SCHAUM'S OUTLINE OF

THEORY AND PROBLEMS

OF

STRENGTH OF MATERIALS

Fourth Edition

•

WILLIAM A. NASH, Ph.D.
Professor of Civil Engineering
University of Massachusetts

•

SCHAUM'S OUTLINE SERIES
McGRAW-HILL

New York San Francisco Washington, D.C. Auckland Bogotá Caracas Lisbon London Madrid Mexico City Milan Montreal New Delhi San Juan Singapore Sydney Tokyo Toronto

This book is dedicated by the author to his parents, William A. Nash and Rose Nash, for their years of patient guidance toward his career.

WILLIAM A. NASH is Professor of Civil Engineering at the University of Massachusetts, Amherst. He received his B.S. and M.S. from the Illinois Institute of Technology and his Ph.D. from the University of Michigan. He served as Structural Research Engineer at the David Taylor Research Center of the Navy Department in Washington, D.C., and was a faculty member at the University of Florida for 13 years prior to his present affiliation. He has had extensive consulting experience with the U.S. Air Force, the U.S. Navy Department, Lockheed Aerospace Corp., and the General Electric Co. His special areas of interest are structural dynamics and structural stability.

Schaum's Outline of Theory and Problems of
STRENGTH OF MATERIALS

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 PRS PRS 9 0 2 1 0 9 8

ISBN 0-07-046617-3

Sponsoring Editor: Barbara Gilson
Production Supervisor: Pamela Pelton
Editing Supervisor: Maureen B. Walker
Project Supervision: Keyword Publishing Services Ltd

Library of Congress Cataloging-in-Publication Data

Nash, William A.
Schaum's outline of theory and problems of strength of materials /
William A. Nash. -- 4th ed.
p. cm. -- (Schaum's outline series)
includes index.
ISBN 0-07-046617-3
1. Strength of materials--Problems, exercises, etc. 2. Strength of materials--Outlines, syllabi, etc. I. Title.
TA407.4.N37 1998
620.1'12--dc21 98-28410
CIP

McGraw-Hill
A Division of The McGraw-Hill Companies

Preface

This Fourth Edition of *Schaum's Outline of Theory and Problems of Strength of Materials* adheres to the basic plan of the third edition but has several distinctive features.

1. Problem solutions are given in both SI (metric) and USCS units.
2. About fourteen computer programs are offered in either FORTRAN or BASIC for those types of problems that otherwise involve long, tedious computation. For example, beam stresses and deflections are readily determined by the programs given. All of these programs may be utilized on most PC systems with only modest changes in input format.
3. The presentation passes from elementary to more complex cases for a variety of structural elements subject to practical conditions of loading and support. Generalized treatments, such as elastic energy approaches, as well as plastic analysis and design are treated in detail.

The author is much indebted to Kathleen Derwin for preparation of most of the computer programs as well as careful checking of some of the new problems.

WILLIAM A. NASH

Contents

Chapter 1

Tension and Compression

INTERNAL EFFECTS OF FORCES

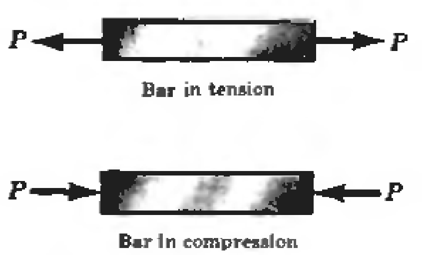
In this book we shall be concerned with what might be called the *internal effects* of forces acting on a body. The bodies themselves will no longer be considered to be perfectly rigid as was assumed in statics; instead, the calculation of the deformations of various bodies under a variety of loads will be one of our primary concerns in the study of strength of materials.

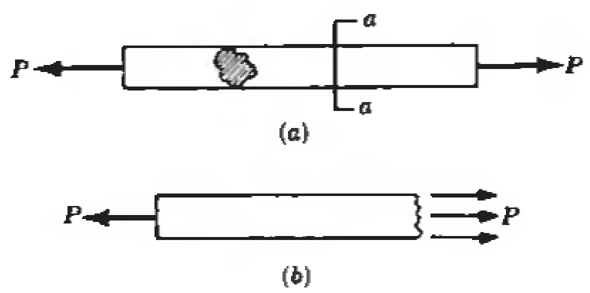
Axially Loaded Bar

The simplest case to consider at the start is that of an initially straight metal bar of constant cross section, loaded at its ends by a pair of oppositely directed collinear forces coinciding with the longitudinal axis of the bar and acting through the centroid of each cross section. For static equilibrium the magnitudes of the forces must be equal. If the forces are directed away from the bar, the bar is said to be in *tension*; if they are directed toward the bar, a state of *compression* exists. These two conditions are illustrated in Fig. 1-1.

Under the action of this pair of applied forces, internal resisting forces are set up within the bar and their characteristics may be studied by imagining a plane to be passed through the bar anywhere along its length and oriented perpendicular to the longitudinal axis of the bar. Such a plane is designated as *a-a* in Fig. 1-2(*a*). If for purposes of analysis the portion of the bar to the right of this plane is considered to be removed, as in Fig. 1-2(*b*), then it must be replaced by whatever effect it exerts upon the left portion. By this technique of introducing a cutting plane, the originally internal forces now become external with respect to the remaining portion of the body. For equilibrium of the portion to the left this "effect" must be a horizontal force of magnitude P. However, this force P acting normal to the cross-section *a-a* is actually the resultant of distributed forces acting over this cross section in a direction normal to it.

At this point it is necessary to make some assumption regarding the manner of variation of these distributed forces, and since the applied force P acts through the centroid it is commonly assumed that they are uniform across the cross section.





Normal Stress

Instead of speaking of the internal force acting on some small element of area, it is better for comparative purposes to treat the normal force acting over a *unit* area of the cross section. The intensity of normal force per unit area is termed the normal *stress* and is expressed in units of force per unit area, e.g., lb/in^2 or N/m^2. If the forces applied to the ends of the bar are such that the bar is in tension, then *tensile stresses* are set up in the bar; if the bar is in compression we have *compressive stresses*. It is essential that the line of action of the applied end forces pass through the centroid of each cross section of the bar.

Test Specimens

The axial loading shown in Fig. 1-2(*a*) occurs frequently in structural and machine design problems. To simulate this loading in the laboratory, a test specimen is held in the grips of either an electrically driven gear-type testing machine or a hydraulic machine. Both of these machines are commonly used in materials testing laboratories for applying axial tension.

In an effort to standardize materials testing techniques the American Society for Testing Materials (ASTM) has issued specifications that are in common use. Only two of these will be mentioned here, one for metal plates thicker than $\frac{3}{16}$ in (4.76 mm) and appearing as in Fig. 1-3, the other for metals over 1.5 in (38 mm) thick and having the appearance shown in Fig. 1-4. As may be seen from these figures, the central portion of the specimen is somewhat smaller than the end regions so that failure will not take place in the gripped portion. The rounded fillets shown are provided so that no stress concentrations will arise at the transition between the two lateral dimensions. The standard gage length over which elongations are measured is 8 in (203 mm) for the specimen shown in Fig. 1-3 and 2 in (57 mm) for that shown in Fig. 1-4.

The elongations are measured by either mechanical or optical extensometers or by cementing an electric resistance-type strain gage to the surface of the material. This resistance strain gage consists of a number of very fine wires oriented in the axial direction of the bar. As the bar elongates, the electrical resistance of the wires changes and this change of resistance is detected on a Wheatstone bridge and interpreted as elongation.

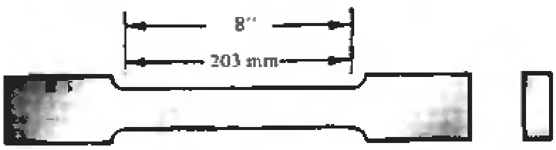


Fig. 1-3

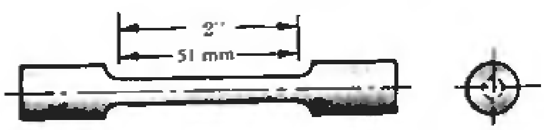


Fig. 1-4

Normal Strain

Let us suppose that one of these tension specimens has been placed in a tension-compression testing machine and tensile forces gradually applied to the ends. The elongation over the gage length may be measured as indicated above for any predetermined increments of the axial load. From these values the elongation per unit length, which is termed *normal strain* and denoted by ϵ, may be found by dividing the total elongation Δ by the gage length L, that is, $\epsilon = \Delta/L$. The strain is usually expressed in units of inches per inch or meters per meter and consequently is dimensionless.

Stress-Strain Curve

As the axial load is gradually increased in increments, the total elongation over the gage length is measured at each increment of load and this is continued until fracture of the specimen takes place. Knowing the original cross-sectional area of the test specimen the *normal stress*, denoted by σ, may be obtained for any value of the axial load by the use of the relation

$$\sigma = \frac{P}{A}$$

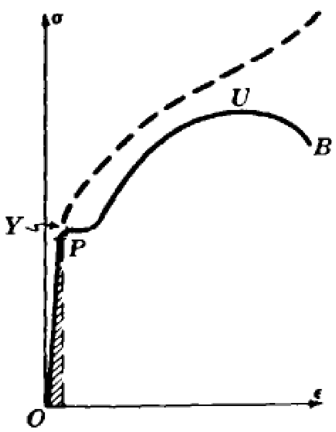


Fig. 1-5

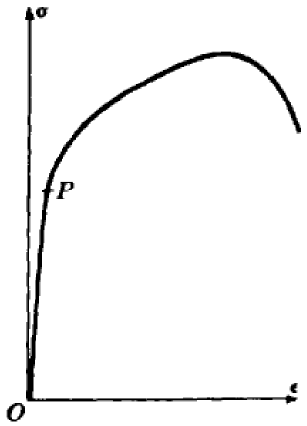


Fig. 1-6

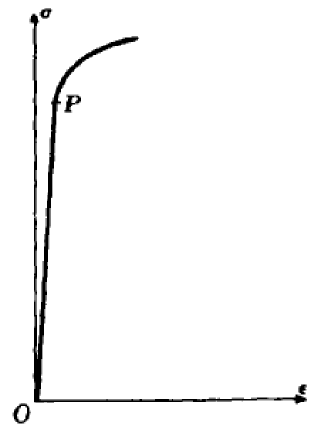


Fig. 1-7

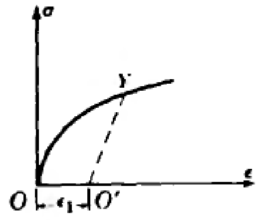


Fig. 1-8

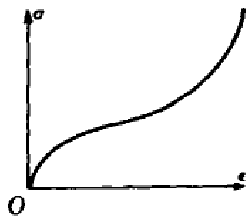


Fig. 1-9

where P denotes the axial load in pounds or Newtons and A the original cross-sectional area. Having obtained numerous pairs of values of normal stress σ and normal strain ϵ, the experimental data may be plotted with these quantities considered as ordinate and abscissa, respectively. This is the *stress-strain curve* or *diagram* of the material for this type of loading. Stress-strain diagrams assume widely differing forms for various materials. Figure 1-5 is the stress-strain diagram for a medium-carbon structural steel, Fig. 1-6 is for an alloy steel, and Fig. 1-7 is for hard steels and certain nonferrous alloys. For nonferrous alloys and cast iron the diagram has the form indicated in Fig. 1-8, while for rubber the plot of Fig. 1-9 is typical.

Ductile and Brittle Materials

Metallic engineering materials are commonly classed as either *ductile* or *brittle* materials. A *ductile material* is one having a relatively large tensile strain up to the point of rupture (for example, structural steel or aluminum) whereas a *brittle material* has a relatively small strain up to this same point. An arbitrary strain of 0.05 in/in (or mm/mm) is frequently taken as the dividing line between these two classes of materials. Cast iron and concrete are examples of brittle materials.

Hooke's Law

For any material having a stress-strain curve of the form shown in Fig. 1-5, 1-6, or 1-7, it is evident that the relation between stress and strain is linear for comparatively small values of the strain. This linear relation between elongation and the axial force causing it (since these quantities respectively differ from the strain or the stress only by a constant factor) was first noticed by Sir Robert Hooke in 1678 and is called *Hooke's law*. To describe this initial linear range of action of the material we may consequently write

$$\sigma = E\epsilon$$

where E denotes the slope of the straight-line portion OP of each of the curves in Figs. 1-5, 1-6, and 1-7.

Modulus of Elasticity

The quantity E, i.e., the ratio of the unit stress to the unit strain, is the *modulus of elasticity* of the material in tension, or, as it is often called, *Young's modulus*.* Values of E for various engineering materials are tabulated in handbooks. A table for common materials appears at the end of this chapter. Since the unit strain ϵ is a pure number (being a ratio of two lengths) it is evident that E has the same units as does the stress, for example lb/in^2, or N/m^2. For many common engineering materials the modulus of elasticity in compression is very nearly equal to that found in tension. *It is to be carefully noted that the behavior of materials under load as discussed in this book is restricted (unless otherwise stated) to the linear region of the stress-strain curve.*

MECHANICAL PROPERTIES OF MATERIALS

The stress-strain curve shown in Fig. 1-5 may be used to characterize several strength characteristics of the material. They are:

Proportional Limit

The ordinate of the point P is known as the *proportional limit*, i.e., the maximum stress that may be developed during a simple tension test such that the stress is a linear function of strain. For a material having the stress-strain curve shown in Fig. 1-8 there is no proportional limit.

Elastic Limit

The ordinate of a point almost coincident with P is known as the *elastic limit*, i.e., the maximum stress that may be developed during a simple tension test such that there is no permanent or residual deformation when the load is entirely removed. For many materials the numerical values of the elastic limit and the proportional limit are almost identical and the terms are sometimes used synonymously. In those cases where the distinction between the two values is evident the elastic limit is almost always greater than the proportional limit.

Elastic and Plastic Ranges

That region of the stress-strain curve extending from the origin to the proportional limit is called the *elastic range*; that region of the stress-strain curve extending from the proportional limit to the point of rupture is called the *plastic range*.

Yield Point

The ordinate of the point Y in Fig. 1-5, denoted by σ_{yp}, at which there is an increase in strain with no increase in stress is known as the *yield point* of the material. After loading has progressed to the point Y, yielding is said to take place. Some materials exhibit two points on the stress-strain curve at which there is an increase of strain without an increase of stress. These are called *upper* and *lower yield points*.

---

*Thomas Young was an English physicist, born in 1773, who worked in a number of areas such as mechanics, light, and heat. Before Young, historians had been unable to decipher stone tablets cut or painted in the characters (hieroglyphics) employed by Egyptians several thousand years B.C. Young, a master of eleven languages, was the first to successfully decipher any of the characters based upon study of the famous Rosetta stone found in 1799. His work, followed by that of Champollion in France, led to complete decipherment of the ancient language.

Ultimate Strength or Tensile Strength

The ordinate of the point U in Fig. 1-5, the maximum ordinate to the curve, is known either as the *ultimate strength* or the *tensile strength* of the material.

Breaking Strength

The ordinate of the point B in Fig. 1-5 is called the *breaking strength* of the material.

Modnlus of Resilience

The work done on a unit volume of material, as a simple tensile force is gradually increased from zero to such a value that the proportional limit of the material is reached, is defined as the *modulus of resilience.* This may be calculated as the area under the stress-strain curve from the origin up to the proportional limit and is represented as the shaded area in Fig. 1-5. The units of this quantity are $\text{in} \cdot \text{lb/in}^3$, or $\text{N} \cdot \text{m/m}^3$ in the SI system. Thus, resilience of a material is its ability to absorb energy in the elastic range.

Modulus of Toughness

The work done on a unit volume of material as a simple tensile force is gradually increased from zero to the value causing rupture is defined as the *modulus of toughness.* This may be calculated as the entire area under the stress-strain curve from the origin to rupture. Toughness of a material is its ability to absorb energy in the plastic range of the material.

Percentage Reduction in Area

The decrease in cross-sectional area from the original area upon fracture divided by the *original* area and multiplied by 100 is termed *percentage reduction in area.* It is to be noted that when tensile forces act upon a bar, the cross-sectional area decreases, but calculations for the normal stress are usually made upon the basis of the original area. This is the case for the curve shown in Fig. 1-5. As the strains become increasingly larger it is more important to consider the instantaneous values of the cross-sectional area (which are decreasing), and if this is done the *true* stess-strain curve is obtained. Such a curve has the appearance shown by the dashed line in Fig. 1-5.

Percentage Elongation

The increase in length (of the gage length) after fracture divided by the initial length and multiplied by 100 is the *percentage elongation.* Both the percentage reduction in area and the percentage elongation are considered to be measures of the *ductility* of a material.

Working Stress

The above-mentioned strength characteristics may be used to select a *working stress.* Frequently such a stress is determined merely by dividing either the stress at yield or the ultimate stress by a number termed the *safety factor.* Selection of the safety factor is based upon the designer's judgment and experience. Specific safety factors are sometimes specified in design codes.

Strain Hardening

If a ductile material can be stressed considerably beyond the yield point without failure, it is said to *strain-harden.* This is true of many structural metals.

The nonlinear stress-strain curve of a brittle material, shown in Fig. 1-8, characterizes several other strength measures that cannot be introduced if the stress-strain curve has a linear region. They are:

Yield Strength

The ordinate to the stress-strain curve such that the material has a predetermined permanent deformation or "set" when the load is removed is called the *yield strength* of the material. The permanent set is often taken to be either 0.002 or 0.0035 in per in or mm per mm. These values are of course arbitrary. In Fig. 1-8 a set ϵ_1 is denoted on the strain axis and the line $O'Y$ is drawn parallel to the initial tangent to the curve. The ordinate of Y represents the yield strength of the material, sometimes called the *proof stress*.

Tangent Modulus

The rate of change of stress with respect to strain is known as the *tangent modulus* of the material. It is essentially an instantaneous modulus given by $E_t = d\sigma/d\epsilon$.

Coefficient of Linear Expansion

This is defined as the change of length per unit length of a straight bar subject to a temperature change of one degree and is usually denoted by α. The value of this coefficient is independent of the unit of length but does depend upon the temperature scale used. For example, from Table 1-1 at the end of this chapter the coefficient for steel is 6.5×10^{-6}/°F but 12×10^{-6}/°C. Temperature changes in a structure give rise to internal stresses, just as do applied loads.

Poisson's Ratio

When a bar is subject to a simple tensile loading there is an increase in length of the bar in the direction of the load, but a decrease in the lateral dimensions perpendicular to the load. The ratio of the strain in the lateral direction to that in the axial direction is defined as *Poisson's ratio*. It is denoted in this book by the Greek letter μ. For most metals it lies in the range 0.25 to 0.35. For cork, μ is very nearly zero. One new and unique material, so far of interest only in laboratory investigations, actually has a *negative* value of Poisson's ratio; i.e., if stretched in one direction it *expands* in every other direction. See Problems 1.19 through 1.24.

General Form of Hooke's Law

The simple form of Hooke's law has been given for axial tension when the loading is entirely along one straight line, i.e., uniaxial. Only the deformation in the direction of the load was considered and it was given by

$$\epsilon = \frac{\sigma}{E}$$

In the more general case an element of material is subject to three mutually perpendicular normal stresses σ_x, σ_y, σ_z, which are accompanied by the strains ϵ_x, ϵ_y, ϵ_z, respectively. By superposing the strain components arising from lateral contraction due to Poisson's effect upon the direct strains we obtain the general statement of Hooke's law:

$$\epsilon_x = \frac{1}{E}[\sigma_x - \mu(\sigma_y + \sigma_z)] \qquad \epsilon_y = \frac{1}{E}[\sigma_y - \mu(\sigma_x + \sigma_z)] \qquad \epsilon_z = \frac{1}{E}[\sigma_z - \mu(\sigma_x + \sigma_y)]$$

See Problems 1.20 and 1.23.

Specific Strength

This quantity is defined as the ratio of the ultimate (or tensile) strength to specific weight, i.e., weight per unit volume. Thus, in the USCS system, we have

$$\frac{\text{lb}}{\text{in}^2} \Big/ \frac{\text{lb}}{\text{in}^3} = \text{in}$$

and, in the SI system, we have

$$\frac{\text{N}}{\text{m}^2} \Big/ \frac{\text{N}}{\text{m}^3} = \text{m}$$

so that in either system specific strength has units of *length*. This parameter is useful for comparisons of material efficiencies. See Problem 1.25.

Specific Modulus

This quantity is defined as the ratio of the Young's modulus to specific weight. Substitution of units indicates that specific modulus has physical units of length in either the USCS or SI systems. See Problem 1.25.

DYNAMIC EFFECTS

In determination of mechanical properties of a material through a tension or compression test, the rate at which loading is applied sometimes has a significant influence upon the results. In general, ductile materials exhibit the greatest sensitivity to variations in loading rate, whereas the effect of testing speed on brittle materials, such as cast iron, has been found to be negligible. In the case of mild steel, a ductile material, it has been found that the yield point may be increased as much as 170 percent by extremely rapid application of axial force. It is of interest to note, however, that for this case the total elongation remains unchanged from that found for slower loadings.

CLASSIFICATION OF MATERIALS

Up to now, this entire discussion has been based upon the assumptions that two characteristics prevail in the material. They are that we have

A *homogeneous material*, one with the same elastic properties (E, μ) at all points in the body

An *isotropic material*, one having the same elastic properties in all directions at any one point of the body.

Not all materials are isotropic. If a material does not possess any kind of elastic symmetry it is called *anisotropic*, or sometimes *aeolotropic*. Instead of having two independent elastic constants (E, μ) as an isotropic material does, such a substance has 21 elastic constants. If the material has three mutually perpendicular planes of elastic symmetry it is said to be *orthotropic*. The number of independent constants is nine in this case. Modern filamentary reinforced *composite materials*, such as shown in Fig. 1-10, are excellent examples of anisotropic substances.

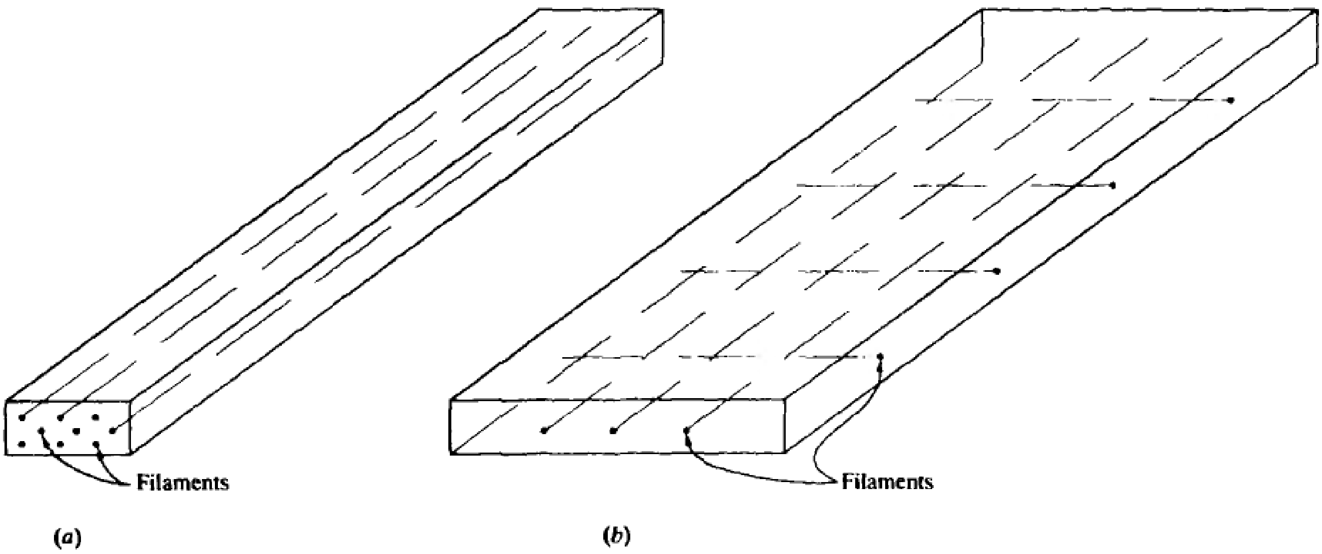


Fig. 1-10 (*a*) Epoxy bar reinforced by fine filaments in one direction; (*b*) epoxy plate reinforced by fine filaments in two directions.

ELASTIC VERSUS PLASTIC ANALYSIS

Stresses and deformations in the plastic range of action of a material are frequently permitted in certain structures. Some building codes allow particular structural members to undergo plastic deformation, and certain components of aircraft and missile structures are deliberately designed to act in the plastic range so as to achieve weight savings. Furthermore, many metal-forming processes involve plastic action of the material. For small plastic strains of low- and medium-carbon structural steels the stress-strain curve of Fig. 1-11 is usually idealized by two straight lines, one with a slope of E, representing the elastic range, the other with zero slope representing the plastic range. This plot, shown in Fig. 1-11, represents a so-called *elastic, perfectly plastic material*. It takes no account of still larger plastic strains occurring in the strain-hardening region shown as the right portion of the stress-strain curve of Fig. 1-5. See Problem 1.26.

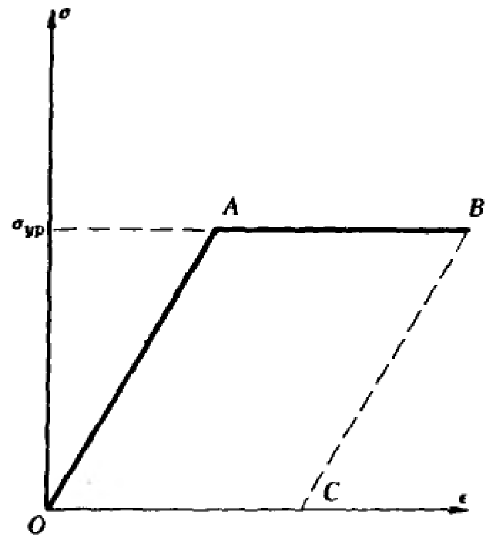


Fig. 1-11

If the load increases so as to bring about the strain corresponding to point B in Fig. 1-11, and then the load is removed, unloading takes place along the line BC so that complete removal of the load leaves a permanent "set" or elongation corresponding to the strain OC.

Solved Problems

1.1. In Fig. 1-12, determine the total elongation of an initially straight bar of length L, cross-sectional area A, and modulus of elasticity E if a tensile load P acts on the ends of the bar.

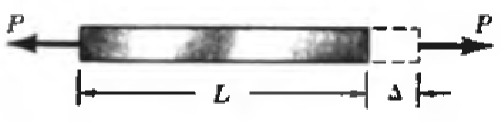


Fig. 1-12

The unit stress in the direction of the force P is merely the load divided by the cross-sectional area, that is, $\sigma = P/A$. Also the unit strain ϵ is given by the total elongation Δ divided by the original length, i.e., $\epsilon = \Delta/L$. By definition the modulus of elasticity E is the ratio of σ to ϵ, that is,

$$E = \frac{\sigma}{\epsilon} = \frac{P/A}{\Delta/L} = \frac{PL}{A\Delta} \qquad \text{or} \qquad \Delta = \frac{PL}{AE}$$

Note that Δ has the units of length, perhaps inches or meters.

1.2. A steel bar of cross section 500 mm^2 is acted upon by the forces shown in Fig. 1-13(a). Determine the total elongation of the bar. For steel, consider $E = 200$ GPa.

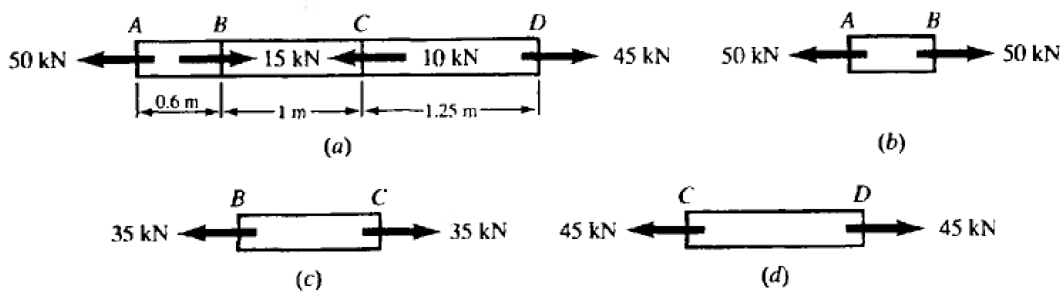


Fig. 1-13

The entire bar is in equilibrium, and hence all portions of it are also. The portion between A and B has a resultant force of 50 kN acting over every cross section and a free-body diagram of this 0.6-m length appears as in Fig. 1-13(b). The force at the right end of this segment must be 50 kN to maintain equilibrium with the applied load at A. The elongation of this portion is, from Problem 1.1:

$$\Delta_1 = \frac{(50{,}000\ \text{N})(0.6\ \text{m})}{(500 \times 10^{-6}\ \text{m}^2)(200 \times 10^9\ \text{N/m}^2)} = 0.0003\ \text{m}$$

The force acting in the segment between B and C is found by considering the algebraic sum of the forces to the left of any section between B and C, i.e., a resultant force of 35 kN acts to the left, so that a tensile force exists. The free-body diagram of the segment between B and C is shown in Fig. 1-13(c) and the elongation of it is

$$\Delta_2 = \frac{(35{,}000\ \text{N})(1\ \text{m})}{(500 \times 10^{-6}\ \text{m}^2)(200 \times 10^9\ \text{N/m}^2)} = 0.00035\ \text{m}$$

Similarly, the force acting over any cross section between C and D must be 45 kN to maintain equilibrium with the applied load at D. The elongation of CD is

$$\Delta_3 = \frac{(45{,}000\ \text{N})(1.25\ \text{m})}{(500 \times 10^{-6}\ \text{m}^2)(200 \times 10^9\ \text{N/m}^2)} = 0.00056\ \text{m}$$

The total elongation is

$$\Delta = \Delta_1 + \Delta_2 + \Delta_3 = 0.00121\ \text{m} \qquad \text{or} \qquad 1.21\ \text{mm}$$

1.3. The pinned members shown in Fig. 1-14(a) carry the loads P and $2P$. All bars have cross-sectional area A_1. Determine the stresses in bars AB and AF.

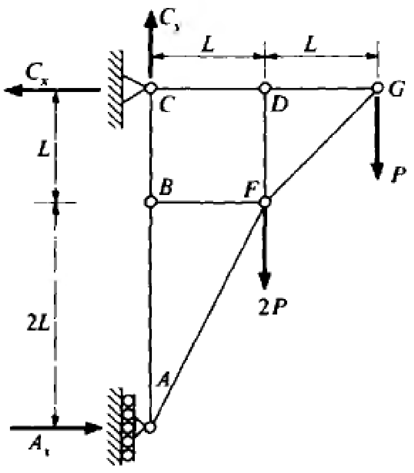


Fig. 1-14(a)

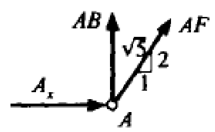


Fig. 1-14(b)

The reactions are indicated by C_x, C_y, and A_x. From statics we have

$$\Sigma M_c = -(2PL) - P(2L) + A_x(3L) = 0; \qquad A_x = \frac{4}{3}P$$

A free-body diagram of the pin at A is shown in Fig. 1.14(b). From statics:

$$\Sigma F_x = \frac{4P}{3} + \frac{1}{\sqrt{5}}(AF) = 0; \qquad AF = -\frac{4P\sqrt{5}}{3}$$

$$\Sigma F_y = (AB) + \frac{2}{\sqrt{5}}(AF) = 0; \qquad AB = -\frac{8}{3}P$$

The bar stresses are

$$\sigma_{AF} = -\frac{4P\sqrt{5}}{3A}; \qquad \sigma_{AB} = -\frac{8P}{3A}$$

1.4. A component of a power generator consists of a torus supported by six tie rods from an overhead central point as shown in Fig. 1-15. The weight of the torus is 2000 N per meter of circumferential length. The point of attachment A is 1.25 m above the plane of the torus. The radius of the middle line of the torus is 0.5 m. Each tie rod has a cross-sectional area of 25 mm^2. Determine the vertical displacement of the torus due to its own weight.

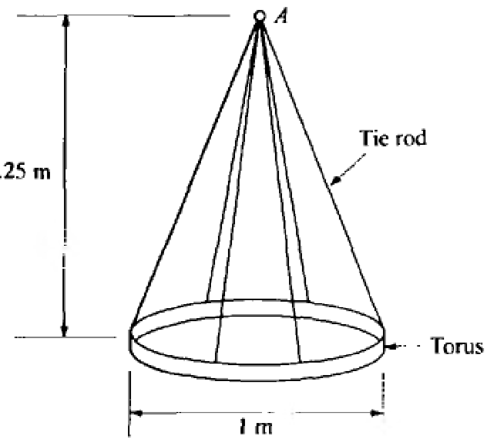


Fig. 1-15

A free-body diagram of the torus appears in Fig. 1-16 where T denotes the tensile force in each rod. Summing forces vertically:

$$6T\left(\frac{1.25}{1.34}\right) - \left(2000\frac{\text{N}}{\text{m}}\right)2\pi(0.5\text{ m}) = 0$$

$$T = 1120\text{ N}$$

Let us examine the deformation of a typical tie rod, such as AB. Figure 1-17 shows how AB elongates an amount BB' given by

$$\Delta = BB' = \frac{TL}{AE} = \frac{(1120\text{ N})(1.34\text{ m})}{(25\text{ mm}^2)\left(\dfrac{\text{m}}{10^3\text{ mm}}\right)^2\left(200\times 10^9\dfrac{\text{N}}{\text{m}^2}\right)} = 0.0003\text{ m} \quad \text{or} \quad 0.3\text{ mm}$$

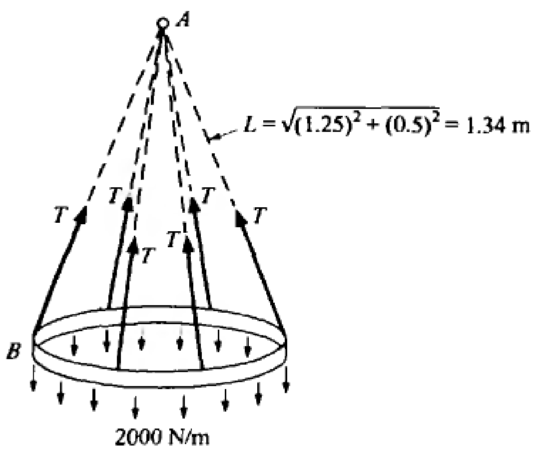


Fig 1-16

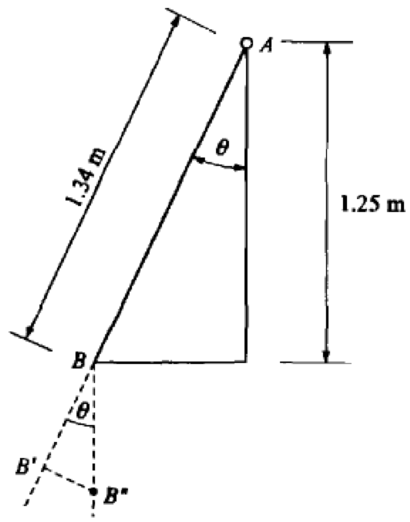


Fig. 1-17

Since B is on the torus, it (B) must move to B'' which is vertical below B. From Fig. 1-16 we have

$$BB'' = (0.3)\frac{1}{\cos\theta} = \frac{0.3}{\left(\frac{1.25}{1.34}\right)} = 0.32\text{ mm}$$

which is the vertical displacement of the rigid torus.

1.5. In Fig. 1-18, determine the total increase of length of a bar of constant cross section hanging vertically and subject to its own weight as the only load. The bar is initially straight.

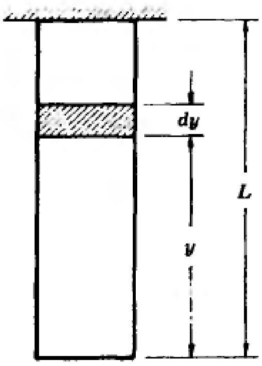


Fig. 1-18

The normal stress (tensile) over any horizontal cross section is caused by the weight of the material below that section. The elongation of the element of thickness dy shown is

$$d\Delta = \frac{Ay\gamma}{AE}dy$$

where A denotes the cross-sectional area of the bar and γ its specific weight (weight/unit volume). Integrating, the total elongation of the bar is

$$\Delta = \int_0^L \frac{Ay\gamma\,dy}{AE} = \frac{A\gamma}{AE}\frac{L^2}{2} = \frac{(A\gamma L)L}{2AE} = \frac{WL}{2AE}$$

where W denotes the total weight of the bar. Note that the total elongation produced by the weight of the bar is equal to that produced by a load of half its weight applied at the end.

1.6. In 1989, *Jason*, a research-type submersible with remote TV monitoring capabilities and weighing 35,200 N was lowered to a depth of 646 m in an effort to send back to the attending surface vessel photographs of a sunken Roman ship offshore from Italy. The submersible was lowered at the end of a hollow steel cable having an area of $452 \times 10^{-6}\text{ m}^2$ and $E = 200$ GPa. The central core of the cable contained the fiber-optic system for transmittal of photographic images to the surface ship. Determine the extension of the steel cable. Due to the small volume of the entire system buoyancy may be neglected, and the effect of the fiber optic cable on the extension is also negligible. (*Note: Jason* was the system that took the first photographs of the sunken *Titanic* in 1986.)

The total cable extension is the sum of the extensions due to (*a*) the weight of *Jason*, and (*b*) the weight of the steel cable. From Problem 1.1, we have for (*a*)

$$\Delta_1 = \frac{PL}{AE} = \frac{(35{,}200\ \mathrm{N})(646\ \mathrm{m})}{(452 \times 10^{-6}\ \mathrm{m}^2)(200 \times 10^9\ \mathrm{N/m}^2)} = 0.252\ \mathrm{m}$$

and from Problem 1.5, we have for (*b*)

$$\Delta_2 = \frac{WL}{2AE}$$

where W is the weight of the cable. W may be found as the volume of the cable

$$(452 \times 10^{-6}\ \mathrm{m}^2)(646\ \mathrm{m}) = 0.292\ \mathrm{m}^3$$

which must be multiplied by the weight of steel per unit volume which, from Table 1-1 at the end of the chapter is 77 kN/m³. Thus, the cable weight is

$$W = (0.292\ \mathrm{m}^3)(77\ \mathrm{kN/m}^3) = 22{,}484\ \mathrm{N}$$

so that the elongation due to the weight of the cable is

$$\Delta_2 = \frac{(22{,}484\ \mathrm{N})(646\ \mathrm{m})}{2(452 \times 10^{-6}\ \mathrm{m}^2)(200 \times 10^9\ \mathrm{N/m}^2)} = 0.080\ \mathrm{m}$$

The total elongation is the sum of the effects,

$$\Delta = \Delta_1 + \Delta_2 = 0.252 + 0.080 = 0.332\ \mathrm{m}$$

1.7. **Two prismatic bars are rigidly fastened together and support a vertical load of 10,000 lb, as shown in Fig. 1-19. The upper bar is steel having specific weight 0.283 lb/in³, length 35 ft, and cross-sectional area 10 in². The lower bar is brass having specific weight 0.300 lb/in³, length 20 ft, and cross-sectional area 8 in². For steel $E = 30 \times 10^6$ lb/in², for brass $E = 13 \times 10^6$ lb/in². Determine the maximum stress in each material.**

The maximum stress in the brass bar occurs just below the junction at section *B-B*. There, the vertical normal stress is caused by the combined effect of the load of 10,000 lb together with the weight of the entire brass bar below *B-B*.

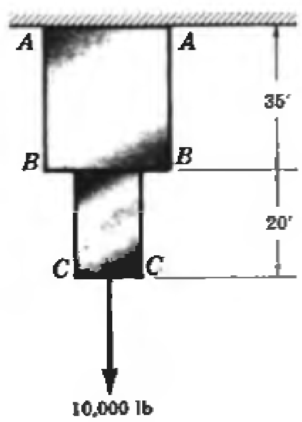


Fig. 1-19

The weight of the brass bar is $W_b = (20 \times 12)(8)(0.300) = 576$ lb.
The stress at this section is

$$\sigma = \frac{P}{A} = \frac{10{,}000 + 576}{8} = 1320 \text{ lb/in}^2$$

The maximum stress in the steel bar occurs at section A-A, the point of suspension, because there the entire weight of the steel and brass bars gives rise to normal stress, whereas at any lower section only a portion of the weight of the steel would be effective in causing stress.
The weight of the steel bar is $W_s = (35 \times 12)(10)(0.283) = 1185$ lb.
The stress across section A-A is

$$\sigma = \frac{P}{A} = \frac{10{,}000 + 576 + 1185}{10} = 1180 \text{ lb/in}^2$$

1.8. A solid truncated conical bar of circular cross section tapers uniformly from a diameter d at its small end to D at the large end. The length of the bar is L. Determine the elongation due to an axial force P applied at each end. See Fig. 1-20.

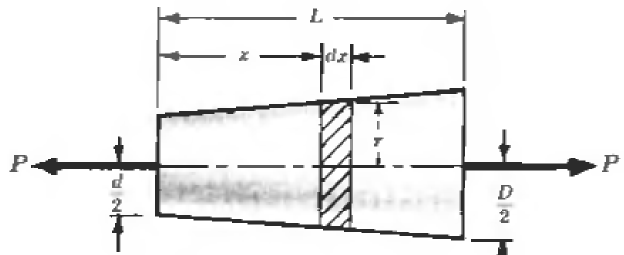


Fig. 1-20

The coordinate x describes the distance from the small end of a disc-like element of thickness dx. The radius of this small element is readily found by similar triangles:

$$r = \frac{d}{2} + \frac{x}{L}\left(\frac{D-d}{2}\right)$$

The elongation of this disc-like element may be found by applying the formula for extension due to axial loading, $\Delta = PL/AE$. For the element, this expression becomes

$$d\Delta = \frac{P\,dx}{\pi\left[\dfrac{d}{2} + \dfrac{x}{L}\left(\dfrac{D-d}{2}\right)\right]^2 E}$$

The extension of the entire bar is obtained by summing the elongations of all such elements over the bar. This is of course done by integrating. If Δ denotes the elongation of the entire bar,

$$\Delta = \int_0^L d\Delta = \int_0^L \frac{4P\,dx}{\pi[d + (x/L)(D-d)]^2 E} = \frac{4PL}{\pi DdE}$$

1.9. Two solid circular cross-section bars, one titanium and the other steel, each in the form of a truncated cone, are joined as shown in Fig. 1-21(a) and attached to a rigid vertical wall at the left. The system is subject to a concentric axial tensile force of 500 kN at the right end, together with an axisymmetric ring-type load applied at the junction of the bars as shown and having a horizontal resultant of 1000 kN. Determine the change of length of the system. For titanium, $E = 110\,\text{GPa}$, and for steel, $E = 200\,\text{GPa}$.

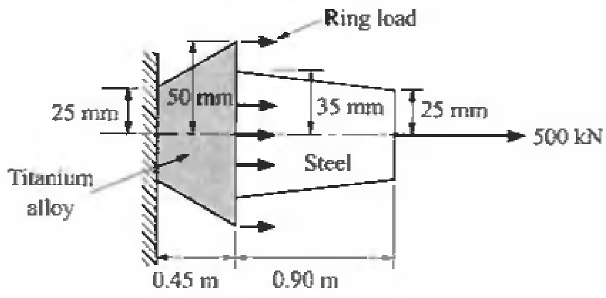


Fig. 1-21(*a*)

A free-body diagram of the system appears as shown in Fig. 1-21(*b*)

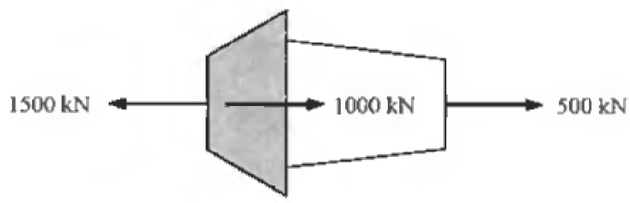


Fig. 1-21(*b*)

and a free-body diagram of each bar is shown in Fig. 1-21(*c*).

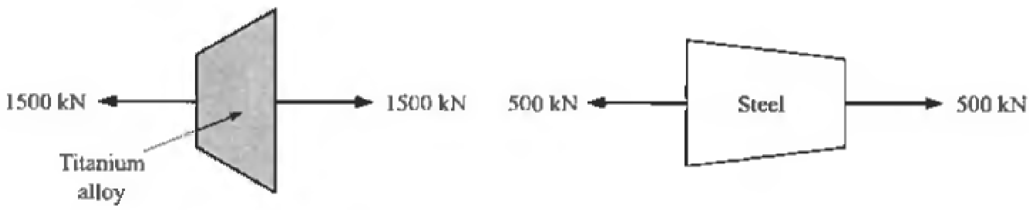


Fig. 1-21(*c*)

We may now apply the result of Problem 1.8 to each bar and obtain

$$\Delta_{Ti} = \frac{4(1{,}500{,}000\ \text{N})\,(0.45\ \text{m})}{\pi(0.10\ \text{m})\,(0.05\ \text{m})\,(110 \times 10^9\ \text{N/m}^2)} = 0.00156\ \text{m}$$

$$\Delta_{ST} = \frac{4(500{,}000\ \text{N})\,(0.90\ \text{m})}{\pi(0.07\ \text{m})\,(0.05\ \text{m})\,(200 \times 10^9\ \text{N/m}^2)} = 0.00082$$

Using superposition,

$$\Delta = \Delta_{ST} + \Delta_{Ti} = 0.00238\ \text{m} \qquad \text{or} \qquad 2.38\ \text{mm}$$

1.10. A large-scale pumping system to lift water consists of a pump of weight W in a circular cylindrical housing (with vertical axis) suspended from an axisymmetric thick-walled tube of variable radial thickness [see Fig. 1-22(*a*)]. Find the variation in outer radius R along the height so that the normal (vertical) stress in the tube is constant. The specific weight of the tube material is γ and the inner radius is R_i, which is constant.

We introduce the coordinate y, with origin at the top of the pump and extending positive upward as shown. Let us consider the free-body diagram of a ring-shaped element of the tube located a distance y above the top of the pump and of height dy as shown in Fig. 1-22(*b*).

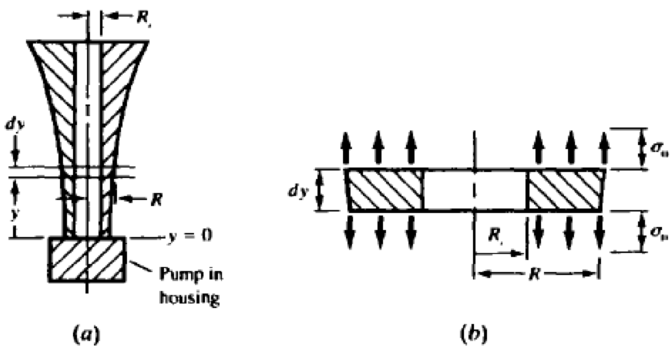


Fig. 1-22

The cross-sectional area of the lower surface of this ring is

$$A = \pi(R^2 - R_i^2) \tag{1}$$

and the area of the upper surface is $(A + dA)$. The weight of the material in the ring is $\gamma A\,dy$. For vertical equilibrium we have

$$\sigma_0(A + dA) - \sigma_0(A) - \gamma A(dy) = 0 \tag{2}$$

Simplifying:

$$\sigma_0(dA) = \gamma A(dy) \tag{3}$$

At the lower end ($y = 0$) of the tube, we denote the tube cross-section area by A_0. Integrating Eq. (3) between the lower end ($y = 0$) and the elevation y, we have

$$\int_{A_0}^{A} \frac{dA}{A} = \int_{y=0}^{y} \frac{\gamma}{\sigma_0}(dy) \tag{4}$$

Thus:

$$\ln\frac{A}{A_0} = \frac{\gamma}{\sigma_0}(y) \tag{5}$$

$$A = A_0 e^{\gamma y/\sigma_0} \tag{6}$$

At $y = 0$, we have for vertical equilibrium

$$\sigma_0 = \frac{W}{A_0} \tag{7}$$

so from (*1*), (*6*), and (*7*) we have the radius at any elevation y as

$$R^2 = R_i^2 + \frac{W}{\pi\sigma_0} e^{\gamma y/\sigma_0} \tag{8}$$

1.11. The pin-connected framework shown in Fig. 1-23(*a*) consists of two identical upper rods AB and AC, two shorter, lower rods BD and DC, together with a rigid horizontal brace BC. All bars have cross-sectional area A and modulus of elasticity E. Determine the vertical displacement of point D due to the action of the vertical load P applied at D as well as the distributed load q per unit length.

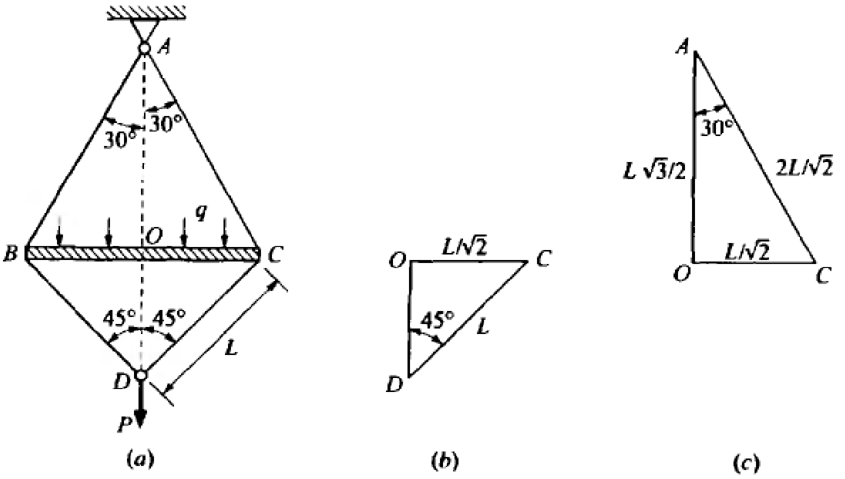


Fig. 1-23

Let us consider a horizontal cutting plane passed through the system slightly above BC. The free-body diagram is shown in Fig. 1-24 where F_2 represents the force in each of the bars AB and AC. From statics:

$$\Sigma F_v = -P - q\left(\frac{L}{\sqrt{2}}\right)(2) + 2F_2 \sin 60^\circ = 0$$

$$F_2 = \frac{P + \left(\frac{2}{\sqrt{2}}\right)qL}{\sqrt{3}} \tag{1}$$

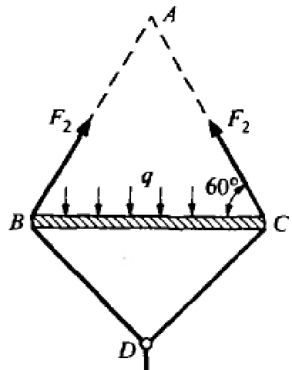


Fig. 1-24

To determine the dropping of bar BC we consider the deformation of bar AB, as shown in Fig. 1-25. The increase of length of AB is given by

$$\Delta_{AB} = \frac{F_2\left(\frac{2L}{\sqrt{2}}\right)}{AE}$$

and the vertical projection of this is

$$B'B'' = \frac{F_2\left(\frac{2L}{\sqrt{2}}\right)}{AE\cos 30^\circ}$$

Substituting F_2 from (*1*), this is

$$BB'' = \frac{4PL}{3\sqrt{2}AE} + \frac{4qL^2}{3AE} \qquad (2)$$

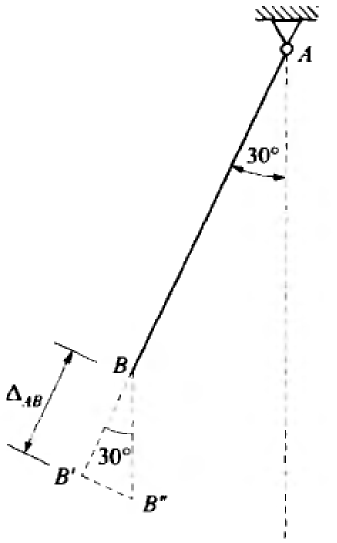


Fig. 1-25

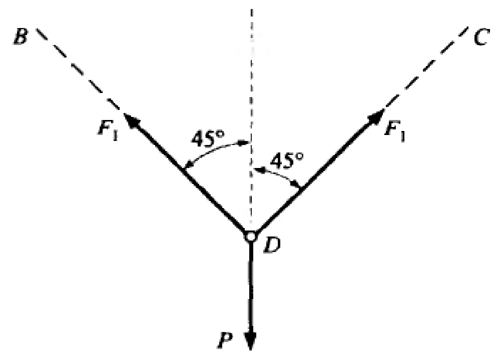


Fig. 1-26

Let us now consider another horizontal plane passed through the system just below BC. The free-body diagram is shown in Fig. 1-26 where F_1 represents the force in each of the bars BD and DC. From statics:

$$\Sigma F_v = -P + 2F_1\cos 45^\circ = 0$$

$$F_1 = \frac{P\sqrt{2}}{2} \qquad (3)$$

We must now determine the lowering of point D due to the action of the load P acting on bars BD and DC (see Fig. 1-27). The increase of length of BD is

$$\frac{F_1 L}{AE}$$

and the vertical projection of this is

$$\frac{F_1 L}{AE\cos 45^\circ}$$

Substituting (*3*), we find the vertical projection to be

$$\frac{PL}{AE} \qquad (4)$$

The actual drop of point D is the sum of (*2*) and (*4*):

$$\Delta_D = \frac{4PL}{3\sqrt{2}AE} + \frac{PL}{AE} + \frac{4qL^2}{3AE}$$

$$= 1.942\frac{PL}{AE} + 1.333\frac{qL^2}{AE} \qquad (5)$$

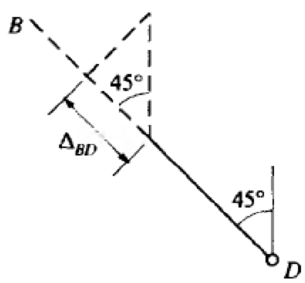


Fig. 1-27

1.12. Consider the system of two pinned end bars AB and CB (which is vertical) subject to the single horizontal force P applied at the pin B (see Fig. 1-28). Bar AB has area A_1, length L_1, and Young's modulus E_1. The corresponding quantities for bar CB are A_2, L_2, and E_2. Determine the horizontal and vertical components of displacement of pin B.

The free-body diagram of the pin is shown in Fig. 1-29(a) where F_1 and F_2 denote the forces bars AB and CB, respectively, exert on that pin. Each of these bar forces has been assumed to be positive in the direction shown; i.e., each bar is assumed to be in tension. Should the equilibrium equations indicate a negative value for either of these bar forces, that would signify that we have assumed the direction incorrectly and that the bar is in compression. Figures 1-29(b) and 1-29(c) indicate the effects that the pin at B exerts on bars AB and CB, respectively. These are of course equal and opposite to the values shown in Fig. 1-29(a).

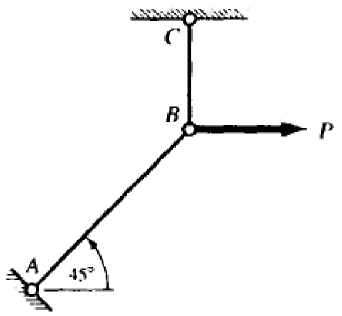


Fig. 1-28

For equilibrium of the pin at B, we have

$$\Sigma F_x = P - F_1 \cos 45^\circ = 0 \qquad (1)$$

$$\Sigma F_y = F_2 - F_1 \sin 45^\circ = 0 \qquad (2)$$

Solving,

$$F_1 = P\sqrt{2} \qquad F_2 = P \qquad (3)$$

which indicates tension in each bar. Let us think of temporarily unlocking the bars at B by removing pin B. Bar AB then stretches an amount BB' and bar CB stretches an amount BB'', as shown in Fig. 1-30. These extensions are found from Problem 1.1 to be

$$BB' = \frac{F_1 L_1}{A_1 E_1} = \frac{P\sqrt{2}L_1}{A_1 E_1} \qquad (4)$$

$$BB'' = \frac{F_2 L_2}{A_2 E_2} = \frac{PL_2}{A_2 E_2} \qquad (5)$$

However, the final position of the pin must be the same after the pin is considered to be reintroduced, so the bar AB must undergo a rigid-body rotation about pin A and bar CB must rotate about pin C. The point B' on AB (extended) must move along a circular arc with center at A, but for the very small deformations that we consider this arc may be replaced by a dotted straight line $B'B'''$ perpendicular to AB'. Likewise point B'' on CB (extended) must move along the horizontal dotted line $B''B'''$ as rotation takes place about pin C. The intersection of these two dotted lines at B''' must be the true, final position of the pin B.

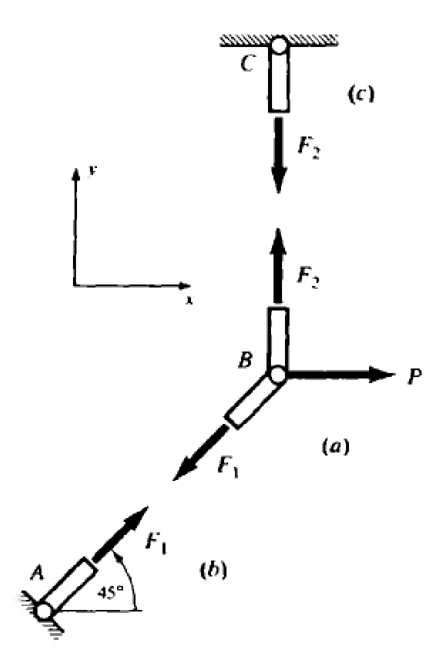


Fig. 1-29

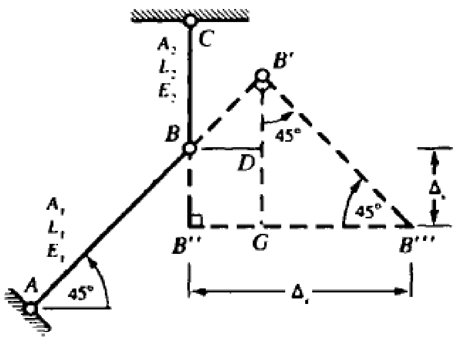


Fig. 1-30

From the geometry of Fig. 1-30 we have

$$BD = BB' \cos 45^\circ = \frac{\sqrt{2}PL_1}{A_1E_1} \cdot \frac{1}{\sqrt{2}} = \frac{PL_1}{A_1E_1} \tag{6}$$

$$B'D = BB' \sin 45^\circ = \frac{PL_1}{A_1E_1} \tag{7}$$

$$B'G = B'D + DG = \frac{PL_1}{A_1E_1} + \frac{PL_2}{A_2E_2} \tag{8}$$

$$GB''' = B'G \qquad (45^\circ \text{ triangle}) \tag{9}$$

$$B''B''' = BD + GB''' = \frac{PL_1}{A_1E_1} + \left(\frac{PL_1}{A_1E_1} + \frac{PL_2}{A_2E_2}\right) \tag{10}$$

$$= \frac{2PL_1}{A_1E_1} + \frac{PL_2}{A_2E_2} = \Delta_x \tag{11}$$

Finally, from Fig. 1-30 the vertical displacement of B is

$$\Delta_v = BB'' = \frac{PL_2}{A_2E_2} \tag{12}$$

1.13. In 1989 a new fiber-optic cable capable of handling 40,000 telephone calls simultaneously was laid under the Pacific Ocean from California to Japan, a distance of 13,300 km. The cable was unreeled from shipboard at a mean temperature of 22°C and dropped to the ocean floor having a mean temperature of 5°C. The coefficient of linear expansion of the cable is $75 \times 10^{-6}/°C$. Determine the length of cable that must be carried on the ship to span the 13,300 km.

The length of cable that must be carried on board ship consists of the 13,300 km plus an unknown length ΔL that will allow for contraction to a final length of 13,300 km when resting on the ocean floor. From the definition of the coefficient of thermal expansion (Chap. 1), we have

$$\Delta L = \alpha L(\Delta T)$$

$$\Delta L = (75 \times 10^{-6}/°C)[13{,}300\text{ km} + \underline{\Delta L}](22 - 5)°C \tag{a}$$

Solving, we find

$$\Delta L = 16.96\text{ km}$$

The percent change of length is thus

$$\frac{(16.96)(100)}{13{,}300 + 16.96} = 0.13\%$$

so that the underlined term in Eq. (*a*) is of minor consequence. Thus, the required length of cable at shipboard temperature is approximately 13,317 km.

1.14. **An elastic bar of variable cross section is loaded by axial tension or compression at its ends as shown in Fig. 1-31. The variation of cross-sectional dimension may be known either analytically or numerically along the dimension in the axial direction. Write a FORTRAN program for change of length of the bar for the cases of (*a*) a bar of solid circular cross section and (*b*) a flat slab of constant thickness *t* as shown in Figs. 1-31(*b*) and 1-31(*c*), respectively. The contour of the bar is described by the equation $y = Ae^{Bx}$, where x is the axial coordinate.**

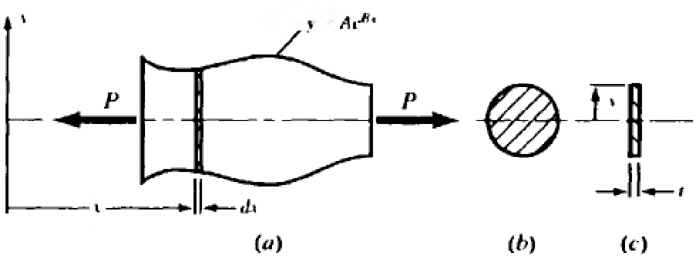


Fig. 1-31

The equation derived in Problem 1.1 may be applied to each subsegment of length dx as shown in Fig. 1-31(*a*). The cross-sectional area of each such subsegment is taken to be constant and we then apply the relation

$$\Delta = \frac{PL}{AE}$$

to this segment, where the length of the segment is dx and A is the cross-sectional area of the segment. Clearly A may be found if the equation $y = y(x)$ for the cross section is known, or, alternatively, measurements may be made at a number of stations along the length of the bar and the area found numerically at each such station.

This approach is represented by the following FORTRAN program which is self-prompting. Tensile loadings are regarded as positive and compressives as negative.

Note that in the equation describing the shape of the bar, $y = Ae^{Bx}$, e represents the base of natural logs, and A and B are parameters of the contour. Note in particular that this A is *not* cross-sectional area.

```
**********************************************************************
                      PROGRAM SLBTEN2(INPUT,OUTPUT)
**********************************************************************
*
*          AUTHOR: KATHLEEN DERWIN
*          DATE  : FEBRUARY 5, 1989
*
*   BRIEF DESCRIPTION:
*      THIS PROGRAM DETERMINES THE CHANGE OF LENGTH OF A BAR DUE
*   TO AXIAL TENSION OR COMPRESSION. THE BAR MAY BE A CONSTANT
*   THICKNESS, VARIABLE WIDTH RECTANGULAR SLAB, OR A SOLID CIRCULAR
*   ROD WITH VARIABLE DIAMETER. IN EITHER CASE THE SHAFT IS CENTRALLY
```

```
*     LOADED BY AN AXIAL FORCE.
*        THE VARYING WIDTH (OF THE SLAB) OR DIAMETER (OF THE ROD) MAY
*     BE DESCRIBED EITHER ANALYTICALLY AS  Y = A*E ^(B*X)  WHERE X IS THE
*     GEOMETRIC AXIS OF THE BAR, OR NUMERICALLY USING THE MAGNITUDE OF
*     Y AT EACH END OF  N SEGMENTS, MEANING N+1 VALUES.
*
*     INPUT:
*        THE USER IS PROMPTED FOR THE TOTAL BAR LENGTH, THE ELASTIC
*     MODULUS, AND THE AXIAL LOAD. THE USER IS THEN ASKED IF THE
*     BAR IS BOUNDED BY A KNOWN FUNCTION, AS WELL AS THE SHAPE OF ITS
*     X-SECTION. FOR THE CASE OF THE SLAB, THE UNIFORM THICKNESS IS
*     ALSO ASKED FOR... IF THE FUNCTION IS KNOWN, THE CONSTANTS ARE
*     THEN PROMPTED AND THE ENDPOINTS OF THE BAR ON THE X-AXIS INPUTTED;
*     ALTERNATELY, THE NUMBER OF SEGMENTS AND MEASURED HEIGHTS/DIAMETERS
*     MUST BE ENTERED.
*
*     OUTPUT:
*        THE TOTAL ELONGATION OF THE BAR IS DETERMINED AND PRINTED.
*
*     VARIABLES:
*        L,T,EM       ---  LENGTH,THICKNESS,ELASTIC MODULUS OF BAR
*        A,B          ---  CONSTANTS OF Y = A*E^(B*X)  GOVERNING BAR BOUNDA
*        X0,XN        ---  ENDPOINTS OF SHAFT ON X-AXIS
*        P            ---  CENTRALLY APPLIED AXIAL LOAD
*        AA(100)      ---  INDIVIDUAL SEGMENT HEIGHTS/DIAMETERS
*        AREA         ---  X-SECTIONAL AREA OF EACH SMALL INCREMENT
*        ANS          ---  DETERMINE IF USER HAS A KNOWN FUNCTION
*        TYPE         ---  DETERMINE BAR X-SECTION
*        DELTA        ---  UNIFORM BAR ELONGATION
*        LEN          ---  LENGTH OF INCREMENTAL ELEMENT
*
***********************************************************************
***********************************************************************
*                           MAIN PROGRAM
***********************************************************************
***********************************************************************
*
*     VARIABLE DECLARATION
*
      REAL I,T,L,EM,A,B,X0,XN,P,DELTA,AA(100),AREA,LEN
      INTEGER ANS,TYPE,NUM,J
*
*          USER INPUT PROMPTS
*
      PRINT*,'ENTER THE TOTAL LENGTH OF THE BAR (IN M OR INCHES):'
      READ*,L
      PRINT*,'ENTER THE ELASTIC MODULUS (IN PASCALS OR PSI) :'
      READ*,EM
      PRINT*,'ENTER THE UNIFORM AXIAL LOAD (IN NEWTONS OR LBS) :'
      READ*,P
      PRINT*,'PLEASE DENOTE THE BAR X-SECTIONAL SHAPE:'
      PRINT*,'ENTER  1--SLAB  ;   2--CIRCULAR ROD'
      READ*,TYPE
*
*          IF A SLAB, PROMPT FOR ITS THICKNESS
*
      IF (TYPE.EQ.1) THEN
         PRINT*,'ENTER THE THICKNESS OF THE SLAB (IN M OR INCHES):'
         READ*,T
      ENDIF
      PRINT*,'DO YOU KNOW THE FUNCTION DESCRIBING THE BAR?'
      PRINT*,'ENTER  1--YES  ;   2--NO'
      READ*,ANS
*
*          IF ANS EQUALS ONE, THE USER KNOWS FUNCTION. PROMPT
*          FOR CONSTANTS AND ENDPOINTS.
*
      IF (ANS.EQ.1) THEN
```

```
         PRINT*,'F(X) = A*E^(B*X)'
         PRINT*,'ENTER A,B:'
         READ*,A,B
         PRINT*,'ENTER THE X-COORDINATE FOR BOTH ENDS OF THE BAR:'
         PRINT*,'(IN M OR INCHES):'
         READ*,X0,XN
*
      AREA = 0
      L=XN-X0
      LEN=L/50
      DO 20 I = X0,XN,LEN
         Y1=(A*(2.71828**(B*I)))*2
         Y2=(A*(2.71828**(B*(I + LEN))))*2
         Y=(Y1+Y2)/2
         IF(TYPE.EQ.1) THEN
            AREA=1/(Y*T) + AREA
         ELSE
            AREA=4/(3.14159*(Y**2)) + AREA
         ENDIF
   20 CONTINUE
*
*        IF ANS EQUALS TWO, THE USER DOES NOT KNOW FUNCTION.
*        PROMPT FOR NUMBER OF SEGMENTS AND MEASURED HEIGHTS/DIAMETERS.
*
      ELSE
         PRINT*,'ENTER THE NUMBER OF SECTIONS TO BE CALCULATED:'
         READ*,NUM
         IF(TYPE.EQ.1) THEN
            PRINT*,'ENTER THE HEIGHTS OF THE ENDS FOR SECTIONS 1 TO N:'
            PRINT*,'(IN M OR INCHES):'
         ELSE
            PRINT*,'ENTER THE DIAMETERS OF THE ENDS FOR SECTIONS 1 TO N:'
            PRINT*,'(IN M OR INCHES):'
         ENDIF
*
*        INPUT MEASURED HEIGHTS/DIAMETERS
*
         DO 30 J=1,NUM+1
            READ*,AA(J)
   30    CONTINUE
*
         AREA = 0
         LEN = L/NUM
         DO 40 J = 1,NUM+1
            Y=(AA(J)+AA(J+1))/2
            IF(TYPE.EQ.1) THEN
               AREA = 1/(Y*T) + AREA
            ELSE
               AREA = 4/(3.14159*(Y**2)) + AREA
            ENDIF
   40    CONTINUE
      ENDIF
*
*        DETERMINING THE ELONGATION OF THE LOADED BAR
*
      DELTA=(P*LEN*AREA)/EM
*
      PRINT 50,DELTA
*
   50 FORMAT(2X,'THE DEFORMATION OF THE BAR IS:',F8.5,' (M OR IN.)')
*
      STOP
      END
```

1.15. A bar of variable solid circular cross section is bounded by the curve $y = 8e^{-0.01x}$ and extends from $x = 0$ to $x = 180$ in. It is subject to an axial tensile load of 100,000 lb as shown in Fig. 1-32. The material is steel, for which $E = 30 \times 10^6$ lb/in^2. Use the FORTRAN program of Problem 1.14 to determine the elongation of the bar.

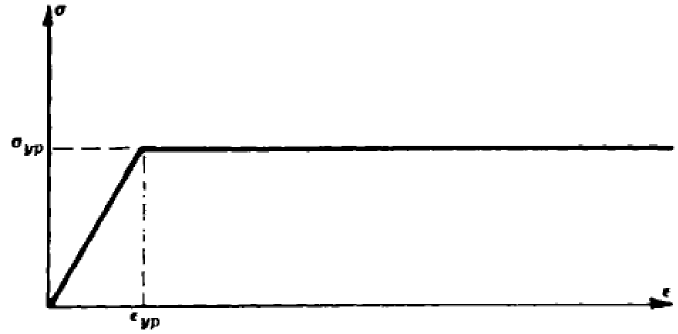


Fig. 1-32

Since the contour is bounded by the curve of the form $y = Ae^{Bx}$, we have $A = 8$ and $B = -0.01$. The bar extends from $x = 0$ to $x = 180$ in and entry of these data into the program of Problem 1.14 leads to an axial elongation of 0.03176 in.

```
run
 ENTER THE TOTAL LENGTH OF THE BAR (IN M OR INCHES):
? 180
 ENTER THE ELASTIC MODULUS (IN PASCALS OR PSI) :
? 30E+6
 ENTER THE UNIFORM AXIAL LOAD (IN NEWTONS OR LBS) :
? 100000
 PLEASE DENOTE THE BAR X-SECTIONAL SHAPE:
 ENTER  1--SLAB  :   2--CIRCULAR ROD
? 2
 DO YOU KNOW THE FUNCTION DESCRIBING THE BAR?
 ENTER  1--YES  ;   2--NO
? 1
 F(X) = A*E^(B*X)
 ENTER A,B:
? 8,-0.01
 ENTER THE X-COORDINATE FOR BOTH ENDS OF THE BAR:
 (IN M OR INCHES):
? 0,180
  THE DEFORMATION OF THE BAR IS:  .03176 (M OR IN)

SRU      0.804 UNTS.
```

1.16. A flat slab of variable depth is bounded by the curve $y = 0.25e^{0.025x}$ and extends from $x = 4$ m to $x = 10$ m as shown in Fig. 1-33. The slab is 10 mm thick and is subject to an axial tensile force of 385 kN. Use the FORTRAN program of Problem 1.14 to determine the elongation of the slab. Take $E = 200$ GPa.

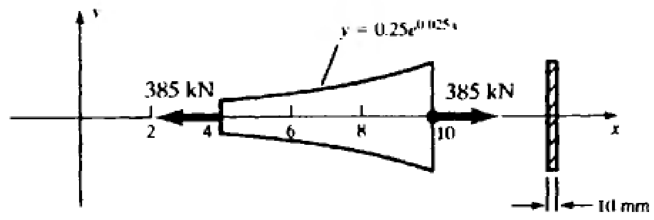


Fig. 1-33

To enter the program of Problem 1.14, we must set $A = 0.25$ and $B = 0.025$. The input data then appear as

```
run
 ENTER THE TOTAL LENGTH OF THE BAR (IN M OR INCHES):
? 6
 ENTER THE ELASTIC MODULUS (IN PASCALS OR PSI) :
? 200E+9
 ENTER THE UNIFORM AXIAL LOAD (IN NEWTONS OR LBS) :
? 385000
 PLEASE DENOTE THE BAR X-SECTIONAL SHAPE:
 ENTER  1--SLAB  :   2--CIRCULAR ROD
? 1
 ENTER THE THICKNESS OF THE SLAB (IN M OR INCHES):
? 0.01
 DO YOU KNOW THE FUNCTION DESCRIBING THE BAR?
 ENTER  1--YES  ;   2--NO
? 1
 F(X) = A*E^(B*X)
 ENTER A,B:
? 0.25,0.025
 ENTER THE X-COORDINATE FOR BOTH ENDS OF THE BAR:
 (IN M OR INCHES):
? 4,10
  THE DEFORMATION OF THE BAR IS:  .00198 (M OR IN)
```

The elongation of the bar is thus 0.00198 m or 1.98 mm.

1.17. Consider two thin rods or wires as shown in Fig. 1-34(a), which are pinned at A, B, and C and are initially horizontal and of length L when no load is applied. The weight of each wire is negligible. A force Q is then applied (gradually) at the point B. Determine the magnitude of Q so as to produce a prescribed vertical deflection δ of the point B.

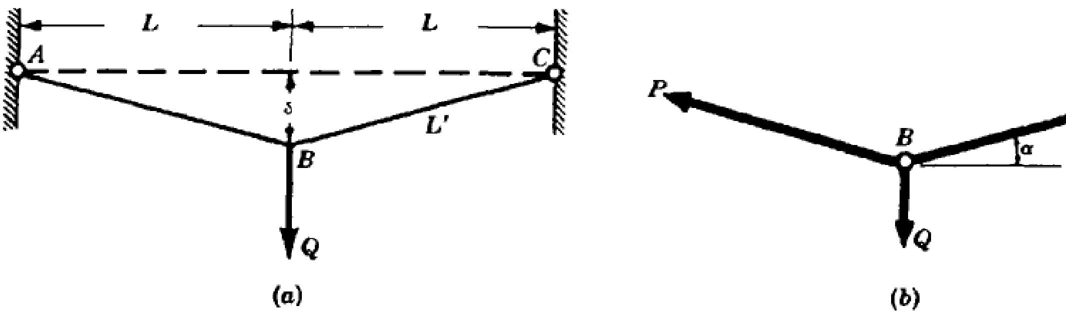


Fig. 1-34

This is an extremely interesting example of a system in which the elongations of all the individual members satisfy Hooke's law and yet for geometric reasons deflection is *not* proportional to force.

Each bar obeys the relation $\Delta = PL/AE$ where P is the axial force in each bar and Δ the axial elongation. Initially each bar is of length L and after the entire load Q has been applied the length is L'. Thus

$$L' - L = \frac{PL}{AE} \qquad (1)$$

The free-body diagram of the pin at B is shown in Fig. 1-34(b). From statics,

$$\Sigma F_v = 2P\sin\alpha - Q = 0 \qquad \text{or} \qquad Q = 2P\left(\frac{\delta}{L'}\right)$$

Using (1),
$$Q = 2\frac{(L'-L)AE}{L}\frac{\delta}{L'} = \frac{2\delta AE}{L}\left(1 - \frac{L}{L'}\right) \tag{2}$$

But
$$(L')^2 = L^2 + \delta^2 \tag{3}$$

Consequently
$$Q = \frac{2\delta AE}{L}\left(1 - \frac{L}{\sqrt{L^2+\delta^2}}\right) \tag{4}$$

Also, from the binomial theorem we have

$$\sqrt{L^2+\delta^2} = L\left(1+\frac{\delta^2}{L^2}\right)^{1/2} = L\left(1+\frac{1}{2}\frac{\delta^2}{L^2}+\cdots\right) \tag{5}$$

and thus
$$1 - \frac{L}{L\left(1+\frac{1}{2}\frac{\delta^2}{L^2}\right)} \approx 1 - \left(1 - \frac{1}{2}\frac{\delta^2}{L^2}\right) = \frac{1}{2}\frac{\delta^2}{L^2} \tag{6}$$

From this we have the approximate relation between force and displacement,

$$Q \approx \frac{2AE\delta}{L}\frac{\delta^2}{2L^2} = \frac{AE\delta^3}{L^3} \tag{7}$$

which corresponds to (4).

Thus the displacement δ is *not* proportional to the force Q even though Hooke's law holds for each bar individually. It is to be noted that Q becomes more nearly proportional to δ as δ becomes larger, assuming that Hooke's law still holds for the elongations of the bars. In this example superposition does *not* hold. The characteristic of this system is that the action of the external forces is *appreciably* affected by the small deformations which take place. In this event the stresses and displacements are not linear functions of the applied loads and superposition does not apply.

Summary: A material must follow Hooke's law if superposition is to apply. But this requirement alone is not sufficient. We must see whether or not the action of the applied loads is affected by small deformations of the structure. If the effect is substantial, superposition does not hold.

1.18. For the system discussed in Problem 1.17, let us consider wires each of initial length 5 ft, cross-sectional area 0.1 in^2, and with $E = 30 \times 16^6$ lb/in^2. For a load Q of 20 lb determine the central deflection δ by both the exact and the approximate relations given there.

The exact expression relating force and deflection is $Q = \frac{2\delta AE}{L}\left(1 - \frac{L}{\sqrt{L^2+\delta^2}}\right)$. Substituting the given numerical values, $20 = \frac{2\delta(0.1)(30\times10^6)}{(60)}\left(1 - \frac{60}{\sqrt{(60)^2+\delta^2}}\right)$. Solving by trial and error we find $\delta = 1.131$ in.

The approximate relation between force and deflection is $Q \approx \frac{AE\delta^3}{L^3}$. Substituting,

$$20 \approx \frac{(0.1)(30\times10^6)\delta^3}{(60)^3} \qquad \text{from which} \qquad \delta \approx 1.129 \text{ in}$$

1.19. A square steel bar 50 mm on a side and 1 m long is subject to an axial tensile force of 250 kN. Determine the decrease in the lateral dimension due to this load. Consider $E = 200$ GPa and $\mu = 0.3$.

The loading is axial, hence the stress in the direction of the load is given by

$$\sigma = \frac{P}{A} = \frac{(250 \times 10^3\,\text{N})}{(0.05\,\text{m})(0.05\,\text{m})} = 100\,\text{MPa}$$

The simple form of Hooke's law for uniaxial loading states that $E = \sigma/\epsilon$. The strain ϵ in the direction of the load is thus $(100 \times 10^6)/(200 \times 10^9) = 5 \times 10^{-4}$.

The ratio of the lateral strain to the axial strain is denoted as Poisson's ratio, i.e.,

$$\mu = \frac{\text{lateral strain}}{\text{axial strain}}$$

The axial strain has been found to be 5×10^{-4}. Consequently, the lateral strain is μ times that value, or $(0.3)(5 \times 10^{-4}) = 1.5 \times 10^{-4}$. Since the lateral strain is 1.5×10^{-4}, the change in a 50 mm length is 7.5×10^{-3} mm, which represents the decrease in the lateral dimension of the bar.

It is to be noted that the definition of Poisson's ratio of two strains presumes that only a single uniaxial load acts on the member.

1.20. Consider a state of stress of an element such that a stress σ_x is exerted in one direction, lateral contraction is free to occur in a second (z) direction, but is completely restrained in the third (y) direction. Find the ratio of the stress in the x-direction to the strain in that direction. Also, find the ratio of the strain in the z-direction to that in the x-direction.

Let us examine the general statement of Hooke's law discussed earlier. If in those equations we set $\sigma_z = 0$, $\epsilon_y = 0$ so as to satisfy the conditions of the problem, then Hooke's law becomes

$$\epsilon_x = \frac{1}{E}[\sigma_x - \mu(\sigma_y + 0)] \qquad (a)$$

$$\epsilon_y = \frac{1}{E}[\sigma_y - \mu(\sigma_x + 0)] = 0 \qquad (b)$$

$$\epsilon_z = \frac{1}{E}[0 - \mu(\sigma_x + \sigma_y)] \qquad (c)$$

From (b),

$$\sigma_y = \mu\sigma_x$$

Consequently, from (a)

$$\epsilon_x = \frac{1}{E}(\sigma_x - \mu^2\sigma_x) = \frac{1-\mu^2}{E}\sigma_x$$

Solving this equation for σ_x as a function of ϵ_x and substituting in (c), we have

$$\epsilon_z = -\frac{\mu}{E}(\sigma_x + \mu\sigma_x) = -\frac{\mu(1+\mu)}{E}\frac{\epsilon_x E}{1-\mu^2} = -\frac{\mu\epsilon_x}{1-\mu}$$

We may now form the ratios

$$\frac{\sigma_x}{\epsilon_x} = \frac{E}{1-\mu^2} \qquad \text{and} \qquad -\frac{\epsilon_z}{\epsilon_x} = \frac{\mu}{1-\mu}$$

The first quantity, $E/(1-\mu^2)$, is usually denoted as the *effective modulus of elasticity* and is useful in the theory of thin plates and shells. The second ratio, $\mu/(1-\mu)$, is called the *effective value of Poisson's ratio.*

1.21. Consider an elemental block subject to uniaxial tension (see Fig. 1-35). Derive approximate expressions for the change of volume per unit volume due to this loading.

The strain in the direction of the forces may be denoted by ϵ_x. The strains in the other two orthogonal directions are then each $-\mu\epsilon_x$. Consequently, if the initial dimensions of the element are dx, dy, and dz then the final dimensions are

$$(1 + \epsilon_x)\,dx \qquad (1 - \mu\epsilon_x)\,dy \qquad (1 - \mu\epsilon_x)\,dz$$

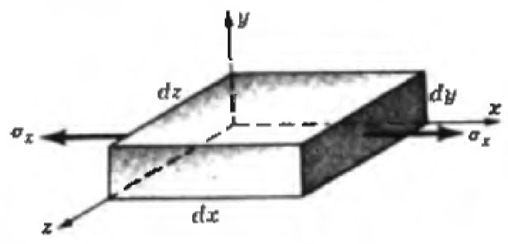


Fig. 1-35

and the volume after deformation is

$$V' = [(1+\epsilon_x)\,dx]\,[(1-\mu\epsilon_x)\,dy]\,[(1-\mu\epsilon_x)\,dz]$$
$$= (1+\epsilon_x)(1-2\mu\epsilon_x)\,dx\,dy\,dz$$
$$= (1-2\mu\epsilon_x+\epsilon_x)\,dx\,dy\,dz$$

since the deformations are so small that the *squares* and *products* of strains may be neglected.

Since the initial volume was $dx\,dy\,dz$, the change of volume per unit volume is

$$\frac{\Delta V}{V} = (1-2\mu)\epsilon_x$$

Hence, for a tensile force the volume increases slightly, for a compressive force it decreases.

Also, the cross-sectional area of the element in a plane normal to the direction of the applied force is given approximately by $A = (1-\mu\epsilon_x)^2\,dy\,dz = (1-2\mu\epsilon_x)\,dy\,dz$.

1.22. A square bar of aluminum 50 mm on a side and 250 mm long is loaded by axial tensile forces at the ends. Experimentally, it is found that the strain in the direction of the load is 0.001. Determine the volume of the bar when the load is acting. Consider $\mu = 0.33$.

From Problem 1.21 the change of volume per unit volume is given by

$$\frac{\Delta V}{V} = \epsilon(1-2\mu) = 0.001(1-0.66) = 0.00034$$

Consequently, the change of volume of the entire bar is given by

$$\Delta V = (50)(50)(250)(0.00034) = 212.5\ \text{mm}^3$$

The original volume of the bar in the unstrained state is $6.25\times10^5\ \text{mm}^3$. Since a tensile force increases the volume, the final volume under load is $6.252125\times10^5\ \text{mm}^3$. Measurements made with the aid of lasers do permit determination of the final volume under load to the indicated accuracy of seven significant figures. Ordinary methods of measurement do not of course lead to such accuracy.

1.23. The general three-dimensional form of Hooke's law in which strain components are expressed as functions of stress components has already been presented. Occasionally it is necessary to express the stress components as functions of the strain components. Derive these expressions.

Given the previous expressions

$$\epsilon_x = \frac{1}{E}[\sigma_x - \mu(\sigma_y+\sigma_z)] \qquad (1)$$

$$\epsilon_y = \frac{1}{E}[\sigma_y - \mu(\sigma_x+\sigma_z)] \qquad (2)$$

$$\epsilon_z = \frac{1}{E}[\sigma_z - \mu(\sigma_x+\sigma_y)] \qquad (3)$$

let us introduce the notation

$$e = \epsilon_x + \epsilon_y + \epsilon_z \tag{4}$$

$$\theta = \sigma_x + \sigma_y + \sigma_z \tag{5}$$

With this notation, (*1*), (*2*), and (*3*) may be readily solved by determinants for the unknowns σ_x, σ_y, σ_z to yield

$$\sigma_x = \frac{\mu E}{(1+\mu)(1-2\mu)}e + \frac{E}{1+\mu}\epsilon_x \tag{6}$$

$$\sigma_y = \frac{\mu E}{(1+\mu)(1-2\mu)}e + \frac{E}{1+\mu}\epsilon_y \tag{7}$$

$$\sigma_z = \frac{\mu E}{(1+\mu)(1-2\mu)}e + \frac{E}{1+\mu}\epsilon_z \tag{8}$$

These are the desired expressions.

Further information may also be obtained from (*1*) through (*5*). If (*1*), (*2*), and (*3*) are added and the symbols e and θ introduced, we have

$$e = \frac{1}{E}(1-2\mu)\theta \tag{9}$$

For the special case of a solid subjected to uniform hydrostatic pressure p, $\sigma_x = \sigma_y = \sigma_z = -p$. Hence

$$e = \frac{-3(1-2\mu)p}{E} \qquad \text{or} \qquad \frac{p}{e} = -\frac{E}{3(1-2\mu)} \tag{10}$$

The quantity $E/3(1-2\mu)$ is often denoted by K and is called the *bulk modulus* or *modulus of volume expansion* of the material. Physically, the bulk modulus K is a measure of the resistance of a material to change of volume without change of shape or form.

We see that the final volume of an element having sides dx, dy, dz prior to loading and subject to strains $\epsilon_x, \epsilon_y, \epsilon_z$ is $(1+\epsilon_x)\,dx\,(1+\epsilon_y)\,dy\,(1+\epsilon_z)\,dz = (1+\epsilon_x+\epsilon_y+\epsilon_z)\,dx\,dy\,dz$.

Thus the ratio of the increase in volume to the original volume is given approximately by

$$e = \epsilon_x + \epsilon_y + \epsilon_z$$

This change of volume per unit volume, e, is defined as the *dilatation*.

1.24. A steel cube is subject to a hydrostatic pressure of 1.5 MPa. Because of this pressure the volume decreases to give a dilatation of -10^{-5}. The Young's modulus of the material is 200 GPa. Determine Poisson's ratio of the material and also the bulk modulus.

From Problem 1.23 for hydrostatic loading the dilatation e is given by Eq. (*10*)

$$e = \frac{-3(1-2\mu)p}{E}$$

Substituting the given numerical values, we have

$$-10^{-5} = \frac{-3(1-2\mu)(1.5\times 10^6\ \text{N/m}^2)}{200\times 10^9\ \text{N/m}^2}$$

from which $\mu = 0.278$. Also from Problem 1.23 the bulk modulus is

$$K = \frac{E}{3(1-2\mu)}$$

which becomes

$$K = \frac{200\times 10^9\ \text{N/m}^2}{3(1-0.556)} = 150\ \text{MPa}$$

1.25. Determine the specific strength and also the specific modulus in the USCS system of (*a*) aluminum alloy, (*b*) titanium alloy, and (*c*) S-glass epoxy. Use materials properties given in Table 1-1.

By definition, specific strength is the ratio of the ultimate stress to the specific weight of the material and specific modulus is the ratio of Young's modulus to the specific weight.

(*a*) From aluminum alloy we have

$$\text{Specific strength} = \frac{80{,}000\ \text{lb/in}^2}{0.0984\ \text{lb/in}^3} = 813{,}000\ \text{in}$$

$$\text{Specific modulus} = \frac{12 \times 10^6\ \text{lb/in}^2}{0.0984\ \text{lb/in}^3} = 122 \times 10^6\ \text{in}$$

(*b*) For titanium alloy we have

$$\text{Specific strength} = \frac{140{,}000\ \text{lb/in}^2}{0.162\ \text{lb/in}^3} = 864{,}200\ \text{in}$$

$$\text{Specific modulus} = \frac{17 \times 10^6\ \text{lb/in}^2}{0.162\ \text{lb/in}^3} = 105 \times 10^6\ \text{in}$$

(*c*) For S-glass epoxy we have

$$\text{Specific strength} = \frac{275{,}000\ \text{lb/in}^2}{0.0766\ \text{lb/in}^3} = 3.6 \times 10^6\ \text{in}$$

$$\text{Specific modulus} = \frac{9.6 \times 10^6\ \text{lb/in}^2}{0.0766\ \text{lb/in}^3} = 125 \times 10^6\ \text{in}$$

Comparison of these specific strengths reveals that the composite material (S-glass epoxy) is much stronger on a unit weight basis than either of the metals, and it also has a slightly higher modulus, indicating greater rigidity than either of the metals.

1.26. Consider a low-carbon square steel bar 20 mm on a side and 1.7 m long having a material yield point of 275 MPa and $E = 200$ GPa. An applied axial load gradually builds up from zero to a value such that the elongation of the bar is 15 mm, after which the load is removed. Determine the permanent elongation of the bar after removal of the load. Assume elastic, perfectly plastic behavior as shown in Fig. 1-36.

Yield begins when the applied load reaches a value of

$$\begin{aligned} P &= \sigma_{yp}\ (\text{area}) \\ &= (275 \times 10^6\ \text{N/m}^2)(0.020\ \text{m})^2 \\ &= 110{,}000\ \text{N} \end{aligned}$$

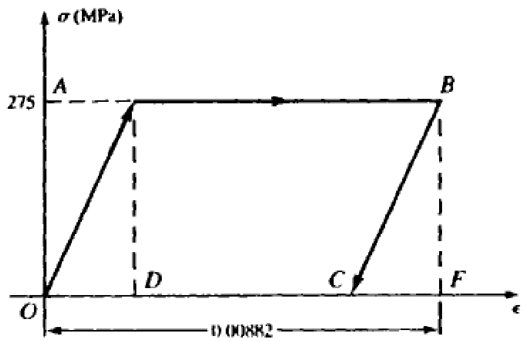


Fig. 1-36

which corresponds to point A of Fig. 1-36. Note that in that figure the ordinate is stress and the abscissa is strain. However, values on each of these axes differ only by constants from those on a force-elongation plot.

When the elongation is 15 mm, corresponding to point B in Fig. 1-36, unloading begins and the axial strain at the initiation of unloading is

$$\frac{15 \text{ mm}}{1700 \text{ mm}} = 0.00882$$

Unloading follows along line BC (parallel to AO) until the horizontal axis is reached, so that OC corresponds to the strain after complete removal of the load. We next find the strain CF—but this is readily found from using the similar triangles OAD and CBF to be

$$E = \frac{\sigma}{\epsilon}$$

$$\epsilon = \frac{275 \times 10^6 \text{ Pa}}{200 \times 10^9 \text{ Pa}} = 1.375 \times 10^{-3}$$

Thus, after load removal the residual strain is

$$OC = OF - CF$$
$$= 0.00882 - 0.00138 = 0.00744$$

The elongation of the 1.7-m long bar is consequently

$$(1.7 \text{ m})(0.00744) = 0.0126 \text{ m} \quad \text{or} \quad 12.6 \text{ mm}$$

Supplementary Problems

1.27. Forces acting in the articulated joints in the human vertebrae may lead to excessive stresses and eventual rupture of the spinal discs. Measurements of the adult disc indicate a surface area of approximately 1000 mm^2. Additional measurements during a lifting exercise indicate that a normal force of 708 N has been developed. Determine the normal stress in the disc. *Ans.* 708 kPa

1.28. Laboratory tests on human teeth indicate that the area effective during chewing is approximately 0.04 in^2 and that the tooth length is about 0.41 in. If the applied load in the vertical direction is 200 lb and the measured shortening is 0.0015 in, determine Young's modulus. *Ans.* 1.37×10^6 lb/in^2

1.29. A hollow right-circular cylinder is made of cast iron and has an outside diameter of 75 mm and an inside diameter of 60 mm. If the cylinder is loaded by an axial compressive force of 50 kN, determine the total shortening in a 600-mm length. Also determine the normal stress under this load. Take the modulus of elasticity to be 100 GPa and neglect any possibility of lateral buckling of the cylinder.
Ans. $\Delta = 0.188$ mm, $\sigma = 31.45$ MPa

1.30. A solid circular steel rod 6 mm in diameter and 500 mm long is rigidly fastened to the end of a square brass bar 25 mm on a side and 400 mm long, the geometric axes of the bars lying along the same line. An axial tensile force of 5 kN is applied at each of the extreme ends. Determine the total elongation of the assembly. For steel, $E = 200$ GPa and for brass $E = 90$ GPa. *Ans.* 0.477 mm

1.31. A high-performance jet aircraft cruises at three times the speed of sound at an altitude of 25,000 m. It has a long, slender titanium body reinforced by titanium ribs. The length of the aircraft is 30 m and the coefficient of thermal expansion of the titanium is 10×10^{-6}/°C. Determine the increase of overall length of the aircraft at cruise altitude over its length on the ground if the temperature while cruising is 500°C above ground temperature. (*Note:* This change of length is of importance since the designer must account for it because it changes the performance characteristics of the system.) *Ans.* 0.150 m

1.32. One of the most promising materials for use as a superconductor is composed of yttrium (a rare earth metal), barium, copper, and oxygen. This material acts as a superconductor (i.e., transmits electricity with essentially no resistance losses) at temperatures up to $-178°C$. If the temperature is then raised to $67°C$, and the coefficient of thermal expansion is $11.0 \times 10^{-6}/°C$, determine the elongation of a 100-m long segment due to this temperature differential. *Ans.* 0.27 m

1.33 A solid circular cross-section bar in the form of a truncated cone is made of aluminum and has the dimensions shown in Fig. 1-37. The bar is loaded by an axial tensile force of 80,000 lb and $E = 10 \times 10^6$ lb/in^2. Find the elongation of the bar. *Ans.* 0.00874 in

Fig. 1-37

1.34. A solid conical bar of circular cross section is suspended vertically, as shown in Fig. 1-38. The length of the bar is L, the diameter of the base is D, the modulus of elasticity is E, and the weight per unit volume is γ. Determine the elongation of the bar due to its own weight.

Ans. $\Delta = \dfrac{\gamma L^2}{6E}$

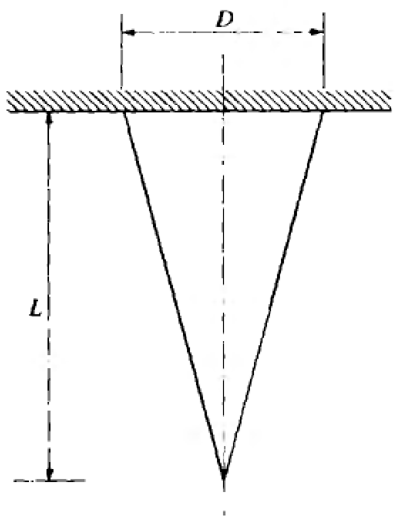


Fig. 1-38

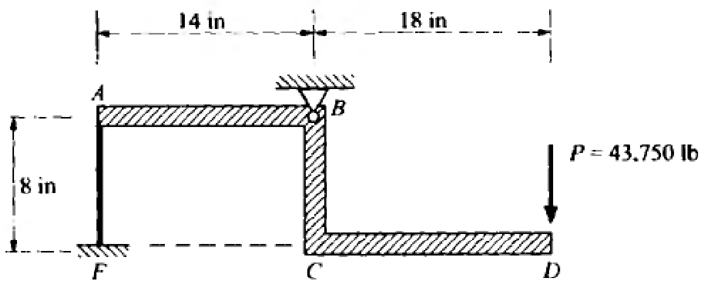


Fig. 1-39

1.35. A Z-shaped rigid bar $ABCD$, shown in Fig. 1-39, is suspended by a pin at B, and loaded by a vertical force P. At A a steel tie rod AF connects the section to a firm ground support at F. Take $E = 30 \times 10^6$ lb/in^2. Determine the vertical deflection at D. *Ans.* 0.099 in

1.36. The rigid bar ABC is pinned at B and at A attached to a vertical steel bar AD which in turn is attached to a larger steel bar DF which is firmly attached to a rigid foundation. The geometry of the system is shown in Fig. 1-40. If a vertical force P of magnitude 40 kN is applied at C, determine the vertical displacement of point C. *Ans.* 9.17 mm

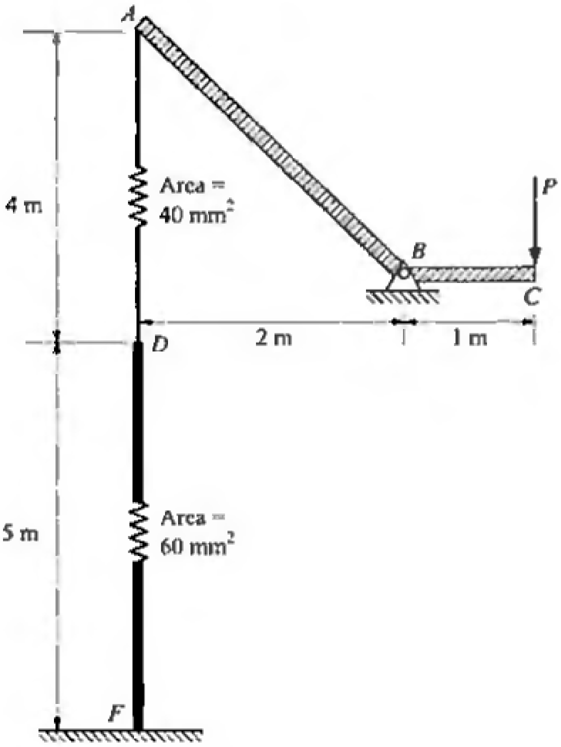


Fig. 1-40

1.37. A body having the form of a solid of revolution supports a load P as shown in Fig. 1-41. The radius of the upper base of the body is r_0 and the specific weight of the material is γ per unit volume. Determine how the radius should vary with the altitude in order that the compressive stress at all cross sections should be constant. The weight of the solid is not negligible. *Ans.* $r = r_0 e^{\gamma\pi r_0^2 y/2P}$

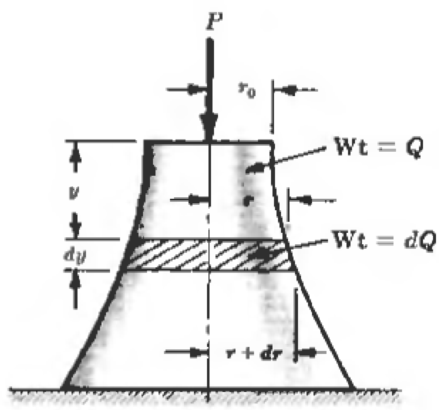


Fig. 1-41

1.38. In Problem 1.12 consider the force P to be 20,000 lb, $A_1 = 1.2\ \text{in}^2$, $L_1 = 5$ ft, $E_1 = 16 \times 10^6\ \text{lb/in}^2$, $A_2 = 1.5\ \text{in}^2$, $L_2 = 4$ ft, and $E_2 = 10 \times 10^6\ \text{lb/in}^2$. Find the horizontal and vertical components of displacement of pin B. *Ans.* $\Delta_x = 0.189$ in; $\Delta_y = 0.064$ in

1.39. In Fig. 1-42, AB, AC, BC, CD, and BD are pin-connected rods. Point B is attached to point E by a spring whose unstretched length is 1 m and whose spring constant is 4 kN/m. Neglecting the weight of all bars and the spring, determine the magnitude of the load W applied at D that makes CD horizontal. *Ans.* 583 N

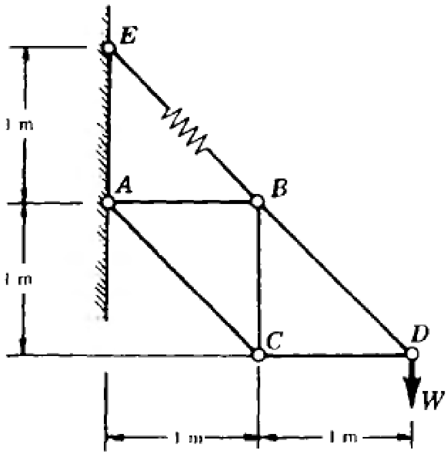


Fig. 1-42

1.40. The steel bars AB and BC are pinned at each end and support the load of 200 kN, as shown in Fig. 1-43. The material is structural steel, having a yield point of 200 MPa, and safety factors of 2 and 3.5 are satisfactory for tension and compression, respectively. Determine the size of each bar and also the horizontal and vertical components of displacement of point B. Take $E = 200$ GPa. Neglect any possibility of lateral buckling of bar BC.
Ans. Area $AB = 1732\ \text{mm}^2$, area $BC = 1750\ \text{mm}^2$, $\Delta_h = 0.37$ mm (to right), $\Delta_v = 1.78$ mm (downward)

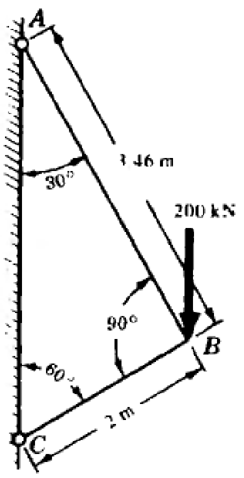


Fig. 1-43

1.41. The two bars AB and CB shown in Fig. 1-44 are pinned at each end and subject to a single vertical force P. The geometric and elastic constants of each are as indicated. Determine the horizontal and vertical components of displacement of pin B.

Ans. $\Delta_x = -\dfrac{PL_1}{\sqrt{3}A_1E_1} + \dfrac{PL_2}{\sqrt{3}A_2E_2}, \; \Delta_y = \dfrac{PL_1}{3A_1E_1} + \dfrac{PL_2}{3A_2E_2}$

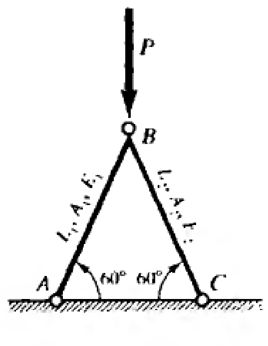


Fig. 1-44

1.42. In Problem 1-41, the bar AB is titanium, having an area of 1000 mm^2, length of 2.4 m, and E_1 = 110 GPa. Bar CB is steel having an area of 400 mm^2, length of 2.4 m, and E_2 = 200 GPa. What are the horizontal and vertical components of displacement of the pin B if P = 600 kN? *Ans.* Δ_x = 2.83 mm, Δ_y = 10.4 mm

1.43. A flat slab of variable width is bounded by the curve $y = 10e^{0.25x}$ and extends from the origin to x = 5 in. It is subject to an axial tensile load of 20,000 lb and the material is steel for which $E = 30 \times 10^6$ lb/in^2. The slab thickness is 0.125 in. Use the FORTRAN program of Problem 1.14 to determine the elongation of the slab. *Ans.* 0.00275 in

1.44. A steel bar of solid circular cross section is bounded by the curve $y = 0.07e^{0.05x}$ and extends from the origin to x = 5 m. It is subject to an axial tensile load of 1.5 MN and Young's modulus is 200 GPa. Use the FORTRAN program of Problem 1.14 to determine the elongation of the bar. *Ans.* 3.24 mm

1.45. Consider a state of stress of an element in which a stress σ_x is exerted in one direction and lateral contraction is completely restrained in each of the other two directions. Find the effective modulus of elasticity and also the effective value of Poisson's ratio.

Ans. eff. mod. = $\dfrac{E(1-\mu)}{(1-2\mu)(1+\mu)}$, eff. Poisson's ratio = 0

1.46. A block of aluminum alloy is 400 mm long and of rectangular cross section 25 by 30 mm. A compressive force P = 60 kN is applied in the direction of the 400-mm dimension and lateral contraction is completely restrained in each of the other two directions. Find the effective modulus of elasticity as well as the change of the 400-mm length. Take E = 75 GPa and Poisson's ratio to be 0.33.
Ans. eff. mod. = 114.5 GPa, change of length = −0.286 mm

1.47. Consider the state of stress in a bar subject to compression in the axial direction. Lateral expansion is restrained to half the amount it would ordinarily be if the lateral faces were load free. Find the effective modulus of elasticity.

Ans. $\dfrac{E(1-\mu)}{1-\mu-\mu^2}$

1.48. A bar of uniform cross section is subject to uniaxial tension and develops a strain in the direction of the force of 1/800. Calculate the change of volume per unit volume. Assume $\mu = 1/3$. *Ans.* 1/2400 (increase)

1.49. A square steel bar is 50 mm on a side and 250 mm long. It is loaded by an axial tensile force of 200 kN. If $E = 200$ GPa and $\mu = 0.3$, determine the change of volume per unit volume. *Ans.* 0.00016

1.50. Consider a low-carbon steel square steel bar 1 in on a side and 70 in long having a material yield point of 40,000 lb/in^2 and a Young's modulus of 30×10^6 lb/in^2. An axial tensile load gradually builds up from zero to a value such that the elongation of the bar is 0.6 in, after which the load is removed. Determine the permanent elongation of the bar. Assume that the material is elastic, perfectly plastic. *Ans.* 0.509 in

1.51. Determine, from Table 1-1, the specific strength and also the specific modulus of (*a*) nickel, and (*b*) boron epoxy composite. Use the SI system.
Ans. (*a*) nickel: specific strength = 3563 to 8736 m, specific modulus = 2.41×10^6 m; (*b*) boron epoxy: specific strength = 71.8×10^3 m, specific modulus = 11.0×10^6 m

Table 1-1. Properties of Common Engineering Materials at 68 °F (20 °C)

Material	Specific weight		Young's modulus		Ultimate stress		Coefficient of linear thermal expansion		Poisson's ratio
	lb/in³	kN/m³	lb/in²	GPa	lb/in²	kPa	10e-6/°F	10e-6/°C	
I. Metals in slab, bar, or block form									
Aluminum alloy	0.0984	27	10–12e6	70–79	45–80e3	310–550	13	23	0.33
Brass	0.307	84	14–16e6	96–110	43–85e3	300–590	11	20	0.34
Copper	0.322	87	16–18e6	112–120	33–55e3	230–380	9.5	17	0.33
Nickel	0.318	87	30e6	210	45–110e3	310–760	7.2	13	0.31
Steel	0.283	77	28–30e6	195–210	80–200e3	550–1400	6.5	12	0.30
Titanium alloy	0.162	44	15–17e6	105–120	130–140e3	900–970	4.5–5.5	8–10	0.33
II. Nonmetallics in slab, bar, or block form									
Concrete (composite)	0.0868	24	3.6e6	25	4000–6000	28–41	6	11	
Glass	0.0955	26	7–12e6	48–83	10,000	70	3–6	5–11	0.23
III. Materials in filamentary (whisker) form: [dia. < 0.001 in (0.025 mm)]									
Aluminum oxide	0.141	38	100–350e6	690–2410	2–4e6	13,800–27,600			
Barium carbide	0.090	25	65e6	450	1e6	6900			
Glass			50e6	345	1–3e6	7000–20,000			
Graphite	0.081	22	142e6	980	3e6	20,000			
IV. Composite materials (unidirectionally reinforced in direction of loading)									
Boron epoxy	0.071	19	31e6	210	198,000	1365	2.5	4.5	
S-glass-reinforced epoxy	0.0766	21	9.6e6	66.2	275,000	1900			
V. Others									
Graphite-reinforced epoxy	0.054	15	15e6	104	190,000	1310			
Kevlar-49 epoxy*	0.050	13.7	12.5e6	86	220,000	1520			

*Tradename of E. I. duPont Co.

Chapter 2

Statically Indeterminate Force Systems Tension and Compression

DEFINITION OF A DETERMINATE FORCE SYSTEM

If the values of all the external forces which act on a body can be determined by the equations of static equilibrium alone, then the force system is *statically determinate*. The problems in Chap. 1 were all of this type.

Example 1

The bar shown in Fig. 2-1 is loaded by the force P. The reactions are R_1, R_2, and R_3. The system is statically determinate because there are three equations of static equilibrium available for the system and these are sufficient to determine the three unknowns.

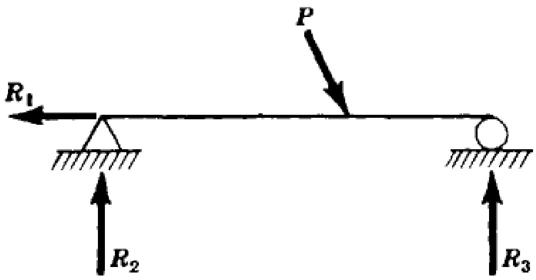


Fig. 2-1

Example 2

The truss $ABCD$ shown in Fig. 2-2 is loaded by the forces P_1 and P_2. The reactions are R_1, R_2, and R_3. Again, since there are three equations of static equilibrium available, all three unknown reactions may be determined and consequently the external force system is statically determinate.

The above two illustrations refer only to external reactions and the force systems may be defined as statically determinate *externally*.

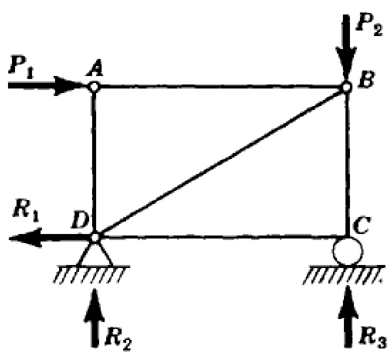


Fig. 2-2

DEFINITION OF AN INDETERMINATE FORCE SYSTEM

In many cases the forces acting on a body cannot be determined by the equations of statics alone because there are more unknown forces than there are equations of equilibrium. In such a case the force system is said to be *statically indeterminate.*

Example 3

The bar shown in Fig. 2-3 is loaded by the force P. The reactions are R_1, R_2, R_3, and R_4. The force system is statically indeterminate because there are four unknown reactions but only three equations of static equilibrium. Such a force system is said to be *indeterminate to the first degree.*

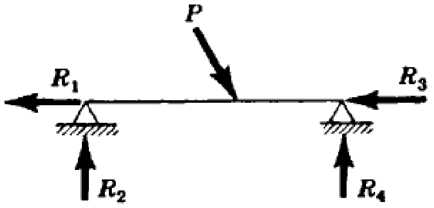


Fig. 2-3

Example 4

The bar shown in Fig. 2-4 is statically indeterminate to the second degree because there are five unknown reactions R_1, R_2, R_3, R_4, and M_1 but only three equations of static equilibrium. Consequently the values of all reactions cannot be determined by use of statics equations alone.

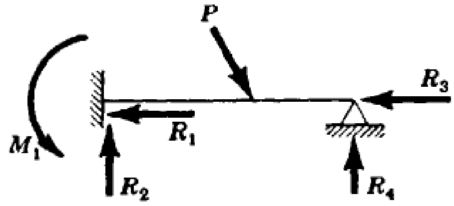


Fig. 2-4

METHOD OF ELASTIC ANALYSIS

The approach that we will consider here is called the *deformation method* because it considers the deformations in the system. Briefly, the procedure to be followed in analyzing an indeterminate system is first to write all equations of static equilibrium that pertain to the system and then *supplement* these equations with additional equations based upon the deformations of the structure. Enough equations involving deformations must be written so that the total number of equations from both statics and deformations is equal to the number of unknown forces involved. See Problems 2.1 through 2.12.

ANALYSIS FOR ULTIMATE STRENGTH (LIMIT DESIGN)

We consider that the stress-strain curve for the material is of the form indicated in Fig. 2-5, i.e., one characterizing an extremely ductile material such as structural steel. Such idealized elastoplastic behavior is a good representation of low-carbon steel. This representation assumes that the material is incapable of developing stresses greater than the yield point.

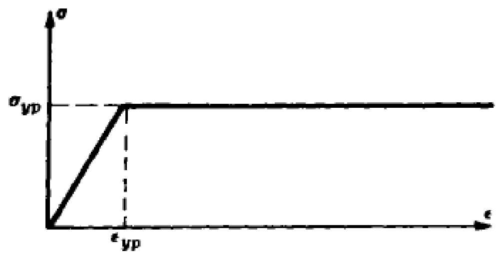


Fig. 2-5

In a statically indeterminate system any inelastic action changes the conditions of constraint. Under these altered conditions the loading that the system can carry usually increases over that predicted on the basis of completely elastic action everywhere in the system. Design of a statically indeterminate structure for that load under which some or all of the regions of the structure reach the yield point and cause "collapse" of the system is termed *limit design*. The *ultimate load* corresponding to such design is of course divided by some factor of safety to determine a *working load*. The term "limit design," when used in this manner, applies only to statically indeterminate structures. For applications, see Problems 2.13 through 2.17.

Solved Problems

Elastic Analysis

In Problems 2.1 through 2.12 it is assumed that the system is acting within the linear elastic range of action of the material.

2.1. In medical (orthopedic) applications it is occasionally necessary to lengthen a main bone of a human leg or arm. This situation may arise if the bone has healed in a wrong configuration after some accident, or alternatively the improper length may be due to a birth defect. One way to accomplish this lengthening is for the surgeon to weaken the bone through the introduction of one or two cuts near the outer surface of the bone, then attach the mechanical system shown in Fig. 2-6 to the exterior of the leg. This system consists of a pair of metallic rings which encircle the leg, with the rings being connected by a pair of parallel brass rods which are threaded at each end. The distance between the rings can be varied over the months of treatment by turning the nut at each end of each rod. Typically, the bone has a cross-sectional area of 1.2 in^2, a modulus of elasticity of 4.6×10^6 lb/in^2, and a length of 8 in. The two brass rods have a total cross-sectional area of 0.05 in^2, a modulus of 13.5×10^6 lb/in^2, and 32 threads per inch. If the nut at the end of the bar is turned $\frac{1}{8}$ of a revolution to stretch the bone, determine the axial stress arising in the bone.

Let us consider a section to be passed through the bone and perpendicular to the axial dimension of the bone. The free-body diagram of the system is shown in Fig. 2-7 where P_{bone} represents the axial force in the bone and P_{rod} is the axial force in each brass bar. For equilibrium:

$$P_{\text{bone}} = P_{\text{rod}} \tag{1}$$

From deformations of the system, we realize that the extension of the bone plus the shortening of each rod is equal to the displacement of the nut along the bar. This latter quantity is $\frac{1}{8}(\frac{1}{32}\text{ in})$. Thus, we have

$$\frac{P_{\text{bone}}(8\text{ in})}{(1.2\text{ in}^2)(4.6\times 10^6\text{ lb/in}^2)} + \frac{P_{\text{rod}}(8\text{ in})}{(0.05\text{ in}^2)(13.5\times 10^6\text{ lb/in}^2)} = (\tfrac{1}{8})(\tfrac{1}{32}\text{ in}) \tag{2}$$

Solving (*1*) and (*2*) we find

$$P_{\text{bone}} = 588\ \text{lb}$$

$$\sigma_{\text{bone}} = \frac{588\ \text{lb}}{1.2\ \text{in}^2} = 490\ \text{lb/in}^2$$

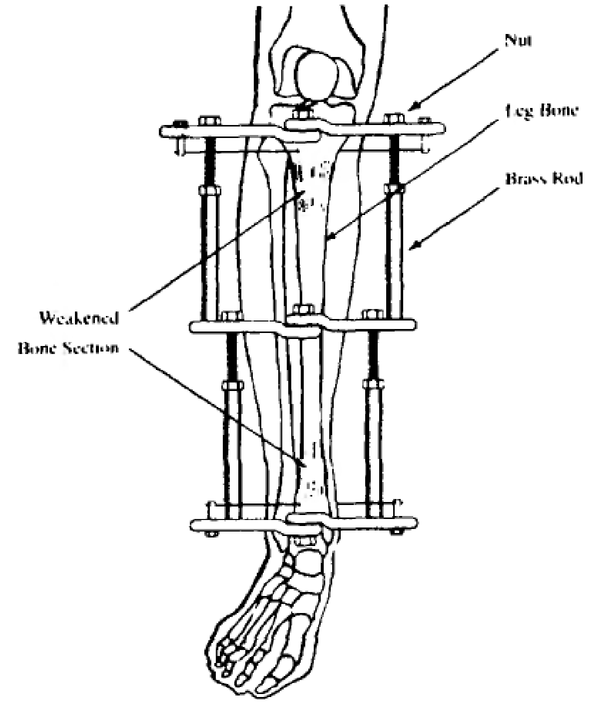


Fig. 2-6

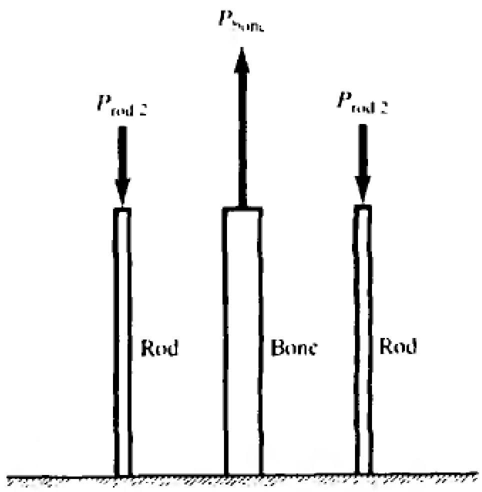


Fig. 2-7

2.2. Consider a steel tube surrounding a solid aluminum cylinder, the assembly being compressed between infinitely rigid cover plates by centrally applied forces as shown in Fig. 2-8(*a*). The aluminum cylinder is 3 in in diameter and the outside diameter of the steel tube is 3.5 in. If $P = 48{,}000$ lb, find the stress in the steel and also in the aluminum. For steel, $E = 30 \times 10^6\ \text{lb/in}^2$ and for aluminum $E = 12 \times 10^6\ \text{lb/in}^2$.

Let us pass a horizontal plane through the assembly at any elevation except in the immediate vicinity of the cover plates and then remove one portion or the other, say the upper portion. In that event the portion that we have removed must be replaced by the effect it exerted upon the remaining portion and that effect consists of vertical normal stresses distributed over the two materials. The free-body diagram of the portion of the assembly below this cutting plane is shown in Fig. 2-8(*b*) where σ_{st} and σ_{al} denote the normal stresses existing in the steel and aluminum respectively.

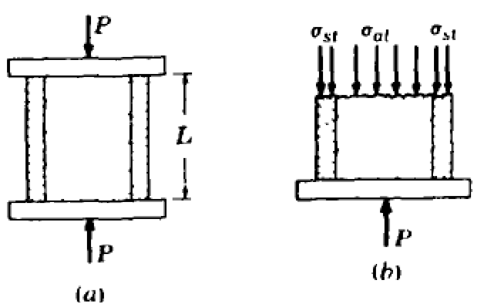


Fig. 2-8

Let us denote the resultant force carried by the steel by P_{st} (lb) and that carried by the aluminum by P_{al}. Then $P_{st} = A_{st}\sigma_{st}$ and $P_{al} = A_{al}\sigma_{al}$ where A_{st} and A_{al} denote the cross-sectional areas of the steel tube and the aluminum cylinder, respectively. There is only one equation of static equilibrium available for such a force system and it takes the form

$$\Sigma F_v = P - P_{st} - P_{al} = 0$$

Thus, we have one equation in two unknowns, P_{st} and P_{al}, and hence the problem is statically indeterminate. In that event we must supplement the available statics equation by an equation derived from the deformations of the structure. Such an equation is readily obtained because the infinitely rigid cover plates force the axial deformations of the two metals to be identical.

The deformation due to axial loading is given by $\Delta = PL/AE$. Equating axial deformations of the steel and the aluminum we have

$$\frac{P_{st}L}{A_{st}E_{st}} = \frac{P_{al}L}{A_{al}E_{al}}$$

or $$\frac{P_{st}L}{(\pi/4)[(3.5)^2-(3)^2](30\times 10^6)} = \frac{P_{al}L}{(\pi/4)(3)^2(12\times 10^6)}$$ from which $P_{st} = 1.23P_{al}$

This equation is now solved simultaneously with the statics equation, $P - P_{st} - P_{al} = 0$, and we find $P_{al} = 0.448P$, $P_{st} = 0.552P$.

For a load of $P = 48{,}000$ lb this becomes $P_{al} = 21{,}504$ lb and $P_{st} = 26{,}496$ lb. The desired stresses are found by dividing the resultant force in each material by its cross-sectional area:

$$\sigma_{al} = \frac{21{,}504}{(\pi/4)(3)^2} = 3050 \text{ lb/in}^2 \qquad \sigma_{st} = \frac{26{,}496}{(\pi/4)[(3.5)^2-(3)^2]} = 1038 \text{ lb/in}^2$$

2.3. The three-bar assembly shown in Fig. 2-9 supports the vertical load P. Bars AB and BD are identical, each of length L and cross-sectional area A_1. The vertical bar BC is also of length L but of area A_2. All bars have the same modulus E and are pinned at A, B, C, and D. Determine the axial force in each of the bars.

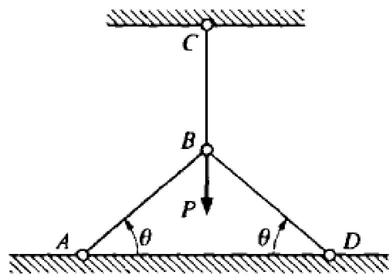


Fig. 2-9

First, we draw a free-body diagram of the pin at B. The forces in each of the bars are represented by P_1 and P_2 as shown in Fig. 2-10. For vertical equilibrium we find:

$$\Sigma F_v = 2P_1 \sin\theta + P_2 - P = 0 \qquad (1)$$

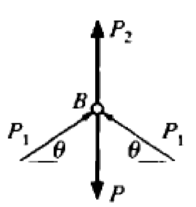


Fig. 2-10

We assume, temporarily, that the pin at B is removed. Next we examine deformations. Under the action of the axial force P_2 the vertical bar extends downward an amount

$$\Delta_1 = \frac{P_2 L}{A_2 E} \tag{2}$$

so that the lower end (originally at B) moves to B' as shown in Fig. 2-11.

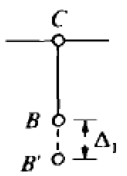


Fig. 2-11

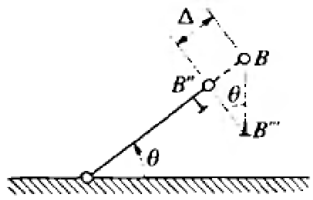


Fig. 2-12

The compressive force in AB causes it to shorten an amount Δ shown as BB'' in Fig. 2-12. The bar AB then rotates about A as a rigid body so that B'' moves to B''' directly below point C. From Fig. 2-12 the vertical component of Δ is

$$BB''' = \frac{P_1 L}{A_1 E \sin\theta}$$

Next, we consider the pin to be reinserted in the system. The points B' and B''' must coincide so that

$$\frac{P_2 L}{A_2 E} = \frac{P_1 L}{A_1 E \sin\theta} \tag{3}$$

Substituting Eq. (3) in Eq. (1) we find

$$P_1 = \frac{P\sin\theta}{2\sin^2\theta + \alpha}$$

$$P_2 = \frac{P\alpha}{2\sin^2\theta + \alpha}$$

where $\alpha = A_2/A_1$.

2.4. Consider the two identical bars AB and AC, each 0.5 m long, each with area A and $E = 200$ GPa. They are pinned at A, B, and C. Bar DF has area $2A$ and $E = 200$ GPa. Bar DF is accidentally made 0.8 mm too short to extend between A and D. Points A and F must be brought together mechanically to form a frame consisting of the two isosceles triangles shown in Fig. 2-13. Find

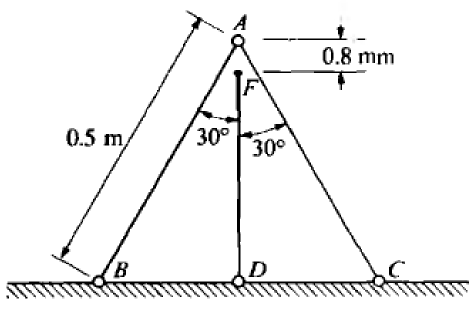


Fig. 2-13

the initial stresses in the bars prior to application of any external loading. The system of bars lies on a frictionless horizontal plane.

It is evident that point A must be forced downward (creating compression in AB and AC) and the end F of the vertical bar must be pulled upward to meet the (lowered) point A. The meeting point of A and F is *not* necessarily midway between the initial locations of A and F. After these two points have met, they are joined by a pin. At this stage there are no external applied loads on the three-bar system. However, there are locked-in stresses in each of the bars.

We may find these initial stresses by designating compressive forces in AB and AC by P_2 and the tensile force in FD by P_1 (Newtons). After these bars have been jointed by a pin, the free-body diagram of that pin appears as shown in Fig. 2-14.

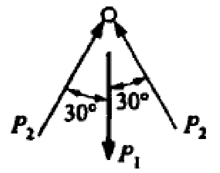


Fig. 2-14

For equilibrium of the pin:

$$2P_2 \cos 30° - P_1 = 0; \qquad \text{or} \qquad P_1 = P_2\sqrt{3} \tag{1}$$

As point A is mechanically forced downward, each of the bars AB and AC shortens an amount

$$\Delta_1 = \frac{P_2(500)}{AE}$$

in the direction of the respective bar. With the pin at A removed, the deformed configuration appears as shown in Fig. 2-15. The vertical component of Δ_1 is given by

$$\frac{P_2(500)}{AE \cos 30°}$$

The deformation of the inclined bars may be visualized (see Fig. 2-15) by realizing that the compressed

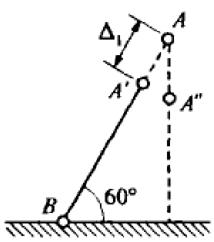


Fig. 2-15

bar AB first shortens as A moves to A', then the entire bar AB rotates as a rigid body about B so that A' moves to A'' actually along a circular arc whose center is at B, but for small angles of rotation the arc may be replaced by the straight line $A'A''$.

The tensile force in bar DF causes the point F in the originally stress-free bar to move vertically upward to F', as shown in Fig. 2-16. F' is the final position of F after the pin has been inserted at the junction of all three bars. The vertical elongation of the bar is

$$\Delta_2 = \frac{P_1(500 \cos 30°)}{2AE} \tag{2}$$

where $2A$ is the cross-sectional area of bar DF.

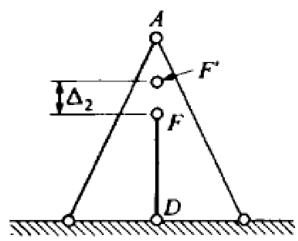


Fig. 2-16

Thus, to close the gap of 0.8 mm between the bars, we must have

$$\frac{P_2(500)}{AE\cos 30°}+\frac{P_1(500\cos 30°)}{2AE}=0.8\text{ mm} \qquad (3)$$

Substituting Eq. (1) in Eq. (3) we find

$$577.4\frac{P_2}{AE}+\frac{(216.5)(P_2\sqrt{3})}{AE}=0.8$$

But $E = 200$ GPa, so solving the above equations for normal stresses in the bars we find

$$\sigma_2=\frac{P_2}{A}=168\text{ MPa}$$

$$\sigma_1=\frac{P_1}{2A}=145.5\text{ MPa}$$

2.5. The composite bar shown in Fig. 2-17(*a*) is rigidly attached to the two supports. The left portion of the bar is copper, of uniform cross-sectional area 12 in^2 and length 12 in. The right portion is aluminum, of uniform cross-sectional area 3 in^2 and length 8 in. At a temperature of 80°F the entire assembly is stress free. The temperature of the structure drops and during this process the right support yields 0.001 in in the direction of the contracting metal. Determine the minimum temperature to which the assembly may be subjected in order that the stress in the aluminum does not exceed 24,000 lb/in^2. For copper $E = 16\times10^6$ lb/in^2, $\alpha = 9.3\times10^{-6}$/°F and for aluminum $E = 10\times10^6$ lb/in^2, $\alpha = 12.8\times10^{-6}$/°F.

It is perhaps simplest to consider that the bar is cut just to the left of the supporting wall at the right and is then free to contract due to the temperature drop ΔT. The total shortening of the composite bar is given by

$$(9.3\times10^{-6})(12)\Delta T+(12.8\times10^{-6})(8)\Delta T$$

according to the definition of the coefficient of linear expansion. It is to be noted that the shape of the cross section has no influence upon the change in length of the bar due to a temperature change.

Even though the bar has contracted this amount, it is still stress free. However, this is not the complete analysis because the reaction of the wall at the right has been neglected by cutting the bar there. Consequently, we must represent the action of the wall by an axial force P applied to the bar, as shown

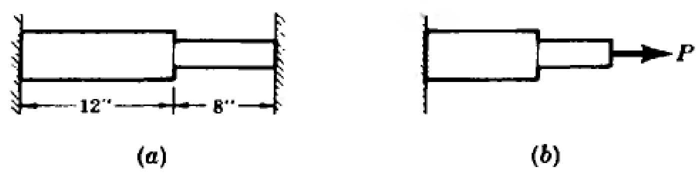


Fig. 2-17

in Fig. 2-17(*b*). For equilibrium, the resultant force acting over any cross section of either the copper or the aluminum must be equal to P. The application of the force P stretches the composite bar by an amount

$$\frac{P(12)}{12(16\times 10^6)}+\frac{P(8)}{3(10\times 10^6)}$$

If the right support were unyielding, we would equate the last expression to the expression giving the total shortening due to the temperature drop. Actually the right support yields 0.001 in and consequently we may write

$$\frac{P(12)}{12(16\times 10^6)}+\frac{P(8)}{3(10\times 10^6)}=(9.3\times 10^{-6})(12)\Delta T+(12.8\times 10^{-6})(8)\Delta T-0.001$$

The stress in the aluminum is not to exceed 24,000 lb/in², and since it is given by the formula $\sigma = P/A$, the maximum force P becomes $P = A\sigma = 3(24{,}000) = 72{,}000$ lb. Substituting this value of P in the above equation relating deformations, we find $\Delta T = 115°$F. Therefore the temperature may drop 115°F from the original 80°F. The final temperature would be −35°F.

2.6. A bar (see Fig. 2-18) in the shape of a solid, truncated cone of circular cross section is situated between two rigid supports which constrain the bar from any change of axial length. The temperature of the entire bar is then raised ΔT. Assume that the cross sections perpendicular to the longitudinal axis of symmetry remain plane and neglect localized end effects due to the end supports. Determine the normal stress at any point in the bar.

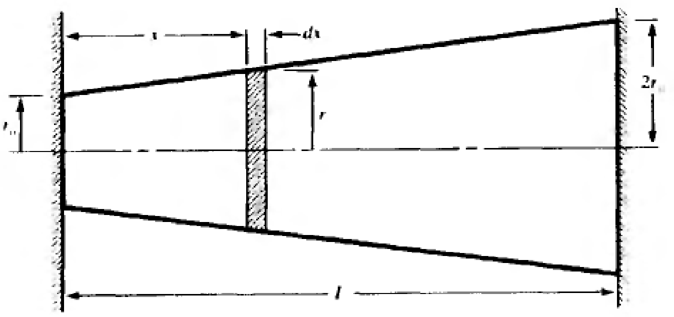


Fig. 2-18

Let us introduce the coordinate system shown in Fig. 2-18 where x denotes the distance of a thin disc from the left end of the bar, and dx is the thickness of the disc in the direction of the x-axis. The radius of this disc is found from geometry to be

$$r = r_0 + \frac{r_0 x}{L}$$

If the support at the right end of the bar is considered to be temporarily removed, the entire bar will expand in length an amount $\alpha(L)(\Delta T)$, where α is the coefficient of thermal expansion of the material.

We may now consider an axial force N to act on the right end of the bar, as shown in Fig. 2-19, to compress the bar back to its original length L. The disc of thickness dx compresses an amount (see Problem 1.1)

$$\frac{N\,dx}{AE}=\frac{N(dx)}{\pi r^2(E)}$$

because of this axial force N (which, for equilibrium, must be constant over any cross section of the bar).

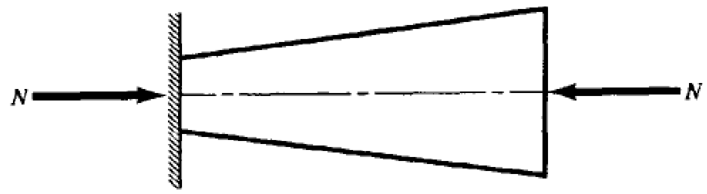


Fig. 2-19

The total compression of the bar due to N is found by summing the changes of length of all discs from $x = 0$ to $x = L$:

$$\int_{x=0}^{x=L} \frac{N(dx)}{\pi r^2 E} = \frac{NL^2}{E\pi r_0^2} \int_0^L \frac{dx}{(L+x)^2}$$

Integrating,

$$\int_0^L \frac{dx}{(L+x)^2} = \frac{1}{2L}$$

and setting the bar extension due to heating equal to bar compression due to the axial force N, we find

$$\alpha(L)(\Delta T) = \frac{NL^2}{E\pi r_0^2}\left(\frac{1}{2L}\right)$$

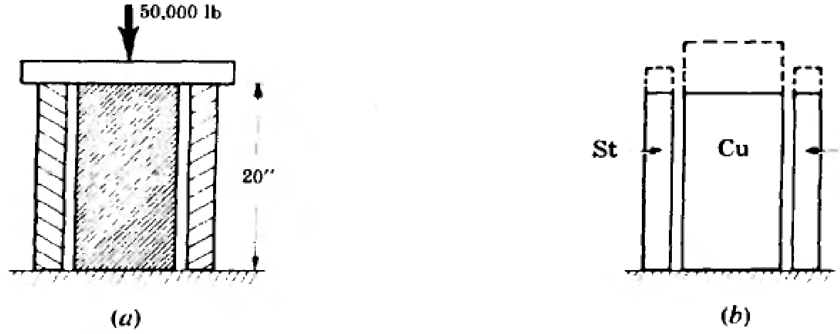
$$N = 2\alpha(\Delta T)E\pi r_0^2$$

The axial normal stress is now found by dividing the force N by the cross-sectional area at any station x,

$$\sigma = \frac{N}{\pi r^2} = \frac{2\alpha(\Delta T)E}{(1 + x/L)^2}$$

2.7. A hollow steel cylinder surrounds a solid copper cylinder and the assembly is subject to an axial loading of 50,000 lb as shown in Fig. 2-20(a). The cross-sectional area of the steel is 3 in^2, while that of the copper is 10 in^2. Both cylinders are the same length before the load is applied. Determine the temperature rise of the entire system required to place all of the load on the copper cylinder. The cover plate at the top of the assembly is rigid. For copper $E = 16 \times 10^6$ lb/in^2, $\alpha = 9.3 \times 10^{-6}$/°F, while for steel $E = 30 \times 10^6$ lb/in^2, $\alpha = 6.5 \times 10^{-6}$/°F.

One method of analyzing this problem is to assume that the load as well as the upper cover plate are removed and that the system is allowed to freely expand vertically because of a temperature rise ΔT. In



(a) (b)

Fig. 2-20

that event the upper ends of the cylinders assume the positions shown by the dashed lines in Fig. 2-20(*b*).

The copper cylinder naturally expands upward more than the steel one because the coefficient of linear expansion of copper is greater than that of steel. The upward expansion of the steel cylinder is $(6.5\times 10^{-6})(20)\Delta T$, while that of the copper is $(9.3\times 10^{-6})(20)\Delta T$.

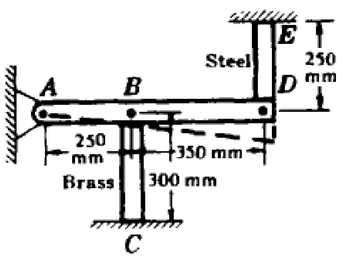
This is not of course the true situation because the load of 50,000 lb has not as yet been considered. If all of this axial load is carried by the copper then only the copper will be compressed and the compression of the copper is given by

$$\Delta_{cu} = \frac{PL}{AE} = \frac{50{,}000(20)}{10(16\times 10^6)}$$

The condition of the problem states that the temperature rise ΔT is just sufficient so that all of the load is carried by the copper. Thus, the expanded length of the copper indicated by the dashed lines in the above sketch will be decreased by the action of the force. The net expansion of the copper is the expansion caused by the rise of temperature minus the compression due to the load. The change of length of the steel is due only to the temperature rise. Consequently we may write

$$(9.3\times 10^{-6})(20)\Delta T - \frac{50{,}000(20)}{10(16\times 10^6)} = (6.5\times 10^{-6})(20)\Delta T \qquad \text{or} \qquad \Delta T = 111°F$$

2.8. The rigid bar AD is pinned at A and attached to the bars BC and ED, as shown in Fig. 2-21(*a*). The entire system is initially stress free and the weights of all bars are negligible. The temperature of bar BC is lowered 25°C and that of the bar ED is raised 25°C. Neglecting any possibility of lateral buckling, find the normal stresses in bars BC and ED. For BC, which is brass, assume $E = 90$ GPa, $\alpha = 20\times 10^{-6}$/°C, and for ED, which is steel, take $E = 200$ GPa and $\alpha = 12\times 10^{-6}$/°C. The cross-sectional area of BC is 500 mm^2 and of ED is 250 mm^2.



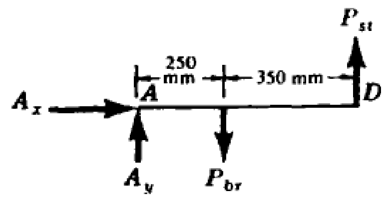


Fig. 2-20

Let us denote the forces on AD by P_{st} and P_{br} acting in the assumed directions shown in the free-body diagram, Fig. 2-21(*b*). Since AD rotates as a rigid body about A (as shown by the dashed line) we have $\Delta_{br}/250 = \Delta_{st}/350$ where Δ_{br} and Δ_{st} denote the axial compression of BC and the axial elongation of DE, respectively.

The total change of length of BC is composed of a shortening due to the temperature drop as well as a lengthening due to the axial force P_{br}. The total change of length of DE is composed of a lengthening due to the temperature rise as well as a lengthening due to the force P_{st}. Hence we have

$$\left(\frac{25}{60}\right)\left[(12\times 10^{-6})(250)(25) + \frac{P_{st}(250)}{(250)(200\times 10^9\times 10^{-6})}\right] = -(20\times 10^{-6})(300)(25) + \frac{P_{br}(300)}{(500)(90\times 10^9\times 10^{-6})}$$

or

$$6.66P_{br} - 2.08P_{st} = 153.0 \times 10^3$$

From statics,

$$\Sigma M_A = 250P_{br} - 600P_{st} = 0$$

Solving these equations simultaneously, $P_{st} = 10.99$ kN and $P_{br} = 26.3$ kN. Using $\sigma = P/A$ for each bar, we obtain $\sigma_{st} = 43.9$ MPa and $\sigma_{br} = 52.6$ kN.

2.9. Consider the statically indeterminate pin-connected framework shown in Fig. 2-22(a). Before the load P is applied the entire system is stress free. Find the axial force in each bar caused by the vertical load P. The two outer bars are identical and have cross-sectional area A_i, while the middle bar has area A_v. All bars have the same modulus of elasticity, E.

The free-body diagram of the pin at A appears as in Fig. 2-22(b) where F_1 and F_2 denote axial forces (lb) in the vertical and inclined bars. From statics we have

$$\Sigma F_v = F_1 + 2F_2 \cos\theta - P = 0$$

This is the only statics equation available since we have made use of symmetry in stating that the forces in the inclined bars are equal. Since it contains two unknowns, F_1 and F_2, the force system is statically

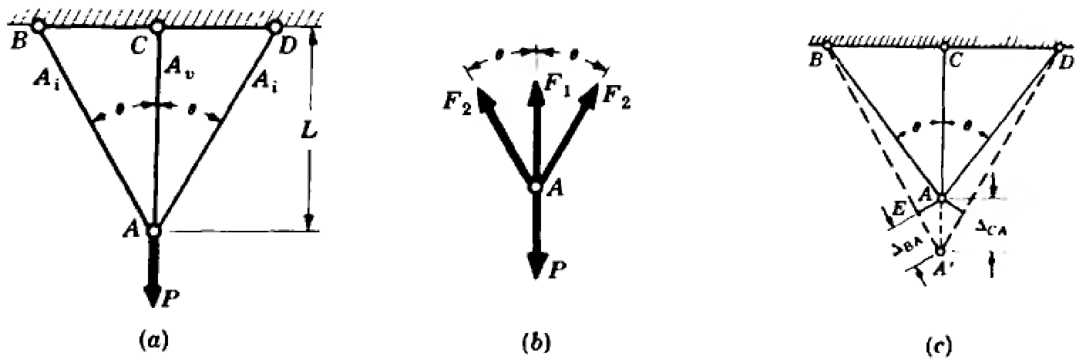


Fig. 2-22

indeterminate. Hence we must examine the deformations of the system to obtain another equation. Under the action of the load P the bars assume the positions shown by the dashed lines in Fig. 2-22(c).

Because the deformations of the system are *small*, the basic geometry is essentially unchanged and the angle $BA'A$ may be taken to be θ. AEA' is a right triangle and AE, which is actually an arc having a radius equal in length to the length of the inclined bars, is perpendicular to BA'. The elongation of the vertical bar is thus represented by AA' and that of the inclined bars by EA'. From this small triangle we have the relation

$$\Delta_{BA} = \Delta_{CA} \cos\theta$$

where Δ_{BA} and Δ_{CA} denote elongations of the inclined and vertical bars, respectively.

Since these bars are subject to axial loading their elongations are given by $\Delta = PL/AE$. From that expression we have

$$\Delta_{BA} = \frac{F_2(L/\cos\theta)}{A_i E} \qquad \text{and} \qquad \Delta_{CA} = \frac{F_1 L}{A_v E}$$

Substituting these in the above equation relating Δ_{BA} and Δ_{CA} we have

$$\frac{F_2 L}{A_i E \cos\theta} = \frac{F_1 L}{A_v E}\cos\theta \qquad \text{or} \qquad F_2 = F_1 \frac{A_i}{A_v}\cos^2\theta$$

Substituting this in the statics equation we find $F_1 + 2F_1(A_t/A_c)\cos^3\theta = P$, or

$$F_1 = \frac{P}{1 + 2(A_t/A_c)\cos^3\theta} \quad \text{and} \quad F_2 = \frac{P\cos^2\theta}{(A_c/A_t) + 2\cos^3\theta} \tag{1}$$

2.10. Two initially horizontal rigid bars AC and DG are pinned at A and G and are also connected by elastic vertical bars BD and CF, each of rigidity AE, as shown in Fig. 2-23. The temperature of bar BD is then raised by an amount ΔT. Determine the force in the two vertical bars.

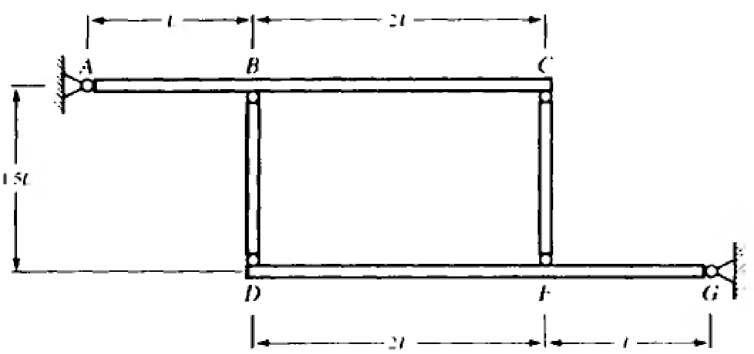


Fig. 2-23

Free-body diagrams of the components, assuming all unknown forces are positive, in tension appear as in Fig. 2-24.

For equilibrium of bar DG, we have

$$+\uparrow \Sigma M_G = -F_2(L) - F_1(3L) = 0 \qquad \therefore F_2 + 3F_1 = 0 \tag{1}$$

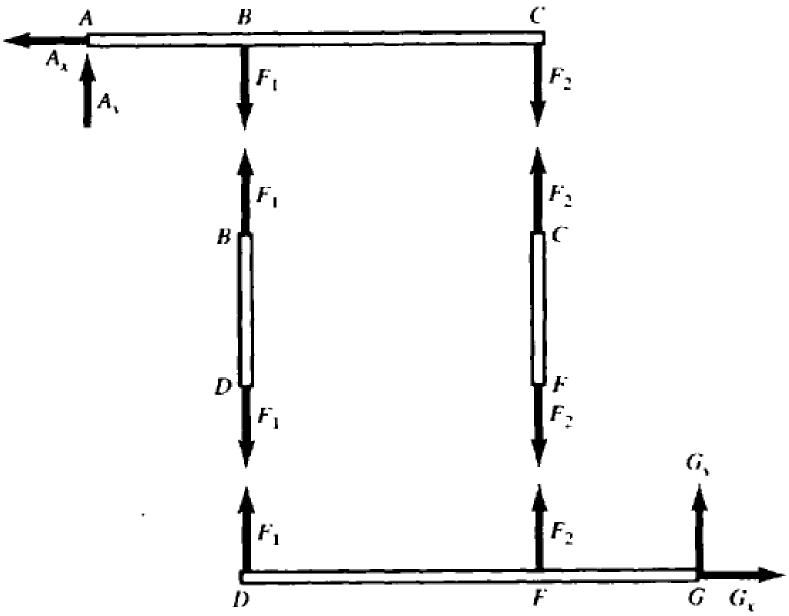


Fig. 2-24

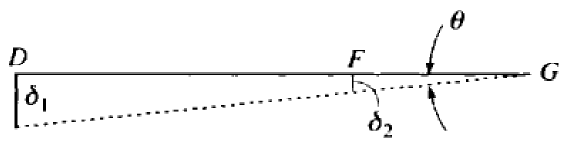


Fig. 2-25

We must now examine deformations of the system. To simplify this analysis, it is permissible to assume that the upper bar AC remains horizontal and that all distortion is due to rigid-body rotation of the lower bar DG about G. This leads to the deformed position of DG as shown by the dotted line in Fig. 2-25. The changes of length of the vertical bars are indicated by Δ_1 and Δ_2 in that figure. From geometry, for a small angle of rotation, we have

$$\theta = \frac{\delta_1}{3L} = \frac{\delta_2}{L}$$

from which

$$\delta_1 = 3\delta_2 \qquad (2)$$

The increase in length of bar BD is due partially to the force F_1 it carries and partially to the increase in temperature. It is

$$\delta_1 = \frac{F_1(1.5L)}{AE} + \alpha(\Delta T)(1.5L) \qquad (3)$$

For bar CF, the increase of length is due only to the force F_2 in it, so we have

$$\delta_2 = \frac{F_2(1.5L)}{AE} \qquad (4)$$

Solving Eqs. (*1*), (*2*), (*3*), and (*4*) simultaneously, we have

$$F_1 = -\frac{\alpha(\Delta T)AE}{10}$$

$$F_2 = \frac{3\alpha(\Delta T)AE}{10}$$

The negative sign accompanying the bar force F_1 indicates that bar BD is in compression, whereas bar CF is in tension.

2.11. A two-dimensional framework consists of two bars AB and BH forming a 30° triangle with pins at A, B, and H together with a horizontal bar GD, as shown in Fig. 2-26. Because of a manufacturing error, the bar GD is slightly short of the length $2L$. All bars have axial rigidity AE. Determine the axial force in bar GD when the gap Δ is closed by mechanical action.

First, let us examine the forces acting at point B. In particular, we apply a horizontal force F at the node B, and a free-body diagram of that node is shown in Fig. 2-27. For horizontal equilibrium, we have

$$\Sigma F_x = F - F_{BA}\cos 30° = 0$$

from which

$$F_{BA} = \frac{F}{\cos 30°} \qquad (1)$$

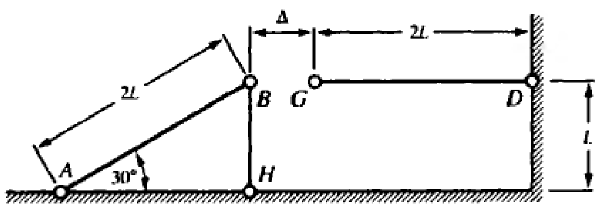


Fig. 2-26

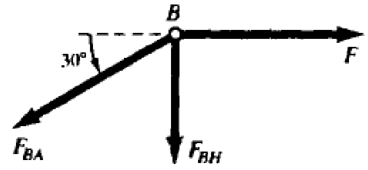


Fig. 2-27

Next, let us examine displacements at the node B. Since we have just found that bar AB is in tension, it will lengthen an amount Δ_{BA}, as shown in Fig. 2-28, where

$$\Delta_{BA} = \frac{F_{BA}(2L)}{AE} \tag{2}$$

The bar AB will then rotate as a rigid body about point A through the circular arc from B'' to B', which for small deformations we approximate as a straight line from B'' to B'. The horizontal projections of BB'' and $B''B'$ are denoted by Δ_3 and Δ_2, respectively. From geometry we have

$$\Delta_2 = \Delta_1 \sin 30^\circ = \Delta_{BA}(\tan 30^\circ)(\sin 30^\circ) \tag{3}$$

$$\Delta_3 = \Delta_{BA} \cos 30^\circ \tag{4}$$

The bar GD is subject to an equal and opposite force F, as shown in Fig. 2-29, and it elongates an amount

$$\frac{F(2L)}{AE} \tag{5}$$

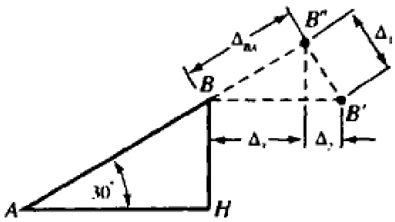


Fig. 2-28

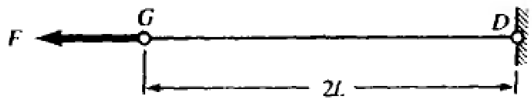


Fig. 2-29

Thus, to bring points B and G together and close the gap, we have

$$\Delta_2 + \Delta_3 + \frac{F(2L)}{AE} = \Delta \tag{6}$$

From Eqs. (*1*) through (*6*), we have the required force in bar GD to close the gap Δ:

$$F = \frac{AE\Delta}{2L(1 + \tan^2 30^\circ)} = \frac{2AE\Delta}{5L}$$

2.12. The rigid horizontal bar ABC is supported by vertical elastic posts and restrained against horizontal movement at A as shown in Fig. 2-30. A vertical load P acts at C. The extensional rigidity of each post is indicated in the figure and each is of length L. Find the axial force in each of the three posts.

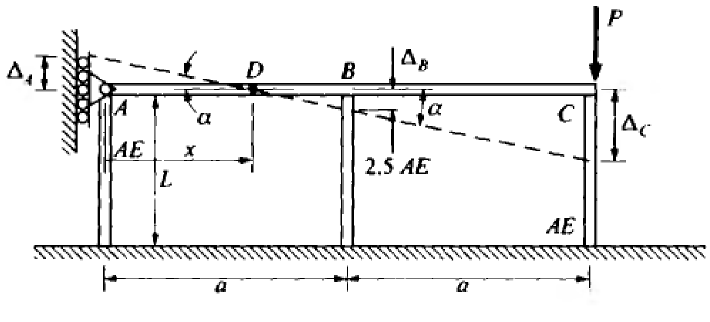


Fig. 2-30

Due to the load P the originally horizontal bar deforms to the configuration indicated by the dotted line in Fig. 2-31. That is, it rotates as a rigid body about some point D (whose location is unknown) through the angle α.

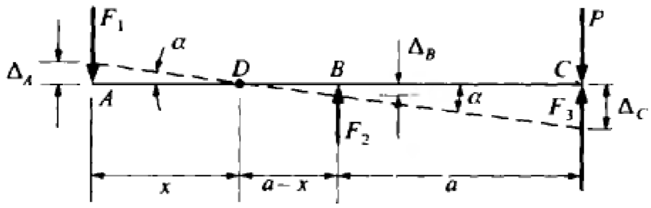


Fig. 2-31

Figure 2-31 shows a free-body diagram of ABC where the forces exerted on ABC by the posts are represented by F_1, F_2, and F_3. The change of length of each post is indicated by Δ in the figure. From the geometry of the deformed system we have

$$\alpha = \frac{\Delta_A}{x} = \frac{(F_1 L/AE)}{x} = \frac{(F_2 L/2.5AE)}{a - x} \tag{1}$$

For this parallel force system there are two equations of static equilibrium. For the first equation we set

$$\Sigma M_C = F_1(2a) - F_2(a) = 0$$

from which

$$F_1 = \frac{F_2}{2} \tag{2}$$

If we now substitute (2) in (1), we find

$$\frac{(F_2 L/2AE)}{x} = \frac{(F_2 L/2.5AE)}{a - x}$$

from which

$$x = \left(\frac{5}{9}\right) a \tag{3}$$

For the second statics equation we write

$$\Sigma M_B = -F_3 a + Pa - F_1 a = 0$$

Thus

$$F_3 = P - F_1 \tag{4}$$

The changes of length of posts C and A are given by

$$\Delta_C = \frac{F_3 L}{AE}; \qquad \Delta_A = \frac{F_1 L}{AE} \tag{5}$$

From the geometry of Fig. 2-31 we have

$$\frac{\Delta_C}{a + (a - x)} = \frac{\Delta_A}{x}$$

and from Eq. (3):

$$\Delta_C = \left(\frac{13}{5}\right)\Delta_A \tag{6}$$

Substituting (5) in (4), we find

$$F_3 = (13/9)\,F_1 \tag{7}$$

and from (2) and (7) we have

$$F_1 = \left(\frac{9}{22}\right)P; \qquad F_2 = \left(\frac{9}{11}\right)P \tag{8}$$

Ultimate Strength (Limit Design)

In each of the following problems the elastoplastic behavior of the material is assumed to follow the idealized stress-strain curve of Fig. 2-32.

The ultimate load, or limit load, determined in each of the following problems is the maximum possible load that can be applied to each system provided the stress-strain curve is of the type indicated and the material has infinite ductility, i.e., the flat region of the curve extends indefinitely to the right.

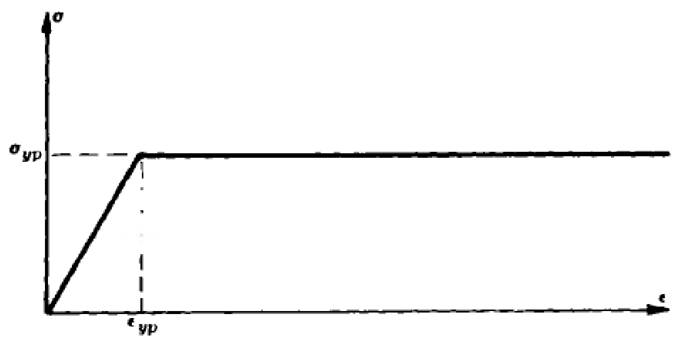


Fig. 2-32

2.13. Consider the system composed of three vertical bars as indicated in Fig. 2-33(a). The outer bars of length L are equally spaced from the central bar and a load P is applied to the rigid horizontal member. Using limit design, determine the ultimate load P. The values of A and E are identical in all three bars.

Let us analyze the action as the load P increases from an initial value of zero, i.e., as it is slowly applied. For equilibrium we have

$$2P_1 + P_2 = P \tag{1}$$

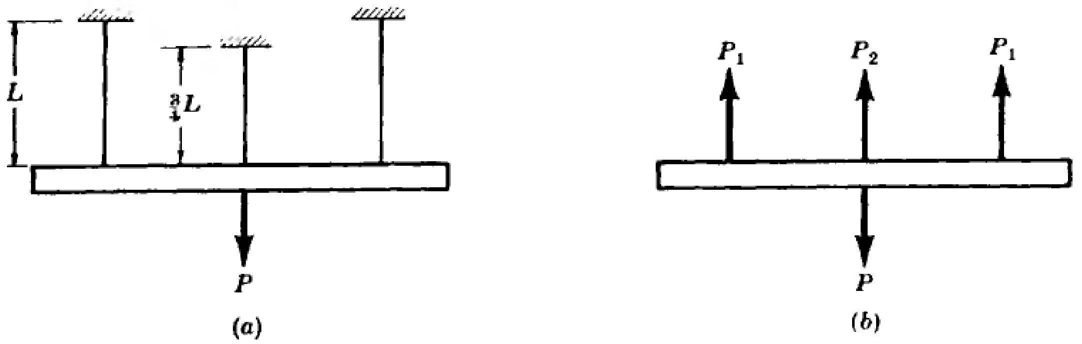


Fig. 2-33

where P_1 represents the force in each of the outer bars and P_2 is the force in the inner bar [see Fig. 2-33(*b*)]. Since the horizontal member is rigid, the vertical elongation of each of the outer bars must equal that of the central bar. Thus

$$\frac{P_1 L}{AE} = \frac{P_2(3L/4)}{AE} \tag{2}$$

or

$$P_1 = \tfrac{3}{4}P_2 \tag{3}$$

Substituting this value in (*1*) we find

$$P_2 = \tfrac{2}{5}P \qquad P_1 = \tfrac{3}{10}P \tag{4}$$

The system thus begins to yield when $P_2 = \sigma_{yp}A$. Thus

$$P_{yp} = \tfrac{5}{2}\sigma_{yp}A$$

From the time of yielding of the central bar, the system deforms as if supported by only the two outside bars (which still act elastically) together with a constant force $\sigma_{yp}A$ supplied by the central bar. The value of P increases until yielding begins in each of the outer bars, i.e., when $P_1 = \sigma_{yp}A$. The ultimate load is thus

$$P_u = 2P_1 + P_2 = 2\sigma_{yp}A + \sigma_{yp}A = 3\sigma_{yp}A$$

It is to be noted that the deformation equation (*2*) is not employed to determine the ultimate load.

2.14. Reconsider Problem 2.9 for the case of three bars of equal cross-sectional area. Determine the ultimate load-carrying capability of the system.

For $A_i = A_v = A$ the force in the vertical bar exceeds that in either inclined bar as indicated by (*1*) of Problem 2.9. Thus, as P increases, the central vertical bar is the first to enter the inelastic range of action and its stiffness (effective value of AE) decreases. Any additional increase in the load P will cause no further increase in F_1 which will remain at the limit value $F_1^* = \sigma_{yp}A$. The central bar can now be replaced by a constant upward vertical force F_1^* and the system is now reduced to a statically determinate system consisting of the two outer bars subject to an applied load $P - F_1^*$. The load P can now be increased until the outer bars also develop the yield stress. It is not necessary to consider deformations of the system; we need look only at the equilibrium relation

$$P = F_1^* + 2F_2 \cos\theta \tag{1}$$

As the load P increases still more, the outer bars also reach the yield point and the force in each of them becomes

$$F_2^* = \sigma_{yp}A \tag{2}$$

The ultimate load thus corresponds to the situation when $F_1^* = F_2^* = \sigma_{yp}A$ and this load is found from (*1*) as

$$P_u = \sigma_{yp}A(1 + 2\cos\theta) \tag{3}$$

This *limit load* should be divided by some safety factor to obtain a *working load.*

2.15. Suppose the three-bar system of Problem 2.9 is to withstand a load $P = 200$ kN. Compare the bar weights required if the design is based upon (*a*) the peak stress just reaching the yield point, and (*b*) ultimate load analysis. Assume that all bars are of identical cross section, that $\theta = 45°$, and take the yield point of the material to be 250 MPa.

(*a*) According to the elastic theory of Problem 2.9, the force in the vertical bar becomes

$$F_1 = \frac{2P}{2+\sqrt{2}} = 117\text{ kN}$$

If the stress in that bar is equal to the yield point, we have a required cross-sectional area of $F_1 = A_1\sigma_{yp}$. Hence

$$117 \times 10^3 = A_1(250) \qquad \text{or} \qquad A_1 = 468\text{ mm}^2$$

(*b*) If the ultimate load analysis of Problem 2.14 is employed, the stresses in all three bars are equal to the yield point and from (*3*) of Problem 2.14 we find a cross-sectional area of

$$200 \times 10^3 = 250A_2[1 + 2(0.707)] \qquad \text{or} \qquad A_2 = 331\text{ mm}^2$$

Ultimate load analysis thus implies a 29 percent saving in cross-sectional area and the same weight saving.

2.16. The frame shown in Fig. 2-34 consists of three pinned end bars AD, BD, and CD. The bars are of identical material and cross section, and the ultimate load-carrying capacity of each is 30 kN. Determine the ultimate vertical load P_u that may be applied to the system at point D.

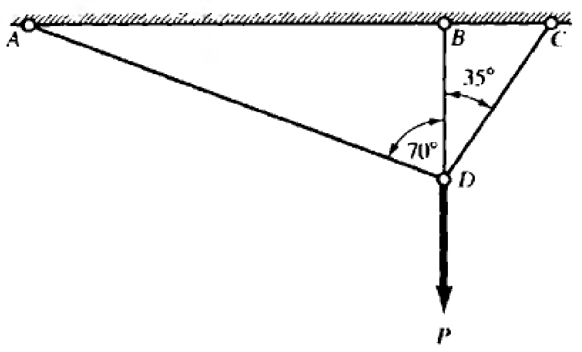


Fig. 2-34

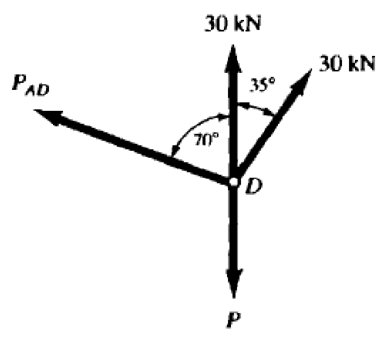


Fig. 2-35

Let us assume that bars BD and CD have reached yield. Examination of a free-body diagram for the node D as shown in Fig. 2-35 leads to

$$\Sigma F_x = 30\sin 35° - P_{AD}\sin 70° = 0$$

$$P_{AD} = 18.3\text{ kN}$$

Thus, bar AD does not yield since the bar force for equilibrium is less than the 30 kN required for yield. Summing vertically for equilibrium we have

$$\Sigma F_y = -P_u + 18.3\cos 70^\circ + 30 + 30\cos 35^\circ = 0$$

$$P_u = 60.9\ \text{kN}$$

2.17. A system composed of a rigid horizontal member AB supported by four bars is indicated in Fig. 2-36(a). The bars have identical cross sections and are made of the same material. Determine the ultimate load P that may be applied to the system.

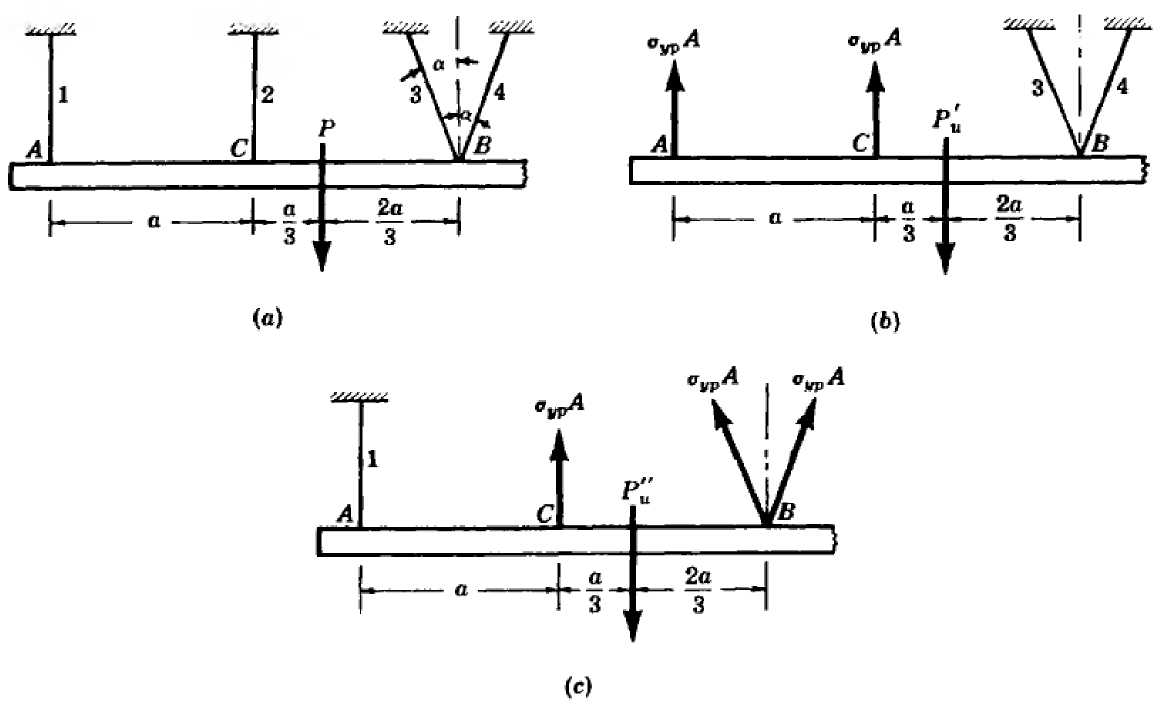


Fig. 2-36

Since the member AB is rigid, it is evident that, upon application of a sufficiently large load P, AB may rotate as a rigid body about either point A or point B. (The ultimate load implies plastic deformation in bar 2; hence it is not necessary to consider rotation about C.) It is necessary to determine the ultimate loads corresponding to these two possibilities and then to select the smaller.

Let us first assume that yielding first begins in bars 1 and 2, in which case their effect can be represented by the two constant forces $\sigma_{yp}A$ as indicated in Fig. 2-36(b). The bars 3 and 4 are still in the elastic range of action and the forces in them are unknown. However, it is not necessary to determine the forces since the ultimate load P'_u may be determined by summing moments about point B:

$$P'_u\left(\frac{2a}{3}\right) - \sigma_{yp}A(a) - \sigma_{yp}A(2a) = 0$$

Solving,

$$P'_u = 4.5\sigma_{yp}A$$

Next, let us consider that yielding begins in bars 2, 3, and 4 as indicated in Fig. 2-36(c). Bar 1 is still in the elastic range of action. Taking moments about point A:

$$(\sigma_{yp}A\cos\alpha)4a + \sigma_{yp}Aa - P''_u\frac{4a}{3} = 0$$

Solving,

$$P_u'' = \tfrac{3}{4}\sigma_{yp}A(1 + 4\cos\alpha)$$

It is evident from inspection of P_u' and P_u'' that, for all values of the angle α, the value of P_u'' is the smaller of the two and thus P_u'' represents the ultimate load. When the applied load reaches this value, the system is essentially converted into a mechanism and the rigid bar rotates about point A. Even in this condition bar 1 is not working to its full capacity.

Supplementary Problems

2.18. Two initially straight bars are joined together and attached to supports as in Fig. 2-37. The left bar is brass for which $E = 90\text{ GPa}$, $\alpha = 20 \times 10^{-6}/°C$, and the right bar is aluminum for which $E = 70\text{ GPa}$, $\alpha = 25 \times 10^{-6}/°C$. The cross-sectional area of the brass bar is 500 mm^2, and that of the aluminum bar is 750 mm^2. Let us suppose that the system is initially stress free and that the temperature then drops 20°C.

(*a*) If the supports are unyielding, find the normal stress in each bar.

(*b*) If the right support yields 0.1 mm, find the normal stress in each bar. The weight of the bars is negligible. *Ans.* (*a*) $\sigma_{br} = 41\text{ MPa}$, $\sigma_{al} = 27.33\text{ MPa}$; (*b*) $\sigma_{br} = 28.4\text{ MPa}$, $\sigma_{al} = 19\text{ MPa}$

Fig. 2-37

2.19 The framework shown in Fig. 2-38 consists of bars AD, AC, BC, and BD pinned at A, B, C, and D, and also a fifth bar CD. The system is loaded by the equal and opposite forces P. All bars are of identical material and cross section. Determine the decrease of the distance between A and B due to these loads.

Ans. $\dfrac{PL\sqrt{2}}{AE}(\sqrt{2} + 1)$

2.20. Refer to the framework shown in Fig. 2-38. Now, instead of the two loads P, the temperature of the entire system is raised by an amount ΔT. Determine the change of distance between A and B in terms of the geometry of the system and the coefficient of thermal expansion α of the material. *Ans.* $L\sqrt{2}\alpha(\Delta T)$

2.21. Refer to Problem 2.6. If the conical bar has a diameter at its small end of 100 mm, a length of 1 m, and is of steel having $E = 200\text{ GPa}$ and a coefficient of thermal expansion of $12 \times 10^{-6}/°C$, determine the maximum axial stress in the bar due to a temperature drop of 20°C. *Ans.* 96 MPa

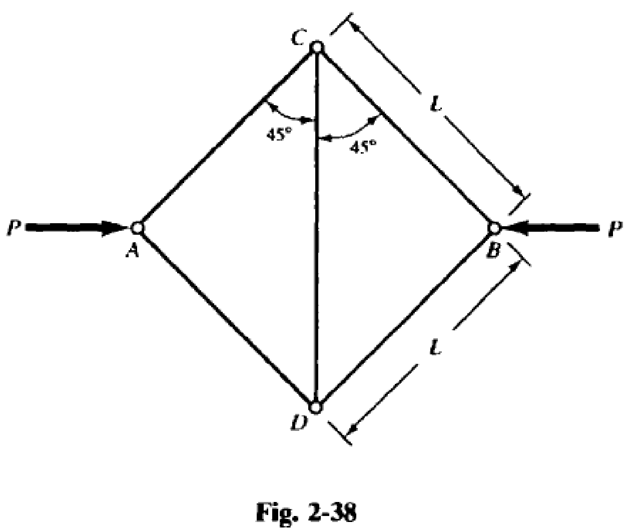


Fig. 2-38

2.22. A compound bar is composed of a strip of copper between two cold-rolled steel plates. The ends of the assembly are covered with infinitely rigid cover plates and an axial tensile load P is applied to the bar by means of a force acting on each rigid plate as shown in Fig. 2-39. The width of all bars is 4 in, the steel plates are each $\frac{1}{4}$ in thick and the copper is $\frac{3}{4}$ in thick. Determine the maximum load P that may be applied. The ultimate strength of the steel is 80,000 lb/in² and that of the copper is 30,000 lb/in². A safety factor of 3 based upon the ultimate strength of each material is satisfactory. For steel $E = 30 \times 10^6$ lb/in² and for copper $E = 13 \times 10^6$ lb/in². *Ans.* $P = 76{,}200$ lb

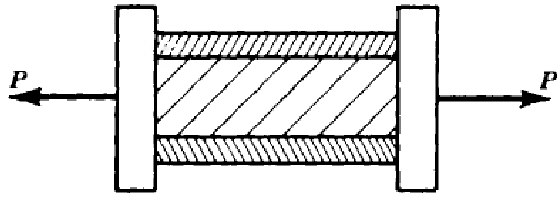


Fig. 2-39

2.23. An aluminum right-circular cylinder surrounds a steel cylinder as shown in Fig. 2-40. The axial compressive load of 200 kN is applied through the infinitely rigid cover plate shown. If the aluminum cylinder is originally 0.25 mm longer than the steel before any load is applied, find the normal stress in each when the temperature has dropped 20 K and the entire load is acting. For steel take $E = 200$ GPa; $\alpha = 12 \times 10^{-6}/°C$, and for aluminum assume $E = 70$ GPa, $\alpha = 25 \times 10^{-6}/°C$.
Ans. $\sigma_{st} = 9$ MPa, $\sigma_{al} = 15.5$ MPa

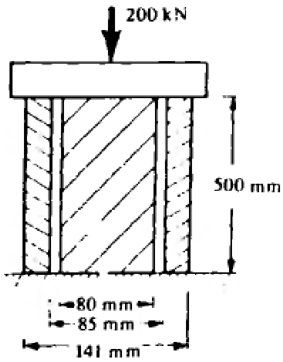


Fig. 2-40

2.24. The rigid horizontal bar AB is supported by three vertical wires as shown in Fig. 2-41 and carries a load of 24,000 lb. The weight of AB is negligible and the system is stress free before the 24,000-lb load is applied. After the load is applied, the temperature of all three wires is raised by 25°F. Find the stress in each wire

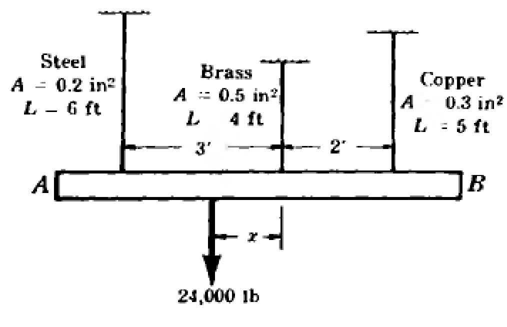


Fig. 2-41

as well as the location of the applied load in order that AB remains horizontal. For the steel wire take $E = 30 \times 10^6\ \text{lb/in}^2$, $\alpha = 6.5 \times 10^{-6}/°\text{F}$, for the brass wire $E = 14 \times 10^6\ \text{lb/in}^2$, $\alpha = 10.4 \times 10^{-6}/°\text{F}$, and for copper $E = 17 \times 10^6\ \text{lb/in}^2$, $\alpha = 9.3 \times 10^{-6}/°\text{F}$. Neglect any possibility of lateral buckling of any of the wires. *Ans.* $\sigma_{st} = 32{,}300\ \text{lb/in}^2$, $\sigma_{br} = 22{,}400\ \text{lb/in}^2$, $\sigma_{cu} = 21{,}400\ \text{lb/in}^2$, $x = 0.273$ ft

2.25. A system consists of two rigid end-plates, tied together by three horizontal bars as shown in Fig. 2-42. Through a fabrication error, the central bar, ②, is $0.0005L$ too short. All bars are of identical cross section and of steel having $E = 210$ GPa. Find the stress in each bar after the system has mechanically been pulled together so that the gap Δ is closed.

Ans $\sigma_1 = -35$ MPa

$\sigma_2 = 70$ MPa

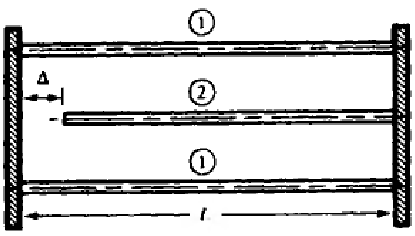


Fig. 2-42

2.26. A structural system consists of three joined bars of different materials and geometries, as shown in Fig. 2-43. Bar ① is aluminum alloy, bar ② is cold rolled brass, and bar ③ is tempered alloy steel. Properties and dimensions of all three are shown in the figure. Initially, the entire system is free of stresses, but then the right support is moved 3 mm to the right whereas the left support remains fixed in space. Determine the stress in each bar due to this 3 mm displacement.

Ans. $\sigma_1 = 223$ MPa

$\sigma_2 = 178$ MPa

$\sigma_3 = 446$ MPa

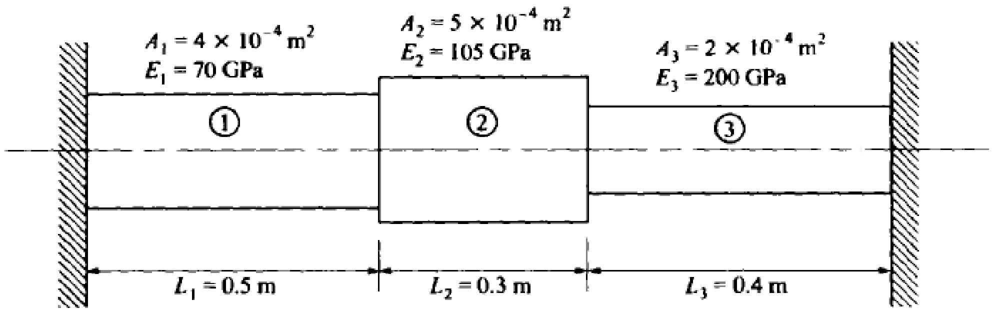


Fig. 2-43

2.27. The bar AC is absolutely rigid and is pinned at A and attached to bars DB and CE as shown in Fig. 2-44. The weight of AC is 50 kN and the weights of the other two bars are negligible. Consider the temperature of both bars DB and CE to be raised 35°C. Find the resulting normal stresses in these two bars. DB is copper for which $E = 90$ GPa, $\alpha = 18 \times 10^{-6}/°\text{C}$, and the cross-sectional area is 1000 mm², while CE is steel for which $E = 200$ GPa, $\alpha = 12 \times 10^{-6}/°\text{C}$, and the cross section is 500 mm². Neglect any possibility of lateral buckling of the bars. *Ans.* $\sigma_{st} = 72$ MPa, $\sigma_{cu} = -21.7$ MPa

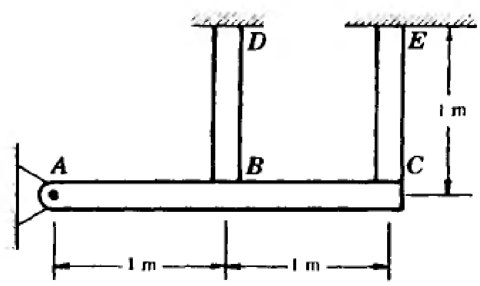


Fig. 2-44

2.28. The three bars shown in Fig. 2-45 support the vertical load of 5000 lb. The bars are all stress free and joined by the pin at A before the load is applied. The load is put on gradually and simultaneously the temperature of all three bars decreases by 15°F. Calculate the stress in each bar. The outer bars are each brass and of cross-sectional area 0.4 in^2. The central bar is steel and of area 0.3 in^2. For brass $E = 13 \times 10^6$ lb/in^2 and $\alpha = 10.4 \times 10^{-6}$/°F and for steel $E = 30 \times 10^6$ lb/in^2 and $\alpha = 6.3 \times 10^{-6}$/°F.
Ans. $\sigma_{br} = 3550$ lb/in^2, $\sigma_{st} = 10{,}000$ lb/in^2

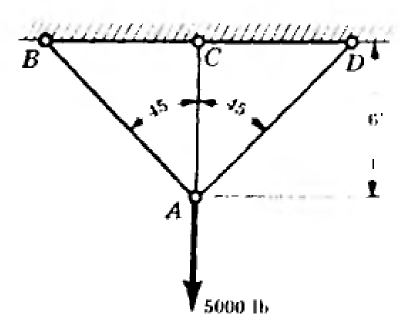


Fig. 2-45

2.29. A framework consists of three pinned bars AD, BD, and CD as shown in Fig. 2-46. The load $F = 8$ kN acts vertically at D. The cross-sectional areas of bars ① and ③ are each $200\sqrt{5}$ mm^2, the area of bar ② is 400 mm^2, $L = 3$ m, the elastic moduli are $E_1 = 200$ GPa, $E_2 = 80$ GPa, and $E_3 = 100$ GPa. Determine the horizontal and vertical components of displacement of point D as well as the axial force in bar ②.
Ans. −0.136 mm, −0.204 mm, 2.182 kN

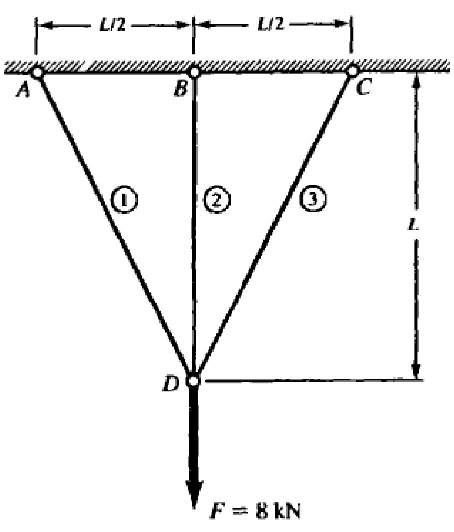


Fig. 2-46

2.30. The rigid bar AD in Fig. 2-47 is pinned at A and supported by a steel rod at D together with a linear spring at B. The bar carries a vertical load of 30 kN applied at C. Determine the vertical displacement of point D. *Ans.* 0.8 mm

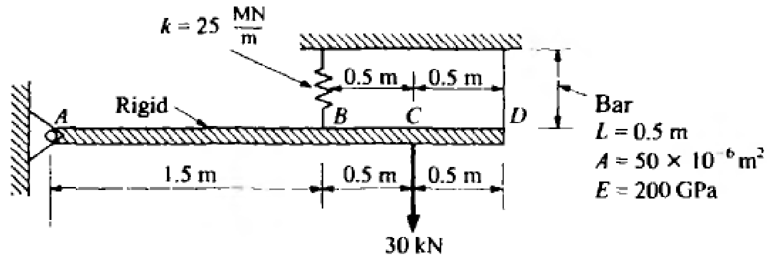


Fig. 2-47

2.31. The curved rigid bar ADB is joined to the two elastic bars OA and OB as shown in Fig. 2-48. For additional strength, it is desired to join bar OC to ADB at the midpoint D. However, through a manufacturing error OC is fabricated 1.8 mm too short. Determine the initial stresses in these three bars when point C is mechanically forced to D and these two points pinned together. The area of each outer bar is three times that of the central bar, and for all bars $E = 200$ GPa.
Ans. Outer bars 43.6 MPa, central bar 75.5 MPa

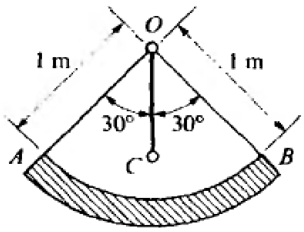


Fig. 2-48

2.32. The five-bar assembly of Fig. 2-49 was found to be slightly defective, i.e., points A and C which ought to have coincided failed to coincide by a distance Δ. After these points had been forced to coincide, the joint at that point was pinned. Determine the forces existing in each bar. All bars have the same cross-sectional area.

Ans. $F_1 = F_2 = F_3 = \left(\dfrac{\sqrt{3}}{2+3\sqrt{3}}\right)\dfrac{\Delta AE}{L} \qquad F_4 = F_5 = -\left(\dfrac{1}{2+3\sqrt{3}}\right)\dfrac{\Delta AE}{L}$

2.33. The rigid bar AB is supported by the four rods shown in Fig. 2-50. The rods are each circular in cross section and of 50 mm diameter. They have a yield point of 300 MPa. Using limit design determine the maximum weight of the bar AB. Assume that the weight is uniformly distributed along the length. *Ans.* 1.38 MN

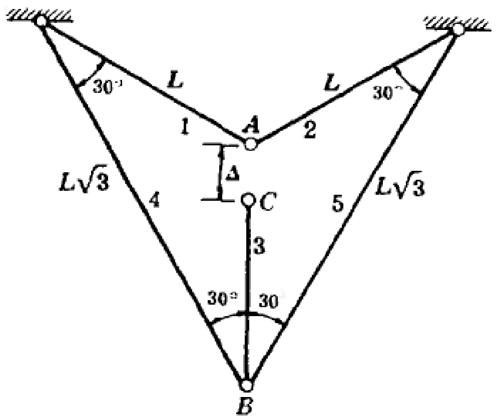


Fig. 2-49

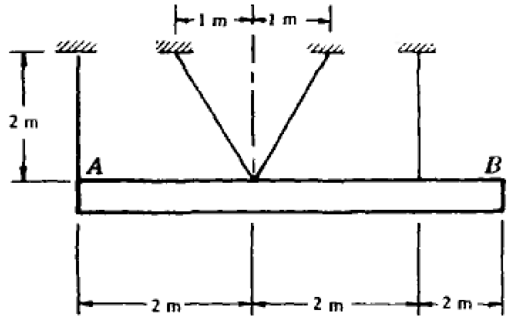


Fig. 2-50

Chapter 3

Thin-Walled Pressure Vessels

In Chaps. 1 and 2 we examined various cases involving uniform normal stresses acting in bars. Another application of uniformly distributed normal stresses occurs in the approximate analysis of thin-walled pressure vessels, such as cylindrical, spherical, conical, or toroidal shells subject to internal or external pressure from a gas or a liquid. In this chapter we will treat only thin shells of revolution and restrict ourselves to axisymmetric deformations of these shells.

NATURE OF STRESSES

The shell of revolution shown in Fig. 3-1 is formed by rotating a plane curve (the meridian) about an axis lying in the plane of the curve. The radius of curvature of the meridian is denoted by r_1 and this of course varies along the length of the meridian. This radius of curvature is defined by two lines perpendicular to the shell and passing through points B and C of Fig. 3-1. Another parameter, r_2, denotes the radius of curvature of the shell surface in a direction perpendicular to the meridian. This radius of curvature is defined by perpendiculars to the shell through points A and B of Fig. 3-1. The center of curvature corresponding to r_2 must lie on the axis of symmetry of the shell although the center for r_1 in general does not lie there. An internal pressure p acting normal to the curved surface of the shell gives rise to *meridional stresses* σ_ϕ and *hoop stresses* σ_θ as indicated in the figure. These stresses are orthogonal to one another and act in the plane of the shell wall.

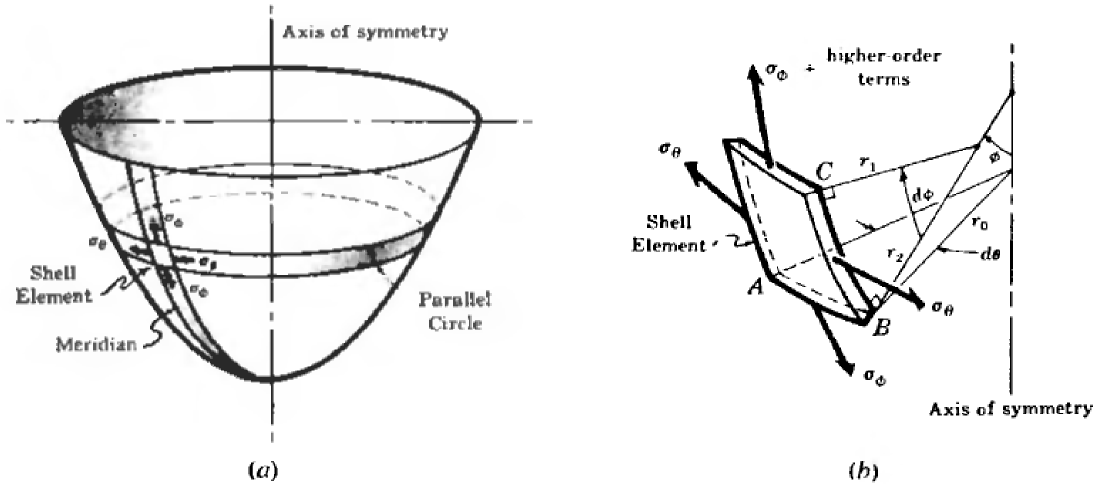


Fig. 3-1

In Problem 3.15 it is shown that

$$\frac{\sigma_\phi}{r_1} + \frac{\sigma_\theta}{r_2} = \frac{p}{h}$$

where h denotes the shell thickness. A second equation may be obtained by consideration of the vertical equilibrium of the entire shell above some convenient parallel circle, as indicated in Problem 3.15. The derivation of the above equation assumes that the stresses σ_ϕ and σ_θ are uniformly distributed over the wall thickness.

Applications of this analysis to cylindrical shells are to be found in Problems 3.1 through 3.6; to spherical shells in Problems 3.7 through 3.11, and 3.16, 3.17; to conical shells in Problem 3.14; and to toroidal shells in Problem 3.18.

LIMITATIONS

The ratio of the wall thickness to either radius of curvature should not exceed approximately 0.10. Also there must be no discontinuities in the structure. The simplified treatment presented here does not permit consideration of reinforcing rings on a cylindrical shell as shown in Fig. 3-2, nor does it give an accurate indication of the stresses and deformations in the vicinity of end closure plates on cylindrical pressure vessels. Even so, the treatment is satisfactory in many design problems.

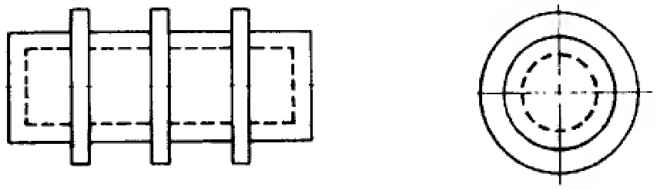


Fig. 3-2

The problems which follow are concerned with stresses arising from a uniform *internal* pressure acting on a thin shell of revolution. The formulas for the various stresses will be correct if the sense of the pressure is reversed, i.e., if external pressure acts on the container. However, it is to be noted that an additional consideration, beyond the scope of this book, must then be taken into account. Not only must the stress distribution be investigated but another study of an entirely different nature must be carried out to determine the load at which the shell will *buckle* due to the compression. A buckling or instability failure may take place even though the peak stress is far below the maximum allowable working stress of the material.

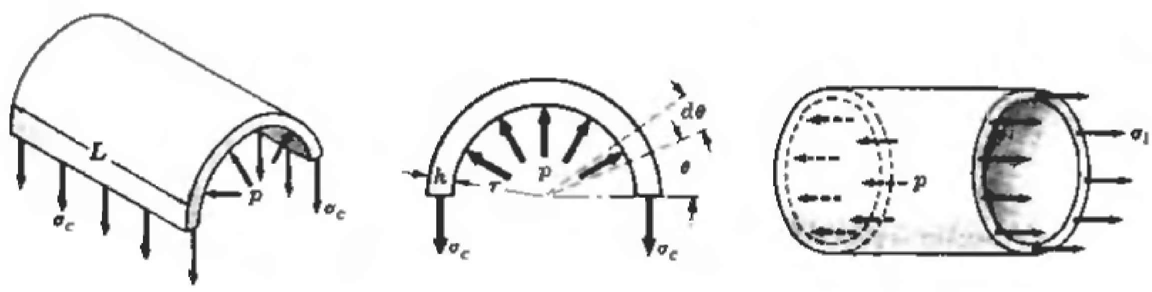
APPLICATIONS

Liquid and gas storage tanks and containers, water pipes, boilers, submarine hulls, and certain airplane components are common examples of thin-walled pressure vessels.

Solved Problems

3.1. Consider a thin-walled cylinder closed at both ends by cover plates and subject to a uniform internal pressure p. The wall thickness is h and the inner radius r. Neglecting the restraining effects of the end-plates, calculate the longitudinal (meridional) and circumferential (hoop) normal stresses existing in the walls due to this loading.

To determine the circumferential stress σ_c let us consider a section of the cylinder of length L to be removed from the vessel. The free-body diagram of half of this section appears as in Fig. 3-3(*a*). Note that the body has been cut in such a way that the originally *internal* effect (σ_c) now appears as an *external* force to this free body. Figure 3-3(*b*) shows the forces acting on a cross section.



(a) (b) (c)

Fig. 3-3

The horizontal components of the radial pressures cancel one another by virtue of symmetry about the vertical centerline. In the vertical direction we have the equilibrium equation

$$\Sigma F_v = -2\sigma_c hL + \int_0^{\pi} pr(d\theta)(\sin\theta)L = 0$$

Integrating,

$$2\sigma_c hL = -prL[\cos\theta]_0^{\pi} \qquad \text{or} \qquad \sigma_c = \frac{pr}{h}$$

Note that the resultant vertical force due to the pressure p could have been obtained by multiplying the pressure by the horizontal *projected area* upon which the pressure acts.

To determine the longitudinal stress σ_l consider a section to be passed through the cylinder normal to its geometric axis. The free-body diagram of the remaining portion of the cylinder is shown in Fig. 3-3(*c*). For equilibrium

$$\Sigma F_h = -p\pi r^2 + 2\pi rh\sigma_l = 0 \qquad \text{or} \qquad \sigma_l = \frac{pr}{2h}$$

Consequently, the circumferential stress is twice the longitudinal stress. These rather simple expressions for stresses are not accurate in the immediate vicinity of the end closure plates.

3.2. The Space Simulator at the Jet Propulsion Laboratory in Pasadena, California, consists of a 27-ft-diameter cylindrical vessel which is 85 ft high. It is made of cold-rolled stainless steel having a proportional limit of 165,000 lb/in^2. The minimum operating pressure of the chamber is 10^{-6} torr, where 1 torr = 1/760 of a standard atmosphere, which in turn is approximately 14.7 lb/in^2. Determine the required wall thickness so that a working stress based upon the proportional limit together with a safety factor of 2.5 will not be exceeded. This solution will neglect the possibility of buckling due to the external pressure, and also the effects of certain hard-load points in the Simulator to which the test specimens are attached.

From Problem 3.1 the significant stress is the circumferential stress, given by $\sigma_c = pr/h$. The pressure to be used for design is essentially the atmospheric pressure acting on the outside of the shell, which is satisfactorily represented as 14.7 lb/in^2 since the internal pressure of 10^{-6} torr is negligible compared to 14.7 lb/in^2. We thus have

$$\frac{165{,}000}{2.5} = \frac{14.7(13.5)(12)}{h} \qquad \text{or} \qquad h = 0.036 \text{ in}$$

3.3. A vertical axis circular cylindrical wine storage tank, fabricated from stainless steel, has total height of 25 ft, a radius of 5 ft, and is filled to a depth of 20 ft with wine. An inert gas occupies the 5-ft height H_0 above the liquid-free surface and is pressurized to a value of p_0 of 12 lb/in^2.

If the working stress in the steel is 28,000 lb/in^2, determine the required wall thickness. The specific weight of the wine is 62.4 lb/ft^3.

If there were no gas pressure above the surface of the wine, the pressure (in any direction) at any depth y below the liquid-free surface is given as $p = \gamma y$, where γ is the specific weight (weight per unit volume) of the wine. This is evident if we consider the pressure on 1 ft^2 of the horizontal cross section a

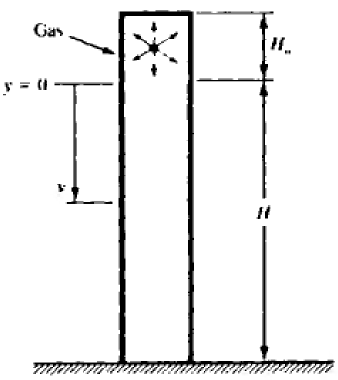


Fig. 3-4

distance y below the liquid surface to be given by the weight of the column of wine above that section divided by the 1-ft^2 area. The total pressure at the base ($y = H$) is thus $(p_0 + \gamma H)$ so that from Problem 3.1 the circumferential stress is

$$\sigma_c = \frac{(p_0 + \gamma H)R}{h} \tag{1}$$

where t is tank wall thickness.

The liquid has zero viscosity, and hence it can exert no tangential shearing stresses on the inside of the tank wall. For vertical equilibrium the upward thrust of the gas pressure p_0 must be balanced by longitudinal stresses σ_l distributed uniformly around the tank wall at the tank bottom as shown in Fig. 3-5. Thus

$$\Sigma F_y = \sigma_L(2\pi R)h - p_0\pi R^2 = 0$$

$$\therefore \sigma_L = \frac{p_0 R}{2h} \qquad \text{(independent of } y\text{)} \tag{2}$$

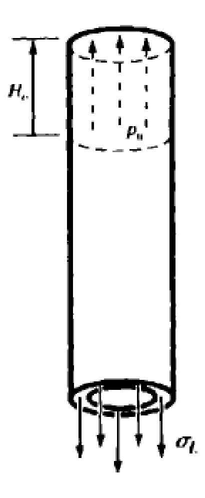


Fig. 3-5

The circumferential stress (*1*) is clearly larger than the longitudinal stress (*2*) and thus controls design. We have from (*1*)

$$\frac{\left[12\ \text{lb/in}^2 + (62.4\ \text{lb/ft}^3)\left(\dfrac{\text{ft}^3}{1728\ \text{in}^3}\right)(240\ \text{in})\right](60\ \text{in})}{h} = 18{,}000\ \text{lb/in}^2$$

from which the thickness is found to be $h = 0.055$ in.

3.4. A vertical axis circular cylindrical liquid storage tank of cross-sectional area A is filled to a depth of 15 m with a liquid whose specific weight (weight per unit volume) γ varies according to the law $\gamma = \gamma_{H_2O}(1 + 0.018z)$, where z is depth below the free surface of the liquid as shown in Fig. 3-6(*a*). The tank is 4 m in radius and is made of steel having a yield point of 240 MPa. The specific weight of water γ_{H_2O} is 9810 N/m³. If a safety factor of 2 is applied, determine the required tank wall thickness.

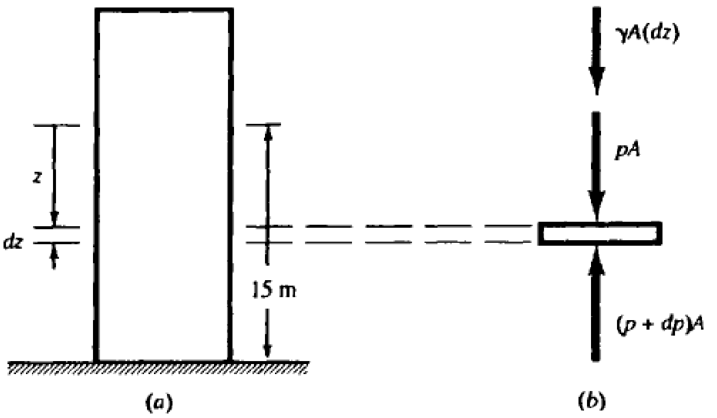


Fig. 3-6

Let us draw a free-body diagram of a thin layer of liquid situated at a distance z below the liquid free surface and of depth dz as shown in Fig. 3-6(*b*). The pressure at the top of the layer is p and at the bottom of the layer is $(p + dp)$. The weight of the layer of liquid is $\gamma A(dz)$ where it must be noted that γ in this problem is a function of z; that is, $\gamma = \gamma(z)$. Note that it is incorrect to use the equation $p = \gamma z$ from Problem 3.3 since its derivation assumes that γ is constant in the liquid whereas here γ varies with depth.

For vertical equilibrium of the element:

$$\Sigma F = (p + dp)A - pA - \gamma A\, dz = 0$$

from which

$$dp = \gamma_{H_2O}(1 + 0.018z)\, dz$$

Integrating:

$$p = \gamma_{H_2O}\left[z + 0.018\frac{z^2}{2}\right] + C$$

To find the constant of integration C, we note that at the liquid-free surface $z = 0$, $p = 0$. Thus, $C = 0$. Thus, the pressure at the tank bottom ($z = 15$ m) is

$$p_{max} = \gamma_{H_2O}[15 + 0.009(15)^2]$$

Since γ_{H_2O} is 9810 N/m³, the peak pressure is

$$p_{max} = [9810\ \text{N/m}^3][15\ \text{m} + 0.009(15\ \text{m})^2] = 167{,}000\ \text{N/m}^2$$

From Problem 3.1, this is the significant pressure that controls design, so

$$\sigma_{max} = \frac{(p_{max})r}{h}$$

$$\frac{240 \times 10^6}{2} \text{N/m}^2 = \frac{(167{,}000 \text{ N/m}^2)(4 \text{ m})}{h}$$

from which the required tank wall thickness is

$$h = 0.0056 \text{ m} \quad \text{or} \quad 5.6 \text{ mm}$$

3.5. Calculate the increase in the radius of the cylinder considered in Problem 3.1 due to the internal pressure p.

Let us consider the longitudinal and circumferential loadings separately. Due to radial pressure *p only*, the circumferential stress is given by $\sigma_c = pr/h$, and because $\sigma = E\epsilon$ the circumferential strain is given by $\epsilon_c = pr/Eh$.

It is to be noted that ϵ_c is a unit strain. The length over which it acts is the circumference of the cylinder which is $2\pi r$. Hence the total elongation of the circumference is

$$\Delta = \epsilon_c(2\pi r) = \frac{2\pi pr^2}{Eh}$$

The final length of the circumference is thus $2\pi r + 2\pi pr^2/Eh$. Dividing this circumference by 2π we find the radius of the deformed cylinder to be $r + pr^2/Eh$, so that the increase in radius is pr^2/Eh.

Due to the axial pressure *p only*, longitudinal stresses $\sigma_l = pr/2h$ are set up. These longitudinal stresses give rise to longitudinal strains $\epsilon_l = pr/2Eh$. As in Chap. 1 an extension in the direction of loading, which is the longitudinal direction here, is accompanied by a decrease in the dimension perpendicular to the load. Thus here the circumferential dimension decreases. The ratio of the strain in the lateral direction to that in the direction of loading was defined in Chap. 1 to be Poisson's ratio, denoted by μ. Consequently the above strain ϵ_l induces a circumferential strain equal to $-\mu\epsilon_l$ and if this strain is denoted ϵ_c' we have $\epsilon_c' = -\mu pr/2Eh$, which tends to decrease the radius of the cylinder as shown by the negative sign.

In a manner exactly analogous to the treatment of the increase of radius due to radial loading only, the decrease of radius corresponding to the strain ϵ_c' is given by $\mu pr^2/2Eh$. The resultant increase of radius due to the internal pressure p is thus

$$\Delta r = \frac{pr^2}{Eh} - \frac{\mu pr^2}{2Eh} = \frac{pr^2}{Eh}\left(1 - \frac{\mu}{2}\right)$$

3.6. A thin-walled cylinder with rigid end closures is fabricated by welding long rectangular plates around a cylindrical form so that the completed pressure vessel has the form shown in Fig. 3-7. The angle that the helix makes with a generator of the shell is 35° at all points. The mean radius of the cylinder is 20 in, the wall thickness is $h = 0.5$ in, and the internal pressure is 400 lb/in^2. Neglect the localized effects at each end due to the end closure plates and determine the normal and shearing stresses acting on the helical weld in the curved plane of the cylinder wall.

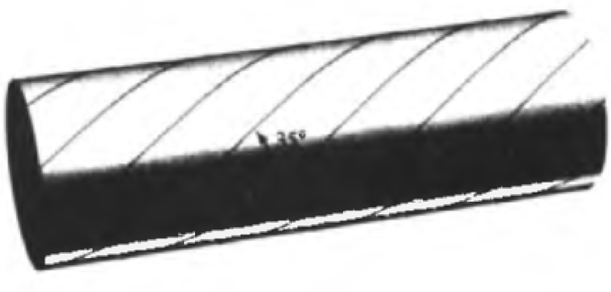


Fig. 3-7

From Problem 3.1 the circumferential and longitudinal stresses in the cylinder are

$$\sigma_c = \frac{pr}{h} = \frac{(400\ \text{lb/in}^2)(20\ \text{in})}{0.5\ \text{in}} = 16{,}000\ \text{lb/in}^2$$

$$\sigma_L = \frac{pr}{2h} = 8000\ \text{lb/in}^2$$

Let us consider a small triangular element to be removed from the cylinder wall, with the element being bounded on its hypotenuse by the weld and along the other two sides by a generator together with a circumference of the shell. The stresses found above (shown by solid vectors) act on the perpendicular sides as shown in Fig. 3-8, and on the inclined side of the element (coinciding with the helical weld) we have the unknown normal stress σ and shearing stress τ. The length of the hypotenuse of the element is taken to be ds, in which case the side along a generator has the length $ds\cos 35°$ and the length in the circumferential direction is $ds\sin 35°$.

It is convenient to introduce n- and t-axes perpendicular to and along the helical weld. The n and t components of the applied stresses are shown in Fig. 3-8 by dotted vectors. For equilibrium in the

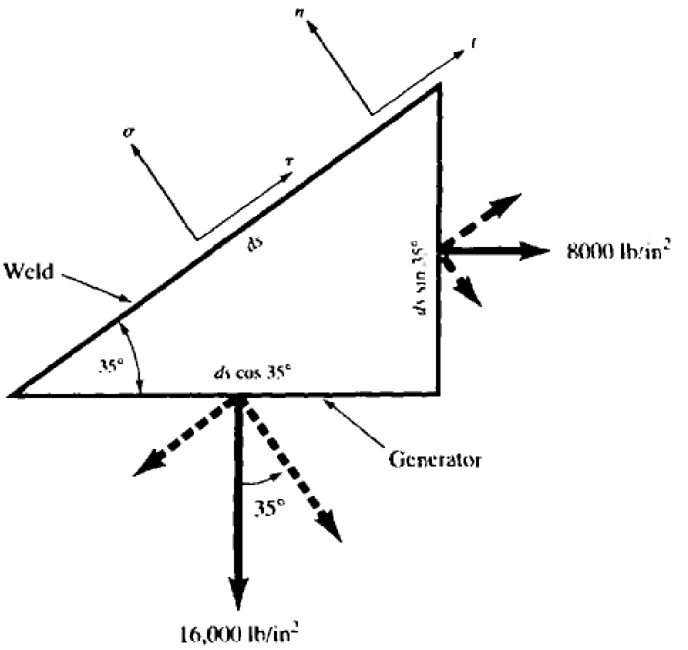


Fig. 3-8

n-direction, we have

$$\Sigma F_n = \sigma(ds)(h) - 8000(ds)(\sin 35°)(h)(\sin 35°) - 16{,}000(ds)(\cos 35°)(h)(\cos 35°) = 0$$

$$\therefore \sigma = 8000\sin^2 35° + 16{,}000\cos^2 35° = 13{,}370\ \text{lb/in}^2$$

Similarly, in the tangential direction (i.e., in the direction along the helix), we have

$$\Sigma F_t = \tau(ds)(h) + 8000(ds)(\sin 35°)(\cos 35°)(h) - 16{,}000(ds)(\cos 35°)(h)(\sin 35°) = 0$$

$$\therefore \tau = (8000)(\sin 35°)(\cos 35°) = 3760\ \text{lb/in}^2$$

3.7. Consider a closed thin-walled spherical shell subject to a uniform internal pressure p. The inside radius of the shell is r and its wall thickness is h. Derive an expression for the tensile stress existing in the wall.

For a free-body diagram, let us consider exactly half of the entire sphere. This body is acted upon by the applied internal pressure p as well as the forces that the other half of the sphere, which has been

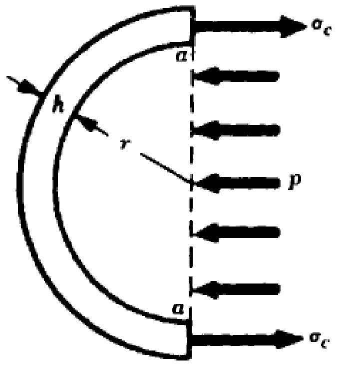


Fig. 3-9

removed, exerts upon the half under consideration. Because of the symmetry of loading and deformation, these forces may be represented by circumferential tensile stresses σ_c as shown in Fig. 3-9.

This free-body diagram represents the forces acting on the hemisphere, the diagram showing only a projection of the hemisphere on a vertical plane. Actually the pressure p acts over the entire inside surface of the hemisphere and in a direction perpendicular to the surface at every point. However, as mentioned in Problem 3.1, it is permissible to consider the force exerted by this same pressure p upon the *projection* of this area which in this case is the vertical circular area denoted by a-a. This is possible because the hemisphere is symmetric about the horizontal axis and the vertical components of the pressure annul one another. Only the horizontal components produce the tensile stress σ_c. For equilibrium we have

$$\Sigma F_h = \sigma_c 2\pi r h - p\pi r^2 = 0 \qquad \text{or} \qquad \sigma_c = \frac{pr}{2h}$$

From symmetry this circumferential stress is the same in all directions at any point in the wall of the sphere.

3.8. A 20-m-diameter spherical tank is to be used to store gas. The shell plating is 10 mm thick and the working stress of the material is 125 MPa. What is the maximum permissible gas pressure p?

From Problem 3.7 the tensile stress in all directions is uniform and given by $\sigma_c = pr/2h$. Substituting:

$$125 \times 10^6 \text{ N/m}^2 = \frac{p(10 \text{ m})}{2(0.010 \text{ m})}$$
$$p = 0.25 \text{ MPa}$$

3.9. The undersea research vehicle *Alvin* has a spherical pressure hull 1 m in radius and shell thickness of 30 mm. The pressure hull is steel having a yield point of 700 MPa. Determine the depth of submergence that would set up the yield point stress in the spherical shell. Consider sea water to have a specific weight of 10.07 kN/m^3.

From Problem 3.7 the compressive stress due to the external hydrostatic pressure is given by $\sigma_c = pr/2h$. The hydrostatic pressure corresponding to yield is thus

$$700 \times 10^6 \text{ N/m}^2 = \frac{p(1 \text{ m})}{2(0.03 \text{ m})} \qquad \text{or} \qquad p = 42 \text{ MPa}$$

Since, as in Problem 3.3, we have $p = \gamma h$, where γ is the specific weight of the sea water, we have

$$42 \times 10^6 \text{ N/m}^2 = (10.07 \times 10^3 \text{ N/m}^3)(h) \qquad \text{or} \qquad h = 4170 \text{ m}$$

It should be noted that this neglects the possibility of buckling of the sphere due to hydrostatic pressure as well as effects of entrance ports on its strength. These factors, beyond the scope of this treatment, result in a true operating depth of 1650 m.

3.10. Find the increase of volume of a thin-walled spherical shell subject to a uniform internal pressure p.

From Problem 3.7 we know that the circumferential stress is constant through the shell thickness and is given by

$$\sigma_c = \frac{pr}{2h}$$

in all directions at any point in the shell. From the two-dimensional form of Hooke's law (see Chap. 1), we have the circumferential strain as

$$\epsilon_c = \frac{1}{E}[\sigma_c - \mu\sigma_c] = \frac{pr}{2Eh}[1-\mu]$$

This strain is the change of length per unit length of the circumference of the sphere, so the increase of length of the circumference is

$$(2\pi r)\cdot\frac{pr}{2Eh}[1-\mu]$$

The radius of the spherical shell subject to internal pressure p is now found by dividing the circumference of the pressurized shell by the factor 2π. Thus the final radius is

$$\left[2\pi r + (2\pi r)\cdot\frac{pr}{2Eh}(1-\mu)\right]\Big/2\pi \qquad (1)$$

or

$$\left[r + \frac{pr^2}{2Eh}(1-\mu)\right] \qquad (2)$$

and the volume of the pressurized sphere is

$$\frac{4}{3}\pi\left[r + \frac{pr^2}{2Eh}(1-\mu)\right]^3 \qquad (3)$$

The desired increase of volume due to pressurization is found by subtracting from (3) the initial volume:

$$\Delta V = \frac{4\pi}{3}\left[r + \frac{pr^2}{2Eh}(1-\mu)\right]^3 - \frac{4}{3}\pi r^3$$

Expanding and dropping terms involving powers of (p/E), which is ordinarily of the order of 1/1000, we see that the increase of volume due to pressurization is

$$\Delta V = \frac{2\pi p r^4}{Eh}(1-\mu)$$

3.11. A thin-walled titanium alloy spherical shell has a 1-m inside diameter and is 7 mm thick. It is completely filled with an unpressurized, incompressible liquid. Through a small hole an additional 1000 cm^3 of the same liquid is pumped into the shell, thus increasing the shell radius. Find the pressure after the additional liquid has been introduced and the hole closed. For this titanium allow $E = 114$ GPa and the tensile yield point of the material to be 830 MPa.

The initial volume of the spherical shell is

$$V = \frac{4}{3}\pi r^3 = \frac{\pi}{6}d^3 \qquad (d = \text{diameter})$$

$$= \frac{\pi}{6}(1\text{ m})^3 = 0.5236\text{ m}^3$$

The volume of liquid pumped in is

$$1000\,\text{cm}^3\left(\frac{\text{m}}{100\,\text{cm}}\right)^3 = \frac{1}{10^3}\,\text{m}^3$$

so that the final volume of the incompressible liquid is

$$0.5236\,\text{m}^3 + 0.001\,\text{m}^3 = 0.5246\,\text{m}^3$$

which is equal to the volume of the expanded shell. The relation between pressure and volume change was found in Problem 3.10 to be

$$\Delta V = \frac{2\pi p r^4}{Eh}(1-\mu)$$

Substituting,

$$0.001\,\text{m}^3 = \frac{(2\pi)p(0.5\,\text{m})^4(0.67)}{(114\times 10^9\,\text{N/m}^2)(0.007\,\text{m})}$$

Solving,

$$p = 3.03\,\text{MPa}$$

It is well to check the normal stress in the titanium shell due to this pressure. From Problem 3.7 we have

$$\sigma = \frac{pr}{2h}$$
$$= \frac{(3.03\times 10^6\,\text{N/m}^2)(0.05\,\text{m})}{2(0.007\,\text{m})} = 109\,\text{MPa}$$

which is well below the yield point of the material.

3.12. Consider a laminated pressure vessel composed of two thin coaxial cylinders as shown in Fig. 3-10. In the state prior to assembly there is a slight "interference" between these shells, i.e., the inner one is too large to slide into the outer one. The outer cylinder is heated, placed on the inner, and allowed to cool, thus providing a "shrink fit." If both cylinders are steel and the mean diameter of the assembly is 100 mm, find the tangential stresses in each shell arising from the shrinking if the initial interference (of diameters) is 0.25 mm. The thickness of the inner shell is 2.5 mm, and that of the outer shell 2 mm. Take $E = 200$ GPa.

There is evidently an interfacial pressure p acting between the adjacent faces of the two shells. It is to be noted that there are no external applied loads. The pressure p may be considered to increase the diameter of the outer shell and decrease the diameter of the inner so that the inner shell may fit inside the outer. The radial expansion of a cylinder due to a radial pressure p was found in Problem 3.5 to be pr^2/Eh. No longitudinal forces are acting in this problem. The increase in radius of the outer shell due to p, plus

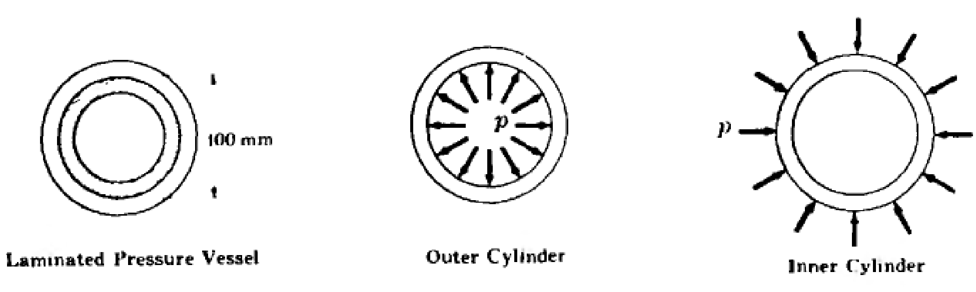


Fig. 3-10

the decrease in radius of the inner one due to p, must equal the initial interference between radii, or 0.25/2 mm. Thus we have

$$\frac{p(0.05\text{ m})^2}{(200\times10^9\text{ N/m}^2)(0.0025\text{ m})}+\frac{p(0.05\text{ m})^2}{(200\times10^9\text{ N/m}^2)(0.002\text{ m})}=\frac{0.125}{1000}\text{ m}$$

$$p = 11.1\text{ MPa}$$

This pressure, illustrated in the above figures, acts between the cylinders after the outer one has been shrunk onto the inner one. In the inner cylinder this pressure p gives rise to a stress

$$\sigma_c=\frac{pr}{h}=\frac{(11.1\times10^6\text{ N/m}^2)(0.05\text{ m})}{(0.0025\text{ m})}=-222\text{ MPa}$$

In the outer cylinder the circumferential stress due to the pressure p is

$$\sigma_c=\frac{pr}{h}=\frac{(11.1\times10^6\text{ N/m}^2)(0.05\text{ m})}{(0.002\text{ m})}=277\text{ MPa}$$

If, for example, the laminated shell is subject to a uniform internal pressure, these shrink-fit stresses would merely be added algebraically to the stresses found by the use of the simple formulas given in Problem 3.1.

3.13. The thin steel cylinder just fits over the inner copper cylinder as shown in Fig. 3-11. Find the tangential stresses in each shell due to a temperature rise of 60 °F. Do not consider the effects introduced by the accompanying longitudinal expansion. This arrangement is sometimes used for storing corrosive fluids. Take

$$E_{st} = 30\times10^6\text{ lb/in}^2 \qquad \alpha_{st} = 6.5\times10^{-6}/°\text{F}$$

$$E_{cu} = 13\times10^6\text{ lb/in}^2 \qquad \alpha_{cu} = 9.3\times10^{-6}/°\text{F}$$

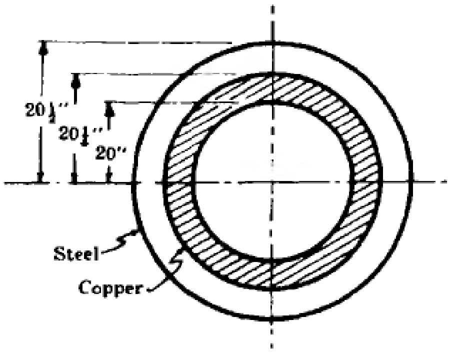


Fig. 3-11

The simplest approach is to first consider the two shells to be separated from one another so that they are no longer in contact.

Due to the temperature rise of 60 °F the circumference of the steel shell increases by an amount $2\pi(20.375)(60)(6.5\times10^{-6}) = 0.0498$ in. Also, the circumference of the copper shell increases an amount $2\pi(20.125)(60)(9.3\times10^{-6}) = 0.0705$ in. Thus the interference between the radii, i.e., the difference in radii, of the two shells (due to the heating) is $(0.0705-0.0498)/2\pi = 0.00345$ in. Again, there are no external loads acting on either cylinder.

However, from the statement of the problem the adjacent surfaces of the two shells are obviously in contact after the temperature rise. Hence there must be an interfacial pressure p between the two surfaces, i.e., a pressure tending to increase the radius of the steel shell and decrease the radius of the copper shell

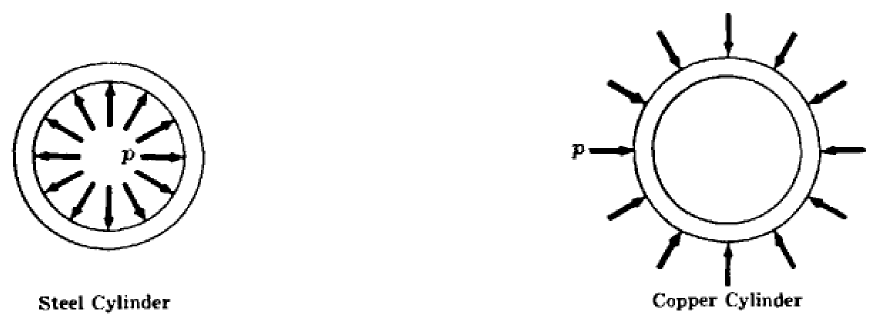


Fig. 3-12

so that the copper shell may fit inside the steel one. Such a pressure is shown in the free-body diagrams of Fig. 3-12.

In Problem 3.5 the change of radius of a cylinder due to a uniform radial pressure p (with no longitudinal forces acting) was found to be pr^2/Eh. Consequently the increase of radius of the steel shell due to p, added to the decrease of radius of the copper one due to p, must equal the interference; thus

$$\frac{p(20.375)^2}{(30\times10^6)(0.25)}+\frac{p(20.125)^2}{(13\times10^6)(0.25)}=0.00345 \qquad \text{or} \qquad p=19.2\ \text{lb/in}^2$$

This interfacial pressure creates the required continuity at the common surface of the two shells when they are in contact. Using the formula for the tangential stress, $\sigma_c = pr/h$, we find the tangential stresses in the steel and copper shells to be, respectively,

$$\sigma_{st}=\frac{19.2(20.375)}{0.25}=1560\ \text{lb/in}^2 \qquad \text{and} \qquad \sigma_{cu}=-\frac{19.2(20.125)}{0.25}=-1550\ \text{lb/in}^2$$

3.14. Consider a thin-walled conical shell containing a liquid whose weight per unit volume is γ [see Fig. 3-13(a)]. The shell is supported around its upper rim and filled with liquid to a depth H. Determine the stresses in the shell walls due to this loading. The geometric axis of the shell is vertical.

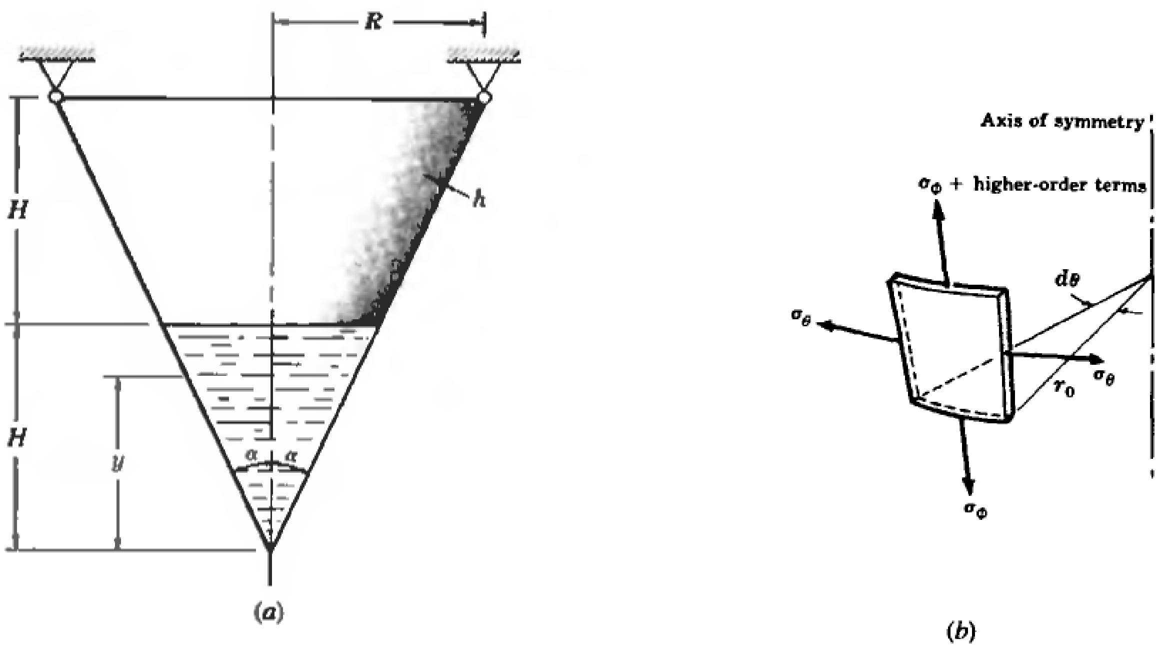


Fig. 3-13

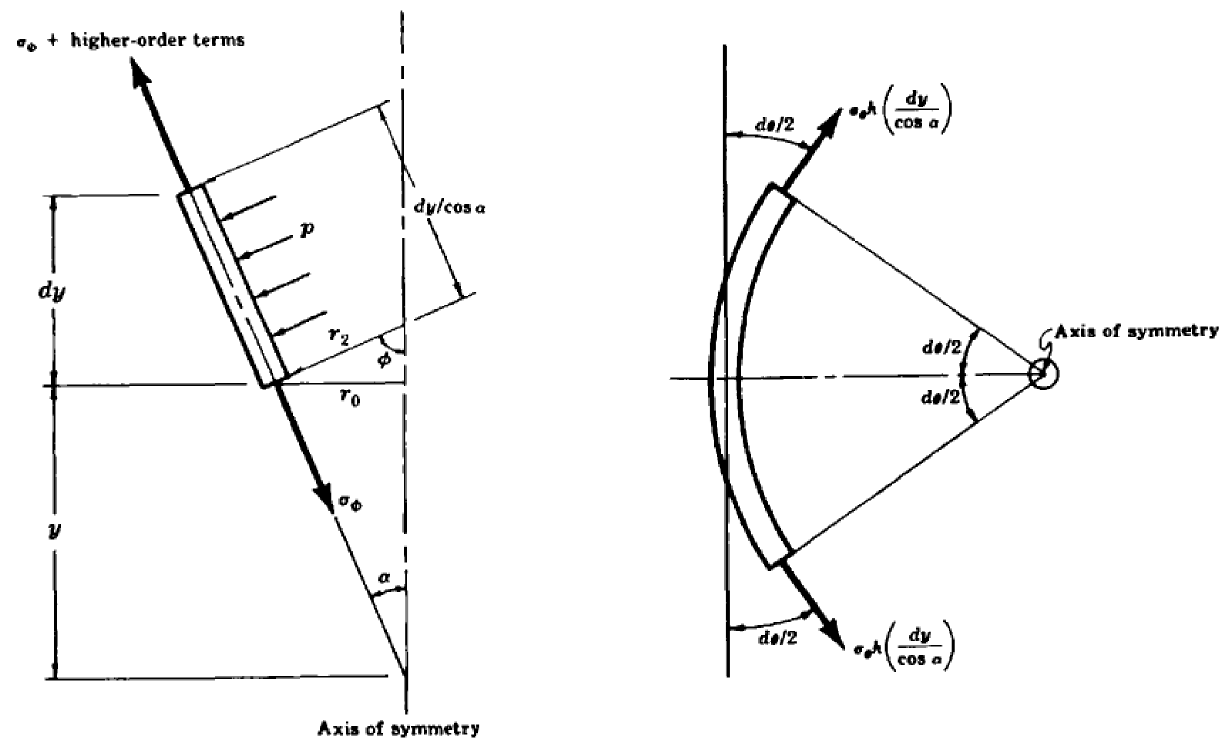
The state of stress in this shell is obviously axisymmetric. It is assumed that the shell thickness h is small compared to H and R. The stresses may be determined by consideration of the equilibrium of a shell element bounded by two closely adjacent parallel circles whose planes are normal to the vertical axis of symmetry of the cone and by two closely adjacent generators of the cone. Such an element, together with the vectors representing the stresses σ_θ in the horizontal direction and σ_ϕ in the direction of a generator, is indicated in Fig. 3-13(b). The quantity σ_θ is called *hoop stress* and σ_ϕ is termed the *meridional stress.*

In the diagram θ represents the angular coordinate measured in a horizontal plane which is normal to the vertical axis of symmetry of the shell. The radius of the cone there is r_0, which is of course a function of the location of the element with respect to its position along the axis of symmetry. Another coordinate useful for defining the geometry of the cone is r_2, which corresponds to the radius of curvature of the shell surface in a direction perpendicular to the generator. This is best illustrated by examining a section of the cone formed by passing a vertical plane through the shell axis as indicated in Fig. 3-14(a) below. It is evident that $r_0 = r_2 \cos\alpha$.

From geometry we have

$$r_0 = y\tan\alpha \qquad \text{and so} \qquad r_2 = \frac{y\tan\alpha}{\cos\alpha}$$

The hoop *stresses* in Fig. 3-13(b) may be visualized more clearly by looking along the axis of symmetry, as shown in Fig. 3-14(b). It is evident that each of the hoop *forces* vectors $\sigma_\theta(dy/\cos\alpha)h$ makes an angle $d\theta/2$ with the tangent to the element. The resultant of these hoop forces is $2\sigma_\theta h(dy/\cos\alpha)\sin(d\theta/2)$ or, since $d\theta/2$ is small, $\sigma_\theta h(dy/\cos\alpha)\,d\theta$ acting in a horizontal plane and directed toward the geometric axis of the shell. From Fig. 3-14(a) we see that this resultant must be multiplied by $\cos\alpha$ to determine the component of this force acting in a direction normal to the shell surface. Also, it is evident that the meridional forces corresponding to Fig. 3-14(a) cancel one another. The liquid exerts a normal pressure



(a) (b)

Fig. 3-14

p as indicated in the figure and it acts over an area $(r_0\,d\theta)(dy/\cos\alpha)$. Thus, for equilibrium of the element in a direction normal to the surface we have

$$\sigma_\theta h\left(\frac{dy}{\cos\alpha}\right)(d\theta)\cos\alpha - pr_0(d\theta)\frac{dy}{\cos\alpha} = 0 \tag{1}$$

or

$$\sigma_\theta = \frac{pr_0}{h\cos\alpha} = \frac{py\tan\alpha}{h\cos\alpha} = \frac{pr_2}{h} \tag{2}$$

This expression holds anywhere in the conical shell. In the lower half, $0<y<H$, we have $p=\gamma(H-y)$, so

$$\sigma_\theta = \frac{\gamma(H-y)y\tan\alpha}{h\cos\alpha} \qquad \text{for} \qquad 0<y<H \tag{3}$$

In the upper half, $H<y<2H$, $p=0$, so $\sigma_\theta=0$ in that region.

The other stress σ_ϕ may be found by considering the vertical equilibrium of the conical shell. For $0<y<H$ the weight of the liquid in the conical region abo plus that in the cylindrical region $abcd$ is held in equilibrium by the forces corresponding to σ_ϕ and we have from Fig. 3-15(a)

$$\sigma_\phi h2\pi y\tan\alpha\cos\alpha - \gamma[\tfrac{1}{3}\pi(y\tan\alpha)^2y + (H-y)\pi(y\tan\alpha)^2] = 0 \tag{4}$$

or

$$\sigma_\phi = \frac{\gamma\tan\alpha}{h\cos\alpha}\left(\frac{Hy}{2}-\frac{y^2}{3}\right) \qquad \text{for} \qquad 0<y<H \tag{5}$$

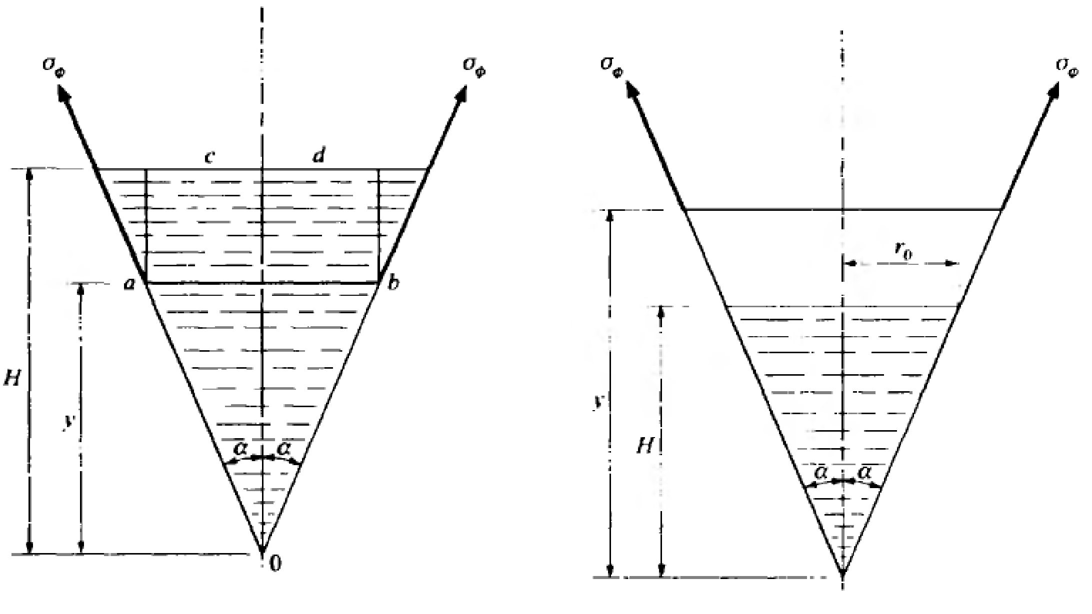
Similarly, for $H<y<2H$, the weight of all the liquid is held in equilibrium by the forces corresponding to σ_ϕ so that from Fig. 3-15(b)

$$\sigma_\phi h(2\pi y)(\tan\alpha)\cos\alpha - \gamma\tfrac{1}{3}\pi r_0^2H = 0 \tag{6}$$

Since $r_0 = H\tan\alpha$ we get

$$\sigma_\phi = \frac{\gamma H^3\tan\alpha}{6hy\cos\alpha} \qquad \text{for} \qquad H<y<2H \tag{7}$$

It is to be observed that the stresses associated with these axisymmetric deformations are statically determinate; i.e., it was not necessary to use any deformation relations to determine the stresses. Thus the relations are valid into the plastic range of action.



(a) (b)

Fig. 3-15

3.15. Determine the hoop stresses and meridional stresses in a thin shell of revolution subject to an internal pressure p.

This problem is readily solved as a generalization of Problem 3.14. The stresses may be determined by consideration of the equilibrium of a shell element bounded by two closely adjacent parallel circles whose planes are normal to the vertical axis of a symmetry of the shell and by two closely adjacent generators, or meridians, of the shell (see Fig. 3-1). This element is analogous to that shown in Fig. 3-13(*b*) of Problem 3.14, except that the vertical sides are curved rather than straight.

The hoop stresses σ_θ and the meridional stresses σ_ϕ thus appear as shown in Fig. 3-16. We now require two radii of curvature to describe this element. We use r_1 to denote the radius of curvature of the meridian and r_2 to denote the radius of curvature of the shell surface in a direction perpendicular to the meridian. The center of curvature corresponding to r_2 must lie on the axis of symmetry although the center for r_1 does not (in general). Figure 3-17(*a*) shows the hoop forces as seen by looking along the axis of symmetry and, analogous to Problem 3.14, they have a horizontal component $2\sigma_\theta h r_1\, d\phi(d\theta/2)$ directed toward the shell axis. This is multiplied by $\sin\phi$ to obtain the component normal to the shell element. The meridional forces appear as in Fig. 3-17(*b*) and they have a component normal to the shell given by $\sigma_\phi h r_0\, d\theta\, d\phi$. The

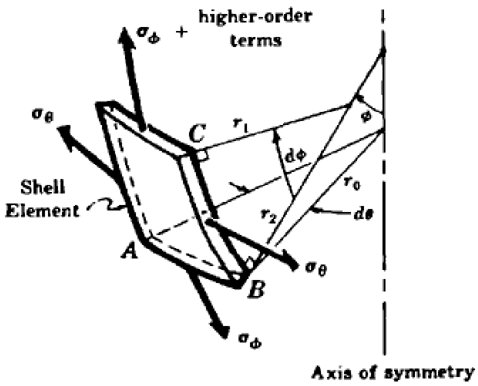
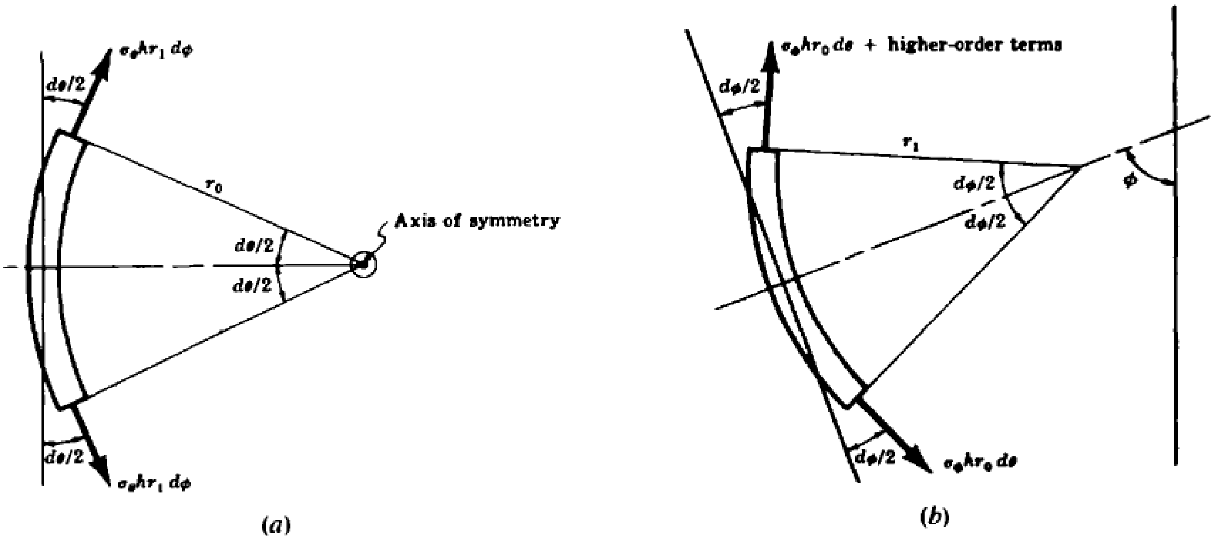


Fig. 3-16



(*a*) (*b*)

Fig. 3-17

pressure p acts over an area $(r_0\,d\theta)(r_1\,d\phi)$ so that the equation of equilibrium in the normal direction becomes

$$\sigma_\theta h r_1\,d\theta\,d\phi \sin\phi + \sigma_\phi h r_0\,d\theta\,d\phi - p r_0\,d\theta\,r_1\,d\phi = 0$$

or, since $r_0 = r_2 \sin\phi$, we get

$$\frac{\sigma_\phi}{r_1} + \frac{\sigma_\theta}{r_2} = \frac{p}{h} \tag{1}$$

This fundamental equation applies to axisymmetric deformations of all thin shells of revolution. A second equation is obtained as in Problem 3.14 by consideration of the vertical equilibrium of the entire shell above some convenient parallel circle. Again, these equations are valid into the plastic range of action.

3.16. Consider a constant-thickness thin-walled spherical dome of radius r loaded only by its own weight q per unit of surface area. The dome is supported by frictionless rollers around its lower boundary as shown in Fig. 3-18(a). Determine meridional and hoop stresses at all points in the system.

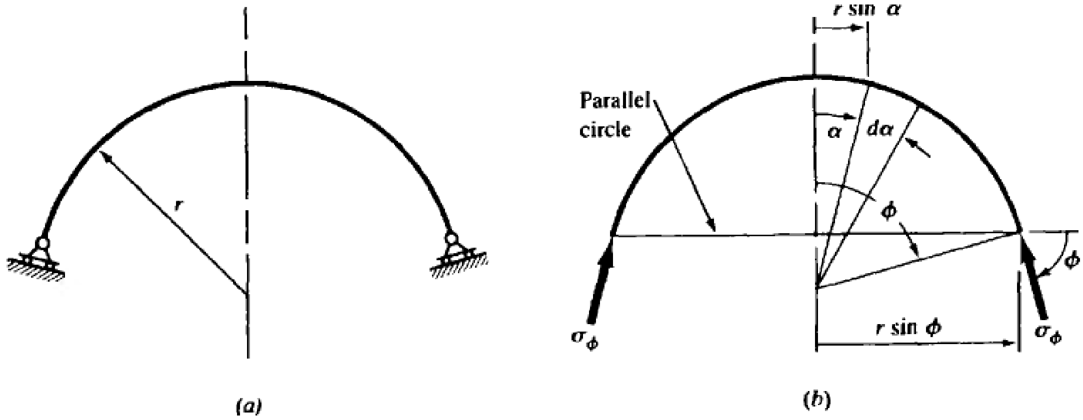


Fig. 3-18

Let us consider the vertical equilibrium of a portion of the dome above some parallel circle defined by the angle ϕ shown in Fig. 3-18(b). The variable angle α is introduced and the weight of the central portion of the dome above the parallel circle is found by considering a ring-shaped element of radius $(r\sin\alpha)$ and meridional length $(r\,d\alpha)$. The weight of the portion of the dome above the parallel circle is

$$\int_{\alpha=0}^{\alpha=\phi} q[2\pi(r\sin\alpha)](r\,d\alpha)$$

which becomes

$$2\pi r^2 q(1-\cos\phi)$$

The meridional stress σ_ϕ is uniformly distributed around the circumference of the parallel circle and has an upward vertical resultant given by

$$2\pi(r\sin\phi)h\sigma_\phi(\sin\phi)$$

For vertical equilibrium of the dome above the parallel circle, we thus have

$$2\pi(r\sin\phi)h\sigma_\phi(\sin\phi) - 2\pi r^2 q(1-\cos\phi) = 0$$

or

$$\sigma_\phi = \frac{rq}{h(1+\cos\phi)} \qquad \text{(compression)} \tag{1}$$

This value, when introduced into Eq. (*1*) of Problem 3.15, leads to a hoop stress σ_θ given by

$$\sigma_\theta = \frac{rq}{h}\left[\frac{1}{1+\cos\phi} - \cos\phi\right] \tag{2}$$

3.17. The spherical dome of Problem 3.16 subtends an opening angle of 120°, has a wall thickness of 100 mm, and a radius of 50 m. It is constructed of concrete having a specific weight of 23.5 kN/m³. Determine meridional and circumferential stresses at (*a*) the apex of the dome, and (*b*) the simply supported rim.

The meridional stress is given by Eq. (*1*) of Problem 3.16. In that equation q denotes weight per unit of surface area. Here, since the specific weight refers to a cube of concrete weighing 23.5 kN, the weight per unit surface area is found by considering the 100-mm thickness to be

$$q = (23{,}500\ \text{N/m}^3)\left(\frac{100}{1000}\right) = 2350\ \text{N/m}^2$$

The meridional stress at the apex, where $\phi = 0°$, is

$$\sigma_\phi = -\frac{(50\ \text{m})(2350\ \text{N/m}^2)}{(0.1\ \text{m})[1+\cos 0°]} = -587{,}500\ \text{N/m}^2 \qquad \text{or} \qquad -0.587\ \text{MPa}$$

and at the rim, where $\phi = 60°$, we have

$$\sigma_\phi = -\frac{(50\ \text{m})(2350\ \text{N/m}^2)}{(0.1\ \text{m})[1+\cos 60°]} = -786{,}000\ \text{N/m}^2 \qquad \text{or} \qquad -0.786\ \text{MPa}$$

The circumferential stress is given by Eq. (*2*) of Problem 3.16. At the apex this is

$$\sigma_\theta = \frac{(50\ \text{m})(2350\ \text{N/m}^2)}{(0.1\ \text{m})}\left[\frac{1}{1+\cos 0°} - \cos 0°\right] = -587{,}500\ \text{N/m}^2 \qquad \text{or} \qquad -0.588\ \text{MPa}$$

and at the rim, where $\phi = 60°$, it is

$$\sigma_\theta = \frac{(50\ \text{m})(2350\ \text{N/m}^2)}{(0.1\ \text{m})}\left[\frac{1}{1+\cos 60°} - \cos 60°\right] = 195{,}000\ \text{N/m}^2 \qquad \text{or} \qquad 0.195\ \text{MPa}$$

Thus the circumferential stress is tensile at the rim and compressive at the apex. From Eq. (*2*) of Problem 3.16, the circumferential stress is zero when

$$\frac{1}{1+\cos\phi_0} - \cos\phi_0 = 0$$

Solving by trial and error, we find $\phi_0 = 51.8°$.

3.18. Thin toroidal shells are sometimes employed as gas storage tanks in boosters for space vehicles. One design considered by the National Aeronautics and Space Administration for possible future use employs a torus of mean diameter $2b = 70$ ft with a cross-section diameter of $2R = 5$ ft as indicated in Fig. 3-19. The internal pressure p is 20 lb/in² and the shell material is 2219 T87 aluminum alloy, having a yield point of 50,000 lb/in² at room temperature. For this material the yield point increases at lower temperatures, reaching 120 percent of the above value at −300 °F. If a safety factor of 1.5 is employed, determine the required wall thickness.

First, we consider the vertical equilibrium of a ring-shaped portion of the toroidal shell above an

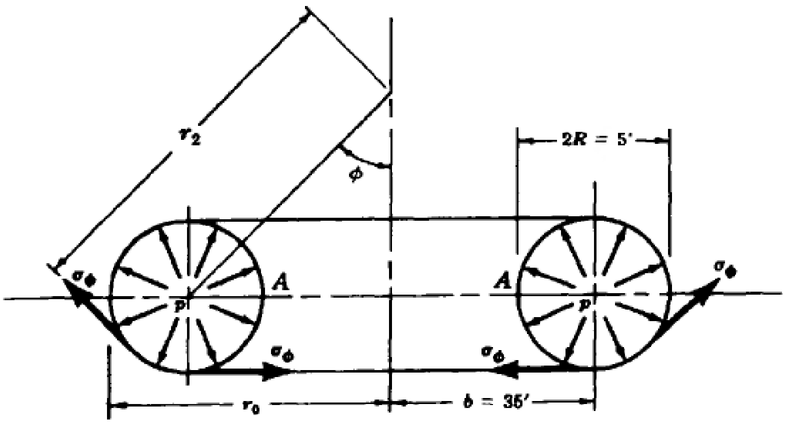


Fig. 3-19

arbitrary plane, as indicated by the angle ϕ. The meridional stress σ_ϕ is readily found by considering the pressure p to act on the horizontal projection of the curved area. Thus

$$2\pi r_0 \sigma_\phi h \sin\phi = \pi p(r_0^2 - b^2)$$

or since $\sin\phi = (r_0 - b)/R$

$$\sigma_\phi = \frac{pR(r_0 + b)}{2\pi r_0 h} \tag{1}$$

From (1) it is evident that the peak value of σ_ϕ occurs at the innermost points A where

$$(\sigma_\phi)_{max} = \frac{pR}{2h}\left(\frac{2b - R}{b - R}\right) \tag{2}$$

If $b = 0$, the torus reduces to a sphere and (2) coincides with the stresses in a sphere as found in Problem 3.7. For the given dimensions we have $R = 30$ in, $b = 420$ in, $p = 20$ lb/in^2, and (2) becomes

$$\frac{50{,}000}{1.5} = \frac{20(30)(840 - 30)}{2h(420 - 30)} \qquad \text{or} \qquad h = 0.0187 \text{ in} \tag{3}$$

If σ_ϕ as given by (1) is substituted into (1) of Problem 3.15 (which holds for axisymmetric deformation of any thin shell of revolution) we obtain, for $r_1 = R$ and $r_2 = (b + R\sin\phi)/\sin\phi$,

$$\sigma_\theta = \frac{pR}{2h} \tag{4}$$

at any point in the toroidal shell. Evidently the peak value of σ_ϕ as given by (2) exceeds the value of σ_θ and hence the maximum value of σ_ϕ controls the design. The required thickness is thus given by (3).

Supplementary Problems

3.19. One proposed design for an energy-efficient automobile involves an on-board tank storing hydrogen (in a special nonvolatile form) which would be released to a fuel cell. The tank is to be cylindrical, 0.4 m in diameter, made of type 302 stainless steel having a working stress in tension of 290 MPa, and closed by hemispherical end caps. The hydrogen would be pressurized to 15 MPa when the tank is initially filled. Determine the required wall thickness of the tank. *Ans.* $h = 5.2$ mm

3.20. A vertical cylindrical gasoline storage tank is 30 m in diameter and is filled to a depth of 15 m with gasoline whose specific gravity is 0.74. If the yield point of the shell plating is 250 MPa and a safety factor of 2.5 is adequate, calculate the required wall thickness at the bottom of the tank. *Ans.* $h = 16.7$ mm

3.21. The research deep submersible *Aluminaut* has a cylindrical pressure hull of outside diameter 8 ft and a wall thickness of 5.5 in. It is constructed of 7079-T6 aluminum alloy, having a yield point of 60,000 lb/in^2. Determine the circumferential stress in the cylindrical portion of the pressure hull when the vehicle is at its operating depth of 15,000 ft below the surface of the sea. Use the mean diameter of the shell in calculations, and consider sea water to weigh 64.0 lb/ft^3. *Ans.* 54,800 lb/in^2

3.22. Derive an expression for the increase of volume per unit volume of a thin-walled circular cylinder subjected to a uniform internal pressure p. The ends of the cylinder are closed by circular plates. Assume that the radial expansion is constant along the length.

Ans. $\dfrac{\Delta V}{V} = \dfrac{pr}{Eh}\left(\dfrac{5}{2} - 2\mu\right)$

3.23. Calculate the increase of volume per unit volume of a thin-walled steel circular cylinder closed at both ends and subjected to a uniform internal pressure of 0.5 MPa. The wall thickness is 1.5 mm, the radius 350 mm, and $\mu = \frac{1}{3}$. Consider $E = 200$ GPa. *Ans.* $\Delta V/V = 10^{-3}$

3.24. Consider a laminated cylinder consisting of a thin steel shell "shrunk" on an aluminum one. The thickness of each is 0.10 in and the mean diameter of the assembly is 4 in. The initial "interference" of the shells prior to assembly is 0.004 in measured on a diameter. Find the tangential stresses in each shell caused by this shrink fit. For aluminum $E = 10 \times 10^6$ lb/in^2 and for steel $E = 30 \times 10^6$ lb/in^2.
Ans. $\sigma_{st} = 7500$ lb/in^2, $\sigma_{al} = -7500$ lb/in^2

3.25. A spherical tank for storing gas under pressure is 25 m in diameter and is made of structural steel 15 mm thick. The yield point of the material is 250 MPa and a safety factor of 2.5 is adequate. Determine the maximum permissible internal pressure, assuming the welded seams between the various plates are as strong as the solid metal. Also, determine the permissible pressure if the seams are 75 percent as strong as the solid metal. *Ans.* $p = 0.24$ MPa, $p = 0.18$ MPa

3.26. A thin-walled spherical shell is subject to a temperature rise ΔT which is constant at all points in the shell as well as through the shell thickness. Find the increase of volume per unit volume of the shell. Let α denote the coefficient of thermal expansion of the material. *Ans.* $3\alpha(\Delta T)$

3.27. A liquid storage tank consists of a vertical axis circular cylindrical shell closed at its lower end by a hemispherical shell as shown in Fig. 3-20. The weight of the system is carried by a ring-like support at the top and the lower extremity is unsupported. A liquid of specific weight γ entirely fills the container. Determine the peak circumferential and meridional stress in the cylindrical region of the assembly, as well as the peak stresses in the hemispherical region.

Ans.

Cylinder: $\sigma_c = \dfrac{\gamma R}{h}(H - R) \qquad \sigma_L = \dfrac{\gamma R}{2h}\left(H - \dfrac{R}{3}\right)$

Hemisphere: $\dfrac{\gamma HR}{2h}$

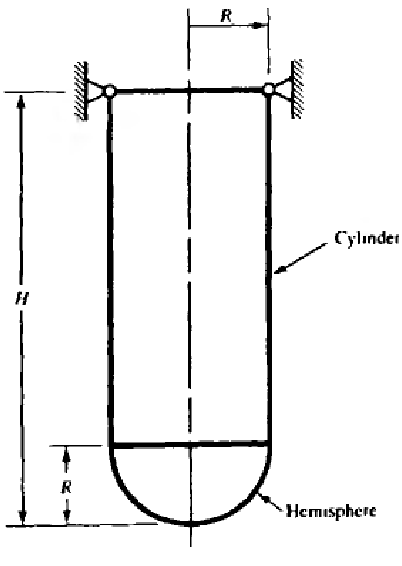


Fig. 3-20

3.28. Reexamine Problem 3.18 with all parameters as indicated there except that the shell material is now Ti-6Al-4V titanium alloy having a yield point of 126,000 lb/in^2 at room temperature. If a safety factor of 1.5 is used, determine the required wall thickness. *Ans.* 0.0074 in

Chapter 4

Direct Shear Stresses

DEFINITION OF SHEAR FORCE

If a plane is passed through a body, a force acting along this plane is called a *shear force* or *shearing force.* It will be denoted by F_s.

DEFINITION OF SHEAR STRESS

The shear force, divided by the area over which acts, is called the *shear stress* or *shearing stress.* It is denoted in this book by τ. Thus

$$\tau = \frac{F_s}{A} \tag{4.1}$$

COMPARISON OF SHEAR AND NORMAL STRESSES

Let us consider a bar cut by a plane *a-a* perpendicular to its axis, as shown in Fig. 4-1. A normal stress σ is perpendicular to this plane. This is the type of stress considered in Chaps. 1, 2, and 3.

A shear stress is one acting *along* the plane, as shown by the stress τ. Hence the distinction between normal stresses and shear stresses is one of *direction.*

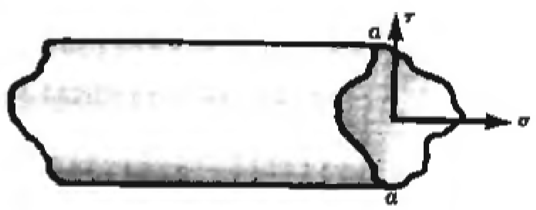


Fig. 4-1

ASSUMPTION

It is necessary to make some assumption regarding the manner of distribution of shear stresses, and for lack of any more precise knowledge it will be taken to be uniform in all problems discussed in this chapter. Thus the expression $\tau = F_s/A$ indicates an average shear stress over the area.

APPLICATIONS

Punching operations (Problem 4.2), wood test specimens (Problem 4.3), riveted joints (Problem 4.5), welded joints (Problem 4.6), and towing devices (Problem 4.10) are common examples of systems involving shear stresses.

DEFORMATIONS DUE TO SHEAR STRESSES

Let us consider the deformation of a plane rectangular element cut from a solid where the forces acting on the element are known to be shearing stresses τ in the directions shown in Fig. 4-2(*a*).

The faces of the element parallel to the plane of the paper are assumed to be load free. Since there are no normal stresses acting on the element, the lengths of the sides of the originally rectangular element will not change when the shearing stresses assume the value τ. However, there will be a distortion of the originally right *angles* of the element, and after this distortion due to the shearing stresses the element assumes the configuration shown by the dashed lines in Fig. 4-2(*b*).

Fig. 4-2

SHEAR STRAIN

The change of angle at the corner of an originally rectangular element is defined as the *shear strain*. It must be expressed in radian measure and is usually denoted by γ.

MODULUS OF ELASTICITY IN SHEAR

The ratio of the shear stress τ to the shear strain γ is called the *modulus of elasticity in shear* and is usually denoted by G. Thus

$$G = \frac{\tau}{\gamma} \tag{4.2}$$

G is also known as the *modulus of rigidity*.

The units of G are the same as those of the shear stress, e.g., lb/in^2 or N/m^2, since the shear strain is dimensionless. The experimental determination of G and the region of linear action of τ and γ will be discussed in Chap. 5. Stress-strain diagrams for various materials may be drawn for shearing loads, just as they were drawn for normal loads in Chap. 1. They have the same general appearance as those sketched in Chap. 1 but the numerical values associated with the plots are of course different.

WELDED JOINTS

In addition to the traditional techniques of gas welding and electric arc welding, the past few decades have seen the emergence of two significant new methods, namely (*a*) electron beam welding and (*b*) laser beam welding.

Electron Beam Welding

In electron beam welding (EBW), coalescence of metals is achieved by having a focused beam of high-velocity electrons striking the surfaces to be joined. This beam of electrons carries a very high energy density that is capable of producing deep, narrow welds. Such welds can be produced much more quickly and with less distortion of the parent metals than with either gas or arc welding. Negative aspects of EBW are (i) surfaces to be joined must be very accurately aligned, and (ii) in certain situations EBW must be done in a partial vacuum. Also, safety precautions must be taken to protect personnel from the electron beam. (See Problem 4.12.)

Laser Beam Welding

In laser beam welding (LBW), joining of metals is carried out by having an optical energy source focused over a very small spot, such as the diameter of a circle ranging from 100 to 1000 μm (0.004 to 0.040 in). The term "laser" is an acronym for light amplification by stimulated emission of radiation. Energy densities of the order of 10^5 watts/cm^2 (6×10^6 watts/in^2) make the laser beam suitable for welding of metals. Laser beams can produce welds of high quality, but precautions must be taken to guard the operators of the laser, particularly with regard to damage to the human eye. One of the first successful applications involved laser welding of thermocouple gages in the Apollo lunar probe in the late 1960s. Types of systems in common use today include lasers of ruby, carbon dioxide, and various rare earth materials. Common commercial applications in the 1990s include sealing of batteries for digital watches and heart pacemakers, sealing of ink cartridges for fountain pens, joining telephone wires in circuits, and a host of other applications in aerospace, automotive, and electronic consumer items. (See Problem 4.13.)

Solved Problems

4.1. Consider the bolted joint shown in Fig. 4-3. The force P is 30 kN and the diameter of the bolt is 10 mm. Determine the average value of the shearing stress existing across either of the planes a-a or b-b.

Lacking any more precise information we can only assume that force P is equally divided between the sections a-a and b-b. Consequently a force of $\frac{1}{2}(30 \times 10^3) = 15 \times 10^3$ N acts across either of these planes over a cross-sectional area

$$\tfrac{1}{4}\pi(10)^2 = 78.6 \text{ mm}^2$$

Thus the average shearing stress across either plane is $\tau = \frac{1}{2}P/A = 15 \times 10^3/78.6 = 192$ MPa.

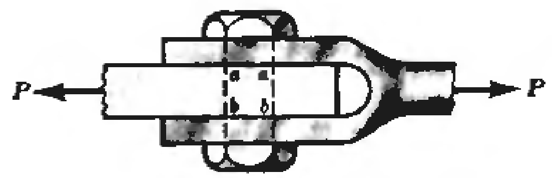


Fig. 4-3

4.2. Low-carbon structural steel has a shearing ultimate strength of approximately 45,000 lb/in^2. Determine the force P necessary to punch a 1-in-diameter hole through a plate of this steel $\frac{3}{8}$ in thick. If the modulus of elasticity in shear for this material is 12×10^6 lb/in^2, find the shear strain at the edge of this hole when the shear stress is 21,000 lb/in^2.

Let us assume uniform shearing on a cylindrical surface 1 in in diameter and $\frac{3}{8}$ in thick as shown in Fig. 4-4. For equilibrium the force P is $P = \tau A = \pi(1)(\frac{3}{8})(45{,}000) = 53{,}100$ lb.

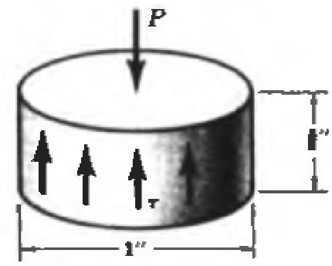


Fig. 4-4

To determine the shear strain γ when the shear stress τ is 21,000 lb/in^2, we employ the definition $G = \tau/\gamma$ to obtain $\gamma = \tau/G = 21{,}000/12{,}000{,}000 = 0.00175$ radian.

4.3. In the wood industries, inclined blocks of wood are sometimes used to determine the *compression-shear* strength of glued joints. Consider the pair of glued blocks A and B which are 1.5 in deep in a direction perpendicular to the plane of the paper. Determine the shearing ultimate strength of the glue if a vertical force of 9000 lb is required to cause rupture of the joint. It is to be noted that a good glue causes a large proportion of the failure to occur in the wood.

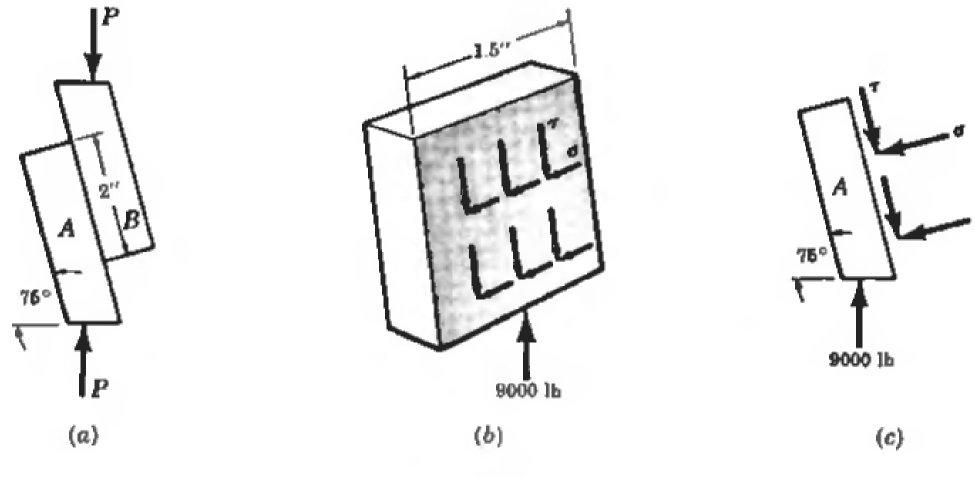


Fig. 4-5

Let us consider the equilibrium of the lower block, A. The reactions of the upper block B upon the lower one consist of both normal and shearing forces appearing as in the perspective and orthogonal views of Figs. 4-5(b) and 4-5(c).

Referring to Fig. 4-5(c) we see that for equilibrium in the horizontal direction

$$\Sigma F_h = \tau(2)(1.5)\cos 75° - \sigma(2)(1.5)\cos 15° = 0 \qquad \text{or} \qquad \sigma = 0.269\tau$$

For equilibrium in the vertical direction we have

$$\Sigma F_v = 9000 - \tau(2)(1.5)\sin 75^\circ - \sigma(2)(1.5)\sin 15^\circ = 0$$

Substituting $\sigma = 0.269\tau$ and solving, we find $\tau = 2900\ \text{lb/in}^2$.

4.4. The shearing stress in a piece of structural steel is 100 MPa. If the modulus of rigidity G is 85 GPa, find the shearing strain γ.

By definition, $G = \tau/\gamma$. Then the shearing strain $\gamma = \tau/G = (100 \times 10^6)/(85 \times 10^9) = 0.00117$ rad.

4.5. A single rivet is used to join two plates as shown in Fig. 4-6. If the diameter of the rivet is 20 mm and the load P is 30 kN, what is the average shearing stress developed in the rivet?

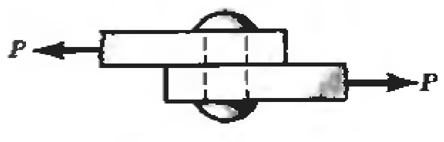


Fig. 4-6

Here the average shear stress in the rivet is P/A where A is the cross-sectional area of the rivet. However, rivet holes are usually 1.5 mm larger in diameter than the rivet and it is customary to assume that the rivet fills the hole completely. Hence the shearing stress is given by

$$\tau = \frac{30{,}000\ \text{N}}{(\pi/4)[0.0215\ \text{m}]^2} = 8.26 \times 10^7\ \text{N/m}^2 \qquad \text{or} \qquad 82.6\ \text{MPa}$$

4.6. One common type of weld for joining two plates is the *fillet weld*. This weld undergoes shear as well as tension or compression and frequently bending in addition. For the two plates shown in Fig. 4-7, determine the allowable tensile force P that may be applied using an allowable working stress of 11,300 lb/in^2 for shear loading as indicated by the Code for Fusion Welding of the American Welding Society. Consider only shearing stresses in the weld. The load is applied midway between the two welds.

The minimum dimension of the weld cross section is termed the *throat*, which in this case is $\frac{1}{2}\sin 45^\circ = 0.353$ in. The effective weld area that resists shearing is given by the length of the weld times the throat dimension, or weld area = 7(0.353) = 2.47 in^2 for each of the two welds. Thus the allowable

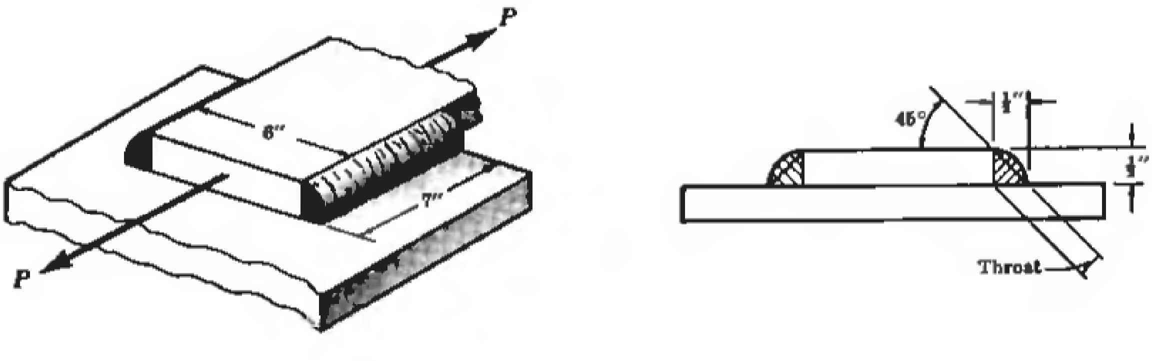


Fig. 4-7

tensile load P is given by the product of the working stress in shear times the area resisting shear, or $P = 11{,}300(2)(2.47) = 56{,}000$ lb.

4.7. Shafts and pulleys are usually fastened together by means of a key, as shown in Fig. 4-8(a). Consider a pulley subject to a turning moment T of 10,000 lb-in keyed by a $\frac{1}{2} \times \frac{1}{2} \times 3$ in key to the shaft. The shaft is 2 in in diameter. Determine the shear stress on a horizontal plane through the key.

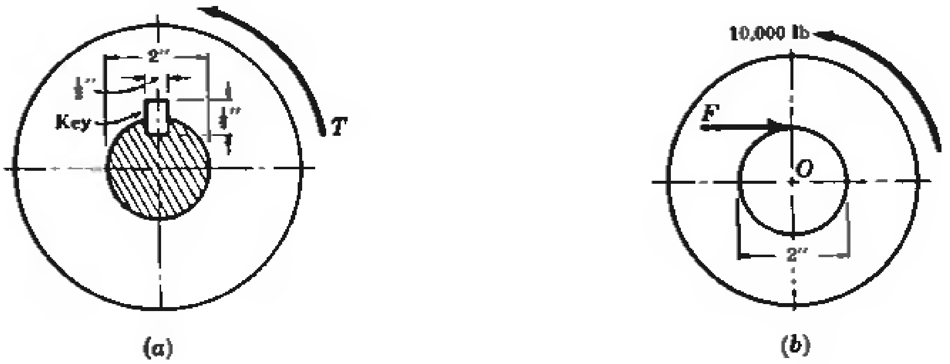


Fig. 4-8

Drawing a free-body diagram of the pulley alone, as shown in Fig. 4-8(b), we see that the applied turning moment of 10,000 lb-in must be resisted by a horizontal tangential force F exerted on the pulley by the key. For equilibrium of moments about the center of the pulley we have

$$\Sigma M_0 = 10{,}000 - F(1) = 0 \qquad \text{or} \qquad F = 10{,}000 \text{ lb}$$

It is to be noted that the shaft exerts additional forces, not shown, on the pulley. These act through the center O and do not enter the above moment equation. The resultant forces acting on the key appear as in Fig. 4-9(a). Actually the force F acting to the right is the resultant of distributed forces acting over the lower half of the left face. The other forces F shown likewise represent resultants of distributed force systems. The exact nature of the force distribution is not known.

The free-body diagram of the portion of the key below a horizontal plane a-a through its midsection is shown in Fig. 4-9(b). For equilibrium in the horizontal direction we have

$$\Sigma F_h = 10{,}000 - \tau(\tfrac{1}{2})(3) = 0 \qquad \text{or} \qquad \tau = 6670 \text{ lb/in}^2$$

This is the horizontal shear stress in the key.

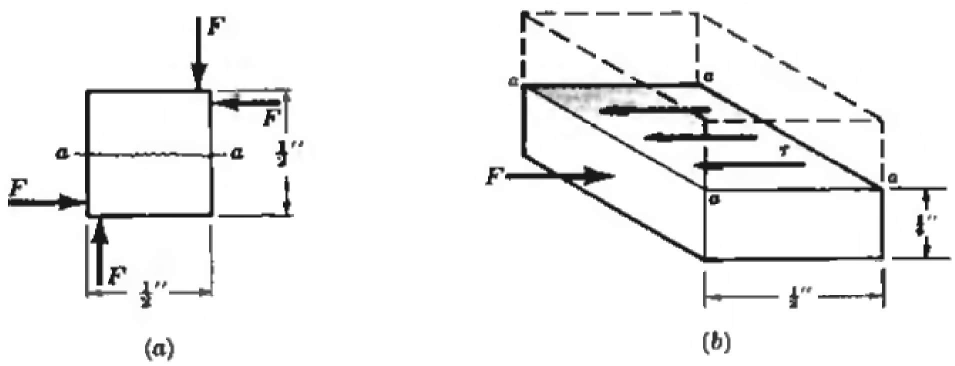


Fig. 4-9

4.8. A lifeboat on a seagoing cruise ship is supported at each end by a stranded steel cable passing over a pulley on a davit anchored to the top deck. The cable at each end carries a tension of 4000 N and the cable as well as the pulley are located in a vertical plane as shown in Fig. 4-10. The pulley may rotate freely about the horizontal circular axle indicated. Determine the diameter of this axle if the allowable transverse shearing stress is 50 MPa.

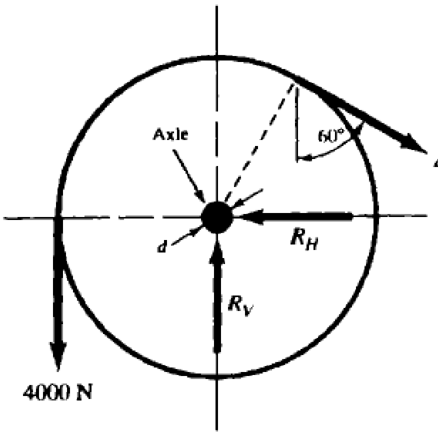


Fig. 4-10

The free-body diagram of the pulley shows not only the cable tensions but also the forces R_H and R_V exerted on the pulley by the circular axle. From statics we have

$$\Sigma F_H = -R_H + 4000 \sin 60° = 0$$

$$R_H = 3464 \text{ N } (\leftarrow)$$

$$\Sigma F_V = R_V - 4000 - 4000 \cos 60° = 0$$

$$R_V = 6000 \text{ N } (\uparrow)$$

The resultant of R_H and R_V is $R = \sqrt{(3464)^2 + (6000)^2} = 6930$ N oriented at an angle θ from the horizontal given by

$$\theta = \arctan \frac{6000 \text{ N}}{3464 \text{ N}}$$

$$= 60°$$

The force exerted by the pulley upon the axle is equal and opposite to that shown in Fig. 4-11. If we assume that the resultant force of 6930 N is uniformly distributed over the cross section of the axle, the transverse shearing stress has the appearance shown in Fig. 4-12. From Eq. (*4.1*), we have

$$50 \times 10^6 \text{ N/m}^2 = \frac{6930 \text{ N}}{(\pi/4)d^2}$$

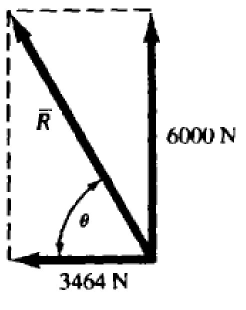


Fig. 4-11

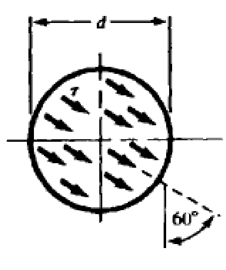


Fig. 4-12

where d is the unknown axle diameter. Solving,

$$d = 13.3 \times 10^{-3}\,\text{m} \qquad \text{or} \qquad 13.3\,\text{mm}$$

4.9. A building that is 60 m tall has essentially the rectangular configuration shown in Fig. 4-13. Horizontal wind loads will act on the building exerting pressures on the vertical face that may be approximated as uniform within each of the three "layers" as shown. From empirical expressions for wind pressures at the midpoint of each of the three layers, we have a pressure of 781 N/m² on the lower layer, 1264 N/m² on the middle layer, and 1530 N/m² on the top layer. Determine the resisting shear that the foundation must develop to withstand this wind load.

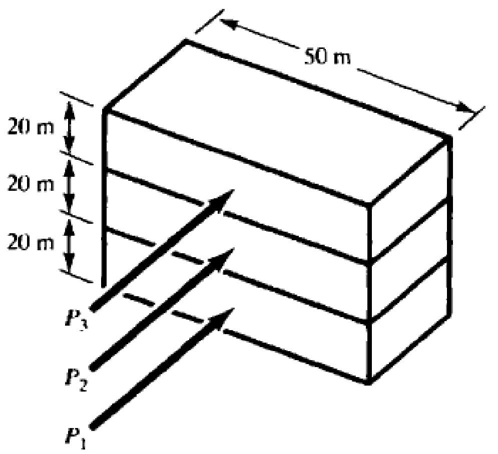


Fig. 4-13

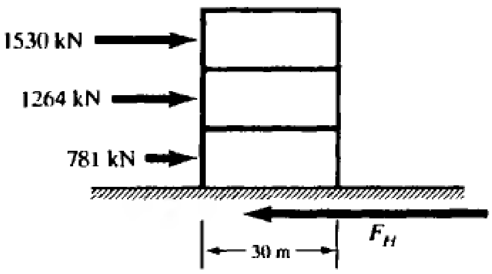


Fig. 4-14

The horizontal forces acting on these three layers are found to be

$$P_1 = (20\,\text{m})(50\,\text{m})(781\,\text{N/m}^2) = 781\,\text{kN}$$

$$P_2 = (20\,\text{m})(50\,\text{m})(1264\,\text{N/m}^2) = 1264\,\text{kN}$$

$$P_3 = (20\,\text{m})(50\,\text{m})(1530\,\text{N/m}^2) = 1530\,\text{kN}$$

These forces are taken to act at the midheight of each layer, so the free-body diagram of the building has the appearance of Fig. 4-14, where F_H denotes the horizontal shearing force exerted by the foundation upon the structure. From horizontal equilibrium, we have

$$\Sigma F_H = 1530 + 1264 + 781 - F_H = 0$$

$$F_H = 3575\,\text{kN}$$

If we assume that this horizontal reaction is uniformly distributed over the base of the structure, the horizontal shearing stress given by Eq. (*4.1*) is

$$\tau = \frac{3575\,\text{kN}}{(30\,\text{m})(50\,\text{m})} = 2.38\,\text{kN/m}^2$$

4.10. In the North Atlantic Ocean, large icebergs (often weighing more than 8000 MN) present a menace to ship navigation. A recently developed technique makes it possible to tow them to acceptable locations. The method involves the use of a remotely operated unmanned submersible vehicle which drills a hole in the iceberg about 30 m below the water surface and then inserts a cylindrical anchor in the hole as shown in Fig. 4-15. The anchor is a cylindrical steel tube of diameter 100 mm and it is secured to the hole in the iceberg by injecting gaseous carbon dioxide through small holes in the tube. This gas quickly freezes and fills the narrow annular space between the outside of the anchor and the inside of the hole in the ice. A connection from the exposed end of the anchor permits a cable to be run to the towing vessel. If the maximum allowable shear stress in the frozen carbon dioxide is 0.5 MPa, determine the minimum length of the cylindrical anchor so that it will not be pulled out from the iceberg under a towing force of 200 kN.

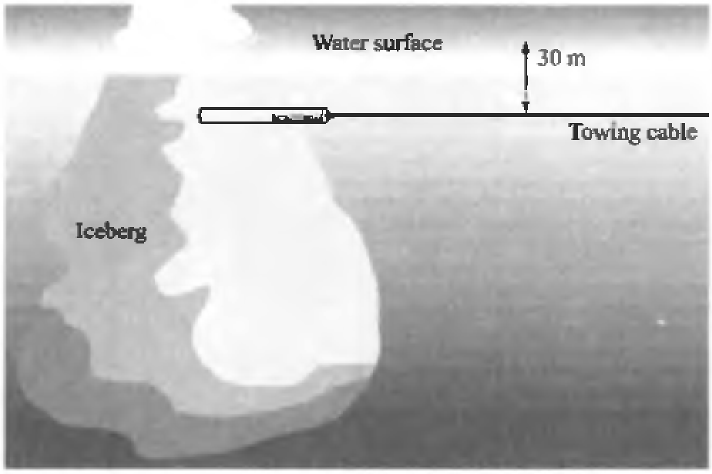


Fig. 4-15

A free-body diagram of the cylindrical tube (anchor) is shown in Fig. 4-16. There, T represents the towing force in the cable attached to the anchor and τ is the shearing stress in the frozen carbon dioxide. It is assumed that τ is uniform at all points along the length L of the anchor as well as around the circumference of the tube. If τ is 0.5 MPa for horizontal equilibrium, we have

$$\Sigma F_H = T - \pi D L \tau = 0$$

$$200{,}000\ \text{N} - \pi(0.1\ \text{m})L(0.5 \times 10^6\ \text{N/m}^2) = 0$$

$$L = 1.27\ \text{m}$$

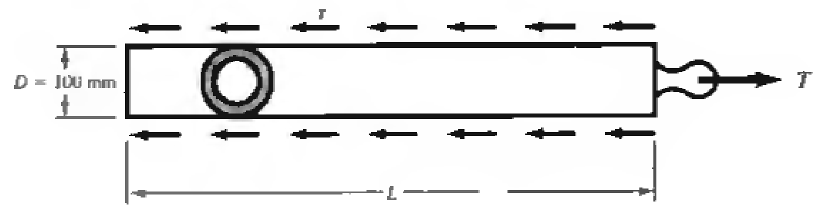


Fig. 4-16

4.11. It is occasionally desirable to design certai.. structural fasteners to be strong in tension yet somewhat weak in transverse shear. One example of this is to be found in contemporary design of four-engine wide-body aircraft. Each engine is attached to the main supporting frame inside the wing [see Fig. 4-17(a)] by aluminum alloy bolts that are adequately strong to support the dead weight of the engine plus additional loads occurring in flight. However, the alloying is such that each bolt can carry only moderate transverse shear in the unlikely event of a "wheels-up" emergency landing so that the engine will be torn free from the wing. If the ultimate transverse shear strength of each bolt is 120 MPa, the bolt diameter 20 mm, and four bolts secure the engine to the wing, determine the horizontal force that must act between the ground and the engine for separation of the engine from the wing to occur.

A free-body diagram of the engine together with the four bolts is shown in Fig. 4-17(*b*). There F_u represents ultimate shearing force in each bolt ($F_u = \tau_u A$), where τ_u represents the ultimate shear stress and A the cross-sectional area of each bolt. Also, F_g represents the force exerted by the ground on the bottom of the engine. Note that the underside of the aircraft fuselage is above the bottom of the engine. We have

$$F_u = \frac{\pi}{4}(0.020\ \text{m})^2(120 \times 10^6\ \text{N/m}^2) = 37.7\ \text{kN}$$

and for horizontal equilibrium (neglecting dynamic effects)

$$\Sigma F_H = F_g - 4F_u = 0$$

$$F_g = 4(37.7) = 151\ \text{kN}$$

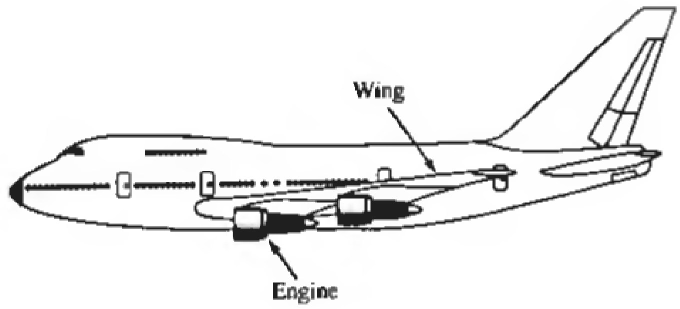


Fig. 4-17(*a*)

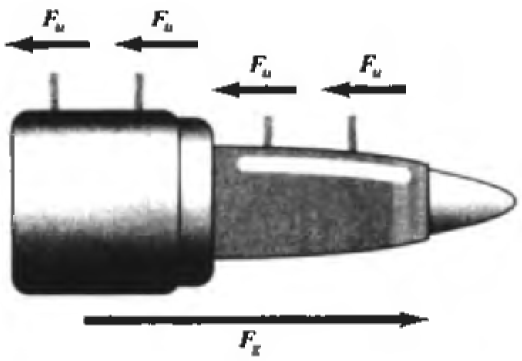


Fig. 4-17(*b*)

4.12. A power reactor has certain of its pressurized components (see Fig. 4-18) made of type 304 stainless steel, 2.5 in thick. Adjacent butt-welded sections are joined by electron beam welding in a partial vacuum using a 200 kW system. The ultimate strength of the parent steel is 160,000 lb/in². If the weld is assumed to be 100 percent efficient, determine the force that may be transmitted through each 14 in wide section. Also, determine the force if 80 percent efficiency is assumed.

For 100 percent effectiveness of the weld we determine the cross-sectional area of the 14 in by 2.5 in section to be (14 in) (2.5 in) = 35 in². The allowable load P is then given by

$$P = (35\ \text{in}^2)(160{,}000\ \text{lb/in}^2) = 5.6 \times 10^6\ \text{lb}$$

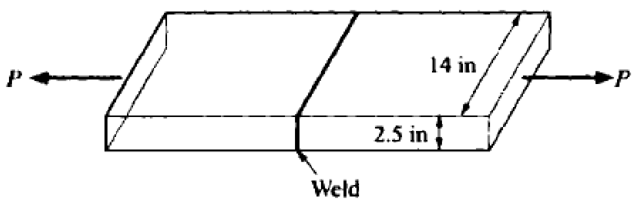


Fig. 4-18

For 80 percent effectiveness of the weld we have the allowable load

$$P = (5.6 \times 10^6 \text{ lb/in}^2)(0.80) = 4.48 \times 10^6 \text{ lb}$$

4.13. Two $\frac{1}{16}$ in thick strips of titanium alloy 1.75 in wide are joined by a 45° laser weld as shown in Fig. 4-19. A 100 kW carbon dioxide laser system is employed to form the joint. If the allowable

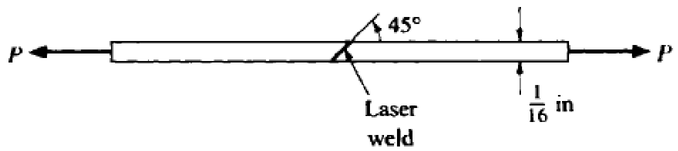


Fig. 4-19

shearing stress in the alloy is 65,000 lb/in² and the joint is assumed to be 100 percent efficient, determine the maximum allowable force P that may be applied.

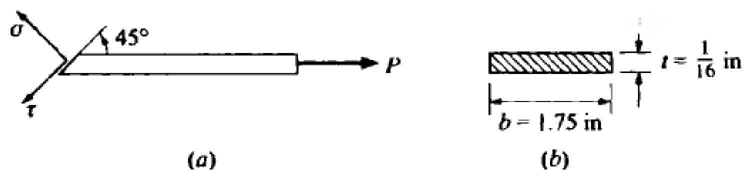


Fig. 4-20

A free-body diagram of the right strip has the form shown in Fig. 4-20. There, σ denotes normal stress in the weld on the 45° plane and τ the shearing stress. These are, of course, forces per unit area on the 45° plane and these must be multiplied by the area of the 45° plane which is $bt/\cos 45°$ where t denotes strip thickness and b the width. For horizontal equilibrium we have

$$\Sigma F_t = \tau\left(\frac{bt}{\cos 45°}\right) - P\cos 45° = 0$$

$$\tau = \frac{P\cos 45°}{bt}$$

$$65{,}000 \text{ lb/in}^2 = \frac{P(1/\sqrt{2})^2}{(1.75 \text{ in})(\frac{1}{16} \text{ in})} \qquad \text{or} \qquad P = 7110 \text{ lb}$$

Supplementary Problems

4.14. In Problem 4.1, if the maximum allowable working stress in shear is 14,000 lb/in^2, determine the required diameter of the bolt in order that this value is not exceeded. *Ans.* $d = 0.585$ in

4.15. A circular punch 20 mm in diameter is used to punch a hole through a steel plate 10 mm thick. If the force necessary to drive the punch through the metal is 250 kN, determine the maximum shearing stress developed in the material. *Ans.* $\tau = 400$ MPa

4.16. In structural practice, steel clip angles are commonly used to transfer loads from horizontal girders to vertical columns. If the reaction of the girder upon the angle is a downward force of 10,000 lb as shown in Fig. 4-21 and if two $\frac{7}{8}$-in-diameter rivets resist this force, find the average shearing stress in each of the rivets. As in Problem 4.5, assume that the rivet fills the hole, which is $\frac{1}{16}$ in larger in diameter than the rivet. *Ans.* 7200 lb/in^2

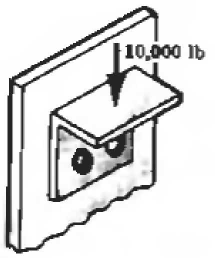


Fig. 4-21

4.17. A pulley is keyed (to prevent relative motion) to a 60-mm-diameter shaft. The unequal belt pulls, T_1 and T_2, on the two sides of the pulley give rise to a net turning moment of 120 N · m. The key is 10 mm by 15 mm in cross section and 75 mm long, as shown in Fig. 4-22. Determine the average shearing stress acting on a horizontal plane through the key. *Ans.* $\tau = 5.33$ MPa

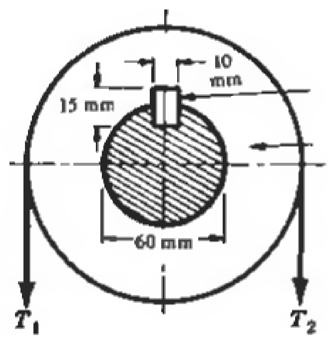


Fig. 4-22

4.18. Consider the balcony-type structure shown in Fig. 4-23. The horizontal balcony is loaded by a total load of 80 kN distributed in a radially symmetric fashion. The central support is a shaft 500 mm in diameter and the balcony is welded at both the upper and lower surfaces to this shaft by welds 10 mm on a side (or leg) as shown in the enlarged view at the right. Determine the average shearing stress existing between the shaft and the weld. *Ans.* 2.5 MPa

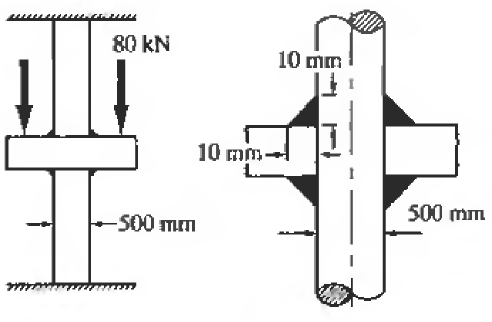


Fig. 4-23

4.19. Consider the two plates of equal thickness joined by two fillet welds as indicated in Fig. 4-24. Determine the maximum shearing stress in the welds. *Ans.* $\tau = 0.707P/ab$

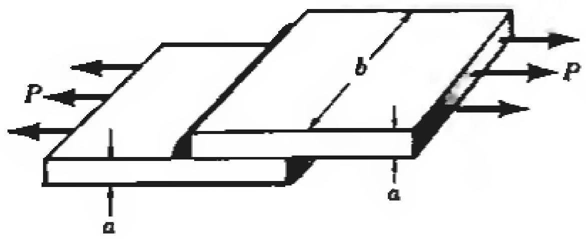


Fig. 4-24

4.20. A copper tube 55 mm in outside diameter and of wall thickness 5 mm fits loosely over a solid steel circular bar 40 mm in diameter. The two members are fastened together by two metal pins each 8 mm in diameter and passing transversely through both members, one pin being near each end of the assembly. At room temperature the assembly is stress free when the pins are in position. The temperature of the entire assembly is then raised 40°C. Calculate the average shear stress in the pins. For copper $E = 90$ GPa, $\alpha = 18 \times 10^{-6}$/°C; for steel $E = 200$ GPa, $\alpha = 12 \times 10^{-6}$/°C. *Ans.* $\tau = 132$ MPa

4.21. The shear strength of human bone is an important parameter when implants must be employed to maintain the desired length of a fractured leg or arm. Substitute animal bone segments are sometimes employed but it is necessary to select a substance having the same transverse shear strength as human bone. For this purpose tests such as shown in Fig. 4-25 are first carried out on the substitute under consideration. If the cross-sectional area of the animal bone is 150 mm^2 and a transverse force $F = 600$ N is required to cause shear fracture, find the mean transverse shear stress at fracture. *Ans.* 2 MPa

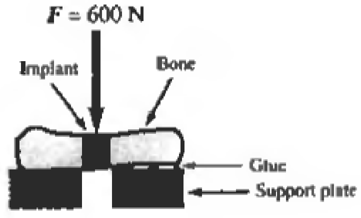


Fig. 4-25

4.22. In automotive as well as aircraft applications, two pieces of thin metal are often joined by a single lap shear joint, as shown in Fig. 4-26. Here, the metal has a thickness of 2.2 mm. The ultimate shearing strength of the epoxy adhesive joining the metals is 2.57×10^4 kPa, the shear modulus of the epoxy is 2.8 GPa, and the epoxy is effective over the 12.7×25.4-mm overlapping area. Determine the maximum axial load P the joint can carry. Neglect the slight bending effect that arises because the metal pieces are not in the same plane. *Ans.* 8290 N

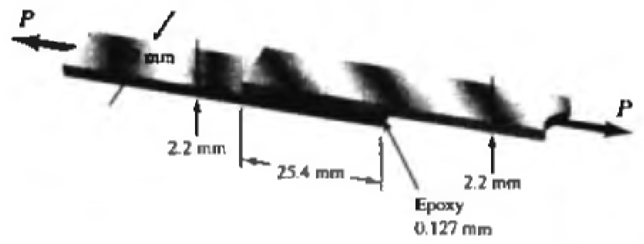


Fig. 4-26

4.23. If the shear modulus of the epoxy in Problem 4.22 is 2.8 GPa, determine the axial displacement of one piece of metal with respect to the other just prior to failure of the epoxy if the epoxy is 0.127 mm thick. *Ans.* 0.0017 mm

Chapter 5

Torsion

DEFINITION OF TORSION

Consider a bar rigidly clamped at one and twisted at the other end by a torque (twisting moment) $T = Fd$ applied in a plane perpendicular to the axis of the bar as shown in Fig. 5-1. Such a bar is in torsion. An alternative representation of the torque is the double-headed vector directed along the axis of the bar.

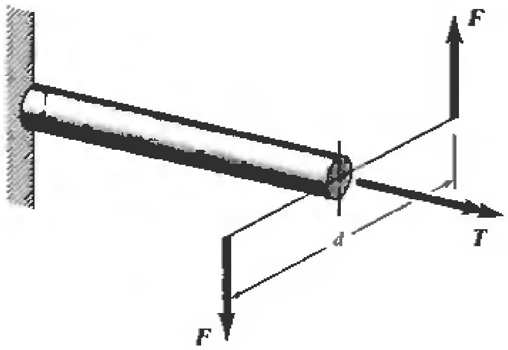


Fig. 5-1

TWISTING MOMENT

Occasionally a number of couples act along the length of a shaft. In that case it is convenient to introduce a new quantity, the *twisting moment*, which for any section along the bar is defined to be the algebraic sum of the moments of the applied couples that lie to one side of the section in question. The choice of side in any case is of course arbitrary.

POLAR MOMENT OF INERTIA

For a hollow circular shaft of outer diameter D_o with a concentric circular hole of diameter D_i the *polar moment of inertia* of the cross-sectional area, usually denoted by J, is given by

$$J = \frac{\pi}{32}(D_o^4 - D_i^4) \tag{5.1}$$

The polar moment of inertia for a solid shaft is obtained by setting $D_i = 0$. See Problem 5.1. This quantity J is a mathematical property of the geometry of the cross section which occurs in the study of the stresses set up in a circular shaft subject to torsion.

Occasionally it is convenient to rewrite the above equation in the form

$$J = \frac{\pi}{32}(D_o^2 + D_i^2)(D_o^2 - D_i^2)$$

$$= \frac{\pi}{32}(D_o^2 + D_i^2)(D_o + D_i)(D_o - D_i)$$

This last form is useful in numerical evaluation of J in those cases where the difference $(D_o - D_i)$ is small. See Problem 5.6.

TORSIONAL SHEARING STRESS

For either a solid or a hollow circular shaft subject to a twisting moment T the *torsional shearing stress* τ at a distance ρ from the center of the shaft is given by

$$\tau = \frac{T\rho}{J} \tag{5.2}$$

This expression is derived in Problem 5.2. For applications see Problems 5.4, 5.5, 5.9, 5.10, and 5.11. This stress distribution varies from zero at the center of the shaft (if it is solid) to a maximum at the outer fibers, as shown in Fig. 5-2. It is to be emphasized that no points of the bar are stressed beyond the proportional limit.

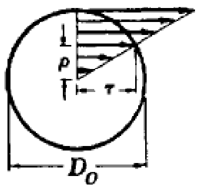


Fig. 5-2

SHEARING STRAIN

If a generator a-b is marked on the surface of the unloaded bar, then after the twisting moment T has been applied this line moves to a-b', as shown in Fig. 5-3. The angle γ, measured in radians, between the final and original positions of the generator is defined as the *shearing strain* at the surface of the bar. The same definition would hold at any interior point of the bar.

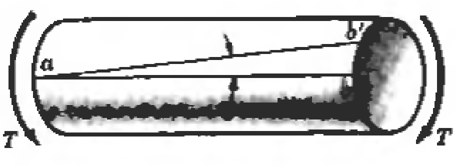


Fig. 5-3

MODULUS OF ELASTICITY IN SHEAR

The ratio of the shear stress τ to the shear strain γ is called the *modulus of elasticity in shear* and, as in Chap. 4, is given by

$$G = \frac{\tau}{\gamma} \tag{5.3}$$

Again the units of G are the same as those of shear stress, since the shear strain is dimensionless.

ANGLE OF TWIST

If a shaft of length L is subject to a constant twisting moment T along its length, then the angle θ through which one end of the bar will twist relative to the other is

$$\theta = \frac{TL}{GJ} \tag{5.4}$$

where J denotes the polar moment of inertia of the cross section. See Fig. 5-4. This equation is derived in Problem 5.3. For applications see Problems 5.5, 5.7, 5.8, 5.11, 5.12, and 5.13. This expression holds only for purely elastic action of the bar.

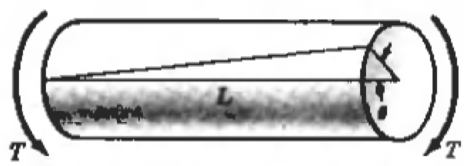


Fig. 5-4

COMPUTER SOLUTION

For a bar of circular cross section and variable diameter, the angle of twist θ is determined by dividing the bar into a number of segments along its length, such that in each segment the diameter may be taken to be constant. This procedure is well suited to computer implementation, and a FORTRAN program for implementing it is given in Problem 5.14. (See also Problem 5.15.)

POWER TRANSMISSION

A shaft rotating with constant angular velocity ω (radians per second) is being acted on by a twisting moment T and hence transmits a power $P = T\omega$. Alternatively, in terms of the number of revolutions per second f, the power transmitted is $P = 2\pi fT$. (See Problems 5.9, 5.10 and 5.11.)

PLASTIC TORSION OF CIRCULAR BARS

As the twisting moment acting on either a solid or hollow circular bar is increased, a value of the twisting moment is finally reached for which the extreme fibers of the bar have reached the yield point in shear of the material. This is the maximum possible elastic twisting moment that the bar can withstand and is denoted by T_e. A further increase in the value of the twisting moment puts the interior fibers at the yield point, with yielding progressing from the outer fibers inward. The limiting case occurs when all fibers are stressed to the yield point in shear and this represents the *fully plastic twisting moment*. It is denoted by T_p. Provided we do not consider stresses greater than the yield point in shear, this is the maximum possible twisting moment the bar can carry. For a solid circular bar subject to torsion it is shown in Problem 5.21 that $T_p = 4T_e/3$.

Solved Problems

5.1. Derive an expression for the polar moment of inertia of the cross-sectional area of a hollow circular shaft. What does this expression become for the special case of a solid circular shaft?

Let D_o denote the outside diameter of the shaft and D_i the inside diameter. Because of the circular symmetry involved, it is most convenient to adopt the polar coordinate system shown in Fig. 5-5.

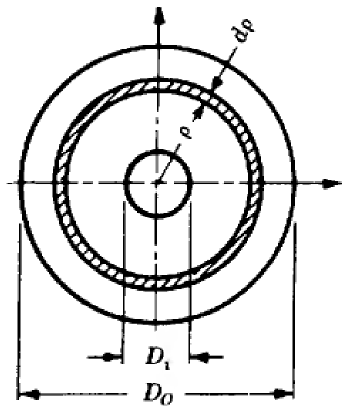


Fig. 5-5

By definition, the polar moment of inertia is given by the integral

$$J = \int_A \rho^2 \, da$$

where A indicates that the integral is to be evaluated over the entire cross-sectional area.

To evaluate this integral we select as an element of area a thin ring-shaped element of radius ρ and radial thickness $d\rho$ as shown. The area of the ring is $da = 2\pi\rho(d\rho)$. Thus

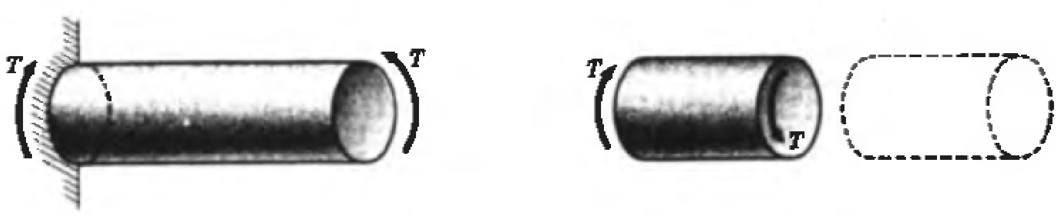
$$J = \int_{1/2D_i}^{1/2D_o} \rho^2(2\pi\rho)\, d\rho = \frac{\pi}{32}[D_o^4 - D_i^4]$$

The units of J are in^4 or m^4. For the special case of a solid circular shaft, the above becomes $J = \pi D^4/32$, where D denotes the diameter of the shaft.

5.2. Derive an expression relating the applied twisting moment acting on a shaft of circular cross section and the shearing stress at any point in the shaft.

In Fig. 5-6(a) the shaft is shown loaded by the two torques T and consequently is in static equilibrium. To determine the distribution of shearing stress in the shaft, let us cut the shaft by a plane passing through it in a direction perpendicular to the geometric axis of the bar.

The free-body diagram of the portion of the shaft to the left of this plane appears as in Fig. 5-6(b). Obviously a torque T must act over the cross section cut by the plane. This is true since the entire shaft



(a) (b)

Fig. 5-6

is in equilibrium, and hence any portion of it also is. The torque T acting on the cut section represents the effect of the right portion of the shaft on the left portion. Since the right portion has been removed, it must be replaced by its effect on the left portion. This effect is represented by the torque T. This torque is of course a resultant of shearing stresses distributed over the cross section. It is now necessary to make certain assumptions in order to determine the nature of the variation of shear stress intensity over the cross section.

One fundamental assumption is that a plane section of the shaft normal to its axis before loads are applied remains plane and normal to the axis after loading. This may be verified experimentally for circular shafts, but this assumption is not valid for shafts of noncircular cross section.

A generator on the surface of the shaft, denoted by O_1A in Fig. 5-7, deforms into the configuration O_1B after torsion has occurred. The angle between these configurations is denoted by α. By definition, the shearing unit strain γ on the surface of the shaft is

$$\gamma = \tan\alpha \approx \alpha$$

where the angle α is measured in radians. From the geometry of the figure,

$$\alpha = \frac{AB}{L} = \frac{r\theta}{L}$$

Hence

$$\gamma = \frac{r\theta}{L}$$

But since a diameter of the shaft prior to loading is assumed to remain a diameter after torsion has occurred, the shearing unit strain at a general distance ρ from the center of the shaft may likewise be written $\gamma_\rho = \rho\theta/L$. Consequently the shearing strains of the longitudinal fibers vary linearly as the distances from the center of the shaft.

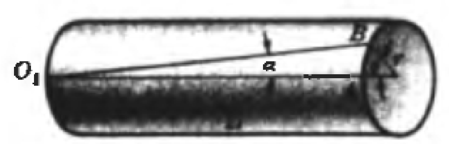


Fig. 5-7

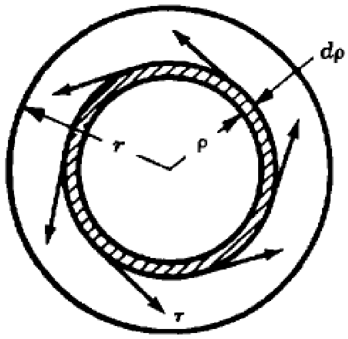


Fig. 5-8

If we assume that we are concerned only with the linear range of action of the material where the shearing stress is proportional to shearing strain, then it is evident that the shearing stresses of the longitudinal fibers vary linearly as the distances from the center of the shaft. Obviously the distribution of shearing stresses is symmetric around the geometric axis of the shaft. They have the appearance shown in Fig. 5-8. For equilibrium, the sum of the moments of these distributed shearing forces over the entire circular cross section is equal to the applied twisting moment. Also, the sum of the moments of these forces is exactly equal to the torque T shown in Fig. 5-6(b) above.

Thus we have

$$T = \int_0^r \tau\rho\, da$$

where da represents the area of the shaded ring-shaped element shown in Fig. 5-8. However, the shearing stresses vary as the distances from the geometric axis; hence

$$\frac{\tau_\rho}{\rho} = \frac{\tau_r}{r} = \text{constant}$$

where the subscripts on the shearing stress denote the distances of the element from the axis of the shaft.

Consequently we may write

$$T = \int_0^r \frac{\tau_\rho}{\rho}(\rho^2)\,da = \frac{\tau_\rho}{\rho}\int_0^r \rho^2\,da$$

since the ratio τ_ρ/ρ is a constant. However, the expression $\int_0^r \rho^2\,da$ is by definition (see Problem 5.1) the polar moment of inertia of the cross-sectional area. Values of this for solid and hollow circular shafts are derived in Problem 5.1. Hence the desired relationship is

$$T = \frac{\tau_\rho J}{\rho} \qquad \text{or} \qquad \tau_\rho = \frac{T\rho}{J}$$

It is to be emphasized that this expression holds *only* if no points of the bar are stressed beyond the proportional limit of the material.

5.3. Derive an expression for the angle of twist of a circular shaft as a function of the applied twisting moment. Assume that the entire shaft is acting within the elastic range of action of the material.

Let L denote the length of the shaft, J the polar moment of inertia of the cross section, T the applied twisting moment (assumed constant along the length of the bar), and G the modulus of elasticity in shear. The angle of twist in a length L is represented by θ in Fig. 5-9.

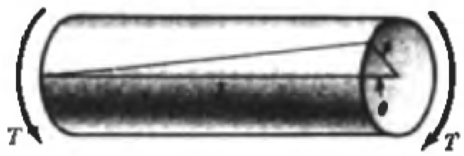


Fig. 5-9

From Problem 5.2 we have at the outer fibers where $\rho = r$:

$$\gamma_r = \frac{r\theta}{L} \qquad \text{and} \qquad \tau_r = \frac{Tr}{J}$$

By definition, the shearing modulus is given by $G = \dfrac{\tau}{\gamma} = \dfrac{Tr/J}{r\theta/L} = \dfrac{TL}{J\theta}$ from which $\theta = \dfrac{TL}{GJ}$. Note that θ is expressed in radians, i.e., it is dimensionless.

Occasionally the angle of twist in a unit length is useful. It is often denoted by ϕ and is given by $\phi = \theta/L = T/GJ$.

5.4. If a twisting moment of 10,000 lb·in is impressed upon a $1\frac{3}{4}$-in-diameter shaft, what is the maximum shearing stress developed? Also, what is the angle of twist in a 4-ft length of the shaft? The material is steel for which $G = 12 \times 10^6$ lb/in^2. Assume entirely elastic action.

From Problem 5.1 the polar moment of inertia of the cross-sectional area is

$$J = \frac{\pi}{32}(D_o)^4 = \frac{\pi}{32}\left(\frac{7}{4}\right)^4 = 0.92\ \text{in}^4$$

The torsional shearing stress τ at any distance ρ from the center of the shaft was shown in Problem 5.2 to be $\tau_\rho = T\rho/J$. The maximum shear stress is developed at the outer fibers and there at $\rho = \frac{7}{8}$ in

$$\tau_{max} = \frac{10{,}000(\frac{7}{8})}{0.92} = 9500\ \text{lb/in}^2$$

Hence the shear stress varies linearly from zero at the center of the shaft to 9500 lb/in^2 at the outer fibers as shown in Fig. 5-10.

The angle of twist θ in a 4-ft length of the shaft is

$$\theta = \frac{TL}{GJ} = \frac{10{,}000(48\text{ in})}{12 \times 10^6(0.92)} = 0.0435 \text{ radian}$$

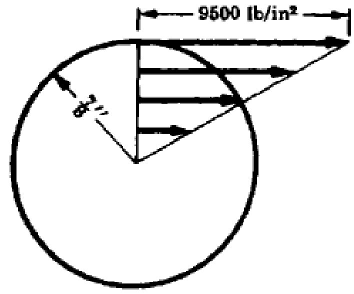


Fig. 5-10

5.5. A hollow steel shaft 3 m long must transmit a torque of 25 kN · m. The total angle of twist in this length is not to exceed 2.5° and the allowable shearing stress is 90 MPa. Determine the inside and outside diameter of the shaft if $G = 85$ GPa.

Let d_o and d_i designate the outside and inside diameters of the shaft, respectively. From Eq. (*5.4*) the angle of twist is $\theta = TL/GJ$, where θ is expressed in radians. Thus, in the 3-m length we have

$$2.5°\left(\frac{\text{rad}}{57.3\text{ deg}}\right) = \frac{(25{,}000\text{ N}\cdot\text{m})(3\text{ m})}{(85 \times 10^9\text{ N/m}^2)(\pi/32)(d_o^4 - d_i^4)}$$

or

$$d_o^4 - d_i^4 = (206 \times 10^{-6})\text{ m}^4$$

The maximum shearing stress occurs at the outer fibers where $\rho = d_o/2$. At these points from Eq. (*5.2*), we have

$$90 \times 10^6\text{ N/m}^2 = \frac{(25{,}000\text{ N}\cdot\text{m})(d_o/2)}{(\pi/32)(d_o^4 - d_i^4)}$$

or

$$d_o^4 - d_i^4 = (1414\, d_o)(10^{-6})\text{ m}^4$$

Comparison of the right-hand sides of these equations indicates that

$$206 \times 10^{-6} = 1414\, d_o(10^{-6})$$

and thus $d = 0.145$ m or 145 mm. Substitution of this value into either of the equations then gives $d_i = 0.125$ m or 125 mm.

5.6. Let us consider a thin-walled tube subject to torsion. Derive an approximate expression for the allowable twisting moment if the working stress in shear is a given constant τ_w. Also, derive an approximate expression for the strength–weight ratio of such a tube. It is assumed the tube does not buckle, and the material is within the elastic range of action.

The polar moment of inertia of a hollow circular shaft of outer diameter D_o and inner diameter D_i is $J = (\pi/32)(D_o^4 - D_i^4)$. If R denotes the outer radius of the tube, then $D_o = 2R$, and further, if t denotes the wall thickness of the tube, then $D_i = 2R - 2t$.

The polar moment of inertia J may be written in the alternate form

$$J = \frac{\pi}{32}[(2R)^4 - (2R - 2t)^4] = \frac{\pi}{2}[R^4 - (R - t)^4] = \frac{\pi}{2}(4R^3t - 6R^2t^2 + 4Rt^3 - t^4)$$

$$= \frac{\pi}{2}R^4\left[4\left(\frac{t}{R}\right) - 6\left(\frac{t}{R}\right)^2 + 4\left(\frac{t}{R}\right)^3 - \left(\frac{t}{R}\right)^4\right]$$

Neglecting squares and higher powers of the ratio t/R, since we are considering a thin-walled tube, this becomes, approximately, $J = 2\pi R^3 t$.

The ordinary torsion formula is $T = \tau_w J/R$. For a thin-walled tube this becomes, for the allowable twisting moment, $T = 2\pi R^2 t\tau_w$.

The weight W of the tube is $W = \gamma LA$ where γ is the specific weight of the material, L the length of the tube, and A the cross-sectional area of the tube. The area is given by

$$A = \pi[R^2 - (R-t)^2] = \pi(2Rt - t^2) = \pi R^2\left[\frac{2t}{R} - \left(\frac{t}{R}\right)^2\right]$$

Again neglecting the square of the ratio t/R for a thin tube, this becomes $A = 2\pi Rt$.

The strength–weight ratio is defined to be T/W. This is given by

$$\frac{T}{W} = \frac{2\pi R^2 t\tau_w}{2\pi RtL\gamma} = \frac{R\tau_w}{L\gamma}$$

The ratio is of considerable importance in aircraft design.

5.7. A solid circular shaft has a slight taper extending uniformly from one end to the other. Denote the radius at the small end by a, that at the large end by b. Determine the error committed if the angle of twist for a given length is calculated using the mean radius of the shaft. The radius at the larger end is 1.2 times that at the smaller end.

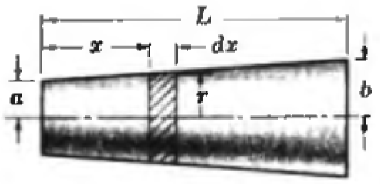


Fig. 5-11

Let us set up a coordinate system with the variable x denoting the distance from the small end of the shaft (see Fig. 5-11). The radius at a section at the distance x from the small end is

$$r = a + \frac{(b-a)x}{L}$$

where L is the length of the bar.

Provided the angle of taper is small, it is sufficient to consider the angle $d\theta$ through which the shaded element of length dx is twisted. This is obtained by applying the expression $\theta = TL/GJ$ to the element of length dx and radius $r = a + [(b-a)x/L]$. For such an element the polar moment of inertia is

$$J = \frac{\pi}{32}D^4 = \frac{\pi}{2}r^4 = \frac{\pi}{2}\left[a + \frac{(b-a)x}{L}\right]^4$$

Thus

$$d\theta = \frac{T\,dx}{G\dfrac{\pi}{2}\left[a + \dfrac{(b-a)x}{L}\right]^4}$$

The angle of twist in the length L is found by integrating the last equation. Thus

$$\theta = \frac{2T}{G\pi}\int_0^L \frac{dx}{\left[a + \dfrac{(b-a)x}{L}\right]^4} = \frac{2T}{G\pi}\left(-\frac{1}{3}\right)\left(\frac{L}{b-a}\right)\left[\frac{1}{\left[a + \dfrac{(b-a)x}{L}\right]^3}\right]_0^L = \frac{2TL}{3G\pi(b-a)}\left(-\frac{1}{b^3} + \frac{1}{a^3}\right)$$

If $b = 1.2a$, this becomes $\theta = 1.40433TL/G\pi a^4$. For a solid shaft of radius $1.1a$

$$\theta_1 = \frac{TL}{G\dfrac{\pi}{2}(1.1a)^4} = \frac{1.36602TL}{G\pi a^4}$$

Using these values of θ and θ_1, we find

$$\text{Percent error} = \frac{0.03831}{1.40433} \times 100 = 2.73\%$$

5.8. Consider two solid circular shafts connected by 2-in- and 10-in-pitch-diameter gears as in Fig. 5-12(a). The shafts are assumed to be supported by the bearings in such a manner that they undergo no bending. Find the angular rotation of D, the right end of one shaft, with respect to A, the left end of the other, caused by the torque of 2500 lb · in applied at D. The left shaft is steel for which $G = 12 \times 10^6$ lb/in^2 and the right is brass for which $G = 5 \times 10^6$ lb/in^2. Assume elastic action.

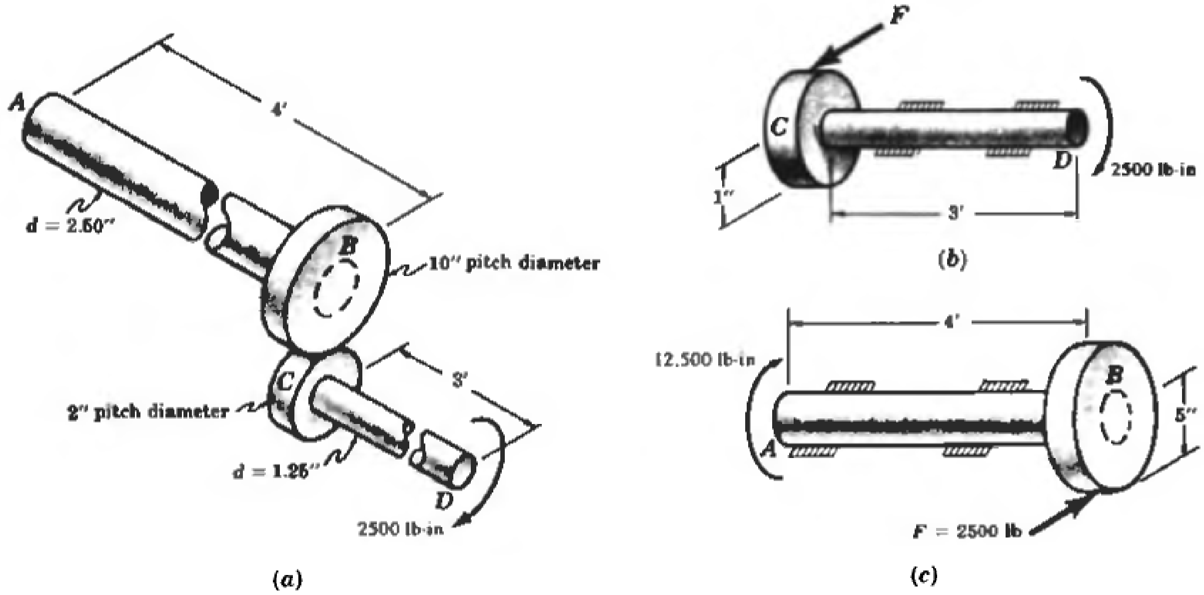


Fig. 5-12

A free-body diagram of the right shaft CD [Fig. 5-12(b)] reveals that a tangential force F must act on the smaller gear. For equilibrium, $F = 2500$ lb.

The angle of twist of the right shaft is

$$\theta_1 = \frac{TL}{GJ} = \frac{2500(36)}{5 \times 10^6 \frac{\pi}{32}(1.25)^4} = 0.0750 \text{ rad}$$

A free-body diagram of the left shaft AB is shown in Fig. 5-12(c). The force F is equal and opposite to that acting on the small gear C. This force F acts 5 in from the center line of the left shaft; hence it imparts a torque of 5(2500) = 12,500 lb · in to the shaft AB. Because of this torque there is a rotation of end B with respect to end A given by the angle θ_2, where

$$\theta_2 = \frac{12{,}500(48)}{12 \times 10^6(\pi/32)(2.5)^4} = 0.0130 \text{ rad}$$

It is to be carefully noted that this angle of rotation θ_2 induces a *rigid-body* rotation of the entire shaft CD because of the gears. In fact, the rotation of CD will be in the same ratio to that of AB as the ratio of the pitch diameters, or 5:1. Thus a rigid-body rotation of 5(0.0130) rad is imparted to shaft CD. Superposed on this rigid body movement of CD is the angular displacement of D with respect to C previously denoted by θ_1.

Hence the resultant angle of twist of D with respect to A is $\theta = 5(0.0130) + 0.075 = 0.140$ rad.

5.9. **A solid circular shaft is required to transmit 200 kW while turning at 1.5 rev/s. The allowable shearing stress is 42 MPa. Find the required shaft diameter.**

In the SI system the time rate of work (power) is expressed in N · m/s. By definition 1 N · m/s is 1 W. Power is thus given by $P = T\omega$, where T is twisting moment and ω is shaft angular velocity in radians/second. Or, alternatively, $P = 2\pi fT$, where f is revolutions per second or hertz. Thus we have

$$200{,}000\ \text{N}\cdot\text{m/s} = 2\pi(1.5\ \text{rev/s})T$$

$$T = 21{,}230\ \text{N}\cdot\text{m}$$

As in Problem 5.2, the outer fiber shearing stresses are maximum and given by

$$\tau = \frac{16T}{\pi d^3}$$

Thus,

$$42 \times 10^6\ \text{N/m}^2 = \frac{16(21{,}230\ \text{N}\cdot\text{m})}{\pi d^3}$$

Solving,

$$d = 138\ \text{mm}$$

5.10. **It is required to transmit 70 hp from a turbine by a solid circular shaft turning at 200 r/min. If the allowable shearing stress is 7000 lb/in², determine the required shaft diameter.**

In the USCS system the time rate of work (i.e., power) is expressed in lb · in/s. By definition 6600 lb · in/s is 1 hp. Power is thus given by $P = T\omega$, where T is the twisting moment and ω is shaft angular velocity in radians/second. Or, alternatively, $P = 2\pi fT$, where f is revolutions per second, usually termed hertz. Here, we have

$$70(6600\ \text{lb}\cdot\text{in/s}) = 2\pi\left(\frac{200\ \text{r}}{1\ \text{min}}\right)\left(\frac{1\ \text{min}}{60\ \text{s}}\right)T$$

from which $T = 22{,}070$ lb · in.

From Eq. (5.2), we have the peak shearing stresses at the outer fibers of the shaft as

$$\tau = \frac{T(d/2)}{J} = \frac{Td/2}{\pi d^4/32} = \frac{16T}{\pi d^3}$$

Thus

$$7000\ \text{lb/in}^2 = \frac{16{,}000\ \text{lb}\cdot\text{in}}{\pi d^3}$$

Solving, $d = 2.52$ in.

5.11. **A solid circular shaft has a uniform diameter of 2 in and is 10 ft long. At its midpoint 65 hp is delivered to the shaft by means of a belt passing over a pulley. This power is used to drive two machines, one at the left end of the shaft consuming 25 hp and one at the right end consuming the remaining 40 hp. Determine the maximum shearing stress in the shaft and also the relative angle of twist between the two extreme ends of the shaft. The shaft turns at 200 r/min and the material is steel for which $G = 12 \times 10^6$ lb/in². Assume elastic action.**

In the left half of the shaft we have 25 hp which corresponds to a torque T_1 given by

$$T_1 = \frac{63{,}000 \times \text{hp}}{n} = \frac{63{,}000(25)}{200} = 7880\ \text{lb}\cdot\text{in}$$

Similarly, in the right half we have 40 hp corresponding to a torque T_2 given by

$$T_2 = \frac{63{,}000(40)}{200} = 12{,}600\ \text{lb}\cdot\text{in}$$

The maximum shearing stress consequently occurs in the outer fibers in the right half and is given by the ordinary torsion formula:

$$\tau_\rho = \frac{T\rho}{J} \qquad \text{or} \qquad \tau = \frac{12{,}600(1)}{(\pi/32)(2)^4} = 8000 \text{ lb/in}^2$$

The angles of twist of the left and right ends relative to the center are, respectively,

$$\theta_1 = \frac{7880(60)}{12\times 10^6(\pi/32)(2)^4} = 0.0250 \text{ rad} \qquad \text{and} \qquad \theta_2 = \frac{12{,}600(60)}{12\times 10^6(\pi/32)(2)^4} = 0.0401 \text{ rad}$$

Since θ_1 and θ_2 are in the same direction, the relative angle of twist between the two ends of the shaft is $\theta = \theta_2 - \theta_1 = 0.015$ rad.

5.12. A circular cross-section bar is clamped at one end, free at the other, and loaded by a uniformly distributed twisting moment of magnitude t per unit length along its length [see Fig. 5-13(*a*)]. The torsional rigidity of the bar is GJ. Find the angle of twist of the free end of the bar.

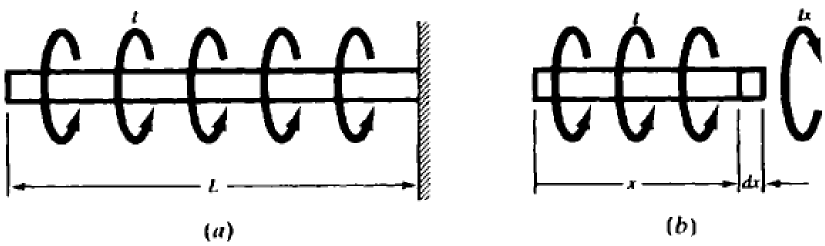


Fig. 5-13

The twisting moment per unit length is denoted by t, and the coordinate x having its origin at the left end is introduced. A free-body diagram of the portion of the bar between the left end and the section x is shown in Fig. 5-13(*b*). An element of length dx is shown in that figure and we wish to determine the angular rotation of the cylindrical element of length dx. For equilibrium of moments about the axis of the bar, a twisting moment tx must act at the right of the section shown. This twisting moment tx imparts to the element of length dx an angular rotation (from Problem 5.3)

$$d\theta = \frac{(tx)\,dx}{GJ}$$

The total rotation of the left end with respect to the right end is found by integration of all such elemental angles of twist to be

$$\theta = \int_{x=0}^{x=L} \frac{(tx)\,dx}{GJ} = \frac{tL^2}{2GJ}$$

5.13. A circular cross-section bar is clamped at one end, free at the other, and loaded by a twisting moment distributed parabolically along the length as shown in Fig. 5-14(*a*). The torsional rigidity of the bar is GJ and the moment intensity is T_0 at the clamped end. Find the angle of twist of the free end of the bar.

Let us introduce a coordinate x having origin at B and extending positive to the left. The equation of a parabola is of the general form

$$t_x = ax^2 + bx + c$$

and for the given loading we have the conditions (a) when $x = 0$, $t_x = 0$, (b) when $x = L$, $t_x = t_0$, and

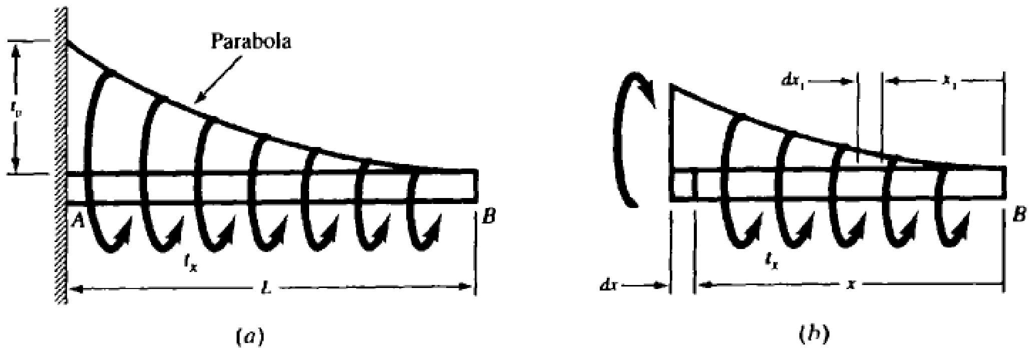


Fig. 5-14

(c) when $x = 0$, $dt_0/dx = 0$. From these conditions we find $a = t_0/L^2$, and $b = c = 0$. Thus, the loading intensity is described by the relation

$$t_x = \frac{t_0}{L^2}x^2$$

A free-body diagram of the portion of the bar between B and a section x is shown in Fig. 5-15(b). An element of length dx is also shown there and we seek to determine the angular rotation of that element. The moment acting on the element dx is found by equilibrium of twisting moments about the geometric axis of the bar to be equal to the sum of the distributed moments to the right of dx. This sum is found by introducing an auxiliary variable x_1 and we have

$$\int_{x_1=0}^{x_1=x} t_x\,dx_1 = \int_{x_1=0}^{x_1=x} \frac{t_0}{L^2}(x_1)^2\,dx_1 = \frac{t_0x^3}{3L^2}$$

From Problem 5.3, the angular rotation of the element dx is

$$d\theta = \frac{t_x\,dx}{GJ}$$

and the total angle of rotation between A and B is found by integration to be

$$\theta = \int_{x=0}^{x=L} d\theta = \int_{x=0}^{x=L} \frac{t_x\,dx}{GJ} = \int_{x=0}^{x=L} \frac{t_0x^2}{GJ}dx = \frac{t_0L^3}{3GJ}$$

5.14. An elastic bar of variable-diameter circular cross section is loaded in torsion at its ends as shown in Fig. 5-15. The variation of diameter may be known analytically, or through measurements at a number of locations along the axial direction. Write a FORTRAN program to give the angle of twist of one end of the bar with respect to the other.

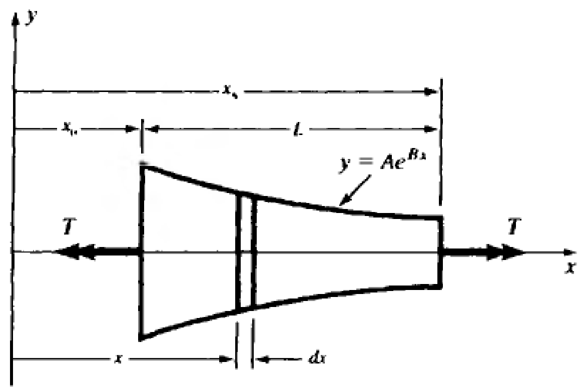


Fig. 5-15

Let us divide the bar of length L into a number of infinitesimal subsegments each of length dx, so that the cross section may be regarded as constant for each such element. Then, we may determine the angular rotation of each such element through use of the equation $\theta = TL/GJ$ from Problem 5.3. For the element of length dx, L is replaced by dx, and J is the polar moment of inertia of the cross section of the segment. This approach is represented by the following FORTRAN program which is applicable to any bar of arbitrarily varying circular cross section where the bar contour is described by the equation

$$y = Ae^{Bx}$$

```
************************************************************************
                      PROGRAM TORSN2(INPUT,OUTPUT)
************************************************************************
*
*               AUTHOR: KATHLEEN DERWIN
*               DATE  : FEBRUARY 5,1989
*
*       BRIEF DESCRIPTION:
*          THIS PROGRAM DETERMINES THE TOTAL ANGLE OF TWIST OF A CIRCULAR
*       ROD DUE TO TORSIONAL LOADING. CONSIDER THE ROD TO BE OF SOLID
*       CIRCULAR CROSS SECTION WITH A VARIABLE DIAMETER,  LOADED
*       BY A UNIFORM TORQUE.
*          THE VARYING  DIAMETER (OF THE ROD) MAY  BE DESCRIBED
*       EITHER ANALYTICALLY AS  Y = A*E   ^(B*X) ,  WHERE X IS THE
*       GEOMETRIC AXIS OF THE ROD, OR NUMERICALLY USING THE MAGNITUDE OF
*       Y AT EACH END OF  N SEGMENTS, MEANING N+1 VALUES.
*
*       INPUT:
*          THE USER IS PROMPTED FOR THE TOTAL SHAFT LENGTH, THE SHEAR
*       MODULUS, AND THE APPLIED TORQUE. THE USER IS THEN ASKED IF THE
*       ROD IS BOUNDED BY A KNOWN FUNCTION...IF THE FUNCTION IS KNOWN, THE
*       CONSTANTS AND THE ENDPOINTS OF THE ROD ON THE X-AXIS ARE INPUTTED;
*       ALTERNATELY, THE NUMBER OF SEGMENTS AND MEASURED  DIAMETERS
*       MUST BE ENTERED.
*
*       OUTPUT:
*          THE TOTAL ANGLE OF TWIST OF THE ROD IS DETERMINED AND PRINTED.
*
*       VARIABLES:
*          L,G          ---   LENGTH,SHEAR MODULUS OF ROD
*          A,B          ---   CONSTANTS OF Y=A*E  ^(B*X) GOVERNING ROD BOUNDAR
*          X0,XN        ---   ENDPOINTS OF SHAFT ON X-AXIS
*          T            ---   CENTRALLY APPLIED TORQUE
*          AA(100)      ---   INDIVIDUAL SEGMENT  DIAMETERS
*          INER         ---   POLAR MOMENT OF INERTIA OF EACH SMALL INCREMENT
*          ANS          ---   DETERMINE IF USER HAS A KNOWN FUNCTION
*         TWIST         ---   UNIFORM ANGLE OF TWIST
*          LEN          ---   LENGTH OF INCREMENTAL ELEMENT
*
****************************************************************************
****************************************************************************
*                          MAIN PROGRAM
****************************************************************************
****************************************************************************
*
*       VARIABLE DECLARATION
*
        REAL I,T,L,G,A,B,X0,XN,TWIST,AA(100),INER,LEN
        INTEGER ANS,NUM,J
*
*            USER INPUT PROMPTS
*
        PRINT*,'ENTER THE TOTAL LENGTH OF THE ROD (IN M OR INCHES):'
        READ*,L
```

```
      PRINT*,'ENTER THE SHEAR MODULUS (IN PASCALS OR PSI) :'
      READ*,G
      PRINT*,'ENTER THE UNIFORM TORQUE (IN N-M OR LB-IN) :'
      READ*,T
      PRINT*,'DO YOU KNOW THE FUNCTION DESCRIBING THE ROD?'
      PRINT*,'ENTER  1--YES  ;   2--NO'
      READ*,ANS
*
*         IF ANS EQUALS ONE, THE USER KNOWS FUNCTION. PROMPT
*         FOR CONSTANTS AND ENDPOINTS.
*
      INER  = 0
      IF (ANS.EQ.1) THEN
         PRINT*,'F(X) = A*E ^(B*X) '
         PRINT*,'ENTER A,B:'
         READ*,A,B
         PRINT*,'ENTER THE X-COORDINATE FOR BOTH ENDS OF THE ROD:'
         PRINT*,'(IN M OR INCHES):'
         READ*,X0,XN
*
      L=XN-X0
      LEN=L/50
      DO 20 I = X0,XN,LEN
         Y1=A*(2.71828**(B*I))
         Y2=A*(2.71828**( B*(I+LEN)))
         Y=(Y1+Y2)/2
         INER =(2./(3.14159*(Y**4)))+INER
   20 CONTINUE
*
*         IF ANS EQUALS TWO, THE USER DOES NOT KNOW FUNCTION.
*         PROMPT FOR NUMBER OF SEGMENTS AND MEASURED DIAMETERS.
*
      ELSE
         PRINT*,'ENTER THE NUMBER OF SECTIONS TO BE CALCULATED:'
         READ*,NUM
         PRINT*,'ENTER THE DIAMETERS OF THE ENDS FOR SECTIONS 1 TO N:'
         PRINT*,'(IN M OR INCHES):'
*
*         INPUT MEASURED  DIAMETERS
*
         DO 30 J=1,NUM+1
            READ*,AA(J)
   30    CONTINUE
*
         LEN = L/NUM
         DO 40 J = 1,NUM+1
            Y=(AA(J)+AA(J+1))/4
            INER =(2./(3.14159*(Y**4)))+INER
   40    CONTINUE
      ENDIF
*
      TWIST = (T*LEN*INER)/G
      TWIST = TWIST*180/3.14159
      PRINT 50,TWIST
*
   50 FORMAT(2X,'THE ANGLE OF TWIST IS:',F9.3,' DEGREES.')
*
      STOP
      END
```

5.15. A solid circular cross-section shaft (see Fig. 5-16) lies along the x-axis and has a contour described by the equation

$$y = 3e^{-0.05x}$$

The contour extends from $x = 0$ to $x = 25$ in. The shear modulus of the material is 12×10^6 lb/in^2 and the shaft is loaded by a twisting moment of 23,000 lb · in at each end. Use the FORTRAN program of Problem 5.14 to determine the angle of twist between the ends.

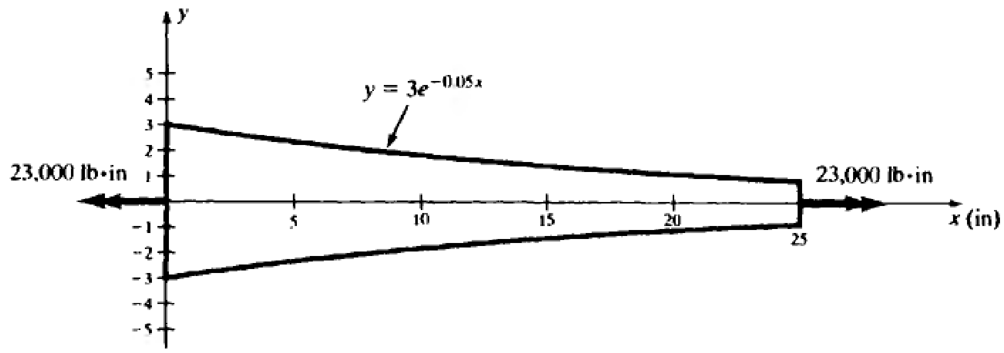


Fig. 5-16

Entering the above data into the program, we have the computer run:

```
 ENTER THE TOTAL LENGTH OF THE ROD (IN M OR INCHES):
? 25
 ENTER THE SHEAR MODULUS (IN PASCALS OR PSI) :
? 12E+6
 ENTER THE UNIFORM TORQUE (IN N-M OR LB-IN) :
? 23000
 DO YOU KNOW THE FUNCTION DESCRIBING THE ROD?
 ENTER  1--YES  ;   2--NO
? 1
 F(X) = A*E^(B*X)
 ENTER A,B:
? 3,-0.05
 ENTER THE X-COORDINATE FOR BOTH ENDS OF THE ROD:
 (IN M OR INCHES):
? 0,25
  THE ANGLE OF TWIST IS:     .703 DEGREES.
```

5.16. A circular cross-section bar is clamped at each end and loaded by the distributed twisting moments of magnitude t_1 per unit length of the bar in one direction in the left region AB and by the same intensity twisting moment but in the opposite direction in the right region BC (see Fig. 5-17). If $t_1 = 30$ N · m per meter of length, $L = 0.7$ m, and the maximum allowable shearing stress is 32 MPa, determine the required diameter of the bar.

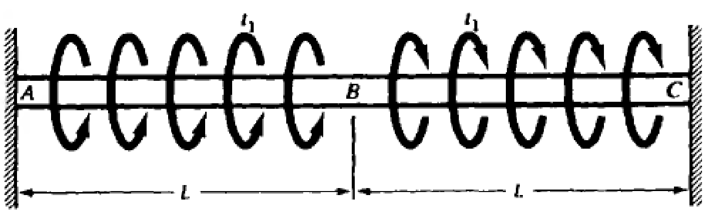


Fig. 5-17

Let us solve this problem by superposition of solutions of two subproblems. These problems are Fig. 5-18(a), labeled I, and Fig. 5-18(b), labeled II.

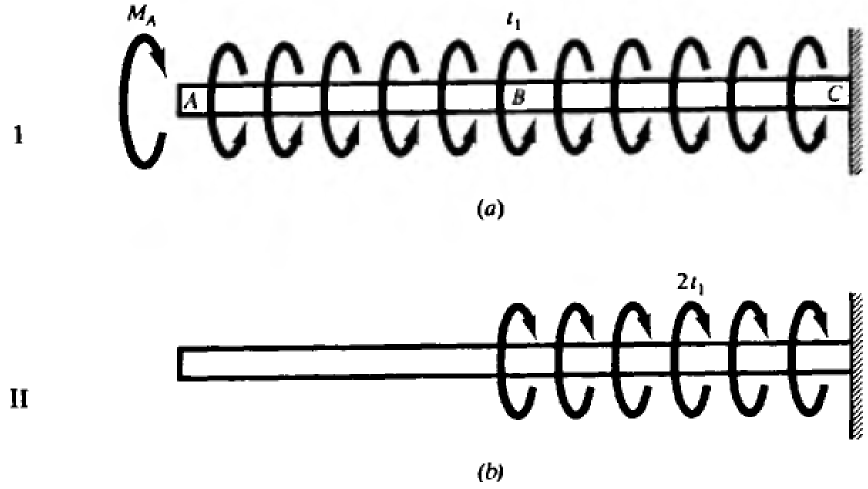


Fig. 5-18

Let us temporarily release the end A of the bar and determine the rotation of A due to an arbitrary end moment M_A plus the two distributed loadings I and II. Using the results of Problems 5.3 and 5.12, we find that the angular rotation at A is given by

$$\theta_A = \frac{t_1(2L)^2}{2GJ} - \frac{M_A(2L)}{GJ} - \frac{(2t_1)L^2}{2GJ}$$

However, since we know that end A is rigidly clamped, $\theta_A = 0$; solving we find

$$M_A = \frac{t_1 L}{2}$$

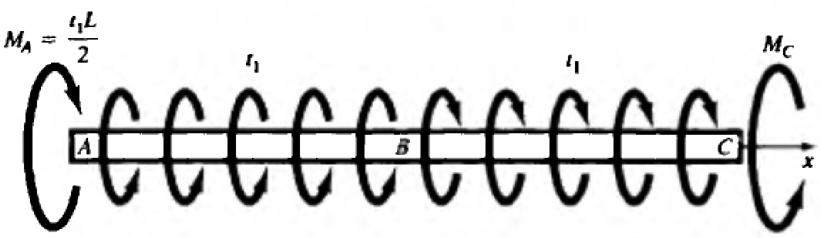


Fig. 5-19

Thus, the free-body diagram of the bar ABC appears as shown in Fig. 5-19.

From Fig. 5-19 the sum of the twisting moments about the x-axis is

$$\Sigma M_x = M_A - M_C + t_1 L - t_1 L = 0$$

which leads to

$$M_C = \frac{t_1 L}{2}$$

Thus, the variation of twisting moment along the length of the bar may be plotted as shown in Fig. 5-20.

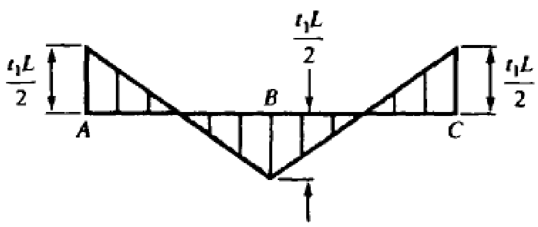


Fig. 5-20

Alternatively, using the vector representation of twisting moment, we see that the free-body diagrams of the left and right regions of ABC appear as shown in Fig. 5-21.

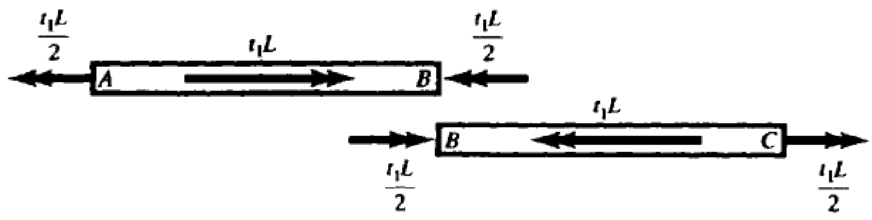


Fig. 5-21

The free-body diagram of AB indicates that there must be a twisting moment $t_1 L/2$ acting as shown at B. By Newton's law, there is an equal and opposite twisting moment acting at the left end of BC. Thus, there is a nonzero moment at the midpoint B, as indicated by Fig. 5-21. It can be shown that the angular rotation of the bar at B is zero.

From Fig. 5-21, the peak torque in the bar is $t_1 L/2$. The maximum shearing stress occurs at the outer fibers of ABC at the ends A and C as well as the midpoint B. The peak stress is, from Eq. (5.2):

$$\tau_{max} = \frac{T(d/2)}{\pi d^4/32}$$

$$32 \times 10^6\ \text{N/m}^2 = \frac{16\left[\dfrac{30\ \text{N}\cdot\text{m}}{1\ \text{m}}\cdot\dfrac{1}{2}\right](0.7\ \text{m})}{\pi d^3}$$

Solving, $d = 17.4$ mm.

5.17. A steel bar ABC, of constant circular cross section and of diameter 80 mm, is clamped at the left end A, loaded by a twisting moment of 6000 N·m at its midpoint B, and elastically restrained against twisting at the right end C (see Fig. 5-22). At end C the bar ABC is attached to vertical steel bars each of 16-mm diameter. The upper bar MN is attached to the end N of a horizontal diameter of the 80-mm bar ABC and the lower bar PQ is attached to the other end Q of this same horizontal diameter, as shown in Fig. 5-22(a). For all materials $E = 200$ GPa and $G = 80$ GPa. Determine the peak shearing stress in bar ABC as well as the tensile stress in bar MN.

Let us consider that bars MN and PQ are temporarily disconnected from the bar ABC. Then, from Problem 5.3 the angle of twist at B relative to A is

$$\theta = \frac{TL}{GJ} = \frac{(6000\ \text{N}\cdot\text{m})(0.75\ \text{m})}{(G)(\pi/32)(0.08\ \text{m})^4}$$

Since no additional twisting moments act between B and C, this same angle of twist due to the 6000-N·m loading exists at C, called θ_C.

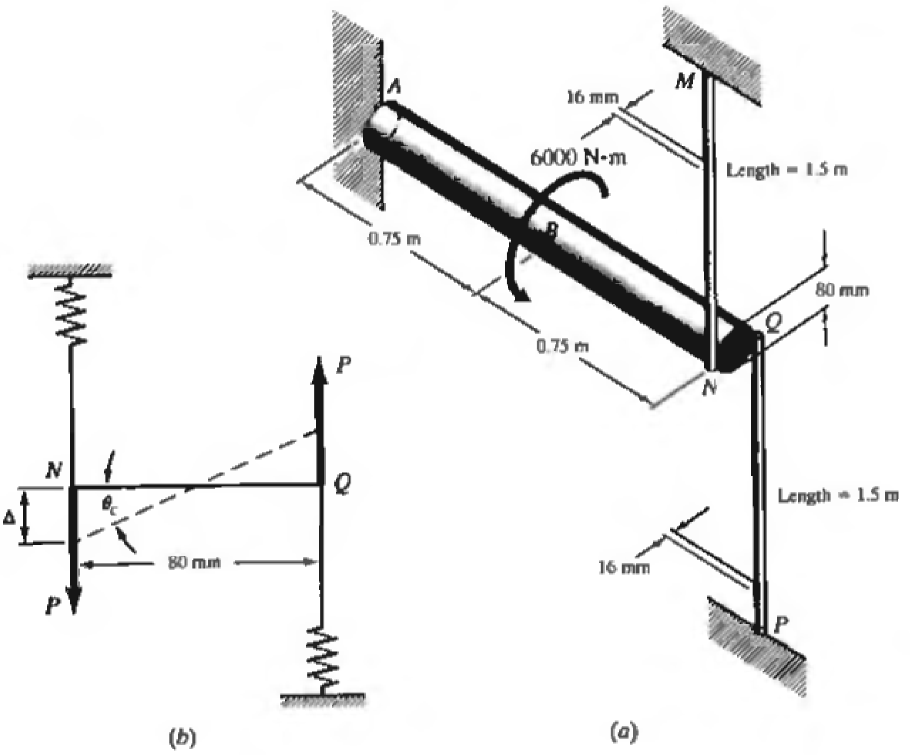


Fig. 5-22

From Fig. 5-22(*b*) the horizontal diameter *NQ* of bar *ABC* must rotate to some true, final position indicated by the dotted line. This is due to extension Δ of each of the vertical bars, which is accompanied by an axial force *P* in each bar. For a small angle of rotation *θ*, we have $\Delta = (0.040\ \text{m})\theta_C$. The axial forces *P* constitute a couple of magnitude $P(0.08\ \text{m}) = T_C$ which must act at the end *C* of bar *ABC* when the vertical bars are once again considered to be attached to the horizontal bar *ABC*. This couple must act in a sense opposite to the 6000-N · m load as shown in Fig. 5-22(*a*) since the elastic vertical bars tend to restrain angular rotation of the end *C*.

The elongation of each vertical bar may be found from Problem 1.1 to be

$$\Delta = \frac{PL}{AE} = \frac{P(1.5\ \text{m})}{(\pi/4)(0.016\ \text{m})^2 E} = \frac{(T_C/0.08)(1.5\ \text{m})}{(\pi/4)(0.016\ \text{m})^2 E}$$

The angular rotation of end *C* of bar *ABC* may now be determined by (*a*) considering the effect of the twisting moments of 6000 N · m and the end load T_C, and by (*b*) considering the angular rotation caused by the axial force *P* in the vertical bars. Thus, for the same rotation of end *C* we have

$$\frac{(6000\ \text{N}\cdot\text{m})(0.75\ \text{m})}{(G)(\pi/32)(0.08\ \text{m})^4} - \frac{T_C(1.5\ \text{m})}{(G)(\pi/32)(0.08)^4} = \frac{(T_C/0.8)(1.5\ \text{m})}{(\pi/4)(0.016\ \text{m})^2(0.04\ \text{m})(E)}$$

Solving, $T_C = 1327\ \text{N}\cdot\text{m}$ and $P = T_C/0.08 = 16{,}587\ \text{N}$. The variation of twisting moment along *ABC*

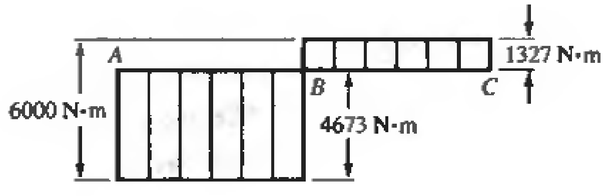


Fig. 5-23

appears as in Fig. 5-23 so that the peak torsional shearing stress occurs at the outer fibers at all points between A and B and is from Problem 5.2

$$\tau_{max} = \frac{16(4673\ \text{N}\cdot\text{m})}{\pi(0.08\ \text{m})^3} = 46.5\ \text{MPa}$$

The axial stress in each of the vertical bars is

$$\sigma = \frac{P}{A} = \frac{16{,}587\ \text{N}}{\pi(0.008\ \text{m})^2} = 82.5\ \text{MPa}$$

5.18. Determine the reactive torques at the fixed ends of the circular shaft loaded by the couples shown in Fig. 5-24(a). The cross section of the bar is constant along the length. Assume elastic action.

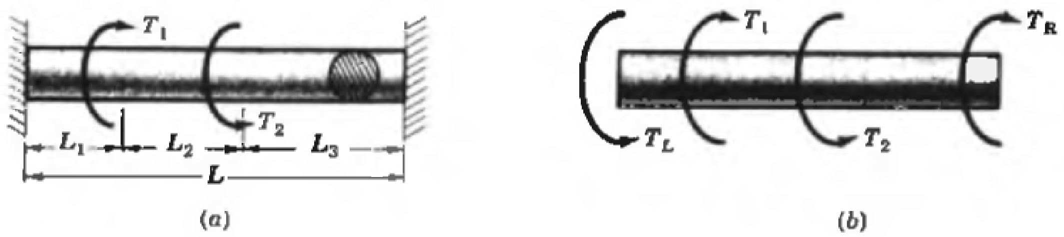


Fig. 5-24

Let us assume that the reactive torques T_L and T_R are positive in the directions shown in Fig. 5-24(b). From statics we have

$$T_L - T_1 + T_2 - T_R = 0 \qquad (1)$$

This is the only equation of static equilibrium and it contains two unknowns. Hence this problem is statically indeterminate and it is necessary to augment this equation with another equation based on the deformations of the system.

The variation of torque with length along the bar may be represented by the plot shown in Fig. 5-25.

The free-body diagram of the left region of length L_1 appears as in Fig. 5-26(a).

Working from left to right along the shaft, the twisting moment in the central region of length L_2 is given by the algebraic sum of the torques to the left of this section, i.e., $T_1 - T_L$. The free-body diagram of this region appears as in Fig. 5-26(b).

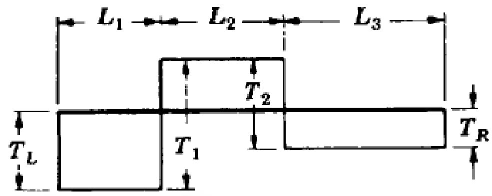


Fig. 5-25

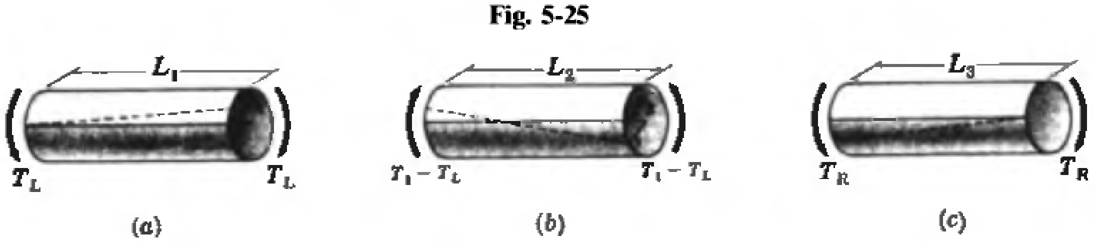


Fig. 5-26

Finally, the free-body diagram of the right region of length L_3 appears as in Fig. 5-26(c).

Let θ_1 denote the angle of twist at the point of application of T_1, and θ_2 the angle at T_2. Then from a consideration of the regions of lengths L_1 and L_3 we immediately have

$$\theta_1 = \frac{T_L L_1}{GJ} \tag{2}$$

$$\theta_2 = \frac{T_R L_3}{GJ} \tag{3}$$

The original position of a generator on the surface of the shaft is shown by a solid line in Fig. 5-26, and the deformed position by a dashed line. Consideration of the central region of length L_2 reveals that the angle of twist of its right end with respect to its left end is $\theta_1 + \theta_2$. Hence, since the torque causing this deformation is $T_1 - T_L$, we have

$$\theta_1 + \theta_2 = \frac{(T_1 - T_L)L_2}{GJ} \tag{4}$$

Solving (*1*) through (*4*) simultaneously, we find

$$T_L = T_1\frac{L_2 + L_3}{L} - T_2\frac{L_3}{L} \qquad \text{and} \qquad T_R = -T_1\frac{L_1}{L} + T_2\frac{L_1 + L_2}{L}$$

It is of interest to examine the behavior of a generator on the surface of the shaft. Originally it was, of course, straight over the entire length L, but after application of T_1 and T_2 it has the appearance shown by the broken line in Fig. 5-27.

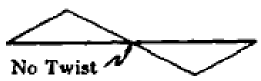


Fig. 5-27

5.19. Consider a composite shaft fabricated from a 2-in-diameter solid aluminum alloy, $G = 4 \times 10^6$ lb/in^2, surrounded by a hollow steel circular shaft of outside diameter 2.5 in and inside diameter 2 in, $G = 12 \times 10^6$ lb/in^2. The two metals are rigidly connected at their juncture. If the composite shaft is loaded by a twisting moment of 14,000 lb · in, calculate the shearing stress at the outer fibers of the steel and also at the extreme fibers of the aluminum. The action is elastic.

Let T_1 = torque carried by the aluminum shaft and T_2 = torque carried by the steel. For static equilibrium of moments about the geometric axis we have

$$T_1 + T_2 = T = 14{,}000$$

where T = external applied twisting moment. This is the only equation from statics available in this problem. Since it contains two unknowns, T_1 and T_2, it is necessary to supplement it with an additional equation coming from the deformations of the shaft. The structure is thus statically indeterminate.

Such an equation is easily found, since the two materials are rigidly joined; hence their angles of twist must be equal. In a length L of the shaft we have, using the formula $\theta = TL/GJ$,

$$\frac{T_1 L}{4 \times 10^6(\pi/32)(2)^4} = \frac{T_2 L}{12 \times 10^6(\pi/32)[(2.5)^4 - (2)^4]} \qquad \text{or} \qquad T_1 = 0.231T_2$$

This equation, together with the statics equation, may be solved simultaneously to yield

$T_1 = 2600$ lb · in (carried by aluminum) and $T_2 = 11{,}400$ lb · in (carried by steel)

The shearing stresses at the extreme fibers of the steel and of the aluminum are, respectively,

$$\tau_2 = \frac{11{,}400(1.25)}{(\pi/32)[(2.5)^4 - (2)^4]} = 6300 \text{ lb/in}^2 \qquad \text{and} \qquad \tau_1 = \frac{2600(1)}{(\pi/32)(2)^4} = 1650 \text{ lb/in}^2$$

5.20. A stepped shaft has the appearance shown in Fig. 5-28. The region AB is Al 2014-T6 alloy, having $G = 28\ \text{GPa}$, and the region BC is steel, having $G = 84\ \text{GPa}$. The aluminum portion is of solid circular cross section 45 mm in diameter, and the steel region is circular of 60-mm outside diameter and 30-mm inside diameter. Determine the peak shearing stress in each material as well as the angle of twist at B where a torsional load of $4000\ \text{N}\cdot\text{m}$ is applied. Ends A and C are rigidly clamped.

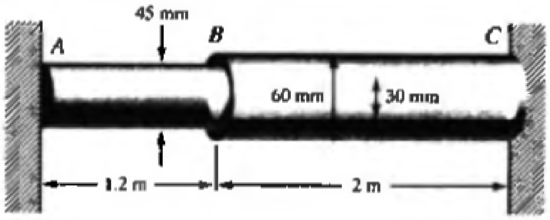


Fig. 5-28

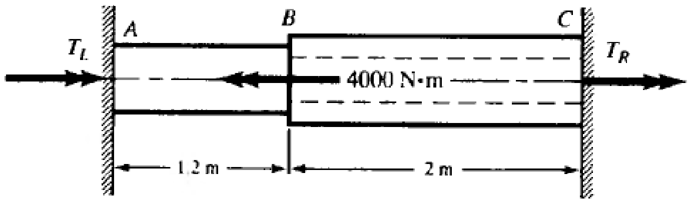


Fig. 5-29

The free-body diagram of the system is shown in Fig. 5-29.

The applied load of $4000\ \text{N}\cdot\text{m}$ as well as the unknown end reactive torques are indicated by the double-headed vectors above. There is only one equation of static equilibrium:

$$\Sigma M_x = T_L + T_R - 4000\ \text{N}\cdot\text{m} = 0$$

Since there are two unknowns T_L and T_R, another equation (based upon deformations) is required. This is set up by realizing that the angular rotation at B is the same if we determine it at the right end of AB or the left end of BC. Using Eq. (5.4), we thus have

$$\frac{T_L(1.2\ \text{m})}{(28\times10^9\ \text{N/m}^2)J_{\text{Al}}} = \frac{T_R(2.0\ \text{m})}{(84\times10^9\ \text{N/m}^2)J_{\text{ST}}} \tag{1}$$

The polar moment of inertia in AB is

$$J_{\text{Al}} = \frac{\pi(0.045\ \text{m})^4}{32} = 0.40\times10^{-6}\ \text{m}^4$$

and in BC it is

$$J_{\text{ST}} = \frac{\pi}{32}[(0.060\ \text{m})^4 - (0.030\ \text{m})^4] = 1.19\times10^{-6}\ \text{m}^4$$

Thus, from the above Eq. (1), we have

$$T_L = 0.187T_R \tag{2}$$

Substituting this relation in Eq. (1), we find

$$T_L = 630\ \text{N}\cdot\text{m} \qquad \text{and} \qquad T_R = 3370\ \text{N}\cdot\text{m}$$

The outer fiber shearing stresses in AB are given by

$$\tau_{AB} = \frac{T\rho}{J} = \frac{(630\ \text{N}\cdot\text{m})(0.0225\ \text{m})}{0.40\times 10^{-6}\ \text{m}^4} = 35.2\ \text{MPa}$$

and in BC by

$$\tau_{BC} = \frac{T\rho}{J} = \frac{(3370\ \text{N}\cdot\text{m})(0.030\ \text{m})}{1.19\times 10^{-6}\ \text{m}^4} = 85.0\ \text{MPa}$$

The angle of twist at B, using parameters of the region AB, is

$$\theta_B = \frac{TL}{GJ} = \frac{(630\ \text{N}\cdot\text{m})(1.2\ \text{m})}{(28\times 10^6\ \text{N/m}^2)(0.40\times 10^{-6}\ \text{m}^4)} = 0.675\times 10^{-3}\ \text{rad} \qquad \text{or} \qquad 0.039^\circ$$

5.21. Consider a bar of solid circular cross section subject to torsion. The material is considered to be elastic-perfectly plastic, i.e., the shear stress-strain diagram has the appearance indicated in Fig. 5-30(*a*). Determine the distance from the center at which plastic flow begins in terms of the twisting moment. Also determine the twisting moment for fully plastic action of the cross section.

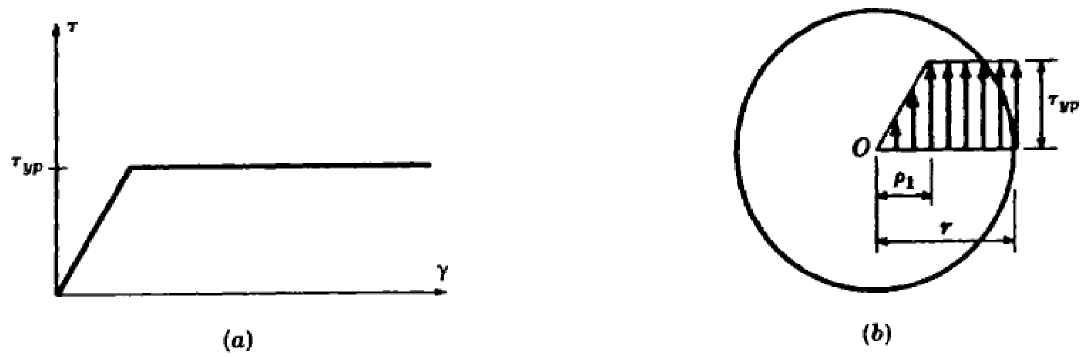


Fig. 5-30

Even though torsion of the bar has caused the outer portion to have yielded it is still realistic to assume that plane sections of the bar normal to its axis prior to loading remain plane after the torques have been applied, and further that a diameter in the section before deformation remains a diameter, or straight line, after deformation. Consequently the shearing strains of the longitudinal fibers vary linearly as the distances from the center of the bar.

Let us assume that plastic action begins at a distance ρ_1 from the center of the bar, so that the stress distribution appears as in Fig. 5-30(*b*). Thus, the shearing stresses vary linearly as the distance of the fiber from the center up to the point ρ_1 after which they are constant and equal to the yield point in shear.

From Fig. 5-30(*b*) we have for $\rho < \rho_1$:

$$\frac{\tau}{\rho} = \frac{\tau_{yp}}{\rho_1} \qquad \text{or} \qquad \tau = \left(\frac{\rho}{\rho_1}\right)\tau_{yp}$$

and for $\rho > \rho_1$: $\tau = \tau_{yp}$ = constant. Thus the twisting moment is

$$T = \int_0^r \tau\rho\, da \tag{1}$$

where da refers to the ring-shaped element shown in Fig. 5-8 of Problem 5.2. Using the above values of shearing stress in the inner elastic region and outer plastic region, we have

$$T = \int_0^{\rho_1} \left(\frac{\rho}{\rho_1}\right) \tau_{yp} \rho \, da + \int_{\rho_1}^{r} \tau_{yp} \rho \, da = \frac{\tau_{yp}}{\rho_1} \int_0^{\rho_1} \rho^2 \, da + \tau_{yp} \int_{\rho_1}^{r} \rho \, da$$

$$= \frac{\tau_{yp}}{\rho_1} \int_0^{\rho_1} \rho^2 2\pi\rho \, d\rho + \tau_{yp} \int_{\rho_1}^{r} \rho 2\pi\rho \, d\rho = \tau_{yp}\left(\frac{\pi}{2} - \frac{2\pi}{3}\right)\rho_1^3 + \frac{2\pi}{3}\tau_{yp} r^3$$

Solving for ρ_1,

$$\rho_1 = \left[4r^3 - \frac{6T}{\pi\tau_{yp}}\right]^{1/3} \tag{2}$$

as the distance from the center at which plastic flow begins. For fully plastic action, that is, $\tau = \tau_{yp}$ at all points of the cross section, we set $\rho_1 = 0$ to obtain the fully plastic twisting moment T_p:

$$T_p = \frac{2}{3}\pi r^3 \tau_{yp} = \frac{4}{3}\frac{J}{r}\tau_{yp} \tag{3}$$

But from Problem 5.2 if only the outer fibers of the bar are stressed to the yield point of the material and all interior fibers are in the elastic range of action we have the maximum possible elastic twisting moment T_e:

$$T_e = \frac{\tau_{yp}}{2}\pi r^3 \tag{4}$$

Comparison of (*3*) and (*4*) indicates that $T_p = 4T_e/3$, that is, fully plastic action permits application of a twisting moment $33\frac{1}{3}$ percent greater than the twisting moment that just causes plastic action to begin in the outer fibers.

5.22. Consider a circular shaft having a concentrically bored hole. Determine the twisting moment that it can carry for fully plastic action.

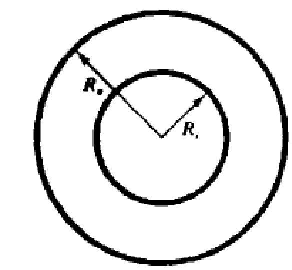


Fig. 5-31

As shown in Fig. 5-31, we denote the outer radius of the shaft by R_o and the inner radius by R_i. The yield point of the material in torsion is denoted by τ_{yp}. We return to Eq. (*1*) of Problem 5.21 and merely change the limits of integration. That is,

$$T = \int_{R_i}^{R_o} \tau_{yp} \rho \, da = \tau_{yp} \int_{R_i}^{R_o} \rho(2\pi\rho \, d\rho)$$

$$= \frac{2\pi}{3}\tau_{yp}[R_o^3 - R_i^3]$$

Note that if we express the fully plastic moment in Eq. (*3*) of Problem 5.21 in terms of J for the solid shaft it is not possible to obtain the correct fully plastic torsional loading for a hollow shaft merely by utilizing $(J_o - J_i)$ where these Js correspond to the outside and inside boundaries of the hollow shaft, respectively. It is necessary to determine the fully plastic load by returning to fundamentals and integrating as shown above.

Supplementary Problems

5.23. If a solid circular shaft of 1.25-in diameter is subject to a torque T of 2500 lb · in causing an angle of twist of 3.12° in a 5-ft length, determine the shear modulus of the material. *Ans.* $G = 11.5 \times 10^6$ lb/in^2

5.24. Determine the maximum shearing stress in a 4-in-diameter solid shaft carrying a torque of 228,000 lb · in. What is the angle of twist per unit length if the material is steel for which $G = 12 \times 10^6$ lb/in^2?
Ans. 18,100 lb/in^2, 0.000755 rad/in

5.25. A propeller shaft in a ship is 350 mm in diameter. The allowable working stress in shear is 50 MPa and the allowable angle of twist is 1° in 15 diameters of length. If $G = 85$ GPa, determine the maximum torque the shaft can transmit. *Ans.* 416 kN · m

5.26. Consider the same shaft described in Problem 5.25 but with a 175-mm axial hole bored throughout its length. The conditions on working stress and angle of twist remain as before. By what percentage is the torsional load-carrying capacity reduced? By what percentage is the weight of the shaft reduced?
Ans. 6 percent, 25 percent

5.27. A compound shaft is composed of a 24-in length of solid copper 4 in in diameter, joined to a 32-in length of solid steel 4.5 in in diameter. A torque of 120,000 lb · in is applied to each end of the shaft. Find the maximum shear stress in each material and the total angle of twist of the entire shaft. For copper $G = 6 \times 10^6$ lb/in^2, for steel $G = 12 \times 10^6$ lb/in^2.
Ans. in the copper, 9520 lb/in^2; in the steel, 6700 lb/in^2; $\theta = 0.027$ rad

5.28. In Fig. 5-32 the vertical shaft and pulley keyed to it may be considered to be weightless. The shaft rotates with a uniform angular velocity. The known belt pulls are indicated and the three pulleys are rigidly keyed to the shaft. If the working stress in shear is 50 MPa, determine the necessary diameter of a solid circular shaft. Neglect bending of the shaft because of the proximity of the bearings to the pulleys. *Ans.* 29 mm

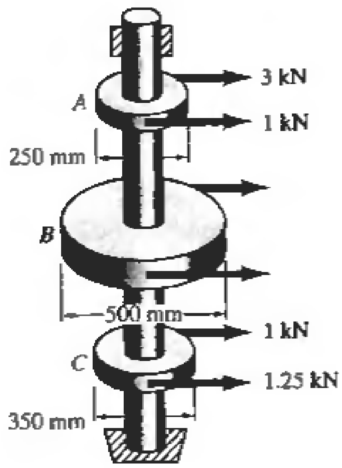


Fig. 5-32

5.29. Determine the reactive torques at the fixed ends of the circular shaft loaded by the three couples shown in Fig. 5-33. The cross section of the bar is constant along the length.
Ans. $T_L = 3600$ lb · in, $T_R = 13,600$ lb · in

5.30. A hollow steel shaft has an outside diameter of 4 in and an inside diameter of 3 in. Determine the maximum torque the shaft can transmit in fully plastic action if the yield point of the material in shear is 22,000 lb/in^2. *Ans.* 214,000 lb · in

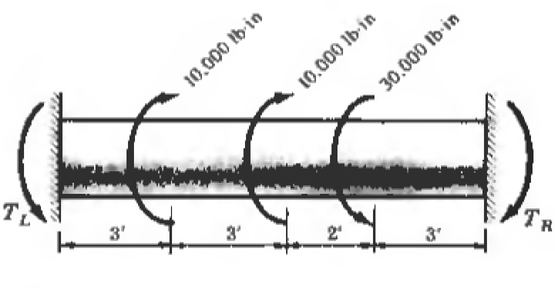


Fig. 5-33

5.31. A bar of circular cross section is clamped at its left end, free at the right, and loaded by a twisting moment t per unit length that is uniformly distributed along the middle third of the bar as shown in Fig. 5-34. Find the angle of twist of the free end of the bar.

Ans. $\dfrac{2}{9}\dfrac{tL^2}{GJ}$

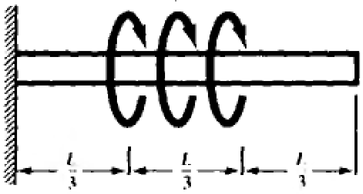


Fig. 5-34

5.32. It is desired to transmit 90 kW by means of a solid circular shaft rotating at 3.5 r/s. The allowable shearing stress is 45 MPa. Find the required shaft diameter. *Ans.* 77.4 mm

5.33. A hollow circular shaft whose outside diameter is three times its inner diameter transmits 110 hp at 120 r/min. If the maximum allowable shearing stress is 6500 lb/in^2, find the required outside diameter of the shaft. *Ans.* 3.58 in

5.34. A solid circular cross-section shaft lies along the x-axis and has a contour described by the equation

$$y = 0.074e^{0.045x}$$

The shaft extends from $x = 0$ to $x = 3$ m. The shear modulus of the material is 83 GPa and the shaft is loaded by a twisting moment of 42,100 N · m at each end. Use the FORTRAN program of Problem 5.14 to determine the angle of twist between the ends of the bar. *Ans.* 2.518°

5.35. A solid circular cross-section shaft lies along the x-axis and has a contour described by the equation

$$y = 8e^{0.01x}$$

The shaft extends from $x = 0$ to $x = 180$ in. The shear modulus of the material is 12×10^6 lb/in^2, and the shaft is loaded by a twisting moment of 65,000 lb · in. Use the FORTRAN program of Problem 5.14 to determine the angle of twist between the ends of the bar. *Ans.* 1.861°

5.36. A solid circular cross-section shaft is clamped at both ends and loaded by a twisting moment t per unit length as shown in Fig. 5-35. Determine the reactive twisting moments at each end of the bar.
Ans. $M_A = \frac{2}{9}tL$, $M_C = \frac{4}{9}tL$

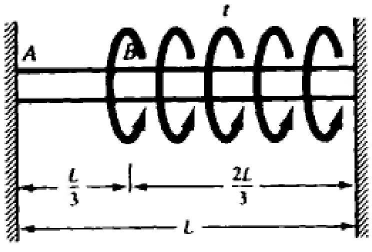


Fig. 5-35

5.37. A solid steel shaft of circular cross section has a length of 300 mm and is tapered from 50-mm diameter at the small end to 100-mm diameter at the large end, as shown in Fig. 5-36. The shaft is subject to a twisting moment of 1000 N · m applied at each end. For $G = 80$ GPa, determine the angle of twist between the ends and the peak shearing stress. *Ans.* 0.48°, 40.7 MPa

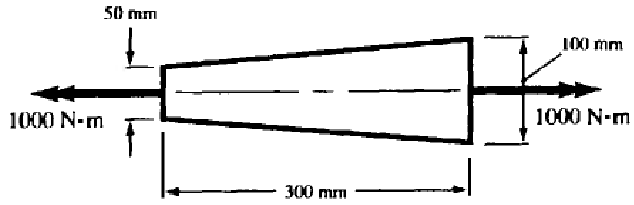


Fig. 5-36

5.38. A circular cross-section steel shaft is of diameter 50 mm over the left 150 mm of length and of diameter 100 mm over the right 150 mm, as shown in Fig. 5-37. Each end of the shaft is loaded by a twisting moment of 1000 N · m. If $G = 80$ GPa, determine the angle of twist between the ends of the shaft as well as the peak shearing stress. *Ans.* 1.09°, 40.7 MPa

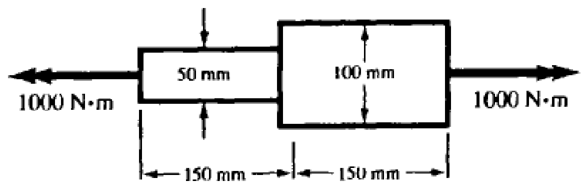


Fig. 5-37

Chapter 6

Shearing Force and Bending Moment

DEFINITION OF A BEAM

A bar subject to forces or couples that lie in a plane containing the longitudinal axis of the bar is called a *beam*. The forces are understood to act perpendicular to the longitudinal axis.

CANTILEVER BEAMS

If a beam is supported at only one end and in such a manner that the axis of the beam cannot rotate at that point, it is called a *cantilever beam*. This type of beam is illustrated in Fig. 6-1. The left end of the bar is free to deflect but the right end is rigidly clamped. The right end is usually said to be "restrained." The reaction of the supporting wall at the right upon the beam consists of a vertical force together with a couple acting in the plane of the applied loads shown.

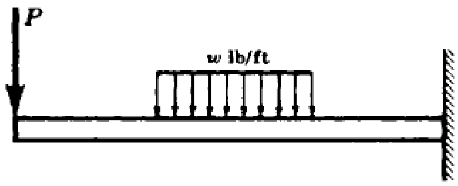


Fig. 6-1

SIMPLE BEAMS

A beam that is freely supported at both ends is called a *simple beam*. The term "freely supported" implies that the end supports are capable of exerting only forces upon the bar and are not capable of exerting any moments. Thus there is no restraint offered to the angular rotation of the ends of the bar at the supports as the bar deflects under the loads. Two simple beams are sketched in Fig. 6-2.

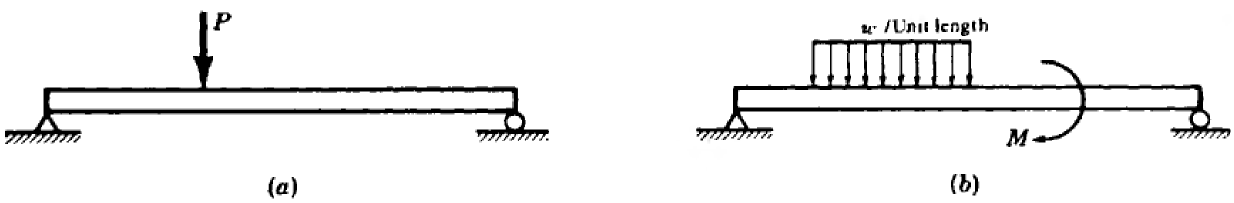


Fig. 6-2

It is to be observed that at least one of the supports must be capable of undergoing horizontal movement so that no force will exist in the direction of the axis of the beam. If neither end were free to move horizontally, then some axial force would arise in the beam as it deforms under load. Problems of this nature are not considered in this book.

The beam of Fig. 6-2(*a*) is said to be subject to a concentrated force; that of Fig. 6-2(*b*) is loaded by a uniformly distributed load as well as a couple.

OVERHANGING BEAMS

A beam freely supported at two points and having one or both ends extending beyond these supports is termed an *overhanging beam*. Two examples are given in Fig. 6-3.

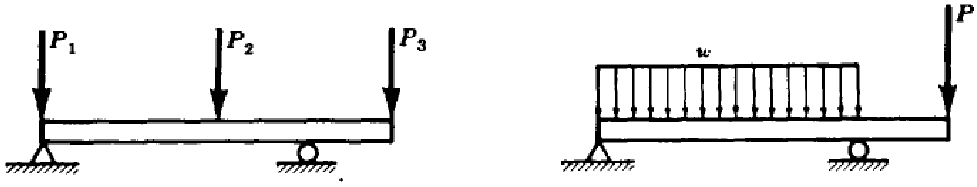


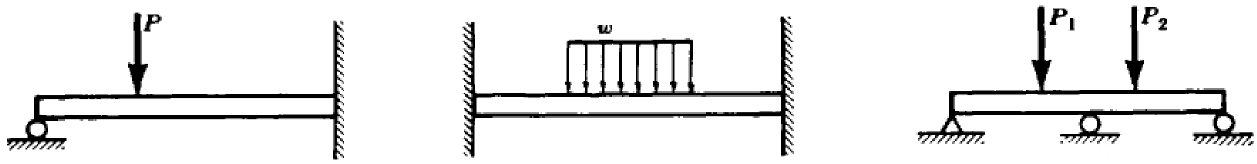
Fig. 6-3

STATICALLY DETERMINATE BEAMS

All the beams considered above, the cantilevers, simple beams, and overhanging beams, are ones in which the reactions of the supports may be determined by use of the equations of static equilibrium. The values of these reactions are independent of the deformations of the beam. Such beams are said to be *statically determinate*.

STATICALLY INDETERMINATE BEAMS

If the number of reactions exerted upon the beam exceeds the number of equations of static equilibrium, then the statics equations must be supplemented by equations based upon the deformations of the beam. In this case the beam is said to be *statically indeterminate*. Examples are shown in Fig. 6-4.



(*a*) (*b*) (*c*)

Fig. 6-4

TYPES OF LOADING

Loads commonly applied to a beam may consist of concentrated forces (applied at a point), uniformly distributed loads, in which case the magnitude is expressed as a certain number of pounds per foot or Newtons per meter of length of the beam, or uniformly varying loads. This last type of load is exemplified in Fig. 6-5.

A beam may also be loaded by an applied couple. The magnitude of the couple is usually expressed in lb · ft or N · m.

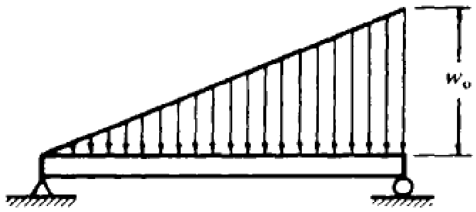


Fig. 6-5

INTERNAL FORCES AND MOMENTS IN BEAMS

When a beam is loaded by forces and couples, internal stresses arise in the bar. In general, both normal and shearing stresses will occur. In order to determine the magnitude of these stresses at any section of the beam, it is necessary to know the resultant force and moment acting at that section. These may be found by applying the equations of static equilibrium.

Example 1

Suppose several concentrated forces act on a simple beam as in Fig. 6-6(*a*).

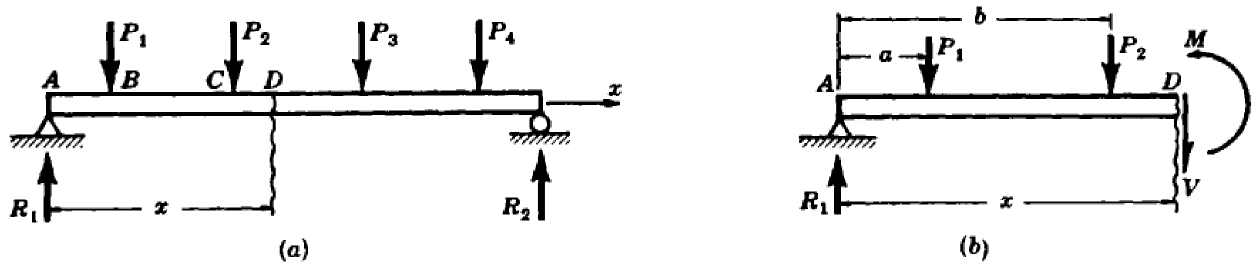


Fig. 6-6

It is desired to study the internal stresses across the section at D, located a distance x from the left end of the beam. To do this let us consider the beam to be cut at D and the portion of the beam to the right of D removed. The portion removed must then be replaced by the effect it exerted upon the portion to the left of D and this effect will consist of a vertical shearing force together with a couple, as represented by the vectors V and M, respectively, in the free-body diagram of the left portion of the beam shown in Fig. 6-6(*b*).

The force V and the couple M hold the left portion of the bar in equilibrium under the action of the forces R_1, P_1, P_2. The quantities V and M are taken to be positive if they have the senses indicated above.

RESISTING MOMENT

The couple M shown in Fig. 6-6(*b*) is called the *resisting moment* at section D. The magnitude of M may be found by use of a statics equation which states that the sum of the moments of all forces about an axis through D and perpendicular to the plane of the page is zero. Thus

$$\Sigma M_0 = M - R_1 x + P_1(x - a) + P_2(x - b) = 0 \qquad \text{or} \qquad M = R_1 x - P_1(x - a) - P_2(x - b)$$

Thus the resisting moment M is the moment at point D created by the moments of the reaction at A and the applied forces P_1 and P_2. The resisting moment M is the resultant couple due to stresses that are distributed over the vertical section at D. These stresses act in a horizontal direction and are tensile in certain portions of the cross section and compressive in others. Their nature will be discussed in detail in Chap. 8.

RESISTING SHEAR

The vertical force V shown in Fig. 6-6(b) is called the *resisting shear* at section D. For equilibrium of forces in the vertical direction,

$$\Sigma F_v = R_1 - P_1 - P_2 - V = 0 \qquad \text{or} \qquad V = R_1 - P_1 - P_2$$

This force V is actually the resultant of shearing stresses distributed over the vertical section at D. The nature of these stresses will be studied in Chap. 8.

BENDING MOMENT

The algebraic sum of the moments of the external forces to one side of the section D about an axis through D is called the *bending moment* at D. This is represented by

$$R_1 x - P_1(x - a) - P_2(x - b)$$

for the loading considered above. The quantity is considered in Problems 6.1 through 6.12. Thus the bending moment is opposite in direction to the resisting moment but is of the same magnitude. It is usually denoted by M also. Ordinarily the bending moment rather than the resisting moment is used in calculations because it can be represented directly in terms of the external loads.

SHEARING FORCE

The algebraic sum of all the vertical forces to one side, say the left side, of section D is called the *shearing force* at that section. This is represented by $R_1 - P_1 - P_2$ for the above loading. The shearing force is opposite in direction to the resisting shear but of the same magnitude. Usually it is denoted by V. It is ordinarily used in calculations, rather than the resisting shear. This quantity is considered in Problems 6.1 through 6.12.

SIGN CONVENTIONS

The customary sign conventions for shearing force and bending moment are represented in Fig. 6-7. Thus a force that tends to bend the beam so that it is concave upward is said to produce a positive bending moment. A force that tends to shear the left portion of the beam upward with respect to the right portion is said to produce a positive shearing force.

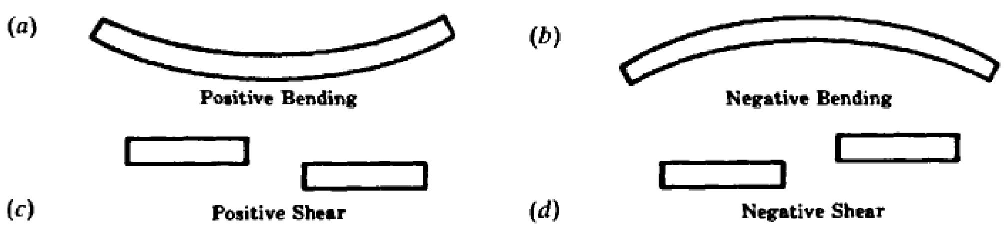


Fig. 6-7

An easier method for determining the algebraic sign of the bending moment at any section is to say that upward external forces produce positive bending moments, downward forces yield negative bending moments.

SHEAR AND MOMENT EQUATIONS

Usually it is convenient to introduce a coordinate system along the beam, with the origin at one end of the beam. It will be desirable to know the shearing force and bending moment at all sections along the beam and for this purpose two equations are written, one specifying the shearing force V as a function of the distance, say x, from one end of the beam, the other giving the bending moment M as a function of x.

SHEARING FORCE AND BENDING MOMENT DIAGRAMS

The plots of these equations for V and M are known as *shearing force* and *bending moment diagrams*, respectively. In these plots the abscissas (horizontals) indicate the position of the section along the beam and the ordinates (verticals) represent the values of the shearing force and bending moment, respectively. Thus these diagrams represent graphically the variation of shearing force and bending moment at any section along the length of the bar. From these plots it is quite easy to determine the maximum value of each of these quantities.

RELATIONS BETWEEN LOAD INTENSITY, SHEARING FORCE, AND BENDING MOMENT

A simple beam with a varying load indicated by $w(x)$ is sketched in Fig. 6-8. The coordinate system with origin at the left end A is established and distances to various sections in the beam are denoted by the variable x.

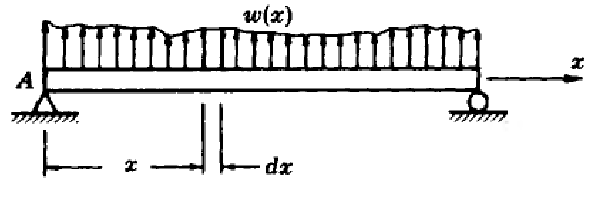


Fig. 6-8

For any value of x the relationship between the load $w(x)$ and the shearing force V is

$$w = \frac{dV}{dx}$$

and the relationship between shearing force and bending moment M is

$$V = \frac{dM}{dx}$$

These relations are derived in Problem 6.1. For applications see Problems 6.3 through 6.7.

SINGULARITY FUNCTIONS

For ease in treating problems involving concentrated forces and concentrated moments we introduce the function

$$f_n(x) = \langle x - a\rangle^n$$

where for $n > 0$ the quantity in pointed brackets is zero if $x < a$ and is the usual $(x - a)^n$ if $x > a$. This is the *singularity* or *half-range* function. Thus, if the argument is positive the pointed brackets behave just as ordinary parentheses. For applications see Problems 6.8 through 6.13.

COMPUTER IMPLEMENTATION

Determination of shearing forces and bending moments in a beam subject to a number of concentrated forces, moments, and distributed loadings is best carried out on a computer. A simple program suitable for PC implementation is given in Problem 6.13 and applications are given in Problems 6.14 and 6.15.

Solved Problems

6.1. Derive relationships between load intensity, shearing force and bending moment at any point in a beam.

Let us consider a beam subject to any type of transverse load of the general form shown in Fig. 6-9(a). Simple supports are illustrated but the following consideration holds for all types of beams. We will isolate from the beam the element of length dx shown and draw a free-body diagram of it. The shearing force V

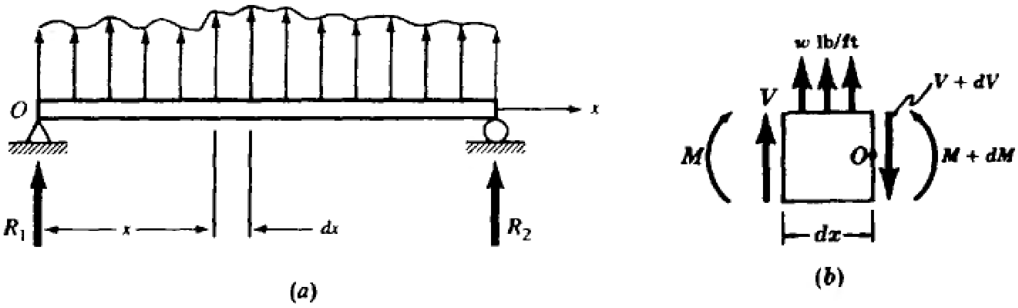


Fig. 6-9

acts on the left side of the element, and in passing through the distance dx the shearing force will in general change slightly to an amount $V + dV$. The bending moment M acts on the left side of the element and it changes to $M + dM$ on the right side. Since dx is extremely small, the applied load may be taken as uniform over the top of the beam and equal to w lb/ft. The free-body diagram of this element thus appears as in Fig. 6-9(b). For equilibrium of moments about O, we have

$$\Sigma M_0 = M - (M + dM) + V\,dx + w\,dx(dx/2) = 0 \qquad \text{or} \qquad dM = V\,dx + \tfrac{1}{2}w(dx)^2$$

Since the last term consists of the product of two differentials, it is negligible compared with the other forms involving only one differential. Hence

$$dM = V\,dx \qquad \text{or} \qquad V = \frac{dM}{dx}$$

Thus the shearing force is equal to the rate of change of the bending moment with respect to x.

This equation will prove to be of considerable value in drawing shearing force and bending moment diagrams for the more complicated types of loading. For example, from this equation it is evident that if the shearing force is positive at a certain section of the beam then the slope of the bending moment diagram is also positive at that point. Also, it demonstrates that an abrupt change in shear, corresponding to a concentrated force, is accompanied by an abrupt change in the slope of the bending moment diagram.

Further, at those points where the shear is zero, the slope of the bending moment diagram is zero. At these points where the tangent to the moment diagram is horizontal, the moment may have a maximum or minimum value. This follows from the usual calculus technique of obtaining maximum or minimum values of a function by equating the first derivative of the function to zero. Thus in Fig. 6-10 if the curves shown represent portions of a bending moment diagram then critical values may occur at points A and B.

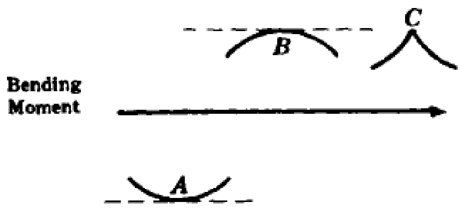


Fig. 6-10

To establish the direction of concavity at a point such as A or B, we may form the second derivative of M with respect to x, that is, d^2M/dx^2. If the value of this second derivative is positive, then the moment diagram is concave upward, as at A, and the moment assumes a minimum value. If the second derivative is negative the moment diagram is concave downward, as at B, and the moment assumes a maximum value.

However, it is to be carefully noted that the calculus method of obtaining critical values by use of the first derivative does not indicate possible maximum values at a cusp-like point in the moment diagram, if one occurs, such as that shown at C. If such a point is present, the moment there must be determined numerically and then compared to other values that are possibly critical.

Lastly, for vertical equilibrium of the element we have

$$w\,dx + V - (V + dV) = 0 \qquad \text{or} \qquad w = \frac{dV}{dx}$$

This relation will be of value in establishing shearing force diagrams.

6.2. For the cantilever beam subject to the uniformly distributed load of w N/m of length, as shown below in Fig. 6-11(a), write equations for the shearing force and bending moment at any point along the length of the bar. Also sketch the shearing force and bending moment diagrams.

It is not necessary to determine the reactions at the supporting wall. We shall choose the axis of the beam as the x-axis of a coordinate system with origin O at the left end of the bar. To determine the shearing force and bending moment at any section of the beam a distance x from the free end, we may replace the

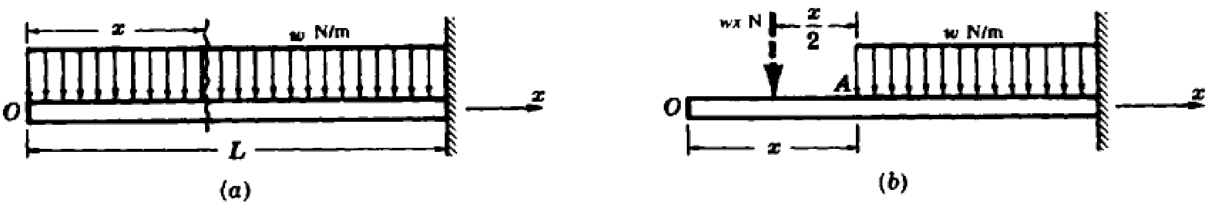


Fig. 6-11

portion of the distributed load to the left of this section by its resultant. As shown by the dashed vector in Fig. 6-11(*b*), the resultant is a downward force of wx N acting midway between O and the section x. Note that none of the load to the right of the section is included in calculating this resultant. Such a resultant force tends to shear the portion of the bar to the left of the section downward with respect to the portion to the right. By our sign convention this constitutes negative shear.

The shearing force at this section x is defined to be the sum of the forces to the left of the section. In this case, the sum is wx N acting downward; hence

$$V = -wx \text{ N}$$

This equation indicates that the shear is zero at $x = 0$ and when $x = L$ it is $-wL$. Since V is a first-degree function of x, the shearing force plots as a straight line connecting these values at the ends of the beam. It has the appearance shown in Fig. 6-12(*a*). The ordinate to this inclined line at any point represents the shearing force at that same point.

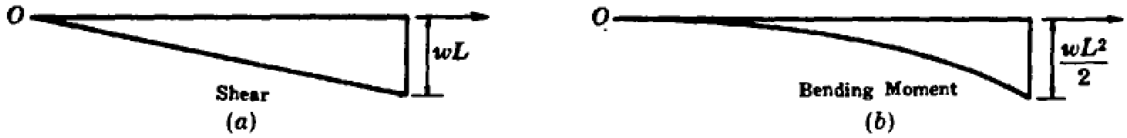


Fig. 6-12

The bending moment at this same section x is defined to be the sum of the moments of the forces to the left of this section about an axis through point A and perpendicular to the plane of the page. This sum of the moments is given by the moment of the resultant, wx N about an axis through A; it is

$$M = -wx\left(\frac{x}{2}\right) \text{N} \cdot \text{m}$$

The minus sign is necessary because downward loads indicate negative bending moments. By this equation the bending moment is zero at the left end of the bar and $-wL^2/2$ at the clamped end when $x = L$. The variation of bending moment is parabolic along the bar and may be plotted as in Fig. 6-12(*b*). The ordinate to this parabola at any point represents the bending moment at that same point.

It is to be noted that a downward uniform load as considered here leads to a bending moment diagram that is concave downward. This could be established by taking the second derivative of M with respect to x, the derivative in this particular case being $-w$. Since the second derivative is negative, the rules of calculus tell us that the curve must be concave downward.

6.3. Consider a simply supported beam 10 ft long and subject to a uniformly distributed vertical load of 120 lb per ft of length, as shown in Fig. 6.13(*a*). Draw shearing force and bending moment diagrams.

The total load on the beam is 1200 lb, and from symmetry each of the end reactions is 600 lb. We shall now consider any cross section of the beam at a distance x from the left end. The shearing force at this

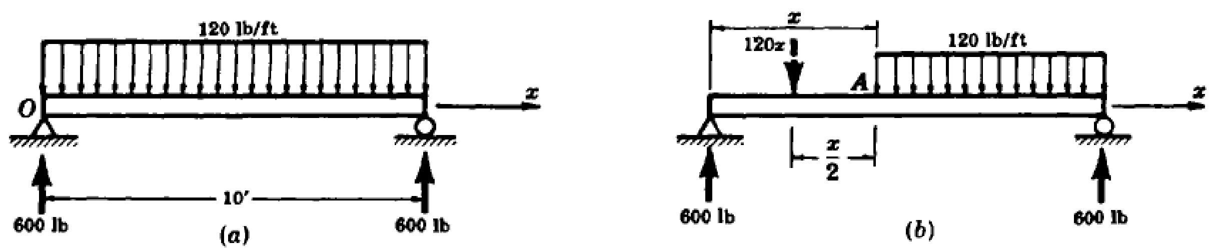


Fig. 6-13

section is given by the algebraic sum of the forces to the left of this section and these forces consist of the 600-lb reaction and the distributed load of 120 lb/ft extending over a length x ft. We may replace the portion of the distributed load to the left of the section at x by its resultant, which is $120x$ lb acting downward as shown by the dashed vector in Fig. 6-13(*b*). None of the load to the right of x is included in this resultant. The shearing force at x is then given by

$$V = 600 - 120x \text{ lb}$$

Since there are no concentrated loads acting on the beam, this equation is valid at all points along its length. Evidently the shearing force varies linearly from $V = 600$ lb at $x = 0$ to $V = 600 - 1200 = -600$ lb at $x = 10$ ft. The variation of shearing force along the length of the bar may then be represented by a straight line connecting these two end-point values. The shear diagram is shown in Fig. 6-14(*a*). The shear is zero at the center of the beam.

The bending moment at the section x is given by the algebraic sum of the moments of the 600-lb reaction and the distributed load of $120x$ lb about an axis through A perpendicular to the plane of the paper. Remembering that upward forces give positive bending moments, we have

$$M = 600x - 120x\left(\frac{x}{2}\right) \text{ lb}\cdot\text{ft}$$

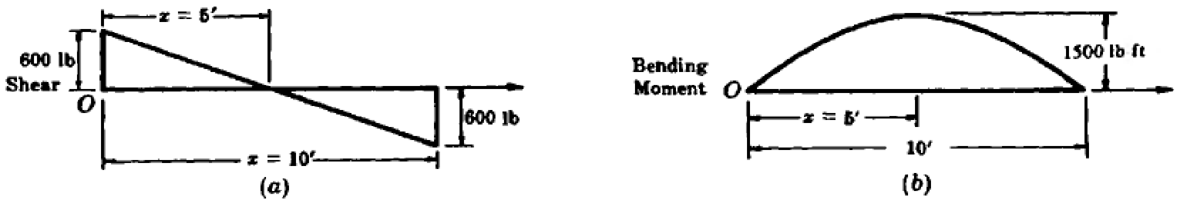


Fig. 6-14

Again, this equation holds along the entire length of the beam. It is to be noted that since the load is uniformly distributed the resultant indicated by the dashed vector acts at a distance $x/2$ from A, i.e., at the midpoint of the uniform load to the left of the section x where the bending moment is being calculated. From the above equation it is evident that the bending moment is represented by a parabola along the length of the beam. Since the bar is simply supported the moment is zero at either end and, because of the symmetry of loading, the bending moment must be a maximum at the center of the beam where $x = 5$ ft. The bending moment at that point is

$$M_{x=5} = 600(5) - 60(5)^2 = 1500 \text{ lb}\cdot\text{ft}$$

The parabolic variation of bending moment along the length of the bar may thus be represented by the ordinates to the bending moment diagram shown in Fig. 6-14(*b*).

6.4. The beam AD in Fig. 6-15 is supported between knife edges at B and C and subject to the end couples indicated. Draw the shearing force and bending moment diagrams.

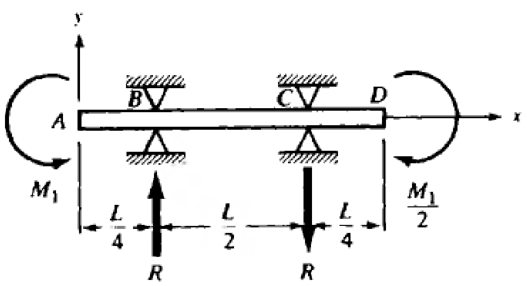


Fig. 6-15

The resultant of the end loadings is a couple

$$M_1 - \frac{M_1}{2} = \frac{M_1}{2}$$

which must be maintained in equilibrium by another couple of that magnitude but oppositely directed. This reactive couple arises from the vertical force reactions R at B and C. The moment of the couple corresponding to these forces must be $M_1/2$ for equilibrium, so we have

$$R \cdot \frac{L}{2} = \frac{M_1}{2}$$

$$R = \frac{M_1}{L}$$

For the coordinate system shown, the shearing force at any point a distance x to the right of A is given by the sum of all vertical forces to the left of x. Thus, for the three regions of the beam we have

$$V = 0 \qquad 0 < x < \frac{L}{4}$$

$$V = \frac{M_1}{L} \qquad \frac{L}{4} < x < \frac{3L}{4}$$

$$V = 0 \qquad \frac{3L}{4} < x < L$$

Analogously, the bending moment at the point x is given by the sum of the moments of all forces and couples to the left of x. Thus, we need the three equations

$$M = -M_1 \qquad 0 < x < \frac{L}{4}$$

$$M = -M_1 + \frac{M_1}{L}\left(x - \frac{L}{4}\right) \qquad \frac{L}{4} < x < \frac{3L}{4}$$

$$M = -M_1 + \frac{M_1}{L} \cdot \frac{L}{2} = -\frac{M_1}{2} \qquad \frac{3L}{4} < x < L$$

Plots of these equations appear in Figs. 6-16(a) and 6-16(b).

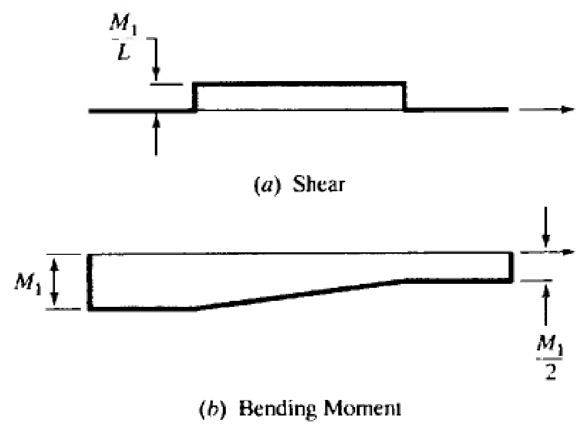


Fig. 6-16

6.5. The simply supported beam shown in Fig. 6-17(a) carries a vertical load that increases uniformly from zero at the left end to a maximum value of 600 lb/ft of length at the right end. Draw the shearing force and bending moment diagrams.

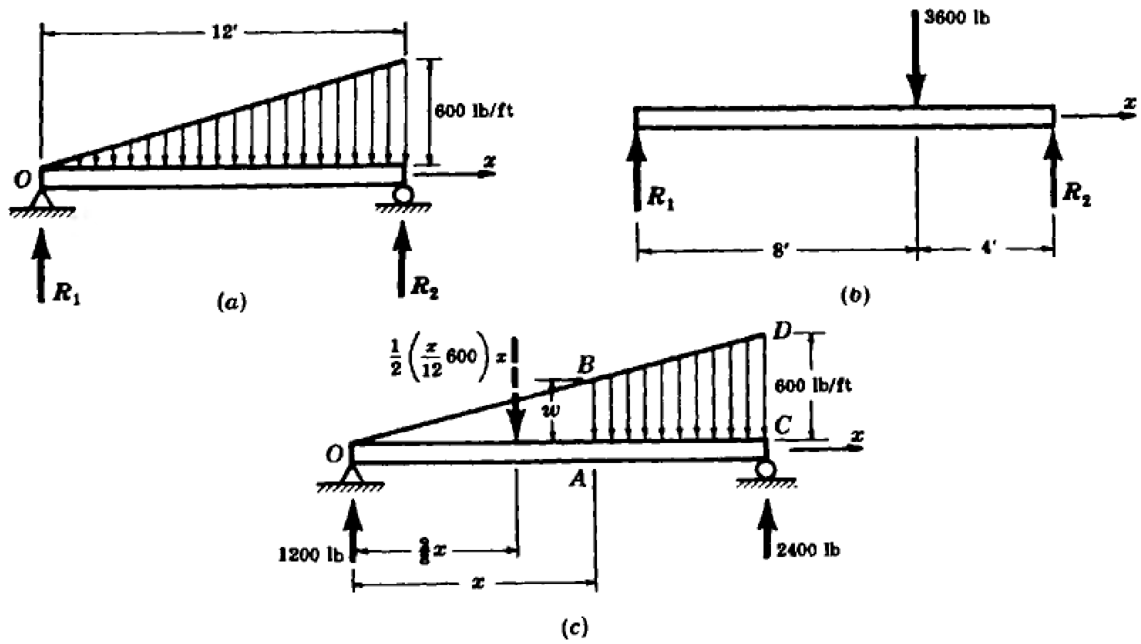


Fig. 6-17

For the purpose of determining the reactions R_1 and R_2 the entire distributed load may be replaced by its resultant which will act through the centroid of the triangular loading diagram. Since the load varies from 0 at the left end to 600 lb/ft at the right end, the average intensity is 300 lb/ft acting over a length of 12 ft. Hence the total load is 3600 lb applied 8 ft to the right of the left support. The free-body diagram to be used in determining the reactions is shown in Fig. 6-17(b). Applying the equations of static equilibrium to this bar, we find $R_1 = 1200$ lb and $R_2 = 2400$ lb.

However, this resultant cannot be used for the purpose of drawing shear and moment diagrams. We must consider the distributed load and determine the shear and moment at a section a distance x from the left end as shown in Fig. 6.17(c). At this section x the load intensity w may be found from the similar triangles OAB and OCD as follows:

$$\frac{w}{x} = \frac{600}{12} \qquad \text{or} \qquad w = \left(\frac{x}{12}\right)600 \text{ lb/ft}$$

The average load intensity over the length x is $\frac{1}{2}(x/12)600$ lb/ft because the load is zero at the left end. The total load acting over the length x is the average intensity of loading multiplied by the length, or $\frac{1}{2}[(x/12)600]x$ lb. This acts through the centroid of the triangular region OAB shown, i.e., through a point located a distance $\frac{2}{3}x$ from O. The resultant of this portion of the distributed load is indicated by the dashed vector in Fig. 6-17(c). No portion of the load to the right of the section x is included in this resultant force.

The shearing force and bending moment at A are now readily found to be

$$V = 1200 - \frac{1}{2}\left(\frac{x}{12}600\right)x = 1200 - 25x^2$$

$$M = 1200x - \frac{1}{2}\left(\frac{x}{12}600\right)x\left(\frac{x}{3}\right) = 1200x - \frac{25}{3}x^3$$

These equations are true along the entire length of the beam. The shearing force thus plots as a parabola, having a value 1200 lb when $x = 0$ and -2400 lb when $x = 12$ ft. The bending moment is a third-degree polynomial. It vanishes at the ends and assumes a maximum value where the shear is zero. This is true because $V = dM/dx$, and hence the point of zero shear must be the point where the tangent to the moment diagram is horizontal. This point of zero shear may be found by setting $V = 0$:

$$0 = 1200 - 25x^2 \qquad \text{or} \qquad x = 6.94 \text{ ft}$$

The bending moment at this point is found by substitution in the general expression given above:

$$M_{x=6.94} = 1200(6.94) - \frac{25}{3}(6.94)^3 = 5520 \text{ lb}\cdot\text{ft}$$

The plots of the shear and moment equations appear in Fig. 6-18.

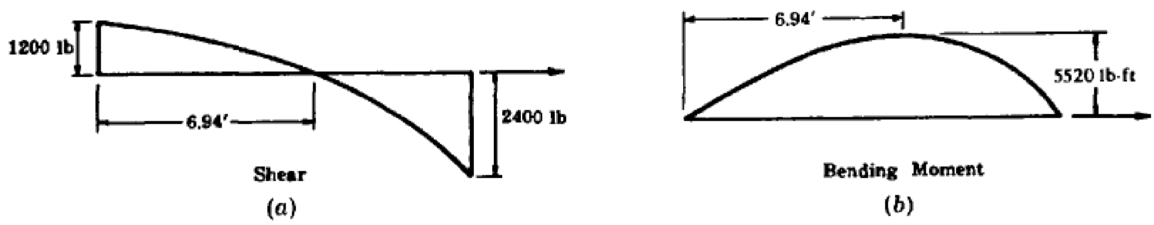


Fig. 6-18

6.6. The cantilever beam AC in Fig. 6-19 is loaded by the uniform load of 600 N/m over the length BC together with the couple of magnitude 4800 N · m at the tip C. Determine the shearing force and bending moment diagrams.

The reactions at A must consist of a vertical shearing force together with a moment to prevent angular rotation. To find these reactions, we write the statics equations

$$\Sigma F_y = R_A - (600 \text{ N/m})(2 \text{ m}) = 0 \qquad (1)$$

$$\circlearrowright \Sigma M_A = M_A - 4800 \text{ N}\cdot\text{m} - (1200 \text{ N})\cdot(3 \text{ m}) = 0 \qquad (2)$$

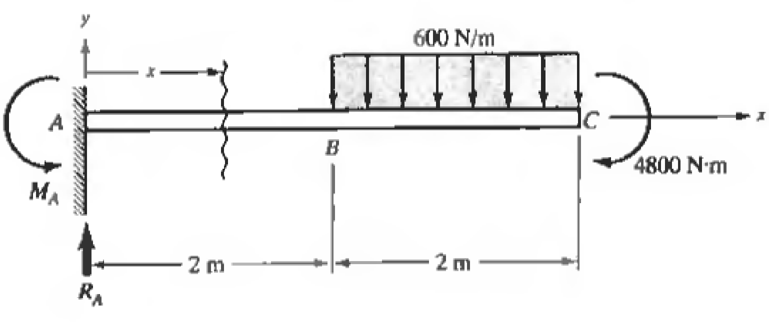


Fig. 6-19

Solving,

$$R_A = 1200 \text{ N} \qquad M_A = 8400 \text{ N}\cdot\text{m}$$

For the coordinate system shown, the shearing force at any point a distance x to the right of A is given by the sum of all forces to the left of x. Thus we must write the two equations

$$V = 1200 \text{ N} \qquad 0 < x < 2 \text{ m} \qquad (3)$$

$$V = 1200 \text{ N} - 600(x - 2)N \qquad 2 < x < 4 \text{ m} \qquad (4)$$

Likewise, the bending moment at this point x is given by the sum of the moments of all forces (and couples) to the left of x about point x. This is given by the two equations

$$M = -8400\ \text{N}\cdot\text{m} + 1200x \qquad 0 < x < 2\ \text{m} \tag{5}$$

$$M = -8400\ \text{N}\cdot\text{m} + 1200x - (600\ \text{N/m})\left[(x-2)\ \text{N}\frac{(x-2)\ \text{N}}{2}\right] \tag{6}$$

Plots of Eqs. (*3*) through (*6*) appear in Fig. 6-20(*a*) and 6-20(*b*), respectively. The nature of the concave region of the bending moment in BC is determined by taking the second derivative of the bending moment Eq. (*6*) in BC:

$$\frac{d^2M}{dx^2} = -600$$

Since this is negative for values of x in BC, the plot in BC of bending moment is concave downward. The bending moment in AB is seen from Eq. (*5*) to be a linear function of x; hence the bending moment in AB plots as a straight line connecting the end couple of $-8400\ \text{N}\cdot\text{m}$ with the bending moment at B of $-6000\ \text{N}\cdot\text{m}$ as determined from Eq. (*6*).

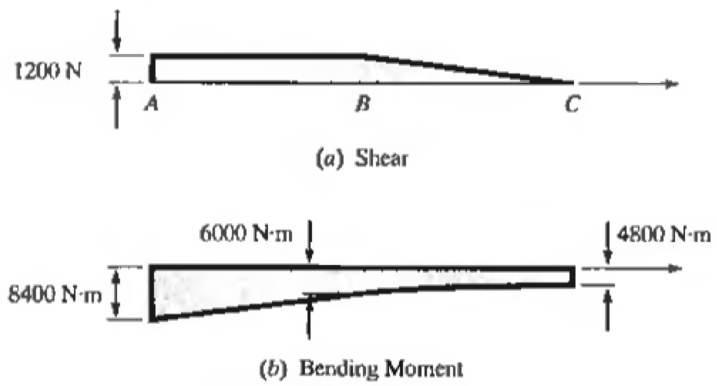


Fig. 6-20

6.7. The beam AC is simply supported at A and C and subject to the uniformly distributed load of 300 N/m plus the couple of magnitude $2700\ \text{N}\cdot\text{m}$ as shown in Fig. 6-21. Write equations for shearing force and bending moment and make plots of these equations.

It is necessary to first determine the reactions from the equilibrium equations

$$+\circlearrowleft \Sigma M_A = 2700\ \text{N}\cdot\text{m} + R_C(6\ \text{m}) - (300\ \text{N/m})(6\ \text{m})(6\ \text{m}) = 0 \tag{1}$$

$$\Sigma F_y = R_A + R_C - (300\ \text{N/m})(6\ \text{m}) = 0 \tag{2}$$

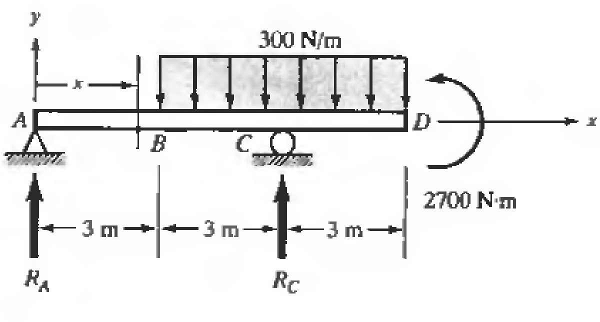


Fig. 6-21

Solving,

$$R_A = 450\ \text{N} \qquad R_C = 1350\ \text{N}$$

For the coordinate x as shown the shearing force a distance x from point A is described by the three relations

$$V = 450\ \text{N} \qquad 0 < x < 3\ \text{m} \tag{3}$$

$$V = [450 - 300(x - 3)]\ \text{N} \qquad 3\ \text{m} < x < 6\ \text{m} \tag{4}$$

$$V = [450 - 300(x - 3) + 1350]\ \text{N} \qquad 6\ \text{m} < x < 9\ \text{m} \tag{5}$$

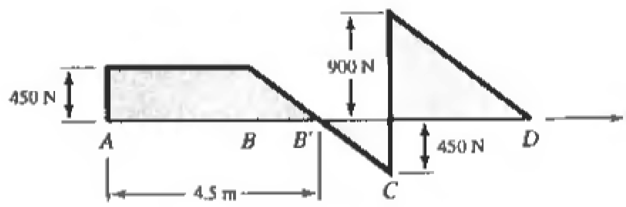
Likewise the bending moment in each of these three regions of the beam is described by

$$M = (450x)\ \text{N}\cdot\text{m} \qquad 0 < x < 3\ \text{m} \tag{6}$$

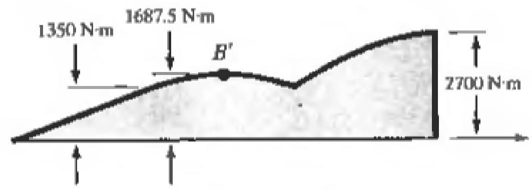
$$M = \left[450x - 300(x - 3)\left(\frac{x-3}{2}\right)\right]\ \text{N}\cdot\text{m} \qquad 3\ \text{m} < x < 6\ \text{m} \tag{7}$$

$$M = \left[450x - 300\frac{(x-3)^2}{2} + 1350(x - 6)\right]\ \text{N}\cdot\text{m} \qquad 6\ \text{m} < x < 9\ \text{m} \tag{8}$$

Plots of these equations appear in Fig. 6-22. In regions BC and CD it is necessary to determine that the second derivative of the bending moment from Eq. (7) and Eq. (8) is negative in each of these regions, and that hence in each case the curvature of the bending moment plot is concave downward.



(a) Shear



(b) Bending Moment

Fig. 6-22

Singularity Functions

The techniques discussed in the preceding problems are adequate if the loadings are continuously varying over the length of the beam. However, if concentrated forces or moments are present, a distinct pair of shearing force and bending moment equations must be written for each region between such concentrated forces or moments. Although this presents no fundamental difficulties, it usually leads to very cumbersome results. As we shall see in a later chapter, these results are particularly unwieldy to work with in dealing with deflections of beams.

At least some compactness of representation may be achieved by introduction of so-called *singularity* or *half-range* functions. Such functions were applied to beam analysis by Macauley in 1919 and this technique of analysis sometimes bears the name of Macauley's method, although the functions were actually used in the 19th century by A. Clebsch. Let us introduce, by definition, the pointed brackets $\langle x - a \rangle$ and define this quantity to be zero if $(x - a) < 0$, that is, $x < a$, and to be simply $(x - a)$ if $(x - a) > 0$, that is, $x > a$. That is, a half-range function is defined to have a value only when the argument is positive. When the argument is positive, the pointed brackets behave just as ordinary parentheses. The singularity function

$$f_n(x) = \langle x - a \rangle^n$$

obeys the integration law

$$\int_{-\infty}^{x} \langle y - a \rangle^n \, dy = \frac{\langle x - a \rangle^{n+1}}{n+1} \qquad \text{for} \qquad n \geq 0$$

The singularity function is very well suited for representation of shearing forces and bending moments in beams subject to loadings of the type discussed in Problems 6.4 through 6.7. This is clear since, say in Problem 6.4 for shearing force, the effect of a single concentrated load is not present (explicitly) in the equation for V for points along the beam to the left of that force, but it immediately appears in the equation for V when one considers values of x to the right of the point of application of the force.

The use of singularity functions for the representations of shearing force and bending moment makes it possible to describe each of these quantities by a single equation along the entire length of the beam, no matter how complex the loading may be. Most important, the singularity function approach leads to simple computer implementation.

6.8. Use singularity functions to write equations for the shearing force and bending moment at any position in the simply supported beam shown in Fig. 6-23.

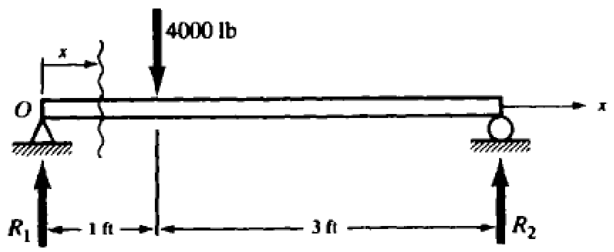


Fig. 6-23

From statics the reactions are easily found to be

$$R_1 = 3000 \text{ lb} \qquad R_2 = 1000 \text{ lb}$$

For the coordinate system shown, with origin at O, we may write

$$V = 3000\langle x \rangle^0 - 4000\langle x - 1 \rangle^0 \text{ lb}$$

which indicates that $V = 3000$ lb if $x < 1$ ft and $V = 3000 - 4000 = -1000$ lb if $x > 1$ ft.

Similarly,

$$M = 3000\langle x \rangle^1 - 4000\langle x - 1 \rangle^1 \text{ lb} \cdot \text{ft} \qquad (2)$$

which tells us that $M = 3000x$ lb · ft if $x < 1$ ft and $M = 3000x - 4000[x - 1]$ lb · ft if $x > 1$ ft.

The relations (*1*) and (*2*) hold for all values of x provided we remember the definition of singularity functions. Use of these equations leads to the shearing force and bending moment diagrams shown in Fig. 6-24.

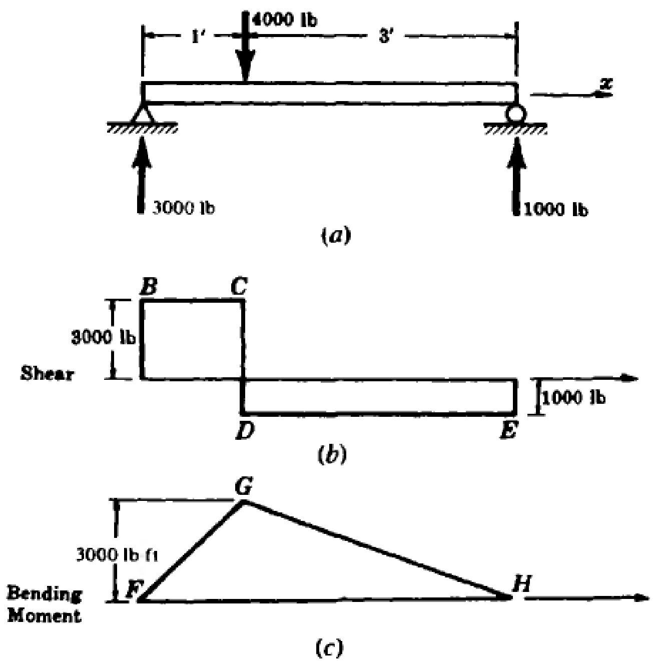


Fig. 6-24

6.9. Consider a cantilever beam loaded only by the couple of 200 lb · ft applied as shown in Fig. 6-25(*a*). Using singularity functions, write equations for the shearing force and bending moment at any position in the beam and plot the shear and moment diagrams.

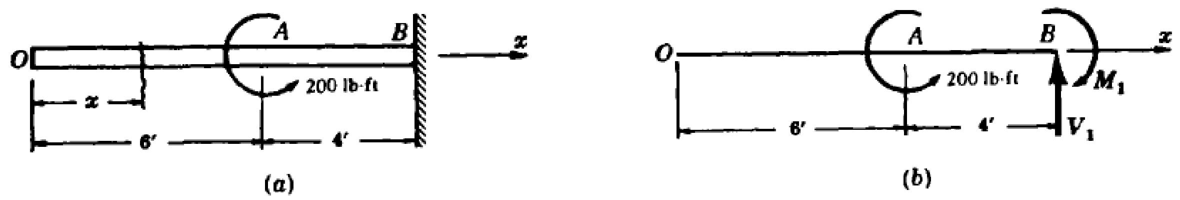


Fig. 6-25

A free-body diagram is shown in Fig. 6-25(*b*), where V_1 and M_1 denote the reactions of the supporting wall. From statics these are found to be $V_1 = 0$, $M_1 = 200$ lb · ft.

We introduce the coordinate system shown in which case the shearing force everywhere is

$$V = 0 \qquad (1)$$

In writing the expression for bending moment, working from left to right it is clear that there is no bending moment to the left of point *A*. At *A* the applied load of 200 lb · ft tends to bend the portion *AB* into a curvature that is concave downward, which according to our sign convention is negative bending. Thus the bending moment anywhere in the beam is

$$M = -200\langle x - 6\rangle^0 \text{ lb} \cdot \text{ft} \qquad (2)$$

Plots of (*1*) and (*2*) appear in Fig. 6-26.

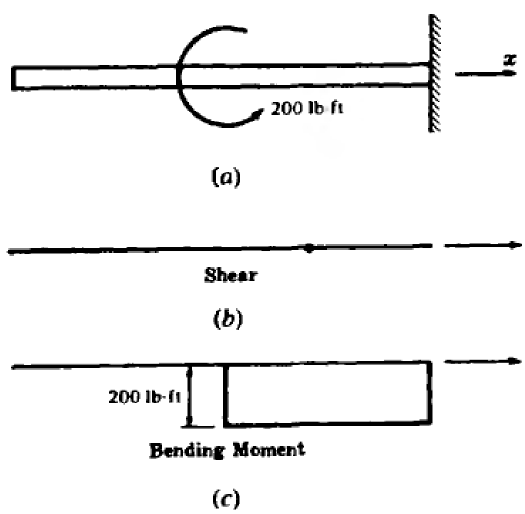


Fig. 6-26

6.10. Consider a cantilever beam loaded by a concentrated force at the free end together with a uniform load distributed over the right half of the beam [see Fig. 6-27(*a*)]. Using singularity functions, write equations for the shearing force and bending moment at any point in the beam and plot the shear and moment diagrams.

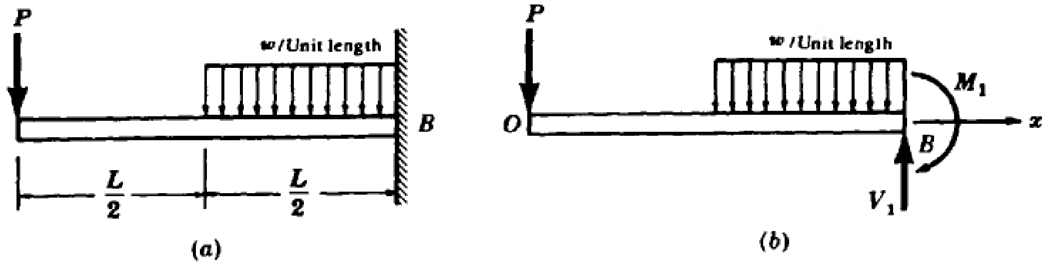


Fig. 6-27

A free-body diagram is shown in Fig. 6-27(*b*). From statics the wall reactions are found to be

$$V_1 = P + \frac{wL}{2} \qquad M_1 = PL + \frac{wL^2}{8}$$

although for the case of a cantilever it is not necessary to find these prior to writing shearing force and bending moment equations.

With the coordinate system shown, with origin at O, the effect of the concentrated force P as well as the distributed load is to produce negative shear according to our shearing force sign convention. Thus we may write

$$V = -P\langle x\rangle^0 - w\left\langle x - \frac{L}{2}\right\rangle^1 \tag{1}$$

which indicates shearing force at any position x if one remembers the definition of the bracketed term.

Likewise, the bending moment at any position x is

$$M = -P\langle x\rangle^1 - \frac{w}{2}\left\langle x - \frac{L}{2}\right\rangle^2 \tag{2}$$

The loaded beam together with plots of the shear and moment equations are shown in Fig. 6-28.

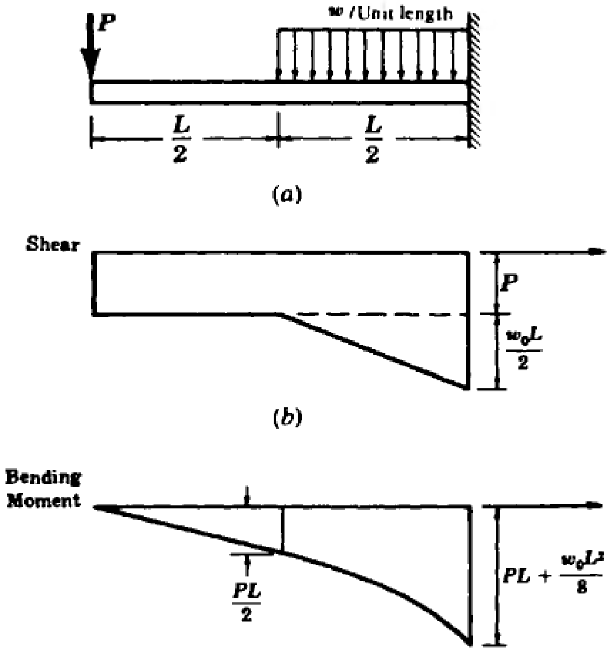


Fig. 6-28

6.11. In Fig. 6-29(*a*) a simply supported beam is loaded by the couple of 1 kN · m. Using singularity functions, write equations for the shearing force and bending moment at any point in the beam and plot the shear and moment diagrams.

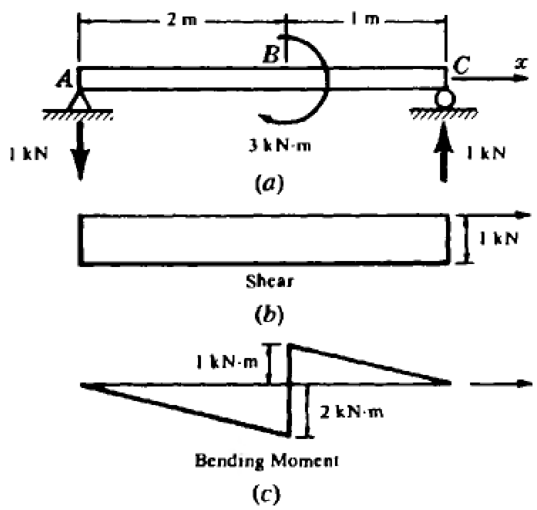


Fig. 6-29

The beam is loaded by one couple, and the only possible manner in which equilibrium may be created is for the reactions R at the supports A and C to constitute another couple. Thus, these reactions appear as in Fig. 6-29(*b*). For equilibrium,

$$\Sigma M_A = 3R - 3 = 0 \qquad \text{from which} \qquad R = 1\ \text{kN}$$

Thus the two forces R shown constitute the reactions necessary for equilibrium.

Inspection of the problem reveals that between A and B the shearing force is negative (according to our sign convention shown in Fig. 6-7) and also the bending moment is negative from the same figure. Just as soon as we consider points on the beam to the right of B, that couple of 3 kN · m tends to produce bending which is concave upward, and thus positive from Fig. 6-7. Therefore the expressions for V and M are

$$V = -(1)\langle x\rangle^0 \quad \text{kN}$$
$$M = -(1)\langle x\rangle^1 + 3\langle x-2\rangle^0 \quad \text{kN}\cdot\text{m}$$

Shear and moment diagrams are plotted in Figs. 6-29(*b*) and 6-29(*c*). From these it is evident that when a couple acts on a bar the bending moment diagram exhibits an abrupt jump or discontinuity at the point where the couple is applied.

6.12. The overhanging beam AE is subject to uniform normal loadings in the regions AB and DE, together with a couple acting at the midpoint C as shown in Fig. 6-30. Using singularity functions, write equations for the shearing force and bending moment at any point in the beam and plot the shear and moment diagram.

To first determine the reactions, we have from statics

$$+\circlearrowright \Sigma M_B = (300\ \text{lb/ft})(1\ \text{ft})(0.5\ \text{ft}) + 150\ \text{lb}\cdot\text{ft} + R_D(3\ \text{ft}) - (300\ \text{lb/ft})(1\ \text{ft})(3.5\ \text{ft}) = 0 \qquad (1)$$

$$\Sigma F_y = -300\ \text{lb} + R_B + R_D - 300\ \text{lb} = 0 \qquad (2)$$

Solving,

$$R_D = 250\ \text{lb} \qquad \text{and} \qquad R_B = 350\ \text{lb} \qquad (3)$$

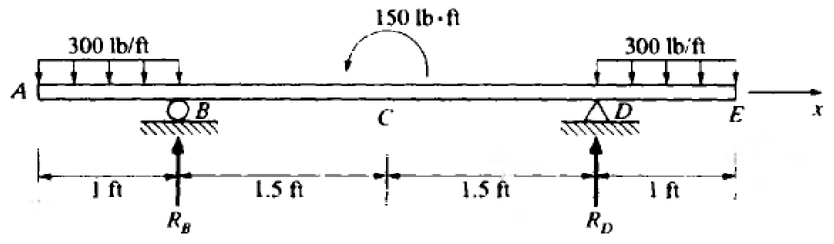


Fig. 6-30

For the coordinate system shown and remembering the definition of the singularity function, we may write

$$V = \overset{①}{-300\langle x\rangle^1} + \overset{②}{300\langle x-1\rangle^1} + \overset{③}{350\langle x-1\rangle^1} + \overset{④}{250\langle x-4\rangle^1} - \overset{⑤}{300\langle x-4\rangle^1} \tag{4}$$

$$M = \overset{⑥}{-300\langle x\rangle^1\frac{\langle x\rangle^1}{2}} + \overset{⑦}{300\langle x-1\rangle^1\frac{\langle x-1\rangle^1}{2}} + \overset{⑧}{350\langle x-1\rangle^1} - \overset{⑨}{150\langle x-2.5\rangle^0}$$

$$+ \overset{⑩}{250\langle x-4\rangle^1} - \overset{⑪}{300\langle x-4\rangle^1\frac{\langle x-4\rangle^1}{2}} \tag{5}$$

Equations (*4*) and (*5*) each contain quantities designated by the numerals circled above the terms. Terms may be interpreted as follows for shearing force V:

I. The shearing force V acting in region OB of Fig. 6-30 is, for any value of the coordinate x in AB, simply the sum of all applied downward normal forces to the left of x, i.e., $300x$, which is term ①. Such forces tend to produce the type of displacement shown in Fig. 6-7(*d*), hence we must prefix the load $300\langle x\rangle$ by a negative sign.

II. Continuing, the first term ① in Eq. (*4*) holds for all values of x ranging from $x = 0$ to $x = 5$ ft. That is, the singularity functions are defined as being zero if the quantity in brackets $\langle\,\rangle$ is negative, but there is no way to specify an upper bound on the coordinate x shown in term ①. Consequently, we must annul the downward 300 lb/ft load to the right of point B and this may be accomplished by adding an upward (positive) uniform load to the right of B, i.e., for all values of $x > 1$ ft, which is term ②. But this upward uniform load has now annulled the actual downward uniform load in region DE. We will return to this shortly.

III. Immediately to the right of B the upward reaction R_B has a shear effect of 350 lb upward so that it tends to produce displacement such as shown in Fig. 6-7(*c*), which we term positive, hence the positive sign in term ③.

IV. The applied couple of 150 lb·ft has no force effect in any direction, hence does not appear in Eq. (*4*).

V. Immediately to the right of D the upward reaction R_D has a shear effect of 250 lb upward so that it tends to produce displacement such as shown in Fig. 6-7(*c*), which we term positive, hence the positive sign in term ④.

VI. As mentioned in (II), the true downward uniform load in DE has temporarily been annulled, hence we must introduce the term ⑤ to return it and make the external loading correct.

Equation (*4*) in terms of singularity functions now correctly specifies the vertical shear at all points on the beam from O to E. A plot of this is given below in Fig. 6-31(*a*).

In a nearly comparable manner, the bending moment from O to E may be written, except that now account must be taken of the applied moment of 150 lb·ft at C. The moment equation is given in (*5*) and a plot of it from O to E appears in Fig. 6-31(*b*).

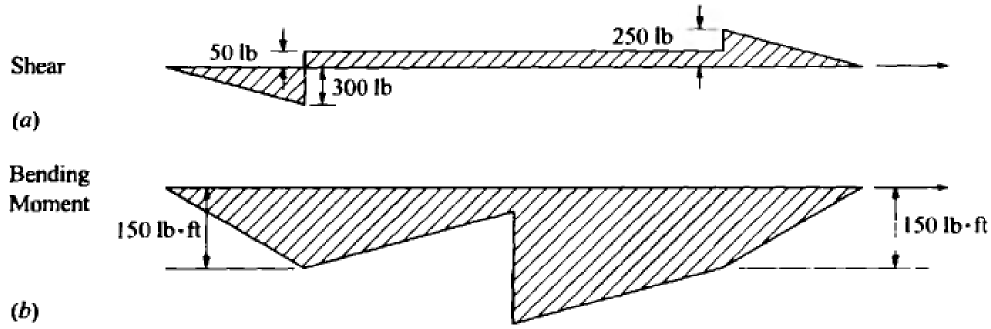


Fig. 6-31

6.13. The simply supported beam AD is subject to a uniform load over the segment BC together with a concentrated force applied at C as shown in Fig. 6-32. Using singularity functions, write equations for the shearing force and bending moment at any point in the beam and plot shear and moment diagrams.

The vertical reactions at A and D must first be determined from statics:

$$+\circlearrowright \Sigma M_A = 4.5R_D - 12\text{ kN}(3.5\text{ m}) - (20\text{ kN})(3.5\text{ m}) = 0$$

$$R_D = 24.89\text{ kN}$$

$$\Sigma F_v = R_A + 24.89\text{ kN} - 12\text{ kN} - 20\text{ kN} = 0$$

$$R_A = 7.11\text{ kN}$$

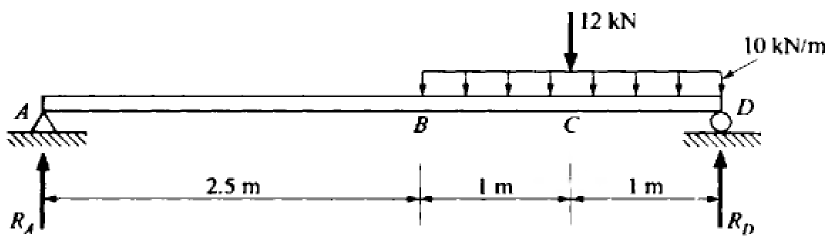


Fig. 6-32

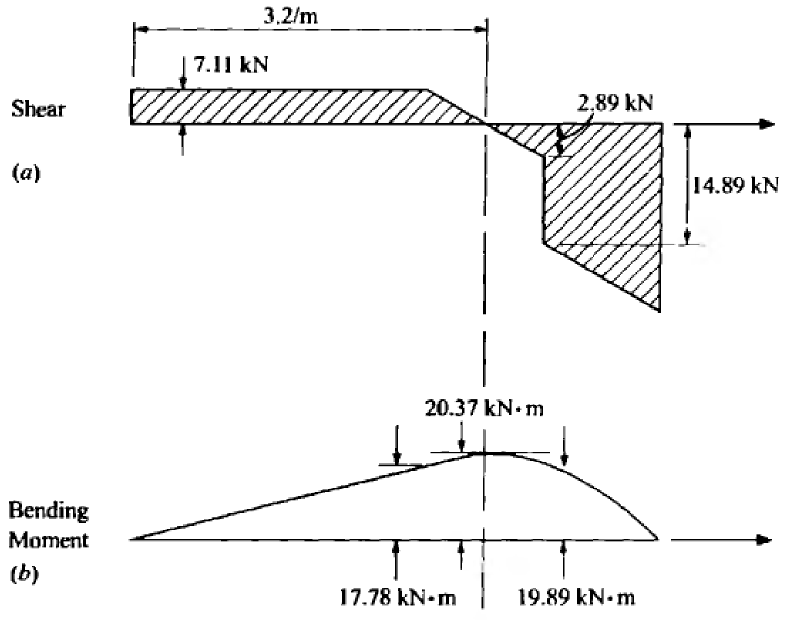


Fig. 6-33

Introducing the coordinate system shown in Fig. 6-30 we can proceed as in Problem 6.12 and write

$$V = 7.11 - 10\langle x - 2.5\rangle^1 - 12\langle x - 3.5\rangle^0$$

$$M = 7.11\langle x\rangle^1 - 10\langle x - 2.5\rangle^1 \frac{\langle x - 2.5\rangle^1}{2} - 12\langle x - 3.5\rangle^1$$

From these equations the shear and moment diagrams may be plotted as shown in Figs. 6-33(*a*) and (*b*).

Computer Implementation

6.14. Consider a straight beam simply supported at any two points. Loading is by a system of concentrated forces, couples, and distributed loads that may (*a*) be uniform along a portion of the beam length, or (*b*) increase (or decrease) linearly. Write a computer program in BASIC to determine shearing force and bending moment at significant locations in the beam.

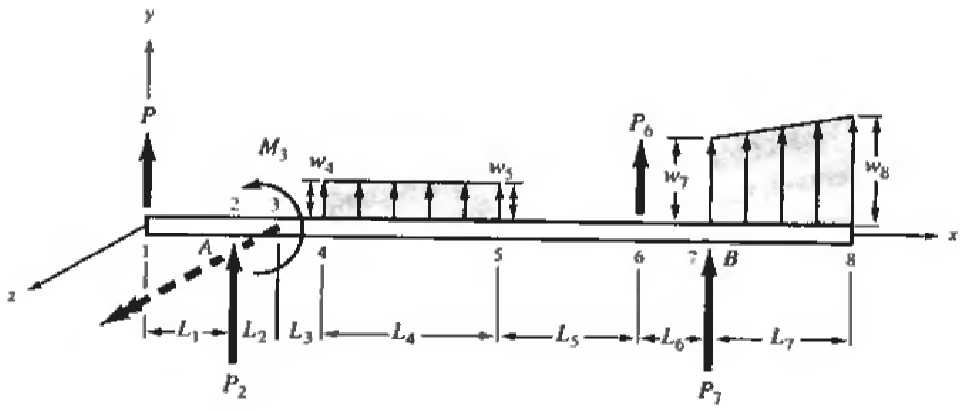


Fig. 6-34

Let us represent the loadings by the terminology of Fig. 6-34. It is first necessary to employ equations of statics to determine the reactions at points A and B. Next, we introduce numbers 1, 2, ... to designate points of application of concentrated forces (including reactions), moments, and left and right end coordinates of distributed loads. Positive directions of all such loads are indicated in Fig. 6-34. The applied moment M_3 is taken positive in the direction indicated because its vector representation (shown by the double-headed vector) is parallel to the z-axis and in the positive direction of that axis.

Use of the method of singularity functions leads to the BASIC program listed below. If more detailed information is needed concerning values of shearing forces and bending moment between number points, one may merely introduce additional points wherever desired.

```
00100 REM     THIS PROGRAM IS DEVELOPED TO EVALUATE THE SHEAR FORCES
00110 REM     AND BENDING MOMENTS.
00120 DIM     S(20), P(21), E(21), D(20,2), T(21,2), B(21,2)
00130 REM
00140 REM     S IS SEGMENT LENGTH
00150 REM     P IS POINT LOAD
00160 REM     E IS EXTERNAL MOMENT
00170 REM     D IS DISTRIBUTED LOAD
00180 REM     T IS SHEAR FORCE
00190 REM     B IS BENDING MOMENT
00200 REM
00210 PRINT   " PROGRAM FOR SHEAR FORCES AND BENDING MOMENTS "
00220 PRINT   " -------------------------------------------- "
00230 PRINT
```

```
00240 PRINT     " PLEASE ENTER THE NUMBER OF SEGMENTS: "
00245 INPUT       N
00250 PRINT
00260 PRINT     " PLEASE ENTER THE LENGTH OF EACH SEGMENT FROM LEFT TO RIGHT.
00270 FOR       I=1 TO N
00280 INPUT     S(I)
00290 NEXT      I
00300 PRINT
00310 PRINT     " PLEASE ENTER THE NUMBER OF POINT LOADS: "
00315 INPUT       N1
00320 PRINT
00330 FOR       I=1 TO N1
00340 PRINT     " LOCATIONS AND LOADS: "
00345 INPUT       I1, P(I1)
00350 NEXT      I
00360 PRINT
00370 PRINT     " ENTER THE NUMBER OF EXTERNAL MOMENTS: "
00375 INPUT       N2
00380 PRINT
00390 FOR       I=1 TO N2
00400 PRINT     " ENTER THE LOCATIONS AND MOMENTS: "
00405 INPUT       L, E(L)
00410 NEXT      I
00420 PRINT
00430 PRINT     " ENTER THE NO. OF DISTRIBUTED LOADED SEGMENTS: "
00435 INPUT      N3
00440 PRINT
00450 FOR       I=1 TO N3
00460 PRINT     " ENTER THE SEGMENT NO., LOADLEFT, LOADRIGHT "
00465 INPUT       N4, D(N4,1), D(N4,2)
00470 NEXT      I
00480 PRINT
00490 LET       T(1,2)=P(1)
00500 LET       B(1,2)=-E(1)
00510 FOR       I=1 TO N
00520 LET       T(I+1,1)=T(I,2)+(D(I,1)+D(I,2))*S(I)/2
00530 LET       T(I+1,2)=T(I+1,1)+P(I+1)
00540 LET       T2=((2*D(I,1)+D(I,2))*S(I)^2)/6
00550 LET       B(I+1,1)=B(I,2)+T(I,2)*S(I)+T2
00560 LET       B(I+1,2)=B(I+1,1)-E(I+1)
00570 NEXT      I
00580 PRINT
00590 PRINT "LOCATION","SHEARLEFT","SHEARRIGHT","MOMENTLEFT","MOMENTRIGHT"
00600 FOR       I=1 TO N+1
00610 PRINT I,T(I,1),T(I,2),B(I,1),B(I,2)
00620 NEXT      I
00630 END
```

"Adapted from a program in *Basic Problems for Applied Mechanics: Statics*, William Weaver, Jr., McGraw-Hill, New York, 1972."

6.15. Use the BASIC program of Problem 6.14 to determine significant shearing forces and bending moments in the simply supported beam shown in Fig. 6-35.

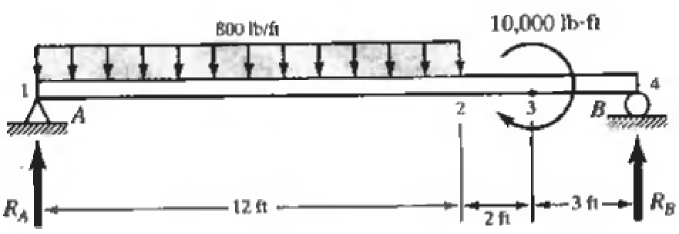


Fig. 6-35

It is first necessary to determine the reactions. From statics,

$$+\uparrow \Sigma M_A = -(9600\ \text{lb})(6\ \text{ft}) - 10{,}000\ \text{lb}\cdot\text{ft} + R_B(17\ \text{ft}) = 0$$

$$\Sigma F_y = R_A + R_B - 9600\ \text{lb} = 0$$

Solving,

$$R_A = 5624\ \text{lb} \qquad R_4 = 3976\ \text{lb}$$

Input to the program is

```
Number of segments:  3
Length of each segment:  12,2,3
Number of point loads (the reactions):  2
Location and magnitude of point loads:  1, 5624
                                        4, 3976
Number of external moments:  1
Location and magnitude of moments:  3, -10,000
Number of segments loaded by distributed load:  1
Segment number, load left, load right:  1, -800, -800
```

The computer output is shown below.

```
 PLEASE ENTER THE NUMBER OF SEGMENTS:
? 3

 PLEASE ENTER THE LENGTH OF EACH SEGMENT FROM LEFT TO RIGHT.
? 12
? 2
? 3

 PLEASE ENTER THE NUMBER OF POINT LOADS:
? 2

 LOCATIONS AND LOADS:
? 1,5624
 LOCATIONS AND LOADS:
? 4,3976

 ENTER THE NUMBER OF EXTERNAL MOMENTS:
? 1

 ENTER THE LOCATIONS AND MOMENTS:
? 3,-10000

 ENTER THE NO. OF DISTRIBUTED LOADED SEGMENTS:
? 1

 ENTER THE SEGMENT NO., LOADLEFT, LOADRIGHT
? 1,-800,-800

LOCATION       SHEARLEFT       SHEARRIGHT       MOMENTLEFT       MOMENTRIGHT
 1              0               5624             0                0
 2             -3976           -3976             9888             9888
 3             -3976           -3976             1936             11936
 4             -3976            0                8                8
```

6.16. A simply supported beam is subject to a uniform load of 2 kN/m over the region shown in Fig. 6-36. Use the BASIC program of Problem 6.14 to determine shearing forces and bending moments at significant points, including the midpoint of the length of the beam.

First, we must determine the end reactions from use of the statics equations. These are readily found to be 2 kN at each end.