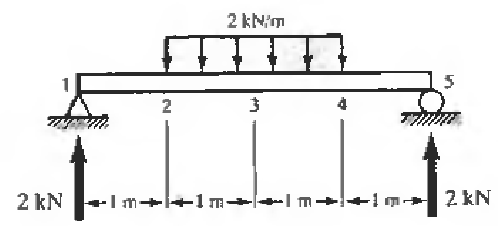


Fig. 6-36

For use of the program, it is necessary to number significant points along the length. These are usually points of application of applied loads. However, here we are asked for the shear and moment at the midpoint of the distributed load. Thus, we introduce an additional numbered point there with the result indicated in Fig. 6-36.

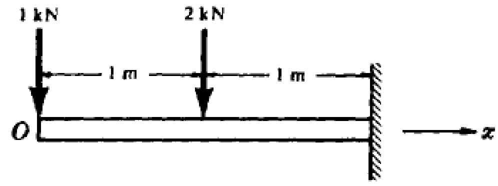
The input and output of the computer program are shown below.

```
 PLEASE ENTER THE NUMBER OF SEGMENTS:
? 4

 PLEASE ENTER THE LENGTH OF EACH SEGMENT FROM LEFT TO RIGHT.
? 1
? 1
? 1
? 1

 PLEASE ENTER THE NUMBER OF POINT LOADS:
? 2

 LOCATIONS AND LOADS:
? 1,2000
 LOCATIONS AND LOADS:
? 5,2000

 ENTER THE NUMBER OF EXTERNAL MOMENTS:
? 0

 ENTER THE NO. OF DISTRIBUTED LOADED SEGMENTS:
? 2

 ENTER THE SEGMENT NO., LOADLEFT, LOADRIGHT
? 2,-2000,-2000
 ENTER THE SEGMENT NO., LOADLEFT, LOADRIGHT
? 3,-2000,-2000

LOCATION        SHEARLEFT       SHEARRIGHT      MOMENTLEFT      MOMENTRIGHT
 1               0               2000            0               0
 2               2000            2000            2000            2000
 3               0               0               3000            3000
 4              -2000           -2000            2000            2000
 5              -2000            0               0               0

SRU       0.129 UNTS.

 RUN COMPLETE.
```

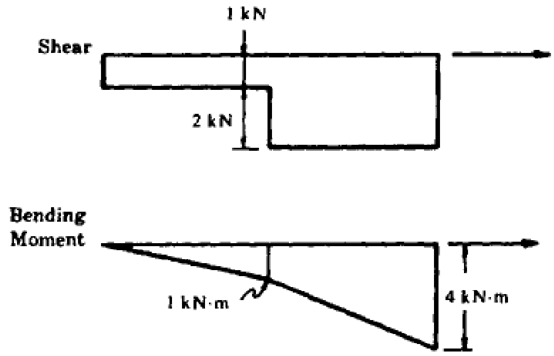
Supplementary Problems

For the cantilever beams loaded as shown in Figs. 6-37 and 6-38, write equations for the shearing force and bending moment at any point along the length of the beam. Also, draw the shearing force and bending moment diagrams.

6.17.



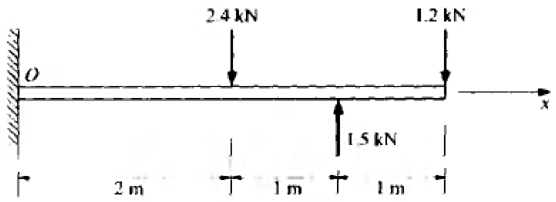
Ans.



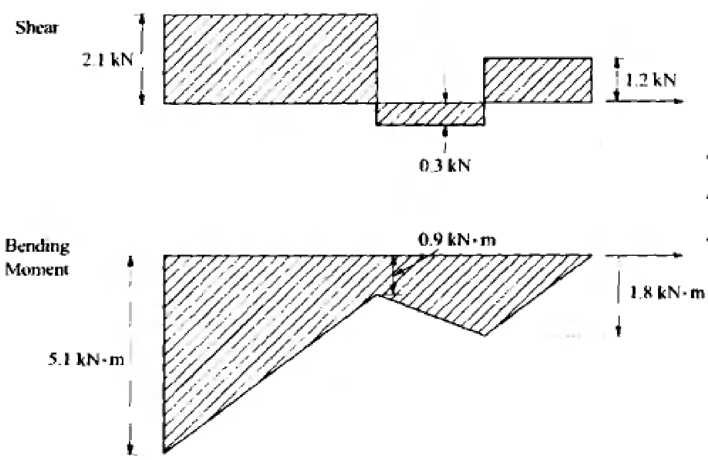
$V = -1$ kN for $0 < x < 1$ m
$V = -3$ kN for $1 < x < 2$ m
$M = -x$ kN·m for $0 < x < 1$ m
$M = -x - 2(x-1)$ kN·m for $1 < x < 2$ m

Fig. 6-37

6.18.



Ans.

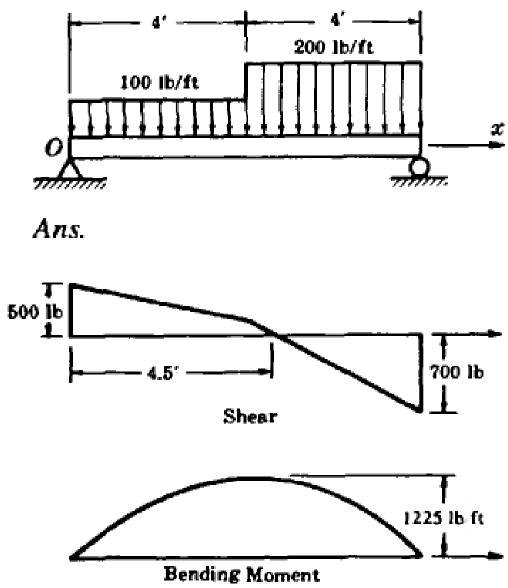


$V = 2.1$ kN for $0 < x < 2$
$V = 2.1 - 2.4 = -0.3$ kN for $2 < x < 3$
$V = 2.1 - 2.4 + 1.5 = 1.2$ for $3 < x < 4$
$M = 2.1x$ kN·m for $0 < x < 2$
$M = 2.1x - 2.4(x-2)$ for $2 < x < 3$
$M = 2.1x - 2.4(x-2) + 1.5(x-3)$ for $3 < x < 4$

Fig. 6-38

For the beams of Problems 6.19 through 6.25 simply supported at the ends and loaded as shown, write equations for the shearing force and bending moment at any point along the length of the beam. Also, draw the shearing force and bending moment diagrams.

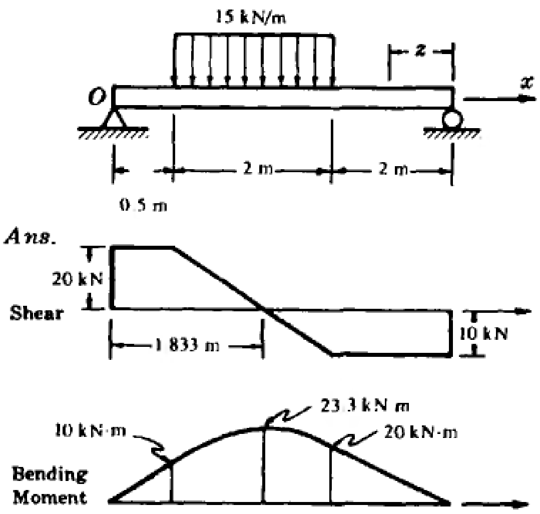
6.19.



$V = 500 - 100x$ lb	for $0 < x < 4$ ft
$V = 100 - 200(x - 4)$ lb	for $4 < x < 8$ ft
$M = 500x - 50x^2$ lb · ft	for $0 < x < 4$ ft
$M = 500x - 400(x - 2) - 100(x - 4)^2$ lb · ft	for $4 < x < 8$ ft

Fig. 6-39

6.20.



$V = 20$ kN	for $0 < x < 0.5$ m
$V = 20 - 15(x - 0.5)$	for $0.5 < x < 2.5$ m
$V = -10$ kN	for $2.5 < x < 4.5$ m
$M = 20x$ kN · m	for $0 < x < 0.5$ m
$M = 20x - 7.5(x - 0.5)^2$ kN · m	for $0.5 < x < 2.5$ m
$M = 10z$ kN · m	for $0 < z < 2$ m

Fig. 6-40

6.21.

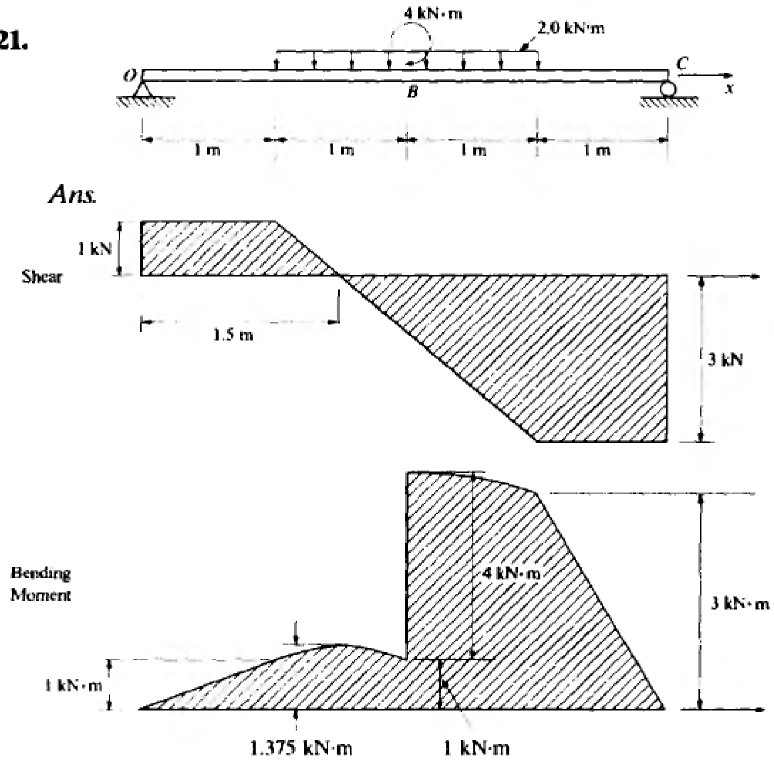


Fig. 6-41

$V = 1\text{ kN}$	for $0 < x < 1$ m
$V = 1 - 2(x-1)$ kN	for $1 < x < 3$
$V = 3$ kN	for $3 < x < 4$
$M = 1x$ kN·m	for $0 < x < 1$ m
$M = 1x - (x-1)\left(\dfrac{x-1}{2}\right)$	for $1 < x < 2$
$M = 1x - 2(x-1)\left(\dfrac{x-1}{2}\right) + 4$	for $2 < x < 3$
$M = 1x - 2(x-1)\left(\dfrac{x-1}{2}\right) + 4$	for $3 < x < 4$

6.22.

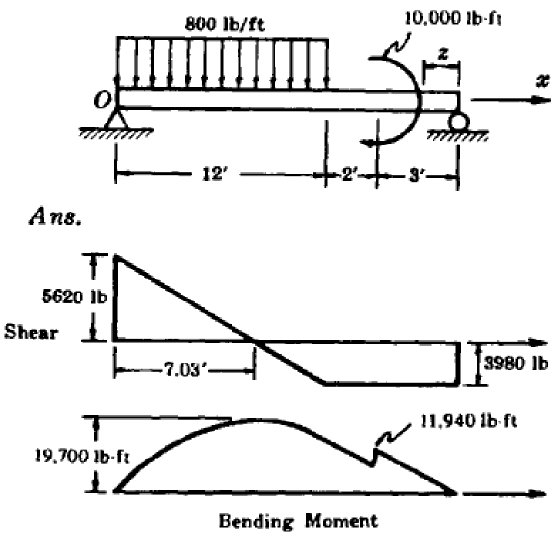
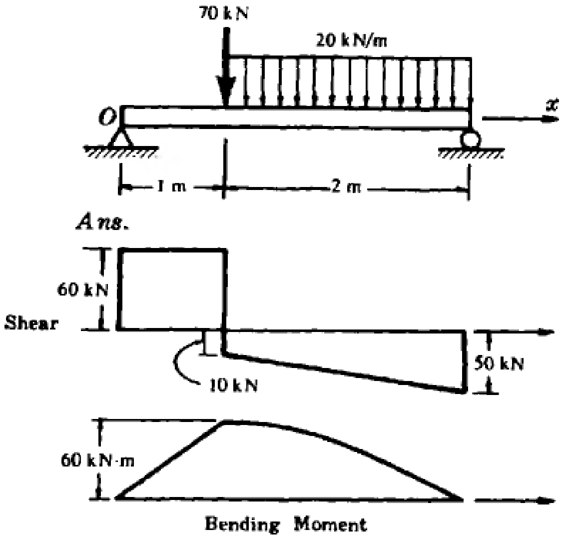


Fig. 6-42

$V = 5620 - 800x$ lb	for $0 < x < 12$ ft
$V = -3980$ lb	for $12 < x < 17$ ft
$M = 5620x - 400x^2$ lb·ft	for $0 < x < 12$ ft
$M = 5620x - 9600(x-6)$ lb·ft	for $12 < x < 14$ ft
$M = 3980z$	for $0 < z < 3$ ft

6.23.



$V = 60\ \text{kN}$ for $0 < x < 1$ m

$V = 60 - 70 - 20(x - 1)\ \text{kN}$ for $1 < x < 3$ m

$M = 60x\ \text{kN} \cdot \text{m}$ for $0 < x < 1$ m

$M = 60x - 70(x - 3) - 10(x - 1)^2\ \text{kN} \cdot \text{m}$ for $1 < x < 3$ m

Fig. 6-43

6.24.

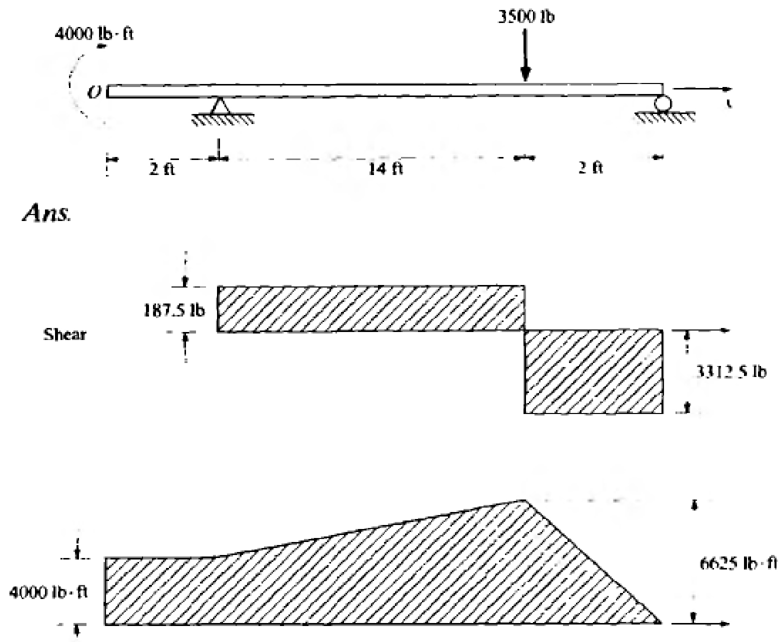


Fig. 6-44

$V = 0$ for $0 < x < 2$ ft

$V = 187.5$ lb for $2 < x < 16$ ft

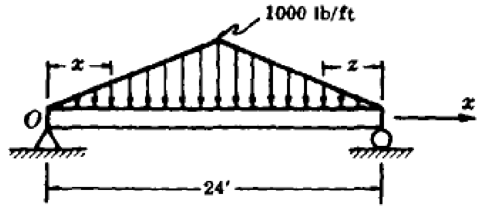
$V = -3312.5$ lb for $16 < x < 18$ ft

$M = 4000\ \text{lb} \cdot \text{ft}$ for $0 < x < 2$ ft

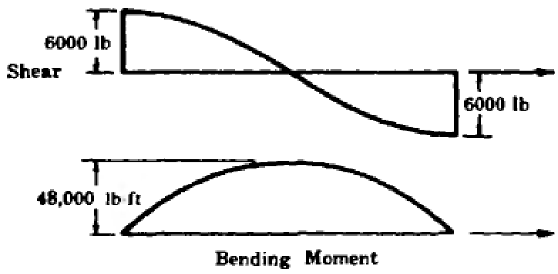
$M = 4000 + 187.5(x - 2)\ \text{lb} \cdot \text{ft}$ for $2 < x < 16$ ft

$M = 4000 + 187.5(x - 2) - 3500(x - 16)$ ft for $16 < x < 18$ ft

6.25.



Ans.



$$V = 6000 - \frac{x^2}{24}(1000)\ \text{lb} \qquad \text{for } 0 < x < 12\ \text{ft}$$

$$V = -6000 + \frac{z^2}{24}(1000)\ \text{lb} \qquad \text{for } 0 < z < 12\ \text{ft}$$

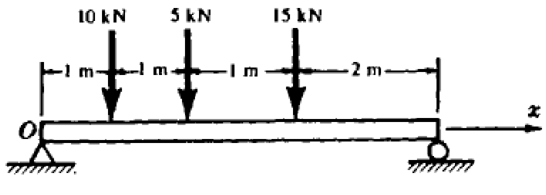
$$M = 6000x - \frac{x^3}{72}(1000)\ \text{lb}\cdot\text{ft} \qquad \text{for } 0 < x < 12\ \text{ft}$$

$$M = 6000z - \frac{z^3}{72}(1000)\ \text{lb}\cdot\text{ft} \qquad \text{for } 0 < z < 12\ \text{ft}$$

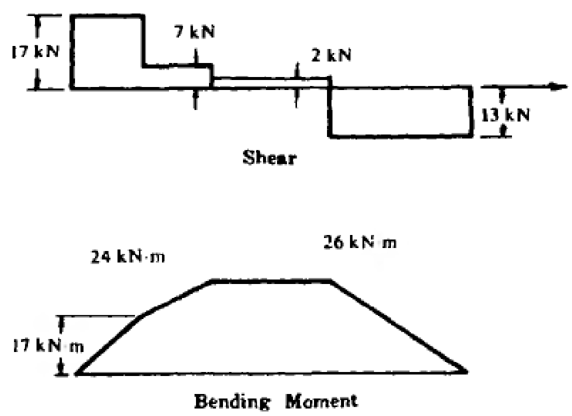
Fig. 6-45

For Problems 6.26 through 6.29 use singularity functions to write the equations for shearing force and bending moment at any point in the beam. Plot the corresponding diagrams.

6.26.



Ans.

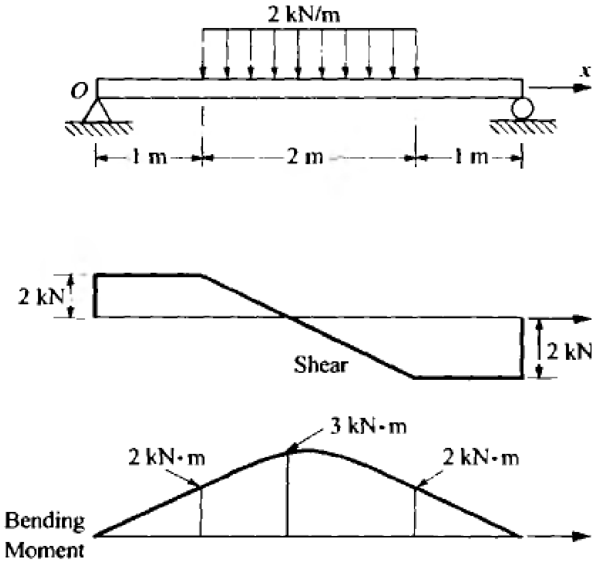


$$V(x) = 17\langle x\rangle^0 - 10\langle x-1\rangle^0 - 5\langle x-2\rangle^0 - 15\langle x-3\rangle^0 \quad \text{kN}$$

$$M(x) = 17\langle x\rangle^1 - 10\langle x-1\rangle^1 - 5\langle x-2\rangle^1 - 15\langle x-3\rangle^1 \quad \text{kN}\cdot\text{m}$$

Fig. 6-46

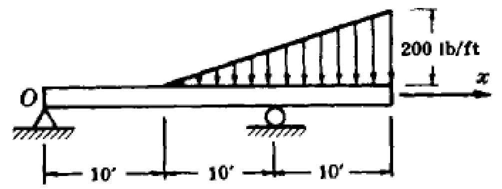
6.27.



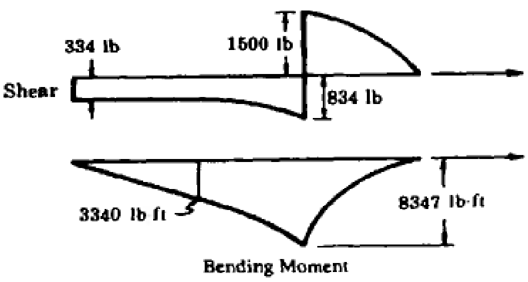
$$V(x) = 2\langle x\rangle^0 - 2\langle x-1\rangle^1 + 2\langle x-3\rangle^1 + 2\langle x-4\rangle^0 \text{ kN}$$
$$M(x) = 2\langle x\rangle^1 - 1\langle x-1\rangle^2 + 1\langle x-3\rangle^2 + 2\langle x-4\rangle^1 \text{ kN}\cdot\text{m}$$

Fig. 6-47

6.28.



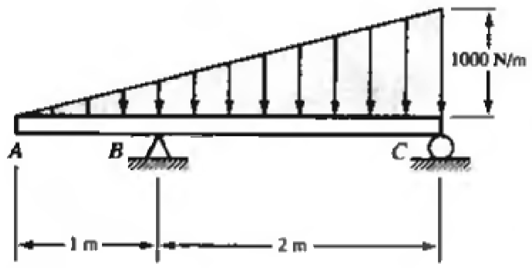
Ans.



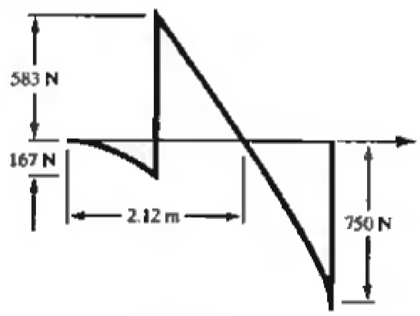
$$V(x) = -334\langle x\rangle^0 - 5\langle x-10\rangle^2 + 2334\langle x-20\rangle^0$$
$$M(x) = -334\langle x\rangle^1 - \tfrac{5}{3}\langle x-10\rangle^3 + 2334\langle x-20\rangle^1$$

Fig. 6-48

6.29.

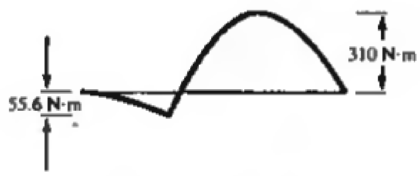


Ans. (*a*)



(*b*) Shear

$$V = -166.7\langle x\rangle^2 + 750\langle x-1\rangle^0$$
$$M = -55.6\langle x\rangle^3 + 750\langle x-1\rangle^1$$



(*c*) Bending Moment

Fig. 6-49

6.30. A simply supported beam is subject to the uniform load together with the couple shown in Fig. 6-50. Use the BASIC program of Problem 6.14 to determine shearing forces and bending moments at significant points along the length of the beam. Draw approximate representations of these results.

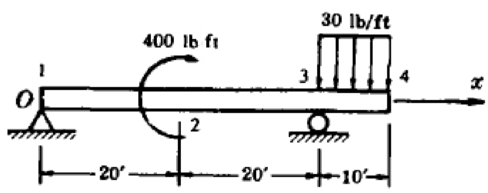


Fig. 6-50

Ans.

LOCATION	SHEARLEFT	SHEARRIGHT	MOMENTLEFT	MOMENTRIGHT
1	0	-47.5	0	0
2	-47.5	-47.5	-950	-550
3	-47.5	300	-1500	-1500
4	0	0	0	0

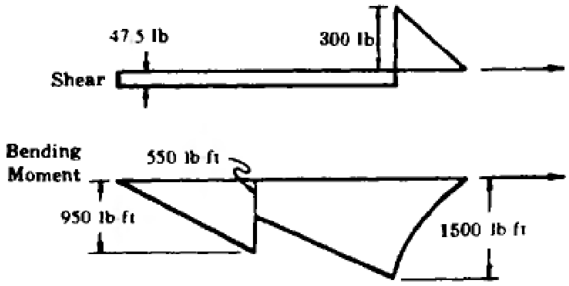


Fig. 6-51

6.31. A simply supported beam is subject to the uniform load together with the couple shown in Fig. 6-52. Use the BASIC program of Problem 6.14 to determine shearing forces and bending moments at significant points along the length of the beam.

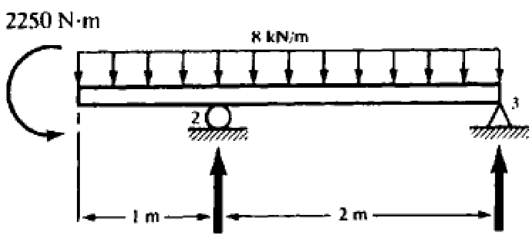


Fig. 6-52

Ans.

LOCATION	SHEARLEFT	SHEARRIGHT	MOMENTLEFT	MOMENTRIGHT
1	0	0	0	-2250
2	-8000	11125	-6250	-6250
3	-4875	0	0	0

Chapter 7

Centroids, Moments of Inertia, and Products of Inertia of Plane Areas

FIRST MOMENT OF AN ELEMENT OF AREA

The first moment of an element of area about any axis in the plane of the area is given by the product of the area of the element and the perpendicular distance between the element and the axis. For example, in Fig. 7-1 the first moment dQ_x of the element da about the x-axis is given by

$$dQ_x = y\,da$$

About the y-axis the first moment is

$$dQ_y = x\,da$$

For applications, see Problems 7.2 and 7.12.

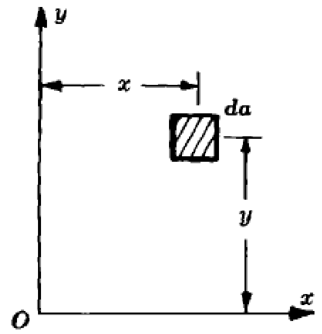


Fig. 7-1

FIRST MOMENT OF A FINITE AREA

The first moment of a finite area about any axis in the plane of the area is given by the summation of the first moments about that same axis of all the elements of area contained in the finite area. This is frequently evaluated by means of an integral. If the first moment of the finite area is denoted by Q_x, then

$$Q_x = \int dQ_x$$

For applications, see Problems 7.1 and 7.3.

CENTROID OF AN AREA

The centroid of an area is defined by the equations

$$\bar{x} = \frac{\int x\,da}{A} = \frac{Q_y}{A} \qquad \bar{y} = \frac{\int y\,da}{A} = \frac{Q_x}{A}$$

where A denotes the area. For a plane area composed of N subareas A_i each of whose centroidal coordinates $\bar{x}_i$ and $\bar{y}_i$ are known, the integral is replaced by a summation

$$\bar{x} = \frac{\sum_{i=1}^{N} \bar{x}_i A_i}{\sum_{i=1}^{N} A_i} \tag{7.1}$$

$$\bar{y} = \frac{\sum_{i=1}^{N} \bar{y}_i A_i}{\sum_{i=1}^{N} A_i} \tag{7.2}$$

For applications see Problems 7.2, 7.3, and 7.12.

The centroid of an area is the point at which the area might be considered to be concentrated and still leave unchanged the first moment of the area about any axis. For example, a thin metal plate will balance in a horizontal plane if it is supported at a point directly under its center of gravity.

The centroids of a few areas are obvious. In a symmetrical figure such as a circle or square, the centroid coincides with the geometric center of the figure.

It is common practice to denote a centroid distance by a bar over the coordinate distance. Thus $\bar{x}$ indicates the x-coordinate of the centroid.

SECOND MOMENT, OR MOMENT OF INERTIA, OF AN ELEMENT OF AREA

The second moment, or moment of inertia, of an element of area about any axis in the plane of the area is given by the product of the area of the element and the square of the perpendicular distance between the element and the axis. In Fig. 7-1, the moment of inertia dI_x of the element about the x-axis is

$$dI_x = y^2\,da$$

About the y-axis the moment of inertia is

$$dI_y = x^2\,da$$

SECOND MOMENT, OR MOMENT OF INERTIA, OF A FINITE AREA

The second moment, or moment of inertia, of a finite area about any axis in the plane of the area is given by the summation of the moments of inertia about that same axis of all of the elements of area contained in the finite area. This, too, is frequently found by means of an integral. If the moment of inertia of the finite area about the x-axis is denoted by I_x, then we have

$$I_x = \int dI_x = \int y^2\,da \tag{7.3}$$

$$I_y = \int dI_y = \int x^2\,da \tag{7.4}$$

For a plane area composed of N subareas A_i each of whose moment of inertia is known about the x- and y-axes, the integral is replaced by a summation

$$I_x = \sum_{i=1}^{N} (I_x)_i \qquad I_y = \sum_{i=1}^{N} (I_y)_i$$

For applications, see Problems 7.4, 7.6, 7.7, 7.8, 7.9, and 7.10.

UNITS

The units of moment of inertia are the fourth power of a length, in^4 or m^4.

PARALLEL-AXIS THEOREM FOR MOMENT OF INERTIA OF A FINITE AREA

The parallel-axis theorem for moment of inertia of a finite area states that the moment of inertia of an area about any axis is equal to the moment of inertia about a parallel axis through the centroid of the area plus the product of the area and the square of the perpendicular distance between the two axes. For the area shown in Fig. 7-2, the axes x_G and y_G pass through the centroid of the plane area. The x- and y-axes are parallel axes located at distances x_1 and y_1 from the centroidal axes. Let A denote the area of the figure, I_{x_G} and I_{y_G} the moments of inertia about the axes through the centroid, and I_x and I_y the moments of inertia about the x- and y-axes. Then we have

$$I_x = I_{x_G} + A(y_1)^2 \tag{7.5}$$

$$I_y = I_{y_G} + A(x_1)^2 \tag{7.6}$$

This relation is derived in Problem 7.5. For applications, see Problems 7.6, 7.8, 7.11, and 7.12.

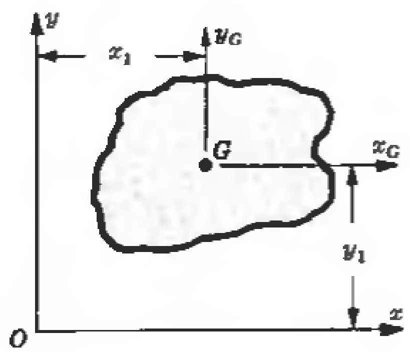


Fig. 7-2

RADIUS OF GYRATION

If the moment of inertia of an area A about the x-axis is denoted by I_x, then the radius of gyration r_x is defined by

$$r_x = \sqrt{\frac{I_x}{A}} \tag{7.7}$$

Similarly, the radius of gyration with respect to the y-axis is given by

$$r_y = \sqrt{\frac{I_y}{A}} \tag{7.8}$$

Since I is in units of length to the fourth power, and A is in units of length to the second power, then the radius of gyration has the units of length, say in or m. It is frequently useful for comparative purposes but has no physical significance. See Problems 7.10 and 7.11.

PRODUCT OF INERTIA OF AN ELEMENT OF AREA

The product of inertia of an element of area with respect to the x- and y-axes in the plane of the area is given by

$$dI_{xy} = xy\,da$$

where x and y are coordinates of the elemental area as shown in Fig. 7-1.

PRODUCT OF INERTIA OF A FINITE AREA

The product of inertia of a finite area with respect to the x- and y-axes in the plane of the area is given by the summation of the products of inertia about those same axes of all elements of area contained within the finite area. Thus

$$I_{xy} = \int xy\,da \tag{7.9}$$

From this, it is evident that I_{xy} may be positive, negative, or zero. For a plane area composed of N subareas A_i each of whose product of inertia is known with respect to specified x- and y-axes, the integral is replaced by the summation

$$I_{xy} = \sum_{i=1}^{N} (I_{xy})_i \tag{7.10}$$

For applications see Problems 7.13 and 7.15.

PARALLEL-AXIS THEOREM FOR PRODUCT OF INERTIA OF A FINITE AREA

The parallel-axis theorem for product of inertia of a finite area states that the product of inertia of an area with respect to the x- and y-axes is equal to the product of inertia about a set of parallel axes passing through the centroid of the area plus the product of the area and the two perpendicular distances from the centroid to the x- and y-axes. For the area shown in Fig. 7.2, the axes x_G and y_G pass through the centroid of the plane area. The x- and y-axes are parallel axes located at distances x_1 and y_1 from the centroidal axes. Let A represent the area of the figure and $I_{x_G y_G}$ be the product of inertia about the axes through the centroid. Then we have

$$I_{xy} = I_{x_G y_G} + A x_1 y_1 \tag{7.11}$$

This relation is derived in Problem 7.14. For applications see Problems 7.15 and 7.16.

PRINCIPAL MOMENTS OF INERTIA

At any point in the plane of an area there exist two perpendicular axes about which the moments of inertia of the area are maximum and minimum for that point. These maximum and minimum values of moment of inertia are termed *principal moments of inertia* and are given by

$$(I_{x_1})_{\max} = \left(\frac{I_x + I_y}{2}\right) + \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2} \tag{7.12}$$

$$(I_{x_1})_{\min} = \left(\frac{I_x + I_y}{2}\right) - \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2} \tag{7.13}$$

These expressions are derived in Problem 7.17. For application, see Problem 7.18.

PRINCIPAL AXES

The pair of perpendicular axes through a selected point about which the moments of inertia of a plane area are maximum and minimum are termed *principal axes*. For application, see Problem 7.16.

The product of inertia vanishes if the axes are principal axes. Also, from the integral defining product of inertia of a finite area, it is evident that if either the x-axis, or the y-axis, or both, are axes of symmetry, the product of inertia vanishes. Thus, axes of symmetry are principal axes.

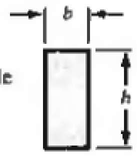
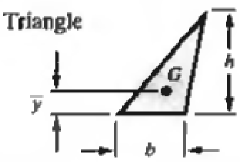
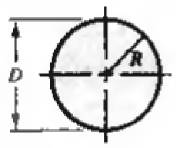
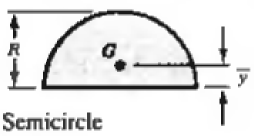
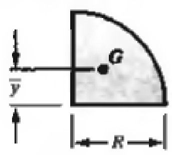
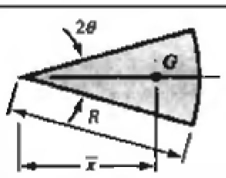
Type of section	Area	Location of centroid
Rectangle  (*a*)	bh	Geometric center
Triangle  (*b*)	$\frac{1}{2}bh$	$\bar{y} = \frac{h}{3}$
Circle  (*c*)	πR^2 or $\frac{\pi}{4}D^2$	Geometric center
 Semicircle (*d*)	$\frac{1}{2}\pi R^2$ or $\frac{\pi}{8}D^2$	$\bar{y} = \frac{4R}{3\pi}$
Quadrant of circle  (*e*)	$\frac{\pi R^2}{4}$	$\bar{y} = \frac{4R}{3\pi}$
 Sector of circle (*f*)	θR^2	$\bar{x} = \frac{2R \sin \theta}{3\theta}$

Fig. 7-3

INFORMATION FROM STATICS

Most texts on statics develop the properties of plane cross-sectional areas shown in Fig. 7-3 that will be needed in the present chapter. Those areas include (*a*) the rectangle, (*b*) the triangle, (*c*) the circle, (*d*) the semicircle, (*e*) the quadrant of a circle, and (*e*) the sector of a circle.

Solved Problems

7.1. The shaded area shown in Fig. 7-4 is bounded by the curves

$$y_1 = \sqrt[3]{x}$$

and

$$y_2 = x^3$$

Determine the y-coordinate of the centroid of this area which ends at (1,1).

We select an element that is horizontal (thus all points in this element have the same "y") and

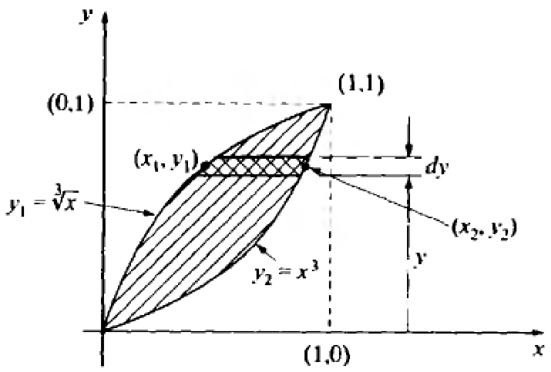


Fig. 7-4

extending from curve y_1 to y_2 as shown in Fig. 7-4. The height of the element is dy. From the definition of the location of the centroid,

$$\bar{y} = \frac{\int y\,da}{A}$$

we can write

$$da = (x_2 - x_1)\,dy$$

in which case we have

$$\bar{y} = \frac{\int_0^1 (x_2 - x_1)(y)(dy)}{\int_0^1 (x_2 - x_1)\,dy}$$

$$= \frac{\int_0^1 (y^{1/3} - y^3)(y)(dy)}{\int_0^1 (y^{1/3} - y^3)\,dy} = \frac{16}{70} = 0.229$$

Although the integrations involved in this problem are simple, for more complex problems one should resort to computers. A number of symbolic operations are available on proprietary software that permit easy and rapid treatments of such computations.

7.2. A circular cross section has a sector having a central angle 2θ removed as shown in Fig. 7-5. Locate the y-coordinate of the centroid of the shaded area.

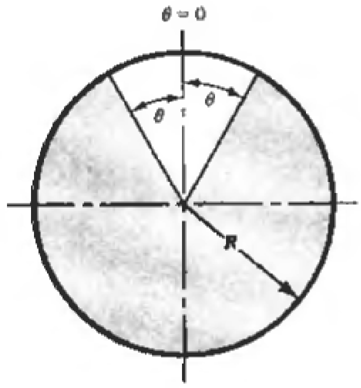


Fig. 7-5

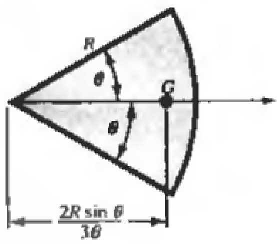


Fig. 7-6

From the summary at the beginning of this chapter, we have for a sector of central angle 2θ the area and centroid given by θR^2 and $2R\sin\theta/3\theta$, respectively (see Fig. 7-6). The area of the entire circle having its centroid at its geometric center is also given in that summary.

By definition the y-coordinate of the centroid of the shaded area in Fig. 7-4 is given by

$$\bar{y} = \frac{\int y\,da}{A} \quad \text{or} \quad \frac{\Sigma\, y\,da}{A}$$

Here we consider the shaded area to be composed of the three components consisting of the lower semicircle ①, the upper semicircle ②, and the sector that has been removed ③. Thus the net shaded area is represented as shown in Fig. 7-7.

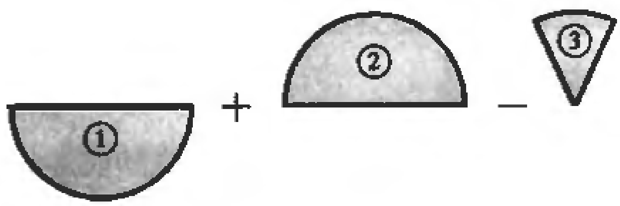


Fig. 7-7

Using these components in the finite summation (*7.1*), we have

$$\bar{y} = \frac{\overset{①}{\dfrac{\pi}{2}R^2\left(-\dfrac{4R}{3\pi}\right)} + \overset{②}{\dfrac{\pi}{2}R^2\left(\dfrac{4R}{3\pi}\right)} - \overset{③}{\theta R^2\left(\dfrac{2R}{3\theta}\sin\theta\right)}}{\pi R^2 - \theta R^2}$$

$$= -\frac{\frac{2}{3}(R\sin\theta)}{(\pi - \theta)}$$

7.3. A thin sheet of metal 600 mm by 1000 mm has its two upper corners folded over along the inclined lines AC and DF as shown in Fig. 7-8. In the regions bounded by the dotted lines, the metal thus becomes doubly thick. Determine the y-coordinate of the centroid of the folded sheet.

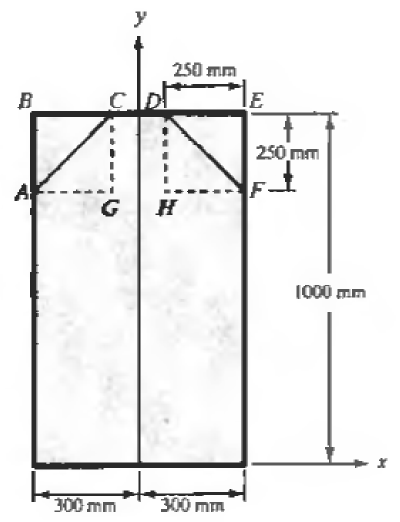


Fig. 7-8

By definition, the y-coordinate of the centroid is

$$\bar{y} = \frac{\int y\,da}{A} \quad \text{or} \quad \frac{\Sigma y_i A_i}{A}$$

where the numerator in each expression represents the first moment of the area about the x-axis. In the numerical evaluation, the triangles ABC and DEF have been removed but replaced by triangles ACG and DFH accounting for the double thickness. Thus we have

$$\bar{y} = \frac{(600)(1000)(500) - \overbrace{2\{\tfrac{1}{2}(250)(250)[1000 - \tfrac{250}{3}]\}}^{\triangle BCA} + \overbrace{2\{\tfrac{1}{2}(250)(250)[750 + \tfrac{250}{3}]\}}^{\triangle AGC}}{(600)(1000)}$$

$$= 491.3 \text{ mm}$$

7.4. Determine the moment of inertia of a rectangle about an axis through the centroid and parallel to the base.

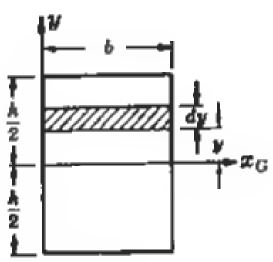


Fig. 7-9

Let us introduce the coordinate system shown in Fig. 7-9. The moment of inertia I_{x_G} about the x-axis passing through the centroid is given by $I_{x_G} = \int y^2\,da$. For convenience it is logical to select an element such that y is constant for all points in the element. The shaded area shown has this characteristic.

$$I_{x_G} = \int_{-h/2}^{h/2} y^2 b\,dy = b\left[\frac{y^3}{3}\right]_{-h/2}^{h/2} = \frac{1}{12}bh^3$$

This quantity has the dimension of a length to the fourth power, perhaps in^4 or m^4.

7.5. Derive the parallel-axis theorem for moments of inertia of a plane area.

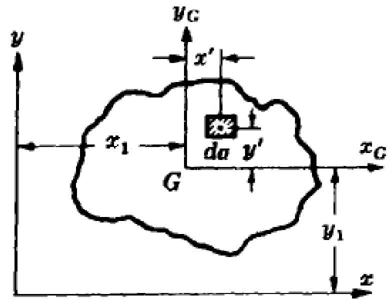


Fig. 7-10

Let us consider the plane area A shown in Fig. 7-10. The axes x_G and y_G pass through its centroid, whose location is presumed to be known. The axes x and y are located at known distances y_1 and x_1, respectively, from the axes through the centroid.

For the element of area da the moment of inertia about the x-axis is given by

$$dI_x = (y_1 + y')^2\,da$$

For the entire area A the moment of inertia about the x-axis is

$$I_x = \int dI_x = \int (y_1 + y')^2\,da = \int (y_1)^2\,da + 2\int y_1 y'\,da + \int (y')^2\,da$$

The first integral on the right is equal to $y_1^2 \int da = y_1^2 A$ because y_1 is a constant. The second integral on the right is equal to $2y_1 \int y'\,da = 2y_1(0) = 0$ because the axis from which y' is measured passes through the centroid of the area. The third integral on the right is equal to I_{x_G}, i.e., the moment of inertia of the area about the horizontal axis through the centroid. Thus

$$I_x = I_{x_G} + A(y_1)^2$$

A similar consideration in the other direction would show that

$$I_y = I_{y_G} + A(x_1)^2$$

This is the parallel-axis theorem for plane areas. It is to be noted that one of the axes involved in each equation must pass through the centroid of the area. In words, this may be stated as follows: The moment of inertia of an area with reference to an axis not through the centroid of the area is equal to the moment of inertia about a parallel axis through the centroid of the area plus the product of the same area and the square of the distance between the two axes.

The moment of inertia always has a positive value, with a minimum value for axes through the centroid of the area in question.

7.6. Find the moment of inertia of a rectangle about an axis coinciding with the base.

The coordinate system shown in Fig. 7-11 is convenient. By definition the moment of inertia about the x-axis is given by $I_x = \int y^2\,da$. For the element shown y is constant for all points in the element. Hence

$$I_x = \int_0^h y^2 b\,dy = b\left[\frac{y^3}{3}\right]_0^h = \frac{1}{3}bh^3$$

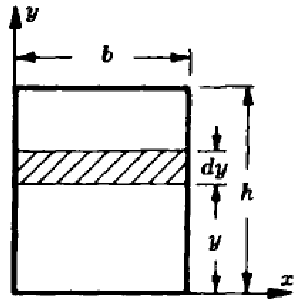


Fig. 7-11

This solution could also have been obtained by applying the parallel-axis theorem to the result obtained in Problem 7-4. This states that the moment of inertia about the base is equal to the moment of inertia about the horizontal axis through the centroid plus the product of the area and the square of the distance between these two axes. Thus

$$I_x = I_{x_G} + A(y_1)^2 = \frac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3}bh^3$$

7.7. Determine the moment of inertia of a triangle about an axis coinciding with the base.

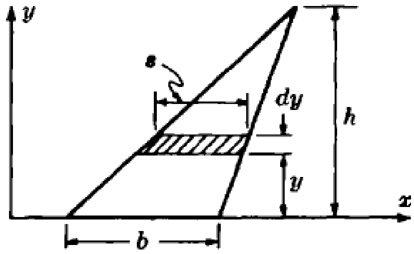


Fig. 7-12

Let us introduce the coordinate system shown in Fig. 7-12. The moment of inertia about the horizontal base is

$$I_x = \int y^2\,da$$

For the shaded element shown the quantity y is constant for all points in the element. Thus

$$I_x = \int_0^h y^2 s\,dy$$

By similar triangles, $s/b = (h - y)/h$, so that

$$I_x = \int_0^h y^2 \frac{b}{h}(h - y)\,dy = \frac{b}{h}\left[h\int_0^h y^2 dy - \int_0^h y^3 dy\right] = \frac{1}{12}bh^3$$

7.8. Determine the moment of inertia of a triangle about an axis through the centroid and parallel to the base.

Let the x_G-axis pass through the centroid and take the x-axis to coincide with the base as shown in Fig. 7-13.

From Fig. 7-3(b) the x_G-axis is located a distance of $h/3$ above the base. Also, the parallel-axis theorem tells us that

$$I_x = I_{x_G} + A(y_1)^2$$

But I_x was determined in Problem 7.7, and A and y_1 ($= h/3$) are known. Hence we may solve for the desired unknown, I_{x_G}. Substituting,

$$\frac{1}{12}bh^3 = I_{x_G} + \frac{1}{2}bh\left(\frac{h}{3}\right)^2 \qquad \text{or} \qquad I_{x_G} = \frac{1}{36}bh^3$$

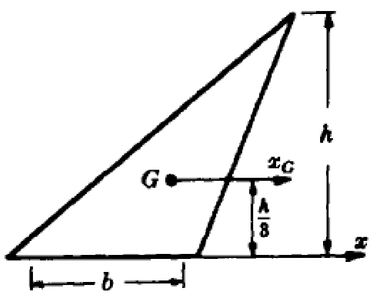


Fig. 7-13

7.9. Determine the moment of inertia of a circle about a diameter.

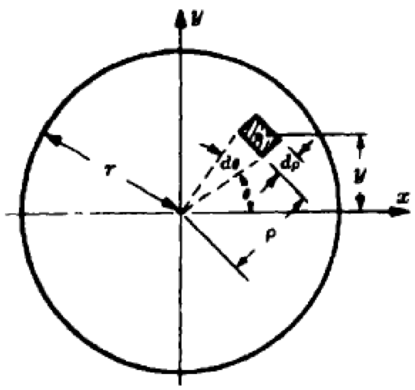


Fig. 7-14

Let us select the shaded element of area shown in Fig. 7-14, and work with the polar coordinate system. The radius of the circle is r.

To find I_x we have the definition $I_x = \int y^2\,da$.

But $y = \rho\sin\theta$ and $da = \rho\,d\theta\,d\rho$. Hence

$$I_x = \int_0^{2\pi}\int_0^r \rho^2\sin^2\theta\rho\,d\theta\,d\rho = \int_0^{2\pi}\sin^2\theta\,d\theta\left[\frac{1}{4}\rho^4\right]_0^r$$

$$= \frac{r^4}{4}\int_0^{2\pi}\sin^2\theta\,d\theta = \frac{\pi r^4}{4}$$

If D denotes the diameter of the circle, then $D = 2r$ and $I_x = \pi D^4/64$. This is half the value of the polar moment of inertia of a solid circular area (see Problem 5.1).

The moment of inertia of a semicircular area about an axis coinciding with its base is

$$I_x = \frac{1}{2}\frac{\pi D^4}{64} = \frac{\pi D^4}{128}$$

7.10. Determine the moment of inertia about both the x- and y-axes as well as the corresponding radii of gyration of the plane area shown in Fig. 7-15.

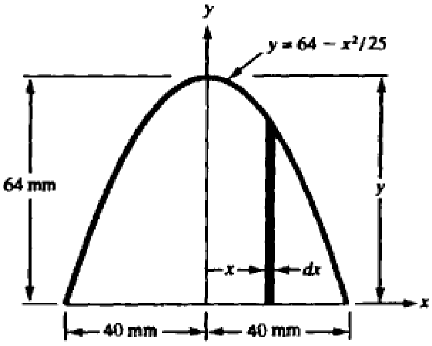


Fig. 7-15

Let us select the shaded element of width dx and altitude y shown in Fig. 7-15. From Problem 7.6 we have the moment of inertia of this element about the x-axis as

$$dI_x = \tfrac{1}{3}bh^3 = \tfrac{1}{3}(dx)y^3$$

Now, we must integrate over all values of x from -40 mm to $+40$ mm to account for all such elements. Thus,

$$I_x = \int dI_x = \frac{1}{3}\int_{x=-40}^{x=40} y^3\,dx$$

$$= \frac{2}{3}\int_{x=0}^{x=40}\left[64 - \frac{x^2}{25}\right]^3 dx$$

$$= 3.197 \times 10^6\ \text{mm}^4$$

The same element may be employed to determine the moment of inertia of the entire area about the y-axis. By definition we have

$$dI_y = x^2\,da$$

which becomes

$$I_y = \int dI_y = \int_{x=-40}^{x=40} x^2 y\,dx$$

$$= 2\int_{x=0}^{x=40} x^2\left(64 - \frac{x^2}{25}\right) dx$$

$$= 1.092 \times 10^6\ \text{mm}^4$$

To determine the radii of gyration, it is first necessary to find the area under the curve. It is given by

$$A = \int y\,dx$$

$$= 2\int_{x=0}^{x=40}\left(64 - \frac{x^2}{25}\right) dx = 3413\ \text{mm}^2$$

from which we have

$$r_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{3.197 \times 10^6\ \text{mm}^4}{3413\ \text{mm}^2}} = 30.6\ \text{mm}$$

$$r_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{1.092 \times 10^6\ \text{mm}^4}{3413\ \text{mm}^2}} = 17.9\ \text{mm}$$

7.11. Two channel sections are attached to a cover plate 16 in long by $\frac{1}{2}$ in thick, as indicated in Fig. 7-16. Locate the centroid of the cross section and determine the moment of inertia and radius of gyration about an axis parallel to the x-axis and passing through the centroid.

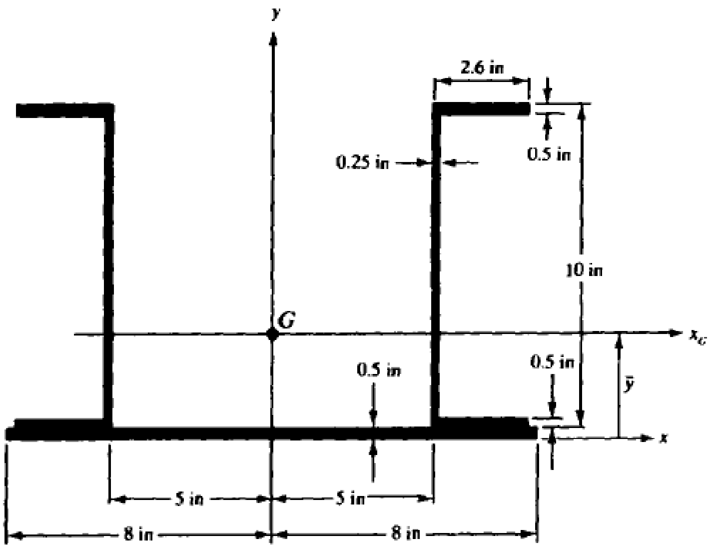


Fig. 7-16

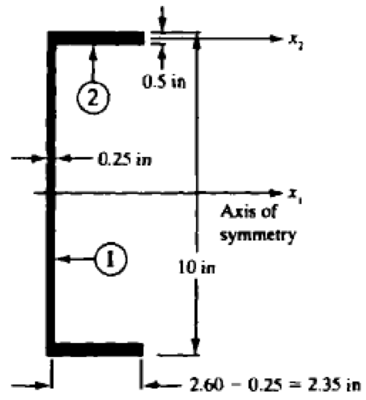


Fig. 7-17

Let us first consider a single channel section, as shown in Fig. 7-17. The area of the cross section is

$$A = 2(\tfrac{1}{2})(2.60 - 0.25) + 10(\tfrac{1}{4}) = 4.85\ \text{in}^2$$

and from Problem 7.4 together with the parallel-axis theorem we have the moment of inertia of the channel about an axis parallel to the x-axis and passing through the centroid of the channel (the x_1-axis) as

$$\begin{aligned} I_{ch} &= \underset{①}{\tfrac{1}{12}(\tfrac{1}{4})(10)^3} + 2\{\underset{②}{\tfrac{1}{12}(2.35)(\tfrac{1}{2})^3} + \underset{③}{(2.35)(\tfrac{1}{2})(5 - \tfrac{1}{4})^2}\} \\ &= 73.90\ \text{in}^4 \end{aligned}$$

where term ① corresponds to the moment of inertia of the vertical rectangle about the x_1-axis, term ② corresponds to the moment of inertia of one horizontal rectangle about the x_2-axis through the centroid of the horizontal rectangle, and term ③ indicates the transfer term from the parallel axis theorem to pass from axis x_2 to axis x_1.

Now, we may write the moment of inertia of the entire assembly about the x-axis by applying the result of Problem 7.6 to the cover plate and applying the parallel axis theorem to I_{ch} to obtain

$$I_x = \tfrac{1}{3}(16)(\tfrac{1}{2})^3 + 2\{73.87 + 4.85(5.5)^2\} = 441.8\ \text{in}^4$$

The centroid of the cross section of the entire assembly is determined from the definition

$$\bar{y} = \frac{\Sigma\, y\, da}{A}$$

$$= \frac{\underset{③}{(16)(\tfrac{1}{2})(\tfrac{1}{4})} + \underset{④}{2[(4.85)(5.5)]}}{(16)(\tfrac{1}{2}) + 2[4.85]} = 3.13\ \text{in}$$

where the terms represented by ③ correspond to the horizontal cover plate and the terms numbered ④ correspond to the channels.

Now that we have located the centroidal axis x_G of the assembly, we may employ the parallel-axis theorem to transfer from the x- to the x_G-axis:

$$I_x = I_{x_G} + A(\bar{y})^2$$

$$441.8\ \text{in}^4 = I_{x_G} + (17.76\ \text{in}^2)(3.13\ \text{in})^2$$

$$I_{x_G} = 268.48\ \text{in}^4$$

The corresponding radius of gyration is

$$r_{x_G} = \sqrt{\frac{I_{x_G}}{A}} = \sqrt{\frac{268.48}{17.76}} = 3.89\ \text{in}$$

7.12. A plane section is in the form of an equilateral triangle, 200 mm on a side. From it is removed another equilateral triangle in such a manner that the width of the remaining section is 30 mm measured perpendicular to the sides of both equilateral triangles, as shown in Fig. 7-18. Determine the location of the centroid of the remaining (shaded) area as well as the moment of inertia about the axis through the centroid and parallel to the x-axis.

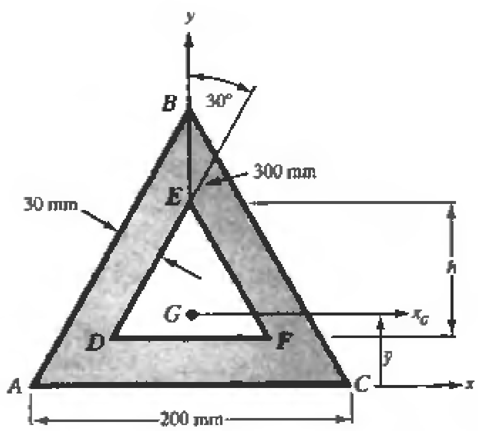


Fig. 7-18

It is necessary to determine the size of the inner triangle that has been removed. From the geometry of Fig. 7-18 it is evident that $BE = 60$ mm because of the 30° angle between BE and BC. Thus the altitude h of the "removed" triangle DEF is

$$h = 200\cos 30 - 30 - 60 = 83.21\ \text{mm}$$

The length of a side of this triangle is

$$DF = \frac{83.21}{0.866} = 96.08\ \text{mm}$$

From symmetry the centroid lies on the y-axis and its location is found by the definition

$$\bar{y} = \frac{\int y\,da}{A} \qquad \text{or} \qquad \frac{\Sigma y\,dA}{A}$$

where the numerator represents the first moment of the area about the x-axis. Using the known location of the centroid of a triangle and its area, as given in the summary at the beginning of this chapter, we have

$$\bar{y} = \frac{\frac{1}{2}(200)(200\cos 30)(\frac{200}{3}\cos 30) - \frac{1}{2}(96.08)(83.21)\{30 + 83.21/3\}}{\frac{1}{2}(200)(200\cos 30) - \frac{1}{2}(96.08)(83.21)}$$

$$= 57.72\ \text{mm}$$

To determine the moment of inertia of the shaded area in Fig. 7-18, we begin by finding the moment of inertia of that area about the x-axis. This is accomplished by taking the moment of inertia of the outer triangle ABC about the x-axis using the result of Problem 7.7, then subtracting the moment of inertia of the inner triangle DEF about that same axis. This latter value is calculated by first determining the moment of inertia of DEF about an axis through the centroid of DEF using the result of Problem 7.8, then employing the parallel-axis theorem to transfer that value to the x-axis. Thus,

$$I_x = \tfrac{1}{12}(200)(200\cos 30)^3 - \{\tfrac{1}{36}(96.08)(83.21)^3 + \tfrac{1}{2}(96.08)(83.21)[30 + 83.21/3]^2\}$$
$$= 71.74 \times 10^6\ \text{mm}^4$$

Utilizing the parallel-axis theorem, we have

$$I_x = I_{x_G} + A(\bar{y})^2$$
$$71.74 \times 10^6\ \text{mm}^4 = I_{x_G} + \{\tfrac{1}{2}(200)(200\cos 30) - \tfrac{1}{2}(96.08)(83.21)\}(57.72\ \text{mm})^2$$
$$I_{x_G} = 27.35 \times 10^6\ \text{mm}^4$$

7.13. Determine the product of inertia of a rectangle with respect to the x- and y-axes indicated in Fig. 7-19.

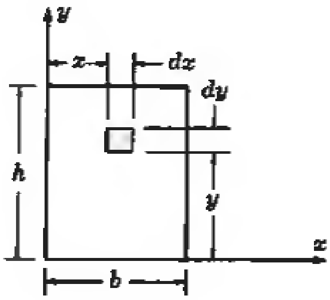


Fig. 7-19

We employ the definition $I_{xy} = \int xy\,da$ and consider the shaded element shown. Integrating,

$$I_{xy} = \int_{y=0}^{y=h}\int_{x=0}^{x=b} xy\,dx\,dy = \int_{y=0}^{y=h}\left[\frac{x^2}{2}\right]_0^b y\,dy$$
$$= \frac{b^2}{2}\left[\frac{y^2}{2}\right]_0^h = \frac{b^2h^2}{4} \tag{1}$$

7.14. Derive the parallel-axis theorem for product of inertia of a plane area.

In Fig. 7-20, the axes x_G and y_G pass through the centroid of the area A. The axes x and y are located the known distances y_1 and x_1, respectively, from the axes through the centroid.

For the element of area da the product of inertia with respect to the x- and y-axes is given by

$$dI_{xy} = (x_1 + x')(y_1 + y')\,dx\,dy$$

For the entire area the product of inertia with respect to the x- and y-axes becomes

$$I_{xy} = \int dI_{xy} = \iint (x_1 + x')(y_1 + y')\,dx\,dy$$
$$= \iint x_1 y_1\,dx\,dy + \iint x' y_1\,dx\,dy + \iint x_1 y'\,dx\,dy + \iint x' y'\,dx\,dy$$

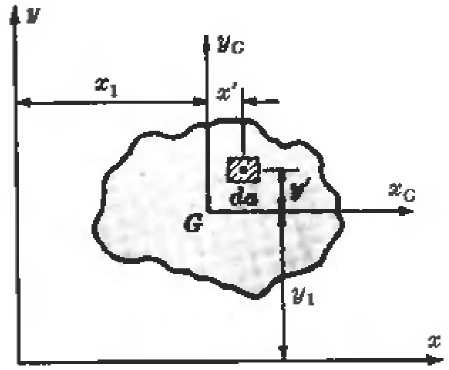


Fig. 7-20

The first integral on the right side equals $x_1 y_1 A$ since x_1 and y_1 are constants. The second and third integrals vanish because x' and y' are measured from the axes through the centroid of the area A. The fourth integral is equal to $I_{x_G y_G}$, that is, the product of inertia of the area with respect to axes through its centroid and parallel to the x- and y-axes. Thus, we have

$$I_{xy} = x_1 y_1 A + I_{x_G y_G} \qquad (1)$$

This is the parallel-axis theorem for product of inertia of a plane area. It is to be noted that the x_G- and y_G-axes must pass through the centroid of the area. Also, x_1 and y_1 are positive only when the x- and y-coordinates have the location relative to the x_G-y_G system indicated in Fig. 7-20. Thus, care must be taken with regard to the algebraic signs of x_1 and y_1.

7.15. Determine I_{xy} for the angle section indicated in Fig. 7-21.

The area may be divided into the component rectangles as shown. For rectangle 1 we have, from (*1*) of Problem 7.13,

$$(I_{xy})_1 = \tfrac{1}{4}(10)^2(125)^2 = 39 \times 10^4 \text{ mm}^4$$

For rectangle 2 we employ (*1*) of Problem 7.14. The product of inertia of rectangle 2 about axes through its centroid and parallel to the x- and y-axes vanishes because these are axes of symmetry. Thus, for rectangle 2, $I_{x_G} = 0$. The parallel-axis theorem of Problem 7.14 thus becomes

$$(I_{xy})_2 = (42.5)(5)(65)(10) = 13.8 \times 10^4 \text{ mm}^4$$

For the entire angle section we thus have

$$I_{xy} = 39 \times 10^4 + 13.8 \times 10^4 = 52.8 \times 10^4 \text{ mm}^4$$

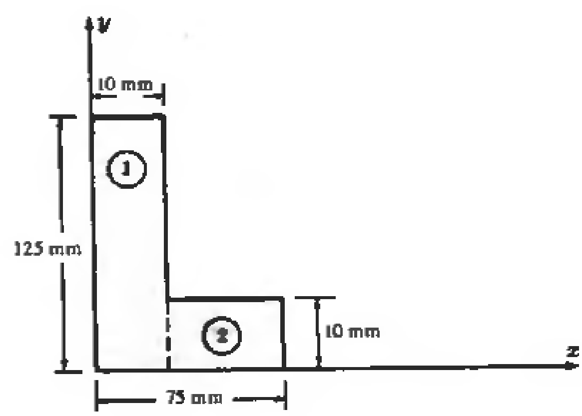


Fig. 7-21

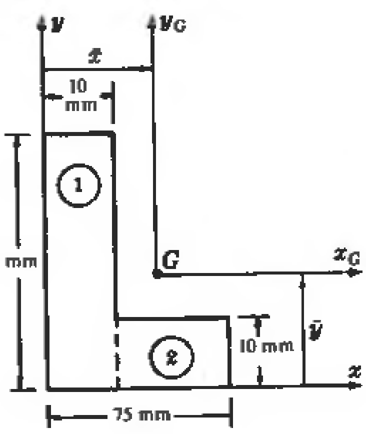


Fig. 7-22

7.16. Determine the product of inertia of the angle section of Problem 7.15 with respect to axes parallel to the x- and y-axes and passing through the centroid of the angle section. See Fig. 7-22.

It is first necessary to locate the centroid of the area, that is, we must find $\bar{x}$ and $\bar{y}$. We have

$$\bar{x} = \frac{125(10)(5) + 65(10)(42.5)}{125(10) + 65(10)} = 17.8\text{ mm}$$

$$\bar{y} = \frac{125(10)(62.5) + 65(10)(5)}{125(10) + 65(10)} = 42.8\text{ mm}$$

Now we employ the parallel-axis theorem of Problem 7.13; that is,

$$I_{xy} = x_1 y_1 A + I_{x_G y_G}$$

In Problem 7.15 we found $I_{xy} = 52.8 \times 10^4\text{ mm}^4$. Thus

$$52.8 \times 10^4 = 17.8(42.8)(1900) + I_{x_G y_G}$$

whence

$$I_{x_G y_G} = -92 \times 10^4\text{ mm}^4$$

7.17. Consider a plane area A and assume that I_x, I_y, and I_{xy} are known. Determine the moments of inertia I_{x_1} and I_{y_1} as well as the product of inertia $I_{x_1 y_1}$ for the set of orthogonal axes x_1-y_1 oriented as shown in Fig. 7-23. Determine also the maximum and minimum values of I_{x1}.

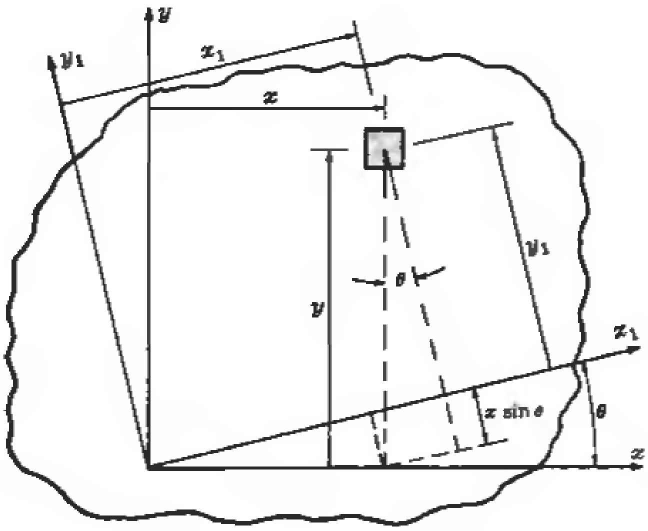


Fig. 7-23

The moment of inertia of the area with respect to the x_1-axis is

$$\begin{aligned} I_{x_1} &= \int y_1^2\,da = \int (y\cos\theta - x\sin\theta)^2\,da \\ &= \cos^2\theta \int y^2\,da + \sin^2\theta \int x^2\,da - 2\sin\theta\cos\theta \int xy\,da \\ &= I_x\cos^2\theta + I_y\sin^2\theta - 2I_{xy}\sin\theta\cos\theta \\ &= I_x\left(\frac{1+\cos 2\theta}{2}\right) + I_y\left(\frac{1-\cos 2\theta}{2}\right) - I_{xy}\sin 2\theta \end{aligned}$$

Or

$$I_{x_1} = \left(\frac{I_x + I_y}{2}\right) + \left(\frac{I_x - I_y}{2}\right)\cos 2\theta - I_{xy}\sin 2\theta \qquad (1)$$

Analogously, I_{y_1} may be obtained from (1) by replacing θ by $\theta + \pi/2$ to yield

$$I_{y_1} = \left(\frac{I_x + I_y}{2}\right) - \left(\frac{I_x - I_y}{2}\right)\cos 2\theta + I_{xy}\sin 2\theta \qquad (2)$$

The value of θ that renders I_{x_1} maximum or minimum is found by setting the derivative of Eq. (1) with respect to θ equal to zero. Thus, since I_x, I_y, and I_{xy} are constants we have from (1)

$$\frac{dI_{x_1}}{d\theta} = -(I_x - I_y)\sin 2\theta - 2I_{xy}\cos 2\theta = 0$$

Solving,

$$\tan 2\theta = -\frac{I_{xy}}{\left(\dfrac{I_x - I_y}{2}\right)} \qquad (3)$$

Fig. 7-24

Equation (3) has the convenient graphical interpretation shown in Cases I and II of Fig. 7-24.

If now the values of 2θ given by (3) are substituted into (1), we obtain

$$(I_{x_1})_{\substack{\max\\ \min}} = \left(\frac{I_x + I_y}{2}\right) \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2} \qquad (4)$$

where the positive sign refers to Case I and the negative sign to Case II. These maximum and minimum values of moment of inertia correspond to axes defined by (3). The maximum and minimum values of moment of inertia are termed *principal moments of inertia* and the corresponding axes are termed *principal axes*.

We may now determine $I_{x_1y_1}$ from

$$\begin{aligned}
I_{x_1y_1} &= \int x_1 y_1\, da \\
&= \int (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)\, da \\
&= \cos^2\theta \int xy\, da - \sin^2\theta \int xy\, da \\
&\quad + \sin\theta\cos\theta \int y^2\, da - \sin\theta\cos\theta \int x^2\, da \\
&= I_{xy}(\cos^2\theta - \sin^2\theta) + (I_x - I_y)\sin\theta\cos\theta \\
&= \left(\frac{I_x - I_y}{2}\right)\sin 2\theta + I_{xy}\cos 2\theta \qquad (5)
\end{aligned}$$

From (5), $I_{x_1y_1}$ vanishes if

$$\tan 2\theta = -\frac{I_{xy}}{\left(\frac{I_x - I_y}{2}\right)}$$

which is identical to condition (3). Since (3) defined principal axes, it follows that the product of inertia vanishes for principal axes.

7.18. A structural aluminum 6 Z 5.42 section has the nominal dimensions indicated in Fig. 7-25. Determine I_x, I_y, I_{xy} and also the maximum and minimum values of the moment of inertia with respect to axes through the point O.

The section may be divided into the component rectangles ①, ②, and ③ as indicated. The result obtained in Problem 7.4, together with the parallel-axis theorem given in Problem 7.5, may be used to determine I_x and I_y:

$$I_x = \tfrac{1}{12}(\tfrac{3}{8})(6)^3 + 2[\tfrac{1}{12}(3\tfrac{1}{8})(\tfrac{3}{8})^3 + (3\tfrac{1}{8})(\tfrac{3}{8})(2\tfrac{13}{16})^2] = 25.27\ \text{in}^4$$

$$I_y = \tfrac{1}{12}(6)(\tfrac{3}{8})^3 + 2[\tfrac{1}{12}(\tfrac{3}{8})(3\tfrac{1}{8})^3 + (\tfrac{3}{8})(3\tfrac{1}{8})(1\tfrac{3}{4})^2] = 9.08\ \text{in}^4$$

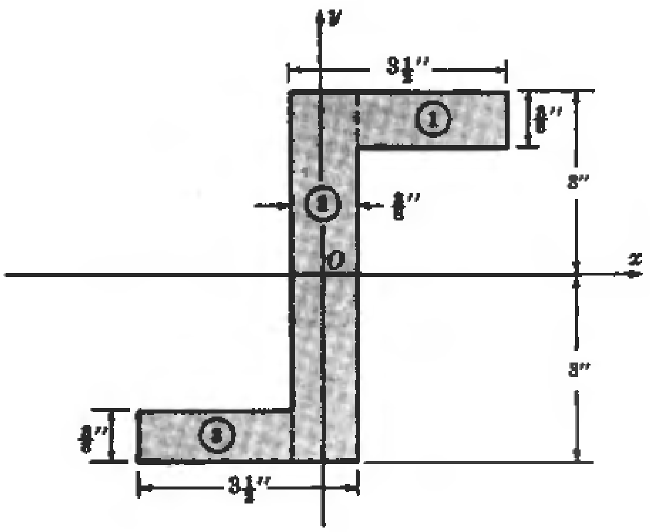


Fig. 7-25

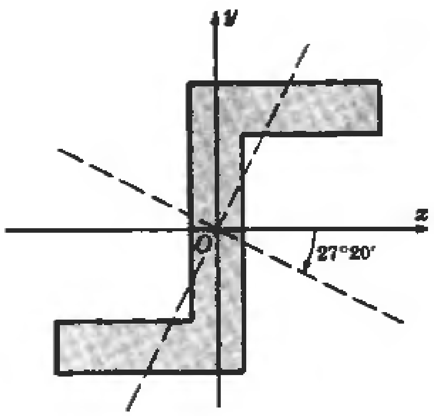


Fig. 7-26

The product of inertia with respect to the x- and y-axes may be determined through use of the parallel-axis theorem for product of inertia as given in Problem 7.14. It is to be noted that the product of inertia of each of the component rectangles about axes through the centroid of each component and parallel to the x- and y-axes vanishes because these are axes of symmetry. Hence, from (1) of Problem 7.14 we have for the entire Z-section

$$I_{xy} = 2[(\tfrac{7}{4})(2\tfrac{13}{16})(3\tfrac{1}{8})(\tfrac{3}{8})] = 11.6\ \text{in}^4$$

The maximum and minimum values of moment of inertia with respect to axes through the point O may be found from (4) of Problem 7.17. From that equation

$$(I_{x_1})_{\substack{\max\\ \min}} = \left(\frac{I_x + I_y}{2}\right) \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2}$$

$$= \left(\frac{25.27 + 9.08}{2}\right) \pm \sqrt{\left(\frac{25.27 - 9.08}{2}\right)^2 + (11.6)^2}$$

$$(I_{x_1})_{\max} = 31.38\ \text{in}^4 \qquad (1)$$

$$(I_{x_1})_{\min} = 2.98\ \text{in}^4 \qquad (2)$$

The orientation of these principal moments of inertia is found from (3) of Problem 7.17 to be

$$\tan 2\theta = -\frac{I_{xy}}{\left(\frac{I_x - I_y}{2}\right)}$$

$$= -\frac{11.6}{\left(\frac{25.27 - 9.08}{2}\right)}$$

$$\theta = -27°20', \qquad -117°20' \tag{3}$$

The principal moments of inertia given in (1) and (2) correspond to the principal axes given by (3). These principal axes are represented by the dashed lines in Fig. 7-26.

Supplementary Problems

7.19. The structural channel section has welded to it a horizontal reinforcing plate as shown in cross section in Fig. 7-27. Determine the y-coordinate of the centroid of the composite section. *Ans.* $\bar{y} = 4.56$ in

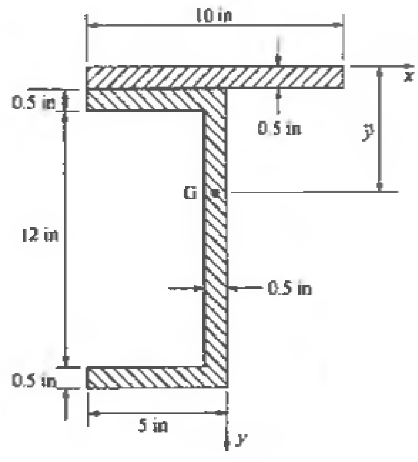


Fig. 7-27

7.20. The shaded area shown in Fig. 7-28 is bounded by a circular arc and a chord. Determine the location of the centroid of the area with respect to the center of the circular arc.

Ans. $\bar{y} = \frac{4R}{3} \cdot \frac{(\sin^3 \theta)}{(2\theta - \sin 2\theta)}$

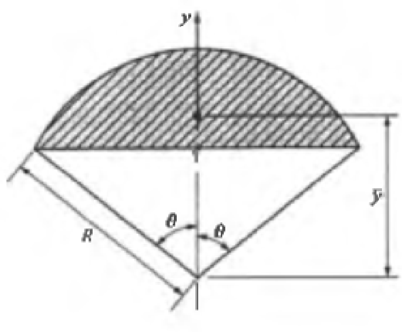


Fig. 7-28

7.21. An area consists of a circle of radius R from which a rectangle of dimensions $a \times 3a$ has been removed, as shown in Fig. 7-29. Determine the moment of inertia of the shaded area about the x- and also the y-axes.

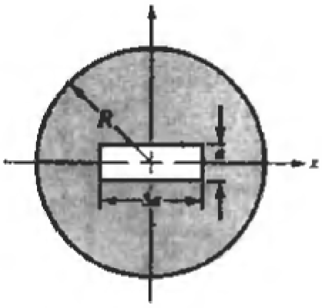


Fig. 7-29

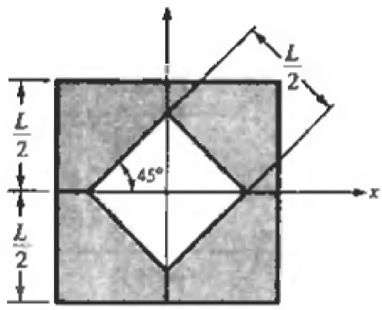


Fig. 7-30

Ans. $I_x = \dfrac{\pi R^4}{4} - \dfrac{a^4}{4}$, $\quad I_y = \dfrac{\pi R^4}{4} - \dfrac{9a^4}{4}$

7.22. The shaded area in Fig. 7-30 results from removing the central square from the outer square. Determine the moment of inertia of the net area about the x-axis. *Ans.* $I_x = 0.0781L^4$

7.23. A thin rectangular sheet has semicircular and also triangular areas removed, as shown in Fig. 7-31. Locate the centroid of the sheet and determine the moment of inertia about the horizontal axis passing through the centroid. *Ans.* $\bar{y} = 370.8$ mm, $I_{x_G} = 9937 \times 10^6\ \text{mm}^4$

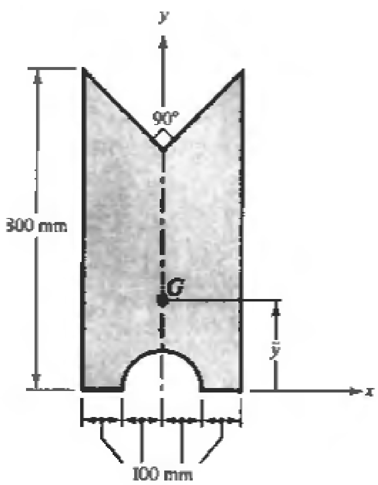


Fig. 7-31

7.24. A trapezoidal area has the dimensions indicated in Fig. 7-32. Determine the location of the centroid as well as the moment of inertia about an axis through the centroid and parallel to the x-axis.
Ans. $\bar{y} = 44.4$ mm. $I_{x_G} = 24.14 \times 10^6\ \text{mm}^4$

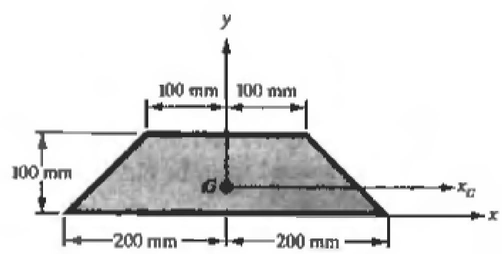


Fig. 7-32

7.25. A thin-walled section ($t \ll a$) has the configuration indicated in Fig. 7-33. Locate the centroid of the cross section and determine the moment of inertia of the area about an axis passing through the centroid and parallel to the x-axis. *Ans.* $\bar{y} = a$, $I_{x_G} = 5.33a^3t + at^3/6$

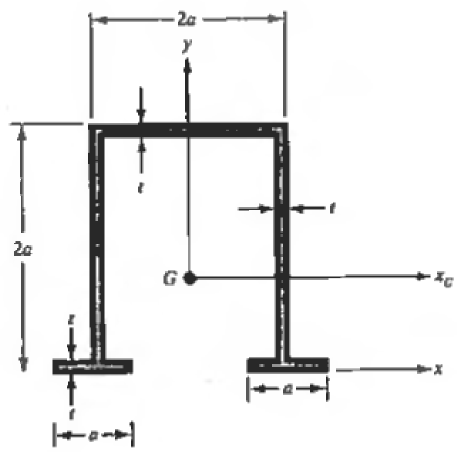


Fig. 7-33

7.26. An area of circular cross section from which three circular holes have been removed is shown in Fig. 7-34. Determine the location of the centroid of the section and the moment of inertia of an axis passing through the centroid and parallel to the x-axis. *Ans.* $\bar{y} = -R/10$, $I_{x_G} = 0.737R^4$

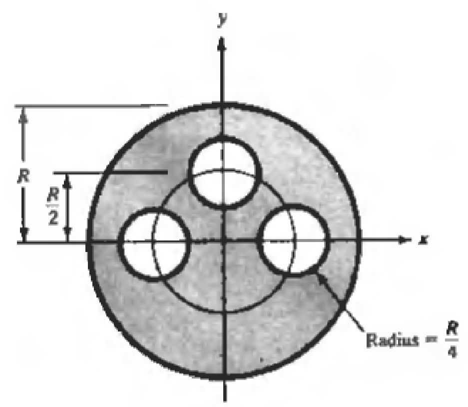


Fig. 7-34

7.27. Determine the moment of inertia of the diamond-shaped figure shown in Fig. 7-35 with respect to the horizontal axis of symmetry. *Ans.* $I_{x_G} = 85.4 \text{ in}^4$

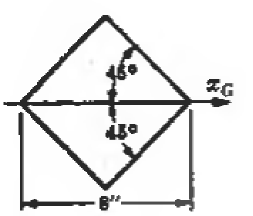


Fig. 7-35

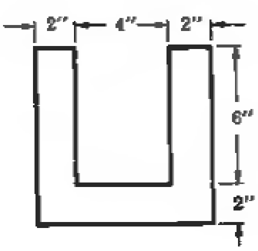


Fig. 7-36

7.28. Determine the moment of inertia of a channel-type section about a horizontal axis through the centroid. Refer to Fig. 7-36. What is the radius of gyration about this same axis?
Ans. $I_{x_G} = 231\ \text{in}^4$, $r_{x_G} = 2.40$ in

7.29. Locate the centroid of the channel-type section shown in Fig. 7-37 and determine the moment of inertia of the cross-sectional area about a horizontal axis through the centroid.
Ans. $\bar{y} = 38.33$ mm, $I_{x_G} = 33 \times 10^6\ \text{mm}^4$

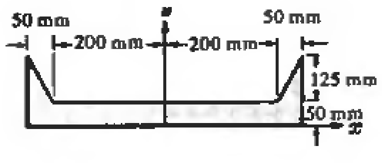


Fig. 7-37

7.30. A plane area has the shape of a parallelogram as shown in Fig. 7-38. The y- and z-axes pass through the centroid of the area. Determine I_y and I_z. *Ans.* $I_y = \frac{1}{12}bh^3$, $I_z = \frac{1}{12}hb(b^2 + c^2)$

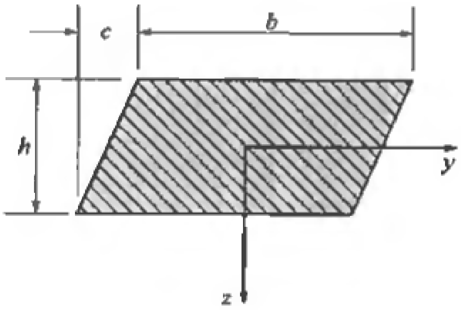


Fig. 7-38

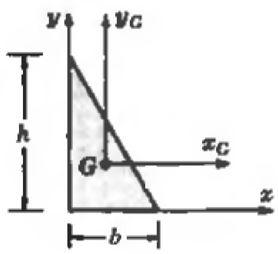


Fig. 7-39

7.31. Determine the product of inertia of a triangle with respect to the x- and y-axes indicated in Fig. 7-39.
Ans. $b^2h^2/24$

7.32. Determine the product of inertia of the triangle shown in Fig. 7-39 with respect to the axes x_G and y_G passing through the centroid. *Ans.* $-b^2h^2/72$

7.33. For the plane area in Fig. 7-40 determine the moments of inertia and product of inertia with respect to the x_G- and y_G-axes passing through the centroid. Also, determine the principal second moments of area with respect to the centroid.
Ans. $I_{x_G} = 400 \times 10^6\ \text{mm}^4$; $I_{y_G} = 147 \times 10^6\ \text{mm}^4$; $I_{x_G y_G} = -58 \times 10^6\ \text{mm}^4$; $(I_{x_1})_{max} = 805 \times 10^6\ \text{mm}^4$; $(I_{x_1})_{min} = 142 \times 10^6\ \text{mm}^4$

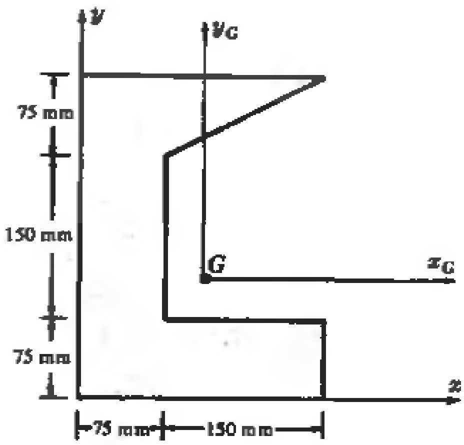


Fig. 7-40

Chapter 8

Stresses in Beams

TYPES OF LOADS ACTING ON BEAMS

Either forces or couples that lie in a plane containing the longitudinal axis of the beam may act upon the member. The forces are understood to act perpendicular to the longitudinal axis, and the plane containing the forces is assumed to be a plane of symmetry of the beam.

EFFECTS OF LOADS

The effects of these forces and couples acting on a beam are (*a*) to impart deflections perpendicular to the longitudinal axis of the bar and (*b*) to set up both normal and shearing stresses on any cross section of the beam perpendicular to its axis. Beam deflections will be considered in Chaps. 9, 10, and 11.

TYPES OF BENDING

If couples are applied to the ends of the beam and no forces act on the bar, then the bending is termed *pure bending*. For example, in Fig. 8-1 the portion of the beam between the two downward forces is subject to pure bending. Bending produced by forces that do not form couples is called *ordinary bending*. A beam subject to pure bending has only normal stresses with no shearing stresses set up in it; a beam subject to ordinary bending has both normal and shearing stresses acting within it.

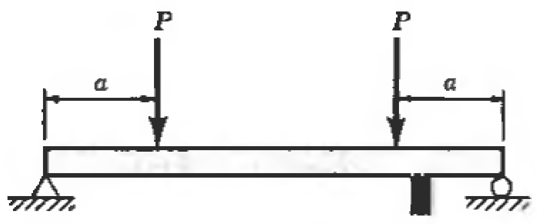


Fig. 8-1

NATURE OF BEAM ACTION

It is convenient to imagine a beam to be composed of an infinite number of thin longitudinal rods or fibers. Each longitudinal fiber is assumed to act independently of every other fiber, i.e., there are no lateral pressures or shearing stresses between the fibers. The beam of Fig. 8-1, for example, will deflect downward and the fibers in the lower part of the beam undergo extension, while those in the upper part are shortened. These changes in the lengths of the fibers set up stresses in the fibers. Those that are extended have tensile stresses acting on the fibers in the direction of the longitudinal axis of the beam, while those that are shortened are subject to compressive stresses.

NEUTRAL SURFACE

There always exists one surface in the beam containing fibers that do not undergo any extension or compression, and thus are not subject to any tensile or compressive stress. This surface is called the *neutral surface* of the beam.

NEUTRAL AXIS

The intersection of the neutral surface with any cross section of the beam perpendicular to its longitudinal axis is called the *neutral axis*. All fibers on one side of the neutral axis are in a state of tension, while those on the opposite side are in compression.

BENDING MOMENT

The algebraic sum of the moments of the external forces to one side of any cross section of the beam about an axis through that section is called the *bending moment* at that section. This concept was discussed in Chap. 6.

ELASTIC BENDING OF BEAMS

The following remarks apply *only if* all fibers in the beam are acting within the elastic range of action of the material.

Normal Stresses in Beams

For any beam having a longitudinal plane of symmetry and subject to a bending moment M at a certain cross section, the normal stress acting on a longitudinal fiber at a distance y from the neutral axis of the beam (see Fig. 8-2) is given by

$$\sigma = \frac{My}{I} \tag{8.1}$$

where I denotes the moment of inertia of the cross-sectional area about the neutral axis. This quantity was discussed in Chap. 7. The derivation of this equation is discussed in detail in Problem 8.1. For applications see Problems 8.2 through 8.18. These stresses vary from zero at the neutral axis of the beam to a maximum at the outer fibers as shown. The stresses are tensile on one side of the neutral axis, compressive on the other. These stresses are also called *bending*, *flexural*, or *fiber stresses*.

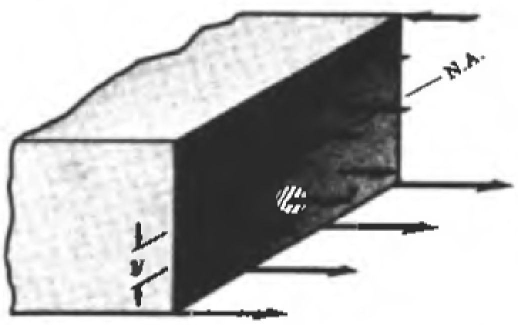


Fig. 8-2

Location of the Neutral Axis

When the beam action is entirely elastic the neutral axis passes through the centroid of the cross section. Hence, the moment of inertia I appearing in the above equation for normal stress is the moment of inertia of the cross-sectional area about an axis through the centroid of the cross section of the beam.

Section Modulus

At the outer fibers of the beam the value of the coordinate y is frequently denoted by the symbol c. In that case the maximum normal stresses are given by

$$\sigma = \frac{Mc}{I} \quad \text{or} \quad \sigma = \frac{M}{I/c} \tag{8.2}$$

The ratio I/c is called the *section modulus* and is usually denoted by the symbol Z. The units are in^3 or m^3. The maximum bending stresses may then be represented as

$$\sigma = \frac{M}{Z} \tag{8.3}$$

This form is convenient because values of Z are available in handbooks for a wide range of standard structural steel shapes. See Problems 8.5, 8.9, and 8.12.

Assumptions

In the derivation of the above expression for normal stresses it is assumed that a plane section of the beam normal to its longitudinal axis prior to loading remains plane after the forces and couples have been applied. Further, it is assumed that the beam is initially straight and of uniform cross section and that the moduli of elasticity in tension and compression are equal. Again, it is to be emphasized that no fibers of the beam are stressed beyond the proportional limit.

Shearing Force

The algebraic sum of all the vertical forces to one side of any cross section of the beam is called the shearing force at that section. This concept was discussed in Chap. 6.

Shearing Stresses in Beams

For any beam subject to a shearing force V (expressed in pounds) at a certain cross section, both vertical and horizontal shearing stresses τ are set up. The magnitudes of the vertical shearing stresses at any cross section are such that these stresses have the shearing force V as a resultant. In the cross section of the beam shown in Fig. 8-3, the vertical plane of symmetry contains the applied forces and the neutral axis passes through the centroid of the section. The coordinate y is measured from the neutral axis. The moment of inertia of the *entire* cross-sectional area about the neutral axis is denoted by I. The shearing stress on all fibers a distance y_0 from the neutral axis is given by the formula

$$\tau = \frac{V}{Ib}\int_{y_0}^{c} y\,da \tag{8.4}$$

where b denotes the width of the beam at the location where the shearing stress is being calculated. This expression is derived in Problem 8.19. For applications see Problems 8.20 through 8.23. The integral in (*8.4*) represents the first moment of the shaded area of the cross section about the neutral axis. This quantity was discussed in detail in Chap. 7. More generally, the integral always represents

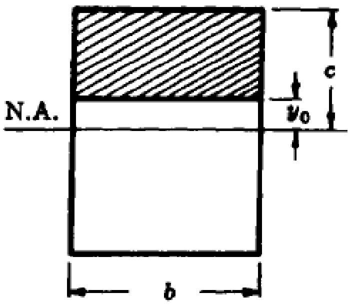


Fig. 8-3

the first moment about the neutral axis of that part of the cross-sectional area of the beam between the horizontal plane on which the shearing stress τ occurs and the outer fibers of the beam, i.e., the area between y_0 and c.

From (*8.4*) it is evident that the maximum shearing stress always occurs at the neutral axis of the beam, whereas the shearing stress at the outer fibers is always zero. In contrast, the normal stress varies from zero at the neutral axis to a maximum at the outer fibers.

In a beam of rectangular cross section the above equation for shearing stress becomes

$$\tau = \frac{V}{2I}\left(\frac{h^2}{4} - y_0^2\right) \tag{8.5}$$

where τ denotes the shearing stress on a fiber at a distance y_0 from the neutral axis and h denotes the depth of the beam. The distribution of vertical shearing stress over the rectangular cross section is thus parabolic, varying from zero at the outer fibers to a maximum at the neutral axis. For application see Problems 8.20 through 8.23.

Both the above equations for shearing stress give the vertical and also the horizontal shearing stresses at a point, as discussed in Problem 8.19, since the intensities of shearing stresses in these two directions are always equal.

PLASTIC BENDING OF BEAMS

The following remarks apply if some or all of the fibers of the beam are stressed to the yield point of the material.

We shall consider a simplified stress-strain curve such as that of Fig. 8-4, where it is assumed that the proportional limit and the yield point coincide. The yield region, i.e., the horizontal plateau of the curve, is assumed to extend indefinitely. This conventionalized representation of ductile material behavior is termed *elastic-perfectly plastic* behavior. Here, σ_{yp} denotes the yield point of the material and ϵ_{yp} represents the strain corresponding to that stress. We shall assume that material properties are identical in tension and compression.

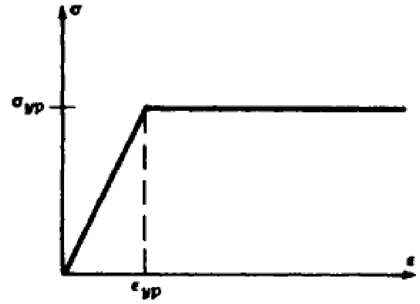


Fig. 8-4

Elastoplastic Action

For sufficiently large bending moments in a beam the interior fibers will be stressed in the elastic range of action, whereas the outer fibers will have reached the yield point of the material. Such a stress distribution may be as indicated in Fig. 8-5.

Fully Plastic Action

As bending moments continue to increase, a limiting case is approached in which all fibers are stressed to the yield point of the material. This stress distribution appears in Fig. 8-6.

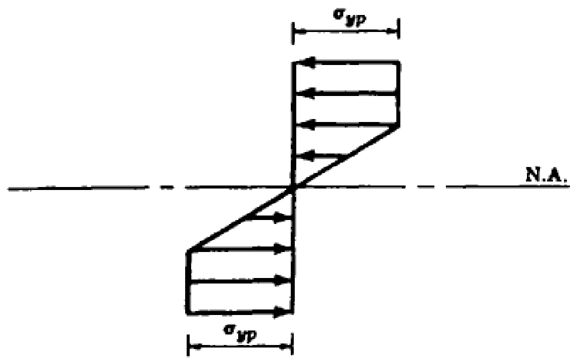


Fig. 8-5

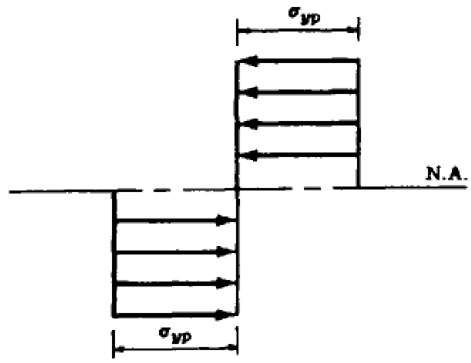


Fig. 8-6

Location of Neutral Axis

When beam action is entirely elastic, the neutral axis passes through the centroid of the cross section. However, as plastic action spreads from the outer fibers inward, the neutral axis shifts from this location to another, which is determined by realizing that the resultant normal force over any cross section vanishes. In the limiting case of fully plastic action, the neutral axis assumes a position such that the total cross-sectional area is divided into two equal parts. This is discussed in Problem 8.29.

Fully Plastic Moment

The bending moment corresponding to fully plastic action is termed the *fully plastic moment* and will be denoted by M_p. For the stress-strain diagram assumed here no greater moment can be developed.

For a beam of rectangular cross section the fully plastic moment is shown in Problem 8.25 to be $M_p = bh^2\sigma_{yp}/4$ where b represents the width of the beam and h its depth.

Solved Problems

Elastic Bending of Beams

8.1. Derive an expression for the relationship between the bending moment acting at any section in a beam and the bending stress at any point in this same section. Assume Hooke's law holds.

The beam shown in Fig. 8-7(a) is loaded by the two couples M and consequently is in static equilibrium. Since the bending moment has the same value at all points along the bar, the beam is said to be in a condition of *pure bending*. To determine the distribution of bending stress in the beam, let us cut the beam by a plane passing through it in a direction perpendicular to the geometric axis of the bar. In this manner the forces under investigation become external to the new body formed, even though they were internal effects with regard to the original uncut body.

Fig. 8-7

The free-body diagram of the portion of the beam to the left of this cutting plane now appears as in Fig. 8-7(*b*). Evidently a moment M must act over the cross section cut by the plane so that the left portion of the beam will be in static equilibrium. The moment M acting on the cut section represents the effect of the right portion of the beam on the left portion. Since the right portion has been removed, it must be replaced by its effect on the left portion and this effect is represented by the moment M. This moment is the resultant of the moments of forces acting perpendicular to the cut cross section and in the plane of the page. It is now necessary to make certain assumptions in order to determine the nature of the variation of these forces over the cross section.

It is convenient to consider the beam to be composed of an infinite number of thin longitudinal rods or fibers. It is assumed that every longitudinal fiber acts independently of every other fiber; that is, there are no lateral pressures or shearing stresses between adjacent fibers. Thus each fiber is subject only to axial tension or compression. Further, it is assumed that a plane section of the beam normal to its axis before loads are applied remains plane and normal to the axis after loading. Finally, it is assumed that the material follows Hooke's law and that the moduli of elasticity in tension and compression are equal.

Let us next consider two adjacent cross sections *aa* and *bb* marked on the side of the beam, as shown in Fig. 8-8. Prior to loading, these sections are parallel to each other. After the applied moments have acted on the beam, these sections are still planes but they have rotated with respect to each other to the positions shown, where O represents the center of curvature of the beam. Evidently the fibers on the upper surface of the beam are in a state of compression, while those on the lower surface have been extended slightly and are thus in tension. The line *cd* is the trace of the surface in which the fibers do not undergo any strain during bending and this surface is called the *neutral surface*, and its intersection with any cross section is called the *neutral axis*. The elongation of the longitudinal fiber at a distance y (measured positive downward) may be found by drawing line *de* parallel to *aa*. If ρ denotes the radius of curvature of the bent beam, then from the similar triangles *cOd* and *edf* we find the strain of this fiber to be

$$\epsilon = \frac{\overline{ef}}{\overline{cd}} = \frac{\overline{de}}{\overline{cO}} = \frac{y}{\rho} \tag{1}$$

Thus, the strains of the longitudinal fibers are proportional to the distance y from the neutral axis.

Since Hooke's law holds, and therefore $E = \sigma/\epsilon$, or $\sigma = E\epsilon$, it immediately follows that the stresses existing in the longitudinal fibers are proportional to the distance y from the neutral axis, or

$$\sigma = \frac{Ey}{\rho} \tag{2}$$

Let us consider a beam of rectangular cross section, although the derivation actually holds for any cross section which has a longitudinal plane of symmetry. In this case, these longitudinal, or bending, stresses appear as in Fig. 8-9.

Let da represent an element of area of the cross section at a distance y from the neutral axis. The stress acting on da is given by the above expression and consequently the force on this element is the product of the stress and the area da, that is,

$$dF = \frac{Ey}{\rho}\,da \tag{3}$$

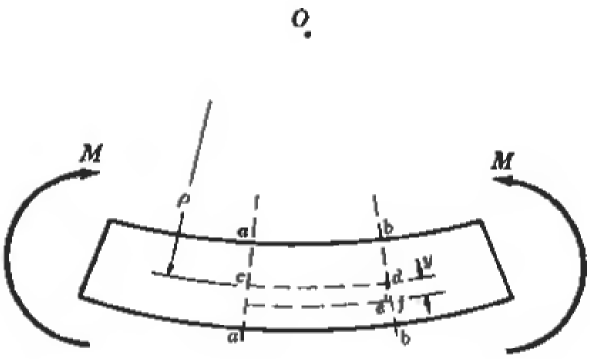


Fig. 8-8

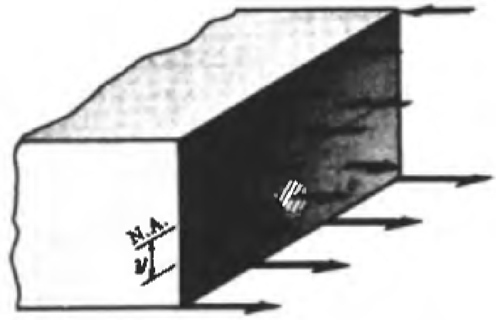


Fig. 8-9

However, the resultant longitudinal force acting over the cross section is zero (for the case of pure bending) and this condition may be expressed by the summation of all forces dF over the cross section. This is done by integration:

$$\int \frac{Ey}{\rho}\, da = \frac{E}{\rho}\int y\, da = 0 \tag{4}$$

Evidently $\int y\, da = 0$. However, this integral represents the first moment of the area of the cross section with respect to the neutral axis, since y is measured from that axis. But, from Chap. 7 we may write $\int y\, da = \bar{y}A$, where $\bar{y}$ is the distance from the neutral axis to the centroid of the cross-sectional area. From this, $\bar{y}A = 0$; and since A is not zero, then $\bar{y} = 0$. Thus the neutral axis always passes through the centroid of the cross section, provided Hooke's law holds.

The moment of the elemental force dF about the neutral axis is given by

$$dM = y\, dF = y\left(\frac{Ey}{\rho}\, da\right) \tag{5}$$

The resultant of the moments of all such elemental forces summed over the entire cross section must be equal to the bending moment M acting at that section and thus we may write

$$M = \int \frac{Ey^2}{\rho}\, da \tag{6}$$

But $I = \int y^2\, da$ and thus we have

$$M = \frac{EI}{\rho} \tag{7}$$

It is to be carefully noted that this moment of inertia of the cross-sectional area is computed with respect to the axis through the centroid of the cross section. But previously we had

$$\sigma = \frac{Ey}{\rho} \tag{8}$$

Eliminating ρ from these last two equations, we obtain

$$\sigma = \frac{My}{I} \tag{9}$$

This formula gives the so-called bending or flexural stresses in the beam. In it, M is the bending moment at any section, I the moment of inertia of the cross-sectional area about an axis through the centroid of the cross section, and y the distance from the neutral axis (which passes through the centroid) to the fiber on which the stress σ acts.

The value of y at the outer fibers of the beam is frequently denoted by c. At these fibers the bending stresses are maximum and there we may write

$$\sigma = \frac{Mc}{I} \tag{10}$$

8.2. A beam is loaded by a couple of 12,000 lb · in at each of its ends, as shown in Fig. 8-10. The beam is steel and of rectangular cross section 1 in wide by 2 in deep. Determine the maximum bending stress in the beam and indicate the variation of bending stress over the depth of the beam.

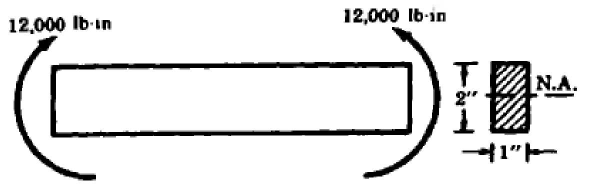


Fig. 8-10

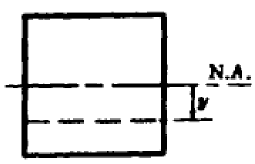


Fig. 8-11

From Problem 8.1, bending takes place about the horizontal neutral axis denoted by N.A. This axis passes through the centroid of the cross section. The moment of inertia of the shaded rectangular cross section about this axis is found by the methods of Chap. 7 to be

$$I = \tfrac{1}{12}bh^3 = \tfrac{1}{12}(1)(2)^3 = 0.667 \text{ in}^4$$

Also from Problem 8.1, the bending stress at a distance y from the neutral axis is given by $\sigma = My/I$, where y is illustrated in Fig. 8-11. Thus, all longitudinal fibers of the beam at the distance y from the neutral axis are subject to the same bending stress given by the above formula.

Since M and I are constant along the length of the bar, evidently the maximum bending stress occurs on those fibers where y takes on its maximum value. These are the fibers along the upper and lower surfaces of the beam, and from inspection it is obvious that for the direction of loading shown the upper fibers are in compression and the lower fibers in tension. For the lower fibers, $y = 1$ in and the maximum bending stress is

$$\sigma = \frac{12{,}000(1)}{0.667} = 18{,}000 \text{ lb/in}^2$$

For the fibers along the upper surface y may be considered to be negative and we have

$$\sigma = \frac{12{,}000(-1)}{0.667} = -18{,}000 \text{ lb/in}^2$$

Thus the peak stresses are 18,000 lb/in^2 in tension for all fibers along the lower surface of the beam and 18,000 lb/in^2 in compression for all fibers along the upper surface. According to the formula $\sigma = My/I$, the bending stress varies linearly from zero at the neutral axis to a maximum at the outer fibers and hence the variation over the depth of the beam may be plotted as in Fig. 8-12.

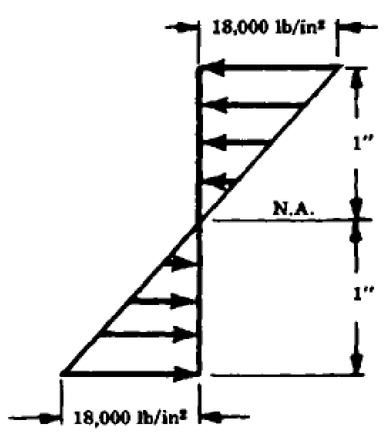


Fig. 8-12

8.3. A beam of circular cross section is 7 in in diameter. It is simply supported at each end and loaded by two concentrated loads of 20,000 lb each, applied 12 in from the ends of the beam. Determine the maximum bending stress in the beam.

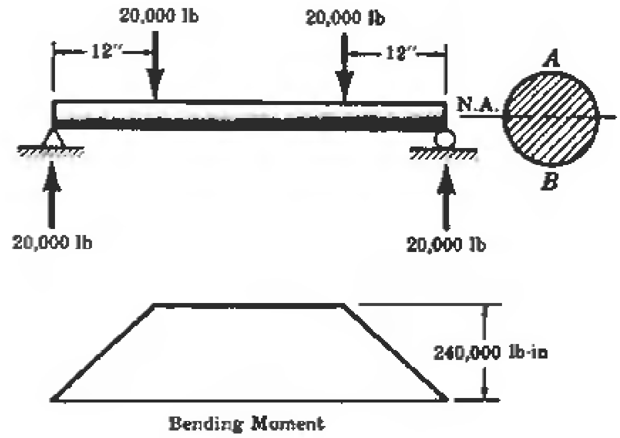


Fig. 8-13

Here the moment is not constant along the length of the beam, as it was in Problem 8.2. The loading is illustrated in Fig. 8-13 together with the bending moment diagram obtained by the methods of Chap. 6. It is to be noted that the portion of the beam between the two downward loads of 20,000 lb is in a condition termed *pure bending* and everywhere in that region the bending moment is equal to $20{,}000(12) = 240{,}000\ \text{lb}\cdot\text{in}$.

From Problem 7.9 the moment of inertia of the shaded circular cross section about the neutral axis, which passes through the centroid of the circle, is $I = \pi D^4/64 = \pi(7)^4/64 = 118\ \text{in}^4$.

The bending stress at a distance y from the horizontal neutral axis shown is $\sigma = My/I$. Evidently the maximum bending stresses occur along the fibers located at the ends of a vertical diameter and designated as A and B. This maximum stress is the same at all such points between the applied loads. At point B, $y = 3.5$ in and the stress becomes

$$\sigma = \frac{240{,}000(3.5)}{118} = 7120\ \text{lb/in}^2\ \text{tension}$$

At point A the stress is $7120\ \text{lb/in}^2$ compression.

8.4. A steel cantilever beam 16 ft 8 in in length is subjected to a concentrated load of 320 lb acting at the free end of the bar. The beam is of rectangular cross section, 2 in wide by 3 in deep. Determine the magnitude and location of the maximum tensile and compressive bending stresses in the beam.

The bending moment diagram for this type of loading, determined by the techniques of Chap. 6, is triangular with a maximum ordinate at the supporting wall, as shown below in Fig. 8-14(a). The maximum bending moment is merely the moment of the 320-lb force about an axis through point B and perpendicular to the plane of the page. It is $-320(200) = -64{,}000\ \text{lb}\cdot\text{in}$.

The bending stress at a distance y from the neutral axis, which passes through the centroid of the cross section, is $\sigma = My/I$ where y is illustrated in Fig. 8-14(b). In this expression I denotes the moment of inertia of the cross-sectional area about the neutral axis and is given by

$$I = \tfrac{1}{12}bh^3 = \tfrac{1}{12}(2)(3)^3 = 4.50\ \text{in}^4$$

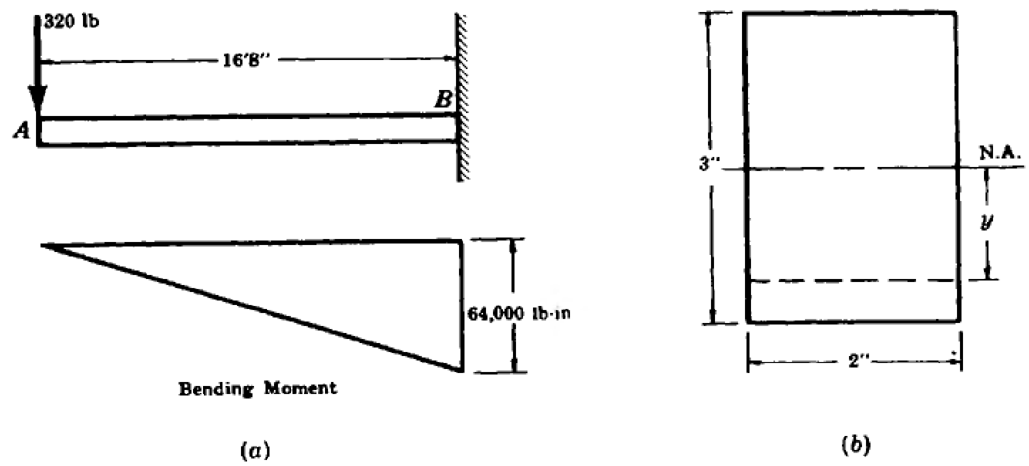


Fig. 8-14

Thus at the supporting wall, where the bending moment is maximum, the peak tensile stress occurs at the upper fibers of the beam and is

$$\sigma = \frac{My}{I} = \frac{(-64{,}000)(-1.5)}{4.50} = 21{,}400 \text{ lb/in}^2$$

It is evident that this stress must be tension because all points of the beam deflect downward. At the lower fibers adjacent to the wall the peak compressive stress occurs and is equal to 21,400 lb/in^2.

8.5. Let us reconsider Problem 8.4 for the case where the rectangular beam is replaced by a commercially available rolled steel section, designated as a W6 × $15\frac{1}{2}$. This standard manner of designation indicates that the depth of the section is 6 in, that it is a so-called wide-flange section, and that it weighs $15\frac{1}{2}$ lb per ft of length. Determine the maximum tensile and compressive bending stresses.

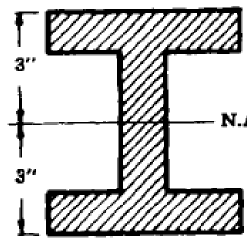


Fig. 8-15

Such a beam has the symmetric cross section shown in Fig. 8-15 and bending takes place about the horizontal neutral axis passing through the centroid. Extensive handbooks listing properties of all available rolled steel shapes are available to designers and abridged tables are presented at the end of this chapter. From that table the moment of inertia about the neutral axis is found to be 28.1 in^4.

The bending stress at a distance y from the neutral axis is given by $\sigma = My/I$. At the outer fibers, $y = c$ and

$$\sigma = \frac{Mc}{I} = \frac{M}{I/c}$$

The ratio I/c is designated as the *section modulus* and is usually denoted by the symbol Z. The units are obviously in^3. From the abridged table we find Z to be 9.7 in^3. Thus if one is concerned only with bending stresses occurring at the outer fibers, which is frequently the case since we are often interested only in maximum stresses, then the section modulus is a convenient quantity to work with, particularly for standard structural shapes.

The stresses in the extreme fibers at the section of the beam immediately adjacent to the wall are thus given by

$$\sigma = \frac{M}{I/c} = \frac{M}{Z} = \frac{64{,}000}{9.7} = 6600 \text{ lb/in}^2$$

Again, since the fibers along the top of the beam are stretching, the stress there will be tension. Along the lower face of the beam the fibers are shortening and there the stress is compressive.

8.6. A cantilever beam 3 m long is subjected to a uniformly distributed load of 30 kN per meter of length. The allowable working stress in either tension or compression is 150 MPa. If the cross section is to be rectangular, determine the dimensions if the height is to be twice as great as the width.

The bending moment diagram for a uniform load acting over a cantilever beam was determined in Problem 6.2. It was found to be parabolic, varying from zero at the free end of the beam to a maximum at the supporting wall. The loaded beam and the accompanying bending moment diagram are shown in Fig. 8-16. The maximum moment at the wall is given by

$$M_{x=3} = -30(3)(1.5) = -135 \text{ kN}\cdot\text{m}$$

It is to be noted that this problem involves the design of a beam, whereas all previous problems in this chapter called for the analysis of stresses acting in beams of known dimensions and subject to various loadings. The only cross section that need be considered for design purposes is the one where the bending moment is a maximum, i.e., at the supporting wall. Thus we wish to design a rectangular beam to resist a bending moment of 135 kN · m with a maximum bending stress of 150 MPa.

Since the cross section is to be rectangular it will have the appearance shown in Fig. 8-17, where the width is denoted by b and the height by $h = 2b$, in accordance with the specifications. The moment of inertia about the neutral axis, which passes through the centroid of the action, is given by

$$I = \tfrac{1}{12}bh^3 = \tfrac{1}{12}b(2b)^3 = \tfrac{2}{3}b^4$$

At the cross section of the beam adjacent to the supporting wall the bending stress in the beam is given by $\sigma = My/I$. The maximum bending stress in tension occurs along the upper surface of the beam, since these fibers elongate slightly, and at this surface $y = -b$ and $\sigma = 150$ MPa. Then

$$\sigma = \frac{My}{I} \qquad \text{or} \qquad 150 = \frac{-135 \times 10^3(10^3)(-b)}{\tfrac{2}{3}b^4}$$

from which $b = 110$ mm and $h = 2b = 220$ mm.

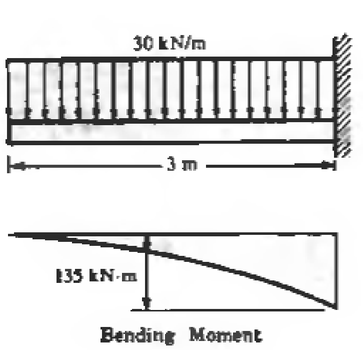


Fig. 8-16

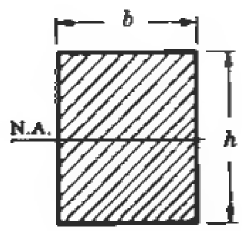
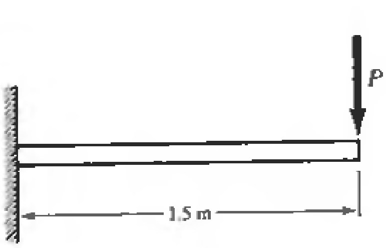
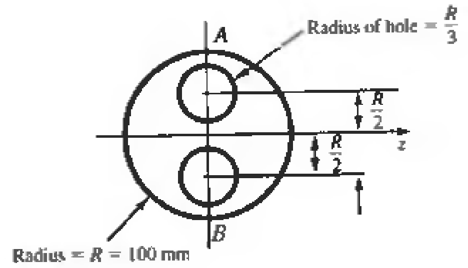


Fig. 8-17

8.7. A cantilever beam is of length 1.5 m, loaded by a concentrated force P at its tip as shown in Fig. 8-18(a), and is of circular cross section ($R = 100$ mm), having two symmetrically placed longitudinal holes as indicated. The material is titanium alloy, having an allowable working stress in bending of 600 MPa. Determine the maximum allowable value of the vertical force P.



(a)



(b)

Fig. 8-18

It is first necessary to determine the section modulus of the beam. From Chap. 7, Problem 7.9, the moment of inertia of a solid circular cross section about a diametral axis z is $\pi R^4/4$. Using this value for the solid section and subtracting the moments of inertia of each of the holes about the same diametral axis z (from the parallel-axis theorem of Chap. 7), we have

$$I = \frac{\pi R^4}{4} - 2\left\{\frac{\pi}{4}\left(\frac{R}{3}\right)^4 + \pi\left(\frac{R}{3}\right)^2\left(\frac{R}{2}\right)^2\right\} = 0.592R^4$$

The section modulus from Eq. (8.3) is

$$Z = \frac{I}{c} = \frac{0.592R^4}{R} = 0.592R^3$$

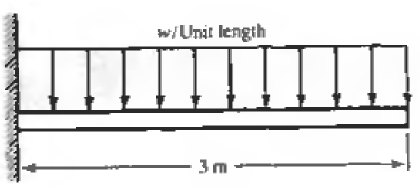
The bending stresses in the uppermost and lowermost fibers, denoted by points A and B, respectively, in Fig. 8-18(b) are, from Eq. (8.3) and using $R = 0.1$ m,

$$\sigma_{max} = \frac{M}{Z}$$

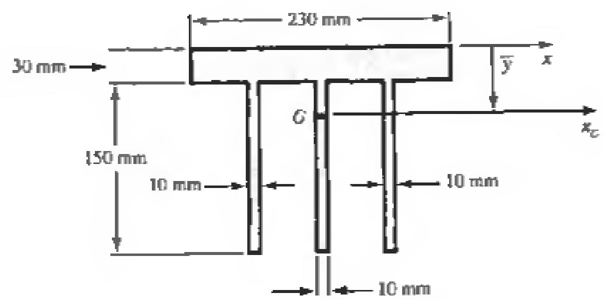
$$600 \times 10^6\ \text{N/m}^2 = \frac{P(1.5\ \text{m})}{0.592R^3}$$

Solving, $P = 237 \times 10^3$ N, or 237 kN.

8.8. The extruded beam shown in Fig. 8-19 is made of 6061-T6 aluminum alloy having an allowable working stress in either tension or compression of 90 MPa. The beam is a cantilever, subject to a uniform vertical load. Determine the allowable intensity of uniform loading.



(a)



(b)

Fig. 8-19

It is first necessary to locate the centroid of the cross section. From the methods of Chap. 7, we have

$$\bar{y} = \frac{(200)(30)(15) + 3(180)(10)(90)}{(200)(30) + 3(180)(10)} = 50.5 \text{ mm}$$

It is next necessary to determine the moment of inertia of the cross section. Let us first work with the x-axis through the top of the flange. From Chap. 7 the moment of inertia of the entire section about that axis is

$$I_x = \tfrac{1}{3}(200\text{ mm})(30\text{ mm})^3 + 3\{\tfrac{1}{3}(10\text{ mm})(180\text{ mm})^3\}$$
$$= 60.12 \times 10^6 \text{ mm}^4$$

and from the parallel axis theorem of Chap. 7 we may now transfer to the x_G axis through the centroid of the cross section to find

$$I_{x_G} = 60.12 \times 10^6 \text{ mm}^4 - (11{,}400 \text{ mm}^2)(50.5 \text{ mm})^2$$
$$= 31.05 \times 10^6 \text{ mm}^4$$

The peak bending moment occurs at the supporting wall and was found in Problem 6.2 to be

$$M_{max} = \frac{wL^2}{2}$$

Next, applying Eq. (*8.1*) to the lowermost fibers (A) of the beam since those are the most distant from the neutral axis through G, we have

$$90 \times 10^6 \text{ N/m}^2 = \frac{[w(3\text{ m})^2][(180 - 50.5)\text{ mm}](1\text{ m}/1000\text{ mm})}{(2)(31.05 \times 10^6 \text{ mm}^4)(1\text{ m}/1000\text{ mm})^4}$$

Solving,

$$w = 4.80 \text{ kN/m}$$

8.9. The simply supported beam AD is loaded by a concentrated force of 80 kN together with a couple of magnitude 30 kN · m, as shown in Fig. 8-20. From Table 8-2 at the end of this chapter select a commercially available steel wide-flange beam capable of carrying these loads if the peak allowable working stress in tension as well as compression is 160 MPa.

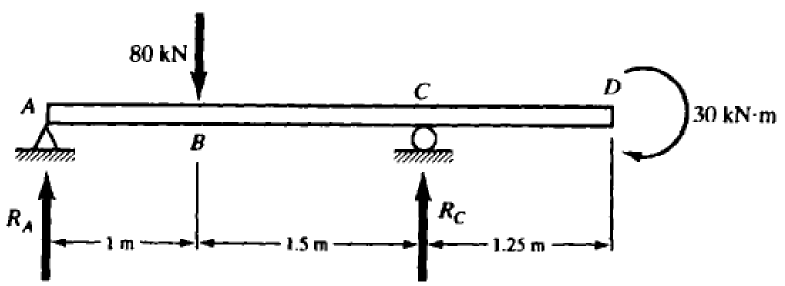


Fig. 8-20

It is first necessary to determine the reactions at A and C from statics. We have

$$+\circlearrowleft \Sigma M_A = -(80\text{ kN})(1\text{ m}) + R_C(2.5\text{ m}) - 30\text{ kN}\cdot\text{m} = 0$$
$$R_C = 44 \text{ kN}$$
$$\Sigma F_y = R_A + 44 - 80 = 0$$
$$R_A = 36 \text{ kN}$$

From the methods of Chap. 6, we can now construct the moment diagram which appears as in Fig. 8-21.

From Eq. (8.3) we have $\sigma_{max} = M/Z$. Substituting,

$$160 \times 10^6 \text{ N/m}^2 = \frac{36 \times 10^3 \text{ N} \cdot \text{m}}{Z}$$

Solving,

$$Z = 225 \times 10^{-6} \text{ m}^3 \quad \text{or} \quad 225 \times 10^3 \text{ mm}^3$$

as the minimum acceptable value of section modulus. From Table 8-2 we see that the W203 × 28 section has a Z value of 262×10^3 mm^3, which is adequate. Undoubtedly a more complete beam listing would indicate other sections with a Z value more nearly equal to the required minimum of 225×10^3 mm^3. Only typical beams are listed in Table 8-2 for the sake of brevity.

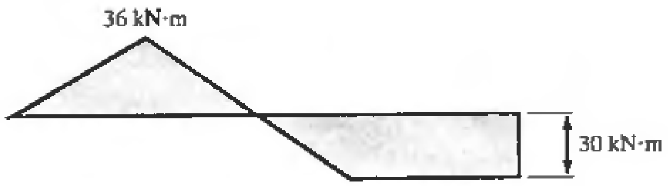


Fig. 8-21

8.10. If a steel wire 0.5 mm in diameter is coiled around a pulley 400 mm in diameter, determine the maximum bending stress set up in the wire. Take $E = 200$ GPa.

Since the radius of curvature of the wire is constant, 200 mm, it is evident from (7) of Problem 8.1, namely $M = EI/R$, that the bending moment M must be constant everywhere in the wire. Thus the wire acts as a beam subject to pure bending. An enlarged sketch of a portion of the wire is shown in Fig. 8-22. For any fiber in the wire at a distance y from the neutral axis, the normal strain was found in (1) of Problem 8.1 to be

$$\epsilon = \frac{y}{R}$$

where R denotes the radius of curvature of the beam at that point.

The maximum strain occurs at the fibers where y assumes its maximum value, that is, $\frac{1}{2}(0.5)$ mm from the neutral axis. The radius of curvature is approximately 200 mm. More accurately, this radius should be measured to the neutral surface of the wire, but the value in that case would only differ from 200 mm by 0.25 mm and this quantity may reasonably be neglected.

Thus the maximum strain at the outer fibers of the wire is

$$\epsilon = \frac{\frac{1}{2}(0.5)}{200} = 0.00125$$

The longitudinal fibers are subject to tensile stresses on one side of the wire and compressive on the other, with no other stresses acting. Hooke's law may then be used to find the stress:

$$\sigma = E\epsilon = (200 \times 10^9)(0.00125) = 250 \text{ MPa}$$

This is the maximum stress in the wire.

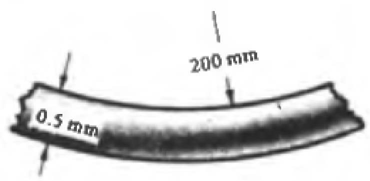


Fig. 8-22

8.11. The simply supported beam shown in Fig. 8-23(a) is subject to a uniformly varying load having a maximum intensity of w N per meter of length at the right end of the bar. If the beam is a wide-flange section having the dimensions shown in Fig. 8-23, determine the maximum load intensity w that may be applied if the working stress is 125 MPa in either tension or compression. Neglect the weight of the beam.

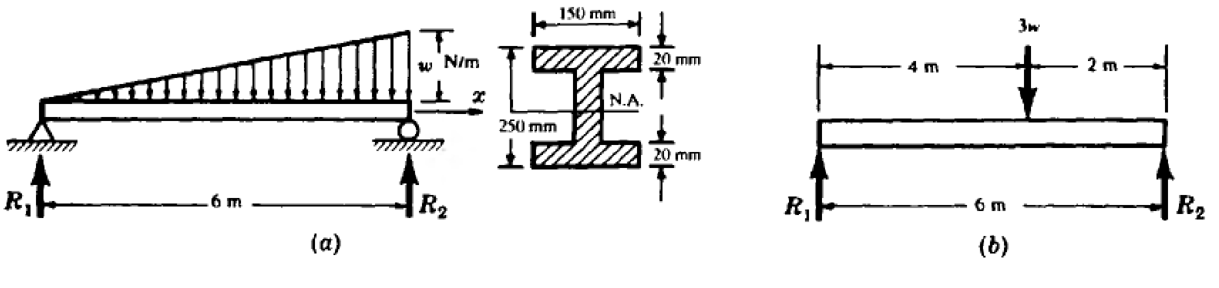


Fig. 8-23

The reactions R_1 and R_2 may readily be determined in terms of the unknown w by replacing the distributed load by its resultant. Since the average value of the distributed load is $w/2$ N/m acting over a length of 6 m, the resultant is a force of magnitude $6(w/2) = 3w$ N acting through the centroid of the triangular loading diagram, that is, 4 m to the right of R_1. This resultant thus appears as in Fig. 8-23(b). From statics we immediately have $R_1 = w$ N and $R_2 = 2w$ N.

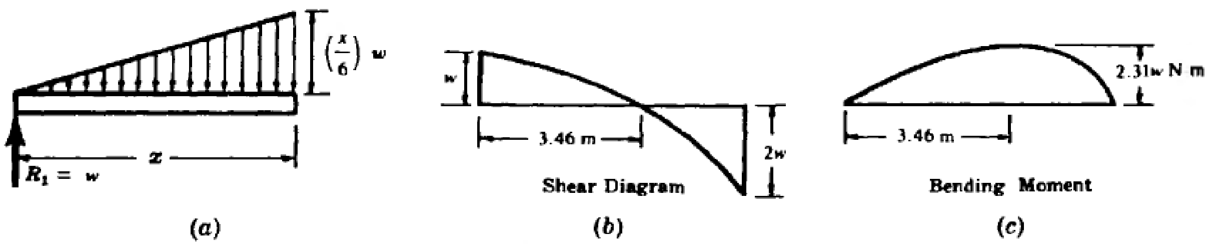


Fig. 8-24

The shearing force and bending moment diagrams for this type of loading were discussed in Problem 6.5. Let us introduce an x-axis coinciding with the beam and having its origin at the left support. Then at a distance x to the right of the left reaction, the intensity of load is found from similar-triangle relationships to be $(x/6)w$ N/m. This portion of the loaded beam between R_1 and the section x appears in Fig. 8-24(a). In accordance with the procedure explained in Problem 6.5, the shearing force V at the section a distance x from the left support is given by

$$V = w - \frac{1}{2}\left(\frac{x}{6}\right)wx = w - \frac{1}{12}wx^2$$

This equation holds for all values of x and from it the shear diagram is readily plotted, as shown in Fig. 8-24(b). The point of zero shear is found by setting

$$w - \tfrac{1}{12}wx^2 = 0 \qquad \text{from which} \qquad x = \sqrt{12} = 3.46\ \text{m}$$

This is also the point where the bending moment assumes its maximum value.

The bending moment M at the section a distance x from the left support is given by

$$M = wx - \frac{1}{2}\left(\frac{x}{6}\right)w\frac{x^2}{3} = wx - \frac{1}{36}wx^3$$

Again, this equation holds for all values of x and from it the bending moment diagram may be plotted as in Fig. 8-24(*c*). At the point of zero shear, $x = 3.46$ m, the bending moment is found by substitution in the above equation to be

$$M_{x=3.46} = 3.46w - \tfrac{1}{36}w(3.46)^3 = 2.31w\ \text{N}\cdot\text{m}$$

This is the maximum bending moment in the beam.

The bending stress on any fiber a distance y from the neutral axis of the beam is given by $\sigma = My/I$. The moment of inertia I of the beam is found from

$$I_x = \frac{150(250)^3}{12} - 2\left[\frac{65(210)^3}{12}\right] = 95 \times 10^6\ \text{mm}^4$$

The maximum tensile stress occurs at the lower fibers of the beam where $y = 125$ mm at the section where the bending moment is a maximum. This stress is 125 MPa, and thus $\sigma = My/I$ becomes

$$125 \times 10^6 = \frac{(2.31w)(0.125)}{95 \times 10^6(10^{-12})} \qquad \text{or} \qquad w = 41\ \text{kN/m}$$

8.12. Determine the section modulus of a beam of rectangular cross section.

Let h denote the depth of the beam and b its width. Bending is assumed to take place about the neutral axis through the centroid of the cross section. The moment of inertia about the neutral axis is $I = bh^3/12$.

At the outer fibers the distance to the neutral axis is $h/2$, and this is commonly denoted by c. The maximum bending stresses at these outer fibers are given by

$$\sigma_{max} = \frac{Mc}{I} = \frac{M}{I/c}$$

The ratio I/c is called the *section modulus* and is usually denoted by Z. Then $\sigma_{max} = M/Z$. For the beam of rectangular cross section,

$$Z = \frac{L}{c} = \frac{bh^3/12}{h/2} = \frac{bh^2}{6}$$

The section modulus Z has units of m^3 or in^3.

8.13. A beam is loaded by one couple at each of its ends, the magnitude of each couple being 5 kN · m. The beam is steel and of T-type cross section with the dimensions indicated in Fig. 8-25(*b*). Determine the maximum tensile stress in the beam and its location, and the maximum compressive stress and its location.

It is first necessary to locate the centroid of the cross-sectional area since the neutral axis is known to pass through the centroid. To do this we introduce the x-y coordinate system shown and use the methods of Chap. 7. The y-coordinate of the centroid is defined by

$$\bar{y} = \frac{\int y\,da}{A}$$

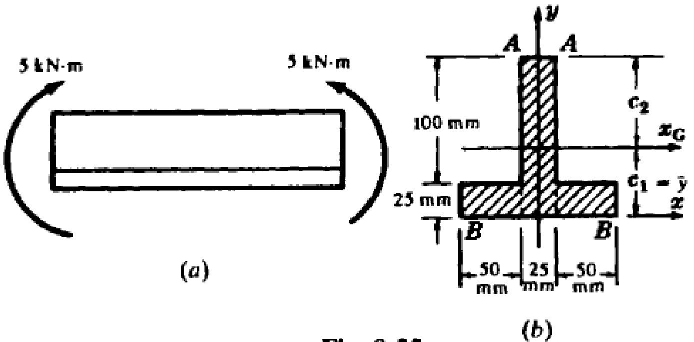


Fig. 8-25

where the numerator of the right side represents the first moment of the entire area about the x-axis. The T-section may be considered to consist of the three rectangles indicated by the dashed lines and this expression becomes

$$\bar{y} = \frac{125(25)(62.5) + 2[50(25)(12.5)]}{125(25) + 2[25(50)]} = 40.3 \text{ mm}$$

Thus, the centroid is located 40.3 mm above the x-axis. The horizontal axis passing through this point is denoted by x_G as shown.

The moment of inertia about the x-axis is given by the sum of the moments of inertia about this same axis of each of the three component rectangles comprising the cross section. Thus

$$I_x = \tfrac{1}{3}(25)(125)^3 + 2[\tfrac{1}{3}50(25)^3] = 16.8 \times 10^6 \text{ mm}^4$$

The moment of inertia about the x_G-axis may now be found by use of the parallel-axis theorem. Thus

$$I_x = I_{x_G} + A(\bar{y})^2 \qquad 16.8 \times 10^6 = I_{x_G} + 5625(40.3)^2 \qquad \text{and} \qquad I_{x_G} = 7.7 \times 10^6 \text{ mm}^4$$

Evidently for the loading shown, the fibers below the x_G-axis are in tension, while the fibers above this axis are in compression. Let c_1 and c_2 denote the distances of the extreme fibers from the neutral axis (x_G) as shown. Obviously $c_1 = 40.3$ mm and $c_2 = 84.7$ mm. The maximum tensile stress occurs in those fibers along B-B and is given by $\sigma = Mc_1/I$, where I denotes the moment of inertia of the entire cross section about the neutral axis passing through the centroid of the cross section. Thus the maximum tensile stress is given by

$$\sigma = \frac{Mc_1}{I} = 5 \times 10^3(10^3)(40.3)/7.7 \times 10^6 = 26.2 \text{ MPa}$$

The maximum compressive stress occurs in those fibers along A-A and is given by $\sigma = Mc_2/I$. To provide a consistent system of algebraic signs, it is necessary to assign a negative value to c_2 since it lies on the side of the x_G-axis opposite to that of c_1. Hence

$$\sigma = \frac{Mc_2}{I} = 5 \times 10^3(10^3)(-84.7)/7.7 \times 10^6 = -55 \text{ MPa}$$

The negative sign indicates that the stress is compressive.

8.14. A simply supported beam is loaded by the couple of 1000 lb · ft as shown in Fig. 8-26. The beam has a channel-type cross section as illustrated. Determine the maximum tensile and compressive stresses in the beam.

The bending moment diagram for this particular loading has been determined in Problem 6.11, where it was found to appear as in Fig. 8-27.

The techniques of Chap. 7 may be employed to locate the centroid as lying 1.5 in above the x-axis and the moment of inertia of the entire cross section about the x_G-axis as 41.6 in^4.

In this problem it is necessary to distinguish carefully between positive and negative bending moments. One method of attack is to consider a cross section of the beam slightly to the left of point B where the 1000 lb · ft couple is applied. According to the bending moment diagram the moment there is -600 lb · ft

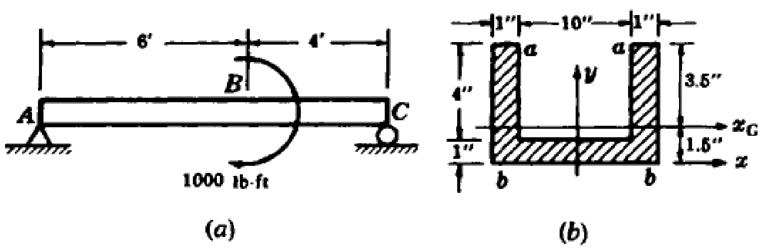


Fig. 8-26

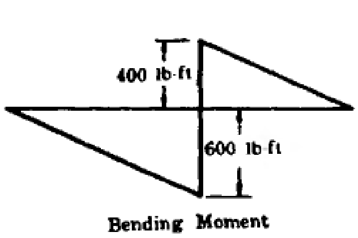


Fig. 8-27

and, according to the sign convention adopted in Chap. 6, since the moment is negative the beam is concave downward at that section, as shown in Fig. 8-28. Thus the upper fibers are in tension and the lower fibers in compression. Along the upper fibers *a-a* the bending stress is given by $\sigma = My/I$. Then

$$\sigma_a = \frac{(-600)(12)(-3.5)}{41.6} = 605 \text{ lb/in}^2$$

Along the lower fibers *b-b* the value of y in the above formula for bending stress must be taken to be positive, and there we have

$$\sigma_b = \frac{(-600)(12)(+1.5)}{41.6} = -260 \text{ lb/in}^2$$

Fig. 8-28

Fig. 8-29

It is next necessary to investigate the bending stresses at a section slightly to the right of point B. There the bending moment is 400 lb · ft and according to the usual sign convention the beam is concave upward at that section, as shown in Fig. 8-29. Here the upper fibers are in compression and the lower fibers in tension. Along the upper fibers *a-a* the bending stress is

$$\sigma'_a = \frac{400(12)(-3.5)}{41.6} = -400 \text{ lb/in}^2$$

Along the lower fibers *b-b* we have

$$\sigma'_b = \frac{400(12)(1.5)}{41.6} = 170 \text{ lb/in}^2$$

The maximum tensile and compressive stresses must now be selected from the above four values. Evidently the maximum tension is 605 lb/in² occurring in the upper fibers just to the left of point B: the maximum compression is 400 lb/in² occurring in the upper fibers also but just to the right of point B.

8.15. Consider the beam with overhanging ends loaded by the three concentrated forces shown in Fig. 8-30. The beam is simply supported and of T-type cross section as shown. The material is gray cast iron having an allowable working stress in tension of 35 MPa and in compression of 150 MPa. Determine the maximum allowable value of P.

From symmetry each of the reactions denoted by R is equal to $P/2$. The bending moment diagram consists of a series of straight lines connecting the ordinates representing bending moments at the points A, B, C, D, and E. At B the bending moment is given by the moment of the force $P/4$ acting at A about an axis through B. Thus

$$M_B = -\left(\frac{P}{4}\right)(1) = \frac{-P}{4} \text{ N} \cdot \text{m}$$

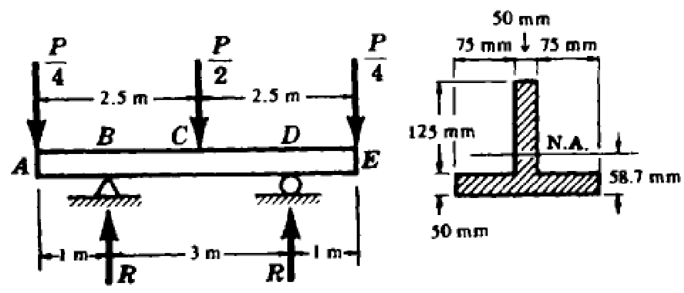


Fig. 8-30

At C the bending moment is given by the sum of the moments of the forces $P/4$ and $R = P/2$ about an axis through C. Thus

$$M_C = -\left(\frac{P}{4}\right)(2.5) + \left(\frac{P}{2}\right)(1.5) = \frac{P}{8}\,\text{N}\cdot\text{m}$$

The bending moment at D is equal to that at B by symmetry and the moment at each of the ends A and E is zero. Hence, the bending moment diagram plots as in Fig. 8-31.

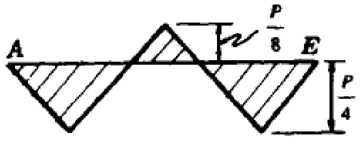


Fig. 8-31

Using the techniques described in Problem 8.13, we find the distance from the lower fibers of the flange to the centroid to be 58.7 mm and the moment of inertia of the area about the neutral axis passing through the centroid to be $40 \times 10^6\ \text{mm}^4$.

It is perhaps simplest to calculate four values of P based upon the various maximum tensile and compressive stresses that may exist at each of the points B and C and then select the minimum of these values. Let us first examine point B. Since the bending moment there is negative, the beam is concave downward at that point, as shown in Fig. 8-32. Evidently the upper fibers are in tension and the lower fibers are subject to compression. We shall first calculate a value of P, assuming that the allowable tensile stress of 35 MPa is realized in the upper fibers. Applying the flexure formula $\sigma = My/I$ to these upper fibers, we find

$$35 \times 10^6 = \frac{(-P/4)(0.116)}{40 \times 10^6(10^{-12})} \qquad \text{or} \qquad P = 48.3\ \text{kN}$$

Next we shall calculate a value of P, assuming that the allowable compressive stress of 150 MPa is set up in the lower fibers. Again applying the flexure formula, we find

$$-150 \times 10^6 = \frac{(-P/4)(0.0587)}{40 \times 10^6(10^{-12})} \qquad \text{or} \qquad P = 410\ \text{kN}$$

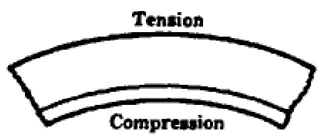


Fig. 8-32

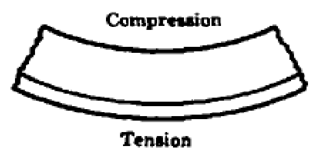


Fig. 8-33

We shall now examine point C. Since the bending moment there is positive, the beam is concave upward at that point and appears as in Fig. 8-33. Here, the upper fibers are in compression and the lower fibers are subject to tension. First we will calculate a value of P, assuming that the allowable tension of 35 MPa is set up in the lower fibers. From the flexure formula we find

$$(35 \times 10^6) = \frac{(P/8)(0.0587)}{40 \times 10^6(10^{-12})} \qquad \text{or} \qquad P = 191\ \text{kN}$$

Last, we shall assume that the allowable compression of 150 MPa is set up in the upper fibers. Applying the flexure formula, we have

$$-150 \times 10^6 = \frac{(P/8)(-0.116)}{40 \times 10^6(10^{-12})} \qquad \text{or} \qquad P = 414\ \text{kN}$$

The minimum of these four values is $P = 48.3$ kN. Thus the tensile stress at the points B and D is the controlling factor in determining the maximum allowable load.

8.16. The cantilever beam ABC supports a uniform load over its right half and is of rectangular cross section with a square cutout as shown in Fig. 8-34. If the maximum permissible stress in either tension or compression is 140 MPa, determine the allowable uniform load w per unit length of the beam.

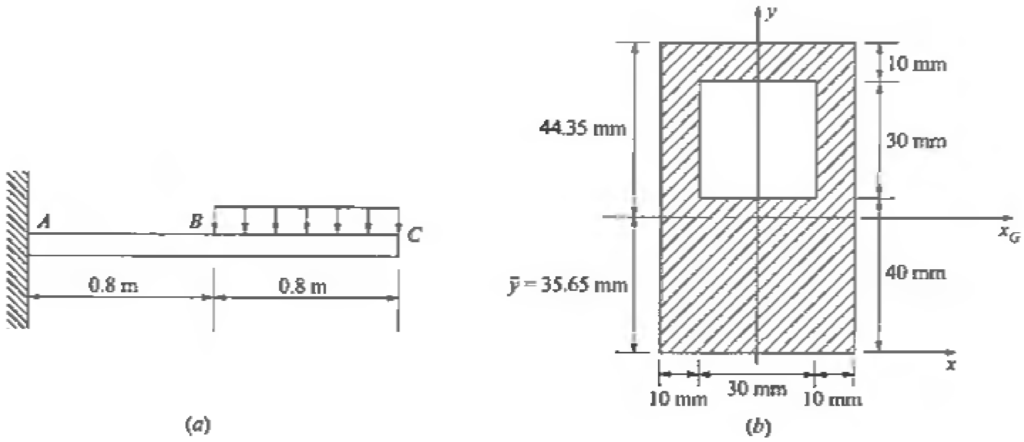


Fig. 8-34

It is first necessary to locate the neutral axis (N.A.) of the beam. For entirely elastic action this passes through the cross section of the beam and is given by (see Chap. 7)

$$\bar{y} = \frac{(80)(50)(40) - (30)(30)(55)}{(80)(50) - (30)(30)} = 35.65 \text{ mm}$$

Also, by the methods of Chap. 7, the moment of inertia about the x-axis is

$$I_x = \tfrac{1}{3}(50)(80)^3 - [\tfrac{1}{12}(30)(30)^3 + (900)(55)^2]$$
$$= (8193.25)(10)^3 \text{ mm}^4$$

Use of the parallel-axis theorem of Chap. 7 leads to the moment of inertia about an axis parallel to x but passing through the centroid, i.e., the x_G axis:

$$I_{x_G} = (8193.25)(10^3)\text{ mm}^4 - [(3100)(35.65)^2] = 4253.39 \times 10^3 \text{ mm}^2$$

The tensile fibers along the top surface of the beam are at a greater distance (44.35 mm) than the compressive fibers along the lower surface (35.65 mm). For these extreme fibers in tension we have

$$\sigma = \frac{Mc}{I}$$

$$140 \times 10^6 \text{ N/m}^2 = \frac{M(44.35 \text{ mm})(\text{m}/1000 \text{ mm})}{4253.39 \times 10^3 \text{ mm}^2 (\text{m}/1000 \text{ mm})^4}$$

Solving,

$$M_{max} = 13{,}372 \text{ N}\cdot\text{m}$$

From the loading conditions, $M_A = M_{max}$, so

$$M_A = M_{max} = (0.8 \text{ m} + 0.4 \text{ m})w(0.8 \text{ m})$$

Solving,

$$w = \frac{(13{,}372 \text{ N}\cdot\text{m})}{(1.2 \text{ m})(0.8 \text{ m})}$$
$$= 13{,}929.6 \text{ N/m}$$

or $$w = 13.93 \text{ kN/m}$$

8.17. The beam shown in Fig. 8-35 is of constant width b but the depth varies in the x-direction and further the depth is symmetric about the x-axis. Loading is due to a vertical force at the tip of the beam where $x = L$ and $y = 0$. Determine the equation of the beam contour $y = h(x)$ so that outer fiber bending stresses are equal to σ_0 at all points on the contour of the beam.

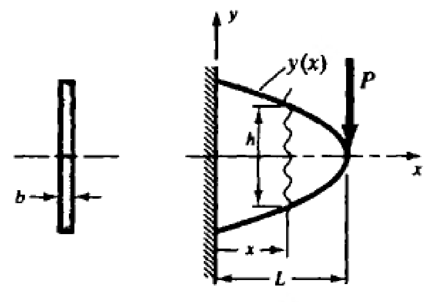


Fig. 8-35

The bending moment equation due to the concentrated load is $-P(L-x)$. From Problem 8.12, the section modulus of any cross section is given by $bh^2/6$. The outer fiber bending stresses along the top surface are, from Eq. (*8.3*),

$$\sigma = \frac{|M|}{Z} = \frac{P(L-x)}{(bh^2/6)} = \frac{6P(L-x)}{bh^2}$$

Since it is specified that this stress must be equal to σ_0 everywhere along the top surface, we have

$$\frac{6P(L-x)}{bh^2} = \sigma_0$$

Solving,

$$h = \sqrt{\frac{6P(L-x)}{b\sigma_0}}$$

This determines the beam contour for constant strength at all points along the length of the beam. This solution neglects the effect of the singular point $(L, 0)$ at the point of load application on stress distribution in the immediate vicinity of the force P.

8.18. A cantilever beam of circular cross section has the dimensions shown in Fig. 8-36. Determine the peak bending stress in the beam due to the concentrated force applied at the tip A.

To express the moment of inertia of the cross section at any point along the length of the beam in terms of the given geometry, we must first determine where the extensions of the top and bottom fibers would meet on the x-axis. From Fig. 8-36 we immediately have from similar triangles:

$$\frac{x_1}{d} = \frac{x_1 + L}{2.5d}$$

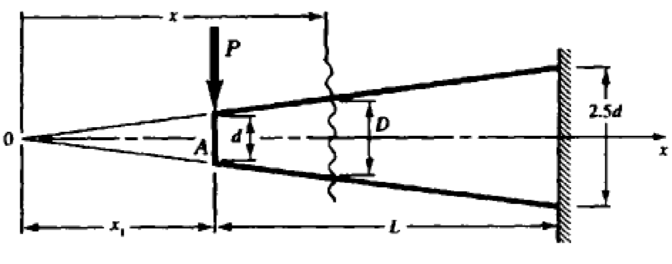


Fig. 8-36

Solving,

$$x_1 = \frac{2L}{3} \tag{1}$$

The bending moment at any station located a distance x from this fictitious point of intersection is

$$M = -P\left[x - \frac{2L}{3}\right] \tag{2}$$

If we designate the beam diameter by D at this location x, we have from geometry

$$\frac{x}{D} = \frac{x_1}{d} \qquad \text{so} \qquad D = \frac{3xd}{2L} \tag{3}$$

so that the cross-sectional moment of inertia at the general location x is

$$I = \frac{\pi D^4}{64} = \frac{\pi}{64}\left[\frac{3d}{2L}x\right]^4 = \left(\frac{81\pi d^4}{(64)(16)L^4}\right)x^4 \tag{4}$$

From Eq. (8.2) we find the outer fiber bending stresses to be

$$\sigma = \frac{M_c}{I} = \frac{P(x - 2L/3)(3xd/2L)}{[81\pi d^4/(64)(16)L^4]x^4} = \frac{256PL^3}{9\pi d^3}\left[\frac{x - 2L/3}{x^3}\right] \tag{5}$$

Note that Eq. (5) indicates that the peak bending stress does *not* occur at the clamped end $x = L$.

To find where the outer fiber stresses reach a maximum value, we take the derivative $d\sigma/dx$ and set it equal to zero to locate the critical value of x. Thus,

$$\frac{d\sigma}{dx} = \left(\frac{256PL^3}{9\pi d^3}\right)\left[\frac{x^3(1) - (x - 2L/3)3x^2}{x^6}\right] = 0 \tag{6}$$

Solving, $x = L$ measured from point 0. Substituting this value of x in Eq. (5), we find the peak outer fiber bending stress to be

$$\sigma = \frac{256PL^3}{9\pi d^3}\left[\frac{L - 2L/3}{L^3}\right] = \left(\frac{256PL^3}{9\pi d^3}\right)\left(\frac{1}{3L^2}\right) = 3.02\frac{PL}{d^3}$$

Note that from Eq. (5) the outer fiber bending stress at the clamped end $x = (L + 2L/3)$ is $1.96\ PL/d^3$, which is less than the peak value.

8.19. In a beam loaded by transverse forces acting perpendicular to the axis of the beam, not only are bending stresses parallel to the axis of the bar produced but shearing stresses also act over cross sections of the beam perpendicular to the axis of the bar. Express the intensity of these shearing stresses in terms of the shearing force at the section and the properties of the cross section.

The theory to be developed applies only to a cross section of rectangular shape. However, the results of this analysis are commonly used to give approximate values of the shearing stress in other cross sections having a plane of symmetry.

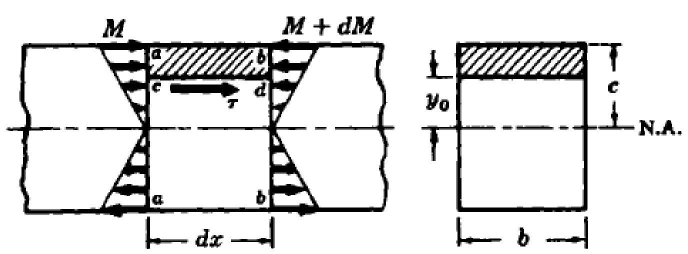


Fig. 8-37

Let us consider an element of length dx cut from a beam as shown in Fig. 8-37. We shall denote the bending moment at the left side of the element by M and that at the right side by $M + dM$, since in general the bending moment changes slightly as we move from one section to an adjacent section of the beam. If y is measured upward from the neutral axis, then the bending stress at the left section a-a is given by

$$\sigma = \frac{My}{I}$$

where I denotes the moment of inertia of the entire cross section about the neutral axis. This stress distribution is illustrated above. Similarly, the bending stress at the right section b-b is

$$\sigma' = \frac{(M + dM)y}{I}$$

Let us now consider the equilibrium of the shaded element $acdb$. The force acting on an area da of the face ac is merely the product of the intensity of the force and the area; thus

$$\sigma\, da = \frac{My}{I} da$$

The sum of all such forces over the left face ac is found by integration to be

$$\int_{y_0}^{c} \frac{My}{I} da$$

Likewise, the sum of all normal forces over the right face bd is given by

$$\int_{y_0}^{c} \frac{(M + dM)y}{I} da$$

Evidently, since these two integrals are unequal, some additional horizontal force must act on the shaded element to maintain equilibrium. Since the top face ab is assumed to be free of any externally applied horizontal forces, then the only remaining possibility is that there exists a horizontal shearing force along the lower face cd. This represents the action of the lower portion of the beam on the shaded element. Let us denote the shearing stress along this face by τ as shown. Also, let b denote the width of the beam at the position where τ acts. Then the horizontal shearing force along the face cd is $\tau b\, dx$. For equilibrium of the element $acdb$ we have

$$\Sigma F_h = \int_{y_0}^{c} \frac{My}{I} da - \int_{y_0}^{c} \frac{(M + dM)y}{I} da + \tau b\, dx = 0$$

Solving,

$$\tau = \frac{1}{Ib} \frac{dM}{dx} \int_{y_0}^{c} y\, da$$

But from Problem 6.1 we have $V = dM/dx$, where V represents the shearing force (in pounds or Newtons) at the section a-a. Substituting,

$$\tau = \frac{V}{Ib} \int_{y_0}^{c} y\, da \qquad (1)$$

The integral in this last equation represents the first moment of the shaded cross-sectional area about the neutral axis of the beam. This area is always the portion of the cross section that is above the level at which the desired shear acts. This first moment of area is sometimes denoted by Q in which case the above formula becomes

$$\tau = \frac{VQ}{Ib} \qquad (2)$$

The units of $\int y\, da$ or of Q are in^3 or m^3.

The shearing stress τ just determined acts horizontally as shown in Fig. 8-37. However, let us consider the equilibrium of a thin element $mnop$ of thickness t cut from any body and subject to a shearing stress

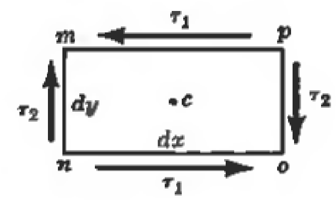


Fig. 8-38

τ_1 on its lower face, as shown in Fig. 8-38. The total horizontal force on the lower face is $\tau_1 t\,dx$. For equilibrium of forces in the horizontal direction, an equal force but acting in the opposite direction must act on the upper face, hence the shear stress intensity there too is τ_1. These two forces give rise to a couple of magnitude $\tau_1 t\,dx\,dy$. The only way in which equilibrium of the element can be maintained is for another couple to act over the vertical faces. Let the shear stress intensity on these faces be denoted by τ_2. The total force on either vertical face is $\tau_2 t\,dy$. For equilibrium of the moments about the center of the element we have

$$\Sigma M_c = \tau_1 t\,dx\,dy - \tau_2 t\,dy\,dx = 0 \qquad \text{or} \qquad \tau_1 = \tau_2$$

Thus we have the interesting conclusion that the shearing stresses on any two perpendicular planes through a point on a body are equal. Consequently, not only are there shearing stresses τ acting horizontally at any point in the beam, but shearing stresses of an equal intensity also act vertically at that same point.

In summary, when a beam is loaded by transverse forces, both horizontal and vertical shearing stresses arise in the beam. The vertical shearing stresses are of such magnitudes that their resultant at any cross section is exactly equal to the shearing force V at that same section.

8.20. A beam of rectangular cross section is simply supported at the ends and subject to the single concentrated force shown in Fig. 8-39(a). Determine the maximum shearing stress in the beam. Also, determine the shearing stress at a point 1 in below the top of the beam at a section 1 ft to the right of the left reaction.

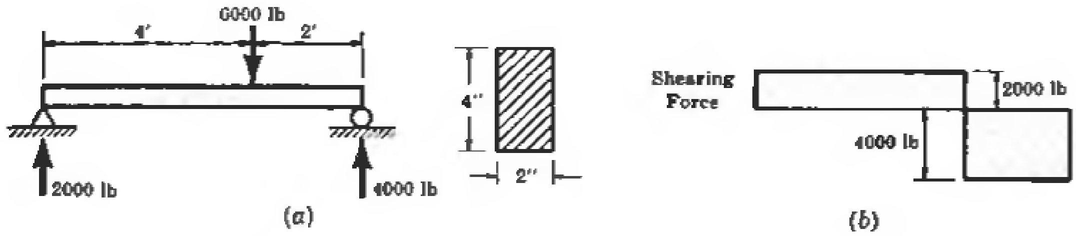


Fig. 8-39

The reactions are readily found from statics to be 2000 lb and 4000 lb as shown. The shearing force diagram for this type of loading appears in Fig. 8-39(b).

From the shear diagram, the shearing force acting at a section 1 ft to the right of the left reaction is 2000 lb. The shearing stress τ at any point in this section a distance y_0 from the neutral axis was shown in Problem 8.19 and also Eq. (*8.5*) to be

$$\tau = \frac{V}{2I}\left(\frac{h^2}{4} - y_0^2\right) \qquad (1)$$

At a point 1 in below the top fibers of the beam, $y_0 = 1$ in. Also, we have $h = 4$ in and $I = bh^3/12 = 2(4)^3/12 = 10.67\ \text{in}^4$. Substituting,

$$\tau_{y_0=1} = \frac{2000}{2(10.67)}\left(\frac{4^2}{4} - 1\right) = 280\ \text{lb/in}^2$$

From Eq. (*1*) it is clear that the peak shearing stress occurs at the neutral axis where $y_0 = 0$. Thus,

$$\tau_{max} = \frac{4000}{2(10.67)}\left(\frac{4^2}{4} - 1\right) = 750\ \text{lb/in}^2$$

Note that for a rectangular cross section this peak shearing stress is 50 percent greater than the average shearing stress, which is given by

$$\tau_{mean} = \frac{4000}{(4)(2)} = 500\ \text{lb/in}^2$$

8.21. Consider the cantilever beam subject to the concentrated load shown in Fig. 8-40. The cross section of the beam is of T-shape. Determine the maximum shearing stress in the beam and also determine the shearing stress 25 mm from the top surface of the beam at a section adjacent to the supporting wall.

The shear force has a constant value of 50 kN at all points along the length of the beam. Because of this simple, constant value the shear diagram need not be drawn.

The location of the centroid and the moment of inertia about the centroidal axis for this particular cross section were determined in Problem 8.15. The centroid was found to be 58.7 mm above the lower surface of the beam and the moment of inertia about a horizontal axis through the centroid was found to be $40 \times 10^6\ \text{mm}^4$.

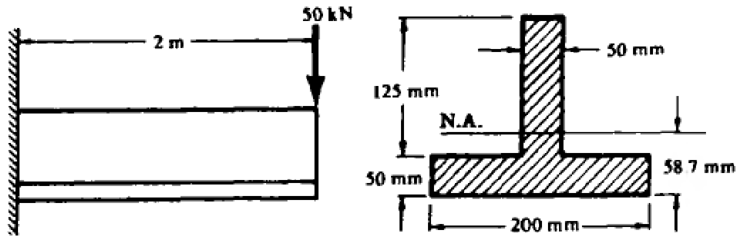


Fig. 8-40

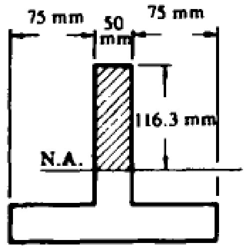


Fig. 8-41

The shearing stress at a distance y_0 from the neutral axis through the centroid was found in Problem 8.19 to be

$$\tau = \frac{V}{Ib}\int_{y_0}^{c} y\,da$$

Inspection of this equation reveals that the shearing stress is a maximum at the neutral axis, since at that point $y_0 = 0$ and consequently the integral assumes the largest possible value. It is not necessary to integrate, however, since the integral is known in this case to represent the first moment of the area between the neutral axis and the outer fibers of the beam about the neutral axis. This area is represented by the shaded region in Fig. 8-41. The value of the integral could also, of course, be found by taking the first moment of the unshaded area below the neutral axis about the line, but that calculation would be somewhat more difficult.

Thus the first moment of the shaded area about the neutral axis is

$$50(116.3)(58.15) = 3.38 \times 10^5\ \text{mm}^3$$

and the shearing stress at the neutral axis, where $b = 50$ mm, is found by substitution in the above general formula to be

$$\tau = \frac{50 \times 10^3}{50(40 \times 10^6)}(3.38 \times 10^5) = 8.45\ \text{MPa}$$

In this formula b was taken to be 50 mm, since that is the width of the beam at the point where the shearing stress is being calculated. Thus the maximum shearing stress is 8.45 MPa and it occurs at all points on the neutral axis along the entire length of the beam, since the shearing force has a constant value along the entire length of the beam.

The shearing stress 25 mm from the top surface of the beam is again given by the formula

$$\tau = \frac{V}{Ib}\int_{y_0}^{c} y\,da$$

Now, the integral represents the first moment of the new shaded area shown in Fig. 8-42, about the neutral axis. Again it is not necessary to integrate to evaluate the integral, since the coordinate of the centroid of this shaded area is known. It is 103.8 mm above the neutral axis. Thus the first moment of this shaded area about the neutral axis is $50(25)(103.8) = 1.3 \times 10^5\ \text{mm}^3$, and the shearing stress 25 mm below the top fibers is

$$\tau = \frac{50 \times 10^3}{50(40 \times 10^6)}(1.3 \times 10^5) = 3.25\ \text{MPa}$$

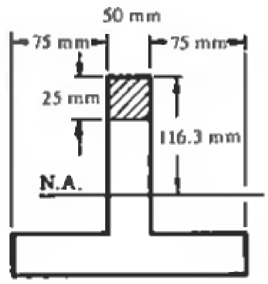


Fig. 8-42

Again, b was taken to be 50 mm since that is the width of the beam at the point where the shearing stress is being evaluated. Since the shearing force is equal to 50 kN everywhere along the length of the beam, the shearing stress 25 mm below the top fibers is 3.25 MPa everywhere along the beam.

8.22. The vertically oriented wide-flange section shown in Fig. 8-43 is loaded by a single horizontal concentrated force of 6.5 kN directed parallel to the z-axis. Determine the horizontal shear stress distribution on a flange at a section 3 m above the lower clamped end in the x-z plane.

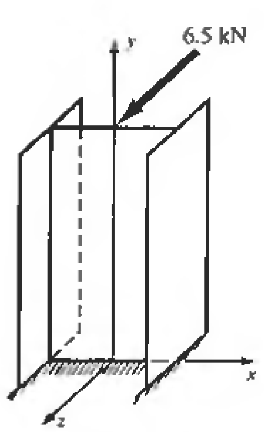


Fig. 8-43

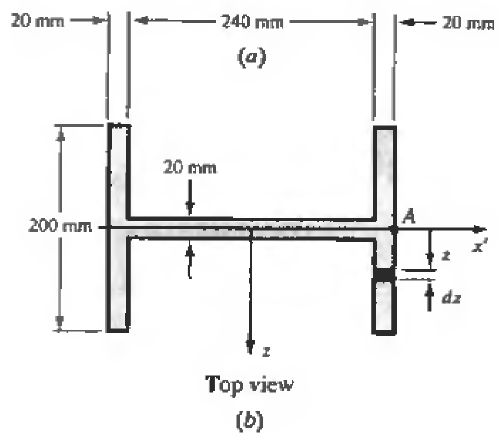


Fig. 8-44

Figure 8-44 shows a typical horizontal cross section parallel to the x-z plane as well as dimensions of the web and flange. The shear stress τ in this plane acts in the z-direction and at a distance z from the x-axis. The specification of 3 m above the x-z plane is unimportant: all that matters is that the equation for shear stress derived in Problem 8.19 does not apply at horizontal sections near either the bottom or top of the vertically oriented bar. To apply Eq. (1) of Problem 8.19 to find τ we must first determine the moment of inertia of the cross section about the x-axis. From the methods of Chap. 7, we find

$$I = \tfrac{1}{12}(2)(20\text{ mm})(200\text{ mm})^3 + \tfrac{1}{12}(240\text{ mm})(20\text{ mm})^2 = 2683 \times 10^4\text{ mm}^4 \tag{1}$$

We next introduce a coordinate z running from the x-y plane in the direction of the z-axis, and appearing as in Fig. 8-44. From Problem 8.19 we have here $V = 6500$ N, and the flange thickness b here is 0.02 m. The integral in Problem 8.19 represents the first moment of the area extending from z to the extreme fibers of the flange—that area is shaded in Fig. 8-44. Thus, we need not integrate and we may evaluate the first moment of the shaded area about the x'-axis by taking the product of the area and the distance of the centroid of the area from the x'-axis: that is,

$$[(0.1 - z)(0.04\text{ m})]\left(\frac{0.1 + z}{2}\text{ m}\right) \qquad \text{or} \qquad (0.02)[(0.1)^2 - z^2]\text{ m}^3$$

Equation (2) of Problem 8.19 now yields the desired shearing stress as

$$\begin{aligned}\tau &= \frac{6500\text{ N}}{(26.83 \times 10^{-6}\text{ m}^4)(0.02\text{ m})(2)}\{(0.02)[(0.1)^2 - z^2]\text{ m}^3\} \\ &= 121.1[(0.1)^2 - z^2](10^6)\end{aligned} \tag{2}$$

At the point A where the value of z is zero, the peak shearing stress is found from Eq. (2) to be

$$\tau_A = (121.1)[(0.1)^2 - 0](10^6) = 1.21 \times 10^6\text{ N/m}^2 \qquad \text{or} \qquad 1.21\text{ MPa}$$

8.23. Consider a beam having an I-type cross section as shown in Fig. 8-45. A shearing force V of 150 kN acts over the section. Determine the maximum and minimum values of the shearing stress in the vertical web of the section.

The shearing stress at any point in the cross section is given by

$$\tau = \frac{V}{Ib}\int_{y_0}^{c} y\,da$$

as derived in Problem 8.19. Here, y_0 represents the location of the section on which τ acts, and is measured from the neutral axis as shown. In this expression, I represents the moment of inertia of the entire cross section about the neutral axis, which passes through the centroid of the section. I is readily calculated by dividing the section into rectangles, as indicated by the dashed lines, and we have

$$I = \tfrac{1}{12}(10)(350)^3 + 2[\tfrac{1}{12}(200)(25)^3 + 200(25)(187.5)^2] = 389 \times 10^6\text{ mm}^4$$

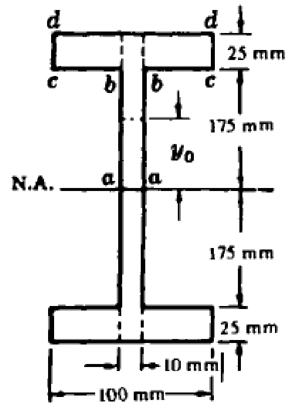


Fig. 8-45

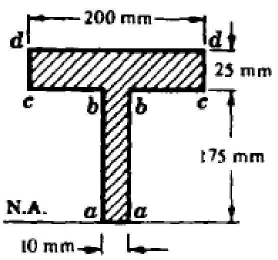


Fig. 8-46

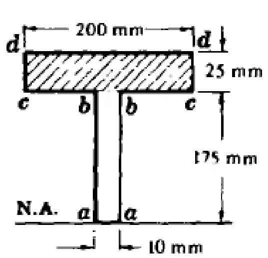


Fig. 8-47

Inspection of the general formula for shearing stress reveals that this stress has a maximum value when $y_0 = 0$, that is, at the neutral axis, since at that point the integral takes on its largest possible value. It is not necessary to integrate to obtain the value of $\int_{y_0}^{c} y\,da$, since this integral is shown to represent the first moment of the area between $y_0 = 0$ (that is, the neutral axis) and the outer fibers of the beam. This area is shaded in Fig. 8-46. For this area we have, taking its first moment about the neutral axis,

$$\int_0^{200} y\,da = 175(10)(87.5) + 200(25)(187.5) = 1.1 \times 10^6\ \text{mm}^3$$

Consequently the maximum shearing stress in the web occurs at the section *a-a* along the neutral axis and by substituting in the general formula for shearing stress is found to be

$$\tau_{max} = \frac{150 \times 10^3}{10(389 \times 10^6)}(1.1 \times 10^6) = 42.4\ \text{MPa}$$

The minimum shearing stress in the web occurs at that point in the web farthest from the neutral axis, i.e., across the section *b-b*. To calculate the shearing stress there, it is necessary to evaluate $\int_{y_0}^{c} y\,da$ for the area between *b-b* and the outer fibers of the beam. This is the shaded area shown in Fig. 8-47. Again, it is not necessary to integrate, since this integral merely represents the first moment of this shaded area about the neutral axis. It is

$$\int_{175}^{200} y\,da = 200(25)(187.5) = 9.375 \times 10^5\ \text{mm}^3$$

The value of b is still 10 mm, since that is the width of the beam at the position where the shearing stress is being calculated. Substituting in the general formula

$$\tau_{min} = \frac{150 \times 10^3}{10(389 \times 10^6)} = (9.375 \times 10^5) = 36.2\ \text{MPa}$$

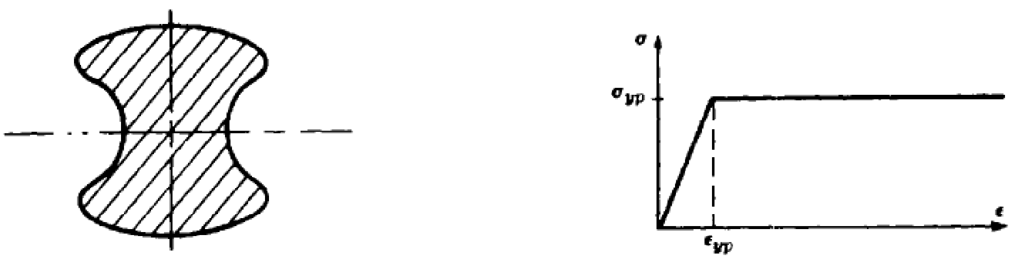
It is to be noted that there is not too great a difference between the maximum and minimum values of shearing stress in the web of the beam. In fact, it is customary to calculate only an approximate value of the shearing stress in the web of such an I-beam. This value is obtained by dividing the total shearing force V by the cross-sectional area of the web alone. This approximate value becomes

$$\tau_{av} = \frac{100 \times 10^3}{(400)(10)} = 37.5\ \text{MPa}$$

A more advanced analysis of shearing stresses in an I-beam reveals that the vertical web resists nearly all of the shearing force V and that the horizontal flanges resist only a small portion of this force. The shear stress in the web of an I-beam is specified by various codes at rather low values. Thus some codes specify 70 MPa, others 90 MPa.

Plastic Bending of Beams

8.24. Consider a beam of arbitrary doubly symmetric cross section, as in Fig. 8-48(*a*), subject to pure bending. The material is considered to be elastic-perfectly plastic, i.e., the stress-strain diagram has the appearance shown in Fig. 8-48(*b*) and stress-strain characteristics in tension and



(*a*) (*b*)

Fig. 8-48

compression are identical. Determine the moment acting on the beam when all fibers a distance y_1 from the neutral axis have reached the yield point of the material.

Even though bending of the beam has caused the outer fibers to have yielded it is still realistic to assume that plane sections of the beam normal to the axis before loads are applied remain plane and normal to the axis after loading. Consequently, normal strains of the longitudinal fibers of the beam still vary linearly with the distance of the fiber from the neutral axis.

As the value of the applied moment is increased, the extreme fibers of the beam are the first to reach the yield point of the material and the normal stresses on all interior fibers vary linearly as the distance of the fiber from the neutral axis, as indicated in Fig. 8-49(*a*). A further increase in the value of the moment puts interior fibers at the yield point, with yielding progressing from the outer fibers inward, as indicated in Fig. 8-49(*b*). In the limiting case when all fibers (except those along the neutral axis) are stressed to the yield point the normal stress distribution appears as in Fig. 8-49(*c*). The bending moment corresponding to Fig. 8-49(*c*) is termed a *fully plastic moment*. For the type of stress-strain curve shown in Fig. 8-48(*b*), no greater moment is possible.

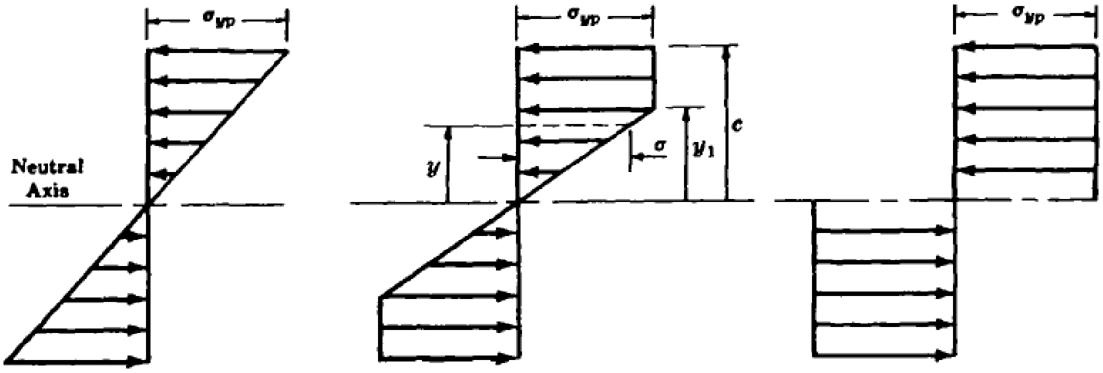


Fig. 8-49

For a beam in pure bending, the sum of the normal forces over the cross section must vanish. Hence, for the doubly symmetric section under consideration, it is evident from inspection of Fig. 8-49(*b*) that the neutral axis must pass through the centroid of such a section; i.e., the area above the neutral axis must be equal to the area below that axis. However, in Problem 8.29 it will be found that for a more general, nonsymmetric cross section the location of the neutral axis after certain of the fibers have yielded is not the same as that found for purely elastic action where the neutral axis passes through the centroid of the cross section.

From Fig. 8-48(*b*) we have for $y < y_1$:

$$\frac{\sigma}{y} = \frac{\sigma_{yp}}{y_1} \qquad \text{or} \qquad \sigma = \frac{y}{y_1}\sigma_{yp}$$

and for $y > y_1 : \sigma = \sigma_{yp} =$ constant. Thus the bending moment is

$$M = \int \sigma y \, da = 2\int_0^{y_1} \frac{y}{y_1}\sigma_{yp} y \, da + 2\int_{y_1}^{c} \sigma_{yp} y \, da$$

$$= \frac{2\sigma_{yp}}{y_1}\int_0^{y_1} y^2 \, da + 2\sigma_{yp}\int_{y_1}^{c} y \, da$$

8.25. For a beam of rectangular cross section determine the moment acting when all fibers a distance y_1 from the neutral axis have reached the yield point of the material.

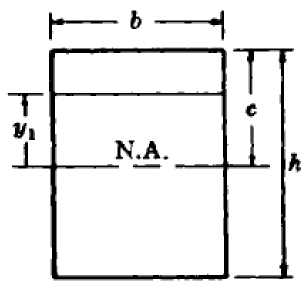


Fig. 8-50

From the result of Problem 8-24 for the geometry indicated in Fig. 8-50 we have

$$M = \frac{2\sigma_{yp}}{y_1}\left(\frac{1}{3}by_1^3\right) + 2\sigma_{yp}b(c - y_1)\left(\frac{c + y_1}{2}\right)$$

$$= \left(bc^2 - \frac{b}{3}y_1^2\right)\sigma_{yp}$$

For the limiting case when $y_1 = 0$ which is indicated by Fig. 8-49(c) of Problem 8.24 the fully plastic moment of this rectangular beam is

$$M_p = bc^2\sigma_{yp} = \frac{bh^2}{4}\sigma_{yp} \tag{1}$$

It is to be noted that the maximum possible elastic moment, i.e., when the extreme fibers have just reached the yield point but all interior fibers are in the elastic range of action as indicated by Fig. 8-49(a), is

$$M_e = \frac{bh^2}{6}\sigma_{yp} \tag{2}$$

Thus, for a rectangular cross section, the fully plastic moment is 50 percent greater than the maximum possible elastic moment.

8.26. Determine the fully plastic moment of a rectangular beam, 1 × 2 in in cross section, of steel with a yield point of 38,000 lb/in^2. Compare this with the maximum possible elastic moment that this same section may carry.

From (*1*) of Problem 8.25, the fully plastic moment is

$$M_p = \frac{1(2)^2}{4}(38{,}000) = 38{,}000 \text{ lb}\cdot\text{in}$$

From (*2*) of that same problem, the maximum possible elastic moment is

$$M_e = \frac{1(2)^2}{6}(38{,}000) = 25{,}400 \text{ lb}\cdot\text{in}$$

It is evident that M_p is 50 percent greater than M_e.

8.27. For a beam of rectangular cross section (Fig. 8-51) determine the relation between the bending moment and the radius of curvature when all fibers at a distance y_1 from the neutral axis have reached the yield point of the material.

As in Problem 8.25, we assume that plane sections before loading remain plane and normal to the beam axis after loading. Because of this, normal strains of the longitudinal fibers vary linearly as the

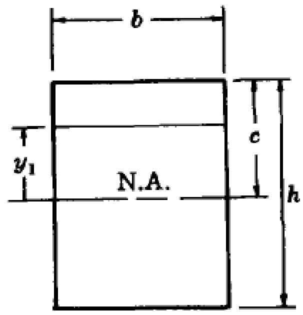


Fig. 8-51

distance of the fibers from the neutral axis. Thus, if ϵ_{yp} denotes the strain of the fibers at a distance y_1 from the neutral axis and ϵ_c represents the outer fiber strain, we have

$$\frac{\epsilon_c}{c} = \frac{\epsilon_{yp}}{y_1} \tag{1}$$

Consideration of the geometry of an originally rectangular element of length dx along the beam axis, as shown in Fig. 8-52(*a*), reveals that after bending it assumes the configuration indicated in Fig. 8-52(*b*). From that sketch we have

$$\frac{1}{R} = \frac{d\theta}{dx} = \frac{\epsilon_c}{c} \tag{2}$$

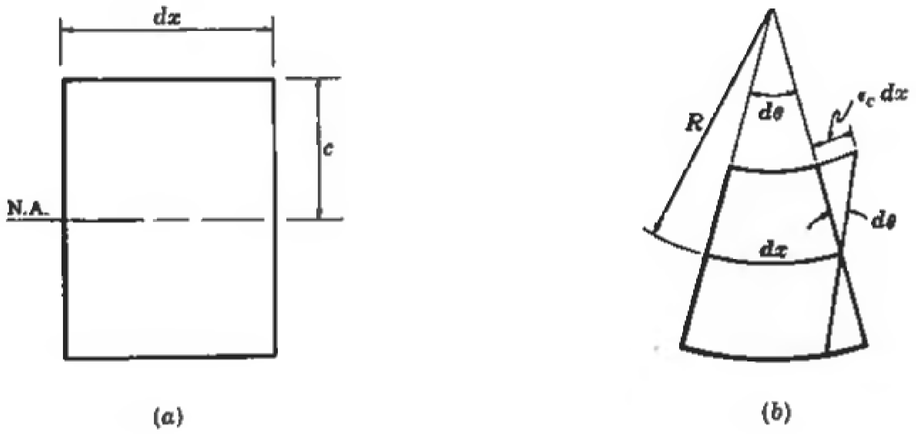


Fig. 8-52

Thus

$$\frac{d\theta}{dx} = \frac{\epsilon_{yp}}{y_1} = \frac{\sigma_{yp}}{Ey_1} \tag{3}$$

since the fibers a distance y_1 from the neutral axis obey Hooke's law: $\sigma_{yp} = E\epsilon_{yp}$. From Problem 8.25, the moment corresponding to these strains is

$$M = \left(bc^2 y_1 - \frac{b}{3}y_1^3\right)\frac{\sigma_{yp}}{y_1} \tag{4}$$

Thus, from (*3*) and (*4*),

$$\frac{d\theta}{dx} = \frac{M}{Eby_1(c^2 - \frac{1}{3}y_1^2)} \tag{5}$$

Finally, from (2) and (5) we have

$$\frac{1}{R} = \frac{M}{EI(M/M_e)\sqrt{3 - 2M/M_e}} \tag{6}$$

where $M_e = bh^2\sigma_{yp}/6$ as in Problem 8.25. This is the desired relation between the bending moment M and the radius of curvature R. Equation (6) plots as shown in Fig. 8-53.

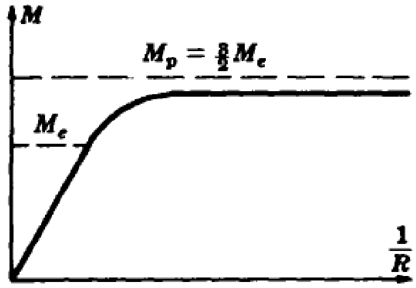


Fig. 8-53

8.28. Consider a beam of rectangular cross section where $b = 25$ mm, $h = 10$ mm. The material is steel for which $\sigma_{yp} = 200$ MPa and $E = 200$ GPa. Determine the radius of curvature corresponding to the maximum possible elastic moment and also the radius of curvature for a moment of $100\ \text{N}\cdot\text{m}$.

From (2) of Problem 8.25, the maximum possible elastic moment is

$$M_e = \frac{0.025(0.01)^2}{6}(200 \times 10^6) = 83\ \text{N}\cdot\text{m}$$

The curvature corresponding to this moment is found from (6) of Problem 8.27 to be

$$\frac{1}{R} = \frac{83}{(200 \times 10^9)[(0.025)(0.01)^3/12]\sqrt{3-2}} = 0.2 \qquad \text{or} \qquad R = 5\ \text{m}$$

The value of y_1 corresponding to a moment of $100\ \text{N}\cdot\text{m}$ may be found from Problem 8.25 to be 4 mm. The curvature corresponding to this is found from (6) of Problem 8.27 to be

$$\frac{1}{R} = \frac{100}{(200 \times 10^9)[(0.025)(0.01)^3/12]\sqrt{3 - 200/83}} = 0.312 \qquad \text{or} \qquad R = 3.2\ \text{m}$$

8.29. Consider the more general case of a beam with a cross section symmetric only about the vertical axis, as shown in Fig. 8-54(*a*). For fully plastic bending [Fig. 8-54(*b*)], determine the location of the neutral axis.

Although the location of the neutral axis is unknown, let us denote the area of that portion of the cross section lying below that axis by A_1 and the area of the portion above that axis by A_2. As shown by Fig. 8-54(*b*), all fibers in A_1 are subject to a tensile stress equal to the yield point of the material and all fibers in A_2 are subject to the same magnitude compressive stress. For horizontal equilibrium of these forces, we have

$$\sigma_{yp}A_1 - \sigma_{yp}A_2 = 0 \tag{1}$$

from which

$$A_1 = A_2 = \frac{A}{2} \tag{2}$$

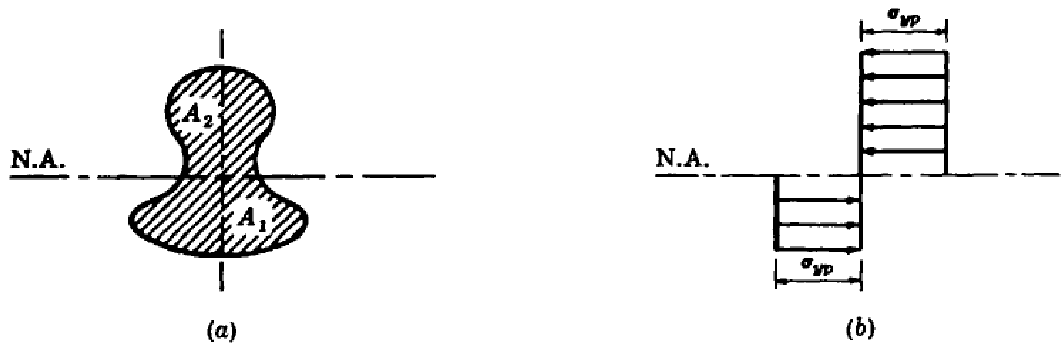


Fig. 8-54

where A is the area of the entire cross section. Thus, for fully plastic action, the neutral axis divides the cross section into two equal parts. This is in contrast to the situation for fully elastic action, where the neutral axis was found in Problem 8.1 to pass through the centroid of the cross section.

Also, the sum of the moments of the tensile and compressive stresses must equal the applied moment M_p, the fully plastic moment. If $\bar{y}_1$ and $\bar{y}_2$ denote the distances from the neutral axis to the centroids of the areas A_1 and A_2, respectively, then from statics

$$\sigma_{yp} A_1 \bar{y}_1 + \sigma_{yp} A_2 \bar{y}_2 = M_p \tag{3}$$

From (2) this becomes

$$\sigma_{yp} \frac{A}{2}(\bar{y}_1 + \bar{y}_2) = M_p \tag{4}$$

or

$$\sigma_{yp} = \frac{M_p}{(A/2)(\bar{y}_1 + \bar{y}_2)} \tag{5}$$

This is frequently written in the form

$$\sigma_{yp} = \frac{M_p}{Z_p} \tag{6}$$

where $Z_p = (A/2)(\bar{y}_1 + \bar{y}_2)$ is termed the *plastic section modulus.*

8.30. For a W8 × 40 wide-flange section of steel having a yield point of 38,000 lb/in^2, determine the fully plastic moment. Compare this with the maximum possible elastic moment that the same section can carry.

From Problem 8.29, the fully plastic moment M_p is given by

$$M_p = \sigma_{yp} Z_p$$

where Z_p is the plastic section modulus. For selected wide-flange sections Z_p is tabulated at the end of this chapter. In particular, for this section it is found to be 39.9 in^3. Thus

$$M_p = 38{,}000(39.9) = 1{,}520{,}000 \text{ lb} \cdot \text{in}$$

The maximum possible elastic moment is $M_e = \sigma_{yp} Z$ where Z is the usual (elastic) section modulus. Thus

$$M_e = 38{,}000(35.5) = 1{,}350{,}000 \text{ lb} \cdot \text{in}$$

The plastic moment is only 12.6 percent greater than the maximum elastic moment for this particular section. In fact, the fully plastic moment usually exceeds the maximum possible elastic moment by approximately 12 to 15 percent for most wide-flange sections.

8.31. Consider the T-section shown in Fig. 8-55(a) in which all fibers in the vertical web at a distance y_1 from the neutral axis have reached the yield point of the material, whereas all other fibers are still in the elastic range of action. Determine the location of the neutral axis and also the moment that corresponds to this stress distribution.

The neutral axis (described by the unknown c_1) may be located by investigating the normal forces over the cross section as shown in Fig. 8-55(b). From geometry

$$\frac{\sigma_0}{5-c_1} = \frac{\sigma_{yp}}{y_1} \quad \text{or} \quad \sigma_0 = \frac{5-c_1}{y_1}\sigma_{yp}$$

$$\frac{\sigma_0'}{4-c_1} = \frac{\sigma_{yp}}{y_1} \quad \text{or} \quad \sigma_0' = \frac{4-c_1}{y_1}\sigma_{yp}$$

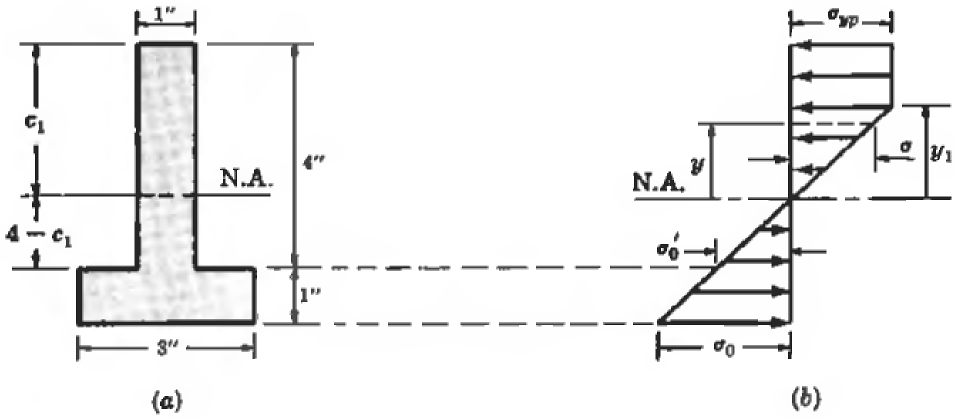


Fig. 8-55

For the resultant normal force to vanish

$$\Sigma F_N = (c_1 - y_1)(1)\sigma_{yp} + y_1(1)\left(\frac{\sigma_{yp}}{2}\right)$$
$$-\left\{\left[\frac{5-c_1}{2y_1}(5-c_1)(3)\sigma_{yp}\right] - \left[\frac{4-c_1}{2y_1}(4-c_1)(2)\sigma_{yp}\right]\right\} = 0$$

from which we obtain the quadratic equation

$$c_1^2 - (2y_1 + 14)c_1 + (y_1^2 + 43) = 0 \tag{1}$$

which determines c_1 for any specified value of y_1. This locates the neutral axis. Note that since y_1 occurred in the denominator in the above derivation, the equation should not be used to locate the neutral axis if $y_1 = 0$. Thus, when the action is entirely elastic the neutral axis passes through the centroid of the cross section. As plastification increases (i.e., as y_1 decreases), the neutral axis shifts to the location indicated by (*1*).

The moment corresponding to the stresses in Fig. 8-55(b) may be found from

$$M = \int \sigma y \, da$$
$$= \int_0^{y_1} \frac{y}{y_1}\sigma_{yp}(y)(1)\,dy + \int_{y_1}^{c_1} \sigma_{yp}(y)(1)\,dy$$
$$+ \int_0^{(5-c_1)} \left(\frac{y}{5-c_1}\right)\left(\frac{5-c_1}{y_1}\right)(\sigma_{yp})(y)(3)\,dy$$
$$- \int_0^{(4-c_1)} \left(\frac{y}{4-c_1}\right)\left(\frac{4-c_1}{y_1}\right)(\sigma_{yp})(y)(2)\,dy$$

or

$$M = \frac{\sigma_{yp}}{y_1}\left[\frac{y_1^3}{3} + \frac{y_1}{2}(c_1^2 - y_1^2) + (5-c_1)^3 - \frac{2}{3}(4-c_1)^3\right] \tag{2}$$

8.32. For the T-section of Problem 8.31 determine the location of the neutral axis when the action is fully plastic over the entire cross section. For fully plastic action determine the moment-carrying capacity and compare this with the maximum possible elastic moment.

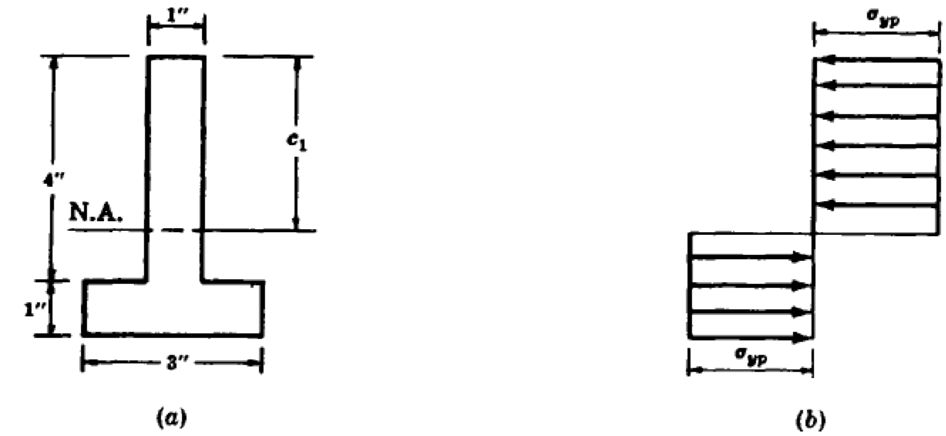


Fig. 8-56

In this case, the normal forces appear as indicated in Fig. 8-56(*b*). For equilibrium of normal forces over the cross section, we have

$$-\sigma_{yp}(1)(c_1)+[\sigma_{yp}(5-c_1)(3)-\sigma_{yp}(4-c_1)(2)]=0$$

from which $c_1 = 3.5$ in. Thus, as mentioned in Problem 8.29, for fully plastic action the neutral axis divides the cross section into two equal parts.

The moment corresponding to this fully plastic action is

$$\begin{aligned} M_p &= \int \sigma y\,da \\ &= \int_0^{c_1} \sigma_{yp}(y)(1)\,dy + \int_0^{(5-c_1)} \sigma_{yp}(y)(3)\,dy - \int_0^{(4-c_1)} \sigma_{yp}(y)(2)\,dy \\ &= \sigma_{yp}[c_1^2 - 7c_1 + 21.5] \end{aligned}$$

For $c_1 = 3.5$ this becomes

$$M_p = 9.25\sigma_{yp}$$

By setting $y_t = c_t$ in (*1*) of Problem 8.31, the neutral axis is located for the case of the maximum possible elastic moment. This location is found to be $c_1 = 3.07$ in (i.e., the neutral axis passes through the centroid of the cross section). The maximum possible elastic moment is found from (*2*) of Problem 8.31 to be

$$M_e = 5.32\sigma_{yp}$$

The fully plastic moment exceeds this value by 74 percent.

8.33. A beam is of square cross section, oriented as shown in Fig. 8-57, and carries a vertical load. If only the extreme top and bottom fibers reach the yield point, determine the maximum allowable elastic bending moment. Also, if the stress reaches yield at all fibers, determine the fully plastic moment.

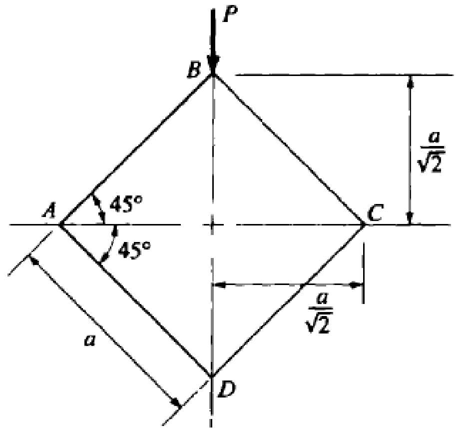


Fig. 8-57

If tensile yield is reached at fiber B and compressive yield at fiber D, the stress distribution over the cross section is given by (see Problem 8.1)

$$\sigma = \frac{Mc}{I} \tag{1}$$

To determine I, we consider the cross section to consist of triangles ABC and ADC. For each of these we have, from Problem 7.7,

$$I' = \tfrac{1}{12}bh^3 \tag{2}$$

So for the entire cross section the moment of inertia is

$$I = 2\left\{\frac{1}{12}\left(\frac{2a}{\sqrt{2}}\right)\left(\frac{a}{\sqrt{2}}\right)^3\right\}$$

$$= \frac{a^4}{24} \tag{3}$$

So from Eq. (1) at the extreme fibers we have

$$\sigma_{yp} = \frac{M_e\left(\frac{a}{\sqrt{2}}\right)}{\left(\frac{a^4}{24}\right)}$$

$$\therefore M_e = \frac{(\sigma_{yp})a^3\sqrt{2}}{12}$$

For fully plastic action over the entire cross section we have the stress distribution shown in Fig. 8-58.

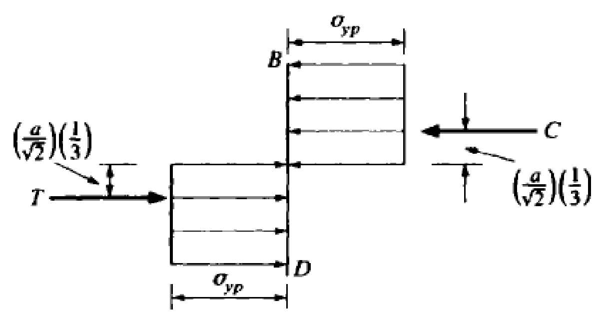


Fig. 8-58

The resultant of the compressive stresses above AC is

$$C = \frac{1}{2}\left(\frac{2a}{\sqrt{2}}\right)\left(\frac{a}{\sqrt{2}}\right)\sigma_{yp} = \frac{a^2}{2}\sigma_{yp}$$

which acts at the centroid of triangle ABC, and the resultant of the tensile stresses below AC is

$$T = \frac{1}{2}\left(\frac{2a}{\sqrt{2}}\right)\left(\frac{a}{\sqrt{2}}\right)\sigma_{yp} = \frac{a^2}{2}\sigma_{yp}$$

acting at the centroid of triangle ADC. These forces form a couple of magnitude

$$M_p = \left(\frac{a^2}{2}\sigma_{yp}\right)\left[2\left(\frac{a}{3\sqrt{2}}\right)\right] = \sigma_{yp}\frac{a^3}{3\sqrt{2}}$$

It is also of interest to form the ratio M_p/M_e:

$$\frac{M_p}{M_e} = \frac{\sigma_{yp}\left(\dfrac{a^3}{3\sqrt{2}}\right)}{\sigma_{yp}\left(\dfrac{a^3\sqrt{2}}{12}\right)} = 2$$

Supplementary Problems

8.34. A beam made of titanium, type Ti-6Al-4V, has a yield point of 120,000 lb/in^2. The beam has 1-in × 2-in rectangular cross section and bends about an axis parallel to the 1-in face. If the maximum bending stress is 90,000 lb/in^2, find the corresponding bending moment. *Ans.* 60,000 lb · in

8.35. A cantilever beam 3 m long carries a concentrated force of 35 kN at its free end. The material is structural steel and the maximum bending stress is not to exceed 125 MPa. Determine the required diameter if the bar is to be circular. *Ans.* 204 mm

8.36. Two $\frac{1}{2}$-in × 8-in cover plates are welded to two channels 10 in high to form the cross section of the beam shown in Fig. 8-59. Loads are in a vertical plane and bending takes place about a horizontal axis. The moment of inertia of each channel about a horizontal axis through the centroid is 78.5 in^4. If the maximum allowable elastic bending stress is 18,000 lb/in^2, determine the maximum bending moment that may be developed in the beam. *Ans.* 1,232,000 lb · in

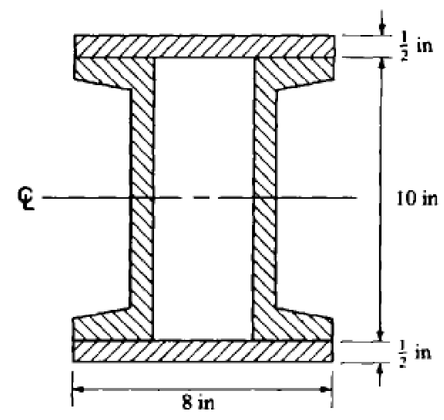


Fig. 8-59

8.37. A 250 mm deep wide-flange section with $I = 61 \times 10^6\ \text{mm}^4$ is used as a cantilever beam. The beam is 2 m long and the allowable bending stress is 125 MPa. Determine the maximum allowable intensity of uniform load that may be carried along the entire length of the beam. *Ans.* 30.5 kN/m

8.38. The beam shown in Fig. 8-60 is simply supported at the ends and carries the two symmetrically placed loads of 60 kN each. If the working stress in either tension or compression is 125 MPa, what is the required moment of inertia of area required for a 250-mm-deep beam? *Ans.* $60 \times 10^6\ \text{mm}^4$

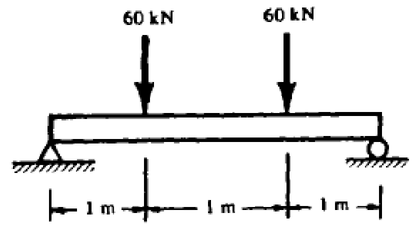


Fig. 8-60

8.39. Consider the simply supported beam subject to the two concentrated forces (60 kN each) shown in Fig. 8-60. Now, the beam is of hollow circular cross section as shown in Fig. 8-61, with an allowable working stress in either tension or compression of 125 MPa. Determine the necessary outer diameter of the beam. *Ans.* 17.4 mm

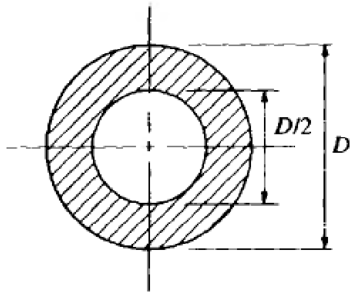


Fig. 8-61

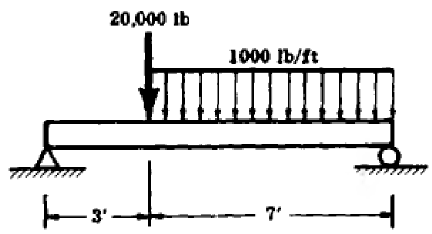


Fig. 8-62

8.40. Consider a simply supported beam carrying the concentrated and uniform loads shown in Fig. 8-62. Select a suitable wide-flange section to resist these loads based upon a working stress in either tension or compression of 20,000 lb/in^2. *Ans.* W12 × 25

8.41. Select a suitable wide-flange section to act as a cantilever beam 3 m long that carries a uniformly distributed load of 30 kN/m. The working stress in either tension or compression is 150 MPa. *Ans.* W305 × 66

8.42. A beam 3 m long is simply supported at each end and carries a uniformly distributed load of 10 kN/m. The beam is of rectangular cross section, 75 mm × 150 mm. Determine the magnitude and location of the peak bending stress. Also, find the bending stress at a point 25 mm below the upper surface at the section midway between supports. *Ans.* 40 MPa, −26.8 MPa

8.43. Reconsider the steel beam of Problem 8-42. Determine the maximum bending stress if now the weight of the beam is considered in addition to the load of 10 kN/m. The weight of steel is 77.0 kN/m^3. *Ans.* 43.6 MPa

8.44. The two distributed loads are carried by the simply supported beam as shown in Fig. 8-63. The beam is a W8 × 28 section. Determine the magnitude and location of the maximum bending stress in the beam.
Ans. 9000 lb/in^2, 5.5 ft from the right support

8.45. A T-beam having the cross section shown in Fig. 8-64 projects 2 m from a wall as a cantilever beam and carries a uniformly distributed load of 8 kN/m, including its own weight. Determine the maximum tensile and compressive bending stresses. *Ans.* +38.5 MPa, −81 MPa

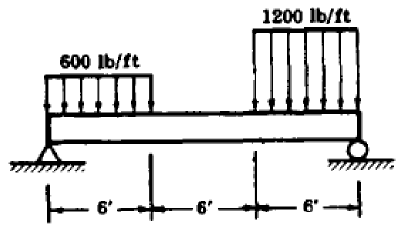


Fig. 8-63

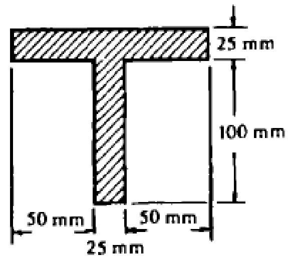


Fig. 8-64

8.46. The simply supported beam *AC* shown in Fig. 8-65(*a*) supports a concentrated load *P*. The beam section is rectangular, 60 mm by 100 mm, with two square cutouts as shown in Fig. 8-65(*b*). If the allowable working stress is 120 MPa, determine the maximum value of *P*. *Ans.* 1.80 kN

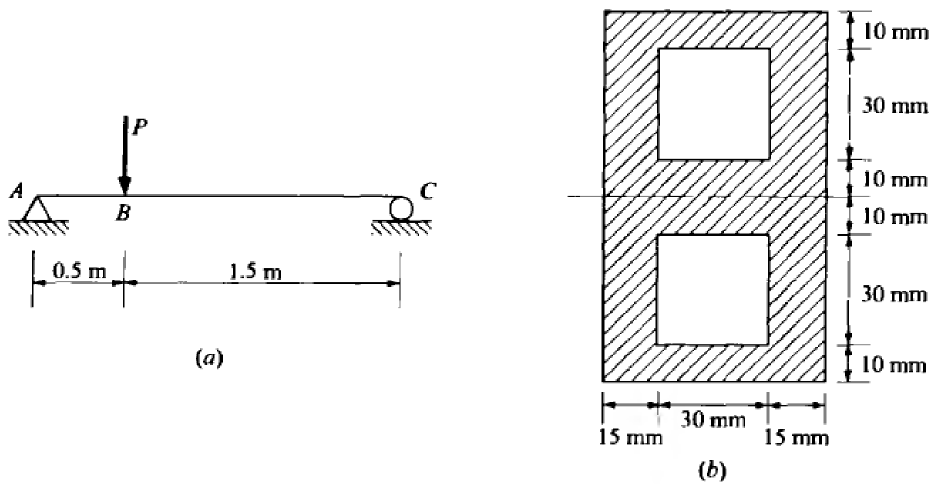


Fig. 8-65

8.47. A simply supported steel beam of channel-type cross section is loaded by both the uniformly distributed load and the couple shown in Fig. 8-66. Determine the maximum tensile and compressive stresses.
Ans. 31.2 MPa, −56.8 MPa

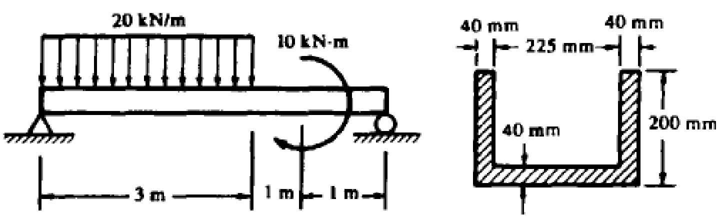


Fig. 8-66

8.48. A beam of circular cross section has the geometry shown in Fig. 8-67 and is subjected to a single concentrated vertical force at its midpoint. Determine the location of the point of maximum bending stress and the value of that stress. *Ans.* $x = L/4$, $\sigma_{max} = 0.377\ PL/d^3$

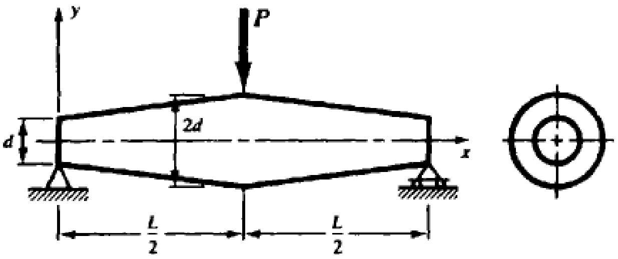


Fig. 8-67

8.49. A channel-shape beam with an overhanging end is loaded as shown in Fig. 8-68. The material is gray cast iron having an allowable working stress of 5000 lb/in^2 in tension and 20,000 lb/in^2 in compression. Determine the maximum allowable value of P. *Ans.* 2400 lb

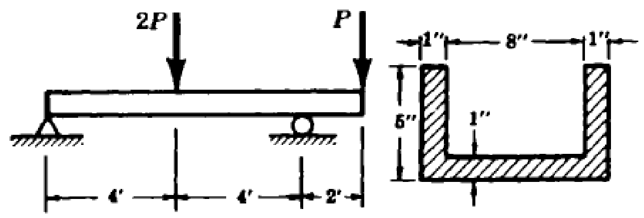


Fig. 8-68

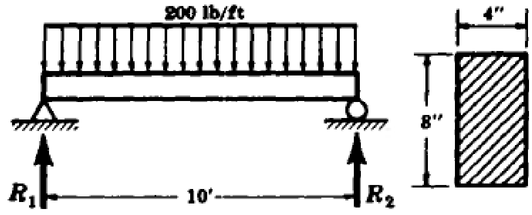


Fig. 8-69

8.50. In Fig. 8-69 the simply supported beam of length 10 ft and cross section 4 in × 8 in carries a uniform load of 200 lb/ft. Neglecting the weight of the beam, find (*a*) the maximum normal stress in the beam, (*b*) the maximum shearing stress in the beam, and (*c*) the shearing stress at a point 2 ft to the right of R_1 and 1 in below the top surface of the beam. *Ans.* (*a*) 705 lb/in^2, (*b*) 47 lb/in^2, (*c*) 12.3 lb/in^2

8.51. Determine (*a*) the maximum bending stress and (*b*) the maximum shearing stress in the simply supported beam shown in Fig. 8-70. *Ans.* (*a*) 22,000 lb/in^2, (*b*) 1660 lb/in^2

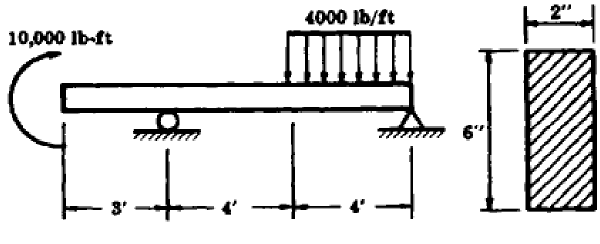


Fig. 8-70

8.52. For a bar of solid circular cross section, determine the amount by which the fully plastic moment exceeds the moment that just causes the yield point to be reached in the extreme fibers. *Ans.* 69.6 percent

8.53. Consider bending of a bar of isosceles triangular cross section (Fig. 8-71). The loads lie in the vertical plane of symmetry. Determine the ratio of the fully plastic moment to the moment that just causes yielding of the extreme fibers. *Ans.* 2.48

8.54. For the T-section shown in Fig. 8-72, determine the location of the neutral axis for fully plastic action. *Ans.* 137.5 mm above the lowest fibers of the section

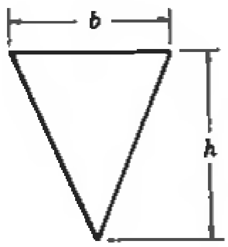


Fig. 8-71

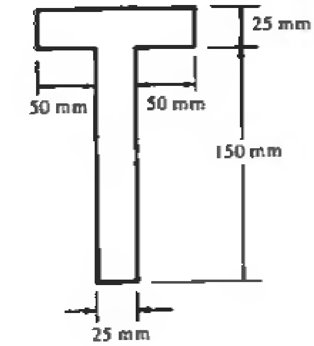


Fig. 8-72

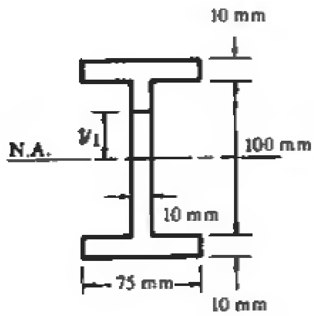


Fig. 8-73

8.55. A bar of solid circular cross section of radius r is subject to bending. By what percent does the bending moment required to cause plastic action at the distance $r/2$ from the neutral axis exceed that required to just cause the yield point to be reached in the extreme fibers? *Ans.* 49.2 percent

8.56. For the section shown in Fig. 8-73 determine the value of y_1 which represents the point where elastic action terminates and plastic flow begins, when the beam is subject to a bending moment of 20 kN · m. Also determine the radius of curvature. Take the yield point of the material to be 200 MPa, and $E = 200$ GPa. *Ans.* $y_1 = 47.4$ mm, $R = 52.6$ m

8.57. A wide-flange section 600 mm high has welded to each of its flanges a 25 mm thick cover plate (see Fig. 8-74). The moment of inertia of the section is $1000 \times 10^6\ \text{mm}^4$. At a particular location along the length of the beam, the transverse shear force is 300 kN. Determine the shear force per unit length existing in each of the four welds. *Ans.* 146 N/mm

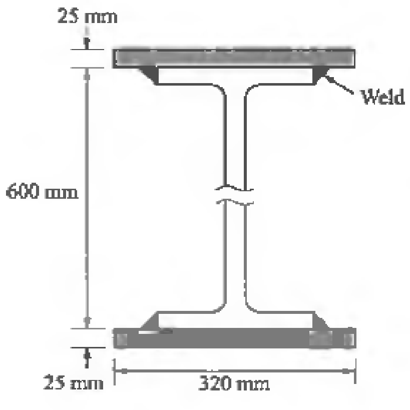


Fig. 8-74

Table 8-1. Properties of Selected Wide-Flange Sections, USCS Units

Designation*	Weight per foot, lb/ft	Area, in^2	I (about x-x axis), in^4	Z, in^3	I (about y-y axis), in^4	Z_p (plastic section modulus), in^3
W 18 × 70	70.0	20.56	1153.9	128.2	78.5	144.7
W 18 × 55	55.0	16.19	889.9	98.2	42.0	111.6
W 12 × 72	72.0	21.16	597.4	97.5	195.3	108.1
W 12 × 58	58.0	17.06	476.1	78.1	107.4	86.5
W 12 × 50	50.0	14.71	394.5	64.7	56.4	72.6
W 12 × 45	45.0	13.24	350.8	58.2	50.0	64.9
W 12 × 40	40.0	11.77	310.1	51.9	44.1	57.6
W 12 × 36	36.0	10.59	280.8	45.9	23.7	51.4
W 12 × 32	32.0	9.41	246.8	40.7	20.6	45.0
W 12 × 25	25.0	7.39	183.4	30.9	14.5	35.0
W 10 × 89	89.0	26.19	542.4	99.7	180.6	114.4
W 10 × 54	54.0	15.88	305.7	60.4	103.9	67.0
W 10 × 49	49.0	14.40	272.9	54.6	93.0	60.3
W 10 × 45	45.0	13.24	248.6	49.1	53.2	55.0
W 10 × 37	37.0	10.88	196.9	39.9	42.2	45.0
W 10 × 29	29.0	8.53	157.3	30.8	15.2	34.7
W 10 × 23	23.0	6.77	120.6	24.1	11.3	33.7
W 10 × 21	21.0	6.19	106.3	21.5	9.7	24.1
W 8 × 40	40.0	11.76	146.3	35.5	49.0	39.9
W 8 × 35	35.0	10.30	126.5	31.1	42.5	34.7
W 8 × 31	31.0	9.12	109.7	27.4	37.0	30.4
W 8 × 28	28.0	8.23	97.8	24.3	21.6	27.1
W 8 × 27	27.0	7.93	94.1	23.4	20.8	23.9
W 8 × 24	24.0	7.06	82.5	20.8	18.2	23.1
W 8 × 19	19.0	5.59	64.7	16.0	7.9	17.7
W 6 × $15\frac{1}{2}$	15.5	4.62	28.1	9.7	9.7	11.3

*The first number after the W is the nominal depth of the section in inches. The second number is the weight in pounds per foot of length.

Table 8-2. Properties of Selected Wide-Flange Sections, SI Units

Designation*	Mass per meter, kg/m	Area, mm^2	I (about x-x axis), 10^6 mm^4	Z, 10^3 mm^3	I (about y-y axis), 10^6 mm^4	Z_p (plastic section modulus), 10^3 mm^3
W 460 × 103	102.9	13,200	479	2100	32.6	2370
W 460 × 81	80.9	10,400	369	1610	17.4	1820
W 305 × 106	105.8	13,600	248	1590	81.0	1770
W 305 × 85	85.3	11,000	198	1280	44.6	1410
W 305 × 74	73.5	9,480	164	1060	23.4	1190
W 305 × 66	66.2	8,530	146	952	20.7	1060
W 305 × 59	58.8	7,580	129	849	18.3	942
W 305 × 53	52.9	6,820	117	750	9.83	840
W 305 × 47	47.0	6,060	102	665	8.55	736
W 305 × 37	36.8	4,760	76.1	505	6.02	572
W 254 × 131	130.8	16,900	225	1630	74.9	1870
W 254 × 79	79.4	10,200	127	988	43.1	1100
W 254 × 72	72.0	9,280	113	893	38.6	986
W 254 × 66	66.2	8,530	103	803	22.1	899
W 254 × 54	54.4	7,010	81.7	652	17.5	736
W 254 × 43	42.6	5,490	65.3	504	6.31	567
W 254 × 34	33.8	4,360	50.0	394	4.69	551
W 254 × 31	30.9	3,990	44.1	352	4.02	394
W 203 × 59	58.8	7,580	60.7	580	20.3	652
W 203 × 51	51.4	6,630	52.5	508	17.6	567
W 203 × 46	45.6	5,870	45.5	448	15.4	497
W 203 × 41	41.2	5,300	40.6	397	8.96	443
W 203 × 40	39.7	5,110	39.0	383	8.63	391
W 203 × 35	35.3	4,550	34.2	340	7.55	378
W 203 × 28	27.9	3,600	26.8	262	3.28	290
W 152 × 23	22.8	2,980	11.7	159	4.02	185

*The first number after the W is the nominal depth of the section in millimeters. The second number is the mass in kilograms per meter of length.

Chapter 9

Elastic Deflection of Beams: Double-Integration Method

INTRODUCTION

In Chap. 8 it was stated that lateral loads applied to a beam not only give rise to internal bending and shearing stresses in the bar, but also cause the bar to deflect in a direction perpendicular to its longitudinal axis. The stresses were examined in Chap. 8 and it is the purpose of this chapter and also Chap. 10 to examine methods for calculating the deflections.

DEFINITION OF DEFLECTION OF A BEAM

The deformation of a beam is most easily expressed in terms of the deflection of the beam from its original unloaded position. The deflection is measured from the original neutral surface to the neutral surface of the deformed beam. The configuration assumed by the deformed neutral surface is known as the elastic curve of the beam. Figure 9-1 represents the beam in its original undeformed state and Fig. 9-2 represents the beam in the deformed configuration it has assumed under the action of the load.

Fig. 9-1

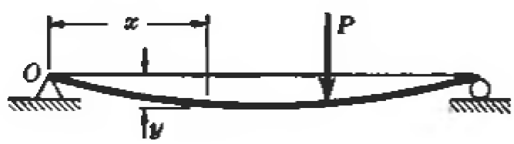


Fig. 9-2

The displacement y is defined as the deflection of the beam. Often it will be necessary to determine the deflection y for every value of x along the beam. This relation may be written in the form of an equation which is frequently called the equation of the deflection curve (or elastic curve) of the beam.

IMPORTANCE OF BEAM DEFLECTIONS

Specifications for the design of beams frequently impose limitations upon the deflections as well as the stresses. Consequently, in addition to the calculation of stresses as outlined in Chap. 8, it is essential that the designer be able to determine deflections. For example, in many building codes the maximum allowable deflection of a beam is not to exceed $\frac{1}{300}$ of the length of the beam. Components of aircraft usually are designed so that deflections do not exceed some preassigned value, else the aerodynamic characteristics may be altered. Thus, a well-designed beam must not only be able to carry the loads to which it will be subjected but it must not undergo undesirably large deflections. Also, the evaluation of reactions of statically indeterminate beams involves the use of various deformation relationships. These will be examined in detail in Chap. 11.

METHODS OF DETERMINING BEAM DEFLECTIONS

Numerous methods are available for the determination of beam deflections. The most commonly used are the following:

1. Double-integration method
2. Method of singularity functions
3. Elastic energy methods

The first method is described in this chapter, the use of singularity functions is discussed in Chap. 10, and elastic energy methods are treated in Chap. 15. It is to be carefully noted that all of these methods apply *only* if all portions of the beam are acting in the *elastic range of action*.

DOUBLE-INTEGRATION METHOD

The differential equation of the deflection curve of the bent beam is

$$EI\frac{d^2y}{dx^2} = M \tag{9.1}$$

where x and y are the coordinates shown in Fig. 9-2. That is, y is the deflection of the beam. This equation is derived in Problem 9.1. In the equation E denotes the modulus of elasticity of the beam and I represents the moment of inertia of the beam cross section about the neutral axis, which passes through the centroid of the cross section. Also, M represents the bending moment at the distance x from one end of the beam. This quantity was defined in Chap. 6 to be the algebraic sum of the moments of the external forces to one side of the section at a distance x from the end about an axis through this section. Usually, M will be a function of x and it will be necessary to integrate (*9.1*) twice to obtain an algebraic equation expressing the deflection of y as a function of x.

Equation (*9.1*) is the basic differential equation that governs the elastic deflection of all beams irrespective of the type of applied loading. For applications, see Problems 9.2 through 9.14 and 9.16 through 9.22.

THE INTEGRATION PROCEDURE

The double-integration method for calculating deflections of beams merely consists of integrating (*9.1*). The first integration yields the slope dy/dx at any point in the beam and the second integration gives the deflection y for any value of x. The bending moment M must, of course, be expressed as a function of the coordinate x before the equation can be integrated. For the cases to be studied here the integrations are extremely simple.

Since the differential equation (*9.1*) is of the second order, its solution must contain two constants of integration. These two constants must be evaluated from known conditions concerning the slope or deflection at certain points in the beam. For example, in the case of a cantilever beam the constants would be determined from the conditions of zero change of slope as well as zero deflection at the built-in end of the beam.

Frequently two or more equations are necessary to describe the bending moment in the various regions along the length of a beam. This was emphasized in Chap. 6. In such a case, (*9.1*) must be written for each region of the beam and integration of these equations yields two constants of integration for each region. These constants must then be determined so as to impose conditions of continuous deformations and slopes at the points common to adjacent regions. See Problems 9.17 through 9.19.

SIGN CONVENTIONS

The sign conventions for bending moment adopted in Chap. 6 will be retained here. The quantities E and I appearing in (*9.1*) are, of course, positive. Thus, from this equation, if M is positive for a certain value of x, then d^2y/dx^2 is also positive. With the above sign convention for bending moments, it is necessary to consider the coordinate x along the length of the beam to be positive to the right and the deflection y to be positive upward. This will be explained in detail in Problem 9.1. With these algebraic signs the integration of (*9.1*) may be carried out to yield the deflection y as a function of x, with the understanding that upward beam deflections are positive and downward deflections negative.

ASSUMPTIONS AND LIMITATIONS

In the derivation of (*9.1*) it is assumed that deflections caused by shearing action are negligible compared to those caused by bending action. Also, it is assumed that the deflections are small compared to the cross-sectional dimensions of the beam and that all portions of the beam are acting in the elastic range. Equation (*9.1*) is derived on the basis of the beam being straight prior to the application of loads. Beams with slight deviations from straightness prior to loading may be treated by modifying this equation as indicated in Problem 9.25.

Solved Problems

9.1. Obtain the differential equation of the deflection curve of a beam loaded by lateral forces.

In Problem 8.1 the relationship

$$M = \frac{EI}{\rho} \tag{1}$$

was derived. In this expression M denotes the bending moment acting at a particular cross section of the beam, ρ the radius of curvature to the neutral surface of the beam at this same section, E the modulus of elasticity, and I the moment of the cross-sectional area about the neutral axis passing through the centroid of the cross section. In this book we will usually be concerned with those beams for which E and I are constant along the entire length of the beam, but in general both M and ρ will be functions of x.

Equation (*1*) may be written in the form

$$\frac{1}{\rho} = \frac{M}{EI} \tag{2}$$

where the left side of Eq. (*2*) represents the curvature of the neutral surface of the beam. Since M will vary along the length of the beam, the deflection curve will be of variable curvature.

Let the heavy line in Fig. 9-3 represent the deformed neutral surface of the bent beam. Originally the beam coincided with the x-axis prior to loading and the coordinate system that is usually found to be most convenient is shown in the sketch. The deflection y is taken to be positive in the upward direction; hence for the particular beam shown, all deflections are negative.

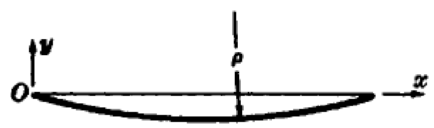


Fig. 9-3

An expression for the curvature at any point along the curve representing the deformed beam is readily available from differential calculus. The exact formula for curvature is

$$\frac{1}{\rho} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}} \tag{3}$$

In this expression, dy/dx represents the slope of the curve at any point; and for small beam deflections this quantity and in particular its square are small in comparison to unity and may reasonably be neglected. This assumption of small deflections simplifies the expression for curvature into

$$\frac{1}{\rho} \approx \frac{d^2y}{dx^2} \tag{4}$$

Hence for small deflections, (2) becomes $d^2y/dx^2 = M/EI$ or

$$EI\frac{d^2y}{dx^2} = M \tag{5}$$

This is the differential equation of the deflection curve of a beam loaded by lateral forces. In honor of its codiscoverers, it is called the Euler-Bernoulli equation of bending of a beam. In any problem it is necessary to integrate this equation to obtain an algebraic relationship between the deflection y and the coordinate x along the length of the beam. This will be carried out in the following problems.

9.2. Determine the deflection at every point of the cantilever beam subject to the single concentrated force P, as shown in Fig. 9-4.

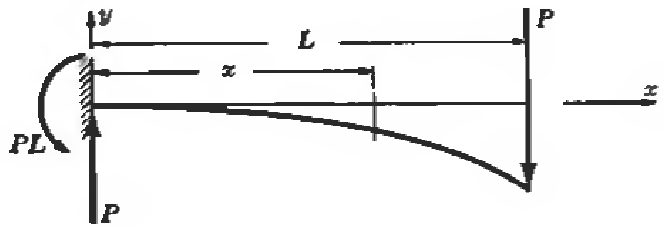


Fig. 9-4

Fig. 9-5

The x-y coordinate system shown is introduced, where the x-axis coincides with the original unbent position of the beam. The deformed beam has the appearance indicated by the heavy line. It is first necessary to find the reactions exerted by the supporting wall upon the bar, and these are easily found from statics to be a vertical force reaction P and a moment PL as shown.

The bending moment at any cross section a distance x from the wall is given by the sum of the moments of these two reactions about an axis through this section. Evidently the upward force P produces a positive bending moment Px, and the couple PL if acting alone would produce curvature of the bar as shown in Fig. 9-5. According to the sign convention of Chap. 6, this constitutes negative bending. Hence the bending moment M at the section x is

$$M = -PL + Px$$

The differential equation of the bent beam is

$$EI\frac{d^2y}{dx^2} = M$$

where E denotes the modulus of elasticity of the material and I represents the moment of inertia of the cross section about the neutral axis. Substituting,

$$EI\frac{d^2y}{dx^2} = -PL + Px \tag{1}$$

This equation is readily integrated once to yield

$$EI\frac{dy}{dx} = -PLx + \frac{Px^2}{2} + C_1 \tag{2}$$

which represents the equation of the slope, where C_1 denotes a constant of integration. This constant may be evaluated by use of the condition that the slope dy/dx of the beam at the wall is zero since the beam is rigidly clamped there. Thus $(dy/dx)_{x=0} = 0$. Equation (*2*) is true for all values of x and y, and if the condition $x = 0$ is substituted we obtain $0 = 0 + 0 + C_1$ or $C_1 = 0$.

Next, integration of (*2*) yields

$$EIy = -PL\frac{x^2}{2} + \frac{Px^3}{6} + C_2 \tag{3}$$

where C_2 is a second constant of integration. Again, the condition at the supporting wall will determine this constant. There, at $x = 0$, the deflection y is zero since the bar is rigidly clamped. Substituting $(y)_{x=0} = 0$ in Eq. (*3*), we find $0 = 0 + 0 + C_2$ or $C_2 = 0$.

Thus Eqs. (*2*) and (*3*) with $C_1 = C_2 = 0$ give the slope dy/dx and deflection y at any point x in the beam. The deflection is a maximum at the right end of the beam ($x = L$), under the load P, and from Eq. (*3*),

$$EIy_{max} = \frac{-PL^3}{3} \tag{4}$$

where the negative value denotes that this point on the deflection curve lies below the x-axis. If only the magnitude of the maximum deflection at $x = L$ is desired, it is usually denoted by Δ_{max} and we have

$$\Delta_{max} = \frac{PL^3}{3EI} \tag{5}$$

9.3. The cantilever beam shown in Fig. 9-4 is 3 m long and loaded by an end force of 20 kN. The cross section is a W203 × 59 steel section, which according to Table 8-2 of Chap. 8 has $I = 60.7 \times 10^{-6}\ \text{m}^4$ and $Z = 580 \times 10^{-6}\ \text{m}^3$. Find the maximum deflection of the beam. Take $E = 200$ GPa.

The maximum deflection occurs at the free end of the beam under the concentrated force and was found in Problem 9.2 to be, by Eq. (*4*),

$$y_{max} = -\frac{PL^3}{3EI} = -\frac{(20{,}000\ \text{N})(3\ \text{m})^3}{3(200 \times 10^9\ \text{N/m}^2)(60.7 \times 10^{-6}\ \text{m}^4)} = -0.0148\ \text{m} \qquad \text{or} \qquad 14.8\ \text{mm}$$

The negative sign of course indicates downward deflection. In the derivation of this deflection formula it was assumed that the material of the beam follows Hooke's law. Actually, from the above calculation alone there is no assurance that the material is not stressed beyond the proportional limit. If it were then the basic beam-bending equation $EI(d^2y/dx^2) = M$ would no longer be valid and the above numerical value would be meaningless. Consequently, in every problem involving beam deflections it is to be emphasized that it is necessary to determine that the maximum bending stress in the beam is below the proportional limit of the material. This is easily done by use of the flexure formula derived in Problem 8.1. According to this formula

$$\sigma = \frac{Mc}{I}$$

where σ denotes the bending stress, M the bending moment, c the distance from the neutral axis to the outer fibers of the beam, and I the second moment of area of the beam cross section about the neutral axis. The maximum bending moment in this problem occurs at the supporting wall and is given by $M_{max} = (20{,}000\ \text{N})(3\ \text{m}) = 60{,}000\ \text{N}\cdot\text{m}$. Using this in the formula for bending stress, we have

$$\sigma_{max} = \frac{M}{Z} = \frac{60{,}000\ \text{N}\cdot\text{m}}{580 \times 10^{-6}\ \text{m}^3} = 103\ \text{MPa}$$

Since this value is below the proportional limit of steel, which is approximately 200 MPa, the use of the beam deflection equation was justifiable.

9.4. Determine the slope of the right end of the cantilever beam loaded as shown in Fig. 9-4. For the beam described in Problem 9.3, determine the value of this slope.

In Problem 9.2 the equation of the slope was found to be

$$EI\frac{dy}{dx} = -PLx + \frac{Px^2}{2}$$

At the free end, $x = L$, and

$$EI\left(\frac{dy}{dx}\right)_{x=L} = -PL^2 + \frac{PL^2}{2}$$

The slope at the end is thus

$$\left(\frac{dy}{dx}\right)_{x=L} = \frac{-PL^2}{2EI}$$

For the beam described in Problem 9.3, this becomes

$$\left(\frac{dy}{dx}\right)_{x=L} = \frac{-(20{,}000\text{ N})(3\text{ m})^3}{2(200\times 10^9\text{ N/m}^2)(60.7\times 10^{-6}\text{ m}^4)}$$

$$= 0.0222\text{ rad} \quad \text{or} \quad 1.27^\circ$$

9.5. Determine the deflection at every point of a cantilever beam subject to the uniformly distributed load w per unit length shown in Fig. 9-6.

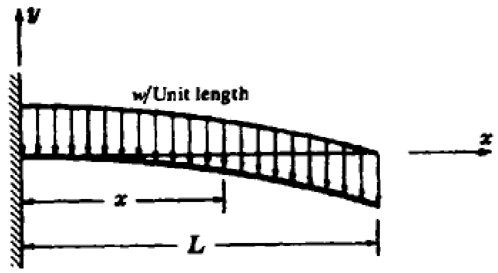


Fig. 9-6

The x-y coordinate system shown is introduced, where the x-axis coincides with the original unbent position of the beam. The deformed beam has the appearance indicated by the heavy line. The equation for the bending moment could be determined in a manner analogous to that used in Problem 9.2, but instead let us seek a slight simplification of that technique. Let us determine the bending moment at the section a distance x from the wall by considering the forces to the right of this section rather than those to the left.

The force of w/unit length acts over the length $L - x$ to the right of this section and hence the resultant force is $w(L - x)$ lb. This force acts at the midpoint of this length of beam to the right of x and thus its moment arm from x is $\frac{1}{2}(L - x)$. The bending moment at the section x is thus given by

$$M = -\frac{w}{2}(L - x)^2$$

the negative sign being necessary since downward loads produce negative bending.

The differential equation describing the bent beam is thus

$$EI\frac{d^2y}{dx^2} = -\frac{w}{2}(L-x)^2 \tag{1}$$

The first integration yields

$$EI\frac{dy}{dx} = \frac{w}{2}\frac{(L-x)^3}{3} + C_1 \tag{2}$$

where C_1 denotes a constant of integration.

This constant may be evaluated by realizing that the left end of the beam is rigidly clamped. At that point, $x = 0$, we have no change of slope and hence $(dy/dx)_{x=0} = 0$. Substituting these values in (2), we find $0 = wL^3/6 + C_1$ or $C_1 = -wL^3/6$. We thus have

$$EI\frac{dy}{dx} = \frac{w}{6}(L-x)^3 - \frac{wL^3}{6} \tag{2'}$$

The next integration yields

$$EIy = -\frac{w}{6}\frac{(L-x)^4}{4} - \frac{wL^3}{6}x + C_2 \tag{3}$$

where C_2 represents a second constant of integration.

At the clamped end, $x = 0$, of the beam the deflection is zero and since (3) holds for all values of x and y, it is permissible to substitute this pair of values in it. Doing this, we obtain

$$0 = \frac{-wL^4}{24} + C_2 \qquad \text{or} \qquad C_2 = \frac{wL^4}{24}$$

The final form of the deflection curve of the beam is thus

$$EIy = -\frac{w}{24}(L-x)^4 - \frac{wL^3}{6}x + \frac{wL^4}{24} \tag{3'}$$

The deflection is a maximum at the right end of the bar ($x = L$) and there we have from (3')

$$EIy_{max} = -\frac{wL^4}{6} + \frac{wL^4}{24} = -\frac{wL^4}{8}$$

where the negative value denotes that this point on the deflection curve lies below the x-axis. The magnitude of the maximum deflection is

$$\Delta_{max} = \frac{wL^4}{8EI} \tag{4}$$

9.6. **A cantilever beam carrying a parabolically distributed load is shown in Fig. 9-7. Determine the equation of the deflected beam as well as the deflection of the tip.**

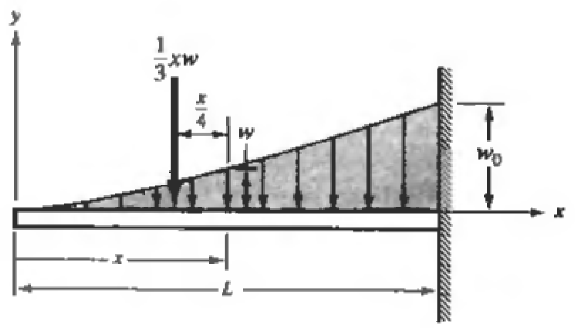


Fig. 9-7

Let us introduce a coordinate system having its origin at the tip of the beam. The intensity of loading at any point x to the right of the tip is, from the properties of a parabola,

$$w = w_0\left(\frac{x}{L}\right)^2 \tag{1}$$

From statics it is known that for any parabolic area such as shown in Fig. 9-8 the area is given by $A = \frac{1}{3}ah$ and the centroid C is located at $x = 3a/4$. Accordingly, it is now possible to determine the bending moment at the point x as the sum of the moments of all loads to the left of x about that point. The resultant of the loading to the left of x is $\frac{1}{3}xw$ and this resultant, shown by the solid arrow in Fig. 9-7, is located a distance $3x/4$ from the tip, or, alternatively, $(x/4)$ from position x. Thus, the bending moment at x is found, with the aid of Eq. (*1*), to be

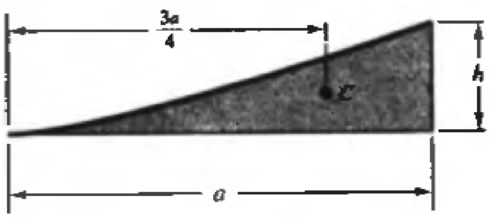


Fig. 9-8

$$-\frac{1}{3}xw\left(\frac{x}{4}\right) \quad \text{or} \quad -\frac{w_0x^4}{12L^2}$$

and the differential equation of the deflection curve is

$$EI\frac{d^2y}{dx^2} = -\frac{w_0x^4}{12L^2} \tag{2}$$

Integrating the first time, we find

$$EI\frac{dy}{dx} = -\frac{w_0}{12L^2}\cdot\frac{x^5}{5} + C_1 \tag{3}$$

When $x = L$, the slope $dy/dx = 0$, so from Eq. (*3*), we have

$$0 = -\frac{w_0L^3}{60} + C_1 \qquad \text{and therefore } C_1 = \frac{w_0L^3}{60}$$

Integrating again, we have

$$EIy = -\frac{w_0}{60L^2}\cdot\frac{x^6}{6} + \frac{w_0L^3}{60}x + C_2 \tag{4}$$

When $x = L$, $y = 0$, so from Eq. (*4*), we have

$$0 = -\frac{w_0L^4}{360} + \frac{w_0L^4}{60} + C_2 \qquad \text{and therefore } C_2 = -\frac{1}{72}w_0L^4$$

The desired equation of the deflected beam is

$$EIy = -\frac{w_0}{360L^2}x^6 + \frac{x_0L^3}{60}x - \frac{1}{72}w_0L^4$$

and the deflection at the tip is

$$EIy]_{x=0} = -\tfrac{1}{72}w_0L^4$$

9.7. Obtain an expression for the deflection curve of the simply supported beam of Fig. 9-9 subject to the uniformly distributed load w per unit length as shown.

The x-y coordinate system shown is introduced, where the x-axis coincides with the original unbent position of the beam. The deformed beam has the appearance indicated by the heavy line. The total load acting on the beam is wL and, because of symmetry, each of the end reactions is $wL/2$. Because of the symmetry of loading, it is evident that the deflected beam is symmetric about the midpoint $x = L/2$.

The equation for the bending moment at any section of a beam loaded and supported as this one is was discussed in Problem 6.3. According to the method indicated there, the portion of the uniform load to the left of the section a distance x from the left support is replaced by its resultant acting at the midpoint of the section of length x. The resultant is wx lb acting downward and hence giving rise to a negative bending moment.

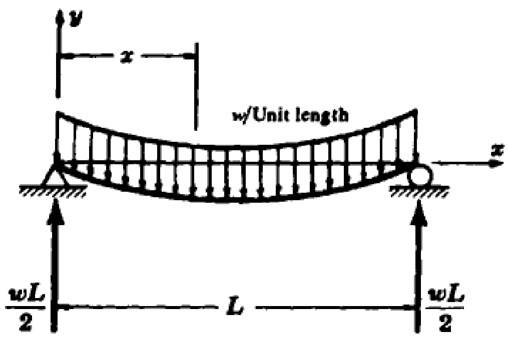


Fig. 9-9

The reaction $wL/2$ gives rise to a positive bending moment. Consequently, for any value of x, the bending moment is

$$M = \frac{wL}{2}x - wx\frac{x}{2}$$

The differential equation of the bent beam is $EI(d^2y/dx^2) = M$. Substituting,

$$EI\frac{d^2y}{dx^2} = \frac{wL}{2}x - \frac{wx^2}{2} \tag{1}$$

Integrating,

$$EI\frac{dy}{dx} = \frac{wL}{2}\frac{x^2}{2} - \frac{w}{2}\frac{x^3}{3} + C_1 \tag{2}$$

It is to be noted that dy/dx represents the slope of the beam. Since the deflected beam is symmetric about the center of the span, i.e., about $x = L/2$, it is evident that the slope must be zero there. That is, the tangent to the deflected beam is horizontal at the midpoint of the beam. This condition enables us to determine C_1. Substituting this condition in (2), we obtain $(dy/dx)_{x=L/2} = 0$,

$$0 = \frac{wL}{4}\frac{L^2}{4} - \frac{w}{6}\frac{L^3}{8} + C_1 \qquad \text{or} \qquad C_1 = -\frac{wL^3}{24}$$

The slope dy/dx at any point is thus given by

$$EI\frac{dy}{dx} = \frac{wL}{4}x^2 - \frac{w}{6}x^3 - \frac{wL^3}{24} \tag{2'}$$

Integrating again, we find

$$EIy = \frac{wL}{4}\frac{x^3}{3} - \frac{w}{6}\frac{x^4}{4} - \frac{wL^3}{24}x + C_2 \tag{3}$$

This second constant of integration C_2 is readily determined by the fact that the deflection y is zero at the left support. Substituting $y_{x=0} = 0$ in (3), we find $0 = 0 - 0 - 0 + C_2$ or $C_2 = 0$.

The final form of the deflection curve of the beam is thus

$$EIy = \frac{wL}{12}x^3 - \frac{w}{24}x^4 - \frac{wL^3}{24}x \tag{3'}$$

The maximum deflection of the beam occurs at the center because of symmetry. Substituting $x = L/2$ in (3'), we obtain

$$EIy_{max} = -\frac{5wL^4}{384}$$

Or, without regard to algebraic sign, we have for the maximum deflection of a uniformly loaded, simply supported beam

$$\Delta_{max} = \frac{5}{384}\frac{wL^4}{EI} \tag{4}$$

9.8. A simply supported beam of length 10 ft and rectangular cross section 1 in × 3 in carries a uniform load of 200 lb/ft. The beam is titanium, type Ti-5Al-2.5Sn, having a yield strength of 115,000 lb/in² and $E = 16 \times 10^6$ lb/in². Determine the maximum deflection of the beam.

From Problem 9.7 the maximum deflection is

$$\Delta_{max} = \frac{5}{384}\frac{wL^4}{EI}$$

Substituting,

$$\Delta_{max} = \frac{5}{384}\frac{(200/12)(120)^4}{(16 \times 10^6)\frac{1}{12}(1)(3)^3} = 1.25 \text{ in}$$

Using the methods of Chap. 8, the maximum bending stress is found to be only 20,000 lb/in², well below the nonlinear range of action of the material. Thus the use of the deflection formula is justified.

9.9. Consider the simply supported beam subject to the two end couples M_1 and M_2 as shown in Fig. 9-10. Determine the equation of the deflection curve and locate the point of peak deflection if $M_1 = 0$.

For equilibrium the resultant of the applied couples, that is, $(M_1 - M_2)$, must be another couple corresponding to the vertical reactions at the ends R_L and R_R. From statics,

$$+\circlearrowleft \Sigma M_0 = -M_1 + M_2 + R_R L = 0$$

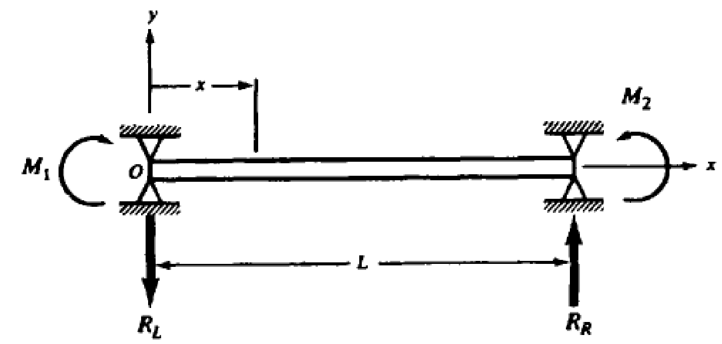


Fig. 9-10

Therefore,

$$R_R = \frac{M_1 - M_2}{L}(\uparrow)$$

$$\Sigma F_v = -R_L + R_R = 0$$

Therefore,

$$R_L = \frac{M_1 - M_2}{L}(\downarrow)$$

The differential equation describing the bent beam is thus

$$EI\frac{d^2y}{dx^2} = M_1 - R_L x \tag{1}$$

Integrating.

$$EI\frac{dy}{dx} = M_1 x - R_L\frac{x^2}{2} + C_1 \tag{2}$$

We have no information concerning the slope anywhere in the beam. Hence it is not possible to determine the constant of integration C_1 at this stage. Let us integrate again:

$$EIy = M_1\frac{x^2}{2} - \frac{R_L}{2}\cdot\frac{x^3}{3} + C_1 x + C_2 \tag{3}$$

We may now determine the two constants of integration through use of the fact that the beam deflection is zero at each end. Accordingly,

When $x = 0$, $y = 0$, so from Eq. (3) we have

$$0 = 0 - 0 + 0 + C_2 \qquad \text{and therefore } C_2 = 0$$

Next, when $x = L$, $y = 0$, so we have from Eq. (3)

$$0 = M_1\frac{L^2}{2} - \frac{R_L}{6}L^3 + C_1 L$$

from which

$$C_1 = -\frac{M_1 L}{3} - \frac{M_2 L}{6}$$

so that the desired equation of the deflection curve is

$$EIy = \frac{M_1}{2}x^2 - \left(\frac{M_1 - M_2}{6L}\right)x^3 - \left(\frac{M_1 L}{3} + \frac{M_2 L}{6}\right)x \tag{4}$$

If $M_1 = 0$, Eq. (4) becomes

$$EIy = \frac{M_2 x^3}{6L} - \frac{M_2 Lx}{6} \tag{5}$$

and

$$EI\frac{dy}{dx} = \frac{M_2 x^2}{2L} - \frac{M_2 L}{6} \tag{6}$$

The point of peak deflection occurs when the slope given by Eq. (6) is zero. Solving Eq. (6) for this value of x,

$$x = \frac{L}{\sqrt{3}} \tag{7}$$

At this point (for $M_1 = 0$) the deflection is given by Eq. (5) to be

$$EIy_{max} = \frac{M_2}{6L}\left(\frac{L}{\sqrt{3}}\right)^3 - \frac{M_2 L}{6}\left(\frac{L}{\sqrt{3}}\right) = -\frac{M_2 L^2\sqrt{3}}{27} \tag{8}$$

Inspection of Eq. (4) for the case $M_1 = M_2 = M$ indicates that

$$EIy = \frac{M}{2}x^2 - \frac{ML}{2}x \tag{9}$$

which indicates a parabolic deflection curve. Yet, Eq. (2) of Problem 9.1 indicates that if M = constant along the length of the beam, the curvature ($1/\rho$) is constant; i.e., the bar bends into a circular arc. The reason for the very slight discrepancy is that Eq. (5) of Problem 9.1, that is,

$$EI\frac{d^2y}{dx^2} = M$$

incorporates the approximation

$$\frac{1}{\rho} \approx \frac{d^2y}{dx^2}$$

as explained in Problem 9.1. In reality the numerical difference between the parabola and the circular arc is very small and in almost all cases may be neglected.

9.10. A simply supported beam is loaded by a couple M_2 as shown in Fig. 9-11. The beam is 2 m long and of square cross section 50 mm on a side. If the maximum permissible deflection in the beam is 5 mm, and the allowable bending stress is 150 MPa, find the maximum allowable load M_2. Take E = 200 GPa.

It is perhaps simplest to determine two values of M_2: one based upon the assumption that the deflection of 5 mm is realized, the other based on the assumption that the maximum bending stress in the bar is 150 MPa. The true value of M_2 is then the minimum of these two values.

Fig. 9-11

Let us first consider that the maximum deflection in the beam is 5 mm. According to Eq. (8), Problem 9.9, we have

$$0.005 = \frac{M_2(2)^2\sqrt{3}}{27(200\times 10^9)(\frac{1}{12})(0.05)(0.05)^3} \qquad \text{or} \qquad M_2 = 2.03\ \text{kN}\cdot\text{m}$$

We shall now assume that the allowable bending stress of 150 MPa is set up in the outer fibers of the beam at the section of maximum bending moment. Referring to Problem 9.9, since $M_1 = 0$, we find the reactions at the ends of the beam are

$$|R| = \frac{M_2}{L}$$

so that they have the appearance shown in Fig. 9-12, and the bending moment diagram for the beam is as shown in Fig. 9-13.

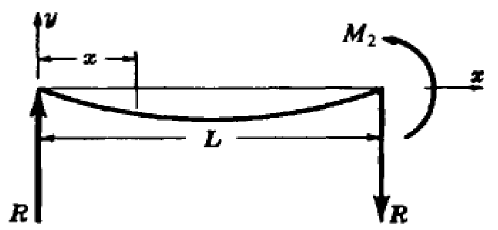


Fig. 9-12

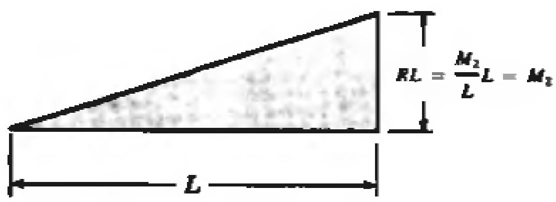


Fig. 9-13

The maximum bending moment in the beam is M_2. Using the usual flexure formula, $\sigma = Mc/I$, we have at the outer fibers of the bar at the right end, i.e., at the section of maximum bending moment,

$$150 \times 10^6 = \frac{M_2(0.025)}{(\frac{1}{12})(0.05)(0.05)^3} \quad \text{or} \quad M_2 = 3.125\ \text{kN}\cdot\text{m}$$

Thus the maximum allowable moment is $M_2 = 2.03\ \text{kN}\cdot\text{m}$.

9.11. A simply supported beam is subjected to the sinusoidal loading shown in Fig. 9-14. Determine the deflection curve of the beam as well as the peak deflection.

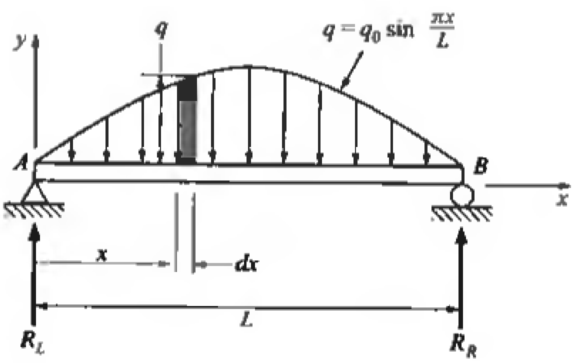


Fig. 9-14

It is first necessary to determine the total load on the beam. Let us consider the shaded element a distance x from the end A and of width dx. If q denotes load per unit length, then the load corresponding to the shaded element is $q\,dx$ and the load on the entire beam is found by integrating:

$$\text{Load} = \int_{x=0}^{x=L} q\,dx = \int_0^L q_0 \sin \frac{\pi x}{L} dx = \frac{2q_0 L}{\pi}$$

From statics, half of this load is carried at each end reaction. Thus,

$$R_L = R_R = \frac{q_0 L}{\pi}$$

The bending moment at the point denoted by x is found as the sum of the moments of all forces to the left of that point. To determine the moment about x of the portion of the sinusoidal load to the left of x, it is necessary to introduce another variable of integration, u, corresponding to a second vertical shaded element of width du, as shown in Fig. 9-15. The variable u must run from $u = 0$ to $u = x$ so as to yield the bending moment due to the sinusoidal load to the left of x.

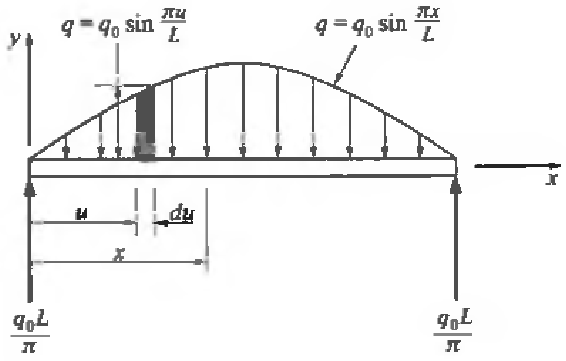


Fig. 9-15

Remembering the contribution that the left support makes to the bending moment, we have

$$M = \frac{q_0 L}{\pi}x - \int_{u=0}^{u=x} q_0\left[\sin\frac{\pi u}{L}\right](du)(x-u)$$

$$= \frac{q_0 L}{\pi}x - q_0\int_{u=0}^{u=x} x\sin\frac{\pi u}{L}\,du + q_0 \overset{(*)}{\int_{u=0}^{u=x}} u\sin\frac{\pi u}{L}\,du \tag{1}$$

In this integration u is a variable and x is to be (temporarily) regarded as a constant. The last integral $(*)$ in Eq. (*1*) must be integrated by parts, remembering that

$$\int \theta(\sin\theta)\,d\theta = \sin\theta - \theta\cos\theta \tag{2}$$

Here,
$$\theta = \frac{\pi u}{L} \qquad d\theta = \frac{\pi}{L}du$$

so that the last integral $(*)$ becomes

$$\int_{u=0}^{u=x} u\sin\frac{\pi u}{L}\,du = \frac{L^2}{\pi^2}\left[\sin\frac{\pi u}{L} - \frac{\pi u}{L}\cos\frac{\pi u}{L}\right]_{u=0}^{u=x}$$

$$= \frac{L^2}{\pi^2}\left[\sin\frac{\pi x}{L}\right] - \frac{Lx}{\pi}\cos\frac{\pi x}{L} \tag{3}$$

The bending moment, Eq. (*1*), is thus

$$M = \frac{q_0 Lx}{\pi} - q_0 x\left(\frac{L}{\pi}\right)\left[-\cos\frac{\pi u}{L}\right]_{u=0}^{u=x} + \frac{q_0 L^2}{\pi^2}\left[\sin\frac{\pi x}{L} - \frac{\pi x}{L}\cos\frac{\pi x}{L}\right]$$

$$= \frac{q_0 L^2}{\pi^2}\sin\frac{\pi x}{L} \tag{4}$$

The differential equation of the deflected beam is thus

$$EI\frac{d^2y}{dx^2} = \frac{q_0 L^2}{\pi^2}\sin\frac{\pi x}{L} \tag{5}$$

Integrating the first time, we have

$$EI\frac{dy}{dx} = -\frac{q_0 L^2}{\pi^2}\left(\frac{L}{\pi}\right)\cos\frac{\pi x}{L} + C_1 \tag{6}$$

As the first boundary condition, from symmetry, when $x = L/2$, $dy/dx = 0$. Substituting in Eq. (*6*), we find $C_1 = 0$. Integrating again,

$$EIy = -\frac{q_0 L^3}{\pi^3}\left(\frac{L}{\pi}\right)\sin\frac{\pi x}{L} + C_2 \tag{7}$$

The second boundary condition is that when $x = 0$, $y = 0$. Substituting in Eq. (7), we have $C_2 = 0$. The equation of the deflected beam is

$$EIy = -\frac{q_0 L^4}{\pi^4} \sin \frac{\pi x}{L} \tag{8}$$

and the peak deflection, at $x = L/2$, is

$$EIy]_{max} = -\frac{q_0 L^4}{\pi^4}$$

9.12. Determine the deflection curve of a simply supported beam subject to the concentrated force P applied as shown in Fig. 9-16.

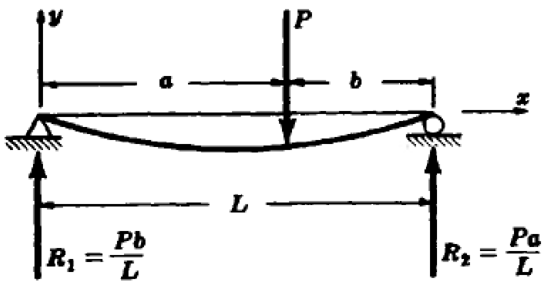


Fig. 9-16

The x-y coordinate system is introduced as shown. The heavy line indicates the configuration of the deformed beam. From statics the reactions are found to be $R_1 = Pb/L$ and $R_2 = Pa/L$.

This problem presents one feature that distinguishes it from the other problems solved thus far in this chapter. Namely, it is essential to consider two different equations describing the bending moment in the beam. One equation is valid to the left of the load P, the other holds to the right of this force. The integration of each equation gives rise to two constants of integration and thus there are four constants of integration to be determined. All problems met thus far have offered only two constants.

In the region to the left of the force P we have the bending moment $M = (Pb/L)x$ for $0 < x < a$. The differential equation of the bent beam thus becomes

$$EI\frac{d^2y}{dx^2} = \frac{Pb}{L}x \qquad \text{for} \qquad 0 < x < a \tag{1}$$

The first integration yields

$$EI\frac{dy}{dx} = \frac{Pb}{L}\frac{x^2}{2} + C_1 \tag{2}$$

No numerical information is available about the slope dy/dx at any point in this region. Since the load is not applied at the center of the beam, there is no reason to believe that the slope is zero at $x = L/2$. However, for the slope of the beam under the point of application of the force P we can write

$$EI\left(\frac{dy}{dx}\right)_{x=a} = \frac{Pba^2}{2L} + C_1 \tag{3}$$

The next integration of (2) yields

$$EIy = \frac{Pb}{2L}\frac{x^3}{3} + C_1x + C_2 \tag{4}$$

At the left support, $y = 0$ when $x = 0$. Substituting these values in *(4)* we immediately find $C_2 = 0$. It is to be noted that it is not permissible to use the condition $y = 0$ at $x = L$ in *(4)* since *(1)* is not valid in that region. We have for the deflection under the point of application of the force P

$$EIy_{x=a} = \frac{Pba^3}{6L} + C_1 a \tag{5}$$

In the region to the right of the force P the bending moment equation is $M = (Pb/L)x - P(x - a)$ for $a < x < L$. Thus

$$EI\frac{d^2y}{dx^2} = \frac{Pb}{L}x - P(x-a) \qquad \text{for} \qquad a < x < L \tag{6}$$

The first integration of this equation yields

$$EI\frac{dy}{dx} = \frac{Pb}{L}\frac{x^2}{2} - \frac{P(x-a)^2}{2} + C_3 \tag{7}$$

Although nothing definite may be said about the slope in this portion of the beam, we have for the slope under the point of application of the force P

$$EI\left(\frac{dy}{dx}\right)_{x=a} = \frac{Pba^2}{2L} + C_3 \tag{8}$$

Under the concentrated load P the slope as given by *(3)* must be equal to that given by *(8)*. Consequently the right sides of these two equations must be equal and we have

$$\frac{Pba^2}{2L} + C_1 = \frac{Pba^2}{2L} + C_3 \qquad \text{or} \qquad C_1 = C_3$$

Equation *(7)* may now be integrated to give

$$EIy = \frac{Pb}{2L}\frac{x^3}{3} - \frac{P(x-a)^3}{6} + C_3 x + C_4 \tag{9}$$

We may write for the deflection under the concentrated load

$$EIy_{x=a} = \frac{Pba^3}{6L} + C_3 a + C_4 \tag{10}$$

The deflection at $x = a$ given by *(5)* must equal that given by *(10)*. Thus the right sides of these two equations are equal and we have

$$\frac{Pba^3}{6L} + C_1 a = \frac{Pba^3}{6L} + C_3 a + C_4$$

Since $C_1 = C_3$, we have $C_4 = 0$.

The condition that $y = 0$ when $x = L$ may now be substituted in *(9)*, yielding

$$0 = \frac{PbL^2}{6} - \frac{Pb^3}{6} + C_3 L \qquad \text{or} \qquad C_3 = \frac{Pb}{6L}(b^2 - L^2)$$

In this manner all four constants of integration are determined. These values may now be substituted in Eqs. *(4)* and *(9)* to give

$$EIy = \frac{Pb}{6L}[x^3 - (L^2 - b^2)x] \qquad \text{for} \qquad 0 < x < a \tag{4'}$$

$$EIy = \frac{Pb}{6L}\left[x^3 - \frac{L}{b}(x-a)^3 - (L^2 - b^2)x\right] \qquad \text{for} \qquad a < x < L \tag{9'}$$

These two equations are necessary to describe the deflection curve of the bent beam. Each equation is valid only in the region indicated.

If the load P acts at the center of the beam, the peak deflection which occurs at $x = L/2$ by symmetry is given by Eq. (4′) as

$$EIy]_{x=L/2} = \frac{P(L/2)}{6L}\left[\left(\frac{L}{2}\right)^3 - \left(L^2 - \left\{\frac{L}{2}\right\}^2\right)\frac{L}{2}\right]$$

$$= -\frac{PL^3}{48} \tag{11}$$

9.13. **The simply supported beam described in Problem 9.12 is 14 ft long and of circular cross section 4 in in diameter. If the maximum permissible deflection is 0.20 in, determine the maximum value of the load P if $a = b = 7$ ft. The material is steel for which $E = 30 \times 10^6$ lb/in².**

The maximum deflection, given by (11) of Problem 9.12, is $\Delta_{max} = PL^3/48EI$. For a circular cross section (see Problem 7.9), $I = \pi D^4/64 = \pi 4^4/64 = 12.6$ in⁴. Also, $L = 14$ ft $= 168$ in. Thus,

$$0.20 = \frac{P(168)^3}{48(30 \times 10^6)(12.6)} \qquad \text{or} \qquad P = 765 \text{ lb}$$

With this load applied at the center of the beam the reaction at each end is 383 lb and the bending moment at the center of the beam is $383(7) = 2681$ lb·ft. This is the maximum bending moment in the beam and the maximum bending stress occurs at the outer fibers at this central section. The maximum bending stress is $\sigma = Mc/I$. Then $\sigma_{max} = 2681(12)(2)/12.6 = 5100$ lb/in². This is below the proportional limit of the material; hence the use of the deflection equation was permissible.

9.14. **Consider the simply supported beam described in Problem 9.12. If the cross section is rectangular, 50 × 100 mm and $P = 20$ kN with $a = 1$ m, $b = 0.5$ m, determine the maximum deflection of the beam. The beam is steel, for which $E = 200$ GPa.**

Since $a > b$, it is evident that the maximum deflection must occur to the left of the load P. It occurs at that point where the slope of the beam is zero.

Differentiating Eq. (4′) of Problem 9.12, we find that the slope in this region is given by

$$EI\frac{dy}{dx} = \frac{Pb}{6L}[3x^2 - (L^2 - b^2)]$$

Setting the slope equal to zero, we find $x = \sqrt{L^2 - b^2/3}$ for the point where the deflection is maximum. The deflection at this point is found by substituting this value of x in (4′):

$$EIy_{max} = \frac{Pb\sqrt{3}}{27L}(L^2 - b^2)^{3/2}$$

For the rectangular section $I = 50(100)^3/12 = 4.167 \times 10^6$ mm⁴. Substituting,

$$y_{max} = \frac{20 \times 10^3(0.5 \times 10^3)[(1.5 \times 10^3)^2 - (0.5 \times 10^3)^2]^{3/2}(\sqrt{3})(10^6)}{27(1.5 \times 10^3)(4.167 \times 10^6)(200 \times 10^9)} = -1.45 \text{ mm}$$

The negative sign indicates that this point on the bent beam lies below the x-axis.

From $\sigma = Mc/I$ the maximum bending stress, which occurs under the load P, is 80 MPa. This is below the proportional limit of steel, so the above deflection equations are valid.

9.15. **The beam AC is simply supported at A and at C is pinned to a cantilever beam CD as shown in Fig. 9-17(a). Both beams have identical flexural rigidities EI. The vertical load of 8 kN acts at point B. Determine the deflection of point B.**

Free-body diagrams of the flexible beams AC and CD appear as in Figs. 9-17(b) and 9-17(c), respectively. For AC, because of symmetry the reaction at C is 4 kN and by Newton's law the equal and opposite force must be exerted at the end C of beam CD as shown in Fig. 9.17(c).

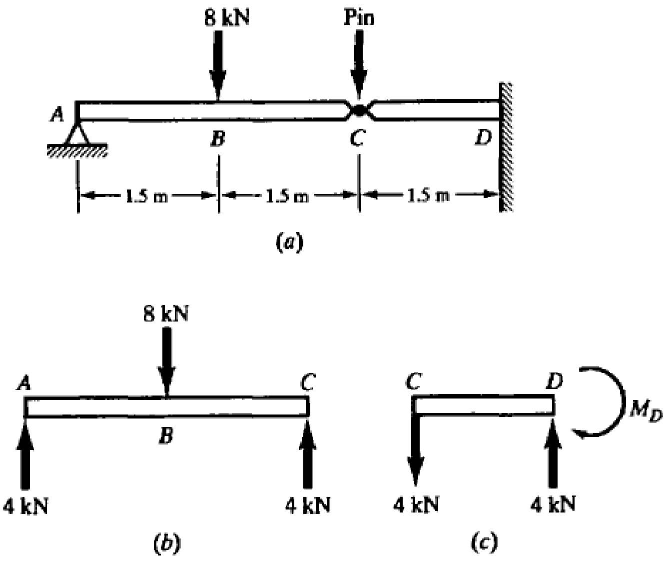


Fig. 9-17

From Problem 9.2 the downward deflection of point C regarded as the tip of beam CD is

$$\Delta_C = \frac{PL^3}{3EI} = \frac{(4\text{ kN})(1.5\text{ m})^3}{3EI} = \frac{4.5}{EI}$$

This same deflection must describe the downward displacement of C regarded as the right end of beam AC. Prior to the deformation of AC due to the 8-kN load, the displacement of point C (on AC) imparts a downward displacement of half that, namely $2.25/EI$ to point B, since the bar during this stage will rotate as a rigid body about A. Then, the deflection of point B due to the 8-kN load must be considered. From Problem 9.12 this is

$$\frac{PL^3}{48EI} = \frac{(8\text{ kN})(3\text{ m})^3}{48EI} = \frac{4.5}{EI}$$

The resultant deflection at point B is thus

$$\Delta_B = \frac{4.5}{EI} + \frac{2.25}{EI} = \frac{6.75}{EI}(\downarrow)$$

9.16. Determine the equation of the deflection curve for a cantilever beam loaded by a uniformly distributed load w per unit length, as well as by a concentrated force P at the free end. See Fig. 9-18.

The deformed beam has the configuration indicated by the heavy line. The x-y coordinate system is introduced as shown. One logical approach to this problem is to determine the reactions at the wall, then

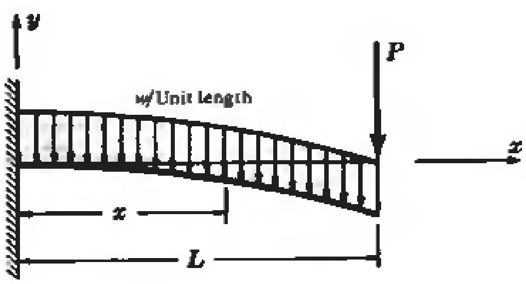


Fig. 9-18

write the differential equation of the bent beam, integrate this equation twice, and determine the constant of integration from the conditions of zero slope and zero deflection at the wall.

Actually this procedure has already been carried out in Problem 9.2 for the case in which only the concentrated load acts on the beam, and in Problem 9.5 when only the uniformly distributed load is acting. For the concentrated force alone the deflection y was found in (*3*) of Problem 9.2 to be

$$EIy = -PL\frac{x^2}{2} + \frac{Px^3}{6} \tag{1}$$

For the uniformly distributed load alone the deflection y was found in (*3'*) of Problem 9.5 to be

$$EIy = -\frac{w}{24}(L-x)^4 - \frac{wL^3}{6}x + \frac{wL^4}{24} \tag{2}$$

It is possible to obtain the resultant effect of these two loads when they act simultaneously merely by adding together the effects of each as they act separately. This is called the *method of superposition*. It is useful in determining deflections of beams subject to a combination of loads, such as we have here. Essentially it consists in utilizing the results of simpler beam-deflection problems to build up the solutions of more complicated problems. Thus it is not an independent method of determining beam deflections.

According to this method the deflection at any point of a beam subject to a combination of loads can be obtained as the sum of the deflections produced at this point by each of the loads acting separately. The final deflection equation resulting from the combination of loads is then obtained by adding the deflection equations for each load.

For the present beam the final deflection equation is given by adding Eqs. (*1*) and (*2*):

$$EIy = -PL\frac{x^2}{2} + \frac{Px^3}{6} - \frac{w}{24}(L-x)^4 - \frac{wL^3}{6}x + \frac{wL^4}{24} \tag{3}$$

The slope dy/dx at any point in the beam is merely found by differentiating both sides of (*3*) with respect to x.

The method of superposition is valid in all cases where there is a linear relationship between each separate load and the separate deflection which it produces.

9.17. Determine the deflection curve of an overhanging beam subject to a uniform load w per unit length and supported as shown in Fig. 9-19.

We replace the distributed load by its resultant of wL acting at the midpoint of the length L. Taking moments about the right reaction, we have

$$\Sigma M_C = R_1 b - \frac{wL^2}{2} = 0 \qquad \text{or} \qquad R_1 = \frac{wL^2}{2b}$$

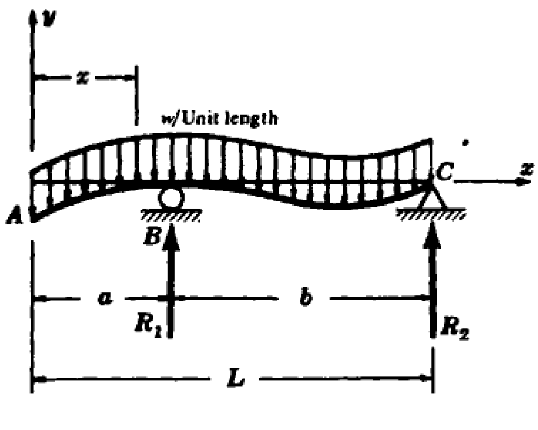


Fig. 9-19

Summing forces vertically, we find

$$\Sigma F_v = \frac{wL^2}{2b} + R_2 - wL = 0$$

or

$$R_2 = wL - \frac{wL^2}{2b}$$

The bending moment equation in the left overhanging region is $M = -wx^2/2$ for $0<x<a$. Consequently the differential equation of the bent beam in that region is

$$EI\left(\frac{d^2y}{dx^2}\right) = \frac{-wx^2}{2} \qquad \text{for} \qquad 0<x<a \tag{1}$$

Two successive integrations yield

$$EI\frac{dy}{dx} = -\frac{w}{2}\frac{x^3}{3} + C_1 \tag{2}$$

$$EIy = -\frac{w}{6}\frac{x^4}{4} + C_1x + C_2 \tag{3}$$

The bending moment equation in the region between supports is $M = -wx^2/2 + R_1(x-a)$. The differential equation of the bent beam in that region is thus

$$EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2} + \frac{wL^2}{2b}(x-a) \qquad \text{for} \qquad a<x<L \tag{4}$$

Two integrations of this equation yield

$$EI\frac{dy}{dx} = -\frac{w}{2}\frac{x^3}{3} + \frac{wL^2}{2b}\frac{(x-a)^2}{2} + C_3 \tag{5}$$

$$EIy = -\frac{w}{6}\frac{x^4}{4} + \frac{wL^2}{4b}\frac{(x-a)^3}{3} + C_3x + C_4 \tag{6}$$

Since we started with two second-order differential equations, (1) and (4), and two constants of integration arose from each, we have four constants C_1, C_2, C_3, and C_4 to evaluate from known conditions concerning slopes and deflections. These conditions are the following:

1. When $x = a$, $y = 0$ in the overhanging region.
2. When $x = a$, $y = 0$ in the region between supports.
3. When $x = L$, $y = 0$ in the region between supports.
4. When $x = a$, the slope given by (2) must be equal to that given by (5); consequently the right sides of these equations must be equal when $x = a$.

Substituting condition (1) in (3), we obtain

$$0 = \frac{-wa^4}{24} + C_1a + C_2 \tag{7}$$

Substituting condition (2) in (6), we find

$$0 = \frac{-wa^4}{24} + C_3a + C_4 \tag{8}$$

Substituting condition (3) in (6), we get

$$0 = \frac{-wL^4}{24} + \frac{wL^2b^2}{12} + C_3L + C_4 \tag{9}$$

Finally, equating slopes at the left reaction by substituting $x = a$ in the right sides of equations (2) and (5), we obtain

$$\frac{-wa^3}{6} + C_1 = \frac{-wa^3}{6} + C_3 \tag{10}$$

Note that there is no reason for assuming the slope to be zero at the left support, $x = a$.

These last four Eqs. (7), (8), (9), (10) may now be solved for the four unknown constants C_1, C_2, C_3, C_4. The solution is found to be

$$C_1 = C_3 = \frac{w(L^4 - a^4)}{24b} - \frac{wL^2b}{12} \tag{11}$$

$$C_2 = C_4 = \frac{wa^4}{24} - \frac{w(L^4 - a^4)a}{24b} + \frac{wL^2ab}{12} \tag{12}$$

The two equations describing the deflection curve of the bent bar are found by substituting these values of the constants in (3) and (6). These equations may be written in the final forms

$$EIy = -\frac{wx^4}{24} + \frac{w(L^4 - a^4)x}{24b} - \frac{wL^2bx}{12} + \frac{wa^4}{24} - \frac{w(L^4 - a^4)a}{24b} + \frac{wL^2ab}{12} \quad \text{for } 0 < x < a \tag{3'}$$

$$EIy = -\frac{wx^4}{24} + \frac{wL^2(x-a)^3}{12b} + \frac{w(L^4 - a^4)x}{24b} - \frac{wL^2bx}{12} + \frac{wa^4}{24} - \frac{w(L^4 - a^4)a}{24b} + \frac{wL^2ab}{12} \quad \text{for } a < x < L \tag{6'}$$

Problem 9.17, although involving relatively simple geometry and loading, is obviously very tedious when solved by the method of double integration. Usually the method is well suited only to situations where a single equation describes the entire deflected beam. Chapter 10 will be based upon use of singularity functions (see Chap. 6) as a much-simplified approach to beam deflections far better adapted to more complex conditions of loading and support than is the straightforward double-integration approach. Also, the singularity function approach is very well adapted to computer implementation, as will be shown in Chap. 10.

9.18. Determine the equation of the deflection curve for the overhanging beam loaded by the two equal forces P shown in Fig. 9-20.

The x-y coordinate system is introduced as shown with the x-axis coinciding with the original unbent position of the bar. The fact that the left end of the bar deflects from the coordinate curve presents no difficulties. For the condition of symmetry it is evident that each support exerts a vertical force P upon the bar.

The bending moment in the left overhanging region is

$$M = -Px \quad \text{for} \quad 0 < x < a$$

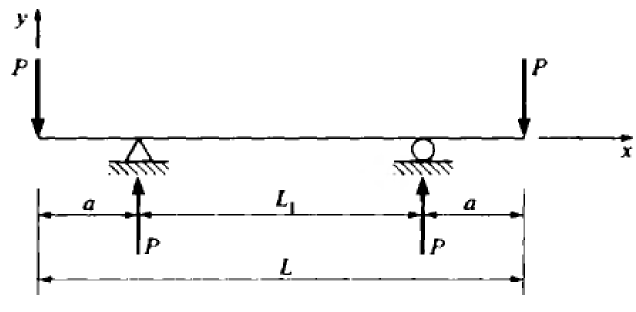


Fig. 9-20

and the differential equation of the bent beam in that region is

$$EI\frac{d^2y}{dx^2} = -Px \qquad \text{for} \qquad 0 < x < a \tag{1}$$

The first integration of this equation yields

$$EI\frac{dy}{dx} = -P_1\frac{x^2}{2} + C_1 \tag{2}$$

Nothing definite is known about the slope dy/dx in this region. In particular, it is to be emphasized that there is no justification for assuming the slope to be zero at the point of support $x = a$. We may denote the slope there by the notation

$$EI\left(\frac{dy}{dx}\right)_{x=a} = -P\left(\frac{a^2}{2}\right) + C_1 \tag{3}$$

The next integration yields

$$EIy = -\frac{P}{2}\left(\frac{x^3}{3}\right) + C_1x + C_2 \tag{4}$$

Since the beam is hinged at the support, it is known that the deflection y is 0 there. Thus, $(y)_{x=a} = 0$. Substituting $y = 0$ when $x = a$ in (4), we find

$$0 = -\frac{Pa^3}{6} + C_1a + C_2 \tag{5}$$

The bending moment in the central region of the beam between supports is $M = -Pa$ and the differential equation of the bent beam in the central region is

$$EI\frac{d^2y}{dx^2} = -Pa \qquad \text{for} \qquad a < x < (L - a) \tag{6}$$

Integrating, we obtain

$$EI\frac{dy}{dx} = -Pax + C_3 \tag{7}$$

Because of the symmetry of loading it is evident that the slope dy/dx must be zero at the midpoint of the bar. Thus $(dy/dx)_{x=L/2} = 0$. Substituting these values in Eq. (7), we find

$$0 = -Pa\left(\frac{L}{2}\right) + C_3 \qquad \text{or} \qquad C_3 = \frac{PaL}{2} \tag{8}$$

Also, from Eq. (7) we may say that the slope of the beam over the left support, $x = a$, is given by substituting $x = a$ in this equation. This yields

$$EI\left(\frac{dy}{dx}\right)_{x=a} = -Pa^2 + \frac{PaL}{2} \tag{9}$$

But the slope dy/dx as given by this expression must be equal to that given by Eq. (3), since the bent bar at that point must have the same slope, no matter which equation is considered. Equating the right sides of Eqs. (3) and (9), we obtain

$$-\frac{Pa^2}{2} + C_1 = -Pa^2 + \frac{PaL}{2} \tag{10}$$

or

$$C_1 = -\frac{Pa^2}{2} + \frac{PaL}{2} \tag{11}$$

Substituting this value of C_1 in Eq. (5), we find

$$0 = -\frac{Pa^3}{6} - \frac{Pa^3}{2} + \frac{Pa^2 L}{2} + C_2 \tag{12}$$

or

$$C_2 = \frac{2Pa^3}{3} - \frac{Pa^2 L}{2}$$

The next integration of Eq. (7) yields

$$EIy = -Pa\frac{x^2}{2} + \frac{PaL}{2}(x) + C_4 \tag{13}$$

Again, it may be said that the deflection y is zero at the left support, where $x = a$. Although this same condition was used previously in obtaining Eq. (5), there is no reason why it should not be used again. In fact, it is essential to use it in order to solve for the constant C_4 in Eq. (13). Thus, substituting the values $(y)_{x=a} = 0$ in Eq. (13), we obtain

$$0 = -\frac{Pa^3}{2} + \frac{Pa^2 L}{2} + C_4 \qquad \text{or} \qquad C_4 = \frac{Pa^3}{2} - \frac{Pa^2 L}{2} \tag{14}$$

Thus two equations were required to define the bending moment in the left and central regions of the beam. Each equation was used in conjunction with the second-order differential equation describing the bent beam, and thus two constants of integration arose from the solution of each of these two equations. It was necessary to utilize four conditions concerning slope and deflection in order to determine these four constants. These conditions were:

(*a*) When $x = a$, $y = 0$ for the overhanging portion of the beam.
(*b*) When $x = a$, $y = 0$ for the central portion of the beam.
(*c*) When $x = L/2$, $dy/dx = 0$ for the central portion of the beam.
(*d*) When $x = a$, the slope dy/dx is the same for the deflection curve on either side of the support.

Finally, the equations of the bent beam may be written in the forms

$$EIy = -\frac{Px^3}{6} - \frac{Pa^2 x}{2} + \frac{PaLx}{2} + \frac{2Pa^3}{3} - \frac{Pa^2 L}{2} \qquad \text{for} \qquad 0 < x < a \tag{15}$$

$$EIy = -\frac{Pax^2}{2} + \frac{PaLx}{2} + \frac{Pa^3}{2} - \frac{Pa^3 L}{2} \qquad \text{for} \qquad a < x < (L-a) \tag{16}$$

Because of the symmetry there is no need to write the equation for the deformed beam in the right overhanging region.

9.19. For the overhanging beam of Problem 9.18, each force P is 4000 lb. The distance a is 3 ft and the length L is 16 ft. The bar is steel and of circular cross section 4 in in diameter. Determine the deflection under each load and also the deflection at the center of the beam. Take $E = 30 \times 10^6$ lb/in^2.

The moment of inertia is given by $I = \pi(4)^4/64 = 12.6$ in^4, according to Problem 7.9 in Chap. 7. Also, we have $a = 3$ ft $= 36$ in, $L = 16$ ft $= 192$ in. The deflection anywhere in the left overhanging region is given by Eq. (15) of Problem 9.18. Under the concentrated force P we have $x = 0$, and substituting these values in Eq. (15) we obtain

$$30 \times 10^6(12.6)\,(y)_{x=0} = \frac{2(4000)\,(36)^3}{3} - \frac{4000(36)^2(192)}{2}$$

or

$$(y)_{x=0} = -0.96 \text{ in}$$

The deflection anywhere in the central portion between supports is given by Eq. (*16*) of Problem 9.18. At the center of the beam we have $x = 8\text{ ft} = 96\text{ in}$ and, as before $a = 36\text{ in}$, $L = 192\text{ in}$, and $P = 4000\text{ lb}$. Substituting in Eq. (*16*), we find

$$(30 \times 10^6)(12.6)(y)_{x=8\text{ ft}} = \frac{-4000(36)(96)^2}{2} + \frac{(4000)(36)(192)(96)}{2} + \frac{4000(36)^3}{2} - \frac{4000(36)^2(192)}{2}$$

Solving

$$y_{x=8\text{ ft}} = 0.69\text{ in}$$

The maximum bending stress occurs at the outer fibers of the bar everywhere between the supports, since the bending moment has the constant value of $4000(3) = 12{,}000\text{ lb}\cdot\text{ft}$ in this region. This maximum stress is given by

$$\sigma = \frac{M_c}{I} = \frac{(12{,}000)(12)(2)}{12.6} = 22{,}800\text{ lb/in}^2$$

This is less than the proportional limit of the material.

9.20. A cantilever beam Fig. 9-21(*a*) lying in a horizontal plane when viewed from the top has the triangular plan form shown in Fig. 9-21(*b*). The side view, Fig. 9-21(*c*), shows the constant thickness h of the beam. Determine the deflection curve of the beam and also the deflection of the tip due to the weight of the beam, which is γ per unit volume.

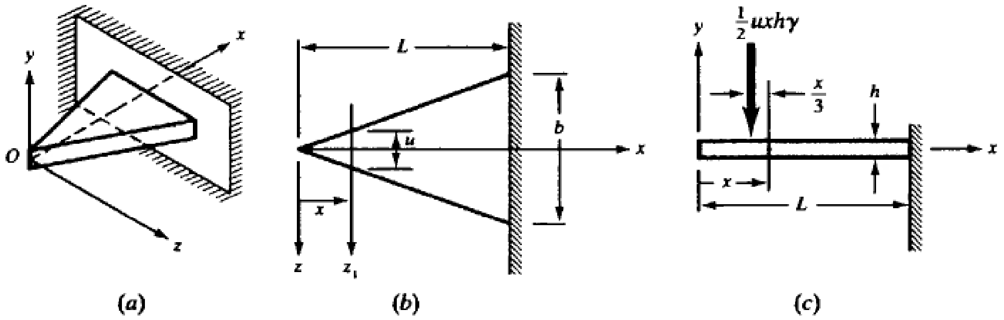


Fig. 9-21

We introduce an x-y-z coordinate system having its origin at point O, the tip of the beam. The location of an arbitrary cross section is denoted by x and the width there is u, as shown in Fig. 9-21(*b*). The overall beam length and base width are denoted by L and b, respectively. From geometry we have

$$u = b\left(\frac{x}{L}\right)$$

and the bending moment at section x is due to the weight of the portion of the triangular beam to the left of x. That weight is

$$\tfrac{1}{2}uxh\gamma$$

and the resultant force corresponding to this weight acts at a distance $x/3$ from the cross-section x, as shown in Fig. 9-21(*c*). Thus, the bending moment at x due to the weight of material to the left of x is

$$M = -\frac{uxh\gamma}{2}\cdot\frac{x}{3} = -\frac{x^2h\gamma}{6}\left(\frac{bx}{L}\right) = -\frac{bh\gamma x^3}{6L} \qquad (1)$$

so that the differential equation of the deflected beam is

$$EI\frac{d^2y}{dx^2} = -\frac{bh\gamma x^3}{6L} \tag{2}$$

However, I is a function of x. Consideration of the cross-section x indicates that I (about an axis z_1 parallel to the z-axis) is

$$I = \frac{1}{12}uh^3 = \frac{1}{12}b\left(\frac{x}{L}\right)h^3$$

so that the differential equation of the beam becomes

$$E\left[\frac{1}{12}b\left(\frac{x}{L}\right)h^3\right]\frac{d^2y}{dx^2} = -\frac{bh\gamma x^3}{6L} \tag{3}$$

or

$$\frac{d^2y}{dx^2} = -\left(\frac{2\gamma}{Eh^2}\right)x^2 \tag{4}$$

Integrating the first time, we obtain

$$\frac{dy}{dx} = -\left(\frac{2\gamma}{Eh^2}\right)\frac{x^3}{3} + C_1 \tag{5}$$

and when $x = L$, $dy/dx = 0$; hence substituting in Eq. (5), we have

$$0 = -\frac{2\gamma L^3}{3Eh^2} + C_1 \qquad \text{and therefore } C_1 = \frac{2\gamma L^3}{3Eh^2}$$

Integrating again, we find

$$y = -\left(\frac{2\gamma}{3Eh^2}\right)\frac{x^4}{4} + \frac{2\gamma L^3}{3Eh^2}x + C_2 \tag{6}$$

As a second boundary condition, when $x = L$, $y = 0$, so from Eq. (6) we find

$$0 = -\frac{2\gamma}{3Eh^2}\cdot\frac{L^4}{4} + \frac{2\gamma L^4}{3Eh^2} + C_2 \qquad \text{and therefore } C_2 = -\frac{\gamma L^4}{2Eh^2}$$

Thus, the equation of the deflected beam is

$$y = -\frac{\gamma}{6Eh^2}x^4 + \frac{2\gamma L^3}{3Eh^2}x - \frac{\gamma L^4}{2Eh^2}$$

which at the tip becomes

$$y\big]_{x=0} = -\frac{\gamma L^4}{2Eh^2}$$

9.21. **A cantilever beam is in the form of a circular truncated cone, of length L, diameter d at the small end, and $2d$ at the large end, as shown in Fig. 9-22. The beam is loaded only by its own weight, which is γ per unit volume. Determine the deflection at the free end.**

From the geometry, we may extend the sloping sides until they intersect at distance x_0 from the left end. By similar triangles we have

$$\frac{d}{x_0} = \frac{2d}{x_0 + L}$$

from which $x_0 = L$. Also,

$$\frac{y}{x} = \frac{d}{2L}$$

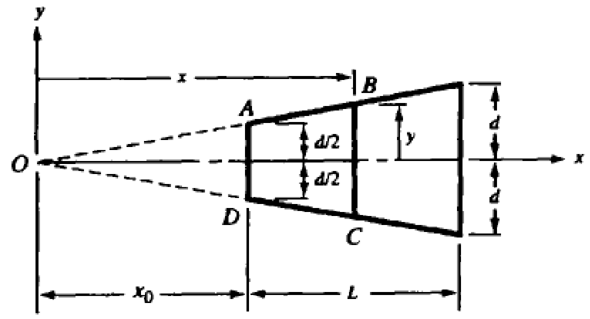


Fig. 9-22

so

$$y = \left(\frac{d}{2L}\right)x$$

The moment of inertia of any circular cross section a distance x from the point O is

$$I = \frac{\pi y^4}{4} = \frac{\pi}{4}\left(\frac{d^4}{16L^4}\right)x^4$$

The differential equation of the deflected beam is given by employing Eq. (5) of Problem 9.1 and using as the bending moment at x the moment of the weight of the solid region $ABCD$ which is found as the moment of the weight of the complete solid cone $OBCO$ about x minus the moment of the cone OAD about that same section. Remembering that the volume of a complete cone is $\frac{1}{3}$ (base) (altitude) and that the center of mass of a solid cone lies $\frac{1}{4}$ the altitude above the base, we have for the equation of the bent beam

$$E\left\{\frac{\pi d^4}{64L^4}\cdot x^4\right\}\frac{d^2y}{dx^2} = -\left\{\frac{1}{3}\pi y^2 x\gamma\left(\frac{x}{4}\right) - \frac{1}{3}\gamma\pi\left(\frac{d}{2}\right)^2 L\left(x - \frac{3}{4}L\right)\right\} \tag{1}$$

This simplifies to the form

$$\frac{d^2y}{dx^2} = \frac{16L^4\gamma\pi}{3\pi d^4 E}\left\{-\frac{d^2}{4L^2} + \frac{Ld^2}{x^3} - \frac{3L^2d^2}{4x^4}\right\} \tag{2}$$

The first integration leads to

$$\frac{dy}{dx} = \frac{16L^4\gamma}{3d^4E}\left\{-\frac{d^2}{4L^2}x + Ld^2\left(-\frac{1}{2x^2}\right) - \frac{3L^2d^2}{4}\left(-\frac{1}{3x^3}\right)\right\} + C_1 \tag{3}$$

As the first boundary condition, when $x = 2L$, $dy/dx = 0$. Substituting in (3), we find

$$C_1 = \frac{19L^3\gamma}{6d^2E}$$

The next integration gives us

$$y = \frac{16L^4\gamma}{3d^4E}\left\{-\frac{d^2}{4L^2}\cdot\frac{x^2}{2} - \frac{Ld^2}{2}\left(-\frac{1}{x}\right) + \frac{L^2d^2}{4}\left(-\frac{1}{2x^2}\right)\right\} + \frac{19L^3\gamma}{6d^2E}x + C_2 \tag{4}$$

and the second boundary condition is that when $x = 2L$, $y = 0$. From Eq. (4) we have

$$C_2 = -\frac{29}{6}\frac{L^4\gamma}{d^2E}$$

The equation of the deflected beam is thus

$$y = \frac{16L^4\gamma}{3d^4E}\left\{-\frac{d^2}{8L^2}x^2 + \frac{Ld^2}{2}\left(\frac{1}{x}\right) - \frac{L^2d^2}{8}\left(\frac{1}{x^2}\right)\right\} + \frac{19L^3\gamma}{6d^2E}x - \frac{29L^4\gamma}{6d^2E} \tag{5}$$

The deflection of the tip is found by setting $x = L$ in Eq. (5) and is

$$y]_{x=L} = -\frac{\gamma L^4}{3d^2 E}$$

9.22. The beam of variable rectangular cross section shown in Fig. 9-23 is simply supported at the ends and loaded by equal magnitude end couples each equal to PL as well as symmetrically placed transverse forces each equal to $1.5P$. The thickness h of the beam is constant. Determine the manner in which the width must vary so that all outer fibers are stressed to the same value σ_0 in both tension and compression. Also determine the central deflection of the beam.

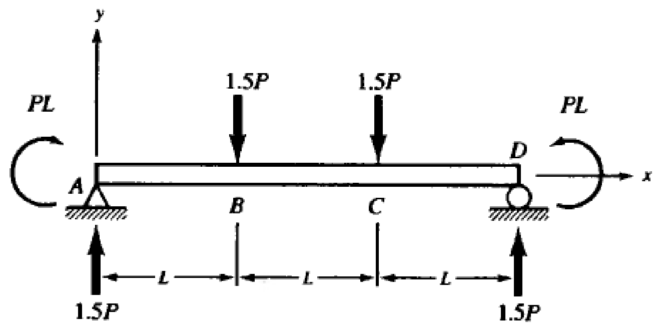


Fig. 9-23

The end reactions are easily found from statics to each be $1.5P$, as shown. The bending moment diagrams corresponding to the force loadings and to the end couples are found by the methods of Chap. 6 and are illustrated in Figs. 9-24(a) and 9-24(b), respectively. The resultant bending moment diagram is found by superposition of these two to be that shown in Fig. 9-24(c).

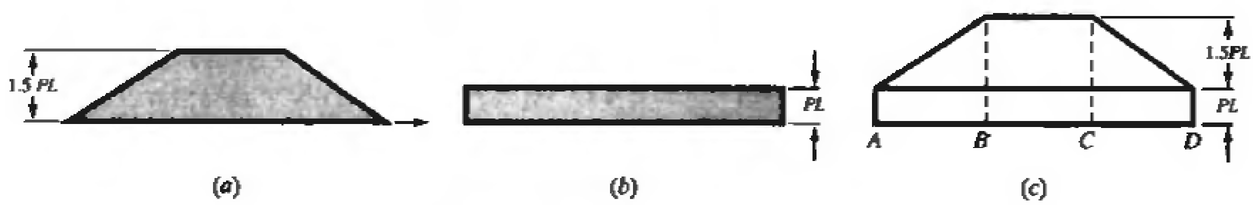


Fig. 9-24

The outer fiber bending stresses in each of the regions AB and BC are found for the rectangular cross section through use of the results of Problems 8.1 and 8.12 to be

$$\sigma_Z = \frac{M_c}{I} = \frac{M}{I/c} = \frac{M}{Z} = \frac{6M}{bh^2} \tag{1}$$

where for the rectangular bar

$$Z = \frac{bh^2}{6} \tag{2}$$

Figure 9-24(c) together with Eq. (1) indicates that in the region BC (since the bending moment is constant) the beam width must also be constant. In that region the cross section must withstand a maximum bending moment of $2.5PL$ and the value of the outer fiber bending stresses is

$$\sigma_0 = \frac{6(2.5PL)}{b_{max}h^2} \tag{3}$$

Solving, we find the maximum width everywhere in BC to be

$$b_{max} = \frac{15PL}{\sigma_0 h^2} \tag{4}$$

In the end region AB, the bending moment from Fig. 9-24(c) is

$$M = PL + 1.5PL\left(\frac{x}{L}\right) \qquad \text{for } 0 < x < L \tag{5}$$

where x is measured positive to the right from the support at A. Since $x = 0$ at A, the width of the beam there must be sufficient to withstand the bending moment PL. Thus, for the outer fiber bending stresses at $x = 0$ to have the magnitude σ_0, we have

$$\sigma_0 = \frac{6M}{b_{min}h^2} = \frac{6PL}{b_{min}h^2}$$

Solving,

$$b_{min} = \frac{6PL}{\sigma_0 h^2} \tag{6}$$

The same width b_{min} must also exist at the right end $x = 3L$ by symmetry. Equation (5) indicates a linear variation of bending moment between A and B so that the width increases linearly from A to B. The resulting constant outer fiber bending stress beam thus appears as shown in Fig. 9-25.

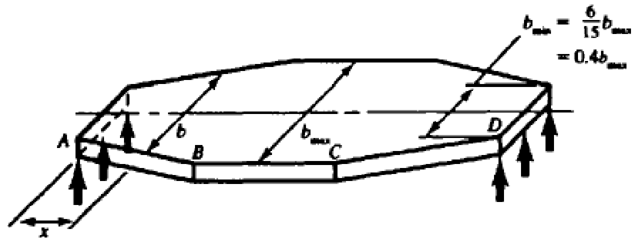


Fig. 9-25

To find the peak deflection, which, because of symmetry, obviously occurs at the midpoint of BC where $x = 3L/2$, we must write the differential equations for bending in regions AB and BC. Because of symmetry of loading and support, there is no need to consider CD since its behavior is symmetric to that of AB. First,

In AB:

$$M = 1.5Px + PL$$

and

$$\sigma_0 = \frac{Mc}{I} = \frac{(PL + 1.5Px)(h/2)}{\frac{1}{12}bh^3} \tag{7}$$

Thus,

$$b = \frac{(PL + 1.5Px)(6)}{\sigma_0 h^2} \tag{8}$$

where b denotes the width of the bar at a distance x from A as indicated in Fig. 9-25. The moment of inertia of the cross section a distance x from A is thus

$$\frac{1}{12}\left[\frac{(PL + 1.5Px)(6)}{\sigma_0 h^2}\right]h^3 \tag{9}$$

The differential equation of the bent beam in AB is

$$E\left[\frac{(PL + 1.5Px)h}{2\sigma_0}\right]\frac{d^2y}{dx^2} = 1.5Px + PL \tag{10}$$

or

$$\frac{d^2y}{dx^2} = \frac{2\sigma_0}{Eh} = \text{constant} \tag{11}$$

Integrating

$$\frac{dy}{dx} = \left(\frac{2\sigma_0}{Eh}\right)x + C_1 \tag{12}$$

Integrating a second time

$$y = \frac{2\sigma_0}{Eh}\cdot\frac{x^2}{2} + C_1x + C_2 \tag{13}$$

As a boundary condition, when $x = 0$, $y = 0$; hence $C_2 = 0$ from Eq. (*13*). Also, when $x = L$, the deflection from Eq. (*13*) is

$$y]_{x=L} = \frac{2\sigma_0}{Eh}\cdot\frac{L^2}{2} + C_1L \tag{14}$$

and the slope at $x = L$ is, from (*12*)

$$\left.\frac{dy}{dx}\right]_{x=L} = \frac{2\sigma_0 L}{Eh} + C_1 \tag{15}$$

In BC, $M = 2.5PL$, and since the width b_{max} in BC is constant, the moment of inertia anywhere in BC is

$$\tfrac{1}{12}b_{max}h^3 \tag{16}$$

so the bent beam in BC is described by the equation

$$E\left[\frac{b_{max}h^3}{12}\right]\frac{d^2y}{dx^2} = 2.5PL \tag{17}$$

or

$$\frac{d^2y}{dx^2} = \frac{30PL}{Eb_{max}h^3} = \text{constant} \tag{18}$$

Integrating,

$$\frac{dy}{dx} = \frac{30PLx}{Eb_{max}h^2} + C_3 \tag{19}$$

As a boundary condition, from symmetry we know that at $x = 3L/2$, $dy/dx = 0$. Hence from (*19*) we have

$$C_3 = -\frac{45PL^2}{Eb_{max}h^3}$$

Integrating again,

$$y = \left(\frac{30PL}{Eb_{max}h^3}\right)\frac{x^2}{2} - \left(\frac{45PL^2}{Eb_{max}h^3}\right)x + C_4 \tag{20}$$

When $x = L$, the deflections are represented by Eqs. (*14*) and (*20*), leading to

$$\frac{2\sigma_0 L^2}{2Eh} + C_1L = -\frac{30PL^3}{Eb_{max}h^3} + C_4 \tag{21}$$

Finally, equating slopes at $x = L$ as given by Eqs. (*15*) and (*19*), we have

$$\frac{2\sigma_0 L}{Eh} + C_1 = \frac{30PL^2}{Eb_{max}h^3} - \frac{45PL^2}{Eb_{max}h^3} \tag{22}$$

Solving Eqs. (*21*) and (*22*), we find

$$C_1 = -\frac{45PL^2}{Eb_{max}h^2} \qquad \text{and therefore } C_4 = 0$$

Hence in the region *BC* from Eq. (*20*), we have

$$y_{max}\Big]_{x=3L/2} = -\frac{33.75PL^3}{Eb_{max}h^3}$$

9.23. Consider the bending of a cantilever beam which remains in contact with a rigid cylindrical surface as it deflects. The tangent to the cantilever is horizontal at point *A* in Fig. 9-26. Determine the deflection of the tip *B* due to the load *P*.

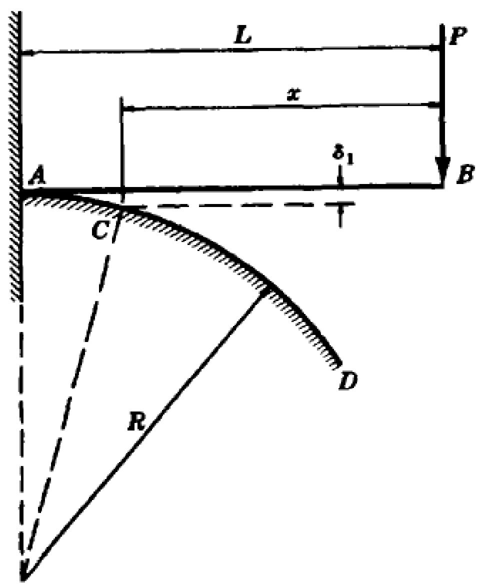


Fig. 9-26

If the curvature of the cantilever at *A* is less than the curvature of the rigid cylindrical surface, then the cantilever touches the surface only at point *A* and the deflection is exactly as found in Problem 9.2. From Problem 9.1, the curvature of the beam at *A* is given by

$$\frac{1}{\rho} = \frac{M}{EI} = \frac{PL}{EI}$$

and thus this curvature must be less than the curvature of the rigid surface, which is $1/R$.

If, however, $1/R = PL/EI$, then the beam comes into contact with the surface to the right of point *A*. We shall denote by P^* the limiting value of the load given by $P^* = EI/RL$. For $P > P^*$ some region *AC* of the beam will be in contact with the surface and at point *C* the curvature of the rigid surface $1/R$ is equal to the curvature of the beam, that is, $Px/EI = 1/R$ from which $x = EI/PR$.

The deflection at the tip *B* may now be found as the sum of

1. The deflection of *C* from the tangent at *A*, which is given by δ_1 in the diagram and is found from the relation

$$(R + \delta_1)^2 = R^2 + (L - x)^2$$

to be approximately

$$\delta_1 = \frac{(L - x)^2}{2R}$$

2. The deflection of the portion of the beam of length x acting as a simple cantilever, given by

$$\delta_2 = \frac{Px^3}{3EI} = \frac{(EI)^2}{3P^2R^3}$$

3. The deflection owing to the rotation at point C, given by

$$\delta_3 = \frac{x(L-x)}{R} = \frac{EI}{PR^2}\left(L - \frac{EI}{PR}\right)$$

The desired deflection at the tip is thus

$$\delta = \delta_1 + \delta_2 + \delta_3 = \frac{L^3}{2R} - \frac{(EI)^2}{6P^2R^3}$$

9.24. A thermostat consists of two strips of different materials of equal thickness bonded together at their interface. Frequently this configuration takes the form of a cantilever beam, as in Fig. 9-27. If E_1 and E_2 denote the Young's moduli and α_1 and α_2 denote the coefficients of linear expansion of the two materials, each of thickness h, determine the deflection of the end of the cantilever assembly due to a temperature rise T.

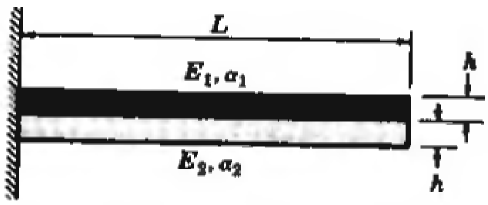


Fig. 9-27

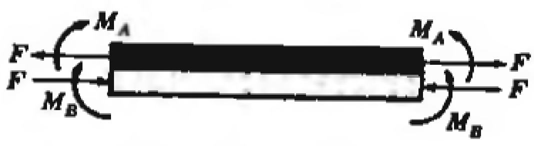


Fig. 9-28

Let b represent the width of the assembly. As in Problem 8.1, we shall assume that a plane section prior to deformation remains plane after deformation. The resultant normal forces F acting over each strip must be numerically equal since no external forces are applied along the length of the beam. Thus a cross section at any station along the length has Fig. 9-28 as its free-body representation.

The normal strain in the lower fibers of the top strip is found as the sum of (*a*) the strain due to the normal load, F/E_1bh; (*b*) the strain due to bending, which is $M_A(h/2)/E_1I$ from Problem 8.1; and (*c*) the strain due to the temperature rise, which is $\alpha_1 T$ as mentioned in Chap. 1. The sum of these strains must be the same as the strain in the upper fibers of the lower strip. Thus

$$\frac{F}{E_1bh} + \frac{M_A(h/2)}{E_1I} + \alpha_1 T = \frac{-F}{E_2bh} - \frac{M_B(h/2)}{E_2I} + \alpha_2 T \qquad (1)$$

The curvatures at this interface must also be equal. Thus, from Problem 9.1,

$$\frac{1}{R_1} = \frac{M_A}{E_1I} \quad \text{and} \quad \frac{1}{R_2} = \frac{M_B}{E_1I} \qquad (2)$$

and since $R_1 = R_2$, we have

$$M_A = \left(\frac{E_1}{E_2}\right)M_B \qquad (3)$$

From statics it is evident that

$$M_A + M_B = Fh \qquad (4)$$

from which

$$M_B = \frac{Fh}{1 + (E_1/E_2)} \qquad M_A = \frac{Fh}{1 + (E_2/E_1)} \qquad (5)$$

Substituting (*5*) in (*1*), we find

$$F = \frac{(\alpha_2 - \alpha_1)TbhE_1E_2(E_1 + E_2)}{E_1^2 + E_2^2 + 14E_1E_2} \tag{6}$$

and from (*5*) we get

$$M_A = \frac{(\alpha_2 - \alpha_1)Tbh^2E_1^2E_2}{E_1^2 + E_2^2 + 14E_1E_2} \tag{7}$$

We may now use the result obtained in Problem 9.23 for the deflection δ of a point on a cylindrical surface (which represents the interface, since in pure bending the assembly deforms into a circular configuration according to Problem 9.1) and express the deflection δ of the end of the assembly as

$$\delta = \frac{L^2}{2R} \tag{8}$$

Substituting from Eq. (*2*),

$$\delta = \frac{M_A L^2}{2E_1 I}$$

From (*7*) we then get

$$\delta = \frac{6(\alpha_2 - \alpha_1)TE_1E_2L^2}{h(E_1^2 + E_2^2 + 14E_1E_2)}$$

9.25. A beam has a slight initial curvature such that the initial configuration (which is stress free) is described by the relation $y_0 = Kx^3$. The beam is rigidly clamped at the origin and is subjected to a concentrated force at its extreme end, as shown in Fig. 9-29. As the force is increased, the beam deflects downward and the region near the clamped end comes in contact with the rigid horizontal plane. If the value of the applied force is P, determine the length of the beam in contact with the horizontal plane and the vertical distance of the extreme end from the plane.

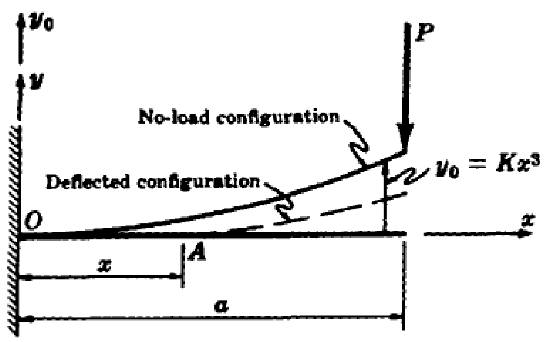


Fig. 9-29

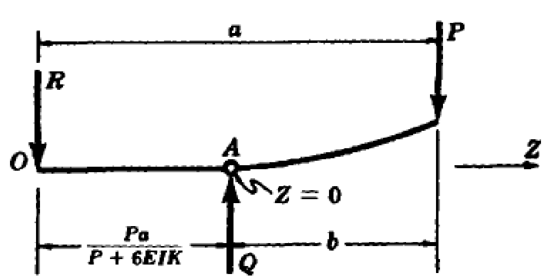


Fig. 9-30

The initial curvature may be determined from the expression $y_0 = Kx^3$ so that the bending moment arising from straightening the portion of the beam near the support is readily found to be $EI(d^2y_0/dx^2) = 6EIKx$, where x is the length of beam in contact with the horizontal plane. If this expression for moment is equated to the moment of the applied load about the point of contact, that is, $P(a - x)$, we have

$$6EIKx = P(a - x) \qquad \text{whence } x = \frac{Pa}{P + 6EIK}$$

Since the beam is considered to be weightless, there is no normal force between the beam and the rigid horizontal plane between the clamp at O and the extreme point of contact at A. The beam is flat between O and A. A free-body diagram of the deformed beam thus appears as in Fig. 9.30. A simple statics equation for equilibrium of moments about point A indicates that the clamp exerts a downward force equal to $6EIK$. For vertical equilibrium there is a concentrated force reaction $Q = P + 6EIK$ acting on the beam at the extreme point of contact A.

We now seek the equation of the deflection curve in the region to the right of point A. In Problem 9.1, Eq. (5) indicated that for an initially straight beam bending moment M is proportional to the curvature, d^2y/dx^2. However, in the present problem it is necessary to modify (5) to say that the bending moment M is proportional to the *change of curvature* since the beam is not initially straight. Thus, the Euler-Bernoulli equation for the portion of the beam to the right of point A is

$$EI\left(\frac{d^2y_0}{dx^2} - \frac{d^2y}{dZ^2}\right) = P(b - Z)$$

where a new coordinate Z has been introduced. This coordinate runs along the x-axis but has its origin at point A. It is important to note that, as the beam deflects, the curvature decreases from its original value; hence the quantity in parentheses on the left side of the equation is positive. Accordingly, the right side must be written as positive. This does not contradict our previous sign convention of downward forces giving negative moments since it was applied to *initially straight* beams. If we substitute $EI(d^2y_0/dx^2) = 6EIKx$, the last equation becomes

$$EI\frac{d^2y}{dZ^2} = 6EIK\left[\frac{Pa}{P + 6EIK} + Z\right] - Pb + PZ$$

Integrating twice and imposing the boundary conditions that $y = dy/dZ = 0$ at $Z = 0$, we obtain the desired deflection

$$EIy_{Z-b} = \frac{36(EIKa)^3}{(P + 6EIK)^2}$$

9.26. The bar ABC in Fig. 9-31 has flexural rigidity $E(3I)$ in region AB and flexural rigidity EI in region BC. The bar is pinned at A, supported by a roller at B, and subject to an applied bending moment M_0 at the free end. Determine the vertical deflection at B.

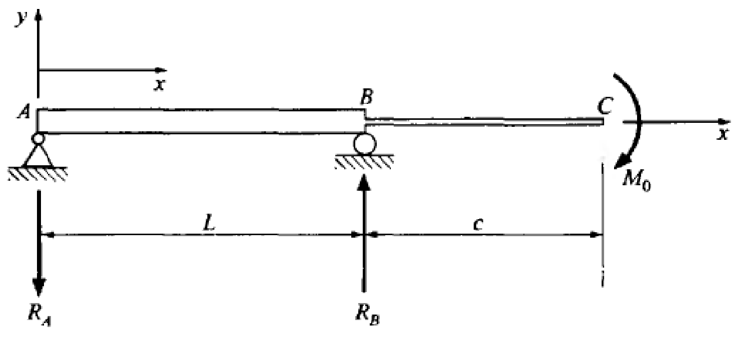


Fig. 9-31

Let us introduce the x-y coordinate system shown, where x may designate a cross section in either AB or BC. It is first necessary to determine the reactions from statics, viz.,

$$+ \circlearrowright \Sigma M_B = -M_0 + R_A L = 0 \qquad \therefore R_A = \frac{M_0}{L}(\downarrow)$$

$$\Sigma F_y = -R_A + R_B = 0 \qquad \therefore R_B = \frac{M_0}{L}(\uparrow)$$

We first write the differential equation of the deflected bar in region AB:

$$E(3I)\frac{d^2y}{dx^2} = -R_A x \qquad \text{for} \qquad 0 < x < L$$

Integrating,

$$E(3I)\frac{dy}{dx} = -R_A\frac{x^2}{2} + C_1 \tag{1}$$

Integrating again,

$$E(3I)y = -\frac{R_A}{2}\cdot\frac{x^3}{3} + C_1x + C_2 \tag{2}$$

As the first boundary condition we have: When $x = 0$, $y = 0$. Substituting in Eq. (2), we have

$$C_2 = 0$$

As a second boundary condition we have: When $x = L$, $y = 0$, and using $R_A = M_0/L$ we have

$$0 = -\frac{M_0}{L}\cdot\frac{L^3}{6} + C_1L + C_2$$

Thus,

$$C_1 = \frac{M_0L}{6}$$

Next, we write the differential equation of the deflected beam in region BC:

$$EI\frac{d^2y}{dx^2} = -R_Ax + R_B(x - L) \qquad \text{for} \qquad L < x < (L + c)$$

$$= -\frac{M_0x}{L} + \frac{M_0x}{L} - R_BL$$

$$= -M_0$$

This result could also have been obtained by taking moments of applied loads to the right of any section designated by "x" in BC.

Integrating,

$$EI\frac{dy}{dx} = -M_0x + C_3 \tag{3}$$

Integrating again,

$$EIy = -M_0\cdot\frac{x^2}{2} + C_3x + C_4 \tag{4}$$

As a third boundary condition at $x = L$, $y = 0$ in Eq. (4), so from (4)

$$0 = -\frac{M_0L^2}{2} + C_3L + C_4 \tag{5}$$

As the fourth boundary condition at $x = L$ the slopes dy/dx as given by Eqs. (1) and (3) must be equal. This leads to

$$\frac{1}{3EI}\left[\frac{R_AL^2}{2} + \frac{M_0L}{6}\right] = \frac{1}{EI}[-M_0L + C_3] \tag{6}$$

Solving Eq. (6) for C_3, then (5) for C_4, we find

$$C_3 = \tfrac{8}{9}M_0L; \qquad C_4 = -\tfrac{7}{18}M_0L^2$$

The equations of the deflected beam are thus

$$E(3I)y = -\frac{M_0}{6L}x^3 + \frac{M_0 L}{6}x \qquad \text{for} \qquad 0 < x < L \tag{7}$$

$$EIy = -\frac{M_0}{2}x^2 + \frac{8}{9}M_0 Lx - \frac{7}{18}M_0 L^2 \qquad \text{for} \qquad L < x < (L + C) \tag{8}$$

When $x = (L + C)$, we have from Eq. (*8*) the desired tip deflection:

$$[y]_{x=L+C} = \frac{M_0}{EI}\left[\frac{(L+C)^2}{2} + \frac{8}{9}L(L+C) - \frac{7}{18}L^2\right]$$

$$= -\frac{M_0 C}{EI}\left(\frac{L}{9} + \frac{C}{2}\right) \tag{9}$$

Supplementary Problems

9.27. The cantilever beam loaded as shown in Problem 9.2 is made of a titanium alloy, having $E = 105$ GPa. The load P is 20 kN, $L = 4$ m, and the moment of inertia of the beam cross section is $104 \times 10^6\ \text{mm}^4$. Find the maximum deflection of the beam. *Ans.* -39 mm

9.28. Consider the simply supported beam loaded as shown in Problem 9.12. The length of the beam is 20 ft, $a = 15$ ft, the load $P = 1000$ lb, and $I = 150\ \text{in}^4$. Determine the deflection at the center of the beam. Take $E = 30 \times 10^6\ \text{lb/in}^2$. *Ans.* -0.044 in

9.29. Refer to Fig. 9-32. Determine the deflection at every point of the cantilever beam subject to the single moment M_1 shown. *Ans.* $EIy = -M_1 x^2/2$

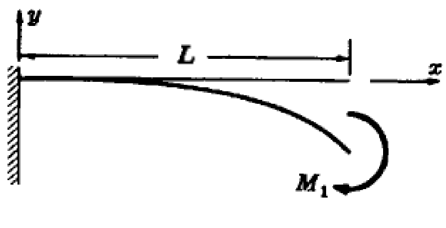


Fig. 9-32

9.30. The cantilever beam described in Problem 9.29 is of circular cross section, 5 in in diameter. The length of the beam is 10 ft and the applied moment is 5000 lb·ft. Determine the maximum deflection of the beam. Take $E = 30 \times 10^6\ \text{lb/in}^2$. *Ans.* -0.469 in

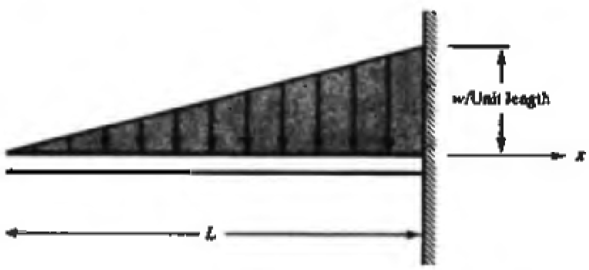


Fig. 9-33

9.31. Refer to Fig. 9-33. Find the equation of the deflection curve for the cantilever beam subject to the uniformly varying load shown.

Ans. $EIy = -\frac{wx^5}{120L} + \frac{wL^3x}{24} - \frac{wL^4}{30}$

9.32. A cantilever beam is loaded by the sinusoidal load indicated in Fig. 9-34. Determine the deflection of the tip of the beam. *Ans.* $EIy]_{x=0} = -0.07385q_0L^4$

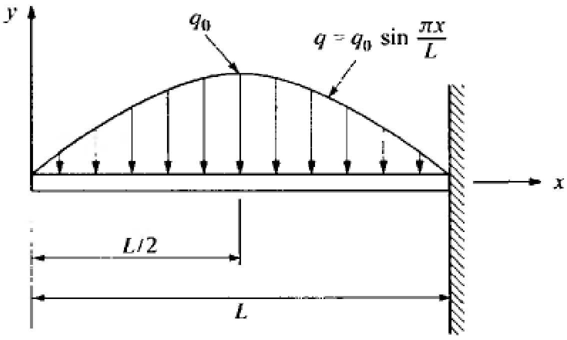


Fig. 9-34

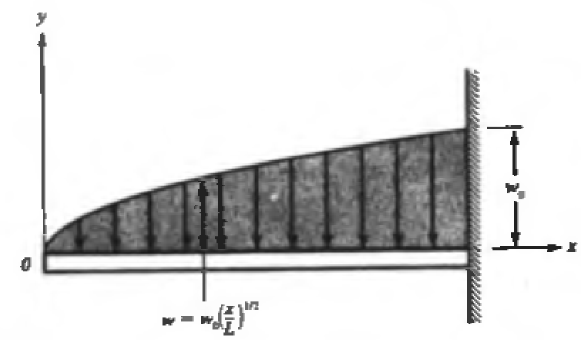
9.33. A cantilever beam carrying a parabolically distributed load is shown in Fig. 9-35. Determine the equation of the deflected beam as well as the deflection at the tip.

Ans. $y]_{x=0} = -\frac{56}{945}w_0L^4, EIy = -\frac{16}{945}\frac{w_0}{L^{1/2}}x^{9/2} + \frac{8}{105}x_0L^3x - \frac{56}{945}w_0L^4$

9.34. The cross section of the cantilever beam loaded as shown in Fig. 9-33 is rectangular, 50 × 75 mm. The bar, 1 m long, is aluminum for which $E = 65$ GPa. Determine the permissible maximum intensity of loading if the maximum deflection is not to exceed 5 mm and the maximum stress is not to exceed 50 MPa.
Ans. $w = 14.1$ kN/m

9.35. Refer to Fig. 9-36. Determine the equation of the deflection curve for the simply supported beam supporting the load of uniformly varying intensity.

Ans. $EIy = \frac{wL}{2}\left(-\frac{x^5}{60L^2} + \frac{x^3}{18} - \frac{7L^2x}{180}\right)$



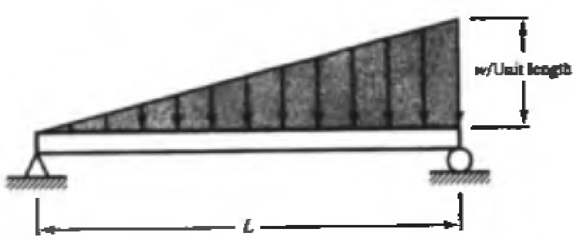


Fig. 9-35 **Fig. 9-36**

9.36. Determine the equation of the deflection curve for the cantilever beam loaded by the concentrated force P as shown in Fig. 9-37.

Ans. $EIy = -\frac{P}{6}(a-x)^3 - \frac{Pa^2}{2}x + \frac{Pa^3}{6}$ for $0<x<a$; $EIy = -\frac{Pa^2}{2}x + \frac{Pa^3}{6}$ for $a<x<L$

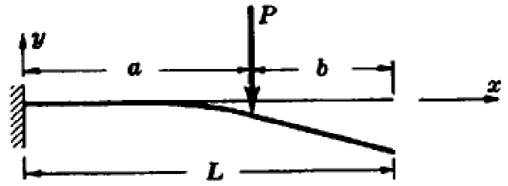


Fig. 9-37

9.37. For the cantilever beam of Fig. 9-37, take $P = 5$ kN, $a = 2$ m, and $b = 1$ m. The beam is of equilateral triangular cross section, 150 mm on a side, with a vertical axis of symmetry. Determine the maximum deflection of the beam. Take $E = 200$ GPa. *Ans.* -12.8 mm

9.38. The cantilever beam shown in Fig. 9-38 is subjected to a uniform load w per unit length over its right half BC. Determine the equations of the deflection curve as well as the maximum deflection.

Ans. $EIy = \frac{wLx^3}{12} - \frac{3wL^2x^2}{16}$ for $0<x<\frac{L}{2}$

$$EIy = -\frac{w(L-x)^4}{24} - \frac{7wL^3x}{48} + \frac{15wL^4}{384} \quad \text{for} \quad \frac{L}{2}<x<L$$

$$\Delta_{max} = \frac{41}{384}\left(\frac{wL^4}{EI}\right)$$

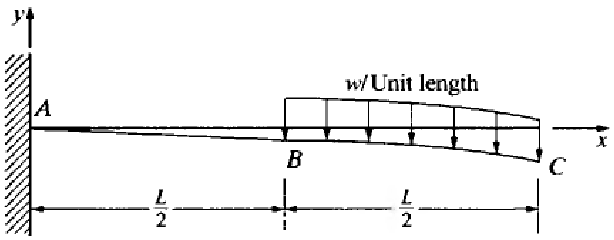


Fig. 9-38

9.39. The simply supported overhanging beam supports the load w per unit length as shown in Fig. 9-39. Find the equations of the deflection curve of the beam. Take coordinates at the level of the supports.

Ans. $EIy = -\frac{wx^4}{24} + \frac{wL^3x}{48} - \frac{wLx}{4}\left(\frac{L}{2}-a\right) + \frac{wa^4}{24} - \frac{waL^3}{48} + \frac{wLa}{4}\left(\frac{L}{2}-a\right)^2$ for $0<x<a$

$$EIy = -\frac{wx^4}{24} + \frac{wL(x-a)^3}{12} + \frac{wL^3x}{48} - \frac{wLx}{4}\left(\frac{L}{2}-a\right)^2$$

$$+\frac{wa^4}{24} - \frac{waL^3}{48} + \frac{wLa}{4}\left(\frac{L}{2}-a\right)^2 \quad \text{for} \quad a<x<(a+b)$$

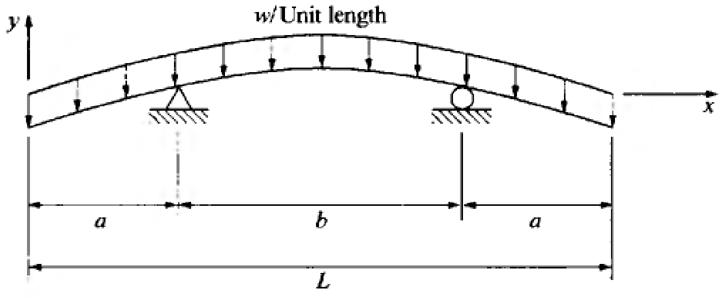


Fig. 9-39

9.40. A simply supported beam with overhanging ends is loaded by the uniformly distributed loads shown in Fig. 9-40. Determine the deflection of the midpoint of the beam with respect to an origin at the level of the supports.

Ans. $\dfrac{wa^2(L-2a)^2}{16EI}$ (above level of supports)

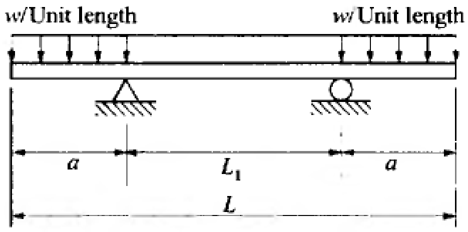


Fig. 9-40

9.41. For the beam described in Problem 9.40, determine the deflection of one end of the beam with respect to an origin at the level of the supports.

Ans. $\dfrac{wa^3L}{4EI} - \dfrac{3wa^4}{8EI}$ (below level of supports)

9.42. The overhanging beam is loaded by the uniformly distributed load as well as the concentrated force shown in Fig. 9-41. Determine the deflection of point A of the beam.

Ans. $\dfrac{-wa^3b}{3EI} + \dfrac{Pab^2}{4EI} - \dfrac{wa^4}{8EI}$ (below level of supports)

9.43. Figure 9-42 shows a cantilever beam in the form of a circular cone whose length L is large compared to the base diameter D. If the only force acting is its own weight, which is γ per unit volume, determine the equation of the deflection curve.

Ans. $y = -\dfrac{2\gamma L^2}{45ED^2}(x^3 + 2L^3 - 3L^2x)$

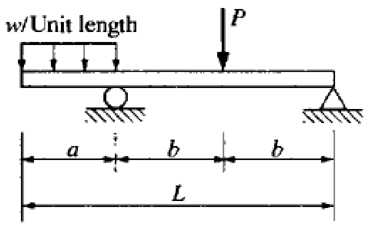


Fig. 9-41

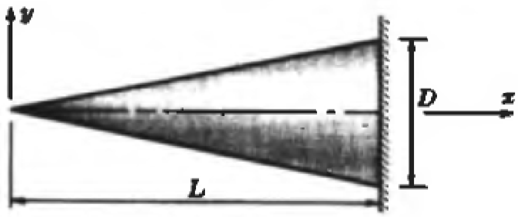


Fig. 9-42

9.44. For the overhanging beam treated in Problem 9.17 consider the uniform load to be 120 lb/ft, $a = 3$ ft, and $b = 12$ ft. The bar has a 3-in × 4-in rectangular cross section. Determine the maximum deflection of the beam. Take $E = 30 \times 10^6$ lb/in^2. *Ans.* -0.10 in at $x = 110.4$ in

9.45. A cantilever beam when viewed from the top [see Fig. 9-43(*a*)] has a triangular configuration. The thickness h of the beam is constant, as shown in the side view Fig. 9-43(*b*). Determine the deflection of the beam due to a concentrated load P at the tip. Neglect the weight of the beam. *Ans.* $y]_{x=0} = -6PL^3/Ebh^3$

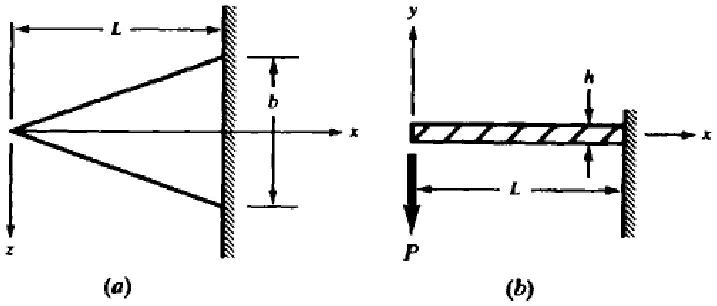


Fig. 9-43

9.46. A cantilever beam when viewed from the top has the configuration indicated in Fig. 9-44(*a*) and is of constant thickness h, as indicated in Fig. 9-44(*b*). Find the equation of the deflection curve as the beam bends under the action of the concentrated force P at the tip. Neglect the weight of the beam.

Ans. $$y = \left[-\frac{16P(L-x)^{11/4}}{77} - \frac{4}{9}PL^{7/4}x + \frac{16PL^{11/4}}{77}\right]\left(\frac{6L^{1/4}}{Eh^3a^{1/2}}\right)$$

$$y]_{x=L} = -\frac{24PL^3}{11Eh^3a^{1/2}}$$

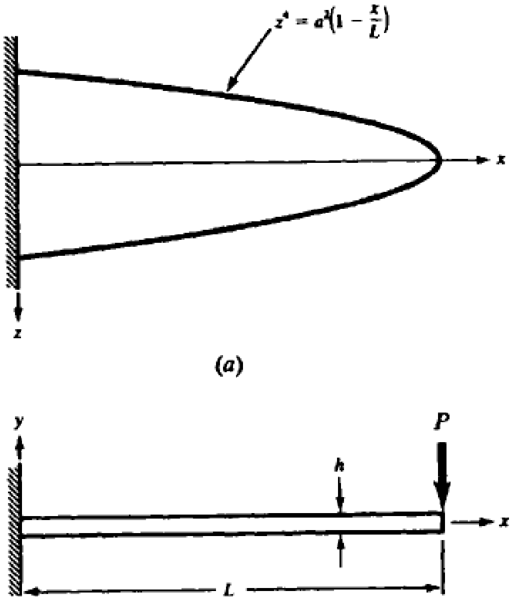


Fig. 9-44

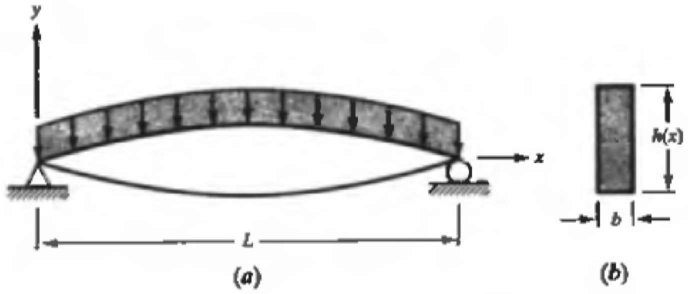


Fig. 9-45

9.47. A simply supported beam of length L is subjected to a uniformly distributed loading w per unit length. The width b of the beam is constant and the height varies in such a manner that all outer fibers along both the top and lower surfaces are subject to the same magnitude normal stress σ_0. Determine the variation of height of the beam as a function of x, as shown in Fig. 9-45(b). Also determine the maximum deflection of the beam.

Ans. $h = \dfrac{2h_{max}\sqrt{Lx - x^2}}{L}$, $y_{max} = -0.0178\dfrac{wL^4}{Eb(h_{max})^3}$

9.48. The cantilever beam of variable cross section shown in Fig. 9-46 is in the form of a wedge of constant width b. The midplane of the wedge lies in the horizontal plane x-z. Find the deflection of the tip of the beam due to its own weight γ per unit volume. *Ans.* $y]_{x=0} = -\gamma L^4/Eh^2$

9.49. Two solid rigid cylinders I and II have their geometric axes in a horizontal plane spaced a distance L apart, as shown in Fig. 9-47. A beam of flexural rigidity EI is then placed across the tops of the cylinders and loaded by a centrally applied vertical force P. The beam deflects (dotted line) and is tangent to each of the cylinders at the points designated as A. Determine the angle θ describing this point of contact.

Ans. $\theta = \dfrac{PL^2}{16EI}\left(1 - \dfrac{PLR}{4EI}\right)$

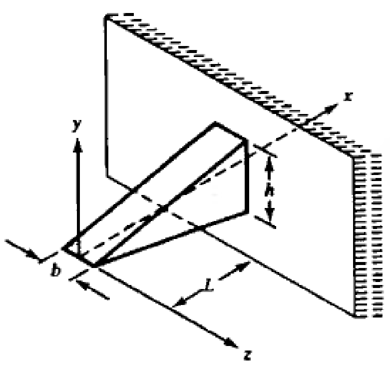


Fig. 9-46

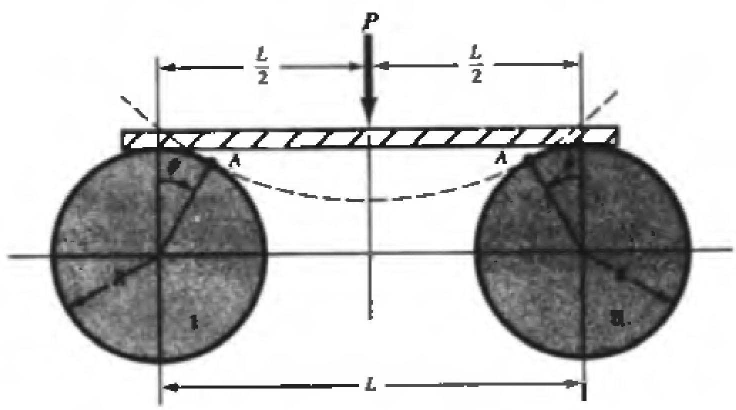


Fig. 9-47

Chapter 10

Elastic Deflection of Beams: Method of Singularity Functions

In Chap. 9 we found the elastic deflections of transversely loaded beams through direct integration of the second-order Euler-Bernoulli equation. As we saw, the approach is direct but may become very lengthy even for relatively simple engineering situations.

A more expedient approach is based upon the use of the singularity functions introduced in Chap. 6. The method is direct and may be applied to a beam subject to any combination of concentrated forces, moments, and distributed loads. One must only remember the definition of the singularity function given in Chap. 6; i.e., the quantity $\langle x-a\rangle$ vanishes if $x<a$ but is equal to $(x-a)$ if $x>a$.

There are several possible approaches for using singularity functions for the determination of beam deflections. Perhaps the simplest is to employ the approach of Chap. 6 in which the bending moment is written in terms of singularity functions in the form of one equation valid along the entire length of the beam. Two integrations of this equation lead to the equation for the deflected beam in terms of two constants of integration which must be determined from boundary conditions. As noted in Chap. 6, integration of the singularity functions proceeds directly and in the same manner as simple power functions. Thus, the approach is direct and avoids the problem of the determination of a pair of constants corresponding to each region of the beam (between loads) as in the case of double integration exemplified in Chap. 9.

Most important, the singularity function approach leads directly into a computerized approach for the determination of beam deflections. See Problems 10.16, 10.17, and 10.18.

Solved Problems

10.1. Using singularity functions, determine the deflection curve of the cantilever beam subject to the loads shown in Fig. 10-1.

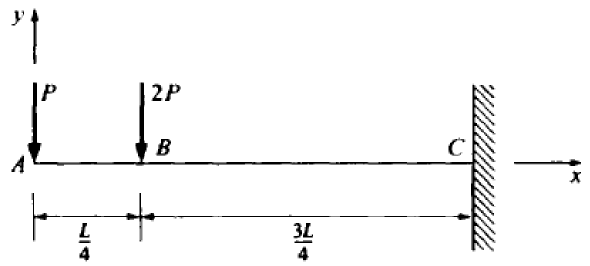


Fig. 10-1

In this case it is not necessary to determine the reactions of the wall supporting the beam at C. From the techniques of Chap. 6 we find the bending moment along the entire length of the beam to be given by

$$M = -P\langle x\rangle^1 - 2P\left\langle x-\frac{L}{4}\right\rangle^1 \tag{1}$$

where the angular brackets have the meanings given in the section "Singularity Functions" of Chap. 6, pages 135–136. Thus, the differential equation for the bent beam is

$$EI\frac{d^2y}{dx^2} = -P\langle x\rangle^1 - 2P\left\langle x-\frac{L}{4}\right\rangle^1 \tag{2}$$

The first integration yields

$$EI\frac{dy}{dx} = -P\frac{\langle x\rangle^2}{2} - 2P\frac{\left\langle x - \frac{L}{4}\right\rangle^2}{2} + C_1 \tag{3}$$

where C_1 is a constant of integration. The next integration leads to

$$EIy = -\frac{P}{2}\frac{\langle x\rangle^3}{3} - 2P\frac{\left\langle x - \frac{L}{4}\right\rangle^3}{2(3)} + C_1\langle x\rangle + C_2 \tag{4}$$

where C_2 is a second constant of integration. These two constants may be determined from the boundary conditions:

(*a*) When $x = L$, $dy/dx = 0$, so from (*3*):

$$0 = -\frac{PL^2}{2} - P\left(\frac{3L}{4}\right)^2 + C_1 \tag{5}$$

(*b*) When $x = L$, $y = 0$, so from (*4*):

$$0 = -\frac{PL^3}{6} - \frac{P}{3}\left(\frac{3L}{4}\right)^3 + C_1L + C_2 \tag{6}$$

Solving (*5*) and (*6*),

$$C_1 = \frac{17}{16}PL^2; \qquad C_2 = -\frac{145}{192}PL^3 \tag{7}$$

The desired deflection curve is thus

$$EIy = -\frac{P}{6}\langle x\rangle^3 - \frac{P}{3}\left\langle x - \frac{L}{4}\right\rangle^3 + \frac{17}{16}PL^2\langle x\rangle - \frac{145}{192}PL^3 \tag{8}$$

For example, the deflection at point B where $x = L/4$ is found from (*8*) to be

$$EIy]_{x=L/4} = -\frac{P}{6}\left(\frac{L}{4}\right)^3 - 0 + \frac{17}{16}PL^2\left(\frac{L}{4}\right) - \frac{145}{192}PL^3$$

or $$y]_{x=L/4} = -\frac{94.5PL^3}{192EI} \quad \text{or} \quad -\frac{0.492PL^3}{EI}$$

10.2. The cantilever beam ABC shown in Fig. 10-2 is subject to a uniform load w per unit length distributed over its right half, together with a concentrated couple $wL^2/2$ applied at C. Using singularity functions, determine the deflection curve of the beam.

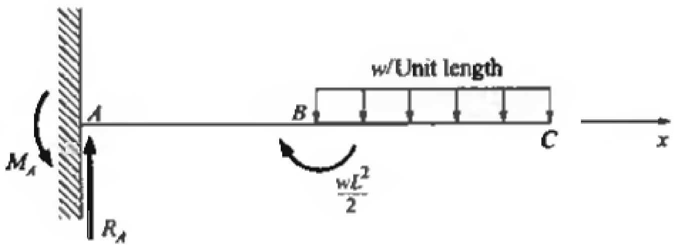


Fig. 10-2

It is first necessary to find from statics the shear and moment reactions exerted by the wall on the beam at A. From statics we have

$$+\circlearrowright \Sigma M_A = M_A - \frac{wL^2}{2} - w\left(\frac{L}{2}\right)\left(\frac{3L}{4}\right) = 0$$

$$M_A = \frac{7wL^2}{8}$$

$$\Sigma F_v = R_A - w\left(\frac{L}{2}\right) = 0; \qquad R_A = \frac{wL}{2}$$

By the singularity function approach we may write the bending moment along the entire length of the beam as

$$M = \frac{wL}{2}\langle x\rangle^1 - \frac{7wL^2}{8}\langle x\rangle^0 + \frac{wL^2}{2}\left\langle x - \frac{L}{2}\right\rangle^0 - w\left\langle x - \frac{L}{2}\right\rangle^1 \frac{\left\langle x - \frac{L}{2}\right\rangle^1}{2} \tag{1}$$

where, again, the singularity functions are as defined in Chap. 6. Thus the differential equation of the bent beam is

$$EI\frac{d^2y}{dx^2} = \frac{wL}{2}\langle x\rangle^1 - \frac{7wL^2}{8}\langle x\rangle^0 + \frac{wL^2}{2}\left\langle x - \frac{L}{2}\right\rangle^0 - w\left\langle x - \frac{L}{2}\right\rangle^1 \frac{\left\langle x - \frac{L}{2}\right\rangle^1}{2} \tag{2}$$

Integrating,

$$EI\frac{dy}{dx} = \frac{wL}{2}\frac{\langle x\rangle^2}{2} - \frac{7wL^2}{8}\langle x\rangle + \frac{wL^2}{2}\frac{\left\langle x - \frac{L}{2}\right\rangle^1}{1} - \frac{w}{2}\frac{\left\langle x - \frac{L}{2}\right\rangle^3}{3} + C_1 \tag{3}$$

The first boundary condition is: When $x = 0$, $dy/dx = 0$. Substituting in (3), we find $C_1 = 0$.

Integrating again,

$$EIy = \frac{wL}{4}\frac{\langle x\rangle^3}{3} - \frac{7wL^2}{8}\frac{\langle x\rangle^2}{2} + \frac{wL^2}{2}\frac{\left\langle x - \frac{L}{2}\right\rangle^2}{2} - \frac{w}{6}\frac{\left\langle x - \frac{L}{2}\right\rangle^4}{4} + C_2 \tag{4}$$

The second boundary condition is: When $x = 0$, $y = 0$. Substituting in (4), we find $C_2 = 0$.

Thus, the desired deflection equation is

$$EIy = \frac{wL}{12}\langle x\rangle^3 - \frac{7wL^2}{16}\langle x\rangle^2 + \frac{wL^2}{4}\left\langle x - \frac{L}{2}\right\rangle^2 - \frac{w}{24}\left\langle x - \frac{L}{2}\right\rangle^4 \tag{5}$$

This yields the deflection at the tip to be

$$EIy]_{x=L} = \frac{wL^4}{12} - \frac{7wL^4}{16} + \frac{wL^2}{4}\left(\frac{L}{2}\right)^2 - \frac{w}{24}\left(\frac{L}{2}\right)^4$$

or

$$y]_{x=L} = -\frac{113}{384EI}$$

10.3. Consider a simply supported beam subject to a uniform load distributed over a portion of its length, as indicated in Fig. 10-3. Use singularity functions to determine the deflection curve of the beam.

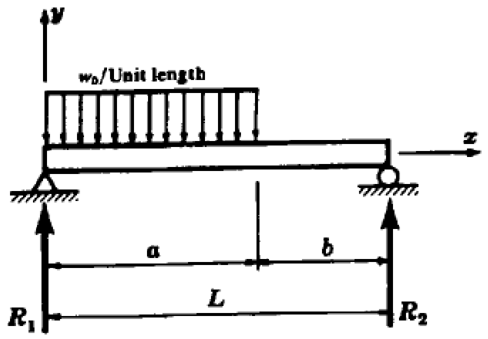


Fig. 10-3

From statics the reactions are found to be

$$R_1 = \frac{w_0}{2L}(L^2 - b^2)$$

$$R_2 = w_0 a - \frac{w_0}{2L}(L^2 - b^2)$$

The bending moment at any point x along the length of the beam is

$$M = R_1 x - \frac{w_0}{2}\langle x\rangle^2 + \frac{w_0}{2}\langle x - a\rangle^2 \tag{1}$$

Note that the last term on the right is required to cancel the distributed load represented by the term

$$-\frac{w_0}{2}\langle x\rangle^2$$

for all values of x greater than $x = a$. Thus

$$EI\frac{d^2y}{dx^2} = M = R_1\langle x\rangle^1 - \frac{w_0}{2}\langle x\rangle^2 + \frac{w_0}{2}\langle x - a\rangle^2 \tag{2}$$

Integrating,

$$EI\frac{dy}{dx} = \frac{R_1}{2}\langle x\rangle^2 - \frac{w_0}{6}\langle x\rangle^3 + \frac{w_0}{6}\langle x - a\rangle^3 + C_1 \tag{3}$$

Finally,

$$EIy = \frac{R_1}{6}\langle x\rangle^3 - \frac{w_0}{24}\langle x\rangle^4 + \frac{w_0}{24}\langle x - a\rangle^4 + C_1 x + C_2 \tag{4}$$

To determine C_1 and C_2, we impose the boundary conditions that $y = 0$ at $x = 0$ and $x = L$. From (4) we thus find

$$C_1 = \frac{w_0 L^3}{24} - \frac{w_0 b^4}{24L} - \frac{w_0 L}{12}(L^2 - b^2)$$
$$C_2 = 0$$

The deflection curve is accordingly

$$EIy = \frac{w_0}{12L}(L^2 - b^2)\langle x\rangle^3 - \frac{w_0}{24}\langle x\rangle^4 + \frac{w_0}{24}\langle x - a\rangle^4 + \left[-\frac{w_0 L^3}{24} - \frac{w_0 b^4}{24L} + \frac{w_0 L b^2}{12}\right]x \tag{5}$$

10.4. Consider the overhanging beam shown in Fig. 10-4. Determine the equation of the deflection curve using singularity functions.

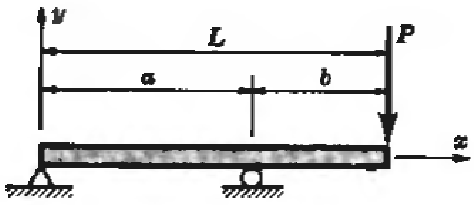


Fig. 10-4

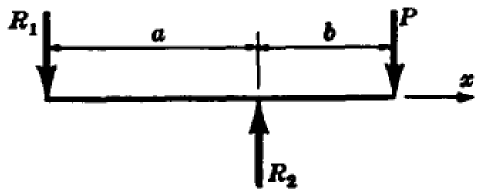


Fig. 10-5

From statics the reactions are first found to be $R_1 = Pb/a$ and $R_2 = P[1 + (b/a)]$, acting as indicated in Fig. 10-5. The bending moment at any point x along the entire length of the beam is

$$M(x) = -R_1\langle x\rangle^1 + R_2\langle x-a\rangle^1 \tag{1}$$

Thus

$$EI\frac{d^2y}{dx^2} = M = -R_1\langle x\rangle^1 + R_2\langle x-a\rangle^1 \tag{2}$$

from which

$$EI\frac{dy}{dx} = -\frac{R_1}{2}\langle x\rangle^2 + \frac{R_2}{2}\langle x-a\rangle^2 + C_1 \tag{3}$$

$$EIy = -\frac{R_1}{6}\langle x\rangle^3 + \frac{R_2}{6}\langle x-a\rangle^3 + C_1x + C_2 \tag{4}$$

The boundary conditions are $y = 0$ at $x = 0$ and $x = a$. From these conditions, C_1 and C_2 are found from (*4*) to be

$$C_1 = \frac{Pab}{6} \qquad C_2 = 0$$

The deflection curve is thus

$$EIy = -\frac{Pb}{6a}\langle x\rangle^3 + \frac{P}{6}\left(1 + \frac{b}{a}\right)\langle x-a\rangle^3 + \frac{Pabx}{6} \tag{5}$$

10.5. Through the use of singularity functions determine the equation of the deflected cantilever beam subject to the triangular loading together with the couple indicated in Fig. 10-6.

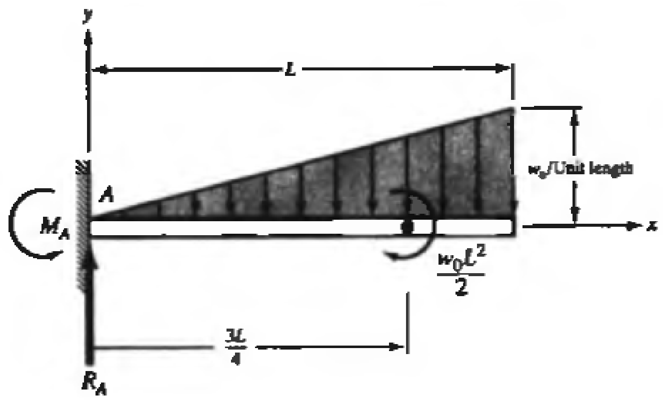


Fig. 10-6

We must first determine the reactions at point A through the use of statics. There will be a vertical shear reaction R_A as well as a moment M_A to prevent angular rotation at point A. From statics

$$\circlearrowright \Sigma M_A = M_A - \frac{w_0 L^2}{2} - \frac{w_0}{2}(L)(\tfrac{2}{3}L) = 0$$

Therefore

$$M_A = \tfrac{5}{6} w_0 L^2$$

$$\Sigma F_y = R_A - \frac{w_0 L}{2} = 0$$

Therefore

$$R_A = \frac{w_0 L}{2}$$

To write the expression for bending moment, let us first examine the contribution from the distributed loading. At any position x to the right of point A, the load intensity from geometry is $w = w_0(x/L)$ and the resultant (shown by the dotted vector in Fig. 10-7) is of magnitude

$$\frac{wx}{2} = w_0 \frac{x^2}{2L}$$

and acts at a point distance $\frac{2}{3}x$ from A. Thus, the moment at x due *only* to the triangular loading is

$$-w_0 \frac{x^2}{2L}\left(\frac{L}{3}\right) \quad \text{or} \quad -\frac{w_0 x^3}{6L}$$

where the negative sign is inserted because this downward loading gives negative bending moment.

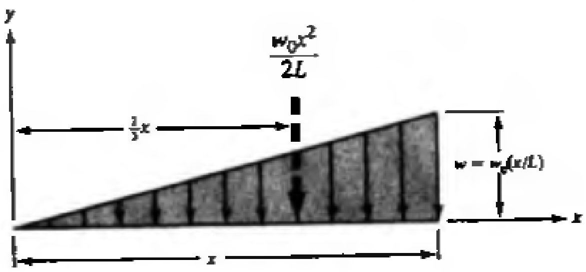


Fig. 10-7

Due to *all* loadings, that is, M_A, R_A, and the triangular load, the bending moment at any location x is

$$M = -\frac{5}{6} w_0 L^2 + \frac{w_0 L}{2} x - \frac{w_0 x^2}{2L} \cdot \frac{x}{3} + \frac{w_0 L^2}{2}\left\langle x - \frac{3L}{4}\right\rangle^0 \tag{1}$$

so that the differential equation of the deflected beam is

$$EI\frac{d^2 y}{dx^2} = -\frac{5}{6} w_0 L^2 + \frac{w_0 L}{2} x - \frac{w_0 x^3}{6L} + \frac{w_0 L^2}{2}\left\langle x - \frac{3L}{4}\right\rangle^0 \tag{2}$$

Integrating the first time, we obtain

$$EI\frac{dy}{dx} = -\frac{5}{6} w_0 L^2 x + \frac{w_0 L}{2} \cdot \frac{x^2}{2} - \frac{w_0}{6L} \cdot \frac{x^4}{4} + \frac{w_0 L^2}{2}\left\langle x - \frac{3L}{4}\right\rangle^1 + C_1 \tag{3}$$

As the first boundary condition, we have $dy^2/dx = 0$ at $x = 0$ which when substituted in Eq. (3) yields $C_1 = 0$. Integrating a second time, we obtain

$$EIy = -\frac{5}{12}w_0L^2x^2 + \frac{w_0L}{4}\frac{x^3}{3} - \frac{w_0}{24L}\cdot\frac{x^5}{5} + \frac{w_0L^2}{4}\left\langle x - \frac{3L}{4}\right\rangle^2 + C_2 \tag{4}$$

The second boundary condition, $y = 0$ at $x = 0$, leads, upon substitution in Eq. (4), to $C_2 = 0$. Thus the beam deflection equation is

$$EIy = -\frac{5}{12}w_0L^2x^2 + \frac{w_0L}{12}x^3 - \frac{w_0}{120L}x^5 + \frac{w_0L^2}{4}\left\langle x - \frac{3L}{4}\right\rangle^2 \tag{5}$$

The deflection at the tip, $x = L$, is found from Eq. (5) to be

$$EIy]_{x=L} = -0.326w_0L^4$$

10.6. Using singularity functions, determine the equation of the deflection curve of the beam simply supported at points B and C and subject to the triangular loading shown in Fig. 10-8.

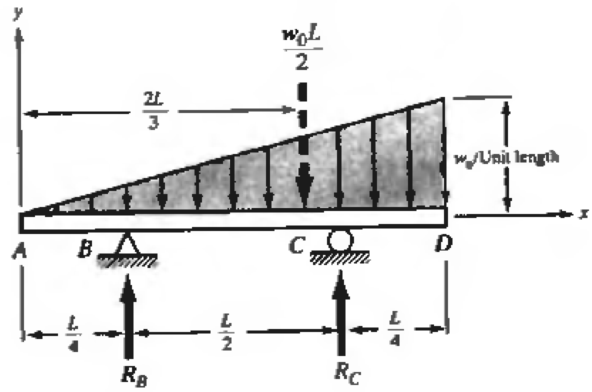


Fig. 10-8

To determine the external vertical reactions at points B and C, we may replace the entire loading by its resultant which acts through the centroid of the triangle. The magnitude of the entire load is the average load per unit length, $w_0/2$, multiplied by the beam length L, or $w_0L/2$. This acts at a distance $2L/3$ from the left end A and is shown by the dotted vector in Fig. 10-8. From statics

$$+\circlearrowright \Sigma M_B = R_C\cdot\frac{L}{2} - \frac{w_0L}{2}\left(\frac{2}{3}L - \frac{L}{4}\right) = 0$$

Therefore

$$R_C = \frac{5w_0L}{12}$$

$$\Sigma F_v = R_B + \frac{5w_0L}{12} - \frac{w_0L}{2} = 0$$

Therefore

$$R_B = \frac{w_0L}{12}$$

At any station x measured from the origin at A, the bending moment in terms of singularity functions is given as the sum of the moments of all forces to the left of that station. Let us examine a portion of the triangular load of horizontal length x. The resultant of that much of the loading is shown by the dotted vector in Fig. 10-9 and the resultant is of magnitude

$$\frac{w}{2}\cdot x = w_0\frac{x}{L}\cdot\frac{x}{2}$$

and acts at a point distance $\frac{2}{3}x$ from A. Thus, the moment at x due *only* to the triangular loading is

$$-w_0\frac{x^2}{2L}\cdot\frac{x}{3} \qquad \text{or} \qquad -\frac{w_0x^3}{6L}$$

where the minus sign is inserted because according to our bending moment sign conventions in Chap. 6 downward loads give rise to negative bending moment.

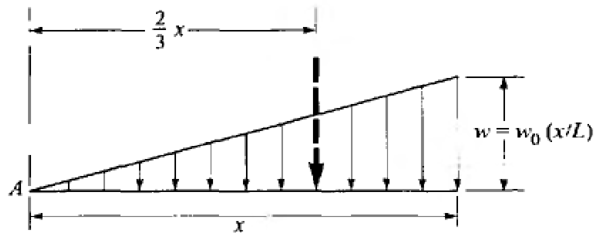


Fig. 10-9

In terms of singularity functions, the bending moment at any station x due to all loadings (including reactions) is

$$M = -\frac{w_0\langle x\rangle^3}{6L} + \frac{w_0L}{12}\left\langle x-\frac{L}{4}\right\rangle + \frac{5w_0L}{12}\left\langle x-\frac{3L}{4}\right\rangle \tag{1}$$

so that the differential equation of the bent beam is

$$EI\frac{d^2y}{dx^2} = -\frac{w_0\langle x\rangle^3}{6L} + \frac{w_0L}{12}\left\langle x-\frac{L}{4}\right\rangle + \frac{5w_0L}{12}\left\langle x-\frac{3L}{4}\right\rangle \tag{2}$$

Integrating the first time, we obtain

$$EI\frac{dy}{dx} = -\frac{w_0}{24L}\langle x\rangle^4 + \frac{w_0L}{24}\left\langle x-\frac{L}{4}\right\rangle^2 + \frac{5w_0L}{24}\left\langle x-\frac{3L}{4}\right\rangle^2 + C_1 \tag{3}$$

and integrating again, we find

$$EIy = -\frac{w_0}{120L}\langle x\rangle^5 + \frac{w_0L}{72}\left\langle x-\frac{L}{4}\right\rangle^3 + \frac{5}{72}w_0L\left\langle x-\frac{3L}{4}\right\rangle^3 + C_1x + C_2 \tag{4}$$

As boundary conditions, when $x = L/4$, $y = 0$, so substituting in Eq. (4) we obtain

$$0 = -\frac{w_0}{120L}\left(\frac{L}{4}\right)^5 + C_1\frac{L}{4} + C_2 \tag{5}$$

Also, when $x = 3L/4$, $y = 0$, and substitution in Eq. (4) yields

$$0 = -\frac{w_0}{120L}\left(\frac{3L}{4}\right)^5 + \frac{w_0L}{72}\left(\frac{L}{2}\right)^3 + C_1\cdot\frac{3L}{4} + C_2 \tag{6}$$

Solving Eqs. (5) and (6), we obtain

$$C_1 = 0.0004666w_0L^3$$
$$C_2 = -0.0001085w_0L^4$$

so that the equation of the deflected beam is

$$EIy = -\frac{w_0}{120L}\langle x\rangle^5 + \frac{w_0 L}{72}\left\langle x - \frac{L}{4}\right\rangle^3 + \frac{5w_0 L}{72}\left\langle x - \frac{3L}{4}\right\rangle^3$$
$$+ 0.0004666w_0 L^3 x - 0.0001085w_0 L^4 \qquad (7)$$

10.7. If the beam subject to triangular loading in Problem 10.6 is a W203 × 40 steel section, of length $L = 4$ m, $I = 39 \times 10^6\ \text{mm}^4$, and $w_0 = 80$ kN/m, determine the deflection at the point D.

Using Eq. (7) of Problem 10.6, we have

$$EI[y]_{x=L} = -\frac{w_0 L^4}{120} + \frac{w_0 L}{72}\left(\frac{3L}{4}\right)^3 + \frac{5w_0 L}{72}\left(\frac{L}{4}\right)^3$$
$$+ 0.0004666w_0 L^4 - 0.0001085w_0 L^4$$
$$= -0.001031w_0 L^4$$
$$y]_{x=L} = -\frac{0.001031w_0 L^4}{EI}$$
$$= -\frac{(0.001031)(80{,}000\ \text{N/m})(4\ \text{m})^4}{(200 \times 10^9\ \text{N/m}^2)(39 \times 10^{-6}\ \text{m}^4)}$$
$$= -0.0027\ \text{m} \quad \text{or} \quad -2.7\ \text{mm}$$

10.8. The beam AD in Fig. 10-10 is simply supported at A and C, loaded by a uniform load from B to D, and also by a couple applied as shown at D. Determine the equation of the deflection curve through the use of singularity functions.

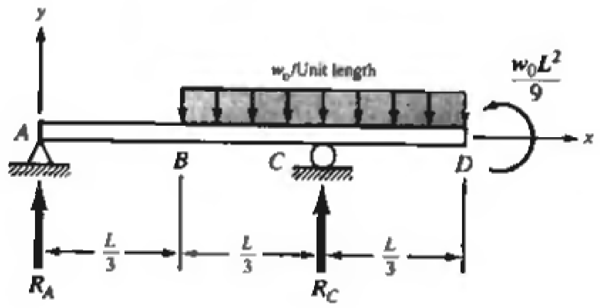


Fig. 10-10

The reactions at A and C are assumed to be positive in the directions shown and are found from the two statics equations to be

$$\circlearrowright \Sigma M_A = R_C\left(\frac{2L}{3}\right) - w_0\left(\frac{2L}{3}\right)^2 + \frac{w_0 L^2}{9} = 0 \qquad (1)$$

$$\Sigma F_y = R_A + R_C - w_0\left(\frac{2L}{3}\right) = 0 \qquad (2)$$

Solving,

$$R_A = \frac{w_0 L}{6} \qquad R_C = \frac{w_0 L}{2}$$

The singularity approach lets us write the equation of the entire deflected beam in the form

$$EI\frac{d^2y}{dx^2} = \frac{1}{6}w_0L\langle x\rangle^1 - \frac{w_0\left\langle x - \frac{L}{3}\right\rangle^2}{2} + \frac{w_0L}{2}\left\langle x - \frac{2L}{3}\right\rangle^1 \tag{3}$$

The applied couple does not appear directly in this equation but its effect is incorporated in the statics equations (*1*) and (*2*). Integrating the first time

$$EI\frac{dy}{dx} = \frac{w_0L}{6}\cdot\frac{\langle x\rangle^2}{2} - \frac{w_0}{2}\frac{\left\langle x - \frac{L}{3}\right\rangle^3}{3} + \frac{w_0L}{2}\frac{\left\langle x - \frac{2L}{3}\right\rangle^2}{2} + C_1\langle x\rangle \tag{4}$$

Integrating the second time

$$EIy = \frac{w_0L}{12}\frac{\langle x\rangle^3}{3} - \frac{w_0}{6}\frac{\left\langle x - \frac{L}{3}\right\rangle^4}{4} + \frac{w_0L}{4}\frac{\left\langle x - \frac{2L}{3}\right\rangle^3}{3} + C_1\frac{\langle x\rangle^2}{2} + C_2 \tag{5}$$

As boundary conditions we have: when $x = 0$, $y = 0$, from which Eq. (*5*) leads to $C_2 = 0$. Also, when $x = 2L/3$, $y = 0$, from which Eq. (*5*) gives us

$$C_1 = -0.03472w_0L^2$$

The required equation of the deflected beam is thus

$$EIy = \frac{w_0L\langle x\rangle^3}{36} - \frac{w_0}{24}\left\langle x - \frac{L}{3}\right\rangle^4 + \frac{w_0L}{12}\left\langle x - \frac{2L}{3}\right\rangle^3 - 0.01736w_0L^2x^2 \tag{6}$$

10.9. In Problem 10-8 if the beam is a steel wide-flange section W203 × 51 (having $I = 52.5 \times 10^6\ \text{mm}^4$ from Table 8-2 of Chap. 8), of length 6 m, and subject to a uniform load over *BD* of intensity 22 kN/m, determine the deflection at point *B*.

From the general equation of the deflection curve, Eq. (*6*) of Problem 10.8, we may write the expression for the deflection at $y = L/3$ as

$$EIy]_{x=L/3} = \frac{w_0L}{36}\cdot\frac{L^3}{27} - 0 + 0 - 0.01736w_0L^2\left(\frac{L^2}{9}\right)$$
$$= -0.0009w_0L^4$$

Substituting,

$$y]_{x=L/3} = \frac{\frac{(22{,}000\ \text{N/m})(6\ \text{m})}{36}\cdot\frac{(6\ \text{m})^3}{27} - 0.01736\left(\frac{22{,}000\ \text{N}}{\text{m}}\right)(6\ \text{m})^2\left(\frac{6\ \text{m}}{9}\right)^2}{(200\times10^9\ \text{N/m}^2)(52.5\times10^{-6}\ \text{m}^4)}$$
$$= -2.44\times10^{-2}\ \text{m} \quad \text{or} \quad -24.4\ \text{mm}$$

10.10. The cantilever beam *AD* is loaded by the applied couples M_1 and $M_1/3$, as shown in Fig. 10-11. Use the method of singularities to determine the equation of the deflected beam.

For static equilibrium, there must be a reactive couple M_A acting at point *A*, as well as possibly a shear-type reactive force R_A. From statics we find

$$+\circlearrowleft \Sigma M_A = M_A - M_1 + \frac{M_1}{3} = 0 \qquad \text{and therefore } M_A = \frac{2}{3}M_1$$

$$\Sigma F_y = R_A = 0$$

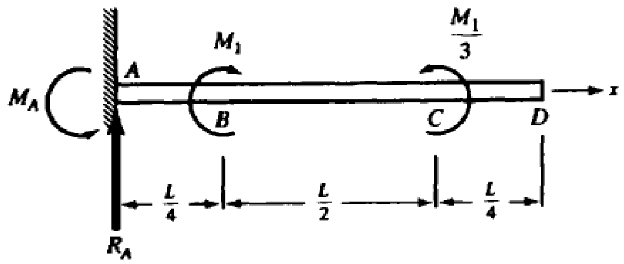


Fig. 10-11

The bending moment for any value of x is

$$M = -\frac{2}{3}M_1\langle x\rangle^0 + M_1\left\langle x - \frac{L}{4}\right\rangle^0 - \frac{M_1}{3}\left\langle x - \frac{3L}{4}\right\rangle^0 \tag{1}$$

so that the differential equation of the deflected beam is

$$EI\frac{d^2y}{dx^2} = -\frac{2}{3}M_1\langle x\rangle^0 + M_1\left\langle x - \frac{L}{4}\right\rangle^0 - \frac{M_1}{3}\left\langle x - \frac{3L}{4}\right\rangle^0 \tag{2}$$

Integrating the first time, we obtain

$$EI\frac{dy}{dx} = -\frac{2}{3}M_1\langle x\rangle^1 + M_1\left\langle x - \frac{L}{4}\right\rangle^1 - \frac{M_1}{3}\left\langle x - \frac{3L}{4}\right\rangle^1 + C_1 \tag{3}$$

and the first boundary condition is that $dy/dx = 0$ when $x = 0$. Hence, $C_1 = 0$.
Integrating a second time

$$EIy = -\frac{2}{3}M_1\frac{\langle x\rangle^2}{2} + \frac{M_1}{2}\left\langle x - \frac{L}{4}\right\rangle^2 - \frac{M_1}{6}\left\langle x - \frac{3L}{4}\right\rangle^2 + C_2 \tag{4}$$

and the second boundary condition is that $y = 0$ when $x = 0$. Hence $C_2 = 0$.
The equation describing the deflected beam is finally

$$EIy = -\frac{M_1\langle x\rangle^2}{3} + \frac{M_1}{2}\left\langle x - \frac{L}{4}\right\rangle^2 - \frac{M_1}{6}\left\langle x - \frac{3L}{4}\right\rangle^2 \tag{5}$$

10.11. The cantilever beam in Problem 10-10 is a steel wide-flange section W254 × 31, having $I = 44.1 \times 10^{-6}\,\text{m}^4$ and a length of 2 m. Determine M_1 if the deflection at point D is to be 3 mm.

We employ Eq. (5) of Problem 10.10 and simplify it for the deflection at $x = L$ to find

$$EIy]_{x=L} = -\frac{M_1L^2}{16}$$

Substituting the given numerical values, we find the tip deflection to be

$$y]_{x=L} = -\frac{M_1(2\,\text{m})^2}{(16)(200 \times 10^9\,\text{N/m}^2)(44.1 \times 10^{-6}\,\text{m}^4)} = 0.003\,\text{m}$$

Solving,

$$M_1 = 106\,\text{kN}\cdot\text{m}$$

10.12. Through the use of singularity functions determine the equation of the deflection curve of the simply supported beam of Fig. 10-12 subject to the couple applied at B plus the linearly varying load in CD.

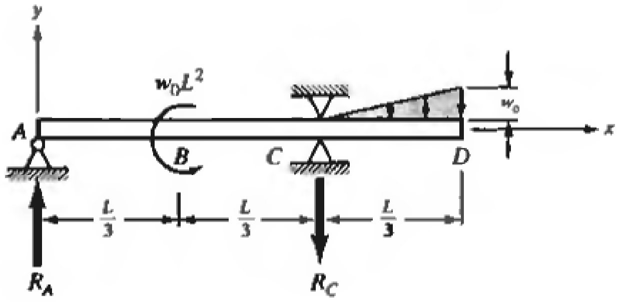


Fig. 10-12

Denoting the reactions at A and C by R_A and R_C assumed positive in the directions indicated and writing the two statics equations for this parallel force system, we obtain

$$+\circlearrowright \Sigma M_A = w_0L^2 - R_C\left(\frac{2L}{3}\right) - \frac{w_0}{2}\left(\frac{L}{3}\right)\left(\frac{2L}{3} + \frac{2}{3}\cdot\frac{L}{3}\right) = 0 \tag{1}$$

$$\Sigma F_y = R_A - R_C - \frac{w_0}{2}\cdot\frac{L}{3} = 0 \tag{2}$$

Solving, $R_A = \frac{13}{9}w_0L$ and $R_C = \frac{23}{18}w_0L$. Since each of these is positive, the assumed directions are correct.

In terms of singularity functions, the differential equation of the deflected beam is

$$EI\frac{d^2y}{dx^2} = \frac{13}{9}w_0L\langle x\rangle^1 - w_0L^2\left\langle x-\frac{L}{3}\right\rangle^0 - \frac{23}{18}w_0L\left\langle x-\frac{2L}{3}\right\rangle^1$$
$$- w_0\frac{\left\langle x-\frac{2L}{3}\right\rangle\left\langle x-\frac{2L}{3}\right\rangle\left(\frac{1}{2}\right)}{\left(\frac{L}{3}\right)}\cdot\frac{\left\langle x-\frac{2L}{3}\right\rangle}{3} \tag{3}$$

where the effect of the triangular loading in CD is represented as the last term in Eq. (3) using the technique for triangular load discussed in Problem 10.6 and illustrated in Fig. 10-9.

Integrating the first time, we obtain

$$EI\frac{dy}{dx} = \frac{13}{9}w_0L\frac{\langle x\rangle^2}{2} - w_0L^2\left\langle x-\frac{L}{3}\right\rangle^1 - \frac{23}{18}w_0L\frac{\left\langle x-\frac{2L}{3}\right\rangle^2}{2} - \frac{w_0}{2L}\frac{\left\langle x-\frac{2L}{3}\right\rangle^4}{4} + C_1 \tag{4}$$

We have no boundary conditions on slope; hence we are unable to determine C_1 at this time. Integrating the second time

$$EIy = \frac{13}{18}w_0L\frac{\langle x\rangle^3}{3} - w_0L^2\frac{\left\langle x-\frac{L}{3}\right\rangle^2}{2} - \frac{23}{36}w_0L\frac{\left\langle x-\frac{2L}{3}\right\rangle^3}{3} - \frac{w_0}{8L}\frac{\left\langle x-\frac{2L}{3}\right\rangle^5}{5} + C_1x + C_2 \tag{5}$$

As boundary conditions, we have $x = 0$ at $y = 0$, so from Eq. (5) we find $C_2 = 0$. Also, when $x = 2L/3$, $y = 0$, from which we have from Eq. (5)

$$0 = \frac{13}{54}w_0L\left(\frac{8L^3}{27}\right) - \frac{w_0L^2}{2}\cdot\frac{L^2}{9} - 0 - 0 + C_1\left(\frac{2L}{3}\right)$$

Solving,

$$C_1 = -0.02366w_0L^3$$

The deflection curve of the bent beam is thus

$$EIy = \frac{13}{54}w_0L\langle x\rangle^3 - \frac{w_0L^2}{2}\left\langle x - \frac{L}{3}\right\rangle^2 - \frac{23}{108}w_0L\left\langle x - \frac{2L}{3}\right\rangle^3 - \frac{w_0}{40L}\left\langle x - \frac{2L}{3}\right\rangle^5 - 0.02366w_0L^3\langle x\rangle$$

10.13. Determine the equation of the deflection curve of the simply supported beam shown in Fig. 10-13(a). Use singularity functions.

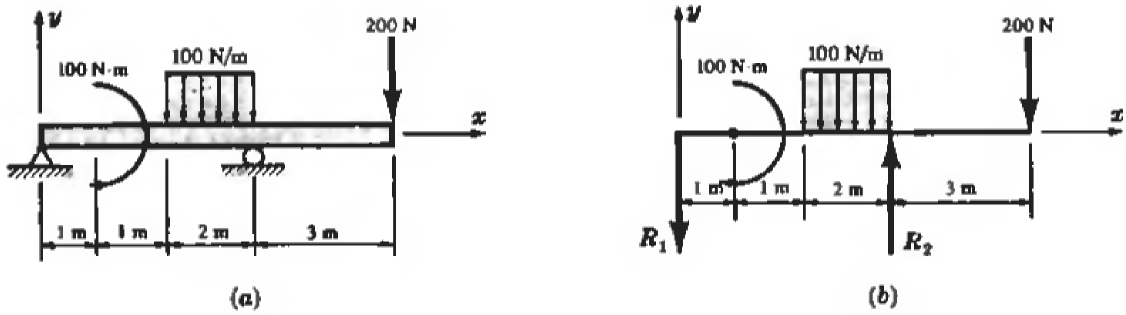


Fig. 10-13

The free-body diagram is shown in Fig. 10-13(b). From statics the reactions are readily found to be $R_1 = 225$ N, $R_2 = 525$ N.

Writing the bending moment corresponding to Fig. 10-13(b) in terms of singularity functions, we have

$$EI\frac{d^2y}{dx^2} = M = -225\langle x\rangle^1 + 100\langle x-1\rangle^0 - \frac{100\langle x-2\rangle^2}{2} + \frac{100\overset{\circledast}{\langle x-4\rangle^2}}{2} + 525\langle x-4\rangle^1 \qquad (1)$$

where the term denoted by ⊛ is necessary to annul the effect of the 100 N/M load to the right of $x = 4$ m.

Integrating,

$$EI\frac{dy}{dx} = -\frac{225}{2}\langle x\rangle^2 + 100\langle x-1\rangle^1 - \frac{50}{3}\langle x-2\rangle^3 + \frac{50}{3}\langle x-4\rangle^3 + \frac{525}{2}\langle x-4\rangle^2 + C_1 \qquad (2)$$

$$EIy = -\frac{225}{6}\langle x\rangle^3 + \frac{100}{2}\langle x-1\rangle^2 - \frac{50}{12}\langle x-2\rangle^4 + \frac{50}{12}\langle x-4\rangle^4 + \frac{525}{6}\langle x-4\rangle^3 + C_1x + C_2 \qquad (3)$$

The boundary conditions are $y = 0$ at $x = 0$, $x = 4$ m. Using these conditions in (3) to determine C_1 and C_2, we find $C_1 = 504$, $C_2 = 0$.

The desired deflection curve is thus

$$EIy = -\frac{225}{6}\langle x\rangle^3 + \frac{100}{2}\langle x-1\rangle^2 - \frac{50}{12}\langle x-2\rangle^4 + \frac{50}{12}\langle x-4\rangle^4 + \frac{525}{6}\langle x-4\rangle^3 + 504x \qquad (4)$$

10.14. The elastic beam AD shown in Fig. 10-14 is simply supported at B and C and subject to an applied couple M_1 at point A together with a uniformly distributed load in the overhanging region CD. Find the equation of the deformed beam as well as the deflection at point A.

From statics the reactions R_B and R_C are found to be

$$R_B = \tfrac{73}{48}wL\,(\downarrow) \qquad R_C = \tfrac{91}{48}wL\,(\uparrow)$$

Using the method of singularity functions, we find that the differential equation of the bent beam is

$$EI\frac{d^2y}{dx^2} = \frac{1}{2}wL^2 - \frac{73}{48}wL\left\langle x - \frac{L}{4}\right\rangle + \frac{91}{48}wL\left\langle x - \frac{5}{8}L\right\rangle - \frac{w}{2}\left\langle x - \frac{5}{8}L\right\rangle^2 \qquad (1)$$

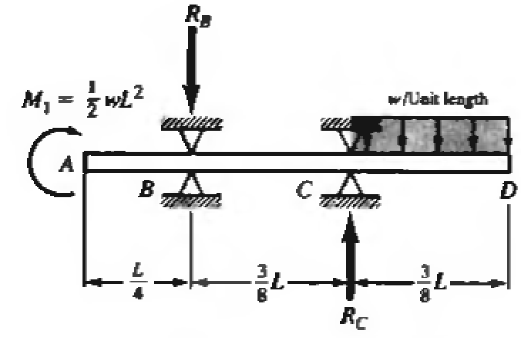


Fig. 10-14

Integrating the first time, we have

$$EI\frac{dy}{dx} = \frac{1}{2}wL^2\langle x\rangle - \frac{73}{96}wL\left\langle x - \frac{L}{4}\right\rangle^2 + \frac{91}{96}wL\left\langle x - \frac{5}{8}L\right\rangle^2 - \frac{w}{6}\left\langle x - \frac{5}{8}L\right\rangle^3 + C_1 \tag{2}$$

Integrating a second time

$$EIy = \frac{1}{2}wL\frac{\langle x\rangle^2}{2} - \frac{73}{288}wL\left\langle x - \frac{L}{4}\right\rangle^3 + \frac{91}{288}wL\left\langle x - \frac{5}{8}L\right\rangle^3 - \frac{w}{24}\left\langle x - \frac{5}{8}L\right\rangle^4 + C_1x + C_2 \tag{3}$$

As boundary conditions to determine C_1 and C_2, we have

First: When $x = L/4$, $y = 0$. Substituting in Eq. (*3*), we have

$$0 = \frac{wL}{4}\cdot\frac{L^2}{16} - 0 + 0 - 0 + C_1\frac{L}{4} + C_2 \tag{4}$$

Second: When $x = 5L/8$, $y = 0$. Substituting in Eq. (*3*), we have

$$0 = \frac{wL^2}{4}\cdot\frac{25}{64}L^2 - \frac{73}{288}wL\cdot\frac{27L^3}{512} + 0 + \frac{5}{8}LC_1 + C_2 \tag{5}$$

Solving Eqs. (*4*) and (*5*), we obtain

$$C_1 = -0.1831wL^3 \qquad C_2 = 0.03015wL^4$$

The equation of the deflected beam, for all values of x, is

$$EIy = -\frac{wL^2}{4}\langle x\rangle^2 - \frac{73}{288}wL\left\langle x - \frac{L}{4}\right\rangle^3 + \frac{91}{288}wL\left\langle x - \frac{5}{8}L\right\rangle^3 - \frac{w}{24}\left\langle x - \frac{5}{8}L\right\rangle^4 - 0.1831wL^3x + 0.03015wL^4 \tag{6}$$

At the left end, $x = 0$, and the deflection there is

$$EIy]_{x=0} = 0.03015w_0L^4$$

10.15. Use singularity functions to determine the equation of the deflection curve of the simply supported beam subject to a uniformly varying load as in Fig. 10-15(*a*). What is the central deflection of the beam?

The free-body diagram with the reactions found from statics is shown in Fig. 10-15(*b*).

If we refer to Problem 10.6, we can write the bending moment at any location x in the form

$$M(x) = +\frac{w_0L}{4}\langle x\rangle^1 - \frac{w_0}{3L}\langle x\rangle^3 + \frac{2w_0}{3L}\left\langle x - \frac{L}{2}\right\rangle^3 \tag{1}$$

where the second term on the right side of (*1*) represents a uniformly varying load extending completely across the beam as indicated by the triangle *OAB* in Fig. 10-16. To remove the portion of this loading represented by triangle *ABD*, we add the third term on the right side, which leaves the true load represented by triangle *ODB*.

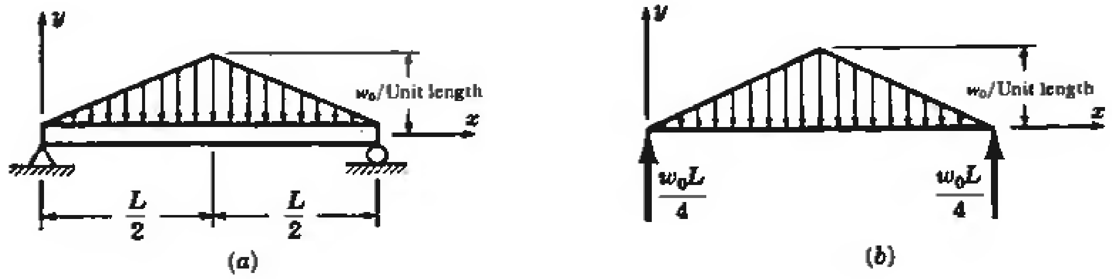


Fig. 10-15

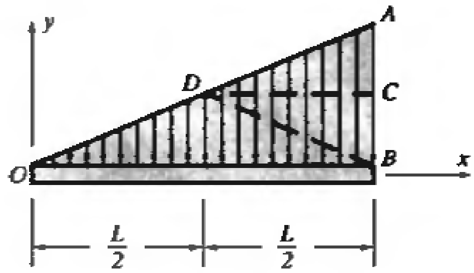


Fig. 10-16

Thus

$$EI\frac{d^2y}{dx^2} = M = +\frac{w_0L}{4}\langle x\rangle^1 - \frac{w_0}{3L}\langle x\rangle^3 + \frac{2w_0}{3L}\left\langle x - \frac{L}{2}\right\rangle^3 \tag{2}$$

from which

$$EI\frac{dy}{dx} = +\frac{w_0L}{8}\langle x\rangle^2 - \frac{w_0}{12L}\langle x\rangle^4 + \frac{w_0}{6L}\left\langle x - \frac{L}{2}\right\rangle^4 + C_1 \tag{3}$$

From symmetry we have as a boundary condition $dy/dx = 0$ at $x = L/2$. From (*3*) we find that $C_1 = -5w_0L^3/192$. Integrating again we get the desired deflection curve.

$$EIy = \frac{w_0L}{24}\langle x\rangle^3 - \frac{w_0}{60L}\langle x\rangle^5 + \frac{w_0}{30L}\left\langle x - \frac{L}{2}\right\rangle^5 - \frac{5}{192}w_0L^3x + C_2 \tag{4}$$

Since $y = 0$ at $x = 0$, it follows that $C_2 = 0$. The central deflection is found from (*4*) to be

$$y = -\frac{w_0L^4}{120EI}$$

Statically Determinate Beams—Computerized Solutions

Problems 10.1 through 10.15 have demonstrated the efficiency of the method of singularity functions for the determination of beam deflections. The technique is very well suited to computer implementation because there is a direct correspondence between the singularity function $\langle x - a\rangle$ defined as

$$\langle x - a\rangle = \begin{cases} 0 & \text{if } x < a \\ (x - a) & \text{if } x > a \end{cases}$$

and the "if" statement in FORTRAN. This feature is utilized extensively in the computerized approach in Problem 10.16.

10.16. Write a FORTRAN program for determination of slope and deflection at selected points along the length of a beam of constant cross section, simply supported at two arbitrary points, and loaded by arbitrary concentrated forces, moments, and uniformly distributed loads.

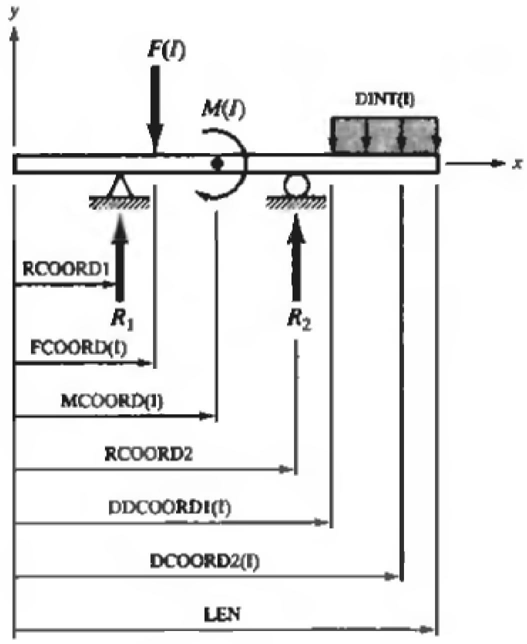


Fig. 10-17

Let us employ the terminology shown in Fig. 10-17. See Table 10-1.

A complete listing of the program based upon numerical solution of the beam bending equation

$$EI\frac{d^2y}{dx^2} = M$$

utilizing singularity functions follows. One must introduce all parameters of beam loading, geometry, and elastic properties. The program will then print out the slope and deflection (with appropriate algebraic sign) at each of the (NUM + 1) points along the length of the beam as well as values of the reactions R_1 and R_2.

Table 10-1

Units	USCS or SI
E	Young's modulus
I	Moment of inertia of beam cross section about the neutral axis
LEN	Length of beam
NF	Number of applied concentrated forces (not including reactions)
NM	Number of applied moments
ND	Number of uniformly distributed loads
NUM	Number of segments into which length of beam is divided for purpose of analysis
RCOORD1	Coordinate locating reaction R
RCOORD2	Coordinate locating reaction R
FCOORD(I)	Coordinate locating applied concentrated force I
FMAG(I)	Magnitude of concentrated force I
MCOORD(I)	Coordinate of locating moment I
MMAG(I)	Magnitude of moment I
DDCOORD1(I)	Left coordinate of distributed load I
DDCOORD2(I)	Right coordinate of distributed load I
MCOORD(I)	Magnitude (load/unit length) of uniformly distributed load I

```
***************************************************************************
          PROGRAM BEND (INPUT,OUTPUT)
***************************************************************************
*
*          AUTHOR: KATHLEEN DERWIN
*          DATE  : JANUARY 29,1989
*
*  BRIEF DESCRIPTION:
*       THIS PROGRAM CONSIDERS THE BENDING OF BEAMS DUE TO CONCENTRATED
*  FORCES, CONCENTRATED MOMENTS, AND UNIFORMLY DISTRIBUTED LOADS. FIRST,
*  THE PIN REACTION FORCES ARE FOUND, AND THEN THE SLOPE AND DEFLECTION OF
*  THE LOADED BEAM AT VARIOUS INCREMENTS ALONG ITS LENGTH ARE DETERMINED.
*  NOTE, THIS PROGRAM WAS DEVELOPED TO CONSIDER GENERAL LOADING, AND THE
*  PINS DO NOT HAVE TO BE AT THE ENDPOINTS OF THE BEAM.
*
*  INPUT:
*     THE USER MUST FIRST ENTER IF USCS OF SI UNITS ARE DESIRED. THEN,
*  THE MOMENT OF INERTIA, YOUNG'S MODULUS, AND THE LENGTH OF THE BEAM
*  ARE ENTERED. FINALLY, THE NUMBER, MAGNITUDE, AND LOCATION OF ALL
*  LOAD TYPES, AND THE NUMBER OF INCREMENTS TO PERFORM THE SLOPE AND
*  DEFLECTION CALCULATIONS  ARE INPUTTED.
*
*  OUTPUT:
*       THE PROGRAM PRINTS THE MAGNITUDE AND SENSE OF THE TWO REACTION
*  FORCES, AS WELL AS THE SLOPE AND DEFLECTION AT SUCCESSIVE INTERVALS
*  ALONG THE BEAM.
*
*  VARIABLES:
*      E,INER,LEN       ---    YOUNG'S MODULUS, MOMENT OF INERTIA, LENGTH
*                              OF BEAM
*     NUM               ---    NUMBER OF INCREMENTS TO DO CALCULATIONS ON
*  RCOORD1,RCOORD2      ---    LOCATION OF THE PINS
*     R1,R2             ---    MAGNITUDE OF THE PIN REACTION FORCES
*  FCOORD(I),FMAG(I) ---       LOCATION AND MAGNITUDE OF CONCENTRATED FORCE
*  DDCOORD1(I),DCOORD2(I)-     LOCATION OF DISTRIBUTED LOADS
*    DINT(I)            ---    INTENSITY OF DISTRIBUTED LOADS
*  MCOORD(I),MMAG(I) ---       LOCATION AND MAGNITUDE OF MOMENTS
*       DX              ---    INCREMENTAL STEP ALONG BEAM (LENGTH/NUM)
*  VV1,VV2,...VV6       ---    THE 'BRACKET TERMS' OF THE SINGULARITY FNCTS
*  SLF(I),DF(I),SLM(I),
*  DM(I),SLD(I),DD(I)---       THE SUMMING ARRAYS FOR SLOPE AND DEFLECTION
*                              DUE TO EACH APPLIED FORCE AT A PARTICULAR PT
*  SLR1,SLR2,DR1,DR2 ---       THE EFFECTS OF THE REACTION FORCES AT A POIN
*  SLFX,SLMX,SLDX       ---    THE TOTAL SLOPE AND DEFLECTION DUE TO BOTH
*  DFX,DMX,DDX                 APPLIED AND REACTIVE FORCES AT A POINT
*     C1,C2             ---    THE CONSTANTS OF INTEGRATION
*  SL(I),D(I)           ---    THE FINAL SLOPE AND DEFLECTION AT ANY POINT
*    NF,NM,ND           ---    THE NUMBER OF CONCENTRATED FORCES (NOT
*                              INCLUDING REACTIONS), APPLIED MOMENTS, AND
*                              UNIFORMLY DISTRIBUTED LOADS
*   FSUM,MSUM           ---    THE SUM OF THE FORCES AND MOMENTS, USED TO
*                              COMPUTE THE REACTIVE FORCES
*  DDIST(I),LOAD(I)     ---    THE DISTANCE EACH DISTRIBUTED LOAD SPANS, AN
*                              THE MAGNITUDE OF THE RESULTING FORCE
*       BIG             ---    GIVES THE LARGEST NUMBER OF ALL FORCE TYPES
*       ANS             ---    DENOTES IF USCS OR SI UNITS ARE DESIRED
*
*
****************************************************************************
**********                      MAIN PROGRAM                    ************
****************************************************************************
*
*          VARIABLE DECLARATIONS
*
      REAL E,INER,LEN,NUM,RCOORD1,RCOORD2,FCOORD(10),FMAG(10),MCOORD(10)
      REAL MMAG(10),DCOORD1(10),DCOORD2(10),DINT(10),DX,X,XX,VV1,VV2
```

```
      REAL VV3,VV4,VV5,VV6,SLF(10),SLD(10),SLM(10),DF(10),DD(10),DM(10)
      REAL SLR1,SLR2,DR1,DR2,R1,R2,SLFX,SLDX,SLMX,DFX,DDX,DMX
      REAL C1,C2,FSUM,MSUM,DDIST(10),LOAD(10),SL(100),D(100)
      INTEGER NF,ND,NM,BIG,ANS
*
*           INITIALIZING VARIABLES TO ZERO
*
      FCOORD(10)=0.0
      FMAG(10)=0.0
      MCOORD(10)=0.0
      MMAG(10)=0.0
      DCOORD1(10)=0.0
      DCOORD2(10)=0.0
      DINT(10)=0.0
      SLF(10)=0.0
      SLD(10)=0.0
      SLM(10)=0.0
      DF(10)=0.0
      DD(10)=0.0
      DM(10)=0.0
      SL(100)=0.0
      D(100)=0.0
      SLFX=0.0
      SLDX=0.0
      SLMX=0.0
      DFX =0.0
      DDX =0.0
      DMX =0.0
*
******     USER INPUT     *****
*
      PRINT*,'PLEASE INDICATE YOUR CHOICE OF UNITS:'
      PRINT*,'1 - USCS'
      PRINT*,'2 - SI'
      PRINT*,' '
      PRINT*,'ENTER 1,2:'
      READ*,ANS
      IF (ANS.EQ.1) THEN
         PRINT*,'PLEASE INPUT ALL DATA IN UNITS OF POUND AND/OR INCH...'
      ELSE
         PRINT*,'PLEASE INPUT ALL DATA IN UNITS OF NEWTON AND/OR METER...
      ENDIF
*

      PRINT*,' '
      PRINT*,'ENTER THE VALUES FOR E,I,LEN,NF,ND,NM,NUM:'
      READ(*,*)E,INER,LEN,NF,ND,NM,NUM
      PRINT*,' '
      PRINT*,'ENTER THE COORDINATES OF THE ALL FORCE TYPES AS DISTANCES'
      PRINT*,'FROM THE LEFT END OF THE BEAM...ALSO, CONSIDER FORCES'
      PRINT*,'DIRECTED DOWNWARD, AND MOMENTS ACTING CLOCKWISE AS POSITIVE
      PRINT*,' '
      PRINT*,'ENTER THE COORDINATES OF THE REACTION POINTS:'
      READ(*,*)RCOORD1,RCOORD2
      IF (NF.GT.0) THEN
      PRINT*,'ENTER THE COORDINATE AND MAGNITUDE OF ALL CONCENTRATED '
      PRINT*,'FORCES:'
      READ(*,*)(FCOORD(I),FMAG(I),I=1,NF)
      ENDIF
      IF (NM.GT.0) THEN
      PRINT*,'ENTER THE COORDINATE AND MAGNITUDE OF ALL CONCENTRATED '
      PRINT*,'MOMENTS:'
      READ(*,*)(MCOORD(I),MMAG(I),I=1,NM)
      ENDIF
      IF (ND.GT.0) THEN
      PRINT*,'ENTER THE FIRST AND SECOND COORDINATE AND THEN INTENSITY '
```

```
      PRINT*,'OF ALL DISTRIBUTED LOADS:'
      READ(*,*)(DCOORD1(I),DCOORD2(I),DINT(I),I=1,ND)
      ENDIF
*
******     END USER INPUT     *****
*
      PRINT*,'  '
      PRINT*,' THE MAGNITUDES OF THE TWO REACTIVE FORCES (LB OR  NEWTONS)
*
******      CALCULATIONS      ******
*
*
*           CALCULATING THE MAGNITUDE AND DIRECTION OF THE PIN REACTION
*           FORCES
*
      FSUM=0.0
      MSUM=0.0
      DO 15 I=1,ND
         DDIST(I)= DCOORD2(I) - DCOORD1(I)
         LOAD(I) = DINT(I)*DDIST(I)
         FSUM = LOAD(I) + FSUM
         MSUM =(((0.5*DDIST(I) + DCOORD1(I)) - RCOORD1) * LOAD(I)) + MSUM
 15   CONTINUE
      DO 20 I = 1,NF
         FSUM = FSUM + FMAG(I)
         MSUM =((FCOORD(I) - RCOORD1)*FMAG(I)) + MSUM
 20   CONTINUE
      DO 30 I = 1,NM
         MSUM = MSUM + MMAG(I)
 30   CONTINUE
      R2 = -(MSUM/(RCOORD2-RCOORD1))
      R1 = -(FSUM+R2)
*
*           PRINTING THE REACTION FORCES
*
      PRINT*,' '
      PRINT*,'R1 = ',R1,'         R2 = ',R2
      PRINT*,' '
*
*           CALCULATING THE LARGEST NUMBER OF EITHER FORCES, DISTRIBUTED
*           LOADS, OR MOMENTS
*
      IF (NF.GE.ND) THEN
         IF (NF.GE.NM) THEN
            BIG=NF
         ELSE
            BIG=NM
         ENDIF
      ELSE
         IF (ND.GE.NM) THEN
            BIG=ND
         ELSE
            BIG=NM
         ENDIF
      ENDIF
*
*
*      THE FOLLOWING SECTION OF THIS PROGRAM PERFORMS THE CALCULATIONS
*      THAT DETERMINE THE SLOPE AND DEFLECTION AT SEVERAL INTERVALS ALONG
*      THE BEAM. THE METHOD OF SINGULARITY FUNCTIONS AND INTEGRATION IS
*      EMPLOYED, AND THE PRINCIPAL OF SUPERPOSITION ALLOWS EACH TYPE OF
*      FORCE TO BE CONSIDERED SEPARATELY AND THEN SUMMED TO PRODUCE
*      THE NET EFFECT ON THE BEAM.
*
      DX=LEN/NUM
      J=1
 10   DO 50 XX=0,LEN,DX
```

```
      X=XX
*
*    THE FUNCTIONS ARE FIRST SOLVED FOR THE INITIAL CONDITIONS OF ZERO
*    DISPLACEMENT AT THE TWO PIN REACTION POINTS, RCOORD1 AND RCOORD2,
*    THAT THE CONSTANTS OF INTEGRATION MAY BE DETERMINED.
*
      IF (J.EQ.1) X=RCOORD1
      IF (J.EQ.2) X=RCOORD2
*
*  EVALUATING THE 'BRACKET TERMS' USED WITH THE SINGULARITY FUNCTIONS
*
      DO 60 I=1,BIG
         VV1=X-FCOORD(I)
         VV2=X-DCOORD1(I)
         VV3=X-DCOORD2(I)
         VV4=X-RCOORD1
         VV5=X-RCOORD2
         VV6=X-MCOORD(I)
*
*      RECALL, WITH SINGULARITY FUNCTIONS IF THE QUANTITY IN THE
*      BRACKETS IS LESS THAN OR EQUAL TO ZERO, THAT TERM MAKES NO
*      CONTRIBUTION TO THE SLOPE AND/OR DEFLECTION AT THAT POINT.
*
         IF (VV1.LE.0) VV1=0
         IF (VV2.LE.0) VV2=0
         IF (VV3.LE.0) VV3=0
         IF (VV4.LE.0) VV4=0
         IF (VV5.LE.0) VV5=0
         IF (VV6.LE.0) VV6=0
*
*     DETERMINING THE SLOPE AND DISPLACEMENT DUE TO EACH FORCE AT A
*     PARTICULAR POINT ON THE BEAM
*
         SLF(I) = FMAG(I)/2*(VV1**2)
         DF(I)  = FMAG(I)/6*(VV1**3)

         SLD(I) = (DINT(I)/6*(VV2**3)) - (DINT(I)/6*(VV3**3))
         DD(I) = (DINT(I)/24*(VV2**4)) - (DINT(I)/24*(VV3**4))

         SLM(I) = MMAG(I)*VV6
         DM(I)  = MMAG(I)/2*(VV6**2)
 60   CONTINUE
*
*     DETERMINING THE SLOPE AND DISPLACEMENT DUE TO THE REACTION FORCE
*     AT A PARTICULAR POINT ON THE BEAM
*
      SLR1 = R1/2 * (VV4**2)
      SLR2 = R2/2 * (VV5**2)
      DR1  = R1/6 * (VV4**3)
      DR2  = R2/6 * (VV5**3)
*
*     SUMMING THE EFFECTS OF ALL FORCE CONTRIBUTIONS OF THE SLOPE AND
*     DISPLACEMENT AT A PARTICULAR POINT ON THE BEAM
*
      DO 40 I=1,BIG
         SLFX= SLFX+ SLF(I)
         SLDX= SLDX+ SLD(I)
         SLMX= SLMX+ SLM(I)
         DFX = DFX + DF(I)
         DDX = DDX + DD(I)
         DMX = DMX + DM(I)
 40   CONTINUE

      SL(J) = SLFX + SLDX + SLMX + SLR1 + SLR2
      D(J)  = DFX + DDX + DMX + DR1 + DR2
      J =J+1
```

```
*
*        SETTING THE SLOPE AND DISPLACEMENT SUMS BACK TO ZERO BEFORE
*        MOVING TO NEXT POINT ON BEAM
*
         SLFX=0.0
         SLDX=0.0
         SLMX=0.0
         DFX =0.0
         DDX =0.0
         DMX =0.0
         IF (J.EQ.3) GO TO 10
*
*        REPEAT THIS PROCEDURE FOR NEXT POINT ON BEAM
   50 CONTINUE
*
*            CALCULATING THE CONSTANTS OF INTEGRATION FROM THE INITIAL
*            CONDITIONS OF ZERO DISPLACEMENT AT THE PINS.
*
      C1 = (D(2) - D(1))/(RCOORD1 - RCOORD2)
      C2 =(-D(1) - (C1*RCOORD1))

      X=0.0
*
*            FINALLY, DETERMINING THE SLOPE AND DISPLACEMENT AT EVERY POIN
*            BY CONSIDERING ALL THE FORCE CONTRIBUTIONS AT EACH RESPECTIVE
*            POINT, AND THE CONSTANTS OF INTEGRATION.
*
      DO 80 I=3,J-1
            SL(I) =(SL(I) + C1)/(E*INER)
            D(I)  =(D(I) + (C1*X) + C2)/(E*INER)
*           PRINT*,SL(I),D(I)
            X=X+DX
   80 CONTINUE
*
*           PRINTING THE SLOPE AND DELECTION AT INCREMENTS ALONG THE BEAM
*
      PRINT 82,'NODE','LOCATION','SLOPE','DEFLECTION'
      IF (ANS.EQ.1) THEN
         PRINT 83
      ELSE
         PRINT 84
      ENDIF
      X=0.0
*
      DO 85 I=3,J-1
         PRINT 90,I-2,X,SL(I),D(I)
         X=X+DX
   85 CONTINUE
*
*          FORMAT STATEMENTS
*
   82 FORMAT(//,2X,A4,5X,A8,5X,A5,6X,A10)
   83 FORMAT(3X,'NO',9X,'IN',8X,'IN/IN',10X,'IN')
   84 FORMAT(3X,'NO',9X,'M',9X,' M/M ',10X,'M')
   90 FORMAT(3X,I2,6X,F8.3,3X,E10.3,4X,E10.3)
      STOP
      END
```

10.17. A beam 12 m long is supported at knife edge reactions and loaded by a concentrated moment of 8000 N·m together with a concentrated force of 8500 N as shown in Fig. 10-18. Use the FORTRAN program of Problem 10.16 to determine the deflection by considering 25 segments along the length of the beam. The beam is of rectangular cross section 60 mm wide and 280 mm high and $E = 200$ GPa.

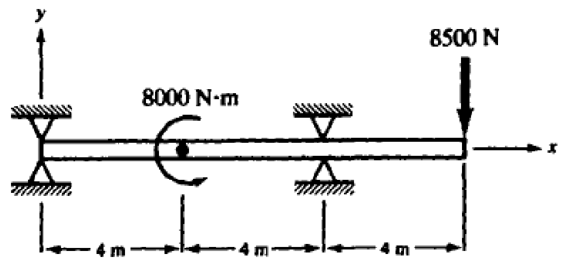


Fig. 10-18

The input into the program is shown in Table 10-2.
Input of these parameters into the program leads to the following output:

```
 PLEASE INDICATE YOUR CHOICE OF UNITS:
 1 - USCS
 2 - SI

 ENTER 1.2:
? 2
 PLEASE INPUT ALL DATA IN UNITS OF NEWTON AND/OR METER...

 ENTER THE VALUES FOR E,I,LEN,NF,ND,NM,NUM:
? 200E+9,109E-6,12,1,0,1,25

 ENTER THE COORDINATES OF ALL THE FORCE TYPES AS DISTANCES
 FROM THE LEFT END OF THE BEAM...ALSO, CONSIDER FORCES
 DIRECTED DOWNWARD, AND MOMENTS ACTING CLOCKWISE AS POSITIVE.

 ENTER THE COORDINATES OF THE REACTION POINTS:
? 0,8
```

Table 10-2

Units	SI
E	200×10^9
I	$\frac{1}{12}$ (0.06 m) (0.28 m)^3 = 109×10^{-6} m^4
LEN	12
NF	1
ND	0
NM	1
NUM	25
RCOORD1	0
RCOORD2	8
FCOORD(I)	12
FMAG(I)	8500
MCOORD(I)	4
MMAG(I)	8000
DCOORD1(I)	0
DCOORD2(I)	0
DMAG(I)	0

```
 ENTER THE COORDINATE AND MAGNITUDE OF ALL CONCENTRATED
 FORCES:
? 12,8500
 ENTER THE COORDINATE AND MAGNITUDE OF ALL CONCENTRATED
 MOMENTS:
? 4,8000

  THE MAGNITUDES OF THE TWO REACTIVE FORCES (LB OR NEWTONS):

 R1 = 5250.          R2 = -13750.

  NODE     LOCATION      SLOPE      DEFLECTION
   NO         M           M/M           M
    1        .000     -.294E-02     .000E+00
    2        .480     -.291E-02    -.140E-02
    3        .960     -.282E-02    -.278E-02
    4       1.440     -.269E-02    -.411E-02
    5       1.920     -.249E-02    -.535E-02
    6       2.400     -.224E-02    -.649E-02
    7       2.880     -.194E-02    -.750E-02
    8       3.360     -.158E-02    -.834E-02
    9       3.840     -.116E-02    -.900E-02
   10       4.320     -.571E-03    -.943E-02
   11       4.800      .132E-03    -.954E-02
   12       5.280      .891E-03    -.929E-02
   13       5.760      .171E-02    -.867E-02
   14       6.240      .257E-02    -.765E-02
   15       6.720      .350E-02    -.619E-02
   16       7.200      .448E-02    -.428E-02
   17       7.680      .552E-02    -.188E-02
   18       8.160      .660E-02     .103E-02
   19       8.640      .763E-02     .445E-02
   20       9.120      .856E-02     .833E-02
   21       9.600      .941E-02     .127E-01
   22      10.080      .102E-01     .174E-01
   23      10.560      .108E-01     .224E-01
   24      11.040      .114E-01     .277E-01
   25      11.520      .119E-01     .333E-01
   26      12.000      .123E-01     .391E-01

 SRU      1.284 UNTS.

 RUN COMPLETE.
```

From the printout we note that the deflection under the 8500-N force is 0.0391 m or 39.1 mm and under the 8000-N · m moment located between nodes 9 and 10 it is approximately −0.0092 m or −9.2 mm.

10.18. A beam 100 in long and of rectangular cross section with $I = 3.375\ \text{in}^4$ is loaded and supported as shown in Fig. 10-19. Use the FORTRAN program of Problem 10.16 to determine the deflections if the beam is represented by 50 segments along its length. Take $E = 30 \times 10^6\ \text{lb/in}^2$.

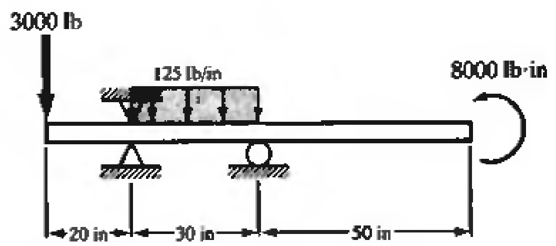


Fig. 10-19

The input to the program is shown in Table 10-3.

Table 10-3

Units	USCS
E	30 X 10^6
I	3.375
LEN	100
NF	1
ND	1
NM	1
NUM	50
RCOORD1	20
RCOORD2	50
FCOORD(I)	0
FMAG(I)	3000
MCOORD(I)	100
MMAG(I)	-8000
DCOORD1(I)	20
DCOORD2(I)	50
DMAG(I)	125

Input of these parameters into the program leads to the following output:

```
run
 PLEASE INDICATE YOUR CHOICE OF UNITS:
 1 - USCS
 2 - SI

 ENTER 1.2:
? 1
 PLEASE INPUT ALL DATA IN UNITS OF POUND AND/OR INCH...

 ENTER THE VALUES FOR E,I,LEN,NF,ND,NM,NUM:
? 30E6,3.375,100,1,1,1,50

 ENTER THE COORDINATES OF ALL THE FORCE TYPES AS DISTANCES
 FROM THE LEFT END OF THE BEAM...ALSO, CONSIDER FORCES
 DIRECTED DOWNWARD, AND MOMENTS ACTING CLOCKWISE AS POSITIVE.

 ENTER THE COORDINATES OF THE REACTION POINTS:
? 20,50
 ENTER THE COORDINATE AND MAGNITUDE OF ALL CONCENTRATED
 FORCES:
? 0,3000
 ENTER THE COORDINATE AND MAGNITUDE OF ALL CONCENTRATED
 MOMENTS:
? 100,-8000
 ENTER THE FIRST AND SECOND COORDINATE AND THEN MAGNITUDE
 OF ALL DISTRIBUTED LOADS:
? 20,50,125
```

THE MAGNITUDES OF THE TWO REACTIVE FORCES (LB OR NEWTONS):

R1 = -7141.666666667 R2 = 391.6666666667

NODE NO	LOCATION IN	SLOPE IN/IN	DEFLECTION IN
1	.000	-.101E-01	.162E+00
2	2.000	-.100E-01	.142E+00
3	4.000	-.983E-02	.122E+00
4	6.000	-.953E-02	.103E+00
5	8.000	-.912E-02	.838E-01
6	10.000	-.859E-02	.661E-01
7	12.000	-.793E-02	.496E-01
8	14.000	-.716E-02	.345E-01
9	16.000	-.628E-02	.210E-01
10	18.000	-.527E-02	.943E-02
11	20.000	-.414E-02	.000E+00
12	22.000	-.304E-02	-.715E-02
13	24.000	-.209E-02	-.123E-01
14	26.000	-.128E-02	-.156E-01
15	28.000	-.605E-03	-.175E-01
16	30.000	-.556E-04	-.181E-01
17	32.000	.380E-03	-.178E-01
18	34.000	.710E-03	-.166E-01
19	36.000	.946E-03	-.150E-01
20	38.000	.110E-02	-.129E-01
21	40.000	.117E-02	-.106E-01
22	42.000	.119E-02	-.826E-02
23	44.000	.114E-02	-.592E-02
24	46.000	.106E-02	-.371E-02
25	48.000	.933E-03	-.172E-02
26	50.000	.784E-03	.000E+00
27	52.000	.626E-03	.141E-02
28	54.000	.468E-03	.250E-02
29	56.000	.310E-03	.328E-02
30	58.000	.152E-03	.374E-02
31	60.000	-.617E-05	.389E-02
32	62.000	-.164E-03	.372E-02
33	64.000	-.322E-03	.323E-02
34	66.000	-.480E-03	.243E-02
35	68.000	-.638E-03	.131E-02
36	70.000	-.796E-03	-.123E-03
37	72.000	-.954E-03	-.187E-02
38	74.000	-.111E-02	-.394E-02
39	76.000	-.127E-02	-.632E-02
40	78.000	-.143E-02	-.902E-02
41	80.000	-.159E-02	-.120E-01
42	82.000	-.174E-02	-.154E-01
43	84.000	-.190E-02	-.190E-01
44	86.000	-.206E-02	-.230E-01
45	88.000	-.222E-02	-.273E-01
46	90.000	-.238E-02	-.319E-01
47	92.000	-.253E-02	-.368E-01
48	94.000	-.269E-02	-.420E-01
49	96.000	-.285E-02	-.475E-01
50	98.000	-.301E-02	-.534E-01
51	100.000	-.317E-02	-.596E-01

SRU 1.305 UNTS.

RUN COMPLETE.

Supplementary Problems

10.19. The cantilever beam *ABC* is loaded by a uniformly distributed load *w* per unit length over the right half *BC* as shown in Fig. 10-20. Use singularity functions to determine the deflection curve of the bent beam. Also, determine the deflection at the tip *C*.

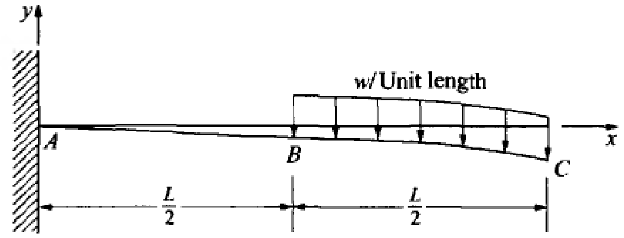


Fig. 10-20

Ans. $EIy = \frac{wL}{12}\langle x\rangle^3 - \frac{3}{8}wL^2\frac{\langle x\rangle^2}{2} - \frac{w}{24}\left\langle x - \frac{L}{2}\right\rangle^4$

$EIy]_{x=L} = -\frac{41}{384}$

10.20. Consider a simply supported beam subject to a uniform load acting over a portion of the beam as indicated in Fig. 10-21. Use singularity functions to determine the equation of the deflection curve.

Ans. $EIy = \frac{wb}{6L}\left(\frac{b}{2}+c\right)\langle x\rangle^3 - \frac{w}{24}\langle x-a\rangle^4 + \frac{w}{24}\langle x-a-b\rangle^4$

$+ \left\{\frac{w}{24L}[(L-a)^4 - (L-c)^4] - \frac{wbL}{6}\left(\frac{b}{2}+c\right)\right\}\langle x\rangle$

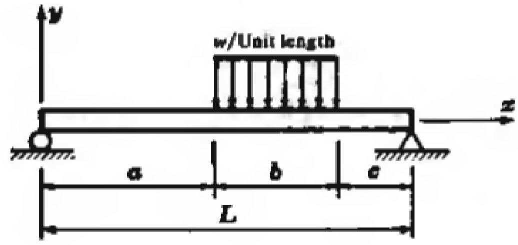


Fig. 10-21

10.21. The beam *ABCD* is pinned at *B*, rests on a roller at *C*, and is subjected to the tip loads each of magnitude *P* as shown in Fig. 10-22. Use the method of singularity functions to determine the deflection curve of the beam, which is symmetric about the midlength of the beam. Also, determine the deflection at point *A*.

Ans. $EIy = -\frac{P}{6}\langle x\rangle^3 + \frac{P}{6}\langle x-a\rangle^3 + \frac{P}{6}\langle x-(a+L_1)\rangle^3 + \left(\frac{PLa}{2} - \frac{Pa^2}{2}\right)\langle x\rangle$

$EIy]_{x=0} = \frac{2}{3}Pa^3 - \frac{PLa^2}{2}$

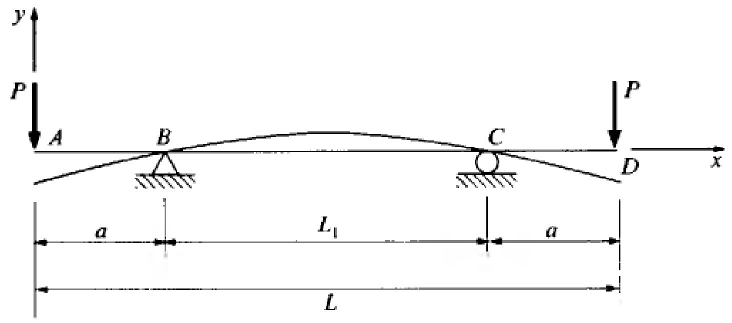


Fig. 10-22

Use singularity functions to determine the equation of the deflected beams in Problems 10.22 through 10.25.

10.22. See Fig. 10-23.

Ans. $EIy = -\frac{wa}{24}\langle x\rangle^3 - \frac{w}{24}\langle x\rangle^4 + \frac{w}{24}\langle x-a\rangle^4 + \frac{wa^2}{2}\langle x-a\rangle^2 + \frac{9}{24}wa\langle x-2a\rangle^3 - \frac{wa}{6}\langle x-3a\rangle^3 + \frac{11}{48}wa^3\langle x\rangle^1$

10.23. See Fig. 10-24.

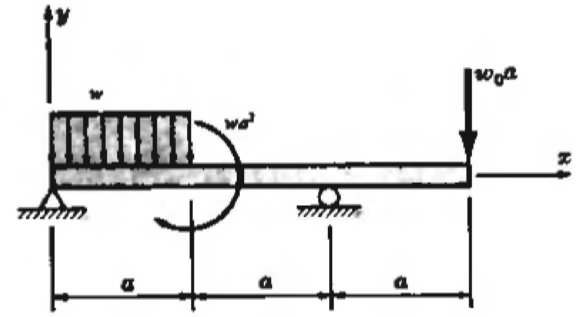


Fig. 10-23

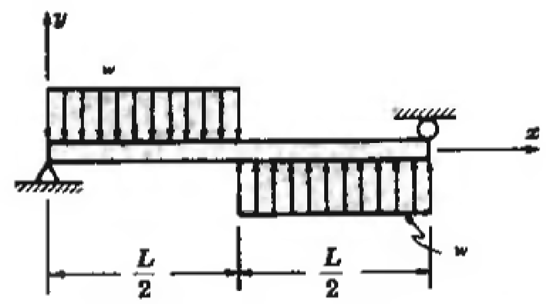


Fig. 10-24

Ans. $EIy = \frac{wL}{24}\langle x\rangle^3 - \frac{w}{24}\langle x\rangle^4 + \frac{w}{12}\left\langle x-\frac{L}{2}\right\rangle^4 - \frac{wL^3}{192}\langle x\rangle^1$

10.24. See Fig. 10-25.

Ans. $EIy = \frac{w_0L}{24}\langle x\rangle^3 - \frac{w_0}{24}\langle x\rangle^4 + \frac{w_0}{60L}\langle x\rangle^5 - \frac{w_0}{10L}\left\langle x-\frac{L}{2}\right\rangle^5 - \frac{3}{192}w_0L^3x$

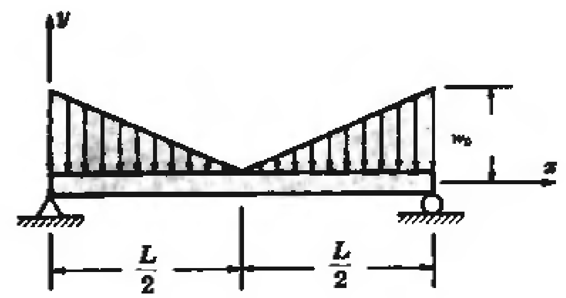


Fig. 10-25

10.25. See Fig. 10-26.

Ans. $EIy = -\frac{850}{3}\langle x\rangle^3 + 3300\langle x-3\rangle^2 - \frac{500}{12}\langle x-6\rangle^4 + \frac{500}{12}\langle x-9\rangle^4 + \frac{2350}{3}\langle x-9\rangle^3 + 10{,}175x$

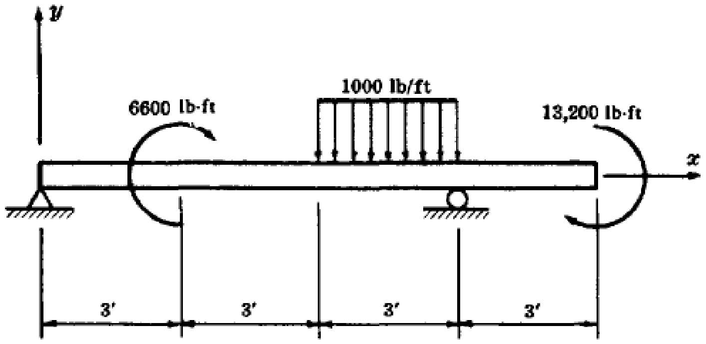


Fig. 10-26

10.26. The beam AC in Fig. 10-27 is 15 ft long, 3 in × 4 in in rectangular cross section, is subject to a uniform load of 120 lb/ft, and has $E = 30 \times 10\ \text{lb/in}^2$. Use the FORTRAN program of Problem 10.16 to determine (*a*) the deflection at the left end of the beam and (*b*) the maximum deflection of the beam.
Ans. (*a*) 0.065 in, (*b*) −0.10 in at $x = 110$ in

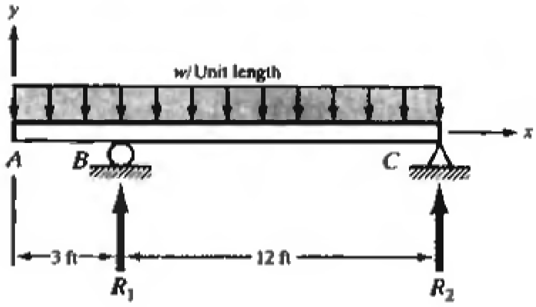


Fig. 10-27

10.27. Through the use of singularity functions, determine the equation of the deflection curve of the beam simply supported at B and C and subject to the triangular loading shown in Fig. 10-28.

Ans. $EIy = -\frac{w_0\langle x\rangle^5}{180L} + \frac{w_0 L}{16}\left\langle x - \frac{L}{2}\right\rangle^3 - 0.02050 w_0 L^3 x + 0.01042 w_0 L^4$

10.28. The beam shown in Fig. 10-29 is simply supported and subject to a concentrated force, the moment, and the uniformly distributed load indicated. The material has $E = 200$ GPa and the beam cross section has $I = 20 \times 10^{-6}\ \text{m}^4$. Use the FORTRAN program of Problem 10.16 to determine the deflection under the point of application of the 4200-N force. *Ans.* 19.8 mm

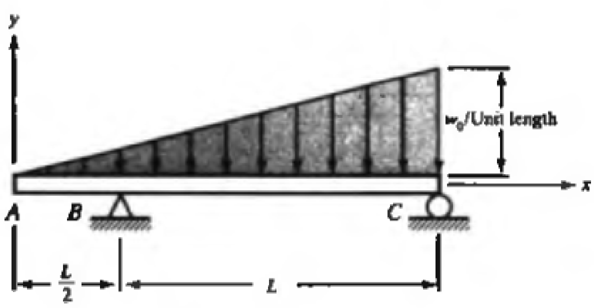


Fig. 10-28

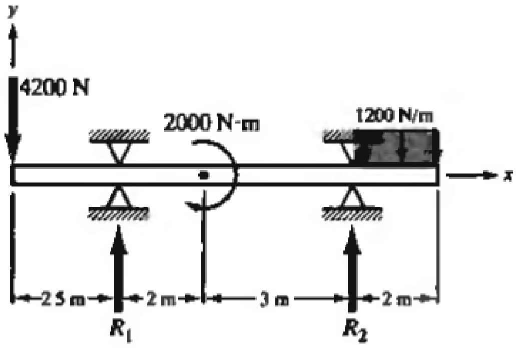


Fig. 10-29

Chapter 11

Statically Indeterminate Elastic Beams

STATICALLY DETERMINATE BEAMS

In Chaps. 8, 9, and 10 the deflections and stresses were determined for beams having various conditions of loading and support. In the cases treated it was always possible to completely determine the reactions exerted upon the beam merely by applying the equations of static equilibrium. In these cases the beams are said to be *statically determinate*.

STATICALLY INDETERMINATE BEAMS

In this chapter we shall consider those beams where the number of unknown reactions exceeds the number of equilibrium equations available for the system. In such a case it is necessary to supplement the equilibrium equations with additional equations stemming from the deformations of the beam. In these cases the beams are said to be *statically indeterminate*.

TYPES OF STATICALLY INDETERMINATE BEAMS

Several common types of statically indeterminate beams are illustrated below. Although a wide variety of such structures exists in practice, the following four diagrams will illustrate the nature of an indeterminate system. For the beams shown below the reactions of each constitute a parallel force system and hence there are two equations of static equilibrium available. Thus the determination of the reactions in each of these cases necessitates the use of additional equations arising from the deformation of the beam.

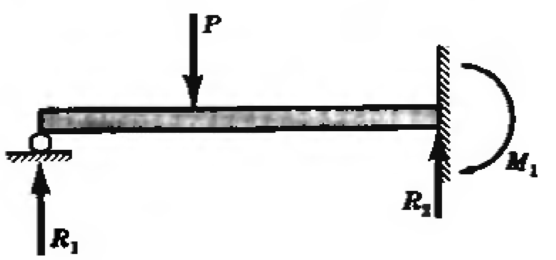


Fig. 11-1

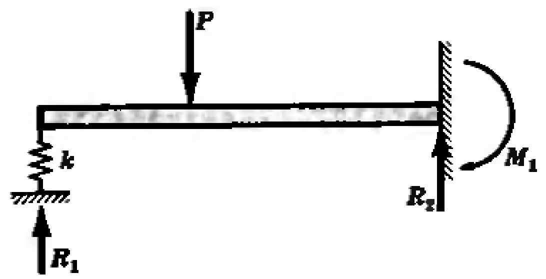


Fig. 11-2

In the case (Fig. 11-1) of a beam fixed at one end and supported at the other, sometimes termed a *supported cantilever*, we have as unknown reactions R_1, R_2, and M_1. The two statics equations must be supplemented by one equation based upon deformations. For applications, see Problems 11.1 and 11.3.

In Fig. 11-2 the beam is fixed at one end and has a flexible springlike support at the other. In the case of a simple linear spring the flexible support exerts a force proportional to the beam deflection at that point. The unknown reactions are again R_1, R_2, and M_1. The two statics equations must be supplemented by one equation stemming from deformations. For applications see Problems 11.2 and 11.16.

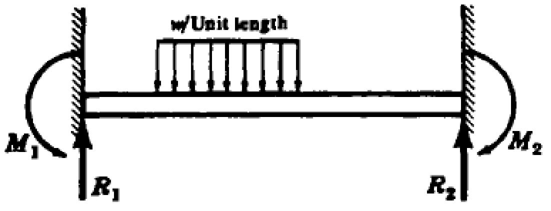


Fig. 11-3

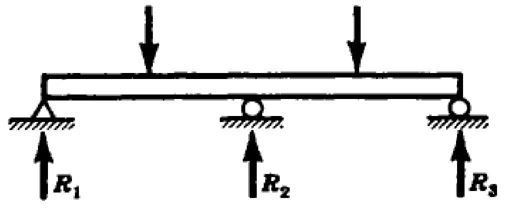


Fig. 11-4

As shown in Fig. 11-3, a beam fixed or clamped at both ends has the unknown reactions R_1, R_2, M_1, and M_2. The two statics equations must be supplemented by two equations arising from the deformations. For applications, see Problems 11.4, 11.6, and 11.12.

In Fig. 11-4 the beam is supported on three supports at the same level. The unknown reactions are R_1, R_2, and R_3. The two statics equations must be supplemented by one equation based upon deformations. A beam of this type that rests on more than two supports is called a *continuous beam.*

Solved Problems

11.1. A beam is clamped at A, simply supported at B, and subject to the concentrated force shown in Fig. 11-5. Determine all reactions.

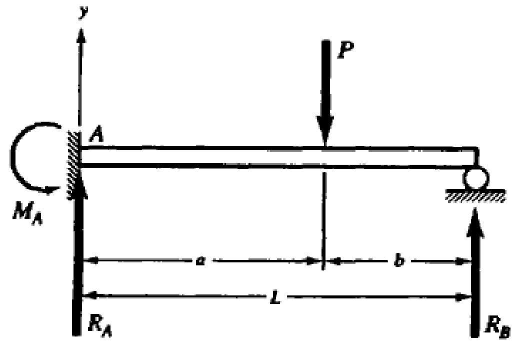


Fig. 11-5

The reactions are R_A, R_B, and M_A. From statics we have

$$+\circlearrowright \Sigma M_A = M_A - Pa + R_b L = 0 \tag{1}$$

$$\Sigma F_y = R_A + R_B - P = 0 \tag{2}$$

Thus there are two equations in the three unknowns R_A, R_B, and M_A. We can supplement the statics equations with an equation stemming from deformations using the method of singularity functions to describe the bent beam. This is

$$EI\frac{d^2y}{dx^2} = R_A\langle x\rangle - M_A\langle x\rangle^0 - P\langle x-a\rangle \tag{3}$$

Integrating the first time, we have

$$EI\frac{dy}{dx} = R_A\frac{\langle x\rangle^2}{2} - M_A\langle x\rangle - \frac{P}{2}\langle x-a\rangle^2 + C_1 \tag{4}$$

The first boundary condition is that at $x = 0$, $dy/dx = 0$, and thus $C_1 = 0$. Integrating again,

$$EIy = \frac{R_A}{2}\frac{\langle x\rangle^3}{3} - M_A\frac{\langle x\rangle^2}{2} - \frac{p}{2}\frac{\langle x-a\rangle^3}{3} + C_2 \tag{5}$$

The second boundary condition is that at $x = 0$, $y = 0$, and we find $C_2 = 0$.

The third boundary condition is that at $x = L$, $y = 0$. Substituting in Eq. (5), we have

$$0 = \frac{R_A L^3}{6} - \frac{M_A L^2}{2} - \frac{Pb^3}{6} \tag{6}$$

Simultaneous solution of the three equations (1), (2), and (6) leads to

$$R_A = \frac{Pb}{2L^3}(3L^2 - b^2)$$

$$R_B = \frac{Pa^2}{2L^3}(2L + b)$$

$$M_A = \frac{Pb}{2L^2}(L^2 - b^2)$$

11.2. The beam AB in Fig. 11-6 is clamped at A, spring supported at B, and loaded by the uniformly distributed load w per unit length. Prior to application of the load, the spring is stress free. The spring constant is 345 kN/m. To determine the flexural rigidity EI of the beam, an experiment is conducted without the uniform load w and also without the spring being present. In this experiment it is found that a vertical force of 10,000 N applied at end B deflects that point 50 mm. The spring is then attached to the beam at B and a uniform load of magnitude 5 kN/m is applied between A and B. Determine the deflection of point B under these conditions.

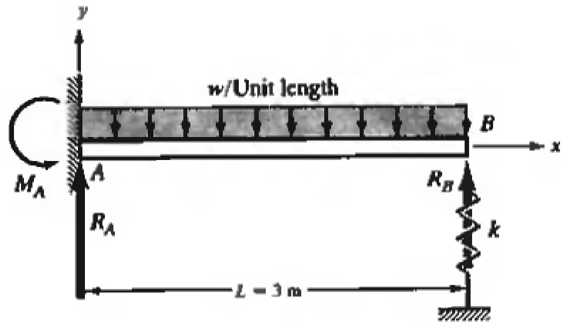


Fig. 11-6

The forces acting on the beam when it is uniformly loaded as well as spring supported at its tip are shown in Fig. 11-6. The force R_B represents the force exerted by the spring on the beam. The differential equation of the bent beam in terms of singularity functions is

$$EI\frac{d^2y}{dx^2} = -M_A\langle x\rangle^0 + R_A\langle x\rangle^1 - \frac{w}{2}\langle x\rangle^2 \tag{1}$$

Integrating the first time, we find

$$EI\frac{dy}{dx} = -M_A\langle x\rangle^1 + \frac{R_A}{2}\langle x\rangle^2 - \frac{w}{6}\langle x\rangle^3 + C_1 \tag{2}$$

Now, invoking the boundary condition that when $x = 0$, $dy/dx = 0$, we find from Eq. (2) that $C_1 = 0$. The second integration yields

$$EIy = -\frac{M_A}{2}\langle x\rangle^2 + \frac{R_A}{6}\langle x\rangle^3 - \frac{w}{24}\langle x\rangle^4 + C_2 \tag{3}$$

and the second boundary condition is that $x = 0$ when $y = 0$, so from Eq. (3) we have $C_2 = 0$. From Eq. (3) we have the deflection at B due to the uniform load plus the presence of the spring to be given by

$$EI[y]_{x=L} = -\frac{M_A L^2}{2} + \frac{R_A L^3}{6} - \frac{wL^4}{24} \tag{4}$$

But for linear action of the spring we have the usual relation

$$R_B = -k[y]_{x=L} = +k\,\Delta_B \tag{5}$$

Also, from statics for this parallel force system we have the two equilibrium equations

$$+\circlearrowright \Sigma M_A = M_A + R_B L - \frac{wL^2}{2} = 0 \tag{6}$$

$$\Sigma F_y = R_A + R_B - (5000\ \text{N/m})(3\ \text{m}) = 0 \tag{7}$$

Simultaneous solution of Eqs. (4), (6), and (7) indicates that

$$R_A\left(\frac{EI}{k} + \frac{L^3}{3}\right) = \frac{EIwL}{k} + \frac{5wL^4}{24} \tag{8}$$

The flexural rigidity EI is easily found by consideration of the experimental evidence. The tip deflection of a tip-loaded cantilever beam is

$$\frac{PL^3}{3EI}$$

which becomes, for this experiment,

$$0.050\ \text{m} = \frac{(10{,}000\ \text{N})(3\ \text{m})^3}{3EI}$$

from which

$$EI = 1.8 \times 10^6\ \text{N}\cdot\text{m}^2 \tag{9}$$

If this value together with the spring constant of 345,000 N/m is substituted in Eq. (8), we find that $R_A = 11{,}440$ N. From Eq. (7) we find that $R_B = 3560$ N, so that the spring equation (5) indicates the displacement of point B to be

$$\Delta_B = \frac{3560\ \text{N}}{345{,}000\ \text{N/m}} = 0.01032\ \text{m} \quad \text{or} \quad 10.3\ \text{mm} \tag{10}$$

11.3. Consider the overhanging beam shown in Fig. 11-7. Determine the magnitude of the supporting force at B.

There are two statics equations

$$\circlearrowright \Sigma M_A = M_1 + R_2 a - \frac{w(a+b)^2}{2} = 0 \tag{1}$$

$$\Sigma F_v = R_1 + R_2 - w(a+b) = 0 \tag{2}$$

Let us employ the method of singularity functions to write the differential equation of the bent beam

$$EI\frac{d^2y}{dx^2} = -M_1\langle x\rangle^0 + R_1\langle x\rangle^1 - \frac{w}{2}\langle x\rangle^2 + R_2\langle x-a\rangle^1 \tag{3}$$

Note that in (1) a negative sign is assigned to M_1 since, as we work from left to right starting at the origin A, the reactive moment M_1 tends to bend the portion of the beam to the right of A into a configuration having curvature concave downward, which is negative according to the bending moment sign convention given in Chap. 6.

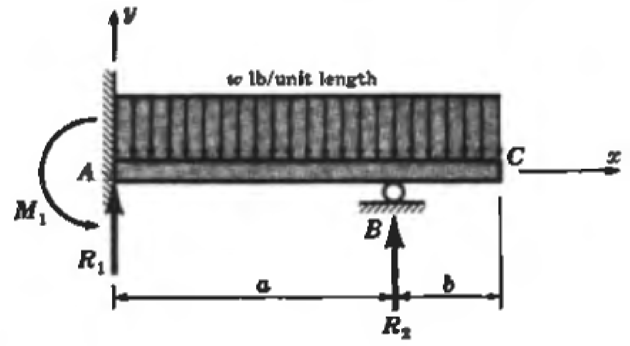


Fig. 11-7

Integrating

$$EI\frac{dy}{dx} = -M_1\langle x\rangle^1 + \frac{R_1}{2}\langle x\rangle^2 - \frac{w}{6}\langle x\rangle^3 + \frac{R_2}{2}\langle x-a\rangle^2 + C_1 \qquad (4)$$

But when $x = 0$, $dy/dx = 0$; hence $C_1 = 0$. Integrating again,

$$EIy = -\frac{M_1}{2}\langle x\rangle^2 + \frac{R_1}{6}\langle x\rangle^3 - \frac{w}{24}\langle x\rangle^4 + \frac{R_2}{6}\langle x-a\rangle^3 + C_2 \qquad (5)$$

But when $x = 0$, $y = 0$, so that $C_2 = 0$.

Since the support at point B is unyielding, y must vanish in (5) when $x = a$. Substituting, we find

$$0 = -\frac{M_1a^2}{2} + \frac{R_1a^3}{6} - \frac{wa^4}{24} \qquad \text{from which } M_1 = R_1\frac{a}{3} - \frac{wa^2}{12}$$

Solving this in conjunction with the statics equations, we find

$$R_1 = \frac{5}{8}wa - \frac{3wb^2}{4a} \qquad R_2 = \frac{3}{8}wa + wb + \frac{3wb^2}{4a}$$

11.4. The clamped end beam is loaded as shown in Fig. 11-8 by a couple M_0. Determine all reactions.

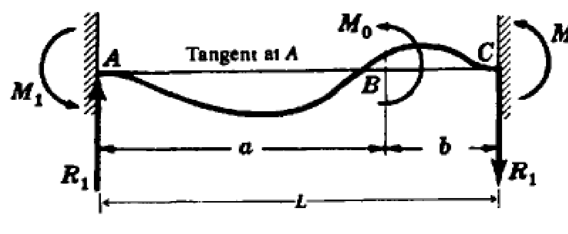


Fig. 11-8

Under the action of the couple, the initially straight beam bends into the configuration shown by the curved line. Tangents to the deformed configuration remain horizontal at ends A and B and of course there is zero vertical displacement at each of these ends. This gives rise to the reactions shown in which the vertical (shear) reactions are of equal magnitude for vertical equilibrium. This leaves only one equation from statics, namely,

$$+\circlearrowright \Sigma M_A = -M_1 - M_2 - M_0 + R_1(a+b) = 0 \qquad (1)$$

This equation contains R_1, M_1, and M_2 as unknowns. Since there are no more statics equations available, we must supplement Eq. (1) with two additional equations stemming from deformations of the system. We employ the method of singularity functions and write the bending moment at any point along the length of the beam as

$$M = -M_1\langle x\rangle^0 + R_1\langle x\rangle - M_0\langle x-a\rangle^0 \tag{2}$$

The differential equation of the bent beam is thus

$$EI\frac{d^2y}{dx^2} = -M_1\langle x\rangle^0 + R_1\langle x\rangle - M_0\langle x-a\rangle^0 \tag{3}$$

Integrating the first time, we obtain

$$EI\frac{dy}{dx} = -M_1\langle x\rangle + R_1\frac{\langle x\rangle^2}{2} - M_0\frac{\langle x-a\rangle^1}{1} + C_1 \tag{4}$$

As the first boundary condition, when $x = 0$, $dy/dx = 0$; hence from (4) we have $C_1 = 0$. Integrating again

$$EIy = -M_A\frac{\langle x\rangle^2}{2} + \frac{R_1}{2}\cdot\frac{\langle x\rangle^3}{3} - M_0\frac{\langle x-a\rangle^2}{2} + C_2 \tag{5}$$

The second boundary condition states that when $x = 0$, $y = 0$. Substituting these values in Eq. (5), we find $C_2 = 0$.

The third boundary condition is that when $x = L$, $dy/dx = 0$. Thus from Eq. (4) we have

$$0 = -M_1L + \frac{R_1L^2}{2} - M_0b \tag{6}$$

The fourth and last boundary condition is that when $x = L$, $y = 0$. From Eq. (5) we obtain

$$0 = -\frac{M_1}{2}L^2 + \frac{R_1}{2}\cdot\frac{L^3}{6} - M_0\frac{b^2}{2} \tag{7}$$

It is now possible to solve Eqs. (1), (6), and (7) simultaneously to obtain the desired reactions

$$R_1 = \frac{6M_0ab}{L^3}$$

$$M_1 = \frac{M_0(2ab-b^2)}{L^2} \tag{8}$$

$$M_2 = \frac{M_0(2ab-a^2)}{L^2}$$

There may have been a temptation to say that the deflection under the point of application of the couple, at B, is zero. There is no reason for making such an assumption and, in fact, we may now return to the deflection Eq. (5) and calculate the deflection at $x = a$ and find that it is

$$EI[y]_{x=a} = \frac{M_0a^2(2ab-b^2)}{2L^2} + \frac{M_0a^4b}{L^3} \tag{9}$$

which is clearly nonzero.

11.5. The horizontal beam shown in Fig. 11-9(a) is simply supported at the ends and is connected to a composite elastic vertical rod at its midpoint. The supports of the beam and the top of the copper rod are originally at the same elevation, at which time the beam is horizontal. The temperature of both vertical rods is then decreased 40°C. Find the stress in each of the vertical rods. Neglect the weight of the beam and of the rods. The cross-sectional area of the copper rod is 500 mm^2, $E_{cu} = 100$ GPa, and $\alpha_{cu} = 20\times10^{-6}$/°C. The cross-sectional area of the aluminum rod is 1000 mm^2, $E_{al} = 70$ GPa, and $\alpha_{al} = 25\times10^{-6}$/°C. For the beam, $E = 10$ GPa and $I = 400\times10^6$ mm^4.

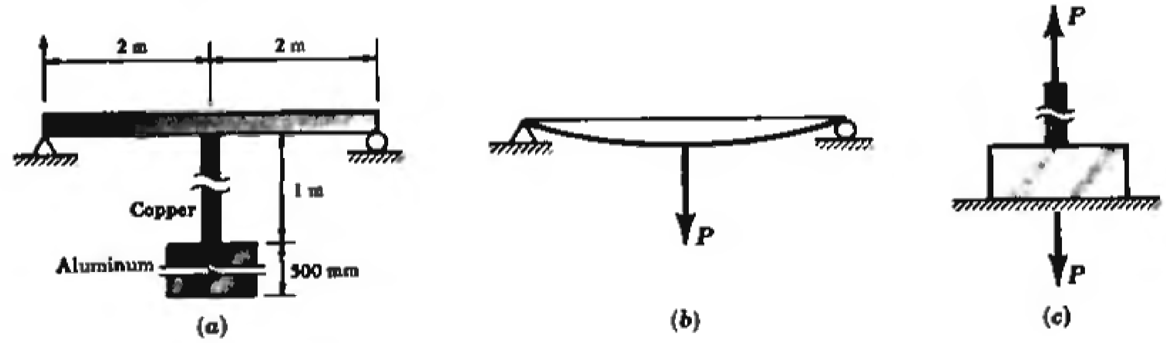


Fig. 11-9

A free-body diagram of the horizontal beam appears as in Fig. 11-9(*b*). Here, P denotes the force exerted upon the beam by the copper rod. Since this force is initially unknown, there are three forces acting upon the beam, but only two equations of equilibrium for a parallel force system; hence the problem is statically indeterminate. It will thus be necessary to consider the deformations of the system.

A free-body diagram of the two vertical rods appears as in Fig. 11-9(*c*). The simplest procedure is temporarily to cut the connection between the beam and the copper rod, and then allow the vertical rods to contract freely because of the decrease in temperature. If the horizontal beam offers no restraint, the copper rod will contract an amount

$$\Delta_{cu} = (20 \times 10^{-6})(10^3)(40) = 0.8\ \text{mm}$$

and the aluminum rod will contract by an amount

$$\Delta_{al} = (25 \times 10^{-6})(500)(40) = 0.5\ \text{mm}$$

However, the beam exerts a tensile force P upon the copper rod and the same force acts in the aluminum rod as shown in Fig. 11-9(*c*). These axial forces elongate the vertical rods and this elongation (see Problem 1.1) is

$$\frac{P(10^3)(10^6)}{500(100 \times 10^9)} + \frac{P(500)(10^6)}{10^3(70 \times 10^6)}$$

The downward force P exerted by the copper rod upon the horizontal beam causes a vertical deflection of the beam. In Problem 9.12 this central deflection was found to be $\Delta = PL^3/48EI$.

Actually, of course, the connection between the copper rod and the horizontal beam is not cut in the true problem and we realize that the resultant shortening of the vertical rods is exactly equal to the downward vertical deflection of the midpoint of the beam. This change of length of the vertical rods is caused partially by the decrease in temperature and partially by the axial force acting in the rods. For the shortening of the rods to be equal to the deflection of the beam we must have

$$(0.8 + 0.5) - \left[\frac{P(10^3)(10^6)}{500(100 \times 10^9)} + \frac{P(500)(10^6)}{10^3(70 \times 10^9)}\right] = \frac{P(4 \times 10^3)^3(10^6)}{48(10 \times 10^9)(400 \times 10^6)}$$

Solving, $P = 3.61$ kN; then,

$$\sigma_{cu} = 3.61 \times 10^3/500 = 7.22\ \text{MPa} \qquad \text{and} \qquad \sigma_{al} = 3.61 \times 10^3/1000 = 3.61\ \text{MPa}$$

11.6. The beam of flexural rigidity EI shown in Fig. 11-10 is clamped at both ends and subjected to a uniformly distributed load extending along the region BC of length $0.6L$. Determine all reactions.

At end A as well as C the supporting walls exert bending moments M_A and M_C plus shearing forces R_A and R_C as shown. For such a plane, parallel force system there are two equations of static equilibrium