing the multiplications in (1)
(After substituting β's and γ's from (2))

$$\varepsilon_x = \left(\frac{-h}{3}u_1 + \frac{h}{3}u_2 + \frac{4h}{3}u_4 - \frac{4h}{3}u_5\right)\frac{1}{bh}$$

$$\varepsilon_y = \left(\frac{-b}{3}v_1 + \frac{b}{3}v_3 + \frac{4b}{3}v_4 - \frac{4b}{3}v_6\right)\frac{1}{bh}$$

$$\gamma_{xy} = \left(\frac{-b}{3}u_1 - \frac{h}{3}v_1 + \frac{h}{3}v_2 + \frac{b}{3}u_3 + \frac{4b}{3}u_4 + \frac{4h}{3}v_4 - \frac{4h}{3}v_5 - \frac{4b}{3}u_6\right)\frac{1}{bh}$$

$$\varepsilon_x = \frac{h}{3}\,[-u_1 + u_2 + 4u_4 - 4u_5]\left(\frac{1}{bh}\right)$$

$$\varepsilon_y = \frac{b}{3}\,[-v_1 + v_3 + 4v_4 - 4v_6]\left(\frac{1}{bh}\right)$$

$$\gamma_{xy} = \left\{\frac{b}{3}[-u_1 + u_3 + 4u_4 - 4u_6] + \frac{h}{3}[-v_1 + v_2 + 4v_4 - 4v_5]\right\}\left(\frac{1}{bh}\right)$$

Stresses

$$\{\sigma\} = [D]\{\varepsilon\}$$

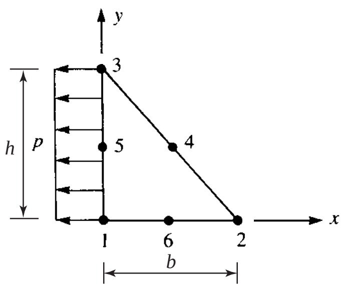
$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E}{1-v^2}\begin{bmatrix}1 & v & 0\\ v & 1 & 0\\ 0 & 0 & \frac{1-v}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

$$\sigma_x = \frac{E}{1-v^2}\left[\frac{h}{3}(-u_1 + u_2 + 4u_4 - 4u_5) + v\,\frac{b}{3}(-v_1 + v_3 + 4v_4 - 4v_6)\right]\frac{1}{bh}$$

$$\sigma_y = \frac{E}{1-v^2}\left[v\,\frac{h}{3}(-u_1 + u_2 + 4u_4 - 4u_5) + \frac{b}{3}(-v_1 + v_3 + 4v_4 - 4v_6)\right]\frac{1}{bh}$$

$$\tau_{xy} = \frac{E}{2(1+v)}\left[\frac{b}{3}(-u_1 + u_3 + 4u_4 - 4u_6) + \frac{h}{3}(-v_1 + v_2 + 4v_4 - 4v_5)\right]\frac{1}{bh}$$

8.3



The equation is $\{f_s\} = \int_s [N_s]^T\{T\}\,ds$ (1)

$$\{T\} = \begin{Bmatrix}p_x\\ p_y\end{Bmatrix} = \begin{Bmatrix}p\\ 0\end{Bmatrix} \text{ …. is the surface traction} \tag{2}$$

$$[N] = \begin{bmatrix}N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0\\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6\end{bmatrix} \tag{3}$$

Substituting (2) and (3) in Equation (1), we have

$$\{f_s\} = \int_0^t \int_0^h \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \\ \vdots & \vdots \\ N_6 & 0 \\ 0 & N_6 \end{bmatrix} \begin{Bmatrix} p \\ 0 \end{Bmatrix} dy\, dz \quad \begin{matrix} \text{at } x = 0 \\ y = y \end{matrix}$$

$$\{f_s\} = t \int_0^h \begin{Bmatrix} N_1 p \\ 0 \\ N_2 p \\ 0 \\ N_3 p \\ 0 \\ \vdots \\ N_6 p \\ 0 \end{Bmatrix} dy \quad \begin{matrix} \text{at } x = 0 \\ y = y \end{matrix} \tag{4}$$

From Section 8.2 for this particular element we have

$$N_1 = 1 - \frac{3x}{b} - \frac{3y}{h} + 2x^2 + 4xy + \frac{2y^2}{h^2}$$

$$N_2 = \frac{-x}{b} + \frac{2x^2}{b^2},\ N_3 = \frac{-y}{h} + \frac{2y^2}{h^2}$$

$$N_4 = \frac{4xy}{bh},\ N_5 = \frac{4y}{h} - \frac{4xy}{bh} - \frac{4y^2}{h^2} \tag{5}$$

$$N_6 = \frac{4x}{b} - \frac{4x^2}{b^2} - \frac{4xy}{bh}$$

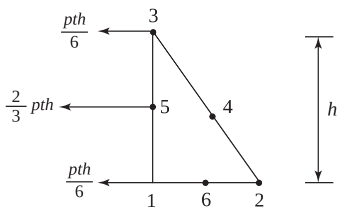
Substitute (5) into (4) and evaluating N's at $x = 0$, $y = y$, we have

$$\{f_s\} = t\int_0^h \begin{Bmatrix} \left(1-\frac{3y}{h}+\frac{2y^2}{h^2}\right)p \\ 0 \\ 0 \\ 0 \\ \left(\frac{-y}{h}+\frac{2y^2}{h^2}\right)p \\ 0 \\ 0 \\ 0 \\ \left(\frac{4y}{h}-\frac{4y^2}{h^2}\right)p \\ 0 \\ 0 \\ 0 \end{Bmatrix} dy$$

$$f_{s1x} = pt\left(y-\frac{3y^2}{2h}+\frac{2y^3}{3h^2}\right)\Bigg|_0^h = \frac{pth}{6}$$

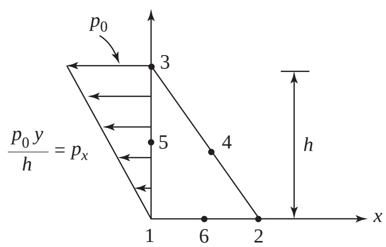
$$f_{s3x} = \left(\frac{-y^2}{2h}+\frac{2y^3}{3h^2}\right)pt = \frac{pth}{6}$$

$$f_{s5x} = pt\left(\frac{4y^2}{2h}-\frac{4y^3}{3h^2}\right)\Bigg|_0^h = \frac{2\ pth}{3}$$



Nodal equivalent forces

8.4



$$\{f_s\} = \int_s [N_s]^T \{T\} ds \tag{1}$$

$$\{T\} = \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} = \begin{Bmatrix} \frac{p_0 y}{h} \\ 0 \end{Bmatrix} \tag{2}$$

$$[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 \end{bmatrix} \quad (3)$$

Substituting (2) and (3) in (1)

$$\{f_s\} = \int_0^t \int_0^h \left\{ \begin{matrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \\ \vdots & \\ N_6 & 0 \\ 0 & N_6 \end{matrix} \right\} \left\{ \begin{matrix} \frac{p_0 y}{h} \\ 0 \end{matrix} \right\} dy\, dz \quad \begin{matrix} \text{at } x = 0 \\ y = y \end{matrix}$$

$$\{f_s\} = t \int_0^h \left\{ \begin{matrix} N_1 \frac{p_0 y}{h} \\ 0 \\ N_2 \frac{p_0 y}{h} \\ 0 \\ N_3 \frac{p_0 y}{h} \\ 0 \\ N_4 \frac{p_0 y}{h} \\ 0 \\ N_5 \frac{p_0 y}{h} \\ 0 \\ N_6 \frac{p_0 y}{h} \\ 0 \end{matrix} \right\} dy \quad \begin{matrix} \text{at } x = 0 \\ y = y \end{matrix} \qquad (4)$$

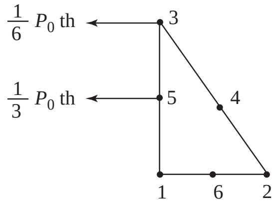
(Shape functions are same as in Problem 8.3). Upon substituting shape functions into (4) evaluating the N's at $x = 0$, $y = y$, we have

$$\{f_s\} = t \int_0^h \left\{ \begin{matrix} \left(1 - \frac{3y}{h} + \frac{2y^2}{h^2}\right) \frac{p_0 y}{h} \\ 0 \\ 0 \\ 0 \\ \left(\frac{-y}{h} + \frac{2y^2}{h^2}\right) \frac{p_0 y}{h} \\ 0 \\ 0 \\ 0 \\ \left(\frac{4y}{h} - \frac{4y^2}{h^2}\right) \frac{p_0 y}{h} \\ 0 \\ 0 \\ 0 \end{matrix} \right\} dy$$

$$f_{s1x} = \frac{p_0\,t}{h}\left(\frac{y^2}{2} - \frac{3y^3}{3h} + \frac{2y^4}{4h^2}\right)\Bigg|_0^h = 0$$

$$f_{s3x} = \frac{p_0\,t}{h}\left(\frac{-\,y^3}{3h} + \frac{2y^4}{4h^2}\right)\Bigg|_0^h = \frac{p_0\,th}{6}$$

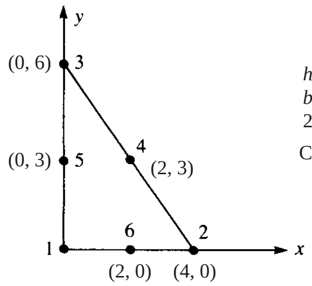
$$f_{s5x} = \frac{p_0\,t}{h}\left(\frac{4y^3}{3h} - \frac{4y^4}{4h^2}\right)\Bigg|_0^h = \frac{p_0\,th}{3}$$



Note: Different result in Problem 6.9

Nodal equivalent forces

8.5 (a)



$h = 6$ in.
$b = 4$ in.
$2A = 24$ in.^{2}
Centroid at $\left(\frac{4}{3}, 2\right)$

(a)

$$\{\varepsilon\} = [B]\,\{d\}$$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 & \beta_4 & 0 & \beta_5 & 0 & \beta_6 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 & 0 & \gamma_4 & 0 & \gamma_5 & 0 & \gamma_6 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 & \gamma_4 & \beta_4 & \gamma_5 & \beta_5 & \gamma_6 & \beta_6 \end{bmatrix}$$

Element is oriented as in Section 9.2

$\therefore$ β's and γ's as in Section 9.2

$$\beta_1 = -\,3h + \frac{4hx}{6} + 4y = 6x + 4y - 18$$

$$\beta_2 = -\,h + \frac{4hx}{6} = 6x - 6,\ \beta_3 = 0$$

$$\beta_4 = 4y,\ \beta_5 = -\,4y$$

$$\beta_6 = 4h - \frac{8hx}{6} - 4y = -\,12x - 4y + 24$$

$$\gamma_1 = -\,3b + 4x + \frac{4by}{h} = 4x + \frac{8}{3}\,y - 12$$

$$\gamma_2 = 0$$

$$\gamma_3 = -\,b + \frac{4by}{h} = \frac{8}{3}\,y - 4$$

$$\gamma_4 = 4x$$

$$\gamma_5 = 4b - 4x - \frac{8by}{h} = -4x - \frac{16}{3}y + 16$$

$$\gamma_6 = -4x$$

$$\therefore \quad 2A\,\varepsilon_x = \beta_2 u_2 + \beta_4 u_4 + \beta_6 u_6$$

$$= 0.001\,(6x - 6) + 0.0002\,(4y) + 0.0005\,(-12x - 4y + 24)$$

$$2A\,\varepsilon_x = -0.0012y + 0.006$$

$$\therefore \quad \varepsilon_x = -5 \times 10^{-5}y + 2.5 \times 10^{-4}$$

$$2A\,\varepsilon_y = \gamma_3 v_3 + \gamma_4 v_4 + \gamma_5 v_5 + \gamma_6 v_6$$

$$= 0.0002\left(\frac{8}{3}y - 4\right) + 0.0001\,(4x) + 0.0001\left(-4x - \frac{16}{3}y + 16\right) + 0.001\,(-4x)$$

$$2A\,\varepsilon_y = -0.004x + 0.0008$$

$$\therefore \quad \varepsilon_y = -1.67 \times 10^{-4}x + 3.33 \times 10^{-5}$$

$$2A\,\gamma_{xy} = 0.002\,(6x - 6) + 0.0005\left(\frac{8}{3}y - 4\right) + 0.0002\,(4x) + 0.0001(4y)$$

$$+ 0.0001\,(-4y) + 0.0005\,(-4x) + 0.001\,(-12x - 4y + 24)$$

$$2A\,\gamma_{xy} = -0.0012x - 0.00267y + 0.01$$

$$\therefore \quad \gamma_{xy} = -5 \times 10^{-5}x - 1.11 \times 10^{-4}y + 4.167 \times 10^{-4}$$

Evaluate stresses at centroid

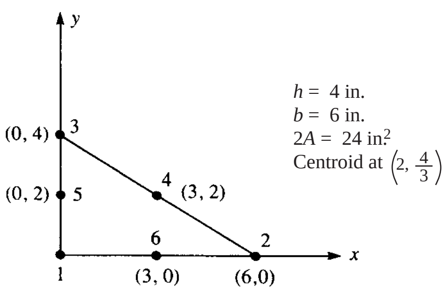
$$\{\sigma\} = [D]\,\{\varepsilon\}$$

$$\{\varepsilon\}\Big|_{\left(\frac{4}{3},2\right)} = \begin{Bmatrix} 0.00015 \\ -1.89 \times 10^{-4} \\ 0.000128 \end{Bmatrix}$$

$$\{\sigma\} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{Bmatrix} 1.5 \times 10^{-4} \\ -1.89 \times 10^{-4} \\ 1.28 \times 10^{-4} \end{Bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} 3288 \\ -4848 \\ 1536 \end{Bmatrix} \text{ psi}$$

(b)



(b)

Using expressions for β's and γ's from part (a) with $h = 4$ in. and $b = 6$ in. now

$$\beta_1 = \frac{8}{3}x + 4y - 12,\ \beta_2 = \frac{8}{3}x - 4,\ \beta_3 = 0$$

$$\beta_4 = 4y,\ \beta_5 = -4y,\ \beta_6 = \frac{-16}{3}x - 4y + 16$$

$$\gamma_1 = 4x + 6y - 18,\ \gamma_2 = 0,\ \gamma_3 = 6y - 6$$

$$\gamma_4 = 4x,\ \gamma_5 = -4x - 12y + 24,\ \gamma_6 = -4x$$

$$2A\,\varepsilon_x = 0.001\left(\frac{8}{3}x - 4\right) + 0.0002\,(4y) + 0.0005\left(-\frac{16}{3}x - 4y + 16\right)$$

$$2A\,\varepsilon_x = -0.0012y + 0.004$$

$$\therefore \quad \varepsilon_x = -5\times 10^{-5}y + 1.67\times 10^{-4}$$

$$2A\,\varepsilon_y = 0.0002\,(6y - 6) + 0.0001\,(4x)$$
$$+ 0.0001\,(-4x - 12y + 24) + (0.001)\,(-4x)$$

$$2A\,\varepsilon_y = -0.004\,x + 0.0012$$

$$\therefore \quad \varepsilon_y = -1.67\times 10^{-4}x + 5\times 10^{-5}$$

$$2A\,\gamma_{xy} = 0.002\left(\frac{8}{3}x - 4\right) + 0.0005\,(6y - 6)$$
$$+ 0.0002\,(4x) + 0.0001\,(4y) + 0.0001\,(-4y)$$
$$+ 0.0005\,(-4x) + 0.001\left(-\frac{16}{3}x - 4y + 16\right)$$

$$2A\,\gamma_{xy} = -0.0012x - 0.001y + 0.005$$

$$\therefore \quad \gamma_{xy} = -5\times 10^{-5}x - 4.167\times 10^{-5}y - 2.083\times 10^{-4}$$

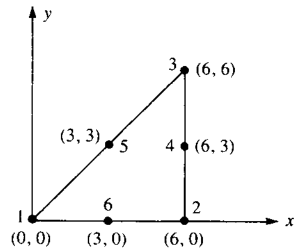
$$\{\sigma\} = [D]\{\varepsilon\} \text{ at centroid } \left(2, \frac{4}{3}\right)$$

$$\{\varepsilon\}\Big|_{(2,\,\frac{4}{3})} = \begin{Bmatrix} 0.0001 \\ -0.000284 \\ -0.0000527 \end{Bmatrix}$$

$$\{\sigma\} = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}\begin{Bmatrix} 0.0001 \\ -0.000284 \\ -0.0000527 \end{Bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} 928 \\ -8288 \\ 632 \end{Bmatrix} \text{ psi}$$

8.6



Using Equation (8.1.14) in (8.1.13)

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 2x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & x & 2y \\ 0 & 0 & 1 & 0 & x & 2y & 0 & 1 & 0 & 2x & y & 0 \end{bmatrix} [X]^{-1} \{d\} \qquad (1)$$

where by Equation (8.1.7) $\{a\} = [X]^{-1}\{d\}$ and

$$\{a\} = \left[\begin{array}{cccccc:cccccc} 1 & 0 & 0 & 0 & 0 & 0 & & & & & & \\ 1 & 6 & 0 & 36 & 0 & 0 & & & & & & \\ 1 & 6 & 6 & 36 & 36 & 36 & & & \underline{O}_{6\times 6} & & & \\ 1 & 6 & 3 & 36 & 18 & 9 & & & & & & \\ 1 & 3 & 3 & 9 & 9 & 9 & & & & & & \\ 1 & 3 & 0 & 9 & 0 & 0 & & & & & & \\ \hdashline & & & & & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & 1 & 6 & 0 & 36 & 0 & 0 \\ & & & & & & 1 & 6 & 6 & 36 & 36 & 36 \\ & & \underline{O}_{6\times 6} & & & & 1 & 6 & 3 & 36 & 18 & 9 \\ & & & & & & 1 & 3 & 3 & 9 & 9 & 9 \\ & & & & & & 1 & 3 & 0 & 9 & 0 & 0 \end{array}\right]^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{Bmatrix} \qquad (2)$$

Using computer, we invert $[X]$ in Equation (2) and reorder $\{d\}$ to normal form $[u_1\, v_1\, u_2\, v_2 \ldots]^T = \{d\}^T$

$\therefore \qquad [X]^{-1}\{d\} =$

$$\left[\begin{array}{cccccc:cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 0 & -0.167 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.667 & 0 \\ 0 & 0 & 0.167 & 0 & -0.167 & 0 & 0 & 0 & 0.667 & 0 & -0.667 & 0 \\ 0.056 & 0 & 0.056 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.111 & 0 \\ 0 & 0 & -0.111 & 0 & 0 & 0 & 0.111 & 0 & -0.111 & 0 & -0.111 & 0 \\ 0 & 0 & 0.056 & 0 & -0.056 & 0 & 0.111 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & -0.167 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.667 \\ 0 & 0 & 0 & 0.167 & 0 & -0.167 & 0 & 0 & 0 & 0.667 & 0 & -0.667 \\ 0 & 0.056 & 0 & 0.056 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.111 \\ 0 & 0 & 0 & -0.111 & 0 & 0 & 0 & 0.111 & 0 & -0.111 & 0 & 0.111 \\ 0 & 0 & 0 & 0.056 & 0 & -0.056 & 0 & -0.111 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_6 \\ v_6 \end{Bmatrix} \tag{3}$$

at centroid ($x = 4$, $y = 2$)

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 8 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 4 \\ 0 & 0 & 1 & 0 & 4 & 4 & 0 & 1 & 0 & 8 & 2 & 0 \end{bmatrix} [X]^{-1} \{d\} \tag{4}$$

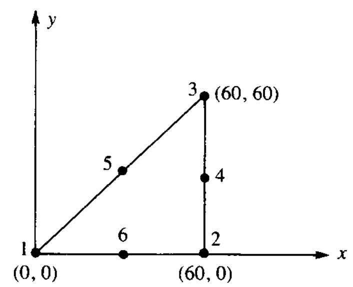
Multiplying matrices in Equation (4) yields

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} (-0.052\, u_1 + 0.059\, u_2 + 0.222\, u_4 - 0.222\, u_5 + 0.001\, u_6) \\ (-0.053\, v_2 - 0.391\, v_3 + 0.223\, v_5 - 0.223\, v_6) \\ (-0.052\, v_1 + 0.059\, v_2 + 0.222\, v_4 - 0.222\, v_5 + 0\, v_6 - \\ 0.053\, u_2 - 0.391\, u_3 + 0.223\, u_5 - 0.223\, u_6) \end{Bmatrix}$$

Then

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1 - v^2} \begin{Bmatrix} \varepsilon_x + v\, \varepsilon_y \\ \varepsilon_y + v\, \varepsilon_x \\ \left(\frac{1-v}{2}\right) \gamma_{xy} \end{Bmatrix}$$

8.7



$$u_1 = u(0, 0) = a_1 \tag{1}$$

$$u_2 = u(60, 0) = a_1 + 60\, a_2 + 3600\, a_4 \tag{2}$$

$$u_3 = u(60, 60) = a_1 + 60\, a_2 + 60\, a_3 + 3600\, a_4 + 3600\, a_5 + 3600\, a_6 \tag{3}$$

$$u_4 = u(60, 30) = a_1 + 60\, a_2 + 30\, a_3 + 3600\, a_4 + 1800\, a_5 + 800\, a_6 \tag{4}$$

$$u_5 = u(30, 30) = a_1 + 30\, a_2 + 30\, a_3 + 900\, a_4 + 900\, a_5 + 900\, a_6 \tag{5}$$

$$u_6 = u(30, 0) = a_1 + 30\, a_2 + 900\, a_4 \tag{6}$$

By (1) $\Rightarrow a_1 = u_1$

By (2) – 2 (6) $\Rightarrow a_4 = \dfrac{u_2 - 2u_6 + u_1}{1800}$

By – (2) + 4 (6) $\Rightarrow a_2 = \dfrac{4u_6 - u_2 + 3u_1}{60}$

By 2(4) – (3) $\Rightarrow a_6 = \dfrac{u_2 + u_3 - 2u_4}{1800}$

(4) – (5) $\Rightarrow a_5 = \dfrac{-u_2 + u_4 - u_5 + u_6}{900}$

(4) $\Rightarrow a_3 = \dfrac{u_2 - u_3 + 4u_5 - 4u_6}{60}$

Can verify by substituting all a's into Equation (3)

$$\therefore \quad u = u_1 + \left(\frac{4u_6 - u_2 + 3u_1}{60}\right) x + \left(\frac{u_2 - u_3 + 4u_5 - 4u_6}{60}\right) y + \left(\frac{u_2 - 2u_6 + u_1}{1800}\right) x^2 + \left(\frac{-u_2 + u_4 - u_5 + u_6}{900}\right) xy + \left(\frac{u_2 + u_3 - 2u_4}{1800}\right) y^2$$

$\therefore$ Shape functions are

$$N_1 = 1 - \frac{3x}{60} + \frac{x^2}{1800} \quad \text{(From all } u_1 \text{ coefficient)}$$

$$N_2 = \frac{-x}{60} + \frac{y}{60} + \frac{x^2}{1800} - \frac{xy}{900} + \frac{y^2}{1800}$$

$$N_3 = \frac{-y}{60} + \frac{y^2}{1800}$$

$$N_4 = \frac{xy}{900} - \frac{2y^2}{1800}$$

$$N_5 = \frac{4y}{60} - \frac{xy}{900}$$

$$N_6 = \frac{4x}{60} - \frac{4y}{60} - \frac{2x^2}{1800} + \frac{xy}{900}$$

$$2A = 2\left(\frac{1}{2}\right)(60)(60) = 3600$$

$$\beta_1 = 2A\left(\frac{\partial N_1}{\partial x}\right) = 3600\left(-\frac{3}{60} + \frac{2x}{1800}\right) = -180 + 4x$$

$$\beta_2 = 3600\left[-\frac{1}{60} + \frac{2x}{1800} - \frac{y}{900}\right] = -60 + 4x - 4y$$

$$\beta_3 = 0, \ \beta_4 = 3600\left(\frac{y}{900}\right) = 4y$$

$$\beta_5 = 3600\left(\frac{-y}{900}\right) = -4y$$

$$\beta_6 = 3600\left[\frac{4}{60} - \frac{4x}{1800} + \frac{y}{900}\right] = 240 - 8x + 4y$$

$$\gamma_1 = 2A\,\frac{\partial N_1}{\partial y} = 0,\ \gamma_5 = 3600\left(\frac{4}{60} - \frac{x}{900}\right) = 240 - 4x$$

$$\gamma_2 = 3600\left(\frac{1}{60} - \frac{x}{900} + \frac{2y}{1800}\right) = 60 - 4x + 4y$$

$$\gamma_3 = 3600\left(-\frac{1}{60} + \frac{2y}{1800}\right) = -60 + 4y$$

$$\gamma_4 = 3600\left(\frac{x}{900} - \frac{4y}{1800}\right) = 4x - 8y$$

$$\gamma_6 = 3600\left(\frac{-4}{60} + \frac{x}{900}\right) = -240 + 4x$$

Chapter 9

9.1

(a)

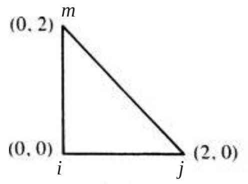


Figure 9.1a

$$[k] = 2\,\pi\, \underline{r}\, A\, [\bar{B}]^T [D]\, [\bar{B}]$$

$$r_i = 0,\ z_i = 0,\ r_j = 2,\ z_j = 0,\ r_m = 0,\ z_m = 2$$

$$\alpha_i = r_j\, z_m - z_j\, r_m = 2.2 - 0 = 4$$

$$\alpha_j = r_m\, z_i - z_m\, r_i = 0.0 - 2.0 = 0$$

$$\alpha_m = r_i\, z_j - z_i\, r_j = 0.0 - 0.2 = 0$$

$$\beta_i = z_j - z_m = 0 - 2 = -2$$

$$\beta_j = z_m - z_i = 2 - 0 = 2$$

$$\beta_m = z_i - z_j = 0 - 0 = 0 \qquad \bar{r} = \frac{1}{3}(2) = \frac{2}{3}$$

$$\gamma_i = r_m - r_j = 0 - 2 = -2 \qquad \bar{z} = \frac{1}{3}(2) = \frac{2}{3}$$

$$\gamma_j = r_i - r_m = 0 - 0 = 0 \qquad A = \frac{1}{2}\,(2)\,(2) = 2$$

$$\gamma_m = r_j - r_i = 2 - 0 = 2$$

$$[\bar{B}] = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ 2 & 0 & 2 & 0 & 2 & 0 \\ -2 & -2 & 0 & 2 & 2 & 0 \end{bmatrix}$$

$$[D] = \frac{30 \times 10^6}{(1+0.25)(1-0.5)}\begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$[\bar{B}]^T\ [D] = \frac{30\times 10^6}{2.5}\begin{bmatrix} -1 & 0 & 1 & -0.5 \\ -0.5 & -1.5 & -0.5 & -0.5 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 1.5 & 0.5 \\ 0.5 & 1.5 & 0.5 & 0 \end{bmatrix}$$

$$[k] = 25.1327\times 10^6 \begin{bmatrix} 5 & 1 & 0 & -1 & 1 & 0 \\ 1 & 4 & -2 & -1 & -2 & -3 \\ 0 & -2 & 8 & 0 & 4 & 2 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 1 & -2 & 4 & 1 & 4 & 1 \\ 0 & -3 & 2 & 0 & 1 & 3 \end{bmatrix}$$

(b)

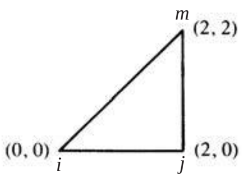


Figure 9.1b

$r_i = 0,\ z_i = 0,\ r_j = 2,\ z_j = 0,\ r_m = 2,\ z_m = 2$

$\alpha_i = 4,\ \alpha_j = 0,\ \alpha_m = 0$

$\beta_i = -2,\ \beta_j = 2,\ \beta_m = 0$

$\gamma_i = 0,\ \gamma_j = -2,\ \gamma_m = 2$

$\bar{r} = 2\times\frac{2}{3} = 1.333,\ \bar{z} = \frac{2}{3},\ A = 2$

$$[\bar{B}] = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & -2 & 2 & 2 & 0 \end{bmatrix}$$

$$[D] = \frac{30\times 10^6}{(1+0.25)\left(1-\frac{0.5}{2}\right)}\begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$[\bar{B}]^T[D] = \frac{30\times 10^6}{2.5}\begin{bmatrix} -1.25 & -0.25 & 0.25 & 0 \\ 0 & 0 & 0 & -0.5 \\ 1.75 & 0.75 & 1.25 & -0.5 \\ -0.5 & -1.5 & -0.5 & 0.5 \\ 0.25 & 0.25 & 0.75 & 0.5 \\ 0.5 & 1.5 & 0.5 & 0 \end{bmatrix}$$

$$[k] = 50.265 \times 10^6 \begin{bmatrix} 2.75 & 0 & -2.25 & 0.5 & 0.25 & -0.5 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ -2.25 & 1 & 5.75 & -2.5 & 0.25 & 1.5 \\ 0.5 & -1 & -2.5 & 4 & 0.5 & -3 \\ 0.25 & -1 & 0.25 & 0.5 & 1.75 & 0.5 \\ -0.5 & 0 & 1.5 & -3 & 0.5 & 3 \end{bmatrix}$$

(c) $E = 30 \times 10^6 \ \dfrac{\text{lb}}{\text{in.}^2}$ $\qquad v = 0.25$ (Mathcad used here)

Triangle coordinate definition

$i = \begin{pmatrix} 0 \text{ in.} \\ 0 \text{ in.} \end{pmatrix} \quad \begin{matrix} r = 0 \\ z = 1 \end{matrix}$ $\qquad$ This defines an array variable

x coordinate is the top

y coordinate is the bottom

$j = \begin{pmatrix} 2 \text{ in.} \\ 0 \text{ in.} \end{pmatrix}$ $\qquad$ Area of triangle

$\dfrac{1}{2}$ base $\times$ height

$m = \begin{pmatrix} 1 \text{ in.} \\ 2 \text{ in.} \end{pmatrix}$ $\qquad A = \dfrac{1}{2}\ (j_r - i_r)\ (m_z - i_z) \quad A = 2 \text{ in.}^2$

Develop stiffness matrix

$\alpha_i = j_r\, m_z - j_z\, m_r \quad \alpha_i = 4 \text{ in.}^2 \quad \beta_i = j_z - m_z \quad \beta_i = -2 \text{ in.} \quad \gamma_i = m_r - j_r \quad \gamma_i = -1 \text{ in.}$

$\alpha_j = m_r\, i_z - m_z\, i_r \quad \alpha_j = 0 \quad \beta_j = m_z - i_z \quad \beta_j = 2 \text{ in.} \quad \gamma_j = i_r - m_r \quad \gamma_j = -1 \text{ in.}$

$\alpha_m = i_r\, j_z - i_z\, j_r \quad \alpha_m = 0 \quad \beta_m = i_z - j_z \quad \beta_m = 0 \text{ in.} \quad \gamma_m = j_r - i_r \quad \gamma_m = 2 \text{ in.}$

Evaluate $[B]$ at centroid of element

$r_{\text{bar}} = \dfrac{i_r + j_r + m_r}{3} \qquad z_{\text{bar}} = \dfrac{i_z + j_z + m_z}{3} \qquad r_{\text{bar}} = 1 \text{ in.} \qquad z_{\text{bar}} = 0.667 \text{ in.}$

$$[B_i] = \frac{1}{2A}\begin{pmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \frac{\alpha_i}{r_{\text{bar}}} + \beta_i + \frac{\gamma_i z_{\text{bar}}}{r_{\text{bar}}} & 0 \\ \gamma_i & \beta_i \end{pmatrix} \qquad [B_j] = \frac{1}{2A}\begin{pmatrix} \beta_j & 0 \\ 0 & \gamma_j \\ \frac{\alpha_j}{r_{\text{bar}}} + \beta_j + \frac{\gamma_j z_{\text{bar}}}{r_{\text{bar}}} & 0 \\ \gamma_j & \beta_j \end{pmatrix}$$

$$[B_m] = \frac{1}{2A}\begin{pmatrix} \beta_m & 0 \\ 0 & \gamma_m \\ \frac{\alpha_m}{r_{\text{bar}}} + \beta_m + \frac{\gamma_m z_{\text{bar}}}{r_{\text{bar}}} & 0 \\ \gamma_m & \beta_m \end{pmatrix}$$

$B = \text{augment}(B_i, B_j, B_m)$

Gradient matrix at centroid of element

$$[B] = \begin{pmatrix} -0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & -0.25 & 0 & -0.25 & 0 & 0.5 \\ 0.3333 & 0 & 0.3333 & 0 & 0.3333 & 0 \\ -0.25 & -0.5 & -0.25 & 0.5 & 0.5 & 0 \end{pmatrix} \frac{1}{\text{in.}}$$

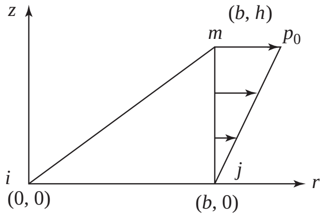
$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{pmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{pmatrix}$$ Axisymmetric stress constitutive matrix

$$[D] = \begin{bmatrix} 3.6\times10^7 & 1.2\times10^7 & 1.2\times10^7 & 0 \\ 1.2\times10^7 & 3.6\times10^7 & 1.2\times10^7 & 0 \\ 1.2\times10^7 & 1.2\times10^7 & 3.6\times10^7 & 0 \\ 0 & 0 & 0 & 1.2\times10^7 \end{bmatrix} \frac{\text{lb}}{\text{in.}^2}$$

$[k] = 2\pi\ r_{\text{bar}}\ A\ [B]^T\ [D]\ [B]$ Axisymmetric element stiffness matrix

$$[k] = \begin{bmatrix} 1.225\times10^8 & 2.513\times10^7 & -5.341\times10^7 & -1.257\times10^7 & 6.283\times10^6 & -1.257\times10^7 \\ 2.513\times10^7 & 6.597\times10^7 & -1.257\times10^7 & -9.425\times10^6 & -5.027\times10^7 & -5.655\times10^7 \\ -5.341\times10^7 & -1.257\times10^7 & 2.231\times10^8 & -5.027\times10^7 & 5.655\times10^7 & 6.283\times10^7 \\ -1.257\times10^7 & -9.425\times10^6 & -5.027\times10^7 & 6.597\times10^7 & 2.513\times10^7 & -5.655\times10^7 \\ 6.283\times10^6 & -5.027\times10^7 & 5.655\times10^7 & 2.513\times10^7 & 8.796\times10^7 & 2.513\times10^7 \\ -1.257\times10^7 & -5.655\times10^7 & 6.283\times10^7 & -5.655\times10^7 & 2.513\times10^7 & 1.131\times10^8 \end{bmatrix} \frac{\text{lb}}{\text{in.}}$$

9.2



$$\{f_s\} = \int_s [N_s]^T \begin{Bmatrix} p_r \\ p_z \end{Bmatrix} ds = \int_s \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix}_{\substack{@\ \text{Evaluated} \\ r=b \\ z=z}} \begin{Bmatrix} \frac{p_0 z}{h} \\ 0 \end{Bmatrix} 2\pi r dz$$

Now $\quad N_i = \dfrac{1}{2A}\ (\alpha_i + \beta_i r + \gamma_i z),\ N_j = \dfrac{1}{2A}\ (\alpha_j + \beta_j r + \gamma_j z)$

$$N_m = \frac{1}{2A}(\alpha_m + \beta_m r + \gamma_m z)$$

$$r_i = 0,\ r_j = b,\ r_m = b,\ z_i = 0,\ z_j = 0,\ z_m = h$$

$$\alpha_i = r_j\, z_m - z_j\, r_m = bh - 0b = bh,\ \alpha_j = r_m\, z_i - z_m\, r_i = 0$$

$$\beta_i = z_j - z_m = 0 - h = -h,\ \beta_j = z_m - z_i = h - 0 = h$$

$$\gamma_i = r_m - r_j = b - b = 0,\ \gamma_j = r_i - \gamma_m = 0 - b = -b$$

$$\alpha_m = r_i\, z_j - z_i\, r_j = 0,\ \beta_m = z_i - z_j = 0 - 0 = 0$$

$$\gamma_m = r_j - r_i = b - 0 = b$$

So the shape functions evaluated at $r = b$ and $z = z$

$$N_i = \frac{1}{bh}\ (bh + (-\,h)\, b + 0\ \ z) = 0$$

$$N_j = \frac{1}{bh}\ (0 + h\, b + (-\, b)z) = \frac{1}{bh}\ (hb - bz)$$

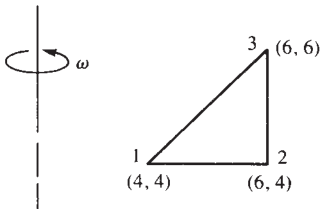
$$N_m = \frac{1}{bh}\ (0 + 0\ b + b\ z) = \frac{1}{bh}\ (bz)$$

$$\{f_s\} = \int_0^h \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{bh}(hb-bz) & 0 \\ 0 & \frac{1}{bh}(hb-bz) \\ \frac{1}{bh}(bz) & 0 \\ 0 & \frac{1}{bh}(bz) \end{bmatrix} \begin{Bmatrix} \frac{p_0\, z}{h} \\ 0 \end{Bmatrix} 2\pi\ dz$$

$$= \frac{2\pi b}{bh}\int_0^h \begin{Bmatrix} 0 \\ 0 \\ p_0bz - \frac{p_0bz^2}{h} \\ 0 \\ \frac{p_0bz^2}{h} \\ 0 \end{Bmatrix} dz = \frac{2\pi}{h}\begin{Bmatrix} 0 \\ 0 \\ \frac{p_0bz^2}{2} - \frac{p_0bz^3}{3h} \\ 0 \\ \frac{p_0bz^3}{3h} \\ 0 \end{Bmatrix}\Bigg|$$

$$= \frac{2\pi}{h}\begin{Bmatrix} 0 \\ 0 \\ \frac{p_0\,bh^2}{6} \\ 0 \\ \frac{p_0\,bh^2}{3} \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{s1x} \\ f_{s1y} \\ f_{s2x} \\ f_{s2y} \\ f_{s3x} \\ f_{s3y} \end{Bmatrix} = 2\pi b\begin{Bmatrix} 0 \\ 0 \\ \frac{p_0\,h}{6} \\ 0 \\ \frac{p_0\,h}{3} \\ 0 \end{Bmatrix}$$

9.3



Equation to be evaluated is $\{f_B\} = \dfrac{2\pi\bar{r}A}{3}\begin{Bmatrix} \bar{R}_B \\ Z_B \\ \bar{R}_B \\ Z_B \\ \bar{R}_B \\ Z_B \end{Bmatrix}$

$$\bar{r} = 4 + 2 \times \frac{2}{3} \quad \Rightarrow \bar{r} = 5.333 \text{ in.}$$

$$Z_B = 0.283 \ \frac{\text{lb}}{\text{in.}^3}$$

$$\bar{R}_B = w^2 \rho \ \bar{r} = \left[20 \frac{\text{rev.}}{\text{min}}\left(2\pi\frac{\text{rad}}{\text{rev}}\right)\frac{1 \text{ min}}{60 \ s}\right]^2 \frac{\left(0.283 \frac{\text{lb}}{\text{in.}^3}\right)}{\left(32.2 \times 12 \frac{\text{lb}}{\text{in.}^3}\right)} \ [5.333 \text{ m}]$$

$$\bar{R}_B = 0.01712 \ \frac{\text{lb}}{\text{in.}^3}$$

$$\frac{2\pi\bar{r}A}{3} = \frac{2}{3}\pi (5.333 \text{ in.}) (2 \text{ in.}^2) = 22.34 \text{ in.}^3$$

So

$$f_{B1r} = (22.34)\ (0.01712) = 0.382 \text{ lb}$$
$$f_{B1z} = (–22.34)\ (0.283) = – 6.32 \text{ lb}$$
$$f_{B2r} = (22.34)\ (0.01712) = 0.382 \text{ lb}$$
$$f_{B3z} = (– 22.34)\ (0.283) = – 6.32 \text{ lb}$$
$$f_{B3r} = (22.34)\ (0.01712) = 0.382 \text{ lb}$$
$$f_{B3z} = (– 22.34)\ (0.283) = – 6.32 \text{ lb}$$

9.4

(a) Element Figure 9.4 a

The equation to be evaluated is $\{\sigma\} = [D]\ [B]\ \{d\}$

$r_i = 0,\ z_i = 0,\ r_j = 2,\ z_j = 0,\ r_m = 1,\ z_m = 3$

$\alpha_i = 6,\ \alpha_j = 0,\ \alpha_m = 0,\ \beta_i = – 3,\ \beta_j = 3,\ \beta_m = – 3,$

$\gamma_i = – 1,\ \gamma_j = – 1,\ \gamma_m = 2$

$\bar{r} = 1,\ \bar{z} = 1,\ A = \dfrac{1}{2}\ (3)\ (2) = 3 \text{ in.}^2$

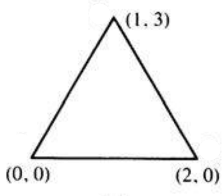


Figure 9.4a

$$[\bar{B}] = \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 & 0 & -3 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ 2 & 0 & 2 & 0 & -1 & 0 \\ -1 & -3 & -1 & 3 & 2 & -3 \end{bmatrix}$$

$$[D] = \frac{30 \times 10^6}{(1 + 0.25)(1 - 0.5)}\begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$\begin{Bmatrix}\sigma_r\\ \sigma_z\\ \sigma_\theta\\ \tau_{rz}\end{Bmatrix} = \frac{30\times10^6}{(1.25)(0.5)}\begin{bmatrix}0.75 & 0.25 & 0.25 & 0\\ 0.25 & 0.75 & 0.25 & 0\\ 0.25 & 0.25 & 0.75 & 0\\ 0 & 0 & 0 & 0.25\end{bmatrix}\left(\frac{1}{6}\right)$$

$$\times\begin{bmatrix}-3 & 0 & 3 & 0 & -3 & 0\\ 0 & -1 & 0 & -1 & 0 & 2\\ 2 & 0 & 2 & 0 & -1 & 0\\ -1 & -3 & -1 & 3 & 2 & -3\end{bmatrix}\begin{Bmatrix}1\\2\\5\\6\\0\\0\end{Bmatrix}\times10^{-4}$$

$$\begin{Bmatrix}\sigma_r\\ \sigma_z\\ \sigma_\theta\\ \tau_{rz}\end{Bmatrix} = \begin{Bmatrix}80.0\times10^2\\ -17.8\times10^{-10}\\ 80.0\times10^2\\ 12.0\times10^2\end{Bmatrix}\text{psi}$$

(b) Element Figure 9.4b

$r_i = 1,\ z_i = 0,\ r_j = 3,\ z_j = 0,\ r_m = 3,\ z_m = 3$

$\alpha_i = 9,\ \alpha_j = -3,\ \alpha_m = 0,\ \beta_i = -3,\ \beta_j = 3,\ \beta_m = 0$

$\gamma_i = 0,\ \gamma_j = -2,\ \gamma_m = 2$

$\bar{r} = 2.333,\ \bar{z}_m = 0.666,\ A = 3$

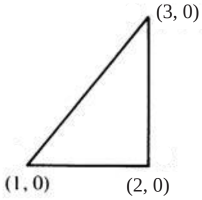


Figure 9.4 b

$$[\bar{B}] = \frac{1}{6}\begin{bmatrix}-3 & 0 & 3 & 0 & 0 & 0\\ 0 & 0 & 0 & -2 & 0 & 2\\ 0.858 & 0 & 1.14 & 0 & 0.512 & 0\\ 0 & -3 & -2 & 3 & 2 & 0\end{bmatrix}$$

$$[\bar{D}] = \frac{30\times10^6}{(1+0.25)(1-0.5)}\begin{bmatrix}0.75 & 0.25 & 0.25 & 0\\ 0.25 & 0.75 & 0.25 & 0\\ 0.25 & 0.25 & 0.75 & 0\\ 0 & 0 & 0 & 0.25\end{bmatrix}$$

$$\{\sigma\} = [D]\ [\bar{B}]\ \{d\}$$

$$\begin{Bmatrix}\sigma_r\\ \sigma_z\\ \sigma_\theta\\ \tau_{rz}\end{Bmatrix} = \begin{Bmatrix}5830\\ -3770\\ 3090\\ 400\end{Bmatrix}\text{psi}$$

(c) $u_1 = 0.0001$ in. $w_1 = 0.0002$ in.

$u_2 = 0.0005$ in. $w_2 = 0.0006$ in.

$u_3 = 0.0$ in. $w_3 = 0.0$ in.

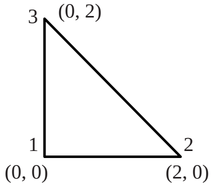


Figure 9.4c

$$d = \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix} \begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = [D]\,[B]\,\{d\} = \begin{Bmatrix} 9.6\times 10^3 \\ 2.4\times 10^3 \\ 1.2\times 10^4 \\ 1.8\times 10^3 \end{Bmatrix} \frac{\text{lb}}{\text{in.}^2}$$

9.5 By Equation (9.1.35)

$$\{f_{si}\} = \int_{z_j}^{z_m} \frac{1}{2A} \begin{bmatrix} \alpha_j + \beta_j r_j + \gamma_j z & 0 \\ 0 & \alpha_j + \beta_j\ r_j + \gamma_j z \end{bmatrix} \begin{Bmatrix} p_r \\ p_z \end{Bmatrix} 2\pi r_j\, dz$$

Now

$$\alpha_j = r_m z_i - r_i z_m = r_j z_j - r_i z_m \qquad \text{Since } r_j = r_m$$

$$\beta_j = z_m - z_j \qquad \gamma_j = r_i - r_j \qquad z_i = z_m$$

$$A = \frac{1}{2}\,(r_j - r_i)\,(z_m - z_j)$$

$$\therefore \quad \{f_{sj}\} = \int_{z_j}^{z_m} \frac{2\pi r_j}{2A} \begin{Bmatrix} p_r[r_j z_j - r_i z_m + (z_m - z_j) r_j + (r_i - r_j) z \\ p_z[r_j z_j - r_i z_m + (z_m - z_j) r_j + (r_i - r_j) z \end{Bmatrix} dz$$

Integrating, we obtain

$$\{f_{sj}\} = \frac{2\pi r_j}{2A} \begin{Bmatrix} P_r\left[(r_j z_j - r_i z_m)(z_m - z_j) + r_j (z_m - z_j)^2 + (r_i - r_j)\frac{(z_m^2 - z_j^2)}{2}\right] \\ P_z\left[(r_j z_j - r_i z_m)(z_m - z_j) + r_j (z_m - z_j)^2 + (r_i - r_j)\frac{(z_m^2 - z_j^2)}{2}\right] \end{Bmatrix}$$

Factoring out $z_m - z_j$ and simplifying

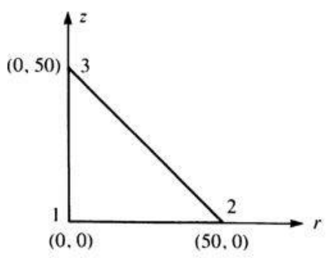
$$\{f_{si}\} = \frac{2\pi r_j (z_m - z_j)}{2A} \begin{Bmatrix} p_r\left[\frac{-r_j z_j}{2} - \frac{r_i z_m}{2} + \frac{r_j z_j}{2} + \frac{r_i z_m}{2}\right] \\ p_z\left[\frac{-r_j z_j}{2} - \frac{r_j z_m}{2} + \frac{r_j z_j}{2} + \frac{r_i z_m}{2}\right] \end{Bmatrix}$$

$$= \frac{2\pi r_j (z_m - z_j)}{2A} \begin{Bmatrix} \left[p_r\left[\frac{1}{2}(z_m - z_j)(r_j - r_i)\right]\right] \\ p_z\left[\frac{1}{2}(z_m - z_j)(r_j - r_i)\right) \end{Bmatrix}$$

$$= \frac{2\pi r_j (z_m - z_j)}{2A} \begin{Bmatrix} p_r A \\ p_z A \end{Bmatrix}$$

$$\{f_{sj}\} = \frac{2\pi r_j (z_m - z_j)}{2}$$

9.6 (a)



(a)

$E = 210$ GPa
$\upsilon = 0.25$
$A = \frac{1}{2}(50)(50)$
$A = 1250$ mm^2
$\bar{r} = 16.67$ mm
$\bar{z} = 16.67$ mm

$\alpha_i = 50\,(50) - (0)\,(0) = 2500\text{ mm}^2$

$\beta_i = -50\text{ mm},\ \gamma_i = -50\text{ mm}$

$\beta_j = 50\text{ mm},\ \gamma_j = 0\text{ mm}$

$\beta_m = 0\text{ mm},\ \gamma_m = 50\text{ mm}$

$$[\bar{B}] = \frac{1}{2(1250)}\begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 0 & -50 & 0 & 0 & 0 & 50 \\ 50 & 0 & 50 & 0 & 50 & 0 \\ -50 & -50 & 0 & 50 & 50 & 0 \end{bmatrix}\frac{1}{\text{mm}}$$

$$[D] = \frac{210\times10^9}{(1.25)(0.5)}\begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}\frac{\text{N}}{\text{m}^2}$$

$$[D]\,[\bar{B}] = \frac{210\times10^9}{1250(1.25)}\begin{bmatrix} -25 & -12.5 & 50 & 0 & 12.5 & 12.5 \\ 0 & -37.5 & 25 & 0 & 12.5 & 37.5 \\ 25 & -12.5 & 50 & 0 & 37.5 & 12.5 \\ -12.5 & -12.5 & 0 & 12.5 & 12.5 & 0 \end{bmatrix}$$

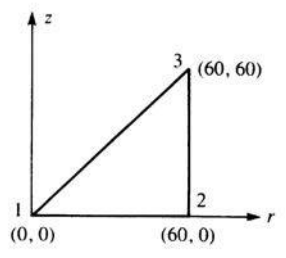
$$[k] = 2\pi\,\bar{r}\,A\,[\bar{B}]^T\,[D]\,[\bar{B}]$$

$$= \frac{2\pi(0.01667\text{ m})(1250)(210\times10^9)\times10^{-6}}{(1250)(1.25)}\ [\bar{B}]^T\,[D]\,[\bar{B}]$$

Multiplying $[\bar{B}]^T$ times $[D]\,[\bar{B}]$, we obtain

$$[k] = 7.039\times10^6\begin{bmatrix} 3125 & 625 & 0 & -625 & 625 & 0 \\ & 2500 & -1250 & -625 & -1250 & -1875 \\ & & 5000 & 0 & 2500 & 1250 \\ & & & 625 & 625 & 0 \\ & & & & 2500 & 625 \\ \text{Symmetry} & & & & & 1875 \end{bmatrix}\frac{\text{N}}{\text{m}}$$

9.6 (b)



(b)

$\bar{r} = 40$ mm
$\bar{z} = 20$ mm

$A = 1800\text{ mm}^2$

$\beta_i = -60\text{ mm},\ \beta_j = 60\text{ mm},\ \beta_m = 0$

$\gamma_i = 0,\ \gamma_j = -60\text{ mm},\ \gamma_m = 60\text{ mm}$

$$[k] = 2\pi\,\bar{r}\,A\,[\bar{B}]^T[D]\,[\bar{B}]$$

$$[\bar{B}] = \frac{1}{2(1800)}\begin{bmatrix} -60 & 0 & 60 & 0 & 0 & 0 \\ 0 & 0 & 0 & -60 & 0 & 60 \\ 30 & 0 & 30 & 0 & 30 & 0 \\ 0 & -60 & -60 & 60 & 60 & 0 \end{bmatrix}\frac{1}{\text{mm}}$$

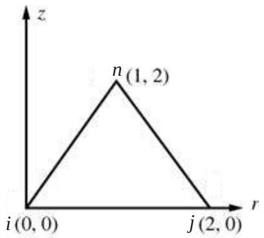
$[D]$ as in 9.6 (a)

$$[D]\,[\bar{B}] = \frac{210\times10^9}{1.25(1800)}\begin{bmatrix} -37.5 & 0 & 52.5 & -15 & 7.5 & 15 \\ -7.5 & 0 & 22.5 & -45 & 7.5 & 45 \\ 7.5 & 0 & 37.5 & -15 & 22.5 & 15 \\ 0 & -15 & -15 & 15 & 15 & 0 \end{bmatrix}$$

$$[k] = \frac{2\pi(0.04\,\text{m})(\cancel{1800})(210\times10^9)}{(1.25)(1800)(2)(\cancel{1800})}[\bar{B}]^T[D][\bar{B}]$$

$$[k] = 11.73\times10^6\begin{bmatrix} 2475 & 0 & -2025 & 450 & 225 & -450 \\ & 900 & 900 & -900 & -900 & 0 \\ & & 5175 & -2250 & 225 & 1350 \\ & & & 3600 & 450 & -2700 \\ & & & & 1575 & 450 \\ \text{Symmetry} & & & & & 2700 \end{bmatrix}$$

(c)



$r_i = 0$ $\quad z_i = 0$

$r_j = 0.002$ $\quad z_j = 0$

$r_m = 0.001$ $\quad z_m = 0.002$

$$A = \frac{1}{2}(r_j - r_i)(z_m - z_i),\ E = 210\times10^9,\ z = \frac{1}{3}\times0.002,\ r = \frac{0.002}{2},\ v = 0.25$$

$$\alpha_i = r_j z_m - z_j r_m \qquad \alpha_j = r_m z_i - z_m r_i \qquad \alpha_m = r_i z_j - z_i r_j$$

$$\beta_i = z_j - z_m \qquad \beta_j = z_m - z_i \qquad \beta_m = z_i - z_j$$

$$\gamma_i = r_m - r_j \qquad \gamma_j = r_i - r_m \qquad \gamma_m = r_j - r_i$$

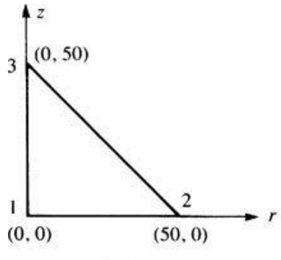
$$[B] = \frac{1}{2A}\begin{pmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \frac{\alpha_i}{r}+\beta_i+\frac{\gamma_i z}{r} & 0 & \frac{\alpha_j}{r}+\beta_j+\frac{\gamma_j z}{r} & 0 & \frac{\alpha_m}{r}+\beta_m+\frac{\gamma_m z}{r} & 0 \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{pmatrix}$$

$$[D] = \frac{E}{(1+v)(1-2v)}\begin{pmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} \end{pmatrix}$$

$$[k] = 2\pi r\, A\, [B]^T [D]\, [B]$$

$$[k] = \begin{pmatrix} 8.577\times10^8 & 1.759\times10^8 & -3.738\times10^8 & -8.796\times10^7 & 4.398\times10^7 & -8.796\times10^7 \\ 1.759\times10^8 & 4.618\times10^8 & -8.796\times10^7 & -6.597\times10^7 & -3.519\times10^8 & -3.958\times10^8 \\ -3.738\times10^8 & -8.796\times10^7 & 1.561\times10^9 & -3.519\times10^8 & 3.958\times10^8 & 4.398\times10^8 \\ -8.796\times10^7 & -6.597\times10^7 & -3.519\times10^8 & 4.618\times10^8 & 1.759\times10^8 & -3.958\times10^8 \\ 4.398\times10^7 & -3.519\times10^8 & 3.958\times10^8 & 1.759\times10^8 & 6.158\times10^8 & 1.759\times10^8 \\ -8.796\times10^7 & -3.958\times10^8 & 4.398\times10^8 & -3.958\times10^8 & 1.759\times10^8 & 7.917\times10^8 \end{pmatrix}$$

9.7 (a)



(a)

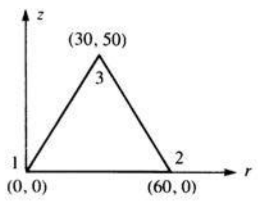
From Problem 9.6 (a), we have $[D]\,[\bar{B}]$

$$\therefore \quad \{\sigma\} = [D]\,[\bar{B}]\{d\}$$

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{210\times10^3\,\text{MPa}}{(1250)(1.25)}\begin{bmatrix} -25 & -12.5 & 50 & 0 & 12.5 & 12.5 \\ 0 & -37.5 & 25 & 0 & 12.5 & 37.5 \\ 25 & -12.5 & 50 & 0 & 37.5 & 12.5 \\ -12.5 & -12.5 & 0 & 12.5 & 12.5 & 0 \end{bmatrix}\begin{Bmatrix} 0.05 \\ 0.03 \\ 0.02 \\ 0.02 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} -84 \\ -84 \\ 252 \\ -101 \end{Bmatrix}\ \text{MPa}$$

(b)



(b)

From Problem 9.6 (b), we have $[D]\ [\bar{B}]$

$$\therefore \quad \begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{210\times 10^3}{(1500)(1.25)} \begin{bmatrix} -37.5 & 0 & 52.5 & -15 & 7.5 & 15 \\ -7.5 & 0 & 22.5 & -45 & 7.5 & 45 \\ 7.5 & 0 & 37.5 & -15 & 22.5 & 15 \\ 0 & -15 & -15 & 15 & 15 & 0 \end{bmatrix} \begin{Bmatrix} 0.05 \\ 0.03 \\ 0.02 \\ 0.02 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -103 \\ -103 \\ 112 \\ -73 \end{Bmatrix} \text{MPa}$$

(c) $\quad u_i = 0.00005 \qquad w_i = 0.00003$

$u_j = 0.00002 \qquad w_j = 0.00002$

$u_m = 0 \qquad w_m = 0$

$$\{d\} = \begin{pmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{pmatrix}$$

$$\begin{pmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{pmatrix} = [D]\,[B]\,\{d\}$$

$$\sigma_r = -\,2.87\times 10^9 \text{ Pa}$$

$$\sigma_z = -\,2.45\times 10^9 \text{ Pa}$$

$$\sigma_\theta = 3.57\times 10^9 \text{ Pa}$$

$$\tau_{rz} = -\,1.89\times 10^9 \text{ Pa}$$

9.8 No, not in general, as the axisymmetric elements are rings, not plane quadrilaterals or triangles. So axisymmetric nodes are actually nodal circles whereas plane stress elements have node points.

9.9 No, the element circumferential strain is a function of *r* and *z* (see Equation (9.1.15).

9.10 Make $u_r = 0$ for all nodes acting on the axis of symmetry.

9.11 How would you evaluate circumferential strain ε_θ at $r = 0$?

From text Equation (9.1.15)

$$\varepsilon_\theta = \frac{a_1}{r} + a_2 + \frac{a_3 z}{r} \qquad (1)$$

$$\varepsilon_r = a_2 \qquad (2)$$

Also from text Equation (9.1.1e)

$$\varepsilon_\theta = \frac{u}{r} \qquad (3)$$

$$\varepsilon_r = \frac{\partial u}{\partial r} \tag{4}$$

$$\therefore \quad u = \varepsilon_\theta\, r \tag{5}$$

Substituting (1) into (5)

$$\Rightarrow \quad u = \left[\frac{a_1}{r} + a_2 + \frac{a_3\, z}{r}\right] r = a_1 + a_2 r + a_3 z \tag{6}$$

Partial of (6) with reference to r

$$\Rightarrow \quad \frac{\partial u}{\partial r} = a_2 \qquad \text{Compare to (2)} \tag{7}$$

$$\therefore\ \varepsilon_\theta\big|_{r=0} = \varepsilon_r = a_2 \text{ as stated in problem statement}$$

9.12 What will be the stresses σ_r and σ_θ at $r = 0$?

From Equation (9.1.2)

$$\sigma_r = \frac{E}{(1+v)(1-2v)}\left(\overset{a_2}{\overset{\|}{\varepsilon_r}}(1-v) + \overset{a_6}{\overset{\|}{\varepsilon_z}}(v) + \overset{a_2}{\overset{\|}{\varepsilon_\theta}}(v)\right)$$

$$= \frac{E}{(1+v)(1-2v)}\,(a_z - \cancel{a_2 v} + a_6 v + \cancel{a_2 v})$$

$$\sigma_r = \boxed{\frac{E}{(1+v)(1-2v)}\ (a_2 + a_6\, v)}$$

$$\therefore \quad \sigma_\theta = \frac{E}{(1+v)(1-2v)}\left(\overset{a_2}{\overset{\|}{\varepsilon_r}}(v) + \overset{a_6}{\overset{\|}{\varepsilon_z}}(v) + \overset{a_2}{\overset{\|}{\varepsilon_\theta}}(1-v)\right)$$

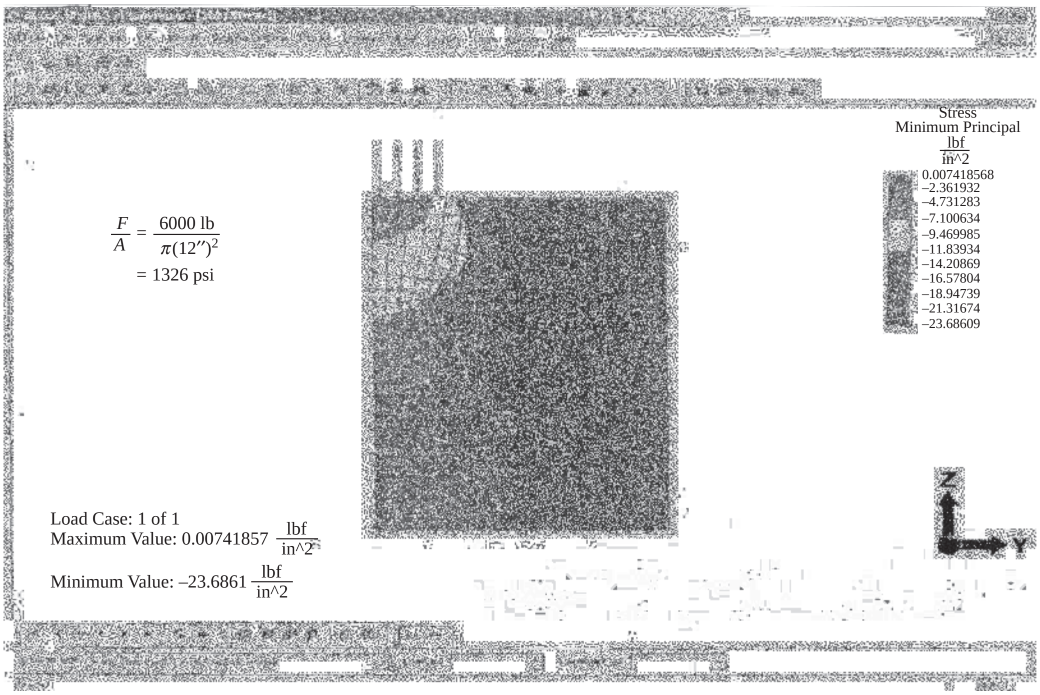
$$= \frac{E}{(1+v)(1-2v)}\,(\cancel{a_2 v} + a_6 v + a_2 - \cancel{a_2 v})$$

$$\sigma_\theta = \boxed{\frac{E}{(1+v)(1-2v)}\ (a_2 + a_6\, v)}$$

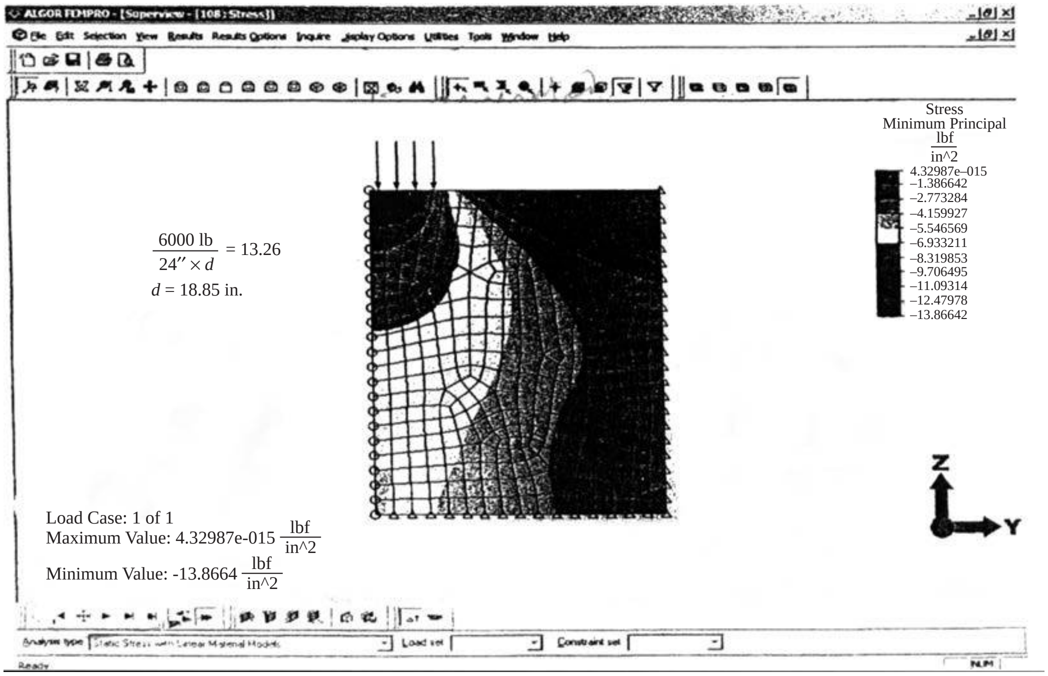
$$\therefore\ \boxed{\text{at } \ r = 0, \qquad \sigma_\theta = \sigma_r}$$

9.13

Axisymmetric model pressure load of 13.26 $\frac{\text{lb}}{\text{in.}^2}$

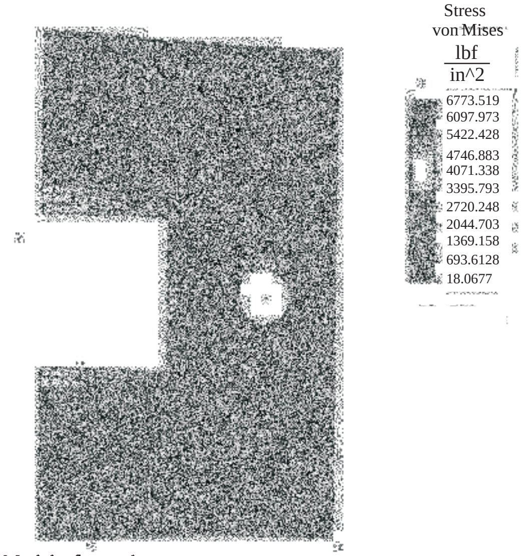


Plane stress with a thickness of 18.85 inches



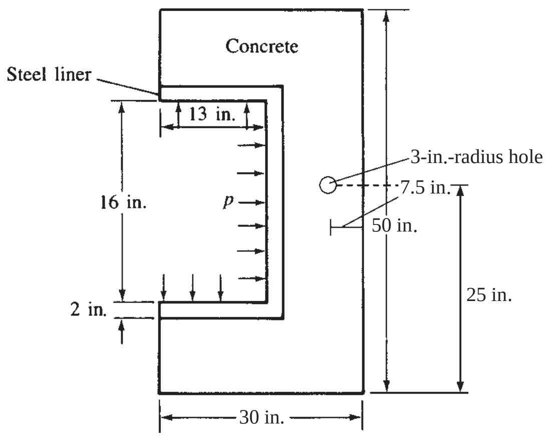
9.14 von Mises stresses (with filleted corners)

von Mises stresses (with filleted corners)

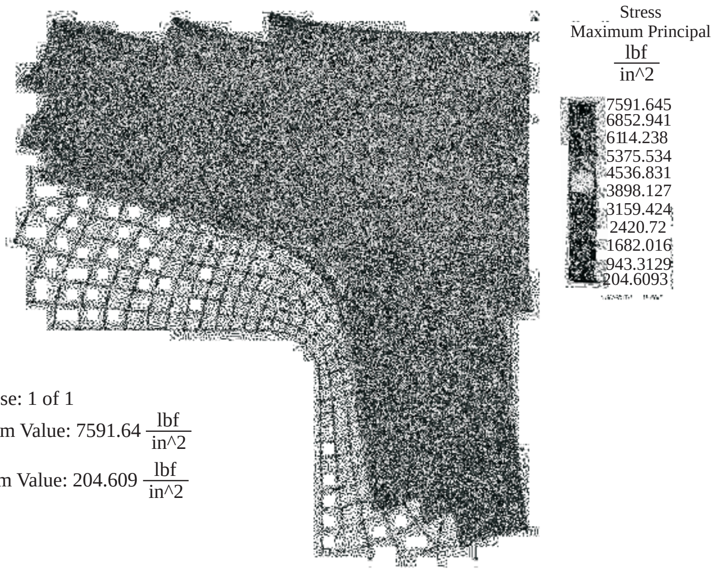


Load Case: 1 of 1
Maximum Value: 6773.52 lbf/in^2
Minimum Value: 18.0677 lbf/in^2

Model of a nuclear reactor

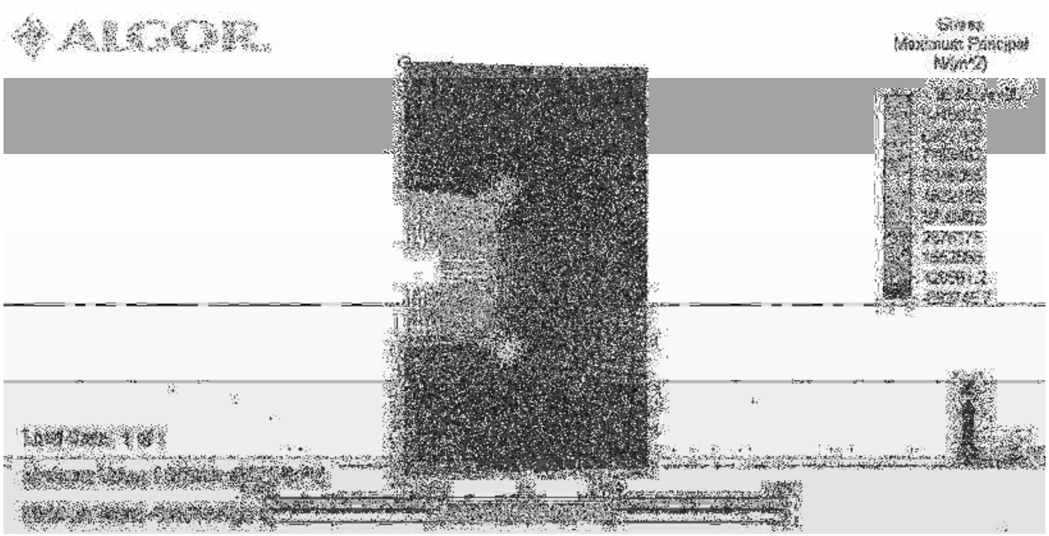


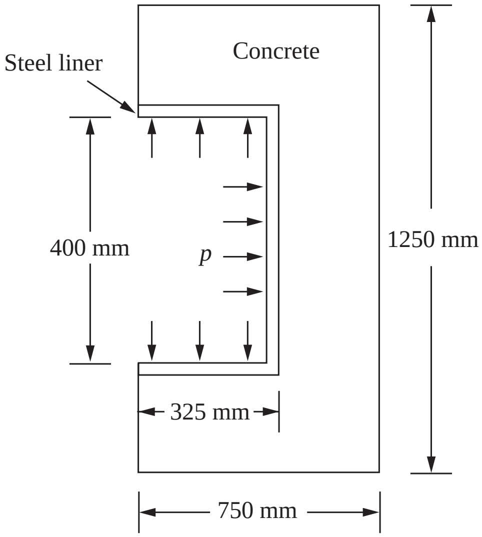
Zoomed maximum principal stresses in filleted corner



Load Case: 1 of 1
Maximum Value: 7591.64 lbf/in^2
Minimum Value: 204.609 lbf/in^2

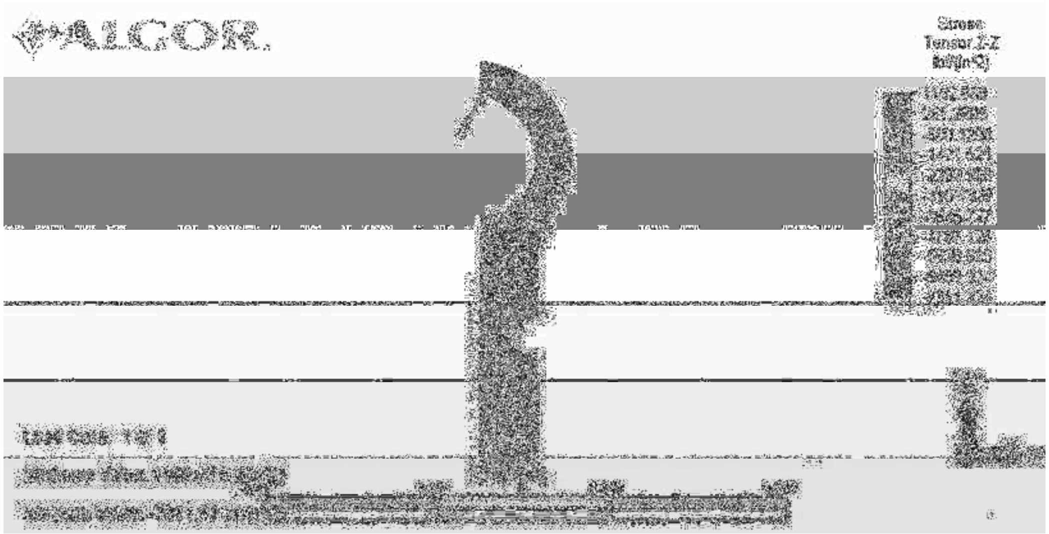
9.15





Note: Without the arc (inside radius), we have a 90° re-entrant corner where stress is approaching infinity. We have a singularity in the linear-elastic solution based on linear theory of elasticity. Therefore, we need the arc as in good practice or elastic-plastic model where an upper bound on the corner stress is the yield strength of the material.

9.18



9.19

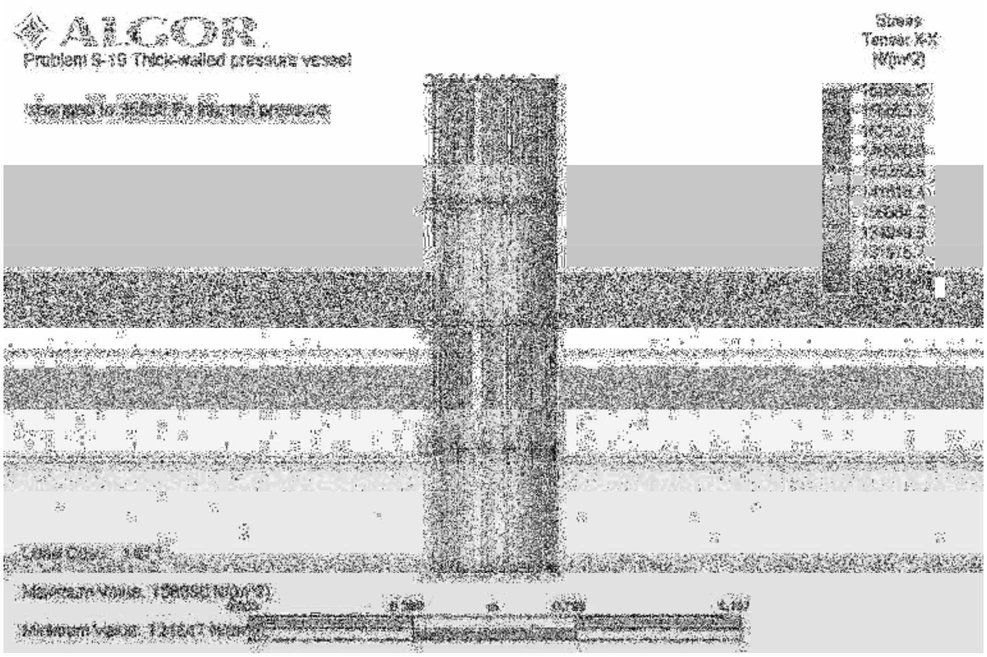


Figure 1 Thick walled open-ended cylinder

Theoretical Solution for hoop stress at inner radius

$$q = 35 \times 10^6 \quad a = 1.5 \quad b = 1.2 \quad r = 1.2$$

$$\sigma_\theta = \frac{qb^2}{r^2} \cdot \frac{a^2 + r^2}{a^2 - b^2}$$

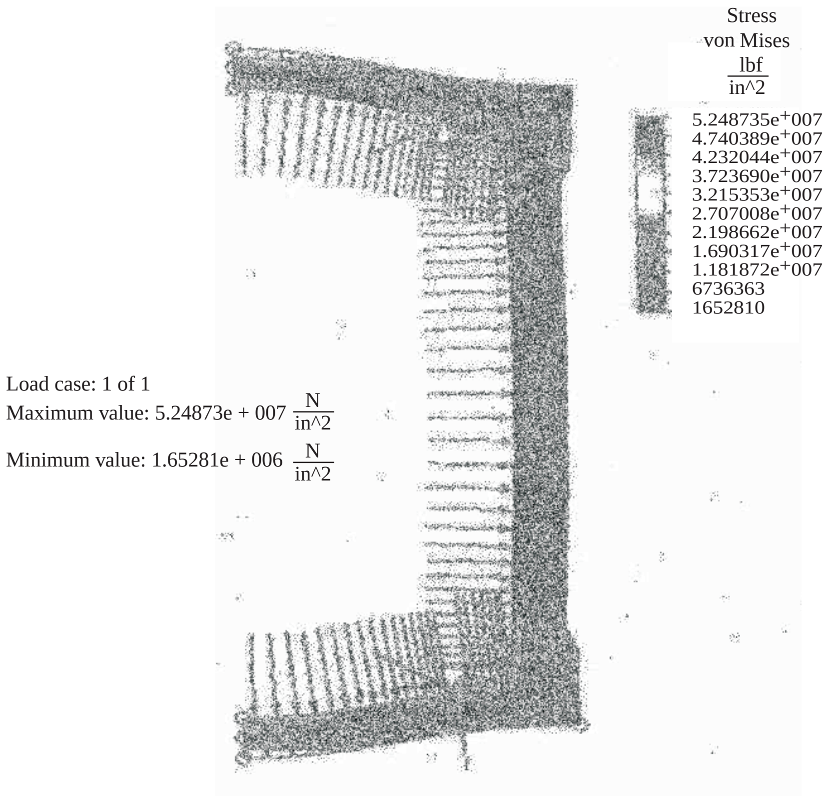
$$\sigma_\theta = 1.594 \times 10^8 \text{ Pa}$$

	Algor Results	Theoretical Results
Hoop Stress	159.5 MPa	159.4MPa
Maximum Principal Stress	159.5 MPa	-
Minimum Principal Stress	–35 MPa	-
Deflection in *y*-direction	0.93mm	-

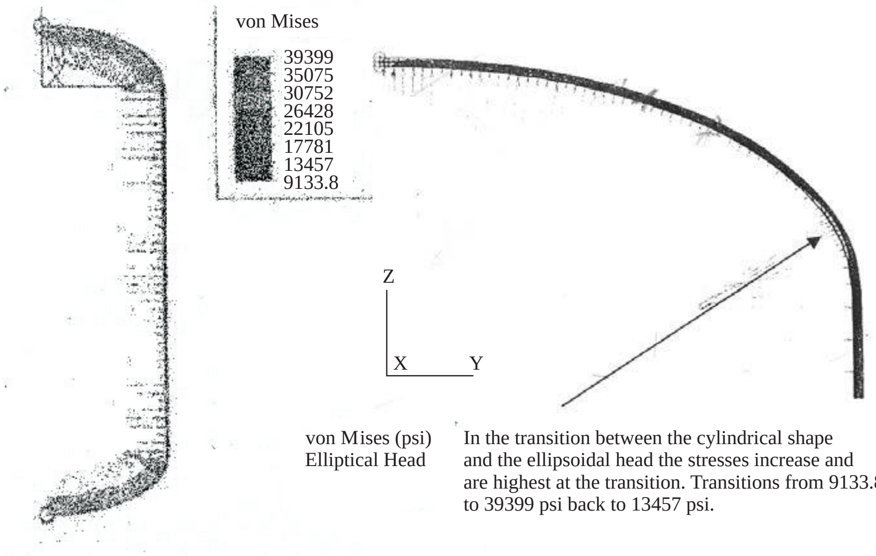
The Algor results for hoop stress and the theoretical solution for hoop stress are very close which proves that the Algor model is correct. The pipe has a very minimum internal and external deflection, less than 1mm on the inner radius. The stresses are also manageable at 159 MPa.

9.20 A steel cylindrical pressure vessel with flat plate end caps is shown in the figure with vertical axis of symmetry. Addition of thickened sections helps to reduce stress concentrations in the corners. Analyze the design and identify the most critically stressed regions. Note that inside sharp re-entrant corners produce infinite stress concentration zones, so refining the mesh in these regions will not help you get a better answer unless you use an inelastic theory or place small fillet radii there. Recommend any design changes in your report. Let the pressure inside be 1000 kPa.

503 elements and 645 nodes. Stresses are highest at sharp corners and the middle of the top and bottom of the pressure vessel. The design is acceptable as the von Mises stresses do not reach the yield strength of the material.

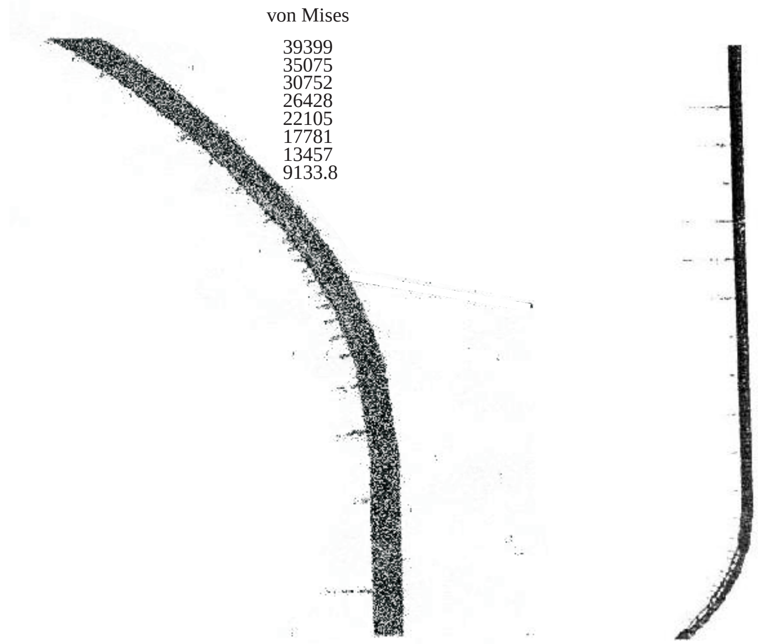


9.22



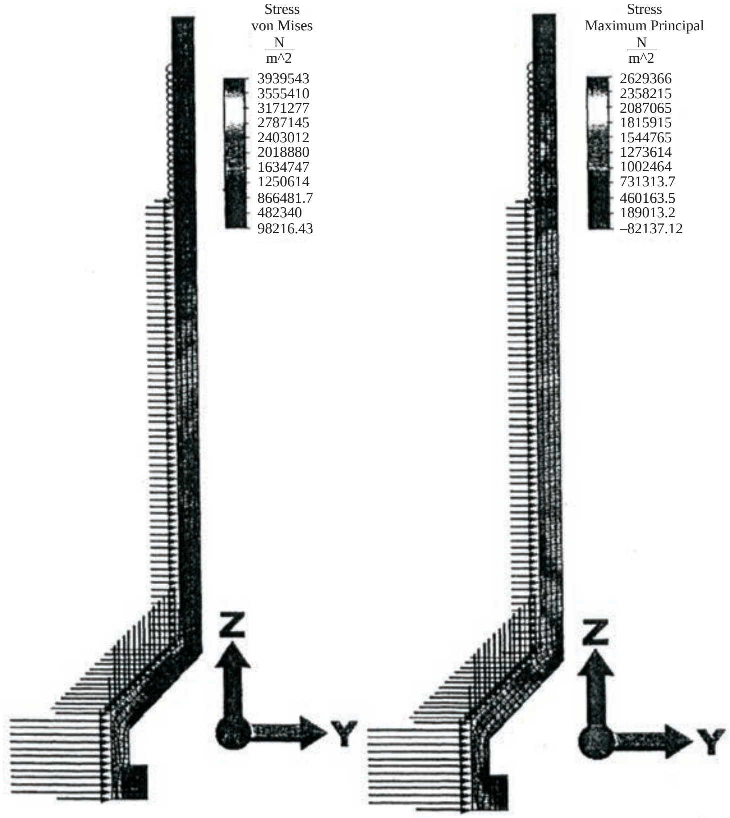
von Mises (psi)
Elliptical Head

In the transition between the cylindrical shape and the ellipsoidal head the stresses increase and are highest at the transition. Transitions from 9133.8 psi to 39399 psi back to 13457 psi.



The recommended head shape of the hemispherical ends versus the ellipsoidal ends would be the hemispherical ends due to a lower stress concentration at the transition between the head and the cylindrical body.

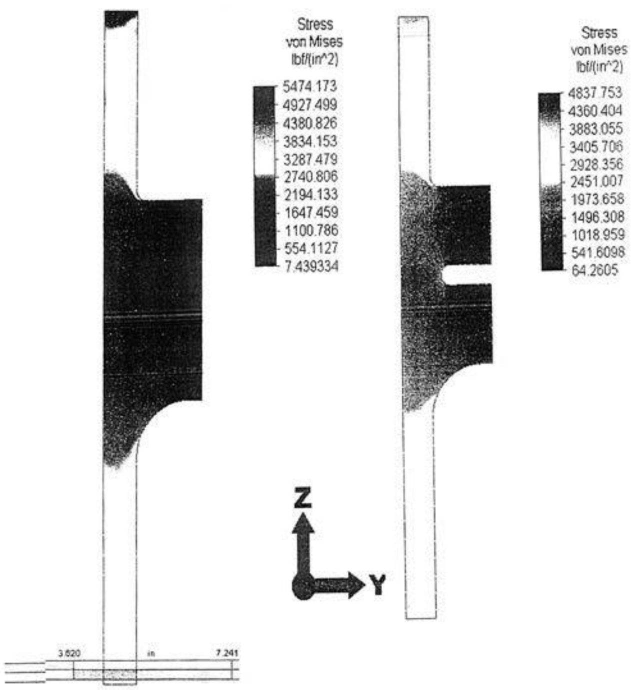
9.23 According to the von Mises stress analysis, the average stress through the glass is around $\frac{1}{5}$ of the tensile strength of the glass. If the maximum force used with this syringe is 45 N, the design should be fine. However, if 45 N is the normal operating force which may increase, I would recommend analyzing this again with a safety factor of 4 (180 N force) to make sure it will still be under 5 MPa. As for the maximum principal stresses, they are well below the tensile strength of the glass and do not appear to be an issue.



Another analysis with a safety factor of 4 (28.64789 $\frac{\text{N}}{\text{radian}}$) reveals that this syringe is still within the tensile strength of glass is all areas. With this information, I would conclude that the syringe design is indeed safe with this material specification.

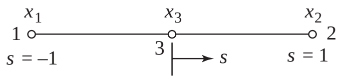
9.25 Steel hole punch

No side groove | With side groove



Chapter 10

10.1



$$J = \frac{dx}{ds} \qquad\qquad u = a_1 + a_2 s + a_3 s^2$$

$$x = a_1 + a_2 s + a_3 s^2$$

$$x_1 = a_1 + a_2(-1) + a_3(-1)^2 \tag{1}$$

$$x_2 = a_1 + a_2(1) + a_3(1)^2 \tag{2}$$

(1) – (2) gives

$$x_1 - x_2 = -2a_2,\ a_2 = \frac{x_2 - x_1}{2} \tag{3}$$

$$x_3 = a_1 + a_2(0) + a_3(0)$$

$$\therefore \quad a_1 = x_3 \tag{4}$$

(3) and (4) into (1)

$$a_3 = x_1 - x_3 + \frac{x_2 - x_1}{2} = \frac{x_1 + x_2 - 2x_3}{2}$$

$$\therefore \quad x = x_3 + \frac{x_2 + x_1}{2}\, s + \frac{(x_1 + x_2 - 2x_3)}{2} s^2 \tag{5}$$

$$J = \frac{dx}{ds} = \frac{x_2 + x_1}{2} + (x_1 + x_2 - 2x_3)s \tag{6}$$

Now $\quad x_3 = \dfrac{x_1 + x_2}{2}$ for x_3 at $s = 0$

and $\quad x_2 - x_1 = L$

$$\therefore \quad J = \frac{L}{2} + [x_1 + x_2 - (x_1 + x_2)]s$$

$$J = \frac{L}{2}$$

10.2 Using Equation (10.1.1 b)

(a)

$$s = [x - \frac{(x_1 + x_2)}{2}]\left(\frac{2}{x_2 - x_1}\right)$$

At $A \Rightarrow x = x_A = 14$ in.

$$s = [14 - \frac{(10 + 20)}{2}]\left(\frac{2}{20 - 10}\right)$$

$$s = (14 - 15)\,\frac{2}{10} = -\frac{1}{5} = -0.2$$

By Equation (10.1.5)

$$N_1 = \frac{1+0.2}{2} = 0.6,\ N_2 = \frac{1-0.2}{2} = 0.4$$

By Equation (10.1.1 b)

(b)

At $A = 7$ in.

$$s = \left[7 - \left(\frac{5+10}{2}\right)\right]\left(\frac{2}{10-5}\right)$$

$$s = [7 - 7.5]\left(\frac{2}{5}\right)$$

$$s = -0.2$$

By Equation (10.1.5)

$$N_1 = \frac{1-(-0.2)}{2} = 0.6$$

$$N_2 = \frac{1-(0.2)}{2} = 0.4$$

10.3 (a) Using Equation (10.1.1 b)

$$x = x_A = 40 \text{ mm}$$

$$s = \left[40 - \left(\frac{20+60}{2}\right)\right]\left(\frac{2}{60-20}\right)$$

$$s = [40 - 40]\left(\frac{2}{40}\right)$$

$$s = 0$$

$$N_1 = \frac{1-0}{2} = \frac{1}{2},\ N_2 = \frac{1+0}{2} = \frac{1}{2}$$

(b)

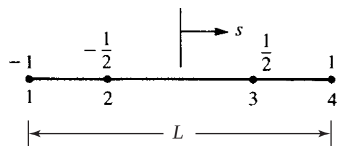
$$s = \left[20 - \left(\frac{10+30}{2}\right)\right]\left(\frac{2}{30-10}\right)$$

$$s = [20 - 20]\ \frac{2}{20}$$

$$s = 0$$

$$N_1 = \frac{1\quad 0}{2} = \frac{1}{2},\ N_2 = \frac{1+0}{2} = \frac{1}{2}$$

10.4



(1) $$u = a_1 + a_2 s + a_3 s^2 + a_4 s^3 \quad \text{(A)}$$

$$x = a_1 + a_2 s + a_3 s^2 + a_4 s^3 \quad \text{(B)}$$

$$x_1 = a_1 + a_2(-1) + a_3(-1)^2 + a_4(-1)^3 \quad (1)$$

$$x_2 = a_1 + a_2\left(\frac{-1}{2}\right) + a_3\left(\frac{-1}{2}\right)^2 + a_4\left(\frac{-1}{2}\right)^3 \quad (2)$$

$$x_3 = a_1 + a_2\left(\frac{1}{2}\right) + a_3\left(\frac{1}{2}\right)^2 + a_4\left(\frac{1}{2}\right)^3 \quad (3)$$

$$x_4 = a_1 + a_2(1) + a_3(1)^2 + a_4(1)^3 \quad (4)$$

$$(1) + (4) \Rightarrow x_1 + x_4 = 2a_1 + 2a_3 \quad (5)$$

$$(2) + (3) \Rightarrow x_2 + x_3 = 2a_1 + \frac{a_3}{2} \quad (6)$$

(5) – (6) gives

$$x_1 + x_4 - (x_2 + x_3) = 2a_1 + 2a_3 - \left(2a_1 + \frac{a_3}{2}\right)$$

or $$a_3 = \frac{2}{3}(x_1 + x_4 - x_2 - x_3) \quad (7)$$

(7) into (5)

$$x_1 + x_4 = 2a_1 + 2\left(\frac{2}{3}\right)(x_1 + x_4 - x_2 - x_3)$$

or $$a_1 = \frac{-\frac{1}{3}(x_1 + x_4) + \frac{4}{3}(x_2 + x_3)}{2} \quad (8)$$

$$(1) - (4) \Rightarrow x_1 - x_4 = -2a_2 - 2a_4 \quad (9)$$

$$(2) - (3) \Rightarrow x_2 - x_3 = -a_2 - \frac{a_4}{4} \quad (10)$$

(9) – 2 (10) gives

$$x_1 - x_4 - 2(x_2 - x_3) = \frac{-3a_4}{2}$$

$$\therefore \quad a_4 = \frac{2}{3}[2(x_2 - x_3) - (x_1 - x_4)] \quad (11)$$

(11) into (9) yields

$$a_2 = \frac{\frac{1}{3}(x_1 - x_4) - \frac{8}{3}(x_2 - x_3)}{2} \quad (12)$$

Substituting (7), (8), (11) and (12) into (B)

$$x = \frac{4(x_2 + x_3) - (x_1 + x_4)}{6} + \frac{[(x_1 - x_4) - 8(x_2 - x_3)]s}{6}$$

$$+ \frac{4(x_1 + x_4 - x_2 - x_3)}{6} s^2 + \frac{[8(x_2 - x_3) - 4(x_1 - x_4)]s^3}{6} \quad (13)$$

Combine like x_1, x_2, x_3 and x_4 coefficients

$$x = \left(-\frac{2}{3}s^3 + \frac{2}{3}s^2 + \frac{s}{6} - \frac{1}{6}\right)x_1 + \left(\frac{4}{3}s^3 - \frac{2}{3}s^2 - \frac{4}{3}s + \frac{2}{3}\right)x_2$$

$$+ \left(-\frac{4}{3}s^3 - \frac{2}{3}s^2 + \frac{4}{3}s + \frac{2}{3}\right)x_3 + \left(\frac{2}{3}s^3 + \frac{2}{3}s^2 - \frac{s}{6} - \frac{1}{6}\right)x_4 \qquad (14)$$

By (14) then

$$\{x\} = [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}$$

$$\therefore \quad N_1 = \frac{-2}{3}s^3 + \frac{2}{3}s^2 + \frac{s}{6} - \frac{1}{6}$$

$$N_2 = \frac{4}{3}s^3 - \frac{2}{3}s^2 - \frac{4}{3}s + \frac{2}{3}$$

$$N_3 = \frac{-4}{3}s^3 - \frac{2}{3}s^2 + \frac{4}{3}s + \frac{2}{3}$$

$$N_4 = \frac{2}{3}s^3 + \frac{2}{3}s^2 - \frac{s}{6} - \frac{1}{6}$$

(2) $$\frac{du}{ds} = \left[-2s^2 + \frac{4}{3}s + \frac{1}{6} \quad 4s^2 - \frac{4}{3}s - \frac{4}{3} \quad -4s^2 - \frac{4}{3}s + \frac{4}{3} \quad 2s^2 + \frac{4}{3}s - \frac{1}{6}\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Differentiating (13)

$$\frac{dx}{ds} = \left(-2s^2 + \frac{4}{3}s + \frac{1}{6}\right)x_1 + \left(4s^2 + \frac{4}{3}s - \frac{4}{3}\right)x_2$$

$$+ \left(-4s^2 - \frac{4}{3}s + \frac{4}{3}\right)x_3 + \left(2s^2 + \frac{4}{3}s - \frac{1}{6}\right)x_4$$

Simplifying

$$= 2s^2(x_4 - x_1) + \frac{4}{3}s(x_4 + x_1) - \frac{1}{6}(x_4 - x_1)$$

$$-4s^2(x_3 - x_2) - \frac{4}{3}s(x_3 + x_2) + \frac{4}{3}(x_3 - x_2)$$

$$= 2s^2 L + \frac{8}{3}s\frac{(x_4 + x_1)}{2} - \frac{1}{6}L - 4s^2\left(\frac{L}{2}\right) - \frac{8}{3}s\left(\frac{x_3 + x_2}{2}\right) + \frac{4}{3}\left(\frac{L}{2}\right)$$

$$= 2s^2 L + \frac{8}{3}s\,x_c - \frac{L}{6} - 2s^2 L - \frac{8}{3}s\,x_c + \frac{2}{3}L$$

$$\frac{dx}{ds} = \frac{L}{2}$$

Now

$$\frac{du}{dx} = \frac{\frac{du}{ds}}{\frac{dx}{ds}} \quad \text{and} \quad \frac{du}{dx} = \varepsilon_x = [B]\{d\}$$

$$\therefore \quad \varepsilon_x = \left[\frac{-12s^2+8s+1}{3L} \;\; \frac{12s^2-4s-4}{\frac{3L}{2}} \;\; \frac{-12s^2-4s+4}{\frac{3L}{2}} \;\; \frac{12s^2+8s-1}{3L}\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\therefore \quad [B] = \left[\frac{-12s^2+8s+1}{3L} \;\; \frac{12s^2-4s-4}{\frac{3L}{2}} \;\; \frac{-12s^2-4s+4}{\frac{3L}{2}} \;\; \frac{12s^2+8s-1}{3L}\right]$$

10.5 (a) Using Equation (10.5.6)

$$x = x_A = 13 = 15 + \left(\frac{20-10}{2}\right)s + \frac{10+20-2\,(15)}{2}s^2$$

$$\therefore\ 0s^2 + 5s + 2 = 0$$

$$s = \frac{-2}{5} = -0.4$$

$$N_1 = \frac{s(s-1)}{2} = \frac{-0.4\,(-0.4-1)}{2}$$

$$= 0.28$$

$$N_2 = \frac{s(s+1)}{2} = \frac{-0.4\,(-0.4+1)}{2}$$

$$= -0.12$$

$$N_3 = 1 - s_2 = 1 - (-0.4)^2$$

$$= 0.84$$

$$\Sigma N\text{'s} = 0.28 - 0.12 + 0.84 = 1$$

$$u = a_1 + a_2 s + a_3 s^2$$

$$u_1 = 0.006 = a_1 + a_2(-1) + a_3(-1)^2$$

$$u_3 = 0 = a_1 + a_2(0) + a_3(0)$$

$$u_2 = -0.006 = a_1 + a_2(1) + a_3(1)^2$$

$$\therefore \quad a_3 = 0,\ a_2 = -0.006,\ a_1 = 0$$

$$\therefore \quad u = -0.006s \quad \text{and} \quad s = -0.4 \quad \text{at} \quad x_A = 13$$

$$\therefore \quad u = -0.006\,(-0.4) = 0.0024 \text{ in.}$$

$$\varepsilon_x = \frac{2s-1}{L}u_1 + \frac{2s+1}{L}u_2 - \frac{4s}{L}u_3$$

$$\varepsilon_x = \frac{2s-1}{L}(0.006) + \frac{2s+1}{L}(-0.006) = 0$$

$$\varepsilon_x = \frac{-0.12}{L}\ (L = 10^{11})$$

$$\therefore \quad \varepsilon_x = -0.012\ \frac{\text{in.}}{\text{in.}}$$

10.6 (a) Using Equation (10.5.6)

$$x = x_A = 1.5 = 1 + \left(\frac{2-0}{2}\right)s + \left(\frac{0+2-2\,(1)}{2}\right)s^2$$

$$1.5 = 1 + s + 0s^2$$

$$s - 0.5 = 0$$

$$s = 0.5$$

$$N_1 = \frac{s(s-1)}{2} = \frac{0.5(0.5-1)}{2}$$

$$= -0.125$$

$$N_2 = \frac{s(s+1)}{2} = \frac{0.5(0.5+1)}{2}$$

$$= 0.375$$

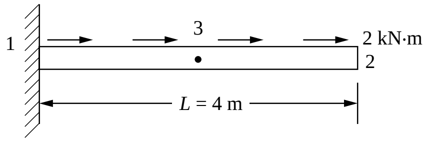
$$N_3 = 1 - s^2 = 1 - 0.5^2$$

$$= 0.75$$

$$\Sigma N\text{'s} = -0.125 + 0.375 + 0.75$$

$$= 1.0$$

10.8



$A = 2 \times 10^{-4}$ m^2, $E = 205$ GPa
$L = 4$ m

$$\{F\} = [K]\{d\} \qquad \text{(A)}$$

where by Equation (10.5.22)

$$[K] = \frac{AE}{L}\begin{bmatrix} 4.67 & 0.667 & -5.33 \\ & 4.67 & -5.33 \\ \text{Symmetry} & & 10.67 \end{bmatrix} \qquad (1)$$

By Equation (10.5.9) for N's and using Equation (10.1.21)

$$\{f_s\} = \int_{-1}^{1} [N_s]^T \{T\}\, dx\,,\ \{T\} = \left\{2\,\frac{\text{kN}}{\text{m}}\right\},\ dx = \frac{L}{2}\,ds$$

$$\{f_s\} = \int_{-1}^{1} \begin{Bmatrix} \dfrac{s(s-1)}{2} \\ \dfrac{s(s+1)}{2} \\ 1 - s^2 \end{Bmatrix} \left(2\,\frac{\text{kN}}{\text{m}}\right) \frac{L}{2}\,ds \qquad (2)$$

Upon integrating Equation (2)

$$\{f_s\} = \begin{Bmatrix} \left(\dfrac{s^3}{6} - \dfrac{s^2}{4}\right)_{-1}^{1} \\ \left(\dfrac{s^3}{6} - \dfrac{s^2}{4}\right)_{-1}^{1} \\ \left(s - \dfrac{s^3}{3}\right)_{-1}^{1} \end{Bmatrix} (2)\ \frac{4}{2} \qquad (3)$$

$$\{f_s\} = \begin{Bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{4}{3} \end{Bmatrix} (4) = \begin{Bmatrix} \frac{4}{3} \\ \frac{4}{3} \\ \frac{16}{3} \end{Bmatrix} \text{kN} \tag{4}$$

Using Equations (4) and (1) in (A) and applying boundary condition $u_1 = 0$,

$$\begin{Bmatrix} \frac{4}{3} \\ \frac{16}{3} \end{Bmatrix} = \begin{bmatrix} 2.393 & -2.732 \\ -2.732 & 5.468 \end{bmatrix} \times 10^4 \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \tag{5}$$

Solving (5) for u_2 and u_3

$$u_2 = 3.885 \times 10^{-4} \text{ m}, u_3 = 2.916 \times 10^{-4} \text{ m} \tag{6}$$

Stress in bar

$$E = 205 \times 10^9$$

At $s = -1$ or $x = 0$

$$\sigma_1 = E \begin{bmatrix} \frac{2(-1)-1}{4} & \frac{2(-1)+1}{4} & \frac{-4(-1)}{4} \end{bmatrix} \begin{Bmatrix} 0 \\ 3.885 \times 10^{-4} \\ 2.916 \times 10^{-4} \end{Bmatrix}, \quad \underbrace{\sigma_1 = (3.987 \times 10^7)}_{\overset{||}{39.87 \text{ MPa}}} \frac{\text{N}}{\text{m}}$$

At $s = 1$ or $x = L$

$$\sigma_2 = E \begin{bmatrix} \frac{2 \times 1 - 1}{4} & \frac{2+1}{4} & \frac{-4}{4} \end{bmatrix} \begin{Bmatrix} 0 \\ 3.885 \times 10^{-4} \\ 2.916 \times 10^{-4} \end{Bmatrix}, \sigma_2 = (-4.357 \times 10^4)$$

$$-0.04357 \text{ MPa}$$

Note small number compared to stress at fixed end.

10.9

$$x = \frac{1}{4} [(1-s)(1-t) x_1 + (1+s)(1-t) x_2 + (1+s)(1+t) x_3 + (1-s)(1+t) x_4]$$

$$y = \frac{1}{4} [(1-s)(1-t) y_1 + (1+s)(1-t) y_2 + (1+s)(1+t) y_3 + (1-s)(1+t) y_4]$$

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \text{ Equation (10.2.10)}$$

$$\frac{\partial x}{\partial s} = \frac{1}{4} [-x_1 + tx_1 + x_2 - tx_2 + x_3 + t x_3 - x_4 - tx_4]$$

$$\frac{\partial x}{\partial t} = \frac{1}{4} [-x_1 + sx_1 - x_2 - sx_2 + x_3 + sx_3 + x_4 - sx_4]$$

$$\frac{\partial y}{\partial s} = \frac{1}{4}\,[-y_1 + ty_1 + y_2 - ty_2 + y_3 + ty_3 - y_4 - ty_4]$$

$$\frac{\partial y}{\partial t} = \frac{1}{4}\,[-y_1 + sy_1 - y_2 - sy_2 + y_3 + sy_3 + y_4 - sy_4]$$

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

where

$$J_{11} = \frac{1}{4}\,(-x_1 + tx_1 + x_2 - tx_2 + x_3 + tx_3 - x_4 - tx_4)$$

$$J_{12} = \frac{1}{4}\,(-y_1 + ty_1 + y_2 - ty_2 + y_3 + ty_3 - y_4 - ty_4)$$

$$J_{21} = \frac{1}{4}\,(-x_1 + sx_1 - x_2 - sx_2 + x_3 + sx_3 + x_4 - sx_4)$$

$$J_{22} = \frac{1}{4}\,(-y_1 + sy_1 - y_2 - sy_2 + y_3 + sy_3 + y_4 - sy_4)$$

Find determinate $|J|$

$$|J| = J_{11}J_{22} - J_{21}J_{12}$$

Multiplying and collecting terms

$$\begin{aligned} |J| = \frac{1}{16}\,[&2x_1y_2 - 2tx_1y_2 + 2tx_1y_3 - 2sx_1y_3 + 2sx_1y_4 - 2x_1y_4 - 2x_2y_1 + 2tx_2y_3 \\ &+ 2x_2y_3 + 2sx_2y_3 - 2sx_2y_4 - 2tx_2y_4 + 2sx_3y_1 - 2tx_3y_1 - 2x_3y_2 - 2sx_3y_2 \\ &+ 2x_3y_4 + 2tx_3y_4 + 2x_4y_1 - 2sx_4y_1 + 2sx_4y_2 + 2x_4y_2 + 2tx_4y_2 - 2x_4y_3 \\ &- 2tx_4y_3] \end{aligned}$$

Factor out x_i's

$$\begin{aligned} |J| = \frac{1}{8}\,[&x_1(y_2 - ty_2 + ty_3 - sy_3 + sy_4 - y_4) \\ &+ x_2(-y_1 + ty_1 + y_3 + sy_3 - sy_4 - ty_4) + x_3(sy_1 - ty_1 - y_2 - sy_2 + y_4 + ty_4) \\ &+ x_4(y_1 - sy_1 + sy_2 + ty_2 - y_3 - ty_3)] \end{aligned}$$

$$|J| = \frac{1}{8}\,[x_1\ x_2\ x_3\ x_4] \begin{bmatrix} y_2 - ty_2 + ty_3 - sy_3 + sy_4 - y_4 \\ -y_1 + ty_1 + y_3 + sy_3 - sy_4 - ty_4 \\ sy_1 - ty_1 - y_2 - sy_2 + y_4 + ty_4 \\ y_1 - sy_1 + sy_2 + ty_2 - y_3 - ty_3 \end{bmatrix}$$

$$|J| = \frac{1}{8}\,[x_1\ x_2\ x_3\ x_4] \begin{bmatrix} y_1(0) + y_2(1-t) + y_3(t-s) + y_4(s-1) \\ y_1(-1+t) + y_2(0) + y_3(0) + y_4(1+t) \\ y_1(s-t) + y_2(-1-s) + y_3(0) + y_4(1+t) \\ y_1(1-s) + y_2(s+t) + y_3(-1-t) + y_4(0) \end{bmatrix}$$

$$|J| = \frac{1}{8}\,[x_1\ x_2\ x_3\ x_4] \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ -1+t & 0 & s+1 & -s-t \\ s-t & -1-s & 0 & 1+t \\ 1-s & s+t & -1-t & 0 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

10.10

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix}$$

x and y from Problem 10.9

$$\frac{\partial x}{\partial s} = \frac{1}{4}\,(-1)\,(1-t)x_1 + \frac{1}{4}\,(1-t)x_2 + \frac{1}{4}\,(1+t)x_3 + \frac{1}{4}\,(-1)\,(1+t)x_4$$

$$= N_{1,s}\,x_1 + N_{2,s}\,x_2 + N_{3,s}\,x_3 + N_{4,s}\,x_4$$

$$\frac{\partial x}{\partial t} = \frac{1}{4}\,(-1)\,(1-s)x_1 + \frac{1}{4}\,(-1)\,(1+s)x_2 + \frac{1}{4}\,(1+s)x_3 + \frac{1}{4}\,(1-s)x_4$$

$$= N_{1,t}\,x_1 + N_{2,t}\,x_2 + N_{3,t}\,x_3 + N_{4,t}\,x_4$$

$$\frac{\partial y}{\partial s} = \frac{1}{4}\,(-1)\,(1-t)y_1 + \frac{1}{4}\,(1-t)y_2 + \frac{1}{4}\,(1+t)y_3 + \frac{1}{4}\,(-1)\,(1+t)y_4$$

$$= N_{1,s}\,y_1 + N_{2,s}\,y_2 + N_{3,s}\,y_3 + N_{4,s}\,y_4$$

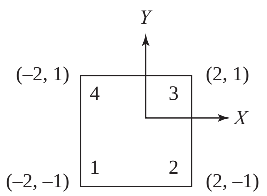
$$\frac{\partial y}{\partial t} = \frac{1}{4}\,(-1)\,(1-s)y_1 + \frac{1}{4}\,(-1)\,(1+s)y_2 + \frac{1}{4}\,(1+s)y_3 + \frac{1}{4}\,(1-s)y_4$$

$$= N_{1,t}\,y_1 + N_{2,t}\,y_2 + N_{3,t}\,y_3 + N_{4,t}\,y_4$$

$$[J] = \begin{bmatrix} N_{1,s} & N_{2,s} & N_{3,s} & N_{4,s} \\ N_{1,t} & N_{2,t} & N_{3,t} & N_{4,t} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

10.11 (a)

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \begin{pmatrix} x_1 = -z,\ x_2 = z \\ x_3 = z,\ x_4 = -z \end{pmatrix}$$



$$\frac{\partial x}{\partial s} = \frac{1}{4}(-1)(1-t)(-2) + \frac{1}{4}(1-t)(2) + \frac{1}{4}(1+t)(2) + \frac{1}{4}(-1)(1+t)(-2) = z$$

$$\frac{\partial x}{\partial t} = \frac{1}{4}\,(-1)\,(1-s)(-2) + \frac{1}{4}\,(-1)\,(1+s)\,(2) + \frac{1}{4}\,(1+s)\,(2)$$

$$+ \frac{1}{4}(1-s)\,(-2) = 0$$

$y_1 = 1,\ y_2 = -1$

$y_3 = 1,\ y_4 = 1$

$$\frac{\partial y}{\partial s} = \frac{1}{4}\,(-1)\,(1-t)\,(-1) + \frac{1}{4}\,(1-t)\,(-1) + \frac{1}{4}\,(1+t)\,(1)$$

$$+ \frac{1}{4}\,(-1)\,(1+t)\,(1) = 0$$

$$\frac{\partial y}{\partial t} = \frac{1}{4}\ (-1)\,(1-s)\,(-1) + \frac{1}{4}\ (-1)\,(1+s)\,(-1) + \frac{1}{4}\ (1+s)\,(1)$$

$$+\ \frac{1}{4}\ (1-s)\,(1) = 1$$

$$|[J]| = \left|\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right| = 2 - 0 = 2$$

By Equation (10.2.22)

$$|[J]| = \frac{1}{8}\ [-2\ \ 2\ \ 2\ -2] \times \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \begin{Bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{Bmatrix} \qquad \text{(A)}$$

Symplying by multiplying the matrices in Equation (A) yields

$|[J]| = 2$ also

and $\quad |[J]| = \dfrac{A}{4}$ as

$A = 4 \times 2 = 8$ (area of element)

$\therefore \quad = |[J]|\ \dfrac{8}{4} = 2$

10.12

By Equation (10.2.18)

$$[B(s, t)] = \frac{1}{|[J]|}\ [B_1]\quad [B_2]\quad [B_3]\quad [B_4]$$

By Equation (10.2.3)

$$x = \frac{1}{4}\ [(1-s)\,(1-t)x_1 + (1+s)\,(1-t)x_2 + (1+s)\,(1+t)x_3 + (1-s)$$
$$(1+t)x_4] \qquad (1)$$

$$y = \frac{1}{4}\ [(1-s)\,(1-t)y_1 + (1+s)\,(1-t)y_2 + (1+s)\,(1+t)y_3 + (1-s)$$
$$(1+t)y_4]$$

By Equation (10.2.16)

$$[D'] = \frac{1}{|[J]|}\begin{bmatrix} \frac{\partial y}{\partial t}\frac{\partial(\,)}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\,)}{\partial t} & 0 \\ 0 & \frac{\partial x}{\partial s}\frac{\partial(\,)}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\,)}{\partial s} \\ \frac{\partial x}{\partial s}\frac{\partial(\,)}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\,)}{\partial s} & \frac{\partial y}{\partial t}\frac{\partial(\,)}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\,)}{\partial t} \end{bmatrix} \qquad (2)$$

Let

$$a = \frac{\partial y}{\partial t} = \frac{1}{4}\ [y_1(s-1) + y_2(-1-s) + y_3(1+s) + y_4(1-s)]$$

$$b = \frac{\partial y}{\partial s} = \frac{1}{4}\ [y_1(t-1) + y_2(1-t) + y_3(1+t) + y_4(-1-t)]$$

$$c = \frac{\partial x}{\partial s} = \frac{1}{4}\ [x_1(t-1) + x_2(1-t) + x_3(1+t) + x_4(-1-t)]$$

$$d = \frac{\partial x}{\partial t} = \frac{1}{4}\ [x_1(s-1) + x_2(-1-s) + x_3(1+s) + x_4(1-s)]$$

By Equation (10.3.5)

$$N_1 = \frac{(1-s)(1-t)}{4},\ N_2 = \frac{(1+s)(1-t)}{4}$$

$$N_3 = \frac{(1+s)(1+t)}{4},\ N_4 = \frac{(1-s)(1+t)}{4}$$

$$[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

Let

$$N_{i,s} = \frac{\partial N_i}{\partial s}, \quad N_{i,t} = \frac{\partial N_i}{\partial t}$$

Now

$$[B] = [D'][N]$$

Using $[D']$ from Equation (2) above, we obtain

$$[B] = \frac{1}{|[J]|} \left[\begin{array}{c:c} a\frac{\partial(\ )}{\partial s} - b\frac{\partial(\ )}{\partial t} & 0 \\ 0 & c\frac{\partial(\ )}{\partial t} - d\frac{\partial(\ )}{\partial s} \\ c\frac{\partial(\ )}{\partial t} - d\frac{\partial(\ )}{\partial s} & a\frac{\partial(\ )}{\partial s} - b\frac{\partial(\ )}{\partial t} \end{array}\right] \begin{bmatrix} \frac{(1-s)(1-t)}{4} & 0 \cdots 0 \\ 0 & \frac{(1-s)(1-t)}{4} \ \cdots \end{bmatrix}$$

$$[B] = \frac{1}{|[J]|} \begin{array}{cccc} \text{col}(1) & \text{col}(2) & \cos(7) & \cos(8) \\ \left[\begin{array}{c} \frac{a(t-1)}{4} - \frac{b(s-1)}{4} \\ 0 \\ \frac{c(s-1)}{4} - \frac{d(t-1)}{4} \end{array}\right. & \begin{array}{c} 0 \quad \cdots \\ \frac{c(s-1)}{4} - \frac{d(t-1)}{4} \cdots \\ \frac{a(t-1)}{4} - \frac{b(s-1)}{4} \cdots \end{array} & \begin{array}{c} \cdots \frac{a(1-t)}{4} - \frac{b(1-s)}{4} \\ 0 \\ \cdots \frac{c(1-s)}{4} - \frac{d(-1-t)}{4} \end{array} & \left.\begin{array}{c} 0 \\ \frac{c(1-s)}{4} - \frac{d(-1-t)}{4} \\ \frac{a(-1-t)}{4} - \frac{b(1-s)}{4} \end{array}\right] \end{array}$$

By Equation (10.2.19)

$$[B] = \frac{1}{|[J]|} = [B_1] \quad [B_2] \quad [B_3] \quad [B_4]$$

where the submatrices are

$$[B_i] = \begin{bmatrix} a(N_{i,s}) - b(N_{i,t}) & 0 \\ 0 & c(N_{i,t}) - d(N_{i,s}) \\ c(N_{i,t}) - d(N_{i,s}) & a(N_{i,s}) - b(N_{i,t}) \end{bmatrix}$$

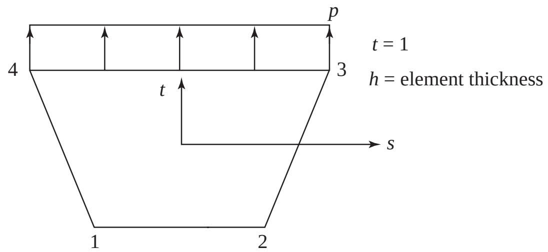
where, for instance

$$N_{1,s} = \frac{\partial N_1}{\partial s} = \frac{t-1}{4}$$

$$N_{1,t} = \frac{\partial N_1}{\partial t} = \frac{s-1}{4}$$

etc.

10.13



$$\{f_s\} = \int_{-1}^{1} [N_s]^T \{T\}\, h\, \frac{L}{2}\, ds$$

At $\quad t = 1$

$$\begin{Bmatrix} f_{s3s} \\ f_{s3t} \\ f_{s4s} \\ f_{s4t} \end{Bmatrix} = \int_{-1}^{1} \begin{bmatrix} N_3 & 0 & N_4 & 0 \\ 0 & N_3 & 0 & N_4 \end{bmatrix}^T \begin{Bmatrix} p_s \\ p_t \end{Bmatrix} h\, \frac{L}{2}\, ds$$

$$p_s = 0,\ p_t = p$$

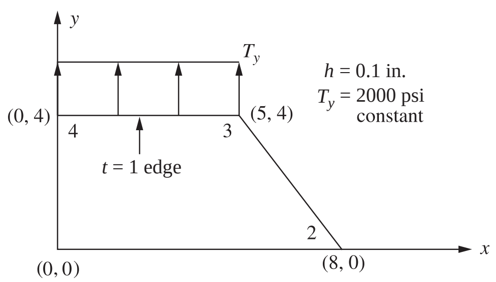
$$\therefore \quad \{f_s\} = \int_{-1}^{1} \begin{Bmatrix} 0 \\ N_3 p \\ 0 \\ N_4 p \end{Bmatrix} h\, \frac{L}{2}\, ds$$

$$\{f_s\} = \int_{-1}^{1} \begin{Bmatrix} 0 \\ \frac{(1+s)\,(1+t)}{4}\, p \\ 0 \\ \frac{(1-s)\,(1+t)}{4}\, p \end{Bmatrix}_{t=1} h\, \frac{L}{2}\, ds$$

$$= \left. \begin{bmatrix} 0 \\ \frac{ps}{2} + \frac{ps^2}{4} \\ 0 \\ \frac{ps}{2} - \frac{ps^2}{4} \end{bmatrix} \right|_{-1}^{1} \frac{Lh}{2} = \begin{matrix} f_{s3s} = 0 \\ f_{s3t} = \dfrac{pLh}{2} \\ f_{s4s} = 0 \\ f_{s4t} = \dfrac{pLh}{2} \end{matrix}$$

10.14

(a)



$$\{f_s\} = \int_{-1}^{1} [N_s]^T \{T\} \frac{L}{2}\, h\, dt$$

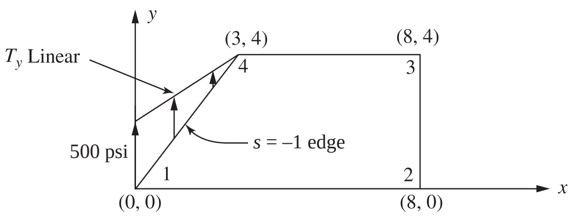
$$L = 5 \text{ in.}, \quad p_t = 2000 \text{ psi}, \quad p_s = 0$$

$$N_3 = \frac{1+s}{2} \text{ and } N_4 = \frac{1-s}{2} \text{ for } t = 1$$

$$\therefore \quad \begin{Bmatrix} f_{s3s} \\ f_{s3t} \\ f_{s4s} \\ f_{s4t} \end{Bmatrix} = \int_{-1}^{1} \begin{bmatrix} N_3 & 0 \\ 0 & N_3 \\ N_4 & 0 \\ 0 & N_4 \end{bmatrix} \begin{Bmatrix} p_s = 0 \\ p_t = 2000 \end{Bmatrix} h \frac{L}{2}\, dt$$

$$= \begin{Bmatrix} 0 \\ \int_{-1}^{1} \left(\frac{1+s}{2}\right)(2000)\,(0.1)\left(\frac{5}{2}\right) dt \\ 0 \\ \int_{-1}^{1} \left(\frac{1-s}{2}\right)(2000)\,(0.1)\left(\frac{5}{2}\right) dt \end{Bmatrix} \begin{Bmatrix} 0 \\ 500 \\ 0 \\ 500 \end{Bmatrix} \text{lb}$$

(b)



$h = 0.1$ in.
$s = -1$
$L = 5$ in.

$$\begin{Bmatrix} f_{s1s} \\ f_{s1t} \\ f_{s4s} \\ f_{s4t} \end{Bmatrix} = \int_{-1}^{1} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_4 & 0 \\ 0 & N_4 \end{bmatrix} \begin{Bmatrix} p_s = 0 \\ p_t = -250t + 250 \end{Bmatrix} h \frac{L}{2}\, dt$$

$$f_{s1s} = f_{s4s} = 0 \quad N_1 = \frac{1-t}{2},\ N_4 = \frac{1+t}{2} \text{ for } s = -1$$

$$f_{s1t} = \int_{-1}^{1} \left(\frac{1-t}{2}\right) (-250\,t + 250)\,(0.1)\left(\frac{5}{2}\right) dt$$

$$= 83.33 \text{ lb}$$

$$f_{s4t} = \int_{-1}^{1} \left(\frac{1+t}{2}\right) (-250\,t + 250)\,(0.1)\left(\frac{5}{2}\right) dt$$

$$= 41.67 \text{ lb}$$

10.15

(a) $\int_{-1}^{1} \cos\frac{s}{2}\, ds$ Use Table 10.1

$$I = \sum_{i=1}^{3} W_i \cos\frac{s_i}{2} = W_1 \cos\frac{s_1}{2} + W_2 \cos\frac{s_2}{2} + W_3 \cos\frac{s_3}{2}$$

$$= \frac{5}{9}\cos\left(\frac{0.7746}{2}\right) + \frac{8}{9}\cos(0) + \frac{5}{9}\cos\left(\frac{-0.7746}{2}\right)$$

$$I = 1.918 \qquad \text{(Analytical } I = 1.918\text{)}$$

That is

$$\int_{-1}^{1} \cos\frac{s}{2}\,ds = 2\sin\frac{s}{2}\Big|_{-1}^{1} = 2\sin\frac{1}{2} - 2\sin\left(-\frac{1}{2}\right)$$

$$= 4\sin\frac{1}{2} = 4(0.47)$$

$$= 1.918$$

(b) $\int_{-1}^{1} s^2\,ds$

$$I = \sum_{i=1}^{3} W_i s_i^2 = W_1 s_1^2 + W_2 s_2^2 + W_3 s_3^2$$

$$= \frac{5}{9}(0.7746)^2 + \frac{8}{9}(0)^2 + \frac{5}{9}(-0.7746)^2$$

$$I = 0.667 \qquad \text{(Analytical } I = 0.667\text{)}$$

(c) $$\int_{-1}^{1} s^4\,ds = \frac{5}{9}(0.7746)^4 + \frac{8}{9}(0)^4 + \frac{5}{9}(-0.7746)^4$$

$$= 0.400$$

(d) $$\int_{-1}^{1} \frac{\cos s}{1-s^2}\,ds = \frac{5}{9}\left(\frac{\cos(0.7746)}{1-0.7746^2}\right) + \frac{8}{9}\left(\frac{\cos 0}{1-0^2}\right) + \frac{5}{9}\,\frac{\cos(-0.7746)}{1-(-0.7746)^2} = 2.873$$

(Exact is 3.86)

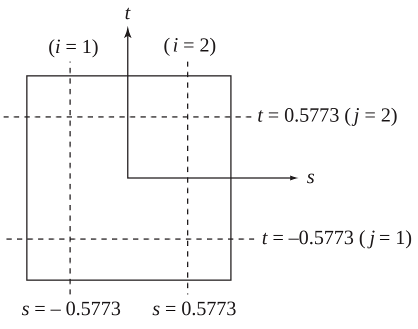
(e) $$\int_{-1}^{1} s^3\,ds = \frac{5}{9}(0.7746)^3 + \frac{8}{9}(0)^3 + \frac{5}{9}(-0.7746)^3 = 0$$

(f) $$\int_{-1}^{1} s\cos s\,ds = \left(\frac{5}{9}\right)(0.7746)\cos(0.7746) + \left(\frac{8}{9}\right)(0)\cos(0) + \frac{5}{9}(-0.7746)\cos(-0.7746)$$

$$= 0.30756 + 0 - 0.30756$$

$$= 0$$

10.16



$$[k] = [B]^T(s_1, t_1)\,[D]\,[B](s_1, t_1)\,|[J](s_1, t_1)|\,h\,W_1 W_1$$
$$+ [B]^T(s_2, t_1)\,[D]\,[B](s_2, t_1)\,|[J](s_2, t_1)|\,h\,W_2 W_1$$

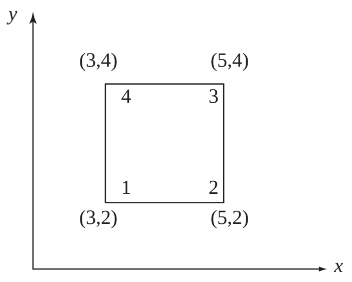
$$+ [B]^T (s_1, t_2) [D] [B] (s_1, t_2) \mid [J] (s_1, t_2) \mid h\, W_1\, W_2$$
$$+ [B]^T (s_2, t_2) [D] [B] (s_2, t_2) \mid [J] (s_2, t_2) \mid h\, W_2\, W_2$$

where

$$s_1 = -0.5773,\ s_2 = 0.5773$$
$$t_1 = -0.5773,\ t_2 = 0.5773$$

Using computer program

(a)



ENTER THE GAUSS POINTS S AND T FOR POINT 1
–0.57735, –0.57735
ENTER THE WEIGHT FOR POINT 1
1.0
ENTER THE NODAL VALUES X AND Y FOR POINT 1
3.0, 2.0
ENTER THE GAUSS POINTS S AND T FOR POINT 2
0.57735, –0.57735
ENTER THE WEIGHT FOR POINT 2
1.0
ENTER THE NODAL VALUES X AND Y FOR POINT 2
5.0, 2.0
ENTER THE GAUSS POINTS S AND T FOR POINT 3
0.57735, 0.57735
ENTER THE WEIGHT FOR POINT 3
1.0
ENTER THE NODAL VALUES X AND Y FOR POINT 3
5.0, 4.0
ENTER THE GAUSS POINTS S AND T FOR POINT 4
–0.57734, 0.57734
ENTER THE WEIGHT FOR POINT 4
1.0
ENTER THE NODAL VALUES X AND Y FOR POINT 4
3.0, 4.0
ENTER THE VALUE FOR YOUNGS MODULUS
30000000.0
ENTER THE VALUE FOR POISSONS RATIO
0.25
ENTER THE VALUE FOR THE THICKNESS, *h*
1.0

THE GAUSS VALUES S AND T AND WEIGHTS ARE

POINT	S	T	WEIGHT
1	–5.773500E–001	–5.773500E–001	1.0000000
2	5.773500E–001	–5.773500E–001	1.0000000
3	5.773500E–001	5.773500E–001	1.0000000
4	–5.773400E–001	5.773400E–001	1.0000000

THE NODAL COORDINATE VALUES ARE

NODE	X	Y
1	3.0000000	2.0000000
2	5.0000000	2.0000000
3	5.0000000	4.0000000
4	3.0000000	4.0000000

THE ELEMENT PARAMETERS ARE

YOUNGS MODULUS	POISSON'S RATIO	THICKNESS
30000000.0000000	2.500000E–001	1.0000000

DO YOU WISH TO VIEW THE VALUES OF J (Y/N)?
THE VALUES OF I J I ARE
THE VALUE OF J 1
1.0000000
THE VALUE OF J 2
1.0000000
THE VALUE OF J 3
1.0000000
THE VALUE OF J 4
1.0000000
DO YOU WITH TO VIEW THE B MATRIX (Y/N)?
THE B MATRIX VALUES ARE ($[B]_{3\times 8}$)

$$\left.\begin{matrix} -1.0566\text{ E}-1 & 0 & 1.0566\text{ E}-1 & 0 & 0 & -3.943\text{ E}-1 \\ & & & & 0 & -1.0566\text{ E}-1 \end{matrix}\right\}$$

$$\left.\begin{matrix} -3.943\text{ E}-1 & -1.0566\text{ E}-1 & -1.0566\text{ E}-1 & & 1.0566\text{ E}-1 \\ & 3.943\text{ E}-1 & 0 & -3.943\text{ E}-1 & 0 \end{matrix}\right\}$$

$$\left.\begin{matrix} 0 & 1.0566\text{ E}-1 & 0 & 3.943\text{E}-1 & 1.0566\text{ E}-1 \\ & 3.943\text{ E}-1 & & 3.943\text{ E}-1 & -3.943\text{ E}-1 \end{matrix}\right\}$$

$\}$ = one row of the 3×8 $[B]$

DO YOU WISH TO VIEW THE D MATRIX (Y/N)?
THE VALUES OF THE D MATRIX ARE

32000000.0000000	8000000.0000000	0.0000000
8000000.0000000	32000000.0000000	0.0000000
0.0000000	0.0000000	12000000.0000000

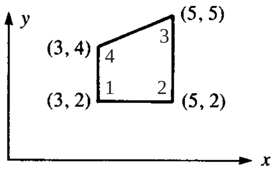
DO YOU WISH TO VIEW THE K MATRIX (Y/N)?
THE K MATRIX VALUES ARE

14666660.0000000	5000015.0000000	–8666672.0000000	–1000005.0000000
5000015.0000000	14666610.0000000	1000001.0000000	1333353.0000000
–8666672.0000000	1000001.0000000	14666690.0000000	–5000011.0000000
–1000005.0000000	1333353.0000000	–5000011.0000000	14666690.0000000
–7333369.0000000	–4999991.0000000	1333353.0000000	1000001.0000000

–4999981.0000000	–7333369.0000000	–1000005.0000000	–8666672.0000000
1333383.0000000	–1000025.0000000	–7333369.0000000	5000015.0000000
999970.6000000	–8666592.0000000	5000015.0000000	–7333369.0000000

–7333369.0000000	–4999981.0000000	1333383.0000000	999970.6000000
–4999991.0000000	–7333369.0000000	–1000025.0000000	–8666592.0000000
1333353.0000000	–1000005.0000000	–7333369.0000000	5000015.0000000
1000001.0000000	–8666672.0000000	5000015.0000000	–7333369.0000000
14666610.0000000	5000015.0000000	–8666592.0000000	–1000025.0000000
5000015.0000000	14666660.0000000	999970.6000000	1333383.0000000
–8666592.0000000	999970.6000000	14666580.0000000	–4999961.0000000
–1000025.0000000	1333383.0000000	–4999961.0000000	14666580.0000000

(b)



THE K MATRIX VALUES ARE

8-1 Column	8-2
14990860.0000000	3483641.0000000
3483641.0000000	13670480.0000000
–11385300.0000000	370145.5000000
–1626729.0000000	–565783.1000000
–4661870.0000000	–4327808.0000000
–4309764.0000000	–5451907.0000000
1056312.0000000	474022.0000000
2452853.0000000	–7652789.0000000
8-3	8-4
–11385300.0000000	–1626729.0000000
370145.6000000	–565783.1000000
19631590.0000000	–6267461.0000000
–6267461.0000000	14127760.0000000
4403077.0000000	2237624.0000000
222705.5000000	–5114752.0000000
–12649370.0000000	5656566.0000000
5674610.0000000	–8447218.0000000
8-5	8-6
–4661870.0000000	–4309764.0000000
–4327808.0000000	–5451907.0000000
4403076.0000000	222705.4000000
2237624.0000000	–5114752.0000000
11571920.0000000	3807131.0000000
3807131.0000000	11105380.0000000
–11313130.0000000	279927.3000000
–1716948.0000000	–538717.6000000
8-7	8-8
1056312.0000000	2452853.0000000
474021.9000000	–7652789.0000000

–12649370.0000000	5674610.0000000
5656566.0000000	–8447218.0000000
–11313130.0000000	–1716948.0000000
279927.4000000	–538717.6000000
22906180.0000000	–6410515.0000000
–6410515.0000000	16638720.0000000

10.18

$$[B\,(s,t)] = \frac{1}{|[J]|}\ [B_1]\quad [B_2]\quad [B_3]\quad [B_4]\quad [B_5]\quad [B_6]\quad [B_7]\quad [B_8]$$

$$N_{1,\,s} = \frac{1}{4}\ (1-t)\,(s+t+1) - \frac{1}{4}\ (1-s)\,(1-t)$$

$$N_{2,\,s} = \frac{1}{4}\ (1-t)\,(s-t-1) + \frac{1}{4}\ (1+s)\,(1-t)$$

$$N_{3,\,s} = \frac{1}{4}\ (1+t)\,(s+t-1) - \frac{1}{4}\ (1+s)\,(1+t)$$

$$N_{4,\,s} = \frac{1}{4}\ (1+t)\,(s-t+1) - \frac{1}{4}\ (1-s)\,(1+t)$$

$$N_{5,\,s} = (t-1)s$$

$$N_{6,\,s} = \frac{1}{2}\ (1-t^2)$$

$$N_{7,\,s} = -\,(1+t)s$$

$$N_{8,\,s} = \frac{1}{2}\ (t^2-1)$$

$$N_{1,\,t} = \frac{1}{4}\ (1-s)\,(s+t+1) + \frac{1}{4}\ (1-s)\,(t-1)$$

$$N_{2,\,t} = \frac{1}{4}\ (1+s)\,(t+1-s) - \frac{1}{4}\ (1+s)\,(1-t)$$

$$N_{3,\,t} = \frac{1}{4}\ (1+s)\,(s+t-1) + \frac{1}{4}\ (1+s)\,(1+t)$$

$$N_{4,\,t} = \frac{1}{4}\ (1-s)\,(-s+t-1) + \frac{1}{4}\ (1-s)\,(1+t)$$

$$N_{5,\,t} = \frac{1}{2}\ (1+s)\,(s-1)$$

$$N_{6,\,t} = -\,(1+s)\,t$$

$$N_{7,\,t} = \frac{1}{2}\ (1-s^2)$$

$$N_{8,\,t} = (s-1)\,t$$

$$|[J]| = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial y}{\partial s}\frac{\partial x}{\partial t}$$

$$= [N_{1,s}\, x_1 + N_{2,s}\, x_2 + \ldots + N_{8,s}\, x_8]$$

$$\times [N_{1,t}\, y_1 + N_{2,t}\, y_2 + \ldots + N_{8,t}\, y_8]$$

$$- [N_{1,s}\, y_1 + N_{2,s}\, y_2 + \ldots + N_{8,s}\, y_8]$$

$$\times [N_{1,t}\, x_1 + N_{2,t}\, x_2 + \ldots + N_{8,t}\, x_8]$$

$$[B_i] = \begin{bmatrix} a\,(N_{i,s}) - b(N_{i,t}) & 0 \\ 0 & c\,(N_{i,t}) - d\,(N_{i,s}) \\ c\,(N_{i,t}) - d\,(N_{i,s}) & a\,(N_{i,s}) - b(N_{i,t}) \end{bmatrix}$$

where

$$a = \frac{\partial y}{\partial t} = N_{1,t}\, y_1 + N_{2,t}\, y_2 + \ldots + N_{8,t}\, y_8$$

$$b = \frac{\partial y}{\partial s} = N_{1,s}\, y_1 + N_{2,s}\, y_2 + \ldots + N_{8,s}\, y_8$$

$$c = \frac{\partial x}{\partial s} = N_{1,s}\, x_1 + N_{2,s}\, x_2 + \ldots + N_{8,s}\, x_8$$

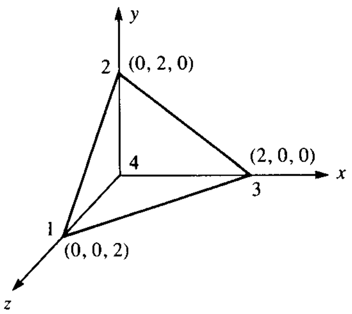
$$d = \frac{\partial x}{\partial t} = N_{1,t}\, x_1 + N_{2,t}\, x_2 + \ldots + N_{8,t}\, x_8$$

10.21 The 2-pt rule works as we have a 2^{nd} order in s for the integrand see Equation (10.6.19) and for integrand of order $2n - 1 = 2 \times 2 - 1 = 3$ we get exact solution.

Chapter 11

11.1

(a)



$$[B] = \frac{1}{6V}\begin{bmatrix} \beta_1 & 0 & 0 & \beta_2 & 0 & 0 & \beta_3 & 0 & 0 & \beta_4 & 0 & 0 \\ 0 & \gamma_1 & 0 & 0 & \gamma_2 & 0 & 0 & \gamma_3 & 0 & 0 & \gamma_4 & 0 \\ 0 & 0 & \delta_1 & 0 & 0 & \delta_2 & 0 & 0 & \delta_3 & 0 & 0 & \delta_4 \\ \gamma_1 & \beta_1 & 0 & \gamma_2 & \beta_2 & 0 & \gamma_3 & \beta_3 & 0 & \gamma_4 & \beta_4 & 0 \\ 0 & \delta_1 & \gamma_1 & 0 & \delta_2 & \gamma_2 & 0 & \delta_3 & \gamma_3 & 0 & \delta_4 & \gamma_4 \\ \delta_1 & 0 & \beta_1 & \delta_2 & 0 & \beta_2 & \delta_3 & 0 & \beta_3 & \delta_4 & 0 & \beta_4 \end{bmatrix}$$

By Equations (12.2.4) to (12.2.8)

$$\beta_1 = -\begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0,\ \beta_2 = \begin{vmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0,\ \beta_3 = -\begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 4$$

$$\beta_4 = \begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -4,\ \gamma_1 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

$$\gamma_2 = -\begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 4,\ \gamma_3 = \begin{vmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0,\ \gamma_4 = -\begin{vmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{vmatrix} = -4$$

$$\delta_1 = -\begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 4,\ \delta_2 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix},\ \delta_3 = -\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

$$\delta_4 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix} = -4$$

$$6V = \begin{vmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = (1)\,(-1)^2 \begin{vmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} + 2\,(-1)^3 \begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 8$$

$$[B] = \frac{1}{8}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 & 0 & -4 & -4 & 0 \\ 0 & 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & -4 & -4 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -4 & 0 & -4 \end{bmatrix}$$

Problem 11-1: '*B*' matrix for tetrahedral solid element

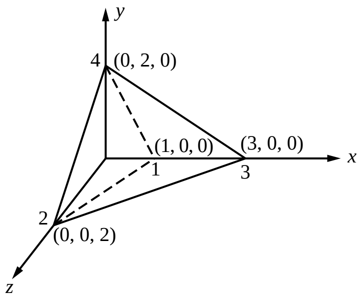
(b)

$x_1 = 1 \quad y_1 = 0 \quad z_1 = 0$

$x_2 = 0 \quad y_2 = 0 \quad z_2 = 2$

$x_3 = 3 \quad y_3 = 0 \quad z_3 = 0$

$x_4 = 0 \quad y_4 = 2 \quad z_4 = 0$



Geometry description (m)

$$\alpha_1 = \begin{pmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{pmatrix} \quad \beta_1 = -\begin{pmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{pmatrix} \quad \gamma_1 = \begin{pmatrix} 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{pmatrix} \quad \delta_1 = -\begin{pmatrix} 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{pmatrix}$$

$$\alpha_2 = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{pmatrix} \quad \beta_2 = \begin{pmatrix} 1 & y_1 & z_1 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{pmatrix} \quad \gamma_2 = -\begin{pmatrix} 1 & x_1 & z_1 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{pmatrix} \quad \delta_2 = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{pmatrix}$$

$$\alpha_3 = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{pmatrix} \quad \beta_3 = -\begin{pmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_4 & z_4 \end{pmatrix} \quad \gamma_3 = \begin{pmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_4 & z_4 \end{pmatrix} \quad \delta_3 = -\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_4 & y_4 \end{pmatrix}$$

$$\alpha_4 = -\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \quad \beta_4 = \begin{pmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{pmatrix} \quad \gamma_4 = -\begin{pmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \end{pmatrix} \quad \delta_4 = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}$$

$$V = \frac{1}{6}\left|\begin{pmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{pmatrix}\right| \qquad V = 1.333$$

$$[B_1] = \frac{1}{6V}\begin{pmatrix} |\beta_1| & 0 & 0 \\ 0 & |\gamma_1| & 0 \\ 0 & 0 & |\delta_1| \\ |\gamma_1| & |\beta_1| & 0 \\ 0 & |\delta_1| & |\gamma_1| \\ |\delta_1| & 0 & |\beta_1| \end{pmatrix} \quad [B_2] = \frac{1}{6V}\cdot\begin{pmatrix} |\beta_2| & 0 & 0 \\ 0 & |\gamma_2| & 0 \\ 0 & 0 & |\delta_2| \\ |\gamma_2| & |\beta_2| & 0 \\ 0 & |\delta_2| & |\gamma_2| \\ |\delta_2| & 0 & |\beta_2| \end{pmatrix}$$

$$[B_3] = \frac{1}{6V}\begin{pmatrix} |\beta_3| & 0 & 0 \\ 0 & |\gamma_3| & 0 \\ 0 & 0 & |\delta_3| \\ |\gamma_3| & |\beta_3| & 0 \\ 0 & |\delta_3| & |\gamma_3| \\ |\delta_3| & 0 & |\beta_3| \end{pmatrix} \quad [B_4] = \frac{1}{6V}\begin{pmatrix} |\beta_4| & 0 & 0 \\ 0 & |\gamma_4| & 0 \\ 0 & 0 & |\delta_4| \\ |\gamma_4| & |\beta_4| & 0 \\ 0 & |\delta_4| & |\gamma_4| \\ |\delta_4| & 0 & |\beta_4| \end{pmatrix}$$

$[B]$ = augment ($[B_1]$, $[B_2]$, $[B_3]$, $[B_4]$)

$[B]$ =

−0.5	0	0	0	0	0	0.5	0	0	0	0	0
0	−0.75	0	0	0	0	0	0.25	0	0	0.5	0
0	0	−0.75	0	0	0.5	0	0	0.25	0	0	0
−0.75	−0.5	0	0	0	0	0.25	0.5	0	0.5	0	0
0	−0.75	−0.75	0	0.5	0	0	0.25	0.25	0	0	0.5
−0.75	0	−0.5	0.5	0	0	0.25	0	0.5	0	0	0

11.2(a) Use Equation (11.2.18) and substitute $[B]$ from 11.1 (a) and $[D]$ from Equation (11.1.5) into Equation (11.2.18)

$$\therefore \quad [k] = [B]^T [D] [B] V$$

[B] =

	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0	0	0	0	0	0.5	0	0	–0.5	0	0
2	0	0	0	0	0.5	0	0	0	0	0	–0.5	0
3	0	0	0.5	0	0	0	0	0	0	0	0	–0.5
4	0	0	0	0.5	0	0	0	0.5	0	–0.5	–0.5	0
5	0	0.5	0	0	0	0	0	0	0.5	0	–0.5	–0.5
6	0.5	0	0	0	0	0	0	0	0.5	–0.5	0	–0.5

$v = 0.3 \qquad E = 30 \times 10^6$

$$[D] = \frac{E}{(1+v)(1-2v)} \begin{pmatrix} 1-v & v & v & 0 & 0 & 0 \\ 0 & 1-v & v & 0 & 0 & 0 \\ 0 & 0 & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{pmatrix}$$

$$[k] = [B]^T\ [D]\ [B]\ V$$

$[k] =$

	1	2	3	4	5	6	7	8	9	10	11	12
1	3.846	0	0	0	0	0	0	0	3.846	–3.846	0	–3.846
2	0	3.846	0	0	0	0	0	0	3.846	0	–3.846	–3.846
3	0	0	13.462	0	0	0	0	0	0	0	0	13.462
4	0	0	0	3.846	0	0	0	3.846	0	– 3.846	– 3.846	0
5	0	0	5.769	0	13.462	0	0	0	0	0	13.462	–5.769
6	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	5.769	0	5.769	0	13.462	0	0	13.462	–5.769	–5.769
8	0	0	0	3.846	0	0	0	3.846	0	–3.846	–3.846	0
9	3.846	3.846	0	0	0	0	0	0	7.692	–3.846	–3.846	–7.692
10	–3.846	0	–5.769	–3.846	–5.769	0	13.462	–3.846	–3.846	21.154	9.615	9.615
11	0	–3.846	–5.769	–3.846	13.462	0	0	–3.846	–3.846	3.846	21.154	9.615
12	–3.846	–3.846	13.462	0	0	0	0	0	–7.692	3.846	3.846	21.154

$$\times 10^6\ \frac{\text{lb}}{\text{in.}}$$

(b) Evaluate the stiffness matrix for the element shown

$$[D] = \frac{E}{(1+v)(1-2v)} \begin{pmatrix} 1-v & v & v & 0 & 0 & 0 \\ 0 & 1-v & v & 0 & 0 & 0 \\ 0 & 0 & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{pmatrix}$$

$$[k] = [B]^T\ [D]\ [B]\ V$$

$[k] =$

	0	1	2	3	4	5	6	7	8	9	10	11
0	$3.077\cdot10^7$	$1.442\cdot10^7$	$1.442\cdot10^7$	$-5.769\cdot10^6$	0	$-5.769\cdot10^6$	$-1.923\cdot10^7$	$-8.654\cdot10^6$	$-8.654\cdot10^6$	$-5.769\cdot10^6$	$-5.769\cdot10^6$	0
1	$1.442\cdot10^7$	$4.279.10^7$	$2.163\cdot10^7$	0	$-5.769\cdot10^6$	$-8.654\cdot10^6$	$-1.058\cdot10^7$	$-1.683\cdot10^7$	$-7.212\cdot10^6$	$-3.846\cdot10^6$	$-2.019\cdot10^7$	$-5.769\cdot10^6$
2	$1.442\cdot10^7$	$2.163\cdot10^7$	$4.279\cdot10^7$	$-3.846\cdot10^6$	$-5.769\cdot10^6$	$-2.019\cdot10^7$	$-1.058\cdot10^7$	$-7.212\cdot10^6$	$-1.683\cdot10^7$	0	$-8.654\cdot10^6$	$-5.769\cdot10^6$
3	$-5.769\cdot10^6$	0	$-3.846\cdot10^6$	$3.846\cdot10^6$	0	0	$1.923\cdot10^6$	0	$3.846\cdot10^6$	0	0	0
4	0	$-5.769\cdot10^6$	$-5.769\cdot10^6$	0	$3.846\cdot10^6$	0	0	$1.923\cdot10^6$	$1.923\cdot10^6$	0	0	$3.846\cdot10^6$
5	$-5.769\cdot10^6$	$-8.654\cdot10^6$	$-2.019\cdot10^7$	0	0	$1.346\cdot10^7$	$5.769\cdot10^6$	$2.885\cdot10^6$	$6.731\cdot10^6$	0	$5.769\cdot10^6$	0
6	$-1.923\cdot10^7$	$-1.058\cdot10^7$	$-1.058\cdot10^7$	$1.923\cdot10^6$	0	$5.769\cdot10^6$	$1.538\cdot10^7$	$4.808\cdot10^6$	$4.808\cdot10^6$	$1.923\cdot10^6$	$5.769\cdot10^6$	0
7	$-8.654\cdot10^6$	$-1.683\cdot10^7$	$-7.212\cdot10^6$	0	$1.923\cdot10^6$	$2.885\cdot10^6$	$4.808\cdot10^6$	$8.173\cdot10^6$	$2.404\cdot10^6$	$3.846\cdot10^6$	$6.731\cdot10^6$	$1.923\cdot10^6$
8	$-8.654\cdot10^6$	$-7.212\cdot10^6$	$-1.683\cdot10^7$	$3.846\cdot10^6$	$1.923\cdot10^6$	$6.731\cdot10^6$	$4.808\cdot10^6$	$2.404\cdot10^6$	$8.173\cdot10^6$	0	$2.885\cdot10^6$	$1.923\cdot10^6$
9	$-5.769\cdot10^6$	$-3.846.10^6$	0	0	0	0	$1.923\cdot10^6$	$3.846\cdot10^6$	0	$3.846\cdot10^6$	0	0
10	$-5.769\cdot10^6$	$-2.019\cdot10^7$	$-8.654\cdot10^6$	0	0	$5.769\cdot10^6$	$5.769\cdot10^6$	$6.731\cdot10^6$	$2.885\cdot10^6$	0	$1.346\cdot10^7$	0
11	0	$-5.769\cdot10^6$	$-5.769\cdot10^6$	0	$3.846\cdot10^6$	0	0	$1.923\cdot10^6$	$1.923\cdot10^6$	0	0	$3.846\cdot10^6$

11.3 (a) $\{\varepsilon\} = [B]\,\{d\}$

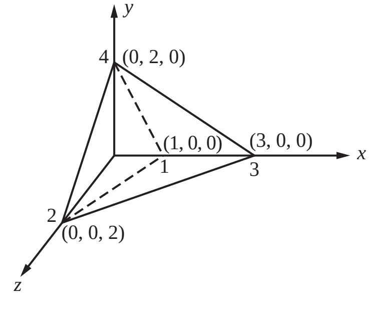
$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{Bmatrix} = \frac{1}{8}\begin{bmatrix}0&0&0&0&0&0&4&0&0&-4&0&0\\ 0&0&0&0&4&0&0&0&0&0&4&0\\ 0&0&4&0&0&0&0&0&0&0&0&-4\\ 0&0&0&4&0&0&0&4&0&-4&-4&0\\ 0&4&0&0&0&4&0&0&0&0&-4&-4\\ 4&0&0&0&0&0&0&0&4&-4&0&-4\end{bmatrix}$$

$$\times\begin{Bmatrix}0.005\\ 0.0\\ 0.0\\ 0.001\\ 0.0\\ 0.001\\ 0.005\\ 0.0\\ 0.0\\ -0.001\\ 0.0\\ 0.005\end{Bmatrix} = \begin{Bmatrix}0.003\\ 0.0\\ -0.0025\\ 0.001\\ -0.002\\ 0.0005\end{Bmatrix}\frac{\text{in.}}{\text{in.}}$$

$\{\sigma\} = [D]\,\{\varepsilon\}$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{Bmatrix} = \frac{30\times10^3}{(1+0.3)(1-2(0.3))}\begin{bmatrix}0.7&0.3&0.3&0&0&0\\ 0.3&0.7&0.3&0&0&0\\ 0.3&0.3&0.7&0&0&0\\ 0&0&0&0.2&0&0\\ 0&0&0&0&0.2&0\\ 0&0&0&0&0&0.2\end{bmatrix}\begin{Bmatrix}0.003\\ 0.0\\ -0.0025\\ 0.001\\ -0.002\\ 0.0005\end{Bmatrix} = \begin{Bmatrix}77.9\\ 8.65\\ -49.0\\ 11.5\\ -23.1\\ 5.77\end{Bmatrix}\text{ ksi}$$

(b)



Nodal displacements (in.)

$u_1 = 0.005 \quad v_1 = 0 \quad w_1 = 0$

$u_2 = 0.001 \quad v_2 = 0 \quad w_2 = 0.001$

$u_3 = 0.005 \quad v_3 = 0 \quad w_3 = 0$

$u_4 = -0.001 \quad v_4 = 0 \quad w_4 = 0.005$

$$\{d_1\} = \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} \quad \{d_2\} = \begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix} \quad \{d_3\} = \begin{pmatrix} u_3 \\ v_3 \\ w_3 \end{pmatrix} \quad \{d_4\} = \begin{pmatrix} u_4 \\ v_4 \\ w_4 \end{pmatrix}$$

$$\{d\} = \text{stack}\,(\{d_1\}, \{d_2\}, \{d_3\}, \{d_4\})$$

Material properties

$$E = 30 \times 10^6 \qquad v = 0.3$$

Element strain matrix ([B] from P11.b)

$$\{\varepsilon\} = [B]\,\{d\}$$

$$\varepsilon = \begin{pmatrix} 0 \\ 0 \\ 5\times10^{-4} \\ -3\times10^{-3} \\ 2.5\times10^{-3} \\ -2\times10^{-3} \end{pmatrix} \frac{\text{in.}}{\text{in.}}$$

Determine constitutive matrix

$$[D] = \frac{E}{(1+v)(1-2v)} \begin{pmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{pmatrix}$$

Determine element stresses

$$\{\sigma\} = [D]\,\{\varepsilon\}$$

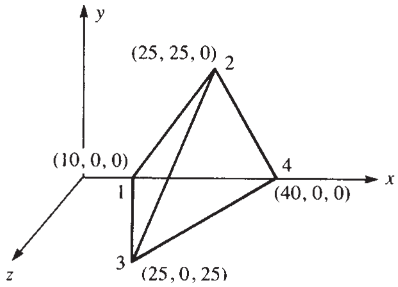
$$\{\sigma\} = \begin{Bmatrix} 8.654\times10^3 \\ 8.654\times10^3 \\ 2.019\times10^4 \\ -3.462\times10^4 \\ 2.885\times10^4 \\ -2.308\times10^4 \end{Bmatrix} \text{(psi)}$$

11.4 The strains and stress are constant in the 4-noded tetrahedral element.

11.5 Use Equation (11.2.10) for the shape functions $N_1 - N_4$. Substitute $[N]_{3\,\times 12}$ from Equation (11.2.9) and $\{X\}$ from Equation (11.2.20a) into Equation (11.2.19), $\{f_b\}_{12\,\times\,1}$ = $\iint_V \int [N]^T_{12\times3} \begin{Bmatrix} X_b \\ Y_b \\ Z_b \end{Bmatrix}$ to show at node i, $\{f_{b_i}\} = \frac{V}{4}\begin{Bmatrix} X_b \\ Y_b \\ Z_b \end{Bmatrix}$.

11.6

(a)



$$\beta_1 = -\begin{vmatrix} 1 & 25 & 0 \\ 1 & 0 & 25 \\ 1 & 0 & 0 \end{vmatrix} = -625, \qquad \beta_2 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 25 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

$$\beta_3 = -\begin{vmatrix} 1 & 0 & 0 \\ 1 & 25 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0, \qquad \beta_4 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 25 & 0 \\ 1 & 0 & 25 \end{vmatrix} = 625$$

$$\gamma_1 = \begin{vmatrix} 1 & 25 & 0 \\ 1 & 25 & 25 \\ 1 & 40 & 0 \end{vmatrix} = -375, \qquad \gamma_2 = -\begin{vmatrix} 1 & 10 & 0 \\ 1 & 0 & 25 \\ 1 & 40 & 0 \end{vmatrix} = 750$$

$$\gamma_3 = \begin{vmatrix} 1 & 10 & 0 \\ 1 & 25 & 0 \\ 1 & 40 & 0 \end{vmatrix} = 0, \qquad \gamma_4 = -\begin{vmatrix} 1 & 10 & 0 \\ 1 & 25 & 0 \\ 1 & 25 & 25 \end{vmatrix} = -375$$

$$\delta_1 = -\begin{vmatrix} 1 & 25 & 25 \\ 1 & 25 & 0 \\ 1 & 40 & 0 \end{vmatrix} = -375, \ \delta_2 = \begin{vmatrix} 1 & 10 & 0 \\ 1 & 25 & 0 \\ 1 & 40 & 0 \end{vmatrix} = 0$$

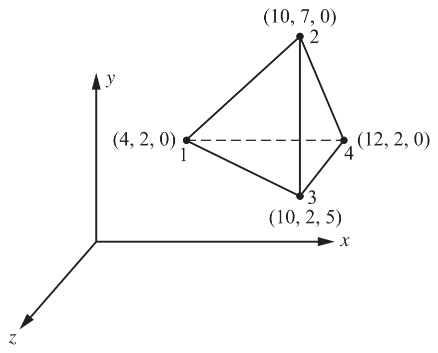
$$\delta_3 = -\begin{vmatrix} 1 & 10 & 0 \\ 1 & 25 & 25 \\ 1 & 40 & 0 \end{vmatrix} = 750, \ \delta_4 = \begin{vmatrix} 1 & 10 & 0 \\ 1 & 25 & 25 \\ 1 & 25 & 0 \end{vmatrix} = -375$$

$$6V = \begin{vmatrix} 1 & 10 & 0 & 0 \\ 1 & 25 & 25 & 0 \\ 1 & 25 & 0 & 25 \\ 1 & 40 & 0 & 0 \end{vmatrix} = 1\,(-1)^2 \begin{vmatrix} 25 & 25 & 0 \\ 25 & 0 & 25 \\ 40 & 0 & 0 \end{vmatrix}$$

$$+\,10(-1)^3 \begin{vmatrix} 1 & 25 & 0 \\ 1 & 0 & 25 \\ 1 & 0 & 0 \end{vmatrix} = 18750$$

$$[B] = \frac{1}{18750}\begin{bmatrix} -625 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 625 & 0 & 0 \\ 0 & -375 & 0 & 0 & 750 & 0 & 0 & 0 & 0 & 0 & -375 & 0 \\ 0 & 0 & -375 & 0 & 0 & 0 & 0 & 0 & 750 & 0 & 0 & -375 \\ -375 & -625 & 0 & 750 & 0 & 0 & 0 & 0 & 0 & -375 & 625 & 0 \\ 0 & -375 & -375 & 0 & 0 & 750 & 0 & 750 & 0 & 0 & -375 & -375 \\ -375 & 0 & -625 & 0 & 0 & 0 & 750 & 0 & 0 & -375 & 0 & 625 \end{bmatrix}$$

(b)



$x_1 = 4 \qquad x_2 = 10 \qquad x_3 = 10 \qquad x_4 = 12$

$y_1 = 2 \qquad y_2 = 7 \qquad y_3 = 2 \qquad y_4 = 2$

$z_1 = 0 \qquad z_2 = 0 \qquad z_3 = 5 \qquad z_4 = 0$

$$v = \frac{1}{6}\left|\begin{pmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{pmatrix}\right|$$

$$\alpha_1 = \left|\begin{pmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{pmatrix}\right| \quad \beta_1 = -\left|\begin{pmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{pmatrix}\right| \quad \gamma_1 = \left|\begin{pmatrix} 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{pmatrix}\right| \quad \delta_1 = -\left|\begin{pmatrix} 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{pmatrix}\right|$$

$$\alpha_2 = -\left|\begin{pmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{pmatrix}\right| \quad \beta_2 = \left|\begin{pmatrix} 1 & y_1 & z_1 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{pmatrix}\right| \quad \gamma_2 = -\left|\begin{pmatrix} 1 & x_1 & z_1 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{pmatrix}\right| \quad \delta_2 = \left|\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{pmatrix}\right|$$

$$\alpha_3 = \left|\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{pmatrix}\right| \quad \beta_3 = -\left|\begin{pmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_4 & z_4 \end{pmatrix}\right| \quad \gamma_3 = \left|\begin{pmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_4 & z_4 \end{pmatrix}\right| \quad \delta_3 = -\left|\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_4 & y_4 \end{pmatrix}\right|$$

$$\alpha_4 = -\left|\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}\right| \quad \beta_4 = \left|\begin{pmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{pmatrix}\right| \quad \gamma_4 = -\left|\begin{pmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \end{pmatrix}\right| \quad \delta_4 = \left|\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}\right|$$

$$[B_1] = \frac{1}{6V}\begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \gamma_1 & 0 \\ 0 & 0 & \delta_1 \\ \gamma_1 & \beta_1 & 0 \\ 0 & \delta_1 & \gamma_1 \\ \delta_1 & 0 & \beta_1 \end{pmatrix} [B_2] = \frac{1}{6V}\begin{pmatrix} \beta_2 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \delta_2 \\ \gamma_2 & \beta_2 & 0 \\ 0 & \delta_2 & \gamma_2 \\ \delta_2 & 0 & \beta_2 \end{pmatrix} [B_3] = \frac{1}{6V}\begin{pmatrix} \beta_3 & 0 & 0 \\ 0 & \gamma_3 & 0 \\ 0 & 0 & \delta_3 \\ \gamma_3 & \beta_3 & 0 \\ 0 & \delta_3 & \gamma_3 \\ \delta_3 & 0 & \beta_3 \end{pmatrix}$$

$$[B_4] = \frac{1}{6v}\begin{pmatrix} \beta_4 & 0 & 0 \\ 0 & \gamma_4 & 0 \\ 0 & 0 & \delta_4 \\ \gamma_4 & \beta_4 & 0 \\ 0 & \delta_4 & \gamma_4 \\ \delta_4 & 0 & \beta_4 \end{pmatrix}$$

$$[B] = ([B_1]\ [B_2]\ [B_3]\ [B_4])$$

$$[B] = \left[\begin{pmatrix} -0.125 & 0 & 0 \\ 0 & -0.05 & 0 \\ 0 & 0 & -0.05 \\ -0.05 & -0.125 & 0 \\ 0 & -0.05 & -0.05 \\ -0.05 & 0 & -0.125 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0 \\ 0.2 & 0 & 0 \\ 0 & 0 & 0.2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.2 \\ 0 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0.2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0.125 & 0 & 0 \\ 0 & -0.15 & 0 \\ 0 & 0 & -0.15 \\ -0.15 & 0.125 & 0 \\ 0 & -0.15 & -0.15 \\ -0.15 & 0 & 0.125 \end{pmatrix}\right]$$

11.7 **(a)** $\{\varepsilon\} = [B]\ \{d\}$

(see $[B]$ from Problem 11.6 above)

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = [B] \begin{Bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.01 \\ 0.02 \\ 0.01 \\ 0.02 \\ 0.01 \\ 0.005 \\ 0.0 \\ 0.01 \\ 0.01 \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.0006 \\ 0.0 \\ 0.000733 \\ 0.0004 \\ 0.00113 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{210\times 10^9}{(1+0.3)(1-2(0.3))}\begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0.3 & 0 & 0 & 0 \\ 0.3 & 0.3 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.0006 \\ 0 \\ 0.00073 \\ 0.0004 \\ 0.00113 \end{Bmatrix}$$

$$= \begin{Bmatrix} 72.7 \\ 169.6 \\ 72.7 \\ 59.2 \\ 32.3 \\ 91.5 \end{Bmatrix} \text{MPa}$$

11.7 (b)

$$\begin{array}{lll} u_1 = 0 & v_1 = 0 & w_1 = 0 \\ u_2 = 0.00001 & v_2 = 0.00002 & w_2 = 0.00001 \\ u_3 = 0.00002 & v_3 = 0.00001 & w_3 = 0.000005 \\ u_4 = 0 & v_4 = 0.00001 & w_4 = 0.00001 \end{array}$$

$$[D] = \frac{E}{(1+v)(1-2v)} \begin{pmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{pmatrix}$$

$[B] =$

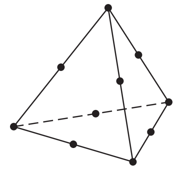
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	–125	0	0	0	0	0	0	0	0	125
1	0	–50	0	0	200	0	0	0	0	0
2	0	0	–50	0	0	0	0	0	200	0
3	–50	–125	0	200	0	0	0	0	0	–150
4	0	–50	–50	0	0	200	0	200	0	0
5	–50	0	–125	0	0	0	200	0	0	–150

$$\text{disp} = \begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{pmatrix} \quad \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{pmatrix} = [D]\ [B]\ \{\text{disp}\}$$

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{pmatrix} = \begin{pmatrix} 1.963 \times 10^3 \\ 5.236 \times 10^3 \\ 1.309 \times 10^3 \\ 2.127 \times 10^3 \\ 654.48 \\ 3.436 \times 10^3 \end{pmatrix}$$

11.8

$$\begin{aligned} u &= a_1 + a_2 x + a_3 y + a_4 z \\ &\quad + a_5 xy + a_6 xz + a_7 yz \\ &\quad + a_8 x^2 + a_9 y^2 + a_{10} z^2 \end{aligned}$$

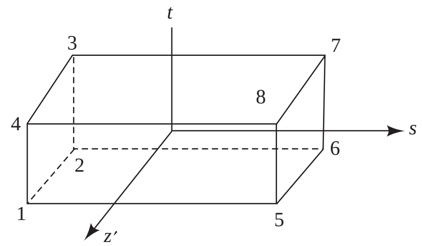


Similar expressions for v and w.

Figure P11-8

11.9 Loads must be in the y-z plane (in the plane of the plane elements).

11.10



Using Equation (11.3.3) and Figure 11.5

$$N_i = \frac{(1+ss_i)(1+tt_i)(1+z'z_i')}{8}$$

$$N_1 = \frac{(1-s)(1-t)(1+z')}{8}, \ (s_1 = -1,\ t_1 = -1,\ z_1 = 1)$$

$$N_2 = \frac{(1-s)(1-t)(1-z')}{8}, \ (s_2 = -1,\ t_2 = -1,\ z'_2 = -1)$$

$$N_3 = \frac{(1-s)(1+t)(1-z')}{8}, \ (s_3 = -1,\ t_3 = 1,\ z_3 = -1)$$

$$N_4 = \frac{(1-s)(1+t)(1+z')}{8}, \ (s_4 = -1,\ t_4 = 1,\ z'_4 = 1)$$

$$N_5 = \frac{(1+s)(1-t)(1+z')}{8}, \ (s_5 = 1,\ t_5 = -1,\ z'_5 = 1)$$

$$N_6 = \frac{(1+s)(1+t)(1-z')}{8}, \ (s_6 = 1,\ t_6 = -1,\ z'_6 = -1)$$

$$N_7 = \frac{(1+s)(1+t)(1-z')}{8}, \ (s_7 = 1,\ t_7 = 1,\ z'_7 = -1)$$

$$N_8 = \frac{(1+s)(1+t)(1+z')}{8}, (s_8 = 1, t_8 = 1, z'_8 = 1)$$

11.11 Quadratic hexahedral element (see Figure 11.6)
By Equation (11.3.11)

$$N_i = \frac{(1+ss_i)(1+tt_i)(1+z'z'_i)}{8}\ (ss_i + tt_i + z'z'_i - 2)$$

Node 1

$$s_1 = -1, t_1 = -1, z'_1 = 1$$

$$N_1 = \frac{(1-s)(1-t)(1+z')}{8}\ (-s - t + z' - 2)$$

Node 2

$$s_2 = -1, t_2 = -1, z'_2 = -1$$

$$N_2 = \frac{(1-s)(1-t)(1-z')}{8}\ (-s - t - z' - 2)$$

Node 3

$$s_3 = -1, t_3 = 1, z'_3 = -1$$

$$N_3 = \frac{(1-s)(1+t)(1-z')}{8}\ (-s + t - z' - 2)$$

Node 4

$$s_4 = -1, t_4 = 1, z'_4 = 1$$

$$N_4 = \frac{(1-s)(1+t)(1+z')}{8}\ (-s + t + z' - 2)$$

Node 5

$$s_5 = 1, t_5 = -1, z'_5 = 1$$

$$N_5 = \frac{(1+s)(1-t)(1+z')}{8}\ (s - t + z' - 2)$$

Node 6

$$s_6 = 1, t_6 = -1, z'_6 = -1$$

$$N_6 = \frac{(1+s)(1-t)(1-z')}{8}\ (s - t - z' - 2)$$

Node 7

$$s_7 = 1, t_7 = 1, z'_7 = -1$$

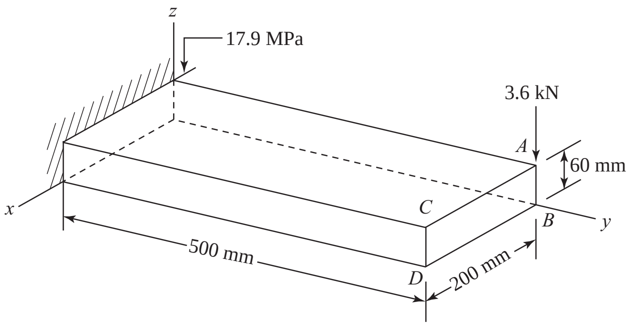
$$N_7 = \frac{(1+s)(1+t)(1-z')}{8}\ (s + t - z' - 2)$$

Node 8

$$s_8 = 1,\ t_8 = 1,\ z'_8 = 1$$

$$N_8 = \frac{(1+s)(1+t)(1+z')}{8}\ (s + t + z' - 2)$$

11.13



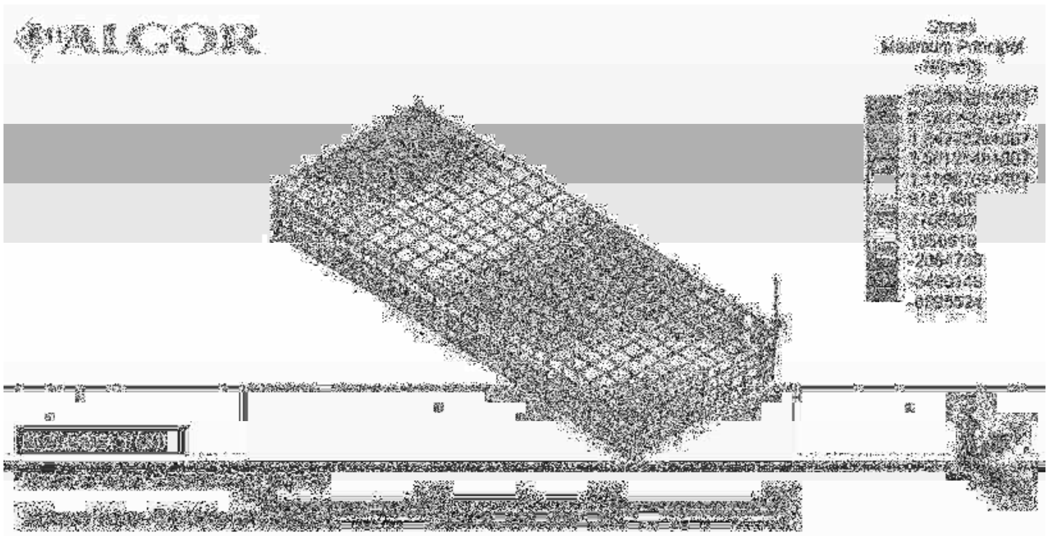
Outer free face corner node deflections (z-direction)

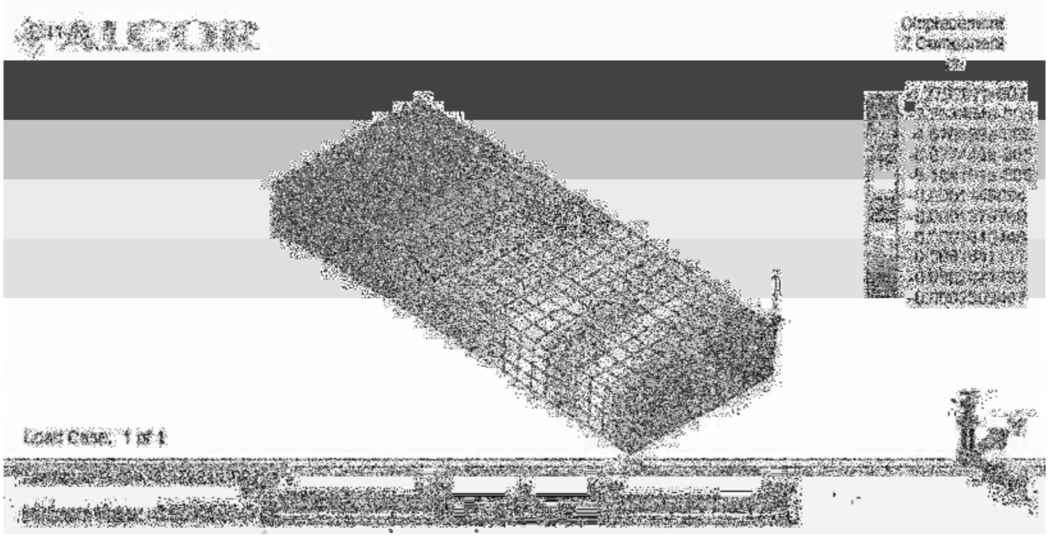
Pt A = – 0.000231 m Pt B = – 0.000224 m Pt C = – 0.000187 m Pt D = – 0.000187 m

Mechanics of materials, $\delta = \dfrac{PL^3}{3EI} = -\,0.000208$ m, where $I = \dfrac{(0.2)\,(0.06)^3}{12} = 3.6 \times 10^{-6}\ \text{m}^4$

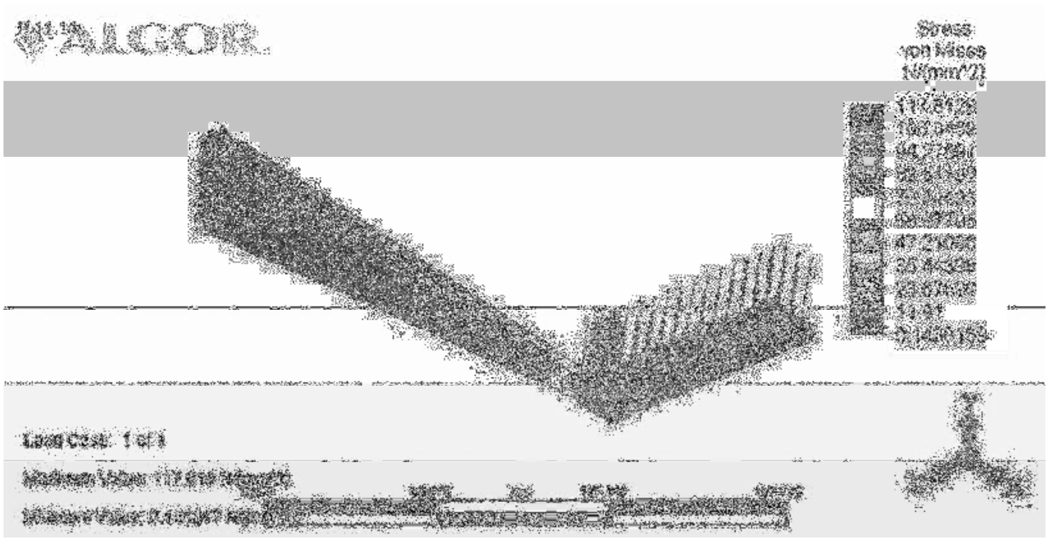
$L = 0.5$ m, $P = 3600$ N, $E = 200 \times 10^9\ \dfrac{\text{N}}{\text{m}^2}$

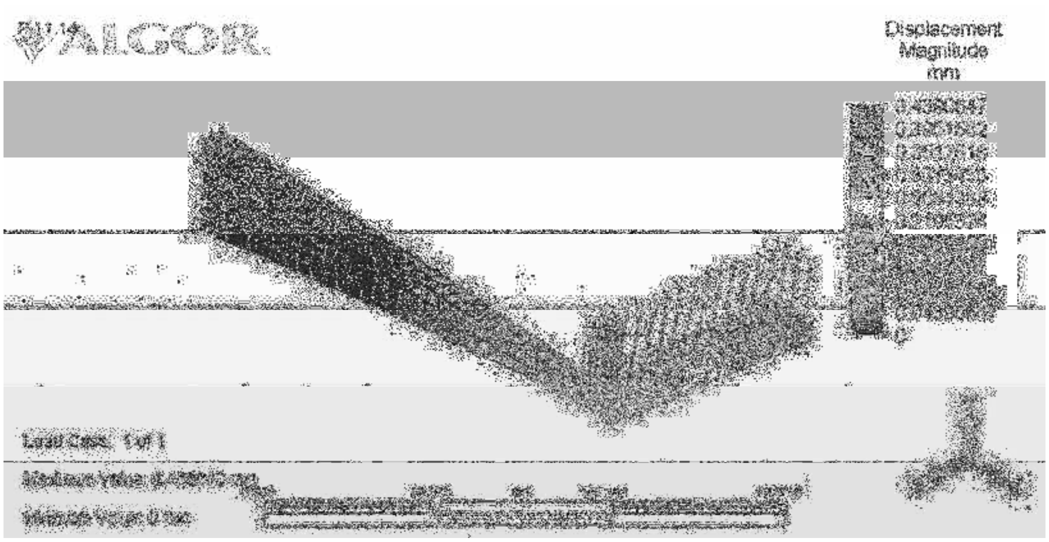
The corner node answers from computer program Algor. Note the classical mechanics of materials solution gives the maximum deflection for a load applied through the centroid not offset.





11.14 (Load replaced with concentrated end load)





11.15

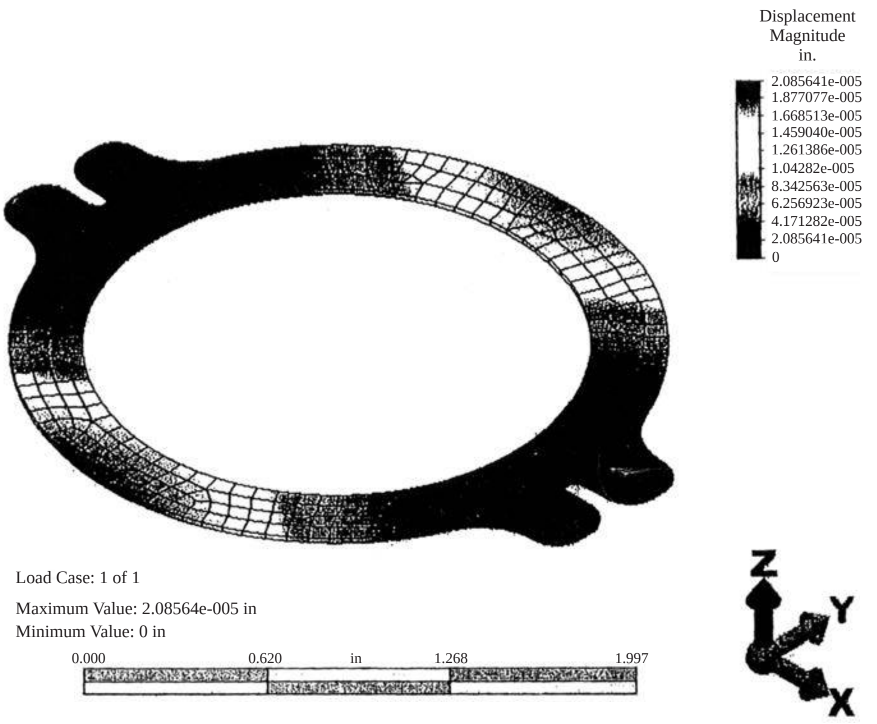


Figure 5 Flap valve with maximum displacement of 2.09×10^{-5} in.

Conclusion and recommendations

The applied pressure of 2.318 psi on the annular region of the flap valve with the clip ears supported on both sides, resulted in a maximum von Mises stress of 31,001 psi. The safety factor with the applied pressure is 2.0. It is recommended to use a pressure no greater than 2,318 psi during the operation of the compressor with this flap valve.

11.16

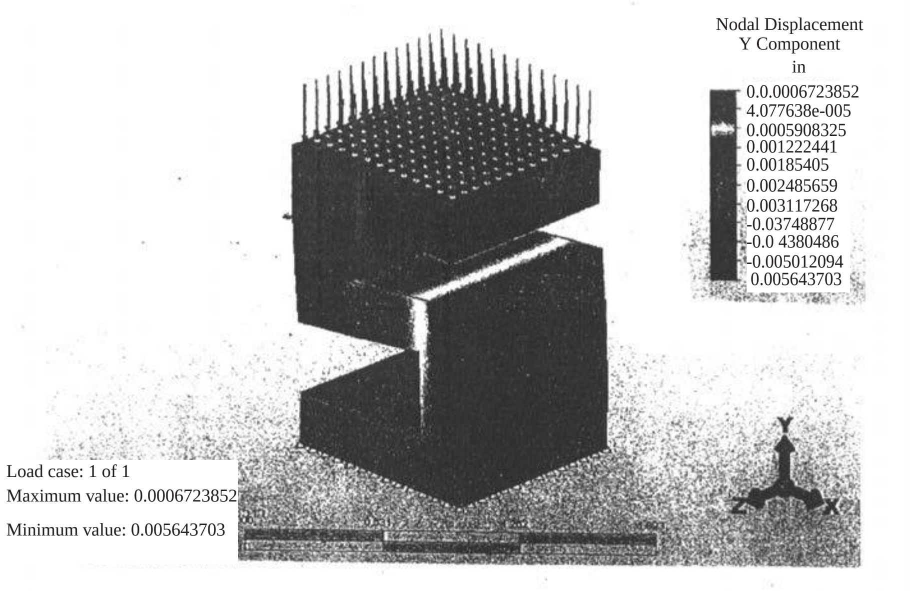


Figure 1 Displacement of our designed s-block

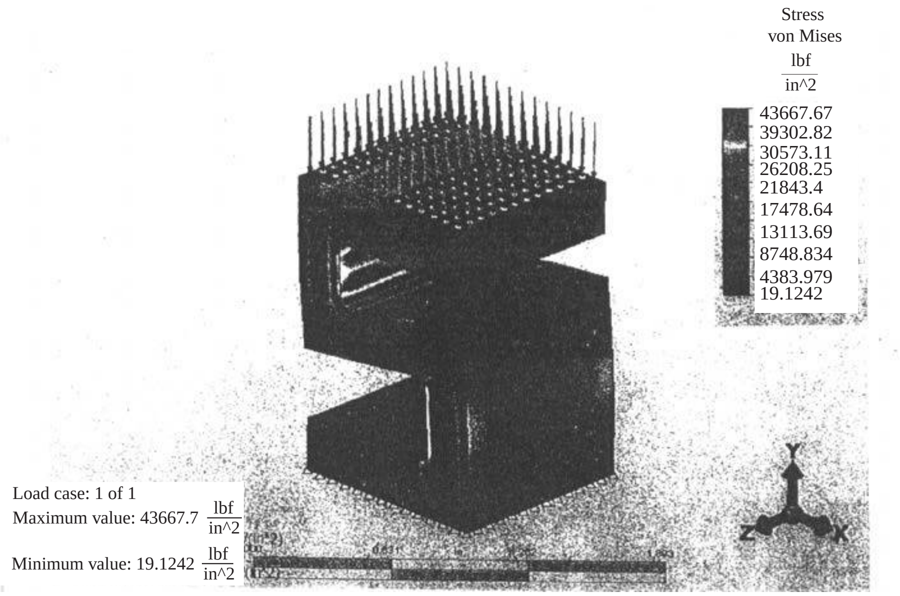


Figure 2 von Mises stress on our designed s-block
(Final thickness = 0.25 in.)

11.17

Determine the thickness of the device such that the maximum deflection is 0.1 in. vertically.

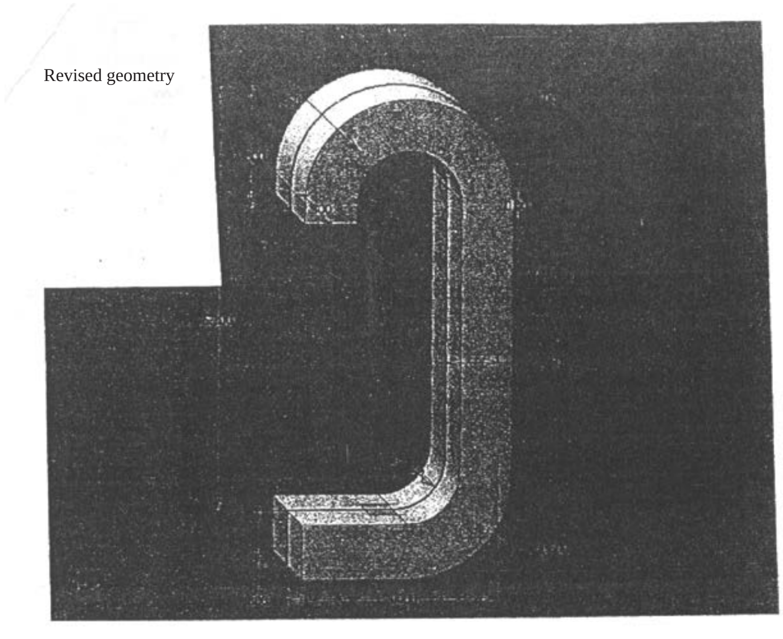


Figure 1 Inventor Model Dimensions
(Final thickness = 0.75 in.)

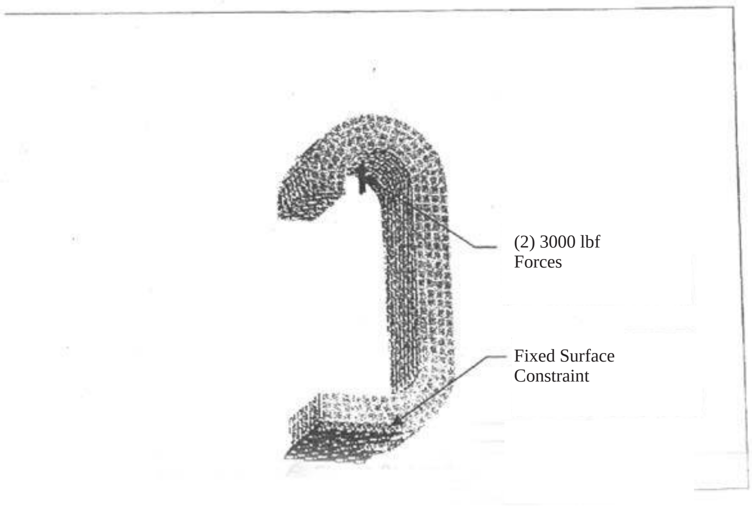


Figure 2 Loads and Constraints

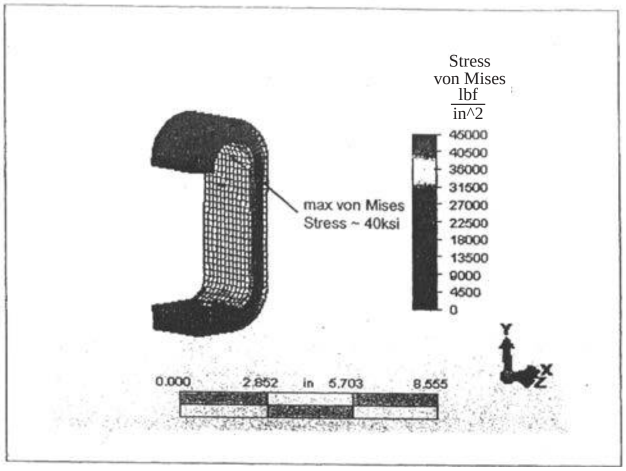


Figure 3 von Mises Stress Distribution (0.75 in. thick)

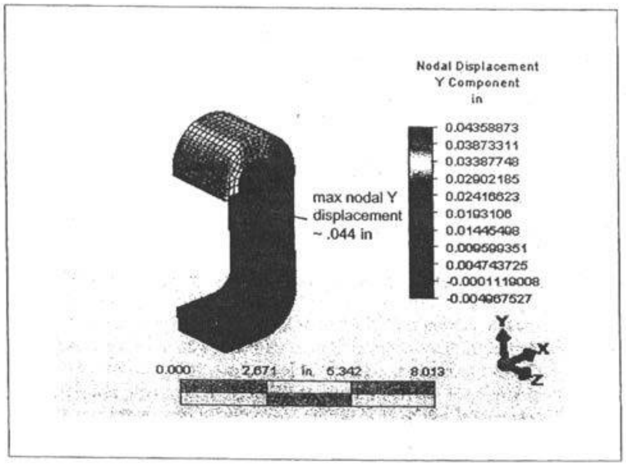


Figure 4 Vertical Displacement Distributions
Maximum displacement = 0.0436 in.

11.18

Model Variables

Variable	Value
Material	1035 quenched & tempered steel
Modulus of Elasticity	200 MPa
Force	150 N
Yield Strength	615 MPa
Maximum von Mises Stress	758 MPa
Maximum Displacement	4.13 mm

Algor Results

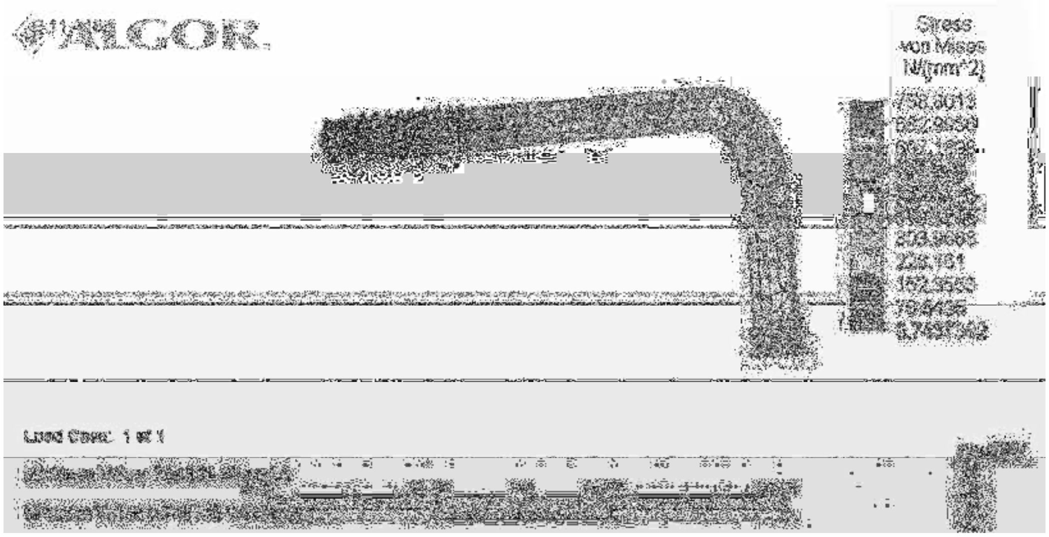


Figure 4 von Mises Stress (MPa)

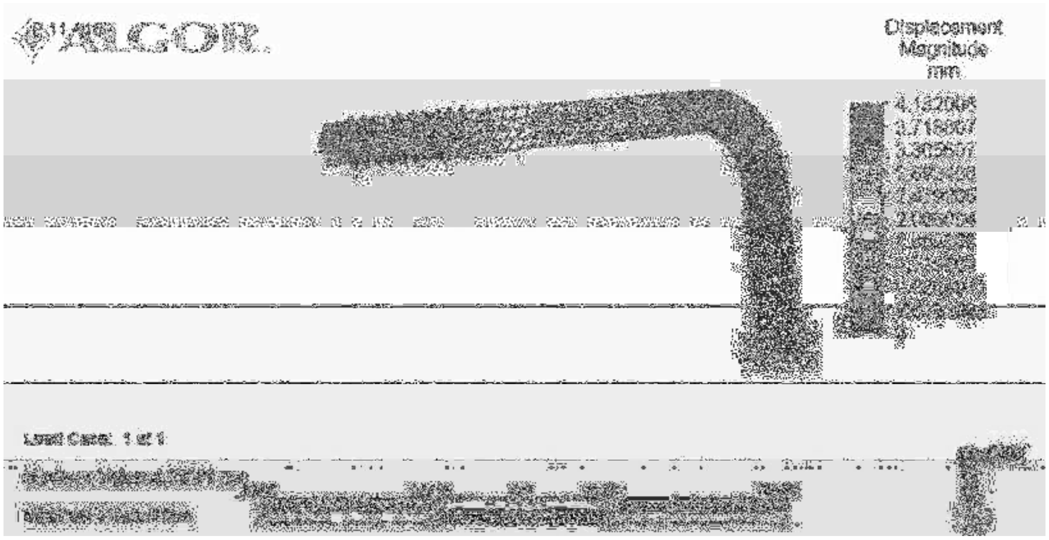
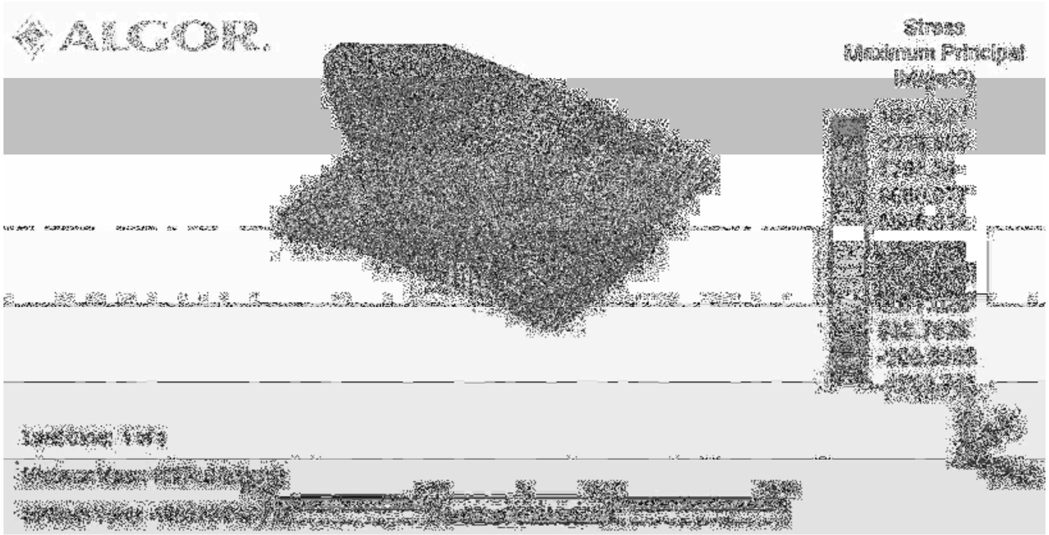


Figure 5 Displacement (mm)

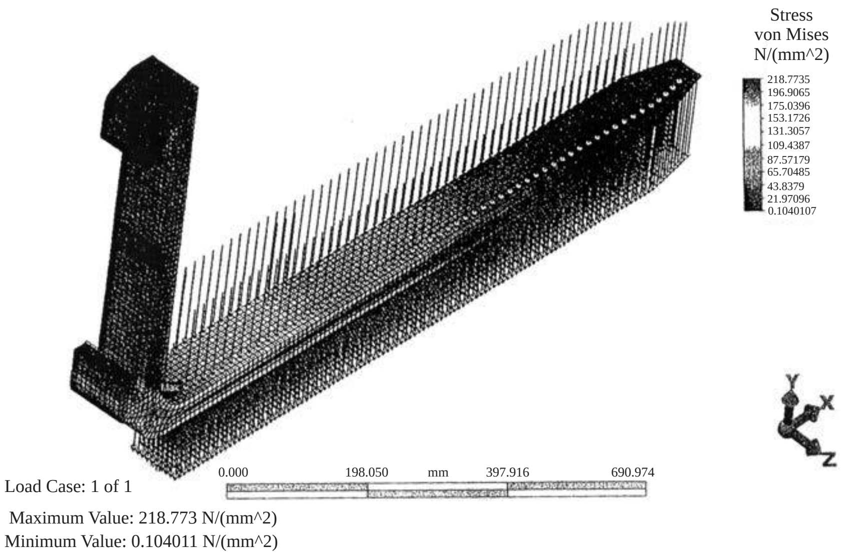
11.19



Maximum principal stress is 10080 psi.

11.20

von Mises Stress



The yield strength of AISI 4130 is approximately 360.69 MPa (from eFunda). Converting the maximum von Mises stress of 218.8 $\frac{\text{N}}{\text{mm}^2}$ into MPa gives approximately 218.8 MPa. This is over 100 MPa below the yield strength, so this will not fail due to static loading.

11.21

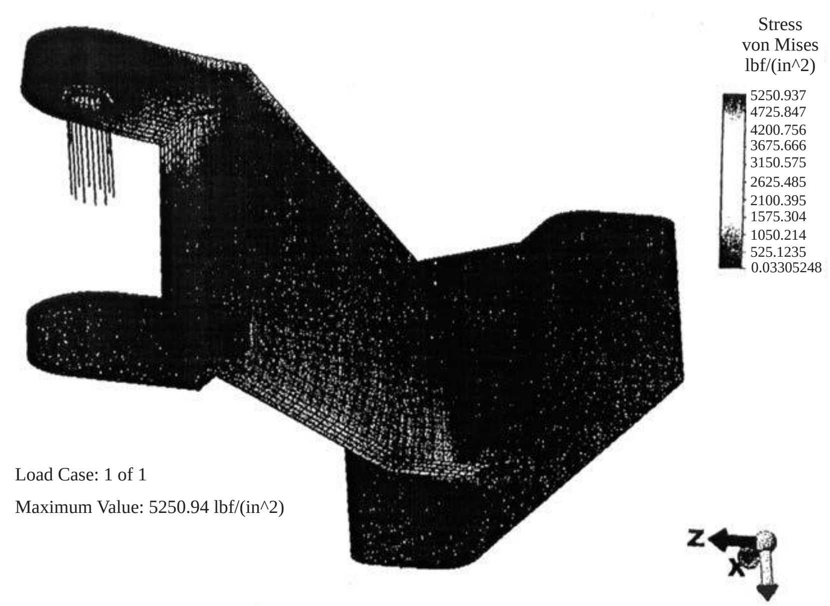


Figure 1 von Mises Stress of part.

With yield strength of 6,000 psi and a max von Mises stress of 5250 psi, it is getting close to failing due to the total weight of the entire car placed on one wheel. Under normal operation conditions the actual weight placed on a front wheel is less than one quarter of the entire car weight.

11.22

With 6,282 elements Algor calculates higher stresses. Figure 7 shows the von Mises stress for analysis with 6282 elements.

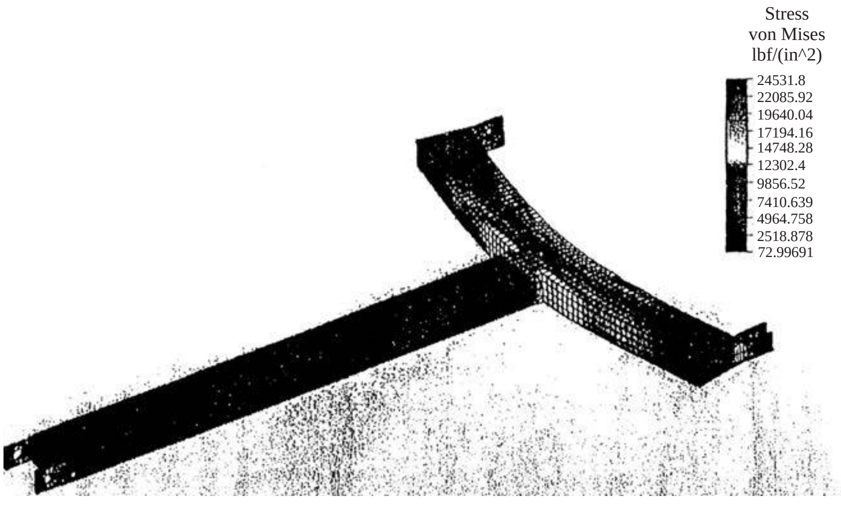


Figure 7

The yield strength of the material is 53,700 psi and the maximum von Mises stress is 24,530 psi so the hitch has a factor of safety of 2.18 when analyzed with 6282 elements.

11.23

Solution: C bracket

Maximum von Mises stress = 1681.273 $\frac{\text{lb}}{\text{in.}^2}$

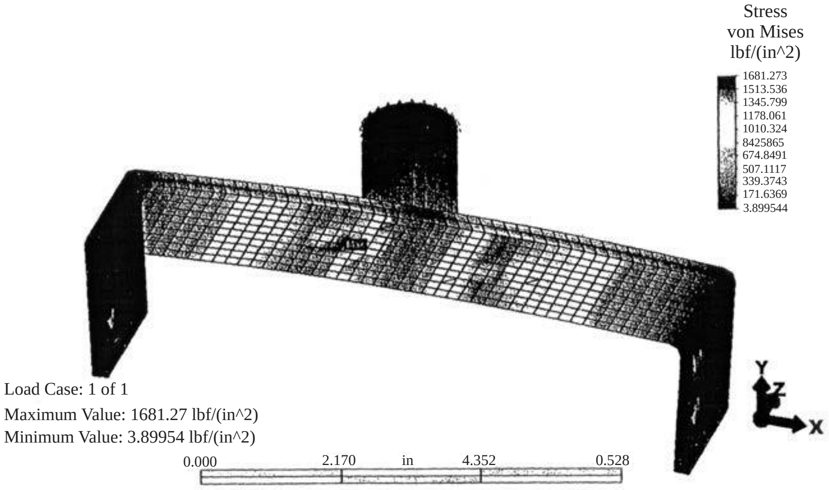


Figure 2 von Mises Stress

Maximum deflection = 0.0041 in.

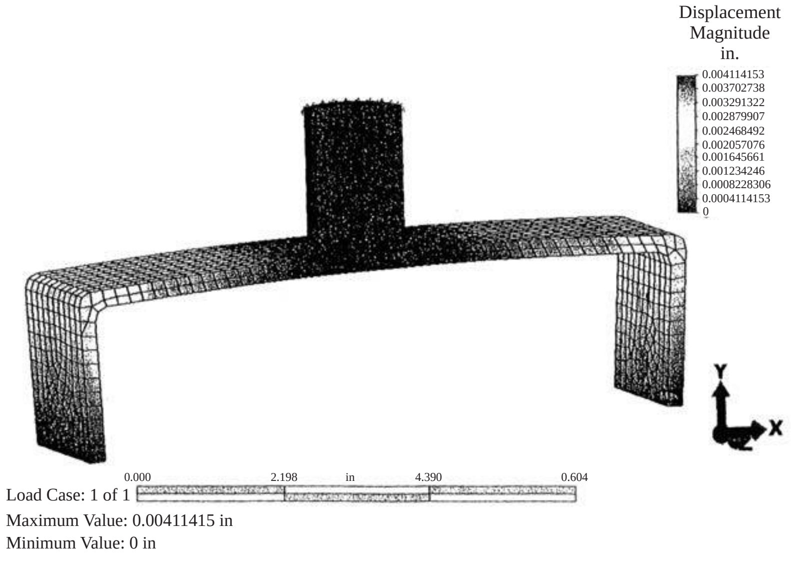


Figure 3 Maximum Deflection

11.24

The von Mises plot of the loader under loading is shown below in Figure 2. I am looking for the load that caused the loader to break. For this to happen, there is no factor of safety, However, based on the von Mises plot, you can see that much of the loader has a near infinite safety factor.

Figure 2 von Mises Plot

I found that a load of 2200 pounds applied on three sides of the right lower arm (left in the above figure) in a counter clockwise direction caused the loader to fail. This loader put the loader under 44251 psi, which is just over the yield strength. It appears it failed roughly six to eight inches above the end of the arm. This is very near where the loader broke this spring.

11.25

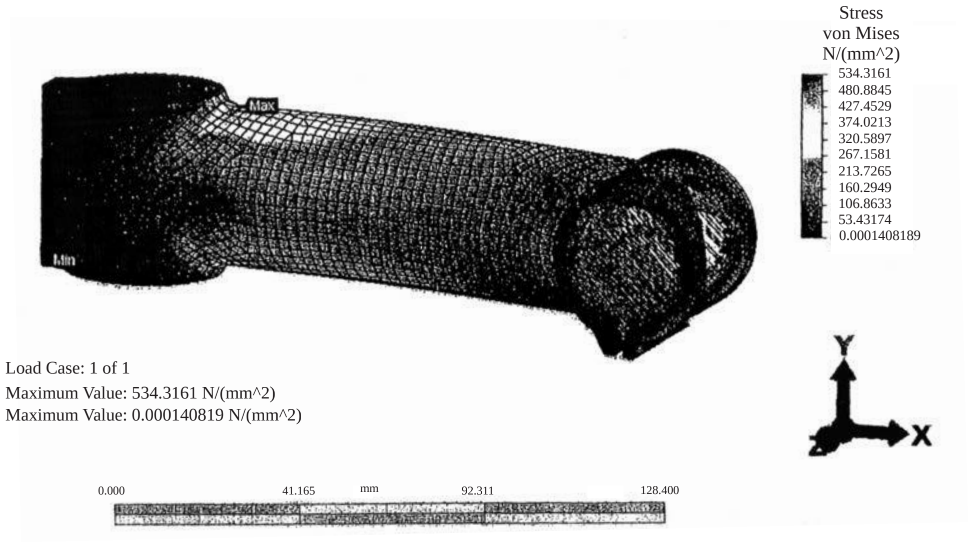
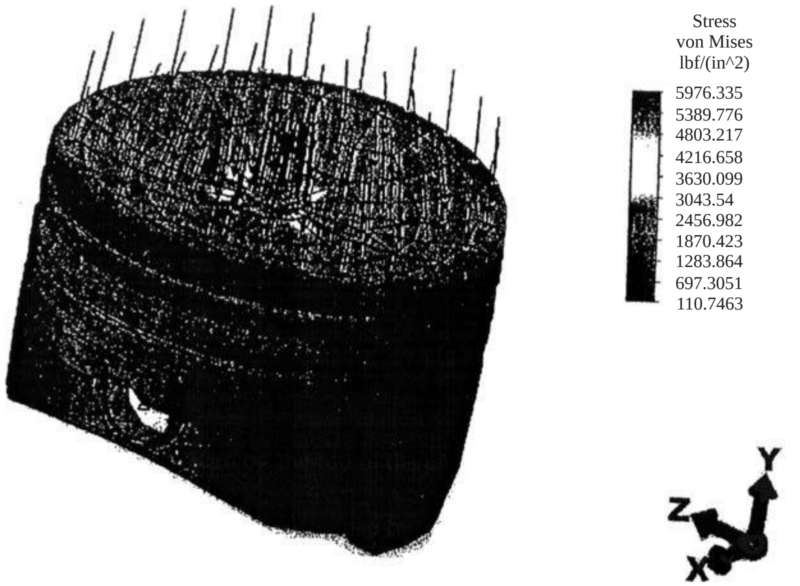


Figure 3 von Mises plot due to 250 lb rider (maximum stress is 534.3 MPa)

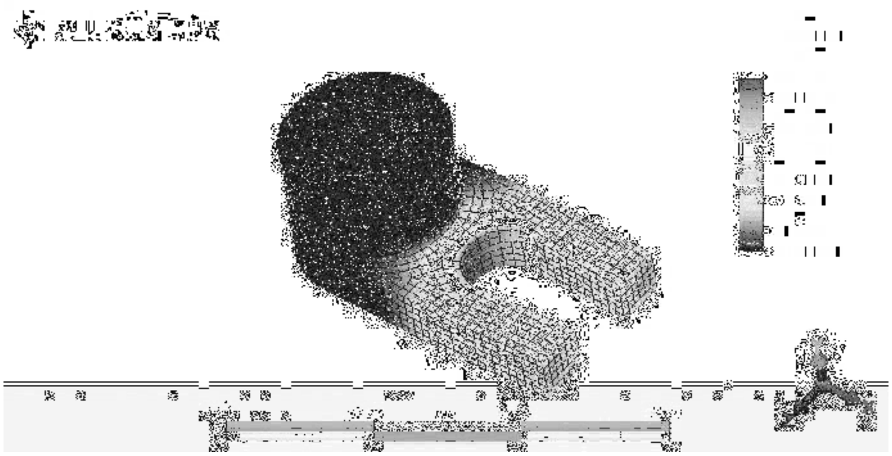
11.26

The load of 185 psi placed on the top of the piston head with the wrist-pin hole fixed on the top side to represent resistance on the piston head by the connecting rod. This is ample constraint because in a real internal combustion engine, the pistons are not frozen, but allowed to travel up and down with minimal friction resistance.



The maximum von Mises stress of 5976 psi is less than the yield strength of the material (σ_y = 44,200 psi)

11.27



(Largest von Mises stress is 186 psi) located inside surface of hole.

Chapter 12

Solve these problems using the plate element from a computer program.

12.1 A square steel plate of dimensions 20 in. by 20 in. with thickness of 0.1 is clamped all around. The plate is subjected to a uniformly distributed loading of 1 $\frac{\text{lb}}{\text{in.}^2}$. Using a 2 by 2 mesh and then a 4 by 4 mesh, determine the maximum deflection and maximum stress in the plate. Compare the finite element solution to the classical one in [1].

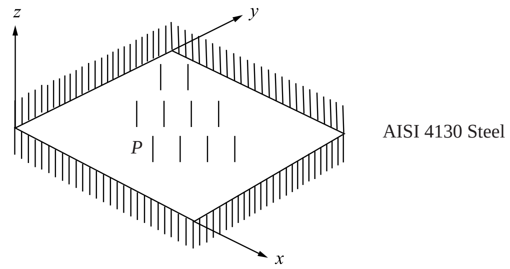
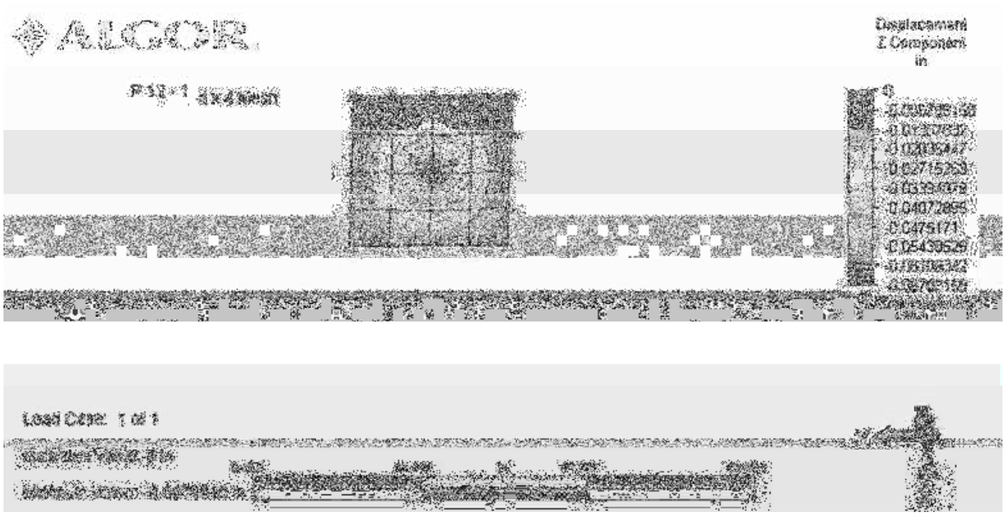


Figure P12–1 AISI 4130 Steel



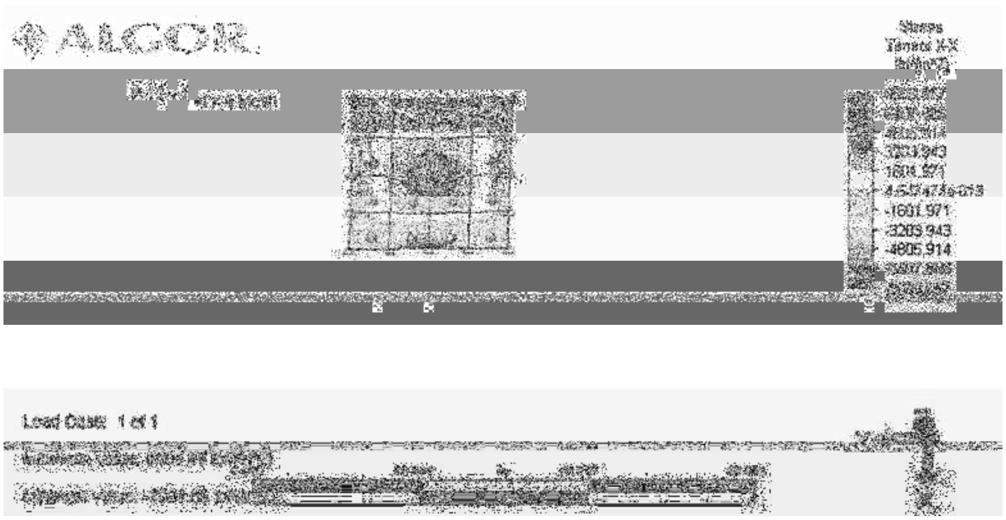


Plate bending

$20'' \times 20'' \times 0.1''$ 4130 steel plate

Clamped all the way around with a 1 psi load.

Mesh size	Minimum stress	Maximum stress	Maximum deflection
2×2	3440.01 psi	3573.76 psi	0.0785 in.
4×4	668.06 psi	7046.06 psi	0.06788 in.

Analytical solution: (A.C. ugural, plates and shells McGraw Hill)

$$W_{\max} = \frac{Pa^4}{D}\ (0.00126) = \frac{(1\text{ psi})\,(20'')^4}{2.7470 \times 10^3}\ (0.00126) = 0.07339 \text{ in.}$$

where

$$D = \frac{Et^3}{12\,(1-v^2)} = \frac{(30 \times 10^6)\,(0.1)^3}{12\,(1-3^2)} = \frac{0.03 \times 10^6}{(0.91)\,12} = 2.747 \times 10^3\ \frac{\text{lb}}{\text{in.}}$$

$$\sigma_{\max} = \frac{6\,M_{\max}}{t^2} = \frac{\sigma\,(0.0513\,p_0\,a^2)}{t^2}$$

$$= 12312 \text{ psi}$$

12.2 An *L*-shaped plate with thickness 0.1 in. is made of ASTM A-36 steel. Determine the deflection under the load and the maximum principal stress and its location using the plate element. Then model the plate as a grid with two beam elements with each beam having the stiffness of each *L*-portion of the plate and compare your answer.

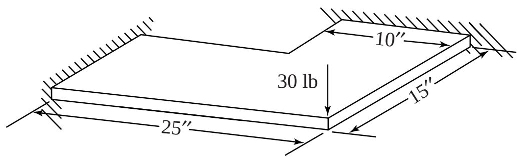


Figure P12–2

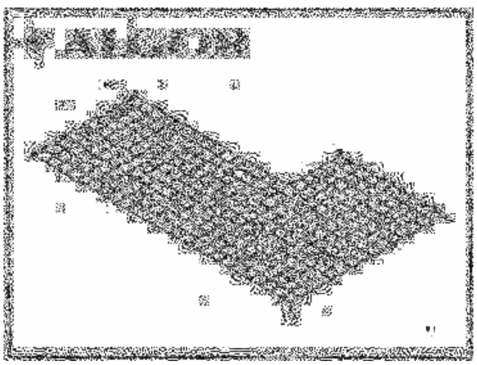


Figure 1: Plate Bending Mesh with Boundary Conditions and Nodal Force

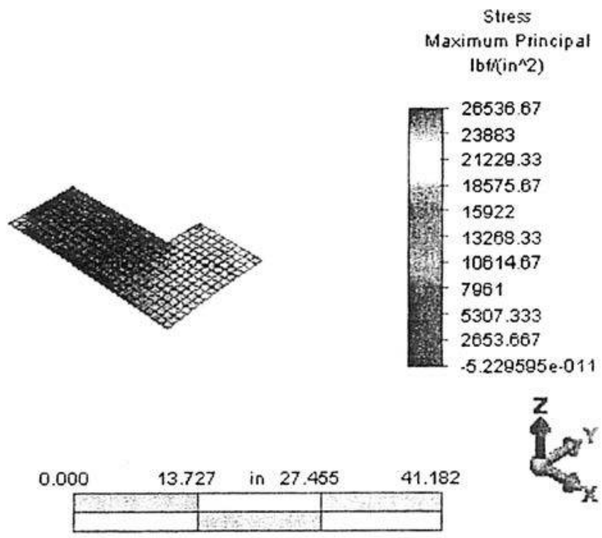


Figure 2: Plate Bending Maximum Principal stresses (psi)

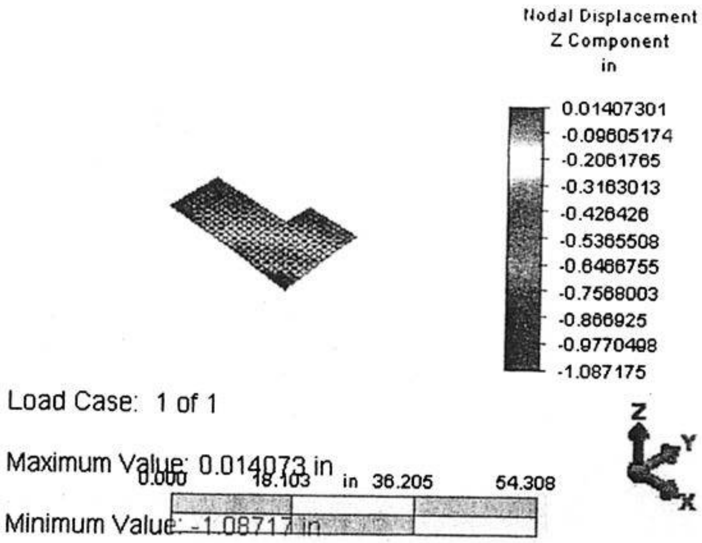


Figure 3 Nodal Z Displacement (in.)

12.3 A square simply supported 20 in. by 20 in. steel plate with thickness 0.15 in. has a round hole of 4 in. diameter drilled through its center. The plate is uniformly loaded with a load of 2 $\frac{\text{lbf}}{\text{in.}^2}$. Determine the maximum principal stress in the plate.

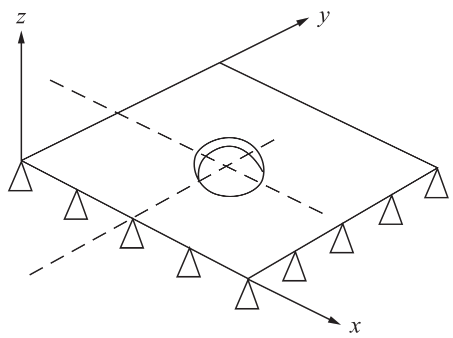
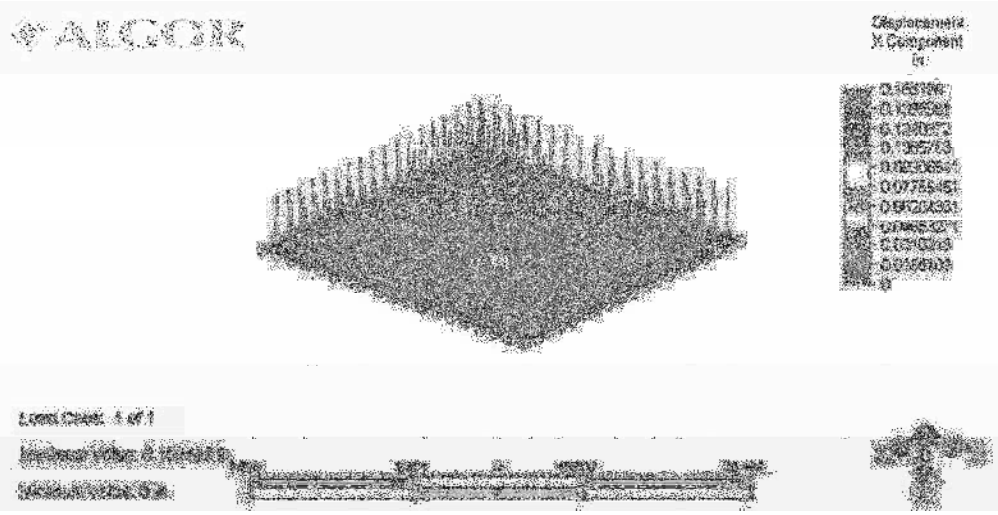
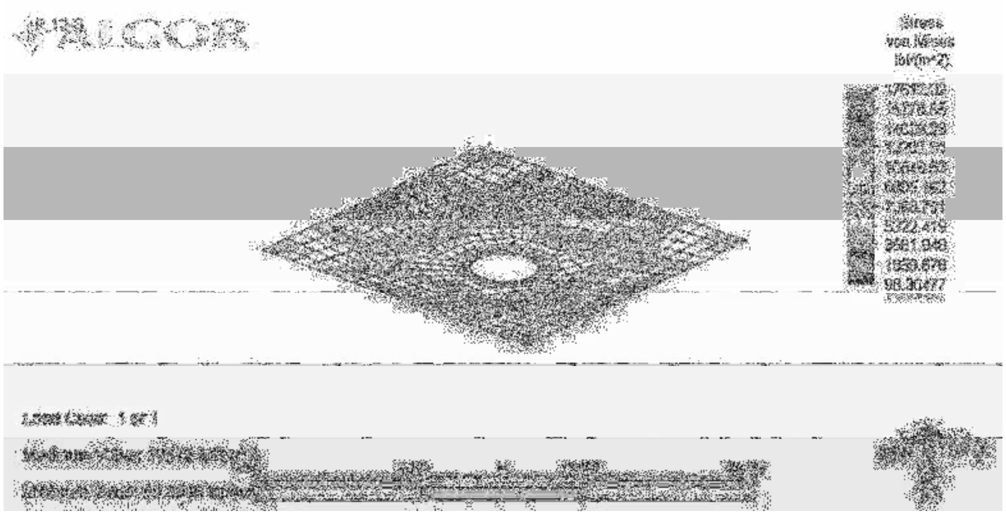


Figure P12–3





From the Algor analysis, the maximum principal stress was determined to be 10152.22 psi. The precision of the von Mises stress was very close to 0.1, therefore, the results were deemed feasible.

	Stress (psi)		Displacement (in.)			
	von Mises	Maximum Principal	X	Y	Z	Magnitude
Maximum Value	9041.787	10152.22	–0.0468	0	0	–0.0468

12.4 A *C*-channel section structural steel beam of 2 in. wide flanges, 3 in. depth and thickness of both flanges and web of 0.25 in. is loaded as shown with 100 lb acting in the y direction on the free end. Determine the free end deflection and angle of twist. Now move the load in the z direction until the rotation (angle of twist) becomes zero. This distance is called the shear center (the location where the force can be placed so that the cross section will bend but not twist).

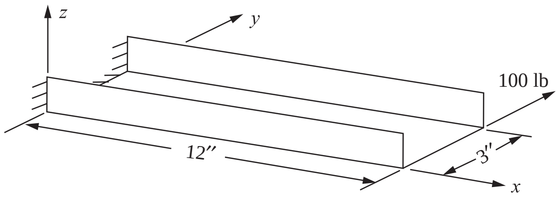
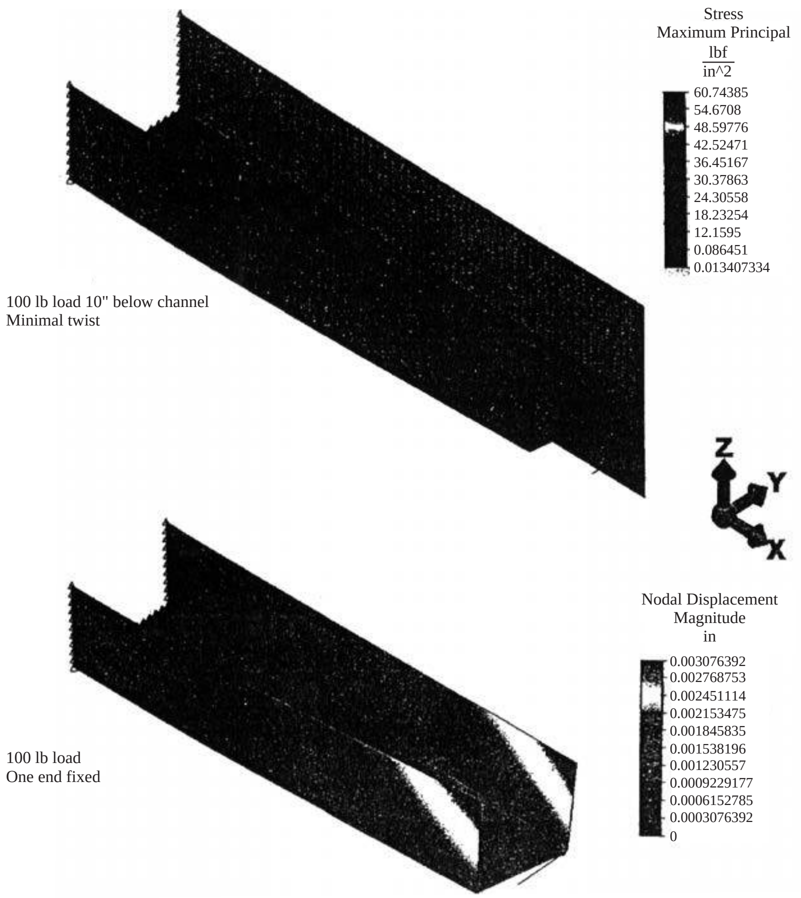


Figure P12–4



Need to apply load at shear center. See Table 5-1, Chapter 5, (P-263).

12.5 For the simply supported structural steel $W\ 14 \times 61$ wide flange beam shown, compare the plate element model results with the classical beam bending results for deflection and bending stress. The beam is subjected to a central vertical load of 22 kip. The cross-sectional area is 17.9 in.^{2}, depth is 13.89 in., flange width is 9.995 in., flange thickness of 0.645 in., web thickness of 0.375 in., and moment of inertia about the strong axis of 640 in.^{4}

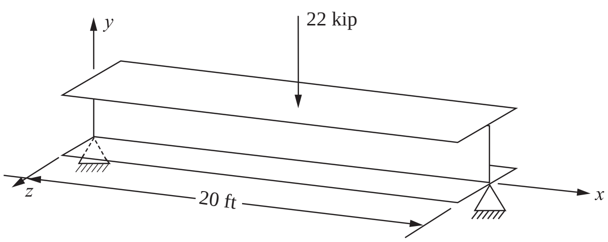
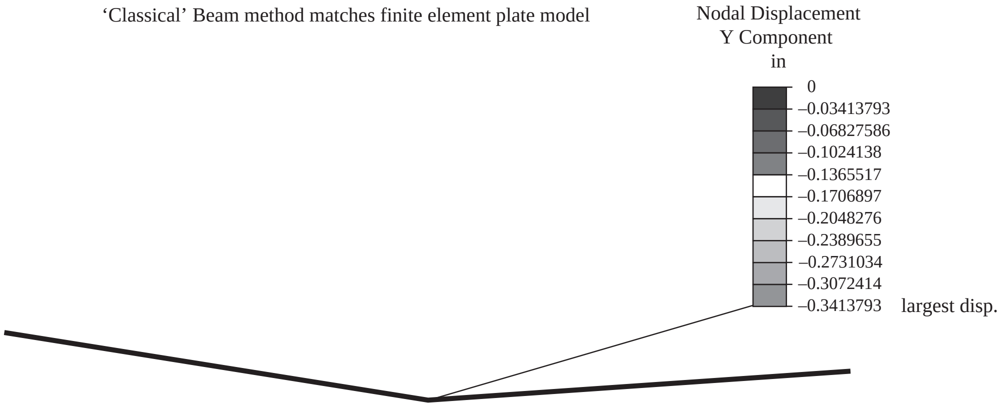


Figure P12–5

$$\delta = \frac{PL^3}{48EI} = \frac{(22\,K)(240'')^3}{48.29 \times 10^3 \text{ksi} \times 640 \text{ in.}^4}$$

$$= -0.34 \text{ in.}$$

'Classical' Beam method matches finite element plate model



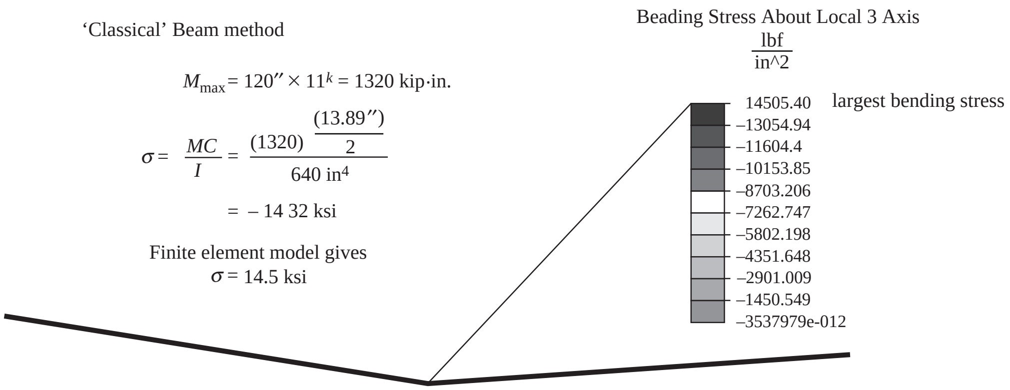
'Classical' Beam method

$M_{max} = 120'' \times 11^k = 1320$ kip·in.

$$\sigma = \frac{MC}{I} = \frac{(1320)\dfrac{(13.89'')}{2}}{640 \text{ in}^4}$$

$= -14\ 32$ ksi

Finite element model gives
$\sigma = 14.5$ ksi



12.6 For the structural steel plate structure shown, determine the maximum principal stress and its location. If the stresses are unacceptably high, recommend any design changes. The initial thickness of each plate is 0.25 in. The left and right edges are simply supported. The load is a uniformly applied pressure of 10 $\frac{\text{lb}}{\text{in.}^2}$ over the top plate.

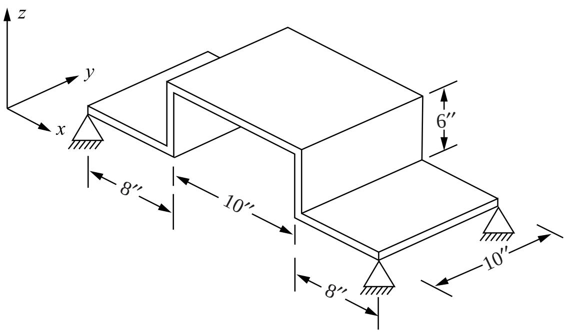
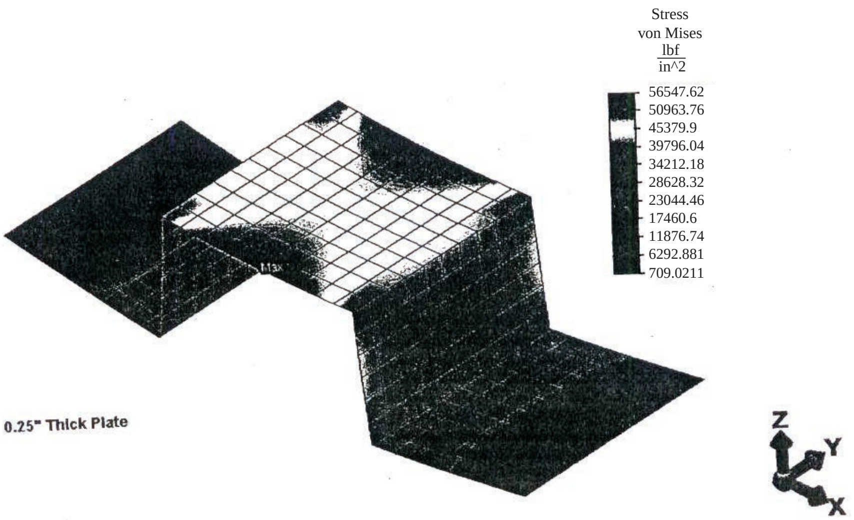


Figure: P12–6



This problem was a plate element analysis problem with plates that I assumed were made from ASTM A-36 steel. We were asked to determine the maximum principal stress and its location for the structure. We were asked to start with a plate thickness of 0.25 in., and if the stresses were found to be unacceptably high, we were to recommend design changes. A comparison of the results for a range of thickness is as follows.

	Maximum von Mises Stress $\left(\dfrac{\text{lb}}{\text{in}^{\wedge 2}}\right)$
0.25 in. thick plate	56,547.62
0.375 in. thick plate	25,194.28
0.50 in. thick plate	14,205.43

ASTM A-36 steel has the following strength properties.

- S_t = 58 to 80 ksi
- S_y = 36 ksi minimum

Therefore, the stress levels in the structure would lead to failure with the 0.25 in. thick plate using the MDET (maximum distortional energy theory). The 0.375 in. thick plate could be used allowing a factor of safety equal to 1.43 using the MDET. This is a small safety factor, and I would recommend using the 0.50 in. thick plate to construct the structure, which allows a more comfortable safety factor of 2.53 using the MDET.

12.7 Design a steel box structure 4 ft wide by 8 ft long made of plates to be used to protect construction workers while working in a trench. That is, determine a recommended thickness of each plate. The depth of the structure must be 8 ft. Assume the loading is from a side load acting along the long sides due to a wet soil (density of 62.4 $\frac{\text{lb}}{\text{ft}^3}$) and varies linearly with the depth. The allowable deflection of the plate type structure is 1 in. and the allowable stress is 20 ksi.

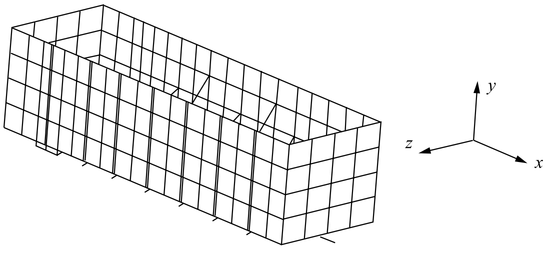


Figure P12–7

Solution method

I first attempted to solve the problem by examining the long side and short side of the box individually, rather than as a welded assembly. It was felt that once believable results were obtained from modeling each side as an individual plate, the model could be expanded to include the entire box.

However, Algor appears to have a glitch so that its 'surface hydrostatic pressure' feature does not work. Therefore, I was unable to easily apply a linearly varying pressure along the surface of a plate.

To get around this problem, I tried to break the side plate of the box into 4 vertically stacked sections (different Algor parts) and then applying a different pressure to the surface of each section so that it approximated the linear pressure distribution of the soil. However, Algor failed to recognize the applied surface pressure on 3 of the 4 sections. So this model was deficient also.

To solve the problem, I was forced to apply loads directly to the nodes. To facilitate the selection of the nodes and to make the nodal forces regular and repeatable, the plate was manually meshed into a convenient rectangular pattern. Four different nodal loads were used to approximate the linear pressure distribution.

Results

For the short side of the box (48″ wide × 96″ deep), it was found that a $\frac{3''}{8}$ thick plate was as thin as could be used and still meet the given design criteria. Using simply supported boundary conditions, the maximum von Mises stress was 19.9 ksi and the maximum deflection was 0.76″. The maximum stress was located at the bottom corners, which is typically an area of high stress for rectangular plates.

These results were than compared to equations found in Roark's 'Formulas for Stress and Strain'. From Roark, the maximum stress was 18.2 ksi and the maximum deflection was 0.68 in.

The same analysis was performed for the long side of the box (96″ square). It was found that a 0.75″ plate was right at the margins of acceptability for deflection. The Algor and hand calculations for both sides of the box are summarized below.

	Algor model		Hand Calc.	
	von Mises (ksi)	Deflection (in.)	von Mises (ksi)	Deflection (m)
Long side of box ($\frac{3}{4}''$ thk $\times$ 96″ sq)	14.5	1.06	11.36	0.93
Short side of box ($\frac{3}{8}''$ thk $\times$ 48″ $\times$ 96″ dp)	19.9	0.76	18.18	0.68

Comments

For both sides, the Algor model predicted higher stress and higher deflection than the hand calculations. I suspect the difference in results can be attributed to two sources. One, the stepwise variation of pressure that I was forced to use in the Algor model to approximate the hydrostatic pressure. The other is due to the coarseness of the mass.

Nodal loads to approximate hyprostatic pressure variation as function of depth

$$q_0 = \gamma\ a = \text{Maximum pressure}$$

$$= 62.4\ \frac{\text{lb}}{\text{ft}^3}\ 8\,\text{ft}\left(\frac{1\,\text{ft}}{12\text{ in.}}\right)^2$$

$$= 3.467 \text{ psi soil pressure at } 8' \text{ depth}$$

Assume linear pressure profile

$$x = \text{depth}$$

$$q(x) = q_0 - q_0\ \frac{x}{a}$$

$$= q_0\left(1 - \frac{x}{a}\right)$$

For Step 1

$$q\left(x = \frac{1}{8}a\right) = q_0 \frac{7}{8} = 3.04 \text{ psi}$$

For Step 2

$$q\left(x = \frac{3}{8}a\right) = \frac{5}{8} q_0 = 2.17 \text{ psi}$$

For Step 3

$$q\left(x = \frac{5}{8}a\right) = \frac{3}{8} q_0 = 1.30 \text{ psi}$$

For Step 4

$$q\left(x = \frac{7}{8}a\right) = \frac{1}{8} q_0 = 0.43 \text{ psi}$$

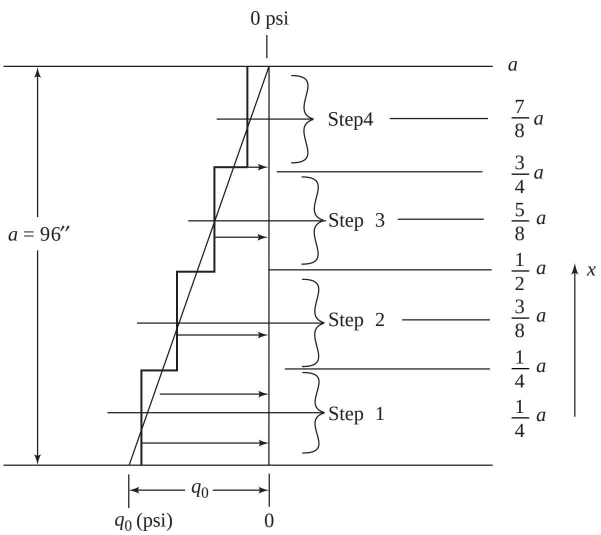
For a $g'' \times g''$ mesh, the area supported by a interior node is, $A_I = g^2$

For a $g'' \times g''$ mesh, the area supported by a perimeter node is $A_P = \frac{g^2}{2}$

Let $g = 4'' \Rightarrow$ nodal force for step 1 interior node $= (3.04)\,(4)^2 = \frac{48.64 \text{ lb}}{\text{node}}$

Let $g = 4'' \Rightarrow$ nodal force for step 1 perimeter node $= (3.04)\,\frac{4^2}{2} = 24.32\,\frac{\text{lb}}{\text{node}}$

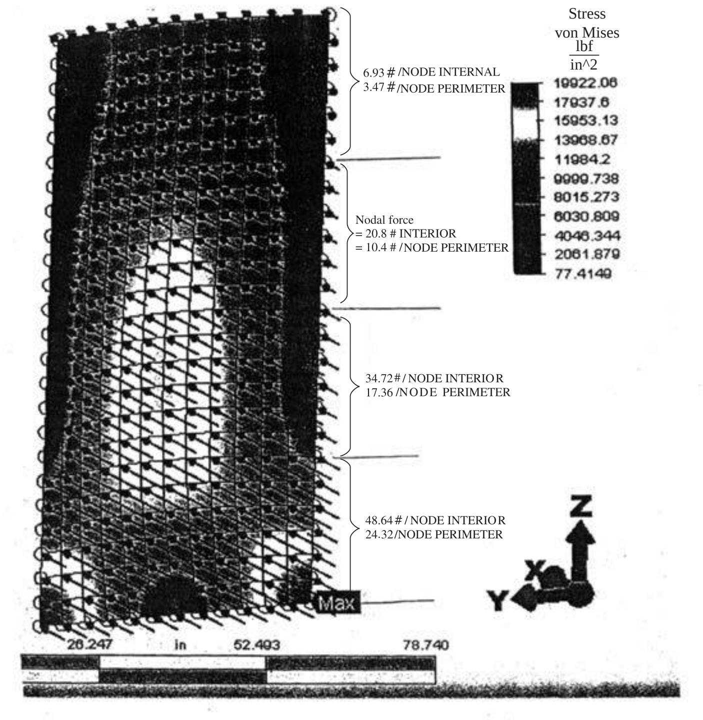
etc. …



Manual mesh
Stepwise approximation of soil pressure
Simple support boundary conditions
Short side

$48''$ wide $\times$ $96''$ Deep $\times$ $\frac{7''}{8}$ thick

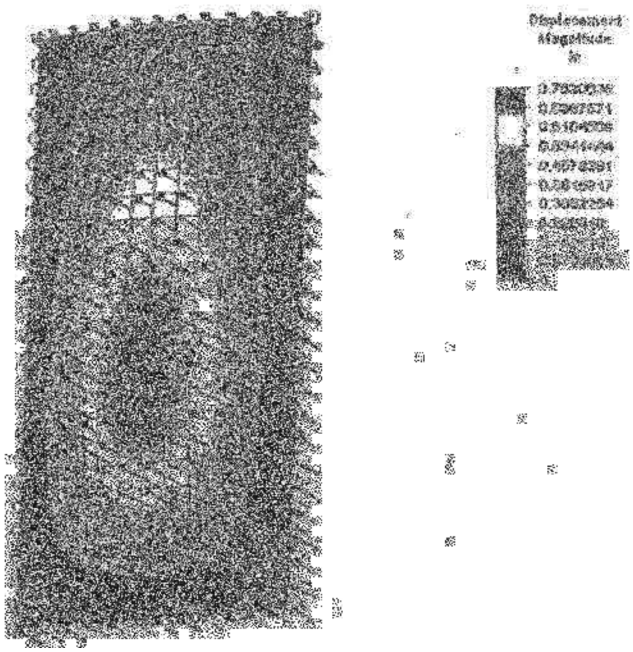
$4''$ square mesh



Manual mesh
Stepwise approximation of soil pressure
Simple support boundary conditions
Short side of box

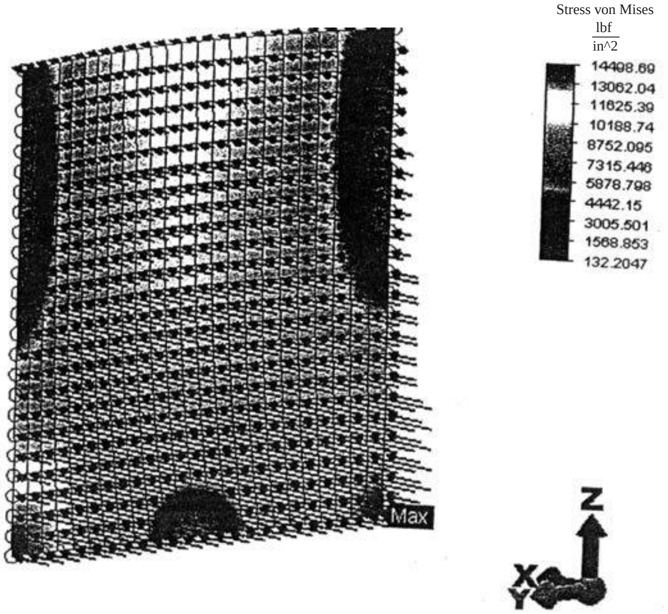
$48'' \text{ wide} \times 96'' \text{ deep} \times \dfrac{3''}{8} \text{ thick}$

$4''$ square mesh

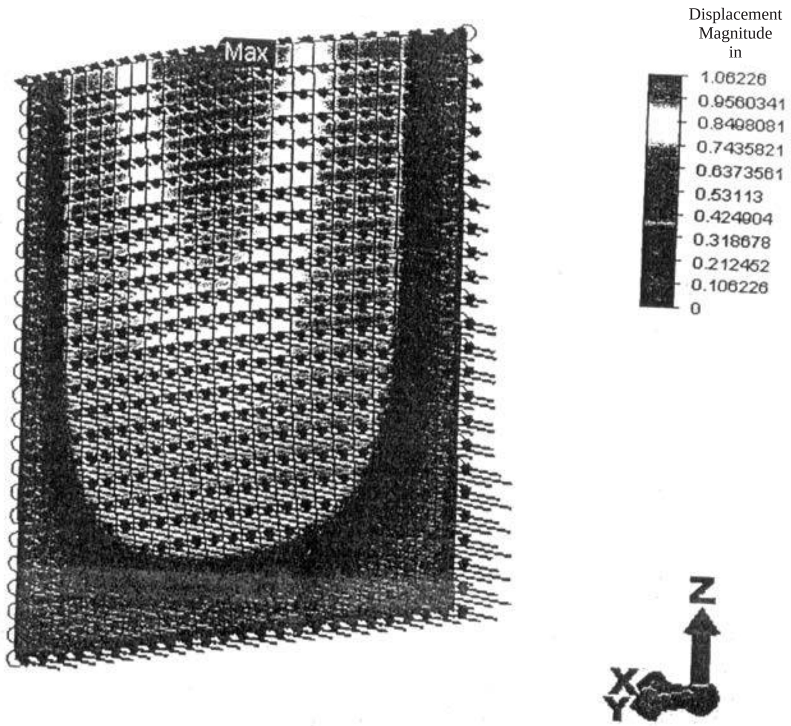


Manual $4''$ square mesh
Stepwise approximation of soil pressure

Simple boundary conditions
Long side (96″ square)
0.75″ thick



Manual 4″ square mesh
Stepwise approximation of soil pressure
Simple boundary conditions
Long side (96″ square)
0.75″ thick



12.8 Determine the maximum deflection and maximum principal stress of the circular plate shown in Figure P12–8. The plate is subjected to a uniform pressure p = 700 kPa and fixed along its outer edge. Let E = 200 GPa, v = 0.3, radius r = 500 mm, and thickness t = 12 mm.

Analytical solution

$$W_{\max} = \frac{Pr^4}{64\,D} = \frac{(700{,}000)\,(0.5)^4}{64\,(2.013\times 10^6)} = 0.0217 \text{ m}$$

$$D = 0.091\,E t^3$$

$$= 0.091\,(200 \times 10^9)\,(0.012)^3$$

$$= 2.013 \times 106 \text{ lb}\cdot\text{in.}$$

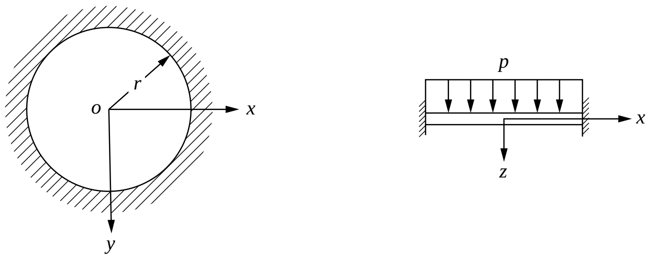
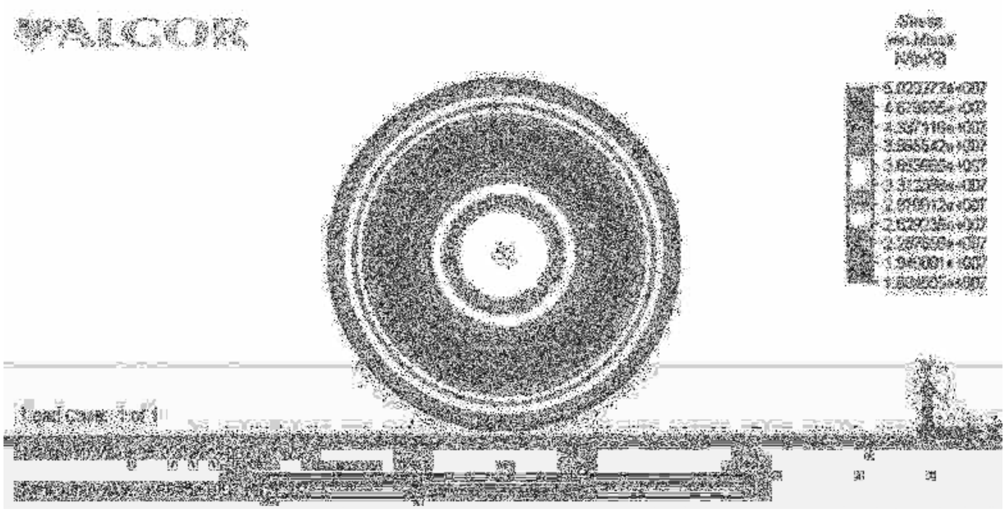


Figure P12–8

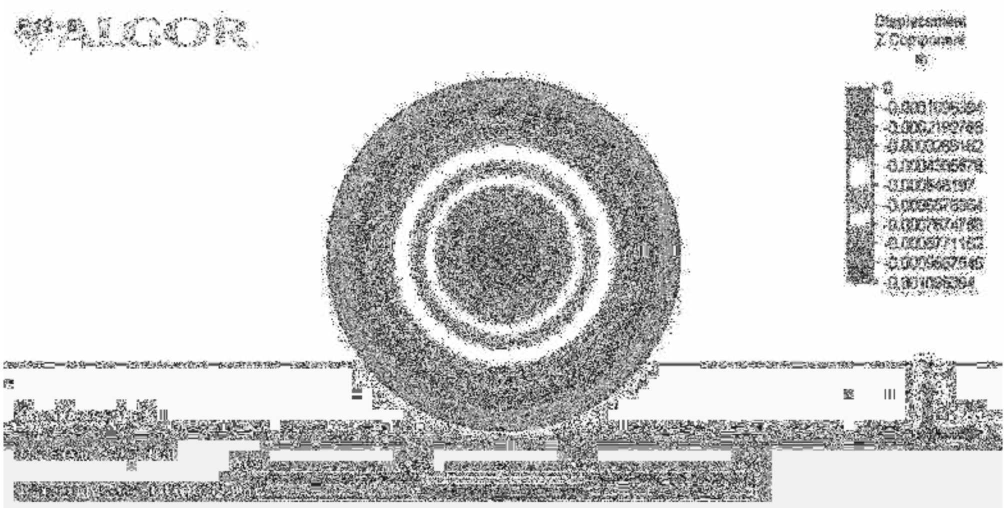


Algor

$$W_{\max} = -0.001096 \text{ m}$$

$$= -1.096 \times 10^{-3} \text{ m}$$

Compares closely to analytical solution



12.9 Determine the maximum deflection and maximum stress for the plate shown. The plate is fixed along three sides. A uniform pressure of 70 kPa is applied to the surface. The plate is made of steel with E = 200 GPa, v = 0.3, and t = 12 mm and sides equal to a = 0.75 m and b = 1 m.

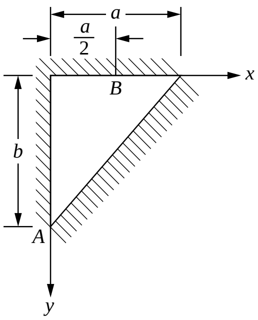
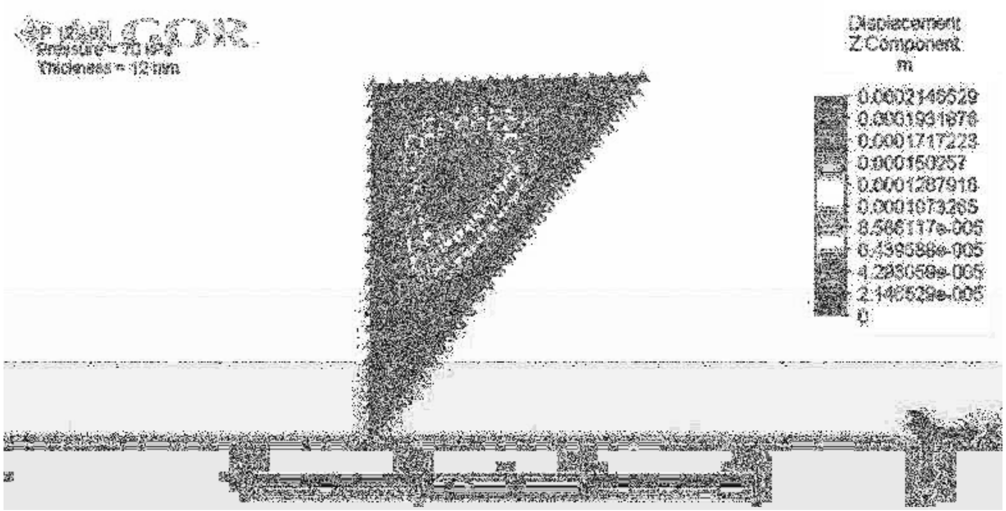
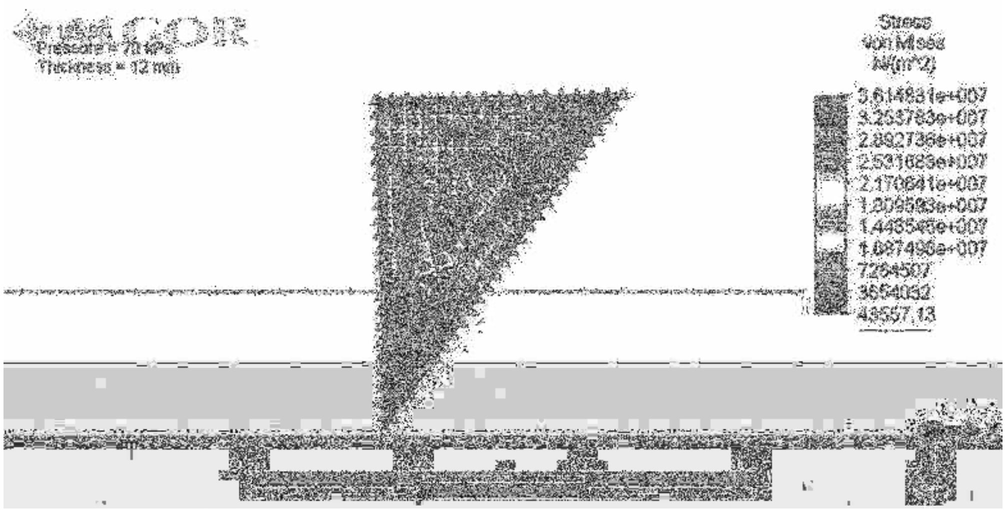


Figure P12–9





12.11 A square steel plate 2 m by 2 m and 10 mm thick at the bottom of a tank must support salt water at a height of 3 m, as shown in Figure P12–11. Assume the plate to be built in (fixed all around). The plate allowable stress is 100 MPa. Let E = 200 GPa, v = 0.3 for the steel properties. The weight density of salt water is 10.054 $\frac{\text{kN}}{\text{m}^3}$. Determine the maximum principal stress in the plate and compare to the yield strength.

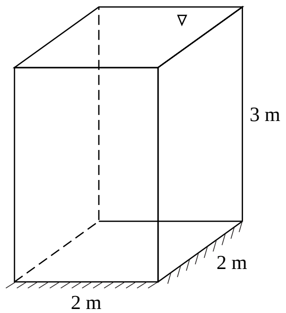
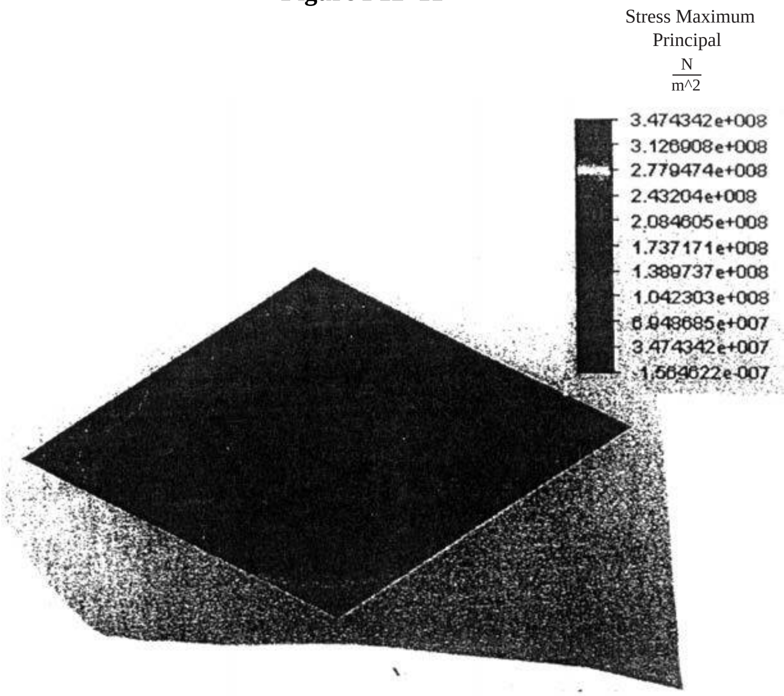
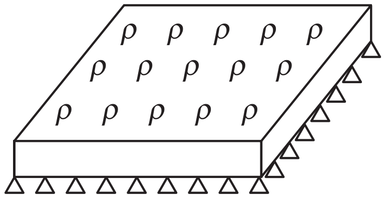


Figure P12–11



$$F_{\text{tot}} = (V)\,(\rho) = (3\,\cancel{m})(2\,\cancel{m})(2\,\cancel{m})(10.054E^3\,\frac{\text{N}}{\cancel{m^3}}) = 120{,}648\text{ N}$$

$$P = \frac{F}{A_{\text{plate}}} = \frac{120{,}648\,\text{N}}{(2\text{m})(2\text{m})} = 30{,}162\,\frac{\text{N}}{\text{m}^2}$$



Find

Maximum principal stress = ? $347.43\ \frac{\text{MN}}{\text{m}^2}$

Safe or not safe = ?

Since stress allowable = 100 MPa < actual stress 347.43 MPa

Tank plate is not safe.

12.12 A stockroom floor carries a uniform load of $p = 80\ \frac{\text{lb}}{\text{ft}^2}$ over half the floor as shown in Figure P12–12. The floor has opposite edges clamped and remaining edges and mid-span simply supported. The dimensions are 40 ft by 20 ft. The floor thickness is 6 in. The floor is made of reinforced concrete with $E = 3 \times 10^6$ psi and $v = 0.25$. Determine the maximum deflection and maximum principal stress in the floor.

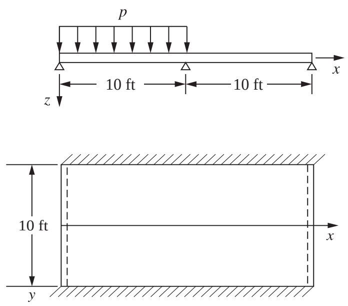
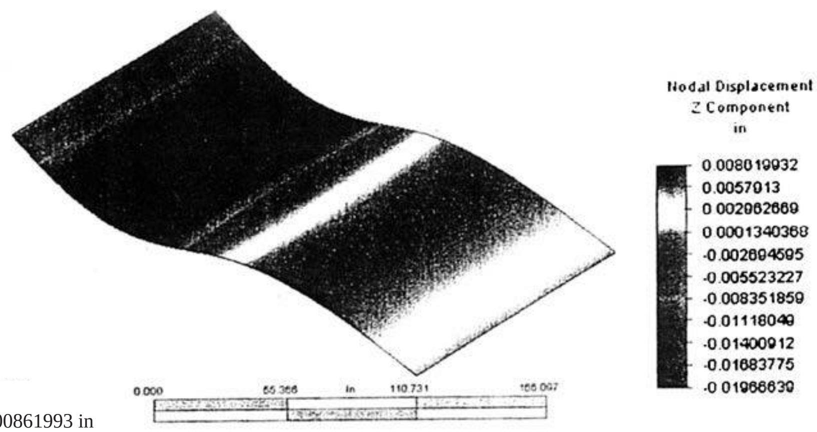
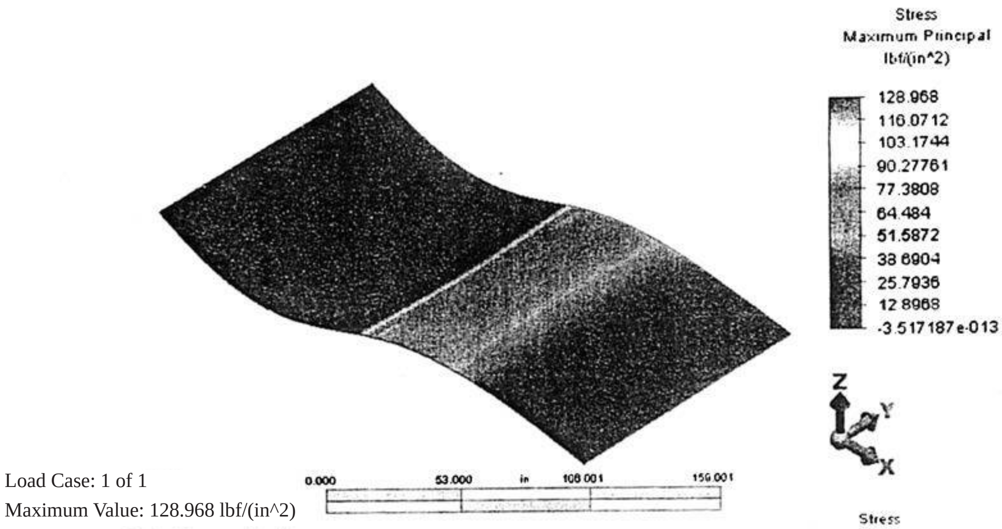


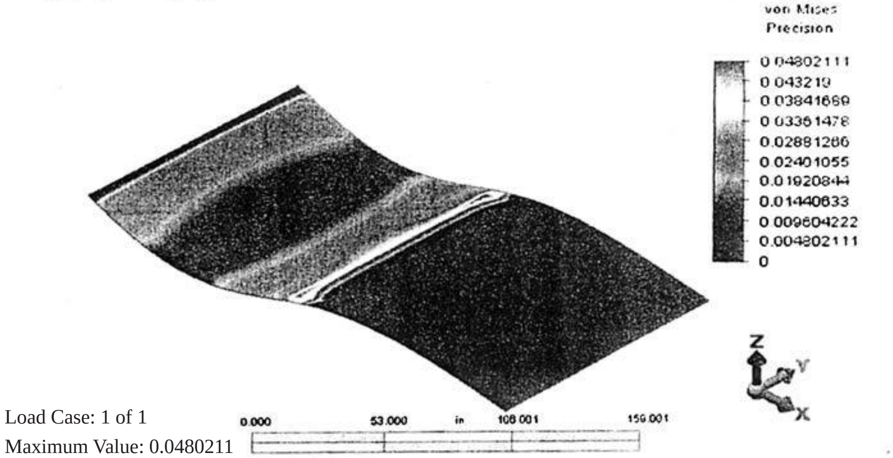
Figure P12–12



Load Case: 1 of 1

Maximum Value: 0.00861993 in

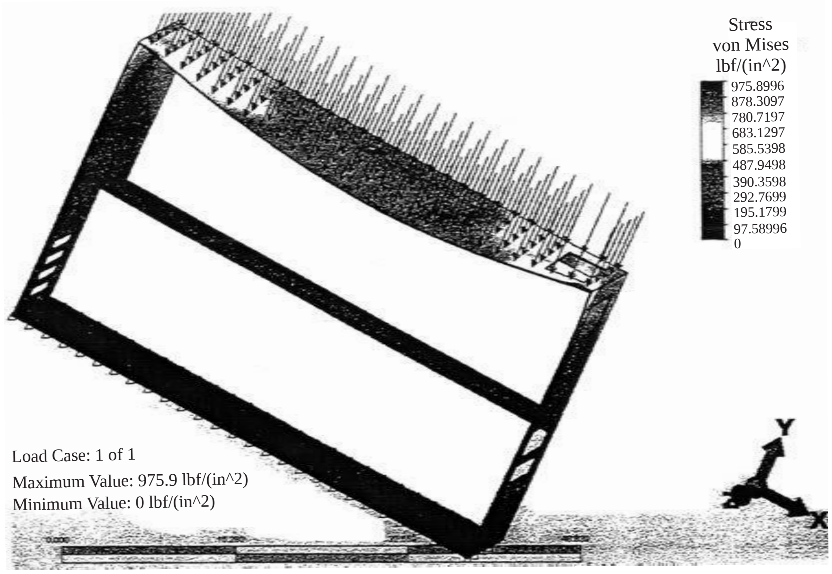


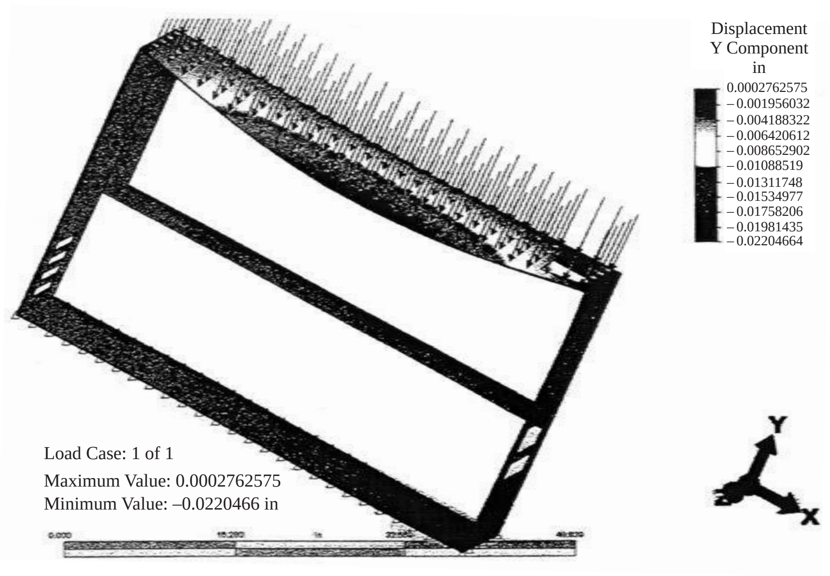


12.13

Material used: AISI 4130 Steel

Uniform load of 0.1 psi applied to top surface.



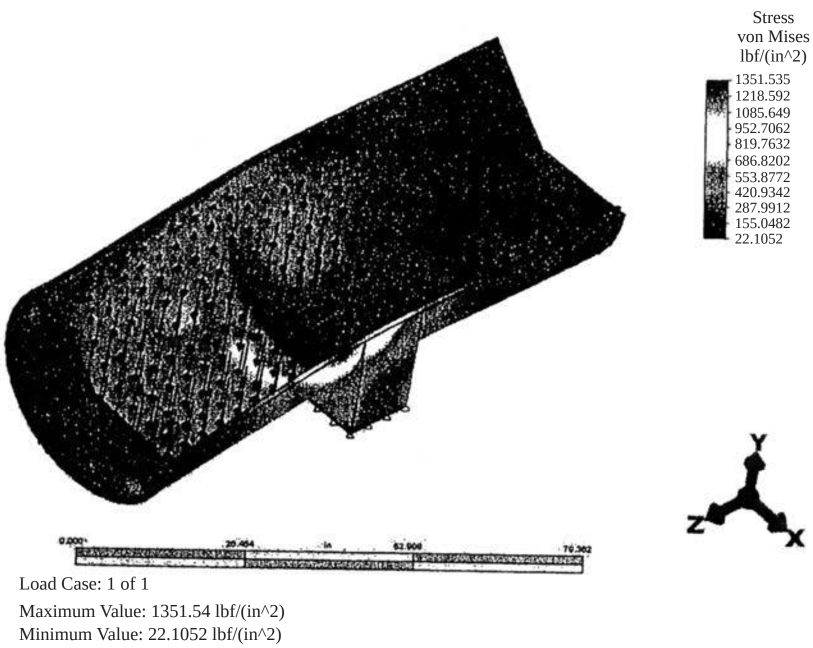


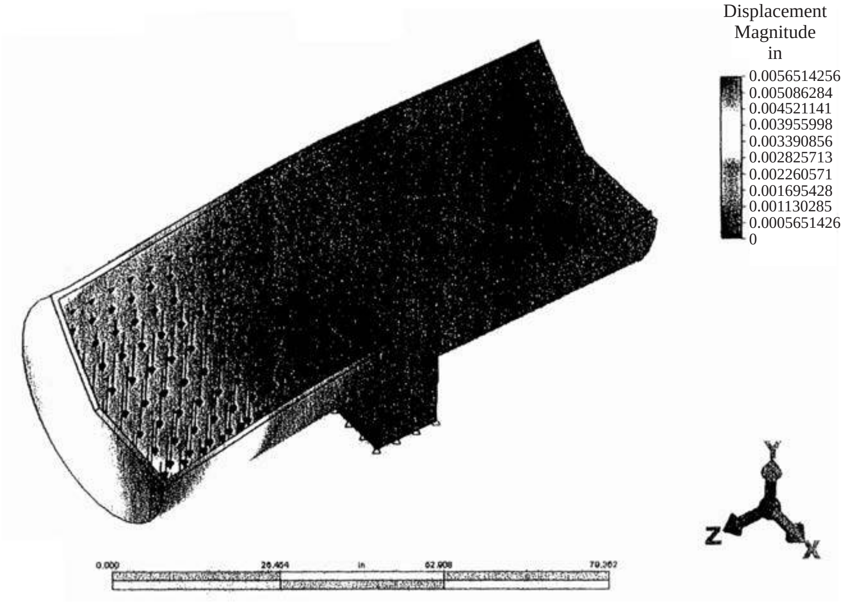
12.15

Model variables

Variable	Value
Material	1010 cold rolled
Modulus of Elasticity	29×10^6 psi
Maximum von Mises Stress	1351 psi
Maximum Displacement	0.00565 in.

Algor results





Load Case: 1 of 1
Maximum Value: 0.00565143 in
Minimum Value: 0 in

Figure 6 Displacement (mm)

12.16

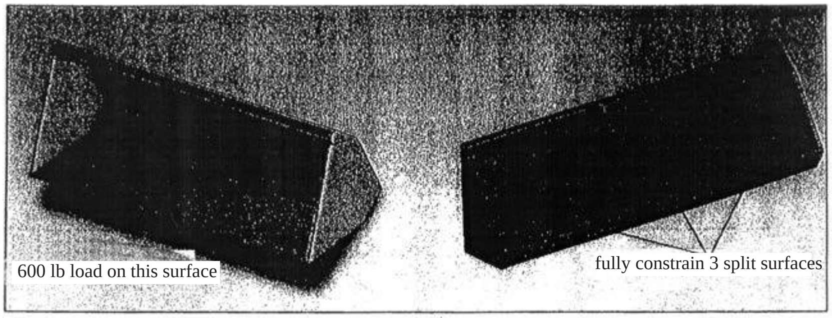


Figure 2 The Boundary Conditions on the bucket.

The results of the analysis are shown below in Figure 3.

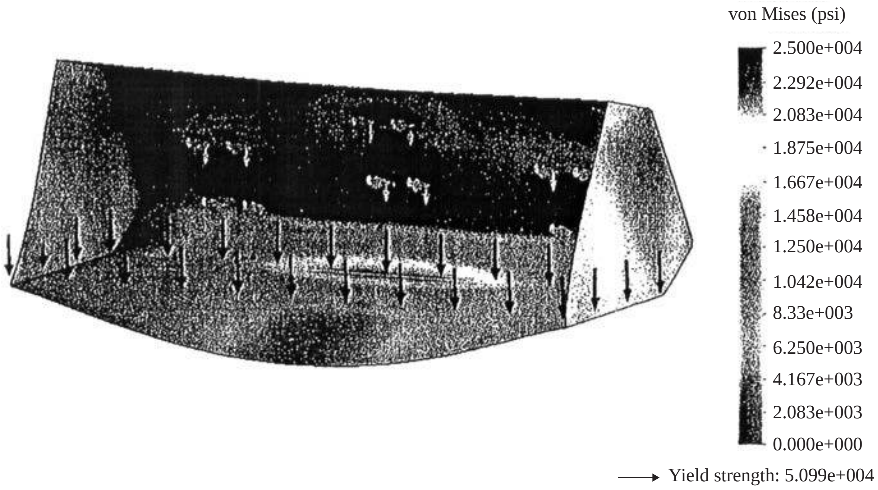
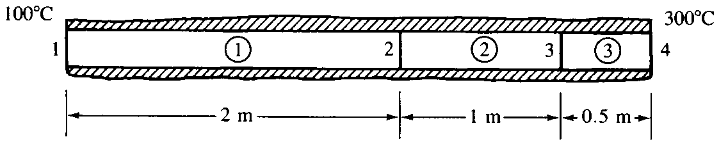


Figure 3 The von Mises stress in psi for the bucket plate analysis.

Chapter 13

13.1



Element $[K]$'s

$$[k^{(1)}] = \frac{(0.1\text{m}^2)(200\,\frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}})}{2\text{m}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}10 & -10\\ -10 & 10\end{bmatrix}\frac{\text{W}}{{}^\circ\text{C}}$$

$$[k^{(2)}] = \frac{(0.1\text{m}^2)(100\,\frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}})}{1\text{m}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}10 & -10\\ -10 & 10\end{bmatrix}$$

$$[k^{(3)}] = \frac{(0.1\text{m}^2)(50\,\frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}})}{0.5\text{m}} = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}10 & -10\\ -10 & 10\end{bmatrix}$$

$$q^* = 0 \qquad Q = 0$$

$$\therefore \quad \{f^{(1)}\} = \{f^{(2)}\} = \{f^{(3)}\} = 0$$

Assemble equations

$$\begin{bmatrix}10 & -10 & 0 & 0\\ -10 & 20 & -10 & 0\\ 0 & -10 & 20 & -10\\ 0 & 0 & -10 & 10\end{bmatrix}\begin{Bmatrix}t_1\\ t_2\\ t_3\\ t_4\end{Bmatrix} = \begin{Bmatrix}F_1\\ 0\\ 0\\ F_4\end{Bmatrix}$$

Boundary conditions $t_1 = 100°\text{C}$, $t_y = 300°\text{C}$

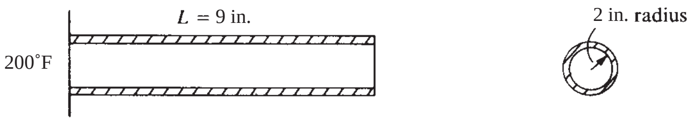
$$\therefore \quad \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 20 & -10 & 0\\ 0 & -10 & 20 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}t_1\\ t_2\\ t_3\\ t_4\end{Bmatrix} = \begin{Bmatrix}100\\ 1000\\ 3000\\ 300\end{Bmatrix}$$

Solving

$$t_2 = 166.7°\text{C}$$

$$t_3 = 233.3°\text{C}$$

13.2



$$[k^{(1)}] = [k^{(2)}] = \frac{\pi(2'')^2\, 3\,\frac{\text{Btu}}{\text{h}\cdot\text{in.}\cdot{}^\circ\text{F}}}{3\text{ in.}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 12.57 & -12.57 \\ -12.57 & 12.57 \end{bmatrix}\frac{\text{Btu}}{\text{h}\cdot{}^\circ\text{F}}$$

Now there is convection through right end

$$[k^{(3)}] = \begin{bmatrix} 12.57 & -12.57 \\ -12.57 & 12.57 \end{bmatrix} + \left(1\frac{\text{Btu}}{\text{kip}\cdot\text{in.}^2\cdot{}^\circ\text{F}}\right)\pi(2'')^2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[k^{(3)}] = \begin{bmatrix} 12.57 & -12.57 \\ -12.57 & 25.13 \end{bmatrix}$$

$$\text{Now } q^* = 0, \quad Q = 0 \quad \therefore \quad \{f^{(1)}\} = \{f^{(2)}\} = 0$$

$$\{f^{(3)}\} = h\,T_\infty\,A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (1)\,(0°\text{F})\,\pi\,2^2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assemble equations and boundary condition $t_1 = 200°\text{F}$

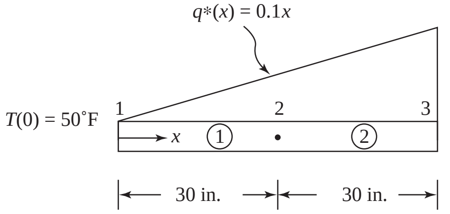
$$\begin{bmatrix} 12.57 & -12.57 & 0 & 0 \\ & 25.13 & -12.57 & 0 \\ & & 25.13 & -12.57 \\ \text{Symmetry} & & & 25.13 \end{bmatrix}\begin{Bmatrix} t_1 = 200°\text{F} \\ t_2 \\ t_3 \\ t_4 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Solve Equations (2-4) of above

$$\begin{bmatrix} 25.13 & -12.57 & 0 \\ & 25.13 & -12.57 \\ \text{Symmetry} & & 25.13 \end{bmatrix}\begin{Bmatrix} t_2 \\ t_3 \\ t_4 \end{Bmatrix} = \begin{Bmatrix} 2513 \\ 0 \\ 0 \end{Bmatrix}$$

$$t_2 = 150°\text{F},\ t_3 = 100°\text{F},\ t_4 = 50°\text{F}$$

13.3



$$\frac{AK_{xx}}{L} = \frac{(2\text{ in.}^2)\,(3\,\frac{\text{Btu}}{\text{h}\cdot\text{in.}\cdot{}^\circ\text{F}})}{30\text{ in.}} = \frac{1}{5}\ \frac{\text{Btu}}{\text{h}\cdot{}^\circ\text{F}}$$

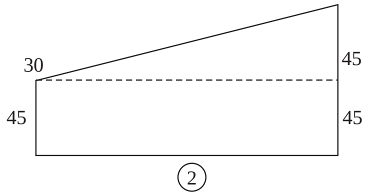
$$h = 0 \quad \therefore \quad \frac{hPL}{6} = 0, \quad hA = 0, \quad h\,T_\infty\,P\,L = 0$$

$$[k^{(1)}] = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = [k^{(2)}]$$

$$\{f_q^{(1)}\} = \int_0^L q^*\,[N]^T\,dx = \int_0^{L\,=\,30}(0.1x)\begin{Bmatrix} 1 - \frac{X}{L} \\ \frac{X}{L} \end{Bmatrix}dx$$

$$= 0.1 \begin{Bmatrix} \frac{L^2}{6} \\ \frac{L^2}{3} \end{Bmatrix} \frac{\text{Btu}}{\text{h}} = \begin{Bmatrix} 15 \\ 30 \end{Bmatrix}$$

Then



$$\{f_q^{(2)}\} = \begin{Bmatrix} 15+45 \\ 30+45 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 75 \end{Bmatrix}$$

Solve $\{F\} = [K]\{t\}$

Heat flow

$$\begin{Bmatrix} 15+F_1 \\ 30+60 \\ 75 \end{Bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} t_1 = 50°\text{F} \\ t_2 \\ t_3 \end{Bmatrix} \qquad \text{(A)}$$

Solve the 2nd and 3rd equations

$$90 + 10 = \frac{1}{5}(2\,t_2 - t_3) \qquad (1)$$

$$+ 75 = \frac{1}{5}(-\,t_2 + t_3) \qquad (2)$$

Adding (1) and (2)

$$175 = \frac{2}{5}\,t_2 - \frac{1}{5}\,t_2$$

$$175 = \frac{1}{5}\,t_2$$

$$t_2 = 875°\text{F}$$

Back-substitution into (1) yields

$$t_3 = 1750 - 500$$

$$t_3 = 1250\ °\text{F}$$

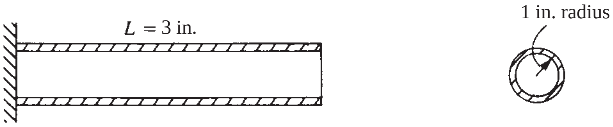
Solve for F_1 using Equation (A) above

$$15 + F_1 = \frac{1}{5}(t_1 - t_2)$$

$$F_1 = -15 + \frac{1}{5}(50 - 875)$$

$$= -180\ \frac{\text{Btu}}{\text{h}} = \text{heat flow out left end.}$$

13.4



$$Q = 10000 \frac{\text{Btu}}{\text{h}\cdot\text{ft}^3}$$

$$[k^{(1)}] = [k^{(2)}] = \frac{AK_{xx}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{\pi\left(\frac{1}{12}\right)^2 12}{\left(12\,\frac{\text{in.}}{\text{ft}}\right)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \pi\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\frac{\text{Btu}}{\text{h}\cdot{}^\circ\text{F}}$$

$$[k^{(3)}] = [k^{(1)}] + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \pi\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 100\,\pi\left(\frac{1}{12}\right)^2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[k^{(3)}] = \pi\begin{bmatrix} 1 & -1 \\ -1 & 1.694 \end{bmatrix}$$

$$\{f^{(1)}\} = \frac{QAL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \frac{(10000)\pi\left(\frac{1}{12}\right)^2\frac{1}{12}}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= \pi\begin{Bmatrix} 2.894 \\ 2.894 \end{Bmatrix}$$

$$\{f^{(2)}\} = \{f^{(1)}\}$$

$$\{f^{(3)}\} = \{f^{(1)}\} + \{f_{h_{\text{end}}}\} = \pi\begin{Bmatrix} 2.894 \\ 2.894 \end{Bmatrix} + h\,T_\infty\,A\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$= \pi\begin{Bmatrix} 2.894 \\ 2.894 \end{Bmatrix} + (100)\,(100)\,\pi\left(\frac{1}{12}\right)^2\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\{f^{(3)}\} = \pi\begin{Bmatrix} 2.894 \\ 2.894 \end{Bmatrix} + \pi\begin{Bmatrix} 0 \\ 69.4 \end{Bmatrix}$$

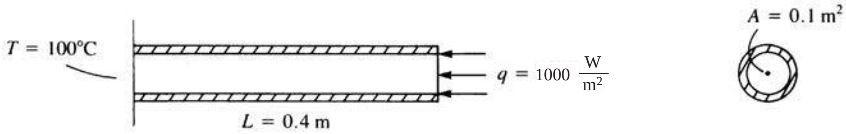
Assemble global equations

$$\pi\begin{bmatrix} 1 & -1 & 0 & 0 \\ & 2 & -1 & 0 \\ & & 2 & -1 \\ & & & 1.694 \end{bmatrix}\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{Bmatrix} = \begin{Bmatrix} 2.894 \\ 2.894+2.894 \\ 2.894+2.894 \\ 2.894+69.4 \end{Bmatrix}$$

Solving simultaneously

$t_1 = 151°\text{F}$, $t_2 = 148°\text{F}$, $t_3 = 140°\text{F}$, $t_4 = 125°\text{F}$,

13.5



$$\frac{AK_{xx}}{L} = \frac{(0.1\text{m}^2)(6\,\frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}})}{0.1\,\text{m}} = 6\ \frac{\text{W}}{{}^\circ\text{C}}$$

$$[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = \frac{AK_{xx}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 6\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\ \frac{\text{W}}{{}^\circ\text{C}}$$

$[k^{(4)}] = [k^{(1)}]$ also

$$\{f^{(1)}\} = \{f^{(2)}\} = \{f^{(3)}\} = 0 \text{ as } Q = 0,\ q^* = 0$$

$$\{f^{(4)}\} = qA\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \left(1000\frac{\text{W}}{\text{m}^2}\right)(0.1\text{m}^2)\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$= 100\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}\text{W}$$

Assemble global equations

$$\begin{bmatrix} 6 & -6 & 0 & 0 & 0 \\ & 12 & -6 & 0 & 0 \\ & & 12 & -6 & 0 \\ & & & 12 & -6 \\ \text{Symmetry} & & & & 6 \end{bmatrix}\begin{Bmatrix} t_1 = 100^\circ\text{C} \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ 0 \\ 0 \\ 0 \\ 100 \end{Bmatrix} \qquad (1)$$

Now $t_1 = 100^\circ$C into Equation (1)

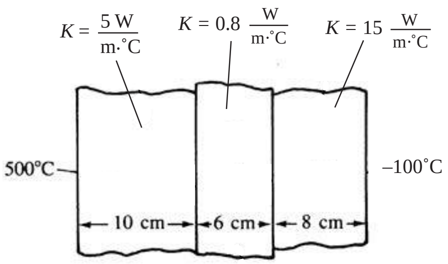
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 12 & -6 & 0 & 0 \\ 0 & -6 & 12 & -6 & 0 \\ 0 & 0 & -6 & 12 & -6 \\ 0 & 0 & 0 & -6 & 6 \end{bmatrix}\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 600 \\ 0 \\ 0 \\ 100 \end{Bmatrix} \qquad (2)$$

Solving Equations (2-5) of Equation (2)

$$t_2 = 116.7^\circ\text{C},\ t_3 = 133.3^\circ\text{C}$$

$$t_4 = 150^\circ\text{C},\ t_5 = 166.7^\circ\text{C}$$

13.6



Area = A (Can use unit A)

$$[k^{(1)}] = \frac{A(5)}{(0.1\text{m})}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 50 & -50 \\ -50 & 50 \end{bmatrix}\frac{\text{W}}{{}^\circ\text{C}}$$

$$[k^{(2)}] = \frac{A(0.8)}{(0.06\,\text{m})}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 13.3 & -13.3 \\ -13.3 & 13.3 \end{bmatrix}$$

$$[k^{(3)}] = \frac{A(15)}{(0.08\,\text{m})}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 187.5 & -187.5 \\ -187.5 & 187.5 \end{bmatrix}$$

$$\{f\}\text{'s} = 0$$

Assemble global equations with $t_1 = 500°\text{C}$ and $t_4 = 100°\text{C}$

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = A\begin{bmatrix} 50 & -50 & 0 & 0 \\ & 50+13.3 & -13.3 & 0 \\ & & 13.3+187.5 & -187.5 \\ \text{Symmetry} & & & 187.5 \end{bmatrix}\begin{Bmatrix} t_1 = 500\,°\text{C} \\ t_2 \\ t_3 \\ t_4 = 100\,°\text{C} \end{Bmatrix}$$

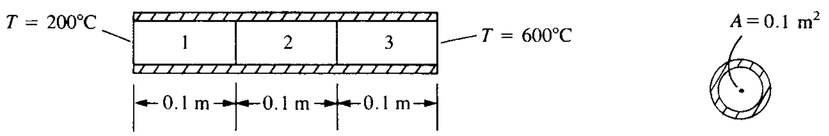
Solving the 2nd and 3rd equations above

$$t_2 = 420.5°\text{C},\ t_3 = 121.2°\text{C}$$

$$q^{(3)} = -K_{xx}\,\frac{(100-121.2)°\text{C}}{0.08\,\text{m}}$$

$$q^{(3)} = 3975\ \frac{\text{W}}{\text{m}^2}$$

13.7



$$\frac{AK_{xx}^{(1)}}{L} = \frac{(0.1\text{m}^2)5}{0.1\text{m}} = 5$$

$$[k^{(1)}] = \frac{AK_{xx}^{(1)}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 5\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(2)}] = 10\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},\quad [k^{(3)}] = 15\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Assemble global equations

$$\begin{bmatrix} 5 & -5 & 0 & 0 \\ & 15 & -10 & 0 \\ & & 25 & -15 \\ \text{Symmetry} & & & 15 \end{bmatrix}\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ 0 \\ 0 \\ F_{4x} \end{Bmatrix} \qquad (1)$$

Boundary conditions

$$t_1 = 200°\text{C},\ t_4 = 600°\text{C}$$

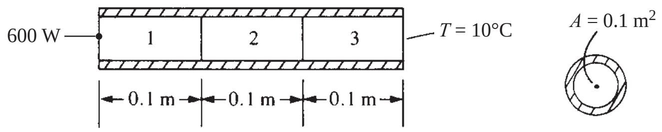
$\therefore$ Equation (1) becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 15 & -10 & 0 \\ & & 25 & 0 \\ \text{Symmetry} & & & 1 \end{bmatrix}\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{Bmatrix} = \begin{Bmatrix} 200 \\ 5(200) \\ 15(600) \\ 600 \end{Bmatrix} \qquad (2)$$

Solving the 2nd and 3rd of Equations (2)

$$t_2 = 418.2°\text{C},\quad t_3 = 527.3°\text{C}$$

13.8 A composite wall is shown below. For element 1, let $K_{xx} = 5\ \frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}}$, for element 2 let $K_{xx} = 10\ \frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}}$, for element 3 let $K_{xx} = 15\ \frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}}$. The left end has a heat source of 600 W applied to it. The right end is held at 10°C. Determine the left end temperature and the interface temperatures and the heat flux through element 3.



$$[k^{(1)}] = \frac{(0.1)(5)}{0.1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}$$

$$[k^{(2)}] = \frac{(0.1)(10)}{0.1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix}$$

$$[k^{(3)}] = \frac{(0.1)(15)}{0.1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

$$\{f^{(2)}\} = \{f^{(3)}\} = 0 \quad \{f^{(1)}\} = 600\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 600 \\ 0 \\ 0 \\ F_{4x} \end{bmatrix} = \begin{bmatrix} 5 & -5 & 0 & 0 \\ -5 & 15 & -10 & 0 \\ 0 & -10 & 25 & -15 \\ 0 & 0 & -15 & 15 \end{bmatrix}\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{Bmatrix}$$

$400 = 5t_1 - 5t_2$ $\qquad t_1 = 230°\text{C}$

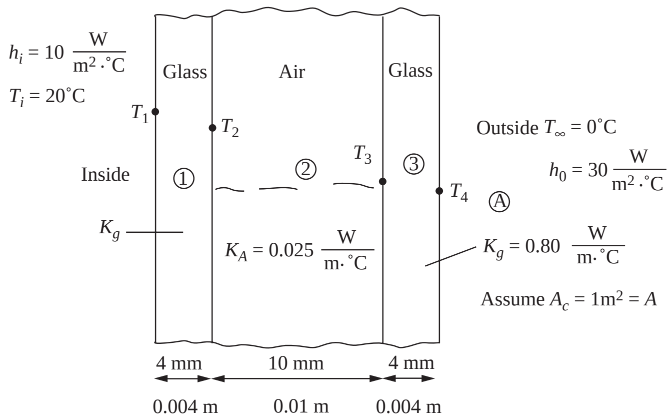
$0 = -5t_1 + 15t_2 - 10\,t_3$ $\qquad t_2 = 110°\text{C}$

$0 = -10t_2 + 25t_3 - 15\,(10)$ $\qquad t_3 = 50°\text{C}$

$F_{yx} = 15\,(50) + 15\,(10) = -\,600\text{ W}$

$$q_x = -15\left[-\frac{1}{0.1}\ \frac{1}{0.1}\right]\begin{Bmatrix} 50 \\ 10 \end{Bmatrix} = 6000\,\frac{\text{W}}{\text{m}^2}$$

13.9



Find T_1 T_2, T_3, T_4, Q (heat transfer through the double pane) (use $A = 1\ \text{cm}^2$)

$$[k^{(1)}] = \frac{AK_g}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + h_i A \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1\text{m}^2\left(0.80\,\frac{\text{W}}{\text{m}\cdot\text{K}}\right)}{0.004\,\text{m}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 10\frac{\text{W}}{\text{m}^2\cdot{}^\circ\text{C}}(1\text{m}^2)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 210 & -200 \\ -200 & 200 \end{bmatrix}\frac{\text{W}}{\text{K}}$$

$$[k^{(2)}] = \frac{AK_A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1\text{m}^2\left(0.025\,\frac{\text{W}}{\text{m}\cdot\text{K}}\right)}{0.01\,\text{m}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2.5 & -2.5 \\ -2.5 & 2.5 \end{bmatrix}\frac{\text{W}}{\text{K}}$$

$$[k^{(3)}] = \frac{AK_g}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + h_0 A \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1\text{m}^2\left(0.80\,\frac{\text{W}}{\text{m}\cdot\text{K}}\right)}{0.004\,\text{m}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 30\frac{\text{W}}{\text{m}^2\cdot{}^\circ\text{C}}(1\text{m}^2)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 200 & -200 \\ -200 & 230 \end{bmatrix}$$

$$\{f^{(1)}\} = h_i\, T_\infty\, A \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = 10\frac{\text{W}}{\text{m}^2\cdot\text{K}}\ (293\ \text{K})\ (1\text{m}^2)\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = 2930\ \text{W}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$\{f^{(2)}\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

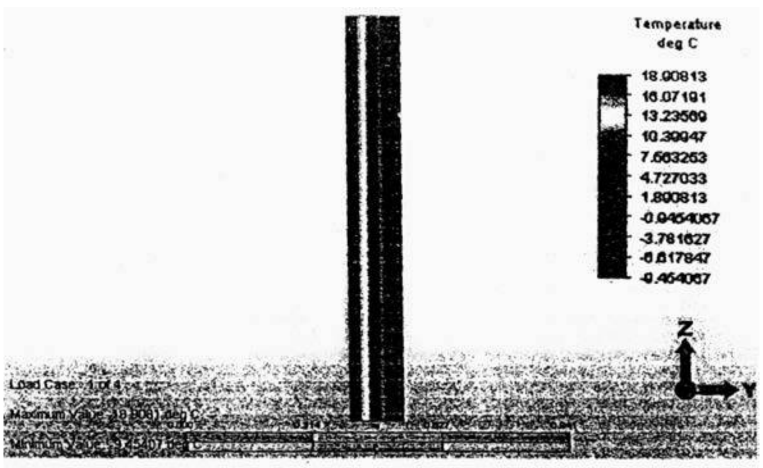
$$\{f^{(3)}\} = h_0\, T_\infty\, A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = 30\ \frac{\text{W}}{\text{m}^2\cdot\text{K}}\ (273\ \text{K})\ (1\text{m}^2)\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = 8190\ \text{W}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

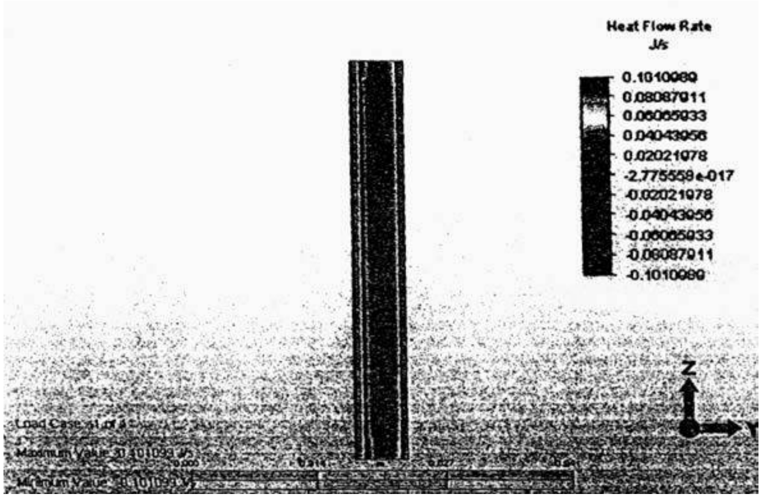
$$\{F\} = [K]\,[T]$$

$$\begin{Bmatrix} F_1 = 2930\,\text{W} \\ F_2 = 0 \\ F_3 = 0 \\ F_4 = 8190\ \text{W} \end{Bmatrix} = \begin{bmatrix} 210 & -200 & 6 & 0 \\ -200 & 200+2.5 & -2.5 & 0 \\ 0 & -2.5 & 2.5+200 & -200 \\ 0 & 0 & -200 & 230 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix}$$

$$T_1 = 289.3\,K = 16.3^\circ\text{C}$$
$$T_2 = 289.1\,K = 16.1^\circ\text{C}$$
$$T_3 = 279.4\,K = 1.4^\circ\text{C}$$
$$T_4 = 274.2\,K = 1.2^\circ\text{C}$$

13.10





13.11

Unit definition

Given in problem statement (Mathcad used to solve this one)

$$L_{\text{total}} = 20\ \text{cm}$$

$$K_{xx} = 15\ \frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}}$$

$$h_1 = 50\ \frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}}$$

$$h_2 = 80\ \frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}}$$

$$T_{\text{left end}} = 100^\circ\text{C}$$

$$T_{\text{inf}} = 20^\circ\text{C}$$

$$\text{dia} = 0.5\ \text{cm}$$

$$\text{radius} = \frac{\text{dia}}{2}$$

$$P = \pi\ \text{dia} \qquad A = \pi\ \text{radius}^2$$

$$L = \frac{L_{\text{total}}}{4}$$

$$P = 0.016\ \text{m}$$

$$A = 0.00002\ \text{m}^2$$

$$\text{radius} = 0.0025\ \text{m}$$

$$M = 150\ \frac{\text{W}}{\text{m}^3\cdot{}^\circ\text{C}}$$

$$y_1(x) = M\,x + h_1$$

$$k_{h_1} = \begin{bmatrix} P\int_0^{0.05\text{m}} y_1(x)\left(1-\frac{x}{L}\right)^2 dx & P\int_0^{0.05\text{m}} y_1(x)\left(\frac{x}{L}-\frac{x^2}{L^2}\right)dx \\ P\int_0^{0.05\text{m}} y_1(x)\left(\frac{x}{L}-\frac{x^2}{L^2}\right)dx & P\int_0^{0.05\text{m}} y_1(x)\left(\frac{x^2}{L^2}\right)dx \end{bmatrix}$$

Develop stiffness matricies for each element

$$x = \frac{L_{\text{total}}}{4}$$

$$[k_{h_1}] = \begin{pmatrix} 0.0136 & 0.007 \\ 0.007 & 0.0146 \end{pmatrix} \frac{\text{W}}{°\text{C}}$$

$$[k_1] = \frac{AK_{xx}}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + [k_{h_1}]$$

$$[k_1] = \begin{pmatrix} 0.0195 & 0.0011 \\ 0.0011 & 0.0205 \end{pmatrix} \frac{\text{W}}{°\text{C}}$$

$$y_2(x) = Mx + y_1(x)$$

$$[k_{h_2}] = \begin{bmatrix} P\int_0^{0.05\text{m}} y_2(x)\left(1-\frac{x}{L}\right)^2 dx & P\int_0^{0.05\text{m}} y_2(x)\left(\frac{x}{L}-\frac{x^2}{L^2}\right)dx \\ P\int_0^{0.05\text{m}} y_2(x)\left(\frac{x}{L}-\frac{x^2}{L^2}\right)dx & P\int_0^{0.05\text{m}} y_2(x)\left(\frac{x^2}{L^2}\right)dx \end{bmatrix}$$

$$[k_{h_2}] = \begin{pmatrix} 0.0141 & 0.0075 \\ 0.0075 & 0.016 \end{pmatrix} \frac{\text{W}}{°\text{C}}$$

$$[k_2] = \frac{AK_{xx}}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + [k_{h_2}]$$

$$[k_2] = \begin{pmatrix} 0.02 & 0.0016 \\ 0.0016 & 0.0219 \end{pmatrix} \frac{\text{W}}{°\text{C}}$$

$$y_3(x) = M\,x + y_2(x)$$

$$[k_{h_3}] = \begin{bmatrix} P\int_0^{0.05\text{m}} y_3(x)\left(1-\frac{x}{L}\right)^2 dx & P\int_0^{0.05\text{m}} y_3(x)\left(\frac{x}{L}-\frac{x^2}{L^2}\right)dx \\ P\int_0^{0.05\text{m}} y_3(x)\left(\frac{x}{L}-\frac{x^2}{L^2}\right)dx & P\int_0^{0.05\text{m}} y_3(x)\left(\frac{x^2}{L^2}\right)dx \end{bmatrix}$$

$$[k_{h_3}] = \begin{pmatrix} 0.015 & 8.018\times10^{-3} \\ 8.018\times10^{-3} & 0.018 \end{pmatrix} \frac{\text{W}}{°\text{C}}$$

$$[k_3] = \frac{AK_{xx}}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + [k_{h_3}]$$

$$[k_3] = \begin{pmatrix} 0.02 & 2.127\times10^{-3} \\ 2.127\times10^{-3} & 0.023 \end{pmatrix} \frac{\text{W}}{°\text{C}}$$

$$y_4(x) = Mx + y_3(x)$$

$$[k_{h_4}] = \begin{bmatrix} P\int_0^{0.05\text{m}} y_4(x)\left(1-\frac{x}{L}\right)^2 dx & P\int_0^{0.05\text{m}} y_4(x)\left(\frac{x}{L}-\frac{x^2}{L^2}\right)dx \\ P\int_0^{0.05\text{m}} y_4(x)\left(\frac{x}{L}-\frac{x^2}{L^2}\right)dx & P\int_0^{0.05\text{m}} y_4(x)\left(\frac{x^2}{L^2}\right)dx \end{bmatrix}$$

$$[k_{h_4}] = \begin{pmatrix} 0.0151 & 0.0085 \\ 0.0085 & 0.019 \end{pmatrix} \frac{\text{W}}{°\text{C}}$$

$$[k_4] = \frac{AK_{xx}}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + [k_{h_4}] + h_2\,A\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[k_4] = \begin{pmatrix} 0.0209 & 0.0026 \\ 0.0026 & 0.0264 \end{pmatrix} \frac{\text{W}}{°\text{C}}$$

$$[K] = \begin{pmatrix} k_{1_{0,0}} & k_{1_{0,1}} & 0 & 0 & 0 \\ k_{1_{0,0}} & k_{1_{1,1}} + k_{2_{0,0}} & k_{2_{0,1}} & 0 & 0 \\ 0 & k_{2_{1,0}} & k_{2_{1,1}} + k_{3_{0,0}} & k_{3_{0,1}} & 0 \\ 0 & 0 & k_{3_{1,0}} & k_{3_{1,1}} + k_{4_{0,0}} & k_{4_{0,1}} \\ 0 & 0 & 0 & k_{4_{1,0}} & k_{4_{1,1}} \end{pmatrix}$$

Global $[K]$

$$[K] = \begin{pmatrix} 0.0195 & 0.0011 & 0 & 0 & 0 \\ 0.0011 & 0.0404 & 0.0016 & 0 & 0 \\ 0 & 0.0016 & 0.0424 & 0.0021 & 0 \\ 0 & 0 & 0.0021 & 0.0443 & 0.0026 \\ 0 & 0 & 0 & 0.0026 & 0.0264 \end{pmatrix} \frac{\text{W}}{°\text{C}}$$

Element force matricies for each element

$$y_1(x) = 57.5 \ \frac{\text{W}}{\text{m}^2 \cdot °\text{C}} \qquad h_{1\text{ave}} = \frac{h_1 + y_1(x)}{2} \qquad h_{1\text{ave}} = 53.75 \ \frac{\text{W}}{\text{m}^2 \cdot °\text{C}}$$

$$y_2(x) = 65 \frac{\text{W}}{\text{m}^2 \cdot °\text{C}} \qquad h_{2\text{ave}} = \frac{y_1(x) + y_2(x)}{2} \qquad h_{2\text{ave}} = 61.25 \ \frac{\text{W}}{\text{m}^2 \cdot °\text{C}}$$

$$y_3(x) = 72.5 \ \frac{\text{W}}{\text{m}^2 \cdot °\text{C}} \qquad h_{3\text{ave}} = \frac{y_2(x) + y_3(x)}{2} \qquad h_{3\text{ave}} = 68.75 \ \frac{\text{W}}{\text{m}^2 \cdot °\text{C}}$$

$$y_4(x) = 80 \ \frac{\text{W}}{\text{m}^2 \cdot °\text{C}} \qquad h_{4\text{ave}} = \frac{y_3(x) + y_4(x)}{2} \qquad h_{4\text{ave}} = 76.25 \ \frac{\text{W}}{\text{m}^2 \cdot °\text{C}}$$

$$\{f_1\} = \frac{h_{1\text{ave}} T_{\text{inf}} \, P \, L}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad f_1 = \begin{pmatrix} 6.188 \\ 6.188 \end{pmatrix} \text{W}$$

$$\{f_2\} = \frac{h_{2\text{ave}} T_{\text{inf}} \, P \, L}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad f_2 = \begin{pmatrix} 7.051 \\ 7.051 \end{pmatrix} \text{W}$$

$$\{f_3\} = \frac{h_{3\text{ave}} T_{\text{inf}} \, P \, L}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad f_3 = \begin{pmatrix} 7.914 \\ 7.914 \end{pmatrix} \text{W}$$

$$\{f_4\} = \left[\frac{h_{4\text{ave}} T_{\text{inf}} \, P \, L}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] + \left[h_2 \, T_{\text{inf}} \, A \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \qquad f_4 = \begin{pmatrix} 8.778 \\ 9.238 \end{pmatrix} \text{W}$$

$$F_1 = f_{1_{0,0}}$$

$$F_2 = f_{1_{1,0}} + f_{2_{0,0}}$$

$$F_3 = f_{2_{1,0}} + f_{3_{0,0}}$$

$$F_4 = f_{3_{1,0}} + f_{4_{0,0}}$$

$$F_5 = f_{4_{1,0}}$$

$$F_1 = 6.188 \text{ W}, F_2 = 13.239 \text{ W}, F_3 = 14.966 \text{ W}, F_4 = 16.692 \text{ W}, F_5 = 9.238 \text{ W}$$

Set up equations to solve for temperature

$$[K_{\text{mod}}] = \begin{pmatrix} k_{1_{1,1}} + k_{2_{0,0}} & k_{2_{0,1}} & 0 & 0 \\ k_{2_{1,0}} & k_{2_{1,1}} + k_{3_{0,0}} & k_{3_{0,1}} & 0 \\ 0 & k_{3_{1,0}} & k_{3_{1,1}} + k_{4_{0,0}} & k_{4_{0,1}} \\ 0 & 0 & k_{4_{1,0}} & k_{4_{1,1}} \end{pmatrix}$$

Guess

$$t_2 = 75°\text{C}$$
$$t_3 = 60°\text{C}$$
$$t_4 = 50°\text{C}$$
$$t_5 = 25°\text{C}$$

Given

$$\begin{bmatrix} F_2 + (-K_{1,0}\, T_{\text{left end}}) \\ F_3 \\ F_4 \\ F_5 \end{bmatrix} = K_{\text{mod}} \begin{pmatrix} t_2 \\ t_3 \\ t_4 \\ t_5 \end{pmatrix}$$

$$\begin{pmatrix} t_2 \\ t_3 \\ t_4 \\ t_5 \end{pmatrix} = \text{Find}\,(t_2, t_3, t_4, t_5)$$

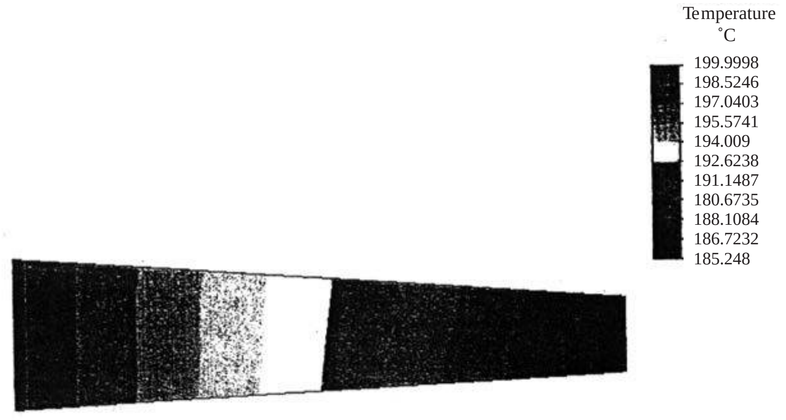
$$\begin{pmatrix} t_2 \\ t_3 \\ t_4 \\ t_5 \end{pmatrix} = \begin{pmatrix} 30.717 \\ 51.077 \\ 69.113 \\ 42.348 \end{pmatrix} °\text{C}$$

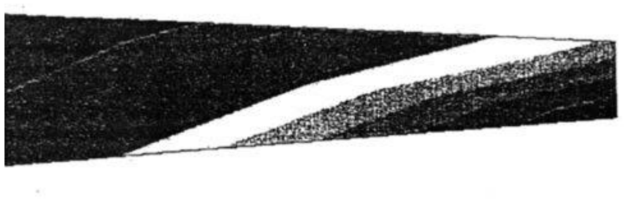
$$F_1 = (K_{0,\,0}\, K_{0,}\, 1) \begin{pmatrix} T_{\text{left end}} \\ t_2 \end{pmatrix}$$

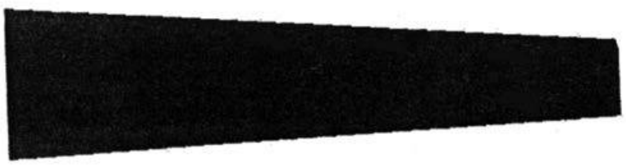
$$F_1 = 7.614 \text{ W}$$

13.12 A tapered aluminum fin ($k = \frac{200\,\text{W}}{\text{m}\cdot°\text{C}}$) shown in Figure P13-12, has a circular cross-section with base diameter of 1 cm and tip diameter of 0.5 cm. The base is maintained at 200°C and looses heat by convection to the surrounding at $T_\infty = 10°\text{C}$, $h = 150 \frac{\text{W}}{\text{m}^2\cdot°\text{C}}$. The tip of the fin is insulated. Assume one-dimensional heat flow and determine the temperatures at the quarter points along the fin. What is the rate of heat loss in Watts through each element? Use four elements with an average cross-sectional area for each element.

(Algor results)







13.13

$$[k^{(1)}] = \frac{AK_{xx}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ Use unit area, } A = 1\text{ ft}^2$$

$$= \frac{(1\,\text{ft}^2)\left(0.10\,\dfrac{\text{Btu}}{\text{h}\cdot\text{ft}\cdot{}^\circ\text{F}}\right)}{\dfrac{0.50\text{ in.}}{12\text{ in.}}(1\,\text{ft})}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 1.5\,\frac{\text{Btu.1ft}^2}{\text{h}\cdot\text{ft}^2\cdot{}^\circ\text{F}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$[k^{(1)}] = \begin{bmatrix} 3.9 & -2.4 \\ -2.4 & 2.4 \end{bmatrix}$$

$$[k^{(2)}] = \frac{(1\,\text{ft}^2)(0.02)}{\dfrac{5\text{ in.}}{12\text{ in.}}(1\,\text{ft})}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0.048 & -0.048 \\ -0.048 & 0.048 \end{bmatrix}$$

$$[k^{(3)}] = \frac{(1\,\text{ft}^2)(0.8)}{\dfrac{0.5\text{ in.}}{12\text{ in.}}(1\,\text{ft})}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 4\,\frac{\text{Btu}}{\text{h}\cdot\text{ft}^2\cdot{}^\circ\text{F}}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[k^{(3)}] = \begin{bmatrix} 19.2 & -19.2 \\ -19.2 & 23.2 \end{bmatrix}$$

$$\{f^{(1)}\} = h\,T_\infty A\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \left(1.5\,\frac{\text{Btu}}{\text{h}\cdot\text{ft}^2\cdot{}^\circ\text{F}}\right)(1\,\text{ft}^2)(65^\circ\text{F})\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 97.5 \\ 0 \end{Bmatrix}$$

$$\{f^{(3)}\} = h\,T_\infty A\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \left(4.0\,\frac{\text{Btu}}{\text{h}\cdot\text{ft}^2\cdot{}^\circ\text{F}}\right)(1\text{ft}^2)(0^\circ\text{F})\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\{f^{(2)}\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

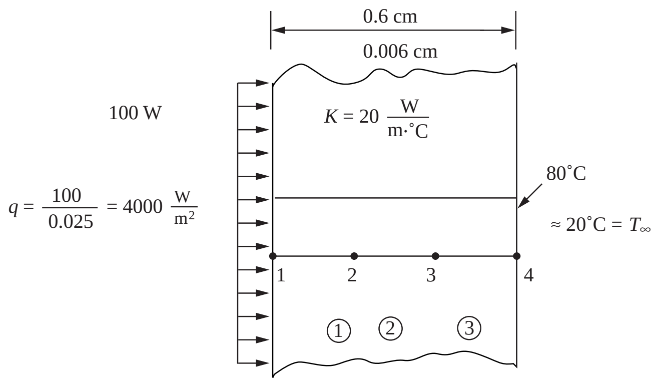
$$\{F\} = [K]\,\{t\}$$

$$\begin{Bmatrix} 97.5 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 3.9 & -2.4 & 0 & 0 \\ -2.4 & 2.448 & -0.048 & 0 \\ 0 & -0.048 & 19.248 & -19.2 \\ 0 & 0 & -19.2 & 23.2 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{Bmatrix}$$

Solving for t's

$$t_1 = 63.05°\text{F}, \quad t_2 = 61.83°\text{F}, \quad t_3 = 0.884°\text{F}, \quad t_4 = 0.731°\text{F}$$

13.15 Base plate of an iron is 0.6 cm thick. The plate is subjected to 600 W of power over a base surface area of 250 cm^2 resulting in a uniform flux generated on the inside surface. The thermal conductivity of the metal base plate is $k = 20\ \frac{\text{W}}{\text{m}\cdot°\text{C}}$. The outside ambient temperature of plate is 20°C at steady state. Assume 1-D heat transfer through the plate thickness. Using 3 elements, model the plate to determine the temperatures at the inner surface and interior $\frac{1}{3}$ points.



From Mathcad solution

$A = 0.025$ $\qquad Kxx = 20$ $\qquad h = 20$ $\qquad L = 20$

$q = \dfrac{100}{\text{A}}$ $\qquad q = 4 \times 10^3$ $\qquad T_{\text{inf}} = 20$ $\qquad qA = 100$

$$[k_1] = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad k_1 = A\frac{K_{xx}}{L}k_1 \qquad [k_1] = \begin{pmatrix} 25 & -25 & 0 & 0 \\ -25 & 25 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[k_{2c}] = A\frac{K_{xx}}{L}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad k_2 = k_2c \qquad [k_2] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 25 & -25 & 0 \\ 0 & -25 & 25 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[k_{3c}] = A\frac{K_{xx}}{L}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \qquad [k_{3h}] = hA\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad hA = 0.5$$

$[k_3] = [k_{3c}] + [k_{3h}]$ $\qquad [k_3] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 25 & -25 \\ 0 & 0 & -25 & 25.5 \end{pmatrix}$

$\{f_q\} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} q\,A \qquad \{f_q\} = \begin{pmatrix} 100 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \{f_h\} = h\,T_{\text{inf}}\;A \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \{f_h\} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 10 \end{pmatrix}$

$\{f\} = \{f_q\} + \{f_h\} \qquad \{f\} = \begin{pmatrix} 100 \\ 0 \\ 0 \\ 10 \end{pmatrix} \qquad [K] = [k_1] + [k_2] + [k_3]$

$$[k] = \begin{pmatrix} 25 & -25 & 0 & 0 \\ -25 & 50 & -25 & 0 \\ 0 & -25 & 50 & -25 \\ 0 & 0 & -25 & 25.5 \end{pmatrix}$$

$\text{temps} = 1\text{ solve }(k, f) \qquad \text{temps} = \begin{pmatrix} 232 \\ 228 \\ 224 \\ 220 \end{pmatrix} \qquad \text{temps}_1 = 228$

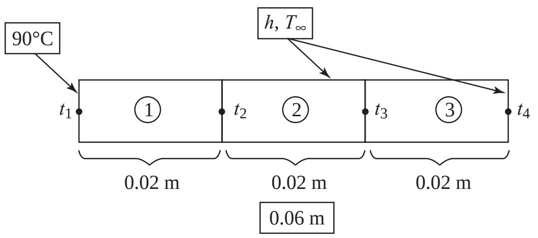
$$q_1 = -\;Kxx\left(\frac{-1}{L}\;\;\frac{1}{L}\right)\cdot\begin{pmatrix} \text{temps}_1 \\ \text{temps}_2 \end{pmatrix}$$

Remember to use the left bracket key to get the subscript temps 1 and temps 2.

$$q_1 = 4 \times 10^3$$

13.16 A hot surface of a plate is cooled by attaching fins (called pin fins) to it. The surface of the plate (left end of the fin) is at 90°C. The typical fin is 6 cm (0.06 m) long and has a cross-sectional area of 5×10^{-6} m^2 with a perimeter of 0.006 m. The fin is made of copper with $k = 400\ \frac{\text{W}}{\text{m}\cdot°\text{C}}$. The temperature of the surrounding air is $T_\infty = 20°\text{C}$ with heat transfer coefficient on the surface (including the right end) estimated to be $10\ \frac{\text{W}}{\text{m}^2\cdot°\text{C}}$.

Use three elements in your model to estimate the temperature distribution along the fin length.



$$[k^{(1)}] = \frac{AK_{xx}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{5\times10^{-6}\cdot400}{0.02}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$+\frac{10\cdot0.006\cdot0.02}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[k^{(1)}] = \begin{bmatrix} 0.1004 & -0.0998 \\ -0.0998 & 0.1004 \end{bmatrix} \frac{\text{W}}{{}^\circ\text{C}} = [k^{(2)}]$$

$$[k^{(3)}] = [k^{(1)}] + hA \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.1004 & -0.0998 \\ -0.0998 & 0.1004 \end{bmatrix} + 10.5 \times 10^{-6} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad [k^{(3)}] = \begin{bmatrix} 0.1004 & -0.0998 \\ -0.0998 & 0.10045 \end{bmatrix} \left(\frac{\text{W}}{{}^\circ\text{C}}\right)$$

$$\{f^{(1)}\} = \frac{h T_\infty PL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \frac{10 \cdot 20^\circ\text{C} \cdot 0.006\,\text{m} \cdot 0.02\,\text{m}}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.012 \\ 0.012 \end{Bmatrix} \text{W}$$

$$\{f^{(2)}\} = \{f^{(1)}\}$$

$$\{f^{(3)}\} = \{f^{(1)}\} + h\,T_\infty A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \{f^{(1)}\} + 10.20^\circ\text{C} \cdot 5 \times 10^{-6}\,\text{m}^2 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

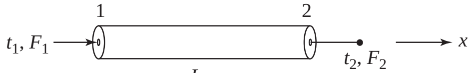
$$\{f^{(3)}\} = \begin{Bmatrix} 0.012 \\ 0.012 \end{Bmatrix} \text{W}$$

$$\begin{bmatrix} 0.1004 & -0.0998 & 0 & 0 \\ & 2(0.1004) & -0.0998 & 0 \\ & & 2(0.1004) & -0.0998 \\ \text{Symmetry} & & & 0.10045 \end{bmatrix} \begin{Bmatrix} t_1 = 90^\circ\text{C} \\ t_2 \\ t_3 \\ t_4 \end{Bmatrix} = \begin{Bmatrix} 0.012 + h \\ 2(0.012) \\ 2(0.012) \\ 0.013 \end{Bmatrix}$$

Solving Equations (2-4),

$$t_2 = 87.95^\circ\text{C}, \quad t_3 = 86.72^\circ\text{C}, \quad t_4 = 86.24^\circ\text{C}$$

13.17



F's in. Btu's, t's in °F

Fourier's law

$$q = -K_{xx} \frac{dT}{dx} \left(\frac{\text{Btu}}{\text{ft}^2}\right) \tag{1}$$

Want to link thermal inputs F_1 and F_2 to nodal temperatures t_1 and t_2

$$[T] = [N]\{t\} = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} \tag{2}$$

$$\frac{d[T]}{dx} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} = [B]\{t\} \tag{3}$$

Total heat input at node 1 is

$$F_1 = q\,A \tag{4}$$

and at node 2

$F_2 = -\,qA$ (negative sign accounts for the positive direction of F_2 being an output at node 2) (5)

Using Equations (1) and (3) in (4) and (5), we have

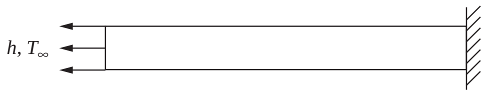
$$F_1 = -\,K_{xx}\,[B]\,\{t\}\,A \qquad F_2 = K_{xx}\,[B]\,\{t\}\,A \tag{6}$$

or

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{K_{xx}A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} \tag{7}$$

$$\therefore\ [k] = \frac{K_{xx}A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{element conductivity matrix}$$

13.18



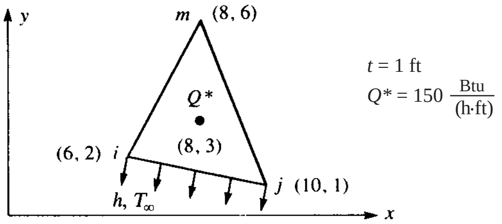
See Equation (13.4.28)

Now convection from left end

$$[K_{h_{\text{left}}}] = hA\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\{f_{h_{\text{left}}}\} = h\,T_\infty\,A\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

13.19



$$[k] = [B]^T\,[D]\,[B]\,t\,A + \int_s h[N]^T\,[N]\,ds$$

$$\alpha_i = x_j\,y_m - x_m\,y_j = (10)\,(6) - (8)\,(1) = 52$$

$$\alpha_j = x_m\,y_i - x_i\,y_m = (8)\,(2) - (6)\,(6) = -\,20$$

$$\alpha_m = x_i\,y_j - x_j\,y_i = (6)\,(1) - (10)\,(2) = -\,14$$

$$\beta_i = y_j - y_m = 1 - 6 = -5$$

$$\beta_j = y_m - y_i = 6 - 2 = 4$$

$$\beta_m = y_i - y_j = 2 - 1 = 1$$

$$\gamma_i = x_m - x_j = 8 - 10 = -\,2$$

$$\gamma_j = x_i - x_m = 6 - 8 = -\,2$$

$$\gamma_m = x_j - x_i = 10 - 6 = 4$$

$$2A = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_i & y_i \\ 1 & x_m & y_m \end{vmatrix} = \begin{vmatrix} 1 & 6 & 2 \\ 1 & 10 & 1 \\ 1 & 8 & 6 \end{vmatrix} = 18 \text{ ft}^2$$

Conduction part of $[k] = [k_c]$

$$[k_c] = \frac{1}{2A}\begin{bmatrix} \beta_i & \gamma_i \\ \beta_i & \gamma_j \\ \beta_m & \gamma_m \end{bmatrix}\begin{bmatrix} K_{xx} & 0 \\ 0 & K_{yy} \end{bmatrix}\frac{1}{2A}\begin{bmatrix} \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} t\,A$$

$$= \frac{1}{18}\begin{bmatrix} -5 & -2 \\ 4 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}\frac{1}{18}\begin{bmatrix} -5 & 4 & 1 \\ -2 & -2 & 4 \end{bmatrix} (1 \text{ ft})\,(9 \text{ ft}^2)$$

$$[k_c] = \begin{bmatrix} 12.08 & -6.67 & -5.42 \\ & 8.33 & -1.67 \\ \text{Symmetry} & & 7.08 \end{bmatrix}$$

Convection part of $[k] = [k_h]$

$$L_{ij} = 4.123 \text{ ft}$$

$$[k_h] = \frac{hL_{ij}t}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{(20)(4.123)(1)}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 27.49 & 13.74 & 0 \\ & 27.49 & 0 \\ \text{Symmetry} & & 0 \end{bmatrix}$$

$$\text{Total } [k] = [k_c] + [k_h]$$

$$[k] = \begin{bmatrix} 39.57 & 7.08 & -5.42 \\ & 35.82 & -1.67 \\ \text{Symmetry} & & 7.08 \end{bmatrix} \frac{\text{Btu}}{\text{h}\cdot{}^\circ\text{F}}$$

Force matrix

$$\{f\} = \int_V [N]^T Q\, dV + \int_{S_2} [N]^T q\, ds + \int_{S_3} [N]^T h\, T_\infty ds$$

$$q = 0$$

For point source $Q^* = \bar{Q}$

$$\{f_Q\} = \bar{Q}t\begin{Bmatrix} N_i \\ N_j \\ N_m \end{Bmatrix}\Bigg|_{\substack{x=8 \\ y=3}}$$

$$N_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y)\Big|_{\substack{x=8 \\ y=3}}$$

$$N_i = \frac{1}{18}(52 + (-5)\,(8) + (-2)\,(3)) = 0.333$$

$$N_j = \frac{1}{18}(-20 + (4)(8) + (-2)(3)) = 0.333$$

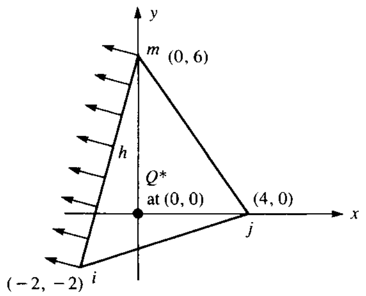
$$N_m = \frac{1}{18}(-14 + (1)(8) + (4)(3)) = 0.333$$

$$\therefore \ \{f_Q\} = 150\ (1)\begin{Bmatrix}0.333\\0.333\\0.333\end{Bmatrix} = \begin{Bmatrix}50\\50\\50\end{Bmatrix}\ \frac{\text{Btu}}{\text{h}}$$

$$\{f_h\} = \frac{hT_\infty L_{ij} t}{2}\begin{Bmatrix}1\\1\\0\end{Bmatrix} = \frac{(20)(70)(4.123)1}{2}\begin{Bmatrix}1\\1\\0\end{Bmatrix}$$

$$\{f_h\} = \begin{Bmatrix}2886\\2886\\0\end{Bmatrix} \ \therefore \ \ \{f\} = \begin{Bmatrix}50+2886\\50+2886\\50+0\end{Bmatrix} = \begin{Bmatrix}2936\\2936\\50\end{Bmatrix}\frac{\text{Btu}}{\text{h}}$$

13.20



$$[k] = t\,A\,[B]^T\,[D]\,[B] + \frac{h\,L_{im}}{6}\begin{bmatrix}2 & 0 & 1\\0 & 0 & 0\\1 & 0 & 2\end{bmatrix} \qquad (1)$$

$$2A = \begin{vmatrix}1 & -2 & -2\\1 & 4 & 0\\1 & 0 & 6\end{vmatrix} = 44$$

$$L_{im} = 8.246 \text{ m}$$

$$\beta_i = -6,\ \beta_j = 8,\ \beta_m = -2$$

$$\gamma_i = -4,\ \gamma_j = -2,\ \gamma_m = 6$$

$$N_i = \frac{1}{44}\,[24 + (-6)(0) + 0] = 0.545$$

$$N_j = \frac{1}{44}\,[12 + 8(0) + 0] = 0.278$$

$$N_m = \frac{1}{44}\,[8 + (-2)(0) + 0] = 0.181$$

By (1)

$$[K] = \frac{1}{4(22)}\begin{bmatrix} -6 & -4 \\ 8 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}\begin{bmatrix} -6 & 8 & -2 \\ -4 & -2 & 6 \end{bmatrix}$$

$$+ \frac{20(8.246)}{6}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$[k] = \begin{bmatrix} 63.9 & -6.82 & 25.45 \\ & 11.6 & -4.77 \\ \text{Symmetry} & & 61.82 \end{bmatrix} \frac{\text{W}}{°\text{C}}$$

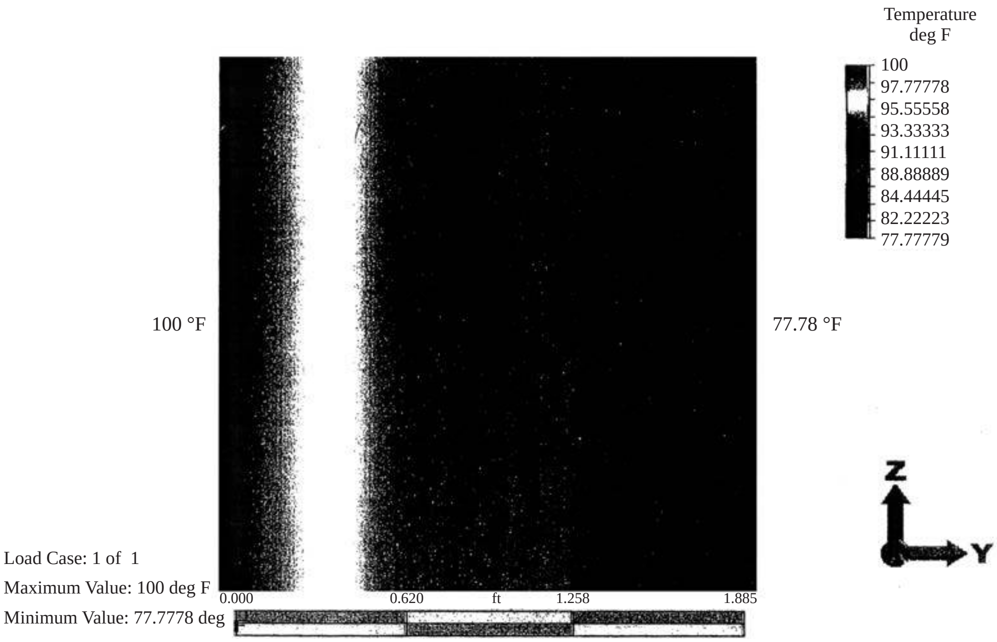
$$\{f\} = \int_v [N]^T\big|_{(0,0)}\ Q^*\,dV + \int_{S_3} hT_\infty [N]^T_{\text{Along } S_3}\,ds$$

$$= Q^* \times \begin{Bmatrix} N_i \\ N_j \\ N_m \end{Bmatrix}_{(0,0)} + \frac{hT_\infty L_{im}}{2}\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

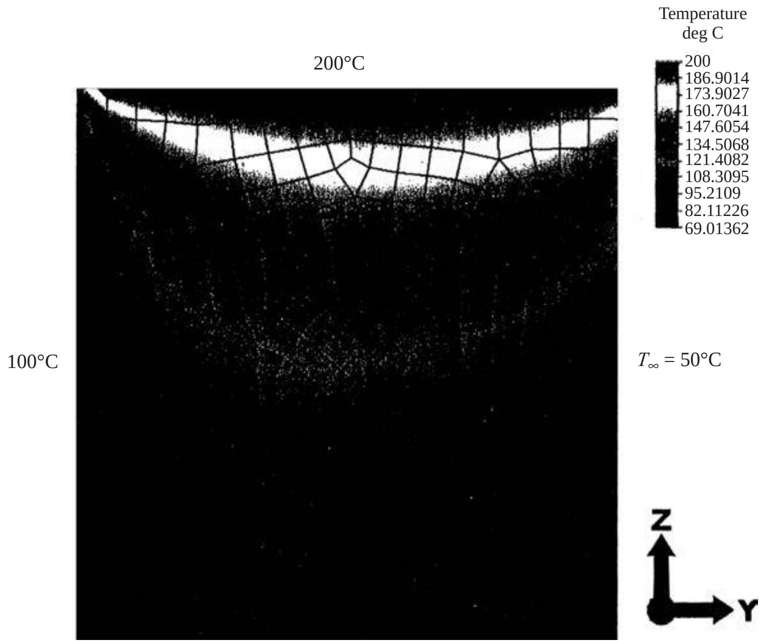
$$= 100\begin{Bmatrix} 0.545 \\ 0.273 \\ 0.183 \end{Bmatrix} + \frac{(20)(15)(8.246)}{2}\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

$$\{f\} = \begin{Bmatrix} 1291 \\ 27.3 \\ 1254 \end{Bmatrix} \text{W}$$

13.21



13.22 For the square plate in figure P13-22, determine the temperature distribution. Let $K_{xx} = K_{yy} = 10\,\frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}}$, and $h = 20\,\frac{\text{W}}{\text{m}^2\cdot{}^\circ\text{C}}$. The temperature along the left side is maintained at 100 °C and that along the top side is maintained at 200 °C.



The maximum temperature is along the top edge of the plate and is 200 °C. The smallest temperature is at the lower right edge of the plate and is 69.01 °C.

13.23

HEAT—Problem 13–23

KXX = 1.0 *KYY* = 1.0

CONVECTION COEFF = 0.0

FLUID TEMPERATURE = 0.0

SEMI–BANDWIDTH = 4

NEL	NODE	NUMBER		X(1)	Y(1)
1	1	2	3	0.0000	2.0000
2	2	5	3	0.0000	0.0000
3	3	5	4	0.7500	1.0000
4	1	3	4	0.0000	2.0000
5	4	5	6	1.5000	2.0000
6	5	e	6	1.5000	0.0000
7	6	8	7	2.2500	1.0000
8	4	6	7	1.5000	2.0000

X(2)	Y(2)	X(3)	Y(3)
0.0000	0.0000	0.7500	1.0000
1.5000	0.0000	0.7500	1.0000

1.5000	0.0000	1.5000	2.0000
0.7500	1.0000	1.5000	2.0000
1.5000	0.0000	2.2500	1.0000
3.0000	0.0000	2.2500	1.0000
3.0000	0.0000	3.0000	2.0000
2.2500	1.0000	3.0000	2.0000

PRESCRIBED NODAL TEMPERATURE VALUES

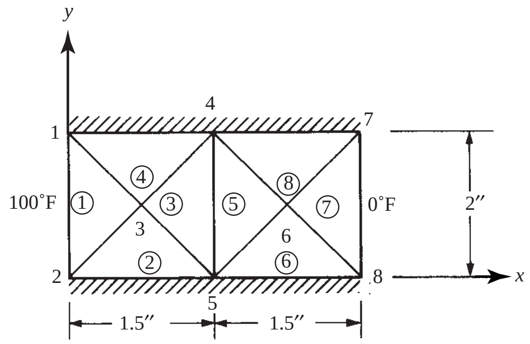
1	0.10000E+03
2	0.10000E+03
7	0.00000E+00
8	0.00000E+00

RESULTING NODAL TEMPERATURE VALUES

1	0.10000E+03	2	0.10000E+03
3	0.75000E+02	4	0.50000E+02
5	0.50000E+02		
6	0.25000E+02	7	0.00000E+00
8	0.00000E+00		

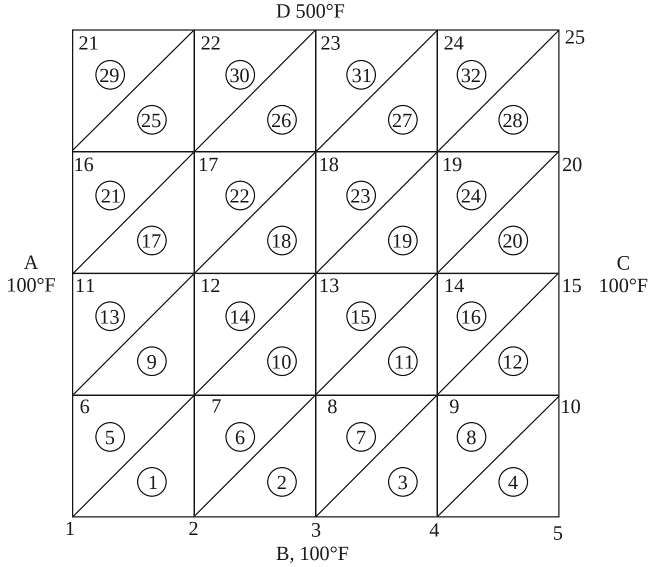
ELEMENT RESULTANTS

ELEMENT	GRAD (X)	GRAD (Y)	AVE TEMP
1	–0.3333E+02	0.0000E+00	0.9167E+02
2	–0.3333E+02	0.0000E+00	0.7500E+02
3	–0.3333E+02	–0.1907E–05	0.5833E+02
4	–0.3333E+02	–0.1907E–05	0.7500E+02
5	–0.3333E+02	–0.1907E–05	0.4167E+02
6	–0.3333E+02	0.0000E+00	0.2500E+02
7	–0.3333E+02	0.0000E+00	0.8333E+01
8	–0.3333E+02	–0.1907E+05	0.2500E+02



5	0.20000E + 03
10	0.10647E + 03
15	0.15418E + 03
20	0.85000E + 02
25	0.13588E + 03
30	0.16754E + 03
35	0.13246E + 03
40	0.85000E + 02

13.24

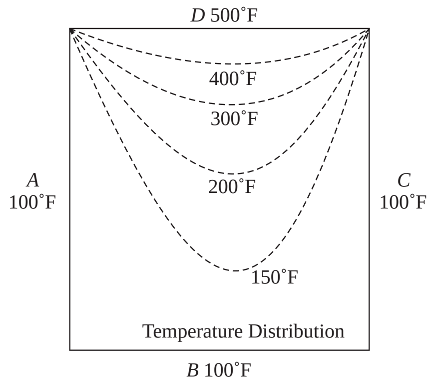


PRESCRIBED NODAL VALUES

1	0.10000E+03	15	0.10000E+03
2	0.10000E+03	16	0.10000E+03
3	0.10000E+03	20	0.10000E+03
4	0.10000E+03	21	0.50000E+03
5	0.10000E+03	22	0.50000E+03
6	0.10000E+03	23	0.50000E+03
10	0.10000E+03	24	0.50000E+03
11	0.10000E+03	25	0.50000E+03

NODAL VALUES, LOADING CASE 1

1	0.10000E+03	2	0.10000E+03
3	0.10000E+03	4	0.10000E+03
5	0.10000E+03		
6	0.10000E+03	7	0.12857E+03
8	0.13929E+03	9	0.12857E+03
10	0.10000E+03		
11	0.10000E+03	12	0.17500E+03
13	0.20000E+03	14	0.17500E+03
15	0.10000E+03		
16	0.10000E+03	17	0.27143E+03
18	0.31071E+03	19	0.27143E+03
20	0.10000E+03		
21	0.50000E+03	22	0.50000E+03
23	0.50000E+03	24	0.50000E+03
25	0.50000E+03		



13.25 Same model as in Problem 13.24

Now convection from right side and bottom.

Convection from side 1 of element 1

Convection from side 1 of element 2

Convection from side 1 of element 3

Convection from side 1 of element 4

Convection from side 2 of element 4

Convection from side 2 of element 12

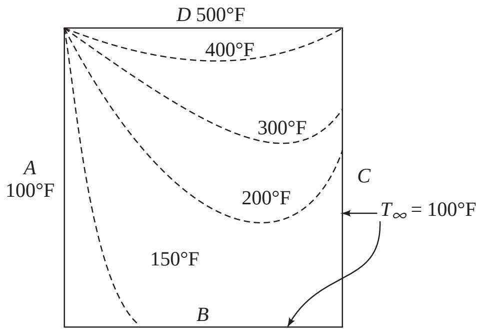
Convection from side 2 of element 20

Convection from side 2 of element 28

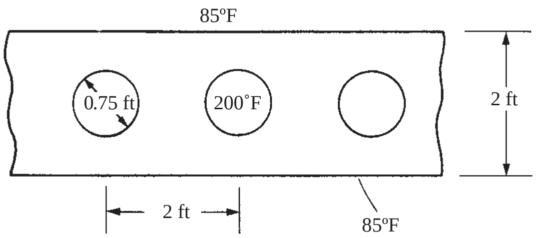
ELEMENT RESULTANTS

ELEMENT	GRAD(X)	GRAD(Y)	AVE TEMP
1	0.1618E+03	0.6971E+02	0.1328E+03
2	0.9794E+02	0.1099E+03	0.1659E+03
3	–0.1227E+00	0.1318E+03	0.1759E+03
4	0.4868E+02	–0.5679E+02	0.1473E+03
5	0.2315E+03	0.0000E+00	0.1193E+03
6	0.1381E+03	0.6971E+02	0.1636E+03
7	0.2178E+02	0.1099E+03	0.1851E+03
8	–0.6407E+02	0.1318E+03	0.1815E+03
9	0.2315E+03	0.1631E+03	0.1522E+03
10	0.1381E+03	0.2262E+03	0.1997E+03
11	0.2178E+02	0.2176E+03	0.2142E+03
12	–0.6407E+02	–0.5051E+02	0.1830E+03
13	0.3946E+03	0.0000E+00	0.1329E+03
14	0.2012E+03	0.1631E+03	0.2018E+03
15	0.1319E+02	0.2262E+03	0.2312E+03
16	–0.3322E+03	0.2176E+03	0.2064E+03
17	0.3946E+03	0.3565E+03	0.1955E+03
18	0.2012E+03	0.4142E+03	0.2667E+03
19	0.1319E+02	0.4841E+03	0.2915E+03
20	–0.3322E+03	0.1633E+03	0.3072E+03
21	0.7512E+03	0.0000E+00	0.1626E+03

22	0.2589E+03	0.3565E+03	0.2797E+03
23	0.8312E+02	0.4142E+03	0.3249E+03
24	0.6042E+02	0.4841E+03	0.3380E+03
25	0.7512E+03	0.3488E+03	0.2959E+03
26	0.2539E+03	0.5900E+03	0.3801E+03
27	0.8312E+02	0.5068E+03	0.4086E+03
28	0.6042E+02	0.4464E+03	0.4206E+03
29	0.0000E+00	0.1600E+04	0.3667E+03
30	0.0000E+00	0.8488E+03	0.4293E+03
31	0.0000E+00	0.5900E+03	0.4508E+03
32	0.0000E+00	0.5068E+03	0.4579E+03



13.26



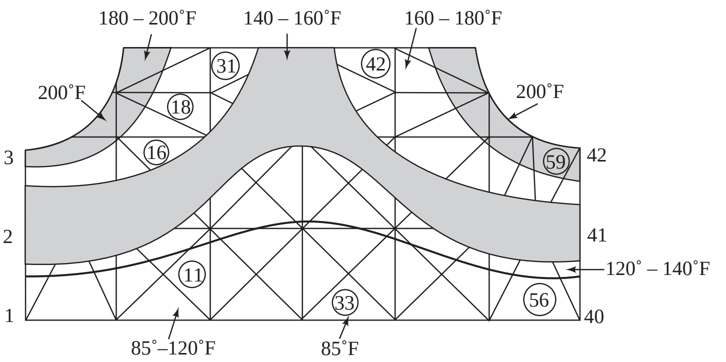
PRESCRIBED NODAL TEMPERATURE VALUES

1	0.85000E+02
3	0.20000E+03
5	0.20000E+03
6	0.85000E+02
9	0.20000E+03
12	0.20000E+03
13	0.85000E+02
20	0.85000E+02
26	0.85000E+02
33	0.20000E+03
34	0.85000E+02
37	0.20000E+03
39	0.20000E+03
40	0.85000E+02
42	0.20000E+03

RESULTING NODAL TEMPERATURE VALUES

1	0.85000E+02	2	0.13937E+03	3	0.20000E+03	4	0.13737E+03
6	0.85000E+02	7	0.13246E+03	8	0.18216E+03	9	0.20000E+03
11	0.14805E+03	12	0.20000E+03	13	0.85000E+02	14	0.12341E+03
16.	0.16387E+03	17	0.16754E+03	18	0.10324E+03	19	0.13588E+03
21	0.11956E+03	22	0.14636E+03	23	0.15741E+03	24	0.10324E+03
26	0.85000E+02	27	0.12341E+03	28	0.15418E+03	29	0.16387E+03
31	0.10647E+03	32	0.14805E+03	33	0.20000E+03	34	0.85000E+02
36	0.18216E+03	37	0.20000E+03	38	0.13737E+03	39	0.20000E+03
41	0.13937E+03	42	0.20000E+03				

Temperature contours



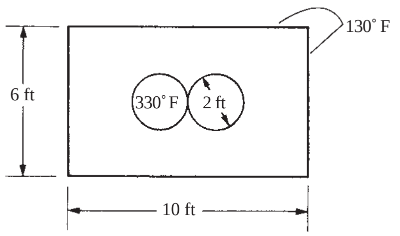
Finite element model
(62 elements, 42 nodes)
(Element resultants are given below)

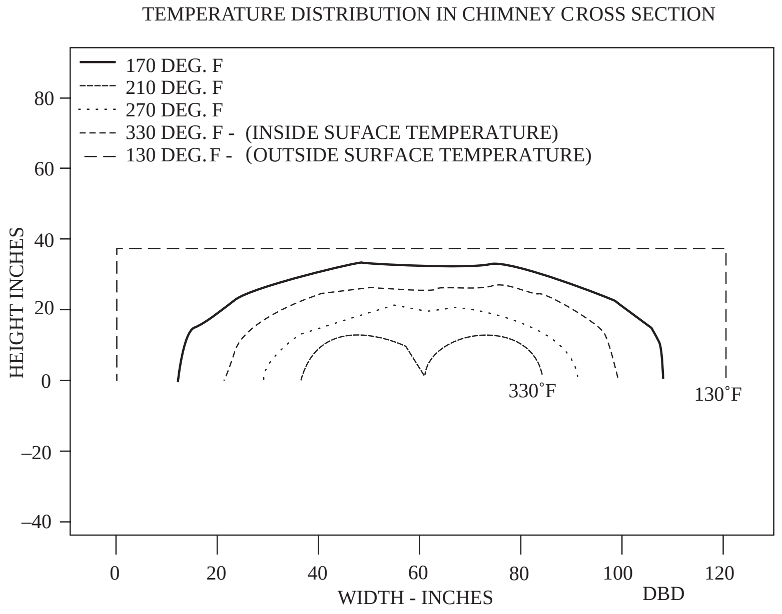
ELEMENT RESULTANTS

ELEMENT	GRAD(X)	GRAD(Y)	AVE TEMP
1	–0.1194E+02	0.1631E+03	0.1206E+03
2	–0.1194E+02	0.2079E+03	0.1589E+03
3	–0.4697E+02	0.1879E+03	0.1791E+03
4	–0.1071E+03	0.1071E+03	0.1941E+03
5	–0.1071E+03	0.1491E+03	0.1715E+03
6	–0.2951E+02	0.1879E+03	0.1566E+03
7	0.2787E–05	0.1571E+03	0.1025E+03
8	–0.2951E+02	0.1424E+03	0.1183E+03
9	–0.1357E+02	0.1424E+03	0.1080E+03
10	–0.2092E–04	0.1288E+03	0.9216E+02
11	–0.1357E+02	0.1152E+03	0.1050E+03
12	–0.2714E+02	0.1288E+03	0.1208E+03
13	–0.5554E+02	0.1491E+03	0.1542E+03
14	–0.2714E+02	0.1207E+03	0.1346E+03
15	–0.5554E+02	0.9231E+02	0.1419E+03
16	–0.8394E+02	0.1207E+03	0.1615E+03
17	–0.8394E+02	0.1071E+03	0.1788E+03
18	–0.1084E+03	0.5818E+02	0.1727E+03

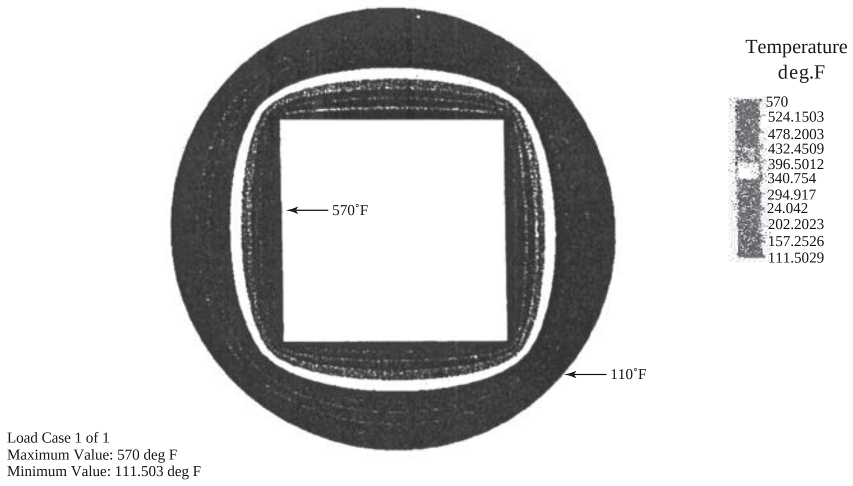
19	–0.1084E+03	0.2197E+02	0.1771E+03
20	–0.1113E+03	0.2783E+02	0.1892E+03
21	–0.5775E+01	0.1152E+03	0.1039E+03
22	0.2190E–04	0.1095E+03	0.9108E+02
23	–0.5775E+01	0.1037E+03	0.1026E+03
24	–0.1155E+02	0.1095E+03	0.1154E+03
25	–0.1751E+02	0.9231E+02	0.1378E+03
26	–0.1155E+02	0.8635E+02	0.1263E+03
27	–0.1751E+02	0.8039E+02	0.1339E+03
28	–0.2347E+02	0.8635E+02	0.1455E+03
29	–0.2347E+02	0.5818E+02	0.1548E+03
30	–0.3597E+02	0.3317E+02	0.1559E+03
31	–0.3037E+02	0.2197E+02	0.1629E+03
32	0.5775E+01	0.1037E+03	0.1026E+03
33	0.6646E–05	0.1095E+03	0.9108E+02
34	0.5775E+01	0.1152E+03	0.1039E+03
35	0.1155E+02	0.1095E+03	0.1154E+03
36	0.1751E+02	0.8039E+02	0.1339E+03
37	0.1155E+02	0.8635E+02	0.1263E+03
38	0.1751E+02	0.9231E+02	0.1378E+03
39	0.2347E+02	0.8635E+02	0.1455E+03
40	0.2347E+02	0.5818E+02	0.1548E+03
41	0.3597E+02	0.3317E+02	0.1559E+03
42	0.3037E+02	0.2197E+02	0.1629E+03
43	0.1357E+02	0.1152E+03	0.1050E+03
44	0.3809E–05	0.1288E+03	0.9216E+02
45	0.1357E+02	0.1424E+03	0.1080E+03
46	0.2715E+02	0.1288E+03	0.1208E+03
47	0.5554E+02	0.9231E+02	0.1419E+03
48	0.2715E+02	0.1207E+03	0.1346E+03
49	0.5554E+02	0.1491E+03	0.1542E+03
50	0.8394E+02	0.1207E+03	0.1615E+03
51	0.8394E+02	0.1071E+03	0.1788E+03
52	0.1084E+03	0.5818E+02	0.1727E+03
53	0.1084E+03	0.2197E+02	0.1771E+03
54	0.1113E+03	0.2783E+02	0.1892E+03
55	0.2951E+02	0.1424E+03	0.1183E+03
56	0.9467E–06	0.1571E+03	0.1025E+03
57	0.1194E+02	0.1631E+03	0.1206E+03
58	0.1194E+02	0.2079E+03	0.1589E+03
59	0.4697E+02	0.1879E+03	0.1791E+03
60	0.2951E+02	0.1879E+03	0.1566E+03
61	0.1071E+03	0.1491E+03	0.1715E+03
62	0.1071E+03	0.1071E+03	0.1941E+03

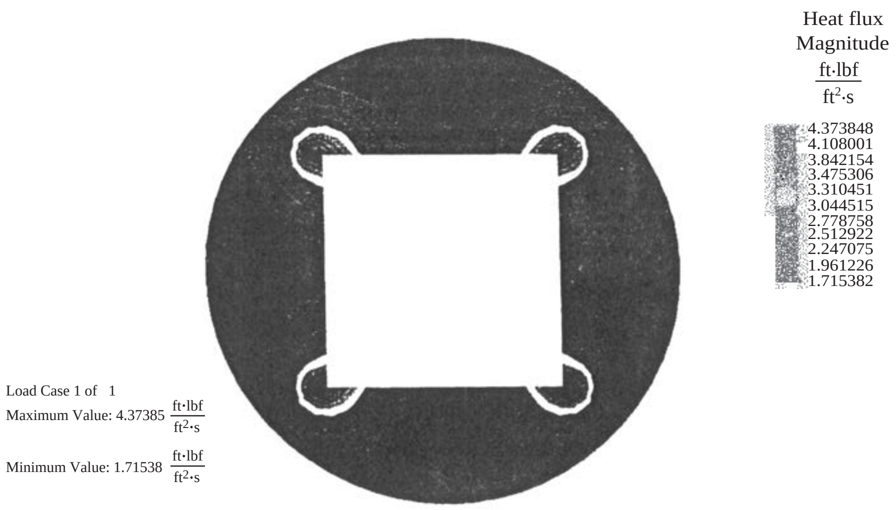
13.27


130˚ F
6 ft
330˚ F
2 ft
10 ft

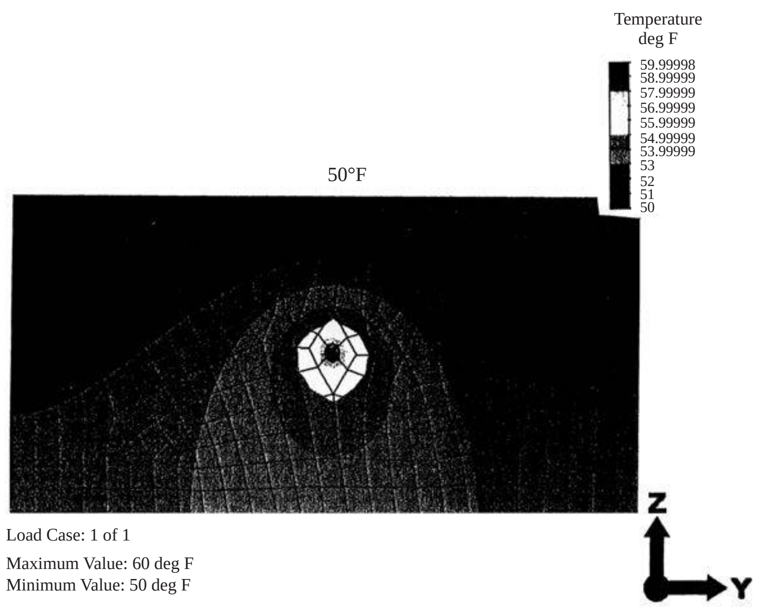

TEMPERATURE DISTRIBUTION IN CHIMNEY CROSS SECTION
170 DEG. F
210 DEG. F
270 DEG. F
330 DEG. F - (INSIDE SUFACE TEMPERATURE)
130 DEG. F - (OUTSIDE SURFACE TEMPERATURE)
HEIGHT INCHES
WIDTH - INCHES
330˚F
130˚F
DBD
80
60
40
20
0
–20
–40
0
20
40
60
80
100
120

13.28

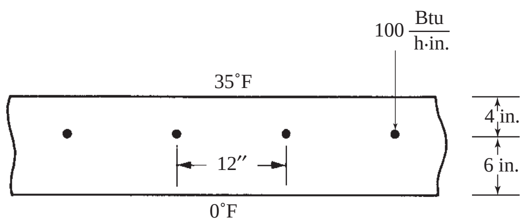

Temperature
deg.F
570
524.1503
478.2003
432.4509
396.5012
340.754
294.917
24.042
202.2023
157.2526
111.5029
570˚F
110˚F
Load Case 1 of 1
Maximum Value: 570 deg F
Minimum Value: 111.503 deg F

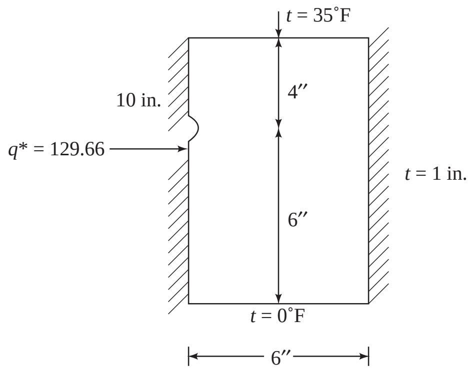


13.29 The temperature distribution of the earth is shown below with a 60 °F oil pipe 15 ft under the earth's surface at 50 °F.



13.30



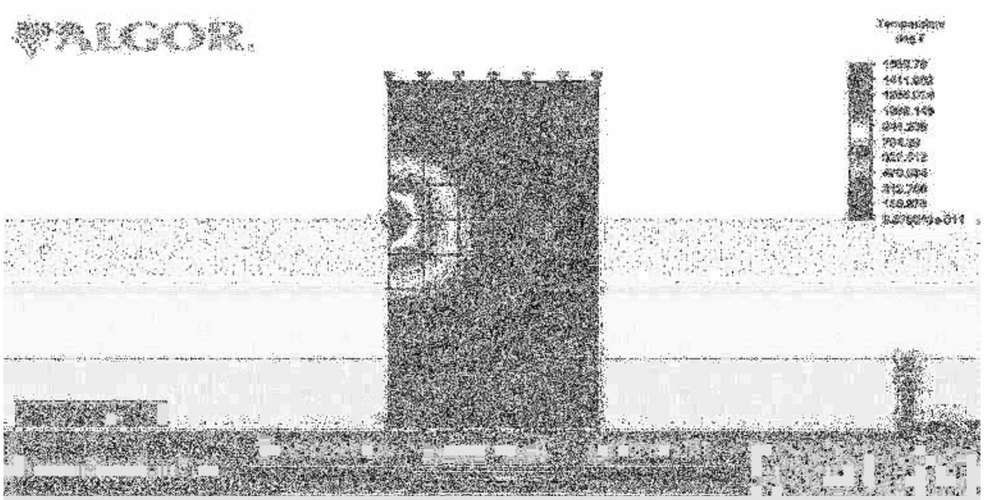


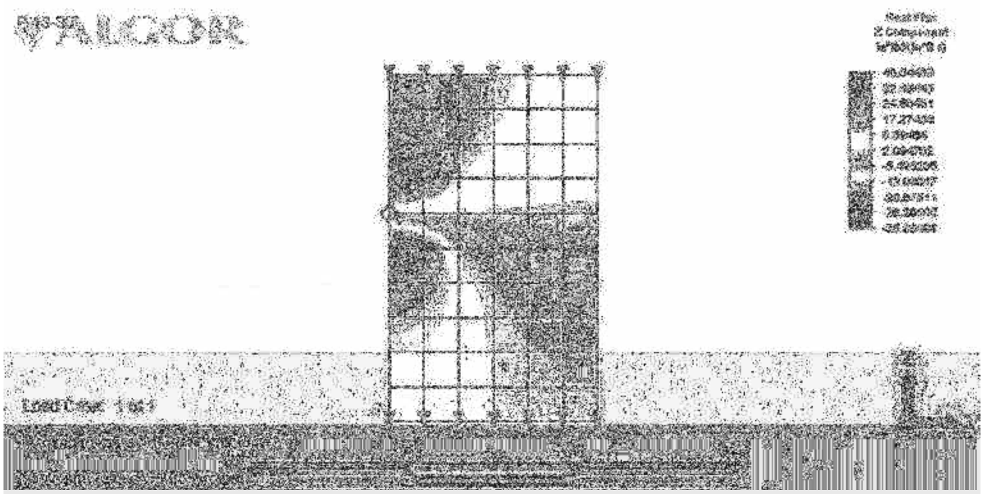
$$K = 0.50\ \frac{\text{Btu}}{\text{h}\cdot\text{ft}\cdot{}^\circ\text{F}}\ \frac{778\ \text{lb}\cdot\text{ft}}{1\ \text{Btu}}\ \frac{1\ \text{h}}{3600\ \text{s}} = 0.108056\ \frac{\text{in.}\cdot\text{lb}}{\text{s}\cdot\text{in.}\cdot{}^\circ\text{F}}$$

$$q^* = 50\ \frac{\cancel{\text{Btu}}}{\cancel{\text{hr}}}\left[\frac{1\ \cancel{\text{hr}}}{60\ \cancel{\text{min}}}\right]\left[\frac{1\ \cancel{\text{min}}}{60\text{s}}\right]\left[\frac{778\text{lbf}\ \cancel{\text{ft}}}{1\ \cancel{\text{Btu}}}\right]\left[\frac{12\ \text{in}}{1\ \cancel{\text{ft}}}\right] = 129.66\ \frac{\text{lbf}\cdot\text{in.}}{\text{s}}$$

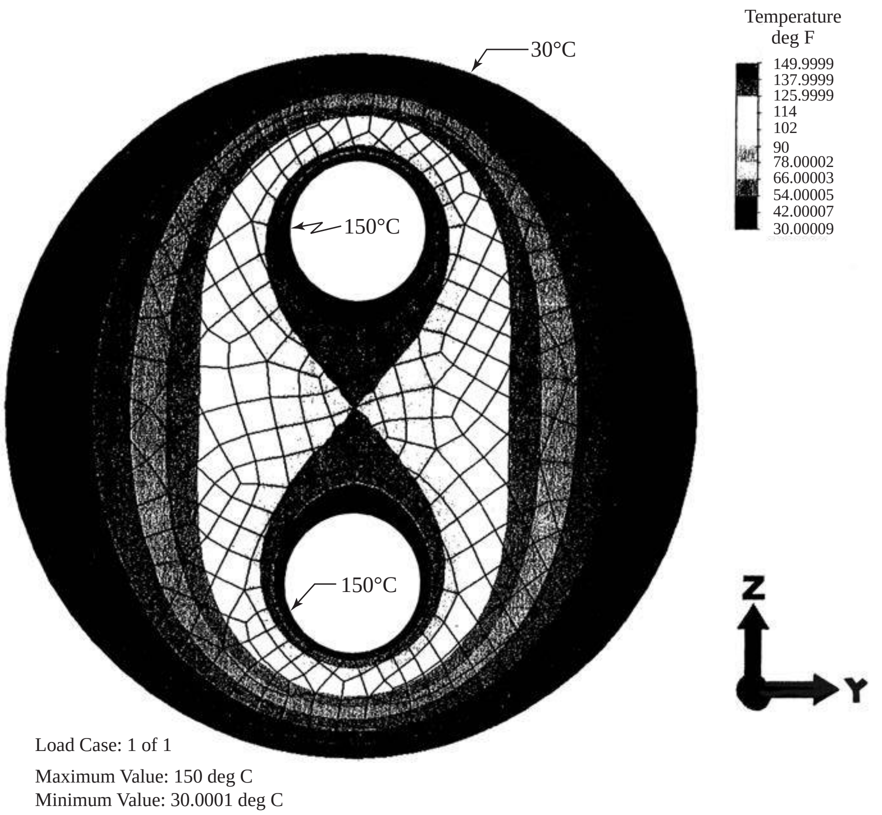
Heat source

$$q^* = 129.66\ \frac{\text{lbf}\cdot\text{in.}}{\text{s}}$$

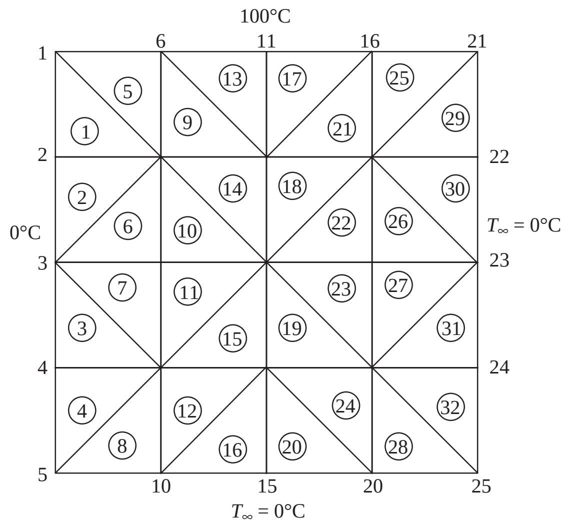




13.31



13.32



KXX = 10.0 KYY = 10.0

CONVECTION COEFF = 10.0

FLUID TEMPERATURE = 0.0

SEMI-BANDWIDTH = 7

NEL	NODE	NUMBER		X(1)	Y(1)	X(2)	Y(2)	X(3)	Y(3)
1	1	2	7	0.0000	1.0000	0.0000	0.8000	0.2000	0.8000
2	2	3	8	0.0000	0.8000	0.0000	0.5000	0.2000	0.5000

3	3	4	8	0.0000	0.5000	0.0000	0.2000	0.2000	0.5000
4	4	5	9	0.0000	0.2000	0.0000	0.0000	0.2000	0.2000
5	1	7	6	0.0000	1.0000	0.2000	0.8000	0.2000	1.0000
6	2	8	7	0.0000	0.8000	0.2000	0.5000	0.2000	0.8000
7	4	9	8	0.0000	0.2000	0.2000	0.2000	0.2000	0.5000
8	5	10	9	0.0000	0.0000	0.2000	0.0000	0.2000	0.2000
9	6	7	12	0.2000	1.0000	0.2000	0.8000	0.5000	0.8000
10	7	8	13	0.2000	0.8000	0.2000	0.5000	0.5000	0.5000
11	8	9	13	0.2000	0.5000	0.2000	0.2000	0.5000	0.5000
12	9	10	14	0.2000	0.2000	0.2000	0.0000	0.5000	0.2000
13	6	12	11	0.2000	1.0000	0.5000	0.8000	0.5000	1.0000
14	7	13	12	0.2000	0.8000	0.5000	0.5000	0.5000	0.8000
15	9	14	13	0.2000	0.2000	0.5000	0.2000	0.5000	0.5000
16	10	15	14	0.2000	0.0000	0.5000	0.0000	0.5000	0.2000
17	11	12	16	0.5000	1.0000	0.5000	0.8000	0.8000	1.0000
18	12	13	17	0.5000	0.8000	0.5000	0.5000	0.8000	0.8000
19	13	14	19	0.5000	0.5000	0.5000	0.2000	0.8000	0.2000
20	14	15	20	0.5000	0.2000	0.5000	0.0000	0.8000	0.0000
21	12	17	16	0.5000	0.8000	0.8000	0.8000	0.8000	1.0000
22	13	18	17	0.5000	0.5000	0.8000	0.5000	0.8000	0.8000
23	13	19	18	0.5000	0.5000	0.8000	0.2000	0.8000	0.5000
24	14	20	19	0.5000	0.2000	0.8000	0.0000	0.8000	0.2000
25	16	17	21	0.8000	1.0000	0.8000	0.8000	1.0000	1.0000
26	17	18	22	0.8000	0.8000	0.8000	0.5000	1.0000	0.8000
27	18	19	24	0.8000	0.5000	0.8000	0.2000	1.0000	0.2000
28	19	20	25	0.8000	0.2000	0.8000	0.0000	1.0000	0.0000
29	17	22	21	0.8000	0.8000	1.0000	0.8000	1.0000	1.0000
30	18	23	22	0.8000	0.5000	1.0000	0.5000	1.0000	0.8000
31	18	24	23	0.8000	0.5000	1.0000	0.2000	1.0000	0.5000
32	19	25	24	0.8000	0.2000	1.0000	0.0000	1.0000	0.2000

CONVECTION FROM SIDE 1 OF ELEMENT 8
CONVECTION FROM SIDE 1 OF ELEMENT 16
CONVECTION FROM SIDE 2 OF ELEMENT 20
CONVECTION FROM SIDE 2 OF ELEMENT 28
CONVECTION FROM SIDE 3 OF ELEMENT 29
CONVECTION FROM SIDE 3 OF ELEMENT 30
CONVECTION FROM SIDE 2 OF ELEMENT 31
CONVECTION FROM SIDE 2 OF ELEMENT 32

PRESCRIBED NODAL TEMPERATURE VALUES

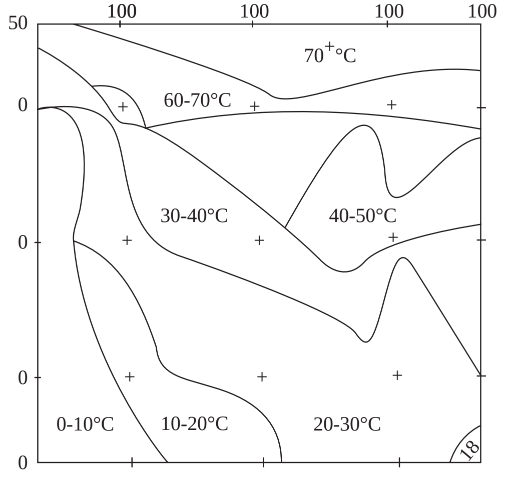
2	0.00000E+00
3	0.00000E+00
4	0.00000E+00
5	0.00000E+00
6	0.10000E+03
11	0.10000E+03
16	0.10000E+03
21	0.10000E+03

RESULTING NODAL TEMPERATURE VALUES

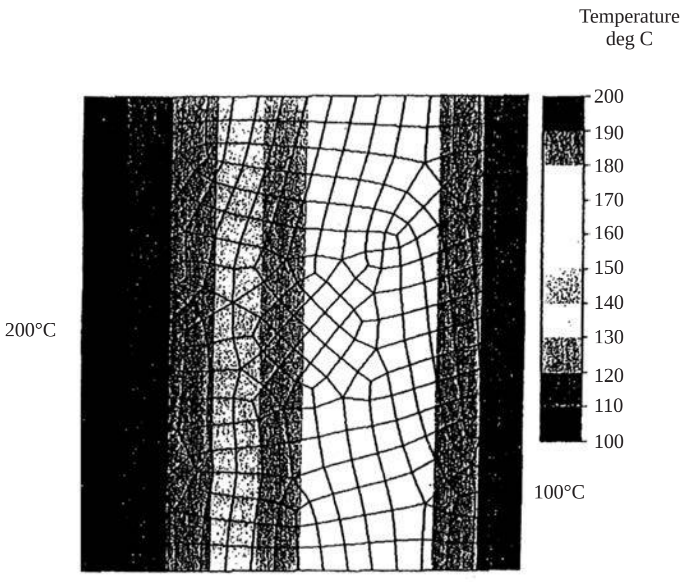
1	0.50000E+02	2	0.00000E+00	3	0.00000E+00	4	0.00000E+00
6	0.10000E+03	7	0.47910E+02	8	0.21010E+02	9	0.11377E+02
11	0.10000E+03	12	0.68538E+02	13	0.38137E+02	14	0.22992E+02
16	0.10000E+03	17	0.69017E+02	18	0.40007E+02	19	0.25421E+02
21	0.10000E+03	22	0.66669E+02	23	0.39117E+02	24	0.22699E+02
10	0.85870E+01	15	0.18000E+02	20	0.20040E+02	25	0.17598E+02

ELEMENT RESULTANTS

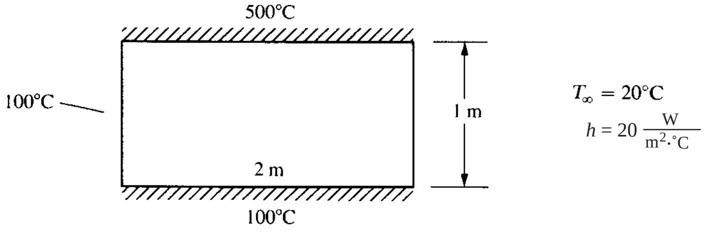
ELEMENTS	GRAD(X)	GRAD(Y)	AVE TEMP
1	0.2395E+03	0.2500E+03	0.3264E+02
2	0.1051E+03	0.0000E+00	0.7003E+01
3	0.1051E+03	0.0000E+00	0.7003E+01
4	0.5688E+02	0.0000E+00	0.3792E+01
5	0.2500E+03	0.2605E+03	0.6597E+02
6	0.2395E+03	0.8967E+02	0.2297E+02
7	0.5688E+02	0.3211E+02	0.1080E+02
8	0.4293E+02	0.1395E+02	0.6655E+01
9	0.6876E+02	0.2605E+03	0.7215E+02
10	0.5709E+02	0.8967E+02	0.3569E+02
11	0.5709E+02	0.3211E+02	0.2351E+02
12	0.3872E+02	0.1395E+02	0.1432E+02
13	0.2862E–04	0.1573E+03	0.8951E+02
14	0.6876E+02	0.1013E+03	0.5153E+02
15	0.3872E+02	0.5048E+02	0.2417E+02
16	0.3138E+02	0.2496E+02	0.1653E+02
17	0.3336E–04	0.1573E+03	0.8951E+02
18	0.1595E+01	0.1013E+03	0.5856E+02
19	0.8097E+01	0.5048E+02	0.2885E+02
20	0.6800E+01	0.2496E+02	0.2034E+02
21	0.1595E+01	0.1549E+03	0.7918E+02
22	0.6234E+01	0.9670E+02	0.4905E+02
23	0.6234E+01	0.4862E+02	0.3452E+02
24	0.8097E+01	0.2691E+02	0.2282E+02
25	– 0.3310E–04	0.1549E+03	0.8967E+02
26	– 0.1174E+02	0.9670E+02	0.5856E+02
27	– 0.1361E+02	0.4862E+02	0.2938E+02
28	– 0.1221E+02	0.2691E+02	0.2102E+02
29	– 0.1174E+02	0.1667E+03	0.7856E+02
30	– 0.4451E+01	0.9184E+02	0.4860E+02
31	– 0.4451E+01	0.5473E+02	0.3394E+02
32	– 0.1361E+02	0.2550E+02	0.2191E+02

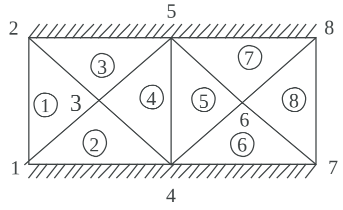


13.34



13.35





KXX = 10.0 KYY = 10.0

CONVECTION COEFF = 20.0

FLUID TEMPERATURE = 20.0

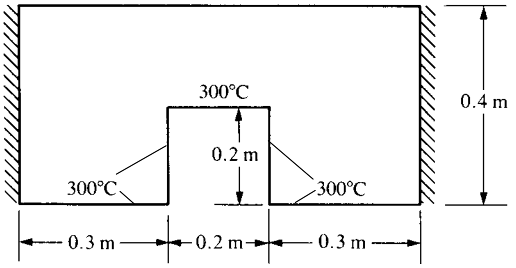
SEMI-BANDWIDTH = 4

NEL	NODE	NUMBER		X(I)	Y(1)	X(2)	Y(2)	X(3)	Y(3)
1	1	3	2	0.0000	0.0000	0.5000	0.5000	0.0000	1.0000
2	1	4	3	0.0000	0.0000	1.0000	0.0000	0.5000	0.5000
3	2	3	5	0.0000	1.0000	0.5000	0.5000	1.0000	1.0000
4	3	4	5	0.5000	0.5000	1.0000	0.0000	1.0000	1.0000
5	4	6	5	1.0000	0.0000	1.5000	0.5000	1.0000	1.0000
6	4	7	6	1.0000	0.0000	2.0000	0.0000	1.5000	0.5000
7	5	6	8	1.0000	1.0000	1.5000	0.5000	2.0000	1.0000
8	6	7	8	1.5000	0.5000	2.0000	0.0000	2.0000	1.0000

ELEMENT RESULTANTS

ELEMENT	GRAD(X)	GRAD(Y)	AVE TEMP
1	0.1000E+03	0.2000E+03	0.2167E+03
2	0.0000E+00	0.3000E+03	0.1500E+03
3	0.2000E+03	0.3000E+03	0.3500E+03
4	0.1000E+03	0.4000E+03	0.2833E+03
5	– 0.1867E+03	0.4000E+03	0.2689E+03
6	0.1333E+02	0.2000E+03	0.1400E+03
7	– 0.3867E+03	0.2000E+03	0.2733E+03
8	– 0.1867E+03	0.7629E–05	0.1444E+03

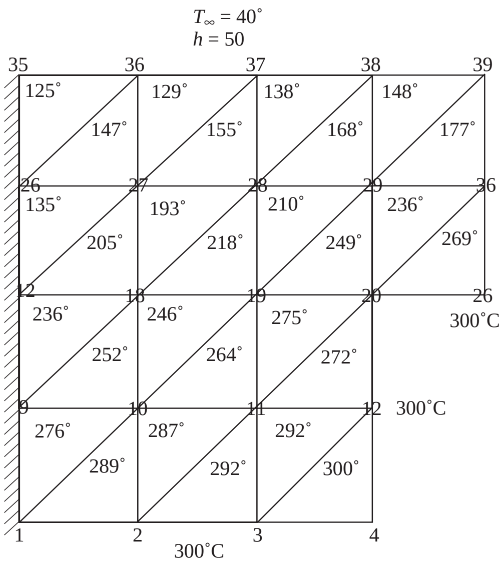
13.36



$T_\infty = 40^\circ$C

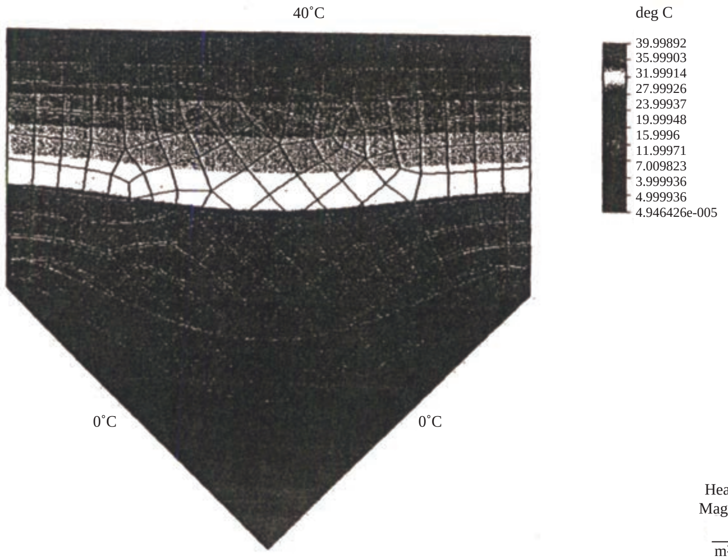
$h = 50\ \frac{\text{W}}{\text{m}^2\cdot^\circ\text{C}}$

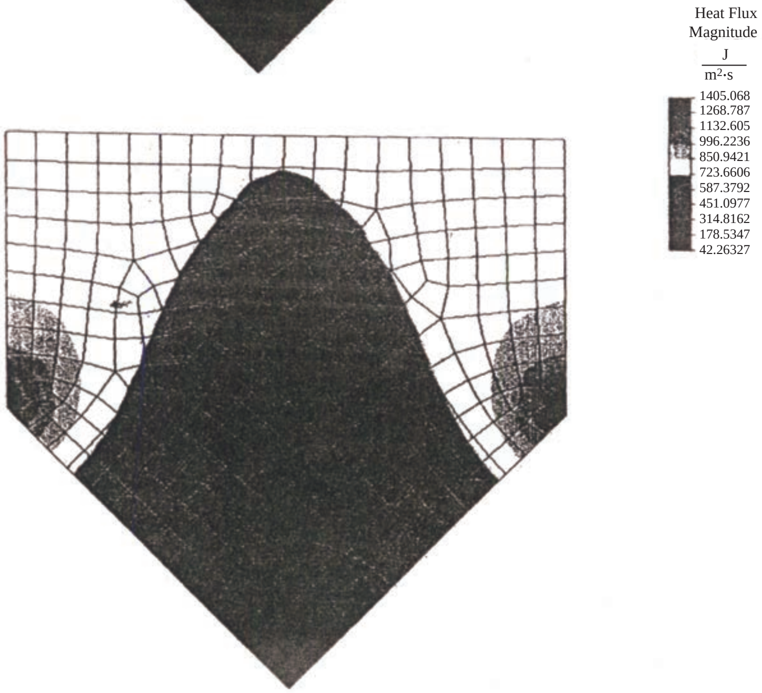
$k = 50\ \frac{\text{W}}{\text{m}\cdot^\circ\text{C}}$



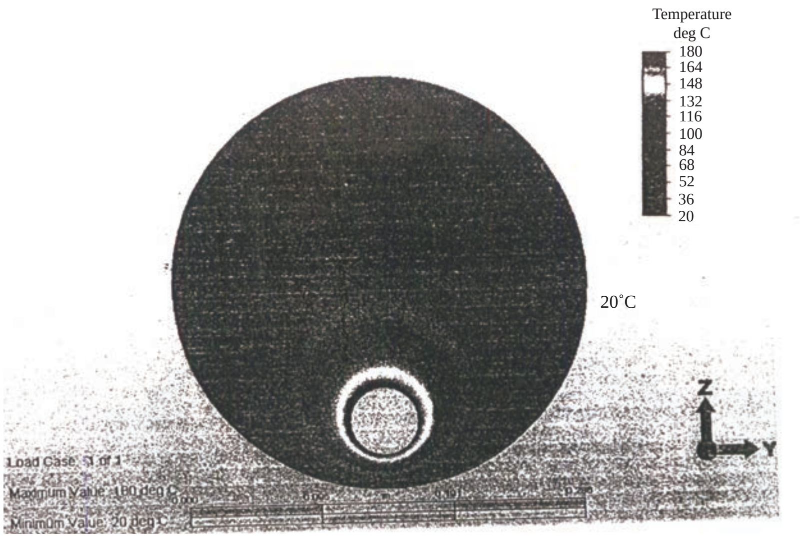
Results showing one-half of model

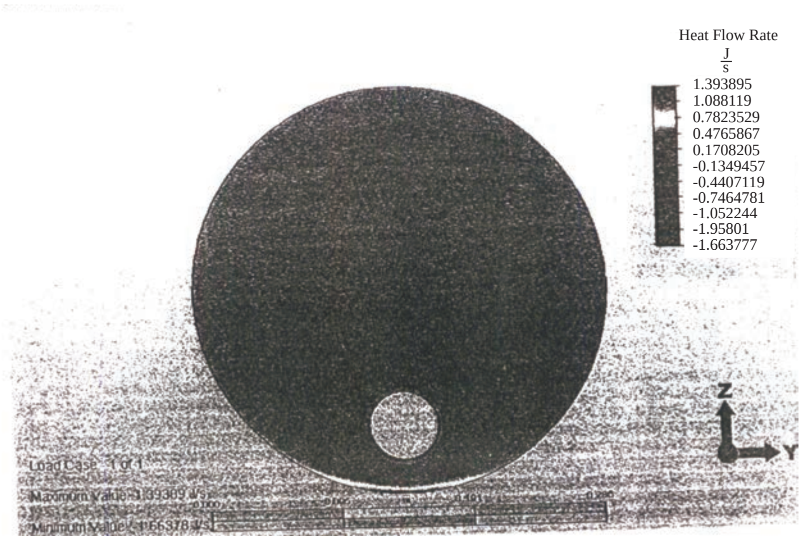
13.37 Determine the temperature distribution and rate of heat flow through the plain carbon steel ingot shown in Figure P13-36. Let $k = 60\,\frac{\text{W}}{\text{m}\cdot\text{K}}$ for the steel. The top surface is held at 40C, while the underside surface is held at 0C. Assume that no heat is lost from the sides.


40˚C
deg C
39.99892
35.99903
31.99914
27.99926
23.99937
19.99948
15.9996
11.99971
7.009823
3.999936
4.999936
4.946426e-005
0˚C
0˚C

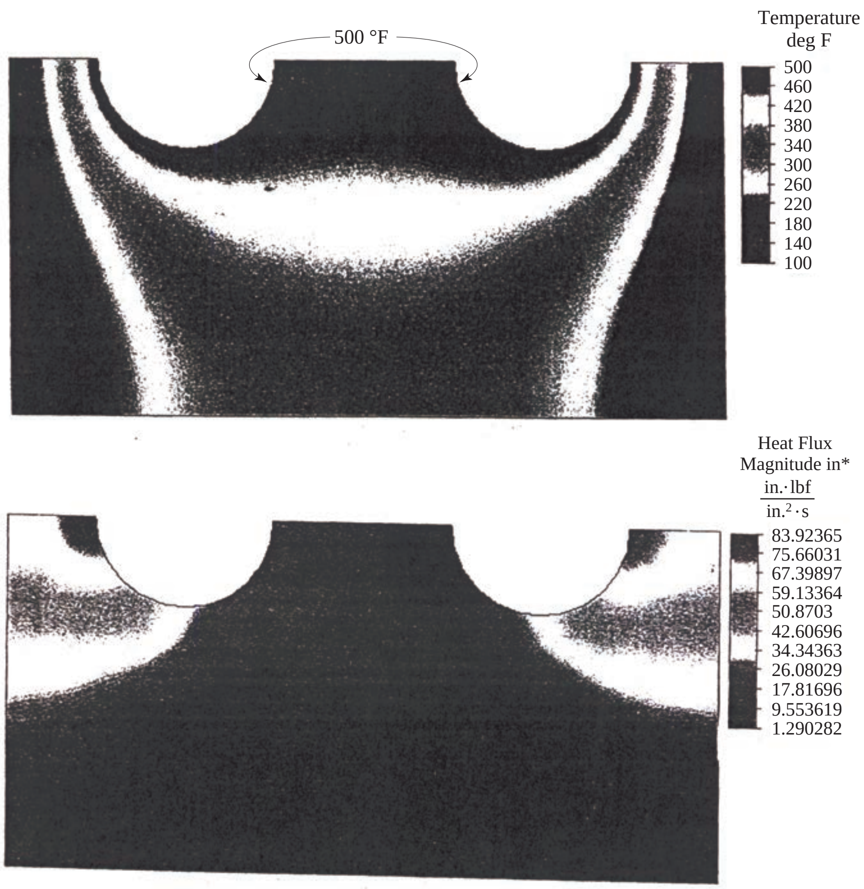

Heat Flux Magnitude
J / m²·s
1405.068
1268.787
1132.605
996.2236
850.9421
723.6606
587.3792
451.0977
314.8162
178.5347
42.26327

13.38

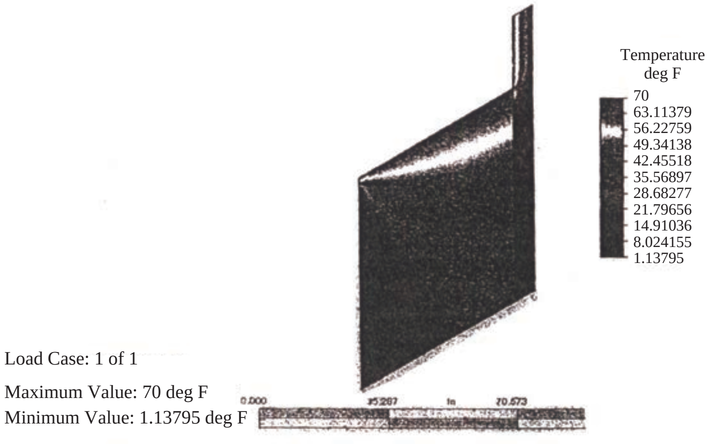

Temperature
deg C
180
164
148
132
116
100
84
68
52
36
20
20˚C
Z
Y
Load Case: 1 of 1
Maximum Value: 180 deg C
Minimum Value: 20 deg C

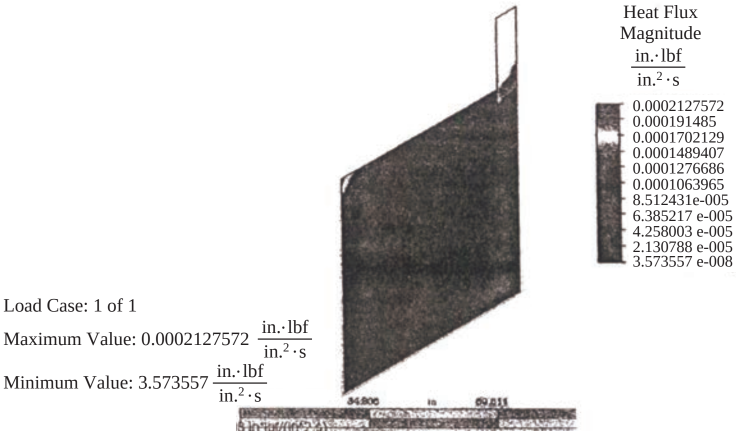


13.39

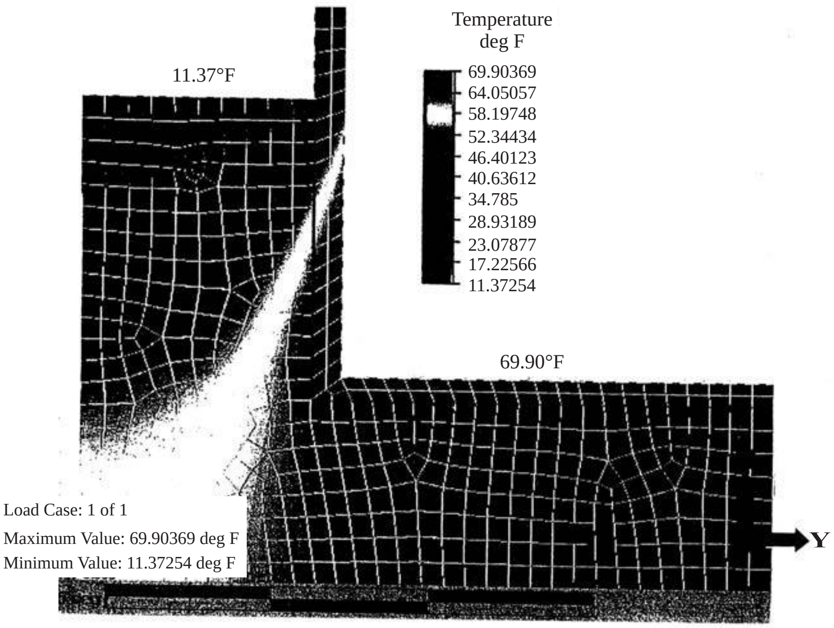


13.40 For the basement wall, determine the temperature distribution and the heat transfer through the wall and soil.

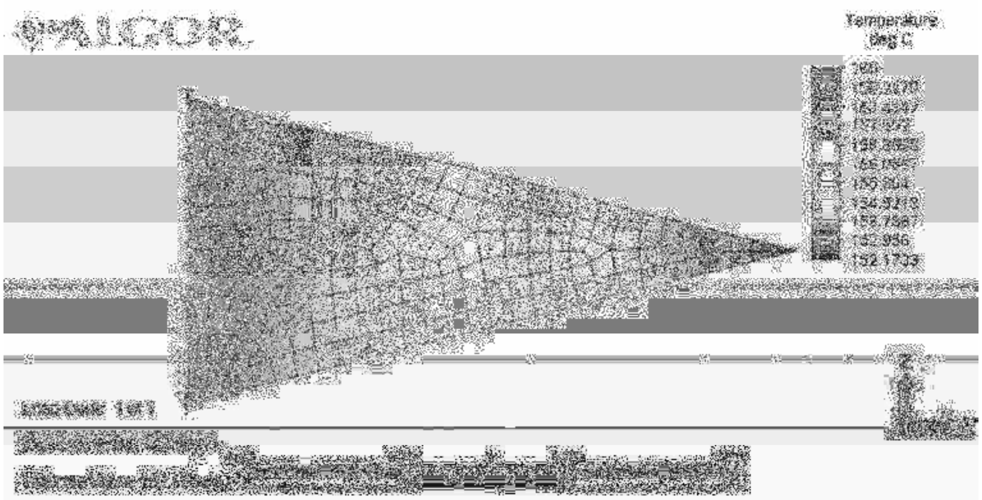

Temperature
deg F
70
63.11379
56.22759
49.34138
42.45518
35.56897
28.68277
21.79656
14.91036
8.024155
1.13795
Load Case: 1 of 1
Maximum Value: 70 deg F
Minimum Value: 1.13795 deg F


Heat Flux
Magnitude
in.·lbf / in.²·s
0.0002127572
0.000191485
0.0001702129
0.0001489407
0.0001276686
0.0001063965
8.512431e-005
6.385217 e-005
4.258003 e-005
2.130788 e-005
3.573557 e-008
Load Case: 1 of 1
Maximum Value: 0.0002127572 in.·lbf / in.²·s
Minimum Value: 3.573557 in.·lbf / in.²·s

13.41


Temperature
deg F
69.90369
64.05057
58.19748
52.34434
46.40123
40.63612
34.785
28.93189
23.07877
17.22566
11.37254
11.37°F
69.90°F
Y
Load Case: 1 of 1
Maximum Value: 69.90369 deg F
Minimum Value: 11.37254 deg F

13.42 Aluminum fins ($k = 170\ \frac{\text{W}}{\text{m}\cdot\text{K}}$) with triangular profiles shown in Figure P13.41 are used to remove heat from a surface with temperature of 160°C. The temperature of the surrounding air is 25°C. The natural convection coefficient is $h = 25\ \frac{\text{W}}{\text{m}^2\cdot\text{K}}$. Determine the temperature distribution throughout and the heat loss from a typical fin.



13.44 The Allen Wrench, shown in Figure 1, is exposed at one end to at temperature of 300 K, while the other end has a heat flux of 10 $\frac{\text{W}}{\text{m}^\wedge 2}$. Determine the temperature distribution throughout the wrench. It has a thermal conductivity of 43.6 $\frac{\text{W}}{\text{m}\cdot\text{K}}$ and a specific heat capacity of 0.000486 $\frac{\text{J}}{\text{g}\cdot\text{K}}$.

Part dimensions

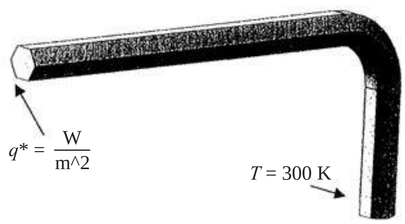


Figure 1: Original Model with forces

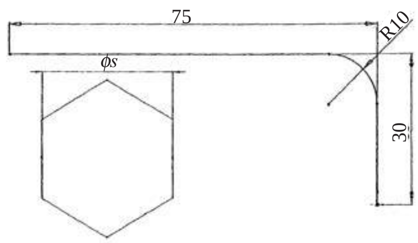


Figure 2: Dimension

Boundary condition

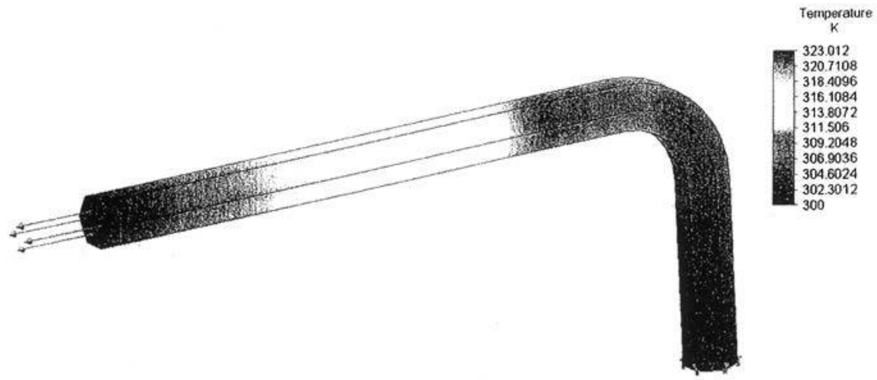
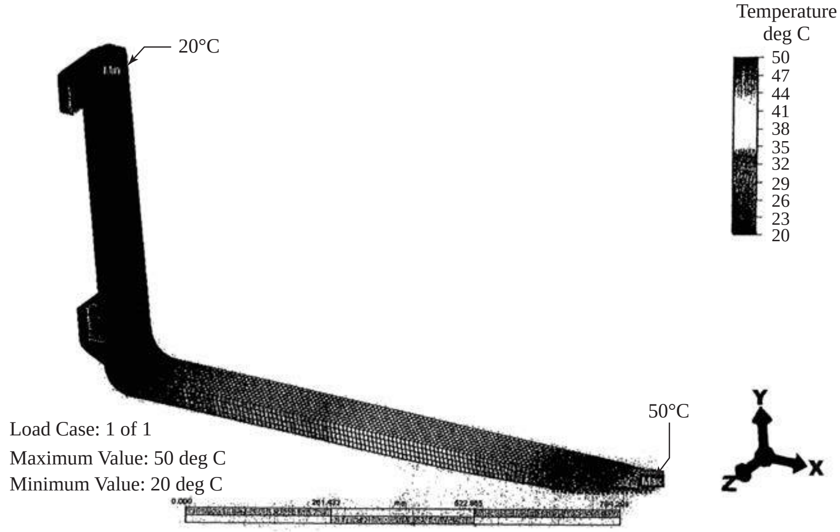
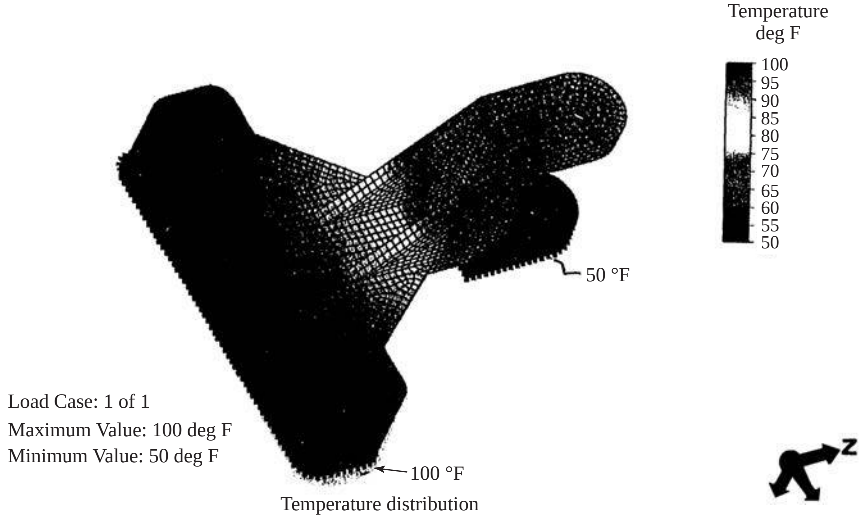


Figure 3: Temperature Distribution (K)

13.45 Temperature distribution



13.46 The thermal aspect of this component is that the base has an applied temperature of 100 °F. This is theoretically due to the warmed aluminum plate being heated by the electric motor. The lower surface of the lower finger has an applied surface temperature of 50 °F.



Temperature distribution

13.48

$$\frac{K_{xx}A}{L} = \frac{(0.017)\,\frac{\pi(1.5'')^2}{4\times 144}}{\frac{10''}{4}/\frac{12''}{1'}} = 0.001\,\frac{\text{Btu}}{\text{h}\cdot{}^\circ\text{F}} \text{ Small so neglect}$$

$$\dot{m}c = 10\,(0.24) = 2.4\,\frac{\text{Btu}}{\text{h}\cdot{}^\circ\text{F}}$$

$$\frac{hPL}{6} = \frac{3}{6}\left(\frac{\pi\,(1.5'')}{\frac{12''}{1'}}\right)\frac{\frac{10''}{4}}{\frac{12''}{1'}} = 0.04096\,\frac{\text{Btu}}{\text{h}\cdot{}^\circ\text{F}}$$

$$hPLT_\infty = (3)\,\pi\,\frac{(1.5)}{12}\left(\frac{\frac{10}{4}}{12}\right)(200^\circ\text{F}) = 49.087\,\frac{\text{Btu}}{\text{h}}$$

$$[k^{(1)}] = \frac{2.4}{2}\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix} + 0.04096\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$$[k^{(1)}] = \begin{bmatrix}-1.118 & 1.24091\\ -1.159 & 1.2818\end{bmatrix}$$

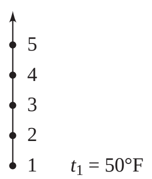
$$[k^{(2)}] = [k^{(3)}] = [k^{(4)}] = [k^{(1)}] \text{ also}$$

$$[K] = \begin{pmatrix}1.2818-1.118 & 1.2409 & 0 & 0\\ -1.159 & 1.2818-1.118 & 1.2409 & 0\\ 0 & -1.159 & 1.2818-1.118 & 1.2409\\ 0 & 0 & -1.159 & 1.2818\end{pmatrix}$$

$$[K] = \begin{pmatrix}0.164 & 1.241 & 0 & 0\\ -1.159 & 0.164 & 1.241 & 0\\ 0 & -1.159 & 0.164 & 1.241\\ 0 & 0 & -1.159 & 1.282\end{pmatrix}$$

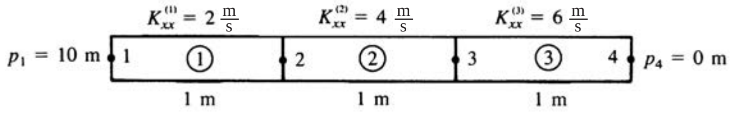
$$\{F\} = \begin{pmatrix}49.087 + 1.159(50)\\ 49.087\\ 49.087\\ 24.5435\end{pmatrix}$$

$$[K^{-1}]\ \{F\} = \begin{pmatrix}64.719\\ 77.715\\ 89.747\\ 100.296\end{pmatrix} = \begin{Bmatrix}t_2\\ t_3\\ t_4\\ t_5\end{Bmatrix}$$



Chapter 14

14.1



Global [K]

$$[K] = 4\frac{\text{m}^2}{\text{s}}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$

$$\{P\} = \begin{Bmatrix} 10 \\ p_2 \\ p_3 \\ 0 \end{Bmatrix}$$

Accounting for the boundary conditions

$p_1 = 10$, $\quad p_4 = 0$, we get

$$\begin{bmatrix} 12 & -8 \\ -8 & 20 \end{bmatrix}\begin{Bmatrix} p_2 \\ p_3 \end{Bmatrix} = \begin{Bmatrix} 40 \\ 0 \end{Bmatrix}$$

Solving

$$p_2 = 4.545 \text{ m}, \quad p_3 = 1.818 \text{ m}$$

$$v_x^{(1)} = -k_{xx}^{(1)}\left[-\frac{1}{L}\ \frac{1}{L}\right]\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$

$$= -2\ [-1\ 1]\begin{Bmatrix} 40 \\ 4.545 \end{Bmatrix} = 10.91\ \frac{\text{m}}{\text{s}}$$

$$v_x^{(2)} = -4\ [-4.545 + 1.818] = 10.91\ \frac{\text{m}}{\text{s}}$$

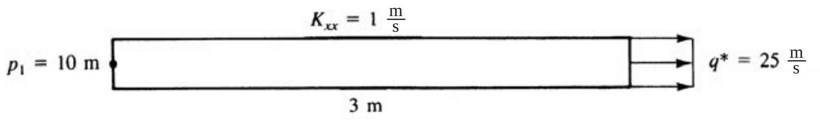
$$v_x^{(3)} = -6(-1.818) = 10.91\ \frac{\text{m}}{\text{s}}$$

$$Q_f^{(1)} = Av_x^{(1)} = 21.82\ \frac{\text{m}^3}{\text{s}}$$

$$Q_f^{(2)} = 21.82\ \frac{\text{m}^3}{\text{s}}$$

$$Q_f^{(3)} = 21.82\ \frac{\text{m}^3}{\text{s}}$$

14.2



$$[\underline{k}^{(1)}] = [k^{(2)}] = [k^{(3)}] = \frac{1\times 2}{1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[f^{(3)}] = q^* A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = -25\ \frac{\text{m}}{\text{s}}\ (2\text{m}^2) \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -50 \end{Bmatrix}$$

Global equations

$$\begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} p_1 = 10 \\ p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -50 \end{Bmatrix}$$

Rewriting

$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} = \begin{Bmatrix} p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 0 \\ -50 \end{Bmatrix}$$

Solving

$$p_2 = -15\text{ m},\quad p_3 = -40\text{ m},\quad p_4 = -65\text{ m}$$

$$v_x^{(1)} = -K_{xx}[B] \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = -\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} 10 \\ -15 \end{Bmatrix}\ (L = 1)$$

$$= 25\ \frac{\text{m}}{\text{s}}$$

$$v_x^{(2)} = -1 \begin{bmatrix} -\frac{1}{1} & \frac{1}{1} \end{bmatrix} \begin{Bmatrix} -15 \\ -40 \end{Bmatrix} = 25\ \frac{\text{m}}{\text{s}}$$

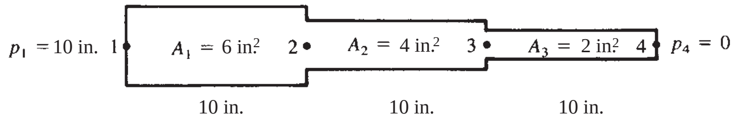
$$v_x^{(3)} = -1 \begin{bmatrix} -\frac{1}{1} & \frac{1}{1} \end{bmatrix} \begin{Bmatrix} -40 \\ -65 \end{Bmatrix} = 25\ \frac{\text{m}}{\text{s}}$$

Volumetric flow rates

$$Q_1 = Q_2 = Q_3 = v_x^{(1)} A_1 = v_x^{(2)} A_2 = v_x^{(3)} A_3$$

$$= 25\,(2) = 50\ \frac{\text{m}^3}{\text{s}}$$

14.3



$$[k^{(1)}] = 0.6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(2)}] = 0.4 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(3)}] = 0.2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0.6 & -0.6 & 0 & 0 \\ -0.6 & 1.0 & -0.4 & 0 \\ 0 & -0.4 & 0.6 & -0.2 \\ 0 & 0 & -0.2 & 0.2 \end{bmatrix} \begin{Bmatrix} 10\text{ in.} \\ p_2 \\ p_3 \\ p_4 = 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Using the 2nd and 3rd equations above

$$\begin{bmatrix} 1 & -0.4 \\ -0.4 & 0.6 \end{bmatrix} \begin{Bmatrix} p_2 \\ p_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 0 \end{Bmatrix}$$

$$p_2 = 8.182 \text{ in.,} \qquad p_3 = 5.455 \text{ in.}$$

$$v_x^{(1)} = -K_{xx}[B]\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = -1\begin{bmatrix} -\frac{1}{10} & \frac{1}{10} \end{bmatrix}\begin{Bmatrix} 10 \\ 8.182 \end{Bmatrix}$$

$$v_x^{(1)} = 0.182 \ \frac{\text{in.}}{\text{s}}$$

$$v_x^{(2)} = -1\begin{bmatrix} -\frac{1}{10} & \frac{1}{10} \end{bmatrix}\begin{Bmatrix} 8.182 \\ 5.455 \end{Bmatrix}$$

$$v_x^{(2)} = 0.273 \ \frac{\text{in.}}{\text{s}}$$

$$v_x^{(3)} = -\begin{bmatrix} -\frac{1}{10} & \frac{1}{10} \end{bmatrix}\begin{Bmatrix} 5.455 \\ 0 \end{Bmatrix}$$

$$= 0.545 \ \frac{\text{in.}}{\text{s}}$$

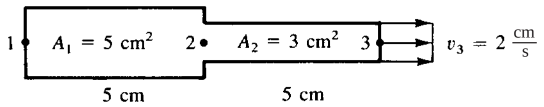
$$Q_f^{(1)} = A\,v_x^{(1)} = (6 \text{ in.}^2)\,(0.182 \ \frac{\text{in.}}{\text{s}})$$

$$= 1.091 \ \frac{\text{in.}^3}{\text{s}}$$

$$Q_f^{(2)} = 1.091 \ \frac{\text{in.}^3}{\text{s}}$$

$$Q_f^{(3)} = 1.091 \ \frac{\text{in.}^3}{\text{s}}$$

14.4



$$[k^{(1)}] = \frac{2\times 5}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$[k^{(2)}] = \frac{2\times 3}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} & -\frac{6}{5} \\ -\frac{6}{5} & \frac{6}{5} \end{bmatrix}$$

$$F_3 = -2\times 3 = -6 \ \frac{\text{cm}^3}{\text{s}}, \quad \text{Negative, as water flows out of right edge}$$

Assemble global equations

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3.2 & -1.2 \\ 0 & -1.2 & 1.2 \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -6 \end{Bmatrix}$$

Now assume $p_1 = 0$

$$\begin{bmatrix} 3.2 & -1.2 \\ -1.2 & 1.2 \end{bmatrix}\begin{Bmatrix} p_2 \\ p_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -6 \end{Bmatrix}$$

Solving for p_2 and p_3

$$p_2 = -3 \qquad p_3 = -8$$

Velocities

$$v_x^{(1)} = -K_{xx}\begin{bmatrix}-\frac{1}{L} & \frac{1}{L}\end{bmatrix}\begin{Bmatrix}p_1\\p_2\end{Bmatrix}$$

$$= -2\begin{bmatrix}-\frac{1}{5} & \frac{1}{5}\end{bmatrix}\begin{Bmatrix}0\\-3\end{Bmatrix}$$

$$v_x^{(1)} = 1.2\ \frac{\text{cm}}{\text{s}} \rightarrow \text{(to right)}$$

$$v_x^{(2)} = -2\begin{bmatrix}-\frac{1}{5} & \frac{1}{5}\end{bmatrix}\begin{Bmatrix}-3\\-8\end{Bmatrix}$$

$$v_x^{(2)} = 2\ \frac{\text{cm}}{\text{s}} \rightarrow \text{(to right)}$$

Flow rates

$$Q_f^{(1)} = v_x^{(1)}\ A = \left(1.2\frac{\text{cm}}{\text{s}}\right)(5\ \text{cm}^2)$$

$$Q_f^{(1)} = 6\ \frac{\text{cm}^3}{\text{s}}$$

$$Q_f^{(2)} = v_x^{(2)}\ A = \left(\frac{2\ \text{cm}}{\text{s}}\right)\ (3\ \text{cm}^2)$$

$$Q_f^{(2)} = 6\ \frac{\text{cm}^3}{\text{s}}$$

14.5

$$[k] = \int_V [B]^T [D] [B]\, dV$$

For 1-D formulation

$$[B] = \begin{bmatrix}-\frac{1}{L} & \frac{1}{L}\end{bmatrix} \qquad [D] = [K_{xx}]$$

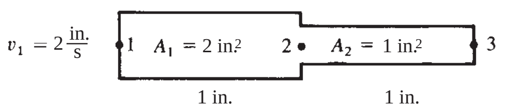
$$\therefore \qquad [k] = \int_v \begin{Bmatrix}-\frac{1}{L}\\ \frac{1}{L}\end{Bmatrix}[K_{xx}]\begin{bmatrix}-\frac{1}{L} & \frac{1}{L}\end{bmatrix} dV$$

For element with constant cross sectional area A

$$[k] = \int_0^L K_{xx}\begin{bmatrix}\frac{1}{L^2} & -\frac{1}{L^2}\\ -\frac{1}{L^2} & \frac{1}{L^2}\end{bmatrix} A\, dx$$

$$[k] = \frac{K_{xx}\ A}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

14.6



$$K_{xx} = \frac{1}{10}\,\frac{\text{in.}}{\text{s}}$$

$$[k^{(1)}] = \frac{2\times 1}{10}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad [k^{(2)}] = \frac{1\times 1}{10}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

'Force' at node 1

$$F_{1x} = \left(\frac{2\text{ in.}}{\text{s}}\right)(2\text{ in.}^2) = 4\,\frac{\text{in.}^3}{\text{s}}$$

Assume $\quad p_3 = 0$

Assemble equations

$$\begin{bmatrix} 0.2 & -0.2 \\ -0.2 & 0.2+0.1 \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 0 \end{Bmatrix}$$

Solving

$$p_1 = 60\text{ in.}, \quad p_2 = 40\text{ in.}$$

Velocities

$$v_x^{(1)} = -K_{xx}\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$

$$v_x^{(1)} = -\frac{1}{10}\begin{bmatrix} -\frac{1}{1} & \frac{1}{1} \end{bmatrix}\begin{Bmatrix} 60 \\ 40 \end{Bmatrix} = 2.0\,\frac{\text{in.}}{\text{s}}$$

$$v_x^{(2)} = -\frac{1}{10}\begin{bmatrix} -\frac{1}{1} & \frac{1}{1} \end{bmatrix}\begin{Bmatrix} 40 \\ 0 \end{Bmatrix} = 4.0\,\frac{\text{in.}}{\text{s}}$$

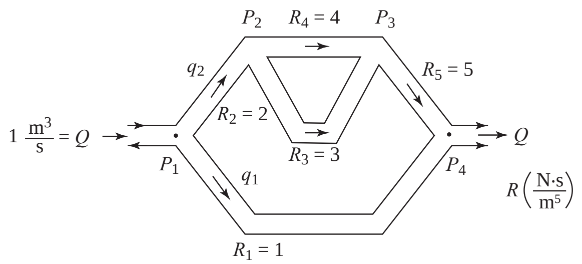
Volumetric flow rates

$$Q_f^{(1)} = v_x^{(1)}\,A^{(1)} = \left(2\,\frac{\text{in.}}{\text{s}}\right)(2\text{ in.}^2) = 4\,\frac{\text{in.}^3}{\text{s}}$$

$$Q_f^{(2)} = v_x^{(2)}\,A^{(2)} = \left(4\,\frac{\text{in.}}{\text{s}}\right)(1\text{ in.}^2) = 4\,\frac{\text{in.}^3}{\text{s}}$$

14.7

(a)



$$[k^{(1)}] = \frac{1}{1}\begin{array}{c} \begin{array}{cc} 1 & 4 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}, \quad [k^{(2)}] = \frac{1}{2}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$[k^{(3)}] = \frac{1}{3}\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}, \quad [k^{(4)}] = \frac{1}{4}\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$[k^{(5)}] = \frac{1}{5}\begin{array}{c} \begin{matrix} 3 & \;\;4 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

Assemble $[k]$'s

$$\begin{array}{c} \begin{matrix} 1 & \quad\quad 2 & \quad\quad 3 \end{matrix} \\ \begin{bmatrix} 1+\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2}+\frac{1}{3}+\frac{1}{4} & -\frac{1}{2}-\frac{1}{4} \\ 0 & -\frac{1}{3}-\frac{1}{4} & \frac{1}{3}+\frac{1}{4}+\frac{1}{5} \end{bmatrix} \\ \frac{\text{m}^5}{\text{N}\cdot\text{s}} \end{array} \begin{array}{c} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix} \\ \frac{\text{N}}{\text{m}^2} \end{array} = \begin{array}{c} \begin{Bmatrix} 1 = Q_1 \\ 0 \\ 0 \end{Bmatrix} \\ \frac{\text{m}^3}{\text{s}} \end{array}$$

Solve in Mathcad for p_1, p_2, p_3

$p_1 = 0.897\,\frac{\text{N}}{\text{m}^2}$, $\quad p_2 = 0.691\,\frac{\text{N}}{\text{m}^2}$, $\quad p_3 = 0.515\,\frac{\text{N}}{\text{m}^2}$

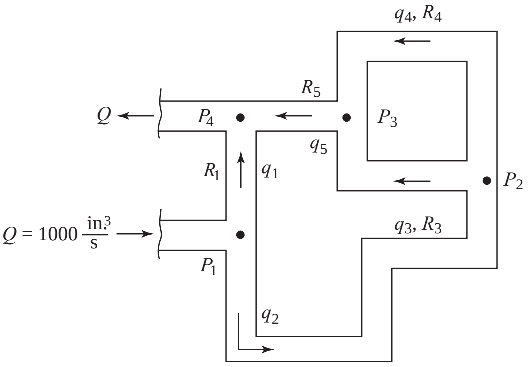
$q_1 = \frac{\Delta P}{R_1} = \frac{P_1 - P_4}{R_1} = 0.897$, $\quad q_2 = \frac{\Delta P}{R_2} = \frac{P_1 - P_2}{R_2} = 0.103$, $\quad q_3 = \frac{P_2 - P_3}{R_3} = 0.059$

$q_4 = \frac{P_2 - P_3}{R_4} = 0.044$, $\quad q_5 = \frac{P_3 - 0}{R_5} = 0.103$ $\quad$ (all qs in $\frac{\text{m}^3}{\text{s}}$)

$q_1 + q_4 = 0.103$ (check equals q_2)

$q_1 + q_5 = 1$ (check equals $Q = 1$)

(b)



$R_1 = 10\ \frac{\text{lb}\cdot\text{s}}{\text{in.}^5}$

$R_2 = 20$

$R_3 = 30$

$R_4 = 40$

$R_5 = 50$

$$[k^{(1)}] = \frac{1}{10}\begin{array}{c} \begin{matrix} 1 & \;\;4 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}, \quad [k^{(2)}] = \frac{1}{20}\begin{array}{c} \begin{matrix} 1 & \;\;2 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$[k^{(3)}] = \frac{1}{30}\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}, \quad [k^{(4)}] = \frac{1}{40}\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$[k^{(5)}] = \frac{1}{50}\begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

Assemble

$$\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{1}{10}+\frac{1}{20} & -\frac{1}{20} & 0 \\ -\frac{1}{20} & \frac{1}{20}+\frac{1}{30}+\frac{1}{40} & -\frac{1}{30}-\frac{1}{40} \\ 0 & -\frac{1}{30}-\frac{1}{40} & \frac{1}{30}+\frac{1}{40}+\frac{1}{50} \end{bmatrix} \\ \frac{\text{in.}^5}{\text{lb}\cdot\text{s}} \end{array} \begin{array}{c} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix} \\ \frac{\text{lb}}{\text{in.}^2} \end{array} = \begin{array}{c} \begin{Bmatrix} 1000 \\ 0 \\ 0 \end{Bmatrix} \\ \frac{\text{in.}^3}{\text{s}} \end{array}$$

Solve for p_1, p_2, p_3 using Mathcad as

$p_1 = 8971$ psi, $p_2 = 6912$ psi, $p_3 = 5147$ psi,

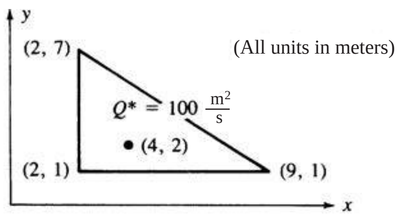
$q_1 = \dfrac{P_1 - P_4 = 0}{R_1} = 897$, $q_2 = \dfrac{P_1 - P_2}{R_2} = 103$, $q_3 = \dfrac{P_2 - P_3}{R_3} = 59$

$q_4 = \dfrac{P_2 - P_3}{R_4} = 44$, $q_5 = \dfrac{P_3 - 0}{R_5} = 103$ (all qs in $\dfrac{\text{in.}^3}{\text{s}}$)

$q_3 + q_4 = 103$ (check equals q_2)

$q_1 + q_5 = 1000$ (check equals Q)

14.8



$$\begin{Bmatrix} f_{Qi} \\ f_{Qs} \\ f_{Qm} \end{Bmatrix} = Q^*\, t \begin{Bmatrix} N_i \\ N_j \\ N_m \end{Bmatrix}\Bigg|_{\substack{x=4 \\ y=2}}$$

$$N_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y)$$

$$\alpha_i = x_i y_m - y_j x_m = 61$$

$$\alpha_j = y_j x_m - x_i y_m = -12$$

$$\alpha_m = x_i y_j - y_i x_j = -7$$

$$\beta_i = y_i - y_m = -6 \quad \gamma_i = x_m - x_j = -7$$

$$\beta_j = y_m - y_i = 6 \qquad \gamma_j = x_i - x_m = 0$$

$$\beta_m = y_i - y_j = 0 \qquad \gamma_m = x_j - x_i = 7$$

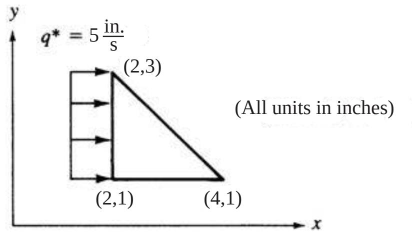
$$N_i\Big|_{\substack{x=4\\y=2}} = \frac{1}{52}\,(61 - 6(4) - 7(2)) = \frac{23}{42}$$

$$N_j\Big|_{\substack{x=4\\y=2}} = \frac{1}{52}\,(-12 + 6(4)) = \frac{12}{42}$$

$$N_m\Big|_{\substack{x=4\\y=2}} = \frac{1}{52}\,(-7 + 7(2)) = \frac{7}{42}$$

$$\therefore \quad \{f_Q\} = \left(100\frac{\text{m}^2}{\text{s}}\right)\left(\frac{I_m}{42}\right)\begin{Bmatrix}23\\12\\7\end{Bmatrix} = \begin{Bmatrix}54.76\\28.57\\16.67\end{Bmatrix}\frac{\text{m}^3}{\text{s}}$$

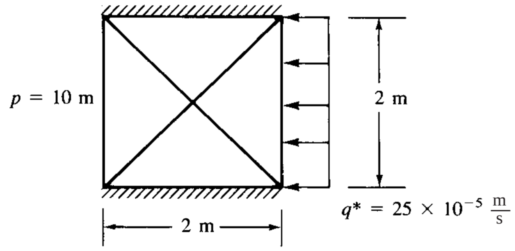
14.9



$$\{f\} = \frac{q^* Lt}{2}\begin{Bmatrix}1\\0\\1\end{Bmatrix}$$

$$\begin{Bmatrix}f_1\\f_2\\f_3\end{Bmatrix} = \frac{5\times 2\times 1}{2}\begin{Bmatrix}1\\0\\1\end{Bmatrix} = \begin{Bmatrix}5\\0\\5\end{Bmatrix}\frac{\text{in.}^3}{\text{s}}$$

14.10



From Equations (14.3.16)

$$[K] = \begin{bmatrix}25 & 0 & 0 & 0 & -25\\ & 25 & 0 & 0 & -25\\ & & 25 & 0 & -25\\ & & & 25 & -25\\ \text{Symmetry} & & & & 100\end{bmatrix}\times 10^{-5}$$

$$F = \begin{Bmatrix} 0 \\ 25 \\ 25 \\ 0 \\ 0 \end{Bmatrix} \times 10^{-5} \ \frac{\text{m}^3}{\text{s}}$$

Boundary conditions $p_1 = p_4 = 10$ m

$$\therefore \quad \begin{bmatrix} 25 & 0 & -25 \\ & 25 & -25 \\ \text{Symmetry} & & 100 \end{bmatrix} \begin{Bmatrix} p_2 \\ p_3 \\ p_5 \end{Bmatrix} = \begin{Bmatrix} 25 \\ 25 \\ 500 \end{Bmatrix}$$

Solving

$$p_2 = 12 \text{ m}, \ p_3 = 12 \text{ m}, \ p_5 = 11 \text{ m}$$

14.11

(33 nodes, 56 elements)

FLUIDS PROBLEM 14–11

BOUNDARY VALUES

NODAL FORCES

LOADING	CASE 1
0	0.00000E+00
LOADING	CASE 2
1	0.50000E+04

PRESCRIBED NODAL VALUES

26	0.50000E+03
27	0.50000E+03
28	0.50000E+03
29	0.50000E+03
30	0.50000E+03
31	0.50000E+03
32	0.50000E+03
33	0.50000E+03

FLUIDS PROBLEM 14–11

NODAL VALUES, LOADING CASE 1

1	– 0.46404E+03	2	0.37597E+02
5	0.14359E+07	6	– 0.26094E+03
9	0.45441E+02	10	0.25835E+03
13	0.25077E+03	14	– 0.57844E+02
17	0.26088E+03	18	0.39845E+03
3	– 0.17302E+02	4	– 0.16157E+03
7	– 0.75913E+02	8	0.13183E+03
11	0.23328E+03	12	0.19417E+03
15	0.16262E+03	16	0.23693E+03
19	0.38851E+03	20	0.37488E+03
21	0.38550E+03	22	0.28400E+03
25	0.39925E+03	26	0.50000E+03
29	0.50000E+03	30	0.50000E+03
33	0.50000E+03		
23	0.35570E+03	24	0.38834E+03
27	0.50000E+03	28	0.50000E+03
31	0.50000E+03	32	0.50000E+03

FLUIDS PROBLEM 14–11

NODAL VALUES, LOADING CASE 2

1	– 0.49446E+03	2	0.23005E+02
5	0.14476E+07	6	– 0.28970E+03
9	0.30588E+02	10	0.25067E+03
13	0.23855E+03	14	– 0.84934E+02
17	0.25296E+03	18	0.39522E+03
21	0.38006E+03	22	0.27361E+03
25	0.39590E+03	26	0.50000E+03
29	0.50000E+03	30	0.50000E+03
33	0.50000E+03		
3	– 0.33072E+02	4	– 0.18077E+03
7	– 0.96780E+02	8	– 0.34748E+01
11	0.22496E+03	12	0.18421E+03
15	0.14912E+03	16	0.22760E+03
19	0.38500E+03	20	0.37068E+03
23	0.34982E+03	24	0.38433E+03
27	0.50000E+03	28	0.50000E+03
31	0.50000E+03	32	0.50000E+03

FLUIDS PROBLEM 14–11

ELEMENT VELOCITY COMPONENTS

ELEMENT	VEL(X)	VEL(Y)
1	– 0.20066E+05	– 0.52061E+04
2	– 0.13173E+05	– 0.12099E+05
3	0.81240E+04	– 0.12099E+05
4	0.81240E+04	– 0.12094E+05
5	0.81240E+04	0.13833E+05
6	0.28676E+04	0.19089E+05
7	– 0.97320E+04	0.19089E+05
8	– 0.20066E+05	0.87555E+04
9	– 0.88300E+04	0.41014E+04
10	– 0.88300E+04	– 0.29504E+04
11	– 0.12222E+05	– 0.19572E+04
12	– 0.49369E+04	– 0.92425E+04
13	– 0.14055E+05	– 0.14229E+05
14	0.74983E+04	– 0.14229E+05
15	0.81240E+04	– 0.12092E+05
16	0.81240E+04	– 0.12097E+05
17	0.81240E+04	– 0.12099E+05
18	0.81240E+04	0.96035E+04
19	0.21425E+04	0.11355E+05
20	0.24686E+04	0.11029E+05
21	– 0.13329E+04	0.89500E+04
22	– 0.43866E+04	0.89500E+04
23	– 0.48610E+04	0.73297E+04
24	– 0.85681E+04	0.36225E+04
25	– 0.56043E+04	0.23942E+04
26	– 0.56043E+04	– 0.21350E+04
27	– 0.67102E+04	– 0.20719E+04
28	– 0.27549E+04	– 0.60273E+04
29	– 0.41023E+04	– 0.72286E+04
30	0.31961E+04	– 0.72286E+04
31	0.33656E+04	– 0.42572E+04
32	0.40359E+04	– 0.35868E+04
33	0.13674E+05	– 0.14397E+05
34	0.13674E+05	0.70190E+04
35	0.33137E+04	0.76100E+04
36	0.27655E+04	0.81583E+04

37	0.40783E+03	0.60563E+04
38	– 0.27156E+04	0.60563E+04
39	– 0.27727E+04	0.50557E+04
40	– 0.55241E+04	0.23043E+04
41	– 0.40618E+04	0.16983E+04
42	– 0.40618E+04	– 0.16833E+04
43	– 0.45923E+04	– 0.17156E+04
44	– 0.18482E+04	– 0.44597E+04
45	– 0.23311E+04	– 0.50048E+04
46	0.20741E+04	– 0.50048E+04
47	0.20397E+04	– 0.44382E+04
48	0.45798E+04	– 0.18980E+04
49	0.86399E+04	– 0.54948E+04
50	0.86399E+04	0.35806E+04
51	0.48159E+04	0.33480E+04
52	0.23920E+04	0.57719E+04
53	0.12355E+04	0.44664E+04
54	– 0.18510E+04	0.44664E+04
55	– 0.18156E+04	0.38847E+04
56	– 0.40301E+04	0.16702E+04

RESULTANT NODAL VALUES

1	– 0.49446E+03
2	0.23005E+02
3	– 0.33072E+02
4	– 0.18077E+03
5	0.14476E+07
6	– 0.28970E+03
7	– 0.96780E+02
8	– 0.34748E+01
9	0.30588E+02
10	0.25067E+03
11	0.22496E+03
12	0.18421E+03
13	0.23855E+03
14	0.84934E+02
15	0.14912E+03
16	0.22760E+03
17	0.25296E+03
18	0.39522E+03
19	0.38500E+03
20	0.37068E+03
21	0.38006E+03
22	0.27361E+03
23	0.34982E+03
24	0.38433E+03
25	0.39590E+03
26	0.50000E+03
27	0.50000E+03
28	0.50000E+03
29	0.50000E+03
30	0.50000E+03
31	0.50000E+03
32	0.50000E+03
33	0.00000E+00

14.12

BOUNDARY VALUES

NODAL FORCES

LOADING CASE 1

0	0.00000E+00

PRESCRIBED NODAL VALUES

1	0.60000E+01
2	0.60000E+01
3	0.60000E+01
6	0.30000E+01
7	0.30000E+01
8	0.30000E+01

NODAL VALUES, LOADING CASE 1

1	0.60000E+01	2	0.60000E+01
5	0.45000E+01	6	0.30000E+01
3	0.60000E+01	4	0.45000E+01
7	0.30000E+01	8	0.30000E+01

14.13 Use $\frac{1}{4}$ of the whole system due to symmetry

(25 nodes, 32 elements)

NODAL FORCES

LOADING CASE 1

0	0.00000E+00	

LOADING CASE 2

25	0.12500E+03	(pumping rate)

PRESCRIBED NODAL VALUES

1	0.10000E+03
6	0.10000E+03
11	0.10000E+03
16	0.10000E+03
21	0.10000E+03

FLUID PROBLEM 14-12

NODAL VALUES, LOADING CASE 1

1	0.10000E+03	2	0.10000E+03
5	0.10000E+03	6	0.10000E+03
9	0.10000E+03	10	0.10000E+03
13	0.10000E+03	14	0.10000E+03
17	0.10000E+03	18	0.10000E+03
21	0.10000E+03	22	0.10000E+03
25	0.10000E+03		
3	0.10000E+03	4	0.10000E+03
7	0.10000E+03	8	0.10000E+03
11	0.10000E+03	12	0.10000E+03
15	0.10000E+03	16	0.10000E+03
19	0.10000E+03	20	0.10000E+03
23	0.10000E+03	24	0.10000E+03

NODAL VALUES, LOADING CASE 2

1	0.10000E+03	2	0.10062E+03
5	0.10202E+03	6	0.10000E+03
9	0.10177E+03	10	0.10209E+03
13	0.10125E+03	14	0.10185E+03
17	0.10063E+03	18	0.10128E+03
21	0.10000E+03	22	0.10063E+03
25	0.10363E+03		

3	0.10121E+03	4	0.10174E+03
7	0.10062E+03	8	0.10122E+03
11	0.10000E+03	12	0.10062E+03
15	0.10231E+03	16	0.10000E+03
19	0.10198E+03	20	0.10277E+03
23	0.10129E+03	24	0.10207E+03

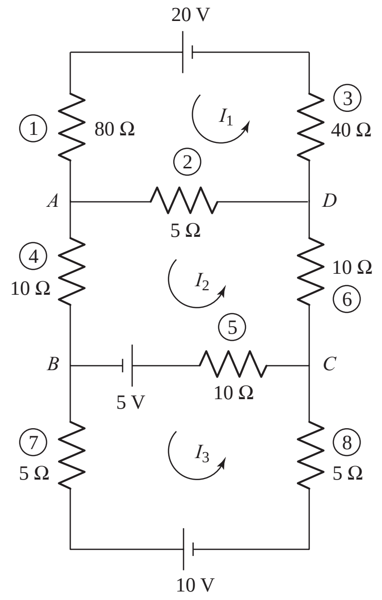
RESULTANT NODAL VALUES

1	0.10000E+03
2	0.10062E+03
3	0.10121E+03
4	0.10174E+03
5	0.10202E+03
6	0.10000E+03
7	0.10062E+03
8	0.10122E+03
9	0.10177E+03
10	0.10209E+03
11	0.10000E+03
12	0.10062E+03
13	0.10125E+03
14	0.10185E+03
15	0.10231E+03
16	0.10000E+03
17	0.10063E+03
18	0.10128E+03
19	0.10198E+03
20	0.10277E+03
21	0.10000E+03
22	0.10063E+03
23	0.10129E+03
24	0.10207E+03
25	0.00000E+00

ELEMENT VELOCITY COMPONENTS

ELEMENT	VEL (X)	VEL (Y)
1	0.12207E–03	0.57220E–05
2	0.79155E–04	0.21935E–04
3	0.26226E–04	0.47684E–05
4	0.17643E–04	0.41962E–04
5	0.11921E–03	0.00000E+00
6	0.92506E–04	0.57220E–05
7	0.12875E–04	0.21935E–04
8	0.33379E–04	0.47684E–05
9	0.12207E–03	0.57220E–05
10	0.94414E–04	– 0.19073E–05
11	0.14305E–04	0.47684E–05
12	0.32902E–04	0.24796E–04
13	0.11921E–03	0.00000E+00
14	0.92506E–04	0.57220E–05
15	0.12875E–04	– 0.19073E–05
16	0.45300E–04	0.47684E–05
17	0.12207E–03	0.57220E–05
18	0.94414E–04	0.28610E–04
19	0.14305E–04	0.35286E–04
20	0.48161E–04	0.95367E–06
21	0.11921E–03	0.00000E+00
22	0.10443E–03	0.57220E–05
23	0.95367E–05	0.28610E–04
24	0.30041E–04	0.35266E–04
25	0.12207E–03	0.57220E–05
26	0.10967E–03	– 0.56267E–04
27	0.17643E–04	– 0.19073E–04
28	0.36240E–04	0.95367E–06
29	0.11921E–03	0.00000E+00
30	0.80585E–04	0.57220E–05
31	0.28133E–04	– 0.56267E–04
32	0.45300E–04	– 0.19073E–04

14.17



$$[k^{(1)}] = \begin{array}{cc} I_1 & 0 \\ \end{array} \begin{bmatrix} 80 & -80 \\ -80 & 80 \end{bmatrix}, \quad [k^{(2)}] = \begin{array}{cc} I_1 & I_2 \end{array} \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}$$

$$[k^{(3)}] = \begin{array}{cc} I_1 & 0 \end{array} \begin{bmatrix} 40 & -40 \\ -40 & 40 \end{bmatrix}, \quad [k^{(4)}] = \begin{array}{cc} I_2 & 0 \end{array} \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix}$$

$$[k^{(5)}] = \begin{array}{cc} I_2 & I_3 \end{array} \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix}, \quad [k^{(6)}] = \begin{array}{cc} I_2 & 0 \end{array} \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix}$$

$$[k^{(7)}] = \begin{array}{cc} I_3 & 0 \end{array} \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}, \quad [k^{(8)}] = \begin{array}{cc} I_3 & 0 \end{array} \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}$$

$$\begin{array}{ccc} I_1 & I_2 & I_3 \end{array}$$
$$\begin{bmatrix} 80+5+40 & -5 & 0 \\ -5 & 5+10+10+10 & -10 \\ 0 & -10 & 10+5+5 \end{bmatrix} \begin{Bmatrix} I_1 \\ I_2 \\ I_3 \end{Bmatrix}$$

$$\begin{bmatrix} 125 & -5 & 0 \\ -5 & 35 & -10 \\ 0 & -10 & 20 \end{bmatrix} \begin{Bmatrix} I_1 \\ I_2 \\ I_3 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 5 \\ -10 \end{Bmatrix}$$

$$I_{AD} = I_1 - I_2$$

$$I_{BC} = I_2 - I_3$$

Using Mathcad

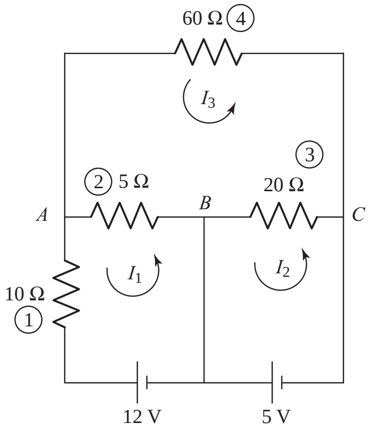
$$[K] = \begin{pmatrix} 125 & -5 & 0 \\ -5 & 35 & -10 \\ 0 & -10 & 20 \end{pmatrix} \qquad [V] = \begin{pmatrix} 20 \\ 5 \\ -10 \end{pmatrix}$$

$$\text{AMPS} = \text{1solve}\,(K, V) \qquad \text{AMPS} = \begin{pmatrix} 0.161 \\ 0.027 \\ -0.487 \end{pmatrix}$$

$$I_{AD} = \text{AMPS}_0 - \text{AMPS}_1 \qquad I_{AD} = 0.134$$

$$I_{BC} = \text{AMPS}_1 - \text{AMPS}_2 \qquad I_{BC} = 0.513$$

14.18



Resistor element stiffness matrices.

$$\begin{array}{cc} & \begin{array}{cc} I_1 & 0 \end{array} \\ [k^{(1)}] = 10 & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{cc} I_1 & I_3 \end{array} \\ [k^{(2)}] = 5 & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} I_2 & I_3 \end{array} \\ [k^{(3)}] = 20 & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{cc} I_3 & 0 \end{array} \\ [k^{(4)}] = 60 & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

Assemble equations

$$\begin{array}{c} \begin{array}{ccc} I_1 & I_2 & I_3 \end{array} \\ \begin{bmatrix} 10+5 & 0 & -5 \\ 0 & 20 & -20 \\ -5 & -20 & 5+20+60 \end{bmatrix} \end{array} \begin{Bmatrix} I_1 \\ I_2 \\ I_3 \end{Bmatrix} = \begin{Bmatrix} -12 \\ -6 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 15 & 0 & -5 \\ 0 & 20 & -20 \\ -5 & -20 & 85 \end{bmatrix} \begin{Bmatrix} I_1 \\ I_2 \\ I_3 \end{Bmatrix} = \begin{Bmatrix} -12 \\ -6 \\ 0 \end{Bmatrix}$$

Solve in Mathcad

$I_{AB} = I_1 - I_3$

$I_{BC} = I_2 - I_3$

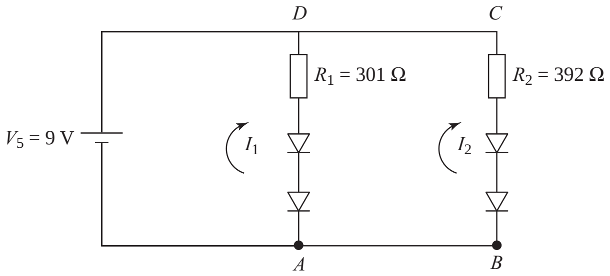
$$[K_1] = \begin{pmatrix} 15 & 0 & -5 \\ 0 & 20 & -20 \\ -5 & -20 & 85 \end{pmatrix} \qquad [V_1] = \begin{pmatrix} -12 \\ -6 \\ 0 \end{pmatrix}$$

$$\text{AMPS}_1 = \text{1solve}(K_1, V_1) \qquad \text{AMPS}_1 = \begin{pmatrix} -0.853 \\ -0.458 \\ -0.158 \end{pmatrix}$$

$$\text{I}_{\text{AB}_1} = \text{AMPS}_{1_0} - \text{AMPS}_{1_2} \qquad \text{I}_{\text{AB}_1} = -0.695$$

$$\text{I}_{\text{BC}_1} = \text{AMPS}_{1_1} - \text{AMPS}_{1_2} \qquad \text{I}_{\text{BC}_1} = -0.3$$

14.19



$$[k^{(1)}] = \begin{array}{cc} I_1 & I_2 \\ \left[\begin{array}{cc} 301 & -301 \\ -301 & 301 \end{array}\right. & \left.\right] \end{array}, \quad [k^{(2)}] = \begin{array}{cc} I_2 & 0 \\ \left[\begin{array}{cc} 392 & -392 \\ -392 & 392 \end{array}\right. & \left.\right] \end{array}$$

$$\begin{array}{cc} I_1 & I_2 \end{array}$$

$$\begin{bmatrix} 301 & -301 \\ -301 & 301+392 \end{bmatrix} \begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix} = \begin{Bmatrix} 9 \\ 0 \end{Bmatrix}$$

And upon solving in Mathcad

$I_{AD} = I_1 - I_2$ if < 0.015 amp R_1 acceptable

$I_{BC} = I_2$ if < 0.015 amp R_2 acceptable

$$[k] = \begin{pmatrix} 301 & -301 \\ -301 & 301+392 \end{pmatrix} \qquad [v] = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$$

I$^{\text{st}}$ iteration

$$\text{AMPS} = \text{1solve}(k, v) \qquad \text{AMPS} = \begin{pmatrix} 0.053 \\ 0.023 \end{pmatrix}$$

$$\text{I}_{\text{AD}} = \text{AMPS}_0 - \text{AMPS}_1 \qquad \text{I}_{\text{AD}} = 0.03$$

$$\text{I}_{\text{BC}} = \text{AMPS}_1 \qquad \text{I}_{\text{BC}} = 0.023$$

Note: amps through diodes must be < 0.015 AMPS, so try larger resistors. Try changing 301 to 392 as well.

$$[k_2] = \begin{pmatrix} 392 & -392 \\ -392 & 784 \end{pmatrix}$$

$$\text{AMPS}_2 = \text{1solve}(k_2, v) \qquad \text{AMPS}_2 = \begin{pmatrix} 0.046 \\ 0.023 \end{pmatrix}$$

No good. Try larger ohm resistor (Try 523 ohms)

$$[k_3] = \begin{pmatrix} 523 & -523 \\ -523 & 1046 \end{pmatrix}$$

$$\text{AMPS}_3 = \text{1solve}\,(k_3, v) \qquad \text{AMPS}_3 = \begin{pmatrix} 0.034 \\ 0.017 \end{pmatrix}$$

$$\text{I}_{\text{AD}_3} = \text{AMPS}_{3_0} - \text{AMPS}_{3_1} \qquad \text{I}_{\text{AD}_3} = 0.03 > 0.015 \quad \therefore \text{ no good}$$

Try 549 ohms for both resistors.

$$[k_4] = \begin{pmatrix} 549 & -549 \\ -549 & 1098 \end{pmatrix}$$

$$\text{AMPS}_4 = \text{1solve}\,(k_4, v) \qquad \text{AMPS}_4 = \begin{pmatrix} 0.033 \\ 0.016 \end{pmatrix}$$

Final iteration

Try 715 for R_1 and 806 for R_2

$$[k_5] = \begin{pmatrix} 715 & -715 \\ -1521 & 1521 \end{pmatrix}$$

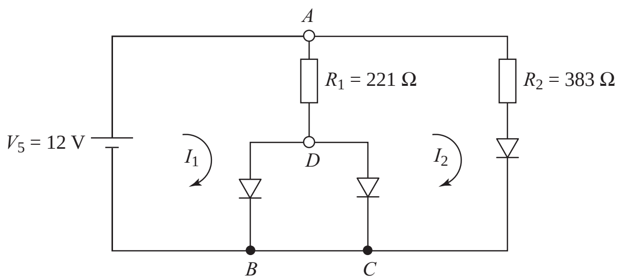
$$\text{AMPS}_5 = \text{1solve}\,(k_5, v) \qquad \text{AMPS}_5 = \begin{pmatrix} 0.024 \\ 0.011 \end{pmatrix}$$

$$\text{I}_{\text{AD}_5} = \text{AMPS}_{5_0} - \text{AMPS}_{5_1} \qquad \text{I}_{\text{AD}_5} = 0.013$$

$$\text{I}_{\text{BC}_5} = \text{AMPS}_{5_1} \qquad \text{I}_{\text{BC}_5} = 0.011 \qquad \text{This works. Now amps} < 0.015 \text{ amps}$$

$\therefore$ Let $R_1 = 715\ \Omega$, $R_2 = 806\ \Omega$

14.20



$$[k^{(1)}] = \begin{array}{c} \begin{matrix} I_1 & I_2 \end{matrix} \\ \begin{bmatrix} 221 & -221 \\ -221 & 221 \end{bmatrix} \end{array}, \quad [k^{(2)}] = \begin{array}{c} \begin{matrix} I_2 & 0 \end{matrix} \\ \begin{bmatrix} 383 & -383 \\ -383 & 383 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 221 & -221 \\ -221 & 221+383 \end{bmatrix} \begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix} = \begin{Bmatrix} 12 \\ 0 \end{Bmatrix}$$

$I_{AD} = I_1 - I_2$ if < 0.015 amp R_1 acceptable

If $I_2 < 0.015$ amp R_2 acceptable

$I_{BD} = I_1$

$I_{CD} = I_2$

Try 2000 Ω for R_1

1270 Ω for R_2

Then $I_1 = 0.015$ amp, $I_2 = 0.00945$ amp

$$[k_{20}] = \begin{pmatrix} 221 & -221 \\ -221 & 221+383 \end{pmatrix} \qquad \{v_{20}\} = \begin{pmatrix} 12 \\ 0 \end{pmatrix}$$

$$\text{AMPS}_{2_0} = \text{1solve}\ (k_{20}, v_{20}) \qquad \text{AMPS}_{2_0} = \begin{pmatrix} 0.086 \\ 0.031 \end{pmatrix}$$

Must increase ohms size so amps < 0.015. Try previous final R's of 715 and 806 ohms.

$$\text{AMPS}_{2_0} = \text{1solve}\ (k_5, v_{20}) \qquad \text{AMPS}_{2_0} = \begin{pmatrix} 0.032 \\ 0.015 \end{pmatrix}$$

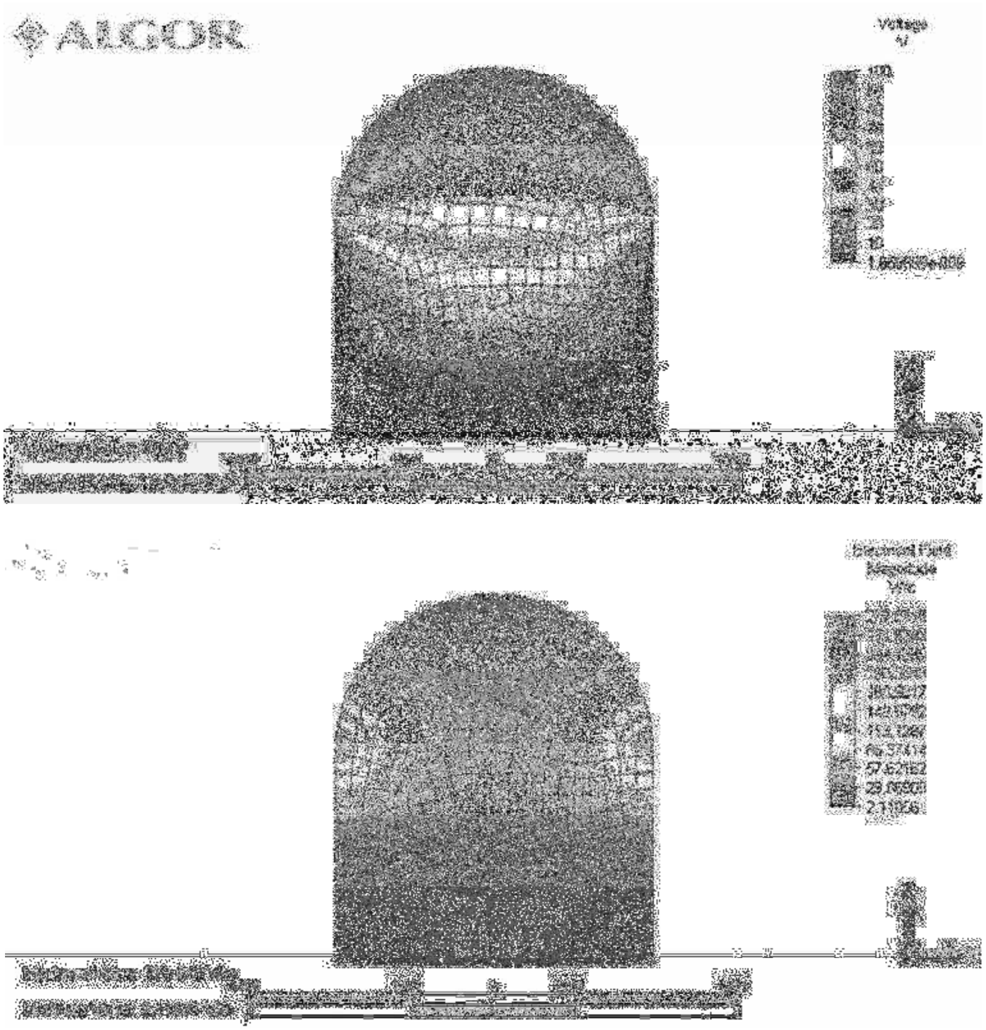
Still amps too large in diodes under $R1$ (Try 20000 ohm for $R1$ and 1270 ohms for $R2$)

$$k_{20}f = \begin{pmatrix} 2000 & -2000 \\ -2000 & 3270 \end{pmatrix}$$

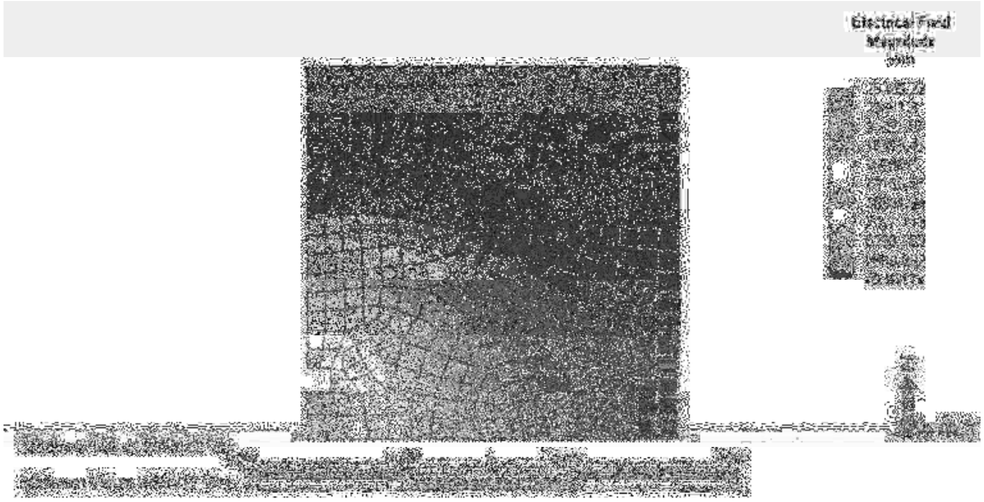
$$\text{AMPS}_{2_{0_f}} = \text{1solve}\ (k_{20}f, v_{20}) \qquad \text{AMPS}_{2_{0_f}} = \begin{pmatrix} 0.015 \\ 9.449 \times 10^{-3} \end{pmatrix}$$

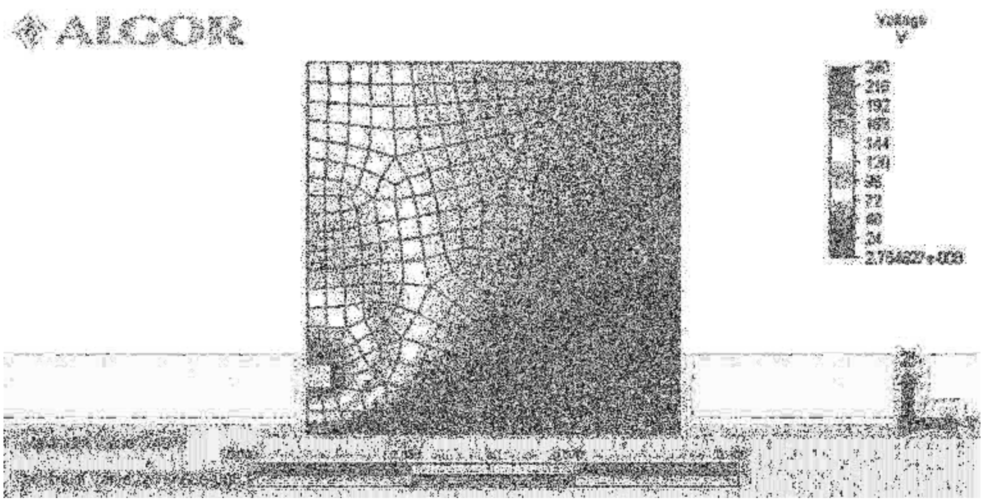
$$\therefore \quad \text{Let } R_1 = 2000\ \Omega, \qquad R_2 = 1270\ \Omega$$

14.22



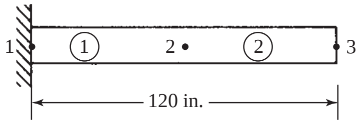
14.23


Electrical Field
Magnitude


ALGOR
Voltage
V
240
216
192
168
144
120
96
72
48
24

Chapter 15

15.1



$$[k^{(1)}] = \frac{AE}{60}\begin{array}{cc} 1 & 2 \\ \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad [k^{(2)}] = \frac{AE}{60}\begin{array}{cc} 2 & 3 \\ \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{f^{(1)}\} = \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix}, \qquad \{f^{(2)}\} = \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix}$$

$\{F\} = [K]\,\{d\}$ becomes

$$\begin{Bmatrix} -E\alpha TA \\ 0 \\ E\alpha TA \end{Bmatrix} = \frac{AE}{60}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \end{Bmatrix}$$

Solving

$$\begin{aligned} u_2 &= \alpha\, T\, L \\ &= (7 \times 10^{-6})\,(50^\circ\text{F})\,(60\text{ in.}) \\ &= 0.021\text{ in.} \end{aligned}$$

$$u_3 = 2\;\alpha\, T\, L = 0.042\text{ in.}$$

Reactions and actual nodal forces

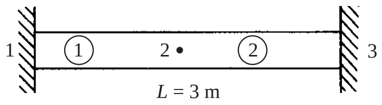
$$\{F\} = [K]\,\{d\} - \{F_0\}$$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\left(\frac{AE}{L}\right)\begin{Bmatrix} 0 \\ \alpha TL \\ 2\alpha TL \end{Bmatrix} - \begin{Bmatrix} -E\alpha TA \\ 0 \\ E\alpha TA \end{Bmatrix}$$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma^{(1)} = \sigma^{(2)} = \frac{0}{4\text{ in.}^2} = 0$$

15.2



$$[k^{(1)}] = \frac{AE}{1.5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad [k^{(2)}] = \frac{AE}{1.5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{f^{(1)}\} = \{f^{(2)}\} = \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix}$$

Global equations

$$\frac{AE}{1.5}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \end{Bmatrix} = \begin{Bmatrix} -E\alpha TA \\ 0 \\ E\alpha TA \end{Bmatrix}$$

Solving

$$\frac{AE}{1.5}\,(2u_2) = 0$$

$$u_2 = 0$$

Forces in elements

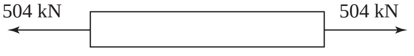
$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix} = \begin{Bmatrix} E\alpha TA \\ -E\alpha TA \end{Bmatrix}$$

$$E\,\alpha\,T\,A = (210\text{ GPa})\,(12\times10^{-6}/°\text{C})\,(-20°\text{C})\times(1\times10^{-2}\text{ m}^2)$$

$$= -504\text{ kN}$$

$$\therefore\ f_{1x}^{(1)} = -504\text{ kN},\ \ f_{2x}^{(1)} = 504\text{ kN}$$

FBD element 1



$$\sigma^{(1)} = \frac{504\text{ kN}}{1\times10^{-2}\text{ m}^2} = 50{,}400\text{ KPa}$$

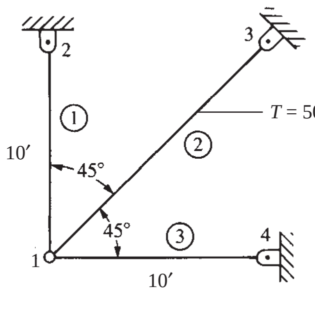
$$= 50.4\text{ MPa}$$

Similarly

$$\sigma^{(2)} = 50.4\text{ MPa}$$

$$F_{1x} = -504\text{ kN},\ F_{3x} = 504\text{ kN}$$

15.3



$\alpha = 7\times10^{-6}/°\text{F}$, $E = 30\times10^{6}$ psi
$A = 2$ in.^{2}, $f_{1x} = -E\alpha TA$

$$\{f\} = [T]^T\,\{f'\}$$

$$\begin{Bmatrix} f_{1x}'^{(2)} \\ f_{3x}'^{(2)} \end{Bmatrix} = \begin{Bmatrix} -21000 \\ 21000 \end{Bmatrix}$$

$$\{f\} = [T]^T\,\{f'\}$$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = \begin{bmatrix} 0.707 & -0.707 & 0 & 0 \\ 0.707 & 0.707 & 0 & 0 \\ 0 & 0 & 0.707 & -0.707 \\ 0 & 0 & 0.707 & 0.707 \end{bmatrix} \begin{Bmatrix} f'_{1x} = -21,000 \\ 0 \\ 21,000 \\ 0 \end{Bmatrix}$$

$f_{1x} = -14,850$ lb, $f_{1y} = -14,850$ lb

$f_{3x} = 14,850$ lb, $f_{3y} = -14,850$ lb

Boundary conditions

$$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$$

$$\begin{Bmatrix} F_{1x} = -14,850 \\ F_{1y} = \ 14,850 \end{Bmatrix} = 500000 \begin{bmatrix} 1.354 & 0.354 \\ 0.354 & 1.354 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

Solving

$$-u_1 = v_1 = -0.01753 \text{ in.}$$

By Equation (14.1.57)

$$\sigma^{(1)} = \frac{E}{L} [-C \quad -S \quad C \quad S] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} - \{0\}$$

$$\sigma^{(1)} = \frac{30 \times 10^6}{120 \text{ in.}} [0 \ -1 \ 0 \ 1] \begin{Bmatrix} 0.01753 \\ -0.01753 \\ 0 \\ 0 \end{Bmatrix}$$

$\sigma^{(1)} = 4350$ psi (T)

$$\sigma^{(2)} = \frac{E}{L} [-C \quad -S \quad C \quad S] \begin{Bmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{Bmatrix} - E\alpha T$$

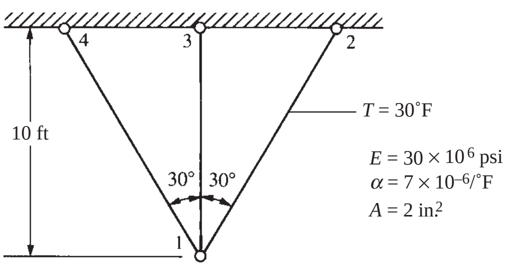
$$= \frac{30 \times 10^6}{120\sqrt{2}} [-0.707 \ -0.707 \ 0.707 \ 0.707] \begin{Bmatrix} 0.01753 \\ -0.01753 \\ 0 \\ 0 \end{Bmatrix}$$

$$= -10500$$

$\sigma^{(2)} = -6150$ psi (C)

$\sigma^{(3)} = 4350$ psi (T)

15.4



$$\{f'^{(1)}\} = \begin{Bmatrix} f'_{1x} \\ f'_{2x} \end{Bmatrix} = \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix} = \begin{Bmatrix} -12600 \\ 12600 \end{Bmatrix} \text{lb}$$

$$\{f\} = [T]^T \{f'\}$$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} f'_{1x} = -12600 \\ f'_{1y} = 0 \\ f'_{2x} = 12600 \\ f'_{2y} = 0 \end{Bmatrix}$$

$$f_{1x} = -\,6300 \text{ lb}$$
$$f_{1y} = -\,10912 \text{ lb}$$
$$f_{2x} = 6300 \text{ lb}$$
$$f_{2y} = 10912 \text{ lb}$$

Boundary conditions

$$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$$

Global equations

$$\begin{Bmatrix} F_{1x} = -6300 \\ F_{1y} = -10912 \end{Bmatrix} = \frac{(2 \text{ in.}^2)(30 \times 10^6)}{120 \text{ in.}} \begin{bmatrix} \frac{\sqrt{3}}{4} & 0 \\ 0 & 1 + \frac{3\sqrt{3}}{4} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$-6300 = 216{,}506\, u_1 \qquad u_1 = -\,0.0291 \text{ in.}$$
$$-10912 = 1{,}149{,}519\, v_1 \qquad v_1 = -0.0095 \text{ in.}$$

$$\sigma^{(1)} = \frac{E}{L}\,[-C \quad -S \quad C \quad S] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} - E\alpha T$$

$$= \frac{30 \times 10^6}{\frac{2(120)}{\sqrt{3}}} \left[-\frac{1}{2} \quad \frac{-\sqrt{3}}{2} \quad \frac{1}{2} \quad \frac{\sqrt{3}}{2} \right] \begin{Bmatrix} -0.0291 \\ -0.0095 \\ 0 \\ 0 \end{Bmatrix}$$

$$-\,30 \times 10^6 \times 7 \times 10^{-6} \times 30°\text{F}$$

$$\sigma^{(1)} = (216506)\,(0.0228) - 6300$$
$$= -\,1370 \text{ psi (C)}$$

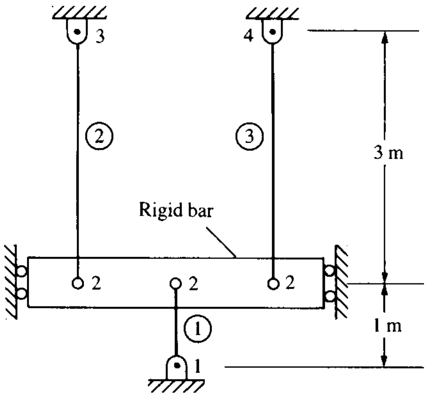
$$\sigma^{(2)} = \frac{30 \times 10^6}{120}\,[0 \;\; -\,1 \;\; 0 \;\; 1] \begin{Bmatrix} -0.0291 \\ -0.0095 \\ 0 \\ 0 \end{Bmatrix} - 0$$

$$\sigma^{(2)} = 2375 \text{ psi (T)}$$

$$\sigma^{(3)} = \frac{30 \times 10^6}{\frac{2(120)}{\sqrt{3}}} \left[\frac{1}{2} \quad -\frac{\sqrt{3}}{2} \quad -\frac{1}{2} \quad \frac{\sqrt{3}}{2} \right] \begin{Bmatrix} -0.0291 \\ -0.0095 \\ 0 \\ 0 \end{Bmatrix} - 0$$

$$\sigma^{(3)} = -\,1370 \text{ psi (C)}$$

15.5



$E = 210$ GPa
$\alpha = 12 \times 10^{-6}/°\text{C}$
$T^{(1)} = 20°\text{C}$
$A = 2 \times 10^{-2}\ \text{m}^2$
$L = 1$ m

$$[k^{(1)}] = \frac{AE}{L^{(1)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad [k^{(2)}] = \frac{AE}{L^{(2)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(3)}] = \frac{AE}{L^{(3)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{f^{(1)}\} = \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix}$$

Global equations

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+\frac{1}{3}+\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -E\alpha TA + F_{1x} \\ E\alpha TA \\ F_{3x} \\ F_{4x} \end{Bmatrix}$$

Solving equation (2) above

$$\frac{AE}{L}\left(\frac{5}{3}\right) u_2 = E\alpha TA$$

$$u_2 = \frac{3\alpha T L}{5} = \frac{3(12\times10^{-6})(20°)(1\text{m})}{5}$$

$$u_2 = 1.44 \times 10^{-4}\ \text{m}$$

Global nodal forces

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \frac{(2\times10^{-2})(210\times10^{9})}{1\text{m}}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}\begin{Bmatrix} 0 \\ 1.44\times10^{-4} \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \\ 0 \\ 0 \end{Bmatrix}$$

$$E\alpha TA = (210\times10^{9})\ (12\times10^{-6})\ (20°)\ (2\times10^{-2})$$

$$= 1{,}008{,}000\ \text{N} = 1{,}008\ \text{kN}$$

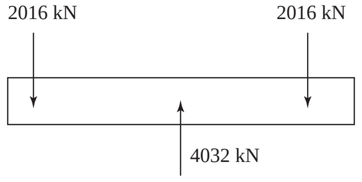
$$F_{1x} = -604.8 + 1{,}008 = 403.2\ \text{kN}$$

$$F_{2x} = 1{,}008 - 1{,}008 = 0$$

$$F_{3x} = -2{,}016\ \text{kN}$$

$F_{4x} = -2{,}016$ kN

FBD



$$\sigma^{(1)} = [C'] \{d\} - E\alpha T$$

$$= \frac{E}{L} [-C \quad -S \quad C \quad S] \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 = 1.44 \times 10^{-4} \\ v_2 = 0 \end{Bmatrix} - E\alpha T$$

$$= \frac{E}{L} (1.44 \times 10^{-4}) - E\alpha T$$

$$= 30.2 - 50.4$$

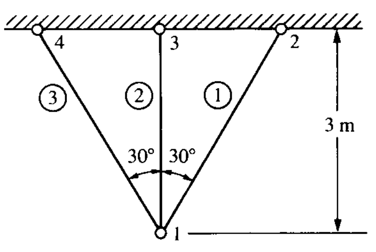
$$\sigma^{(1)} = -20.2 \text{ MPa (C)}$$

$$\sigma^{(2)} = \frac{210 \times 10^{-9}}{3\text{m}} [-1 \quad 0 \quad 1 \quad 0] \begin{Bmatrix} 1.44 \times 10^{-4} \\ 0 \\ 0 \\ 0 \end{Bmatrix} - 0$$

$$= -100.8 \times 10^{5} \frac{\text{N}}{\text{m}^2}$$

$$\sigma^{(2)} = -10.08 \text{ MPa (C)} = \sigma^{(3)}$$

15.6



$E = 70$ GPa $\quad \alpha = 23 \times 10^{-6}/^\circ$C

$A = 4 \times 10^{-2}$ m^2 $\quad T^{(2)} = -20^\circ$C

$$[k^{(1)}] = \frac{AE}{3.46\,\text{m}} \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{4} & -\frac{\sqrt{3}}{4} \\ & \frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\ & & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \text{Symmetry} & & & \frac{3}{4} \end{bmatrix}$$

$$[k^{(2)}] = \frac{AE}{3\,\text{m}}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$[k^{(3)}] = \frac{AE}{3.46\,\text{m}}\begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} \\ & \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} \\ & & \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ & \text{Symmetry} & & \frac{3}{4} \end{bmatrix}$$

Boundary conditions

$$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$$

Thermal forces

$$f_{1x}'^{(2)} = -E\alpha TA, \; f_{3x}'^{(2)} = E\alpha TA$$

Convert to global forces using

$$\{f\} = [T]^T \{f'\}$$

$$\begin{Bmatrix} f_{1x}^{(2)} \\ f_{1y}^{(2)} \\ f_{3x}^{(2)} \\ f_{3y}^{(2)} \end{Bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{Bmatrix} -E\alpha TA \\ 0 \\ E\alpha TA \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ E\alpha TA \\ 0 \\ -E\alpha TA \end{Bmatrix}$$

$$\therefore \quad f_{1y}^{(2)} = E\alpha TA = (70 \times 10^6)\,(23 \times 10^{-6})\,(-20°\text{C}) \times (4 \times 10^{-2}\ \text{m}^2)$$

$$= 1288\ \text{kN}$$

$$f_{3y}^{(2)} = -1288\ \text{kN}$$

Assemble equations $\{F\} = [K]\,\{d\} - \{F_0\}$

$$\frac{AE}{3.46}\begin{bmatrix} 0.5 & 0 \\ 0 & 2.65 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1288 \end{Bmatrix}$$

Solving

$$u_1 = 0$$

$$v_1 = \frac{1288(3.46)}{2.65\left(4 \times 10^{-2}\right)\left(70 \times 10^6\right)} = 6 \times 10^{-4}\ \text{m}$$

Element forces

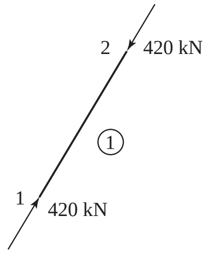
$$\{f'\} = [k']\,\{d'\} - \{f_0'\} = [k']\,[T]^*\,\{d\} - \{f_0'\}$$

$$\begin{Bmatrix} \hat{f}_{1x}^{(1)} \\ \hat{f}_{2x}^{(1)} \end{Bmatrix} = \frac{AE}{3.46}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$= \frac{AE}{3.46}\begin{bmatrix} C & S & -C & -S \\ -C & -S & C & S \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ v_1 = 6\times10^{-4} \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix}$$

$$= \frac{(4\times10^{-2})(70\times10^{6})}{3.46}\begin{Bmatrix} \frac{\sqrt{3}}{2}\times6\times10^{-4} \\ -\frac{\sqrt{3}}{2}\times6\times10^{-4} \end{Bmatrix}$$

$$= \begin{Bmatrix} 420 \\ -420 \end{Bmatrix} \text{ kN}$$



Stresses

$$\underline{\sigma} = [C'] \{d\} - E\,\alpha\,T \quad \text{or} \quad \sigma^{(1)} = \frac{f'_{2x}}{A}$$

Element 1

$$\sigma^{(1)} = \frac{-420}{4\times10^{-2}} = -10.5 \text{ MPa (C)}$$

Element 2

$$\sigma^{(2)} = \frac{70\times10^{6}}{3.0\,\text{m}}\ [0\ \ -1\ \ 0\ \ 1]\begin{Bmatrix} u_1 = 0 \\ v_1 = 6\times10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$-70\times10^{6}\,(23\times10^{-6})\,(-20°\text{C})$$

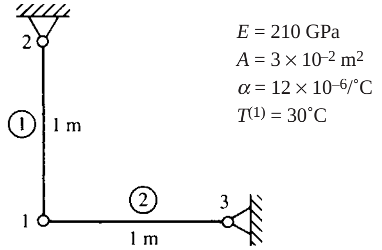
$$= -14000 + 32200$$

$$= 18200 \text{ kPa}$$

$$\sigma^{(2)} = 18.2 \text{ MPa (T)}$$

$$\sigma^{(3)} = \sigma^{(1)} = -10.5 \text{ MPa (C)}$$

15.7



$$E\,\alpha\,T\,A = \left(210\times10^{6}\,\frac{\text{kN}}{\text{m}^2}\right)\times12\times\frac{10^{-6}}{°\text{C}}\times30°\text{C}\times(3\times10^{-2})$$

$$= 2.27\times10^{3} \text{ kN}$$

$$[k^{(1)}] = \frac{AE}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\{f_0'^{(1)}\} = \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix}$$

$$[k^{(2)}] = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left\{f_0'^{(2)}\right\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Transform initial forces to global forces

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{1y}^{(1)} \\ f_{2x}^{(1)} \\ f_{2y}^{(1)} \end{Bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{Bmatrix} -E\alpha TA \\ 0 \\ E\alpha TA \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ E\alpha TA \\ 0 \\ -E\alpha TA \end{Bmatrix}$$

Assemble global equations

$$\frac{\left(3\times10^{-2}\,\text{m}^2\right)\left(210\times10^6\,\frac{\text{kN}}{\text{m}^2}\right)}{1\text{m}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -2.27\times10^3 \end{Bmatrix}$$

where $\quad u_2 = v_2 = u_3 = v_3 = 0$

Solving

$$u_1 = 0 \qquad v_1 = -\,3.6\times10^{-4}\text{ m}$$

Stresses

$$\sigma^{(1)} = \frac{E}{L}\left[-\overset{0}{\overset{\|}{C}} \quad -\overset{-1}{\overset{\|}{S}} \quad \overset{0}{\overset{\|}{C}} \quad \overset{1}{\overset{\|}{S}}\right]\begin{Bmatrix} 0 \\ -3.6\times10^{-4} \\ 0 \\ 0 \end{Bmatrix} - E\alpha T$$

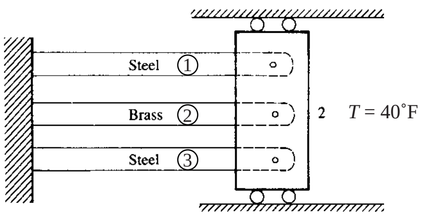
$$= \frac{210\times10^{-9}}{1\text{m}}\ (3.6\times10^{-4}\text{ m}) - 210\times10^9\times12\times10^{-6}\times30$$

$$\sigma^{(1)} = 0$$

$$\sigma^{(2)} = \frac{210\times10^{-9}}{1\text{m}}\ (0) - 0 = 0$$

For statically determinate structure thermal stresses are zero.

15.8



$$[k^{(1)}] = \frac{(2)\left(30\times10^6\right)}{60}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = [k^{(3)}]$$

$$[k^{(2)}] = \frac{2\left(15\times10^{6}\right)}{60}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

Global equations

$$10^{6}\begin{bmatrix} 2.5 & -2.5 \\ -2.5 & 2.5 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} F_{1x} - 15600\times 2 - 12000 \\ 15600\times 2 + 12000 \end{Bmatrix} \qquad (1)$$

where

$$f^{(1)} = -E\alpha TA = -30\times10^{6}\times 6.5\times10^{-6}\times 40\times 2$$

$$= -15600 \text{ lb}$$

$$\{f^{(1)}\} = \begin{Bmatrix} -15600 \\ 15600 \end{Bmatrix} = \{f^{(3)}\}$$

$$f^{(2)} = 15\times10^{6}\times 10\times10^{-6}\times 40\times 2$$

$$= -12000 \text{ lb}$$

$$\{f^{(2)}\} = \begin{Bmatrix} -12000 \\ 12000 \end{Bmatrix} \text{ lb}$$

Solving Equation (1) above

$$u_2 = 0.01728 \text{ in.}$$

Stresses

$$\sigma_{\text{st}} = \frac{AE}{L}\frac{u_2}{A} - E\alpha T$$

$$= \frac{\left(30\times10^{6}\right)}{60}(0.01728) - 30\times10^{6}\ (6.5\times10^{-6})\ 40$$

$$= 8640 - 7800$$

$$\sigma_{\text{st}} = 840 \text{ psi (T)}$$

$$\sigma_{\text{sr}} = \frac{E}{L}\ u_2 - E\alpha T$$

$$= \frac{15\times10^{6}}{60}(0.01728) - 15\times10^{6}\ (10\times10^{-6})\times 40$$

$$= 4320 - 6000 = -1680 \text{ psi}$$

15. 9 A uniform temperature increase of 10°C in each element yields zero stress for this special symmetric arrangement of the truss elements. See the table and figure 1 below showing the stresses from Algor to be zero in each element of the truss.

Note that if the truss is not symmetric as shown in figure 2 and then is uniformly heated, the middle element has a stress of –3.46 MPa in it, while the top element has a stress of 2.83 MPa in it.

**** 3-D truss elements

Number of elements = 3

Number of materials = 1

**** Nodal stresses for 3-D truss elements

El. #	LC	ND	Stress	Force
1	1	I	–1.118E-08	–1.341E-11
1	1	J	1.118E-08	1.341E-11
2	1	I	–1.118E-08	–1.341E-11
2	1	J	1.118E-08	1.341E-11
3	1	I	–1.118E-08	–1.341E-11
3	1	J	1.118E-08	1.341E-11

When we uniformly heat the truss the stresses go to zero for this symmetric structure.

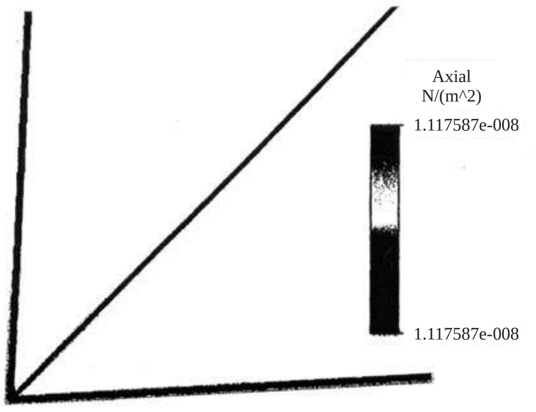


Figure 1

When we uniformly heat the truss the stresses are not zero for this unsymmetric statically indeterminate structure.

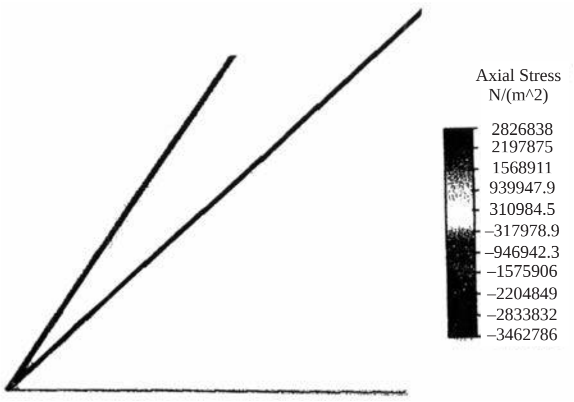
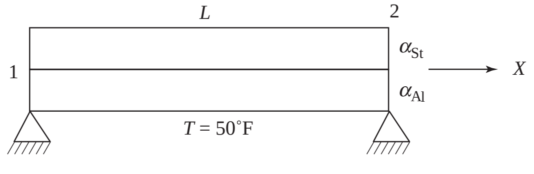


Figure 2

15.10 Bodies that are statically indeterminate will have stress due to uniform temperature change. (see figure 2 in solution to P 15.9) except in special symmetry cases (see figure 1 and table of results in P15.9). Also see, for instance, example 15.1, figure 15-5 and P 15.3, P 15.4 and P 15.6.

15.11



$$[k_{\text{st}}] = \frac{AE_{\text{st}}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},\ [k_{\text{A1}}] = \frac{AE_{\text{Al}}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\frac{A}{L}\begin{bmatrix} E_{\text{st}} + E_{\text{Al}} & -E_{\text{st}} - E_{\text{Al}} \\ -E_{\text{st}} - E_{\text{Al}} & E_{\text{st}} + E_{\text{Al}} \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_{1x} - E_{\text{st}}\,\alpha_{\text{st}}\,TA - E_{\text{Al}}\,\alpha_{\text{Al}}\,TA \\ (E_{\text{st}}\,\alpha_{\text{st}} + E_{\text{A}}\alpha_{\text{Al}})TA \end{Bmatrix}$$

Boundary conditions $u_1 = 0$

$$\frac{\cancel{A}}{L}\,(E_{\text{st}} + E_{\text{A1}})\,u_2 = (E_{\text{st}} + \alpha_{\text{st}} + E_{\text{A1}}\,\alpha_{\text{A1}})\,T\,\cancel{A}$$

$$u_2 = \frac{(E_{\text{st}}\alpha_{\text{st}} + E_{\text{A1}}\alpha_{\text{A1}})\,TL}{E_{\text{st}} + E_{\text{A1}}}$$

$$u_2 = \frac{\left(30\times10^6\times6.5\times10^{-6} + 10\times10^6\times13\times10^{-6}\right)TL}{(30+10)\times10^6}$$

$$u_2 = 8.125\times10^{-6}\,TL$$

$$= (8.125\times10^{-6})\,(50°\text{F})\,L$$

$$u_2 = 406.25\times10^{-6}\,L$$

$$\sigma_{\text{st}} = [C']\,\{d\} - \sigma_T$$

$$= \frac{E_{\text{st}}}{L}\,(-1\quad 0\quad 1\quad 0)\begin{Bmatrix} 0 \\ 0 \\ 406.25\times10^{-6}L \\ 0 \end{Bmatrix} - E_{\text{st}}\,\alpha_{\text{st}}\,T$$

$$= 30\times10^6\times406.25\times10^{-6} - 30\times10^6\times6.5\times10^{-6}\times50$$

$$= 12187.5 - 9750$$

$$\sigma_{\text{st}} = 2437.5 \text{ psi (T)}$$

$$\sigma_{\text{A1}} = 10\times10^6\times406.25\times10^{-6} - 10\times10^6\times13\times10^{-6}\times50$$

$$= 4062.5 - 6500$$

$$\sigma_{\text{A1}} = -\,24375 \text{ psi (C)}$$

$$\sigma_{\text{st}} = -\,\sigma_{A1}$$

15.12 To close gap of 0.005 in.

$$\delta_{\text{gap}} = \alpha_{\text{br}}\,\Delta TL_{\text{br}} + \alpha_{\text{m}}\,\Delta T\,L_{\text{m}}$$

$$\Delta T = \frac{\delta_{\text{gap}}}{\alpha_{\text{br}}L_{\text{br}} + \alpha_{\text{m}}L_{\text{m}}}$$

$$= \frac{0.005\text{ in.}}{11.3\times10^{-6}\times1\text{ in.} + 14.5\times10^{-6}\times1.5\text{ in.}}$$

$$\Delta T = 151.3°\text{F to close gap}$$

$$\delta_{\text{br}_{\text{close}}} = \alpha_{\text{br}}\,\Delta T\,L_{\text{br}}$$

$$= 11.3\times10^{-6}\times151.3°\times1\text{ in.}$$

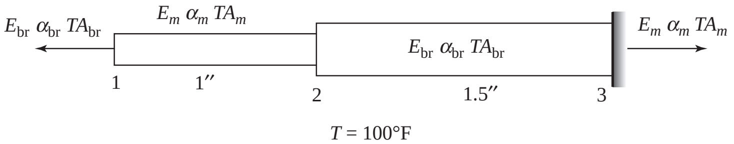
$$= 0.0017097\text{ in.}$$

$$\delta_{\text{m}_{\text{close}}} = \alpha_m\,\Delta_T\,L_m$$

$$= 14.5 \times 10^{-6} \times 151.3 \times 15 \text{ in.}$$

$$= 0.0032908 \text{ in.}$$

(a)



$$[k_{\text{br}}] = \frac{A_{\text{br}} E_{\text{br}}}{L_{\text{br}}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, [k_{\text{m}}] = \frac{A_{\text{m}} E_{\text{m}}}{L_{\text{m}}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{A_{\text{br}}E_{\text{br}}}{L_{\text{br}}} & \frac{-A_{\text{br}}E_{\text{br}}}{L_{\text{br}}} & 0 \\ \frac{-A_{\text{br}}E_{\text{br}}}{L_{\text{br}}} & \frac{A_{\text{br}}E_{\text{br}}}{L_{\text{br}}} + \frac{A_{\text{m}}E_{\text{m}}}{L_{\text{m}}} & \frac{-A_{\text{m}}E_{\text{m}}}{L_{\text{m}}} \\ 0 & \frac{-A_{\text{m}}E_{\text{m}}}{L_{\text{m}}} & \frac{A_{\text{m}}E_{\text{m}}}{L_{\text{m}}} \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} F_{1x} - E_{\text{br}}\alpha_{\text{br}}\ TA_{\text{br}} \\ -E_{\text{m}}\alpha_{\text{m}}\ TA_{\text{m}} + E_{\text{br}}\alpha_{\text{br}}\ TA_{\text{br}} \\ F_{3x} + E_{\text{m}}\alpha_{\text{m}}\ TA_{\text{m}} \end{Bmatrix}$$

$$\left(\frac{A_{\text{br}}E_{\text{br}}}{L_{\text{br}}} + \frac{A_{\text{m}}E_{\text{m}}}{L_{\text{m}}}\right) u_2 = -E_{\text{m}}\ \alpha_{\text{m}}\ TA_{\text{m}} + E_{\text{br}}\ \alpha_{\text{br}}\ TA_{\text{br}}$$

$$u_2 = \left(\frac{-E_{\text{m}}\alpha_{\text{m}}TA_{\text{m}} + E_{\text{br}}\alpha_{\text{br}}TA_{\text{br}}}{\frac{A_{\text{br}}E_{\text{br}}}{L_{\text{br}}} + \frac{A_{\text{m}}E_{\text{m}}}{L_{\text{m}}}}\right)$$

$$u_2 = \left(\frac{-4.5\times10^6 \times 14.5\times10^{-6} \times 0.15 \text{ in.}^2 + 15\times10^6 \times 11.3\times10^{-6} \times 0.1 \text{ in.}^2}{\frac{0.10\times15\times10^6}{1''} + \frac{0.15\times4.5\times10^6}{1.5''}}\right)100$$

$$u_2 = 3.673 \times 10^{-4} \text{ in.} \rightarrow$$

$$\sigma_{\text{br}} = \frac{E_{\text{br}}}{L_{br}}\ u_2 - E_{\text{br}}\,\alpha_{\text{br}}T$$

$$= \frac{E_{\text{br}}}{L_{\text{br}}}\left(\frac{-E_{\text{m}}\alpha_{\text{m}}TA_{\text{m}} + E_{\text{br}}\alpha_{\text{br}}TA_{\text{br}}}{\frac{A_{\text{br}}E_{\text{br}}}{L_{\text{br}}} + \frac{A_{\text{m}}E_{\text{m}}}{L_{\text{m}}}}\right) - E_{\text{br}}\ \alpha_{br}\ T$$

$$= \frac{1.5\times10^6}{1''}\left(\frac{-4.5\times10^6 \times 14.5\times10^{-6} \times 0.15 \text{ in.}^2 + 15\times10^6 \times 11.3\times10^{-6} \times 0.1 \text{ in.}^2}{\frac{0.10\times15\times10^6}{1} + \frac{0.15\times4.5\times10^6}{1.5}}\right)100°$$

$$- 15 \times 10^6 \times 11.3 \times 10^{-6} \times 100$$

$$= 15 \times 10^6 \left(\frac{-9.7875 + 16.95}{(1.5 + 0.45)\times10^6}\right) 100 - 16950$$

$$= \frac{107.4375 \times 100}{1.95} - 16950$$

$$= \frac{10743.75}{1.95} - 16950$$

$$= 5509.6 - 16950$$

$$\sigma_{br} = -11440 \text{ psi (C)}$$

$$F_{br} = (-11440 \text{ psi})(0.1 \text{ in.}^2) = -1144 \text{ lb}$$

$$\sigma_m = [C'] \{d\} - \sigma_m$$

$$= \frac{E_m}{L_m} [-1 \quad 0 \quad 1 \quad 0] \begin{Bmatrix} 3.673 \times 10^{-4} \text{ in.} = u_2 \\ 0 = v_2 \\ 0 = u_3 \\ 0 = v_3 \end{Bmatrix} - E_m \alpha_m T$$

$$= \frac{-4.5 \times 10^6}{1.5 \text{ in.}} (-3.673 \times 10^{-4} \text{ in.}) - 4.5 \times 10^6 \times 14.5 \times 10^{-6} \times 100°\text{F}$$

$$= -1100 - 6525$$

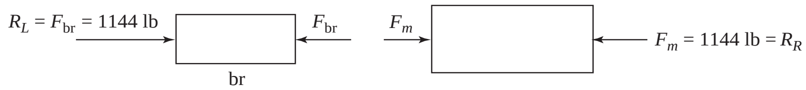
$$\sigma_m = -7625 \text{ psi (C)}$$

Check equations

$$F_m = \sigma_m A_m = (-7625)(0.15 \text{ in.}^2)$$

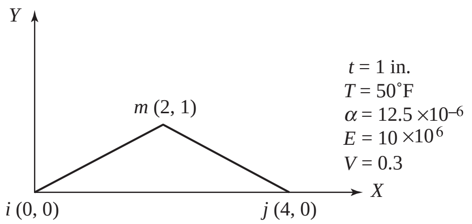
$$F_m = -1144 \text{ lb}$$

Same as $F_b = -1144$ lb. So equations check satisfied.



$\therefore$ Reactions R_L and R_m equal but opposite in direction.

15.13



$$\{f_T\} = \frac{\alpha E t T}{2(1-v)} \begin{Bmatrix} \beta_i \\ \gamma_i \\ \beta_j \\ \gamma_j \\ \beta_m \\ \gamma_m \end{Bmatrix}$$

$\beta_i = y_j - y_m = 0 - 1 = -1 \qquad \gamma_i = x_m - x_j = 2 - 4 = -2$

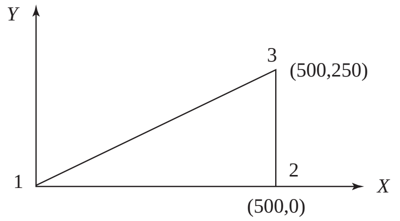
$\beta_j = y_m - y_i = 1 - 0 = 1 \qquad \gamma_j = x_i - x_m = 0 - 2 = -2$

$\beta_m = y_i - y_j = 0 - 0 = 0 \qquad \gamma_m = x_j - x_i = 4 - 0 = 4$

$$\{f_T\} = \frac{(12.5\times10^{-6})(10\times10^{6})(1)(50)}{2(1-0.3)}\begin{Bmatrix}-1\\-2\\1\\-2\\0\\4\end{Bmatrix}$$

$$\{f_T\} = \begin{Bmatrix}-4464\\-8929\\4464\\-8929\\0\\17857\end{Bmatrix}\text{ lb}$$

15.14



$E = 70$ GPa, $\nu = 0.3$

$T = 30$°C, $\alpha = 23\times10^{-6}$/°C $t = 5$ mm

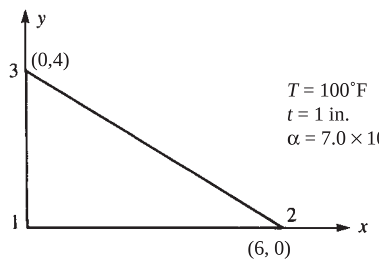
$\beta_1 = -250$ mm, $\beta_2 = 250$ mm, $\beta_3 = 0$

$\gamma_1 = 0$, $\gamma_2 = -500$ mm, $\gamma_3 = 500$ mm

$$\{f_T\} = \frac{(23\times10^{-6})(70\times10^{9})(0.005\,\text{m})30°\text{C}}{2(1-0.3)}\begin{Bmatrix}-0.25\,\text{m}\\0\\0.25\,\text{m}\\-0.50\,\text{m}\\0\\0.50\,\text{m}\end{Bmatrix}$$

$$\{f_T\} = 43125\begin{Bmatrix}-1\\0\\1\\-2\\0\\2\end{Bmatrix}\text{ N}$$

15.15



$\beta_i = y_j - y_m = 0 - 4 = -4, \qquad \gamma_i = x_m - x_j = 0 - 6 = -6$

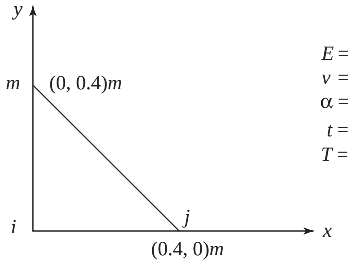
$\beta_j = y_m - y_i = 4 - 0 = 4, \qquad \gamma_j = x_i - x_m = 0 - 0 = 0$

$\beta_m = y_i - y_j = 0 - 0 = 0, \qquad \gamma_m = x_j - x_i = 6 - 0 = 6$

$$\{f_T\} = \frac{(7.0\times10^{-6})(30\times10^{6})(1)(100°\text{F})}{2(1-0.3)}\begin{Bmatrix}-4\\-6\\4\\0\\0\\6\end{Bmatrix}$$

$$= \begin{Bmatrix}-60,000\\-90,000\\60,000\\0\\0\\90,000\text{ lb}\end{Bmatrix}$$

15.16



$E = 210$ GPa
$\nu = 0.25$
$\alpha = 12\times10^{-6}$/°C
$t = 0.01$ m
$T = -20$°C

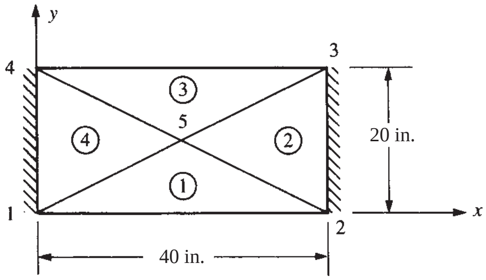
$$\{f_T\} = \frac{\alpha E t T}{2(1-\nu)}\begin{Bmatrix}\beta_i\\\gamma_i\\\beta_j\\\gamma_j\\\beta_m\\\gamma_m\end{Bmatrix}$$

$\beta_i = -0.4$ m $\qquad \gamma_i = -0.4$ m
$\beta_j = 0.4$ m $\qquad \gamma_j = 0$
$\beta_m = 0 \qquad \gamma_m = 0.4$ m

$$\{f_T\} = \frac{(12\times10^{-6})(210\times10^{9})(0.01)(-20°\text{C})}{2(1-0.25)}\begin{Bmatrix}-0.4\\-0.4\\0.4\\0\\0\\0.4\end{Bmatrix}$$

$$\{f_T\} = \begin{Bmatrix}134.4\\134.4\\-134.4\\0\\0\\-134.4\end{Bmatrix}\text{ kN}$$

15.17



Thermal force matrix

Element 1

$$i = 1,\ j = 2,\ m = 5$$

$$\beta_i = y_j - y_m = 0 - 10 = -10, \qquad \gamma_i = x_m - x_j = 20 - 40 = -20$$

$$\beta_j = y_m - y_i = 10 - 0 = 10, \qquad \gamma_j = x_i - x_m = 0 - 20 = -20$$

$$\beta_m = y_i - y_j = 0 - 0 = 0, \qquad \gamma_m = x_j - x_i = 40 - 0 = 40$$

$$\{f_T^{(1)}\} = \frac{(12.5\times10^{-6})(10\times10^{6})(1)(50)}{2(1-0.3)}\begin{Bmatrix}-10\\-20\\10\\-20\\0\\40\end{Bmatrix}$$

$$\{f_T^{(1)}\} = \begin{Bmatrix}-44643\\89286\\44643\\-89286\\0\\178572\end{Bmatrix}$$

Element 2

$$i = 2,\ j = 3,\ m = 5$$

$$\beta_i = 20 - 10 = 10, \qquad \gamma_i = 20 - 40 = -20$$

$$\beta_j = 10 - 0 = 10, \qquad \gamma_j = 40 - 20 = 20$$

$$\beta_m = 0 - 20 = -20, \qquad \gamma_m = 40 - 40 = 0$$

$$\{f_T^{(2)}\} = 4464.3\begin{Bmatrix}10\\-20\\10\\20\\-20\\0\end{Bmatrix} = \begin{Bmatrix}44643\\-89286\\44643\\89286\\-89286\\0\end{Bmatrix}$$

Element 3

$$i = 3,\ j = 4,\ m = 5$$

$$\beta_i = 20 - 10 = 10 \qquad \gamma_j = 20 - 0 = 20$$

$$\beta_j = 10 - 20 = -10 \qquad \gamma_j = 40 - 20 = 20$$

$$\beta_m = 20 - 20 = 0 \qquad \gamma_m = 0 - 40 = -40$$

$$\{f_T^{(3)}\} = 4464.3\begin{Bmatrix} 10 \\ 20 \\ -10 \\ 20 \\ 0 \\ -40 \end{Bmatrix} = \begin{Bmatrix} 44643 \\ 89286 \\ -44643 \\ 89286 \\ 0 \\ -178572 \end{Bmatrix}$$

Element 4

$$i = 4, j = 1, m = 5$$

$$\beta_i = 0 - 10 = -10 \qquad \gamma_i = 20 - 0 = 20$$

$$\beta_j = 10 - 20 = -10 \qquad \gamma_j = 0 - 20 = -20$$

$$\beta_m = 20 - 0 = 20 \qquad \gamma_m = 0 - 0 = 0$$

$$\{f_T^{(4)}\} = 4464.3\begin{Bmatrix} -10 \\ 20 \\ -10 \\ -20 \\ 20 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -44643 \\ 89286 \\ -44643 \\ -89286 \\ 89286 \\ 0 \end{Bmatrix}$$

$$\{F_0\} = [K]\,\{d\}$$

By direct superposition, we have

$$\begin{Bmatrix} -89,286 \\ -178,572 \\ 89,286 \\ -178,572 \\ 89,286 \\ 178,572 \\ -89,286 \\ 178,572 \\ 0 \\ 0 \end{Bmatrix} = \frac{10\times10^6}{4.16}\begin{bmatrix} 3 & 2 & 0.1 & 0.2 & 0 & 0 & -0.1 & -0.2 & -3 & -2 \\ & 6 & -0.2 & 2.6 & 0 & 0 & 0.2 & -2.6 & -2 & -6 \\ & & 3 & -2 & -0.1 & 0.2 & 0 & 0 & -3 & 2 \\ & & & 6 & -0.2 & -2.6 & 0 & 0 & 2 & -6 \\ & & & & 3 & 2 & 0.1 & 0.2 & -3 & -2 \\ & & & & & 6 & -0.2 & 2.6 & -2 & -6 \\ & & & & & & 3 & -2 & -3 & 2 \\ & & & & & & & 6 & 2 & -6 \\ & & & & & & & & 12 & 0 \\ & \text{Symmetry} & & & & & & & & 24 \end{bmatrix}$$

$$\times \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ \vdots \\ \vdots \\ u_5 \\ v_5 \end{Bmatrix}$$

Solving

$$0 = \frac{10\times10^6}{4.16}\; 12\, u_s \Rightarrow u_s = 0$$

$$0 = \frac{10\times10^6}{4.16}\; 24\, v_s \Rightarrow v_s = 0$$

Stresses

$$\{\sigma\} = \{\sigma_L\} - \{\sigma_T\}$$

$$\{\sigma_L\} = [D]\,[B]\,\{d\} = 0 \text{ as } \{d\} = \underline{0}$$

$$\therefore \quad \{\sigma\} = -\{\sigma_T\} = -[D]\,\{\varepsilon_T\}$$

Element 1

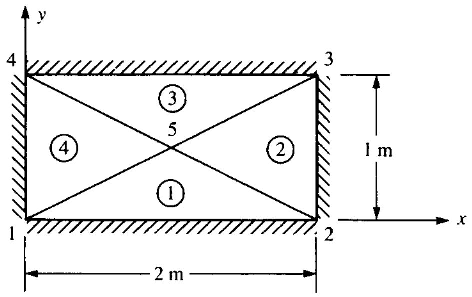
$$\{\sigma\} = -\frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}\begin{Bmatrix} \alpha T \\ \alpha T \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \tau_{xy} \end{Bmatrix} = \frac{-10\times10^6}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{Bmatrix} 6.25\times10^{-4} \\ 6.25\times10^{-4} \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} -8929 \\ -8929 \\ 0 \end{Bmatrix} \text{psi}$$

Since $[D]$ and $\{\varepsilon_T\}$ are same for all elements, all element stresses are equal.

15.18



Based on use of symmetry

$$u_s = v_s = 0 \text{ (Also see solution to Problem 15.17)}$$

$$\therefore \quad \{\sigma\} = \{\sigma_L\} - \{\sigma_T\} = [D]\,[B]\,\{d\} - [D]\,\{\varepsilon_T\}$$

$$\{\sigma\} = -[D]\,\{\varepsilon_T\}$$

All stresses in elements are equal

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{-E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}\begin{Bmatrix} \alpha T \\ \alpha T \\ 0 \end{Bmatrix}$$

$$= \frac{-210\times10^9}{1-0.25^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{0} \end{bmatrix}\begin{Bmatrix} 12\times10^{-6}(-20) \\ 12\times10^{-6}(-20) \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 67.2 \\ 67.2 \\ 0 \end{Bmatrix} \text{ MPa}$$

15.19

For bar with $\alpha = \alpha_0 \left(1 + \frac{X}{L}\right)$

$T =$ constant

By Equation (15.1.18)

$$\{f_T\} = A \int_0^L [B]^T [D]\, \varepsilon_T \, dx$$

L, A, E, T constant $\varepsilon_T = \alpha T$

$$\{f_T\} = A \int_0^L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} E\, \alpha_0 \left(1 + \frac{X}{L}\right) T\, dx$$

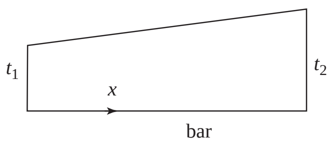
$$= \frac{AE\alpha_0 T}{L} \int_0^L \begin{Bmatrix} -\left(1 + \frac{X}{L}\right) \\ \left(1 + \frac{X}{L}\right) \end{Bmatrix} dx$$

$$= \frac{AE\alpha_0 T}{L} \begin{Bmatrix} -\left(X + \frac{X^2}{2L}\right)\Big|_0^L \\ \left(X + \frac{X^2}{2L}\right)\Big|_0^L \end{Bmatrix}$$

$$= \frac{AE\alpha_0 T}{L} \begin{Bmatrix} -\left(L + \frac{L}{2}\right) \\ \left(L + \frac{L}{2}\right) \end{Bmatrix}$$

$$\{f_T\} = \frac{3\, AE\alpha_0 T}{2} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

15.20



$T = t_1 + t_2\, x$

By Equation (15.1.18)

$$\{f_T\} = A \int_0^L [B]^T [D]\, \{\varepsilon_T\}\, dx$$

$$T = t_1 + t_2 x \qquad [N] = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix}$$

$$\{f_T\} = A \int_0^L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} E\alpha\, [N]\, \{t\}\, dx$$

$$= AE\,\alpha \int_0^L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} \begin{bmatrix} 1-\frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} dx$$

$$= \frac{AE\alpha}{L} \int_0^L \begin{Bmatrix} -1+\frac{x}{L} & \frac{-x}{L} \\ 1-\frac{x}{L} & \frac{x}{L} \end{Bmatrix} \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} dx$$

$$= \frac{AE\alpha}{L} \int_0^L \begin{Bmatrix} \left(-1+\frac{x}{L}\right) t_1 - \frac{x}{L} t_2 \\ \left(1-\frac{x}{L}\right) t_1 + \frac{x}{L} t_2 \end{Bmatrix} dx$$

$$= \frac{AE\alpha}{L} \begin{Bmatrix} \left(-x+\frac{x^2}{2L}\right)\Big|_0^L t_1 - \frac{x^2}{2L} t_2\Big|_0^L \\ \left(x-\frac{x^2}{2L}\right)\Big|_0^L t_1 + \frac{x^2}{2L} t_2\Big|_0^L \end{Bmatrix}$$

$$\{f_T\} = \frac{AE\alpha}{L} \begin{Bmatrix} \left(-L+\frac{L}{2}\right) t_1 - \frac{L}{2} t_2 \\ \left(L-\frac{L}{2}\right) t_1 + \frac{L}{2} t_2 \end{Bmatrix}$$

For $t_1 = t_2 = T$ (constant temperature over element)

$$\{f_T\} = \frac{AE\alpha}{L} \begin{Bmatrix} -TL \\ TL \end{Bmatrix} = \begin{Bmatrix} -AE\alpha T \\ AE\alpha T \end{Bmatrix}$$

Equation (15.1.18)

15.21

$$\{f_T\} = \int_s [B]^T\ [D]\ \{\varepsilon_T\}\ d\,s \tag{1}$$

$$\{\varepsilon_T\} = \alpha T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} \tag{2}$$

Using centroidal approximation

$$\{f_T\} = \int_s [B]^T\ [D]\ \{\varepsilon_T\}\ d\,s$$

$$= 2\pi\,\bar{r}\ A\ \alpha\,T \left[\bar{B}\right]^T [D] \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix} \tag{3}$$

where for axisymmetric case

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \tag{4}$$

substituting (4) into (3) and multiplying

$$\{f_T\} = \frac{2\pi\bar{r}AE\alpha T\left[\bar{B}\right]^T}{1-2\nu} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix}$$

15.22 Using modified CSFEP to account for thermal stress due to element temperature change.

INPUT TABLE 1.. BASIC PARAMETERS

NUMBER OF NODAL POINTS.	4
NUMBER OF ELEMENTS.	2
NUMBER OF DIFFERENT MATEIRALS.	1
NUMBER OF SURFACE LOAD CARDS	0
1 = PLANE STRAIN, 2 = PLANE STRESS. . . .	2
BODY FORCES (1 = IN – Y DIREC., 0 = NONE)	0

INPUT TABLE 2.. MATERIAL PROPERTIES

MATERIAL NUMBER	MODULUS OF ELASTICITY	POISSON'S RATIO	MATERIAL DENSITY	MATERIAL THICKNESS
1	0.3000E+00	0.3333E+00	0.7800E-01	0.1000E+00
				ALPHA
				0.1000E–04

INPUT TABLE 3.. NODAL POINT DATA

NODAL POINT	TYPE	X	Y	X-DISP. OR LOAD	Y-DISP. OR LOAD
1	3	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	2	0.1000E+01	0.0000E+00	0.0000E+00	0.0000E+00
3	0	0.0000E+00	0.1000E+01	0.0000E+00	0.0000E+00
4	0	0.1000E+01	0.1000E+01	0.0000E+00	0.0000E+00

INPUT TABLE 4.. ELEMENT DATA

ELEMENT	GLOBAL INDICES OF ELEMENT NODES 1	2	3	4	MATERIAL	TEMP.
1	1	2	3	3	1	0.800E+02
2	2	4	3	3	1	0.800E+02

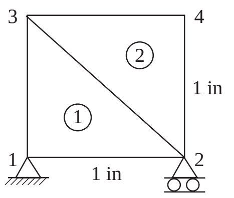
OUTPUT TABLE 1.. NODAL DISPLACEMENTS

NODE	U = X-DISP.	V = Y-DISP.
1	0.00000000E+00	0.00000000E+00
2	0.80000000E–03	0.00000000E+00
3	0.14551920E–10	0.80000000E–03
4	0.80000000E–03	0.80000000E–03

TABLE 2.. STRESSES AT ELEMENT CENTROIDS

ELEMENT	X	Y	SIGMA(X)	SIGMA(Y)	TAU(X, Y)
1	0.33	0.33	0.0000E+00	0.0000E+00	1.6371E–04
2	0.67	0.67	6.5484E–04	1.9645E–03	6.5484E–04
			SIGMA(1)	SIGMA(2)	ANGLE
			1.6371E–04	–1.6371E–04	0.0000E+00
			2.2358E–03	3.8359E–04	6.7500E+01

15.23

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           Verification.5
           4, 2, 2, 0, 2, 0
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           .15E+8, 0.25, 0., 0.1, 5.E–5
           1, 3, 0., 0., 0., 0.
           2, 2, 1., 0., 0., 0.
           3, 0, 0., 1., 0., 0.
           4, 0, 1., 1., 0., 0.
           1, 1, 2. 3, 3. 1. 80
           2, 2, 4, 3, 3, 2. 50.
```

Run using CSFEP modified for temperature changes in elements

verification.5

0 INPUT TABLE 1.. BASIC PARAMETERS

NUMBER OF NODAL POINTS.	4
NUMBER OF ELEMENTS.	2
NUMBER OF DIFFERENT MATERIALS.	2
NUMBER OF SURFACE LOAD CARDS	0
1 = PLANE STRAIN, 2 = PLANE STRESS. . . .	2
BODY FORCE (1 = IN – Y DIREC., 0 = NONE)	0

0 INPUT TABLE 2.. MATERIAL PROPERTIES

MATERIAL NUMBER	MODULUS OF ELASTICITY	POISSON’S RATIO	MATERIAL DENSITY	MATERIAL THICKNESS	ALPHA
1	0.3000E+08	0.3333E+00	0.0000E+00	0.1000E+00	0.1000E–04
2	0.1500E+08	0.2500E+00	0.0000E+00	0.1000E+00	0.5000E–04

INPUT TABLE 3.. NODAL POINT DATA

NODAL POINT	TYPE	X	Y	X-DISP. OR LOAD	Y-DISP. OR LOAD
1	3	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	2	0.1000E+01	0.0000E+00	0.0000E+00	0.0000E+00
3	0	0.0000E+00	0.1000E+01	0.0000E+00	0.0000E+00
4	0	0.1000E+01	0.1000E+01	0.0000E+00	0.0000E+00

0 INPUT TABLE 4.. ELEMENT DATA

	GLOBAL INDICES OR ELEMENT NODES				
ELEMENT	1	2	3	4	MATERIAL TEMP
1	1	2	3	3	1 0.800E+02
2	2	4	3	3	2 0.500E+02

0 OUTPUT TABLE 1.. NODAL DISPLACEMENTS

NODE	U = X-DISP.	V = Y-DISP.
1	0.00000000E+00	0.00000000E+00
2	0.98888970E–03	0.00000000E+00
3	–0.75555370E–03	0.98888970E–03
4	0.13194460E–02	0.20750000E–02

1OUTPUT TABLE 2.. STRESSES AT ELEMENT CENTROIDS

ELEMENT	X	Y	SIGMA(X)	SIGMA(Y)	TAU(X, Y)
1	0.33	0.33	8.5000E+03	8.5000E+03	–8.5000E+03
2	0.67	0.67	–8.5000E+03	–8.5000E+03	8.5000E+03
			SIGMA(1)	SIGMA(2)	ANGLE
			1.7000E+04	–3.9063E–03	0.0000E+00
			–1.9531E–03	–1.7000E+04	4.5000E+01

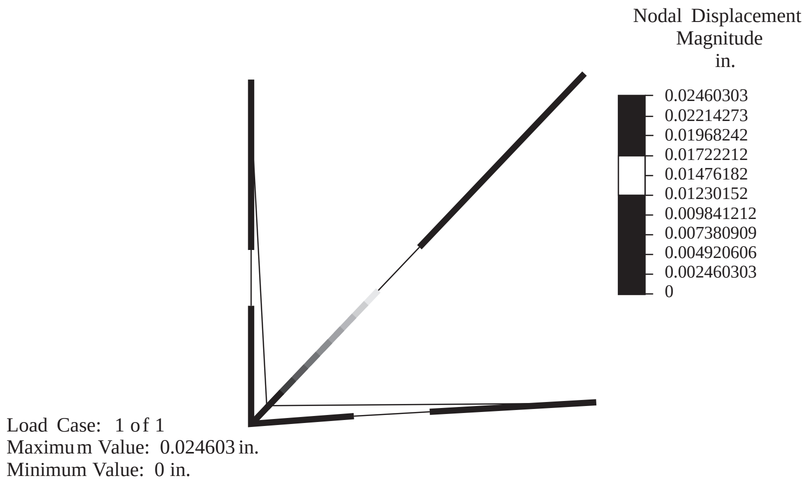
15.24 Solve Problem 15.3 using the Algor Program.

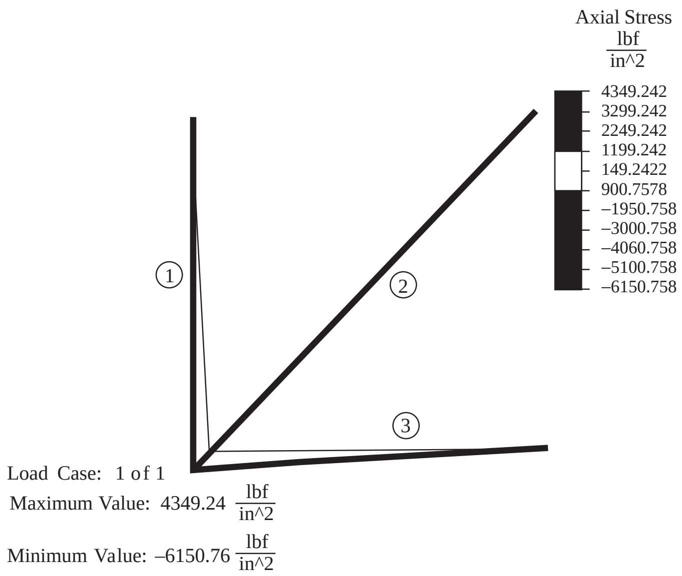
Displacements/Rotations (degrees) of nodes

NODE number	X-translation	Y-translation	Z-translation	X-rotation	Y-rotation	Z-rotation
1	–1.7397E–02	–1.7397E–02	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
3	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
4	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00

**** Nodal stresses for 3-D truss elements

El. #	LC	ND	Stress	Force
1	1	I	–4.349E+03	–8.698E+03
1	1	J	4.349E+03	8.698E+03
2	1	I	–4.349E+03	–8.698E+03
2	1	J	4.349E+03	8.698E+03
3	1	I	6.151E+03	1.230E+04
3	1	J	–6.151E+03	–1.230E+04





15.25 For the plane truss shown in Figure P15-6, bar element 2 is subjected to a uniform temperature drop of T = 20°C. Let E = 70 GPa, A = 4 × 10^–4m^2, and alpha = $23 \times 10^{-6} \dfrac{\frac{\text{mm}}{\text{mm}}}{°\text{C}}$. Determine the stresses in each bar and the displacement of node 1.

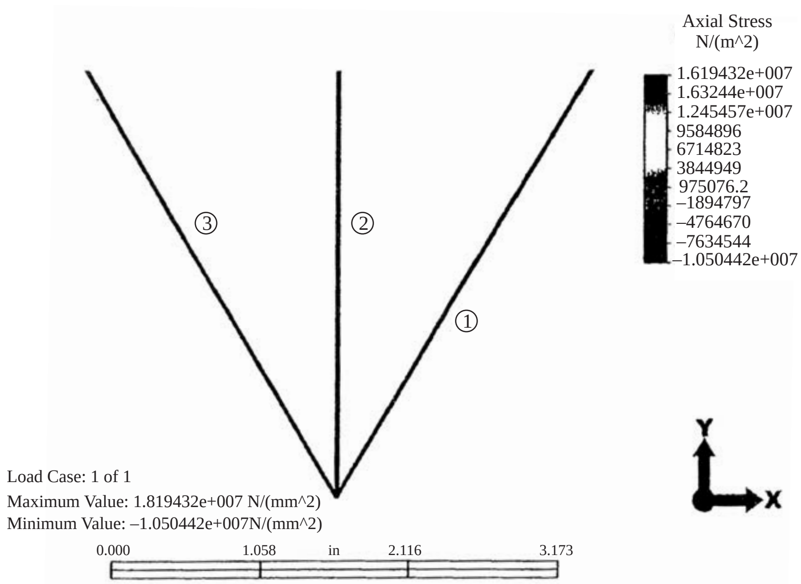


Figure 1 Axial Stress

As shown in Figure 1, the stress in bar 1 and 3 is 10.5 MPa (C) and the stress in bar 2 is 18.2 MPa (T).

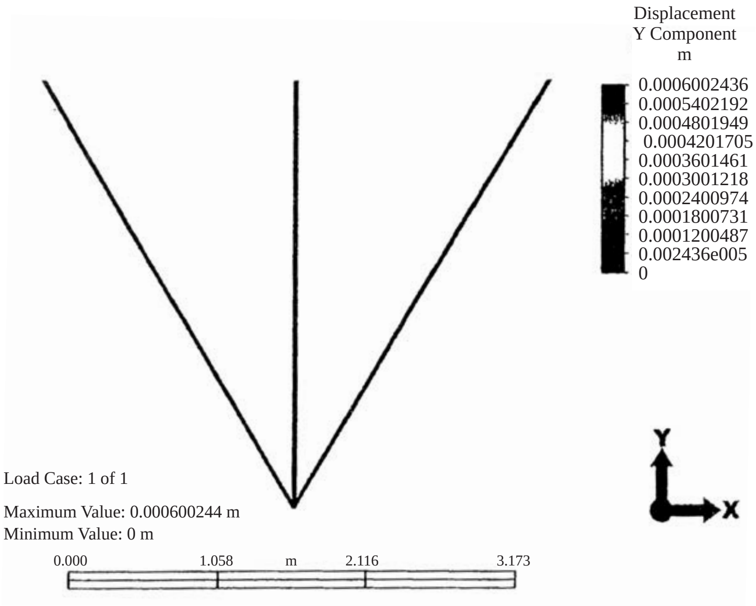
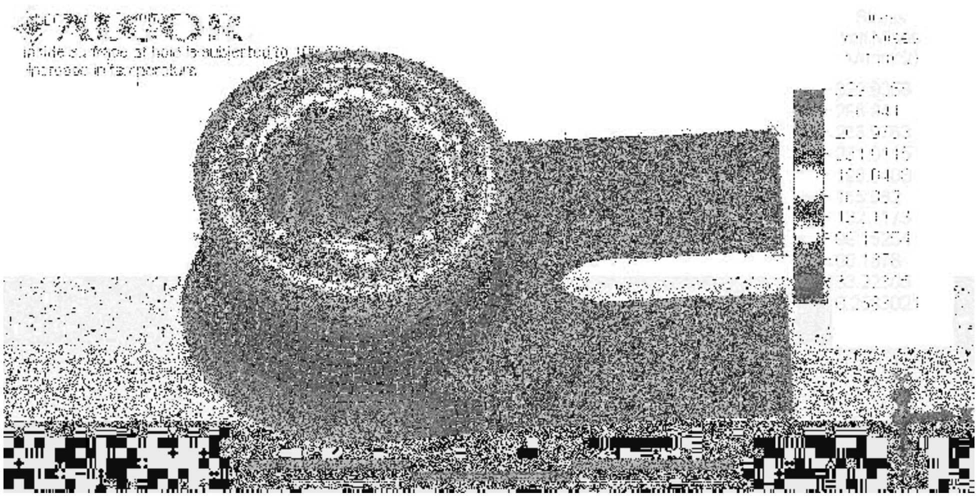


Figure 2 Displacement in Y Direction

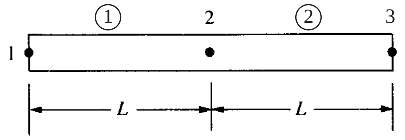
Figure 2 shows the *Y* displacement at node 1 is 0.0006 m in the positive *Y* direction. There is no displacement at node 1 in the *X* direction.

15.27



Chapter 16

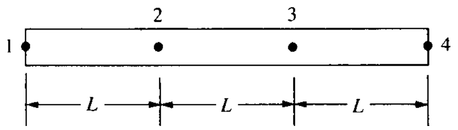
16.1



$$[m^{(1)}] = \frac{\rho AL}{6}, \begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array}, \quad [m^{(2)}] = \frac{\rho AL}{6} \begin{array}{c} \begin{matrix} 2 & 3 \end{matrix} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array}$$

$$[M] = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

16.2



(a) Lumped mass matrix

$$[m^{(1)}] = \frac{\rho AL}{2} \begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}, \quad [m^{(2)}] = \frac{\rho AL}{2} \begin{array}{c} \begin{matrix} 2 & 3 \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

$$[m^{(3)}] = \frac{\rho AL}{2} \begin{array}{c} \begin{matrix} 3 & 4 \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

$$[M] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{\rho AL}{2}$$

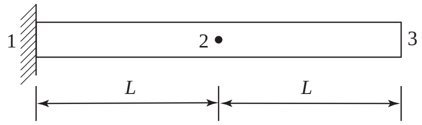
(b) Consistent mass matrix

$$[m^{(1)}] = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad [m^{(2)}] = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[m^{(3)}] = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[M] = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

16.3



$$[K] = \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[M] = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$([K] - \omega^2 [M])\ \{X\} = 0$ with $x_1 = 0$

$$\left(\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho AL}{6}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let $\omega^2 = \lambda$

divide by ρAL and let $\mu = \dfrac{E}{\rho L^2}$

$$\therefore \quad \left|\mu\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\lambda}{6}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} 2\mu - \frac{2}{3}\lambda & -\mu - \frac{\lambda}{6} \\ -\mu - \frac{\lambda}{6} & \mu - \frac{\lambda}{3} \end{vmatrix} = 0$$

$$\left(2\mu - \frac{2}{3}\lambda\right)\left(\mu - \frac{\lambda}{3}\right) - \left(-\mu - \frac{\lambda}{6}\right)^2 = 0$$

$$2\mu^2 - \frac{4}{3}\mu\lambda + \frac{2}{9}\lambda^2 - \mu^2 - \frac{\mu\lambda}{3} - \frac{\lambda^2}{36} = 0$$

$$\mu^2 - \frac{5}{3}\mu\lambda + \frac{7}{36}\lambda^2 = 0$$

or

$$\lambda^2 - \frac{60}{7}\mu\lambda + \frac{36}{7}\mu^2 = 0$$

$$\lambda_{1,2} = \frac{\frac{60}{7}\mu \pm \sqrt{\left(\frac{60}{7}\mu\right)^2 - 4\,(1)\left(\frac{36}{7}\right)\mu^2}}{2}$$

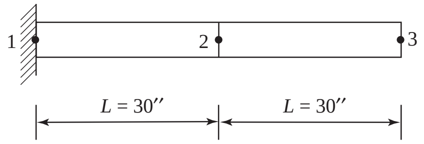
$$= \frac{8.571\,\mu \pm 7.273\,\mu}{2}$$

$$\lambda_1 = 0.649\,\mu \qquad \lambda_2 = 7.922\,\mu$$

$$\therefore \quad \omega_1 = \lambda_1^{\frac{1}{2}} = 0.806\sqrt{\mu}$$

$$\omega_2 = \lambda_2^{\frac{1}{2}} = 2.815\sqrt{\mu}$$

16.4 (a) Two equal length elements



From Problem 16.3 results

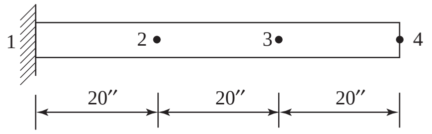
$$\omega_1 = 0.806\,\mu^{\frac{1}{2}},\ \omega_2 = 2.815\,\mu^{\frac{1}{2}}$$

$$\mu = \frac{E}{\rho L^2} = \frac{30\times 10^6}{(0.00073)\,(30)^2} = 45.66\times\frac{10^6}{\text{s}^2}$$

$$\omega_1 = 0.806\,\sqrt{45.66\times 10^6} = 5.446\times 10^3\ \frac{\text{rad}}{\text{s}}$$

$$\omega_2 = 2.815\,\sqrt{45.66\times 10^6} = 19.02\times 10^3\ \frac{\text{rad}}{\text{s}}$$

(b) 3 equal length elements



$$[K] = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\frac{AE}{L}$$

$$[M] = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Now $\quad x_1 = 0 \quad ([K] - \omega^2\,[M])\,\{X\} = 0$

$$\therefore \quad \left|\frac{AE}{L}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2\,\frac{\rho AL}{6}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right| = 0$$

or

$$\left|\mu\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{\lambda}{6}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} 2\mu - \frac{2}{3}\lambda & -\mu - \frac{\lambda}{6} & 0 \\ -\mu - \frac{\lambda}{6} & 2\mu - \frac{2}{3}\lambda & -\mu - \frac{\lambda}{6} \\ 0 & -\mu - \frac{\lambda}{6} & -\mu - \frac{\lambda}{3} \end{vmatrix} = 0$$

$$\left(2\mu-\frac{2}{3}\lambda\right)\left(2\mu-\frac{2}{3}\lambda\right)\left(\mu-\frac{\lambda}{3}\right)-$$

$$\left[\left(-\mu-\frac{\lambda}{6}\right)\left(-\mu-\frac{\lambda}{6}\right)\left(2\mu-\frac{2}{3}\lambda\right)+\left(-\mu-\frac{\lambda}{6}\right)\left(-\mu-\frac{\lambda}{6}\right)\left(\mu-\frac{\lambda}{3}\right)\right]=0$$

$$\left(\mu-\frac{\lambda}{3}\right)\left[4\mu^2-\frac{8}{3}\mu\lambda+\frac{4}{9}\lambda^2-3\mu^2-\mu\lambda-\frac{\lambda^2}{12}\right]=0$$

$$\left(\mu-\frac{\lambda}{3}\right)=0,\ \lambda_1=3\mu$$

or

$$\mu^2-\frac{11}{3}\mu\lambda+\frac{13\lambda^2}{36}=0$$

$$\lambda_{2,3}=\frac{132\mu\pm\sqrt{(132\mu)^2-4(13)36\mu^2}}{2(13)}$$

$$\lambda_2=9.873\ \mu,\quad \lambda_3=0.2805\ \mu$$

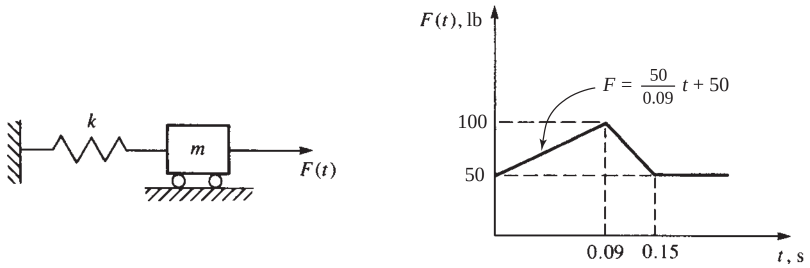
$$\mu=\frac{E}{\rho L^2}=\frac{30\times10^6}{(0.00073)(20'')^2}=1.0274\times10^8$$

$$\omega_1=\sqrt{\lambda_1}=\sqrt{0.2805\mu}=5.368\times10^3\ \frac{\text{rad}}{\text{s}}$$

$$\omega_2=\sqrt{3\mu}=17.556\times10^3\ \frac{\text{rad}}{\text{s}}$$

$$\omega_3=\sqrt{9.873\mu}=31.85\times10^3\ \frac{\text{rad}}{\text{s}}$$

16.5



$$t_0=0\ d_0=d_{0\text{dot}}=0$$

$$d_{0\text{dotdot}}=\frac{1}{2}(50-0)=25\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{-1}=0-0+\frac{(0.03)^2}{2}\times25=0.01125\text{ ft}$$

$$t_1=0.03\text{ s}$$

$$d_1=\frac{1}{2}\ \{(0.03)^2(50)+(2\times2-0.03^2\times2000)\,0-2\times0.01125\}$$

$$=0.01125\text{ ft}$$

$$d_2 = \frac{1}{2}\{(0.03)^2\,(66.67) + (2 \times 2 - 0.03^2 \times 2000 \times 0.01125\}$$

$$d_2 = 0.04238 \text{ ft}$$

$$d_{1\text{dotdot}} = \frac{1}{2}\{66.67 - 2000 \times 0.01125\}$$

$$d_{2\text{dotdot}} = 22.09\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{1\text{dot}} = \frac{0.04238 - 0}{2\,(0.03)} = 0.71\ \frac{\text{ft}}{\text{s}}$$

$$t_2 = 0.06 \text{ s}$$

$$d_3 = \frac{1}{2}\{(0.03)^2\,(83.33) + (2 \times 2 - (0.03)^2 \times 2000) \times (0.04238) - 2 \times 0.01125\}$$

$$d_3 = 0.07287 \text{ ft}$$

$$d_{2\text{dotdot}} = \frac{1}{2}\{83.33 - 2000 \times 0.04238\}$$

$$d_{2\text{dotdot}} = -\,0.715\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{2\text{dot}} = \frac{0.07287 - 0.01125}{2(0.03)} = 1.03\ \frac{\text{ft}}{\text{s}}$$

$$t_3 = 0.09 \text{ s}$$

$$d_4 = \frac{1}{2}\{0.03^2 \times 100 + (2 \times 2 - 0.03^2 \times 2000) \times (0.07287) - 2 \times 0.04238\}$$

$$d_4 = 0.08278 \text{ ft}$$

$$d_{3\text{dotdot}} = \frac{1}{2}(100 - 2000 \times 0.07287)$$

$$d_{3\text{dotdot}} = -\,22.87\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{3\text{dot}} = \frac{0.08278 - 0.04238}{2(0.03)} = 0.67\ \frac{\text{ft}}{\text{s}}$$

$$t_4 = 0.12 \text{ s}$$

$$d_5 = \frac{1}{2}\{(0.03)^2\,75 + (2 \times 2 - 0.03^2 \times 2000)\,(0.08278) - 2 \times 0.07287\}$$

$$d_5 = 0.05194 \text{ ft}$$

$$d_{4\text{dotdot}} = \frac{1}{2}(75 - 2000 \times 0.08278)$$

$$d_{4\text{dotdot}} = -\,45.28\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{4\text{dot}} = \frac{0.05194 - 0.07287}{2(0.03)} = -\,0.35\ \frac{\text{ft}}{\text{s}}$$

$$t_5 = 0.15 \text{ s}$$

$$d_6 = \frac{1}{2}\{(0.03)^2\,50 + (2 \times 2 - 0.03^2 \times 2000)\,(0.5194) - 2 \times 0.08278\}$$

$$d_6 = -\,3.146 \times 10^{-3} \text{ ft}$$

$$d_{5\text{dotdot}} = \frac{1}{2}\{50 - 2000\ (0.05194)\}$$

$$d_{5\text{dotdot}} = -\,26.94\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{5\text{dot}} = \frac{-\,3.146\times 10^{-3} - 0.08278}{2\,(0.03)} = -\ 1.43\,\frac{\text{ft}}{\text{s}}$$

Summary

t, s	$F(t)$ lb	d_i, ft	$d_{i\text{dotdot}}\ \frac{\text{ft}}{\text{s}^2}$	$d_{i\text{dot}}\ \frac{\text{ft}}{\text{s}}$
0	50	0	25	0
0.03	66.67	0.01125	22.09	0.71
0.06	83.33	0.04238	–0.715	1.03
0.09	100	0.07287	–22.87	0.67
0.12	75	0.08278	–45.28	–0.35
0.15	50	0.05194	–26.94	–1.43

(b) By Newmark's method

$$\beta = \frac{1}{6},\ \gamma = \frac{1}{2} \qquad M = 2\text{ slugs}$$

$$F = \frac{50}{0.09}\,t + 50 \qquad K = 2000\ \frac{\text{lb}}{\text{ft}}$$

$$F(0.03) = 555.6\ (0.03) + 50 = 66.67\text{ lb}$$

$$F'_{i+1} = F_{i+1} + \frac{M}{\beta\,\Delta t^2}\left[d_i + \Delta t\, d_{i\text{dot}} + \left(\frac{1}{2} - \beta\right)\Delta t^2\, d_{i\text{dotdot}}\right]$$

$$F'_i = 66.67 + \frac{2}{\frac{1}{6}(0.03)^2}\left[0 + (0.03)(0) + \left(\frac{1}{2} - \frac{1}{6}\right)(0.03)^2\ d_{0\text{dotdot}}\right]$$

and $$d_{0\text{dotdot}} = M^{-1}\,(F_0 - K\,d_0) = \frac{50 - 2000\,(0)}{2} = 25\ \frac{\text{ft}}{\text{s}^2}$$

$$\therefore\quad F'_i = 66.67 + \frac{2}{\frac{1}{6}(0.03)^2}\left[\left(\frac{1}{2} - \frac{1}{6}\right)(0.03)^2\ (25)\right]$$

$$= 166.67\text{ lb}$$

$$\therefore\quad d_1 = \frac{F'_1}{K'} = \frac{166.67}{K'}$$

and

$$K' = K + \frac{1}{\beta\,(\Delta t)^2}\ M$$

$$= 2000 + \frac{1}{\left(\frac{1}{6}\right)(0.03)^2}\ (2) = 15333$$

$$\therefore\qquad d_1 = \frac{166.67}{15333} = 0.01087\text{ ft}$$

$$d_{1\text{dotdot}} = \frac{1}{\beta\,(\Delta t)^2}\left[d_1 - d_0 - \Delta t\,(d_{0\text{dot}}) - (\Delta t)^2\left(\frac{1}{2} - \beta\right)d_{0\text{dotdot}}\right]$$

$$= \frac{1}{\left(\frac{1}{6}\right)(0.03)^2}\left[0.01087 - 0 - 0 - (0.03)^2\left(\frac{1}{2} - \frac{1}{6}\right)25\right]$$

$$d_{1\text{dotdot}} = 22.47\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{1\text{dot}} = d_{0\text{dot}} + \Delta t\,[(1-\gamma)\,d_{0\text{dotdot}} + r\,d_{1\text{dot}}]$$

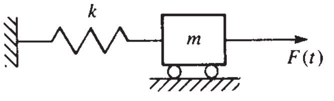
$$= 0 + (0.03)\left[\left(1 - \frac{1}{2}\right)25 + \frac{1}{2}(22.47)\right]$$

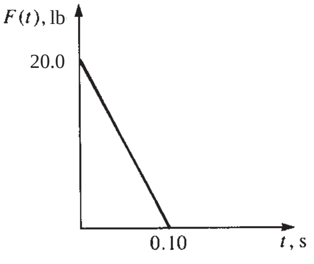
$$d_{1\text{dot}} = 0.71205\ \frac{\text{ft}}{\text{s}}$$

Table below summarizes the results using Newmark's method

Time	Displacement	Velocity	Acceleration	Force	Force prime
0	0	0	25	50	
0.03	0.01087	0.711957	22.46377	66.66667	166.6667
0.06	0.039319	1.084121	2.347196	83.33333	602.8986
0.09	0.069606	0.825234	–19.6063	100	1067.297
0.12	0.081832	–0.13384	–44.3317	75	1254.753
0.15	0.059363	–1.31425	–34.3627	50	910.2281
0.18	0.011632	–1.62917	13.3684	50	178.3512

16.6





(a) Using central difference

$$d_0 = 0, \quad d_{0\text{dot}} = 0$$

$$\Delta t = 0.02\ \text{s}$$

Step 1 $\quad t = 0.02\text{s} \qquad F_1 = 16$ lb

$$[M] = m = 2\ \text{slugs},\ [M^{-1}] = \frac{1}{2}$$

$$[K] = k = 1200\ \frac{\text{lb}}{\text{ft}}$$

$$d_{0\text{dotdot}} = \frac{1}{2}[20 - 1200\,(0)] = 10\ \frac{\text{ft}}{\text{s}^2}$$

$$\{d_{-1}\} = 0 - (0.02)\,(0) + \frac{0.02^2}{2}(10) = 0.002\ \text{ft}$$

$$\{d_1\} = \frac{1}{2}\,[(0.02)^2\,(20) + \{2(2) + (0.02)^2\,(1200)\}\,(0) - 2\,(0.002)]$$

$$= 0.002\ \text{ft}$$

$$\{d_2\} = \frac{1}{2}\ [0.02^2\ (16) + \{2(2) - 0.02^2\ (1200)\}\ (0.002) - 2(0)]$$

$$= 0.00672 \text{ ft}$$

$$d_{1\text{dotdot}} = \frac{1}{2}\ (16 - 1200\ (0.002)) = 6.8\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{1\text{dot}} = \frac{0.00672 - 0}{2(0.02)} = 0.168\ \frac{\text{ft}}{\text{s}}$$

Step 2 $\quad t = 0.04$ s $\quad F_2 = 12$ lb

$$d_3 = \frac{1}{2}[0.02^2\ (12) + \{2(2) - 0.02^2(1200)\}\ (0.00672) - 2\ (0.002)^2]$$

$$d_3 = 0.01223 \text{ ft}$$

$$d_{2\text{dotdot}} = \frac{1}{2}\ (12 - 1200\ (0.00672)) = 1.968\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{2\text{dot}} = \frac{0.01223 - 0.002}{2(0.02)} = 0.2558\ \frac{\text{ft}}{\text{s}}$$

Step 3 $\quad t = 0.06$ s $\quad F_3 = 8$ lb

$$d_4 = \frac{1}{2}\ [0.02^2\ (8) + 3.52\ (0.01223) - 2\ (0.00672)]$$

$$= 0.0164 \text{ ft}$$

$$d_{3\text{dotdot}} = \frac{1}{2}\ (8 - 1200\ (0.01223)) = -\,3.338\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{3\text{dot}} = \frac{0.0164 - 0.00672}{2(0.02)} = 0.242\ \frac{\text{ft}}{\text{s}}$$

Step 4 $\quad t = 0.08$ s $\quad F_4 = 4$ lb

$$d_5 = \frac{1}{2}\ [0.02^2\ (4) + 3.52\ (0.0164) - 2\ (0.01223)$$

$$= 0.01743 \text{ ft}$$

$$d_{4\text{dotdot}} = \frac{1}{2}\ (4 - 1200\ (0.01640)) = -\,7.84\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{4\text{dot}} = \frac{0.01743 - 0.01223}{2(0.02)} = 0.13\ \frac{\text{ft}}{\text{s}}$$

Step 5 $\quad t = 0.10$ s $\quad F_5 = 0$

$$d_6 = \frac{1}{2}\ [0.02^2(0) + 3.52\ (0.01743) - 2\ (0.01640)]$$

$$= 0.01428 \text{ ft}$$

$$d_{5\text{dotdot}} = \frac{1}{2}\ (0 - 1200\ (0.01743)) = -10.46\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{5\text{dot}} = \frac{0.01428 - 0.01640}{2(0.02)} = -0.053\ \frac{\text{ft}}{\text{s}}$$

Summary

t, s	d, ft	d_{dot}, $\frac{\text{ft}}{\text{s}}$	d_{dotdot}, $\frac{\text{ft}}{\text{s}^2}$
0	0	0	10
0.02	0.002	0.168	6.8
0.04	0.00672	0.2558	1.968
0.06	0.01223	0.242	–3.338
0.08	0.01640	0.130	–7.89
0.10	0.01743	–0.053	–10.46

(b) Newmark's time integration method (Mathcad solution)

$$F_0 = 20\text{ lb} \qquad K = 1200\ \frac{\text{lb}}{\text{ft}} \qquad M = 2\text{ slug} \qquad M = 2\ \frac{\text{lb}\cdot\text{s}^2}{\text{ft}} \qquad \Delta t = 0.02\text{s}$$

Assume $\beta = \frac{1}{6}$ $\gamma = \frac{1}{2}$ for linear acceleration within each time step

At time $t = 0$

$$d_0 = 0\text{ ft} \qquad d_{0\text{dot}} = 0\ \frac{\text{ft}}{\text{s}}$$

Acceleration at $t = 0$

$$d_{0\text{dotdot}} = \frac{F_0 - K\,d_0}{M} \qquad d_{0\text{dotdot}} = 10\ \frac{\text{ft}}{\text{s}^2}$$

Displacement at $t = 0.02$

$$K_{\text{prime}} = K + \frac{1}{\beta\,(\Delta t)^2}\,M \qquad K_{\text{prime}} = 2.6 \times 10^3\ \frac{\text{lb}}{\text{in.}}$$

$$F_1 = \frac{4}{5}\ F_0 \qquad F_1 = 16\text{ lb}$$

$$F_{1\text{prime}} = F_1 + \frac{M}{\beta(\Delta t)^2}\left[d_0 + (\Delta t)\,d_{0\text{dot}} + \left(\frac{1}{2} - \beta\right)(\Delta t)^2\ d_{0\text{dotdot}}\right]$$

$$F_{1\text{prime}} = 56\text{ lb}$$

$$d_1 = \frac{F_{1\text{prime}}}{K_{\text{prime}}} \qquad d_1 = 1.795 \times 10^{-3}\text{ ft}$$

Acceleration at $t = 0.02$

$$d_{1\text{dotdot}} = \frac{1}{\beta\,(\Delta t)^2}\left[d_1 - d_0 - (\Delta t)\,d_{0\text{dot}} - (\Delta t)^2\left(\frac{1}{2} - \beta\right)d_{0\text{dotdot}}\right]$$

$$d_{1\text{dotdot}} = 6.923\ \frac{\text{ft}}{\text{s}^2}$$

Velocity at $t = 0.02$

$$d_{1\text{dot}} = d_{0\text{dot}} + (\Delta t)\ [(1 - \gamma)\ d_{0\text{dotdot}} + \gamma\ d_{1\text{dotdot}}] \qquad d_{1\text{dot}} = 0.169\ \frac{\text{ft}}{\text{s}}$$

Displacement at $t = 0.04$ $\quad F_2 = \frac{3}{5}\ F_0 \quad F_2 = 12\text{ lb}$

$$F_{2\text{prime}} = F_2 + \frac{M}{\beta(\Delta t)^2}\left[d_1 + (\Delta t)\, d_{1\text{dot}} + \left(\frac{1}{2} - \beta\right)(\Delta t)^2\, d_{1\text{dotdot}}\right]$$

$F_{2\text{prime}} =$ 195.07 lb

$$d_2 = \frac{F_{2\text{prime}}}{K_{\text{prime}}} \qquad d_2 = 6.252 \times 10^{-3}\ \text{ft}$$

Acceleration at $t =$ 0.04

$$d_{2\text{dotdot}} = \frac{1}{\beta\,(\Delta t)^2}\left[d_2 - d_1 - (\Delta t)\, d_{1\text{dot}} - (\Delta t)^2\left(\frac{1}{2} - \beta\right) d_{1\text{dotdot}}\right]$$

$$d_{2\text{dotdot}} = 2.249\ \frac{\text{ft}}{\text{s}^2}$$

Velocity at $t = 0.04$

$$d_{2\text{dot}} = d_{1\text{dot}} + (\Delta t)\,[(1-\gamma)\, d_{1\text{dotdot}} + \gamma\, d_{2\text{dotdot}}]$$

$$\text{d}_{2\text{dot}} = 0.0261\ \frac{\text{ft}}{\text{s}}$$

Displacement at $t =$ 0.06

$$F_3 = \frac{2}{5}\ F_0 \qquad F_3 = 8\ \text{lb}$$

$$F_{3\text{prime}} = F_3 + \frac{M}{\beta\,(\Delta t)^2}\left[d_2 + (\Delta t)\, d_{2\text{dot}} + \left(\frac{1}{2} - \beta\right)(\Delta t)^2\, d_{2\text{dotdot}}\right]$$

$F_{3\text{prime}}$ =361.136 lb

$$d_3 = \frac{F_{3\text{prime}}}{K_{\text{prime}}} \qquad d_3 = 0.012\ \text{ft}$$

Acclertaion at $t = 0.06$

$$d_{3\text{dotdot}} = \frac{1}{\beta\,(\Delta t)^2}\left[d_3 - d_2 - (\Delta t)\, d_{2\text{dot}} - (\Delta t)^2\left(\frac{1}{2} - \beta\right) d_{2\text{dotdot}}\right]$$

$$d_{3\text{dotdot}} = -\,2.945\ \frac{\text{ft}}{\text{s}^2}$$

Velocity at $t =$ 0.06

$$d_{3\text{dot}} = d_{2\text{dot}} + (\Delta t)\,[(1-\gamma)\, d_{2\text{dotdot}} + \gamma\, d_{3\text{dotdot}}]$$

$$d_{3\text{dot}} = 0.254\ \frac{\text{ft}}{\text{s}}$$

Displacement at $t =$ 0.08

$$F_4 = \frac{1}{5}\ F_0 \qquad F_4 = 4\ \text{lb}$$

$$F_{4\text{prime}} = F_4 + \frac{M}{\beta\,(\Delta t)^2}\left[d_3 + (\Delta t)\, d_{3\text{dot}} + \left(\frac{1}{2} - \beta\right)(\Delta t)^2\, d_{3\text{dotdot}}\right]$$

$F_{4\text{prime}} =$ 491.85 lb

$$d_4 = \frac{F_{4\text{prime}}}{K_{\text{prime}}} \qquad d_4 = 0.016\ \text{ft}$$

Acceleration at $t =$ 0.08

$$d_{4\text{dotdot}} = \frac{1}{\beta\,(\Delta t)^2}\left[d_4 - d_3 - (\Delta t)\, d_{3\text{dot}} - (\Delta t)^2\left(\frac{1}{2} - \beta\right) d_{3\text{dotdot}}\right]$$

$$d_{4\text{dotdot}} = -\,7.459\ \frac{\text{ft}}{\text{s}^2}$$

Velocity at $t = 0.08$

$$d_{4dot} = d_{3dot} + (\Delta t)\,[(1-\gamma)\,d_{3dotdot} + \gamma\, d_{4dotdot}] \qquad d_{4dot} = 0.15\ \frac{\text{ft}}{\text{s}}$$

Displacement at $t = 0.10$

$$F_5 = \frac{0}{5}\,F_0 \qquad F_5 = 0\ \text{lb}$$

$$F_{5prime} = F_5 + \frac{M}{\beta\,(\Delta t)^2}\left[d_4 + (\Delta t)\,d_{4dot} + \left(\frac{1}{2} - \beta\right)(\Delta t)^2\,d_{4dotdot}\right]$$

$$F_{5prime} = 533.071\ \text{lb}$$

$$d_5 = \frac{F_{5prime}}{K_{prime}} \qquad d_5 = 0.017\ \text{ft}$$

Acceleration at $t = 0.10$

$$d_{5dotdot} = \frac{1}{\beta\,(\Delta t)^2}\left[d_5 - d_4 - (\Delta t)\,d_{4dot} - (\Delta t)^2\left(\frac{1}{2} - \beta\right)d_{4dotdot}\right]$$

$$d_{5dotdot} = -\,10.251\ \frac{\text{ft}}{\text{s}^2}$$

Velocity at $t = 0.10$

$$d_{5dot} = d_{4dot} + (\Delta t)\,[(1-\gamma)\,d_{4dotdot} + \gamma\, d_{5dotdot}] \qquad d_{5dot} = -\,0.027\ \frac{\text{ft}}{\text{s}}$$

(c) Wilson’s method

$$F_0 = 20\ \text{lb} \qquad K = 1200\ \frac{\text{lb}}{\text{ft}} \qquad M = 2\ \text{slug} \qquad M = 2\,\frac{\text{lb}\cdot\text{s}^2}{\text{ft}} \quad \Delta t = 0.02\text{s}$$

Assume $\Theta = 1$

At time $t = 0$

$$d_0 = 0\ \text{ft} \qquad d_{0dot} = 0\,\frac{\text{ft}}{\text{s}}$$

Acceleration at $t = 0$

$$d_{0dotdot} = \frac{F_0 - K\,d_0}{M} \qquad d_{0dotdot} = 10\ \frac{\text{ft}}{\text{s}^2}$$

Displacement at $t = 0.02$

$$K_{prime} = K + \frac{6}{(\Theta\,\Delta t)^2}\,M \qquad K_{prime} = 2.6\times 10^3\ \frac{\text{lb}}{\text{in.}}$$

$$F_1 = \frac{4}{5}\,F_0 \qquad F_1 = 16\ \text{lb}$$

$$F_{1prime} = F_1 + \frac{M}{(\Theta\,\Delta t)^2}\,[6\,d_0 + 6\,\Theta\,(\Delta t)\,d_{0dot} + 2\,(\Theta\,\Delta t)^2\,d_{0dotdot}]$$

$$F_{1prime} = 56\ \text{lb}$$

$$d_1 = \frac{F_{1prime}}{K_{prime}} \qquad d_1 = 1.795\times 10^{-3}\ \text{ft}$$

Acceleration at $t = 0.02$

$$d_{1dotdot} = \frac{6}{\Theta^2 (\Delta t)^2} (d_1 - d_0) - \frac{6}{\Theta (\Delta t)} d_{0dot} - 2\, d_{0dotdot} \qquad d_{1dotdot} = 6.923 \frac{\text{ft}}{\text{s}^2}$$

Velocity at $t = 0.02$

$$d_{1dot} = \frac{3}{\Theta (\Delta t)} (d_1 - d_0) - 2\, d_{0dot} - \frac{\Theta (\Delta t)}{2} d_{0dotdot} \qquad d_{1dot} = 0.169 \frac{\text{ft}}{\text{s}}$$

Displacement at $t = 0.04$

$$F_2 = \frac{3}{5} F_0 \qquad F_2 = 12 \text{ lb}$$

$$F_{2prime} = F_2 + \frac{3}{\Theta^2 (\Delta t)^2} [6\, d_1 + 6\, \Theta\, (\Delta t)\, d_{1dot} + 2\, (\Theta\, \Delta t)^2\, d_{1dotdot}]$$

$$F_{2prime} = 195.077 \text{ lb}$$

$$d_2 = \frac{F_{2prime}}{K_{prime}} \qquad d_2 = 6.252 \times 10^{-3} \text{ ft}$$

Acceleration at $t = 0.04$

$$d_{2dotdot} = \frac{6}{\Theta^2 (\Delta t)^2} (d_2 - d_1) - \frac{6}{\Theta (\Delta t)} d_{1dot} - 2\, d_{1dotdot} \qquad d_{2dotdot} = 2.249 \frac{\text{ft}}{\text{s}^2}$$

Velocity at $t = 0.04$

$$d_{2dot} = \frac{3}{\Theta (\Delta t)} (d_2 - d_1) - 2\, d_{1dot} - \frac{\Theta\, \Delta t}{2} d_{1dotdot} \qquad d_{2dot} = 0.261 \frac{\text{ft}}{\text{s}}$$

Displacement at $t = 0.06$

$$F_3 = \frac{2}{5} F_0 \qquad F_3 = 8 \text{ lb}$$

$$F_{3prime} = F_3 + \frac{M}{\theta^2 (\Delta t)^2} [6\, d_2 + 6\, \Theta\, (\Delta t)\, d_{2dot} + 2\, (\Theta\, \Delta t)^2\, d_{2dotdot}]$$

$$F_{3prime} = 361.136 \text{ lb}$$

$$d_3 = \frac{F_{3prime}}{K_{prime}} \qquad d_3 = 0.012 \text{ ft}$$

Acceleration at $t = 0.06$

$$d_{3dotdot} = \frac{6}{\Theta^2 (\Delta t)^2} (d_3 - d_2) - \frac{6}{\Theta (\Delta t)} d_{2dot} - 2\, d_{2dotdot} \qquad d_{3dotdot} = -2.945 \frac{\text{ft}}{\text{s}^2}$$

Velocity at $t = 0.06$

$$d_{3dot} = \frac{3}{\Theta\, \Delta t} (d_3 - d_2) - 2\, d_{2dot} - \frac{\Theta\, \Delta t}{2} d_{2dotdot} \qquad d_{3dot} = 0.254 \frac{\text{ft}}{\text{s}}$$

Displacement at $t = 0.08$

$$F_4 = \frac{1}{5} F_0 \qquad F_4 = 4 \text{ lb}$$

$$F_{4prime} = F_4 + \frac{M}{(\Theta\, \Delta t)^2} [6\, d_3 + 6\, \Theta\, (\Delta t)\, d_{3dot} + 2\, (\Theta\, \Delta t)^2\, d_{3dotdot}]$$

$F_{4\text{prime}} = 491.865 \text{ lb}$

$$d_4 = \frac{F_{4\text{prime}}}{K_{\text{prime}}} \qquad d_4 = 0.016 \text{ ft}$$

Acceleration at $t = 0.08$

$$d_{4\text{dotdot}} = \frac{6}{\Theta^2 (\Delta t)^2}(d_4 - d_3) - \frac{6}{\Theta(\Delta t)} d_{3\text{dot}} - 2\, d_{3\text{dotdot}}$$

$$d_{4\text{dotdot}} = -7.459 \frac{\text{ft}}{\text{s}^2}$$

Velocity at $t = 0.08$

$$d_{4\text{dot}} = \frac{3}{\Theta\, \Delta t}(d_4 - d_3) - 2\, d_{3\text{dot}} - \frac{\Theta\, \Delta t}{2} d_{3\text{dotdot}} \qquad d_{4\text{dot}} = 0.15 \frac{\text{ft}}{\text{s}}$$

Displacement at $t = 0.10$

$$F_5 = \frac{0}{5} F_0 \qquad F_5 = 0 \text{ lb}$$

$$F_{5\text{prime}} = F_5 + \frac{M}{(\Theta\ \ \Delta t)^2}[6\, d_4 + 6\, \Theta\, (\Delta t)\, d_{4\text{dot}} + 2\, (\Theta\, \Delta t)^2\ d_{4\text{dotdot}}]$$

$F_{5\text{prime}} = 533.071 \text{ lb}$

$$d_5 = \frac{F_{5\text{prime}}}{K_{\text{prime}}} \qquad d_5 = 0.017 \text{ ft}$$

Acceleration at $t = 0.10$

$$d_{5\text{dotdot}} = \frac{6}{\Theta^2 (\Delta t)^2}(d_5 - d_4) - \frac{6}{\Theta(\Delta t)} d_{4\text{dot}} - 2\, d_{4\text{dotdot}}$$

$$d_{5\text{dotdot}} = -10.251 \frac{\text{ft}}{\text{s}^2}$$

Velocity at $t = 0.10$

$$d_{5\text{dot}} = \frac{3}{\Theta\, \Delta t}(d_5 - d_4) - 2\, d_{4\text{dot}} - \frac{\Theta\, \Delta t}{2} d_{4\text{dotdot}} \qquad d_{5\text{dot}} = -0.027 \frac{\text{ft}}{\text{s}}$$

Newmark's time integration method. Wilson's method (Linear acceleration)

Assume linear acceleration within each time step $\Theta = 1$

$\beta = 0.167$ $K' = 31200$

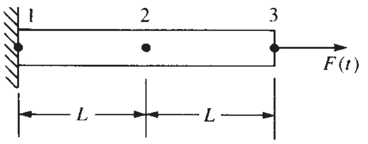
$\gamma = 0.5$

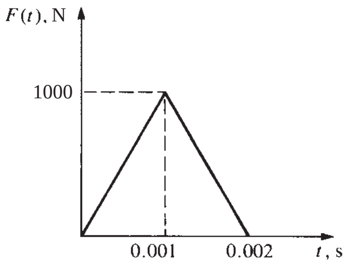
$K' = 31200$

Summary table Newmark and Wilson same results.

di	*time*	$F(t)$ (lb)	d_i (ft)	d_i *velocity* $(\frac{\text{ft}}{\text{s}})$	d_i *accel* $(\frac{\text{ft}}{\text{s}^2})$	F'
0	0	20	0	0	10	
1	0.02	16	0.00179	0.169	6.923	56
2	0.04	12	0.00625	0.261	2.249	195
3	0.06	8	0.01157	0.254	–2.945	361
4	0.08	4	0.01576	0.150	–7.459	492
5	0.1	0	0.01709	– 0.027	–10.251	533

16.7





Use time step $\Delta t = 2.5 \times 10^{-4}$ s

$$[u_1] = 0,\ u_{1\text{dot}} = u_{1\text{dotdot}} = 0$$

$$\{d_{0\text{dotdot}}\} = [M^{-1}]\,(\{F_0\} - [K]\{d_0\})$$

$$\{d_{0\text{dotdot}}\} = \frac{\rho AL}{2}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$t = 0.00025 \text{ s}$$

$$\{d_1\} = \frac{F_1'}{[K']} = [K']^{-1}\ F_i'$$

$$[K'] = [K] + \frac{1}{\beta\,\Delta t^2}\ [M]$$

$$= \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \frac{\rho AL}{2\beta \Delta t^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[K']^{-1} = \begin{bmatrix} 1.319 \times 10^{-7} & 1.040 \times 10^{-8} \\ 1.040 \times 10^{-8} & 2.637 \times 10^{-7} \end{bmatrix}$$

$$[F_1'] = \begin{Bmatrix} 0 \\ 250 \end{Bmatrix} + \frac{\rho AL}{2\beta \Delta t^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left[\{d_0\} + \Delta t\{d_{0\text{dot}}\} + \left(\frac{1}{2} - \beta\right)\{d_{0\text{dotdot}}\}\right]$$

$$[F_1'] = \begin{Bmatrix} 0 \\ 250 \end{Bmatrix}$$

$$\therefore\quad \{d_1\} = \begin{Bmatrix} 2.6 \times 10^{-6} \\ 6.593 \times 10^{-5} \end{Bmatrix} \text{ in.}$$

$$\{d_{1\text{dotdot}}\} = \frac{1}{\beta \Delta t^2}\left[\begin{Bmatrix} 2.6 \times 10^{-6} \\ 6.593 \times 10^{-5} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \Delta t\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \Delta t^2\left(\frac{1}{2} - \beta\right)\left\{\cancelto{0}{d_{0\text{dotdot}}}\right\}\right]$$

$$\{d_{1\text{dotdot}}\} = \begin{Bmatrix} 249.56 \\ 6328.8 \end{Bmatrix} \frac{\text{in.}}{\text{s}^2}$$

Computer program solution

Lumped mass contribution

3504000	0	0
0	7008000	0
0	0	3504000
Stiffness		
300000	–300000	0
–300000	600000	–300000
0	–300000	300000
3804000	–300000	0

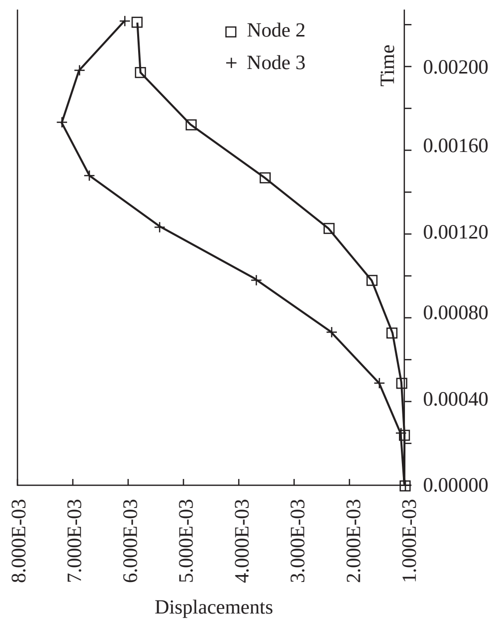
$$K_{\text{prime}} = \begin{matrix} -300000 & 7608000 & -300000 \\ 0 & -300000 & 3804000 \end{matrix}$$

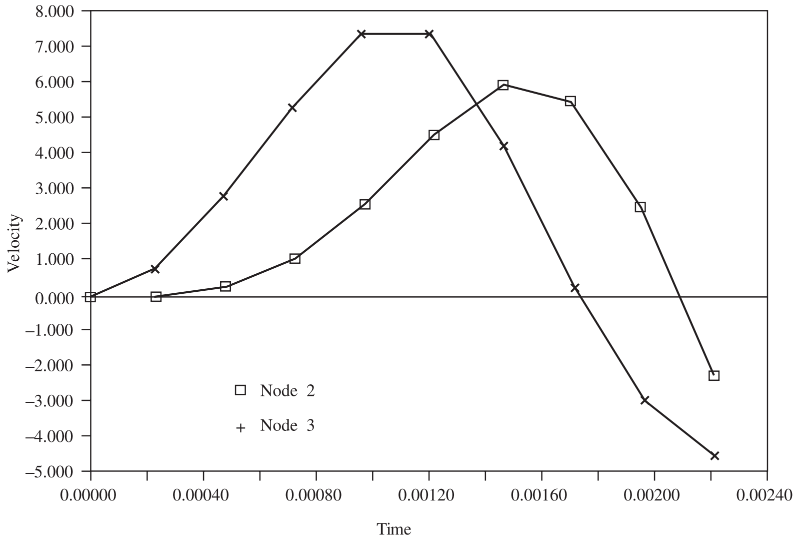
Inverse of $[K]_{\text{prime}}$ (after introducing boundary condition)

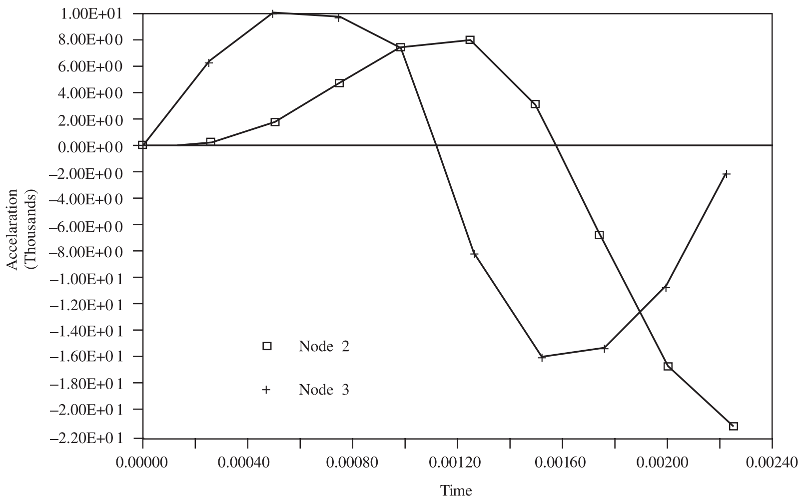
1.319E–07	1.040E–08
1.040E–08	2.637E–07

			Displacement (inches)		Velocity $\left(\frac{\text{in.}}{\text{s}}\right)$	
Step	Time	Node:	2	3	2	3
0	0		0	0	0	0
1	0.00025		2.600E–06	6.593E–05	0.031	0.791
2	0.0005		3.402E–05	4.985E–04	0.284	2.817
3	0.00075		1.901E–04	1.510E–03	1.085	5.265
4	0.001		6.364E–04	3.102E–03	2.605	7.369
5	0.00125		1.528E–03	5.009E–03	4.548	7.252
6	0.0015		2.865E–03	6.487E–03	5.941	4.235
7	0.00175		4.345E–03	7.050E–03	5.488	0.302
8	0.002		5.400E–03	6.694E–03	2.537	–2.951
9	0.00225		5.460E–03	5.713E–03	–2.248	–4.540

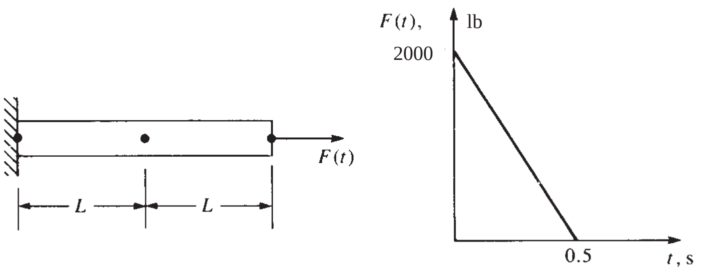
Acceleration $\left(\frac{\text{in.}}{\text{s}^2}\right)$		F prime		Force	
2	3	2	3	2	3
0	0	0	0	0	0
249.560	6328.830	0.000	250.000	0	250
1768.905	9881.174	109.307	1886.014	0	500
4641.891	9701.659	993.395	5686.004	0	750
7519.349	7128.285	3910.631	11610.659	0	1000
8027.620	–8065.184	10121.342	18596.407	0	750
3112.573	–16071.468	19848.150	23815.881	0	500
–6735.088	–15389.137	30938.266	25515.703	0	250
–16876.200	–10634.974	39078.413	23845.304	0	0
–21398.085	–2081.767	39826.784	20095.845	0	0







16.8



$$[K] = \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = 10^4 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[M] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 50 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$[K]\{d\} + [M]\{d_{\text{dotdot}}\} = \{F(t)\}$

Find proper time step

$[M]\{d_{\text{dotdot}}\} + [K]\{d\} = 0$

$([K] - \omega^2 [M])\{d'\} = 0$

$d_1' = 0$

$$\left(10^4 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} 50\right) \begin{Bmatrix} u_2' \\ u_3' \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} 400 - 2\omega^2 & -200 \\ -200 & 200 - \omega^2 \end{vmatrix} = 0$$

$$\omega^2 = 300 \pm 100\sqrt{13}$$

$$\omega = 25.7 \ \frac{\text{rad}}{\text{s}}$$

$$\Delta t \leqq \frac{3}{4}\left(\frac{2}{\omega_{\max}}\right) = \frac{3}{4}\left(\frac{2}{25.7}\right) = 0.058 \text{ s}$$

$$\therefore \ \text{use } \Delta t = 0.05 \text{ s} \qquad \beta = \frac{1}{6}, \gamma = \frac{1}{2}$$

Step 1 $\quad t = 0.05$ s $\qquad F_0 = 2000$ lb

$d_0 = 0$

$$\{d_{0\text{dotdot}}\} = [M^{-1}](\{F_0\} - [K]\{d_0\})$$

$$= \frac{1}{50}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 2000 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 40 \end{Bmatrix}$$

$$[K'] = [K] + \frac{1}{\beta \Delta t^2}[M] = \begin{bmatrix} 20000 & -10000 \\ -10000 & 10000 \end{bmatrix} + 2400 \times 50 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[K'] = \begin{bmatrix} 260{,}000 & -10{,}000 \\ -10{,}000 & 130{,}000 \end{bmatrix}$$

$$\{F_1'\} = \{F_1\} + \frac{[M]}{\beta \Delta t^2}\left(\{d_0\} + \Delta t\{d_{0\text{dot}}\} + \frac{1}{3}\Delta t^2\{d_{0\text{dotdot}}\}\right)$$

$$= \begin{Bmatrix} 0 \\ 1800 \end{Bmatrix} + 2 \times 50 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 40 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 5800 \end{Bmatrix}$$

$$\{d_1\} = [K']^{-1}\{F_1'\} = \frac{1}{10^4} \times \frac{1}{3.37}\begin{bmatrix} 13 & 1 \\ 1 & 26 \end{bmatrix} \begin{Bmatrix} 0 \\ 5800 \end{Bmatrix}$$

$$\{d_1\} = \begin{Bmatrix} 0.001721 \\ 0.0448 \end{Bmatrix} \text{in.}$$

$$\{d_{1\text{dotdot}}\} = \frac{1}{\beta \Delta t^2}\left[\{d_1\} - \{d_0\} - \Delta t\{d_{0\text{dot}}\} - \Delta t^2 \frac{1}{3}\{d_{0\text{dotdot}}\}\right]$$

$$= 2400\left(\begin{Bmatrix}0.001721\\0.0448\end{Bmatrix} - 0.05^2 \times \frac{1}{3} \times \begin{Bmatrix}0\\40\end{Bmatrix}\right)$$

$$= \begin{Bmatrix}4.131\\27.39\end{Bmatrix} \frac{\text{in.}}{\text{s}^2}$$

$$\{d_{1\text{dot}}\} = \{d_{0\text{dot}}\} + \Delta t\left(\frac{1}{2}\{d_{0\text{dotdot}}\} + \frac{1}{2}\{d_{1\text{dotdot}}\}\right)$$

$$\{d_{1\text{dot}}\} = 0.05 \times \frac{1}{2}\left(\begin{Bmatrix}0\\40\end{Bmatrix} + \begin{Bmatrix}4.131\\27.39\end{Bmatrix}\right) = \begin{Bmatrix}0.103\\1.685\end{Bmatrix}$$

Step 2 $\quad t = 0.1$ s

$$\{F_2\} = \begin{Bmatrix}0\\2000(1-2(0.1))\end{Bmatrix} = \begin{Bmatrix}0\\1600\end{Bmatrix}$$

$$\{F_2'\} = \begin{Bmatrix}0\\1600\end{Bmatrix} + 2400 \times 50 \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\left(\begin{Bmatrix}0.001721\\0.0448\end{Bmatrix} + 0.05\begin{Bmatrix}0.103\\1.685\end{Bmatrix} + \frac{1}{3}(0.05)^2\begin{Bmatrix}4.131\\27.39\end{Bmatrix}\right)$$

$$\{F_2'\} = \begin{Bmatrix}0\\1600\end{Bmatrix} + 120000\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{Bmatrix}0.01031\\0.1518\end{Bmatrix} = \begin{Bmatrix}2475\\19818\end{Bmatrix}$$

$$\{d_2\} = [K']^{-1}\{F_2'\} = \frac{1}{337 \times 10^4}\begin{bmatrix}13 & 1\\1 & 26\end{bmatrix}\begin{Bmatrix}2475\\19818\end{Bmatrix}$$

$$\{d_2\} = \begin{Bmatrix}0.01544\\0.1536\end{Bmatrix} \text{in.}$$

$$\{d_{2\text{dotdot}}\} = 2400\left[\begin{Bmatrix}0.01544\\0.1536\end{Bmatrix} - \begin{Bmatrix}0.00172\\0.0448\end{Bmatrix} - 0.05\begin{Bmatrix}0.103\\1.685\end{Bmatrix} - \frac{0.05^2}{3}\begin{Bmatrix}4.131\\27.39\end{Bmatrix}\right]$$

$$\{d_{2\text{dotdot}}\} = \begin{Bmatrix}12.27\\4.37\end{Bmatrix} \frac{\text{in.}}{\text{s}^2}$$

$$\{d_{2\text{dot}}\} = \begin{Bmatrix}0.103\\1.685\end{Bmatrix} + 0.05 \times \frac{1}{2}\left[\begin{Bmatrix}4.131\\27.39\end{Bmatrix} + \begin{Bmatrix}12.27\\4.37\end{Bmatrix}\right]$$

$$= \begin{Bmatrix}0.513\\0.2479\end{Bmatrix} \frac{\text{in.}}{\text{s}}$$

Step 3 $\quad t = 0.15$ s

$$\{F_3\} = \begin{Bmatrix}0\\2000(1-2(0.15))\end{Bmatrix} = \begin{Bmatrix}0\\1400\end{Bmatrix}$$

$$\{F_3'\} = \begin{Bmatrix}0\\1400\end{Bmatrix} + 12 \times 10^4 \begin{Bmatrix}2 & 0\\0 & 1\end{Bmatrix} \times \left(\begin{Bmatrix}0.01544\\0.1536\end{Bmatrix} + 0.05\begin{Bmatrix}0.513\\2.479\end{Bmatrix} + \frac{0.05^2}{3}\begin{Bmatrix}12.27\\4.37\end{Bmatrix}\right)$$

$$= \begin{Bmatrix}12316\\35143\end{Bmatrix}$$

$$\{d_3\} = \frac{1}{337\times10^4}\begin{bmatrix}13 & 1\\ 1 & 26\end{bmatrix}\begin{matrix}12316\\35143\end{matrix} = \begin{Bmatrix}0.0579\\0.2745\end{Bmatrix}\text{ in.}$$

$$\{d_{3\text{dotdot}}\} = 2400\left(\begin{Bmatrix}0.0579\\0.2745\end{Bmatrix} - \begin{Bmatrix}0.01544\\0.1536\end{Bmatrix} - 0.05\begin{Bmatrix}0.513\\2.479\end{Bmatrix} - \frac{0.05^2}{3}\begin{Bmatrix}12.27\\4.37\end{Bmatrix}\right)$$

$$= \begin{Bmatrix}15.80\\-16.06\end{Bmatrix}\frac{\text{in.}}{\text{s}^2}$$

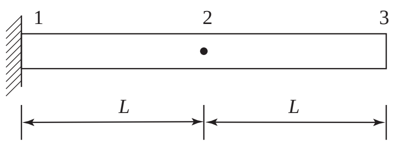
$$\{d_{3\text{dot}}\} = \begin{Bmatrix}0.513\\2.479\end{Bmatrix} + \frac{0.05}{2}\left(\begin{Bmatrix}12.27\\4.37\end{Bmatrix} + \begin{Bmatrix}15.80\\-16.06\end{Bmatrix}\right)$$

$$= \begin{Bmatrix}1.2149\\2.1868\end{Bmatrix}\frac{\text{in.}}{\text{s}}$$

Steps 4 and 5 follow similar procedures as above
Table below summarizes results

t, s	F, lb	d_i, in. Node 2	Node 3	$\dot{d}_i$, $\frac{\text{in.}}{\text{s}}$		$d_{i\text{dot}}$, $\frac{\text{in.}}{\text{s}^2}$	
0	2000	0	0	0	0	0	40
0.05	1800	0.00172	0.0448	0. 103	1.685	4.131	27.39
0.10	1600	0.01544	0.1536	0.513	2.479	12.27	4.37
0.15	1400	0.0579	0.2745	1.2149	2.187	15.80	– 16.06
0.20	1200	0.1356	0.3616	1.836	1.255	9.042	– 21.20
0.25	1000	0.2323	0.401	1.905	0.383	–6.376	– 13.71

16.11



Global stiffness matrix

$$[k] = \frac{EI}{L^3}\begin{array}{c} \begin{matrix} v_1 & \phi_1 & v_2 & \phi_2 & v_3 & \phi_3 \end{matrix} \\ \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0\\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0\\ -12 & -6L & 24 & 0 & -12 & 6L\\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2\\ 0 & 0 & -12 & -6L & 12 & -6L\\ 0 & 0 & 6L & 2L^2 & -6L & 4L \end{bmatrix} \end{array}$$

Lumped mass matrix

$$
\begin{array}{cc}
 & \begin{array}{cccccc} v_1 & \phi_1 & v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\
[m] = \dfrac{\rho AL}{2} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
$$

Invoking boundary conditions $v_1 = \phi_1 = 0$

$$\frac{EI}{L^3}\begin{bmatrix} 24 & 0 & -12 & 6L \\ 0 & 8L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} - \omega^2 \frac{\rho AL}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0$$

Let $\omega^2 = \lambda$ and divide ρAL

and $\mu = \dfrac{EI}{\rho A}$

(a)
$$\begin{vmatrix} \frac{24\mu}{L^4} - \lambda & 0 & \frac{-12\mu}{L^4} & \frac{6\mu}{L^2} \\ 0 & \frac{8\mu}{L^2} & \frac{-6\mu}{L^3} & \frac{2\mu}{L^2} \\ \frac{-12\mu}{L^4} & \frac{-6\mu}{L^3} & \frac{12\mu}{L^4} - \frac{\lambda}{2} & \frac{-6\mu}{L^3} \\ \frac{6\mu}{L^3} & \frac{2\mu}{L^2} & \frac{-6\mu}{L^3} & \frac{4\mu}{L^2} \end{vmatrix} = 0$$

$$\left(\frac{24\mu}{L^4} - \lambda\right)(-1)^2 \begin{vmatrix} \frac{8\mu}{L^2} & \frac{-6\mu}{L^3} & \frac{-2\mu}{L^2} \\ \frac{-6\mu}{L^3} & \frac{12\mu}{L^3} - \frac{\lambda}{2} & \frac{-6\mu}{L^3} \\ \frac{2\mu}{L^2} & \frac{-6\mu}{L^3} & \frac{4\mu}{L^2} \end{vmatrix} + \left(\frac{-12\mu}{L^4}\right)(-1)^{1+3} \begin{vmatrix} 0 & \frac{8\mu}{L^2} & \frac{2\mu}{L^2} \\ \frac{-12\mu}{L^4} & \frac{-6\mu}{L^3} & \frac{-6\mu}{L^3} \\ \frac{6\mu}{L^3} & \frac{2\mu}{L^2} & \frac{4\mu}{L^2} \end{vmatrix}$$

$$+ \frac{6\mu}{L^3}(-1)^{1+4} \begin{vmatrix} 0 & \frac{8\mu}{L^2} & \frac{-6\mu}{L^3} \\ \frac{-12\mu}{L^4} & \frac{-6\mu}{L^3} & \left(\frac{12\mu}{L^4} - \frac{\lambda}{2}\right) \\ \frac{6\mu}{L^3} & \frac{2\mu}{L^2} & \frac{-6\mu}{L^3} \end{vmatrix} = 0$$

$$\left(\frac{24\mu}{L^4} - \lambda\right)\left[\frac{32\mu^2}{L^4}\left(\frac{12\mu}{L^4} - \frac{\lambda}{2}\right) + \frac{36\mu^2}{L^6}\frac{2\mu}{L^2} + \frac{36\mu^2}{L^6}\frac{2\mu}{L^2}\right.$$

$$\left. - \left\{\left(\frac{12\mu}{L^4} - \frac{\lambda}{2}\right)\frac{4\mu^2}{L^4} + \frac{36\mu^2}{L^6}\frac{8\mu}{L^2} + \frac{36\mu^2}{L^6}\frac{4\mu}{L^2}\right\}\right]$$

$$-\frac{12\mu}{L^4}\left[\frac{-36\mu^2}{L^6}\frac{8\mu}{L^2}\ \frac{-12\mu}{L^2}\frac{4\mu^2}{L^4} - \left\{\frac{-36\mu^2}{L^6}\frac{2\mu}{L^2}\ \frac{-12\mu}{L^4}\frac{32\mu^2}{L^4}\right\}\right]$$

$$\frac{-6\mu}{L^3}\left[\left(\frac{12\mu}{L^4} - \frac{\lambda}{2}\right)\frac{48\mu^2}{L^5} + \frac{12\mu}{L^4}\frac{12\mu^2}{L^5} - \left\{\frac{6\mu}{L^3}\frac{36\mu^2}{L^6} + \frac{48\mu^2}{L^5}\frac{12\mu}{L^4}\right\}\right] = 0$$

$$\left(\frac{24\mu}{L^4}-\lambda\right)\left[\left(\frac{12\mu}{L^4}-\frac{\lambda}{2}\right)\frac{28\mu^2}{L^4}-\frac{288\mu^3}{L^8}\right]-\frac{12\mu}{L^4}\left[\frac{120\mu^3}{L^8}\right]$$

$$-\frac{6\mu}{L^3}\left[\left(\frac{12\mu}{L^4}-\frac{\lambda}{2}\right)\frac{48\mu^2}{L^5}-\frac{648\mu^3}{L^9}\right]=0$$

$$2\left(\frac{12\mu}{L^4}-\frac{\lambda}{2}\right)^2\frac{28\mu^2}{L^4}-2\left(\frac{12\mu}{L^4}-\frac{\lambda}{2}\right)\frac{288\mu^3}{L^6}+\frac{2448\mu^4}{L^{12}}-\left(\frac{12\mu}{L^4}-\frac{\lambda}{2}\right)\left(\frac{288\mu^3}{L^8}\right)=0$$

$$\left(\frac{12\mu}{L^4}-\frac{\lambda}{2}\right)^2\frac{56\mu^2}{L^4}-\left(\frac{12\mu}{L^4}-\frac{\lambda}{2}\right)\frac{864\mu^3}{L^8}+\frac{2448\mu^4}{L^{12}}=0$$

Let $\quad \dfrac{12\mu}{L^4}-\dfrac{\lambda}{2}=x$

$$\frac{56\mu^2}{L^4}x^2-\frac{864\mu^3}{L^8}x+\frac{2448\mu^4}{L^{12}}=0$$

$$56x^2-\frac{864\mu}{L^4}x+\frac{2448\mu^2}{L^8}=0$$

$$x_{1,2}=\frac{\frac{864\mu}{L^4}\pm\sqrt{\left(\frac{864\mu}{L^4}\right)^2-4(56)\left(\frac{2448\mu^2}{L^8}\right)}}{2(56)}=\frac{\frac{864\mu}{L^4}\pm\sqrt{\frac{198144\mu^2}{L^8}}}{112}$$

$$x_1=\frac{\frac{864\mu}{L^4}+\frac{445.1\mu}{L^4}}{112}\quad\Rightarrow\quad x_1=\frac{11.69\mu}{L^4}$$

$$x_2=\frac{\frac{864\mu}{L^4}-\frac{445.1\mu}{L^4}}{112}\quad\Rightarrow\quad x_2=\frac{3.74\mu}{L^4}$$

$$x_1=\frac{12\mu}{L^4}-\frac{\lambda_1}{2}=\frac{11.69\mu}{L^4}\Rightarrow\quad \lambda_1=\frac{0.62\mu}{L^4}$$

$$x_2=\frac{12\mu}{L^4}-\frac{\lambda_2}{2}=\frac{3.74\mu}{L^4}\Rightarrow\quad \lambda_2=\frac{16.52\mu}{L^4}$$

$$\omega_1^2=\frac{0.62\mu}{L^4}=\frac{0.62\,EI}{\rho AL^4}$$

$$\omega_1=\sqrt{\frac{0.62EI}{\rho AL^4}}\quad\Rightarrow\quad \boxed{\omega_1=\frac{0.787}{L^2}\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}}}$$

$$\omega_2^2=\frac{16.52EI}{\rho A\,L^4}\quad\Rightarrow\quad \omega_2=\sqrt{\frac{16.52\,EI}{\rho A\,L^4}}$$

$$\Rightarrow\quad \boxed{\omega_2=\frac{4.06}{L^2}\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}}}$$

The exact solution from simple beam theory yields

$$\omega_1=\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}}\left(\frac{1.875}{2L}\right)^2=\frac{0.879}{L^2}\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}}$$

(a) $$\omega_2=\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}}\left(\frac{4.694}{2L}\right)^2=\frac{5.5}{L^2}\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}}$$

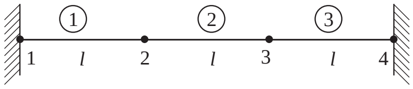
Note: L = one-half actual length of beam in the analysis.

Expressing answers in terms of full length $l = 2L$, we obtain

$$\omega_1 = \frac{0.787}{\left(\frac{l}{2}\right)^2}\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}} = \frac{3.15}{l^2}\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}}$$

and $$\omega_2 = \frac{16.24}{l^2}\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}}$$

(b)



$$[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ \text{Symmetry} & & & 4l^2 \end{bmatrix}$$

$$[m^{(1)}] = [m^{(2)}] = [m^{(3)}] = \frac{\rho Al}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Boundary conditions $v_1 = \phi_1 = v_4 = \phi_4 = 0$

$$|\,[K] - \omega^2\,[M]\,| = 0$$

$$\left|\frac{EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6l \\ & 8l^2 & -6l & 2l^2 \\ & & 24 & 0 \\ \text{Symmetry} & & & 8l^2 \end{bmatrix} - \rho Al\omega^2\begin{bmatrix} 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 1 & 0 \\ \text{Symmetry} & & & 0 \end{bmatrix}\right| = 0$$

Let $$\mu = \frac{EI}{\rho Al^4}$$

$$\begin{vmatrix} 24\mu - \omega^2 & 0 & -12\mu & 6l\mu \\ & 8l^2\mu & -6l\mu & 2l^2\mu \\ & & 24\mu - \omega^2 & 0 \\ \text{Symmetry} & & & 8l^2\mu \end{vmatrix} = 0$$

Rewrite 4×4 determinent as

$$\begin{vmatrix} 24\mu - \omega^2 & 0 & -12\mu & 6l\mu \\ -24l\mu & 0 & -6l\mu & -30l^2\mu \\ 6\mu & 0 & 24\mu - \omega^2 & 24l\mu \\ 6l\mu & 2l^2\mu & 0 & 8l^2\mu \end{vmatrix} = 0$$

$$\therefore \quad \begin{vmatrix} 24\mu-\omega^2 & -12\mu & 6l\mu \\ -24l\mu & -6l\mu & -30l^2\mu \\ 6\mu & 24\mu-\omega^2 & 24l\mu \end{vmatrix} = 0$$

Evaluate

$$972\ l\mu^2 - 192\ L\mu\omega^2 + 5\ l\omega^4 = 0$$

$$\omega^2 = \frac{192\ l\mu \pm \sqrt{(192l\mu)^2 - 4\times 5l^2\mu^2\ 972}}{2(10l)} = \frac{192\ l\mu \pm 132l\mu}{10l}$$

$$\omega_1{}^2 = 6\mu \qquad \omega_2{}^2 = 32.4\ \mu$$

$$\omega_1 = 2.45\sqrt{\frac{EI}{\rho A\left(\frac{l}{3}\right)^4}} = \frac{22.04}{l^2}\sqrt{\frac{EI}{\rho A}}$$

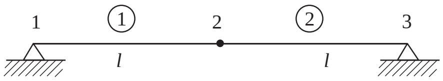
$$\omega_2 = 5.69\sqrt{\frac{EI}{\rho A\left(\frac{l}{3}\right)^4}} = \frac{51.23}{l^2}\sqrt{\frac{EI}{\rho A}}$$

Let $3l = L$ L = whole length

$$\therefore \quad \omega_1 = \frac{22.04}{\left(\frac{L}{3}\right)^2}\sqrt{\frac{EI}{\rho A}} = \frac{198.4}{L^2}\sqrt{\frac{EI}{\rho A}}$$

$$\omega_2 = \frac{51.23}{\left(\frac{L}{3}\right)^2}\sqrt{\frac{EI}{\rho A}} = \frac{461.07}{L^2}\sqrt{\frac{EI}{\rho A}}$$

(c)



Boundary conditions $v_1 = v_5 = 0$

$$[K] = \frac{EI}{l^3}\begin{array}{c} \begin{bmatrix} 4l^2 & -6l & 2l^2 & 0 \\ & 24 & 0 & 6l \\ & & 8l^2 & 2l^2 \\ & & & 4l^2 \end{bmatrix} \end{array}\begin{array}{c} \phi_1 \\ v_2 \\ \phi_2 \\ \phi_3 \end{array}$$

$$[M] = \frac{\rho Al}{2}\begin{bmatrix} & 0 & 0 & 0 \\ 0 & & 0 & 0 \\ 0 & 0 & & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{array}{c} \phi_1 \\ v_2 \\ \phi_2 \\ \phi_3 \end{array}$$

$$I = \frac{1}{3}\ \rho A\frac{l^3}{8}$$

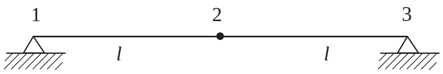
$$|\,[K] - \omega^2\,[M]\,| = 0$$

$$\left| \frac{\rho AE}{24} \begin{bmatrix} 4l^2 & -6l & 2l^2 & 0 \\ & 24 & 0 & 6l \\ & & 8l^2 & 2l^2 \\ & & & 4l^2 \end{bmatrix} - \omega^2 \frac{\rho AE}{24} \begin{bmatrix} 0 & 0 & 0 & 0 \\ & \frac{24l}{E} & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 2l^2 & -3l & l^2 & 0 \\ & 12\left(1 - \frac{l\omega^2}{E}\right) & 0 & 3l \\ & & 4l^2 & l^2 \\ \text{Symmetry} & & & 2l^2 \end{vmatrix} = 0$$

$$\omega = \frac{2.45}{l^2} \sqrt{\frac{EI}{\rho A}}$$

(d)



Boundary conditions $v_1 = \phi_1 = v_3 = 0$

$$[K] = \frac{EI}{l^3} \begin{bmatrix} 24 & 0 & 6l \\ 0 & 8l^2 & 2l^2 \\ 6l & 2l^2 & 4l^2 \end{bmatrix}$$

$$[M] = \frac{\rho AL}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left| \frac{EI}{l^2} \begin{bmatrix} 24 & 0 & 6l \\ & 8l^2 & 2l^2 \\ & & 4l^2 \end{bmatrix} - \omega^2 \frac{\rho AL}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right| = 0$$

Divide by ρAl and let $\omega^2 = \lambda$ and $\dfrac{EI}{\rho AL^4} = \mu$

Determinant becomes

$$\begin{vmatrix} 24\mu - \lambda & 0 & 6l\mu \\ & 8l^2\mu & 2l^2\mu \\ & & 4l^2\mu \end{vmatrix} = 0$$

Simplifying

$$672 - 288\, l^4 \mu^3 = 28\, l^4 \mu^2 \lambda$$

$$\lambda = 13.71\, \mu$$

$$\omega = \sqrt{\lambda} = 3.703 \sqrt{\mu}, \; 2l = L$$

$$\omega = \frac{14.81}{L^2}\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}}$$

16.13

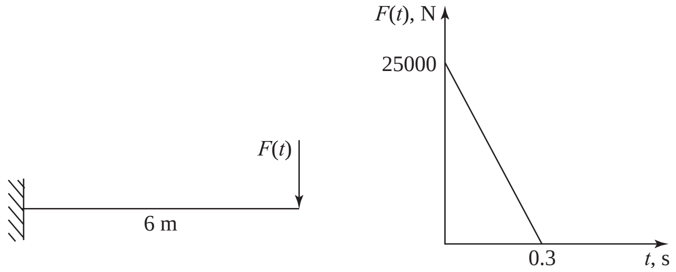
Problem 16.13 using DFRAME
2, 3
1, 1, 1, 0, 0, 0, 0, 0, 0
2, 0, 0, 0, 4, 0, 0, – 1.0, 0
3, 0, 1, 0, 8, 0, 0, 0, 0
1, 1, 2, 210*e*+9, 2*e*–2, 4*e*–4, 1.352*e*–9
2, 2, 3, 210*e*+9, 2*e*–2, 4*e*–4, 1.352*e*–9
0.03, 12
0.25, 0.50
2
–16666.667, 5000, 0, 0, 0.3

Using DFRAME that properly calculates initial acceleration.

		Displacement		Velocity		Acceleration	
Node	Time	‘Y’	‘Z’	‘Y’	‘Z’	‘Y’	‘Z’
1	0	0	0	0	0	0	0
1	0.03	0	–0.00045	0	–0.03016	0	–2.011
1	0.06	0	0.000047	0	0.06349	0	8.254
1	0.09	0	–0.00040	0	–0.09365	0	–18.73
1	0.12	0	0.000095	0	0.127	0	33.44
1	0.15	0	–0.00035	0	–0.1571	0	–52.38
1	0.18	0	0.000142	0	0.1905	0	75.56
1	0.21	0	–0.00030	0	–0.2206	0	–103
1	0.24	0	0.000190	0	0.254	0	134.6
1	0.27	0	–0.00026	0	–0.2841	0	–170.5
1	0.3	0	0.000238	0	0.3175	0	210.6
1	0.33	0	–0.00023	0	–0.3492	0	–255
1	0.36	0	0.000238	0	0.381	0	303.7
2	0	0	0	0	0	–4.6E+13	0
2	0.03	–0.00120	0	–0.08051	0	4.6E+13	0
2	0.06	0.000127	1.2E–21	0.1695	8.3E–20	–4.6E+13	5.5E–18
2	0.09	–0.00107	1.0E–20	–0.25	5.0E–19	4.6E+13	2.2E–17
2	0.12	0.000254	2.5E–21	0.3389	–1.0E–18	–4.6E+13	–1.2E–16
2	0.15	–0.00095	0	–0.4194	8.3E–19	4.6E+13	2.4E–16
2	0.18	0.000381	0	0.5084	–8.3E–19	–4.6E+13	–3.5E–16
2	0.21	–0.00082	–1.0E–20	–0.5889	1.7E–19	4.6E+13	4.2E–16
2	0.24	0.000507	0	0.6778	5.0E–19	–4.6E+13	–4.0E–16
2	0.27	–0.00069	–5.0E–21	–0.7583	–8.3E–19	4.6E+13	3.1E–16
2	0.3	0.000634	0	0.8473	1.2E–18	–4.6E+13	–1.8E–16
2	0.33	–0.00063	–5.0E–21	–0.932	–1.5E–18	4.6E+13	–5.9E–31
2	0.36	0.000634	0	1.017	1.8E–18	–4.6E+13	2.2E–16
3	0	0	0	0	0	0	0
3	0.03	0	0.000452	0	0.03016	0	2.011
3	0.06	0	–0.00004	0	–0.06349	0	–8.254
3	0.09	0	0.000404	0	0.09365	0	18.73
3	0.12	0	–0.00009	0	–0.127	0	–33.44
3	0.15	0.	0.000357	0	0.1571	0	52.38
3	0.18	0	–0.00014	0	–0.1905	0	–75.56
3	0.21	0	0.000309	0	0.2206	0	103

3	0.24	0	–0.00019	0	–0.254	0	–134.6
3	0.27	0	0.000261	0	0.2841	0	170.5
3	0.3	0	–0.00023	0	–0.3175	0	–210.6
3	0.33	0	0.000238	0	0.3492	0	255
3	0.36	0	–0.00023	0	–0.381	0	–303.7

16.14 Note that even though damping data was not given in the problem. This solution includes damping.

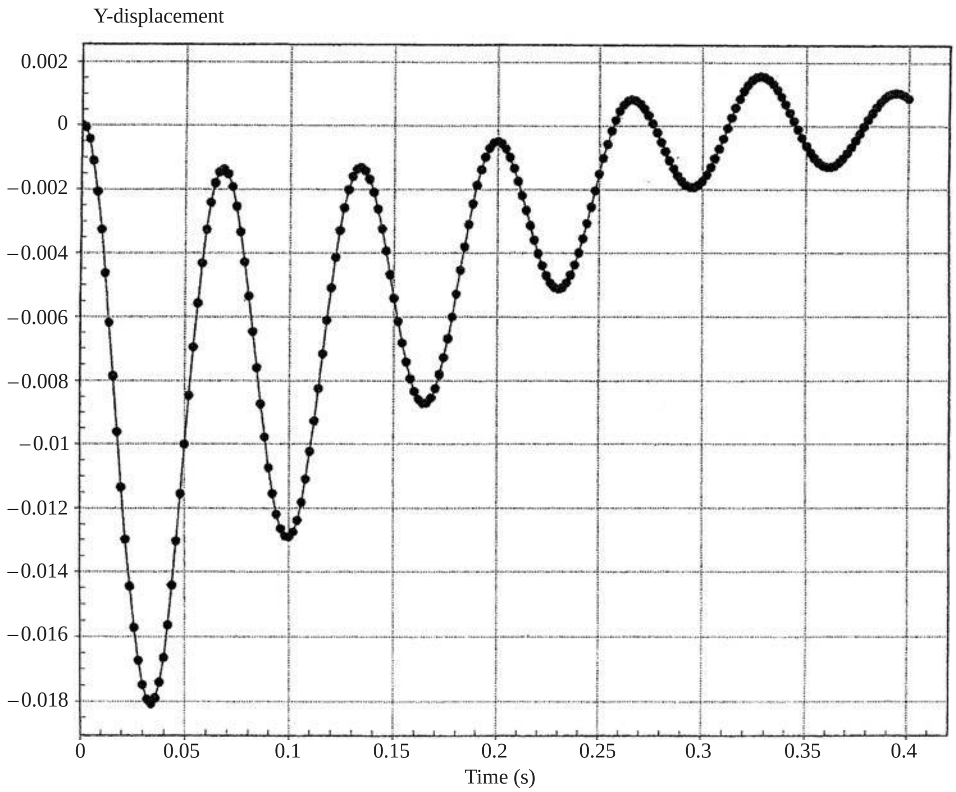


Let the time step or increment = 0.002 s and the number of time steps = 200.
Also use the following data in the Algor program

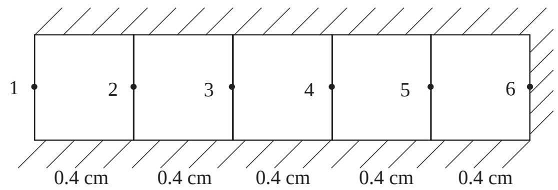
$\rho = 7800\,\dfrac{\text{kg}}{\text{m}^3}$, $E = 210$ GPa, $Y = 0.3$, $\sigma_{ys} = 250$ MPa

$A = 2 \times 10^{-2}\ \text{m}^2$, $\text{J1} = 16 \times 10^{-4}\ \text{m}^4$, $\text{I2} = \text{I3} = 8 \times 10^{-4}\ \text{m}^4$, $\text{S2} = \text{S3} = 16 \times 10^{-4}\ \text{m}^3$

Damping is to be included so use mass damping coefficient $C_m = \alpha = 3.00$ and stiffness damping coefficient $C_k = \beta = 0.001$



16.17



$K = 2\ \frac{\text{W}}{(\text{m}\cdot{}^\circ\text{C})}$

$\rho = 2500\ \frac{\text{kg}}{\text{m}^3}$

$C = 800\ \frac{\text{W}\cdot\text{s}}{(\text{kg}\cdot{}^\circ\text{C})}$

$T_1^0 = 200^\circ\text{C}$ $T_1' = 0^\circ\text{C}$ (sudden temperature change to $T_1' = 0^\circ\text{C}$)

Using Equation (16.8.16)

$$\left(\frac{1}{\Delta t}[M]+\beta[K]\right)\{T_i\}_{+1} = \left(\frac{1}{\Delta t}[M]-(1-\beta)[K]\right)\{T_i\} + (1-\beta)\{F_i\} + \beta\{F_i\}_{+1}$$

$h = 0$ assume unit area

$$[K] = \frac{AK}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{(l\text{m}^2)2\frac{\text{W}}{(\text{m}\cdot{}^\circ\text{C})}}{0.004\text{ m}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 500\ \frac{\text{W}}{{}^\circ\text{C}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$M = \frac{C\rho AL}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{\left(800\frac{\text{W}\cdot\text{s}}{\text{kg}\cdot{}^\circ\text{C}}\right)\left(2500\frac{\text{kg}}{\text{m}^3}\right)(l\text{m}^2)(0.4\text{ cm})}{2\left(100\frac{\text{cm}}{\text{m}}\right)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 4000\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{\text{W}\cdot\text{s}}{{}^\circ\text{C}}$$

$$\text{Let } [C] = \left(\frac{1}{\Delta t}[M]+\beta[K]\right)^{-1}\left\{\frac{1}{\Delta t}[M]-(1-\beta)[K]\right\}$$

Then

$$\{T_i\}_{+1} = C\{T_i\} \text{ as } \{F_i\} = 0, \{F_i\}_{+1} = 0$$

For $[T_1]$ ($t = 8$ s) and eliminating 1st row and column for boundary condition $t_1 = 0$ we have

$$\{T_1\} = \begin{Bmatrix} t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \end{Bmatrix} = C'\begin{Bmatrix} 200 \\ 200 \\ 200 \\ 200 \\ 200 \end{Bmatrix}$$

where $[C']$ is $[C]$ with 1st row and column deleted

with row and column one deleted

$$[K] = \begin{bmatrix} 1000 & -500 & 0 & 0 & 0 \\ & 1000 & -500 & 0 & 0 \\ & & 1000 & -500 & 0 \\ & & & 1000 & -500 \\ \text{Symmetry} & & & & 500 \end{bmatrix}$$

$$[M] = \begin{bmatrix} 8000 & 0 & 0 & 0 & 0 \\ & 8000 & 0 & 0 & 0 \\ & & 8000 & 0 & 0 \\ & & & 8000 & 0 \\ \text{Symmetry} & & & & 4000 \end{bmatrix}$$

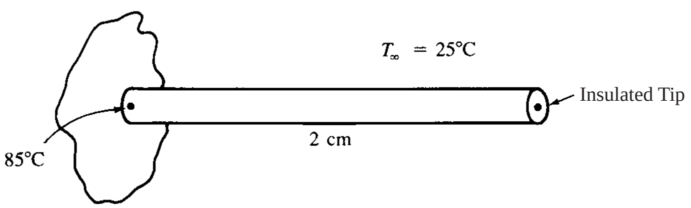
$$\frac{M}{\Delta t} + \beta[K] = \begin{bmatrix} 1666.7 & -333.3 & 0 & 0 & 0 \\ & 1666.7 & -333.3 & 0 & 0 \\ & & 1666.7 & -333.3 & 0 \\ & & & 1666.7 & -333.3 \\ \text{Symmetry} & & & & 833.3 \end{bmatrix}$$

$$\frac{[M]}{\Delta t} - (1-\beta)\,[K] = \begin{bmatrix} 666.7 & 166.7 & 0 & 0 & 0 \\ & 666.7 & 166.7 & 0 & 0 \\ & & 666.7 & 166.7 & 0 \\ & & & 666.7 & 166.7 \\ \text{Symmetry} & & & & 333.3 \end{bmatrix}$$

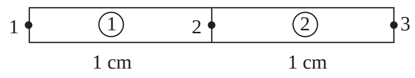
$$[C'] = \begin{bmatrix} 0.5247 & 0.2139 & 0.04472 & 0.00972 & 0.00194 \\ & 0.4839 & 0.2057 & 0.04472 & 0.00894 \\ & & 0.4839 & 0.2139 & 0.0428 \\ & & & 0.5247 & 0.2049 \\ \text{Symmetry} & & & & 0.4820 \end{bmatrix}$$

	Node / t_1 s	Temperatures, °C 1	2	3	4	5	6
	0	200	200	200	200	200	200
1	8	0	159	191	198	199.6	199.8
2	16	0	135	178	193	198.2	199.1
3	24	0	120	165	187	195.5	197.5
4	32	0	109	155	180	191.7	194.8
5	40	0	101	146	173	187.1	191.1
6	48	0	94	138	167	182.0	186.7
7	56	0	88	131	160	176.5	181.6
8	64	0	84	125	154	170.8	176.3
9	72	0	79	119	148	165.1	170.7
10	80	0	76	114	142	159.3	165.0

16.18



Two element solutions



0.4 cm diameter

$$[K] = \frac{AK}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{h\rho L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\frac{\text{W}}{{}^\circ\text{C}}$$

$$= (0.12567 \text{ cm}^2)\left(400\frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}}\right)\left(\frac{1\text{ m}}{100\text{ cm}}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$+\left(150\frac{\text{W}}{\text{m}^2\cdot{}^\circ\text{C}}\right)\frac{(1.2567\text{ cm})(1\text{ cm})}{6\left(100\,\frac{\text{cm}}{\text{m}}\right)^2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5027 & -0.5027 \\ -0.5027 & 0.5027 \end{bmatrix} + \begin{bmatrix} 0.00628 & 0.00314 \\ 0.00314 & 0.00628 \end{bmatrix}$$

Using consistent mass

$$[m] = \frac{c\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \left(375\frac{\text{W}\cdot\text{s}}{\text{kg}\cdot{}^\circ\text{C}}\right)\left(8900\frac{\text{kg}}{\text{m}^3}\right)\left(0.12567\frac{\text{cm}^2}{100^2}\right)\frac{\frac{1\text{ cm}}{100}}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1398 & 0.0699 \\ 0.0699 & 0.1398 \end{bmatrix}\frac{\text{W}\cdot\text{s}}{{}^\circ\text{C}}$$

$$\{f\} = \frac{hT_\infty PL}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.23561 \\ 0.23561 \end{bmatrix}\text{W}$$

Global $[K]$ and $[M]$

$$[K] = \begin{bmatrix} 0.50893 & -0.49951 & 0 \\ & 1.01786 & -0.49951 \\ \text{Symmetry} & & 0.50893 \end{bmatrix}$$

$$[M] = \begin{bmatrix} 0.1398 & 0.06991 & 0 \\ & 0.2796 & 0.06991 \\ \text{Symmetry} & & 0.1398 \end{bmatrix}$$

$$\{F\} = \begin{Bmatrix} 0.2356 \\ 0.4712 \\ 0.2356 \end{Bmatrix}$$

Using Equation (16.8.16) with

$$[T_0] = \begin{Bmatrix} 25^\circ\text{C} \\ 25^\circ\text{C} \\ 25^\circ\text{C} \end{Bmatrix} \text{ (initial temperature of rod)}$$

Suddenly end (left) temperature becomes 85°C ($\beta = \frac{2}{3}$)

For $t = 0.1$ s

$$\left(\frac{1}{\Delta t}[M] + \beta[K]\right)\{T_1\} = \left(\frac{1}{\Delta t}[M] - (1-\beta)[K]\right)\{T_0\} + \{F_0\}$$

As $\{F_i\} = \{F_i\}_{+1}$ for all time t

$$\begin{bmatrix} 1.7374 & 0.3660 & 0 \\ & 3.4747 & 0.3660 \\ \text{Symmetry} & & 1.7374 \end{bmatrix} \begin{Bmatrix} 85°\text{C} \\ t_2 \\ t_3 \end{Bmatrix} = \begin{bmatrix} 1.228 & 0.865 & 0 \\ & 2.457 & 0.865 \\ & & 1.228 \end{bmatrix} \begin{Bmatrix} 25 \\ 25 \\ 25 \end{Bmatrix} + \begin{Bmatrix} 0.2356 \\ 0.4712 \\ 0.2356 \end{Bmatrix}$$

Since $t_1 = 85°$ boundary condition adjust equations

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3.4747 & 0.366 \\ 0 & 0.366 & 1.37 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 105.17 \times 85(0.3660) \\ 52.585 \end{Bmatrix}$$

Solving

$$t_1 = 85°, \quad t_2 = 18.536°\text{C}, \quad t_3 = 26.362°\text{C}$$

For $t = 0.2$ s

$$\begin{bmatrix} 1.737 & 0.366 & 0 \\ & 3.475 & 0.366 \\ \text{Symmetry} & & 1.737 \end{bmatrix} \begin{Bmatrix} 85 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{bmatrix} 1.228 & 0.865 & 0 \\ & 2.457 & 0.865 \\ & & 1.228 \end{bmatrix} \begin{Bmatrix} 85 \\ 18.536 \\ 26.362 \end{Bmatrix} + \begin{Bmatrix} 0.2356 \\ 0.4712 \\ 0.2356 \end{Bmatrix}$$

Solving

$$\begin{bmatrix} 1 & 0 & 0 \\ & 3.475 & 0.366 \\ & & 1.737 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 141.93 - 0.865(85) \\ 48.427 \end{Bmatrix}$$

$$t_2 = 29.613°\text{C}, \quad t_3 = 21.635°\text{C}$$

For $t = 0.3$ s

$$\begin{bmatrix} 1 & 0 & 0 \\ & 3.475 & 0.366 \\ & & 1.737 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 134.0 \\ 52.687 \end{Bmatrix}$$

$$t_2 = 36.404°\text{C}, \quad t_3 = 22.662°\text{C}$$

For $t = 0.4$ s

$$\begin{bmatrix} 1 & 0 & 0 \\ & 3.475 & 0.366 \\ & & 1.737 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 152.0 \\ 59.58 \end{Bmatrix}$$

$$t_2 = 41.03°\text{C}, \quad t_3 = 25.655°\text{C}$$

For $t = 0.5$ s

$$\begin{bmatrix} 1 & 0 & 0 \\ & 3.475 & 0.366 \\ & & 1.737 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 165.9 \\ 67.26 \end{Bmatrix}$$

$$t_2 = 44.665°\text{C}, \quad t_3 = 29.31°\text{C}$$

For $t = 0.6$ s

$$\begin{bmatrix} 1 & 0 & 0 \\ & 3.475 & 0.366 \\ & & 1.737 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 178.03 \\ 74.90 \end{Bmatrix}$$

$$t_2 = 47.75°\text{C}, \quad t_3 = 33.06°\text{C}$$

For $t = 0.7$ s

$$\begin{bmatrix} 1 & 0 & 0 \\ & 3.475 & 0.366 \\ & & 1.737 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 188.85 \\ 82.17 \end{Bmatrix}$$

$$t_2 = 50.48°\text{C}, \quad t_3 = 36.67°\text{C}$$

For $t = 0.8$ s

$$\begin{bmatrix} 1 & 0 & 0 \\ & 3.475 & 0.366 \\ & & 1.737 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 198.7 \\ 88.97 \end{Bmatrix}$$

$$t_2 = 52.96°\text{C}, \quad t_3 = 40.06°\text{C}$$

For $t = 0.9$ s

$$\begin{bmatrix} 1 & 0 & 0 \\ & 3.475 & 0.366 \\ & & 1.737 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 207.7 \\ 95.28 \end{Bmatrix}$$

$$t_2 = 55.22°\text{C}, \ t_3 = 43.22°\text{C}$$

For $t = 1.0$ s

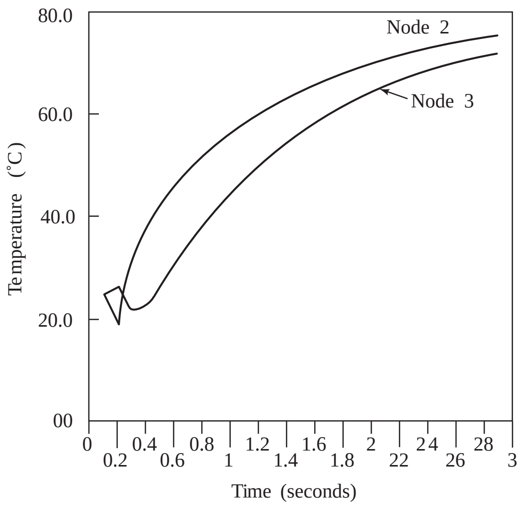
$$\begin{bmatrix} 1 & 0 & 0 \\ & 3.475 & 0.366 \\ & & 1.737 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 215.99 \\ 101.11 \end{Bmatrix}$$

$$t_2 = 57.30°\text{C}, \ t_3 = 46.14°\text{C}$$

etc. (A computer solution follows)

Time	NODE		
(s)	1	2	3
0	25	25	25
0.1	85	18.53611	26.36189
0.2	85	29.61303	21.63526
0.3	85	36.18435	22.42717
0.4	85	40.72491	25.30428
0.5	85	44.27834	28.85201
0.6	85	47.29072	32.49614
0.7	85	49.95809	36.01157
0.8	85	52.37152	39.31761
0.9	85	54.57756	42.39278
1	85	56.60353	45.23933
1.1	85	58.46814	47.86852
1.2	85	60.1859	50.29457

1.3	85	61.76908	52.53218
1.4	85	63.22852	54.59557
1.5	85	64.574	56.49814
1.6	85	65.81448	58.25235
1.7	85	66.95818	59.86974
1.8	85	68.01265	61.36096
1.9	85	68.98485	62.73586
2	85	69.88121	64.0035
2.1	85	70.70765	65.17226
2.2	85	71.46961	66.24984
2.3	85	72.17214	67.24336
2.4	85	72.81986	68.15938
2.5	85	73.41705	69.00393
2.6	85	73.96766	69.78261
2.7	85	74.47531	70.50053
2.8	85	74.94336	71.16246
2.9	85	75.3749	71.77274
3	85	75.77277	72.33542



Appendix A

A.1

(a) $[A] + [B] = \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -3 & 12 \end{bmatrix}$

(b) $[A] + [C]$, Nonsense, $[A]$ and $[C]$ not same order

(c) $[A]\,[C]^T$, Nonsense, columns $[A] \neq$ rows $[C]^T$

(d) $[D]\,[E] = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$

$$= \begin{Bmatrix} 3(1)+1(2)+2(3) \\ 1(1)+4(2)+0(2) \\ 2(1)+0(2)+3(3) \end{Bmatrix} = \begin{Bmatrix} 11 \\ 9 \\ 11 \end{Bmatrix}$$

(e) $[D]\,[C]$, Nonsense, columns $[D] \neq$ rows $[C]$

(f) $[C]\,[D] = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 3(3)+(1)(1)+0 & 3+4+0 & 6+0+0 \\ -3+0+6 & -1+0+0 & -2+0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 7 & 6 \\ 3 & -1 & 7 \end{bmatrix}$$

A.2 $[A] = \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix}$ $\quad [A]^{-1} = \dfrac{[C]^T}{|[A]|}$

$$C_{11} = (-1)^{1+1}(4) = 4, \qquad C_{12} = (-1)^{1+2}(-1) = 1$$

$$C_{21} = (-1)^{2+1}(0) = 0, \qquad C_{22} = (-1)^{2+2}(1) = 1$$

$$[C] = \begin{bmatrix} 4 & 1 \\ 0 & 1 \end{bmatrix}$$

$$|[A]| = A_{11}C_{11} + A_{12}C_{12}$$

$$= (1)(4) + (0)(1) = 4$$

$$[C]^T = \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\therefore\ [A]^{-1} = \frac{\begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}}{4} = \begin{bmatrix} 1 & 0 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Verify by multiplying $[A]\,[A]^{-1} = [I]$

A.3 $[D]^{-1} = \dfrac{[C]^T}{|[D]|}$

$$[D] = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 12 & -3 & -8 \\ -3 & 5 & 2 \\ -8 & 2 & 11 \end{bmatrix}$$

$$|[D]| = 12(3) + (-3)(1) + (-8)(2) = 17$$

$$[D]^{-1} = \frac{1}{17}\begin{bmatrix} 12 & -3 & -8 \\ -3 & 5 & 2 \\ -8 & 2 & 11 \end{bmatrix}$$

A.4 Nonsense

A.5 $[B] = \begin{bmatrix} 2 & 0 \\ -2 & 8 \end{bmatrix}$

(1) $\left[\begin{array}{cc:cc} 2 & 0 & 1 & 0 \\ -2 & 8 & 0 & 1 \end{array}\right]$ divide 1st row by 2

(2) $\left[\begin{array}{cc:cc} 1 & 0 & \frac{1}{2} & 0 \\ -2 & 8 & 0 & 1 \end{array}\right]$ multiply 1st row by 2 and add to row 2

(3) $\left[\begin{array}{cc:cc} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 8 & 1 & 1 \end{array}\right]$ divide 2nd row by 8

(4) $\left[\begin{array}{cc:cc} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{8} & \frac{1}{8} \end{array}\right]$ $\therefore [B]^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{8} & \frac{1}{8} \end{bmatrix}$

A.6 $[D]^{-1}$ by row reduction

$\left[\begin{array}{ccc:ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$ divide row 1 by 3

$\left[\begin{array}{ccc:ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 1 & 4 & 0 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$ subtract row 1 from 2

$\left[\begin{array}{ccc:ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 3\frac{2}{3} & \frac{-2}{3} & \frac{-1}{3} & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$. multiply row 1 by 2 and subtract from row 3

$\left[\begin{array}{ccc:ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 3\frac{2}{3} & \frac{-2}{3} & \frac{-1}{3} & 1 & 0 \\ 0 & \frac{-2}{3} & \frac{5}{3} & \frac{-2}{3} & 0 & 1 \end{array}\right]$. multiple row 2 by $\frac{2}{11}$ and add to row 3

$\left[\begin{array}{ccc:ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 3\frac{2}{3} & \frac{-2}{3} & \frac{-1}{3} & 1 & 0 \\ 0 & 0 & \frac{51}{33} & \frac{-24}{33} & \frac{2}{11} & 1 \end{array}\right]$. multiply row 2 by $\frac{3}{11}$ and row 3 by $\frac{33}{51}$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{-2}{11} & \frac{-1}{11} & \frac{3}{11} & 0 \\ 0 & 0 & 1 & \frac{-24}{51} & \frac{6}{51} & \frac{33}{51} \end{array}\right]\text{. multiply row 3 by } \tfrac{2}{11} \text{ and add to row 2}$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & \frac{-3}{17} & \frac{5}{17} & \frac{2}{17} \\ 0 & 0 & 1 & \frac{-24}{51} & \frac{6}{51} & \frac{33}{51} \end{array}\right]\text{. multiply row 3 by } \tfrac{2}{3} \text{ and subtract from row 1}$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & 0 & \frac{11}{17} & \frac{-4}{51} & \frac{-22}{51} \\ 0 & 1 & 0 & \frac{-3}{17} & \frac{5}{17} & \frac{2}{17} \\ 0 & 0 & 1 & \frac{-24}{51} & \frac{6}{51} & \frac{33}{51} \end{array}\right]\text{. multiply row 2 by } \tfrac{1}{3} \text{ and subtract from row 1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{12}{17} & \frac{-3}{17} & \frac{-8}{17} \\ 0 & 1 & 0 & \frac{-3}{17} & \frac{5}{17} & \frac{2}{17} \\ 0 & 0 & 1 & \frac{-8}{17} & \frac{2}{17} & \frac{11}{17} \end{array}\right]$$

$$\therefore [D]^{-1} = \frac{1}{17}\begin{bmatrix} 12 & -3 & -8 \\ -3 & 5 & 2 \\ -8 & 2 & 11 \end{bmatrix}$$

A.7 Show that $([A]\,[B])^T = [B]^T\,[A]^T$ by using

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad [B] = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$([A]\,[B]) = \begin{bmatrix} a_{11}(b_{11})+a_{12}(b_{21}) & a_{11}(b_{12})+a_{12}(b_{22}) & a_{11}(b_{13})+a_{12}(b_{23}) \\ a_{21}(b_{11})+a_{22}(b_{21}) & a_{21}(b_{12})+a_{22}(b_{22}) & a_{21}(b_{13})+a_{22}(b_{23}) \end{bmatrix}$$

$$([A]\,[B])^T = \begin{bmatrix} a_{11}(b_{11})+a_{12}(b_{21}) & a_{21}(b_{11})+a_{22}(b_{21}) \\ a_{11}(b_{12})+a_{12}(b_{22}) & a_{21}(b_{12})+a_{22}(b_{22}) \\ a_{11}(b_{13})+a_{12}(b_{23}) & a_{21}(b_{13})+a_{22}(b_{23}) \end{bmatrix}$$

$$[B]^T = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \\ b_{13} & b_{23} \end{bmatrix} \qquad [A]^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$[B]^T\,[A]^T = \begin{bmatrix} b_{11}(a_{11})+b_{21}(a_{12}) & b_{11}(a_{21})+b_{21}(a_{22}) \\ b_{12}(a_{11})+b_{22}(a_{12}) & b_{12}(a_{21})+b_{22}(a_{22}) \\ b_{13}(a_{11})+b_{23}(a_{12}) & b_{13}(a_{21})+b_{23}(a_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(b_{11})+a_{12}(b_{21}) & a_{21}(b_{11})+a_{22}(b_{21}) \\ a_{11}(b_{12})+a_{12}(b_{22}) & a_{21}(b_{12})+a_{22}(b_{22}) \\ a_{11}(b_{13})+a_{12}(b_{23}) & a_{21}(b_{13})+a_{22}(b_{23}) \end{bmatrix}$$

Answer : $([A]\,[B])^T = [B]^T\,[A]^T$

A.8 $[T] = \begin{bmatrix} C & S \\ -S & C \end{bmatrix}$

$$[C] = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \qquad [C]^T = \begin{bmatrix} C & -S \\ S & C \end{bmatrix}$$

$$|\,[T]\,| = C^2 + S^2 = 1$$

$$[T]^{-1} = \frac{[C]^T}{|[T]|} = \begin{bmatrix} C & -S \\ S & C \end{bmatrix}$$

and

$$[T]^T = \begin{bmatrix} C & -S \\ S & C \end{bmatrix}$$

$$\therefore \quad [T]^T = [T]^{-1} \text{ and } T \text{ is an orthogonal matrix}$$

A.9 Show $\{X\}^T [A] \{X\}$ is symmetric. Given

$$\{X\} = \begin{bmatrix} x & y \\ 1 & x \end{bmatrix}, [A] = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$$\{X\}^T = \begin{bmatrix} x & 1 \\ y & x \end{bmatrix}$$

$$\{X\}^T [A] \{X\} = \begin{bmatrix} x & 1 \\ y & x \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x & y \\ 1 & x \end{bmatrix}$$

$$= \begin{bmatrix} ax+b & bx+c \\ ay+bx & by+cx \end{bmatrix}\begin{bmatrix} x & y \\ 1 & x \end{bmatrix}$$

$$= \begin{bmatrix} ax^2+bx+bx+c & axy+by+bx^2+cx \\ axy+bx^2+by+cx & ay^2+bxy+bxy+cx^2 \end{bmatrix}$$

as the 1–2 term = 2–1 term $\{X\}^T [A] \{X\}$ is symmetric.

A.10 Evaluate $[K] = \int_0^L [B]^T \; E \, [B] \, dx$, $[B] = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$

$$[K] = \int_0^L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} E \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} dx$$

$$[K] = \int_0^L \begin{bmatrix} \frac{1}{L^2} & \frac{-1}{L^2} \\ \frac{-1}{L^2} & \frac{1}{L^2} \end{bmatrix} E \, dx$$

$$[K] = E\begin{bmatrix} \frac{1}{L} & \frac{1}{L} \\ \frac{-1}{L} & \frac{1}{L} \end{bmatrix} = \frac{E}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(Should multiply by A to get actual $[K]$ for a bar)

A.11 The following integral represents the strain energy in a bar

$$U = \frac{A}{2}\int_0^L \{d\}^T \; [B]^T [D] \, [B] \, \{d\} \, dx$$

where $\{d\} = \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$ $\quad [B] = \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix}$ $\quad [D] = E$

Show that $\frac{dU}{d\{d\}}$ yields $[k] \{d\}$, where $[k]$ is the bar stiffness matrix given by

$$[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{d\}^T = [d_1\ d_2] \qquad [B]^T = \begin{Bmatrix} \frac{-1}{L} \\ \frac{1}{L} \end{Bmatrix}$$

$$U = \frac{A}{2}\int_0^L \{d\}^T\ [B]^T\ [D]\ [B]\ \{d\}\ dx$$

$$U = \frac{AL}{2}\ \{d\}^T\ [B]^T\ [D]^T\ [B]\ \{d\} = \frac{AL}{2}\ [d_1 \quad d_2]\ \begin{Bmatrix} \frac{-1}{L} \\ \frac{1}{L} \end{Bmatrix}[E]\begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$U = \frac{AL}{2}\left[\frac{d_2 - d_1}{L}\right][E]\begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$U = \frac{AEL}{2}\left[\frac{d_2 - d_1}{L}\right]\begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \frac{AEL}{2}\begin{bmatrix} \frac{d_1 - d_2}{L^2} & \frac{-d_1 + d_2}{L^2} \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$U = \frac{AEL}{2}\left[\frac{d_1^2 - d_1 d_2 - d_1 d_2 + d_2^2}{L^2}\right] = \frac{AE}{2L}\ [\ d_1^2 - 2d_1 d_2 + \ d_2^2\ ]$$

$$\frac{dU}{d\{d\}} = \begin{Bmatrix} \frac{\partial U}{\partial d_1} \\ \frac{\partial U}{\partial d_2} \end{Bmatrix} = \begin{Bmatrix} \frac{AE}{2L}(2d_1 - 2d_2) \\ \frac{AE}{2L}(2d_2 - 2d_1) \end{Bmatrix} = \frac{AE}{L}\begin{Bmatrix} d_1 - d_2 \\ -d_2 - d_1 \end{Bmatrix} = \frac{AE}{L}\begin{Bmatrix} d_1 - d_2 \\ -d_1 + d_2 \end{Bmatrix}$$

$$= \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$\frac{dU}{d\{d\}} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = [k]\ \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = [k]\ \{d\} \text{ knowing that } [k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Thus $\quad \dfrac{dU}{d\{d\}} = [k]\ \{d\}$

Appendix B

B.1 By Cramer's Rule

$$\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 5 \\ 12 \end{Bmatrix}$$

$$x_1 = \frac{\left|\{d\}^{(1)}\right|}{|[a]|} = \frac{\begin{vmatrix} 5 & 3 \\ 12 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix}} = \frac{-5-36}{-13} = 3.15$$

$$x_2 = \frac{\left|\{d\}^{(2)}\right|}{|[a]|} = \frac{\begin{vmatrix} 1 & 5 \\ 4 & 12 \end{vmatrix}}{-13} = \frac{12-20}{-13} = 0.62$$

B.2 By Inverse method

$$\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 5 \\ 12 \end{Bmatrix}$$

$$C_{11} = (-1)^{1+1}|-1| = -1, \; C_{12} = (-1)^{1+2}|4| = -4$$

$$C_{21} = (-1)^{2+1}|3| = -3, \; C_{22} = (-1)^{2+2}|1| = 1$$

$$[C] = \begin{bmatrix} -1 & -4 \\ -3 & 1 \end{bmatrix}, \qquad [C]^T = \begin{bmatrix} -1 & -3 \\ -4 & 1 \end{bmatrix}$$

$$|[a]| = a_{11}\,C_{11} + a_{12}\,C_{12} = 1(-1) + 3(-4) = -13$$

$$[a]^{-1} = \frac{\left|[C]^T\right|}{|[a]|} = -\frac{1}{13}\begin{bmatrix} -1 & -3 \\ -4 & -1 \end{bmatrix}$$

$$\therefore \quad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = -\frac{1}{13}\begin{bmatrix} -1 & -3 \\ -4 & 1 \end{bmatrix}\begin{Bmatrix} 5 \\ 12 \end{Bmatrix} = -\frac{1}{13}\begin{Bmatrix} -5-36 \\ -20+12 \end{Bmatrix}$$

$$x_1 = \frac{-41}{-13} = 3.15, \qquad x_2 = \frac{-8}{-13} = 0.62$$

B.3 By Gauss elimination

$$\begin{bmatrix} 1 & -4 & -5 \\ 0 & 3 & 4 \\ -2 & -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ -1 \\ -3 \end{Bmatrix}$$

Mutliply row 1 by $-\left(\frac{-2}{1}\right)$ and add to row 3

$$\begin{bmatrix} 1 & -4 & -5 \\ 0 & 3 & 4 \\ 0 & -9 & -8 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ -1 \\ 5 \end{Bmatrix}$$

Multliply row 2 by $-\left(\frac{-9}{3}\right)$ and add to row 3

$$\begin{bmatrix} 1 & -4 & -5 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ -1 \\ 2 \end{Bmatrix} \qquad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

By (3)

$$\therefore \quad 4x_3 = 2$$

$$x_3 = \frac{1}{2}$$

By (2)

$$3x_2 + 4x_3 = -1$$

$$x_2 = \frac{-1-2}{3} = -1$$

By (1)

$$x_1 - 4(-1) - 5\left(\frac{1}{2}\right) = 4$$

$$x_1 = \frac{5}{2}$$

B.4 $2x_1 + x_2 - 3x_3 = 11$

$4x_1 - 2x_2 + 3x_3 = 8$

$-2x_1 + 2x_2 - x_3 = -6$

$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & 11 \\ 4 & -2 & 3 & 8 \\ -2 & 2 & 1 & -6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{-3}{2} & \frac{11}{2} \\ 0 & -4 & 9 & -14 \\ 0 & 3 & -4 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{-3}{2} & \frac{11}{2} \\ 0 & 1 & \frac{-9}{4} & \frac{7}{2} \\ 0 & 0 & \frac{11}{4} & \frac{-11}{2} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{-3}{2} & \frac{11}{2} \\ 0 & 1 & \frac{-9}{4} & \frac{7}{2} \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & \frac{5}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\therefore \quad x_1 = 3, \quad x_2 = -1, \quad x_3 = -2$$

B.5 $x_1 = 2y_1 - y_2 \qquad z_1 = -x_1 - x_2$

$x_2 = y_1 - y_2 \qquad z_2 = 2x_1 + x_2$

(a) $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}$

$\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$

(b) $z_1 = -(2y_1 - y_2) - (y_1 - y_2)$

$z_1 = -3y_1 + 2y_2$

$z_2 = 2(2y_1 - y_2) + (y_1 - y_2)$

$z_2 = 5y_1 - 3y_2$

$\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix}$

(c) $\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$

$\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = 9 - 10 = -1$

$[C] = \begin{bmatrix} -3 & -5 \\ -2 & -3 \end{bmatrix}$

$[C]^T = \begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$

$\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}$

B.6 $\{X\}^T = (1, 1, 1, 1, 1)$

First iteration

$$2x_1 - x_2 = -1 \Rightarrow x_1 = \frac{1}{2}(x_2 - 1)$$

$$= \frac{1}{2}(1 - 1) = 0$$

$$6x_2 = x_1 + x_3 + 4 \Rightarrow x_2 = \frac{1}{6}(x_1 + x_3 + 4)$$

$$= \frac{1}{6}(0 + 1 + 4)$$

$$= 0.833$$

$$4x_3 = 2x_2 + x_4 + 4 \Rightarrow x_3 = \frac{1}{4}(2x_2 + x_4 + 4)$$

$$x_3 = \frac{1}{4}(2(0.833) + 1 + 4)$$

$$= 1.667$$

$$4x_4 = x_3 + x_5 + 6 \Rightarrow x_4 = \frac{1}{4}(x_3 + x_5 + 6)$$

$$x_4 = \frac{1}{4}(1.667 + 1 + 6)$$

$$= 2.167$$

$$2x_5 = x_4 - 2 \Rightarrow x_5 = \frac{1}{2}(x_4 - 2)$$

$$= \frac{1}{2}(2.167 - 2)$$

$$= 0.083$$

2nd iteration

$$x_1 = \frac{1}{2}(x_2 - 1) = \frac{1}{2}(0.833 - 1) = -0.084$$

$$x_2 = \frac{1}{6}(x_1 + x_3 + 4) = \frac{1}{6}(-0.084 + 1.667 + 4)$$

$$= 0.93$$

$$x_3 = \frac{1}{4}(2x_2 + x_4 + 4) = \frac{1}{4}(2(0.93) + 2.167 + 4)$$

$$= 2.007$$

$$x_4 = \frac{1}{4}(x_3 + x_5 + 6) = \frac{1}{4}(2.007 + 0.083 + 6)$$

$$= 2.023$$

$$x_5 = \frac{1}{2}(x_4 - 2) = \frac{1}{2}(2.023 - 2)$$

$$= 0.011$$

3rd iteration	4th iteration	5th iteration
$x_1 = -0.035$	$x_1 = -0.003$	$x_1 = 0$
$x_2 = 0.995$	$x_2 = 1.000$	$x_2 = 1$
$x_3 = 2.003$	$x_3 = 2.001$	$x_3 = 2$
$x_4 = 2.004$	$x_4 = 2.001$	$x_4 = 2$
$x_5 = 0.002$	$x_5 = 0.000$	$x_5 = 0$

B.7 By Gauss-Seidel

Initial guess $x_1 = 1$ $x_2 = 1$

1st iteration (Reorder equations so a_{ii} largest)

$$4x_1 - x_2 = 12 \Rightarrow x_1 = \frac{1}{4}(12 + x_2)$$

$$= \frac{1}{4}(12 + 1) = 3.25$$

$$x_1 + 3x_2 = 5 \Rightarrow x_2 = \frac{1}{3}(5 - x_1)$$

$$= \frac{1}{3}(5 - 3.25) = 0.583$$

2nd iteration

$$x_1 = 3.146$$
$$x_2 = 0.618$$

3rd iteration

$$x_1 = 3.155$$
$$x_2 = 0.615$$

4th iteration

$$x_1 = 3.154$$
$$x_2 = 0.615$$

5th iteration

$$x_1 = 3.154$$
$$x_2 = 0.615$$

B.8

(a) $2x_1 - 4x_2 = 2$

$$-9x_1 + 12x_2 = -6$$

$$|[a]| \neq 0$$

$$\begin{vmatrix} 2 & -4 \\ -9 & 12 \end{vmatrix} = 24 + 36 = 60 \neq 0$$

$\therefore$ unique solution

(b) $10x_1 + x_2 = 0$

$$5x_1 + \frac{1}{2}x_2 = 3$$

$$\begin{vmatrix} 10 & 1 \\ 5 & \frac{1}{2} \end{vmatrix} = 0$$

Non existent solution

(c) $2x_1 + x_2 + x_3 = 6$

$$3x_1 + x_2 - x_3 = 4$$
$$5x_1 + 2x_2 + 2x_3 = 8$$

$$\begin{vmatrix} 2 & 1 & 1 \\ 3 & 1 & -1 \\ 5 & 2 & 2 \end{vmatrix} \neq 0$$

$\therefore$ unique solution

(d) $x_1 + x_2 + x_3 = 4$

$$2x_1 + 2x_2 + 2x_3 = 2$$
$$3x_1 + 3x_2 + 3x_3 = 3$$
$$|[a]| = 0$$

$\therefore$ Non unique

all lines are parallel to each other

B.9 1st Figure $n_d = 2$, $m = 3$

$$n_b = n_d\,(m + 1) = 2\,(3 + 1) = 8$$

2nd Figure $n_d = 2$, $m = 5$

$$n_b = 2(5 + 1) = 12$$

Appendix D

D.1 Using table D–1

(a) Load case 1

$$P = 10 \text{ kip} \quad L = 20 \text{ ft}$$

$$f_{1y} = f_{2y} = \frac{-10}{2} = -5 \text{ kip}$$

$$m_1 = -m_2 = \frac{-10\,(20)}{8} = -25 \text{ kip}\cdot\text{ft}$$

(b) Load case 3

$$P = 5 \text{ kip} \quad L = 20 \text{ ft} \quad \alpha = \frac{1}{4}$$

$$f_{1y} = f_{2y} = -5 \text{ kip}$$

$$m_1 = -m_2 = -\frac{1}{4}\left(1-\frac{1}{4}\right)(5)\,(20)$$

$$= -18.75 \text{ kip}\cdot\text{ft}$$

(c) Load case 4

$$w = 1000 \ \frac{\text{lb}}{\text{ft}} \quad L = 30 \text{ ft}$$

$$f_{1y} = f_{2y} = \frac{-(1000)\,(30)}{2} = -15000 \text{ lb}$$

$$m_1 = -m_2 = \frac{-(1000)\,(30)^2}{12} = -75000 \text{ lb}\cdot\text{ft}$$

(d) Load cases 1 and 7

$$P = 5 \text{ kip}, \quad L = 20 \text{ ft} \quad w = 2\ \frac{\text{kip}}{\text{ft}}$$

$$f_{1y} = \frac{-5}{2} - \frac{13(20)(2)}{32} = -18.75 \text{ kip}$$

$$f_{2y} = \frac{-5}{2} - \frac{3(20)(2)}{32} = -6.25 \text{ kip}$$

$$m_1 = \frac{-5(20)}{8} - \frac{11(2)(20)^2}{192} = -12.5 - 45.83$$

$$= -58.33 \text{ kip}\cdot\text{ft}$$

$$m_2 = \frac{5(20)}{8} - \frac{5(2)(20)^2}{192} = 12.5 + 20.83$$

$$= 33.33 \text{ kip}\cdot\text{ft}$$

(e) Load case 5

Note: Switch nodes 1 and 2

$$w = 2000\ \frac{\text{lb}}{\text{ft}} \qquad L = 20\text{ ft}$$

$$f_{1y} = \frac{-3(2000)20}{20} = -6000\text{ lb}$$

$$f_{2y} = \frac{-7(2000)(20)}{20} = -14{,}000\text{ lb}$$

$$m_1 = \frac{(2000)(20)^2}{30} = -26{,}667\text{ lb}\cdot\text{ft}$$

$$m_2 = \frac{(2000)(20)^2}{20} = 40{,}000\text{ lb}\cdot\text{ft}$$

(f) Load case 2

$$P = 5\text{ kN}, \quad L = 7\text{ m}, \quad a = 5\text{ m}, \quad b = 2\text{ m}$$

$$f_{2y} = \frac{-5(2)^2\,[7+2(5)]}{7^3} = -0.99\text{ kN}$$

$$f_{1y} = \frac{-5(5)^2\,[7+2(2)]}{7^3} = -4.01\text{ kN}$$

$$m_2 = \frac{-5(5)\,(2)^2}{7^2} = -2.04\text{ kN}\cdot\text{m}$$

$$m_1 = \frac{5(5)^2\,(2)}{7^2} = 5.10\text{ kN}\cdot\text{m}$$

(g) Load case 6

$$w = 4\ \frac{\text{kN}}{\text{m}},\ L = 6\text{ m}$$

$$f_{1y} = f_{2y} = \frac{-4(6)}{4} = -6\text{ kN}$$

$$m_1 = -\,m_2 = \frac{-5(4)(6)^2}{96} = -7.5\text{ kN}\cdot\text{m}$$

(h) Load case 4

$$w = 5\ \frac{\text{kN}}{\text{m}},\ L = 4\text{ m}$$

$$f_{1y} = f_{2y} = \frac{-5(4)}{2} = -10\text{ kN}$$

$$m_1 = -\,m_2 = \frac{-5(4)^2}{12} = -6.67\text{ kN}\cdot\text{m}$$