Next we observe that the ordinate of point $f$ is given by

$$\tau = \overline{nf} = \overline{cf}\sin(2\theta_p - 2\theta) = \overline{cf}(\sin 2\theta_p \cos 2\theta - \cos 2\theta_p \sin 2\theta)$$
$$= \sqrt{[\tfrac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2}\,(\sin 2\theta_p \cos 2\theta - \cos 2\theta_p \sin 2\theta)$$

Again, substituting the values of $\tau_{xy}$ and $\frac{1}{2}(\sigma_y - \sigma_x)$ from (*1*) into this equation, we find

$$\tau = \tau_{xy}\cos 2\theta + \tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta$$

But this is exactly the shearing stress on a plane inclined at an angle $\theta$ to the $x$-axis as derived in (*2*) of Problem 16.13.

Hence the coordinates of point $f$ on Mohr's circle represent the normal and shearing stresses on a plane inclined at an angle $\theta$ to the $x$-axis.

**16.15.** A plane element is subject to the stresses shown in Fig. 16-42. Determine (*a*) the principal stresses and their directions, (*b*) the maximum shearing stresses and the directions of the planes on which they occur.

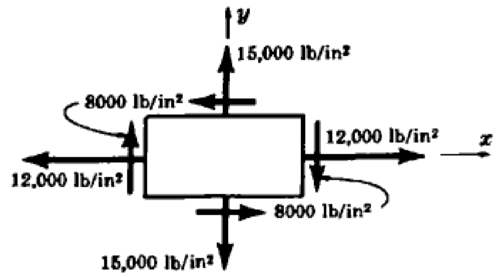


**Fig. 16-42**

(*a*) In accordance with the notation of Problem 16.13, we have $\sigma_x = 12{,}000\ \text{lb/in}^2$, $\sigma_y = 15{,}000\ \text{lb/in}^2$, and $\tau_{xy} = 8000\ \text{lb/in}^2$. The maximum normal stress is, by (*5*) of Problem 16.13,

$$\sigma_{max} = \tfrac{1}{2}(\sigma_x + \sigma_y) + \sqrt{[\tfrac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2}$$
$$= \tfrac{1}{2}(12{,}000 + 15{,}000) + \sqrt{[\tfrac{1}{2}(12{,}000 - 15{,}000)]^2 + (8000)^2}$$
$$= 13{,}500 + 8150 = 21{,}650\ \text{lb/in}^2$$

The minimum normal stress is given by (*6*) of Problem 16.13 to be

$$\sigma_{min} = \tfrac{1}{2}(\sigma_x + \sigma_y) - \sqrt{[\tfrac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2} = 13{,}500 - 8150 = 5350\ \text{lb/in}^2$$

From (*3*) of Problem 16.13 the directions of the principal planes on which these stresses of $21{,}650\ \text{lb/in}^2$ and $5350\ \text{lb/in}^2$ occur are given by

$$\tan 2\theta_p = -\frac{\tau_{xy}}{\tfrac{1}{2}(\sigma_x - \sigma_y)} = -\frac{8000}{\tfrac{1}{2}(12{,}000 - 15{,}000)} = 5.33$$

Then $2\theta_p = 79°24', 259°24'$ and $\theta_p = 39°42', 129°42'$.

To determine which of the above principal stresses occurs on each of these planes, we return to (*1*) of Problem 16.13, namely,

$$\sigma = \tfrac{1}{2}(\sigma_x + \sigma_y) - \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

and substitute $\theta = 39°42'$ together with the given values of $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ to obtain

$$\sigma = \tfrac{1}{2}(12{,}000 + 15{,}000) - \tfrac{1}{2}(12{,}000 - 15{,}000)\cos 79°24' + 8000\sin 79°24' = 21{,}650\ \text{lb/in}^2$$

Thus an element oriented along the principal planes and subject to the above principal stresses appears as in Fig. 16-43. The shearing stresses on these planes are zero.

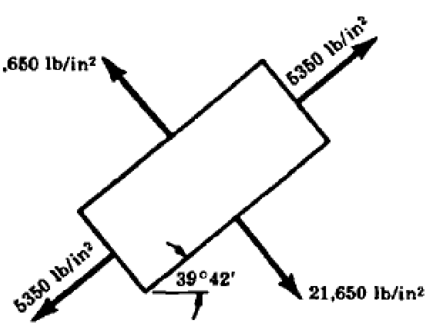


Fig. 16-43

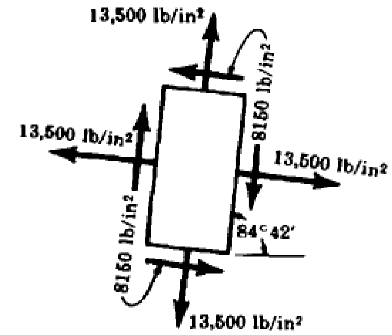


Fig. 16-44

(*b*) The maximum and minimum shearing stresses were found in (*8*) of Problem 16.13 to be

$$\tau_{\substack{\max \\ \min}} = \pm\sqrt{[\tfrac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2}$$
$$= \pm\sqrt{[\tfrac{1}{2}(12{,}000 - 15{,}000)]^2 + (8000)^2} = \pm 8150 \text{ lb/in}^2$$

From (*7*) of Problem 16.13 the planes on which these maximum shearing stresses occur are defined by the equation

$$\tan 2\theta_s = \frac{\tfrac{1}{2}(\sigma_x - \sigma_y)}{\tau_{xy}} = -0.188$$

Then $2\theta_s = 169°24', 349°24'$ and $\theta_s = 84°42', 174°42'$. Evidently these planes are located 45° from the planes of maximum and minimum normal stress.

To determine whether the shearing stress is positive or negative on the 84°42′ plane, we return to (*2*) of Problem 16.13, namely,

$$\tau = \tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta$$

and substitute $\theta = 84°42'$ together with the given values of $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ to obtain

$$\tau = \tfrac{1}{2}(12{,}000 - 15{,}000)\sin 169°24' + 8000\cos 169°24' = -8150 \text{ lb/in}^2$$

The negative sign indicates that the shearing stress is directed oppositely to the assumed positive direction shown in Fig. 16-36. Finally, the normal stresses on these planes of maximum shearing stress are found from (*9*) of Problem 16.13 to be

$$\sigma = \tfrac{1}{2}(\sigma_x + \sigma_y) = \tfrac{1}{2}(12{,}000 + 15{,}000) = 13{,}500 \text{ lb/in}^2$$

The orientation of the element for which the shearing stresses are maximum is as in Fig. 16-44.

**16.16.** A plane element is subject to the stresses shown in Fig. 16-45. Using Mohr's circle, determine (*a*) the principal stresses and their directions and (*b*) the maximum shearing stresses and the directions of the planes on which they occur.

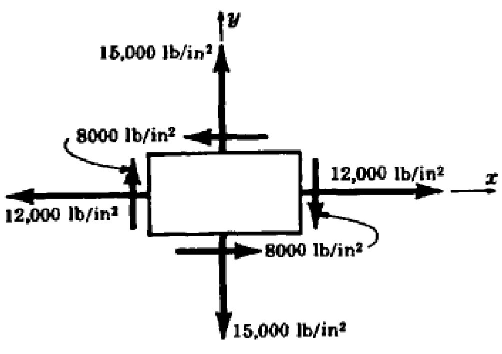


Fig. 16-45

The procedure for the construction of Mohr's circle was outlined in Problem 16.14. Following the instructions there, we realized that the shearing stresses on the vertical faces of the given element are positive, whereas those on the horizontal faces are negative. Thus the stress condition of $\sigma_x = 12{,}000\ \text{lb/in}^2$, $\tau_{xy} = 8000\ \text{lb/in}^2$ existing on the vertical faces of the element plots as point $b$ in Fig. 16-46. The stress condition of $\sigma_y = 15{,}000\ \text{lb/in}^2$, $\tau_{xy} = -8000\ \text{lb/in}^2$ existing on the horizontal faces plots as point $d$. Line $\overline{bd}$ is drawn, its midpoint $c$ is located, and a circle of radius $\overline{cb} = \overline{cd}$ is drawn with $c$ as a center. This is Mohr's circle. The endpoints of the diameter $\overline{bd}$ represent the stress conditions existing in the element if it has the original orientation of Fig. 16-45.

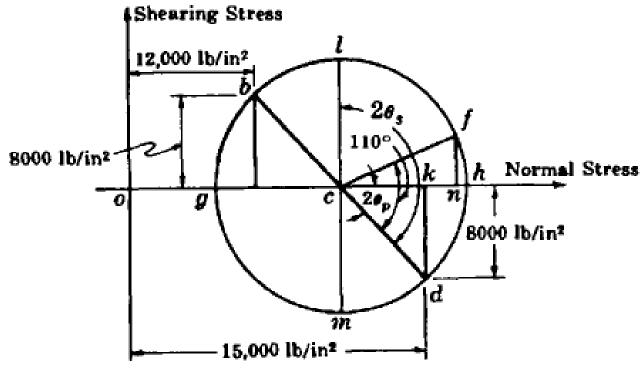


Fig. 16-46

(a) The principal stresses are represented by points $g$ and $h$, as demonstrated in Problem 16.14. The principal stress may be determined either by direct measurement from Fig. 16-46 or by realizing that the coordinate of $c$ is 13,500, that $\overline{ck} = 1500$, and that $\overline{cd} = \sqrt{(1500)^2 + (8000)^2} = 8150$. Thus the minimum principal stress is

$$\sigma_{\min} = \overline{og} = \overline{oc} - \overline{cg} = 13{,}500 - 8150 = 5350\ \text{lb/in}^2$$

Also, the maximum principal stress is

$$\sigma_{\max} = \overline{oh} = \overline{oc} + \overline{ch} = 13{,}500 + 8150 = 21{,}650\ \text{lb/in}^2$$

The angle $2\theta_p$ is given by $\tan 2\theta_p = 8000/1500 = 5.33$ from which $\theta_p = 39°42'$. This value could also be obtained by measurement of $\angle dck$ in Mohr's circle. From this it is readily seen that the principal stress represented by point $h$ acts on a plane oriented 39°42′ from the original $x$-axis. The principal stresses thus appear as in Fig. 16-47. It is evident that the shearing stresses on these planes are zero, since points $g$ and $h$ lie on the horizontal axis of Mohr's circle.

(b) The maximum shearing stress is represented by $\overline{cl}$ in Mohr's circle. This radius has already been found to represent $8150\ \text{lb/in}^2$. The angle $2\theta_s$ may be found either by direct measurement from the above plot or simply by adding 90° to the angle $2\theta_p$, which has already been determined. This leads to $2\theta_s = 169°24'$ and $\theta_s = 84°42'$. The shearing stress represented by point $l$ is positive; hence on this 84°42′ plane the shearing stress tends to rotate the element in a clockwise direction.

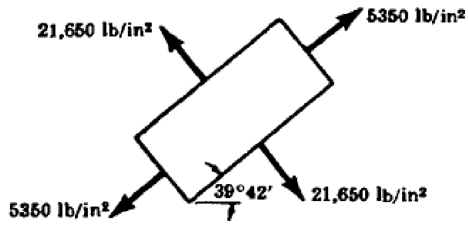


Fig. 16-47

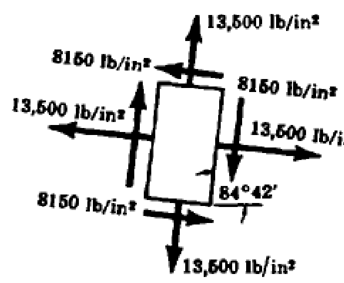


Fig. 16-48

Also, from Mohr's circle the abscissa of point $l$ is 13,500 lb/in$^2$ and this represents the normal stress occurring on the planes of maximum shearing stress. The maximum shearing stresses thus appear as in Fig. 16-48.

**16.17.** For the element discussed in Problem 16.16, determine the normal and shearing stresses on a plane making an angle of 55° measured counterclockwise from the positive end of the $x$-axis.

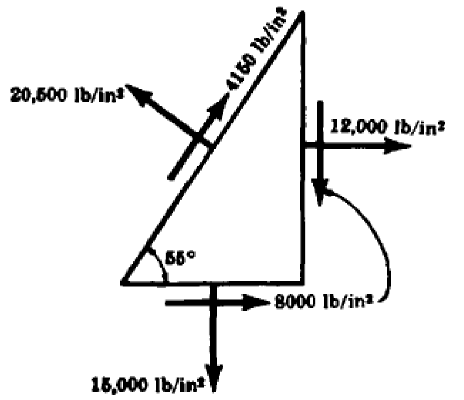


**Fig. 16-49**

According to the properties of Mohr's circle discussed in Problem 16.14, we realize that the endpoints of the diameter $\overline{bd}$ represent the stress conditions occurring on the original $x$-$y$ plane. On any plane inclined at an angle $\theta$ to the $x$-axis the stress conditions are represented by the coordinates of a point $f$, where the radius $\overline{cf}$ makes an angle of $2\theta$ with the original diameter $\overline{bd}$. This angle $2\theta$ appearing in Mohr's circle is measured in the same direction as the angle representing the inclined plane, namely, counterclockwise.

Hence in the Mohr's circle appearing in Problem 16.16, we merely measure a counterclockwise angle of 2(55°) = 110° from line $\overline{cd}$. This locates point $f$. The abscissa of point $f$ represents the normal stress on the desired 55° plane and may be found either by direct measurement or by realizing that

$$\overline{on} = \overline{oc} + \overline{cn} = 13{,}500 + 8150\cos(110° - 79°24') = 20{,}500 \text{ lb/in}^2$$

The ordinate of point $f$ represents the shearing stress on the desired 55° plane and may be found from the relation

$$\overline{fn} = 8150\sin(110° - 79°24') = 4150 \text{ lb/in}^2$$

The stresses acting on the 55° plane may thus be represented as in Fig. 16-49.

**16.18.** A plane element is subject to the stresses shown in Fig. 16-50. Determine (*a*) the principal stresses and their directions and (*b*) the maximum shearing stresses and the directions of the planes on which they occur.

(*a*) In accordance with the notation of Problem 16.13, $\sigma_x = -75$ MPa, $\sigma_y = 100$ MPa, and $\tau_{xy} = -50$ MPa. The maximum normal stress is given by (5) of Problem 16.13 to be

$$\begin{aligned}\sigma_{max} &= \tfrac{1}{2}(\sigma_x + \sigma_y) + \sqrt{[\tfrac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2} \\ &= \tfrac{1}{2}(-75 + 100) + \sqrt{[\tfrac{1}{2}(-75 - 100)]^2 + (-50)^2} \\ &= 12.5 + 100.8 = 113.3 \text{ MPa}\end{aligned}$$

The minimum normal stress is given by (6) of Problem 16.13 to be

$$\sigma_{min} = \tfrac{1}{2}(\sigma_x + \sigma_y) - \sqrt{[\tfrac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2} = 12.5 - 100.8 = -88.3 \text{ MPa}$$

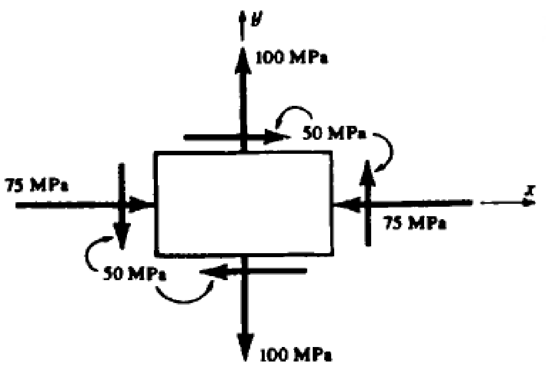


Fig. 16-50

From (3) of Problem 16.13 the directions of the principal planes on which these stresses of 113.3 MPa and −88.3 MPa occur are given by

$$\tan 2\theta_p = -\frac{\tau_{xy}}{\frac{1}{2}(\sigma_x - \sigma_y)} = -\frac{-50}{\frac{1}{2}(-75-100)} = -0.571$$

Then $2\theta_p = 150°15', 330°15'$ and $\theta_p = 75°8', 165°8'$.

To determine which of the above principal stresses occurs on each of these planes, we return to (1) of Problem 16.13, namely,

$$\sigma = \tfrac{1}{2}(\sigma_x + \sigma_y) - \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

and substitute $\theta = 75°8'$ together with the given values of $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ to obtain

$$\sigma = \tfrac{1}{2}(-75+100) - \tfrac{1}{2}(-75-100)\cos 150°15' - 50\sin 150°15' = 88.3\text{ MPa}$$

Consequently an element oriented along the principal planes and subject to the above principal stresses appears as in Fig. 16-51. The shearing stresses on these planes are zero.

(b) The maximum and minimum shearing stresses were found in (8) of Problem 16.13 to be

$$\tau_{\substack{max\\min}} = \pm\sqrt{[\tfrac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2} = \pm\sqrt{[\tfrac{1}{2}(-75-100)]^2 + (-50)^2} = \pm 100.8\text{ MPa}$$

From (7) of Problem 16.13, the planes on which these maximum shearing stresses occur are defined by

$$\tan 2\theta_s = \frac{\frac{1}{2}(\sigma_x - \sigma_y)}{\tau_{xy}} = 1.75$$

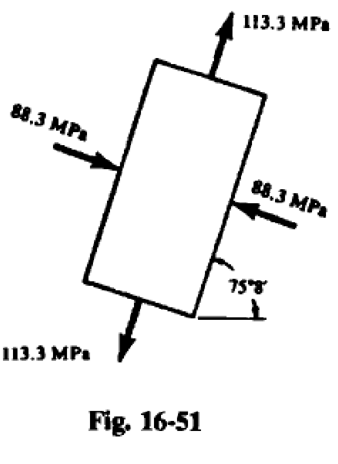


Fig. 16-51

Then $2\sigma_s = 60°15', 240°15'$ and $\theta_s = 30°8', 120°8'$. It is apparent that these planes are located 45° from the planes of maximum and minimum normal stress.

To determine whether the shearing stress is positive or negative on the 30°8′ plane, we return to (2) of Problem 16.13, namely,

$$\tau = \tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta$$

and substitute $\theta = 30°8'$ together with the given values of $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ to obtain

$$\tau = \tfrac{1}{2}(-75 - 100)\sin 60°15' - 50\cos 60°15' = -100.8\text{ MPa}$$

The negative sign indicates that the shearing stress on the 30°8′ plane is directed oppositely to the assumed positive direction shown in Fig. 16-36. The normal stresses on these planes of maximum shearing stress were found in (9) of Problem 16.13 to be

$$\sigma = \tfrac{1}{2}(\sigma_x + \sigma_y)$$
$$= \tfrac{1}{2}(-75 + 100) = 12.5\text{ MPa}$$

Consequently, the orientation of the element for which the shearing stresses are a maximum appears as in Fig. 16-52.

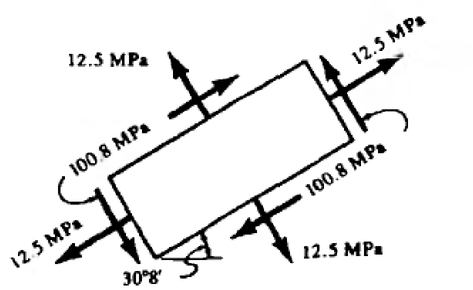


**Fig. 16-52**

**16.19.** A plane element is subject to the stresses shown in Fig. 16-53. Using Mohr's circle, determine (a) the principal stresses and their directions and (b) the maximum shearing stresses and the directions of the planes on which they occur.

Again we refer to Problem 16.14 for the procedure for constructing Mohr's circle. In accordance with the sign convention outlined there, the shearing stresses on the vertical faces of the element are negative, those on the horizontal faces positive. Thus the stress condition of $\sigma_x = -75$ MPa, $\tau_{xy} = -50$ MPa existing on the vertical faces of the element plots as point $b$ in Fig. 16-54. The stress condition of $\sigma_y = 100$ MPa, $\tau_{xy} = 50$ MPa existing on the horizontal faces plots as point $d$. Line $\overline{bd}$ is drawn, its midpoint $c$ is located, and a circle of radius $\overline{cb} = \overline{cd}$ is drawn with $c$ as a center. This is Mohr's circle. The endpoints of the

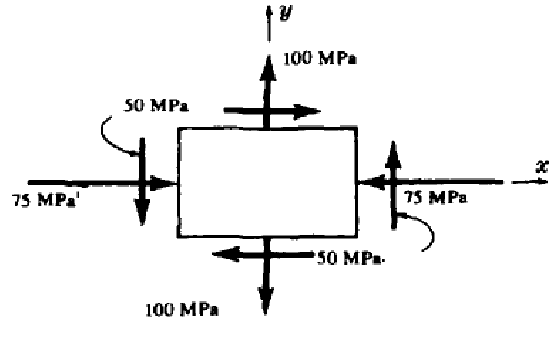


**Fig. 16-53**

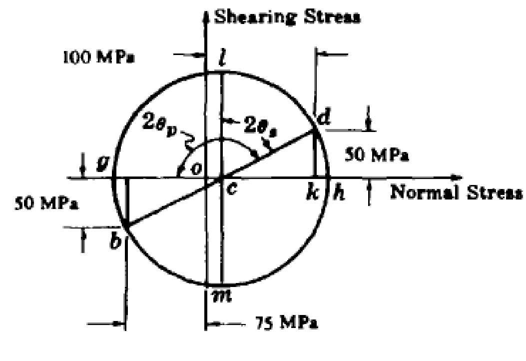


**Fig. 16-54**

diameter $\overline{bd}$ represent the stress conditions existing in the element if it has the original orientation shown above.

(*a*) The principal stresses are represented by points $g$ and $h$, as shown in Problem 16.14. They may be found either by direct measurement from the above diagram or by realizing that the coordinate of $c$ is 12.5, that $ck = 87.5$, and that $\overline{cd} = \sqrt{(87.5)^2 + (50)^2} = 100.8$ MPa. Thus the minimum principal stress is

$$\sigma_{\min} = \overline{og} = \overline{oc} - \overline{cg} = 12.5 - 100.8 = 88.3 \text{ MPa}$$

Also, the maximum principal stress is

$$\sigma_{\max} = \overline{oh} = \overline{oc} + \overline{ch} = 12.5 + 100.8 = 113.3 \text{ MPa}$$

The angle $2\theta_p$ is given by $\tan 2\theta_p = -50/87.5 = -0.571$ from which $\theta_p = 75°8'$. This value could also be obtained by measurement of $\angle dcg$ in Mohr's circle. From this it is readily seen that the principal stress represented by point $g$ acts on a plane oriented 75°8′ from the original $x$-axis. The principal stresses thus appear as in Fig. 16-55. Since the ordinates of points $g$ and $h$ are each zero, the shearing stresses on these planes are zero.

(*b*) The maximum shearing stress is represented by $\overline{cl}$ in Mohr's circle. This radius has already been found to represent 100.8 MPa. The angle $2\theta_s$ may be found either by direct measurement from the above plot or simply by subtracting 90° from the angle $2\theta_p$ which has already been determined. This leads to $2\theta_s = 60°15'$ and $\theta_s = 30°8'$. The shearing stress represented by point $l$ is positive, hence on this 30°8′ plane the shearing stress tends to rotate the element in a clockwise direction.

Also, from Mohr's circle the abscissa of point $l$ is 12.5 MPa and this represents the normal stress occurring on the planes of maximum shearing stress. The maximum shearing stresses thus appear as in Fig. 16-56.

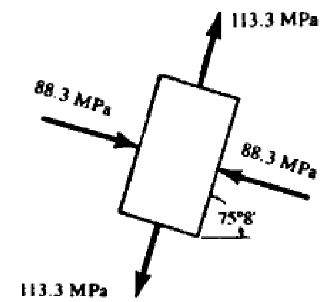


**Fig. 16-55**

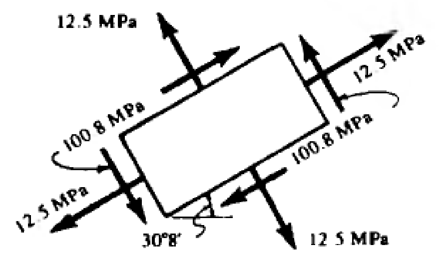


**Fig. 16-56**

**16.20.** Develop a FORTRAN program to indicate the principal stresses as well as their directions for an element subject to the stresses shown in Fig. 16-36.

The input to the program consists of the two normal stresses and one shearing stress, as indicated in Fig. 16-36. The normal stresses, for purposes of developing a program, are, as before, taken to be positive if tensile. The simplest sign convention for shearing stresses is to regard the horizontally directed shears as positive if they tend to produce clockwise rotation of the element, i.e., opposite to the convention associated with Problem 16.13. In Problem 16.13 we found the principal stresses to be given by Eqs. (*5*) and (*6*) and their directions by Eq. (*3*). The desired program is listed below.

```
*************************************************************************
                         PROGRAM STRES2D (INPUT,OUTPUT)
*************************************************************************
*
*          AUTHOR: KATHLEEN DERWIN
*          DATE  : JANUARY 26,1989
*
*   BRIEF DESCRIPTION:
*      THIS FORTRAN PROGRAM  MAY BE USED TO SOLVE A SIMPLE 2-D STRESS
```

```
*  PROBLEM WHERE THE USER IS PROMPTED FOR THE STRESS CONDITIONS FOR A
*  SINGLE OR SET OF POINTS, AND THE PRINCIPAL STRESS  AND ROTATING ANGLE
*  ARE CALCULATED.
*
*  INPUT:
*     THE USER WILL BE ASKED TO INPUT THE NUMBER OF STRESS SETS AND THE
*  NORMAL AND SHEAR STRESSES AT EACH POINT.
*
*  OUTPUT:
*     THE PRINCIPAL STRESSES AND ROTATING ANGLE FOR EACH SET OF PTS. WIL
*  BE PRINTED.
*
*  VARIABLES:
*     X(100),Y(100),S(100)  --- NORMAL AND SHEAR STRESS ARRAYS
*           NUM             --- THE NUMBER OF STRESS SETS
*           PI              --- 3.14159
*
*  SUBROUTINES CALLED:
*     PRINCIP --- CALCULATES THE PRINCIPAL STRESSES AND THE ROTATING
*                 ANGLE FOR A SINGLE OR SET OF POINTS.
*
**************************************************************************
**********                  MAIN PROGRAM                      ************
**************************************************************************
*
*          VARIABLE DECLARATIONS
*
       REAL X(100),Y(100),S(100),PI
       INTEGER NUM
*
      PI = 3.14159
*
*           USER INPUT
*
      PRINT*,'PLEASE ENTER THE NUMBER OF STRESS SETS:'
      READ*,NUM
      DO 10 N=1,NUM
         PRINT*,'PLEASE ENTER THE NORMAL STRESSES IN THE X,Y DIRECTIONS'
         PRINT*,'AND THE SHEAR STRESS;'
         READ*,X(N),Y(N),S(N)
 10   CONTINUE
*
*           CALLING SUBROUTINE PRINCIP TO CALCULATE THE PRINCIPAL
*           STRESSES AND THE ROTATING ANGLE
*
      CALL PRINCIP(X,Y,S,NUM)
*
      STOP
      END
**************************************************************************
         SUBROUTINE PRINCIP(XX,YY,SS,NUM)
*
*      THIS SUBROUTINE WILL EVALUATE THE PRINCIPAL STRESSES AND ROTATING
*      ANGLE FOR A SINGLE OR SET OF POINTS.
*
*          VARIABLE DECLARATIONS
*
      REAL PI,XX(100),YY(100),SS(100),P1(100),P2(100),T(100)
      INTEGER NUM
*
*          CALCULATIONS
*
      PI = 3.14159
      DO 15 N=1,NUM
         A=((XX(N)-YY(N))/2.0)**2
         B=SQRT(A+(SS(N)**2))
```

```
         C=(XX(N)+YY(N))/2.0
         P1(N)=C+B
         P2(N)=C-B
         A1=2*SS(N)/(XX(N)-YY(N))
         T(N)=90*ATAN(A1)/PI
         IF (XX(N).EQ.YY(N)) THEN
            T(N) = 45.0
         ENDIF
 15   CONTINUE
*
*           PRINTING OUTPUT
*
      PRINT 30
      DO 20 N=1,NUM
         PRINT 40,N,XX(N),YY(N),SS(N),P1(N),P2(N),T(N)
 20   CONTINUE
*
*            FORMAT STATEMENTS
*
 30   FORMAT(/,2X,'NO.',5X,'SIGXX',7X,'SIGYY',7X,'SIGXY',7X,'SIG(1)',
+            7X,'SIG(2)',7X,'THETA',/)
 40   FORMAT(2X,I2,3X,5(F9.2,3X),F9.2)
*
*           END SUBROUTINE PRINCIP
*
      RETURN
      END
```

**16.21.** Use the FORTRAN program of Problem 16.20 to determine principal stresses and their directions for an element subject to the stresses indicated in Fig. 16-57.

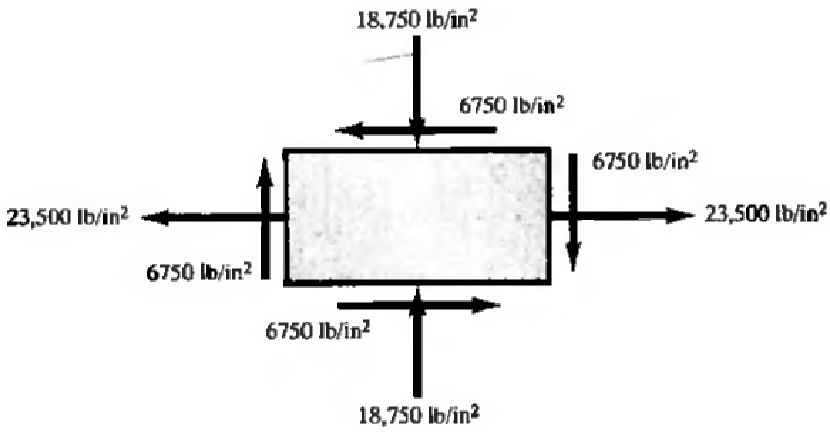


**Fig. 16-57**

If we use the notation of Problem 16.20 together with the directions of stresses shown in Fig. 16-57, we have $\sigma_x = 23{,}500\ \text{lb/in}^2$, $\sigma_y = -18{,}750\ \text{lb/in}^2$, and $\tau_{xy} = -6750\ \text{lb/in}^2$. Substituting these values into the self-prompting program of Problem 16.20, we get the following computer run.

```
 READY.
run
 PLEASE ENTER THE NUMBER OF STRESS SETS:
? 1
 PLEASE ENTER THE NORMAL STRESSES IN THE X,Y DIRECTIONS
 AND THE SHEAR STRESS:
? 23500,-18750,-6750
```

| NO. | SIGXX | SIGYY | SIGXY | SIG(1) | SIG(2) | THETA |
|---|---|---|---|---|---|---|
| 1 | 23500.00 | -18750.00 | -6750.00 | 24552.20 | -19802.20 | -8.86 |

```
SRU      0.734 UNTS.
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## Supplementary Problems

**16.22.** A bar of uniform cross section 50 mm × 75 mm is subject to an axial tensile force of 500 kN applied at each end of the bar. Determine the maximum shearing stress existing in the bar. *Ans.* 66.7 MPa

**16.23.** In Problem 16.22 determine the normal and shearing stresses acting on a plane inclined at 11° to the line of action of the axial loads. *Ans.* 4.87 MPa, 24.97 MPa

**16.24.** A square steel bar 1 in on a side is subject to an axial compressive load of 8000 lb. Determine the normal and shearing stresses acting on a plane inclined at 30° to the line of action of the axial loads. The bar is so short that the possibility of buckling as a column may be neglected.
*Ans.* $\sigma = -2000 \text{ lb/in}^2$, $\tau = -3460 \text{ lb/in}^2$

**16.25.** Rework Problem 16.24 by use of Mohr's circle.
*Ans.* See Fig. 16.58. $\sigma = \overline{ko} = -2000 \text{ lb/in}^2$, $\tau = \overline{dk} = 3460 \text{ lb/in}^2$

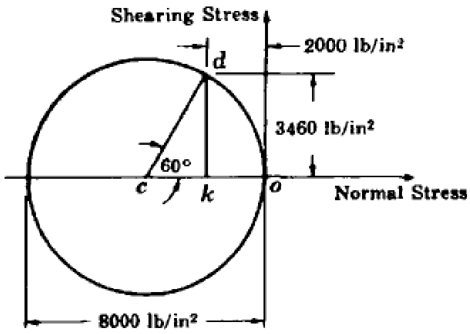



**Fig. 16-58**

**16.26.** A plane element in a body is subject to the stresses $\sigma_x = 20$ MPa, $\sigma_y = 0$, and $\tau_{xy} = 30$ MPa. Determine analytically the normal and shearing stresses existing on a plane inclined at 45° to the $x$-axis.
*Ans.* $\sigma = 40$ MPa, $\tau = 10$ MPa

**16.27.** A plane element is subject to the stresses $\sigma_x = 50$ MPa and $\sigma_y = 50$ MPa. Determine analytically the maximum shearing stress existing in the element. *Ans.* 0

**16.28.** A plane element is subject to the stresses $\sigma_x = 12{,}000 \text{ lb/in}^2$ and $\sigma_y = -12{,}000 \text{ lb/in}^2$. Determine analytically the maximum shearing stress existing in the element. What is the direction of the planes on which the maximum shearing stresses occur? *Ans.* $12{,}000 \text{ lb/in}^2$ at 45°

**16.29.** For the element described in Problem 16.28 determine analytically the normal and shearing stresses acting on a plane inclined at 30° to the $x$-axis. *Ans.* $\sigma = -6000 \text{ lb/in}^2$, $\tau = 10{,}400 \text{ lb/in}^2$

**16.30.** Draw Mohr's circle for a plane element subject to the stresses $\sigma_x = 8000$ lb/in$^2$ and $\sigma_y = -8000$ lb/in$^2$. From Mohr's circle determine the stresses acting on a plane inclined at 20° to the $x$-axis.
*Ans.* See Fig. 16-59. $\sigma = \overline{on} = -6130$ lb/in$^2$, $\tau = \overline{nf} = -5130$ lb/in$^2$

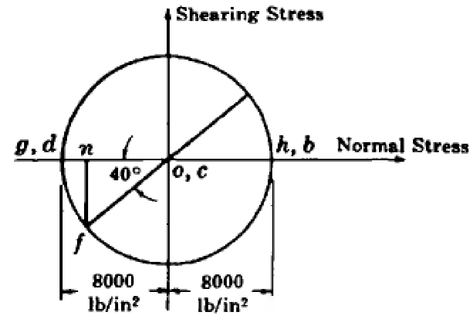



Fig. 16-59

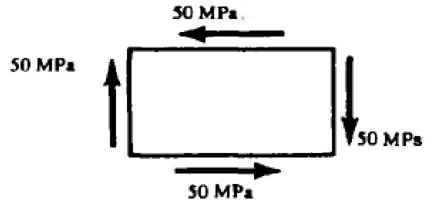



Fig. 16-60

**16.31.** A plane element removed from a thin-walled cylindrical shell loaded in torsion is subject to the shearing stresses shown in Fig. 16-60. Determine the principal stresses existing in this element and the directions of the planes on which they occur. *Ans.* 50 MPa at 45°

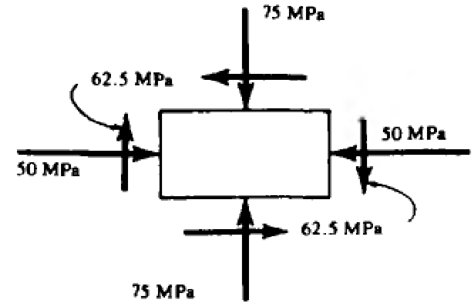



Fig. 16-61

**16.32.** A plane element is subject to the stresses shown in Fig. 16-61. Determine analytically ($a$) the principal stresses and their directions and ($b$) the maximum shearing stresses and the directions of the planes on which they act.
*Ans.* ($a$) $\sigma_{max} = 1.2$ MPa at 50°40′, $\sigma_{min} = -126.2$ MPa at 140°40′; ($b$) $\tau_{max} = 63.7$ MPa at 5°40′

**16.33.** Rework Problem 16.32 by the use of Mohr's circle. *Ans.* See Fig. 16-62.

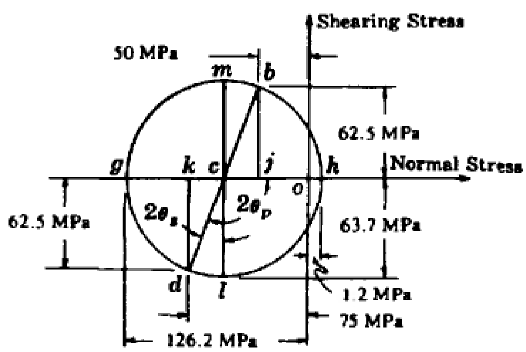



Fig. 16-62

**16.34.** A plane element is subject to the stresses indicated in Fig. 16-63. Use the FORTRAN program of Problem 16.20 to determine principal stresses together with their orientation.
*Ans.* SIG(1): 198.12; SIG(2): 66.88; THETA: 24.82

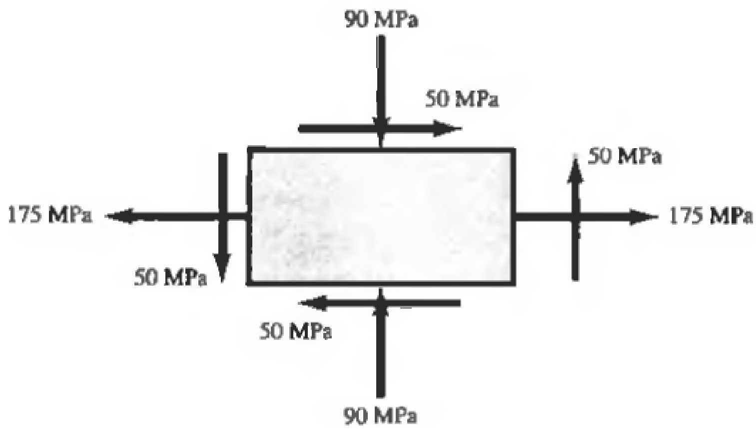



**Fig. 16-63**

**16.35.** A plane element is subject to the stresses indicated in Fig. 16-64. Use the FORTRAN program of Problem 16.20 to determine principal stresses together with their orientation.
*Ans.* SIG(1): 20,388.68; SIG(2): −31,738.68; THETA: 14.20

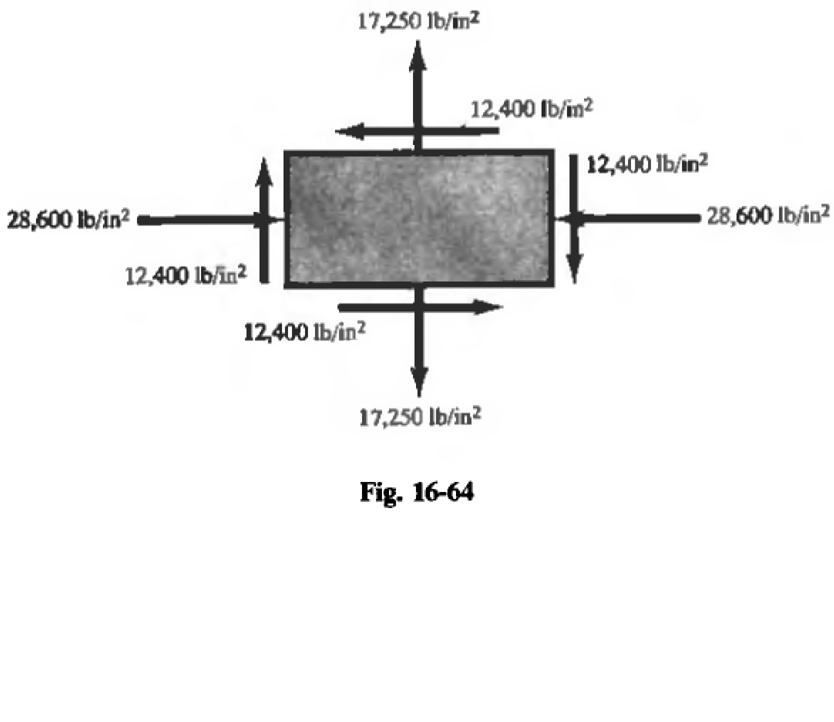



**Fig. 16-64**

# Chapter 17

# Members Subject to Combined Loadings; Theories of Failure

## AXIALLY LOADED MEMBERS SUBJECT TO ECCENTRIC LOADS

In Chaps. 1 and 2, where we considered straight bars subject to either tensile or compressive loads, it was always required that the action line of the applied force pass through the centroid of the cross section of the member. In the present chapter we shall consider those cases where the action line of the applied force acting on a bar in either tension or compression does *not* pass through the centroid of the cross section. A typical example of such an eccentric loading is shown in Fig. 17-1. For those cross sections of the bar that are perpendicular to the direction of the load, the resultant stress at any point is the sum of the direct stress due to a concentric load of equal magnitude $P$ plus a bending stress due to a couple of moment $Pe$. This first stress is found from the expression derived in Chap. 1, namely, $\sigma = P/A$. The second stress is found from the formula for bending stress presented in Chap. 8, namely, $\sigma = My/I$. An application may be found in Problem 17.1.

## CYLINDRICAL SHELLS SUBJECT TO COMBINED INTERNAL PRESSURE AND AXIAL TENSION

In Chap. 3 we considered the stresses arising in a thin-walled cylindrical shell subject to uniform internal pressure. There it was shown that a longitudinal stress given by $\sigma = pr/2t$, as well as a circumferential stress given by $\sigma = pr/t$, exists because of the internal pressure $p$. If in addition an axial tension $P$ is acting simultaneously with the internal pressure, then there arises an additional longitudinal stress given by $\sigma = P/A$ where $A$ denotes the cross-sectional area of the shell. The resultant stress in the longitudinal direction is thus the algebraic sum of these two longitudinal stresses, and the resultant stress in the circumferential direction is equal to that due to the internal pressure.

## CYLINDRICAL SHELLS SUBJECT TO COMBINED TORSION AND AXIAL TENSION/COMPRESSION

In Chap. 5 we considered the stresses arising in a thin-walled cylindrical shell subject to torsion. There it was shown that a shearing stress given by $\tau_{xy} = T\rho/J$ exists on cross sections perpendicular to the axis of the cylinder. If in addition an axial tension $P$ is acting simultaneously with the torque, then there arises a longitudinal stress given by $\sigma = P/A$. This loading is illustrated in Fig. 17-2. In this case the stresses due to these two loadings are acting in different directions and use must be made of the results obtained in Chap. 16. In this manner it will be possible to obtain the principal stresses due to these two loads acting simultaneously. For an application see Problem 17.2.

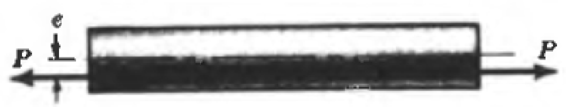



**Fig. 17-1**

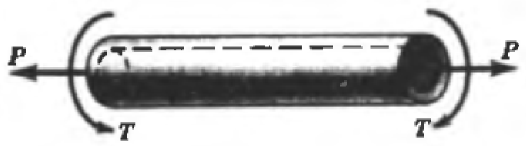



**Fig. 17-2**

## CIRCULAR SHAFT SUBJECT TO COMBINED AXIAL TENSION AND TORSION

This loading is illustrated in Fig. 17-3. Due to the axial tensile force $P$, there exists a uniform longitudinal tensile stress given by $\sigma = P/A$, where $A$ denotes the cross-sectional area of the bar. From Chap. 5 we know that there exists a torsional shearing stress over any cross section perpendicular to the axis given by $\tau_{xy} = T\rho/J$. Again, the stresses due to these two loadings are acting in different directions and the results of Chap. 16 must be employed to obtain the values of the principal stresses at any point or to obtain the state of stress on any plane inclined at some angle to a generator of the shaft.

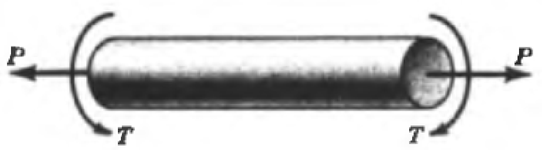



Fig. 17-3


Fig. 17-4

## CIRCULAR SHAFT SUBJECT TO COMBINED BENDING AND TORSION

This loading is illustrated in Fig. 17-4. Again from Chap. 5 we know that there exists a torsional shearing stress over any cross section perpendicular to the axis given by $\tau_{xy} = T\rho/J$. From Chap. 8 we know that there also exists a bending stress perpendicular to this cross section, i.e., in the direction of the axis of the shaft, given by $\sigma = My/I$. Since these stresses are acting in different directions the results of Chap. 16 must be employed to obtain the values of the principal stresses at any point in the shaft or to obtain the state of stress on any plane inclined to a generator of the shaft. For applications see Problem 17.3.

## DESIGN OF MEMBERS SUBJECT TO COMBINED LOADINGS

So far we have discussed only *analysis*, i.e., determination of principal stresses in a member subject to combined loadings. The inverse problem, i.e., *design* of a member to withstand combined loads, is somewhat more complex and must necessarily be related to experimentally determined mechanical properties of the materials. Because such properties cannot be determined for all possible combinations of loadings, the mechanical characteristics are usually determined in very simple tensile, compressive, or shear tests. The problem then arises as to how to relate the strength of an elastic body subject to combined loadings to these known strength characteristics under the simpler loading conditions. Relations between strength under various combined loads and simple mechanical properties of the material are termed *theories of failure*. Many such theories are available but we shall discuss only the three most commonly used, one applicable to brittle materials and two suitable for use in design of ductile members.

## MAXIMUM NORMAL STRESS THEORY

This theory states that failure of the material subject to biaxial or triaxial stresses occurs when the maximum normal stress reaches the value at which failure occurs in a simple tension test on the same material. Failure is usually defined as either yielding or fracture — whichever occurs first. This theory is in good agreement with experimental evidence on brittle materials. For applications, see Problems 17.9 and 17.10.

### MAXIMUM SHEARING STRESS THEORY

This theory states that failure of the material subject to biaxial or triaxial stresses occurs when the maximum shearing stress reaches the value of the shearing stress at failure in a simple tension or compression test on the same material. The theory is widely used for design of ductile materials. For applications see Problem 17.11.

### HUBER–VON MISES–HENCKY (MAXIMUM ENERGY OF DISTORTION) THEORY

For an element subject to the principal stresses $\sigma_1$, $\sigma_2$, $\sigma_3$ this theory states that yielding begins when

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 = 2(\sigma_{yp})^2$$

where $\sigma_{yp}$ is the yield point of the material. This theory is in excellent agreement with experiments on ductile materials. For applications see Problem 17.12.

## Solved Problems

**17.1.** The rectangular block shown in Fig. 17-5 has its axis of symmetry oriented vertically, is clamped at its lower base, and is subject to a concentric compressive force of 220 kN together with a couple $M$ at point $C$, the midpoint of the top cross section. If the peak allowable compressive stress is 180 MPa, determine the allowable magnitude of the couple.

The compressive force gives rise to a compressive stress that is uniform over any horizontal cross section. From Chap. 1 this vertically directed stress is

$$\sigma_1 = \frac{P}{A} = \frac{220{,}000\ \text{N}}{(0.07\ \text{m})(0.05\ \text{m})} = 62.86\ \text{MPa}$$

The couple (located in the $x$-$y$ plane) gives rise to bending about the $z$-axis (as a neutral axis) and from Chap. 8 creates a compressive stress everywhere to the right of the $z$-axis. At point $A$ this is given by

$$\sigma_2 = \frac{Mc}{I} = \frac{M(0.035\ \text{m})}{\frac{1}{12}(0.05\ \text{m})(0.07\ \text{m})^3}$$

The resultant compressive stress at $A$ is $(\sigma_1 + \sigma_2)$ and since this must not exceed 180 MPa, we have at $A$

$$180 \times 10^6\ \text{N/m}^2 = 62.86 \times 10^6\ \text{N/m}^2 + \frac{M(0.035\ \text{m})}{\frac{1}{12}(0.05\ \text{m})(0.07\ \text{m})^3}$$

Solving,

$$M = 4.70\ \text{kN} \cdot \text{m}$$

**17.2.** Consider a hollow cylindrical shell of outer radius $R_o = 140$ mm and inner radius $R_i = 125$ mm. It is subject to an axial compressive force of 68 kN together with a torque of 35 kN · m, as shown in Fig. 17-6. Determine the principal stresses as well as the peak shearing stress in the shell.

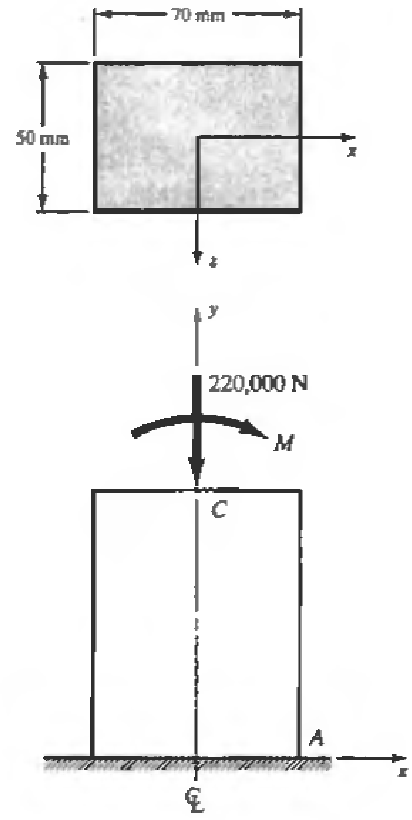



Fig. 17-5


Fig. 17-6

The 68-kN force produces a uniformly distributed compressive stress given by

$$\sigma_1 = \frac{-68{,}000\ \mathrm{N}}{\pi[(0.140\ \mathrm{m})^2 - (0.125\ \mathrm{m})^2]} = -5.44\ \mathrm{MPa}$$

as shown in Fig. 17-7. The torsional shearing stresses due to the 35-kN · m torque were found in Problem 5.2 to be $\tau = T\rho/J$. Here, the polar moment of inertia is

$$J = \frac{\pi}{2}[(0.140\ \mathrm{m})^4 - (0.125)^4] = 0.0002199\ \mathrm{m}^4$$

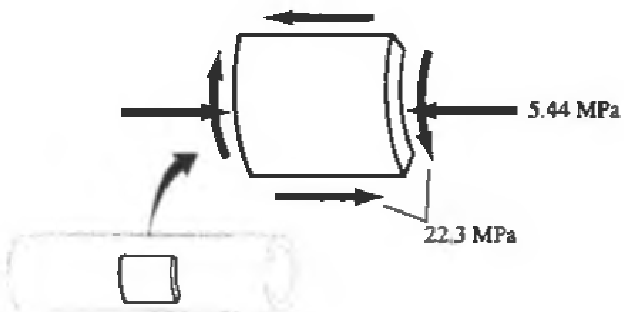



Fig. 17-7

If the approximate expression of Problem 5.6 is used, we find 0.0002191 $m^4$. Thus, the shearing stresses at the outer fibers of the shell are given by

$$\tau = \frac{T\rho}{J} = \frac{(35{,}000\ \mathrm{N\cdot m})(0.140\ \mathrm{m})}{0.0002199} = 22.3\ \mathrm{MPa}$$

and these are shown in Fig. 17-7.

From Problem 16.13 the principal stresses are found to be

$$\sigma = \frac{-5.44+0}{2} \pm \sqrt{\left(-\frac{5.44-0}{2}\right)^2 + (22.3)^2}$$

$$\sigma_{max} = 19.75\ \mathrm{MPa}$$

$$\sigma_{min} = -25.19\ \mathrm{MPa}$$

and the peak shearing stress is 22.47 MPa.

**17.3.** Consider a hollow circular shaft whose outside diameter is 3 in and whose inside diameter is equal to one-half the outside diameter. The shaft is subject to a twisting moment of 20,000 lb · in as well as a bending moment of 30,000 lb · in. Determine the principal stresses in the body. Also, determine the maximum shearing stress.

The twisting moment gives rise to shearing stresses that attain their peak values in the outer fibers of the shaft. From Problem 5.2 these shearing stresses are given by $\tau_{xy} = T\rho/J$. From Problem 5.1 it is seen that for the hollow circular area

$$J = \frac{\pi}{32}(D_o^4 - D_i^4) = \frac{\pi}{32}[3^4 - (1.5)^4] = 7.46\ \mathrm{in}^4$$

where $D_o$ denotes the outer diameter of the section and $D_i$ represents the inner diameter. At the outer fibers the torsional shearing stresses are thus

$$\tau_{xy} = \frac{T\rho}{J} = \frac{20{,}000(1.5)}{7.46} = 4000\ \mathrm{lb/in^2}$$

Let the bending moments lie in a vertical plane. Then the upper and lower fibers of the beam are subject to the peak bending stresses. These are found from the expression $\sigma_x = My/I$. The moment of inertia $I$ for the hollow circular cross section may be seen from Problem 7.9 to be

$$I = \frac{\pi}{64}(D_o^4 - D_i^4) = \frac{\pi}{64}[3^4 - (1.5)^4] = 3.73\ \mathrm{in}^4$$

Substituting,

$$\sigma_x = \frac{My}{I} = \frac{30{,}000(1.5)}{3.73} = 12{,}000\ \mathrm{lb/in^2}$$

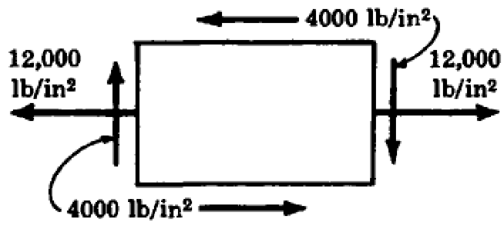



**Fig. 17-8**

Thus an element located at the lower extremity of the shaft is subject to the stresses shown in Fig. 17-8. From Problem 16.7 the principal stresses for this element are

$$\sigma_{max} = \tfrac{1}{2}\sigma_x + \sqrt{(\tfrac{1}{2}\sigma_x)^2 + (\tau_{xy})^2} = 12{,}000/2 + \sqrt{(12{,}000/2)^2 + (4000)^2} = 13{,}200 \text{ lb/in}^2$$

$$\sigma_{min} = \tfrac{1}{2}\sigma_x + \sqrt{(\tfrac{1}{2}\sigma_x)^2 + (\tau_{xy})^2} = 12{,}000/2 - \sqrt{(12{,}000/2)^2 + (4000)^2} = -1200 \text{ lb/in}^2$$

These stresses occur on planes defined by (3) of Problem 16.7:

$$\tan 2\theta_p = -\frac{\tau_{xy}}{\tfrac{1}{2}\sigma_x} = -\frac{4000}{12{,}000/2} = -\frac{2}{3} \qquad \text{or} \qquad \theta_p = 73°10',\ 163°10'$$

Substituting in (1) of Problem 16.7 and letting $\theta = 73°10'$, we have

$$\sigma = 12{,}000/2 - (12{,}000/2)\cos 146°20' + 4000 \sin 146°20' = 13{,}200 \text{ lb/in}^2$$

Thus the maximum tensile stress is 13,200 lb/in$^2$, occurring on a plane oriented 73°10′ to the geometric axis of the shaft. The other principal stress, $\sigma_{min} = -1200$ lb/in$^2$, occurs on a plane oriented 163°10′ to the axis.

The maximum shearing stress is given by (8) of Problem 16.7. It is

$$\tau = \pm\sqrt{(\tfrac{1}{2}\sigma_x)^2 + (\tau_{xy})^2} = \pm\sqrt{(12{,}000/2)^2 + (4000)^2} = \pm 7200 \text{ lb/in}^2$$

and occurs on planes oriented at 45° to the planes found above on which the principal stresses act.

**17.4.** The thick-walled cylindrical shell shown in Fig. 17-9 has its axis of symmetry oriented vertically. It is clamped at its lower extremity and subject to the three concentrated forces indicated. Determine the normal stresses at points $A$, $B$, $C$, and $D$.

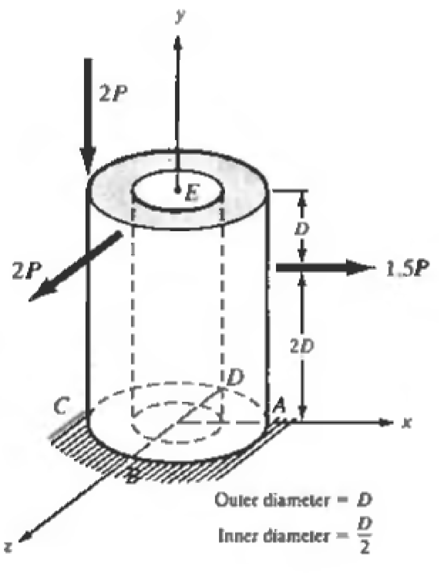



**Fig. 17-9**

Let us look down the $z$-axis toward the $x$-$y$ plane. Also, let us introduce two forces, each of magnitude $2P$, at the center $E$ of the top surface. The force system in the $x$-$y$ plane for this set of three forces thus appears as in Fig. 17-10($a$). The two forces included within the dotted lines constitute a couple of magnitude $(2P)(D/2) = PD$, so that the loading on the top surface (corresponding to the original force $2P$) may be considered to consist of a central downward force of magnitude $2P$ together with a couple of magnitude $PD$, as shown in Fig. 17-10($b$). The total loading on the shell thus consists of the concentric force $2P$, the couple $PD$, and the two concentrated forces of magnitudes $1.5P$ and $2P$.

The effects of these four forces are:

(*a*) The central downward force $2P$ gives rise to uniform compressive stresses over any horizontal cross section.

(*b*) The couple $PD$ shown in Fig. 17-10(*b*) gives rise to bending about an axis parallel to the $z$-axis as a neutral axis.

(*c*) The force $1.5P$ gives rise to bending about an axis parallel to the $z$-axis as a neutral axis.

(*d*) The force $2P$ gives rise to bending about an axis parallel to the $x$-axis as a neutral axis.

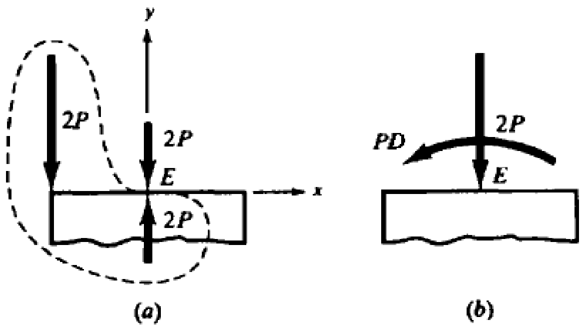



**Fig. 17-10**

From the geometry of the cross section, we find $A = 0.589D^2$ in and $I_x = I_z = 0.0460D^4$ in$^4$. From effect (*a*), we have

$$\sigma_1 = \frac{P}{A} = -\frac{2P}{0.589D^2} = -3.396\frac{P}{D^2}$$

From (*b*), the bending stresses are

$$\sigma'_A = \frac{Mc}{I} = \frac{(PD)(D/2)}{0.0460D^4} = 10.87\frac{P}{D^2}$$

$$\sigma'_C = \frac{Mc}{I} = -\frac{(PD)(D/2)}{0.0460D^4} = -10.87\frac{P}{D^2}$$

From (*c*), the bending stresses are

$$\sigma''_A = \frac{Mc}{I} = -\frac{(1.5P)(2D)(D/2)}{0.0460D^4} = -32.61\frac{P}{D^2}$$

$$\sigma''_C = \frac{Mc}{I} = \frac{(1.5P)(2D)(D/2)}{0.0460D^4} = 32.61\frac{P}{D^2}$$

These stresses appear at $A$ and $C$ as shown in Fig. 17-11, for which

$$\sigma_A = -3.396\frac{P}{D^2} + 10.87\frac{P}{D^2} - 32.61\frac{P}{D^2} = -25.14\frac{P}{D^2}$$

$$\sigma_C = -3.396\frac{P}{D^2} - 10.87\frac{P}{D^2} + 32.61\frac{P}{D^2} = -18.34\frac{P}{D^2}$$

From effect (*d*), we have the bending as

$$\sigma'''_B = \frac{Mc}{I} = \frac{-(2P)(3D)(D/2)}{0.046D^4} = -65.22\frac{P}{D^2}$$

$$\sigma'''_D = \frac{Mc}{I} = \frac{(2P)(3D)(D/2)}{0.046D^4} = 65.22\frac{P}{D^2}$$

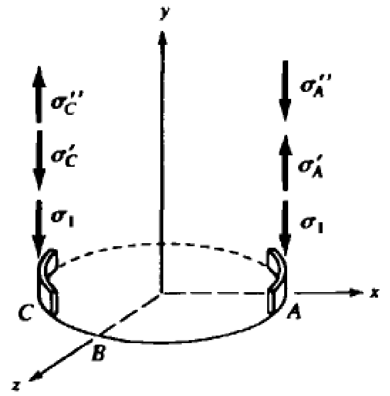



**Fig. 17-11**

To these values must be added the direct stresses so that the resultant vertical normal stresses at $B$ and $D$ are

$$\sigma_B = -3.396\frac{P}{D^2} - 65.22\frac{P}{D^2} = -68.62\frac{P}{D^2}$$

$$\sigma_D = -3.396\frac{P}{D^2} + 65.22\frac{P}{D^2} = 61.82\frac{P}{D^2}$$

**17.5.** The shaft shown in Fig. 17-12($a$) rotates with constant angular velocity. The belt pulls create a state of combined bending and torsion. Neglect the weights of the shaft and pulleys and assume that the bearings can exert only concentrated force reactions. The diameter of the shaft is 1.25 in. Determine the principal stresses in the shaft.

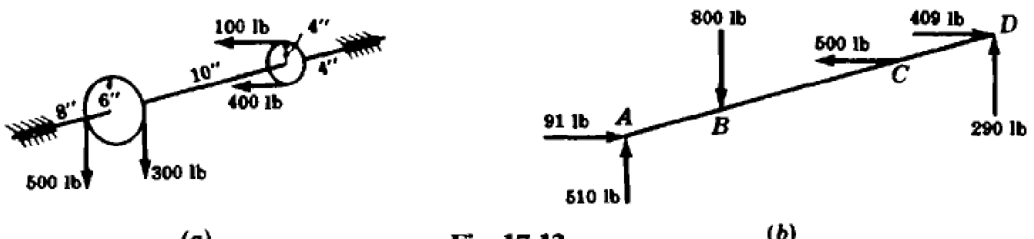



**Fig. 17-12**

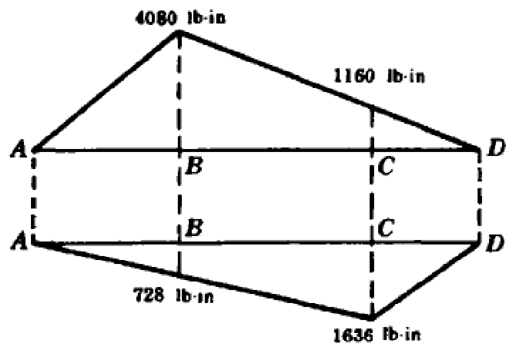



**Fig. 17-13**

The transverse forces acting on the shaft are not parallel and the bending moments caused by them must be added vectorially to obtain the resultant bending moment. This vector addition need be carried out at only a few apparently critical points along the length of the shaft. The loads causing bending, together with the reactions they produce, are shown above in Fig. 17-12($b$). They are considered as passing through the axis of the shaft. The upper and lower shaded portions of Fig. 17-13, respectively, represent the bending moment diagrams for a vertical and for a horizontal plane.

The resultant bending moments at $B$ and $C$ are

$$M_B = \sqrt{(4080)^2 + (728)^2} = 4140 \text{ lb}\cdot\text{in}$$

$$M_C = \sqrt{(1160)^2 + (1636)^2} = 2000 \text{ lb}\cdot\text{in}$$

The twisting moment between the two pulleys is constant and equal to

$$T = (400 - 100)(4) = 1200 \text{ lb}\cdot\text{in}$$

Since the torque is the same at $B$ and $C$, the critical element lies at the outer fibers of the shaft at point $B$. The maximum bending stress is given by

$$\sigma_x = \frac{My}{I} = \frac{(4140)(1.25/2)}{\pi(1.25)^4/64} = 21{,}500 \text{ lb/in}^2$$

The maximum shearing stress, occurring at the outer fibers of the shaft, is given by

$$\tau_{xy} = \frac{T\rho}{J} = \frac{1200(1.25/2)}{\pi(1.25)^4/32} = 3100 \text{ lb/in}^2$$

The principal stresses were found in Problem 16.13 to be

$$\sigma_{max} = \tfrac{1}{2}\sigma_x + \sqrt{(\tfrac{1}{2}\sigma_x)^2 + (\tau_{xy})^2} = 21{,}500/2 + \sqrt{(21{,}500/2)^2 + (3100)^2} = 22{,}000 \text{ lb/in}^2$$

$$\sigma_{min} = \tfrac{1}{2}\sigma_x - \sqrt{(\tfrac{1}{2}\sigma_x)^2 + (\tau_{xy})^2} = 21{,}500/2 - \sqrt{(21{,}500/2)^2 + (3100)^2} = -400 \text{ lb/in}^2$$

**17.6.** Discuss a failure criterion for *brittle* materials.

The criterion which is in best agreement with experimental evidence was advanced by the English engineer W. J. M. Rankine and is termed the *maximum normal stress theory*. It states that failure of the material (i.e., either yielding or fracture — whichever occurs first) occurs when the maximum normal stress reaches the value at which failure occurs in a simple tension test on the same material. Alternatively, if the loading is compressive, failure occurs when the minimum normal stress reaches the value at which failure occurs in a simple compression test. Evidently this criterion considers only the greatest (or smallest) of the principal stresses and disregards the influence of the other principal stresses.

**17.7.** Discuss the *maximum shearing stress* failure criterion for *ductile* materials.

This criterion is in good agreement with experimental evidence, provided the yield point of the material in tension is equal to that in compression. It was advanced first by C. A. Coulomb in 1773 and later by H. Tresca in 1864; in fact, it is often called the *Tresca criterion*. The criterion states that failure of the material subject to biaxial or triaxial stress occurs when the maximum shearing stress at any point reaches the value of the shearing stress at failure in a simple tension or compression test on the same material. In Problem 16.13 it was shown that the maximum shear stress is one-half the difference between the maximum and minimum principal stresses and always occurs on a plane inclined at 45° to the principal planes. Thus, if $\sigma_{yp}$ denotes the yield point of the material in simple tension or compression, then the corresponding maximum shear stress is $\sigma_{yp}/2$. Accordingly, the maximum shearing stress criterion may be formulated as

$$\frac{\sigma_{max} - \sigma_{min}}{2} = \frac{\sigma_{yp}}{2}$$

or

$$\sigma_{max} - \sigma_{min} = \sigma_{yp} \tag{1}$$

where $\sigma_{max}$ and $\sigma_{min}$ are maximum and minimum principal stresses, respectively. It is to be observed that judgment must be used in analysis of three-dimensional situations to determine which of the three principal stresses lead to the greatest difference on the left-hand side of (*1*).

**17.8.** Discuss the *Huber–von Mises–Hencky* failure criterion for *ductile* materials.

This theory was advanced by M. T. Huber in Poland in 1904 and independently by R. von Mises in Germany in 1913 and H. Hencky in 1925. It is in even better agreement with experimental evidence concerning failure of ductile materials subject to biaxial or triaxial stresses than the maximum shearing stress theory discussed in Problem 17.7.

Development of this widely accepted criterion first necessitates determination of the strain energy per unit volume in a simple tension specimen. If the axial tensile stress arising in this test is $\sigma_1$ and the corresponding axial strain is $\epsilon_1$, then the work done on a unit volume of the test specimen is the product of the mean value of force per unit area, that is, $\sigma_1/2$, times the displacement in the direction of the force, or $\epsilon_1$. The work is thus $U = \sigma_1\epsilon_1/2$ and this work is stored as internal strain energy.

The strain energy per unit volume in an element subject to triaxial *principal stresses* $\sigma_1$, $\sigma_2$, $\sigma_3$ is readily found by superposition (since energy is a scalar quantity) to be

$$U = \tfrac{1}{2}\sigma_1\epsilon_1 + \tfrac{1}{2}\sigma_2\epsilon_2 + \tfrac{1}{2}\sigma_3\epsilon_3 \qquad (a)$$

where $\epsilon_1$, $\epsilon_2$, $\epsilon_3$ are the normal strains in the directions of the principal stresses, respectively. If the strains are expressed in terms of the stresses according to the relations given in Problem 1.23, Eq. (*a*) becomes

$$U = \frac{1}{2E}[(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - 2\mu(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3)] \qquad (b)$$

The triaxial principal stresses may be represented as in Fig. 17-14(*a*). Alternatively, this general state of stress may be represented as the sum of the two triaxial states shown in Figs. 17-14(*b*) and 17-14(*c*).

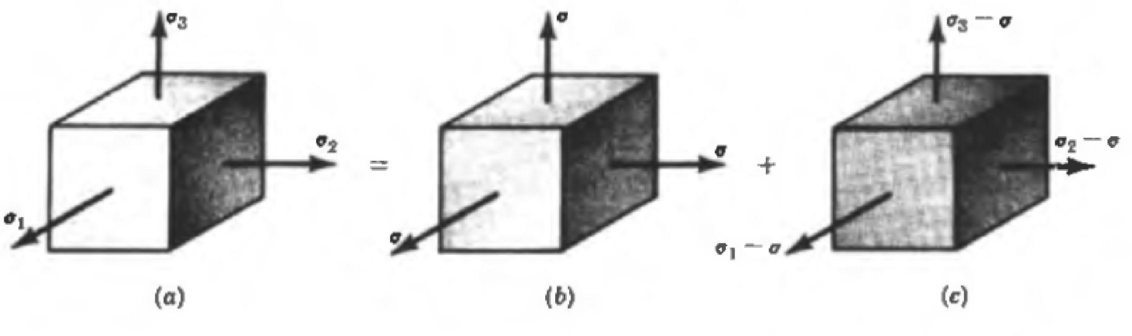



**Fig. 17-14**

The strain energy $U$ given by Eq. (*b*) may be resolved into two components, one portion $U_v$ corresponding to a change of volume with no distortion of the element, the other, $U_d$, corresponding to distortion of the element with no change of volume. The stresses indicated in Fig. 17-14(*c*) represent *distortion only* with no change of volume, provided the expression for *dilatation* given in Problem 1.23 is set equal to zero. Thus

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = \frac{1}{E}[(\sigma_1 - \sigma) - \mu(\sigma_2 + \sigma_3 - 2\sigma) + (\sigma_2 - \sigma) - \mu(\sigma_1 + \sigma_3 - 2\sigma)$$
$$+ (\sigma_3 - \sigma) - \mu(\sigma_1 + \sigma_2 - 2\sigma)] = 0 \qquad (c)$$

Solving (*c*), we find

$$\sigma = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \qquad (d)$$

for the uniform stresses in Fig. 17-14(*b*) which correspond to change of volume with no distortion. The normal strains corresponding to the stresses given in (*d*) are readily found from the three-dimensional form of Hooke's law given in Problem 1.23 to be

$$\epsilon = \frac{(1 - 2\mu)\sigma}{E} \qquad (e)$$

Thus, the internal strain energy corresponding to the unit volume indicated in Fig. 17-14($b$) is found by substituting the expressions ($d$) and ($e$) in ($a$), with $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$ and $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon$, to obtain

$$U_v = 3\left(\frac{\sigma\epsilon}{2}\right) = \frac{1-2\mu}{6E}(\sigma_1 + \sigma_2 + \sigma_3)^2 \tag{f}$$

The strain energy corresponding to *distortion only*, with no change of volume, is now found to be

$$U_d = U - U_v = \frac{1+\mu}{6E}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2] \tag{g}$$

The Huber–von Mises–Hencky theory assumes that failure takes place when the internal strain energy of distortion given by ($g$) is equal to that at which failure occurs in a simple tension test. In such a test $\sigma_2 = \sigma_3 = 0$, $\sigma_1 = \sigma_{yp}$ and the right side of ($g$) becomes

$$\frac{1+\mu}{6E}[2\sigma_{yp}^2] \tag{h}$$

Equating the right side of ($g$) to ($h$), we find

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 = 2\sigma_{yp}^2 \tag{i}$$

as the criterion for failure. This is sometimes called the *maximum energy of distortion theory*. It assumes that $U_v$ is ineffective in causing failure.

**17.9.** A thin-walled cylindrical pressure vessel is subject to an internal pressure of 5 MPa. The mean radius of the cylinder is 400 mm. If the material has a yield point of 300 MPa and a safety factor of 3 is employed, determine the required wall thickness using ($a$) the maximum normal stress theory, and ($b$) the Huber–von Mises–Hencky theory.

The stresses determined in Problem 3.1 are principal stresses. Thus we have

$$\sigma_1 = \sigma_c = \frac{pr}{h} = \frac{5(400)}{h} = \frac{2000}{h}$$

$$\sigma_2 = \sigma_l = \frac{pr}{2h} = \frac{5(400)}{2h} = \frac{1000}{h}$$

The third principal stress varies from zero at the outside of the shell to the value $-p$ at the inside. It is customary to neglect this third component in thin-shell design, so we shall assume that $\sigma_3 = 0$.

($a$) Using the maximum normal stress theory we have

$$\frac{2000}{h} = \frac{300}{3} \qquad \text{from which} \qquad h = 20\,\text{mm}$$

($b$) Using the Huber-von Mises-Hencky theory we have, from ($i$) of Problem 17.8,

$$\left(\frac{2000}{h} - \frac{1000}{h}\right)^2 + \left(\frac{1000}{h} - 0\right)^2 + \left(\frac{2000}{h} - 0\right)^2 = 2\left(\frac{300}{3}\right)^2$$

whence $h = 17.3$ mm.

**17.10.** The solid circular shaft in Fig. 17-15($a$) is subject to belt pulls at each end and is simply supported at the two bearings. The material has a yield point of 250 MPa. Determine the required diameter of the shaft using the maximum normal stress theory together with a safety factor of 3.

The bearing reactions, which are in a vertical plane, are denoted by $R_B$ and $R_C$ in the free-body diagram, Fig. 17-15($b$). From statics it is found that $R_B = 2.83$ kN and $R_C = 3.67$ kN. The variation of

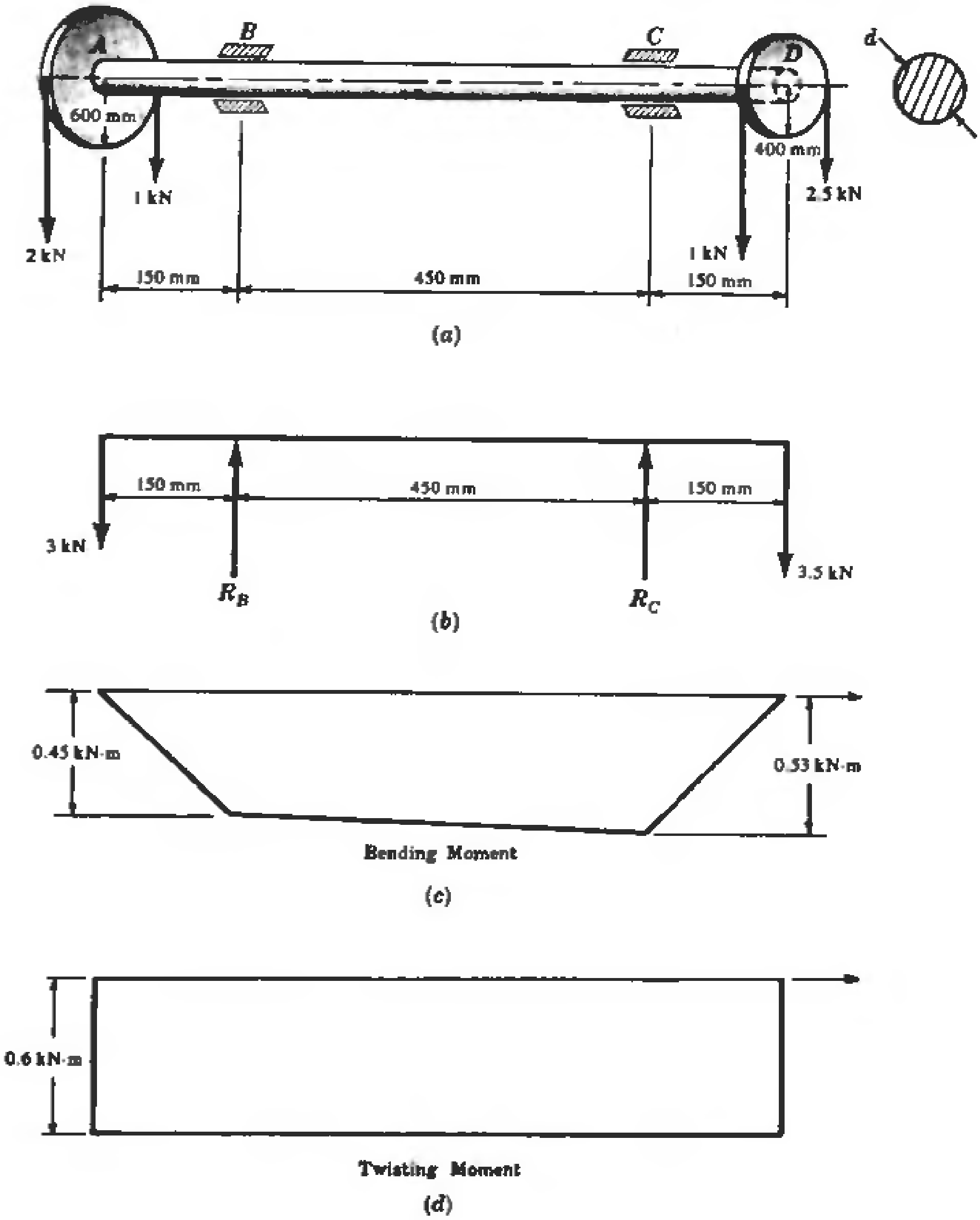



**Fig. 17-15**

bending moment along the length of the shaft is shown in Fig. 17-15(*c*). Similarly, the twisting moment along the length of the shaft may be depicted as a constant, as in Fig. 17-15(*d*).

Evidently the shaft is most critically stressed at its outer fibers at point *C*, where a top view of the uppermost element indicates the stresses $\sigma_x$ and $\tau_{xy}$ shown in Fig. 17-16. The normal stress $\sigma_x$ arises because of bending action, and is found from Problem 8.1 to be

$$\sigma_x = \frac{Mc}{I} = \frac{(0.53 \times 10^3)(10^3)(d/2)}{\pi d^4/64} = \frac{5.4 \times 10^6}{d^3}\,\text{MPa} \tag{a}$$

The other normal stresses, $\sigma_y$ and $\sigma_z$, are zero. The shearing stresses $\tau_{xy}$ arise from the torsion due to the unequal belt pulls, and are found from Problem 5.2 to be

$$\tau_{xy} = \frac{Tr}{J} = \frac{(0.6 \times 10^3)(10^3)(d/2)}{\pi d^4/32} = \frac{3.06 \times 10^6}{d^3}\,\text{MPa} \tag{b}$$

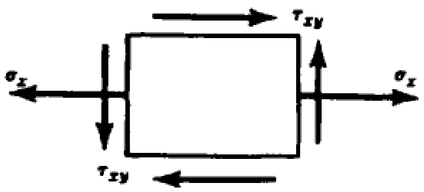



**Fig. 17-16**

According to the maximum normal stress theory, yielding of the shaft occurs when the maximum normal stress reaches the value at which yielding occurs in a simple tensile test. The maximum normal stress is found as the maximum principal stress of Problem 16.13 to be

$$\sigma_{\max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} \tag{c}$$

Substituting the results of (*a*) and (*b*) into (*c*), and introducing the safety factor of 3, yields

$$\frac{250}{3} = \frac{5.4 \times 10^6 + 0}{2d^3} + \sqrt{\left(\frac{5.4 \times 10^6 - 0}{2d^3}\right)^2 + \left(\frac{3.06 \times 10^6}{d^3}\right)^2}$$

from which $d = 43$ mm.

**17.11.** For the shaft loaded as in Problem 17.10 determine the required diameter using the maximum shearing stress theory together with a safety factor of 3.

The maximum normal stress is given in (*c*) of Problem 16.13. The minimum normal stress is given by

$$\sigma_{\min} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} \tag{a}$$

It is to be carefully noted that the difference between the $\sigma_{\max}$ and $\sigma_{\min}$ indicated above leads to the *greatest* possible difference, since the third principal stress is zero and $\sigma_{\min}$ is evidently negative. Substituting in (*1*) of Problem 17.7, we have

$$2\sqrt{\left(\frac{5.4 \times 10^6 - 0}{2d^3}\right)^2 + \left(\frac{3.06 \times 10^6}{d^3}\right)^2} = \frac{250}{3} \qquad \text{or} \qquad d = 46\text{ mm}$$

**17.12.** For the shaft loaded as in Problem 17.10 determine the required diameter using the Huber–von Mises–Hencky theory together with a safety factor of 3.

The criterion is expressed by (*i*) of Problem 17.8, where $\sigma_1$, $\sigma_2$, and $\sigma_3$ are principal stresses. We take these principal stresses to be

$$\sigma_1 = \sigma_{\max} = \left(\frac{5.4 \times 10^6 + 0}{2d^3}\right) + \sqrt{\left(\frac{5.4 \times 10^6 - 0}{2d^3}\right)^2 + \left(\frac{3.06 \times 10^6}{d^3}\right)^2} = \frac{6.8 \times 10^6}{d^3}$$

$$\sigma_2 = 0$$

$$\sigma_3 = \sigma_{\min} = \left(\frac{5.4 \times 10^6 + 0}{2d^3}\right) - \sqrt{\left(\frac{5.4 \times 10^6 - 0}{2d^3}\right)^2 + \left(\frac{3.06 \times 10^6}{d^3}\right)^2} = -\frac{1.4 \times 10^6}{d^3}$$

Substituting in (*i*) of Problem 17.8, we have

$$\left[\frac{6.8 \times 10^6}{d^3} - 0\right]^2 + \left[0 - \left(\frac{1.4 \times 10^6}{d^3}\right)\right]^2 + \left[\frac{6.8 \times 10^6}{d^3} - \left(\frac{1.4 \times 10^6}{d^3}\right)\right]^2 = 2\left(\frac{250}{3}\right)^2$$

Solving, $d = 45$ mm.

## Supplementary Problems

**17.13.** A short block is loaded by a comprehensive force of 1.5 MN. The force is applied with an eccentricity of 60 mm, as shown in Fig. 17-17. The block is 300 mm in cross section. Determine the stresses at the outer fibers $m$ and $n$. *Ans.* $\sigma_m = -36.7$ MPa, $\sigma_n = +3.3$ MPa

**17.14.** In Problem 17.13 how large an eccentricity must exist if the resultant stress at fiber $m$ is to be zero? *Ans.* 50 mm

**17.15.** A short block is loaded by a compressive force of 500 kN acting 50 mm from one axis and 75 mm from another axis of a 200-mm × 200-mm cross section, as shown in Fig. 17-18. Determine the peak tensile and compressive stresses in the cross section. *Ans.* 34.75 MPa, −59.0 MPa

**17.16.** The hollow rectangular block shown in Fig. 17-19 has its vertical axis of symmetry parallel to the $y$-direction, is clamped at its lower extremity, and is subject to a single vertical concentrated load $P = 180$ kN as indicated. Determine the resultant vertical stress at point $A$ lying at the remote corner of the lower extremity of the block. *Ans.* −111.9 MPa

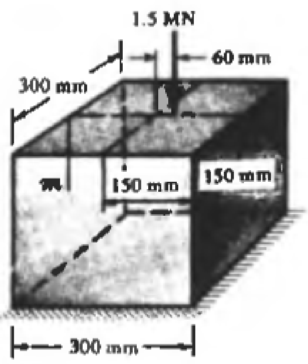



Fig. 17-17

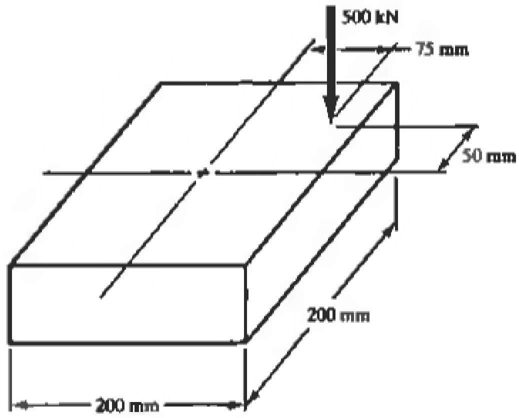



Fig. 17-18

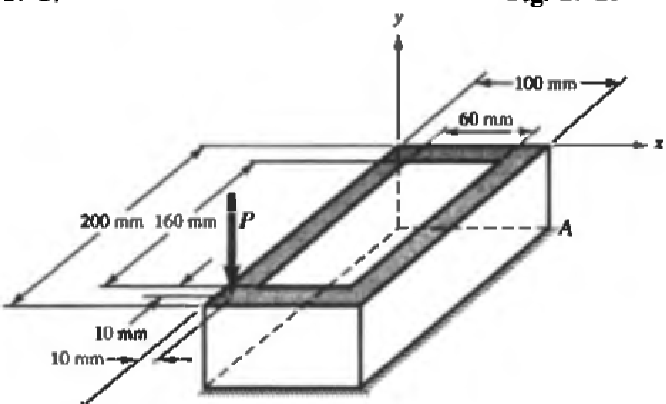



Fig. 17-19

**17.17.** In Problem 17.2, if the axial compressive force is 200 kN, find the allowable torque if the allowable shearing stress is 100 MPa. *Ans.* 1570 kN · m

**17.18.** A thin-walled cylinder is 10 in in diameter and of wall thickness 0.10 in. The cylinder is subject to a uniform internal pressure of 100 lb/in². What additional axial tension may act simultaneously without the maximum tensile stress exceeding 20,000 lb/in²? *Ans.* 55,000 lb

**17.19.** A thin-walled cylindrical shell is subject to an axial compression of 50,000 lb together with a torsional moment of 30,000 lb · in. The diameter of the cylinder is 12 in and the wall thickness 0.125 in. Determine the principal stresses in the shell. Also determine the maximum shearing stress. Neglect the possibility of buckling of the shell. *Ans.* $\sigma_{max} = 120\ \text{lb/in}^2$, $\sigma_{min} = -10{,}680\ \text{lb/in}^2$, $\tau = 5400\ \text{lb/in}^2$

**17.20.** A shaft 2.50 in in diameter is subject to an axial tension of 40,000 lb together with a twisting moment of 35,000 lb · in. Determine the principal stresses in the shaft. Also determine the maximum shearing stress. *Ans.* $\sigma_{max} = 16{,}180\ \text{lb/in}^2$, $\sigma_{min} = -8020\ \text{lb/in}^2$, $\tau = 12{,}100\ \text{lb/in}^2$

**17.21.** Consider a solid circular shaft subject to a twisting moment of 20,000 lb · in together with a bending moment of 30,000 lb · in. The diameter of the shaft is 3 in. Determine the principal stresses, as well as the maximum shearing stress in the shaft. *Ans.* $\sigma_{max} = 12{,}450\ \text{lb/in}^2$, $\sigma_{min} = -1150\ \text{lb/in}^2$, $\tau = 6800\ \text{lb/in}^2$

**17.22.** The shaft shown in Fig. 17-20 rotates with constant angular velocity and is subject to combined bending and torsion due to the indicated belt pulls. The weights of the shaft and pulleys may be neglected and the bearings can exert only concentrated force reactions. The diameter of the shaft is 1.75 in. Determine the principal stresses in the shaft. *Ans.* $\sigma_{max} = 16{,}600\ \text{lb/in}^2$, $\sigma_{min} = -750\ \text{lb/in}^2$

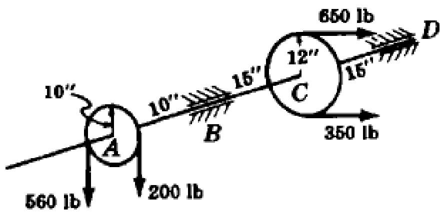



**Fig. 17-20**

**17.23.** Consider a thin-walled cylindrical pressure vessel with mean diameter 150 mm subject to a twisting moment of 1 kN · m together with an internal pressure of 3 MPa. If the allowable working stress in tension is 150 MPa, determine the wall thickness as required by the maximum normal stress theory.
*Ans.* 1.55 mm

**17.24.** For Problem 17.23 determine the wall thickness as required by the maximum shearing stress theory.
*Ans.* 1.55 mm

**17.25.** For Problem 17.23 determine the wall thickness as required by the Huber–von Mises–Hencky theory.
*Ans.* 1.34 mm

# Index

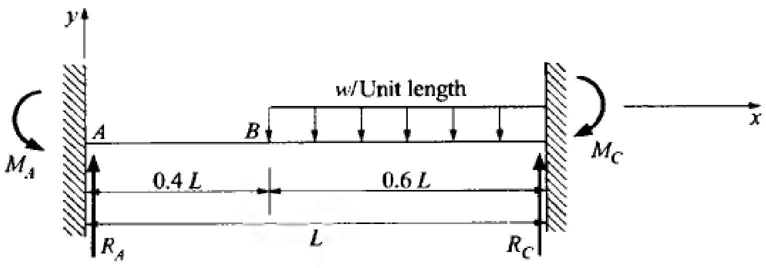



**Fig. 11-10**

and we must supplement these equations with additional relations stemming from beam deformations. The bending moment along the length $ABC$ is conveniently written in terms of singularity functions:

$$EI\frac{d^2y}{dx^2} = -M_A\langle x\rangle^0 + R_A\langle x\rangle - \frac{w\langle x-0.4L\rangle^2}{2} \tag{1}$$

Integrating,

$$EI\frac{dy}{dx} = -M_A\langle x\rangle^1 + R_A\frac{\langle x\rangle^2}{2} - \frac{w}{2}\frac{\langle x-0.4L\rangle^3}{3} + C_1 \tag{2}$$

where $C_1$ is a constant of integration. As the first boundary condition, we have: when $x = 0$, the slope $dy/dx = 0$. Substituting in Eq. (2), we have

$$0 = -0 + 0 - 0 + C_1 \qquad \text{for} \qquad C_1 = 0$$

As the second boundary condition, when $x = L$, $dy/dx = 0$. Substituting in Eq. (2), we find

$$0 = -M_A L + \frac{R_A L^3}{2} - \frac{w}{6}(0.6L)^3 \tag{3}$$

Next, integrating Eq. (2), we find

$$EIy = -M_A\frac{\langle x\rangle^2}{2} + \frac{R_A}{2}\frac{\langle x\rangle^3}{3} - \frac{w}{6}\frac{\langle x-0.4L\rangle^4}{4} + C_2 \tag{4}$$

The third boundary condition is: when $x = 0$, $y = 0$, so from Eq. (4) we have $C_2 = 0$. The fourth boundary condition is: when $x = L$, $y = 0$, so from Eq. (4) we have

$$0 = -\frac{M_A L^2}{2} + \frac{R_A L^3}{6} - \frac{w}{24}(0.6L)^4 \tag{5}$$

The expressions for $M_A$ given in Eqs. (3) and (5) may now be equated to obtain a single equation containing $R_A$ as an unknown. Solving this equation, we find

$$R_A = wL\left\{(0.6)^3 - \frac{(0.6)^4}{2}\right\}$$
$$= 0.1512wL$$

Substituting this value in Eq. (3), we find $M_A = 0.0396wL^2$.

From statics we have

$$\Sigma F_y = -(0.6L)w + 0.1512wL + R_C = 0 \qquad \therefore R_C = 0.4488wL$$

and $$+\circlearrowright\ \Sigma M_A = -0.0396wL^2 - M_C + (0.4488wL)(L) - [w(0.6L)](0.7L) = 0$$

$$\therefore M_C = 0.0684wL^2$$

**11.7.** The beam in Fig. 11-11 of flexural rigidity $EI$ is clamped at $A$, supported between knife edges at $B$, and loaded by a vertical force $P$ at the unsupported tip $C$. Determine the deflection at $C$.

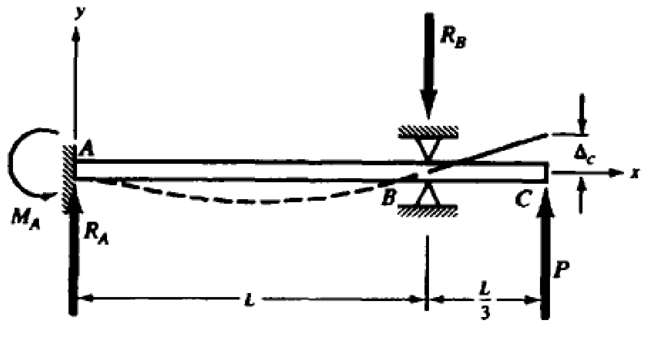



**Fig. 11-11**

The reactions at $A$ are the moment $M_A$ and shear force $R_A$ as shown in Fig. 11-11. From statics we have

$$+\circlearrowleft\ \Sigma M_A = M_A + P\left(\frac{4L}{3}\right) - R_B(L) = 0 \qquad (1)$$

$$\Sigma F_y = R_A + P - R_B = 0 \qquad (2)$$

These two equations contain the three unknowns $M_A$, $R_A$, and $R_B$. Thus, we must supplement these two statics equations with another equation arising from deformation of the beam. Using the $x$-$y$ coordinate system shown, the differential equation of the deformed beam in terms of singularity functions is

$$EI\frac{d^2y}{dx^2} = -M_A\langle x\rangle^0 + R_A\langle x\rangle^1 - R_B\langle x - L\rangle^1 \qquad (3)$$

The first integration yields

$$EI\frac{dy}{dx} = -M_A\langle x\rangle^1 + R_A\frac{\langle x\rangle^2}{2} - R_B\frac{\langle x - L\rangle^2}{2} + C_1 \qquad (4)$$

where $C_1$ is a constant of integration. The first boundary condition is that when $x = 0$, $dy/dx = 0$; hence from (4), $C_1 = 0$. The next integration yields

$$EIy = -M_A\frac{\langle x\rangle^2}{2} + \frac{R_A}{2}\frac{\langle x\rangle^3}{2} - \frac{R_B}{2}\frac{\langle x - L\rangle^3}{3} + C_2 \qquad (5)$$

where the constant $C_2$ is determined from the second boundary condition $x = 0$, $y = 0$, leading to $C_2 = 0$. The third boundary condition arises from the fact that there is no deflection at $B$; that is, when $x = L$, $y = 0$. Substituting in Eq. (5), we find

$$0 = -\frac{M_AL^2}{2} + \frac{R_AL^3}{6} - 0 \qquad (6)$$

Solving Eqs. (1), (2), and (6) simultaneously, we have

$$R_A = \frac{3M_A}{L} = \frac{P}{2} \qquad M_A = \frac{PL}{6} \qquad R_B = \frac{3P}{2} \qquad (7)$$

If we now introduce these values into Eq. (5) and also set $x = 4L/3$ (point $C$), we have

$$EI\Delta_C = 0.0401PL^3 \qquad (8)$$

**11.8.** In Problem 11.7 if the beam is a $W6 \times 15\frac{1}{2}$ steel wide-flange section of length 10 ft, determine the force $P$ required to deflect the tip $C$ 0.2 in.

From Eq. (8) of Problem 11.7, we have the tip deflection $\Delta_C$ as

$$EI\Delta_C = 0.0401PL^3$$

For this structural shape, we have from Table 8-1 that $I = 28.1\ \text{in}^4$. Substituting

$$\left(30 \times 10^6 \frac{\text{lb}}{\text{in}^2}\right)(28.1\ \text{in}^4)(0.2\ \text{in}) = 0.0401P(120\ \text{in})^3$$

Solving, $P = 2430$ lb.

**11.9.** The beam of flexural rigidity $EI$ in Fig. 11-12 is clamped at end $A$, supported at $C$, and loaded by the couple at $B$ together with the load uniformly distributed over the region $BC$. Determine all reactions.

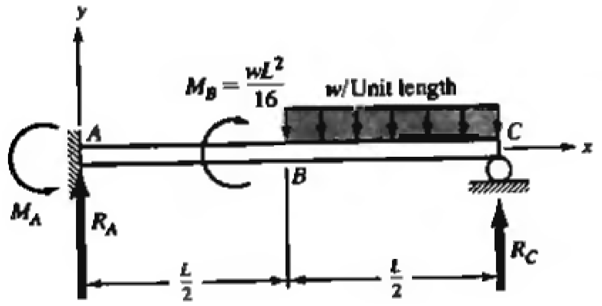



Fig. 11-12

The reactions at the left support $A$ consist of the moment $M_A$ plus the shear force $R_A$. From statics, for this parallel force system, we have two equations of equilibrium

$$+\circlearrowleft\ \Sigma M_A = M_A - \frac{wL^2}{16} - \left(w\frac{L}{2}\right)\left(\frac{3L}{4}\right) + R_C(L) = 0 \tag{1}$$

$$\Sigma F_y = R_A + R_C - \frac{wL}{2} = 0 \tag{2}$$

These two equations contain the three unknowns $M_A$, $R_A$, and $R_C$. Accordingly we must supplement the two statics equations with another equation stemming from deformations of the system.

For the $x$-$y$ coordinate system shown, the differential equation of the bent beam written in terms of singularity functions is

$$EI\frac{d^2y}{dx^2} = -M_A\langle x\rangle^0 + R_A\langle x\rangle^1 + M_B\left\langle x - \frac{L}{2}\right\rangle^0 - \frac{w}{2}\left\langle x - \frac{L}{2}\right\rangle^2 \tag{3}$$

Integrating the first time, this becomes

$$EI\frac{dy}{dx} = -M_A\langle x\rangle^1 + R_A\frac{\langle x\rangle^2}{2} + M_B\left\langle x - \frac{L}{2}\right\rangle - \frac{w}{2}\frac{\langle x - L/2\rangle^3}{3} + C_1 \tag{4}$$

where $C_1$ is a constant of integration. As the first boundary condition, when $x = 0$, $dy/dx = 0$. Substituting these values in Eq. (4), we find $C_1 = 0$. Integrating the second time, we find

$$EIy = -M_A\frac{\langle x\rangle^2}{2} + \frac{R_A}{2}\frac{\langle x\rangle^3}{3} + M_B\frac{\langle x - L/2\rangle^2}{2} - \frac{w}{6}\frac{\langle x - L/2\rangle^4}{4} + C_2 \tag{5}$$

where $C_2$ is the second constant of integration. As a second boundary condition, we have at point $A$, $x = 0$, $y = 0$, and so from Eq. (5) we see that $C_2 = 0$. The third boundary condition is that at point $C$ when $x = L$, $y = 0$. Substituting these values in Eq. (5), we have

$$-\frac{M_A L^2}{2} + \frac{R_A L^3}{6} + \frac{M_B}{2}\cdot\frac{L^2}{4} - \frac{w}{24}\cdot\left(\frac{L}{2}\right)^4 = 0 \tag{6}$$

Solving Eqs. (*1*), (*2*), and (*6*) simultaneously, we find

$$M_A = \tfrac{3}{64}wL^2 \qquad R_A = \tfrac{7}{64}wL \qquad R_C = \tfrac{25}{64}wL \tag{7}$$

**11.10.** In Problem 11.9 if the beam is titanium having a Young's modulus of 110 GPa, with a rectangular cross section 20 mm × 30 mm, is 2 m long, and carries the uniform load in *BC* of 960 N/m, determine the deflection at the midpoint *B*.

From Eq. (*5*) of Problem 11.9 we have the deflection at the midpoint *B* as

$$EIy]_{x=L/2} = -\frac{M_A}{2}\left(\frac{L}{2}\right)^2 + \frac{R_A}{6}\left(\frac{L}{2}\right)^3$$

$$= -\frac{3}{64}wL^2\left(\frac{L^2}{8}\right) + \frac{7}{64}wL\left(\frac{L^3}{48}\right)$$

$$= -\frac{11}{(48)(64)}wL^4 = -0.00358wL^4 \tag{1}$$

For this beam

$$I = \tfrac{1}{12}(0.020\ \text{m})(0.030\ \text{m})^3 = 0.045 \times 10^{-6}\ \text{m}^4$$

so that Eq. (*1*) becomes

$$(110 \times 10^9\ \text{N/m}^2)(0.045 \times 10^{-6}\ \text{m}^4)[y]_{x=L/2} = -0.00358(960\ \text{N/m})(2\ \text{m})^4$$

Solving,

$$y]_{x=L/2} = -11.1\ \text{mm}$$

**11.11.** The beam *AB* of flexural rigidity *EI* is simply supported at *A*, rigidly clamped at end *B*, and subject to the load of uniformly varying intensity shown in Fig. 11-13. Determine the reactions developed at *A* and *B* by the use of the method of singularity functions.

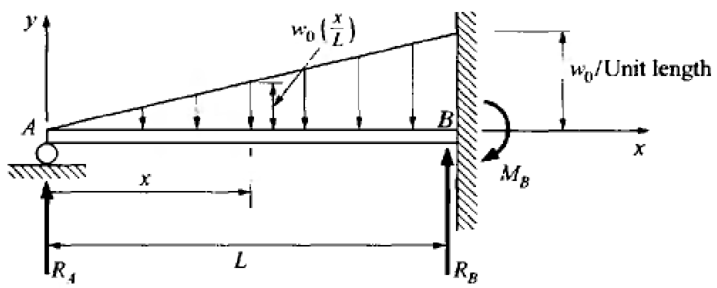



**Fig. 11-13**

Let us denote the vertical force reaction at *A* by $R_A$, that at *B* by $R_B$, and the moment exerted by the wall on the beam at *B* by $M_B$, as shown in Fig. 11-13. A related problem is 10.5 in this book. Following the procedure discussed there, we write the contribution to bending moment of the distributed loading at any point a distance *x* to the right of *A*:

$$M = R_A\langle x\rangle - w_0\left(\frac{x}{L}\right)\langle x\rangle\left(\frac{1}{2}\right)\left(\frac{x}{3}\right)$$

Thus,

$$EI\frac{d^2y}{dx^2} = R_A\langle x\rangle - \frac{w_0\langle x\rangle^3}{6L} \tag{1}$$

Integrating the first time,

$$EI\frac{dy}{dx} = R_A\frac{\langle x\rangle^2}{2} - \frac{w_0}{6L}\cdot\frac{\langle x\rangle^4}{4} + C_1 \tag{2}$$

When $x = L$, $dy/dx = 0$, so from Eq. (2)

$$0 = R_A\frac{L^2}{2} - \frac{w_0 L^3}{24} + C_1 \tag{3}$$

Integrating a second time,

$$EIy = \frac{R_A}{2}\frac{\langle x\rangle^3}{3} - \frac{w_0}{24L}\frac{\langle x\rangle^5}{5} + C_1 x + C_2 \tag{4}$$

When $x = L$, $y = 0$, so we have from Eq. (4)

$$0 = \frac{R_A L^3}{6} - \frac{w_0 L^4}{120} + C_1 L + C_2 \tag{5}$$

Also, when $x = 0$, $y = 0$, so from Eq. (4), $C_2 = 0$.

From Eqs. (3) and (5) we have

$$C_1 = \frac{w_0 L^3}{24} - \frac{R_A L^2}{2} = -\frac{R_A L^2}{6} + \frac{w_0 L^3}{120} \tag{6}$$

Solving,

$$R_A = \tfrac{1}{10} w_0 L \tag{7}$$

The two statics equations for such a force system are

$$\Sigma F_y = R_A + R_B - \frac{w_0 L}{2} = 0$$

$$+\circlearrowright\ \Sigma M_B = -R_A L - M_0 + \left(\frac{w_0}{2}\right)(L)\left(\frac{L}{3}\right) = 0$$

Solving,

$$R_B = \tfrac{2}{5} w_0 L$$
$$M_B = \tfrac{1}{15} w_0 L^2$$

**11.12.** The beam $AC$ in Fig. 11-14 is rigidly clamped at both ends and loaded by a concentrated force $P$ at point $B$. Determine all reactions, the deflection at $B$, and the maximum deflection occurring to the left of point $B$. Take $a > b$.

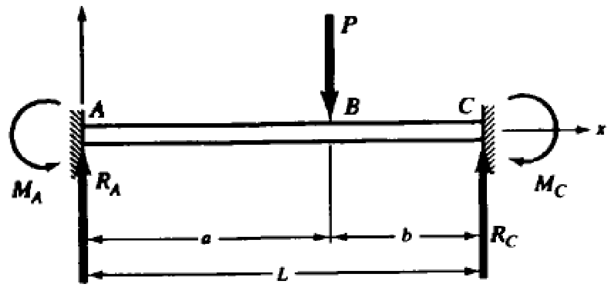



Fig. 11-14

The end moment and shear reactions are shown in Fig. 11-14. From statics we have the two equations

$$+\circlearrowright \Sigma M_A = M_A - Pa + R_C L - M_C = 0 \tag{1}$$

$$\Sigma F_y = R_A + R_C - P = 0 \tag{2}$$

Next, writing the differential equation of the deflected beam in terms of singularity functions,

$$EI\frac{d^2y}{dx^2} = -M_A\langle x\rangle^0 + R_A\langle x\rangle^1 - P\langle x-a\rangle \tag{3}$$

Integrating the first time, we obtain

$$EI\frac{dy}{dx} = -M_A\langle x\rangle^1 + R_A\frac{\langle x\rangle^2}{2} - \frac{P\langle x-a\rangle^2}{2} + C_1 \tag{4}$$

As the first boundary condition, when $x = 0$, the slope $dy/dx = 0$. Substituting these values in Eq. (4), we obtain $C_1 = 0$.

Integrating again, we find

$$EIy = -M_A\frac{\langle x\rangle^2}{2} + \frac{R_A}{2}\frac{\langle x\rangle^3}{3} - \frac{P}{2}\frac{\langle x-a\rangle^3}{3} + C_2 \tag{5}$$

The second boundary condition is that when $x = 0$, $y = 0$. Substituting these values in Eq. (5), we find $C_2 = 0$. The equation of the deflected beam is consequently

$$EIy = -\frac{M_A}{2}\langle x\rangle^2 + \frac{R_A}{6}\langle x\rangle^3 - \frac{P}{6}\langle x-a\rangle^3 \tag{6}$$

Now, apply the boundary conditions at point $C$. The slope there is zero; hence from Eq. (4) we obtain the equation

$$-M_A L + \frac{R_A}{2}L^2 - \frac{Pb^2}{2} = 0 \tag{7}$$

The deflection $y = 0$ at $x = L$; hence we have from Eq. (6) the relation

$$-\frac{M_A}{2}L^2 + \frac{R_A}{6}L^3 - \frac{Pb^3}{6} = 0 \tag{8}$$

We may now solve Eqs. (1), (2), (7), and (8) simultaneously to find the end reactions

$$R_A = \frac{Pb^2}{L^3}(3a+b) \qquad M_A = \frac{Pab^2}{L^2}$$
$$R_C = \frac{Pa^2}{L^3}(a+3b) \qquad M_C = \frac{Pa^2b}{L^2} \tag{9}$$

The deflection at $B$ under the point of application of the load $P$ is found by setting $x = a$ in Eq. (6):

$$EI[y]_{x=a} = -\frac{M_A}{2}a^2 + \frac{R_A}{6}a^3 = -\frac{Pa^3b^3}{3L^3} \tag{10}$$

To determine the maximum deflection of the beam for our case of $a > b$, we consider the deflected bar as shown in Fig. 11-15, from which it is evident that the point of horizontal tangency to the beam occurs to the left of $B$; that is, we are concerned with $x < a$ in Eq. (4) so that the slope in region $AB$ is given by

$$EI\frac{dy}{dx} = -M_A\langle x\rangle^1 + \frac{R_A}{2}\langle x\rangle^2 \tag{11}$$

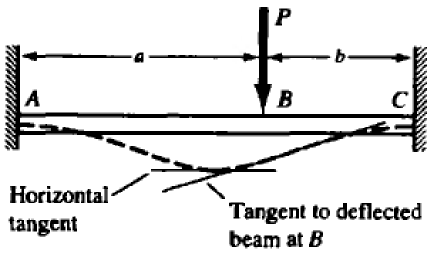



**Fig. 11-15**

which we set equal to zero to find the value $x_1$. This leads to a horizontal tangent at the value of $x_1$ given by

$$x_1 = \frac{2aL}{(3a+b)} \tag{12}$$

Substituting this $x_1$ in Eq. (6) and remembering that $x < a$, we obtain

$$EI[y]_{\max} = -\frac{2Pa^3b^2}{3(3a+b)^2} \tag{13}$$

**11.13.** In Problem 11.12 the beam has $a = 6$ ft, $b = 3$ ft, and is of circular cross section 2.5 in in diameter. The applied load is $P = 6000$ lb. Determine the deflection under the point of application of the load as well as the maximum deflection of the beam. Take $E = 30 \times 10^6$ lb/in$^2$.

The moment of inertia of the cross section is

$$I = \frac{\pi}{64}D^4 = \frac{\pi}{64}(2.5\text{ in})^4 = 1.917\text{ in}^4$$

The deflection under the point of application of the load is given by Eq. (10) of Problem 11.12 to be

$$y]_{x=a} = -\frac{Pa^3b^3}{3EIL^3}$$

Substituting,

$$y]_{x=a} = \frac{-(6000\text{ lb})(72\text{ in})^3(36\text{ in})^3}{3(30\times 10^6\text{ lb/in}^2)(1.917\text{ in}^4)(108\text{ in})^3} = -0.480\text{ in}$$

The location of the point of maximum deflection is given by Eq. (12) to be

$$x_1 = \frac{2aL}{3a+b} = \frac{2(6\text{ ft})(9\text{ ft})}{18\text{ ft}+3\text{ ft}} = 5.14\text{ ft}$$

and the desired maximum deflection is found from Eq. (13) to be

$$\begin{aligned} y]_{\max} &= -\frac{2Pa^3b^2}{3(3a+b)^2EI} \\ &= \frac{2(6000\text{ lb})(72\text{ in})^3(36\text{ in})^3}{2[(3)(72\text{ in})+(36\text{ in})]^2(30\times 10^6\text{ lb/in}^2)(1.917\text{ in}^4)} \\ &= -0.522\text{ in} \end{aligned}$$

**11.14.** The initially horizontal beam $ABC$ in Fig. 11-16 is clamped at $C$ and supported on a smooth roller at $B$. A uniform load $w$ per unit length acts over the entire length of the beam. After

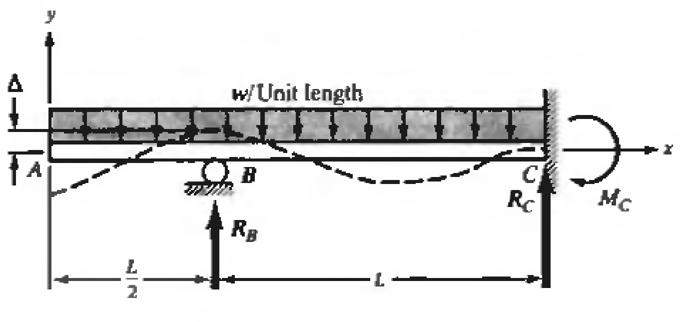



**Fig. 11-16**

application of the load, the reaction at $B$ is mechanically displaced upward an amount $\Delta$ so that the beam then has the configuration shown by the dotted line. Determine the reaction $R_B$ after this displacement has been imposed.

The beam reactions are $R_B$, $R_C$, and a moment $M_C$. Using the method of singularity functions, we have the equation of the bent beam,

$$EI\frac{d^2y}{dx^2} = -w\frac{\langle x\rangle^2}{2} + R_B\left\langle x - \frac{L}{2}\right\rangle \tag{1}$$

Integrating the first time, we obtain

$$EI\frac{dy}{dx} = -\frac{w}{2}\frac{\langle x\rangle^3}{3} + \frac{R_B}{2}\left\langle x - \frac{L}{2}\right\rangle^2 + C_1 \tag{2}$$

For the first boundary condition, we know that, when $x = 3L/2$, $dy/dx = 0$. Substituting in Eq. (2)

$$0 = -\frac{w}{6}\left(\frac{27L^3}{8}\right) + R_B\frac{L^2}{8} + C_1$$

from which

$$C_1 = \frac{9}{16}wL^3 - \frac{R_BL^2}{8} \tag{3}$$

Integrating a second time,

$$EIy = -\frac{w}{6}\frac{\langle x\rangle^4}{4} + \frac{R_B}{2}\frac{\langle x - L/2\rangle^3}{3} + \left(\frac{9}{16}wL^3 - R_B\frac{L^2}{8}\right)\langle x\rangle + C_2 \tag{4}$$

For the second boundary condition, when $x = 3L/2$, $y = 0$. Substituting in Eq. (4), we have

$$wL^4\left[-\frac{27}{(8)(16)} + \frac{27}{32}\right] + R_B\left[\frac{L^3}{6} - \frac{3L^3}{16}\right] + C_2 = 0$$

from which

$$C_2 = -\frac{81}{128}wL^4 + \frac{1}{48}R_BL^3$$

The third and last boundary condition stems from the imposed displacement at point $b$; that is, when $x = L/2$, $y = \Delta$. Substituting these values in Eq. (4), we have

$$EI\Delta = -\frac{w}{24}\left(\frac{L^4}{16}\right) + 0 + \left(\frac{9}{16}wL^3 - R_B\frac{L^2}{8}\right)\left(\frac{L}{2}\right) - \frac{81}{128}wL^4 + \frac{1}{48}R_BL^3$$

Solving for $R_B$, we obtain

$$R_B = \frac{3EI\Delta}{L^3} - 272w_0L$$

**11.15.** The horizontal beam $AB$ shown in Fig. 11-17 is clamped at $A$, subjected to a uniformly distributed load $w$ per unit length, and supported at $B$ in such a manner that it is free to deflect vertically but is completely restrained against rotation at that point. Determine the vertical deflection at $B$ after the beam has deflected as shown by the dotted line.

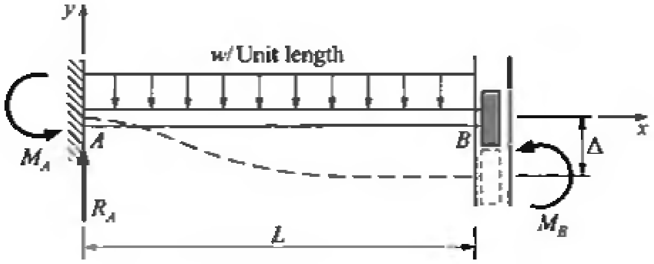



**Fig. 11-17**

The equation of the deflected beam is

$$EI\frac{d^2y}{dx^2} = -M_A\langle x\rangle^0 + R_A\langle x\rangle^1 - \frac{w\langle x\rangle^2}{2} \tag{1}$$

Integrating the first time, we find

$$EI\frac{dy}{dx} = -M_A\langle x\rangle^1 + R_A\frac{\langle x\rangle^2}{2} - \frac{w}{2}\frac{\langle x\rangle^3}{3} + C_1 \tag{2}$$

The first boundary condition is that when $x = 0$, $dy/dx = 0$. Substituting these values in Eq. (2), we find that $C_1 = 0$. Integrating again,

$$EIy = -M_A\frac{\langle x\rangle^2}{2} + \frac{R_A}{2}\frac{\langle x\rangle^3}{3} - \frac{w}{6}\frac{\langle x\rangle^4}{4} + C_2 \tag{3}$$

Imposing the boundary condition that $x = 0$ at $y = 0$, we have $C_2 = 0$.

The third boundary condition is that, when $x = L$, $dy/dx = 0$. Substituting these values in Eq. (2), we obtain the equation

$$0 = -M_A L + \frac{R_A L^2}{2} - \frac{wL^3}{6} \tag{4}$$

From statics, we have the two equilibrium equations

$$+\circlearrowright\ \Sigma M_A = M_A + M_B - \frac{wL^2}{2} = 0 \tag{5}$$

$$\Sigma F_y = R_A - wL = 0 \tag{6}$$

Solving Eqs. (4), (5), and (6) simultaneously, we have

$$R_A = wL$$

$$M_A = \tfrac{2}{3}wL^2$$

$$M_B = \tfrac{1}{6}wL^2$$

Substitution of these values in Eq. (3) leads to

$$EI[y]_{x=L} = -\frac{wL^2}{3}\cdot\frac{L^2}{2} + \frac{wL}{2}\cdot\frac{L^3}{3} - \frac{wL^4}{24}$$

or

$$y]_{x=L} = -\frac{wL^4}{24EI}$$

**11.16.** The cantilever beam $AB$ in Fig. 11-18 is clamped at $B$ and supported through a hinge by a partially submerged (in water) pontoon at $A$. The beam is of flexural rigidity $EI$ and length $L$. It is loaded by a vertical concentrated force $F$ at $A$. Determine the reactive moment at $B$.

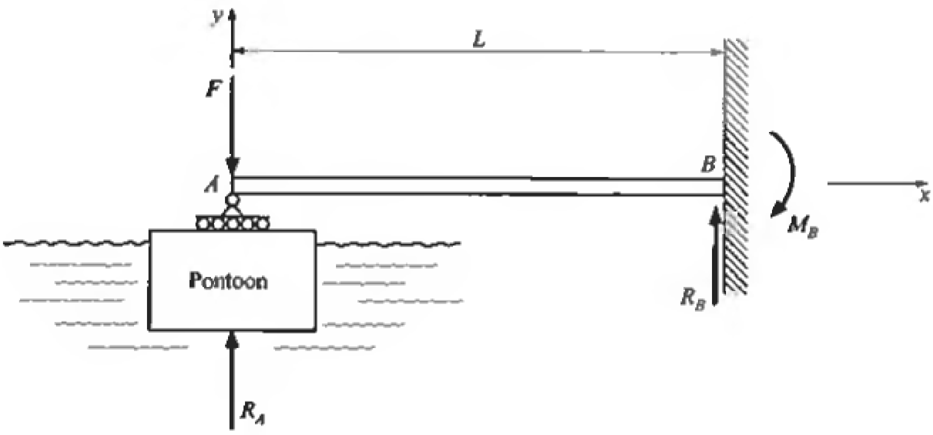



**Fig. 11-18**

When the force $F$ is applied, the pontoon submerges a distance $\Delta$. According to the law of Archimedes, the pontoon is buoyed up by a force $R_A$ of magnitude equal to the weight of the additional water displaced during the movement through $\Delta$. If the cross-sectional area of the pontoon is $A_0$ and the weight of the water per unit volume is $\gamma$, then

$$A_0 \Delta \gamma = -R_A \tag{1}$$

For the coordinate system shown in Fig. 11-18, we have

$$EI\frac{d^2y}{dx^2} = R_A x - Fx \tag{2}$$

Integrating the first time, we have

$$EI\frac{dy}{dx} = R_A\frac{x^2}{2} - F\frac{x^2}{2} + C_1 \tag{3}$$

As a boundary condition, we have $dy/dx = 0$ when $x = L$, so from Eq. (3)

$$C_1 = \frac{FL^2}{2} - \frac{R_A L^2}{2}$$

Integrating a second time,

$$EIy = \frac{R_A}{2}\cdot\frac{x^3}{3} = \frac{F}{2}\cdot\frac{x^3}{3} + \left(\frac{FL^2}{2} - \frac{R_A L^2}{2}\right)x + C_2 \tag{4}$$

As a second boundary condition we have $y = 0$ when $x = L$, so from Eq. (4)

$$C_2 = \frac{R_A L^3}{2} - \frac{FL^3}{3}$$

The equation of the deflected beam $AB$ is thus

$$EIy = \frac{R_A}{6}x^3 - \frac{F}{6}x^3 + \left(\frac{FL^2}{2} - \frac{R_A L^2}{2}\right)x + \frac{R_A L^3}{3} - \frac{FL^3}{3} \tag{5}$$

We seek the deflection $y$ at $x = 0$. From Eq. (1), it is

$$y = \Delta = -\frac{R_A}{A_0\gamma}$$

Accordingly at $x = 0$ from Eq. (5) we have

$$-\frac{R_A EI}{A_0 \gamma} = \frac{R_A L^3}{3} - \frac{FL^3}{3}$$

Solving,

$$R_A = \frac{\left(\frac{FL^3}{3}\right)}{\frac{L^3}{3} + \frac{EI}{A_0 \gamma}} \tag{6}$$

From statics,

$$+\circlearrowright \Sigma M_B = R_A L - FL + M_B = 0 \tag{7}$$

Solving Eqs. (6) and (7) simultaneously we have

$$M_B = \frac{3FLEI}{L^3 A_0 \gamma + 3EI}$$

## Supplementary Problems

**11.17.** A clamped-end beam is supported at the right end, clamped at the left, and carries the two concentrated forces shown in Fig. 11-19. Determine the reaction at the wall and the reaction at the right end of the beam.
*Ans.* $4P/3$ acting upward at left end, $PL/3$ acting counterclockwise at left end, $2P/3$ acting upward at right end

**11.18.** Determine the deflection under the point of application of the force $P$ located a distance $L/3$ from the right end of the beam described in Problem 11.17. *Ans.* $7PL^3/486EI$

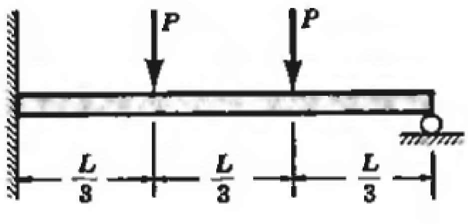



**Fig. 11-19**

**11.19.** The beam of Problem 11.17 is of titanium Ti-4Al-3Mo-IV (STA) with a tensile ultimate strength of 175,000 lb/in² at room temperature. If the cross section is 2 in × 5 in and a safety factor of 1.4 is employed, determine the maximum allowable value of each load $P$. *Ans.* 17,400 lb

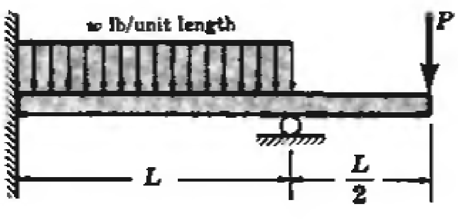



**Fig. 11-20**

**11.20.** A clamped-end beam is supported at an intermediate point and loaded as shown in Fig. 11-20. Determine the various reactions.
*Ans.* $\frac{5}{8}wL - \frac{3}{4}P$ upward at left end, $\frac{1}{8}wL^2 - \frac{1}{4}PL$ counterclockwise at left end, $\frac{3}{8}wL + \frac{7}{4}P$ upward at support

**11.21.** A clamped-end beam is supported at the right end, clamped at the left, and carries the load of uniformly varying intensity, as indicated in Fig. 11-21. Determine the moment exerted by the support on the beam. *Ans.* $7wL^2/120$

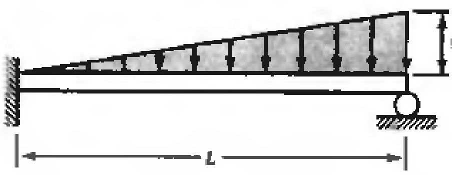



**Fig. 11-21**

**11.22.** The beam shown in Fig. 11-22 is clamped at the left end, supported at the right, and loaded by a couple $M_0$. Determine the reaction at the right support. *Ans.* $3M_0a(a + 2b)/2(a + b)^3$

**11.23.** For the beam shown in Fig. 11-22, determine the deflection under the point of application of the applied moment $M_0$. *Ans.* $M_0a^2b(a^2 - 2b^2)/4(a + b)^3 EI$

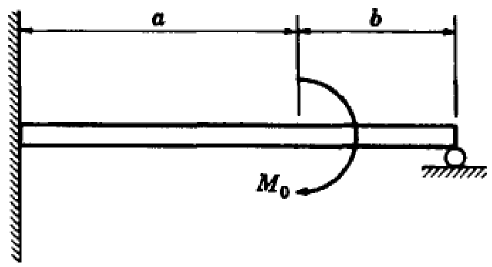



**Fig. 11-22**

**11.24.** In Fig. 11-23 $AB$ and $CD$ are cantilever beams with a roller $E$ between their end points. A load of 5 kN is applied as shown. Both beams are made of steel for which $E = 200$ GPa. For beam $AB$, $I = 20 \times 10^6\ \text{mm}^4$; for $CD$, $I = 30 \times 10^6\ \text{mm}^4$. Find the reaction at $E$. *Ans.* 398 N

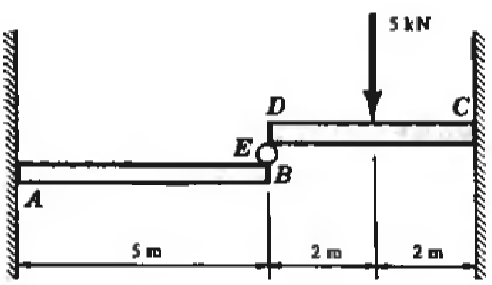



**Fig. 11-23**

**11.25.** The straight elastic beam $AB$ in Fig. 11-24 is a W152 × 23 wide-flange section having $I = 11.7 \times 10^6\ \text{mm}^4$. Member $CD$ is a vertical steel wire of 3-mm-diameter circular cross section and length 4 m. Both the beam and the wire are steel for which $E = 200$ GPa. Prior to the application of any load to the beam, due to a fabrication error, the end $D$ of the wire is 5 mm above the tip $B$ of the beam. The end $D$ of the wire and the tip $B$ of the beam are then mechanically pulled together and joined. Determine the axial stress in the bar prior to the application of any load to the beam. *Ans.* 106 MPa

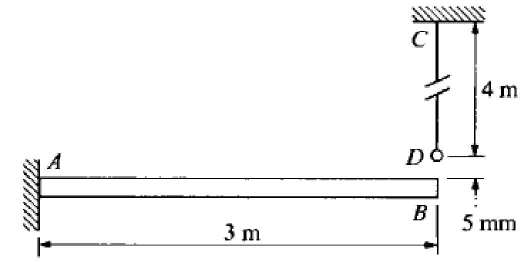



**Fig. 11-24**

**11.26.** A beam is clamped at both ends and supports a uniform load over its right half, as shown in Fig. 11-25. Determine all reactions.
*Ans.* $3wL/32$ acting upward at left end, $5wL^2/192$ acting counterclockwise at left end, $13wL/32$ acting upward at right end, $11wL^2/192$ acting clockwise at right end

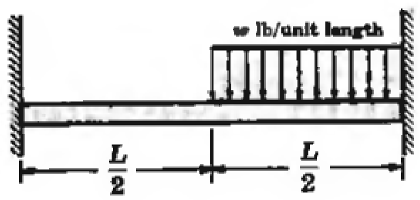



**Fig. 11-25**

**11.27.** Determine the central deflection of the beam described in Problem 11.26. *Ans.* $wL^4/768EI$

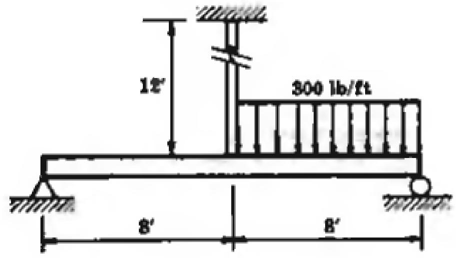



**Fig. 11-26**

**11.28.** A 16-ft beam carries a uniform load over the right half of its span and is supported at the center of the span by a vertical rod, as shown in Fig. 11-26. The rod is steel, 12 ft in length, 0.5 in$^2$ in cross-sectional area, and $E_s = 30 \times 10^6\ \text{lb/in}^2$. The beam is wood 4 in × 8 in in cross section and $E_w = 1.5 \times 10^6\ \text{lb/in}^2$. Determine the stress in the vertical steel rod. *Ans.* 2960 lb/in$^2$

**11.29.** The beam of flexural rigidity $EI$ in Fig. 11-27 is clamped at $A$, supported between knife edges at $B$, and subjected to the couple $M_0$ at its unsupported tip $C$. Determine the deflection of point $C$.
*Ans.* $M_0L^2/4EI$

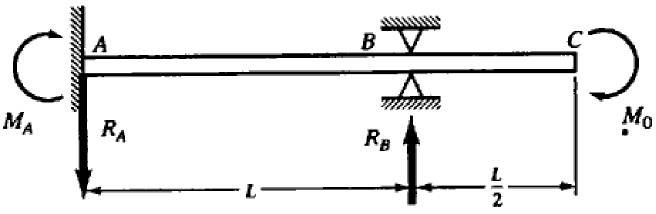



**Fig. 11-27**

**11.30.** The cantilever beam in Fig. 11-28 of length 3 m and rectangular cross section 100 mm × 200 mm has its free end (at no load) 3 mm above the top of a spring whose constant is 150 kN/m. The material is titanium alloy, which has $E = 110$ GPa and a yield point of 900 MPa. A downward force $P$ of 7000 N is applied to the tip of the beam. Find the deformation of the top of the spring under this load. *Ans.* 4.72 mm

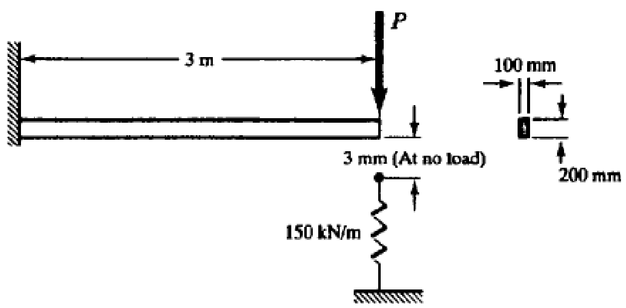



**Fig. 11-28**

**11.31.** A beam $AB$ is clamped at each end and subject to a load of uniformly varying intensity as shown in Fig. 11-29. Determine the moment reactions developed at each end of the beam.
*Ans.* $wL^2/30$ counterclockwise at $A$, $wL^2/20$ clockwise at B

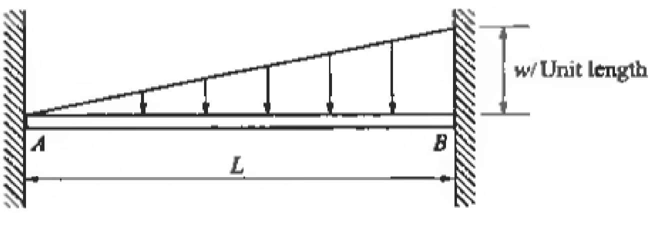



**Fig. 11-29**

**11.32.** The beam $AB$ is pinned at its left end, clamped at the right end, and subjected to the uniformly varying vertical load shown in Fig. 11-30. Determine the vertical reaction at the support at $A$.

*Ans.* $$R_1 = \frac{w_0 L_2}{240L^3}\left[\{10L_2L_3 + 5L_3^2 + 15(L_2 + L_3)^2\}L - L_3^3 - 2L_3^2(L_2 + L_3) - 3L_3^2(L_2 + L_3) - 4(L_2 + L_3)^3\right]$$

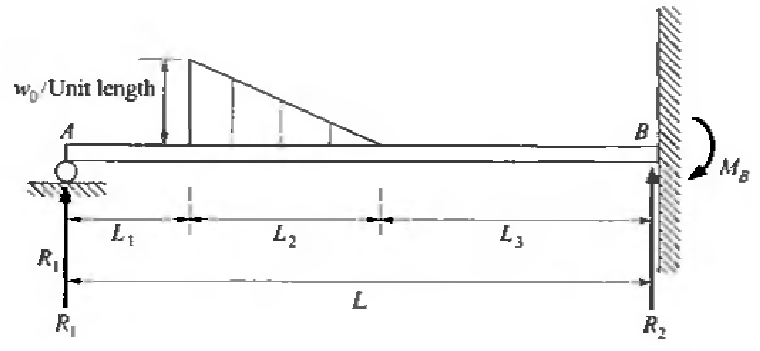



Fig. 11-30

**11.33.** The two-span continuous beam shown in Fig. 11-31 supports the two concentrated loads shown. Determine the various reactions. *Ans.* $R_A = 668$ lb, $R_B = 12{,}061$ lb, $R_C = 7271$ lb

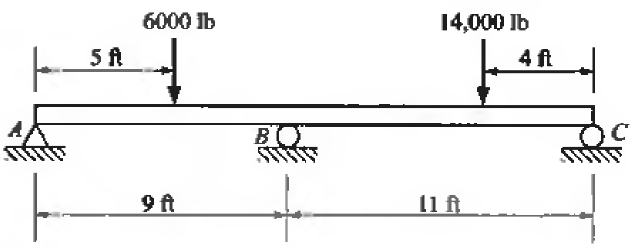



Fig. 11-31

**11.34.** The three-span continuous beam shown in Fig. 11-32 supports a uniformly distributed load in the left and central span, but is unloaded in the right span. Determine the reactions at $A$, $B$, $C$, and $D$.
*Ans.* $R_A = 0.383wL(\uparrow)$, $R_B = 1.20wL$ $(\uparrow)$, $R_C = 0.450wL$ $(\uparrow)$, $R_D = -0.033wL$ $(\downarrow)$

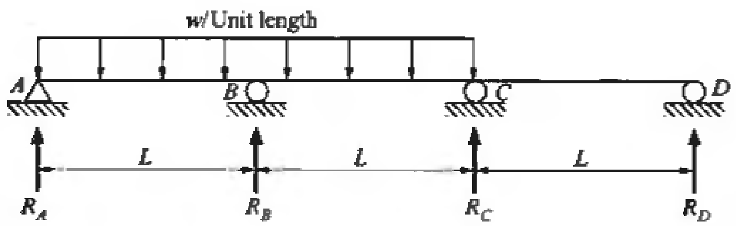



Fig. 11-32

**11.35.** The beam shown in Fig. 11-33 is simply supported at the left and right ends and spring supported at the center. Determine the spring constant so that the bending moment will be zero at the point where the spring supports the beam. *Ans.* $k = 16EI/L^3$

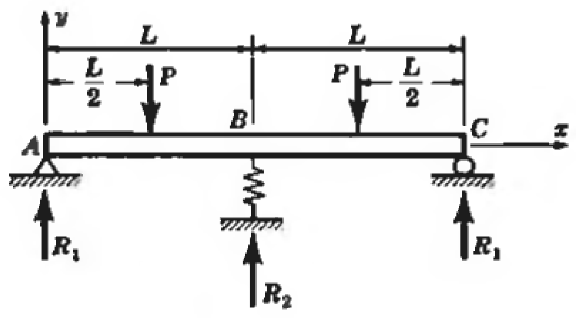



Fig. 11-33

# Chapter 12

# Special Topics in Elastic Beam Theory

## SHEAR CENTER

The simple flexure formula $\sigma = My/I$ determined in Problem 8.1 is valid only if the transverse loads which give rise to bending act in a plane of symmetry of the beam cross section. In this type of loading there is obviously no torsion of the beam. However, in more general cases the beam cross section will have no axes of symmetry and the problem of where to apply transverse loads so that the action is entirely bending with no torsion arises. Every elastic beam cross section possesses a point through which transverse forces may be applied so as to produce *bending only* with *no torsion* of the beam. This point is called the *shear center*. In general, determination of the shear center location is extremely difficult and requires use of the theory of elasticity. However, in this chapter we will be concerned only with beams of *thin-walled open* cross section having a *single axis of symmetry*, with the loads acting in a plane perpendicular to this axis of symmetry. We will locate the shear center of the open cross section on the axis of symmetry of the beam. For applications, see Problems 12.1 through 12.4.

## UNSYMMETRIC BENDING

Frequently beams are of unsymmetric cross section, or even if the cross section is symmetric the plane of the applied loads may not be one of the planes of symmetry. In either of these cases the expression $\sigma = My/I$ derived in Problem 8.1 is not valid for determination of the bending stress. It is convenient to resolve the bending moment into components along the $y$- and $z$-axes of the cross section, as indicated by the double-headed vector representations of these moments in Fig. 12-1.

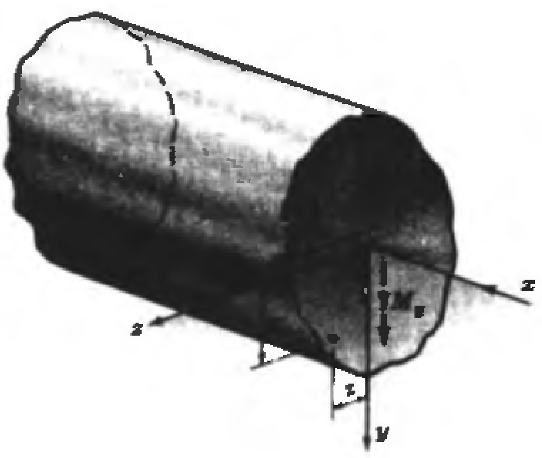



Fig. 12-1

The bending stress at a point located by the coordinates $y, z$ is shown in Problem 12.5 to be

$$\sigma = \frac{(M_z I_y + M_y I_{yz})y + (-M_y I_z - M_z I_{yz})z}{I_y I_z - I_{yz}^2} \tag{12.1}$$

where $I_y$ and $I_z$ denote the moments of inertia about the $y$- and $z$-axes, respectively, and $I_{yz}$ is the product of inertia. These quantities are determined by the methods of Chap. 7. There exists a *neutral axis* and those longitudinal fibers lying on the neutral axis are not subject to any normal stress. However, the neutral axis is usually not perpendicular to the plane of the applied loads nor does it coincide with either of the principal axes. For applications, see Problems 12.6 and 12.7. A computerized approach for determination of bending stresses is offered in Problem 12.8 and examples are offered in Problems 12.9 and 12.10.

## CURVED BEAMS

Occasionally initially curved beams are encountered in machine design and other areas. Here we consider only those elastic beams for which the plane of curvature is also a plane of symmetry of every cross section and the bending loads act in this plane of symmetry. Unlike the case of the initially straight beam, the neutral axis no longer passes through the centroid of the cross section but instead shifts toward the center of curvature of the beam by a distance denoted by $\bar{y}$. The bending stress distribution over the cross section is hyperbolic in nature and in Problem 12.11 it is shown that these stresses are given by

$$\sigma = \frac{My}{A\bar{y}(r+y)} \tag{12.2}$$

where $M$ is the bending moment, $A$ is the cross-sectional area, $r$ is the radius of curvature of the neutral axis, and $y$ denotes the distance of any fiber from the neutral axis. For applications see Problem 12.12. Because of the tedious nature of calculations associated with bending of curved beams, the problem is well suited to computer implementation and a FORTRAN program is developed in Problem 12.13 together with examples in Problems 12.14 and 12.15.

# Solved Problems

### *Shear Center*

**12.1.** Determine the shear center of half of a thin-walled cylindrical section oriented as shown in Fig. 12-2 and subject to a vertical load.

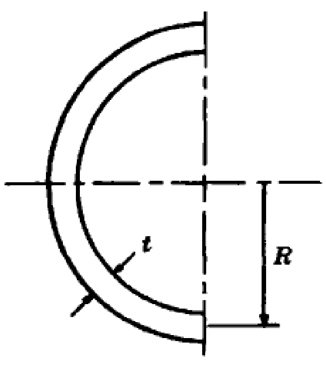



**Fig. 12-2**

Since the beam action is one of bending only with no torsion, it follows that normal stresses are distributed over the cross section in accordance with the flexure formula $\sigma = My/I$. Consequently, according to Problem 8.19, page 198, horizontal shearing stresses acting perpendicular to the plane of the cross section are generated and are determined by the relation

$$\tau = \frac{V}{Ib}\int_{y_0}^{c} y\,da$$

As indicated in Problem 8.19, the presence of these horizontal shearing stresses necessitates the presence of equal intensity shear stresses acting over the vertical cross section. In Fig. 12-3($a$) these shear stresses have been shown as acting tangential to the center line of the cross section and further, for a thin-walled section, it is customary to assume a uniform distribution of the shear stresses across the thickness $t$. Finally, it is assumed that shearing stresses perpendicular to the circular centerline of the section are negligible. In Fig. 12-3($a$), $V$ denotes the resultant of the distributed shearing stresses and it, of course, acts vertically, since the horizontal components of the various stress vectors above and below the axis of symmetry annul one another.

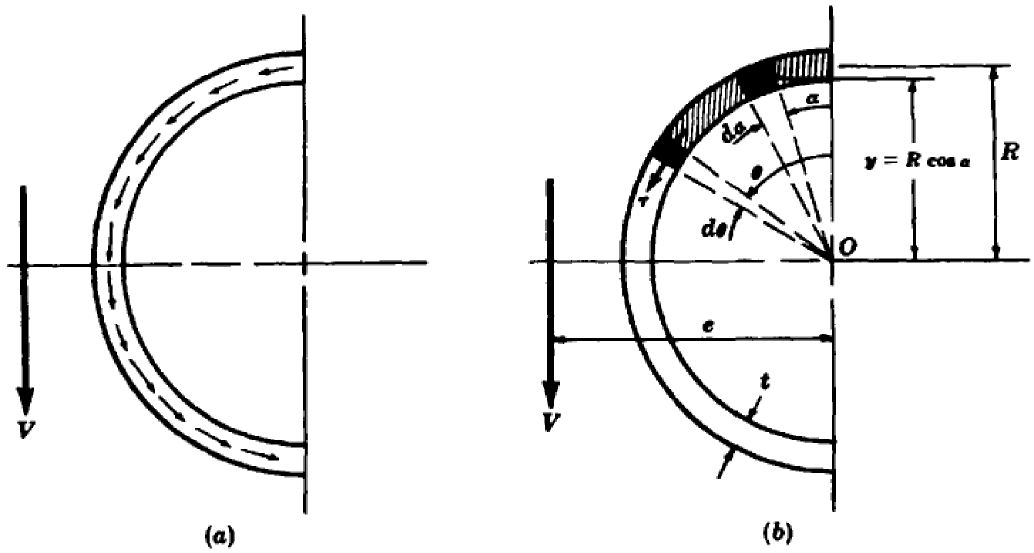



**Fig. 12-3**

Let us examine the shearing stress $\tau$ at an arbitrary point denoted by the angle $\theta$, as indicated in Fig. 12-3($b$). Determination of this stress from the relation

$$\tau = \frac{V}{Ib}\int_{y_0}^{c} y\,da \tag{a}$$

necessitates evaluation of $I$ as well as the integral, which, as explained in Chap. 7, represents the first moment of the shaded area about the axis of symmetry. This is accomplished by introducing an auxiliary variable $\alpha$ $(0 < \alpha < \theta)$ as shown in Fig. 12-3($b$) so that

$$\int_{y_0}^{c} y\,da = \int_0^{\theta} (R\cos\alpha)t(R\,d\alpha) = R^2 t\sin\theta \tag{b}$$

Next, the moment of inertia of the entire cross section about the axis of symmetry is given by

$$I = \int y^2\,da = \int_0^{\pi} (R\cos\theta)^2 tR\,d\theta = \frac{\pi R^3 t}{2} \tag{c}$$

The shearing stress at any point represented by $\theta$ is now found from ($a$), ($b$), and ($c$) to be

$$\tau = \frac{V}{(\pi R^3 t/2)t}[R^2 t\sin\theta] = \frac{2V}{\pi Rt}\sin\theta \tag{d}$$

The moment of these distributed shearing stresses about any point, say $O$, must be equal to the moment of the resultant $V$ about that same point. Thus since $\tau$ acts over an area $t(R\,d\theta)$ we have

$$\int_{\theta=0}^{\theta=\pi}\left(\frac{2V}{\pi Rt}\sin\theta\right)(Rt\,d\theta)R = Ve$$

Thus

$$e = \frac{4R}{\pi}$$

gives the location of the shear center.

**12.2.** Determine the shear center of the "hat"-type thin-walled section indicated in Fig. 12-4. The thickness $t$ is constant throughout the beam.

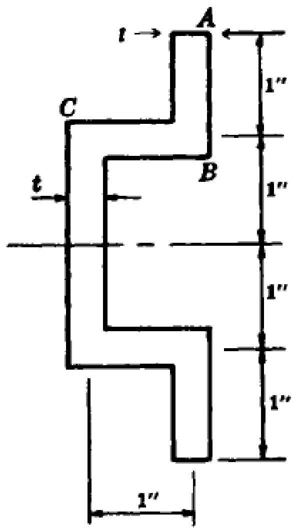



Fig. 12-4

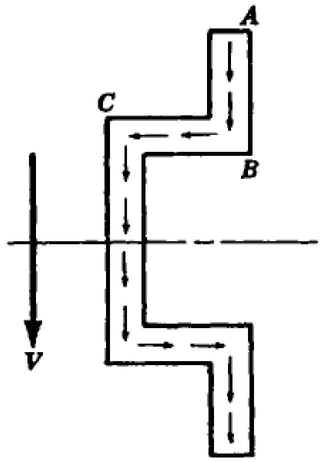



Fig. 12-5

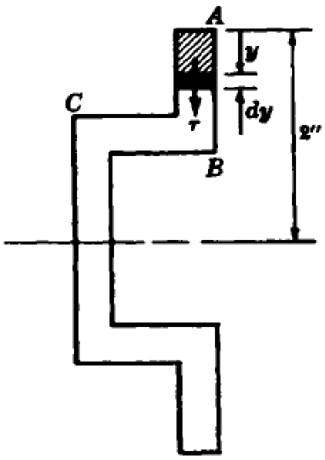



Fig. 12-6

In accordance with the reasoning given in Problem 12.1, the distribution of shear stresses over the cross section appears as in Fig. 12-5. The resultant of the distributed shearing stresses, denoted $V$, acts vertically because the net horizontal effect of the shearing stresses in the two horizontal portions of the "hat" is zero. Let us first examine the shearing stress in the upper vertical member $AB$. At a distance $y$ below the extreme point $A$, as shown in Fig. 12-6, the shearing stress is given by

$$\tau = \frac{V}{It}\int_{y_0}^{c} y\,da \tag{a}$$

The integral represents the first moment of the shaded area about the axis of symmetry and may be readily evaluated as the product of the area, that is, $yt$, and the distance from the centroid of the area to the axis of symmetry, that is, $2 - y/2$. The shear stress at $y$ is thus

$$\tau = \frac{V}{It}\left(2 - \frac{y}{2}\right)yt \tag{b}$$

where it is to be remembered that $V$ and $I$ pertain to the shear force acting over the *entire* cross section and the moment of inertia of the *entire* cross section, respectively. The resultant shear force $V_1$ acting over the vertical region $AB$, as indicated in Fig. 12-7, is found by integration to be

$$V_1 = \int_{y=0}^{y=1}\tau t\,dy = \frac{Vt}{I}\int_0^1\left(2y - \frac{y^2}{2}\right)dy = \frac{5}{6}\frac{Vt}{I} \tag{c}$$

Let us next examine the shearing stress in the upper horizontal member $BC$. At a distance $x$ from point $B$, as indicated in Fig. 12-8, the shearing stress is given by Eq. ($a$), where now the integral represents the first moment of the shaded area in Fig. 12-8 about the axis of symmetry. By inspection the integral has the value $(1)(t)(1.5)+(x)(t)(1)$ and the shear stress at $x$ is thus

$$\tau=\frac{V}{It}[1.5t+xt] \tag{d}$$

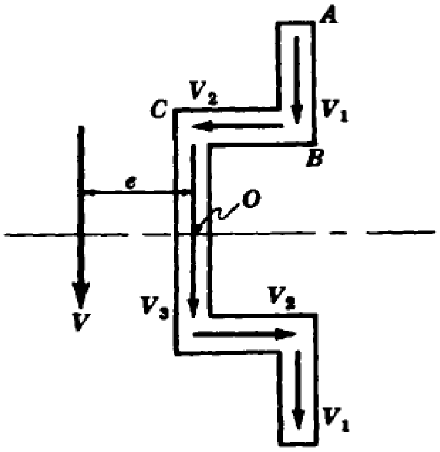



Fig. 12-7

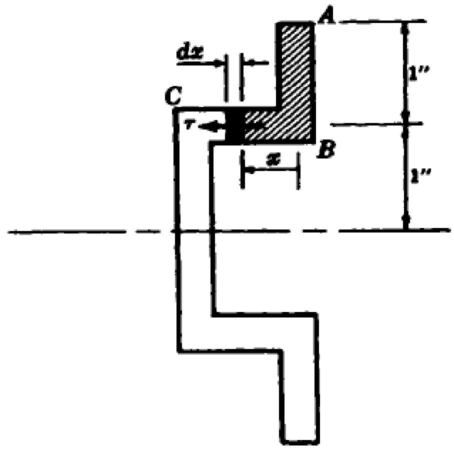



Fig. 12-8

where $V$ and $I$ again pertain to the resultant shear over the *entire* cross section and the moment of inertia of the *entire* cross section, respectively. The resultant shear force $V_2$, as indicated in Fig. 12-7, is found to be

$$V_2=\int_{x=0}^{x=1}\tau t\,dx=\frac{Vt}{I}\int_0^1(1.5+x)\,dx=\frac{3Vt}{2I} \tag{e}$$

Since the entire section is thin walled it is customary to use only nominal dimensions and thus neglect any slight duplication of areas at the intersections of the various members.

Because of symmetry the forces on the lower members are identical to those just found. The sum of the moments of these forces about any point, such as $O$ in Fig. 12-7, must equal the moment of the resultant $V$ about that same point. Thus, we have $-2V_1(1)+2V_2(1)=Ve$ or

$$e=\frac{4t}{3I} \tag{f}$$

Finally, $I$ may be calculated by the methods of Chap. 7 to be

$$I=\frac{1}{12}(t)(4)^3+2[(1)(t)(1)^2]=\frac{22t}{3} \tag{g}$$

The shear center from ($f$) thus becomes

$$e=\frac{4t}{3(22t/3)}=\frac{4}{22}=0.182\text{ in} \tag{h}$$

Note that by choosing the moment center at $O$ it is not necessary to determine $V_3$.

**12.3.** Determine the shear center of a thin-walled rectangular section in which there is a narrow longitudinal slit (see Fig. 12-9). The thickness $t$ is constant.

Observe that this section corresponds to the "hat" section of Problem 12-2 except that the outstanding flanges of the "hat" are turned toward the axis of symmetry here. The distribution of shear stresses appears

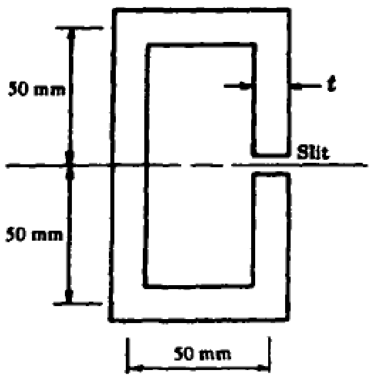



Fig. 12-9

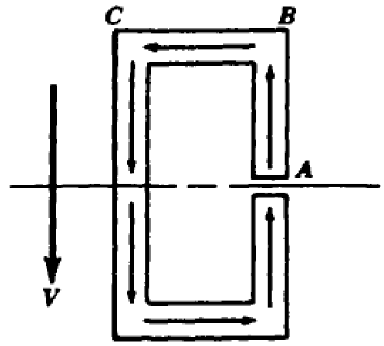



Fig. 12-10

as indicated in Fig. 12-10 and the vertical force $V$ denotes the resultant of these distributed shearing stresses. Let us first examine the shearing stress in the vertical member $AB$. See Fig. 12-11. At a distance $z$ above the axis of symmetry (assuming the slit to be of negligible thickness) the shearing stress is again given by

$$\tau = \frac{V}{Ib}\int_{y_0}^{c} y\,da \tag{a}$$

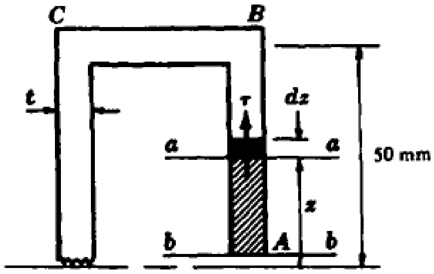



Fig. 12-11

where it is of utmost importance to observe that the integral represents the first moment of the area lying *between* the section $a$-$a$ where the shear stress is desired and the extreme fibers $b$-$b$ of the section. This is true even though fibers $b$-$b$ lie *closer* to the axis of symmetry than $a$-$a$. This statement follows from the derivation of the above equation as given in Chap. 7. The integral is evaluated as the product of the area, that is, $zt$, and the distance from the centroid of the area to the axis of symmetry, that is, $z/2$. The shear stress at $z$ is thus

$$\tau = \frac{V}{It}\left[zt\frac{z}{2}\right] = \frac{Vz^2}{2I} \tag{b}$$

The resultant shear force $V_1$ acting over the vertical region $AB$, indicated in Fig. 12-12, is found by integration to be

$$V_1 = \int_{z=0}^{z=50} \tau t\,dz = \int_0^{50} \frac{Vz^2}{2I} t\,dz = 2.08 \times 10^4 \frac{Vt}{I} \tag{c}$$

Let us next examine the shearing stress in the upper horizontal member $BC$. At a distance $x$ from point $B$, as indicated in Fig. 12-13, the shearing stress is given by Eq. ($a$) where the integral represents the first moment of the shaded area in Fig. 12-13 about the axis of symmetry. From ($a$),

$$\tau = \frac{V}{It}[(x)(t)50 + (50)(t)(25)] = \frac{50V}{I}(x + 25) \tag{d}$$

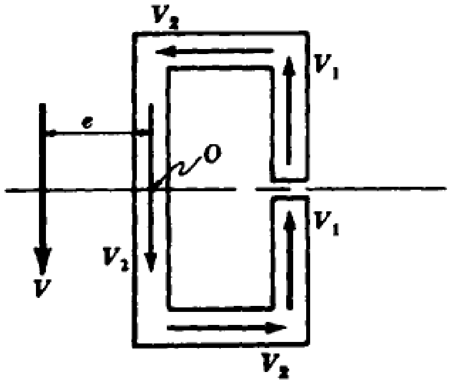



Fig. 12-12

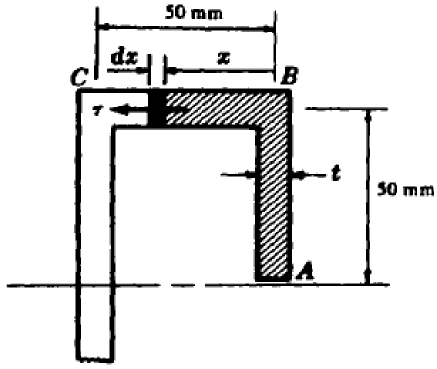



Fig. 12-13

The resultant shear force $V_2$ acting over the horizontal member $BC$, as indicated in Fig. 12-12, is found by integration to be

$$V_2 = \int_{x=0}^{x=50} \tau t\,dx = \int_0^{50} \frac{50V}{I}(x+25)\,dx = 1.25\times 10^5\frac{Vt}{I} \tag{e}$$

From Fig. 12-12 the sum of the moments of the forces $V_1$, $V_2$, and $V_3$ about any point, such as $O$, must equal the moment of the resultant about that point. Thus $2(50V_1) + 2(50V_2) = Ve$.

Substituting from ($c$) and ($e$),

$$2.1\times 10^5\frac{Vt}{I} + 1.25\times 10^7\frac{Vt}{I} = Ve \tag{f}$$

$$e = 1.46\times 10^7\frac{t}{I} \tag{g}$$

The second moment of area is given by

$$I = 2[\tfrac{1}{12}(t)(100)^3] + 2[50t(50)^2] = 4.167\times 10^5 t$$

Thus
$$e = \frac{1.46\times 10^7 t}{4.167\times 10^5 t} = 35 \text{ mm}$$

which locates the shear center.

**12.4.** Determine the shear center of the thin-walled section indicated in Fig. 12-14. The thickness $t$ is constant.

The distribution of shear stresses appears as in Fig. 12-15 where the vertical force $V$ denotes the resultant of these distributed shearing stresses. Let us first determine the shearing stress in the horizontal member $AB$. At a distance $x$ from point $A$, as indicated in Fig. 12-16, the shearing stress is found to be

$$\tau = \frac{V}{Ib}\int_{y_0}^{c} y\,da \tag{a}$$

or
$$\tau = \frac{V}{It}[(x)(t)(3)] = \frac{3Vx}{I} \tag{b}$$

The resultant shear force $V_1$ acting over $AB$, as indicated in Fig. 12-17, is found by integration to be

$$V_1 = \int_{x=0}^{x=2} \tau t\,dx = \int_0^2 \frac{3Vxt}{I}\,dx = \frac{6Vt}{I} \tag{c}$$

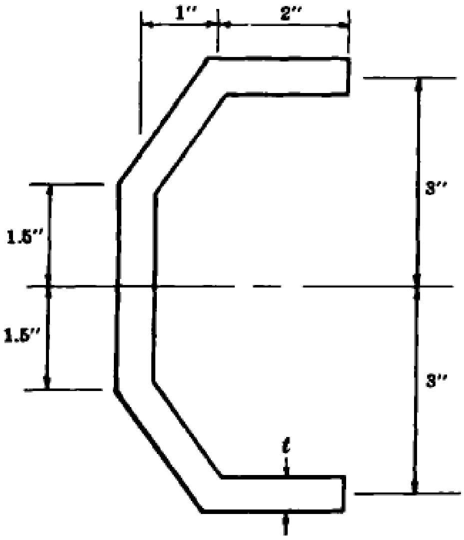



Fig. 12-14

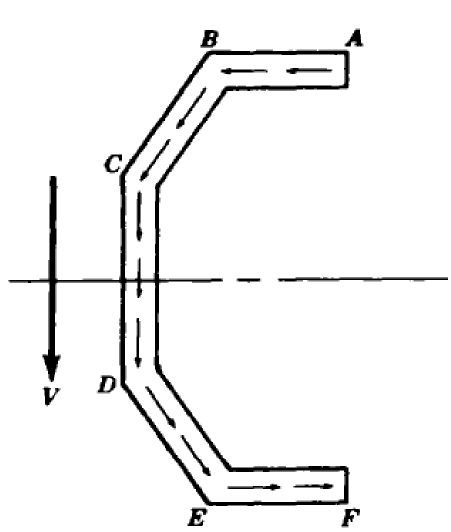



Fig. 12-15

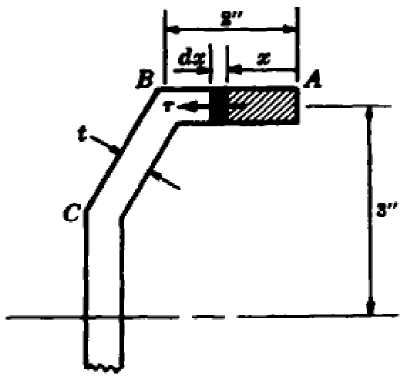



Fig. 12-16

The shearing stress in the inclined member $BC$ at a distance $y$ from point $B$, as indicated in Fig. 12-18, is again given by Eq. ($a$), where the integral represents the first moment of the shaded area in Fig. 12-18 about the axis of symmetry. For the inclined portion of that area, it is simplest to integrate through introduction of an auxiliary variable $u$ as indicated. Thus

$$\tau = \frac{V}{It}\left[(2)(t)(3) + \int_{u=0}^{u=y} [1.5 + (1.80 - u)\sin 56°20']t\,du\right]$$

$$= \frac{V}{I}(6 + 3y - 0.416y^2) \qquad (d)$$

The resultant shear force $V_2$ acting over the inclined member $BC$ in Fig. 12-17 is found by integration to be

$$V_2 = \int_{y=0}^{y=1.80} \tau t\,dy$$

$$= \int_0^{1.80} \frac{Vt}{I}(6 + 3y - 0.416y^3)\,dy = \frac{14.85Vt}{I} \qquad (e)$$

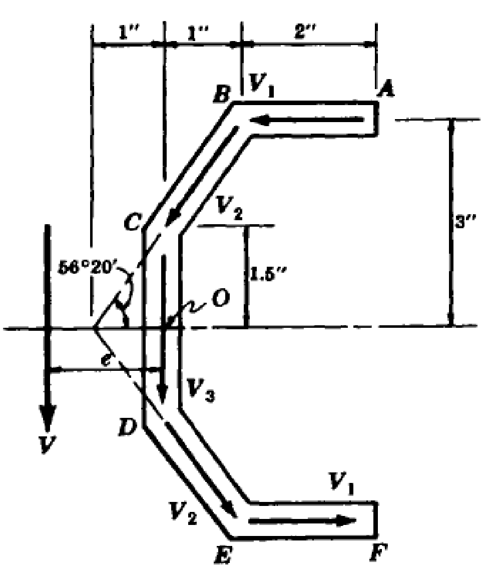



Fig. 12-17

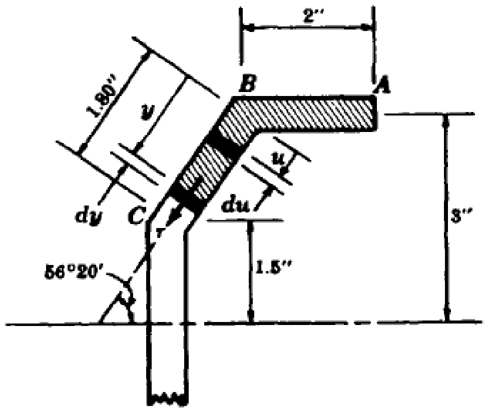



Fig. 12-18

From Fig. 12-17 the sum of the moments of the forces $V_1$, $V_2$, and $V_3$ about any point, such as $O$, must equal the moment of the resultant about that point. Thus

$$2(3V_1) + 2(V_2 \sin 56°20')(1) = Ve$$

Substituting from ($c$) and ($e$),

$$e = \frac{60.8t}{I} \tag{f}$$

The moment of inertia is given by

$$I = \frac{1}{12}(t)(3)^3 + 2[(2)(t)(3)^2] + 2\int_{u=0}^{u=1.80} [1.5 + (1.80 - u)\sin 56°20']^2 t\, du$$

We then have

$$e = \frac{60.8t}{57.2t} = 1.06 \text{ in}$$

which locates the shear center.

### *Unsymmetric Bending*

**12.5.** Consider a beam of arbitrary unsymmetric cross section subject to pure bending, as indicated in Fig. 12-19($a$). Derive an expression for the relationship between the bending moment and the bending stress at any point in this section. Assume Hooke's law holds.

It is convenient to resolve the moment $M$, which acts in a plane oblique to the $y$- and $z$-axes (through the centroid), into moment components about those axes. These components are designated as $M_y$ and $M_z$ and their positive directions are indicated by the double-headed vectors in Fig. 12-19($b$).

As in Problem 8.1 it is reasonable to assume that cross sections that were plane prior to bending remain plane after application of the loads. However, in the general case being considered here there is one radius of curvature $\rho_z$ in the $x$-$y$ plane and another $\rho_y$ in the $x$-$z$ plane. Thus, for a longitudinal fiber of area $da$ as indicated in Fig. 12-19($b$) the normal strain, analogous to ($1$) of Problem 8.1, is given by

$$\varepsilon = \frac{y}{\rho_z} + \frac{z}{\rho_y} \tag{1}$$

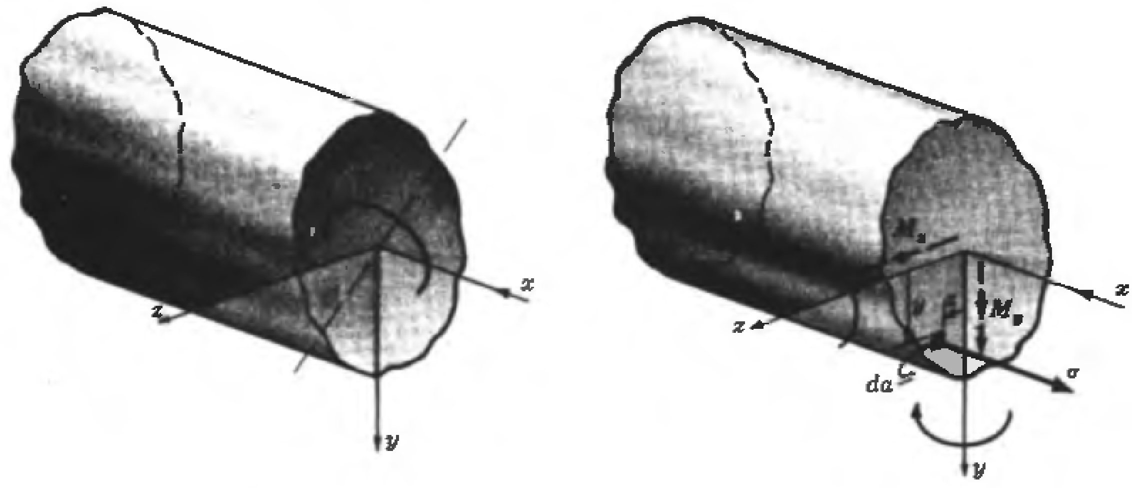



**Fig. 12-19**

Since Hooke's law holds, we immediately have

$$\sigma = \frac{Ey}{\rho_z} + \frac{Ez}{\rho_y} \tag{2}$$

and this longitudinal, or bending, stress is indicated in the figure.

The resultant longitudinal force acting over the cross section is zero (for the case of pure bending) and this condition may be expressed as

$$\int_A \sigma\, da = 0 \qquad \text{or} \qquad \int_A \left(\frac{Ey}{\rho_z} + \frac{Ez}{\rho_y}\right) da = 0$$

where the integration is extended over the cross-sectional area $A$. Since $\rho_y$ and $\rho_z$ are constant over the cross section, we have

$$\frac{E}{\rho_z}\int_A y\, da + \frac{E}{\rho_y}\int_A z\, da = 0 \tag{3}$$

This equation is satisfied if the integrals vanish. This implies taking the origin of the $y$-$z$ coordinate system to coincide with the centroid of the cross section.

From Fig. 12-19($b$) it is evident that

$$M_z = \int_A \sigma y\, da = \int_A \left(\frac{Ey^2}{\rho_z} + \frac{Eyz}{\rho_y}\right) da$$

$$= \frac{E}{\rho_z}\int_A y^2\, da + \frac{E}{\rho_y}\int_A yz\, da$$

where the first integral represents the moment of inertia of the cross-sectional area about the $z$-axis and the second integral (as mentioned in Chap. 7) represents the product of inertia of the same area about the $y$- and $z$-axes. Using the notation of Chap. 7, this last equation becomes

$$M_z = \frac{EI_z}{\rho_z} + \frac{EI_{yz}}{\rho_y} \tag{4}$$

Also from Fig. 12-19($b$) we have

$$M_y = -\int_A \sigma z\, da = -\int_A \left(\frac{Eyz}{\rho_z} + \frac{Ez^2}{\rho_y}\right) da$$

$$= -\frac{EI_{yz}}{\rho_z} + \frac{EI_y}{\rho_y} \tag{5}$$

Equations (4) and (5) may be solved for $\rho_y$ and $\rho_z$ to yield

$$\frac{1}{\rho_y} = \frac{-M_y I_z - M_z I_{yz}}{E(I_y I_z - I_{yz}^2)} \tag{6}$$

$$\frac{1}{\rho_z} = \frac{M_z I_y + M_y I_{yz}}{E(I_y I_z - I_{yz}^2)} \tag{7}$$

Substituting (6) and (7) in (2) yields the bending stress

$$\sigma = \frac{(M_z I_y + M_y I_{yz})y + (-M_y I_z - M_z I_{yz})z}{I_y I_z - I_{yz}^2} \tag{8}$$

Equation (8) is termed the *generalized flexure formula* and holds for an elastic beam of arbitrary cross section with bending loads in an arbitrary plane. For the special case $M_y = I_{yz} = 0$ (implying that the $y$- and $z$-axes are principal axes and that bending takes place only about the $z$-axis) (8) reduces to $\sigma = M_z y / I_z$, which is equivalent to (9) of Problem 8.1.

The equation of the neutral axis is readily found by setting the stress from (8) equal to zero, since by definition the fibers along the neutral axis are free of longitudinal stress. Thus

$$\frac{y}{z} = \frac{M_y I_z + M_z I_{yz}}{M_z I_y + M_y I_{yz}} = \tan\alpha \tag{9}$$

where $\alpha$ denotes the angle of inclination of the neutral axis as indicated in Fig. 12-20. In general the neutral axis is *not* perpendicular to the plane of the applied moments nor does it coincide with either of the principal axes.

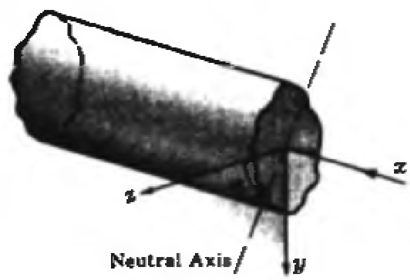



**Fig. 12-20**

**12.6.** The rectangular beam of Fig. 12-21 is subject to loads that create a bending moment of 2000 lb · ft acting in a plane oriented at 30° to the $y$-axis. Determine the peak tensile and compressive stresses in the beam.

The vector representation of the 2000 lb · ft moment is indicated by the solid double-headed vector in Fig. 12-22, together with its moment components (dashed vectors) in the $y$- and $z$-directions. This convenient vector representation enables us to find the components as

$$M_y = 2000 \sin 30° = 1000 \text{ lb} \cdot \text{ft} \qquad M_z = 2000 \cos 30° = 1732 \text{ lb} \cdot \text{ft}$$

From Problem 7.3, we have

$$I_y = \tfrac{1}{12}(6)(3)^3 = 13.5 \text{ in}^4 \qquad I_z = \tfrac{1}{12}(3)(6)^3 = 54 \text{ in}^4$$

Also, since the $y$- and $z$-axes are axes of symmetry, they are principal axes of the cross section and, from Chap. 7, the product of inertia with respect to these axes vanishes: $I_{yz} = 0$.

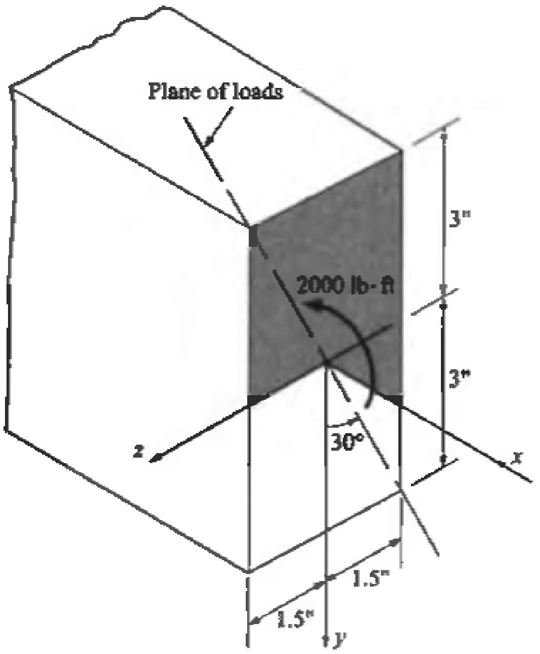



Fig. 12-21

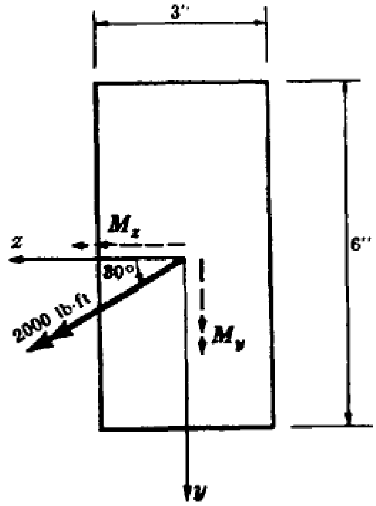



Fig. 12-22

The angle of inclination of the neutral axis (which passes through the centroid) is given by (9) of Problem 12.5 to be

$$\tan\alpha = \frac{M_y I_z + M_z I_{yz}}{M_z I_y + M_y I_{yz}}$$

$$= \frac{(1000)(54) + (1732)(0)}{(1732)(13.5) + 1000(0)} = 2.31$$

$$\alpha = 66°40'$$

As mentioned in Problem 12.5, there is no reason to expect the neutral axis, as indicated in Fig. 12-23, to be normal to the plane of the loads.

In Problem 12.5, it was assumed that plane sections remain plane during bending. The originally plane section rotates about the neutral axis indicated in Fig. 12-23 and since both strains as well as stresses vary as the distance from the neutral axis it is evident that the peak tensile stress occurs at point $B$ and the peak compressive stress occurs at $A$, i.e., *at those points most remote from the neutral axis.* Substituting the

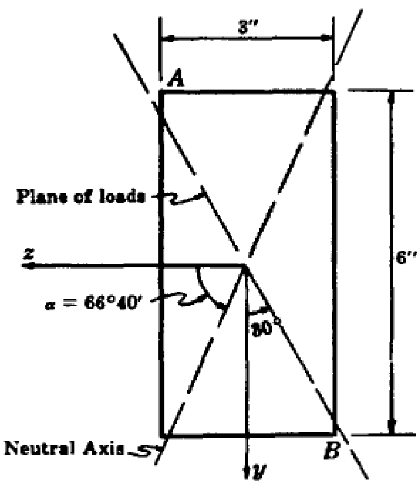



Fig. 12-23

coordinates of these points and the values of the moment components in (8) of Problem 12.5, we obtain

$$\sigma_B = \frac{[(1732)(12)(13.5)+0](3)+[-(1000)(12)(54)-0](-15)}{(13.5)(54)-0} = 2480\ \text{lb/in}^2$$

$$\sigma_A = \frac{[(1732)(12)(13.5)+0](-3)+[-(1000)(12)(54)-0](1.5)}{(13.5)(54)-0} = -2480\ \text{lb/in}^2$$

**12.7.** The structural angle section designated as L127 × 127 × 22.2 has the dimensions and centroidal axis indicated in Fig. 12.24. The values of the cross-sectional properties with respect to the centroidal axis of the section are $I_y = I_z = 7.41 \times 10^{-6}\ \text{m}^4$ and $I_{yz} = -4.201 \times 10^{-6}\ \text{m}^4$. For a loading $M_y = 0$, $M_z = 10\ \text{kN}\cdot\text{m}$, find the angle of inclination of the neutral axis and the bending stress at point $A$.

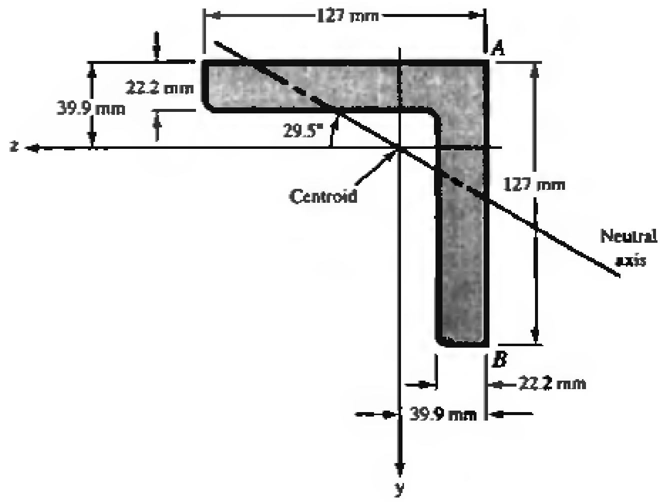



**Fig. 12-24**

The angle of inclination of the neutral axis is given by Eq. (9) of Problem 12.5 as

$$\tan\alpha = \frac{0 + M_z(-4.201 \times 10^{-6}\ \text{m}^4)}{M_z(7.41 \times 10^{-6}\ \text{m}^4) + 0}$$
$$= -0.567$$
$$\alpha = -29.5°$$

which is shown in Fig. 12-24. The minus sign indicates clockwise rotation from the positive end of the $z$-axis because the positive direction of $\alpha$ was taken to be counterclockwise as indicated in Fig. 12-20.

Point $A$ has coordinates $y = z = -39.9$ mm so that the desired stress at that point from Eq. (8) of Problem 12.5 is

$$\sigma = \frac{[(10{,}000\ \text{N}\cdot\text{m})(7.41\times10^{-6}\ \text{m}^4)-0](0.0399\ \text{m})+[0-(10{,}000\ \text{N}\cdot\text{m})(-4.201\times10^{-6}\ \text{m}^4)(-0.0399\ \text{m})}{(7.41\times10^{-6}\ \text{m}^4)(7.41\times10^{-6}\ \text{m}^4)-(-4.201\times10^{-6}\ \text{m}^4)^2}$$
$$= -124\ \text{MPa}$$

**12.8.** Write a computer program in FORTRAN to determine elastic bending stresses as well as the orientation of the neutral axis in a beam of unsymmetric cross section subject to pure bending as shown in Fig. 12-19.

The desired stress is given by Eq. (8) in Problem 12.5 and the angular orientation of the neutral axis is indicated by Eq. (9) of that problem. The components of moment $M_y$ and $M_z$ have the positive directions shown in Fig. 12-19 and all other symbols are defined in Problem 12.5. The program is

```
*************************************************************************
                             PROGRAM BEND (INPUT,OUTPUT)
*************************************************************************
*
*          AUTHOR: KATHLEEN DERWIN
*          DATE  : JANUARY 27,1989
*
*    BRIEF DESCRIPTION:
*        THIS PROGRAM CONSIDERS A BEAM OF ARBITRARY UNSYMMETRIC CROSS
*    SECTION SUBJECTED TO PURE BENDING. THE GENERALIZED FLEXURE FORMULA
*    HOLDS FOR THIS CASE, AND PROVIDES A RELATIONSHIP BETWEEN THE BENDING
*    MOMENT AND THE BENDING STRESS AT ANY POINT IN THE SECTION. ALSO,
*    THE ANGLE OF INCLINATION OF THE NEUTRAL AXIS CAN BE CALCULATED AS A
*    FUNCTION OF THE BENDING MOMENTS.
*
*    INPUT:
*        THE USER IS FIRST ASKED IF USCS OR SI UNITS WILL BE USED. THEN,
*    THE SECTIONAL PROPERTIES (MOMENTS OF INERTIA IY,IZ,IYZ) ARE INPUTTED,
*    AS WELL AS THE BENDING MOMENTS. FINALLY, THE COORDINATES OF THE POINT
*    WHERE THE BENDING STRESS IS DESIRED ARE ENTERED.
*
*    OUTPUT:
*        THE BENDING STRESS AT ANY POINT ON THE CROSS SECTION MAY BE
*    OBTAINED, AS WELL AS THE ANGLE OF INCLINATION OF THE NEUTRAL AXIS.
*
*    VARIABLES:
*     IY,IZ,IYZ     ---     SECTIONAL PROPERTIES (MOMENTS OF INERTIA)
*      MY,MZ        ---     BENDING MOMENTS
*      SIGMA        ---     BENDING STRESS AT THE DESIRED POINT ON THE SECTI
*     TALPHA      ---     THE TANGENT OF THE ANGLE OF INCLINATION OF THE
*                         NEUTRAL AXIS
*      ALPHA      ---     THE ANGLE OF INCLINATION OF THE NEUTRAL AXIS
*      Y,Z          ---     COORDINATE OF THE POINT WHERE STRESS DETERMINATI
*                           IS DESIRED
*      ANS          ---     DENOTES IF USCS OR SI UNITS ARE TO BE USED
*      UNIT         ---     GIVES THE USCS OR SI UNIT FOR STRESS
*
***********************************************************************
******                        MAIN PROGRAM                        ******
***********************************************************************
*
*          VARIABLE DECLARATIONS
*
      REAL IY,IZ,IYZ,SIGMA,MY,MZ,TALPHA,ALPHA
      INTEGER ANS
      CHARACTER UNIT*4
*
*          USER INPUT
*
      PRINT*,'PLEASE INDICATE YOUR CHOICE OF UNITS:'
      PRINT*,'1 - USCS'
      PRINT*,'2 - SI'
      PRINT*,' '
      PRINT*,'ENTER 1,2'
      READ*,ANS
      PRINT*,' '
      PRINT*,' '
      PRINT*,'NOTE, THE COORDINATE SYSTEM USED HAS THE X-AXIS ORIENTED'
      PRINT*,'SO THAT IT IS POSITIVE INTO THE PAGE AND ACTING AS THE'
      PRINT*,'NEUTRAL AXIS OF THE SECTION. THE POSITIVE Y-AXIS IS DIRECTED'
      PRINT*,'DOWNWARD, WHILE THE POSITIVE Z-AXIS IS TO THE LEFT AS ONE'
```

```
      PRINT*,'FACES THE SECTION. (IT IS A RIGHT HANDED SYSTEM.)'
      PRINT*,' '
      PRINT*,' '
      IF (ANS.EQ.1) THEN
         PRINT*,'PLEASE ENTER THE SECTION PROPERTIES IY,IZ,IYZ ,(IN^4):'
         READ*,IY,IZ,IYZ
         PRINT*,' '
         PRINT*,'PLEASE ENTER THE MAGNITUDE OF THE BENDING MOMENTS MY,MZ'
         PRINT*,'FOLLOWING THE SIGN CONVENTION STATED (LB-FT):'
         READ*,MY,MZ
         MY = MY*12
         MZ = MZ*12
      ELSE
         PRINT*,'PLEASE ENTER THE SECTION PROPERTIES IY,IZ,IYZ ,(MM^4):'
         READ*,IY,IZ,IYZ
         PRINT*,' '
         PRINT*,'PLEASE ENTER THE MAGNITUDE OF THE BENDING MOMENTS MY,MZ'
         PRINT*,'FOLLOWING THE SIGN CONVENTION STATED (KN-M):'
         READ*,MY,MZ
         MY = MY*1E6
         MZ = MZ*1E6
      ENDIF
*
      PRINT*,' '
      PRINT*,'ENTER THE Y AND Z COORDINATES OF THE POINT WHERE STRESS '
      PRINT*,'DETERMINATION IS DESIRED.(FOLLOW THE SIGN CONVENTION STATED
      IF(ANS.EQ.1) THEN
         PRINT*,'Y AND Z ARE DISTANCES IN INCHES FROM THE NEUTRAL AXIS:'
      ELSE
         PRINT*,'Y AND Z ARE DISTANCES IN MILLIMETERS FROM NEUTRAL AXIS:'
      ENDIF
      READ*,Y,Z
*
*             END USER INPUT
*********************************************************
*             CALCULATIONS FOR BENDING STRESS AND THE ANGLE OF INCLINATION
*             AS FUNCTIONS OF THE APPLIED BENDING MOMENTS AND THE SECTION
*             PROPERTIES
*
      SIGMA=(((MZ*IY + MY*IYZ)*Y) +((-MY*IZ - MZ*IYZ)*Z))/(IY*IZ - IYZ**2
      TALPHA =((MY*IZ + MZ*IYZ)/(MZ*IY + MY*IYZ))
      ALPHA  = ATAN(TALPHA)
      ALPHA = ALPHA*180/3.14159
*
*           PRINTING OUTPUT
*
      IF (ANS.EQ.1) THEN
         UNIT = ' PSI'
      ELSE
         UNIT = ' MPA'
      ENDIF
     PRINT 10,'THE BENDING STRESS AT (',Y,',',Z,') IS',SIGMA,UNIT,'.'
     PRINT 20,'THE ANGLE OF INCLINATION OF THE NEUTRAL AXIS IS',ALPHA,'DEG.'
*
*           FORMAT STATEMENTS
*
   10 FORMAT(//,2X,A23,F6.1,A1,F6.1,A4,F10.1,A4,A1)
   20 FORMAT(/,2X,A,F8.2,1X,A)
*
      STOP
      END
```

**12.9.** Rework Problem 12.6 using the FORTRAN program in Problem 12.8.

The self-prompting program is utilized by entering the moment components and sectional properties from Problem 12.6. Consideration of the directions of moment components indicates that the peak tensile stress will occur at point $B$ in Fig. 12-23 and the coordinates of that point are $y = 3$, $z = -1.5$. The printout is

```
run
 PLEASE INDICATE YOUR CHOICE OF UNITS:
 1 - USCS
 2 - SI

 ENTER 1,2
? 1


 NOTE, THE COORDINATE SYSTEM USED HAS THE X-AXIS ORIENTED
 SO THAT IT IS POSITIVE INTO THE PAGE AND ACTING AS THE
 NEUTRAL AXIS OF THE SECTION. THE POSITIVE Y-AXIS IS DIRECTED
 DOWNWARD, WHILE THE POSITIVE Z-AXIS IS TO THE LEFT AS ONE
 FACES THE SECTION. (IT IS A RIGHT HANDED SYSTEM.)


 PLEASE ENTER THE SECTION PROPERTIES IY,IZ,IYZ ,(IN^4):
? 13.5,54,0

 PLEASE ENTER THE MAGNITUDE OF THE BENDING MOMENTS MY,MZ
 FOLLOWING THE SIGN CONVENTION STATED (LB-FT):
? 1000,1732

 ENTER THE Y AND Z COORDINATES OF THE POINT WHERE STRESS
 DETERMINATION IS DESIRED.(FOLLOW THE SIGN CONVENTION STATED)
 Y AND Z ARE DISTANCES IN INCHES FROM THE NEUTRAL AXIS:
? 3,-1.5


  THE BENDING STRESS AT (    3.0,   -1.5) IS     2488.0 PSI.

  THE ANGLE OF INCLINATION OF THE NEUTRAL AXIS IS    66.59 DEG.

SRU       0.895 UNTS.

 RUN COMPLETE.
```

**12.10.** Rework Problem 12.7 using the FORTRAN program of Problem 12.8.

Enter the given cross-sectional properties, moment components, and coordinates of point $A$ indicated in Problem 12.8 into the self-prompting program to obtain the following printout, which agrees with the results of Problem 12.7

```
run
 PLEASE INDICATE YOUR CHOICE OF UNITS:
 1 - USCS
 2 - SI

 ENTER 1,2
? 2


 NOTE, THE COORDINATE SYSTEM USED HAS THE X-AXIS ORIENTED
 SO THAT IT IS POSITIVE INTO THE PAGE AND ACTING AS THE
```

```
 NEUTRAL AXIS OF THE SECTION. THE POSITIVE Y-AXIS IS DIRECTED
 DOWNWARD, WHILE THE POSITIVE Z-AXIS IS TO THE LEFT AS ONE
 FACES THE SECTION. (IT IS A RIGHT HANDED SYSTEM.)


 PLEASE ENTER THE SECTION PROPERTIES IY,IZ,IYZ ,MM^4):
? 7.41E+6,7.41E+6,-4.201E+6

 PLEASE ENTER THE MAGNITUDE OF THE BENDING MOMENTS MY, MZ
 FOLLOWING THE SIGN CONVENTION STATED (KN-M):
? 0.10

 ENTER THE Y AND Z COORDINATES OF THE POINT WHERE STRESS
 DETERMINATION IS DESIRED. (FOLLOW THE SIGN CONVENTION STATED)
 Y AND Z ARE DISTANCES IN MILLIMETERS FROM NEUTRAL AXIS:
? -39.9,-39.9

  THE BENDING STRESS AT ( -39.9, -39.9) IS    -124.3 MPA.

  THE ANGLE OF INCLINATION OF THE NEUTRAL AXIS IS  -29.55 DEG.
```

### *Curved Beams*

**12.11.** Consider the bending of an initially curved elastic beam for which the plane of curvature is also a plane of symmetry of every cross section. The bending loads act in this plane of symmetry. Derive an expression for the relationship between the bending moment and the bending stress at any point in the cross section. Assume Hooke's law holds.

The beam is illustrated in Fig. 12-25, where $R$ denotes the distance from the center of curvature $C$ to the axis through the centroid of the cross section. The bending moment $M$ is taken to be positive in the direction indicated, i.e., when it tends to *increase* the curvature (*decrease* the radius of curvature).

Let us examine the behavior of a part of the beam corresponding to a central angle $d\theta$ before deformation. After deformation, this angle changes to $d\theta + \Delta d\theta$, as shown in Fig. 12-26. Just as in the case of the initially straight beam studied in Problem 8.1, we will assume that plane cross sections originally perpendicular to the geometric axis of the beam remain plane after bending. Thus, the normal section $CD$ prior to loading moves to $C'D'$ after loading. For convenience we shall assume that $AB$ remains fixed in space but this in no way influences the results we will obtain. It will still be assumed that there exists one axis, the neutral axis, for which the longitudinal fibers do not change length, and thus the section $CD$ may be considered to rotate about this neutral axis as indicated in Fig. 12-26. However, there is no reason to believe that the neutral axis coincides with the centroid of the cross section as it did for the initially straight beam in Problem 8.1. In the present problem involving the curved beam, Fig. 12.26 indicates that the *total elongation* of a longitudinal fiber varies as the distance $y$ of the fiber from the neutral axis. The coordinate $y$ is measured positive away from the center of curvature. However, the lengths of these fibers prior to loading are obviously different; hence the *unit* elongations, i.e., normal strains, are *not* proportional to the distances from the neutral axis. This point constitutes the fundamental difference between behavior of a curved beam and behavior of the initially straight beam discussed in Problem 8.1. Since Hooke's law is assumed to hold for this curved beam, it follows that stresses on these fibers are *not* proportional to the distances from the neutral axis.

Let us consider the elongation of the fiber at a distance $y$ from the neutral axis. From Fig. 12-26 this is $y(\Delta d\theta)$. Dividing this elongation by the original length of the fiber, $(r+y)\,d\theta$, yields the normal strain as

$$\epsilon = \frac{y(\Delta\, d\theta)}{(r+y)\,d\theta} \tag{a}$$

where $r$ denotes the radius of curvature of the neutral axis. Since Hooke's law holds, the normal stress is

$$\sigma = \frac{Ey(\Delta\, d\theta)}{(r+y)\,d\theta} \tag{b}$$

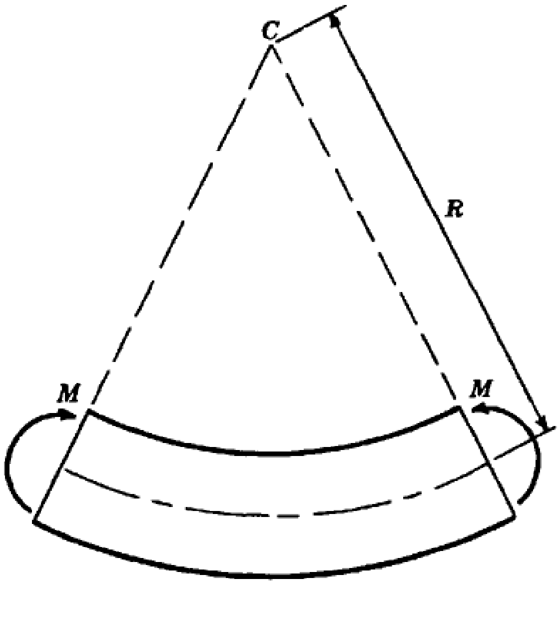



Fig. 12-25

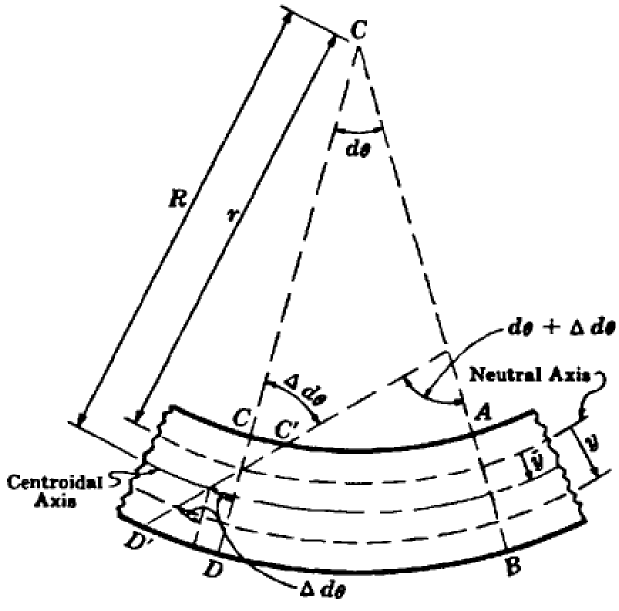



Fig. 12-26

The neutral axis may now be located by requiring the resultant normal force over the cross section to vanish. Thus

$$\int_A \sigma\, da = \int_A \frac{Ey(\Delta\, d\theta)\, da}{(r+y)\, d\theta} = \frac{E(\Delta\, d\theta)}{d\theta} \int_A \frac{y\, da}{(r+y)} = 0 \qquad (c)$$

where the integration is over the entire cross-section area $A$. If $u = r + y$ (i.e., the distance of any fiber from the center of curvature $C$) then ($c$) becomes

$$\int_A \frac{(u-r)\, da}{u} = 0 \qquad \text{or} \qquad r = \frac{A}{\int_A da/u} \qquad (d)$$

where the integral in the denominator represents a mathematical property of the cross-sectional area and is analogous to the moment of inertia that arises in the case of bending of an initially straight beam.

The sum of the moments of the normal forces on the fibers must equal the bending moment:

$$M = \int_A \sigma y\, da = \int_A \frac{Ey^2(\Delta\, d\theta)\, da}{(r+y)\, d\theta} = \frac{E(\Delta\, d\theta)}{d\theta} \int_A \frac{y^2\, da}{r+y}$$

Simplifying,

$$\int_A \frac{y^2\, da}{r+y} = \int_A y\, da - r \int_A \frac{y\, da}{r+y}$$

The first integral represents the first or static moment of the cross-sectional area about the neutral axis, and the second according to ($c$) vanishes. Thus

$$M = \frac{E(\Delta\, d\theta)}{d\theta}[A\bar{y}] \qquad (e)$$

where $\bar{y}$ denotes the distance from the neutral axis to the centroidal axis. Combining ($b$) and ($e$), we find the normal stress on any fiber to be

$$\sigma = \frac{My}{A\bar{y}(r+y)} \qquad (f)$$

From $(f)$ it is evident that the stress distribution across the depth of the curved beam is *hyperbolic*. The maximum stress always occurs at the outer fibers on the concave side of the beam. Further, the neutral axis always lies between the centroidal axis and the center of curvature.

**12.12.** The U-shaped bar of rectangular cross section is loaded by collinear, oppositely directed forces of 9680 N, as shown in Fig. 12-27. The cross-sectional dimensions are 40 mm × 60 mm. The action line of the forces lies 120 mm from the centroid of the cross section. Determine the normal stresses at points $A$ and $B$.

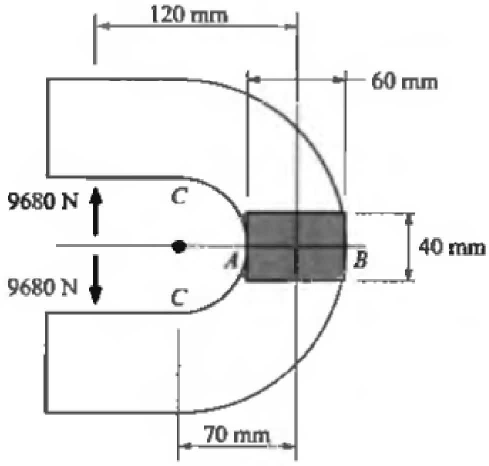



Fig. 12-27

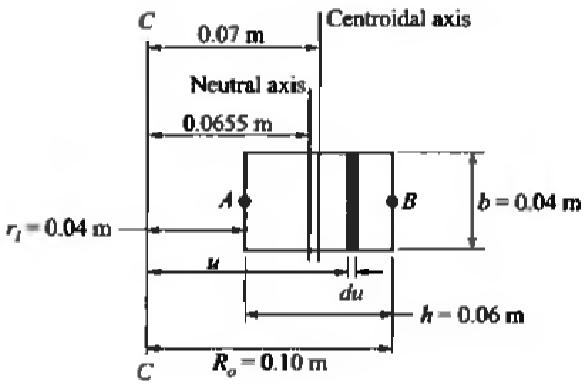



Fig. 12-28

It is first necessary to use Eq. $(d)$ of Problem 12.11 to locate the neutral axis. A horizontal cross section of the system coinciding with points $A$ and $B$ is shown in Fig. 12-28, where the variable $u$ is introduced to carry out the integration in Eq. $(d)$. We have

$$r = \frac{bh}{\int b(du)/u} = \frac{h}{(\ln u)_{r_i}^{R_o}} = \frac{0.6\text{ m}}{\ln(0.1\text{ m}/0.04\text{ m})} = 0.0655\text{ m}$$

as the distance from center of curvature to the neutral axis. The variable $\bar{y}$ is thus $0.07\text{ m} - 0.0655\text{ m} = 0.0045\text{ m}$

The bending stresses are given by Eq. $(f)$ of Problem 12.11, where $M = -(9680\text{ N})(0.12\text{ m}) = -1162\text{ N}\cdot\text{m}$ since the loading tends to decrease the curvature, and thus we must call it negative moment. At point $A$ in Fig. 12-28, we have $y = -0.0255$ m and the bending stress at $A$ is

$$\sigma_A = \frac{(-1162\text{ N}\cdot\text{m})(-0.0255\text{ m})}{(0.06\text{ m})(0.04\text{ m})(0.0045\text{ m})[0.0655\text{ m} - 0.0255\text{ m}]} = 68.6\text{ MPa}$$

At point $B$, we have $y = 0.0345$ m and the bending stress at $B$ is

$$\sigma_B = \frac{(-1162\text{ N}\cdot\text{m})(0.0345\text{ m})}{(0.06\text{ m})(0.04\text{ m})(0.0045\text{ m})[0.0655\text{ m} + 0.0345\text{ m}]} = -37.1\text{ MPa}$$

In addition to these bending stresses, the tensile action of the applied loads on the cross section $A$–$B$ sets up uniform tensile stresses given by

$$\sigma = \frac{P}{A} = \frac{9680\text{ N}}{(0.04\text{ m})(0.06\text{ m})} = 4.03\text{ MPa}$$

The resultant normal stress at point $A$ is thus

$$\sigma'_A = 68.6\text{ MPa} + 4.03\text{ MPa} = 72.63\text{ MPa}$$

and at $B$ it is

$$\sigma'_B = -37.1\ \text{MPa} + 4.03\ \text{MPa} = -33.07\ \text{MPa}$$

**12.13.** Develop a computer program in FORTRAN to determine extreme fiber bending stresses in the curved beam loaded in pure bending as shown in Fig. 12-25.

The general theory given in Problem 12.11 indicates that it is first necessary to determine the location of the neutral axis, which is a distance $r$ from the center of curvature in Fig. 12-26. From Eq. ($d$) of Problem 12.11, $r$ is seen to be a function of the shape of the cross section. From this general expression ($d$) we choose to write a computer program for the three common types of cross section: ($a$) rectangular, ($b$) circular, and ($c$) trapezoidal. The following program carries out the integration of Eq. ($d$) over each of these cross sections, then develops outer fiber stress according to Eq. ($f$) for a pure bending moment loading $M$ as shown in Fig. 12-25, where it must be carefully noted that the moment is negative if it acts so as to reduce the curvature of the beam. The program is

```
************************************************************************
                            PROGRAM CRVBEAM
************************************************************************
*
*            AUTHOR: KATHLEEN DERWIN
*            DATE  : FEBRUARY 5, 1989
*
*   BRIEF DESCRIPTION:
*      THE FOLLOWING PROGRAM CONSIDERS THE BENDING OF AN INITIALLY
*   CURVED ELASTIC BEAM FOR WHICH THE PLANE OF CURVATURE IS ALSO A
*   PLANE OF SYMMETRY AT EVERY CROSS SECTION. THE BENDING LOAD ACTS IN
*   THIS PLANE OF SYMMETRY. THE MAXIMUM BENDING STRESS OCCURS AT THE
*   EXTREME FIBERS OF THE SECTION, AND CAN BE DETERMINED FOR A RECTANGULA
*   CIRCULAR, OR TRAPEZOIDAL CROSS SECTION. NOTE, THE RELATIONSHIP BETWEE
*   THE BENDING MOMENT AND BENDING STRESS INVOLVES TAKING THE NATURAL
*   LOGARITHM OF THE RATIO BETWEEN THE DISTANCE FROM THE CENTER OF CURV-
*   ATURE TO THE OUTER AND INNER EXTREME FIBERS. FOR EXTREMELY THIN
*   CROSS SECTIONS, THIS RATIO MAY BE QUITE CLOSE TO UNITY, IN WHICH
*   CASE THE CALCULATION REQUIRES PRECISION BEYOND THE CAPABILITIES OF
*   MOST COMPUTERS. TO AVOID THIS PROBLEM, A SERIES EXPANSION HAS BEEN
*   EMPLOYED TO APPROXIMATE THE LOGARITHMIC FUNCTION. FOR THE CASE OF
*   THE TRAPEZOIDAL CROSS SECTION, THE LOGARITHMIC FUNCTION IS USED,
*   ASSUMING THAT IF THE BEAM WERE SUFFICIENTLY THIN TO CAUSE PROBLEMS
*   IN THE CALCULATIONS, THE USER COULD APPROXIMATE THE CROSS SECTION
*   AS RECTANGULAR WITH CONSIDERABLE ACCURACY.
*
*   INPUT:
*      THE USER IS FIRST ASKED IF USCS OR SI UNITS ARE DESIRED, AND THEN
*   FOR THE SHAPE OF THE BEAM CROSS SECTION. THEN, DEPENDING ON THE SHAPE
*   OF THE SECTION, THE PHYSICAL DIMENSIONS AND THE DISTANCE FROM THE
*   CENTER OF CURVATURE TO THE INNER FIBERS OF THE SECTION ARE INPUTTED.
*   FINALLY, AFTER THE PROGRAM FINDS THE CENTRAL AXIS LOCATION, THE USER
*   MUST DETERMINE AND ENTER THE BENDING MOMENT BASED ON THE LOADING.
*
*   OUTPUT:
*      THE PROGRAM INITIALLY WILL DETERMINE THE LOCATION OF THE CENTRAL
*   AXIS FOR THE PARTICULAR CROSS SECTION. (FROM THIS INFORMATION, THE
*   USER THEN MUST DETERMINE THE BENDING MOMENT BASED ON THE LOADING.)
*   ULTIMATELY, THE BENDING STRESS AT THE EXTREME FIBERS OF THE CROSS
*   SECTION IS GIVEN.
*
*   VARIABLES:
*     ANS          ---   USER INPUT FOR CHOICE OF UNITS
*     SHAPE        ---   USER INPUT FOR CHOICE OF X-SECTIONAL SHAPE
```

```
*   B,H            ---  DIMENSIONS OF BASE, HEIGHT FOR RECTANGULAR SECTION
*   D              ---  DIAMETER OF CIRCULAR SECTION
*   B1,B2,H        ---  DIMENSIONS OF INNER BASE, OUTER BASE, AND HEIGHT
*                       FOR TRAPEZOIDAL SECTION
*   RI,RO          ---  DISTANCE FROM THE CENTER OF CURVATURE TO THE INNER
*                       AND OUTER FIBERS OF THE SECTION RESPECTIVELY
*   A              ---  AREA OF THE SECTION
*   RR             ---  DISTANCE FROM THE CENTER OF CURVATURE TO THE CENTRA
*                       AXIS OF THE SECTION
*   YBAR           ---  DISTANCE FROM THE CENTRAL AXIS TO THE NEUTRAL AXIS
*   R              ---  THE DISTANCE FROM THE CENTER OF CURVATURE TO THE
*                       NEUTRAL AXIS (THE DIFFERENCE BETWEEN RR AND YBAR)
*   K              ---  A CONSTANT USED FOR THE CASE OF THE CIRCULAR SECTIO
*   YI,YO          ---  THE DISTANCES FROM THE NEUTRAL AXIS TO THE INNER AN
*                       OUTER FIBERS RESPECTIVELY
*   M              ---  THE BENDING MOMENT ACTING ON THE SECTION
*  SIGMAI,SIGMAO---     THE BENDING STRESSES AT THE INNER AND OUTER FIBERS
*  A1,A2,YJ,YK, ---     VARIABLES USED TO FIND THE CENTROID OF TRAPEZOIDAL
*  SUMAY,SUMA,HOLD      SECTION
*   UNIT           ---  CHARACTER VARIABLE DENOTING THE APPROPRIATE UNITS
*
***********************************************************************
**********                      MAIN PROGRAM                ***********
***********************************************************************
*
*         VARIABLE DECLARATION
*
      REAL B,H,D,B1,B2,RI,RO,A,RR,YBAR,R,K,YI,YO,M,SIGMAI,SIGMAO
      REAL A1,A2,YJ,YK,SUMAY,SUMA,HOLD
      INTEGER ANS,SHAPE
      CHARACTER UNIT*7
*
*         USER INPUT
*
      PRINT*,'PLEASE INPUT YOUR CHOICE OF UNITS:'
      PRINT*,'1 - USCS'
      PRINT*,'2 - SI'
      PRINT*,' '
      PRINT*,'ENTER 1,2 :'
      READ*,ANS
      PRINT*,' '
*
      PRINT*,' '
 10   PRINT*,'PLEASE INPUT THE SHAPE OF THE BEAM CROSS SECTION:'
      PRINT*,'1 - RECTANGULAR'         '
      PRINT*,'2 - CIRCULAR'
      PRINT*,'3 - TRAPEZOIDAL'
      PRINT*,' '
      PRINT*,'ENTER 1,2,3:'
      READ*,SHAPE
*
      IF(ANS.EQ.1) THEN
         PRINT*,'PLEASE INPUT THE FOLLOWING DIMENSIONS IN INCHES...'
         UNIT='INCHES.'
      ELSE
         PRINT*,'PLEASE INPUT THE FOLLOWING DIMENSIONS IN METERS...'
         UNIT ='METERS.'
      ENDIF
*
*              PROMPTS FOR THE DIMENSIONS OF THE APPROPRIATE SECTION
*
     PRINT*,' '
     IF (SHAPE.EQ.1) THEN
        PRINT*,'PLEASE INPUT THE DIMENSIONS OF THE BASE AND HEIGHT, '
        PRINT*,'AND THE DISTANCE FROM THE CENTER OF CURVATURE TO THE  '
```

```
            PRINT*,'INNER FIBERS OF THE X-SECTION: (B,H,RI)'
            READ*,B,H,RI
            PRINT*,' '
        ELSEIF (SHAPE.EQ.2) THEN
            PRINT*,'PLEASE INPUT THE DIAMETER AND DISTANCE FROM THE CENTER O
            PRINT*,'CURVATURE TO THE INNER FIBERS OF THE X-SECTION: (D,RI)'
            READ*,D,RI
            PRINT*,' '
        ELSEIF (SHAPE.EQ.3) THEN
           PRINT*,'PLEASE INPUT THE DIMENSIONS OF THE INSIDE, THEN OUTSIDE'
           PRINT*,'BASES, THE HEIGHT, AND THE DISTANCE FROM THE CENTER OF'
           PRINT*,'CURVATURE TO THE INNER FIBERS OF THE X-SECTION:(B1,B2,H,RI)'
           READ*,B1,B2,H,RI
           PRINT*,' '
        ELSE
           PRINT*,'YOU MUST ENTER A 1,2 OR 3!'
           GO TO 10
        ENDIF
*
*             END USER INPUT
*
*        CALCULATIONS --- IN EACH CASE, THE DISTANCE FROM THE CENTER OF
*                         CURVATURE TO THE CENTRAL AND NEUTRAL AXIS IS
*                         FOUND (RR AND  R) ,AND THEN THE DISTANCE FROM
*                         THE NEUTRAL AXIS TO THE EXTREME FIBERS (YI,YO) IS
*                         DETERMINED.
*
         IF (SHAPE.EQ.1) THEN
*
*                  IF SHAPE EQUALS ONE, THEN THE SECTION IS RECTANGULAR
*
            A = B*H
            RO = RI + H
            RR = (H)/2 + RI
            YBAR = H**2/(12*RR)
            R = RR-YBAR
            YI = YBAR -(H/2)
            YO = YBAR +(H/2)
         ELSEIF (SHAPE.EQ.2) THEN
*
*                  IF SHAPE EQUALS TWO, THEN THE SECTION IS CIRCULAR
*
            A = (3.14159/4)*D**2
            RR = RI + (D/2.)
            K = ((D/(2*RR))**2)/4  + ((D/(2*RR))**4)/8
            YBAR = (K*RR)/(1-K)
            R = RR - YBAR
            YI = YBAR -(D/2)
            YO = YBAR +(D/2)
         ELSEIF (SHAPE.EQ.3) THEN
*
*                  IF SHAPE EQUALS THREE, THEN THE SECTION IS TRAPEZOIDAL
*
            A = ((B1 + B2)/2)*H
            RO = RI + H
            HOLD = 0.0
*
*                  FIRST, THE CENTROID OF THE TRAPEZOIDAL SECTION IS FOUND
*
     20     IF (B1.GT.B2) THEN
               A1 = (H/4)*(B1-B2)
               A2 = B2*H
               YJ = H/3.
               YK = H/2.
               SUMA = (2*A1) + A2
```

```
                SUMAY = (A1*YJ*2) + (A2*YK)
             ELSE
                HOLD = B2
                B2 = B1
                B1 = HOLD
                GO TO 20
             ENDIF
             IF (HOLD.EQ.0.) THEN
                RR = RI + (SUMAY/SUMA)
             ELSE
                RR = RI + (H - (SUMAY/SUMA))
             ENDIF
*
             R = ((H**2)*(B1 + B2))/2.
             R = R/(((B1*RO) - (B2*RI))*(LOG(RO/RI)) - H*(B1 - B2))
             YBAR = RR-R
             YI = YBAR - (SUMAY/SUMA)
             YO = YBAR + (H-(SUMAY/SUMA))
          ENDIF
*
*                  ONCE THE CENTRAL AXIS HAS BEEN DETERMINED, THE USER
*                  IS PROMPTED FOR THE BENDING MOMENT WHICH THEY MUST
*                  CALCULATE BASED ON THIS DIMENSION AND THE GIVEN LOAD
*
          PRINT*,'THE DISTANCE FROM THE CENTER OF CURVATURE TO THE CENTRAL'
          PRINT 15,'AXIS OF THE CURVED SECTION IS:',RR,UNIT
          PRINT*,' '
          PRINT*,'GIVEN THIS DIMENSION, THE USER MUST NOW CALCULATE THE'
          PRINT*,'MOMENT ACTING ON THE CROSS SECTION...THE MOMENT IS THE'
          PRINT*,'PRODUCT OF THE APPLIED LOAD AND THE DISTANCE TO THE CENTRAL'
          PRINT*,'AXIS FROM THE POINT OF APPLICATION. NOTE, THE MOMENT IS '
          PRINT*,'NEGATIVE IF IT ACTS TO REDUCE THE CURVATURE!'
          PRINT*,' '
          PRINT*,'PLEASE ENTER THE MOMENT (IN N-M OR LB-IN):'
          READ*,M
          PRINT*,' '
*
*              CALCULATING THE BENDING STRESS AT THE INNER AND OUTER FIBERS
*
          SIGMAI = (M*YI)/(A*YBAR*(R+YI))
          SIGMAO = (M*YO)/(A*YBAR*(R+YO))
*
          IF (ANS.EQ.1) THEN
             UNIT = ' PSI.'
          ELSE
             SIGMAI = SIGMAI/1E6
             SIGMAO = SIGMAO/1E6
             UNIT = ' MPA.'
          ENDIF
*
*                  PRINTING OUTPUT
*
          PRINT*,' '
          PRINT 15,'THE BENDING STRESS AT THE INNER FIBERS IS :',SIGMAI,UNIT
          PRINT 15,'THE BENDING STRESS AT THE OUTER FIBERS IS :',SIGMAO,UNIT
*
       15 FORMAT(1X,A,F11.3,1X,A)
*
          STOP
          END
```

**12.14.** Return to Problem 12.12 and use the FORTRAN program of Problem 12.13 to determine the bending stress at point $A$.

Using the moment loading and geometry of Problem 12.12 we have 68.58, which is in good agreement with the value found in Problem 12.12. Note that the uniform normal stress of 4.03 MPa must be added to this value to obtain the resultant normal stress at $A$. The following computer program yields only the bending effect.

```
run
 PLEASE INPUT YOUR CHOICE OF UNITS:
 1 - USCS
 2 - SI

 ENTER 1,2 :
? 2


 PLEASE INPUT THE SHAPE OF THE BEAM CROSS SECTION:
 1 - RECTANGULAR
 2 - CIRCULAR
 3 - TRAPEZOIDAL

 ENTER 1,2,3:
? 1
 PLEASE INPUT THE FOLLOWING DIMENSIONS IN METERS...

 PLEASE INPUT THE DIMENSIONS OF THE BASE AND HEIGHT,
 AND THE DISTANCE FROM THE CENTER OF CURVATURE TO THE
 INNER FIBERS OF THE X-SECTION: (B,H,RI)
? 0.04,0.06,0.04

 THE DISTANCE FROM THE CENTER OF CURVATURE TO THE CENTRAL
 AXIS OF THE CURVED SECTION IS:       .070 METERS.

 GIVEN THIS DIMENSION, THE USER MUST NOW CALCULATE THE
 MOMENT ACTING ON THE CROSS SECTION...THE MOMENT IS THE
 PRODUCT OF THE APPLIED LOAD AND THE DISTANCE TO THE CENTRAL
 AXIS FROM THE POINT OF APPLICATION. NOTE, THE MOMENT IS
 NEGATIVE IF IT ACTS TO REDUCE THE CURVATURE!

 PLEASE ENTER THE MOMENT (IN N-M OR LB-IN):
? -1162


 THE BENDING STRESS AT THE INNER FIBERS IS :
 68.58
```

**12.15.** Consider a crane hook subject to a vertical load of 5000 lb. The cross section is trapezoidal, as shown in Fig. 12-29. Determine the tensile stress at point $A$ using the computer program of Problem 12.13.

The theory of Problem 12.11 is applicable here but the evaluation of the integral in Eq. $(d)$ of that problem is tedious; hence we employ the FORTRAN program of Problem 12.13 using as input the geometry indicated in Fig. 12-29. The printout first indicates that the distance from the center of curvature to the centroidal axis is 2.287 in and from that we can calculate the acting moment as

$$M = -(1.18\text{ in} + 2.287\text{ in})(5000\text{ lb}) = -17{,}335\text{ lb}\cdot\text{in}$$

Now, using this moment as input in the program, we have the stresses at inner and outer fibers as indicated in the final two lines of the printout.

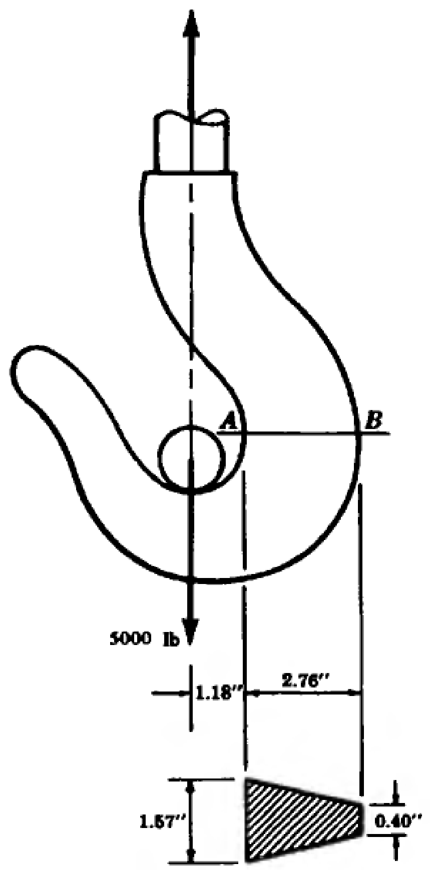



Fig. 12-29

```
run
 PLEASE INPUT YOUR CHOICE OF UNITS:
 1 - USCS
 2 - SI

 ENTER 1,2 :
? 1


 PLEASE INPUT THE SHAPE OF THE BEAM CROSS SECTION:
 1 - RECTANGULAR
 2 - CIRCULAR
 3 - TRAPEZOIDAL

 ENTER 1,2,3:
? 3
 PLEASE INPUT THE FOLLOWING DIMENSIONS IN INCHES...

 PLEASE INPUT THE DIMENSIONS OF THE INSIDE, THEN OUTSIDE
 BASES, THE HEIGHT, AND THE DISTANCE FROM THE CENTER OF
 CURVATURE TO THE INNER FIBERS OF THE X-SECTION:(B1,B2,H,RI)
? 1.57,0.40,2.76,1.18

 THE DISTANCE FROM THE CENTER OF CURVATURE TO THE CENTRAL
 AXIS OF THE CURVED SECTION IS:      2.287 INCHES.

 GIVEN THIS DIMENSION, THE USER MUST NOW CALCULATE THE
 MOMENT ACTING ON THE CROSS SECTION...THE MOMENT IS THE
 PRODUCT OF THE APPLIED LOAD AND THE DISTANCE TO THE CENTRAL
 AXIS FROM THE POINT OF APPLICATION. NOTE, THE MOMENT IS
 NEGATIVE IF IT ACTS TO REDUCE THE CURVATURE!
```

```
 PLEASE ENTER THE MOMENT (IN N-M OR LB-IN):
? -17335


 THE BENDING STRESS AT THE INNER FIBERS IS :  19878.782 PSI.
 THE BENDING STRESS AT THE OUTER FIBERS IS : -12928.359 PSI.

SRU        1.134 UNTS.
```

In addition to these bending stresses, there is a uniformly distributed set of tensile stresses over the cross section $AB$ due to the direct, tensile effect of the 5000-lb load. These stresses are given by

$$\sigma = \frac{P}{A} = \frac{5000\ \text{lb}}{[(1.57 + 0.40)/2\ \text{in}]\,(2.76\ \text{in})} = 1839\ \text{lb/m}^2$$

and must be added to the bending stresses found by the computer program. Thus, the true stress at point $A$ is

$$\sigma'_A = 19.879\ \text{lb/in}^2 + 1839\ \text{lb/in}^2 = 21{,}718\ \text{lb/in}^2 \qquad \text{or} \qquad 21{,}700\ \text{lb/in}^2$$

## Supplementary Problems

**12.16.** Locate the shear center of a thin-walled circular section with a longitudinal slit (Fig. 12-30).
*Ans.* $e = 2R$

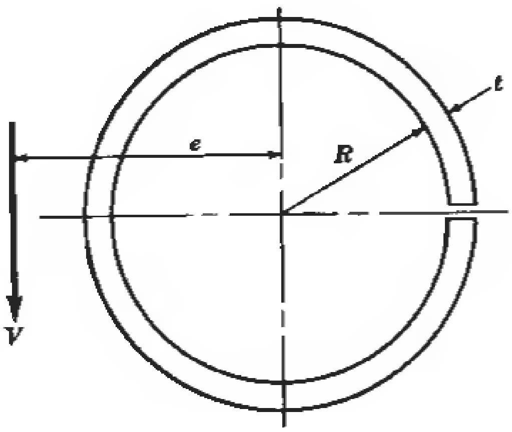



**Fig. 12-30**

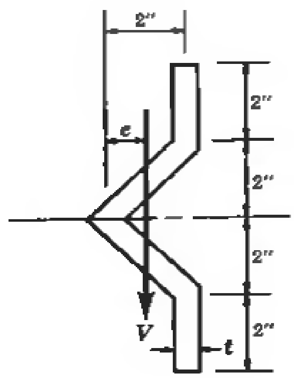



**Fig. 12-31**

**12.17.** Determine the shear center of the thin-walled "hat" section shown in Fig. 12-31. *Ans.* $e = 0.51$ in

**12.18.** Determine the shear center of the thin-walled section indicated in Fig. 12-32. *Ans.* $e = 6.85$ mm

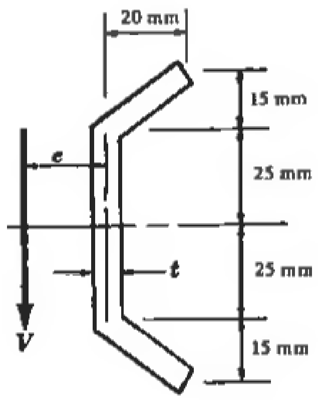



**Fig. 12-32**

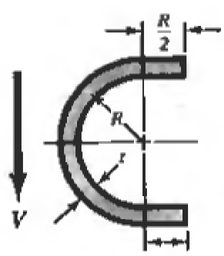



**Fig. 12-33**

**12.19.** Determine the shear center of the thin-walled section shown in Fig. 12-33.
*Ans.* $0.747R$ measured from the centroid

**12.20.** Find the shear center of the thin-walled section shown in Fig. 12-34.
*Ans.* $0.703a$ measured from the centroid

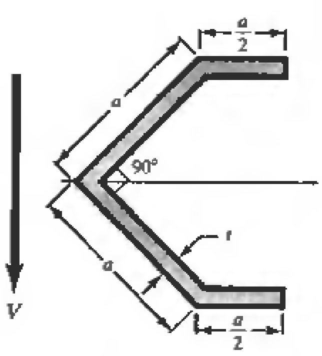



**Fig. 12-34**

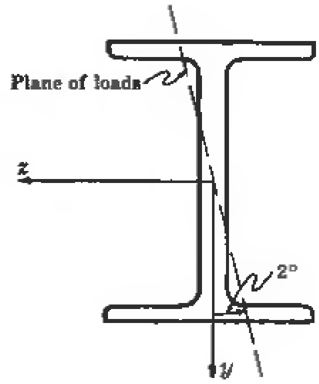



**Fig. 12-35**

**12.21.** A structural steel I-beam 250 mm deep is subjected to a bending moment lying in a plane oriented at 2° to the vertical axis of symmetry of the beam (see Fig. 12-35). Determine the percentage increase in elastic tensile stress over the stress that would exist if the moment acted in the vertical plane of symmetry. For this section $I_z = 57 \times 10^6\ \text{mm}^4$ and $I_y = 3.3 \times 10^6\ \text{mm}^4$. *Ans.* 30 percent

**12.22.** The structural aluminum z-section has the dimensions shown in Fig. 12-36 with cross-sectional properties $I_y = 4.1 \times 10^6\ \text{mm}^4$, $I_z = 10.7 \times 10^6\ \text{mm}^4$, and $I_{yz} = 5.0 \times 10^6\ \text{mm}^4$. The loading has components $M_y = -2.235\ \text{kN}\cdot\text{m}$, $M_z = 4.47\ \text{kN}\cdot\text{m}$. Determine the bending stress at point $A$. *Ans.* $-35.5$ MPa

**12.23.** In Problem 12.7 find the bending stress at point $B$, neglecting the effect of the rounded corner there. Use the FORTRAN program of Problem 12.8 *Ans.* 153.3 MPa

**12.24.** A semicircular bar is of square cross section and is clamped at one end and subject to a load $P$ at the other end, as indicated in Fig. 12-37. The cross section is 4 in on a side and the radius of the bar is 20 in. If the maximum tensile stress at the support is not to exceed 28,000 lb/in$^2$, determine the maximum allowable value of the load $P$. *Ans.* 6460 lb

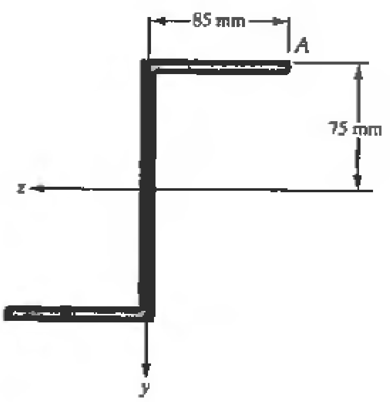



**Fig. 12-36**

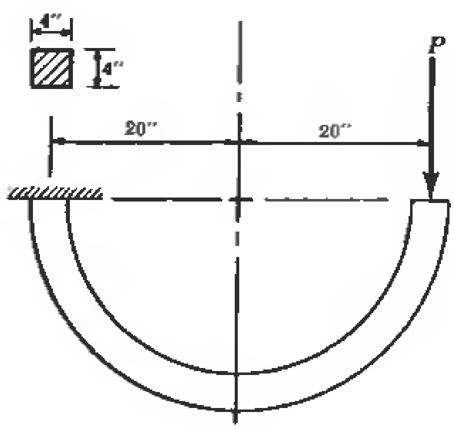



**Fig. 12-37**

# Chapter 13

# Plastic Deformations of Beams

## INTRODUCTION

In certain situations in structural design it is acceptable to permit a modest amount of permanent deformation of the structural element. If this is the case, then it is possible to permit loads greater than indicated by elastic theory, which permits no stress greater than the yield point of the material to develop at any point. This results in more efficient use of the material and is called *plastic design*. Fundamentally, this more efficient design is possible because of the ability of certain materials, such as structural steel, to undergo relatively large plastic deformations after the yield point has been reached. This is illustrated by the horizontal region of the stress-strain diagram shown in Fig. 1-5, page 3.

## PLASTIC HINGE

As the transverse loads on a beam increase, yielding begins at the outer fibers at some critical station along the length of the beam and progresses rather rapidly toward the central fibers at this station. When finally all the fibers on one side of the neutral axis are in a state of tension corresponding to the yield point of the material and all those on the other side are in a state of compression, again at the yield point, then a flowing or *hinging* action occurs at that station and the bending moment transmitted across the *plastic hinge* remains constant. In this book a plastic hinge is denoted by a small, open circle.

## FULLY PLASTIC MOMENT

The bending moment developed at a plastic hinge is termed a *fully plastic moment*. This concept was discussed in Chap. 8.

## LOCATION OF PLASTIC HINGES

In general, plastic hinges form at points of maximum moment. For beams subject to concentrated forces and moments, the peak bending moment must always occur under one of these loadings or at some reaction and thus the plastic hinges must develop first at these points. In the case of distributed loads, the location of the plastic hinges is considerably more difficult to determine and often several possible points must be investigated. This is discussed in Problems 13.8 and 13.9.

## COLLAPSE MECHANISM

When enough plastic hinges have formed in a structure to develop its full plastic load-carrying capacity, then portions of the structure (such as a beam or frame) between hinges may displace without any further increase of load; i.e., the portions between hinges behave as a *mechanism*. Essentially, the hinges allow a kinematic freedom of motion. Under these conditions the shape of the deformed body may be characterized as a straight line between any pair of hinges. Typical representations of collapse mechanisms are shown in Problems 13.2, 13.4, and 13.8 through 13.10.

### LIMIT LOAD

The external load sufficient to cause the structure to behave as a mechanism is termed the *limit load* or *collapse load*. Any design based upon the concept of development of a mechanism is termed *limit design*. All problems in this chapter illustrate computation of the limit load.

## Solved Problems

**13.1.** The simply supported beam $ABC$ in Fig. 13-1($a$) is loaded by a central vertical force of 1200 lb and made of steel having a yield point of 38,000 lb/in$^2$. The beam is of rectangular cross section, as shown in Fig. 13-1($b$), with width $b$, depth $1.6b$, and length $L = 40$ in. Determine $b$ for fully plastic action. Also determine the width $b'$ when only the extreme fibers have reached yield.

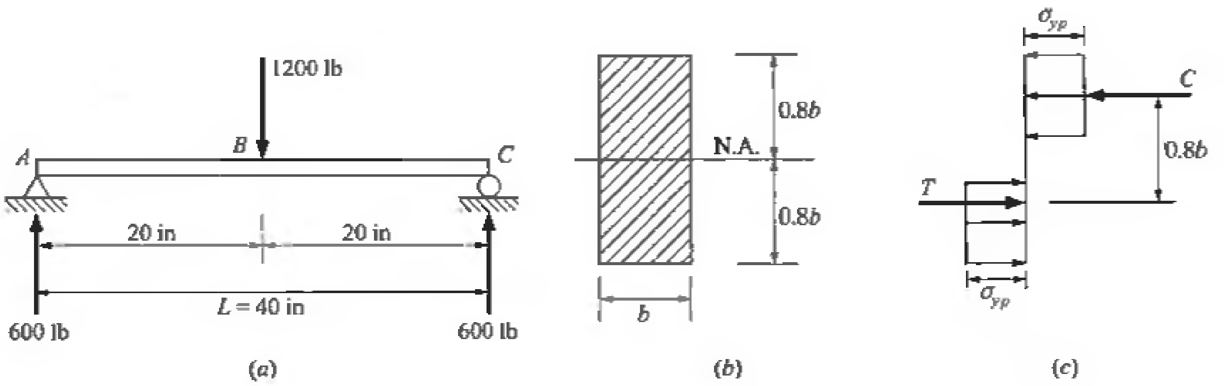



**Fig. 13-1**

The reactions at $A$ and $C$ are each 600 lb by symmetry. The peak bending moment at the midpoint $B$ is given by

$$(600 \text{ lb})(20 \text{ in}) = 12{,}000 \text{ lb}\cdot\text{in}$$

At that time all fibers above the centrally located neutral axis (N.A.) are acting in compression $C$ and those below that axis are in tension $T$, as shown in Fig. 13-1($c$). The location of the action line of each of these forces is shown in Fig. 13-1($c$). The moment resulting from the effect of $T$ and $C$ is

$$M_P = (\sigma_{yp})(0.8b)(b)[0.8b]$$

$$= 0.64b^3\sigma_{yp}$$

$$= 0.64b^3(38{,}000)$$

Thus,

$$0.64b^3(38{,}000) = 12{,}000$$

$$b = 0.79 \text{ in}$$

$$1.6b = 1.26 \text{ in}$$

so that the beam cross-sectional area is 0.995 in$^2$.

From Problem 8.25 for a rectangular cross section, the maximum possible fully elastic moment (i.e., when only the extreme outer fibers have reached the yield point) is given by

$$M_e = \frac{b'(h')^2}{6}\sigma_{yp}$$

Hence,

$$12{,}000 = \frac{b'(1.6b')^2}{6}(38{,}000)$$

Solving,

$$b' = 0.905 \text{ in}$$
$$1.6b' = 1.45 \text{ in}$$

Here, the cross-sectional area is 1.312 in$^2$. The fully elastic moment corresponds to an area of 1.312 in$^2$. Thus, allowing fully plastic action leads to a 24.2 percent reduction of beam weight for any given length. Suitable safety factors, usually specified by building codes, must be introduced into each of the above computations.

**13.2.** Determine the limit load of the simply supported beam shown in Fig. 13-2.

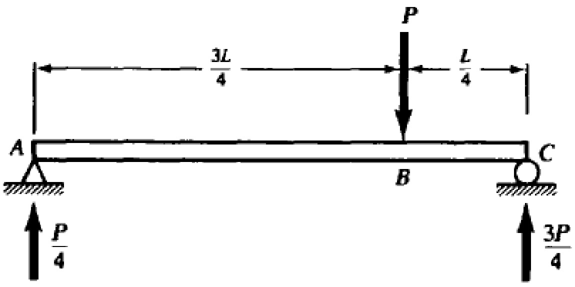



Fig. 13-2

The end reactions at $A$ and $C$ are readily found from statics to be $P/4$ and $3P/4$, respectively, irrespective of whether the beam is in the elastic or plastic state. The peak bending moment occurs under the point of application of $P$ and is thus $(P/4)(3L/4) = 3PL/16$. When this bending moment reaches a value corresponding to fully plastic action of the section of the beam at $B$, which we term $M_p$, a plastic hinge forms at $B$ and the beam continues to deflect without further increase of $P$. This collapse mechanism has the form shown in Fig. 13-3.

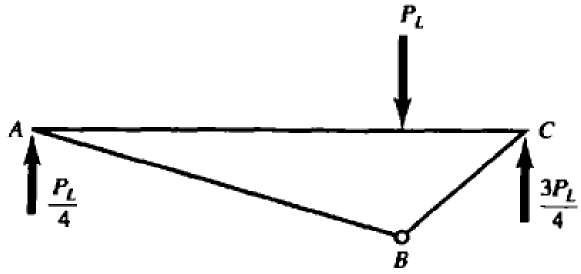



Fig. 13-3

The value of the load $P$ corresponding to this condition is termed the *limit load* $P_L$. The reaction at $A$ is then $(P_L/4)$ and thus the moment at $B$ is

$$\left(\frac{P_L}{4}\right)\left(\frac{3L}{4}\right) = M_p$$

Solving, $P_L = 16M_p/3L$. Dividing $P_L$ by some suitable safety factor gives an allowable working load. This procedure is called *limit design*.

**13.3.** The beam of Problem 13.2 is of rectangular cross section 1.75 in × 3 in. It is titanium, type Ti-8Mn, with a yield point stress of 115,000 lb/in². If the length of the beam is 5 ft, determine the central force $P$ necessary to develop the plastic hinge at $B$.

From Problem 8.25 the fully plastic moment for a rectangular cross section is given by

$$M_p = \sigma_{yp}\frac{bh^2}{4}$$

Substituting,

$$M_p = (115{,}000\ \text{lb/in}^2)\frac{(1.75\ \text{in})(3\ \text{in})^2}{4} = 453{,}000\ \text{lb}\cdot\text{in}$$

Using the result of Problem 13.2,

$$P_L = \frac{16M_p}{3L} = \frac{16(453{,}000\ \text{lb}\cdot\text{in})}{3(60\ \text{in})} = 40{,}300\ \text{lb}$$

This is the limit load of the beam.

From Problem 8.25, the peak elastic moment that this beam could withstand is given by

$$M_e = \sigma_{yp}\frac{bh^2}{6} = 302{,}000\ \text{lb}\cdot\text{in}$$

from which the maximum allowable load $P_e$ based on elastic design is

$$P_e = \frac{16M_e}{3L} = 26{,}850\ \text{lb}$$

Thus use of limit design permits a 50 percent greater load than elastic analysis. However, the designer would want to incorporate some safety factor into the above limit load.

**13.4.** Determine the limit load of a simply supported beam subject to a unformly distributed load. See Fig. 13-4.

According to the methods developed in Chap. 6, the peak bending moment occurs at the midpoint of the length of the beam and is given by $wL^2/8$. For fully plastic action at the midpoint, this moment is denoted by $M_p$. Thus, when the plastic hinge forms at the center, the uniform load has the value $w_L$ (limit load) so that

$$\frac{w_L L^2}{8} = M_p \qquad \text{or} \qquad w_L = \frac{8M_p}{L^2}$$

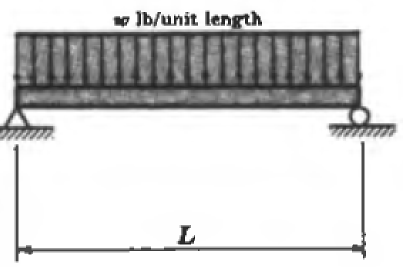



Fig. 13-4

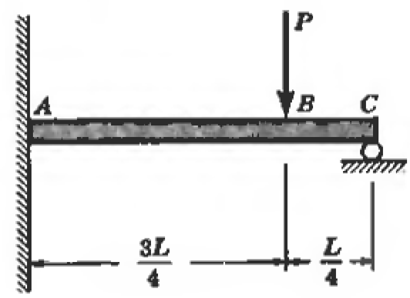



Fig. 13-5

**13.5.** The beam shown in Fig. 13-5 is clamped at the left end, simply supported at the right, and subject to the concentrated load indicated. Determine the magnitude of the limit load $P_L$ corresponding to plastic collapse.

This statically indeterminate beam cannot collapse plastically through formation of a single plastic hinge at $B$ because the region $AB$ is constrained to very small lateral deflections until another hinge forms somewhere along its length. It has been demonstrated in Chap. 6 that significant bending moments in a beam subject to concentrated forces always occur either at the points of application of these forces or where the reactions are applied. In the present case, this would imply the formation of another plastic hinge at $A$. With hinges at $A$ and $B$, we have a so-called *kinematically admissible mechanism* of collapse. The order in which the plastic hinges are formed is of no consequence. The collapse mechanism appears in Fig. 13-6.

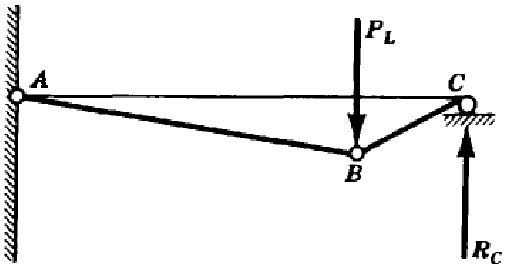



Fig. 13-6

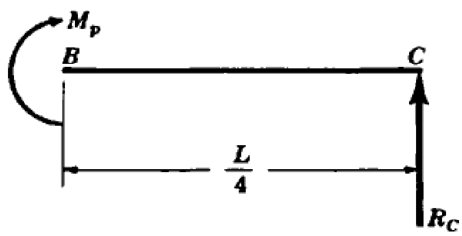



Fig. 13-7

The free-body diagram of the right portion of the beam, extending from $C$ to a point just to the right of the applied load $P$ when that force is the limit load $P_L$, is shown in Fig. 13-7, in which $M_p$ denotes the fully plastic moment at $B$. From statics,

$$M_p - \frac{R_C L}{4} = 0 \qquad \text{or} \qquad R_C = \frac{4M_p}{L} \tag{1}$$

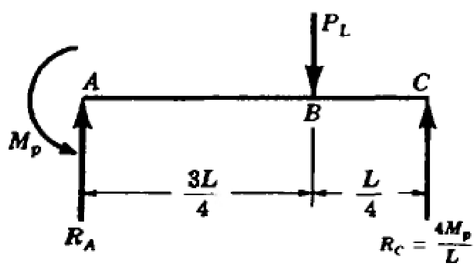



Fig. 13-8

Next, from the free-body diagram of the entire beam (Fig. 13-8), with plastic hinges at $A$ and $B$, we have

$$\Sigma F_v = R_A + \frac{4M_p}{L} - P_L = 0$$

Hence
$$R_A = P_L - \frac{4M_p}{L} \tag{2}$$

$$\Sigma M_C = R_A L - M_p - P_L\left(\frac{L}{4}\right) = 0 \tag{3}$$

Substituting $R_A$ from (2) in (3) yields

$$P_L = \frac{20}{3}\frac{M_p}{L}$$

as the limit load.

**13.6.** The beam described in Problem 13.5 is of hollow circular cross section, as shown in Fig. 13-9, and is of steel having a yield point of 200 MPa. Find the limit load that may be carried if $r = 20$ mm and $L = 2$ m.

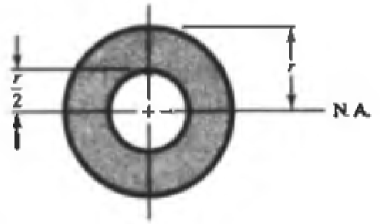



**Fig. 13-9**

For simplicity, let us first find the fully plastic moment $M_p$ for a solid circular cross section of radius $r$. Above the neutral axis (N.A.) there is a uniform normal stress distribution equal to the yield point stress, and the resultant of these stresses acts at the centroid, which is at a distance $(4r/3\pi)$ above the N.A. A like situation exists below the N.A., where the normal stresses are oppositely directed from those above that axis. Thus,

$$M_p = 2\left[\sigma_{yp}\left(\frac{\pi r^2}{2}\right)\left(\frac{4r}{3\pi}\right)\right] = \frac{4r^3}{3}\sigma_{yp}$$

The fully plastic moment for the hollow circular cross section is now given by

$$M_p = \frac{4r^3}{3}\sigma_{yp} - (2)\left[\sigma_{yp}\frac{\pi(r/2)^2}{2}\left\{\frac{4(r/2)}{3\pi}\right\}\right] = \frac{7r^3}{6}\sigma_{yp} \qquad (1)$$

For our parameters,

$$M_p = \tfrac{7}{6}(0.02\text{ m})^3(200\times10^6\text{ N/m}^2) = 1867\text{ N}\cdot\text{m}$$

and from Problem 13.5 we have

$$P_L = \frac{20M_p}{3L} = \frac{20(1867\text{ N}\cdot\text{m})}{3(2\text{ m})} = 6225\text{ N}$$

as the limit load.

**13.7.** The beam described in Problem 13.5 is a wide-flange section having the dimensions indicated in Fig. 13-10. For this section, determine the limit load $P_L$. The material is structural steel with a yield point of 250 MPa and the length of the beam is 2 m.

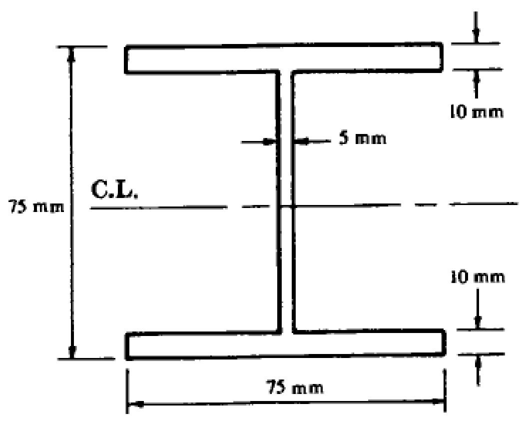



**Fig. 13-10**

As mentioned in Problem 8.29, for fully plastic action, the neutral axis divides the cross-sectional area into two parts of equal area. Here, because of the symmetry, the neutral axis coincides with the centerline (C.L.) and the centroidal distances from that line are

$$\bar{y}_1 = \bar{y}_2 = \frac{(75)(10)(37.5-5)+(27.5)(5)(27.5/2)}{(75)(10)+(27.5)(5)} = 29.6 \text{ mm}$$

The fully plastic moment is thus

$$M_p = \sigma_{yp}\frac{A}{2}(\bar{y}_1+\bar{y}_2) = 250[(75)(10)+(27.5)(5)](29.6+29.6) = 13.13 \text{ kN}\cdot\text{m}$$

The limit load from Problem 13.5 is

$$P_L = \frac{20}{3}\frac{(13.13\times 10^3)}{2} = 43.8 \text{ kN}$$

It is of interest to carry out an elastic analysis of this same beam. In this case the outer fibers are taken to be stressed to the yield point and, of course, the stresses vary linearly over the depth, being zero at the neutral axis. The second moment of area of the cross section is found by the methods of Chap. 7 to be

$$I = \tfrac{1}{12}(75)(75)^3 - \tfrac{1}{12}(70)(55)^3 = 1.67\times 10^6 \text{ mm}^4$$

and the outer fiber stresses are found from

$$\sigma_{yp} = \frac{M_e c}{I} \qquad \text{or} \qquad 250 = \frac{M_e(37.5)}{1.67\times 10^6}$$

and thus the maximum elastic moment $M_e$ that the section can support is $M_e = 11.13$ kN · m. From Problem 11.1 the bending moment at point $A$ is found to be $0.116PL$ while that at point $B$ is $0.159PL$. Using the latter value we can find the maximum load that the beam can support for entirely elastic action to be

$$0.159P_eL = 11.13 \text{ kN}\cdot\text{m} \qquad \text{or} \qquad P_e = 35 \text{ kN}$$

The load $P_L$, corresponding to plastic collapse, exceeds this value by 25 percent.

**13.8.** Determine the limit load of a clamped-end beam carrying a uniformly distributed load (Fig. 13-11).

The collapse mechanism appears in Fig. 13-12, where plastic hinges have formed at points $A$, $B$, and $C$. By virtue of symmetry the shear is zero at the midpoint $C$; hence we may draw the free-body diagram of the left half of the beam as in Fig. 13-13. From statics,

$$\Sigma M_A = 2M_p - w_L\left(\frac{L}{2}\right)\left(\frac{L}{4}\right) = 0$$

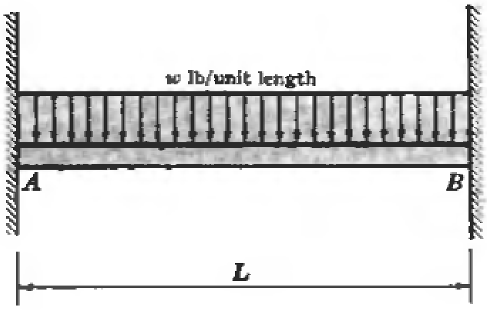



Fig. 13-11

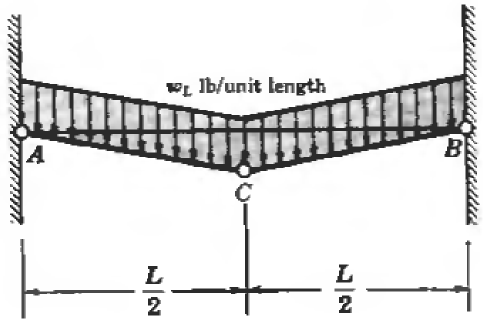



Fig. 13-12

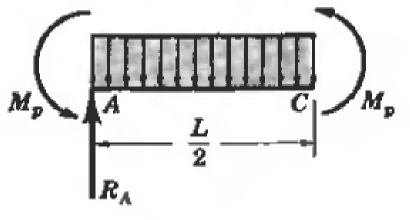



**Fig. 13-13**

The limit load is thus $w_L = 16M_p/L^2$. From considerations similar to Problem 11.6, the permissible load based upon the outer fibers being at the yield point and all interior fibers acting in the elastic range of action is $w_e = 12M_e/L^2$ so that in this case the ratio of limit load $w_L$ to maximum elastic load $w_e$ is $\frac{4}{3}M_p/M_e$. However, the ratio $M_p/M_e$ itself may be significant. For a rectangular cross section it has the value $\frac{3}{2}$, as indicated in Problem 8.25. For such a rectangular bar we then have

$$\frac{w_L}{w_e} = \frac{4}{3}\frac{M_p}{M_e} = \frac{4}{3}\left(\frac{3}{2}\right) = 2$$

indicating that in this particular case, limit design permits application of twice the load permitted by elastic analysis. This rather large variation between the permissible loads is due partially to the indeterminate nature of this beam. It should be noted that there are exceptional cases where the limit load and maximum elastic load coincide even for an indeterminate system.

**13.9.** The beam shown in Fig. 13-14 is clamped at the left end, simply supported at the right, and subject to a uniformly distributed load. Determine the magnitude of this load corresponding to plastic collapse of the beam.

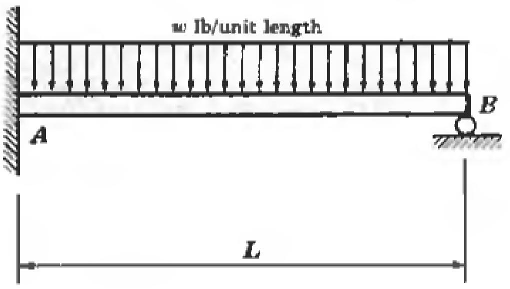



**Fig. 13-14**

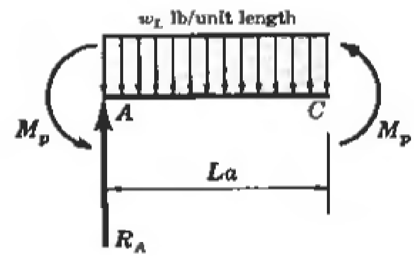



**Fig. 13-15**

This problem is somewhat analogous to Problem 13.5 because the beam cannot collapse plastically through formation of a single plastic hinge but instead, two hinges must form. One of these is obviously at the clamped end $A$ but the location of the other is not immediately apparent. It of course occurs at the position of relative maximum moment (excluding point $A$) but that point is not known. However, since the shear is known to be zero at the point of maximum moment, we may draw the free-body diagram of the left region of the beam of length $La$ and regard $a$ as an unknown. It thus appears as in Fig. 13-15, where $M_p$ denotes the fully plastic moment at each of the two sections.

From statics,

$$\Sigma F_v = R_A - w_L La = 0 \tag{1}$$

$$\Sigma M_A = 2M_p - \frac{w_L L^2 a^2}{2} = 0 \tag{2}$$

Next, let us consider the free-body diagram of the entire beam, as in Fig. 13-16. From statics,

$$\Sigma M_B = -R_A L + \frac{w_L L^2}{2} + M_p = 0 \tag{3}$$

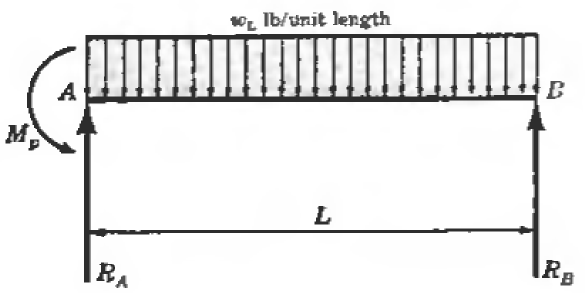



**Fig. 13-16**

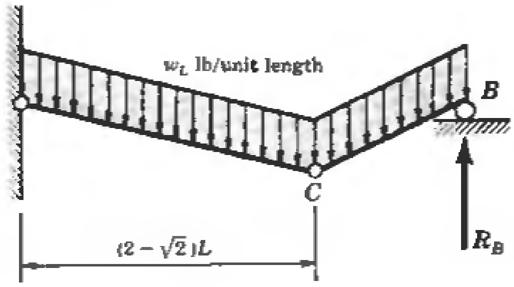



**Fig. 13-17**

Solving (*1*), (*2*), and (*3*) simultaneously we arrive at the single equation

$$a^2 - 4a + 2 = 0$$

for determination of the point of relative maximum moment. Solving, we obtain $a = 2 - \sqrt{2}$, the other root of the quadratic being of no physical significance.

Substituting this value in (*2*), we find

$$w_L = (6 + 4\sqrt{2})\frac{M_p}{L^2}$$

as the limit load. The collapse mechanism appears in Fig. 13-17.

**13.10.** The clamped-end beam is subject to a concentrated force as shown in Fig. 13-18. Determine the magnitude of this load corresponding to plastic collapse of the beam.

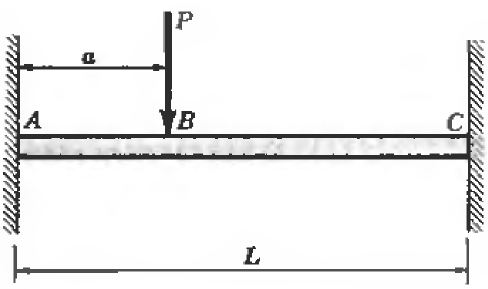



**Fig. 13-18**

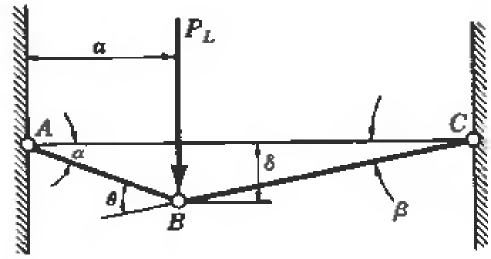



**Fig. 13-19**

The only logical collapse mechanism is that of Fig. 13-19, where plastic hinges form at *A*, *B*, and *C*. From the geometry of triangle *ABC* we have

$$\alpha + \beta = \theta \tag{1}$$

or

$$\frac{\delta}{a} + \frac{\delta}{L-a} = \theta \tag{2}$$

since the deflection $\delta$ is still small compared to $L$ even though plastic collapse has occurred. Solving (*2*) we obtain

$$\delta = \theta a\left(1 - \frac{a}{L}\right) \tag{3}$$

and from geometry we have

$$\alpha = \theta\left(1 - \frac{a}{L}\right) \qquad \beta = \frac{\theta a}{L} \tag{4}$$

This problem could be solved by use of statics equations as employed in Problems 13.5 and 13.8. However, let us introduce another technique which will be well suited to even more complex problems. This involves a consideration of the work done by the load $P_L$ after plastic collapse has occurred. If we assume that the elastic deflection is very small compared to the plastic deflection, then the work done by the load $P_L$ during plastic collapse is $P_L\delta$. It is to be carefully noted that the load assumes the value $P_L$ at the start of the collapse through the deflection $\delta$ and maintains this constant value throughout the collapse process. During the collapse, the beam develops the fully plastic moment $M_p$ at each of the hinge points $A$, $B$, and $C$. The total energy dissipated at these hinges is provided by and is equal to the work done by the load $P_L$.

The work done by the plastic hinge at $A$ is given by $M_p\alpha$, at $B$ it is given by $M_p\theta$ and at $C$ by $M_p\beta$. Thus, equating work done by $P_L$ to the net work done by these three plastic moments, and using (*4*) we have

$$P_L\delta = M_p\theta\left(1-\frac{a}{L}\right) + M_p\theta + M_p\left(\frac{\theta a}{L}\right) \tag{5}$$

Substituting $\delta$ from (*3*) we have as the collapse load

$$P_L = \frac{2M_pL}{a(L-a)}$$

**13.11.** A horizontal beam of rectangular cross section 50 mm × 120 mm is 1.5 m long and hinged at its left end $A$ as shown in Fig. 13-20. The right end $C$ is supported by a vertical bar of the same material, of cross-sectional area 3 cm². The yield point of each material is 200 MPa. The beam is subject to a vertical force $P$ applied at $B$. Determine the limit load $P_L$.

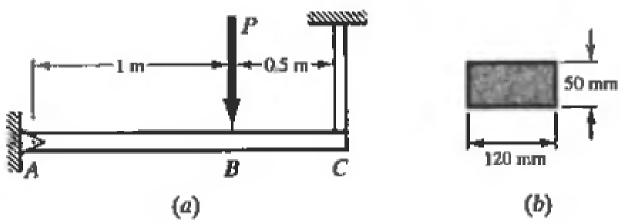



**Fig. 13-20**

It is not clear which yields first, the vertical bar or the horizontal beam $AC$. Let us assume that the vertical bar is the first to yield. The force in it is

$$F_1 = (200\times10^6\ \text{N/m}^2)(3\ \text{cm}^2)(1\ \text{m}/100\ \text{cm})^2 = 6\times10^4\ \text{N}$$

The free-body diagram of the beam is shown in Fig. 13-21.

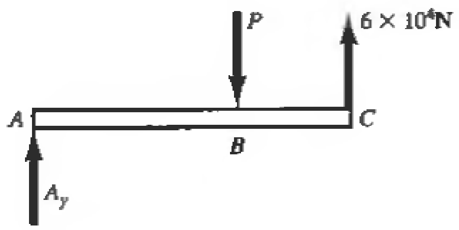



**Fig. 13-21**

For equilibrium,

$$+\uparrow\Sigma M_A = -P(1\text{ m}) + (6\times 10^4\text{ N})(1.5\text{ m}) = 0$$

from which

$$P'_L = 9\times 10^4\text{ N} \quad \text{or} \quad 90\text{ kN}$$

Next, assume that the beam develops a plastic hinge at $B$ with the vertical bar still being entirely elastic. The free-body diagram of the left portion of the beam between $A$ and a point just slightly to the left of $B$ is shown in Fig. 13-22.

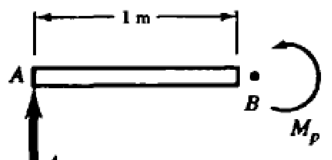



**Fig. 13-22**

For equilibrium of this portion of the beam,

$$+\circlearrowright\Sigma M_B = M_p - A_y(1\text{ m}) = 0$$

For equilibrium of the entire beam $AC$ about point $C$,

$$+\circlearrowright\Sigma M_C = P''_L(0.5\text{ m}) - A_y(1\text{ m}) = 0$$

Solving,

$$P''_L = 3M_p$$

But for a bar of rectangular cross section, the fully plastic moment (see Problem 8.25) is given by

$$M_p = \sigma_{yp}\frac{bh^2}{4}$$

$$= (200\times 10^6\text{ N/m}^2)\frac{(0.12\text{ m})(0.05\text{ m})^2}{4} = 15{,}000\text{ N}\cdot\text{m}$$

Thus,

$$P''_L = 3(15{,}000) = 45{,}000\text{ N} \quad \text{or} \quad 45\text{ kN}$$

Since this load of 45 kN is reached before the load of 90 kN (causing yield of the vertical bar), it is evident that the limit load is 45 kN, which will cause formation of a plastic hinge at $B$ while the vertical bar is still elastic.

**13.12.** Consider the rectangular frame with both bases clamped subject to the two equal loads shown in Fig. 13-23. Determine the magnitude of the loads corresponding to plastic collapse of the frame.

In this situation there are three possible plastic collapse mechanisms. These are shown in Fig. 13-24, where Cases I and II correspond to individual actions of the applied loads and Case III is a composite mechanism formed as a combination of I and II so as to eliminate a plastic hinge at point $B$. We shall determine the collapse loads of each of these three cases and then select the minimum of the three loads as the correct one.

Case I can be treated by the methods of Problem 13.1, so that we immediately have $P_{L1} = 4M_p/L$.

Case II can be treated by the same methods, so for it we have $P_{L2} = 4M_p/L$.

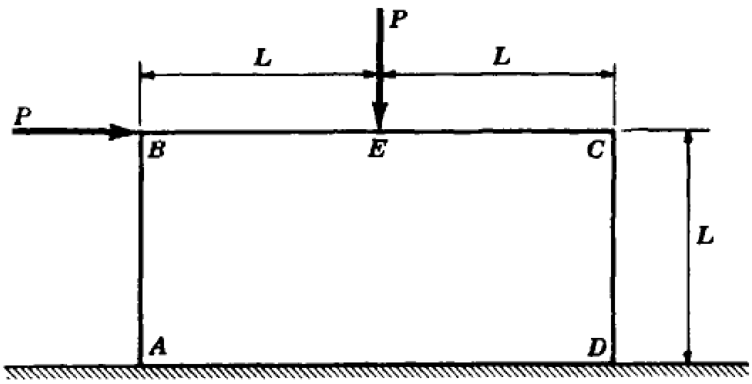



Fig. 13-23

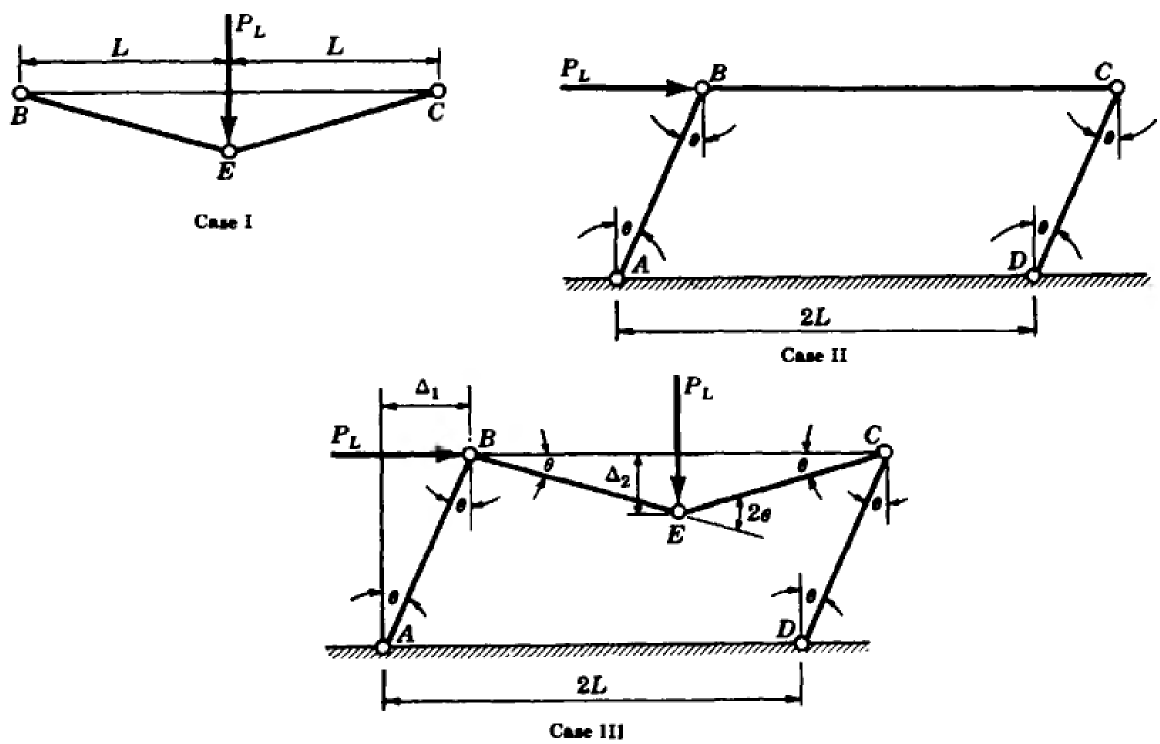



Fig. 13-24

For Case III there are plastic hinges at $A$, $E$, $C$, and $D$, with $B$ constituting a rigid joint. Work-energy balance requires that

$$P_{L3}\Delta_1 + P_{L3}\Delta_2 = [M_p\theta]_A + [M_p(2\theta)]_E + [M_p(2\theta)]_C + [M_p\theta]_D$$

or

$$P_{L3}(L\theta) + P_{L3}(L\theta) = 6M_p\theta$$

from which $P_{L3} = 3M_p/L$.

Thus, the collapse load is $P_L = P_{L3} = 3M_p/L$ and collapse occurs as indicated by the sketch for Case III.

**13.13.** The continuous beam shown in Fig. 13-25($a$) rests on three simple supports and is subject to the single concentrated load indicated. Determine the magnitude of this load for plastic collapse of the beam.

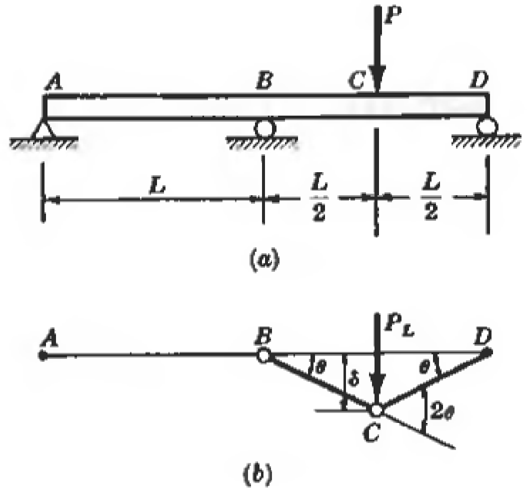



Fig. 13-25

The plastic collapse of such a beam usually occurs in only one of the spans and, in this case, collapse could occur by formation of a mechanism as indicated in Fig. 13-25(*b*), where plastic hinges form at points *B* and *C*.

The work done by the load $P_L$ during plastic collapse is $P_L\,\delta$. The fully plastic moment $M_p$ develops at each of the hinge points *B* and *C*. Work-energy balance requires that

$$P_L\,\delta = [M_p\,\theta]_B + [M_p(2\theta)]_C$$

or

$$P_L\left(\frac{L}{2}\theta\right) = 3M_p\,\theta$$

from which the collapse load is $P_L = 6M_p/L$.

**13.14.** A two-span continuous steel beam supports the concentrated forces indicated in Fig. 13-26(*a*). The beam is of rectangular cross section, 2 in wide by 4 in high, with the yield point of the steel being 38,000 lb/in$^2$. Determine the value of *P* to cause plastic collapse.

Let us first assume that collapse occurs in the span *AC* with the formation of the mechanism indicated in Fig. 13-26(*b*). Fully plastic moments develop at *B* and *C* and the work-energy balance requires that

$$2P_L(10\theta) = [M_p(2\theta)]_B + [M_p\,\theta]_C \qquad \text{or} \qquad P_L = \frac{3M_p}{20}$$

Next, consider the possibility of collapse in the span *CE* with the formation of the mechanism shown in Fig. 13-26(*c*). From the geometry of triangle *CDE* we have

$$\phi = \alpha + \beta$$

But since $\alpha$ is small compared to the span *CE*, this becomes

$$\frac{\delta_1}{8} + \frac{\delta_1}{2} = \phi$$

where $\delta_1$ must of course be in consistent units (i.e., feet). Thus

$$\delta_1 = \tfrac{8}{5}\phi$$

and from geometry

$$\alpha = \tfrac{1}{5}\phi \qquad \beta = \tfrac{4}{5}\phi$$

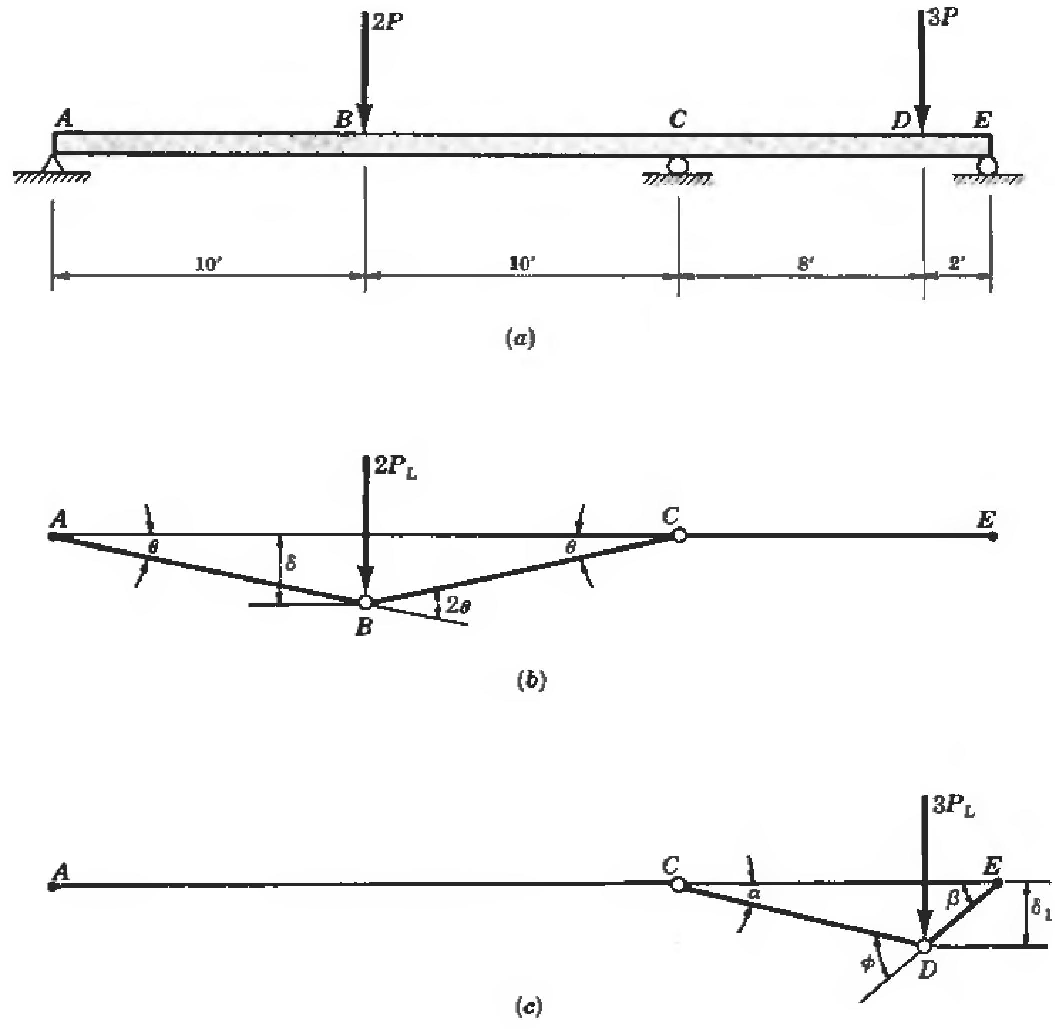



**Fig. 13-26**

In this case fully plastic moments develop at $C$ and $D$ and work-energy balance requires that

$$3P_L(8\alpha) = [M_p\phi]_D + [M_p\alpha]_C \qquad \text{or} \qquad P_L = \frac{M_p}{4}$$

Since this is larger than the $P_L$ found for collapse of the left span, evidently collapse occurs with the formation of the mechanism shown for span $AC$.

Since the fully plastic moment for a rectangular cross section is given by

$$M_p = \sigma_{yp}\left(\frac{bh^2}{4}\right)$$

we find the collapse load to be

$$P_L = \frac{3}{20(12)}(38{,}000)\frac{(2)(4)^2}{4} = 3800 \text{ lb}$$

where the factor of 12 appears in the denominator to render the units consistent.

**13.15.** A simply supported beam of 50-mm × 75-mm rectangular cross section has a yield point stress of 250 MPa and carries the loads indicated in Fig. 13-27($a$). Use the limit design criterion to determine the maximum load $P$.

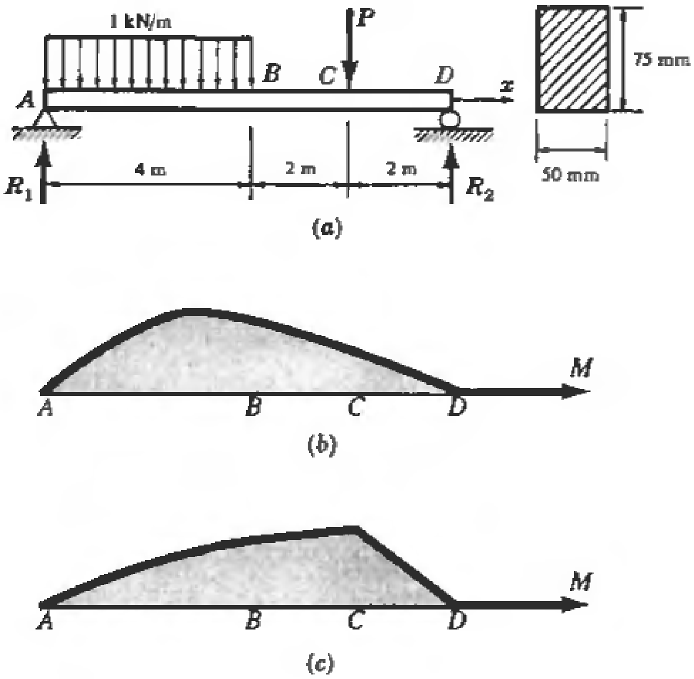



**Fig. 13-27**

From statics the reactions are $R_1 = 3 + (P/4)$ kN and $R_2 = 1 + (3P/4)$ kN.
Fully plastic action of this beam corresponds to a moment of

$$M_p = \sigma_{yp}\frac{bh^2}{4} = 250\frac{(50)(75)^2}{4} = 17.6\ \text{kN}\cdot\text{m}$$

In any problem involving several loads, the location of the first plastic hinge to form is usually not apparent. Here, two possibilities exist. In the first [Fig. 13-27(*b*)], the maximum moment would occur between points $A$ and $B$. If this is the correct form of the moment diagram then the shear must vanish at some point for which $x < 4$. Thus, since

$$V = 3 + \frac{P}{4} - 1x$$

we must find $P$ from the equation

$$0 = 3 + \frac{P}{4} - x \qquad \text{or} \qquad x = 3 + \frac{P}{4}$$

Since $x < 4$ in this consideration, this implies $P < 4$. A simple calculation indicates that $P = 4$ kN cannot develop the fully plastic moment of 17.6 kN · m.

For the second possibility [Fig. 13-27(*c*)], the maximum moment occurs at point $C$. The presence of a plastic hinge at $C$ corresponds to a load $P$, given by

$$\left(1 + \frac{3P}{4}\right)(2) = 17.6\ \text{kN}\cdot\text{m} \qquad \text{or} \qquad P = 10.4\ \text{kN}$$

In this case the moment at $B$ must be less than that at $C$, since the moment diagram must have a common tangent to the two branches meeting at $B$. Hence there is no need to investigate the moment at $B$. Thus $P = 10.4$ kN is the peak load that may be applied according to the limit design criterion.

# Supplemetary Problems

**13.16.** In Problem 6.4 we considered the beam $AD$ supported by knife-edge reactions at $B$ and $C$ as shown in Fig. 13-28($a$). Loading was applied by end bending moments $M_1$ and $M_1/2$ as indicated in that figure. The beam has a T-shaped cross-section as shown in Fig. 13-28($b$), which has previously been considered in Problem 8.32. If the material has a yield point of 39,000 lb/in$^2$, determine the maximum value of applied load for fully plastic action. *Ans.* $M_p = 360{,}750$ lb · in

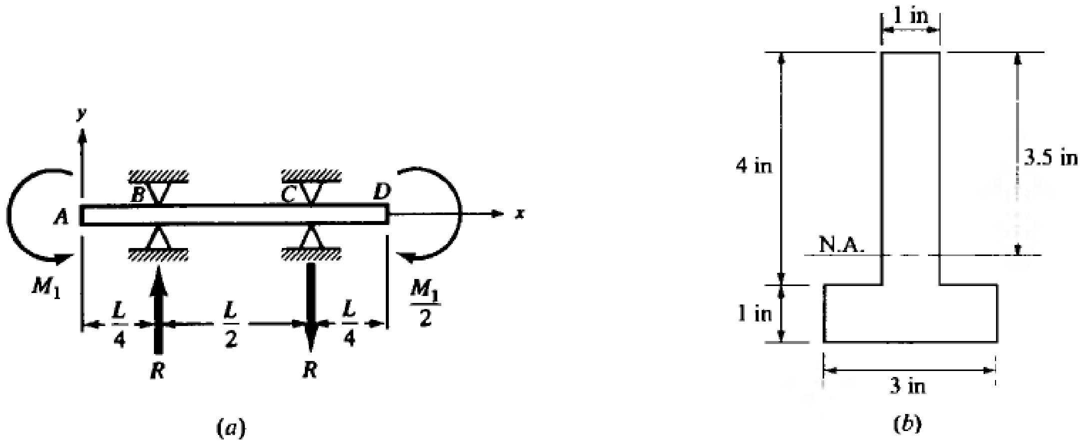



**Fig. 13-28**

**13.17.** Consider again the beam $AD$ and loading shown in Fig. 13-28. The cross section is now a hollow rectangular shape as shown in Fig. 13-29. For a yield point of 39,000 lb/in$^2$, determine the maximum value of applied load for fully plastic action. *Ans.* 546,000 lb · in

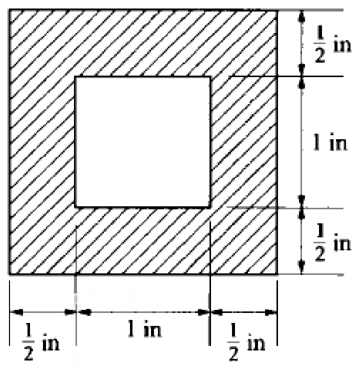



**Fig. 13-29**

**13.18.** Determine the limit load $P$ of the simply supported beam of Fig. 13-30. *Ans.* $P_L = 4.5M_p/L$

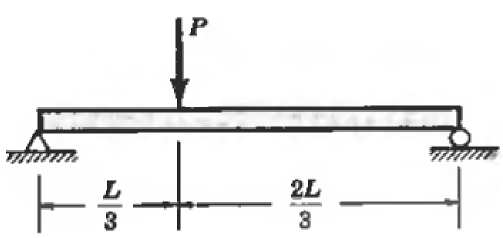



**Fig. 13-30**

**13.19.** The beam of Fig. 13-30 is of rectangular cross section, 25 mm × 50 mm. It is Hy-80 steel with a yield strength of 500 MPa. The length of the beam is 1 m. Determine the limit load when the loading is applied at the third point as indicated. *Ans.* $P_L = 35.2$ kN

**13.20.** The beam of Problem 13.4 is 2 m long and of square cross section 50 mm × 50 mm. It is structural steel with a yield stress of 250 MPa. Determine the limit load. *Ans.* $w_L = 15.6$ kN/m

**13.21.** Determine the magnitude of the limit load $P_L$ for the beam clamped at one end and simply supported at the other (Fig. 13-31).

*Ans.* $P_L = M_p \dfrac{L+x}{(Lx - x^2)}$

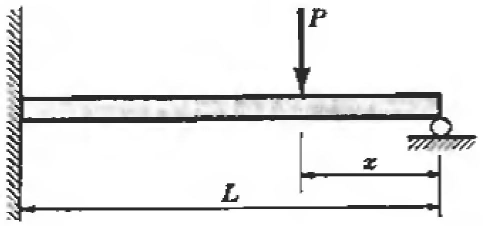



**Fig. 13-31**

**13.22.** In Problem 13.21 determine $x$ so that $P_L$ is a minimum. *Ans.* $x = 0.41L$, $(P_L)_{\min} = 5.64M_p/L$

**13.23.** The simply supported beam $AC$ shown in Fig. 13-32 has a plastic moment $M_p$ and carries the two concentrated loads shown. Determine the limit load $P_L$. *Ans.* $P_L = M_p/2L$

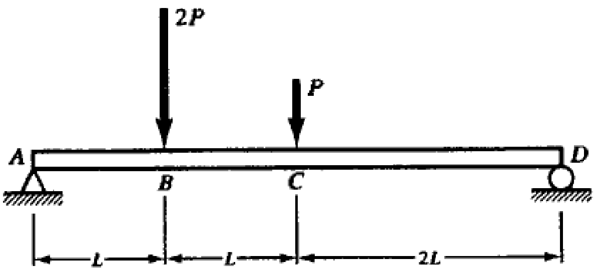



**Fig. 13-32**

Determine the magnitude of the load for plastic collapse of the systems shown in Figs. 13-33 and 13-34.

**13.24.** See Fig. 13-33.

*Ans.* $w_L = (6 + 4\sqrt{2})\dfrac{M_p}{L^2}$

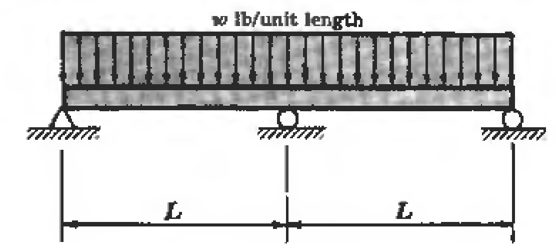



**Fig. 13-33**

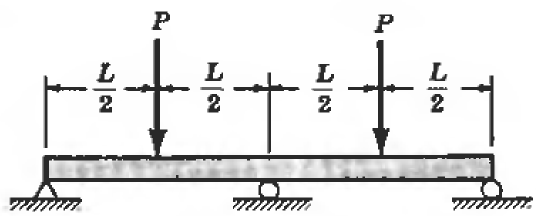



**Fig. 13-34**

**13.25.** See Fig. 13-34.

*Ans.* $P_L = \dfrac{6M_p}{L}$

**13.26.** The continuous beam $ABCD$ is loaded as indicated in Fig. 13-35. Find the ratio $(w_L)L/P_L$ so that the limit load occurs in both $AC$ and $CD$ simultaneously. *Ans.* 2/3

**13.27.** The continuous beam shown in Fig. 13-36 rests on the three simple supports indicated. The span $AC$ has a fully plastic moment $3M_p$ and the lighter span $CD$ has a fully plastic moment $M_p$. A concentrated vertical force acts at the midpoint of $AC$. Find the limit load $P_L$. *Ans.* $P_L = 7M_p/L$

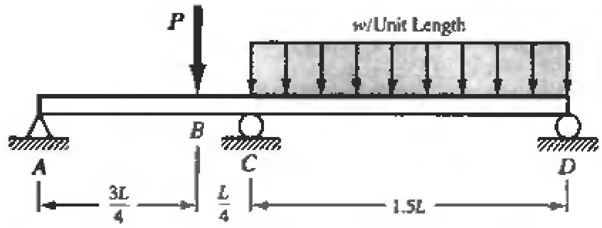



Fig. 13-35

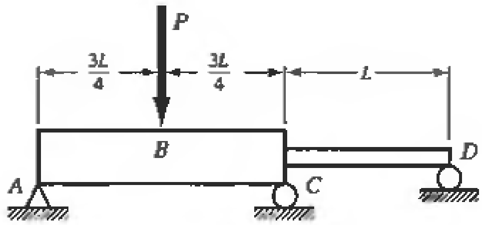



Fig. 13-36

**13.28.** Determine the magnitude of the load $P$ for plastic collapse of the beam shown in Fig. 13-37.

*Ans.* $P_L = \dfrac{6M_p}{L}$

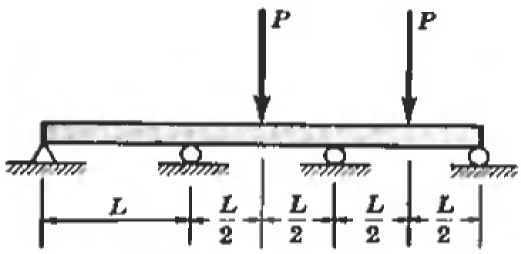



Fig. 13-37

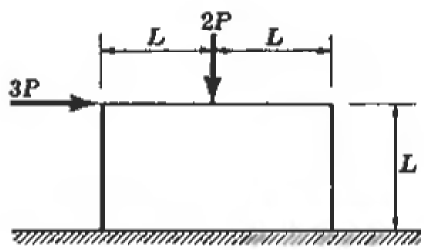



Fig. 13-38

**13.29.** Determine the magnitude of $P$ in Fig. 13-38 for plastic collapse of the rectangular frame having both bases clamped. *Ans.* $P_L = 1.2M_p/L$

**13.30.** Determine the magnitude of $P$ for plastic collapse of the rectangular frame having both bases pinned (Fig. 13-39). *Ans.* $P_L = 4M_p/3L$

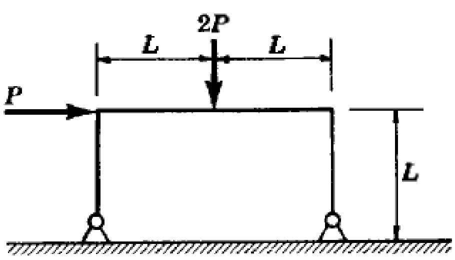



Fig. 13-39

**13.31.** Determine the magnitude of the force $P$ for plastic collapse of the unsymmetric frame having both bases pinned (Fig. 13-40). *Ans.* $P_L = M_p(h_1 + h_2)/h_1 h_2$

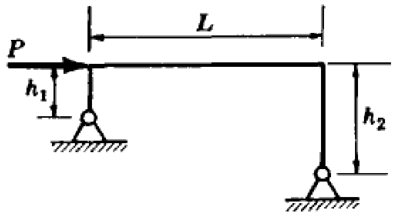



**Fig. 13-40**

**13.32.** See Fig. 13-41. Determine the value of $P$ for plastic collapse of the system.

*Ans.* $P_L = \dfrac{2M_p}{L}$

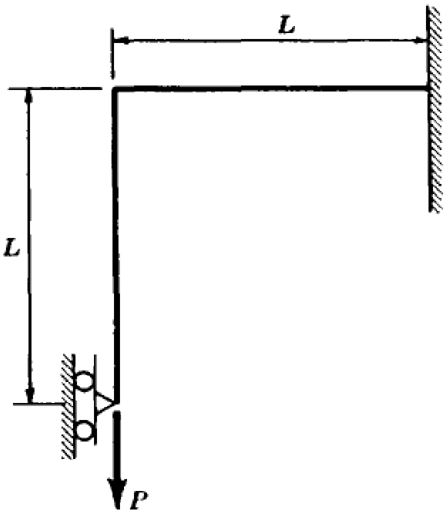



**Fig. 13-41**

# Chapter 14

## Columns

### DEFINITION OF A COLUMN

A long slender bar subject to axial compression is called a *column*. The term "column" is frequently used to describe a vertical member, whereas the word "strut" is occasionally used in regard to inclined bars.

**Examples**

Many aircraft structural components, structural connections between stages of boosters for space vehicles, certain members in bridge trusses, and structural frameworks of buildings are common examples of columns.

### TYPE OF FAILURE OF A COLUMN

Failure of a column occurs by buckling, i.e., by lateral deflection of the bar. In comparison it is to be noted that failure of a short compression member occurs by yielding of the material. Buckling, and hence failure, of a column may occur even though the maximum stress in the bar is less than the yield point of the material. Linkages in oscillating or reciprocating machines may also fail by buckling.

### DEFINITION OF THE CRITICAL LOAD OF A COLUMN

The critical load of a slender bar subject to axial compression is that value of the axial force that is just sufficient to keep the bar in a slightly deflected configuration. Figure 14-1 shows a pin-ended bar in a buckled configuration due to the critical load $P_{cr}$.


**Fig. 14-1**

### SLENDERNESS RATIO OF A COLUMN

The ratio of the length of the column to the minimum radius of gyration of the cross-sectional area is termed the *slenderness ratio of the bar*. This ratio is of course dimensionless. The method of determining the radius of gyration of an area was discussed in Chap. 7.

If the column is free to rotate at each end, then buckling takes place about that axis for which the radius of gyration is a minimum.

## CRITICAL LOAD OF A LONG SLENDER COLUMN

If a long slender bar of constant cross section is pinned at each end and subject to axial compression, the load $P_{cr}$ that will cause buckling is given by

$$P_{cr} = \frac{\pi^2 EI}{L^2} \tag{14.1}$$

where $E$ denotes the modulus of elasticity, $I$ the minimum second moment of area of the cross-sectional area about an axis through the centroid, and $L$ the length of the bar. The derivation of this formula is presented in Problem 14.1.

This formula was first obtained by the Swiss mathematician Leonhard Euler (1707–1783) and the load $P_{cr}$ is called the *Euler buckling load*. As discussed in Problem 14.2, this expression is not immediately applicable if the corresponding axial stress, found from the expression $\sigma_{cr} = P_{cr}/A$, where $A$ represents the cross-sectional area of the bar, exceeds the proportional limit of the material. For example, for a steel bar having a proportional limit of 210 MPa, the above formula is valid only for columns whose slenderness ratio exceeds 100. The value of $P_{cr}$ represented by this formula is a failure load; consequently, a safety factor must be introduced to obtain a design load. Applications of this expression may be found in Problems 14.5 through 14.7.

## INFLUENCE OF END CONDITIONS—EFFECTIVE LENGTH

Equation (*14.1*) may be modified to the form

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} \tag{14.2}$$

where $KL$ is an effective length of the column. For a column pinned at both ends, $K = 1$. If both ends are clamped, $K = 0.5$; for one end clamped and the other pinned, $K = 0.7$. For a column clamped at one end and unsupported at the loaded end, $K = 2$. See Problems 14.1, 14.3, and 14.4.

## DESIGN OF ECCENTRICALLY LOADED COLUMNS

The derivation of the expression leading to the Euler buckling load assumes that the column is loaded perfectly concentrically. If the axial force $P$ is applied with an eccentricity $e$, the peak compressive stress in the bar occurs at the outer fibers at the midpoint of the length of the bar and is given by

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2}\sqrt{\frac{P}{AE}}\right)\right] \tag{14.3}$$

where $c$ is the distance from the neutral axis to the outer fibers, $r$ the radius of gyration, $L$ the length of the column, and $A$ the cross-sectional area. This is the *secant formula* for columns. It is discussed in detail in Problem 14.22.

## INELASTIC COLUMN BUCKLING

The expression for the Euler buckling load may be extended into the inelastic range of action by replacing Young's modulus by the tangent modulus $E_t$. The resulting *tangent-modulus formula* is then

$$P_{cr} = \frac{\pi^2 E_t I}{L^2} \tag{14.4}$$

See Problem 14.9.

## DESIGN FORMULAS FOR COLUMNS HAVING INTERMEDIATE SLENDERNESS RATIOS

The design of compression members having large values of the slenderness ratio proceeds according to the Euler formula presented above together with an appropriate safety factor. For the design of shorter compression members, it is customary to employ any one of the many semiempirical formulas giving a relationship between the yield stress and the slenderness ratio of the bar.

For steel columns, one commonly employed design expression is that due to the American Institute of Steel Construction (AISC), which states that the allowable (working) axial stress on a steel column having slenderness ratio $L/r$ is

$$\sigma_a = \frac{[1-(KL/r)^2]\sigma_{yp}}{\left[\frac{5}{3}+\frac{3(KL/r)}{8C_c}-\frac{(KL/r)^3}{8C_c^3}\right]} \qquad \text{for } \frac{KL}{r} < C_c$$

$$\sigma_a = \frac{\pi^2 E}{(\frac{23}{12})(KL/r)^2} \qquad \text{for } \frac{KL}{r} > C_c \tag{14.5}$$

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_{yp}}} \tag{14.6}$$

where $\sigma_{yp}$ is the yield point of the material and $E$ is Young's modulus. See Problems 14.11, 14.12, 14.13, and 14.14.

Another approach is in the use of the Structural Stability Research Council's (SSRC) equations which give mean axial compressive stress $\sigma_u$ immediately prior to collapse:

$$\begin{aligned}
\sigma_u &= \sigma_{yp} && \text{for } 0 < \lambda < 0.15 \\
\sigma_u &= \sigma_{yp}(1.035 - 0.202\lambda - 0.222\lambda^2) && \text{for } 0.15 \leq \lambda \leq 1.0 \\
\sigma_u &= \sigma_{yp}(-0.111 + 0.636\lambda^{-1} + 0.087\lambda^{-2}) && \text{for } 1.0 \leq \lambda \leq 2.0 \\
\sigma_u &= \sigma_{yp}(0.009 + 0.877\lambda^{-2}) && \text{for } 2.0 \leq \lambda < 3.6 \\
\sigma_u &= \sigma_{yp}\lambda^{-2} \text{ (Euler's curve)} && \text{for } \lambda \geq 3.6
\end{aligned} \tag{14.7}$$

where

$$\lambda = \frac{L}{\pi r}\sqrt{\frac{\sigma_{yp}}{E}} \tag{14.8}$$

No safety factor is present in these equations but of course one must be introduced by the designer. See Problem 14.15.

## COMPUTER IMPLEMENTATION

The design expression advanced by the AISC for allowable (working) stress on a steel column as well as the SSRC's equations giving mean axial compressive stress just prior to collapse are well suited to computer implementation. Problems 14.17 and 14.20, respectively, give FORTRAN programs for each of these recommendations. It is only necessary to input into the self-prompting programs the geometric and materials parameters of the column to obtain its resistance as indicated by each of these sets of relations. For application see Problems 14.18, 14.19, and 14.21.

## BEAM-COLUMNS

Bars subjected to simultaneous axial compression and lateral loading are termed *beam-columns*. An example is given in Problem 14.25.

## BUCKLING OF RIGID SPRING-SUPPORTED BARS

The columns discussed above are *flexible* members, i.e., capable of undergoing lateral bending immediately after buckling. A related type of buckling involves one or more *rigid* bars pinned to fixed supports or to each other and supported by one or more transverse springs. In certain cases the applied loads may cause the bar system to move suddenly to an alternate equilibrium position. This too is a form of instability of the system. See Problem 14.26.

# Solved Problems

**14.1.** Determine the critical load for a long slender pin-ended bar loaded by an axial compressive force at each end. The line of action of the forces passes through the centroid of the cross section of the bar.

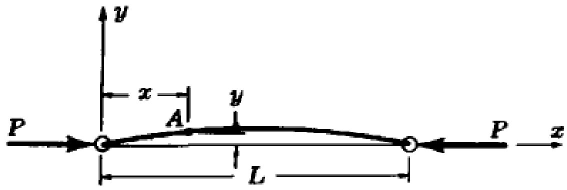



**Fig. 14-2**

The critical load is defined to be that axial force that is just sufficient to hold the bar in a slightly deformed configuration. Under the action of the load $P$ the bar has the deflected shape shown in Fig. 14-2.

It is of course necessary that one end of the bar be able to move axially with respect to the other end in order that the lateral deflection may take place. The differential equation of the deflection curve is the same as that presented in Chap. 9, namely,

$$EI\frac{d^2y}{dx^2} = M \tag{1}$$

Here the bending moment at the point $A$ having coordinates $(x,y)$ is merely the moment of the force $P$ applied at the left end of the bar about an axis through the point $A$ and perpendicular to the plane of the page. It is to be carefully noted that this force produces curvature of the bar that is concave downward, which, according to the sign convention of Chap. 6, constitutes negative bending. Hence the bending moment is $M = -Py$. Thus we have

$$EI\frac{d^2y}{dx^2} = -Py \tag{2}$$

If we set

$$\frac{P}{EI} = k^2 \tag{3}$$

(2) becomes

$$\frac{d^2y}{dx^2} + k^2y = 0 \tag{4}$$

This equation is readily solved by any one of several standard techniques discussed in works on differential equations. However, the solution is almost immediately apparent. We need merely find a function which when differentiated twice and added to itself (times a constant) is equal to zero. Evidently either $\sin kx$ or $\cos kx$ possesses this property. In fact, a combination of these terms in the form

$$y = C \sin kx + D \cos kx \tag{5}$$

may also be taken to be a solution of (4). This may be readily checked by substitution of $y$ as given by (5) into (4).

Having obtained $y$ in the form given in (5), it is next necessary to determine $C$ and $D$. At the left end of the bar, $y = 0$ when $x = 0$. Substituting these values in (5), we obtain

$$0 = 0 + D \qquad \text{or} \qquad D = 0$$

At the right end of the bar, $y = 0$ when $x = L$. Substituting these values in (5) with $D = 0$, we obtain

$$0 = C \sin kL$$

Evidently either $C = 0$ or $\sin kL = 0$. But if $C = 0$ then $y$ is everywhere zero and we have only the trivial case of a straight bar which is the configuration prior to the occurrence of buckling. Since we are not interested in the solution, then we must take

$$\sin kL = 0 \tag{6}$$

For this to be true, we must have

$$kL = n\pi \text{ radians } (n = 1, 2, 3, \ldots) \tag{7}$$

Substituting $k^2 = P/EI$ in (7), we find

$$\sqrt{\frac{P}{EI}}L = n\pi \qquad \text{or} \qquad P = \frac{n^2 \pi^2 EI}{L^2} \tag{8}$$

The smallest value of this load $P$ evidently occurs when $n = 1$. Then we have the so-called first mode of buckling where the critical load is given by

$$P_{cr} = \frac{\pi^2 EI}{L^2} \tag{9}$$

This is called *Euler's buckling load for a pin-ended column*. The deflection shape corresponding to this load is

$$y = C \sin\left(\sqrt{\frac{P}{EI}}x\right) \tag{10}$$

Substituting in this equation from (9), we obtain

$$y = C \sin \frac{\pi x}{L} \tag{11}$$

Thus the deflected shape is in a sine curve. Because of the approximations introduced in the derivation of (1), it is not possible to obtain the amplitude of the buckled shape, denoted by $C$ in (11).

As may be seen from (9), buckling of the bar will take place about that axis in the cross section for which $I$ assumes a minimum value.

Equation (9) may be modified to the form

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} \tag{12}$$

where $KL$ is an effective length of the column, defined to be a portion of the deflected bar between points corresponding to zero curvature. For example, for a column pinned at both ends, $K = 1$. If both ends are rigidly clamped, $K = 0.5$. For one end clamped and the other pinned, $K = 0.7$. In the case of a cantilever-type column loaded at its free end, $K = 2$.

**14.2.** Determine the axial stress in the column considered in Problem 14.1.

In the derivation of the equation $EI(d^2y/dx^2) = M$ used to determine the critical load in Problem 14.1, it was assumed that there is a linear relationship between stress and strain (see Chap. 9). Thus the critical load indicated by (*9*) of Problem 14.1 is correct only if the proportional limit of the material has not been exceeded.

The axial stress in the bar immediately prior to the instant when the bar assumes its buckled configuration is given by

$$\sigma_{cr} = \frac{P_{cr}}{A} \qquad (1)$$

where $A$ represents the cross-sectional area of the bar. Substituting for $P_{cr}$ its value as given by (*9*) of Problem 14.1, we find

$$\sigma_{cr} = \frac{\pi^2 EI}{AL^2} \qquad (2)$$

But from Chap. 7 we know that we may write

$$I = Ar^2 \qquad (3)$$

where $r$ represents the radius of gyration of the cross-sectional area. Substituting this value in (*2*), we find

$$\sigma_{cr} = \frac{\pi^2 EAr^2}{AL^2} = \pi^2 E\left(\frac{r}{L}\right)^2 \qquad (4)$$

or

$$\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2} \qquad (5)$$

The ratio $L/r$ is called the *slenderness ratio* of the column.

Let us consider a steel column having a proportional limit of 210 MPa and $E = 200$ GPa. The stress of 210 MPa marks the upper limit of stress for which (*5*) may be used. To find the value of $L/r$ corresponding to these constants, we substitute in (*5*) and obtain

$$210 \times 10^6 = \frac{\pi^2(200 \times 10^9)}{(L/r)^2} \qquad \text{or} \qquad \frac{L}{r} \approx 100$$

Thus for this material the buckling load as given by (*9*) of Problem 14.1 and the axial stress as given by (*5*) are valid only for those columns having $L/r \geq 100$. For those columns having $L/r < 100$, the compressive stress exceeds the proportional limit before elastic buckling takes place and the above equations are not valid.

Equation (*5*) may be plotted as shown in Fig. 14-3. For the particular values of proportional limit and modulus of elasticity assumed above, the portion of the curve to the left of $L/r = 100$ is not valid. Thus for this material, point $A$ marks the upper limit of applicability of the curve.

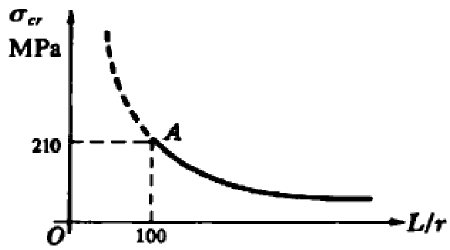



Fig. 14-3

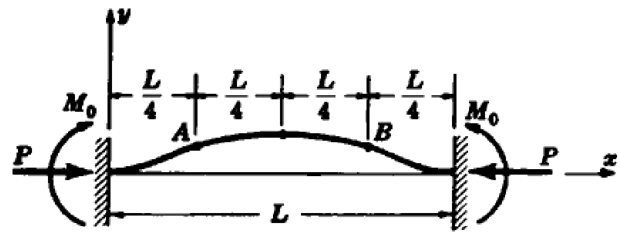



Fig. 14-4

**14.3.** Determine the critical load of a long, slender bar clamped at each end and subject to axial thrust as shown in Fig. 14-4.

Let us introduce the $x$-$y$ coordinate system shown in Fig. 14-4 and let $(x, y)$ represent the coordinates of an arbitrary point on the bar. The bending moment at this point is found as the sum of the moments of the forces to the left of this section about an axis through this point and perpendicular to the plane of the page. Hence at this point we have $M = -Py + M_0$. The differential equation for the bending of the bar is then $EI d^2y/dx^2 = -Py + M_0$, or

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{M_0}{EI} \tag{1}$$

As discussed in texts on differential equations, the solution to (*1*) consists of two parts. The first part is merely the solution of the so-called homogeneous equation obtained by setting the right-hand side of (*1*) equal to zero. We must then solve the equation

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0 \tag{2}$$

But the solution to this equation has already been found in Problem 14.1 to be

$$y = A_1 \cos\left(\sqrt{\frac{P}{EI}}x\right) + B_1 \sin\left(\sqrt{\frac{P}{EI}}x\right) \tag{3}$$

The second part of the solution of (*1*) is given by a so-called particular solution, i.e., any function satisfying (*1*). Evidently one such function is given by

$$y = \frac{M_0}{P}(= \text{constant}) \tag{4}$$

The general solution of (*1*) is given by the sum of the solutions represented by (*3*) and (*4*), or

$$y = A_1 \cos\left(\sqrt{\frac{P}{EI}}x + B_1 \sin \sqrt{\frac{P}{EI}}x\right) + \frac{M_0}{P} \tag{5}$$

Consequently

$$\frac{dy}{dx} = -A_1\sqrt{\frac{P}{EI}}\sin\left(\sqrt{\frac{P}{EI}}x\right) + B_1\sqrt{\frac{P}{EI}}\cos\left(\sqrt{\frac{P}{EI}}x\right) \tag{6}$$

At the left end of the bar we have $y = 0$ when $x = 0$. Substituting these values in (5), we find $0 = A_1 + M_0/P$. Also, at the left end of the bar we have $dy/dx = 0$ when $x = 0$; substituting in (*6*), we obtain $0 = 0 + B_1\sqrt{P/EI}$ or $B_1 = 0$.

At the right end of the bar we have $dy/dx = 0$ when $x = L$; substituting in (*6*), with $B_1 = 0$, we find

$$0 = -A_1\sqrt{\frac{P}{EI}}\sin\left(\sqrt{\frac{P}{EI}}L\right)$$

But $A_1 = -M_0/P$ and since this ratio is not zero, then $\sin(\sqrt{P/EI}\,L) = 0$. This occurs only when $\sqrt{P/EI}\,L = n\pi$ where $n = 1, 2, 3, \ldots$. Consequently

$$P_{cr} = \frac{n^2\pi^2 EI}{L^2} \tag{7}$$

For the so-called first mode of buckling illustrated in Fig. 14-4, the deflection curve of the bent bar has a horizontal tangent at $x = L/2$; that is, $dy/dx = 0$ there. Equation (*6*) now takes the form

$$\frac{dy}{dx} = \frac{M_0}{P}\left(\frac{n\pi}{L}\right)\sin\frac{n\pi x}{L} \tag{6'}$$

and since $dy/dx = 0$ at $x = L/2$, we find

$$0 = \frac{M_0}{P}\left(\frac{n\pi}{L}\right)\sin\frac{n\pi}{2}$$

The only manner in which this equation may be satisfied is for $n$ to assume even values; that is, $n = 2, 4, 6, \ldots$.

Thus for the smallest possible value of $n = 2$, Eq. (7) becomes

$$P_{cr} = \frac{4\pi^2 EI}{L^2}$$

**14.4.** Determine the critical load for a long slender bar clamped at one end, free at the other, and loaded by an axial compressive force applied at the free end.


**Fig. 14-5**

The critical load is that axial compressive force $P$ that is just sufficient to keep the bar in a slightly deformed configuration, as shown in Fig. 14-5. The moment $M_0$ represents the effect of the support in preventing any angular rotation of the left end of the bar.

Inspection of the above deflection curve for the buckled column indicates that the entire bar corresponds to one-half of the deflected pin-ended bar discussed in Problem 14.1. Thus for the column under consideration, the length $L$ corresponds to $L/2$ for the pin-ended column. Hence the critical load for the present column may be found from Eq. (9), Problem 14.1, by replacing $L$ by $2L$. This yields

$$P_{cr} = \frac{\pi^2 EI}{(2L)^2} = \frac{\pi^2 EI}{4L^2}$$

**14.5.** A steel bar of rectangular cross section 40 mm × 50 mm and pinned at each end is subject to axial compression. If the proportional limit of the material is 230 MPa and $E = 200$ GPa, determine the minimum length for which Euler's equation may be used to determine the buckling load.

The minimum second moment of area is $I = \frac{1}{12}bh^3 = \frac{1}{12}(50)(40)^3 = 2.67 \times 10^5\ \text{mm}^4$. Hence the least radius of gyration is

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{2.67 \times 10^5}{(40)(50)}} = 11.5\ \text{mm}$$

The axial stress for such an axially loaded bar was found in Problem 14.2 to be

$$\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2}$$

The minimum length for which Euler's equation may be applied is found by placing the critical stress in the above formula equal to 230 MPa. Doing this, we obtain

$$230 \times 10^6 = \frac{\pi^2(200 \times 10^9)}{(L/11.5)^2} \qquad \text{or} \qquad L = 1.065\ \text{m}$$

**14.6.** Consider again a rectangular steel bar 40 mm × 50 mm in cross section, pinned at each end and subject to axial compression. The bar is 2 m long and $E = 200$ GPa. Determine the buckling load using Euler's formula.

The *minimum* second moment of area of this cross section was found in Problem 14.5 to be $2.67 \times 10^5\ \text{mm}^4$. Applying the expression for buckling load given in (9) of Problem 14.1, we find

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(200 \times 10^9)(10^{-6})(2.67 \times 10^5)}{(2 \times 10^3)^2} = 132\ \text{kN}$$

The axial stress corresponding to this load is

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{132 \times 10^3}{(40)(50)} = 66\ \text{MPa}$$

**14.7.** Determine the critical load for a W10 × 21 section acting as a pinned end column. The bar is 12 ft long and $E = 30 \times 10^6\ \text{lb/in}^2$. Use Euler's theory.

From Table 8-1 of Chap. 8 we find the minimum moment of inertia to be $9.7\ \text{in}^4$. Thus,

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$= \frac{\pi^2(30 \times 10^6\ \text{lb/in}^2)(9.7\ \text{in}^4)}{(144\ \text{in})^2} = 138{,}000\ \text{lb}$$

**14.8.** A long thin bar of length $L$ and rigidity $EI$ is pinned at end $A$, and at the end $B$ rotation is resisted by a restoring moment of magnitude $\lambda$ per radian of rotation at that end. Derive the equation for the axial buckling load $P$. Neither $A$ nor $B$ can displace laterally, but $A$ is free to approach $B$.


Fig. 14-6

The buckled bar is shown in Fig. 14-6, where $M_L$ represents the restoring moment. The differential equation of the buckled bar is

$$EI\frac{d^2y}{dx^2} = Vx - Py$$

or

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{V}{EI}x$$

Let $\alpha^2 = P/EI$. Then

$$\frac{d^2y}{dx^2} + \alpha^2 y = \frac{V}{EI}x$$

The general solution of this equation is easily found to be

$$y = A\sin\alpha x + B\cos\alpha x + \frac{V}{P}x \qquad (1)$$

As the first boundary condition, when $x = 0$, $y = 0$; hence $B = 0$. As the second boundary condition, when $x = L$, $y = 0$; hence from (*1*) we obtain

$$0 = A \sin \alpha L + \frac{VL}{P} \qquad \text{or} \qquad \frac{V}{P} = -\frac{A}{L} \sin \alpha L$$

Thus

$$y = A\left[\sin \alpha x - \frac{x}{L} \sin \alpha L\right] \tag{2}$$

From (*2*) the slope at $x = L$ is found to be

$$\left[\frac{dy}{dx}\right]_{x=L} = A\left[\alpha \cos \alpha L - \frac{1}{L} \sin \alpha L\right] \tag{3}$$

The restoring moment at end $B$ is thus

$$M_L = A\lambda\left[\alpha \cos \alpha L - \frac{1}{L} \sin \alpha L\right] \tag{4}$$

Also, since in general $M = EI(d^2y/dx^2)$, from (2) we have

$$M_L = -A\alpha^2 EI \sin \alpha L \tag{5}$$

Equating expressions (*4*) and (*5*) after carefully noting that as $M_L$ increases $dy/dx$ at that point decreases (necessitating the insertion of a negative sign), we have

$$-A\alpha^2 EI \sin \alpha L = -\left[A\lambda\alpha \cos \alpha L - \frac{A\lambda}{L} \sin \alpha L\right] \tag{6}$$

Simplifying, the equation for determination of the buckling load $P$ becomes

$$\frac{PL}{\lambda} - \alpha L \cot \alpha L + 1 = 0 \tag{7}$$

This equation would have to be solved numerically for specific values of $EI$, $L$, and $\lambda$.

**14.9.** Discuss column behavior when the average applied axial stress in the bar exceeds the proportional limit of the material.

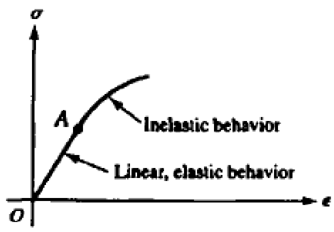



**Fig. 14-7**

The Euler buckling load determined in Problem 14.1 is based upon the assumption that the column everywhere is acting within the linear elastic range of action of the material, shown as $OA$ in Fig. 14-7. In this range the modulus $E$ is the slope of the straight line $OA$. When the stress-strain curve ceases to be linear, i.e., to the right of point $A$, the slope of the curve is called the *tangent modulus* $E_t$ and it varies with strain. This parameter must be determined by materials tests. Under these conditions it is necessary to consider inelastic buckling. One of the earliest approaches to this, still used occasionally, is due to the German engineer Engesser who, in 1889, suggested replacing $E$ in Euler's expression, Eq. (*9*) of Problem 14.1, by the *tangent modulus* $E_t$. In this case the axial stress immediately prior to buckling is given by

$$\sigma_{cr} = \frac{\pi^2 E_t}{(L/r)^2}$$

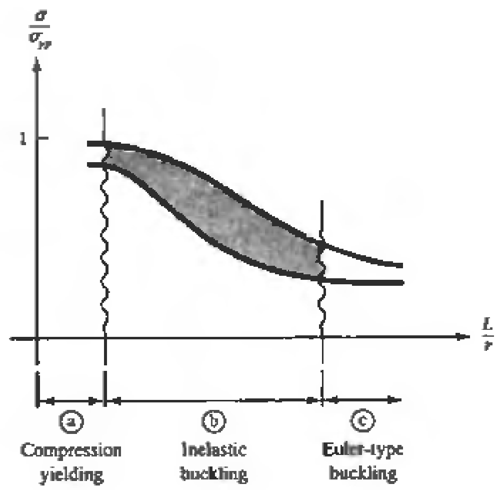



**Fig. 14-8**

This is the *tangent modulus formula* and the load $P_{cr} = A\sigma_{cr}$ is called the *Engesser load*. This approach is simple and easy to use—see Problem 14.10—and indicates a load only slightly less than the inelastic buckling load found experimentally. The theory has certain inconsistencies that will not be discussed here so it is not the best approach to rational column design.

Test results on axially compressed bars usually can be exhibited by the plot shown in Fig. 14-8, where the mean axial stress $\sigma$ just before buckling (divided by the yield point of the material) is shown as a function of the slenderness ratio $L/r$. Experimental results indicate wide scatter, as shown by data points between the two solid curves. The scatter is due to initial geometric deviations from straightness of the bar as well as residual stresses incurred during fabrication. The plot indicates three modes of failure, depending on the value of $L/r$. The first is ⓐ, compressive yielding for very short columns; the second is ⓑ, inelastic buckling for intermediate length bars (which comprise many engineering applications); and the third is ⓒ, Euler-type buckling of very long slender bars. Failures of type ⓐ have been discussed in Chap. 1 and Euler column behavior was treated in Problems 14.1 through 14.7. The rational design of columns corresponding to condition ⓑ is based upon any one of a number of semiempirical approaches discussed in the following problems.

**14.10.** A pinned end column is 275 mm long and has a solid circular cross section. If it must support an axial load of 250 kN, determine the required radius of the rod if the tangent modulus theory is employed and the experimentally determined curve relating tangent modulus to axial stress is that shown in Fig. 14-9.

From Problem 14.9 the load, according to the tangent modulus theory, is given by

$$P_{cr} = (A)\frac{\pi^2 E_t}{(L/r)^2} = \frac{\pi^2 E_t I}{L^2} \qquad (1)$$

For the solid circular cross section of radius $R$, we have $I = \pi R^4/4$ so that (*1*) becomes

$$E_t = \frac{(250{,}000\ \text{N})(0.275\ \text{m})^2}{\pi^2(\pi R^4/4)} = \frac{2439}{R^4}\ \text{N/m}^2 \qquad (2)$$

For any assumed radius $R$ it is easily possible to find the axial stress:

$$\sigma = \frac{P}{A} = \frac{250{,}000}{\pi R^2} \qquad (3)$$

and for any value of $\sigma$ from Fig. 14-9 we can ascertain the corresponding experimentally determined value of $E_t$. Thus, we can solve Eqs. (*2*) and (*3*) by trial and error.

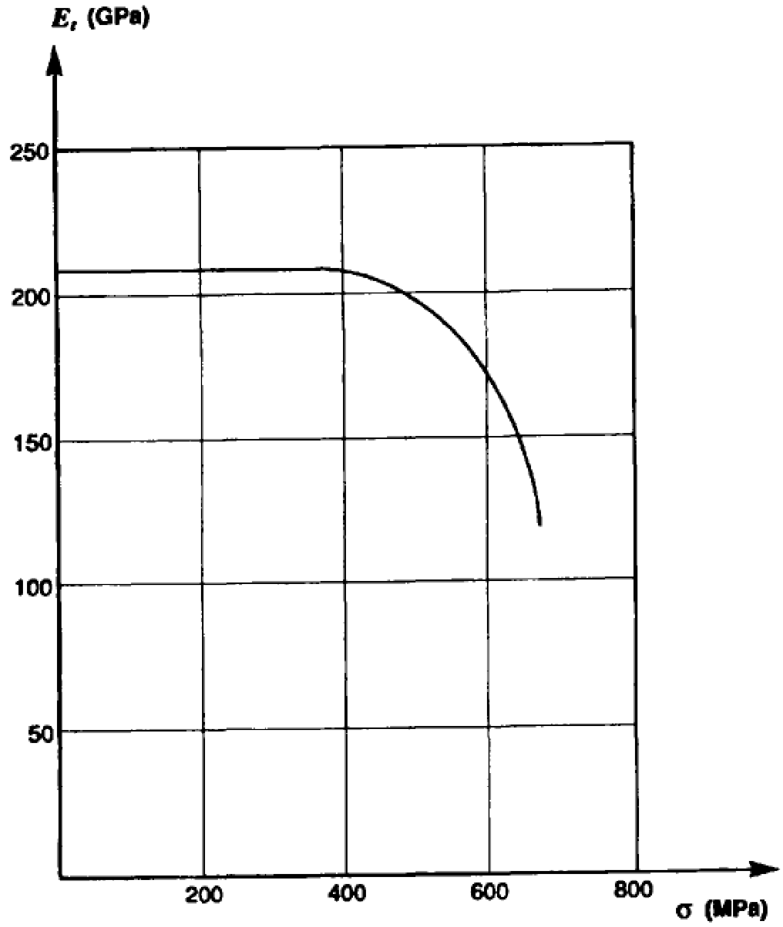



**Fig. 14-9**

Let us try $R = 0.012$ m. From Eq. (*3*)

$$\sigma = \frac{250{,}000}{\pi(0.012\ \text{m})^2} = 553\ \text{MPa}$$

For this value of $\sigma$ from Fig. 14-9, we have $E_t = 175$ GPa. However, from Eq. (*2*) it is

$$E_t = \frac{2439}{(0.012\ \text{m})^4} = 117\ \text{GPa}$$

Clearly these values of $E_t$ do not agree and the assumed radius is too large.

Next, let us try $R = 0.011$ m. From Eq. (*3*)

$$\sigma = \frac{250{,}000}{\pi(0.011\ \text{m})^2} = 658\ \text{MPa}$$

For this value of $\sigma$ from Fig. 14-9, we have $E_t = 125$ GPa. However, from Eq. (*2*) it is

$$E_t = \frac{2439}{(0.011\ \text{m})^4} = 167\ \text{GPa}$$

It is instructive to plot these values as shown in Fig. 14-10. Clearly an acceptable value of radius lies between 0.011 and 0.012 m. Let us try $R = 0.0112$ m. From Eq. (*3*) we have

$$\sigma = \frac{250{,}000}{\pi(0.0112)^2} = 634\ \text{MPa}$$

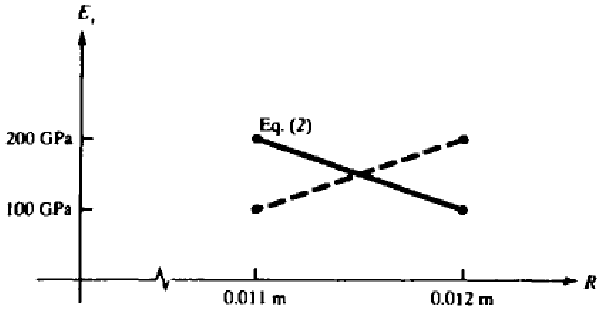



**Fig. 14-10**

and the corresponding value of $E_t$ from Fig. 14-10 is 152 GPa. The value found from Eq. (2) is

$$E_t = \frac{2439}{(0.0112\ \text{m})^4} = 155\ \text{GPa}$$

These two values of $E_t$ are sufficiently close that we may regard the radius of 0.0112 m as acceptable, that is, 11.2 mm.

**14.11.** Discuss design criteria for structural steel columns.

In one approach, advocated by the AISC, the allowable axial compressive stress $\sigma_a$ on a steel column of length $L$, minimum radius of gyration of cross section $r$, material yield point $\sigma_{yp}$, and Young's modulus $E$ is given by the semiempirical relations

$$\sigma_a = \frac{\left[1 - \dfrac{(KL/r)^2}{2C_c^2}\right]\sigma_{yp}}{\left[\dfrac{5}{3} + \dfrac{3(KL/r)}{8C_c} - \dfrac{(KL/r)^3}{8C_c^3}\right]} \qquad \text{for } \frac{KL}{r} < C_c \tag{1}$$

$$\sigma_a = \frac{\pi^2 E}{(\frac{23}{12})(KL/r)^2} \qquad \text{for } \frac{KL}{r} > C_c \tag{2}$$

where

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_{yp}}} \tag{3}$$

Here $K$ is the end fixity coefficient introduced in Problem 14.1. These equations may be used with either the SI or USCS systems of units. In Eqs. (1) and (2) the denominators represent safety factors which clearly increase with increasing values of the slenderness ratio $L/r$.

The second approach, which is perhaps in best agreement with experimental evidence, is due to R. Bjorhovde* who, in 1971, analyzed the behavior of a large number of full-scale test columns all having measured initial imperfections from perfect straightness as well as residual (fabrication) stresses. These columns were relatively light- or medium-weight hot-rolled wide-flange W sections having flange thicknesses less than 2 in (50.8 mm) and material yield points less than approximately 49,000 lb/in (335 MPa). He found that the mean (over the cross section) axial compressive stress $\sigma_u$ just prior to collapse is given by the expressions

$$\begin{aligned}
\sigma_u &= \sigma_{yp} && \text{for } 0 < \lambda < 0.15 \\
\sigma_u &= \sigma_{yp}(1.035 - 0.202\lambda - 0.222\lambda^2) && \text{for } 0.15 \leq \lambda \leq 1.0 \\
\sigma_u &= \sigma_{yp}(-0.111 + 0.636\lambda^{-1} + 0.087\lambda^{-2}) && \text{for } 1.0 \leq \lambda \leq 2.0 \\
\sigma_u &= \sigma_{yp}(0.009 + 0.877\lambda^{-2}) && \text{for } 2.0 \leq \lambda < 3.6 \\
\sigma_u &= \sigma_{yp}\lambda^{-2} \text{ (Euler's curve)} && \text{for } \lambda \geq 3.6
\end{aligned} \tag{4}$$

*R. Bjorhovde and L. Tall, "Minimum Column Strength and Multiple Column Curve Concept," Report 337.29, Lehigh University, Fritz Eng. Lab. Bethlehem, PA, 1971. R. Bjorhovde, "Deterministic and Probabilistic Approaches to the Strength of Steel Columns," Ph.D. dissertation, Lehigh University, Bethlehem, PA, 1972.

where
$$\lambda = \frac{L}{\pi r}\sqrt{\frac{\sigma_{yp}}{E}} \tag{5}$$

These results, known in graphical form as the Structural Stability Research Council Curve No. 2, represent prototype behavior of steel columns in region ⓑ of Fig. 14-8. The equations may be used with either the SI or USCS systems of units. Since the stress $\sigma_u$ in Eqs. (*4*) is that existing just prior to collapse, no safety factor is present but instead must be introduced by the designer. Two comparable sets of equations were given by Bjorhovde for other types of steel sections.

**14.12.** Use the AISC design recommendation discussed in Problem 14.11 to determine the allowable axial load on a W8 × 19 section 10 ft long. The ends are pinned, the yield point is 36,000 lb/in², and $E = 30 \times 10^2$ lb/in.

From Table 8-1 Chap. 8 we have the properties of the cross section as

$$I_{min} = 7.9\ \text{in}^4 \qquad A = 5.59\ \text{in}^2$$

The radius of gyration is found by the method of Chap. 7 to be

$$r = \sqrt{\frac{7.9\ \text{in}^4}{5.59\ \text{in}^2}} = 1.189\ \text{in}$$

Thus,
$$\frac{L}{r} = \frac{(10)(12)}{1.189} = 100.9$$

From Problem 14.11 we have from Eq. (*3*)

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_{yp}}} = \sqrt{\frac{2\pi^2(30 \times 10^6\ \text{lb/in}^2)}{36{,}000\ \text{lb/in}^2}} = 128.26$$

For both ends pinned, $K = 1$ and thus $K(L/r) < C_c$ so that the allowable axial stress is given by Eq. (*1*) of Problem 14.11 to be

$$\sigma_a = \frac{\left[1 - \dfrac{(KL/r)^2}{2C_c^2}\right]\sigma_{yp}}{\dfrac{5}{3} + \dfrac{3(KL/r)}{8C_c} - \dfrac{(KL/r)^3}{8C_c^3}} = \frac{\left[1 - \dfrac{(100.9)^2}{2(128.26)^2}\right](36{,}000)}{\dfrac{5}{3} + \dfrac{3(100.9)}{8(128.26)} - \dfrac{(100.0)^3}{8(128.26)^3}}$$
$$= 13{,}100\ \text{lb/in}^2$$

The allowable axial load is

$$P_a = (5.59\ \text{in}^2)(13{,}100\ \text{lb/in}^2) = 73{,}100\ \text{lb}$$

**14.13.** Reconsider the column of Problem 14.12 but now with a length of 15 ft. Use the AISC design recommendation to determine the allowable axial load. Both ends are pinned.

Now we have $L/r = (15)(12)/1.189 = 151.4$. Thus the increased length (in comparison to that of Problem 14.12) leads to

$$K\frac{L}{r}(= 151.4) > C_c(= 128.26)$$

so that we must compute the allowable axial stress from Eq. (*2*) of Problem 14.11:

$$\sigma_a = \frac{12\pi^2 E}{23(KL/r)^2}$$
$$= \frac{12\pi^2(30 \times 10^6\ \text{lb/in}^2)}{23(151.4)^2} = 6740\ \text{lb/in}^2$$

The allowable axial load is thus

$$P_a = (5.59\ \text{in}^2)(6740\ \text{lb/in}^2) = 37{,}670\ \text{lb}$$

**14.14.** Use the AISC recommendation to determine the allowable axial load on a W203 × 28 section 3 m long. The ends are pinned. The material yield point is 250 MPa and $E = 200$ GPa.

From Table 8-2 of Chap. 8 we have the sectional properties as

$$I_{\min} = 3.28 \times 10^6\ \text{mm}^4 \qquad A = 3600\ \text{mm}^2$$

The radius of gyration is found to be

$$r = \sqrt{\frac{3.28 \times 10^6\ \text{mm}^4}{3600\ \text{mm}^2}} = 30.18\ \text{mm}$$

Thus

$$\frac{L}{r} = \frac{3000\ \text{mm}}{30.18\ \text{mm}} = 99.4$$

From Problem 14.11, Eq. (3), we have

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_{yp}}} = \sqrt{\frac{2\pi^2(200 \times 10^9\ \text{N/m}^2)}{250 \times 10^6\ \text{N/m}^2}} = 125.7$$

For both ends pinned, $K = 1$ and thus $K(L/r) < C_c$ so that the allowable axial stress is given by Eq. (1) of Problem 14.11 to be

$$\sigma_a = \frac{\left[1 - \frac{(KL/r)^2}{2C_c^2}\right]\sigma_{yp}}{\frac{5}{3} + \frac{3(KL/r)}{8C_c} - \frac{(KL/r)^3}{8C_c^3}} = \frac{\left[1 - \frac{(99.4)^2}{2(125.7)^2}\right]250 \times 10^6\ \text{N/m}^2}{\frac{5}{3} + \frac{3(99.4)}{8(125.7)} - \frac{(99.4)^3}{8(125.7)^3}}$$

$$= 90.35\ \text{MPa}$$

The allowable axial load is

$$P = (3600\ \text{mm}^2)\left(\frac{\text{m}}{10^3\ \text{mm}}\right)^2(90.35 \times 10^6\ \text{N/m}^2)$$

$$= 325{,}000\ \text{N} \qquad \text{or} \qquad 325\ \text{kN}$$

**14.15.** Reconsider the column of Problem 14.12 but now use the SSRC recommendation discussed in Problem 14.11 to estimate the maximum load-carrying capacity of the column.

As discussed in Problem 14.11, we must first compute the parameter

$$\lambda = \frac{KL}{r} \cdot \frac{1}{\pi}\sqrt{\frac{\sigma_{yp}}{E}}$$

Here,

$$\lambda = \frac{(1)(10\ \text{ft})(12\ \text{in/ft})}{(1.189)} \cdot \frac{1}{\pi}\sqrt{\frac{36{,}000\ \text{lb/in}^2}{30 \times 10^6\ \text{lb/in}^2}} = 1.113$$

From Problem 14.11, for this value of $\lambda$ we must determine the ultimate (peak) axial stress in the column from the semiempirical relation

$$\sigma_u = \sigma_{yp}\left[-0.111 + \frac{0.636}{\lambda} + \frac{0.087}{\lambda^2}\right]$$

$$= (36{,}000\ \text{lb/in}^2)\left[-0.111 + \frac{0.636}{1.113} + \frac{0.087}{(1.113)^2}\right] = 19{,}000\ \text{lb/in}^2$$

The axial load corresponding to this stress is

$$P_{max} = (5.59\ \text{in}^2)(19{,}000\ \text{lb/in}^2) = 106{,}200\ \text{lb}$$

This load represents the average of actual test values of peak loads that columns of this type were found to carry. It is to be noted that no safety factor is incorporated into these computations, so that the design load for this member is less than the 106,600 lb.

**14.16.** Select a wide-flange section from Table 8-2 of Chap. 8 to carry an axial compressive load of 750 kN. The column is 3.5 m long with a yield point of 250 MPa and a modulus of 200 GPa. Use the AISC specifications. The bar is pinned at each end.

To get a first approximation, let us merely use $P = A\sigma$, from which we have

$$A = \frac{750{,}000\ \text{N}}{250 \times 10^6\ \text{N/m}^2} = 0.0030\ \text{m}^2 \qquad \text{or} \qquad 3000\ \text{mm}^2$$

This tells us that any wide-flange section having an area smaller than 3000 mm is unacceptable.

Next, let us try the W203 × 28 section. From Table 8-2 we find area = 3600 mm$^2$ and $I_{min} = 3.28 \times 10^6\ \text{mm}^4$. The minimum radius of gyration is thus

$$r = \sqrt{\frac{3.28 \times 10^6\ \text{mm}^4}{3600\ \text{mm}^2}} = 30.2\ \text{mm}$$

from which the slenderness ratio is $L/r = 3500/30.2 = 116$.

From Problem 14.11, (Eq. (*3*)), we have

$$C_c = \sqrt{\frac{2\pi^2(200 \times 10^9\ \text{N/m}^2)}{250 \times 10^6\ \text{N/m}^2}} = 125.6$$

Thus, since $K = 1$ for both ends pinned,

$$K\frac{L}{r}(= 116) < C_c(= 125.6)$$

So, we must employ Eq. (*1*) of Problem 14.11. This leads to

$$\sigma_a = \frac{\left[1 - \dfrac{(116)^2}{2(125.6)^2}\right]250}{\left[\dfrac{5}{3} + \dfrac{3(116)}{8(125.6)} - \dfrac{(116)^3}{8(125.6)^3}\right]} = 74.95\ \text{MPa}$$

from which $$P_a = (3600\ \text{mm}^2)\left(\frac{\text{m}}{10^3\ \text{mm}}\right)^2(74.95 \times 10^6\ \text{N/m}^2) = 270{,}000\ \text{N} \qquad \text{or} \qquad 270\ \text{kN}$$

which indicates that this is far too light a section.

Next, let us try the section W254 × 72 having an area of 9280 mm$^2$ and $I_{min} = 38.6 \times 10^6\ \text{mm}^4$. The minimum radius of gyration is found to be

$$r = \sqrt{\frac{38.6 \times 10^6\ \text{mm}^4}{9280\ \text{mm}^2}} = 64.5\ \text{mm}$$

from which the slenderness ratio is 3500/64.5 = 54.26. Again we have

$$K\frac{L}{R}(= 54.26) < C_c(= 125.6)$$

so that we must again use Eq. (*1*) of Problem 14.11 to find the allowable stress which is

$$\sigma_a = \frac{\left[1 - \dfrac{(54.26)^2}{2(125.6)^2}\right]250}{\left[\dfrac{5}{3} + \dfrac{3(54.26)}{8(125.6)} - \dfrac{(54.26)^3}{8(125.6)^3}\right]} = 124.6\ \text{MPa}$$

for which $$P_a = (9280\ \text{mm}^2)\left(\frac{1}{10^3\ \text{mm}}\right)^2(124.6\times 10^6\ \text{N/m}^2) = 1.15\times 10^6\ \text{N} \qquad \text{or} \qquad 1150\ \text{kN}$$

This section is rather heavy, so let us investigate the W254 × 54. Here, the area is 7010 mm$^2$ and $I_{min} = 17.5\times 10^6$ mm$^4$. So, the minimum radius of gyration is found to be 50.0 mm and the slenderness ratio is 3500/50 = 70. Again using Eq. (*1*) of Problem 14.11 we find $\sigma_a = 114$ MPa, from which the allowable load is $P_a = 799$ kN.

Investigation of the next lighter section, W254 × 43, by the above method indicates that it can carry only 478 kN.

Thus, the desired section is the W254 × 54, which can carry an axial load of 799 kN, which is in excess of the 750 kN required. A more complete table of structural shapes might well indicate a slightly lighter section than the W254 × 54.

**14.17.** Develop a FORTRAN program to represent the AISC value of allowable axial load on a steel column as discussed in Problem 14.11.

The symbols are defined in Problem 14.11 and Eqs. (*1*) and (*2*) of that problem indicate allowable axial compressive stress for values of $KL/r$ less than or greater than the dimensionless parameter $C_c$. The program listing is

```
**********************************************************************
          PROGRAM STEELCL (INPUT,OUTPUT)
*                   (AMERICAN INSTITUTE OF STEEL CONSTRUCTION)
**********************************************************************
*
*         AUTHOR: KATHLEEN DERWIN
*         DATE  : JANUARY 24, 1989
*
*  BRIEF DESCRIPTION:
*      ONE APPROACH TO CONSIDERING DESIGN CRITERIA FOR STRUCTURAL
*  STEEL COLUMNS IS GIVEN BY THE A.I.S.C. (AMERICAN INSTITUTE OF
*  STEEL CONSTRUCTION). THIS PROGRAM DETERMINES THE ALLOWABLE AXIAL
*  COMPRESSIVE STRESS AND LOADING OF A STEEL COLUMN USING THE RELATIO
*  DEVELOPED AND ACCEPTED BY THE A.I.S.C.
*
*  INPUT:
*      THE USER IS FIRST ASKED IF USCS OR SI UNITS WILL BE USED. THEN,
*  THE COLUMN LENGTH, THE MINIMUM MOMENT OF INERTIA AND AREA OF THE
*  COLUMN CROSS SECTION, THE MATERIAL YIELD POINT, AND YOUNG'S MODULUS
*  ARE INPUTTED. ALSO, THE END FIXITY COEFFICIENT IS ENTERED.
*
*  OUTPUT:
*      THE ALLOWABLE AXIAL COMPRESSIVE STRESS AND LOADING OF THE COLUMN
*  IS DETERMINED.
*
*  VARIABLES:
*      ANS     ---    DENOTES IF USCS OR SI UNITS ARE DESIRED
*     L,I,A    ---    LENGTH, MIN.MOMENT OF INERTIA, AREA OF COLUMN X-SECT
*    SIGYP,E   ---    YIELD POINT, YOUNG'S MODULUS OF THE MATERIAL
*       R      ---    MIN. RADIUS OF GYRATION AS CALCULATED FROM THE
*                     CROSS-SECTIONAL AREA AND MOMENT OF INERTIA
*      CC      ---    CRITICAL CONSTANT OF THE COLUMN...A FUNCTION OF ITS
*                     PHYSICAL AND MATERIAL PROPERTIES
*     CHECK    ---    THE COLUMN CONSTANT AS CALCULATED FOR THE SPECIFIC
```

```
*                       CASE CONSIDERED. THIS IS COMPARED TO THE CRITICAL
*                       CONSTANT TO DETERMINE WHICH OF TWO RELATIONS TO USE
*     K      ---        END FIXITY COEFFICIENT OF THE COLUMN
* HOLD1,HOLD2---        PARTIAL CALCULATIONS OF THE MORE COMPLICATED FUNCTIO
*                       (USED FOR EASE IN PROGRAMMING)
*    SIGA    ---        ALLOWABLE AXIAL COMPRESSIVE STRESS
*   LOADA    ---        ALLOWABLE AXIAL LOAD
*     PI     ---        3.14159
*
*************************************************************************
******                        MAIN PROGRAM                         *****
*************************************************************************
*
*           VARIABLE DECLARATIONS
*
      REAL L,I,A,SIGYP,E,R,CHECK,CC,K,SIGA,LOADA,PI,HOLD1,HOLD2
      INTEGER ANS
*
      PI = 3.14159
*
*           USER INPUT
*
      PRINT*,'PLEASE INDICATE YOUR CHOICE OF UNITS:'
      PRINT*,'1 - USCS'
      PRINT*,'2 - SI'
      PRINT*,' '
      PRINT*,'ENTER 1,2'
      READ*,ANS
      IF (ANS.EQ.1) THEN
         PRINT*,'PLEASE INPUT ALL DATA IN UNITS OF POUND AND/OR INCH...'
      ELSE
         PRINT*,'PLEASE INPUT ALL DATA IN UNITS OF NEWTON AND/OR METER..
      ENDIF
      PRINT*,' '
      PRINT*,'ENTER COLUMN LENGTH:'
      READ*,L
      PRINT*,'ENTER THE CROSS-SECTIONAL PROPERTIES...'
      PRINT*,'MOMENT OF INERTIA, I:'
      READ*,I
      PRINT*,'AREA:'
      READ*,A
      PRINT*,'ENTER THE MATERIAL YIELD POINT:'
      READ*,SIGYP
      PRINT*,'ENTER THE VALUE FOR YOUNG'S MODULUS:'
      READ*,E
      PRINT*,'FINALLY, ENTER THE END FIXITY COEFFICIENT, K:'
      READ*,K
*
*             END USER INPUT
*
*
******        CALCULATIONS       ******
*
*        MINIMUM RADIUS OF GYRATION
*
      R = (I/A)**0.5
*
*         CRITICAL CONSTANT FOR THIS COLUMN SPECIFICATION
*
      CHECK = (L/R)*K
*
*         THE CRITICAL CONSTANT FOR ALL COLUMNS OF THIS MATERIAL
*
      CC = ((2 * (PI**2) * E)/SIGYP)**0.5
*
*         COMPARE CC AND CHECK TO DETERMINE WHICH RELATION TO USE
*
```

```
      IF (CHECK.LT.CC) THEN
         HOLD1 = (1 - ((CHECK**2)/(2*(CC**2))))*SIGYP
         HOLD2 = ((5./3)+((3*CHECK)/(8*CC)) - ((CHECK**3)/(8*(CC**3))))
      ELSE
         HOLD1 = (PI**2)*E
         HOLD2 = (23./12)*(CHECK**2)
      ENDIF
*
*         THE ALLOWABLE AXIAL STRESS AND LOADING
*
      SIGA = HOLD1/HOLD2
      LOADA = SIGA*A
*
******        PRINTING OUTPUT        ******
*
      PRINT*,' '
      PRINT*,' '
      PRINT*,'AMERICAN INSTITUTE OF STEEL CONSTRUCTION (AISC) STANDARDS:'
      PRINT*,' '
      IF (ANS.EQ.1) THEN
         PRINT 10,SIGA,'PSI.'
         PRINT 20,LOADA,'LB.'
      ELSE
         SIGA=SIGA/1000000.0
         PRINT 10,SIGA,'MPA.'
         PRINT 20,LOADA,'NEWTONS.'
      ENDIF
*
*             FORMAT STATEMENTS
*
 10   FORMAT(2X,'THE ALLOWABLE AXIAL COMPRESSIVE STRESS  IS',F10.1,
+            1X,A4)
 20   FORMAT(2X,'THE ALLOWABLE AXIAL LOAD  IS',F10.1,1X,A)
*
      STOP
      END
```

**14.18.** A pinned end W8 × 19 steel column has a yield point of 33,000 lb/in$^2$ and a modulus of $30 \times 10^6$ lb/in$^2$. The length of the column is 15 ft. Use the FORTRAN program of Problem 14.17 to determine the allowable axial stress and also the load based on AISC specifications.

From Table 8-1 of Chap. 8 we find $I_{min} = 7.9$ in$^4$ and $A = 5.59$ in$^2$. The self-prompting program and computer run is

```
run
 PLEASE INDICATE YOUR CHOICE OF UNITS:
 1 - USCS
 2 - SI

 ENTER 1,2
? 1
 PLEASE INPUT ALL DATA IN UNITS OF POUND AND/OR INCH...

 ENTER COLUMN LENGTH:
? 180
 ENTER THE CROSS-SECTIONAL PROPERTIES...
 MOMENT OF INERTIA, I:
? 7.9
 AREA:
? 5.59
 ENTER THE MATERIAL YIELD POINT:
? 33000
```

```
 ENTER THE VALUE FOR YOUNG'S MODULUS:
? 30E+6
 FINALLY, ENTER THE END FIXITY COEFFICIENT, K:
? 1

 AMERICAN INSTITUTE OF STEEL CONSTRUCTION (AISC) STANDARDS:

  THE ALLOWABLE AXIAL COMPRESSIVE STRESS  IS    6738.2 PSI.
  THE ALLOWABLE AXIAL LOAD  IS   37666.5 LB.

SRU      0.780 UNTS.
```

**14.19.** Consider a pin-ended W305 × 37 column made of steel having a yield point of 270 MPa and a modulus of 200 GPa. The length of the column is 10 m. From Table 8-2 of Chap. 8 we find $I_{min} = 6.02 \times 10^{-6}$ m$^4$ and $A = 4760 \times 10^{-6}$ m$^2$. Use the FORTRAN program of Problem 14.17 to determine the allowable axial stress and load based on AISC specifications.

Using these input data, the computer run is

```
run
 PLEASE INDICATE YOUR CHOICE OF UNITS:
 1 - USCS
 2 - SI

 ENTER 1,2
? 2
 PLEASE INPUT ALL DATA IN UNITS OF NEWTON AND/OR METER...

 ENTER COLUMN LENGTH:
? 10
 ENTER THE CROSS-SECTIONAL PROPERTIES...
 MOMENT OF INERTIA, I:
? 6.02E-6
 AREA:
? 4760E-6
 ENTER THE MATERIAL YIELD POINT:
? 270E+6
 ENTER THE VALUE FOR YOUNG'S MODULUS:
? 200E+9
 FINALLY, ENTER THE END FIXITY COEFFICIENT, K:
? 1

 AMERICAN INSTITUTE OF STEEL CONSTRUCTION (AISC) STANDARDS:

  THE ALLOWABLE AXIAL COMPRESSIVE STRESS  IS      13.0 MPA.
  THE ALLOWABLE AXIAL LOAD  IS   61998.2 NEWTONS.

SRU      0.777 UNTS.
```

**14.20.** Develop a FORTRAN program to represent the SSRC values of mean axial compressive stress just prior to collapse as discussed in Problem 14.11.

The symbols are defined in Problem 14.11 and the Eqs. (*4*) of that problem indicate axial stress just prior to collapse for various values of $\lambda$ given by Eq. (*5*). The program listing is

```
***********************************************************************
            PROGRAM STEELCL (INPUT,OUTPUT)
*                     (BJORHOVDE, STRUCTURAL STABILITY RESEARCH COUNCIL)
***********************************************************************
*
*           AUTHOR: KATHLEEN DERWIN
*           DATE  : JANUARY 24, 1989
*
*   BRIEF DESCRIPTION:
*       ONE APPROACH TO CONSIDERING DESIGN CRITERIA FOR STRUCTURAL
*   STEEL COLUMNS WAS DEVELOPED BY  R. BJORHOVDE, AND IS POSSIBLY
*   IN THE BEST AGREEMENT WITH EXPERIMENTAL EVIDENCE. THE MEAN AXIAL
*   COMPRESSIVE STRESS JUST PRIOR TO COLLAPSE CAN BE OBTAINED FOR THE
*   SPECIFIC COLUMN BY FIRST CALCULATING THE 'COLUMN CONSTANT' AND THEN
*   DETERMINING THE MEAN STRESS AT FAILURE FROM THE APPROPRIATE RELATION.
*
*   INPUT:
*       THE USER IS FIRST ASKED IF USCS OR SI UNITS WILL BE USED. THEN,
*   THE COLUMN LENGTH, THE MINIMUM MOMENT OF INERTIA AND AREA OF THE
*   COLUMN CROSS SECTION, THE MATERIAL YIELD POINT, AND YOUNG'S MODULUS
*   ARE INPUTTED. ALSO, THE END FIXITY COEFFICIENT IS ENTERED.
*
*   OUTPUT:
*       THE MEAN (OVER THE CROSS SECTION) AXIAL COMPRESSIVE STRESS AND
*   THE MEAN PEAK LOADING CONDITIONS ARE DETERMINED.
*
*   VARIABLES:
*       ANS    ---    DENOTES IF USCS OR SI UNITS ARE DESIRED
*      L,I,A   ---    LENGTH, MIN.MOMENT OF INERTIA, AREA OF COLUMN X-SECT
*     SIGYP,E  ---    YIELD POINT, YOUNG'S MODULUS OF THE MATERIAL
*        R     ---    MIN. RADIUS OF GYRATION AS CALCULATED FROM THE
*                     X-SECTIONAL AREA AND MOMENT OF INERTIA
*      LAMDA   ---    CRITICAL CONSTANT OF THE COLUMN...A FUNCTION OF ITS
*                     PHYSICAL AND MATERIAL PROPERTIES
*        K     ---    END FIXITY COEFFICIENT OF THE COLUMN
*       SIGU   ---    MEAN AXIAL COMPRESSIVE STRESS AT FAILURE
*      LOADU   ---    MEAN AXIAL LOAD AT FAILURE
*        PI    ---    3.14159
*
***************************************************************************
******                        MAIN PROGRAM                          *****
***************************************************************************
*
*            VARIABLE DECLARATIONS
*
      REAL L,I,A,SIGYP,E,R,LAMDA,K,SIGU,LOADU,PI
      INTEGER ANS
*
      PI = 3.14159
*
*          USER INPUT
*
      PRINT*,'PLEASE INDICATE YOUR CHOICE OF UNITS:'
      PRINT*,'1 - USCS'
      PRINT*,'2 - SI'
      PRINT*,' '
      PRINT*,'ENTER 1,2'
      READ*,ANS
      IF (ANS.EQ.1) THEN
         PRINT*,'PLEASE INPUT ALL DATA IN UNITS OF POUND AND/OR INCH...'
      ELSE
         PRINT*,'PLEASE INPUT ALL DATA IN UNITS OF NEWTON AND/OR METER..
      ENDIF
      PRINT*,' '
      PRINT*,'ENTER COLUMN LENGTH:'
      READ*,L
```

```
      PRINT*,'ENTER THE CROSS-SECTIONAL PROPERTIES...'
      PRINT*,'MOMENT OF INERTIA, I:'
      READ*,I
      PRINT*,'AREA:'
      READ*,A
      PRINT*,'ENTER THE MATERIAL YIELD POINT:'
      READ*,SIGYP
      PRINT*,'ENTER THE VALUE FOR YOUNG'S MODULUS:'
      READ*,E
      PRINT*,'FINALLY, ENTER THE END FIXITY COEFFICIENT, K:'
      READ*,K
*
*             END USER INPUT
*
*
******         CALCULATIONS         ******
*
*        MINIMUM RADIUS OF GYRATION
*
      R = (I/A)**0.5
*
*      CRITICAL CONSTANT FOR THIS COLUMN SPECIFICATION
*
      LAMDA = ((K*L)/(R*PI))*((SIGYP/E)**0.5)
*
*      MEAN AXIAL COMPRESSIVE STRESS AND LOADING
*
      IF (LAMDA.LT.0.15) THEN
         SIGU = SIGYP
      ELSEIF (LAMDA.GE.0.15 .AND. LAMDA.LT.1.0) THEN
         SIGU = SIGYP*(1.035 - 0.202*LAMDA - 0.222*(LAMDA**2))
      ELSEIF (LAMDA.GE.1.0 .AND. LAMDA.LT.2.0) THEN
         SIGU = SIGYP*(-0.111 + 0.636/LAMDA + 0.0872/(LAMDA**2))
      ELSEIF (LAMDA.GE.2.0 .AND. LAMDA.LT.3.6) THEN
         SIGU = SIGYP*(0.009 + 0.877/(LAMDA**2))
      ELSEIF (LAMDA.GE.3.6) THEN
         SIGU = SIGYP/(LAMDA**2)
      ENDIF
*
      LOADU = SIGU*A
*
******         PRINTING OUTPUT         ******
*
      PRINT*,' '
      PRINT*,' '
      PRINT*,'STRUCTURAL STABILITY RESEARCH COUNCIL (BJORHOVDE) STANDARDS
      PRINT*,' '
      IF (ANS.EQ.1) THEN
         PRINT 10,SIGU,'PSI'
         PRINT 20,LOADU,'LB'
      ELSE
         SIGU=SIGU/1000000.0
         PRINT 10,SIGU,'MPA'
         PRINT 20,LOADU,'NEWTONS'
      ENDIF
*
*             FORMAT STATEMENTS
*
 10   FORMAT(2X,'THE MEAN AXIAL COMPRESSIVE STRESS AT FAILURE IS',F10.1,
+            1X,A3)
 20   FORMAT(2X,'THE MEAN AXIAL LOAD AT FAILURE IS',F10.1,1X,A)
*
      STOP
      END
```

**14.21.** Consider a 3.5-m-long pinned end steel column of wide-flange type W254 × 79. The material has a yield point of 250 MPa and a modulus of 200 GPa. Use the FORTRAN program of Problem 14.20 to determine the mean axial compressive stress just prior to collapse as indicated by the SSRC relations.

The constants of this cross section are found from Table 8-2 of Chap. 8 to be $I = 43.1 \times 10^{-6}\ \text{m}^4$ and $A = 10{,}200 \times 10^{-6}\ \text{m}^2$. Using these values, together with the designated length, yield point, and modulus, the self-prompting program prints as follows:

```
run
 PLEASE INDICATE YOUR CHOICE OF UNITS:
 1 - USCS
 2 - SI

 ENTER 1,2
? 2
 PLEASE INPUT ALL DATA IN UNITS OF NEWTON AND/OR METER...

 ENTER COLUMN LENGTH:
? 3.5
 ENTER THE CROSS-SECTIONAL PROPERTIES...
 MOMENT OF INERTIA, I:
? 43.1E-6
 AREA:
? 10200E-6
 ENTER THE MATERIAL YIELD POINT:
? 250E+6
 ENTER THE VALUE FOR YOUNG'S MODULUS:
? 200E+9
 FINALLY, ENTER THE END FIXITY COEFFICIENT, K:
? 1

 STRUCTURAL STABILITY RESEARCH COUNCIL (BJORHOVDE) STANDARDS:

  THE MEAN AXIAL COMPRESSIVE STRESS AT FAILURE IS     207.8 MPA
  THE MEAN AXIAL LOAD AT FAILURE IS 2119270.2 NEWTONS

SRU      0.786 UNTS.
```

**14.22.** Consider an initially straight, pin-ended column subject to an axial compressive force applied with known eccentricity $e$ (see Fig. 14-11). Determine the maximum compressive stress in the column.

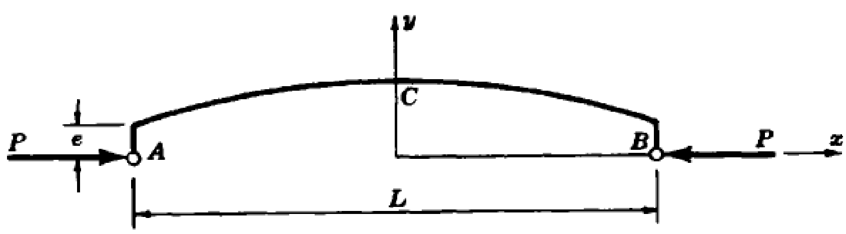



**Fig. 14-11**

The differential equation of the bar in its deflected configuration is

$$EI\frac{d^2y}{dx^2} = -Py$$

which has the standard solution

$$y = C_1 \sin\left(\sqrt{\frac{P}{EI}}x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}x\right)$$

Since $y = e$ at each of the ends $x = -L/2$ and $x = L/2$, the values of the two constants of integration are readily found to be

$$C_1 = 0 \qquad C_2 = \frac{e}{\cos\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)}$$

Thus, the deflection curve of the bent bar is

$$y = \frac{e}{\cos\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)}\cos\left(\sqrt{\frac{P}{EI}}x\right)$$

The maximum value of deflection occurs at $x = 0$, by symmetry, and is

$$y_{max} = e\sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)$$

Introducing the value of the critical load $P_{cr}$ as given by (9) of Problem 14.1, this becomes

$$y_{max} = e\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)$$

Evidently the maximum deflection, which occurs at the center of the bar, becomes very great as the load $P$ approaches the critical value. The phenomenon is one of gradually increasing lateral deflections, not buckling. The maximum compressive stress occurs on the concave side of the bar at $C$ and is given by

$$\sigma_{max} = \frac{P}{A} + \frac{M_{max}c}{I} = \frac{P}{A} + \frac{Pec}{I}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)$$

where $c$ denotes the distance from the neutral axis to the outer fibers of the bar. If we now introduce the radius of gyration $r$ of the cross section, this becomes

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{AE}}\right)\right]$$

This is the *secant formula* for an eccentrically loaded long column. In it, $P/A$ is the average compressive stress. If the maximum stress is specified to be the yield point of the material, then the corresponding average compressive stress which will first produce yielding may be found from the equation

$$\frac{P_{yp}}{A} = \frac{\sigma_{yp}}{1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P_{yp}}{AE}}\right)}$$

For any designated value of the ratio $ec/r^2$, this equation may be solved by trial and error and a curve of $P/A$ versus $L/r$ plotted to indicate the value of $P/A$ at which yielding first begins in the extreme fibers.

**14.23.** Obtain the load-deflection relation for a pin-ended column subject to axial compression and undergoing finite lateral displacements.

The treatment presented in Problem 14.1 is restricted to extremely small lateral deflections because

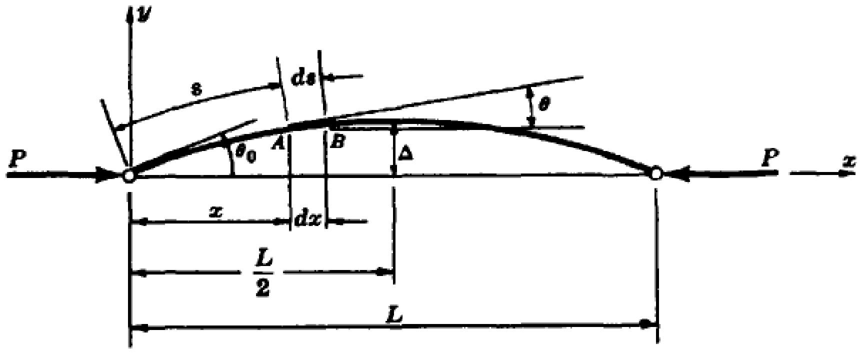



Fig. 14-12

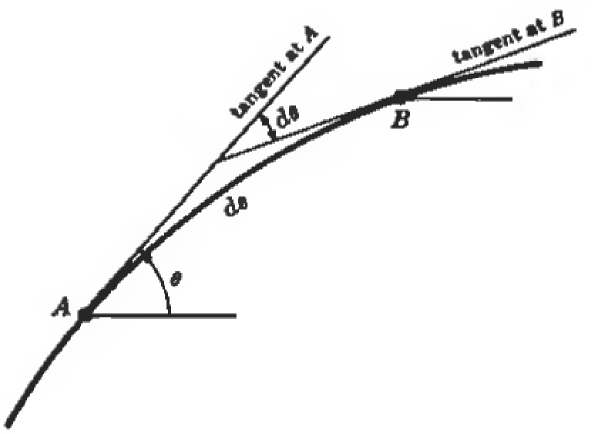



Fig. 14-13

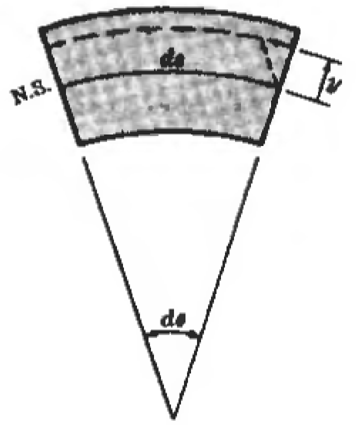



Fig. 14-14

this was the assumption made in deriving Eq. (*1*), the Euler-Bernoulli equation. To obtain a more general representation let us introduce the angular coordinate $\theta$ and arc length $s$, in addition to the $x$- and $y$-coordinates (see Fig. 14-12).

An enlarged view of the deformed bar illustrates the angular coordinates more clearly (Fig. 14-13). Note that $d\theta$ is negative. Let us now examine an element of arc length $ds$ bounded by two adjacent cross sections of the bar. Prior to loading these cross sections are parallel to each other but after the bar has deflected laterally they have the appearance shown in Fig. 14-14 in which they subtend a central angle $d\theta$. In a manner similar to that used in Problem 8.1, we may determine the normal strain of a fiber a distance $y$ from the neutral surface to be

$$\epsilon = \frac{y\,d\theta}{ds} = \frac{\sigma}{E}$$

where $\sigma$ is the longitudinal stress acting on this fiber. But from Problem 8.1 we have $\sigma = My/I$. Thus

$$\frac{y\,d\theta}{ds} = \frac{My}{EI}$$

or, since $M = -Py$ for the bar,

$$\frac{d\theta}{ds} = -\frac{Py}{EI} \tag{1}$$

If we let $a^2 = P/EI$ then

$$\frac{d\theta}{ds} = -\alpha^2 y \tag{2}$$

from which

$$\frac{d^2\theta}{ds^2} = -\alpha^2\frac{dy}{ds} = -\alpha^2 \sin\theta \tag{3}$$

This equation is valid for large, finite lateral deflections of the bar in contrast to (5) of Problem 9.1 which is limited to very small values of deflection. To solve (3), let us multiply through by the integrating factor $2(d\theta/ds)$:

$$2\frac{d\theta}{ds}\frac{d^2\theta}{ds^2} = -2\alpha^2(\sin\theta)\frac{d\theta}{ds} \tag{4}$$

Integrating,

$$\left(\frac{d\theta}{ds}\right)^2 = 2\alpha^2\cos\theta + C_1 \tag{5}$$

When $x = 0$, $\theta = \theta_0$ (the initial slope) and at this same point $y = 0$; hence $d\theta/ds = 0$ from (2). Thus, from (2),

$$0 = 2\alpha^2\cos\theta + C_1$$

so that

$$\frac{d\theta}{ds} = -\sqrt{2}\,\alpha\sqrt{\cos\theta - \cos\theta_0} \tag{6}$$

where the negative square root is taken because $d\theta$ is always negative. This may be transformed to

$$\frac{d\theta}{ds} = -2\alpha\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}} \tag{7}$$

We next introduce the change of variables

$$\sin\frac{\theta}{2} = k\sin\phi \tag{8}$$

where $\phi$ is a parameter assuming the value $\pi/2$ when $x = 0$ and the value 0 when $x = L/2$, from which

$$k = \sin\frac{\theta_0}{2} \tag{9}$$

Then

$$\theta = 2\arcsin(k\sin\phi)$$

and

$$d\theta = \frac{2k\cos\phi\, d\phi}{\sqrt{1 - k^2\sin^2\phi}} \tag{10}$$

From (7), (8), (9), and (10) we have

$$\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = -\alpha\, ds \tag{11}$$

Integrating the last equation and remembering the definition of $\phi$ at its endpoint values,

$$\alpha\int_0^{L/2} ds = -\int_{\pi/2}^{0}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

or

$$\alpha\frac{L}{2} = \int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \tag{12}$$

The right-hand side of (12) is termed the *complete elliptic integral of the first kind* with modulus $k$ and argument $\phi$. Tabulated values of the integral for any specified value of $k$ are readily available; see for example B. O. Peirce, *A Short Table of Integrals*, 4th ed., Ginn, 1957. To employ these tables we must select a value of $\theta_0$ thus fixing $k$ from Eq. (*9*). Then (*12*) may be rewritten in the form

$$P = \frac{4EI}{L^2}\left[\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}\right]^2 \tag{13}$$

to determine the axial load $P$ corresponding to this assumed value of $\theta_0$. To find the maximum deflection occurring at $x = L/2$, we have from geometry

$$\frac{dy}{ds} = \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \tag{14}$$

From (*11*) this becomes

$$\frac{dy}{ds} = -\frac{\alpha\, dy\sqrt{1-k^2\sin^2\phi}}{d\phi} \tag{15}$$

Equating the right sides of (*14*) and (*15*),

$$\frac{-\alpha\, dy\sqrt{1-k^2\sin^2\phi}}{d\phi} = 2k(\sin\phi)\sqrt{1-k^2\sin^2\phi}$$

or

$$\alpha\, dy = -2k\sin\phi\, d\phi \tag{16}$$

Integrating,

$$\alpha y = 2k\cos\phi + C_2$$

When $y = 0$, $\phi = \pi/2$ from which $C_2 = 0$. When $x = L/2$, $\phi = 0$ and $y = y_{max} = \Delta$. Thus $\alpha\Delta = 2k$ or

$$\Delta = \frac{2k}{\alpha} = \frac{2k}{\sqrt{\frac{P}{EI}}} = \frac{kL}{\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}} \tag{17}$$

The procedure is as follows:

1. Select a value of $\theta_0$ and determine $k$ from Eq. (*9*).
2. Ascertain the value of $\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$ from tabulated values in, for example, B. O. Peirce, and then calculate the axial force $P$ corresponding to this value of $\theta_0$ from Eq. (*13*).
3. Calculate the central deflection $\Delta$ from Eq. (*17*).

Results of this computation for selected values appear in Table 14-1 in which the starred value 9.87 ($= \pi^2$) indicates that the simple theory of Problem 14.1 actually gives an exact result if it is assumed that the end slopes are zero.

**Table 14-1**

| $\theta_0$, degrees | $k$ | $\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$ | $\frac{PL^2}{EI}$ | $\frac{\Delta}{L}$ |
|---|---|---|---|---|
| 0 | 0 | $\pi/2$ | 9.87 ($= \pi^2$)* | 0 |
| 40 | 0.342 | 1.6200 | 10.50 | 0.211 |
| 80 | 0.643 | 1.7868 | 12.75 | 0.360 |
| 120 | 0.866 | 2.1565 | 18.56 | 0.403 |
| 160 | 0.985 | 3.1534 | 39.76 | 0.313 |

From the above the progressive states of deformation of the bar are as shown in Fig. 14-15.

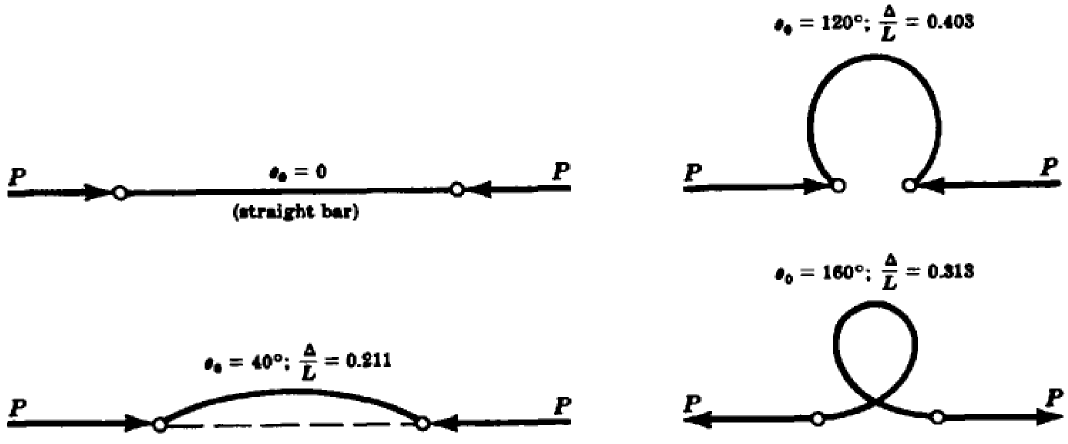



**Fig. 14-15**

This problem was first investigated by L. Euler in 1744 and the shape of the elastic curve is termed the *elastica*. It is only through use of this more exact finite-deflection theory that the amplitude of the lateral deflection may be determined. The approximate small-deflection treatment of Problem 14.1 does not permit determination of this quantity.

**14.24.** A problem that arises in insertion of a fiber-optic cable in a surrounding rigid conduit is that the cable buckles under certain axial "pushing" forces. This situation is represented in Fig. 14-16($a$) by a long slender bar (the cable) having simply supported ends (and represented by a line element) with a clearance $\Delta$ between the bar and the inside of the surrounding rectangular conduit. Assume that the behavior of the cable in this conduit is two-dimensional and determine the behavior of the cable under increasing axial compressive forces $P$.*

From Problem 14.1 when the axial force $P = EI/L^2$, the bar buckles and touches the conduit walls in the central region which is of unknown length $L_2$. For equilibrium of the left region of length $L_1$, there is a concentrated force $R$ acting at the pin, as well as another at $x = L_1$, as indicated in Fig. 14-16($b$). The differential equation of the deformed bar in the region $x \leq L_1$ is

$$EI\frac{d^2y}{dx^2} + Py = Rx$$

where the transverse force $R$ must be considered to be exerted on the bar by the pin at $A$. The solution of this equation is found as the sum of the general solution to the homogeneous equation plus a particular solution to the nonhomogeneous equation, as in Problem 14.3. Thus we have

$$y = A\sin\alpha x + B\cos\alpha x + \frac{Rx}{P} \tag{1}$$

where $\alpha^2 = P/EI$. The boundary conditions for the region of length $L_1$ are ($a$) when $x = 0$, $y = 0$; ($b$) when $x = L_1$, $y = \Delta$; and ($c$) when $x = L_1$, $dy/dx = 0$. From ($a$) we have $B = 0$. From ($b$) and ($c$) we get

$$A\sin\alpha L_1 + \frac{R}{P}L_1 = \Delta \tag{2}$$

$$A\alpha\cos\alpha L_1 + \frac{R}{P} = 0 \tag{3}$$

Since the region of the deformed bar between $x = L_1$ and $x < (L_1 + L_2)$ is in contact with the rigid conduit,

*The author is indebted to Professor V. I. Feodosyev of the Moscow Higher Technical School for suggesting this problem and for his discussions concerning it.

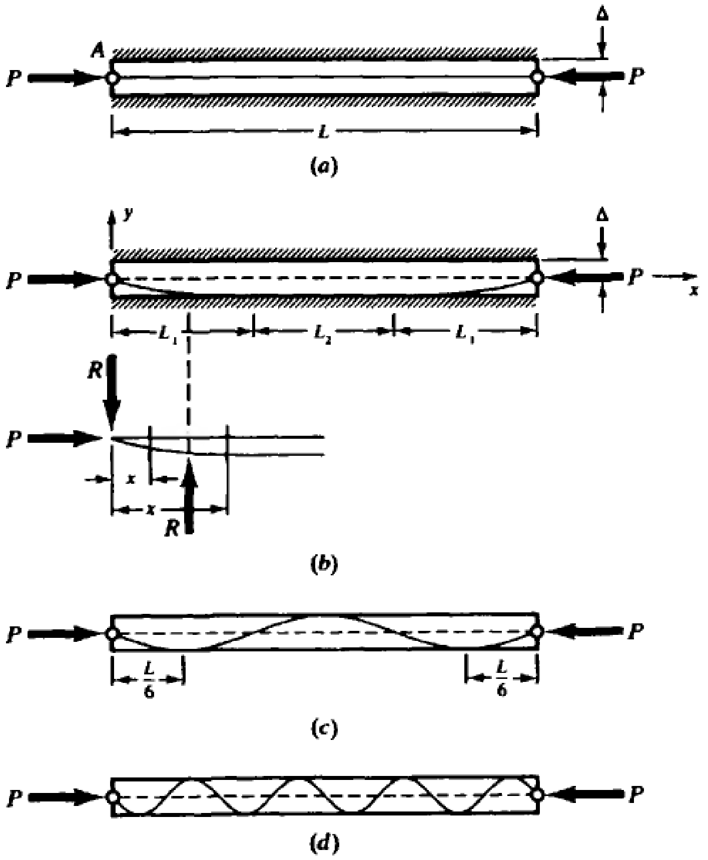



**Fig. 14-16**

the cable is straight in this region, and from Eq. (*5*) of Problem 9.1 the bending moment in that region is zero. Thus the Euler-Bernoulli equation of the beam in this central region of length $L_2$ becomes

$$P\Delta + Rx + R(x - L_1) = 0 \tag{4}$$

from which we have

$$R = P\left(\frac{\Delta}{L_1}\right) \tag{5}$$

From Eq. (*2*) we now have

$$A \sin \alpha L_1 + \Delta = \Delta \tag{6}$$

and thus $\alpha L_1 = \pi$ from which $\alpha = \pi/4$ or $L_1 = \pi/2$. Substituting (*5*) and (*6*) in (*2*), we obtain

$$A = \frac{\Delta}{\pi} \tag{7}$$

Since from (*1*) we have $\alpha^2 = P/EI$, we have from (*6*)

$$\left(\frac{\pi}{4}\right)^2 = \frac{P}{EI}$$

or

$$P = \frac{\pi^2 EI}{L_1} \tag{8}$$

When only the midpoint of the bar of length $L$ is in contact with the interior wall of the conduit, i.e., when $L_1 = L/2$, Eq. (*8*) becomes

$$P = \frac{4\pi^2 EI}{L^2} \tag{9}$$

This indicates that for values of axial force lying between

$$\frac{\pi^2 EI}{L^2} < P < \frac{4\pi^2 EI}{L^2} \tag{10}$$

the flexible fiber-optic cable touches the rigid wall only at the midpoint of the length of the conduit, i.e., at $x = L/2$. Only for values of $P > 4\pi^2 EI/L^2$ is the flexible cable in contact with the conduit interior for a finite length. This is indicated in Fig. 14-16(*b*).

Next, the central region of length $L_2$ may buckle for sufficiently large values of compressive force $P$. The central portion obviously behaves as a clamped end column as shown in Problem 14.3, and it buckles at the load

$$P = \frac{4\pi^2 EI}{L_2^2} \tag{11}$$

into the configuration shown in Fig. 14-16(*c*). But from Fig. 14-16(*b*), we have

$$2L_1 + L_2 = L$$

so from Eq. (*5*) we have

$$L_2 = L - 2\left(\frac{\pi}{\alpha}\right) \tag{12}$$

If we now equate from the values of $P$ from (*8*) and (*12*), we find $L_1 = L/4$, and from (*8*) we find for this value of $L_1$

$$P = \frac{16\pi^2 EI}{L^2} \tag{13}$$

By analyses such as the above, it can be shown that increases in axial force $P$ over that given by Eq. (*13*) lead to the value $L_1 = L/6$, and that configuration is retained until the axial load is

$$P = \frac{36\pi^2 EI}{L^2} \tag{14}$$

Still greater values of axial load will lead to the configuration indicated in Fig. 14-16(*d*). Thus, simple buckling theory has led to the plausible configurations indicated in Fig. 14-16.

**14.25.** Determine the deflection curve of a pin-ended bar subject to combined axial compression $P$ together with a uniform normal loading as shown in Fig. 14-17.

One convenient coordinate system to designate points on the deflected bar is shown in Fig. 14-17. There, the origin is situated at the point of maximum deflection. The bending moment at an arbitrary point $(x, y)$ on the deflected bar is written most easily as the sum of the moment of all forces to the *right* of $(x, y)$

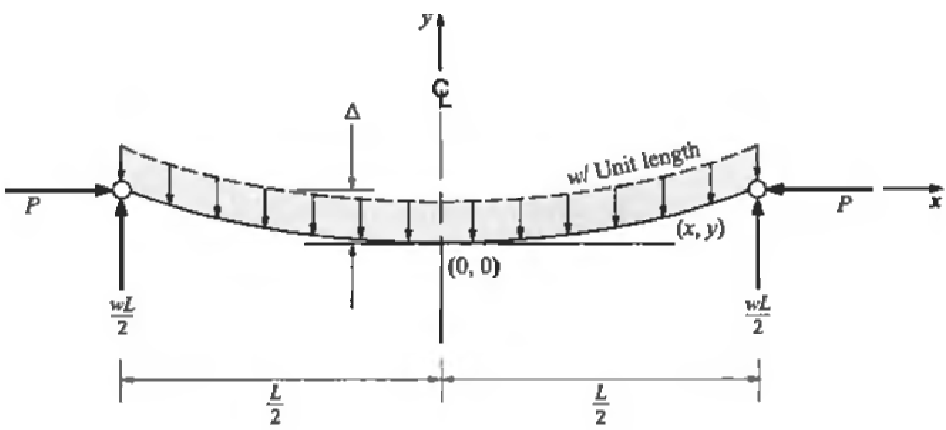



**Fig. 14-17**

and with algebraic signs consistent with the definitions of positive and negative bending introduced in Chap. 6.

The bending moment is thus

$$M = P(\Delta - y) + \frac{wL}{2}\left(\frac{L}{2} - x\right) - w\left(\frac{L}{2} - x\right)\left(\frac{\frac{L}{2} - x}{2}\right) \tag{1}$$

so that the differential equation of the deflected bar is

$$EI\frac{d^2y}{dx^2} = P\Delta - Py + \frac{w}{2}\left(\frac{L^2}{4} - x^2\right) \tag{2}$$

If we introduce the notation

$$n = \sqrt{\frac{P}{EI}}$$

we have the nonhomogeneous differential equation of the bar

$$\frac{d^2y}{dx^2} + n^2y = \frac{w}{2EI}\left(\frac{L^2}{4} - x^2\right) + n^2\Delta^2$$

The solution is given by the usual methods of differential equations as the sum of (*a*) the solution of the corresponding homogeneous equation, and (*b*) any particular solution of the entire nonhomogeneous equation. Thus we may write the solution as

$$y = A\cos nx + B\sin nx - \frac{w}{2P}\left(\frac{L^2}{4} - x^2\right) + \frac{2w}{2n^2P} + \Delta$$

where $A$ and $B$ are constants of integration. These are easily found by realizing that, because of symmetry of the bent bar, the deflection is $\Delta$ at $x = L/2$ and also the bar has a horizontal tangent at $x = 0$. This leads to

$$y = \Delta + \frac{w}{n^2P}\left[\left(\sec\frac{nL}{2}\cos nx - 1\right) - n^2\left(\frac{L^2}{8} - \frac{x^2}{2}\right)\right]$$

as the solution of the nonhomogeneous equation. The peak deflection occurs at the midpoint of the bar (the origin of our coordinate system) and is given by

$$\Delta = \frac{w}{n^2P}\left[\left(\sec\frac{nL}{2} - 1\right) - \frac{n^2L^2}{8}\right]$$

**14.26.** Two identical rigid bars $AB$ and $BC$ are pinned at $B$ and $C$ and supported at $A$ by a pin in a frictionless roller that can only displace vertically. A spring of constant $k$ is attached to bar $BC$, as shown in Fig. 14-18(*a*). Determine the critical load of the system.

A free-body diagram of the entire system of two rigid bars is shown in Fig. 14-18(*b*). The system is shown in a slightly deflected configuration characterized by the angle $\Delta\theta$ corresponding to its buckled shape. Ends $A$ and $C$ are pinned so it is necessary to show two components of pin reaction at each of these points. The spring elongates an amount $a(\Delta\theta)$ and consequently exerts a force $ka(\Delta\theta)$ on bar $BC$. The pin at $B$ is internal to this free body; hence no pin forces should be shown. From statics,

$$+\uparrow\Sigma M_A = C_x(4a) - ka(\Delta\theta)(3a) = 0$$

$$C_x = \frac{3ka(\Delta\theta)}{4}$$

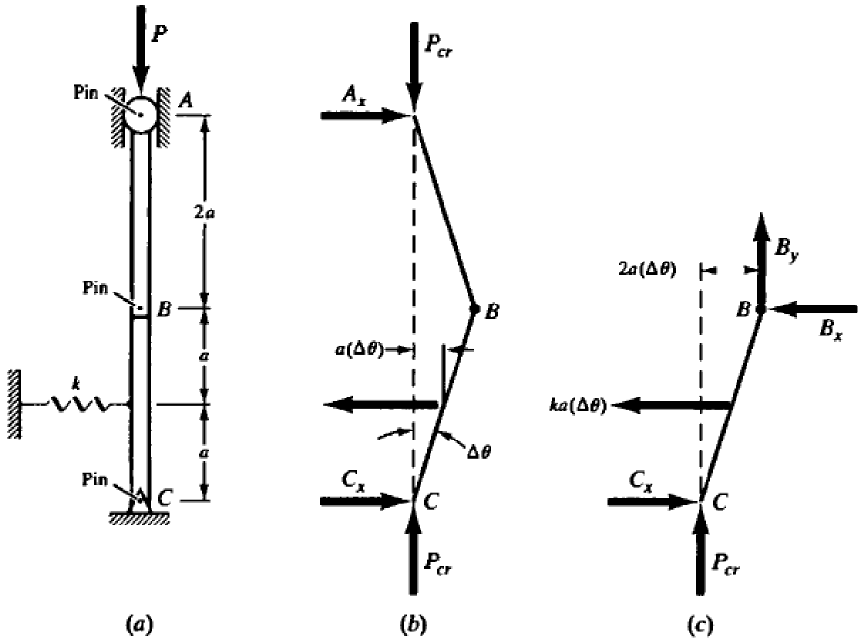



**Fig. 14-18**

Next, consider the free-body diagram of the lower bar $BC$ shown in Fig. 14-18($c$). Now the pin forces at $B$ become external to this free body, and from statics we have

$$+ \circlearrowright \Sigma M_B = \frac{3ka(\Delta\theta)}{4}(2a) - P_{cr}a(2a\,\Delta\theta) - [ka(\Delta\theta)]a = 0$$

$$P_{cr} = \frac{ka}{4}$$

It is impossible to determine $(\Delta\theta)$ by this approach.

## Supplementary Problems

**14.27.** A steel bar of solid circular cross section is 50 mm in diameter. The bar is pinned at each end and subject to axial compression. If the proportional limit of the material is 210 MPa and $E = 200$ GPa, determine the minimum length for which Euler's formula is valid. Also, determine the value of the Euler buckling load if the column has this minimum length. *Ans.* 1.21 m, 412 kN

**14.28.** The column shown in Fig. 14-19 is pinned at both ends and is free to expand into the opening at the upper end. The bar is steel, is 25 mm in diameter, and occupies the position shown at 16 °C. Determine the temperature to which the column may be heated before it will buckle. Take $\alpha = 12 \times 10^{-6}$/°C and $E = 200$ GPa. Neglect the weight of the column. *Ans.* 29.3 °C

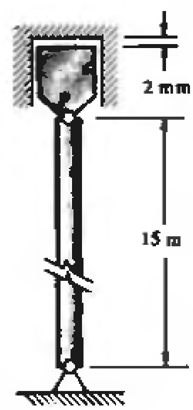



**Fig. 14-19**

**14.29.** A long slender bar $AB$ is clamped at $A$ and supported at $B$ in such a way that transverse displacement is impossible as in Fig. 14-20, but the


**Fig. 14-20**

end of the bar at $B$ is capable of rotating about $B$. Determine the differential equation governing the buckled shape of the bar. *Ans.* $\tan nL = nL$ where $n^2 = P/EI$

**14.30.** A bar of length $L$ is clamped at its lower end and subject to both vertical and horizontal forces at the upper end, as shown in Fig. 14-21. The vertical force $P$ is equal to one-fourth of the Euler load for this bar. Determine the lateral displacement of the upper end of the bar. *Ans.* $16(4 - \pi)RL^3/\pi^3 EI$

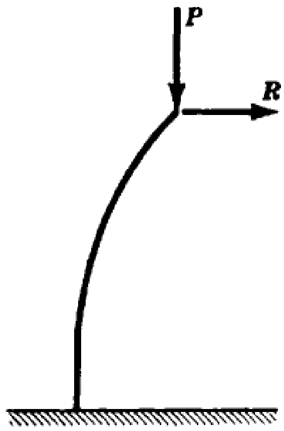



**Fig. 14-21**

**14.31.** A bar of length $L$ and flexural rigidity $EI$ has pinned ends. An axial compressive force

$$P = \frac{\pi^2 EI}{4L^2}$$

is applied to the beam and a bending moment $M$ is applied at one end. Determine the rotational stiffness, i.e., applied moment per radian of rotation at that end of the bar. Rework the problem for the case of an axial tensile force of the same numerical value.

*Ans.* $\dfrac{2.47EI}{L}, \dfrac{3.47EI}{L}$

**14.32.** An initially straight bar $AC$ is pinned at each end and supported at the midpoint $B$ by a spring which resists any lateral movement $\delta$ of $B$ with a lateral force $(kEI/L^3)\delta$. The bar is of length $2L$ and least flexural rigidity $EI$. Equal and opposite thrusts $P$ are applied at the end $C$ as well as at the centroid of the bar at $B$. In any deflected form the line of action of the thrust applied at $B$ remains parallel to the chord $AC$. Determine the minimum buckling load of the system.

*Ans.* $P_{cr} = \beta^2 \dfrac{EI}{L^2}$ where $\beta$ is the smallest positive root of the equation

$$\frac{\beta}{\tan\beta} = \frac{3k + (9 + k)\beta^2 - \beta^4}{3(k - \beta^2)}$$

**14.33.** A long thin bar of length $L$ and rigidity $EI$ is supported at each end in an elastic medium which exerts a restoring moment of magnitude $\lambda$ per radian of angular rotation at the end. Determine the first buckling load of the bar.

*Ans.* $\tan\dfrac{\alpha L}{2} = -\dfrac{P}{\alpha\lambda}$ where $\alpha^2 = \dfrac{P}{EI}$

**14.34.** A long thin bar is pinned at each end and is embedded in an elastic packing which exerts a transverse force on the bar when it deflects laterally. When the transverse deflection at any point is given by $y$, the packing exerts a transverse force per unit length of the bar equal to $ky$. Determine the axial force required to buckle the bar.

*Ans.* $P_{cr} = \dfrac{\pi^2 EI}{L^2}\left(n^2 + \dfrac{kL^4}{n^2\pi^4 EI}\right)$ where $n$ is the integer for which $P_{cr}$ is minimum

**14.35.** Use the AISC formula to determine the allowable axial load on a W10 × 54 column that is 22 ft long. The yield point of the material is 34,000 lb/in$^2$ and the modulus is $30 \times 10^6$ lb/in$^2$. *Ans.* 197,250 lb

**14.36.** Use the AISC formula to determine the allowable axial load on a W254 × 79 column that is 14 m long. The yield point of the material is 250 MPa and the modulus is 200 GPa. *Ans.* 226,500 N

**14.37.** A W12 × 25 pin-ended column made of steel having a yield point of 36,000 lb/in$^2$ and a modulus of $30 \times 10^6$ lb/in$^2$ is 30 ft long. Use the FORTRAN program of Problem 14.17 to determine the allowable axial stress and load based on AISC specifications. *Ans.* 2340 lb/in$^2$, 17,280 lb

**14.38.** A W254 × 79 pin-ended column made of steel having a yield point of 250 MPa and a modulus of 200 GPa is 14 m long. Use the FORTRAN program of Problem 14.17 to determine the allowable axial stress and load based on AISC specifications. *Ans.* 22.2 MPa, 226 kN

**14.39.** Consider a pinned end column 9 m long of wide flange designation W203 × 28. The yield point of the material is 250 MPa and the modulus is 200 GPa. Use the FORTRAN program of Problem 14.20 to determine the mean axial compressive stress as well as axial load just prior to collapse as indicated by the SSRC equations. *Ans.* 21.7 MPa, 78.2 kN

**14.40.** Consider a pinned end column 22 ft long of wide flange designation W10 × 54. The yield point of the steel is 34,000 lb/in$^2$ and the modulus is $30 \times 10^6$ lb/in$^2$. Use the FORTRAN program of Problem 14.20 to determine the mean axial compressive stress as well as load just prior to collapse as indicated by the SSRC equations. *Ans.* 18,200 lb/in$^2$, 289,000 lb

**14.41.** Determine the deflection curve of a pin-ended bar subject to axial compression together with a central transverse force as shown in Fig. 14-22.

*Ans.* $y = \dfrac{Q \sin nx}{2Pn \cos \dfrac{nL}{2}} - \dfrac{Q}{2P}x$ where $n = \sqrt{\dfrac{P}{EI}}$

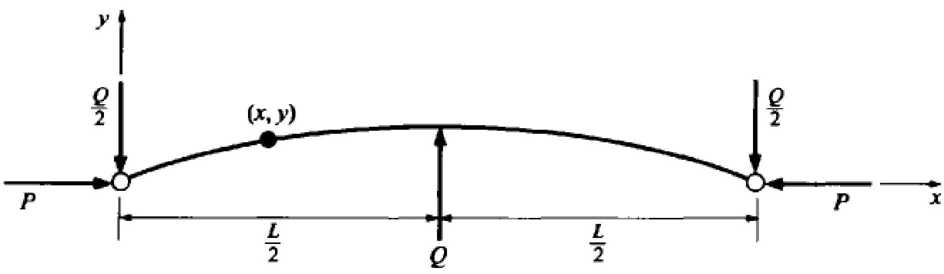



**Fig. 14-22**

**14.42.** A pin-ended bar of flexural rigidity $EI$ is subject to the two transverse loads indicated in Fig. 14-23, each being one quarter of the Euler axial buckling load of the bar and simultaneously the axial loads each being half the Euler buckling load of the bar. Determine the peak transverse deflection of the bar.
*Ans.* $0.008L$

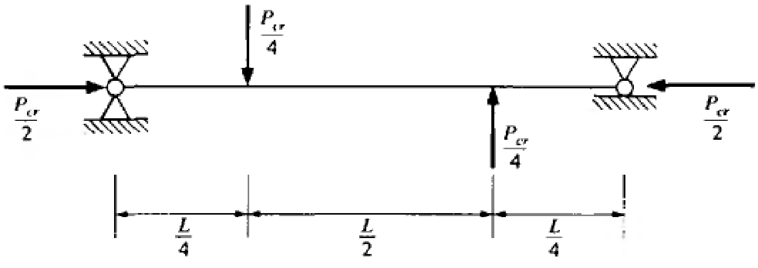



**Fig. 14-23**

**14.43.** The system of two rigid vertical bars $AB$ and $BC$ shown in Fig. 14-24 is pinned at the base $C$ and restrained against lateral motion at the top $A$, but is free to rotate there. The bars are also pinned at $B$. The midpoint $B$ is partially restrained against lateral displacement by the two linear springs, each offering $k$ lb of resistance per inch of lateral movement. The springs are load free prior to application of $P$. Determine the buckling load $P_{cr}$. *Ans.* $P_{cr} = 12k$

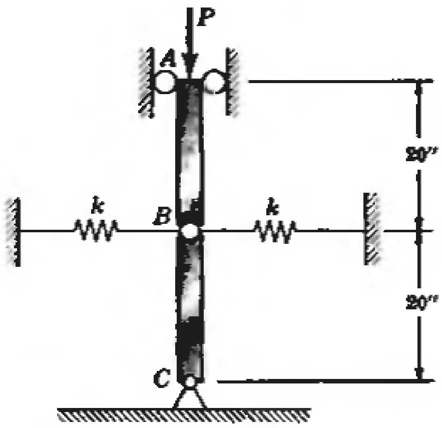



**Fig. 14-24**

**14.44.** The rigid bar $OA$ in Fig. 14-25 is pinned at $O$ and supports a vertical force $P$ at the upper end $A$. Point $A$ is tied back to the ground by a spring of constant $k$. The spring is load free when the rod $OA$ is vertical. Weights of all members are to be neglected. Determine the load $P$ at which the system becomes unstable. *Ans.* $kL/2$

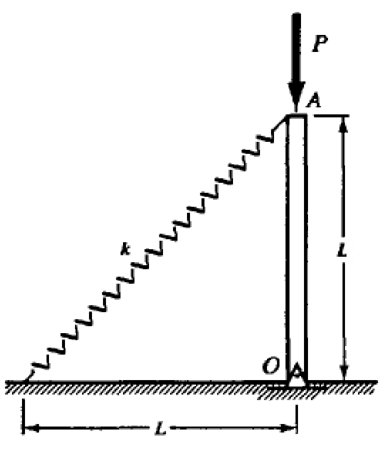



**Fig. 14-25**

**14.45.** The guyed steel mast $AB$ in Fig. 14-26 is pinned at $A$ and braced by a planar system of two thin wires $BC$ and $BD$, as shown in Fig. 14-26. The moment of inertia of the mast is $3.00\ \text{in}^4$ and its height is 50 in. Its modulus of elasticity is $30 \times 10^6\ \text{lb/in}^2$. The wires are each of aluminum having modulus of $10 \times 10^6\ \text{lb/in}^2$ and cross-sectional area $0.10\ \text{in}^2$. The mast is subject to a vertical force $P$ applied at $B$. Determine the magnitude of the buckling load. (*Hint:* It is necessary to consider rigid-body rotation of the mast about $A$ to the configuration $AB'$ as well as independently computing the Euler-type buckling load of the mast into one loop of a sine curve.) *Ans.* $P = 350{,}000\ \text{lb}$

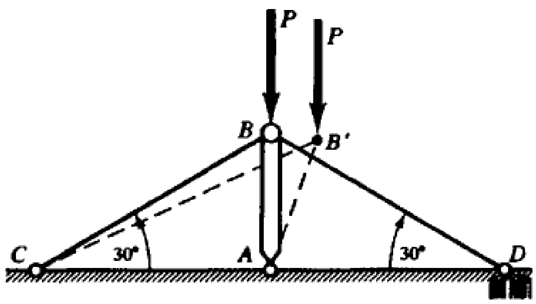



Fig. 14-26

# Chapter 15

## Strain Energy Methods

Thus far in this book various techniques have been discussed for finding deformations and determining values of indeterminate reactions. These techniques have essentially been based upon geometric considerations. There are, however, many types of problems that can be solved more efficiently through techniques based upon relations between the work done by the external forces and the internal strain energy stored within the body during the deformation process. The present chapter will discuss these techniques, which are somewhat more general and more powerful than the various geometric approaches.

### INTERNAL STRAIN ENERGY

When an external force acts upon an elastic body and deforms it, the work done by the force is stored within the body in the form of strain energy. The strain energy is always a scalar quantity. For a straight bar subject to a tensile force $P$, the internal strain energy $U$ is given by

$$U = \frac{P^2 L}{2AE}$$

where $L$ represents the length of the bar, $A$ is its cross-sectional area, and $E$ is Young's modulus. This expression is derived in Problem 15.1.

For a circular bar of length $L$ subject to a torque $T$, the internal strain energy $U$ is given by

$$U = \frac{T^2 L}{2GJ}$$

where $G$ is the modulus of elasticity in shear and $J$ is the polar moment of inertia of the cross-sectional area. This expression is derived in Problem 15.2.

For a bar of length $L$ subject to a bending moment $M$, the internal strain energy $U$ is given by

$$U = \frac{M^2 L}{2EI}$$

where $I$ is the moment of inertia of the cross-sectional area about the neutral axis. This is derived in Problem 15.3.

Note that in each of these expressions the external load always occurs in the form of a squared magnitude, hence each of these energy expressions is always a positive scalar quantity.

### SIGN CONVENTIONS

Strain energy methods are particularly well suited to problems involving several structural members at various angles to one another. The fact that the members may be curved in their planes presents no additional difficulties. One of the great advantages of strain energy methods is that independent coordinate systems may be established for each member without regard for consistency of positive directions of the various coordinate systems. This advantage is essentially due to the fact that the strain energy is always a positive scalar quantity, and hence algebraic signs of external forces need be consistent only within each structural member.

## CASTIGLIANO'S THEOREM

This theorem is extremely useful for finding displacements of elastic bodies subject to axial loads, torsion, bending, or any combination of these loadings. The theorem states that the partial derivative of the total internal strain energy with respect to any external applied force yields the displacement under the point of application of that force in the direction of that force. Here, the terms force and displacement are used in their generalized sense and could either indicate a usual force and its linear displacement, or a couple and the corresponding angular displacement. In equation form the displacement under the point of application of the force $P_n$ is given according to this theorem by

$$\delta_n = \frac{\partial U}{\partial P_n}$$

This theorem is derived in Problem 15.8.

## APPLICATION TO STATICALLY DETERMINATE PROBLEMS

In such problems all external reactions can be found by application of the equations of statics. After this has been done, the deflection under the point of application of any external applied force can be found directly by use of Castigliano's theorem. This is illustrated in Problems 15.9 and 15.10. If the deflection is desired at some point where there is no applied force, then it is necessary to introduce an auxiliary (i.e., fictitious) force at that point and, treating that force just as one of the real ones, use Castigliano's theorem to determine the deflection at that point. At the end of the problem the auxiliary force is set equal to zero. This is illustrated in Problems 15.9, 15.12, 15.13, and 15.19.

## APPLICATION TO STATICALLY INDETERMINATE PROBLEMS

Castigliano's theorem is extremely useful for determining the indeterminate reactions in such problems. This is because the theorem can be applied to each reaction, and the displacement corresponding to each reaction is known beforehand and is usually zero. In this manner it is possible to establish as many equations as there are redundant reactions, and these equations together with those found from statics yield the solution for all reactions. After the values of all reactions have been found, the deflection at any desired point can be found by direct use of Castigliano's theorem. This is illustrated in Problems 15.16 through 15.18.

## ASSUMPTIONS AND LIMITATIONS

Throughout this chapter it is assumed that the material is a linear elastic one obeying Hooke's law. Further, it is necessary that the entire system obey the law of superposition. This implies that certain unusual systems, such as that discussed in Problem 1.17, cannot be treated by the techniques discussed here.

# Solved Problems

**15.1.** Determine the internal strain energy stored within an elastic bar subject to an axial tensile force $P$.

For such a bar the elongation $\Delta$ has been found in Problem 1.1 to be $\Delta = PL/AE$, where $A$ represents the cross-sectional area, $L$ is the length, and $E$ is Young's modulus. The force-elongation diagram will

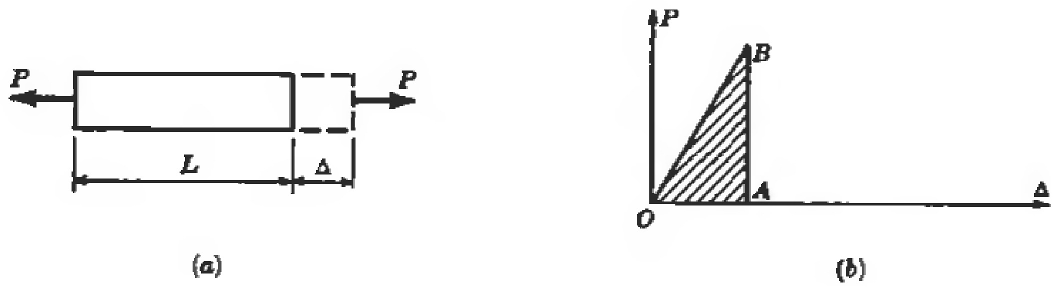



**Fig. 15-1**

consequently be linear, as shown in Fig. 15-1($b$). For any specific value of the force $P$, such as that corresponding to point $B$ in the force-elongation diagram, the force will have done positive work indicated by the shaded area $OBA$. This triangular area is given by $\frac{1}{2}P\Delta$. Replacing $\Delta$ by the value given above, this becomes $P^2L/2AE$. This is the work done by the external force and the work is stored within the bar in the form of internal strain energy, denoted by $U$. Hence

$$U = \frac{P^2 L}{2AE}$$

Essentially, the elastic bar is acting as a spring to store this energy. The same expression for internal strain energy applies if the load is compressive, since the axial force appears as a squared quantity and hence the final result is the same for either a positive or negative force.

If the axial force $P$ varies along the length of the bar, then in an elemental length $dx$ of the bar the strain energy is

$$dU = \frac{P^2\,dx}{2AE}$$

and the energy in the entire bar is found by integrating over the length:

$$U = \int_0^L \frac{P^2\,dx}{2AE}$$

**15.2.** Determine the internal strain energy stored within an elastic bar subject to a torque $T$ as shown in Fig. 15-2($a$).


**Fig. 15-2**

In Problem 5.3, the angle of twist $\theta$ has been found to be $\theta = TL/GJ$, where $G$ is the modulus of elasticity in shear, $L$ is the length, and $J$ is the polar moment of inertia of the cross-sectional area. According to this expression, the relation between torque and angle of twist is a linear one, as shown in Fig. 15-2($b$). When the torque has reached a specific value such as that indicated by point $B$, it will have done positive work indicated by the shaded area $OBA$. This triangular area is given by $\frac{1}{2}T\theta$, or $T^2L/2GJ$. This work done by the external torque is stored within the bar as internal strain energy, denoted by $U$. Hence

$$U = \frac{T^2 L}{2GJ}$$

If the torque $T$ varies along the length of the bar, then in an elemental length $dx$ the strain energy is

$$dU = \frac{T^2\,dx}{2GJ}$$

and in the entire bar it is

$$U = \int_0^L \frac{T^2\,dx}{2GJ}$$

**15.3.** Determine the internal strain energy stored within an elastic bar subject to a pure bending moment $M$.

In Problem 8.1 is shown an initially straight bar subject to the pure bending moment $M$ which deforms it into a circular arc of radius of curvature $\rho$. In Eq. (7) of that problem it was shown that $M = EI/\rho$, where $I$ denotes the moment of inertia of the cross-sectional area about the neutral axis. But the length of the bar, $L$, is equal to the product of the central angle $\theta$ subtended by the circular arc and the radius $\rho$. Thus

$$\frac{M}{EI} = \frac{1}{\rho} = \frac{\theta}{L} \qquad \text{or} \qquad \theta = \frac{ML}{EI}$$

According to this the relation between moment and angle subtended is a linear one, and this is illustrated in Fig. 15-3. When the moment has reached a specific value $M$, such as that indicated by point $B$, it will have done work indicated by the shaded area $OAB$. This area is given by $\frac{1}{2}M\theta$, or $M^2L/2EI$. This work done by the external moment is stored within the bar as internal strain energy, denoted by $U$. Hence

$$U = \frac{M^2 L}{2EI}$$

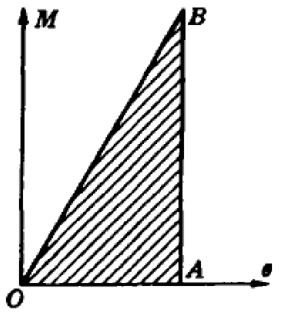



**Fig. 15-3**

If the bending moment $M$ varies along the length of the bar, then in an elemental length $dx$ the strain energy is

$$dU = \frac{M^2\,dx}{2EI}$$

and in the entire bar it is

$$U = \int_0^L \frac{M^2\,dx}{2EI}$$

**15.4.** Consider the two simply supported beams shown in Fig. 15-4. Both are of rectangular cross section and of equal width. The materials are identical. The first beam has constant height along the length, the second has a small groove in the center which reduces the height by one-fifth. The length of the groove along the axis is negligible. The maximum stress in each bar due to

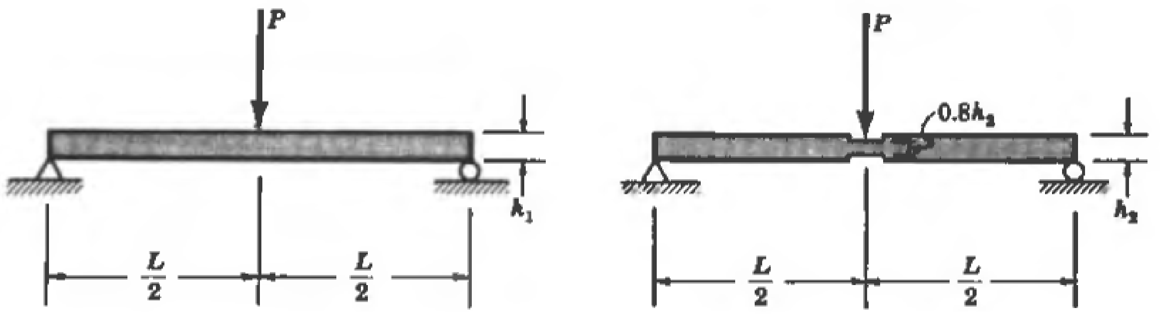



**Fig. 15-4**

the action of the central force $P$ is the elastic limit of the material. Neglecting the effect of stress concentrations, determine the ratio of internal strain energies in the two bars.

For the first bar, the section modulus is

$$Z = \frac{I}{c} = \frac{\frac{1}{12}bh_1^3}{0.5h_1} = 0.167h_1^2 b$$

For the second bar, in the grooved region the section modulus is

$$Z = \frac{I}{c} = \frac{\frac{1}{12}b(0.8h_2)^3}{0.4h_2} = 0.107h_2^2 b$$

and in the thicker region of depth $h_2$ the section modulus is

$$Z = \frac{I}{c} = \frac{\frac{1}{12}bh_2^3}{0.5h_2} = 0.167h_2^2 b$$

In general, for bending we have the bending stress at the outer fibers of a bar given by the relation $\sigma = M/Z$. Since the maximum stresses in each bar are equal, we have

$$0.167h_1^2 b = 0.107h_2^2 b \qquad \text{or} \qquad h_2 = 1.25h_1$$

The strain energy in the first bar is

$$U_1 = \frac{M^2 L}{2EI} = \frac{M^2 L}{2E(\frac{1}{12}bh_1^3)}$$

The strain energy in the second bar, since the groove is of negligible length, is

$$U_2 = \frac{M^2 L}{2E[\frac{1}{12}b(1.25h_1)^3]}$$

The loadings and lengths are identical, hence we need not calculate $M^2 L$ to obtain the desired ratio, which is

$$U_2 : U_1 = 0.512$$

This indicates that a grooved bar is very ineffective in storing internal strain energy. This is an important consideration in the design of bars to withstand dynamic loadings.

**15.5.** Consider a vertical bar of uniform cross section with a flange at the lower end (Fig. 15-5). A weight $W$ is released from the top of the bar and falls freely along the bar until it strikes the flange. Determine the maximum elongation of the bar and also the maximum stress.

To solve this problem we shall introduce several simplifying assumptions: ($a$) the weight of the vertical bar is very small compared to $W$, ($b$) there are no losses of energy due to friction or local distortion, and ($c$) the stress-strain diagram of the material of the bar is the same for dynamic loading as for static. Actually, a more sophisticated treatment would take strain wave propagation in the bar into account, but that is beyond the scope of the present study.

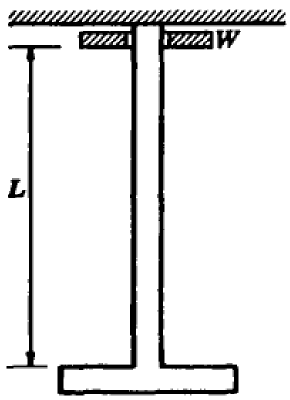



**Fig. 15-5**

The weight $W$ falls through the distance $L$ and after striking the flange extends the bar an unknown amount $\Delta$. At this maximum extension the tension in the bar is maximum and the equation relating work done by $W$ and the internal strain energy of extension at this instant of maximum deformation is

$$W(L+\Delta) = \frac{P^2 L}{2AE} \tag{1}$$

But $\Delta = PL/AE$, and substituting for $P$ in (1), we get

$$W(L+\Delta) = \frac{AE\Delta^2}{2L} \tag{2}$$

The static extension of the bar due to the weight $W$ would be $\Delta_{st} = WL/AE$. If the value of $W$ from this expression is introduced in the above equation and the resulting quadratic equation solved for the unknown extension $\Delta$, we get

$$\Delta = \Delta_{st} + \sqrt{\Delta_{st}^2 + \frac{\Delta_{st}}{g} v^2} \tag{3}$$

where $g$ is the acceleration due to gravity and $v = \sqrt{2gL}$ is the velocity with which $W$ strikes the flange. If the length of the bar, $L$, is very large compared to $\Delta_{st}$, then the above expression becomes approximately

$$\Delta = \sqrt{\frac{\Delta_{st}}{g} v^2} \tag{4}$$

In this case the axial stress is given by

$$\sigma = \frac{P}{A} = \frac{\Delta E}{L} = \frac{E}{L}\sqrt{\frac{\Delta_{st}}{g} v^2} = \sqrt{\frac{Wv^2}{2g}\frac{2E}{AL}} \tag{5}$$

It is of interest to note that in the dynamic case the stress depends upon the length $L$ as well as the Young's modulus $E$. The corresponding static stress does not involve either of these factors.

For the special case of a suddenly applied load $W$ acting on the flange, the length $L$ through which the weight falls may be set equal to zero in (3) to obtain

$$\Delta = 2\Delta_{st} \tag{6}$$

Thus, for this particular problem, a suddenly applied load produces a deflection twice as great as would be produced by a gradually applied load.

**15.6.** A cantilever beam is struck at its tip by a body of weight $W$ falling freely through a height $h$ above the beam, as shown in Fig. 15-6. Neglecting the weight of the beam, determine the total deflection at the tip.

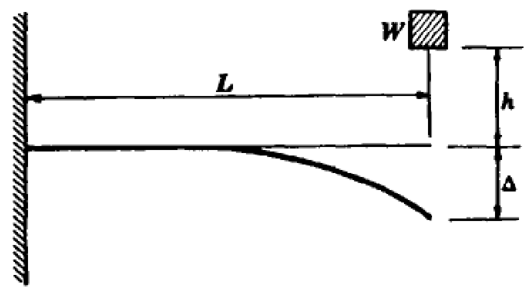



Fig. 15-6

By the time the weight has deflected the tip of the beam to its maximum value, the weight will have done an amount of work given by

$$W(h+\Delta) \tag{1}$$

If we let $P$ denote the force exerted by the weight on the beam at the time of peak deflection, then at this moment the strain energy in the beam is given by $P\Delta/2$. Thus, once the work done by the external force is stored within the beam as internal strain energy we have

$$W(h+\Delta)=\frac{P\Delta}{2} \tag{2}$$

or

$$P=\frac{2W}{\Delta}(h+\Delta) \tag{3}$$

But from Problem 9.2 we know that if this force $P$ acts at the tip of a cantilever beam the deflection at that point is

$$\Delta=\left[\frac{2W}{\Delta}(h+\Delta)\right]\frac{L^3}{3EI} \tag{4}$$

where $I$ is the moment of inertia of the cross section about the neutral axis through the centroid. However, the deflection due to the weight $W$, if it were statically applied, is

$$\Delta_{st}=\frac{WL^3}{3EI} \tag{5}$$

and hence (4) becomes

$$\Delta^2-2\Delta_{st}\Delta-2h\Delta_{st}=0 \tag{6}$$

Solving,

$$\Delta=\Delta_{st}+\sqrt{\Delta_{st}^2+2h\Delta_{st}} \tag{7}$$

where the positive square root is taken so as to obtain the maximum deflection. For the special case of a suddenly applied load at the tip, $h=0$, and (7) yields $\Delta=2\Delta_{st}$. Just as in Problem 15.5, a load suddenly applied produces twice the deflection it would if it were applied gradually.

**15.7.** A simply supported beam is struck at its midpoint by a weight $W = 1$ kN falling freely from a height of $h = 100$ mm above the top of the beam. The beam is 5 m long and of circular cross section 100 mm in diameter. Take $E = 200$ GPa. Determine the maximum deflection of the beam.

The work done by the falling weight in producing the maximum central deflection $\Delta$ is

$$W(h+\Delta) \tag{1}$$

If $P$ denotes the force exerted by the weight on the beam during the moment of maximum deflection, then the strain energy in the beam is $P\Delta/2$. Thus

$$\frac{P\Delta}{2} = W(h + \Delta) \tag{2}$$

or

$$P = \frac{2W(h + \Delta)}{\Delta} \tag{3}$$

But the central deflection of a centrally loaded, simply supported beam is given in Problem 9.12 as

$$\Delta = \frac{PL^3}{48EI} \tag{4}$$

Substituting the above value of $P$, this becomes

$$\Delta = \frac{2W(h + \Delta)}{\Delta}\frac{L^3}{48EI} \tag{5}$$

But the static deflection corresponding to $W$ is $\Delta_{st} = WL^3/48EI$, and hence (5) can be written in the form

$$\Delta^2 - 2\Delta_{st}\Delta - 2h\Delta_{st} = 0 \tag{6}$$

Solving,

$$\Delta = \Delta_{st} + \sqrt{\Delta_{st}^2 + 2h\Delta_{st}} \tag{7}$$

For the beam under consideration,

$$I = \frac{\pi D^4}{64} = 4.9 \times 10^6\,\text{mm}^4$$

The maximum deflection is found from (7) as

$$\Delta_{st} = \frac{(1000)(5)(10^3)^3}{48(200 \times 10^9 \times 10^{-6})(4.9 \times 10^6)} = 2.66\,\text{mm}$$

Thus,

$$\Delta = 2.66 + \sqrt{(2.66)^2 + 2(100)(2.66)} = 25.9\,\text{mm}$$

**15.8.** Derive Castigliano's theorem.

Let us consider a general three-dimensional elastic body loaded by the forces $P_1$, $P_2$, etc. (Fig. 15-7). These would include forces exerted on the body by the various supports. We shall denote the displacement under $P_1$ *in the direction* of $P_1$ by $\Delta_1$, that under $P_2$ in the direction of $P_2$ by $\Delta_2$, etc. If we assume that all

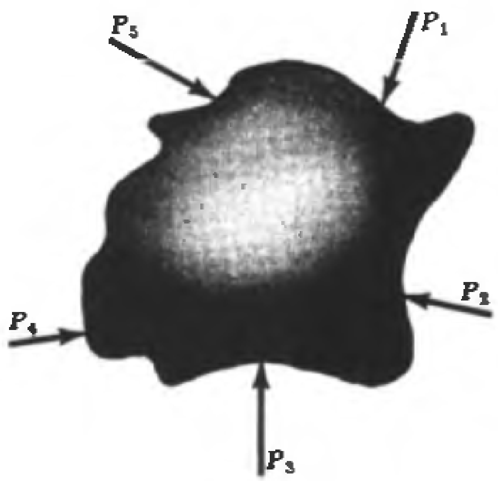



Fig. 15-7

forces are applied simultaneously and gradually increased from zero to their final values given by $P_1$, $P_2$, etc., then the work done by the totality of forces will be

$$U = \frac{P_1}{2}\Delta_1 + \frac{P_2}{2}\Delta_2 + \frac{P_3}{2}\Delta_3 + \cdots \tag{1}$$

This work is stored within the body as elastic strain energy.

Let us now increase the $n$th force by an amount $dP_n$. This changes both the state of deformation and also the internal strain energy slightly. The increase in the latter is given by

$$\frac{\partial U}{\partial P_n} dP_n \tag{2}$$

Thus, the total strain energy after the increase in the $n$th force is

$$U + \frac{\partial U}{\partial P_n} dP_n \tag{3}$$

Let us reconsider this problem by first applying a very small force $dP_n$ alone to the elastic body. Then, we apply the same forces as before, namely, $P_1$, $P_2$, $P_3$, etc. Due to the application of $dP_n$ there is a displacement in the direction of $dP_n$ which is infinitesimal and may be denoted by $d\Delta_n$. Now, when $P_1$, $P_2$, $P_3$, etc., are applied, their effect on the body will not be changed by the presence of $dP_n$ and the internal strain energy arising from application of $P_1$, $P_2$, $P_3$, etc., will be that indicated in (*1*). But as these forces are being applied the small force $dP_n$ goes through the additional displacement $\Delta_n$ caused by the forces $P_1$, $P_2$, $P_3$, etc. Thus, it gives rise to additional work $(dP_n)\Delta_n$ which is stored as internal strain energy and hence the total strain energy in this case is

$$U + (dP_n)\Delta_n \tag{4}$$

Since the final strain energy must be independent of the order in which the forces are applied, we may equate (*3*) and (*4*):

$$U + \frac{\partial U}{\partial P_n} dP_n = U + (dP_n)\Delta_n$$

or

$$\Delta_n = \frac{\partial U}{\partial P_n} \tag{5}$$

This is *Castigliano's theorem*; i.e., the displacement of an elastic body under the point of application of any force, in the direction of that force, is given by the partial derivative of the total internal strain energy with respect to that force. Equations for $U$ are given in Problems 15.1, 15.2, and 15.3 for axial, torsional, and bending loadings, respectively. However, instead of using the integral forms of the equations in those problems, it is usually more convenient to differentiate through the integral signs, and thus for a body subject to combined axial, torsional, and bending effects, we have for the displacement $\Delta_n$ under the force $P_n$

$$\Delta_n = \int \frac{P(\partial P/\partial P_n)\,ds}{AE} + \int \frac{T(\partial T/\partial P_n)\,ds}{GJ} + \int \frac{M(\partial M/\partial P_n)\,ds}{EI}$$

For a body composed of a finite number of elastic subbodies, these integrals are replaced by finite summations, as shown in Problem 15.9.

The term "force" here is used in its most general sense and implies either a true force or a couple. For the case of a couple, Castigliano's theorem gives the angular rotation under the point of application of the couple in the sense of rotation of the couple.

It is important to observe that the above derivation required that we be able to vary the $n$th force, $P_n$, independently of the other forces. Thus, $P_n$ must be statically independent of the other external forces, implying that the energy $U$ must always be expressed in terms of the statically independent forces of the system. Obviously, reactions that can be determined by statics cannot be considered as independent forces.

**15.9.** The bars $AB$ and $CB$ of Fig. 15-8 are pinned at $A$, $C$, and $B$ and subject to the horizontal applied load $P$ acting at $B$. Use Castigliano's theorem to determine the horizontal and vertical components of displacement of pin $B$.

In order to use Castigliano's theorem, we must have a force at $B$ acting in each of the directions in which we seek the displacement. Since the real force $P$ acts horizontally, we must consider that force as well as an auxiliary force $Q$ that we introduce in the vertical direction at $B$. Thus, the free-body diagram of the pin at $B$ appears as shown in Fig. 15-9.

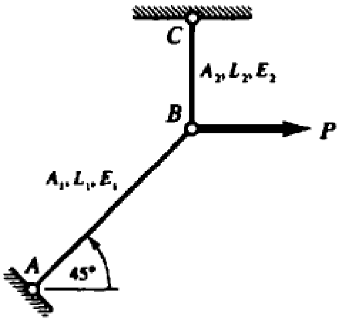



**Fig. 15-8**

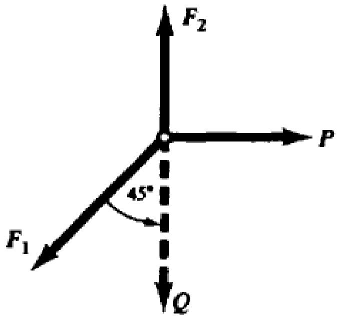



**Fig. 15-9**

For equilibrium we have

$$\Sigma F_x = P - F_1 \sin 45° = 0 \qquad \text{and therefore} \qquad F_1 = P\sqrt{2} \tag{1}$$

$$\Sigma F_y = F_2 - Q - F_1 \cos 45° = 0 \qquad \text{and therefore} \qquad F_2 = P + Q \tag{2}$$

Castigliano's theorem applied to a bar system states that

$$\Delta_x = \sum_{i=1,2} \frac{F_i(\partial F_i/\partial P)L_i}{A_i E_i} \qquad \Delta_y = \sum_{i=1,2} \frac{F_i(\partial F_i/\partial Q)L_i}{A_i E_i} \tag{3}$$

For our bar forces we have

$$F_1 = P\sqrt{2} \qquad \frac{\partial F_1}{\partial P} = \sqrt{2} \qquad \frac{\partial F_1}{\partial Q} = 0$$

$$F_2 = P + Q \qquad \frac{\partial F_2}{\partial P} = 1 \qquad \frac{\partial F_2}{\partial Q} = 1$$

Now that we have taken the partial derivatives with respect to $P$ and $Q$, we may set $Q = 0$.

Substituting in (3),

$$\Delta_x = \frac{(P\sqrt{2})(\sqrt{2})L_1}{A_1 E_1} + \frac{(P)(1)L_2}{A_2 E_2} = \frac{2PL_1}{A_1 E_1} + \frac{PL_2}{A_2 E_2}$$

$$\Delta_y = \frac{(P\sqrt{2})(0)L_1}{A_1 E_1} + \frac{P(1)L_2}{A_2 E_2} = \frac{PL_2}{A_2 E_2}$$

which agree with the results found using a geometric approach in Problem 1.12.

**15.10.** The system shown in Fig. 15-10 consists of a horizontal bar $CDF$ of bending rigidity $EI_1$ and torsional rigidity $GJ_1$ which is rigidly welded at $D$ to bar $DB$ of bending rigidity $EI_2$. At point $B$ the horizontal bar $DB$ is attached to the vertical bar $AB$ of cross-sectional area $A$ and Young's modulus $E$. The support at $C$ permits only rotation in the $x$-$y$ plane about the $z$-axis and the end $F$ is restrained against angular rotation about the $x$-axis and can deflect only vertically. Determine the vertical deflection at $F$ due to the application of the load $P$ acting parallel to the $y$-axis.

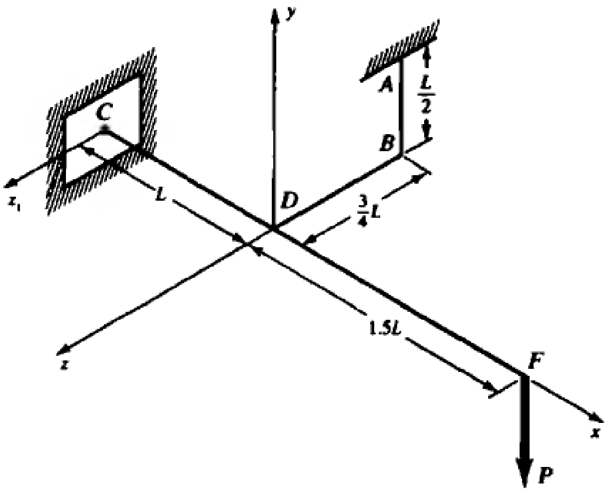



**Fig. 15-10**

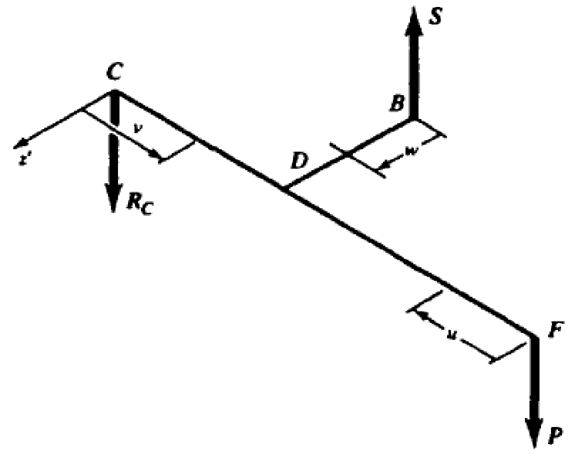



**Fig. 15-11**

A free-body diagram of the bars *CDF* and *DB* is shown in Fig. 15-11, where $R_C$ is the vertical reaction at *C* and *S* is the axial force in bar *AB*. For equilibrium about the $z_1$-axis, we have

$$\Sigma M_{z_1} = S(L) - (2.5L)P = 0$$

$$S = 2.5P$$

and for equilibrium in the *y*-direction

$$-R_C - P + S = 0$$

$$R_C = 1.5P$$

Let us introduce the variables *u*, *v*, and *w* as shown in Fig. 15-11 to denote positions of points in regions *FD*, *CD*, and *BD* of the system. The bending and twisting moments are then given by

In *FD*: $$M = Pu \qquad \frac{\partial M}{\partial P} = u$$

In *CD*: $$M = R_C v = 1.5Pv \qquad \frac{\partial M}{\partial P} = 1.5v$$

In *CDF*: $$T = S\left(\frac{3}{4}L\right) = (2.5P)\left(\frac{3}{4}L\right) = \frac{15}{8}PL \qquad \frac{\partial T}{\partial P} = \frac{15}{8}L$$

In *BD*: $$M = Sw = (2.5P)w \qquad \frac{\partial M}{\partial P} = 2.5w$$

Castigliano's theorem gives the deflection at *F* as the partial derivative of the total internal strain energy with respect to *F*. As indicated by the bending moments in *FD*, *CD*, and *BD*, as well as the twisting moment in *CDF*, and the axial force in *AB*, this becomes

$$\Delta_F = \frac{\partial U}{\partial P} = \int \left[ \overset{\text{(FD)(CD)(BD)}}{\frac{M(\partial M/\partial P)\,ds}{EI}} + \overset{\text{(CDF)}}{\frac{T(\partial T/\partial P)(2.5L)}{GJ}} + \overset{\text{(AB)}}{\frac{S(\partial S/\partial P)(\frac{1}{2}L)}{AE}} \right]$$

where *s* is a coordinate of length used as a variable of integration over the appropriate variable in each of the bars indicated by the circled bar designators above the integrals. The twisting moment is constant in *CDF* and the axial force is constant in *AB*, so there is no need to integrate to obtain the strain energy corresponding to these loads. Substituting the above values of bending and twisting moments and axial force, we find

$$\Delta_F = \overset{(FD)}{\int_0^{1.5L} \frac{(Pu)(u)\,du}{EI_1}} + \overset{(CD)}{\int_0^{L} \frac{(1.5Pv)(1.5v)\,dv}{EI_1}} + \overset{(BD)}{\int_0^{(3/4)L} \frac{(2.5Pw)(2.5w)\,dw}{EI_2}}$$

$$+ \overset{(CDF)}{\frac{(\frac{15}{8}PL)(\frac{15}{8}L)(2.5L)}{GJ_1}} + \overset{(AB)}{\frac{(2.5P)(2.5)(\frac{1}{2}L)}{AE}}$$

$$= 1.875\frac{PL^3}{EI_1} + 0.879\frac{PL^3}{EI_2} + 8.79\frac{PL^3}{GJ_1} + 3.13\frac{PL}{AE}$$

**15.11.** The pin-connected framework shown in Fig. 15-12 consists of two identical upper rods $AB$ and $AC$, two shorter, lower rods $BD$ and $DC$, together with a rigid horizontal brace $BC$. All bars have cross-sectional area $A$ and modulus of elasticity $E$. Determine the vertical displacement of point $D$ due to the action of the vertical load applied there.

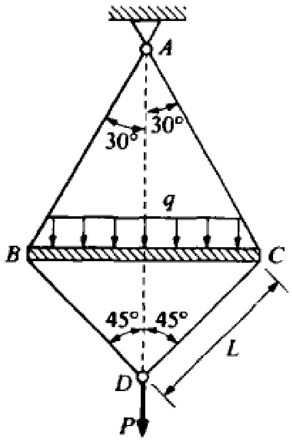



**Fig. 15-12**

This problem was considered by an approach involving the geometry of displacement in Problem 1.11. Let us consider it now using Castigliano's approach. We have already used statics to find bar forces in Problem 1.11, and these are

$$F_{DB} = F_{DC} = \frac{P\sqrt{2}}{2}$$

$$F_{AB} = F_{AC} = \frac{P + \left(\frac{2}{\sqrt{2}}\right)qL}{\sqrt{3}}$$

The deflection of $D$ in the direction of $P$ is given by Castigliano's theorem as

$$\Delta_D = \sum \frac{S\left(\frac{\partial S}{\partial P}\right)L}{AE}$$

Substituting, we have

BD DC

$$\Delta_D = 2\left\{\frac{\left(\frac{P\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)L}{AE}\right\}$$

AB AC

$$+2\left\{\frac{\left(P+\frac{2}{\sqrt{2}}qL\right)\left(\frac{1}{\sqrt{3}}\right)\left(\frac{2L}{\sqrt{2}}\right)}{AE}\right\}$$

$$\Delta_D = 1.942\frac{PL}{AE} + 1.333\frac{qL^2}{AE}$$

which agrees with the result found by the geometric approach in Problem 1.11.

**15.12.** A structure is in the form of one quadrant of a thin circular ring of radius $R$. One end is clamped and the other end is loaded by a vertical force $P$ (see Fig. 15-13). Determine the vertical displacement under the point of application of the force $P$. Consider only strain energy of bending.

From statics, the reactions at the clamped end consist of a vertical force $P$ and a couple $PR$. The bending moment at the section in the ring located by the angle $\theta$ is given by

$$M = PR - P(R - R\cos\theta) = PR\cos\theta \qquad \text{from which} \qquad \frac{\partial M}{\partial P} = R\cos\theta$$

Castigliano's theorem states that the vertical deflection at $A$ is given by

$$\Delta_v = \frac{\partial U}{\partial P} = \int_0^{\pi/2}\frac{M(\partial M/\partial P)R\,d\theta}{EI} = \int_0^{\pi/2}\frac{(PR\cos\theta)(R\cos\theta)R\,d\theta}{EI} = \frac{P\pi R^3}{4EI}$$

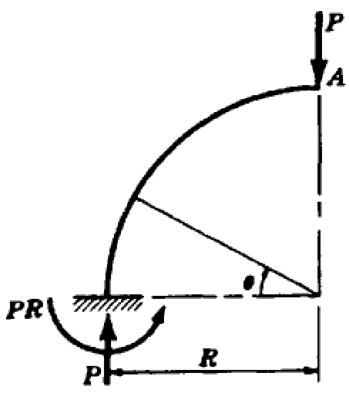



Fig. 15-13

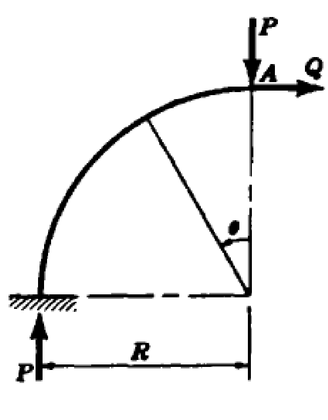



Fig. 15-14

**15.13.** Determine the horizontal displacement of point $A$ in Problem 15.12.

Since there is no horizontal force applied at $A$, we must temporarily introduce an auxiliary force $Q$ shown in Fig. 15-14 in order to be able to use Castigliano's theorem. This time, let us measure $\theta$ from the vertical, making it unnecessary to determine reactions at $B$. Thus, at the section denoted by $\theta$ the bending moment is

$$M = PR\sin\theta + Q(R - R\cos\theta) \qquad \text{from which} \qquad \frac{\partial M}{\partial Q} = R - R\cos\theta$$

The horizontal displacement at $A$ is given by

$$\Delta_h = \frac{\partial U}{\partial Q} = \int_0^{\pi/2} \frac{M(\partial M/\partial Q)R\,d\theta}{EI}$$

Now that the partial derivative has been taken, $Q$ may be set equal to zero, yielding

$$\Delta_h = \int_0^{\pi/2} \frac{(PR\sin\theta)(R - R\cos\theta)R\,d\theta}{EI} = \frac{PR^3}{2EI}$$

**15.14.** A thin circular ring in the form of one quadrant $OA$ of a circle lies in the $x$-$z$ plane and has rigidly attached to it at the point $A$ a straight bar $AB$ also in the $x$-$z$ plane. Both the ring and the bar have bending rigidity $EI$ and torsional rigidity $GJ$. The unsupported end $B$ is loaded by a twisting moment represented by the vector $T_B$ directed parallel to the $x$-axis as shown in Fig. 15-15. Determine the $y$-component of displacement of point $B$.

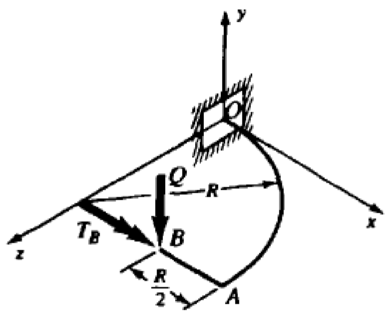



Fig. 15-15

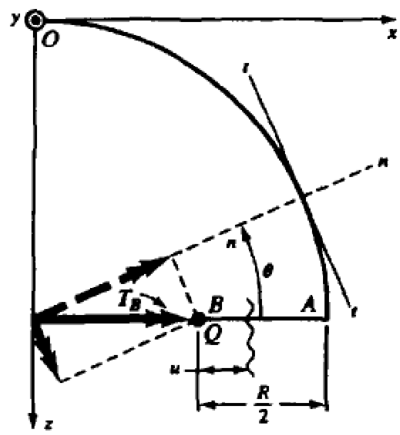



Fig. 15-16

To utilize Castigliano's theorem, we must introduce an auxiliary force $Q$ in the direction of the desired displacement; that is, $Q$ must be directed downward and parallel to the $y$-axis. The view of the system looking from the positive end of the $y$-axis toward the $x$-$z$ plane appears as in Fig. 15-16, where $n$-$n$ and $t$-$t$ denote axes normal and tangential, respectively, to the ring at an arbitrary location denoted by the angle $\theta$. In that figure the applied twisting moment $T_B$ is shown, along with its components oriented in the $n$-$n$ and $t$-$t$ directions. The auxiliary force $Q$ is represented by the tail of its vector representation at $B$ to denote its downward direction.

From Fig. 15-16 we have in the straight bar $BA$

$$M = Qu \qquad \frac{\partial M}{\partial Q} = u$$

$$T = T_B \qquad \frac{\partial T}{\partial Q} = 0$$

In the quadrant $AO$ from the geometry of the figure

$$M = T_B\cos\theta + Q\left(\frac{R}{2}\sin\theta\right)$$

$$\frac{\partial M}{\partial Q} = \frac{R}{2}\sin\theta$$

and $$T = T_B \sin\theta + Q\left(R - \frac{R}{2}\cos\theta\right)$$

$$\frac{\partial T}{\partial Q} = R - \frac{R}{2}\cos\theta$$

Using Castigliano's theorem, we find when we set $Q = 0$ after taking the partial derivatives

$$\Delta_B = \int \frac{M(\partial M/\partial Q)\,ds}{EI} + \int \frac{T(\partial T/\partial Q)\,ds}{GJ}$$

$$\underset{(BA)}{} \quad \underset{(BA)}{} \quad \underset{(AO)}{}$$

$$= 0 + 0 + \int_0^{\pi/2} \frac{(T_B\cos\theta)[(R/2)\sin\theta]R\,d\theta}{EI}$$

$$\underset{(AO)}{}$$

$$+ \int_0^{\pi/2} \frac{(T_B\sin\theta)[R - (R/2)\cos\theta]R\,d\theta}{GJ}$$

$$= \frac{T_B R^2}{4EI} + \frac{T_B R^2}{2GJ}(\pi - 1)$$

**15.15.** A structure consists of a quadrant of a circular ring $OA$, to which is rigidly attached a bar $BA$ which in turn is welded to bar $CB$. These bars all lie in the horizontal plane $x$-$z$, as shown in Fig. 15-17, and all have bending rigidity $EI$ and torsional rigidity $GJ$. Determine the vertical deflection of point $C$ due to the load $P$ applied vertically there.

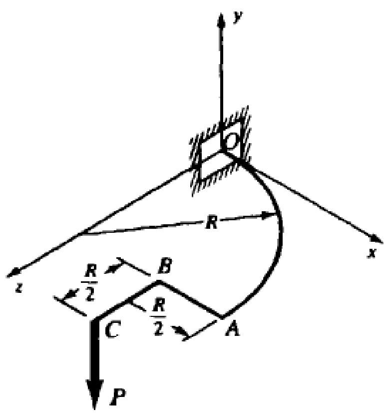



Fig. 15-17

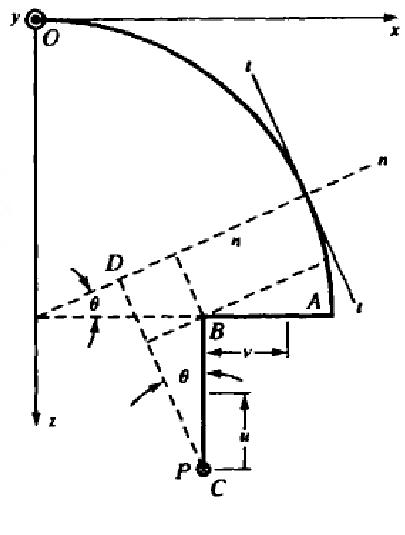



Fig. 15-18

It is first necessary to determine the bending and twisting moments at an arbitrary point in the quadrant $OA$. Let us introduce the coordinate system shown in Fig. 15-18, where $\theta$ denotes the angular coordinate of this arbitrary point. The axes $n$-$n$ and $t$-$t$ represent normal and tangential directions to the circular ring at the point represented by the angle $\theta$. From the geometry of Fig. 15-18, we have the bending moment about $n$-$n$ to be

$$M = M_{n\text{-}n} = P(\overline{CD}) = P\left(\frac{R}{2}\cos\theta + \frac{R}{2}\sin\theta\right)$$

and the twisting moment about $t$-$t$ to be

$$T = T_{t\text{-}t} = P\left[\left(R - \frac{R}{2}\cos\theta\right) + \frac{R}{2}\sin\theta\right]$$

From these equations we thus have in the ring $OA$

$$\frac{\partial M}{\partial P} = \frac{R}{2}(\sin\theta + \cos\theta)$$

$$\frac{\partial T}{\partial P} = \frac{R}{2}(1 + \sin\theta - \cos\theta)$$

Next, in bar $CB$ from Fig. 15-18 we have the bending moment at an arbitrary point represented by $u$ to be

$$M = Pu \qquad \text{so} \qquad \frac{\partial M}{\partial P} = u$$

and the twisting moment $T$ in this bar is zero.

In bar $BA$ the bending moment from Fig. 15-18 is $M = Pv$ and the twisting moment is $T = PR/2$. Thus, for $BA$

$$\frac{\partial M}{\partial P} = v \qquad \frac{\partial T}{\partial P} = \frac{R}{2}$$

By Castigliano's theorem, the deflection of point $C$ due to the force $P$ is

$$\Delta_C = \int \frac{M(\partial M/\partial P)\,ds}{EI} + \int \frac{T(\partial T/\partial P)\,ds}{GJ}$$

$$= \underbrace{\frac{1}{EI}\int_0^{\pi/2} \frac{PR^2}{4}(\sin\theta + \cos\theta)^2 R\,d\theta}_{OA}$$

$$+ \underbrace{\frac{1}{GJ}\int_0^{\pi/4} \frac{PR^2}{4}(1 + \sin\theta - \cos\theta)^2 R\,d\theta}_{OA} + \underbrace{\int_0^{R/2} \frac{(Pu)u\,du}{EI}}_{CB}$$

$$+ \underbrace{\int_0^{R/2} \frac{(Pv)v\,dv}{EI}}_{BA} + \underbrace{\frac{(PR/2)(R/2)(R/2)}{GJ}}_{BA}$$

Therefore

$$\Delta_C = \frac{PR^3}{4EI}\left(\frac{\pi}{2} + \frac{4}{3}\right) + \frac{PR^3}{4GJ}\left(\pi - \frac{1}{2}\right)$$

**15.16.** A thin semicircular ring is hinged at each end and loaded by a central concentrated force $P$, as shown in Fig. 15-19. Determine the horizontal reaction at each hinge.

A free-body diagram of this ring, Fig. 15-20, indicates that the desired reaction $H$ is statically indeterminate. We may formulate the bending moment in the right half of the ring as follows:

$$M = \frac{P}{2}(R - R\cos\theta) - HR\sin\theta \qquad \text{and} \qquad \frac{\partial M}{\partial H} = -R\sin\theta \qquad \text{for} \qquad 0 < \theta < \frac{\pi}{2}$$

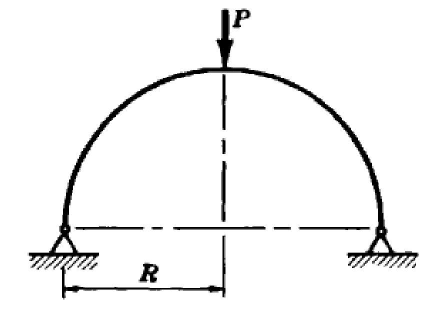



Fig. 15-19

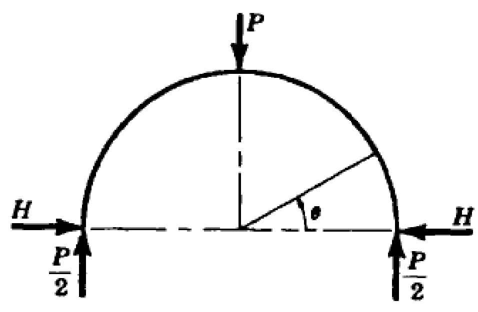



Fig. 15-20

According to Castigliano's theorem, the horizontal displacement at the pin is given by

$$\Delta_H = \frac{\partial U}{\partial H}$$

But we know that this displacement is zero. Taking advantage of the symmetry about the centerline, we may now write

$$0 = \Delta_H = \frac{\partial U}{\partial H} = 2\int_0^{\pi/2} \frac{M(\partial M/\partial H)R\,d\theta}{EI} = \int_0^{\pi/2} \frac{[(P/2)(R - R\cos\theta) - HR\sin\theta](-R\sin\theta)R\,d\theta}{EI}$$

Solving for the unknown $H$: $H = P/\pi$.

**15.17.** In Problem 15.16, determine the vertical displacement under the point of application of the central force $P$.

In almost all statically indeterminate problems it is necessary first to determine the redundant reactions before any displacements can be found. For the present ring this has already been done in Problem 15.16.

In the right half of the ring, the bending moment is

$$M = \frac{P}{2}(R - R\cos\theta) - \frac{P}{\pi}R\sin\theta \qquad \text{for} \qquad 0 < \theta < \frac{\pi}{2}$$

and

$$\frac{\partial M}{\partial P} = \frac{1}{2}(R - R\cos\theta) - \frac{R}{\pi}\sin\theta$$

By Castigliano's theorem, the vertical displacement under the point of application of $P$ is

$$\Delta = \frac{\partial U}{\partial P} = 2\int_0^{\pi/2} \frac{M(\partial M/\partial P)R\,d\theta}{EI}$$

where we have taken advantage of symmetry. Thus

$$\Delta = 2\int_0^{\pi/2} \frac{[(P/2)(R - R\cos\theta) - (PR/\pi)\sin\theta]\,[\frac{1}{2}(R - R\cos\theta) - (R/\pi)\sin\theta]R\,d\theta}{EI} = \frac{PR^3}{EI}\left(\frac{3\pi}{8} + \frac{3}{2\pi} - 1\right)$$

**15.18.** A structure in the form of a thin semicircular ring lies in a horizontal plane, has both ends clamped, and is subjected to a central vertical force $P$, as shown in Fig. 15-21. Determine the various reactions.

The vertical force reactions at $A$ and $C$ are each $P/2$ and the bending moment exerted by the support on the ring at each of these points is found from statics to be $PR/2$. There is also another component of reaction exerted by the support on the ring, i.e., a twisting moment $T_0$ acting at each of the points $A$ and $C$. These two types of moment reaction are best illustrated by the vector representation of moment in Fig. 15-22, where a double-headed arrow indicates a moment in the usual sense of the right-hand rule for vector

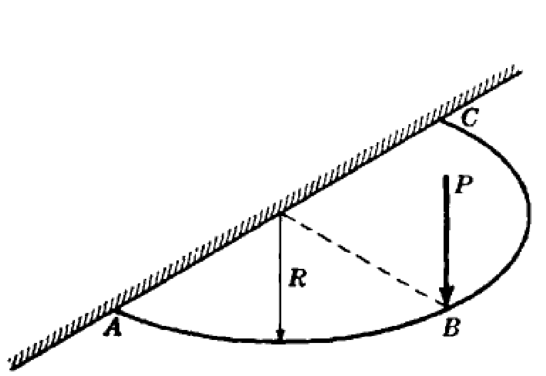



Fig. 15-21

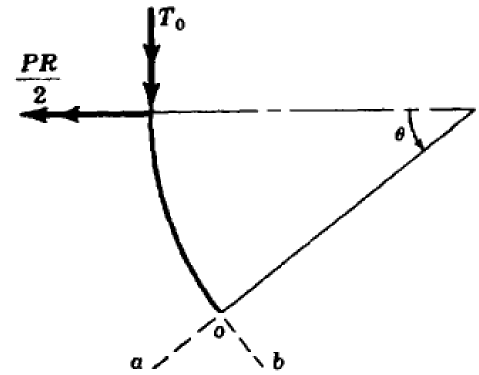



Fig. 15-22

representation of moment. A segment of the ring to the arbitrary point represented by $\theta$ $(0 < \theta < \pi/2)$ is shown and at this cross section given by $\theta$ there is a bending moment about the $oa$-axis given by

$$M = \frac{P}{2} R \sin\theta - \frac{PR}{2}\cos\theta - T_0 \sin\theta$$

There is a twisting moment about the $ob$-axis given by

$$T = \frac{P}{2}(R - R\cos\theta) - \frac{PR}{2}\sin\theta + T_0\cos\theta$$

From these,

$$\frac{\partial M}{\partial T_0} = -\sin\theta \qquad \frac{\partial T}{\partial T_0} = \cos\theta$$

Since the ring is completely restrained at points $A$ and $C$, we may write (taking advantage of symmetry)

$$0 = \phi_A = \phi_C = 2\int_0^{\pi/2} \frac{M(\partial M/\partial T_0)R\,d\theta}{EI} + 2\int_0^{\pi/2} \frac{T(\partial T/\partial T_0)R\,d\theta}{GJ}$$

where $\phi$ is used to denote angular rotation of an arbitrary point of the bar, and $\phi_A$ and $\phi_C$ are the zero values of this quantity at the points $A$ and $C$. Substituting,

$$0 = \int_0^{\pi/2} \frac{\left(\frac{PR}{2}\sin\theta - \frac{PR}{2}\cos\theta - T_0\sin\theta\right)(-\sin\theta)R\,d\theta}{EI}$$
$$+ \int_0^{\pi/2} \frac{\left[\frac{P}{2}(R - R\cos\theta) - \frac{PR}{2}\sin\theta + T_0\cos\theta\right](\cos\theta)R\,d\theta}{GJ}.$$

Solving,

$$T_0 = \frac{\frac{PR}{2}\left(\frac{2-\pi}{EI} + \frac{2-\pi}{GJ}\right)}{\left(\frac{\pi}{EI} - \frac{\pi}{GJ}\right)}$$

**15.19.** The thin rod shown in Fig. 15-23 consists of the straight bar $GFD$ attached to semicircular end bars $BCD$ and $GHJ$, together with two more straight bars $JK$ and $AB$ as indicated. There exists a very small gap $2\Delta_A$ between points $A$ and $K$. Determine the magnitude of this gap when the forces $Q$ are applied.

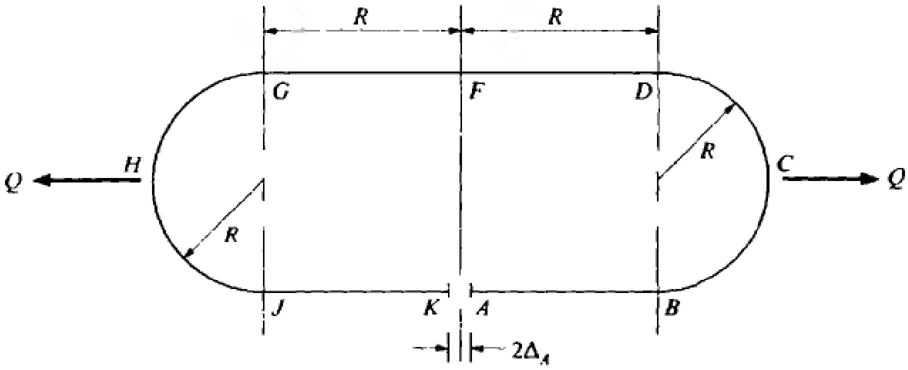



Fig. 15-23

Because of the symmetry of both structure and loading about a centerline extending through $F$, it is possible to examine the structural behavior of only one half of the system, say the right-hand half as shown in Fig. 15-23. Because of the symmetry, point $F$ in Fig. 15-23 behaves as if it were clamped. The real load on this half is $Q$, and to determine displacement at the gap we introduce an auxiliary force $P$ as shown in Fig. 15-24.

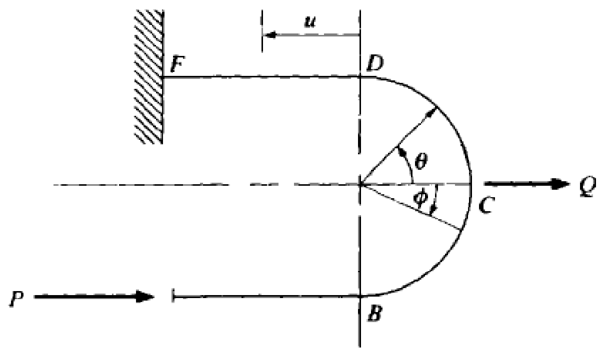



Fig. 15-24

Considering only bending action, in the entire system we have the bending moment in the various regions given by

In $BC$: $$M = P(R - R\sin\phi); \qquad \frac{\partial M}{\partial P} = R(1 - \sin\phi)$$

In $CD$: $$M = P(R + R\sin\theta) + QR\sin\theta$$

$$\frac{\partial M}{\partial P} = R + R\sin\theta$$

In $DF$: $$M = 2PR + QR; \qquad \frac{\partial M}{\partial P} = 2R$$

The deflection at $A$ in the direction of $P$ is given by Castigliano's theorem as

$$\Delta_A = \frac{\partial V}{\partial P} = \int \frac{M\left(\dfrac{\partial M}{\partial P}\right) ds}{EI}$$

Substituting, we have

$$\Delta_A = \overbrace{\int_0^{\pi/2} \frac{P(R - R\sin\phi)R(1-\sin\phi)R\,d\phi}{EI}}^{(BC)}$$

$$+ \overbrace{\int_0^{\pi/2} \frac{[P(R + R\sin\theta) + QR\sin\theta](R + R\sin\theta)R\,d\theta}{EI}}^{(CD)}$$

$$+ \overbrace{\int_0^{R} \frac{(2PR + QR)(du)}{EI}}^{(DF)}$$

This integrates to

$$\Delta_A = \frac{PR^3}{EI}\left[\frac{\pi}{2} - 2 + \frac{\pi}{4}\right] + \frac{R^3}{EI}\left[P\left(\frac{\pi}{2}\right) + P + Q + P + P\left(\frac{\pi}{4}\right) + Q\frac{\pi}{4}\right] + \frac{4PR^3}{EI} + \frac{2QR^3}{EI}$$

Now that the integration has been carried out, we may set $P = 0$ to find

$$\Delta_A = \frac{QR^3}{EI} + \frac{Q\pi R^3}{4EI} + \frac{2QR^3}{EI}$$

The gap at $A$ is twice this because of the deformation of the left half of the system, so that the gap is

$$\frac{QR^3}{2EI}(12 + \pi)$$

**15.20.** The elastic beam $FDCG$ of bending rigidity $EI$ shown in Fig. 15-25 is supported by pinned elastic bars $AB$, $BC$, and $BD$, each of extensional rigidity $AE$. These bars are incapable of resisting bending effects. The load on the system consists of a single concentrated force $P$ applied at the free end $F$. Determine the vertical displacement of $F$.

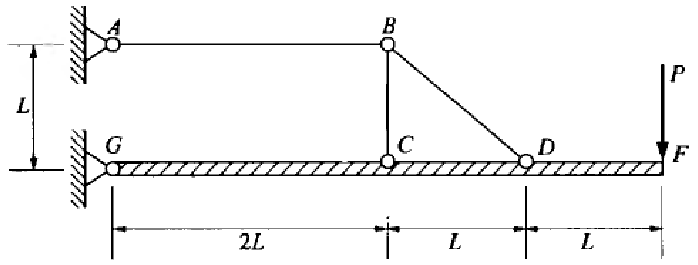



**Fig. 15-25**

This solution is best carried out by Castigliano's method since both bending as well as extensional energies are involved. We must first determine external reactions. A free-body diagram of the system is shown in Fig. 15-26. There is no vertical reaction at $A$ since bar $AB$ is not able to resist transverse (bending) loads. From statics,

$$\Sigma M_A = G_x(L) - P(4L) = 0 \qquad \therefore G_x = 4P$$

$$\Sigma F_H = -A_x + 4P = 0 \qquad \therefore A_x = 4P$$

$$\Sigma F_V = G_y - P = 0 \qquad \therefore G_y = P$$

Thus, bar $AB$ carries a tensile force $4P$.

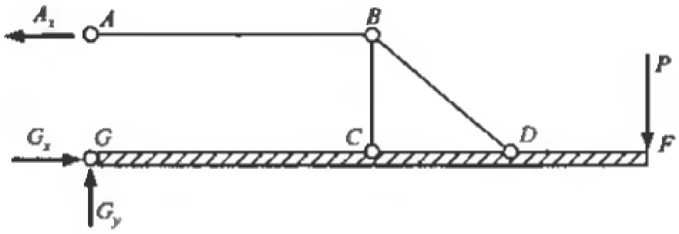



Fig. 15-26

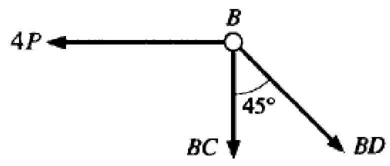



Fig. 15-27

Next, we show in Fig. 15-27 a free-body diagram of the system where a section has been passed through the three bars and axial forces are represented by $BD$ and $BC$ in those bars. From statics,

$$\Sigma F_H = -4P + BD \sin 45° = 0$$

$$\Sigma F_V = -BC - BD \cos 45° = 0$$

Consequently, the axial forces in the three bars are

$$AB = 4P$$

$$BC = -4P$$

$$BD = 4\sqrt{2}P$$

From Problem 15.8 we may determine the deflection at $F$ due to axial loading (only) in these three bars to be

$$\Delta_1 = \sum \frac{S\left(\dfrac{\partial S}{\partial P}\right)L}{AE}$$

$$= \underbrace{\frac{(4P)(4)(2L)}{AE}}_{\text{(AB)}} + \underbrace{\frac{(-4P)(-4)(L)}{AE}}_{\text{(BC)}} + \underbrace{\frac{(4\sqrt{2}P)(4\sqrt{2})(L\sqrt{2})}{AE}}_{\text{(BD)}} \tag{1}$$

$$= \frac{PL}{AE}[24 + 32\sqrt{2}] = 69.2\frac{PL}{AE} \tag{2}$$

Finally, we determine the deflection at $F$ due only to the bending effects in beam $FDCG$. This was shown in Problem 15.8 to be

$$\Delta_2 = \int \frac{M \dfrac{\partial M}{\partial P} ds}{EI} \tag{3}$$

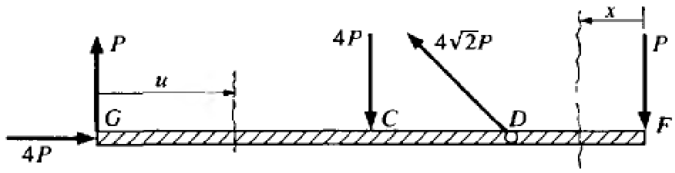



**Fig. 15-28**

Figure 15-28 shows a free-body diagram of the beam with all forces acting upon it. Coordinates $u$ and $x$ are introduced to permit evaluation of the integral in (3). The bending moments are given by

$$FD: \quad M = Px; \quad \frac{\partial M}{\partial P} = x$$

$$DC: \quad M = Px - (4\sqrt{2}P)\left(\frac{1}{\sqrt{2}}\right)(x - L) = -3P_x + 4PL; \quad \frac{\partial M}{\partial P} = -3x + 4L$$

$$GC: \quad M = Pu; \quad \frac{\partial M}{\partial P} = u$$

Thus, for bending effects only, (3) becomes

$$\Delta_2 = \overbrace{\int_{x=0}^{L} \frac{(Px)(x)\,dx}{EI}}^{FD} + \overbrace{\int_{x=L}^{2L} \frac{(-3Px + 4PL)(-3x + 4L)\,dx}{EI}}^{CD}$$

$$+ \overbrace{\int_{u=0}^{u=2L} \frac{(Pu)(u)\,du}{EI}}^{CG}$$

$$= 3.67\frac{PL^3}{EI}$$

The true deflection at $F$ is the sum of $\Delta_1$ and $\Delta_2$:

$$\Delta_F = 69.2\frac{PL}{AE} + 3.67\frac{PL^3}{EI}$$

## Supplementary Problems

**15.21.** A solid conical bar of circular cross section (Fig. 15-29) hangs vertically, subjected only to its own weight, which is $\gamma$ per unit volume. Determine the strain energy stored within the bar.
*Ans.* $U = \pi D^2 L^3 \gamma^2/360E$

**15.22.** The two bars $AB$ and $CB$ of Fig. 15-30 are pinned at each end and subject to a single vertical force $P$. The geometric and elastic constants of each bar are as indicated. Use Castigliano's theorem to determine the horizontal and vertical components of displacement of pin $B$.

*Ans.* $\Delta_x = -\dfrac{PL_1}{\sqrt{3}A_1E_1} + \dfrac{PL_2}{\sqrt{3}A_2E_2}, \quad \Delta_y = \dfrac{PL_1}{3A_1E_1} + \dfrac{PL_2}{3A_2E_2}$

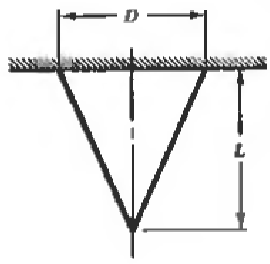



Fig. 15-29

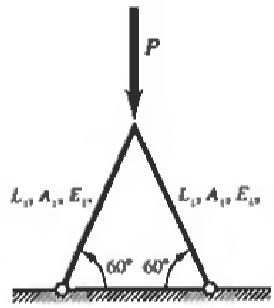



Fig. 15-30

**15.23.** The pin-connected truss shown in Fig. 15-31 is composed of five bars, each of area $A$ and modulus of elasticity $E$. Determine the vertical displacement of point $B$ due to the load $Q$ by equating the work done by $Q$ to the internal strain energy. *Ans.* $\Delta = 2.914QL/AE$

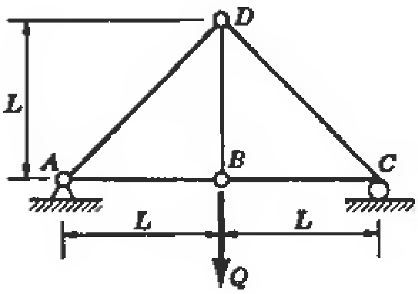



Fig. 15-31

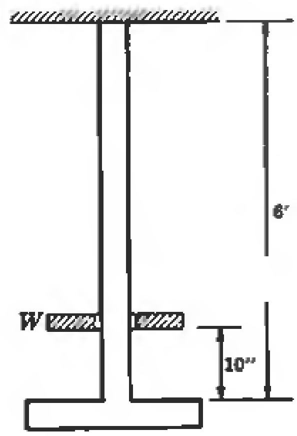



Fig. 15-32

**15.24.** Determine the maximum weight $W$ that can be dropped 10 in onto the flange at the end of the steel bar shown in Fig. 15-32. The bar is 1 in × 2 in in cross section and 6 ft in length. The axial stress is not to exceed 20,000 lb/in². Take $E = 30 \times 10^6$ lb/in². *Ans.* $W = 96$ lb

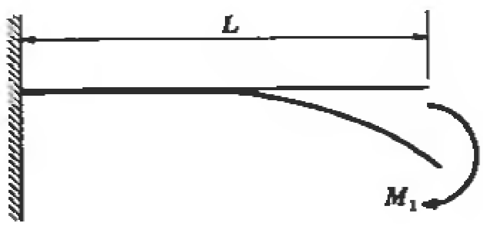



Fig. 15-33

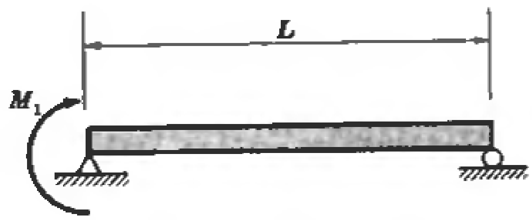



Fig. 15-34

**15.25.** A cantilever beam is loaded by a moment $M_1$ applied at the tip (Fig. 15-33). Determine by Castigliano's theorem the deflection of the tip. *Ans.* $M_1L^2/2EI$

**15.26.** A simply supported beam is loaded by a moment $M_1$ at the left end, as shown in Fig. 15-34. Use Castigliano's theorem to determine the deflection at the midpoint of the bar. *Ans.* $M_1L^2/16EI$

**15.27.** A W203 × 28 steel wide-flange section is used as a cantilever beam of length 4 m. A weight $W$ of 1 kN falls freely through a distance of 150 mm before striking the tip of the beam. Find the maximum deflection of the beam. Take $E = 200$ GPa. Use beam parameters given in Table 8-2 of Chap. 8. *Ans.* 38.8 mm

**15.28.** A structure lies in a vertical plane and is in the form of three quadrants of a thin ring (see Fig. 15-35). One end is clamped, the other is loaded by a vertical force $P$. Determine the horizontal displacement of point $A$. Consider only bending energy. *Ans.* $PR^3/2EI$

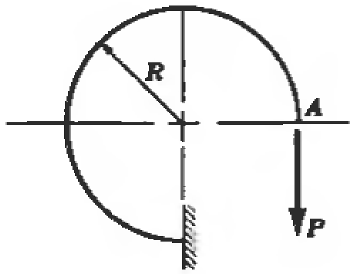



**Fig. 15-35**

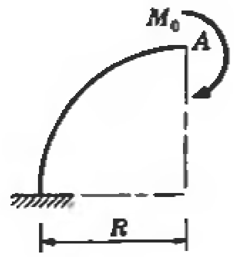



**Fig. 15-36**

**15.29.** A structure is in the form of one quadrant of a thin circular ring of radius $R$. One end is clamped and the other is subject to a couple $M_0$, as shown in Fig. 15-36. Determine the angular rotation, as well as the vertical and horizontal components of displacement of point $A$.

*Ans.* $\dfrac{M_0 \pi R}{2EI}$; $\dfrac{M_0 R^2}{EI}$; $0.571\dfrac{M_0 R^2}{EI}$

**15.30.** The two-sided framework shown in Fig. 15-37 is loaded by a uniformly distributed load $q$ per unit length in region $AB$ together with a couple $M_0$ at the midpoint of $BC$. Determine the vertical displacement of point $A$.

*Ans.* $\dfrac{qL^4}{8EI} + \dfrac{2qL^3 H}{3EI} - \dfrac{M_0 LH}{2EI}$

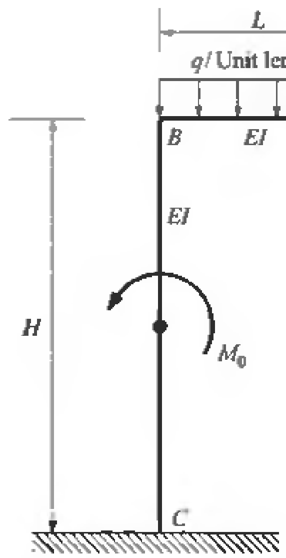



**Fig. 15-37**

**15.31.** The straight bar $AC$ of Fig. 15-38 is rigidly attached at its midpoint $B$ to another rod $BD$ which has end $D$ unsupported but subject to a vertical force $P$. The flexural rigidity of each bar is $EI$ and the torsional rigidity is $GJ$. Bar $AC$ is rigidly clamped at ends $A$ and $C$, and $AC$ and $BD$ lie in a horizontal plane. Determine the deflection under the load $P$.

*Ans.* $\dfrac{3PL^3}{8EI} + \dfrac{PL^3}{4GJ}$

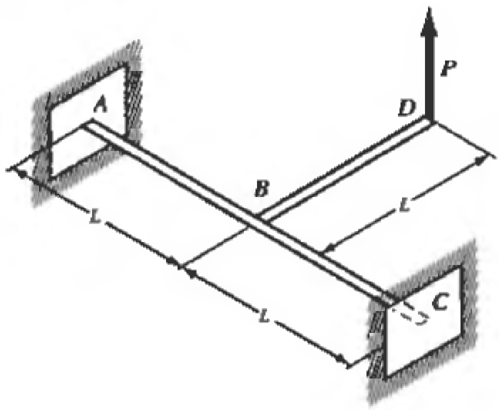



**Fig. 15-38**

**15.32.** Figure 15-39 shows a thin ring in the form of one quadrant of a circle. One end is fixed, the other is free, and the system is loaded by a moment at the midpoint. Determine the vertical component of displacement of point $A$.

*Ans.* $\dfrac{M_0R^2}{\sqrt{2}EI}$

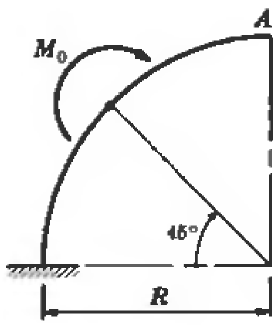



**Fig. 15-39**

**15.33.** The beam of Fig. 15-40 is supported at the left end, clamped at the right end and subject to a concentrated load. Determine the reaction at the left support by Castigliano's theorem. *Ans.* $Pb^2(2L + a)/2L^3$

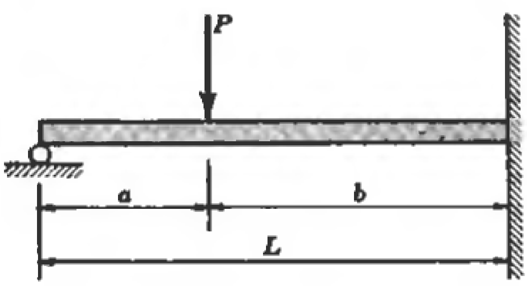



**Fig. 15-40**

**15.34.** A thin ring forms one quadrant of a circle and is loaded as shown in Fig. 15-41. One end is fixed and the other is pinned so as to prevent horizontal and vertical displacements. Find the components of reaction at the pin. *Ans.* $B_v = 0.19M_0/R$, $B_h = 1.12M_0/R$

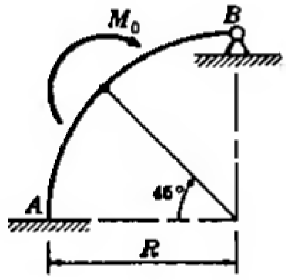



**Fig. 15-41**

**15.35.** A thin ring in the form of one quadrant of a circle lies in a vertical plane and is subject to uniform radial loading, as shown in Fig. 15-42. One end, $A$, is rigidly clamped and the other end, $C$, is unsupported. Determine the horizontal and vertical components of displacement of point $C$.

*Ans.* $\Delta_x = 0.500\dfrac{qR^4}{EI}$, $\Delta_y = 0.36\dfrac{qR^4}{EI}$

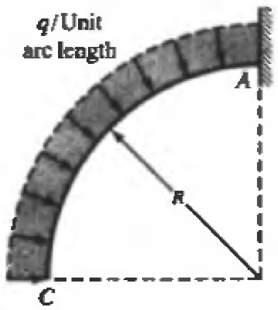



**Fig. 15-42**

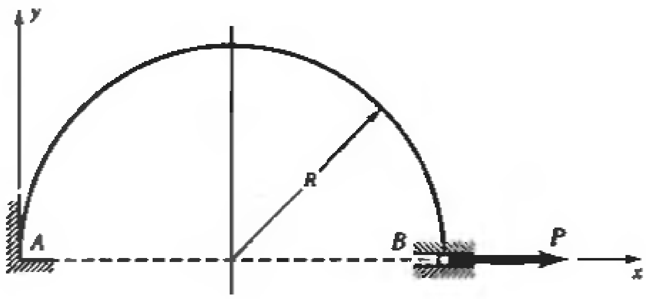



**Fig. 15-43**

**15.36.** A thin semicircular ring (see Fig. 15-43) of bending rigidity $EI$ lies in a vertical plane, is clamped at end $A$, and may move in a horizontal, frictionless guide at end $B$. The load is $P$, applied horizontally at end $B$. Determine the horizontal displacement of end $B$ of the ring. Also, determine the vertical displacement due to the same load at $B$ if the guide is removed.

*Ans.* $\Delta_{B_x} = 0.14\dfrac{PR^3}{EI}$, $\Delta_{B_y} = \dfrac{2PR^3}{EI}$

**15.37.** The structure of Fig. 15-44 is in the form of one quadrant of a thin circular ring $AB$ together with a straight bar $BC$ rigidly joined at $B$ so that $AB$ is tangent to the ring. A load $P$ acts parallel to the $y$-axis at $B$. The end $C$ is unsupported. Determine the $y$-component of displacement of point $C$. The bending rigidity of both regions is $EI$ and the torsional rigidity is $GJ$.

*Ans.* $$\Delta_C = \frac{1}{EI}\left[\frac{\pi}{4}PR^3 + \frac{1}{2}PR^2L\right] + \frac{1}{GJ}\left[\left(\frac{3\pi}{4} - 2\right)PR^3 - \frac{PR^2L}{2}\right]$$

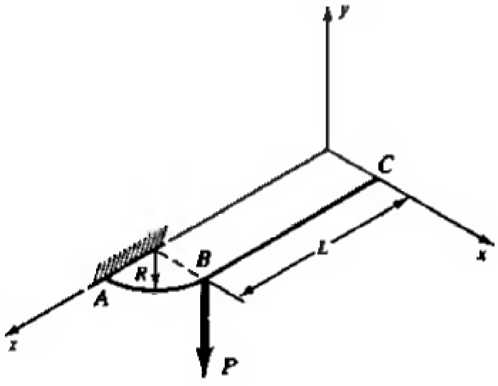



**Fig. 15-44**

**15.38.** The balcony-like structure of Fig. 15-45 is in the form of a semicircular ring, lies in a horizontal plane, and is subject to a twisting moment $T_B$ at its midpoint. Determine the reactive twisting moment at each end $A$ and $C$. *Ans.* $T_B/9\pi$

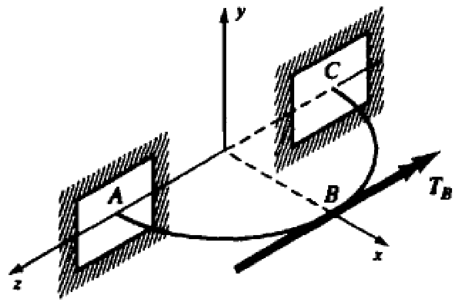



**Fig. 15-45**

**15.39.** A thin ring is subjected to the equal and opposite diametral forces indicated in Fig. 15-46. Determine the bending moment at $A$ and also the increase in diameter of the ring along the diameter $CD$.

*Ans.* $M_A = \dfrac{PR}{2}\left(\dfrac{\pi - 2}{2}\right)$, $\Delta = 0.149\dfrac{PR^3}{EI}$

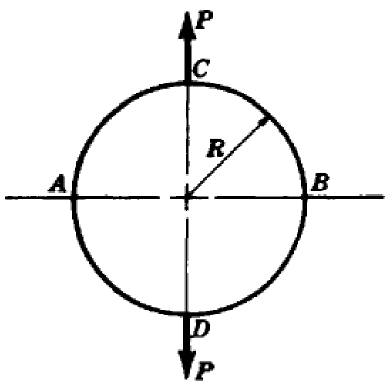



**Fig. 15-46**

**15.40.** A thin ring is loaded by forces which are uniformly distributed along the horizontal projection of the ring (see Fig. 15-47). Determine the decrease in the vertical diameter. *Ans.* $wR^4/6EI$

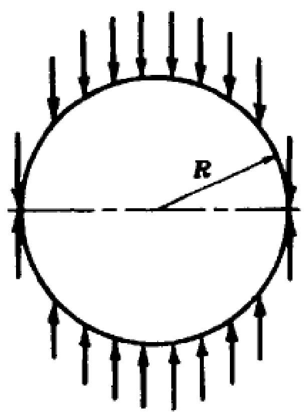



**Fig. 15-47**

**15.41.** A thin semicircular ring shown in Fig. 15-48 of bending rigidity $EI$ lies in a vertical plane; it is clamped at end $A$ and unsupported at $B$. It is loaded by a horizontal force $P$ at end $B$. Determine horizontal and vertical components of displacement of end $B$ of the ring.

*Ans.* $\Delta_{B_x} = \dfrac{PR^3\pi}{2EI}$, $\Delta_{B_y} = \dfrac{2PR^3}{EI}$

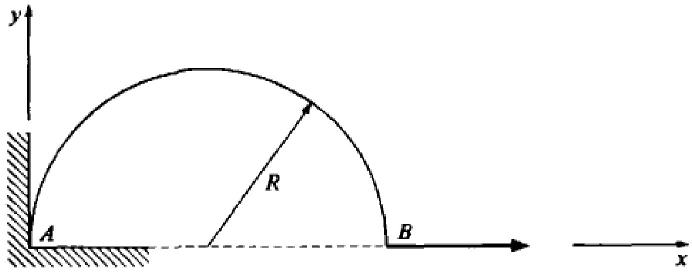



**Fig. 15-48**

**15.42.** A structure in the form of a thin three-sided rectangular frame lies in a horizontal plane, has both ends clamped, and is subject to a central vertical force $P$, as shown in Fig. 15-49. Determine the reactive torque at each support. The frame is of constant cross section throughout.

*Ans.* $\dfrac{Pb^2/EI}{\dfrac{b}{EI}+\dfrac{a}{GJ}}$

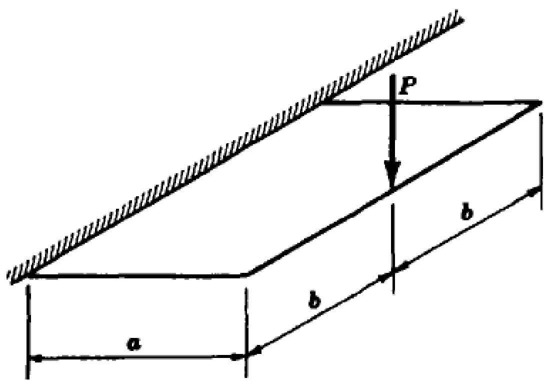



**Fig. 15-49**

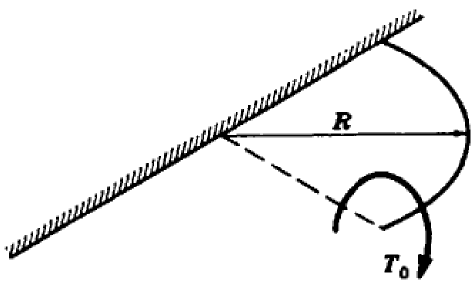



**Fig. 15-50**

**15.43.** A thin structure in the form of one quadrant of a circle (Fig. 15-50) lies in a horizontal plane and is subject to a torque $T_0$ at the free end. The other end is clamped. Determine the vertical displacement of the free end.

*Ans.* $T_0R^2\left(\dfrac{\pi}{4EI}+\dfrac{\pi}{4GJ}-\dfrac{1}{GJ}\right)$

# Chapter 16

## Combined Stresses

### INTRODUCTION

Previously in this book we have considered stresses arising in bars subject to axial loading, shafts subject to torsion, and beams subject to bending, as well as several cases involving thin-walled pressure vessels. It is to be noted that we have considered a bar, for example, to be subject to only *one* loading at a time, such as bending. But frequently such bars are simultaneously subject to several of the previously mentioned loadings, and it is required to determine the state of stress under these conditions. Since normal and shearing stress are vector quantities, considerable care must be exercised in combining the stresses given by the expressions for single loadings as derived in previous chapters. It is the purpose of this chapter to investigate the state of stress on an arbitrary plane through an element in a body subject to several simultaneous loadings.

### GENERAL CASE OF TWO-DIMENSIONAL STRESS

In general if a plane element is removed from a body it will be subject to the normal stresses $\sigma_x$ and $\sigma_y$ together with the shearing stress $\tau_{xy}$, as shown in Fig. 16-1.

### SIGN CONVENTION

For normal stresses, tensile stresses are considered to be positive, compressive stress negative. For shearing stresses, the positive sense is that illustrated in Fig. 16-1.

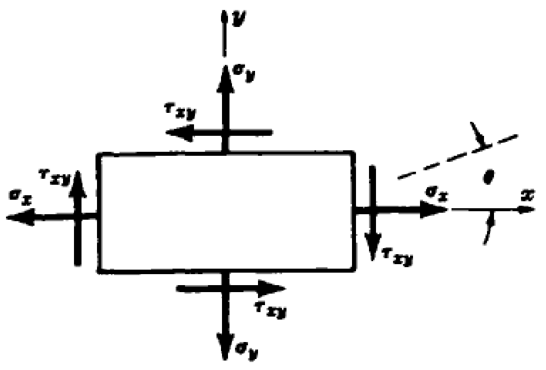



Fig. 16-1

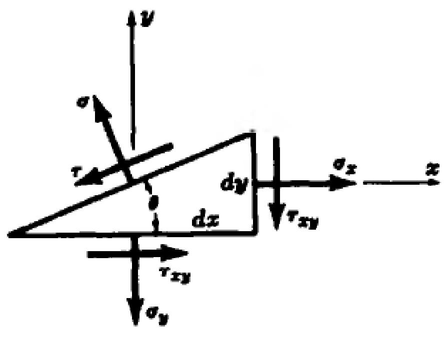



Fig. 16-2

### STRESSES ON AN INCLINED PLANE

We shall assume that the stresses $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ are known. (Their determination will be discussed in Chap. 17.) Frequently it is desirable to investigate the state of stress on a plane inclined at an angle

$\theta$ to the $x$-axis, as shown in Fig. 16-1. The normal and shearing stresses on such a plane are denoted by $\sigma$ and $\tau$ and appear as in Fig. 16-2. In Problem 16.13 it is shown that

$$\sigma = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \tag{16.1}$$

$$\tau = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \tag{16.2}$$

Thus, for any value of $\theta$, $\sigma$ and $\tau$ may be obtained from these expressions. For applications see Problems 16.15, 16.17, and 16.18.

## PRINCIPAL STRESSES

There are certain values of the angle $\theta$ that lead to maximum and minimum values of $\sigma$ for a given set of stresses $\sigma_x$, $\sigma_y$, and $\tau_{xy}$. These maximum and minimum values that $\sigma$ may assume are termed *principal stresses* and are given by

$$\sigma_{max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} \tag{16.3}$$

$$\sigma_{min} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} \tag{16.4}$$

These expressions are derived in Problem 16.13. For applications see Problems 16.15 and 16.18.

## DIRECTIONS OF PRINCIPAL STRESSES; PRINCIPAL PLANES

The angles designated as $\theta_p$ between the $x$-axis and the planes on which the principal stresses occur are given by the equation

$$\tan 2\theta_p = \frac{-\tau_{xy}}{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)} \tag{16.5}$$

This expression also is derived in Problem 16.13. For applications see Problems 16.15 and 16.18. As shown there, we always have two values of $\theta_p$ satisfying this equation. The stress $\sigma_{max}$ occurs on one of these planes, and the stress $\sigma_{min}$ occurs on the other. The planes defined by the angles $\theta_p$ are known as *principal planes*.

## COMPUTER IMPLEMENTATION

For this two-dimensional situation, a simple FORTRAN program may be written to indicate the values of the principal stresses indicated by Eqs. (*16.3*) and (*16.4*) as well as directions of these stresses as given by Eq. (*16.5*). Such a program is developed in Problem 16.20 and an application is found in Problem 16.21.

## SHEARING STRESSES ON PRINCIPAL PLANES

In Problem 16.13 it is demonstrated that the shearing stresses on the planes on which $\sigma_{max}$ and $\sigma_{min}$ occur are always zero, regardless of the values of $\sigma_x$, $\sigma_y$, and $\tau_{xy}$. Thus, an element oriented along the principal planes and subject to the principal stresses appears as in Fig. 16-3.

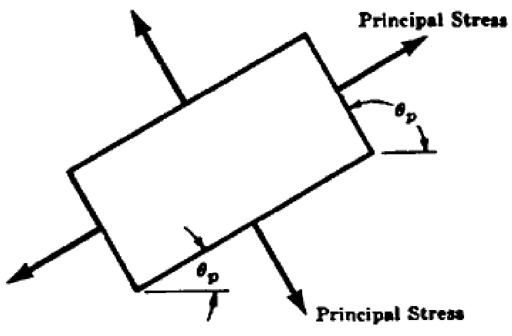



Fig. 16-3

## MAXIMUM SHEARING STRESSES

There are certain values of the angle $\theta$ that lead to a maximum value of $\tau$ for a given set of stresses $\sigma_x$, $\sigma_y$, and $\tau_{xy}$. The maximum and minimum values of the shearing stress are given by

$$\tau_{\substack{\max \\ \min}} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} \tag{16.6}$$

This expression is derived in Problem 16.13. For applications see Problems 16.3, 16.10, 16.18, and 16.19.

## DIRECTIONS OF MAXIMUM SHEARING STRESS

The angles $\theta_s$ between the $x$-axis and the planes on which the maximum shearing stresses occur are given by the equation

$$\tan 2\theta_s = \frac{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)}{\tau_{xy}} \tag{16.7}$$

This expression also is derived in Problem 16.13. For applications see Problems 16.3, 16.10, 16.18, and 16.19. There are always two values of $\theta_s$ satisfying this equation. The shearing stress corresponding to the positive square root given above occurs on one of the planes designated by $\theta_s$, while the shearing stress corresponding to the negative square root occurs on the other plane.

## NORMAL STRESSES ON PLANES OF MAXIMUM SHEARING STRESS

In Problem 16.13, it is demonstrated that the normal stress on each of the planes of maximum shearing stress (which are of course 90° apart) is given by

$$\tau' = \frac{\sigma_x + \sigma_y}{2}$$

Thus an element oriented along the planes of maximum shearing stress appears as in Fig. 16-4. This is illustrated in Problems 16.7, 16.9, and 16.15.

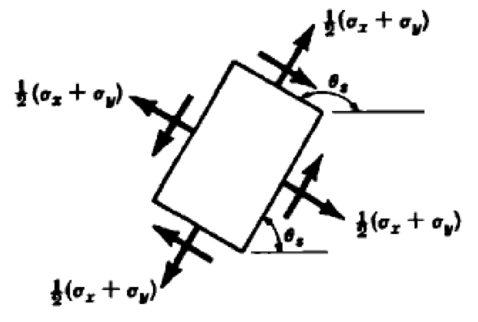



Fig. 16-4

## MOHR'S CIRCLE

All the information contained in the above equations may be presented in a convenient graphical form known as *Mohr's circle*. In this representation normal stresses are plotted along the horizontal axis and shearing stresses along the vertical axis. The stresses $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ are plotted to scale and a circle is drawn through these points having its center on the horizontal axis. Figure 16-5 shows Mohr's circle for an element subject to the general case of plane stress. For applications see Problems 16.4, 16.5, 16.8, 16.10, 16.12, 16.14, 16.16, 16.17, and 16.19.

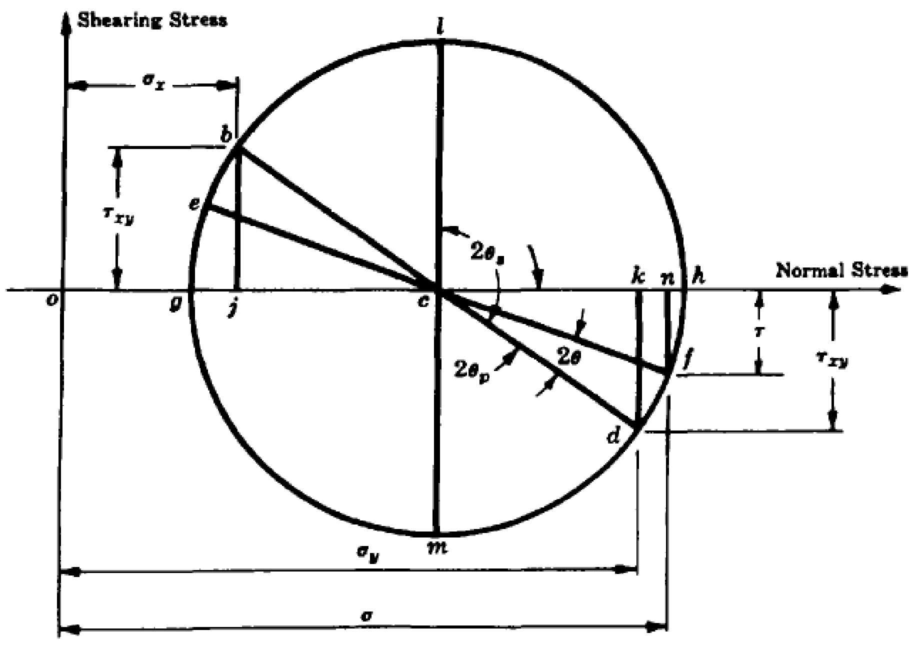



Fig. 16-5

## SIGN CONVENTIONS USED WITH MOHR'S CIRCLE

Tensile stresses are considered to be positive and compressive stresses negative. Thus tensile stresses are plotted to the right of the origin in Fig. 16-5 and compressive stresses to the left. With regard to shearing stresses it is to be carefully noted that a different sign convention exists than is used in connection with the above-mentioned equations. We shall refer to a plane element subject to shearing stresses and appearing as in Fig. 16-6. We shall say that shearing stresses are positive if they tend to rotate the element clockwise, negative if they tend to rotate it counterclockwise. Thus for the above element the shearing stresses on the vertical faces are positive, those on the horizontal faces are negative.

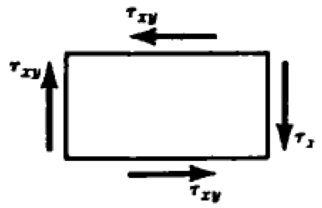



**Fig. 16-6**

## DETERMINATION OF PRINCIPAL STRESSES BY MEANS OF MOHR'S CIRCLE

When Mohr's circle has been drawn as in Fig. 16-5, the principal stresses are represented by the line segments *og* and *oh*. These may either be scaled from the diagram or determined from the geometry of the figure. This is explained in detail in Problem 16.14. For application see Problems 16.4, 16.5, 16.8, 16.10, 16.12, 16.14, 16.16, 16.17, and 16.19.

## DETERMINATION OF STRESSES ON AN ARBITRARY PLANE BY MEANS OF MOHR'S CIRCLE

To determine the normal and shearing stresses on a plane inclined at a counterclockwise angle $\theta$ with the $x$-axis, we measure a counterclockwise angle equal to $2\theta$ from the diameter *bd* of Mohr's circle shown in Fig. 16-5. The endpoints of this diameter *bd* represent the stress conditions in the original $x$-$y$ directions; i.e., they represent the stresses $\sigma_x$, $\sigma_y$, and $\tau_{xy}$. The angle $2\theta$ corresponds to the diameter *ef*. The coordinates of point *f* represent the normal and shearing stresses on the plane at an angle $\theta$ to the $x$-axis. That is, the normal stress $\sigma$ is represented by the abscissa *on* and the shearing stress is represented by the ordinate *nf*. This is discussed in detail in Problem 16.14. For applications see Problems 16.4, 16.5, 16.6, 16.8, 16.14, and 16.17.

# Solved Problems

**16.1.** Let us consider a straight bar of uniform cross section loaded in axial tension. Determine the normal and shearing stress intensities on a plane inclined at an angle $\theta$ to the axis of the bar. Also, determine the magnitude and direction of the maximum shearing stress in the bar.

This is the same elastic body that was considered in Chap. 1, but there the stresses studied were normal stresses in the direction of the axial force acting on the bar. In Fig. 16-7(*a*), *P* denotes the axial force acting on the bar, *A* the area of the cross section perpendicular to the axis of the bar, and from Chap. 1 the normal stress $\sigma_x$ is given by $\sigma_x = P/A$.

Suppose now that instead of using a cutting plane which is perpendicular to the axis of the bar, we pass a plane through the bar at an angle $\theta$ with the axis of the bar. Such a plane *mn* is shown in Fig. 16-7(*b*).

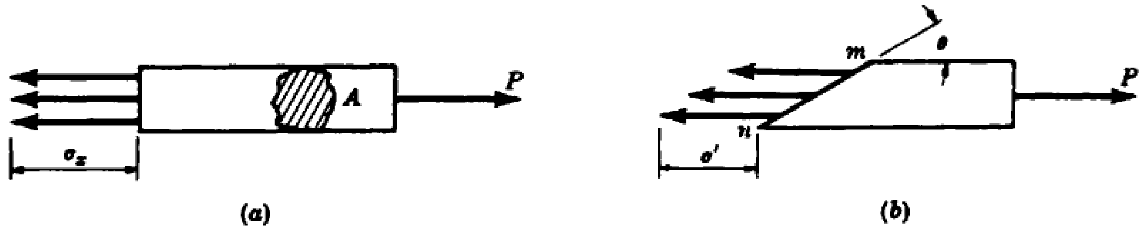



**Fig. 16-7**

Since we must still have equilibrium of the bar in the horizontal direction, there must evidently be distributed horizontal stresses acting over this inclined plane as shown. Let us designate the magnitude of these stresses by $\sigma'$. Evidently the area of the inclined cross section is $A/\sin\theta$ and for equilibrium of forces in the horizontal direction we have

$$\sigma'\left(\frac{A}{\sin\theta}\right) = P \qquad \text{or} \qquad \sigma' = \frac{P\sin\theta}{A}$$

In Fig. 16-8, we consider only a single stress vector $\sigma'$ and resolve it into two components, one normal to the inclined plane $mn$ and one tangential to this plane. We shall label the first of these components $\sigma$ to denote a normal stress, and the second $\tau$ to represent a shearing stress.

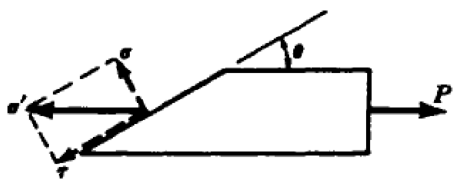



**Fig. 16-8**

Since the angle between $\sigma'$ and $\tau$ is $\theta$, we immediately have the relations

$$\tau = \sigma'\cos\theta \qquad \text{and} \qquad \sigma = \sigma'\sin\theta$$

But $\sigma' = (P\sin\theta)/A$. Substituting this value in the above equations, we obtain

$$\tau = \frac{P\sin\theta\cos\theta}{A} \qquad \text{and} \qquad \sigma = \frac{P\sin^2\theta}{A}$$

But $\sigma_x = P/A$. Hence we may write these in the form

$$\tau = \sigma_x\sin\theta\cos\theta \qquad \text{and} \qquad \sigma = \sigma_x\sin^2\theta$$

Now, employing the trigonometric identities

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \text{and} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

we may write

$$\tau = \tfrac{1}{2}\sigma_x\sin 2\theta \tag{1}$$

$$\sigma = \tfrac{1}{2}\sigma_x(1-\cos 2\theta) \tag{2}$$

These expressions give the normal and shearing stresses on a plane inclined at an angle $\theta$ to the axis of the bar.

**16.2.** A bar of cross section 850 mm$^2$ is acted upon by axial tensile forces of 60 kN applied at each end of the bar. Determine the normal and shearing stresses on a plane inclined at 30° to the direction of loading.

From Problem 16.1 the normal stress on a cross section perpendicular to the axis of the bar is

$$\sigma_x = \frac{P}{A} = \frac{60 \times 10^3}{850} = 70.6\ \text{MPa}$$

The normal stress on a plane at an angle $\theta$ with the direction of loading was found in Problem 16.1 to be $\sigma = \frac{1}{2}\sigma_x(1 - \cos 2\theta)$. For $\theta = 30°$ this becomes

$$\sigma = \tfrac{1}{2}(70.6)(1 - \cos 60°) = 17.65\ \text{MPa}$$

The shearing stress on a plane at an angle $\theta$ with the direction of loading was found in Problem 16.1 to be $\tau = \frac{1}{2}\sigma_x \sin 2\theta$. For $\theta = 30°$ this becomes

$$\tau = \tfrac{1}{2}(70.6)(\sin 60°) = 30.6\ \text{MPa}$$

These stresses together with the axial load of 60 kN are represented in Fig. 16-9.

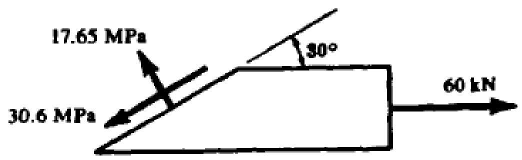



**Fig. 16-9**

**16.3.** Determine the maximum shearing stress in the axially loaded bar described in Problem 16.2.

The shearing stress on a plane at an angle $\theta$ with the direction of the load was shown in Problem 16.1 to be $\tau = \frac{1}{2}\sigma_x \sin 2\theta$. This is maximum when $2\theta = 90°$, that is, when $\theta = 45°$. For this loading we have $\sigma_x = 70.6$ MPa and when $\theta = 45°$ the shear stress is

$$\tau = \tfrac{1}{2}(70.6) \sin 90° = 35.3\ \text{MPa}$$

That is, the maximum shearing stress is equal to one-half of the maximum normal stress.

The normal stress on this 45° plane may be found from the expression

$$\sigma = \tfrac{1}{2}\sigma_x(1 - \cos 2\theta) = \tfrac{1}{2}(70.6)(1 - \cos 90°) = 35.3\ \text{MPa}$$

**16.4.** Discuss a graphical representation of Eqs. (*1*) and (*2*) of Problem 16.1.

According to these equations the normal and shearing stresses on a plane inclined at an angle $\theta$ to the direction of loading are given by

$$\sigma = \tfrac{1}{2}\sigma_x(1 - \cos 2\theta) \qquad \text{and} \qquad \tau = \tfrac{1}{2}\sigma_x \sin 2\theta$$

To represent these relations graphically it is customary to introduce a rectangular cartesian coordinate system, plotting normal stresses as abscissas and shearing stresses as ordinates.

Let us proceed by first laying off to some convenient scale the normal stress $\sigma_x$ (taken to be tensile) along the positive horizontal axis. The midpoint of this line segment, point $c$ in Fig. 16-10, serves as the center of a circle whose diameter is $\sigma_x$. The radius of this circle, denoted by $\overline{oc}$, $\overline{ch}$, and $\overline{cd}$, is $\frac{1}{2}\sigma_x$. The angle $2\theta$ is measured positive in a counterclockwise direction from the radial line $\overline{oc}$. From the figure we immediately have the relations

$$\overline{kd} = \tau = \tfrac{1}{2}\sigma_x \sin 2\theta \qquad \overline{ok} = \overline{oc} - \overline{kc} = \tfrac{1}{2}\sigma_x - \tfrac{1}{2}\sigma_x \cos 2\theta = \sigma = \tfrac{1}{2}\sigma_x(1 - \cos 2\theta)$$

It is to be noted that the scales used in the horizontal and vertical directions are equal.

Thus the abscissa and ordinate of point $d$ represent, respectively, the normal stress and the shearing stress acting on a plane at an angle $\theta$ with the axis of the bar subject to tension. In plotting this diagram tensile stresses are regarded as positive in algebraic sign and compressive stresses are taken to be negative.

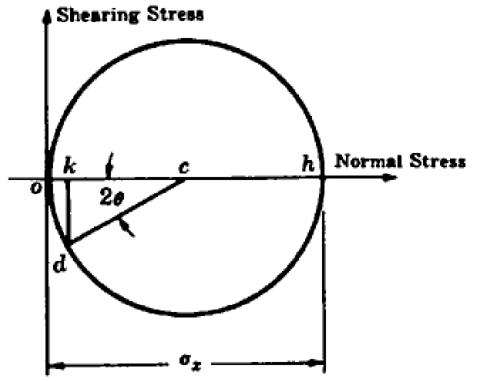



Fig. 16-10

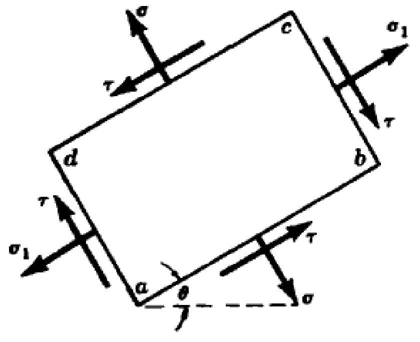



Fig. 16-11

Let us return to Problem 16.1 and examine a free-body diagram (Fig. 16-11) of an element taken from the surface of the inclined section on which the stresses $\sigma$ and $\tau$ act. We shall consider shearing stresses to be positive if they tend to rotate the element clockwise, negative if they tend to rotate the element counterclockwise. This sign convention is used only in this graphical representation, not in the analytical treatment of Problem 16.1. Since the shearing stresses found in Problem 16.1 were actually those acting on face *dc* of the above element, they should be regarded as negative. Hence in the circular diagram representing normal and shearing stresses in Fig. 16-10, the shearing stress on plane *dc* appears as an ordinate $\overline{kd}$ plotted in the negative sense.

This diagram, termed *Mohr's circle* as noted earlier, was first presented by O. Mohr in 1882. It represents the variation of normal and shearing stresses on all inclined planes passing through a given point in the body. It is a convenient graphical representation of Eqs. (*1*) and (*2*) of Problem 16.1.

**16.5.** Consider again the axially loaded bar discussed in Problem 16.2. Use Mohr's circle to determine the normal and shearing stresses on the 30° plane.

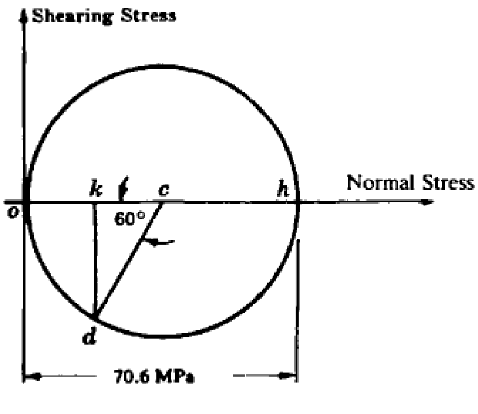



Fig. 16-12

In Fig. 16-12, the normal stress of 70.6 MPa is laid off along the horizontal axis to some convenient scale and a circle is drawn with this line as a diameter. The angle $2\theta = 2(30°) = 60°$ is measured counterclockwise from $\overline{oc}$. The coordinates of the point *d* are

$$\overline{kd} = \tau = -\tfrac{1}{2}(70.6)\sin 60° = -30.6\text{ MPa}$$

$$\overline{ok} = \sigma = \overline{oc} - \overline{kc} = \tfrac{1}{2}(70.6) - \tfrac{1}{2}(70.6)\cos 60° = 17.65\text{ MPa}$$

The negative sign accompanying the value of the shearing stress indicates that the shearing stress on this 30° plane tends to rotate an element bounded by this plane in a counterclockwise direction. This is in agreement with the direction of the shearing stress illustrated in Fig. 16-9.

**16.6.** A bar of cross section 1.3 in² is acted upon by axial compressive forces of 15,000 lb applied to each end of the bar. Using Mohr's circle, find the normal and shearing stresses on a plane inclined at 30° to the direction of loading. Neglect the possibility of buckling of the bar.

The normal stress on a cross-section perpendicular to the axis of the bar is

$$\sigma_x = \frac{P}{A} = \frac{-15{,}000}{1.3} = -11{,}500 \text{ lb/in}^2$$

We shall first lay off this compressive normal stess to some convenient scale along the negative end of the horizontal axis. The midpoint of the line segment, point $c$ in Fig. 16-13, serves as the center of a circle whose diameter is 11,500 lb/in² to the scale chosen.

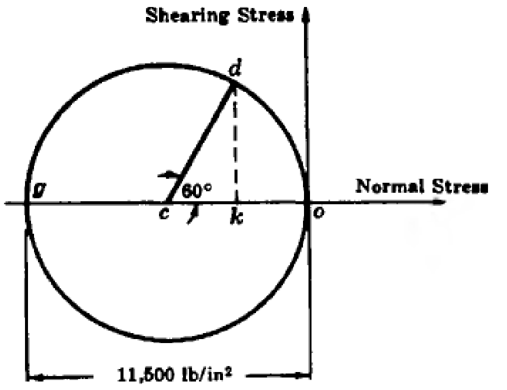



Fig. 16-13

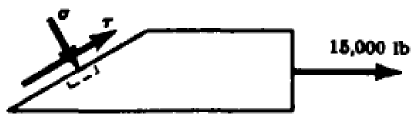



Fig. 16-14

The angle $2\theta = 2(30°) = 60°$ with the vertex at $c$ is measured counterclockwise from $\overline{co}$ as shown. The abscissa of point $d$ represents the normal stress and the ordinate the shearing stress on the desired 30° plane. The coordinates of point $d$ are

$$\overline{kd} = \tau = \tfrac{1}{2}(11{,}500)\sin 60° = 4940 \text{ lb/in}^2$$

$$\overline{ok} = \sigma = \overline{oc} - \overline{ck} = \tfrac{1}{2}(11{,}500) - \tfrac{1}{2}(11{,}500)\cos 60° = 2870 \text{ lb/in}^2$$

It is to be noted that line segment $\overline{ok}$ lies to the left of the origin of coordinates; hence this normal stress is compressive.

The positive algebraic sign accompanying the shearing stress indicates that the shearing stress on the 30° plane tends to rotate an element (denoted by dashed lines in Fig. 16-14) bounded by this plane in a clockwise direction. The directions of the normal and shearing stresses together with the axial load of 15,000 lb are shown in the figure.

**16.7.** Consider a plane element removed from a stressed elastic body and subject to the normal and shearing stresses $\sigma_x$ and $\tau_{xy}$, respectively, as shown in Fig. 16-15. (*a*) Determine the normal and shearing stress intensities on a plane inclined at an angle $\theta$ to the normal stress $\sigma_x$. (*b*) Determine the maximum and minimum values of the normal stress that may exist on inclined planes and find the directions of these stresses. (*c*) Determine the magnitude and direction of the maximum shearing stress that may exist on an inclined plane.

(*a*) The desired normal and shearing stresses acting on an inclined plane are internal quantities with respect to the element shown in Fig. 16-15. We shall follow the customary procedure of cutting this element with a plane in such a manner as to render the desired stresses external to the new body; that is, we will cut the originally rectangular element along the plane inclined at an angle $\theta$ with the $x$-axis and thus obtain a triangular element as shown in Fig. 16-16. The normal and shearing stresses, designated as $\sigma$ and $\tau$, respectively, represent the effect of the remaining portion of the

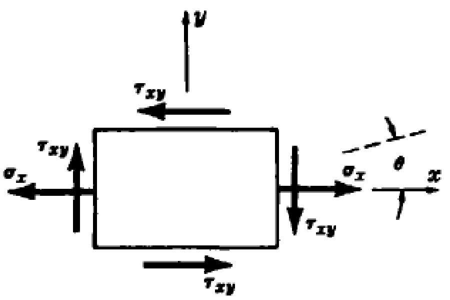



Fig. 16-15

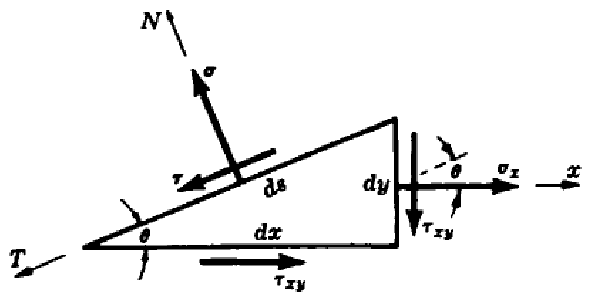



Fig. 16-16

originally rectangular block that has been removed. Consequently, the problem reduces to finding the unknown stresses $\sigma$ and $\tau$ in terms of the known stresses $\sigma_x$ and $\tau_{xy}$. It is to be observed that in the free-body diagram of the triangular element, the vectors indicate stresses acting on the various faces of the element and not forces. Each of these stresses is assumed to be uniformly distributed over the area upon which it acts. The thickness of the element perpendicular to the plane of the paper is denoted by $t$.

Let us introduce $N$- and $T$-axes normal and tangent to the inclined plane, as shown in Fig. 16-16. First, we shall sum forces in the $N$-direction. For equilibrium we have

$$\Sigma F_N = \sigma t\,ds - \sigma_x t\,dy \sin\theta - \tau_{xy} t\,dy \cos\theta - \tau_{xy} t\,dx \sin\theta = 0$$

But $dy = ds \sin\theta$, $dx = ds\cos\theta$. Substituting these relations in the equilibrium equation above, we find

$$\sigma(ds) = \sigma_x(ds)\sin^2\theta + 2\tau_{xy}(ds)\sin\theta\cos\theta$$

Next, employing the identities $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ and $\sin 2\theta = 2\sin\theta\cos\theta$, we obtain

$$\sigma = \tfrac{1}{2}\sigma_x(1-\cos 2\theta) + \tau_{xy}\sin 2\theta = \tfrac{1}{2}\sigma_x - \tfrac{1}{2}\sigma_x\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (1)$$

Thus the normal stress $\sigma$ on any plane inclined at an angle $\theta$ with the $x$-axis is known as a function of $\sigma_x$, $\tau_{xy}$, and $\theta$.

Next we shall consider the equilibrium of the forces acting on the triangular element in the $T$-direction. This leads to the equation

$$\Sigma F_T = \tau t\,ds - \sigma_x t\,dy\cos\theta + \tau_{xy} t\,dy \sin\theta - \tau_{xy} t\,dx\cos\theta = 0$$

Substituting $dy = ds\sin\theta$ and $dx = ds\cos\theta$, we obtain

$$\tau(ds) = +\sigma_x(ds)\sin\theta\cos\theta - \tau_{xy}(ds)\sin^2\theta + \tau_{xy}(ds)\cos^2\theta$$

Employing the identities $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$, this becomes

$$\tau = \tfrac{1}{2}\sigma_x\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (2)$$

Thus the shearing stress $\tau$ on any plane inclined at an angle $\theta$ with the $x$-axis is known as a function of $\sigma_x$, $\tau_{xy}$, and $\theta$.

(b) To determine the maximum value that the normal stress $\sigma$ may assume as the angle $\theta$ varies, we shall differentiate Eq. (1) with respect to $\theta$ and set this derivative equal to zero. Thus

$$\frac{d\sigma}{d\theta} = +\sigma_x\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

The values of $\theta$ leading to maximum and minimum values of the normal stress are consequently

$$\tan 2\theta_p = \frac{-\tau_{xy}}{\frac{1}{2}\sigma_x} \qquad (3)$$

The planes defined by the angles $\theta_p$ are called *principal planes*. The normal stresses that exist on these planes are designated as *principal stresses*. They are the maximum and minimum values that the normal stress may assume in the element under consideration. The values of the principal stresses

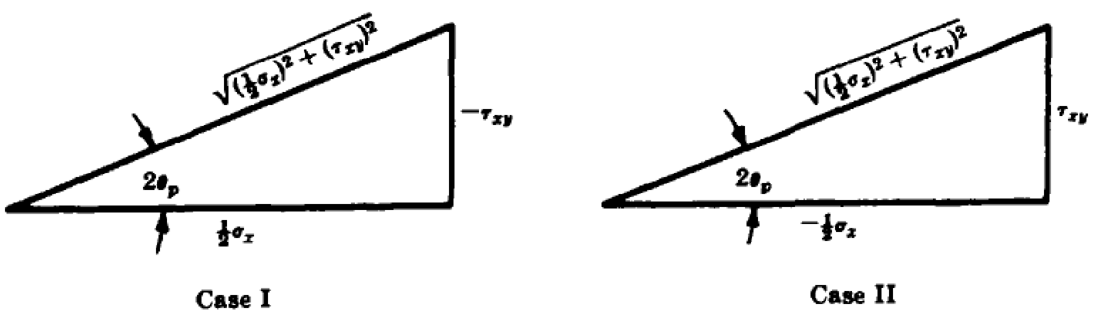



**Fig. 16-17**

may easily be found by interpreting Eq. (*3*) graphically, as in Fig. 16-17. Evidently the tangent of either of the angles designated as $2\theta_p$ has the value given in (*3*). Thus there are two solutions to (*3*), and consequently two values of $2\theta_p$ (differing by 180°) and also two values of $\theta_p$. These values of $\theta_p$ differ by 90°. It is to be noted that the triangles of Fig. 16-17 bear no direct relationship to the triangular element whose free-body diagram was considered earlier.

The values of $\sin 2\theta_p$ and $\cos 2\theta_p$ as found from Fig. 16-17 may now be substituted in (*1*) to yield the maximum and minimum values of the normal stresses. Observing that

$$\sin 2\theta_p = \frac{\mp\tau_{xy}}{\sqrt{(\frac{1}{2}\sigma_x)^2 + (\tau_{xy})^2}} \qquad \cos 2\theta_p = \frac{\pm\frac{1}{2}\sigma_x}{\sqrt{(\frac{1}{2}\sigma_x)^2 + (\tau_{xy})^2}}$$

where the upper signs pertain to Case I and the lower signs to Case II, we obtain from (*1*)

$$\sigma = \tfrac{1}{2}\sigma_x \mp \tfrac{1}{2}\sigma_x \frac{\frac{1}{2}\sigma_x}{\sqrt{(\frac{1}{2}\sigma_x)^2 + (\tau_{xy})^2}} \mp \frac{(\tau_{xy})^2}{\sqrt{(\frac{1}{2}\sigma_x)^2 + (\tau_{xy})^2}} = \tfrac{1}{2}\sigma_x \pm \sqrt{(\tfrac{1}{2}\sigma_x)^2 + (\tau_{xy})^2} \tag{4}$$

The maximum normal stress is

$$\sigma_{\max} = \tfrac{1}{2}\sigma_x + \sqrt{(\tfrac{1}{2}\sigma_x)^2 + (\tau_{xy})^2} \tag{5}$$

The minimum normal stress is

$$\sigma_{\min} = \tfrac{1}{2}\sigma - \sqrt{(\tfrac{1}{2}\sigma_x)^2 + (\tau_{xy})^2} \tag{6}$$

The stresses given by (*5*) and (*6*) are the principal stresses and they occur on the principal planes defined by (*3*). By substituting one of the values of $\theta_p$ from (*3*) into (*1*), one may readily determine which of the two principal stresses is acting on that plane. The other principal stress naturally acts on the other principal plane.

By substituting the values of the angles $2\theta_p$ as given by (*3*) and Fig. 16-17 into (*2*), it is readily seen that the shearing stresses $\tau$ on the principal planes are zero.

(*c*) To determine the maximum value the shearing stress $\tau$ may assume as the angle $\theta$ varies, we shall differentiate Eq. (*2*) with respect to $\theta$ and set this derivative equal to zero. Thus

$$\frac{d\tau}{d\theta} = \sigma_x \cos 2\theta - 2\tau_{xy} \sin 2\theta = 0$$

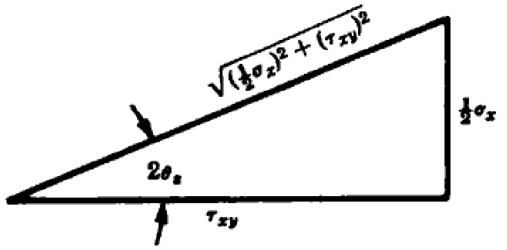
The values of $\theta$ leading to maximum values of the shearing stress are consequently

$$\tan 2\theta_s = \frac{\frac{1}{2}\sigma_x}{\tau_{xy}} \tag{7}$$

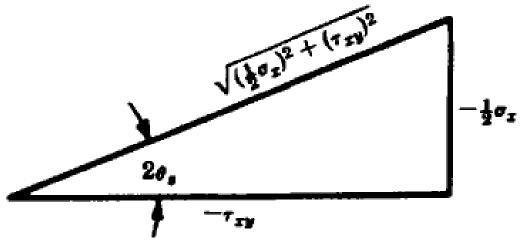
The planes defined by the two solutions to this equation are the planes of maximum shearing stress.

Again, a graphical interpretation of (*7*) is convenient. The two values of the angle $2\theta_s$ satisfying this equation may be represented as in Fig. 16-18. We see that

$$\sin 2\theta_s = \frac{\pm\frac{1}{2}\sigma_x}{\sqrt{(\frac{1}{2}\sigma_x)^2 + (\tau_{xy})^2}} \qquad \cos 2\theta_s = \frac{\pm\tau_{xy}}{\sqrt{(\frac{1}{2}\sigma_x)^2 + (\tau_{xy})^2}}$$



Case I



Case II

Fig. 16-18

where the upper (positive) signs pertain to Case I and the lower (negative) signs apply to Case II. Substituting these values in (2), we obtain

$$\tau_{\substack{max\\min}} = \frac{1}{2}\sigma_x \frac{\pm\frac{1}{2}\sigma_x}{\sqrt{(\frac{1}{2}\sigma_x)^2+(\tau_{xy})^2}} + (\tau_{xy})\frac{\pm\tau_{xy}}{\sqrt{(\frac{1}{2}\sigma_x)^2+(\tau_{xy})^2}} = \pm\sqrt{(\frac{1}{2}\sigma_x)^2+(\tau_{xy})^2} \qquad (8)$$

Here the positive sign represents the maximum shearing stress, the negative sign the minimum shearing stress.

If we compare (3) and (7), it is evident that the angles $2\theta_p$ and $2\theta_s$ differ by 90°, since the tangents of these angles are the negative reciprocals of one another. Hence the planes defined by the angles $\theta_p$ and $\theta_s$ differ from one another by 45°; that is, the planes of maximum shearing stress are oriented 45° from the planes of maximum normal stress.

It is also of interest to determine the normal stresses on the planes of maximum shearing stress. These planes are defined by (7). If we now substitute these values of $\sin 2\theta_s$ and $\cos 2\theta_s$ in (1) for the normal stress, we find

$$\sigma = \frac{1}{2}\sigma_x - \frac{1}{2}\sigma_x\frac{\pm\tau_{xy}}{\sqrt{(\frac{1}{2}\sigma_x)^2+(\tau_{xy})^2}} + (\tau_{xy})\frac{\pm\frac{1}{2}\sigma_x}{\sqrt{(\frac{1}{2}\sigma_x)^2+(\tau_{xy})^2}} = \frac{1}{2}\sigma_x \qquad (9)$$

Thus on each plane of maximum shearing stress we have a normal stress of magnitude $\frac{1}{2}\sigma_x$.

**16.8.** Discuss a graphical representation of the analysis presented in Problem 16.7.

For given values of $\sigma_x$ and $\tau_{xy}$ proceed as follows:

1. Introduce a rectangular coordinate system in which normal stresses are represented along the horizontal axis and shearing stresses along the vertical axis. The scales used on these two axes must be equal.
2. With reference to the original rectangular element considered in Problem 16.7 and reproduced in Fig. 16-19, we shall introduce the sign convention that shearing stresses are positive if they tend to rotate the element clockwise, negative if they tend to rotate it counterclockwise. Here the shearing stresses on the vertical faces are positive, those on the horizontal faces are negative. Also, tensile stresses are considered to be positive and compressive stresses negative.
3. We first locate point $b$ by laying out $\sigma_x$ and $\tau_{xy}$ to their given values. The shear stress $\tau_{xy}$ on the vertical faces on which $\sigma_x$ acts is positive; hence this value is plotted as positive in Fig. 16-20. This is drawn on

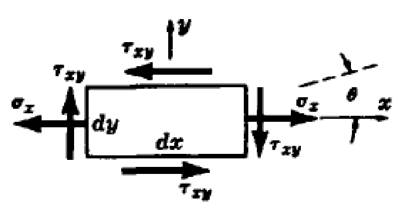



Fig. 16-19

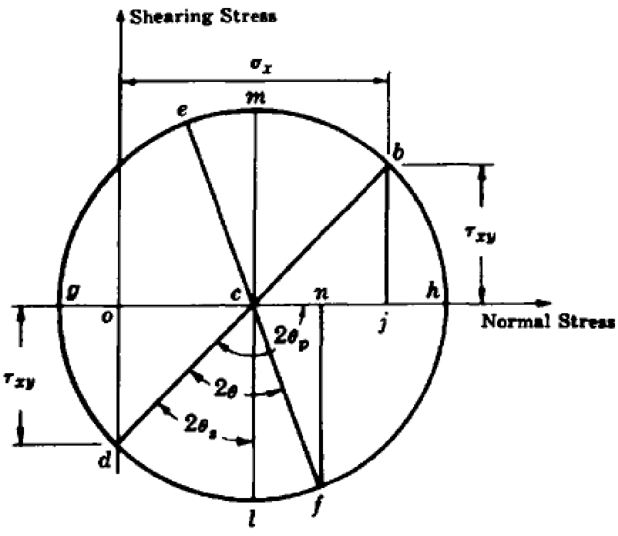



Fig. 16-20

the assumption that $\sigma_x$ is a tensile stress, although the treatment presented here is valid if $\sigma_x$ is compressive.

4. We next locate point $d$ in a similar manner by laying off $\tau_{xy}$ on the negative side of the vertical axis. Actually, this point $d$ corresponds to the negative shearing stresses $\tau_{xy}$ existing on the horizontal faces of the element together with a zero normal stress acting on those same faces.
5. Next, we draw line $\overline{bd}$, locate the midpoint $c$, and draw a circle having its center at $c$ and radius equal to $\overline{cb}$. This is known as Mohr's circle.

We shall first show that the points $g$ and $h$ along the horizontal diameter of the circle represent the principal stresses. To do this we note that the point $c$ lies at a distance $\frac{1}{2}\sigma_x$ from the origin of the coordinate system. From the right-triangle relationship we have

$$(\overline{cd})^2 = (\overline{oc})^2 + (\overline{od})^2 \qquad \text{or} \qquad \overline{cd} = \sqrt{(\tfrac{1}{2}\sigma_x)^2 + (\tau_{xy})^2}$$

Also, we have $\overline{cd} = \overline{ch} = \overline{cg}$. Hence, the $x$-coordinate of point $h$ is $\overline{oc} + \overline{ch}$ or

$$\tfrac{1}{2}\sigma_x + \sqrt{(\tfrac{1}{2}\sigma_x)^2 + (\tau_{xy})^2}$$

But this expression is exactly the maximum principal stress, as found in (5) of Problem 16.7. Likewise the $x$-coordinate of point $g$ is $\overline{oc}$-$\overline{cg}$. But this quantity is negative; hence $\overline{og}$ lies to the left of the origin, and point $g$ symbolizes a compressive stress. This stress becomes

$$\tfrac{1}{2}\sigma_x - \sqrt{(\tfrac{1}{2}\sigma_x)^2 + (\tau_{xy})^2}$$

But this expression is exactly the minimum principal stress, as found in (6) of Problem 16.7. Consequently the points $g$ and $h$ represent the principal stresses existing in the original element. We see that the tangent of $\angle ocd$ is $\tau_{xy}/(\frac{1}{2}\sigma_x)$. But from (3) of Problem 16.7, $\tan 2\theta_p = -\tau_{xy}/\frac{1}{2}\sigma_x$; and by comparison of these two relations we see that $\angle hcd = 2\theta_p$, since $\tan(180° - \theta) = -\tan\theta$. Thus a counterclockwise rotation from the diameter $\overline{bd}$ (corresponding to the stresses in the $x$- and $y$-directions) leads us to the diameter $\overline{gh}$, representing the principal planes, on which the principal stresses occur. The principal planes lie at an angle $\theta_p$ from the $x$-direction.

Thus Mohr's circle is a convenient device for finding the principal stresses, since one can merely establish the circle for a given set of stresses $\sigma_x$ and $\tau_{xy}$ then measure $\overline{og}$ and $\overline{oh}$. These abscissas represent the principal stresses to the same scale used in plotting $\sigma_x$ and $\tau_{xy}$.

It is now apparent that the radius of Mohr's circle, represented by $\overline{cd} = \sqrt{(\frac{1}{2}\sigma_x)^2 + (\tau_{xy})^2}$, corresponds to the maximum shearing stress, as found in (8) of Problem 16.7. Actually, the shearing stress on any plane is represented by the ordinate to Mohr's circle; hence we should consider the radial lines $\overline{cl}$ and $\overline{cm}$ as representing the maximum shearing stresses. The angle $dcl$ is evidently $2\theta_s$ and hence it is apparent that the double angle between the planes of maximum normal stress and the planes of maximum shearing stress ($\angle lch$) is 90°; thus the planes of maximum shearing stress are oriented 45° from the planes of maximum normal stress.

Evidently the endpoints of the diameter $\overline{bd}$ represent the stresses acting in the original $x$- and $y$-directions. We shall now demonstrate that the endpoints of any other diameter, such as $\overline{ef}$ (at any angle $2\theta$ with $\overline{bd}$), represent the stresses on a plane inclined at an angle $\theta$ to the $x$-axis. To do this we note that the abscissa of point $f$ is given by

$$\begin{aligned}\sigma = \overline{oc} + \overline{cn} &= \tfrac{1}{2}\sigma_x + \overline{cf}\cos(2\theta_p - 2\theta)\\ &= \tfrac{1}{2}\sigma_x + \overline{cf}(\cos 2\theta_p \cos 2\theta + \sin 2\theta_p \sin 2\theta)\\ &= \tfrac{1}{2}\sigma_x + \sqrt{(\tfrac{1}{2}\sigma_x)^2 + (\tau_{xy})^2}\,(\cos 2\theta_p \cos 2\theta + \sin 2\theta_p \sin 2\theta)\end{aligned}$$

But from inspection of triangle $cod$ appearing in Mohr's circle it is evident that

$$\sin 2\theta_p = \frac{\tau_{xy}}{\sqrt{(\tfrac{1}{2}\sigma_x)^2 + (\tau_{xy})^2}} \qquad \text{and} \qquad \cos 2\theta_p = \frac{-\tfrac{1}{2}\sigma_x}{\sqrt{(\tfrac{1}{2}\sigma_x)^2 + (\tau_{xy})^2}} \tag{1}$$

Substituting the values of $\tau_{xy}$ and $\frac{1}{2}\sigma_x$ from these two equations into the previous equation, we find

$$\sigma = \tfrac{1}{2}\sigma_x - \tfrac{1}{2}\sigma_x \cos 2\theta + \tau_{xy}\sin 2\theta$$

But this is exactly the normal stress on a plane inclined at an angle $\theta$ to the $x$-axis as derived in (*1*) of Problem 16.7.

Next we observe that the ordinate of point $f$ is given by

$$\begin{aligned}\tau = \overline{nf} &= \overline{cf}\sin(2\theta_p - 2\theta)\\ &= \sqrt{(\tfrac{1}{2}\sigma_x)^2 + (\tau_{xy})^2}\,(\sin 2\theta_p \cos 2\theta - \cos 2\theta_p \sin 2\theta)\end{aligned}$$

Again, substituting the values of $\tau_{xy}$ and $\frac{1}{2}\sigma_x$ from Eqs. (*1*) into this equation, we find

$$\tau = \tfrac{1}{2}\sigma_x \sin 2\theta + \tau_{xy}\cos 2\theta$$

But this is exactly the shearing stress on a plane inclined at an angle $\theta$ to the $x$-axis as derived in (*2*) of Problem 16.7.

Hence the coordinates of point $f$ on Mohr's circle represent the normal and shearing stresses on a plane inclined at an angle $\theta$ to the $x$-axis.

**16.9.** A plane element in a body is subjected to a normal stress in the $x$-direction of 12,000 lb/in$^2$, as well as a shearing stress of 4000 lb/in$^2$, as shown in Fig. 16-21. (*a*) Determine the normal and shearing stress intensities on a plane inclined at an angle of 30° to the normal stress. (*b*) Determine the maximum and minimum values of the normal stress that may exist on inclined planes and the directions of these stresses. (*c*) Determine the magnitude and direction of the maximum shearing stress that may exist on an inclined plane.

(*a*) In accordance with the notation of Problem 16.7, we have $\sigma_x$ = 12,000 lb/in$^2$ and $\tau_{xy}$ = 4000 lb/in$^2$. From (*1*) of Problem 16.7, the normal stress on a plane inclined at an angle $\theta$ to the $x$-axis is

$$\sigma = \tfrac{1}{2}\sigma_x - \tfrac{1}{2}\sigma_x \cos 2\theta + \tau_{xy}\sin 2\theta$$

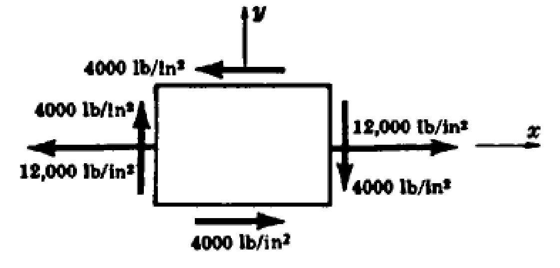



Fig. 16-21

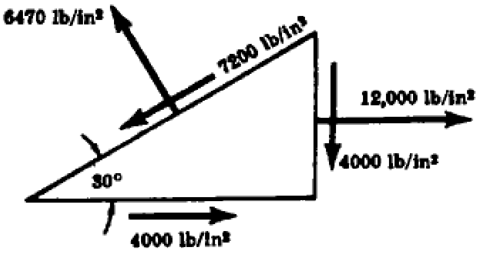



Fig. 16-22

Substituting the above values of $\sigma_x$ and $\tau_{xy}$, when $\theta = 30°$ this becomes

$$\sigma = \tfrac{1}{2}(12{,}000) - \tfrac{1}{2}(12{,}000)\cos 60° + 4000 \sin 60° = 6470\ \text{lb/in}^2$$

From (*2*) of Problem 16.7, the shearing stress on any plane inclined at an angle $\theta$ to the $x$-axis is

$$\tau = \tfrac{1}{2}\sigma_x \sin 2\theta + \tau_{xy} \cos 2\theta$$

Substituting the above values of $\sigma_x$ and $\tau_{xy}$, when $\theta = 30°$ this becomes

$$\tau = \tfrac{1}{2}(12{,}000)\sin 60° + 4000 \cos 60° = 5200 + 2000 = 7200\ \text{lb/in}^2$$

The positive directions of the normal and shearing stresses on an inclined plane were illustrated in Fig. 16-16. In accordance with this sign convention the stresses on the 30° plane appear as in Fig. 16-22.

(*b*) The values of the principal stresses, that is, the maximum and minimum values of the normal stresses existing in this element, were given by (*5*) and (*6*) of Problem 16.7. From (*5*) for the maximum normal stress, we have

$$\sigma_{max} = \tfrac{1}{2}\sigma_x + \sqrt{(\tfrac{1}{2}\sigma_x)^2 + \tau_{xy})^2} = 6000 + \sqrt{(6000)^2 + (4000)^2} = 13{,}220\ \text{lb/in}^2$$

From (*6*) for the minimum normal stress, we have

$$\sigma_{min} = \tfrac{1}{2}\sigma_x - \sqrt{(\tfrac{1}{2}\sigma_x)^2 + (\tau_{xy})^2} = 6000 - \sqrt{(6000)^2 + (4000)^2} = -1220\ \text{lb/in}^2$$

The directions of the planes on which these principal stresses occur were found in (*3*) of Problem 16.7 to be

$$\tan 2\theta_p = -\frac{\tau_{xy}}{\tfrac{1}{2}\sigma_x} = -\frac{4000}{6000} = -\frac{2}{3}$$

Since the tangent of the angle $2\theta_p$ is negative, the two values of $2\theta_p$ lie in the second and fourth quadrants. In the second quadrant, $2\theta_p = 146°20'$; in the fourth quadrant, $2\theta_p' = 326°20'$. Consequently we have the principal planes defined by $\theta_p = 73°10'$ and $\theta_p' = 163°10'$. If $\theta_p = 73°10'$, together with the given values of $\sigma_x$ and $\tau_{xy}$, is now substituted in (*1*) of Problem 16.7, we find

$$\sigma = \tfrac{1}{2}\sigma_x - \tfrac{1}{2}\sigma_x \cos 2\theta + \tau_{xy} \sin 2\theta = 6000 - 6000 \cos 146°20' + 4000 \sin 146°20'$$

$$= 6000 - 6000(-0.833) + 4000(0.554) = 13{,}220\ \text{lb/in}^2$$

Thus the principal stress of 13,220 lb/in² occurs on the principal plane oriented at 73°10′ to the $x$-axis. The principal stresses thus appear as in Fig. 16-23. As stated in Problem 16.7, the shearing stresses on these principal planes are zero.

(*c*) The values of the maximum and minimum shearing stresses were found in (*8*) of Problem 16.7 to be

$$\tau_{\substack{max \\ min}} = \pm\sqrt{(\tfrac{1}{2}\sigma_x)^2 + (\tau_{xy})^2} = \pm\sqrt{(6000)^2 + (4000)^2} = \pm 7220\ \text{lb/in}^2$$

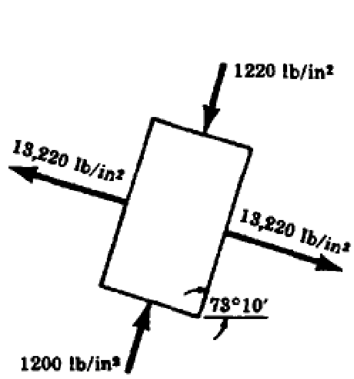



Fig. 16-23

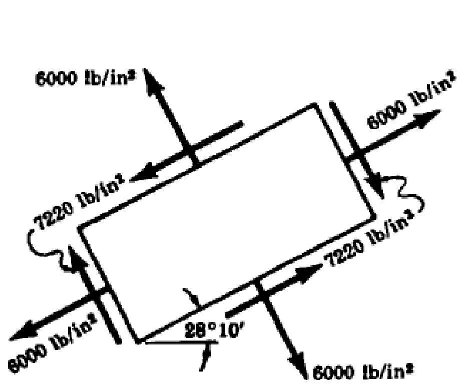



Fig. 16-24

The directions of the planes on which these maximum shearing stresses occur were found in (7) of Problem 16.7 to be given by

$$\tan 2\theta_s = \frac{\frac{1}{2}\sigma_x}{\tau_{xy}} = \frac{6000}{4000} = \frac{3}{2}$$

The angles $2\theta_s$ are consequently in the first and third quadrants, since the tangent is positive. Thus we have $2\theta_s = 56°20'$ and $2\theta_s' = 236°20'$, or $\theta_s = 28°10'$ and $\theta_s' = 118°10'$. The shearing stress on any plane inclined at an angle $\theta$ with the $x$-axis was found in (2) of Problem 16.7 to be

$$\tau = \tfrac{1}{2}\sigma_x \sin 2\theta + \tau_{xy}\cos 2\theta$$

Substituting $\sigma_x = 12{,}000\ \text{lb/in}^2$, $\tau_{xy} = 4000\ \text{lb/in}^2$, and $\theta = 28°10'$, we find

$$\tau = \tfrac{1}{2}(12{,}000)\sin 56°20' + 4000\cos 56°20' = +7220\ \text{lb/in}^2$$

Thus the shearing stress on the 28°10′ plane is positive. The positive sense of shearing stress was shown in Fig. 16-6.

The normal stresses on the planes of maximum shearing stress are found from (9) of Problem 16.7 to be

$$\sigma = \tfrac{1}{2}\sigma_x = \tfrac{1}{2}(12{,}000) = 6000\ \text{lb/in}^2$$

This normal stress acts on each of the planes of maximum shearing stress, as shown in Fig. 16-24.

**16.10.** A plane element is subject to the stresses shown in Fig. 16-25. Using Mohr's circle, determine (*a*) the principal stresses and their directions and (*b*) the maximum shearing stresses and the directions of the planes on which they occur.

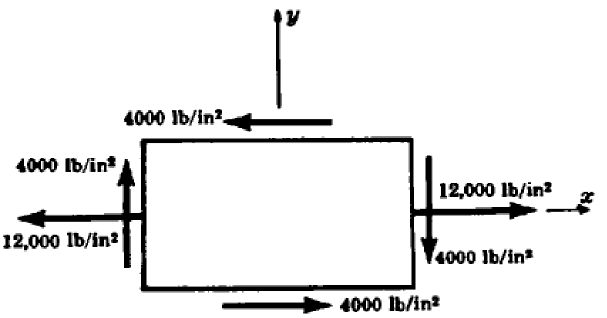



Fig. 16-25

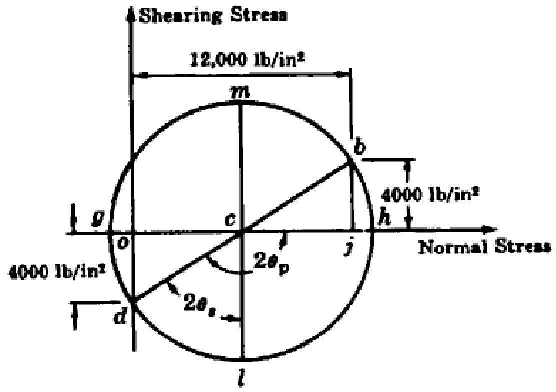



Fig. 16-26

Following the procedure for the construction of Mohr's circle outlined in Problem 16.8, we realize that the shearing stress on the vertical faces of the given element are positive, whereas those on the horizontal faces are negative. Thus the stress condition of $\sigma_x = 12{,}000\ \text{lb/in}^2$, $\tau_{xy} = 4000\ \text{lb/in}^2$ existing on the vertical faces of the element plots as point $b$ in Fig. 16-26. The stress condition of $\tau_{xy} = -4000\ \text{lb/in}^2$ together with a zero normal stress on the horizontal faces plots as point $d$. Line $\overline{bd}$ is drawn, its midpoint $c$ is located, and a circle of radius $\overline{cb} = \overline{cd}$ is drawn with $c$ as a center. This is Mohr's circle. The endpoints of the diameter $\overline{bd}$ represent the stress conditions existing in the element if it has the original orientation shown above.

(*a*) The principal stresses are represented by points $g$ and $h$, as shown in Problem 16.8. The principal stresses may be determined either by direct measurement from Fig. 16-26 or by realizing that the coordinate of $c$ is 6000, and that $\overline{cd} = \sqrt{(6000)^2 + (4000)^2} = 7220$. Therefore the minimum principal stress is

$$\sigma_{min} = \overline{og} = \overline{oc} - \overline{cg} = 6000 - 7200 = -1220\ \text{lb/in}^2$$

Also, the maximum principal stress is

$$\sigma_{max} = \overline{oh} = \overline{oc} + \overline{ch} = 6000 + 7220 = 13{,}220 \text{ lb/in}^2$$

The angle $2\theta_p$ designated above is given by

$$\tan 2\theta_p = -\frac{4000}{6000} = -\frac{2}{3} \qquad \text{or} \qquad \theta_p = 73°10'$$

This value could also be obtained by measurement of $\angle dch$ in Mohr's circle. From this it is readily seen that the principal stress represented by point $h$ acts on a plane oriented 73°10′ from the original $x$-axis. The principal stresses thus appear as in Fig. 16-27($a$). It is evident from Mohr's circle that the shearing stresses on these planes are zero, since points $g$ and $h$ lie on the horizontal axis of Mohr's circle.

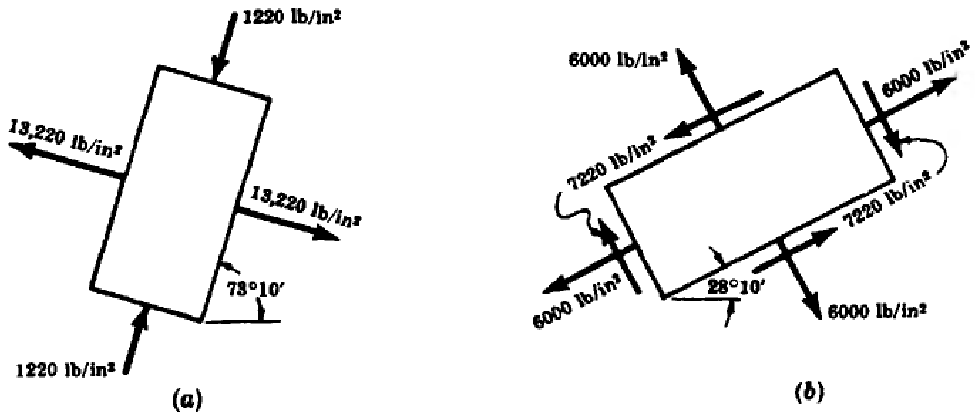



Fig. 16-27

($b$) The maximum shearing stress is represented by $\overline{cl}$ in Mohr's circle. This radius has already been found to be equal to 7220 lb/in². The angle $2\theta_s$ may be found either by direct measurement from Fig. 16-26 or simply by subtracting 90° from the angle $2\theta_p$, which has already been determined. This leads to $2\theta_s = 56°20'$ and $\theta_s = 28°10'$. The shearing stress represented by point $l$ is negative; hence on this 28°10′ plane the shearing stress tends to rotate the element in a counterclockwise direction. Also, from Mohr's circle the abscissa of point $l$ is 6000 lb/in² and this represents the normal stress occurring on the planes of maximum shearing stress. The maximum shearing stresses thus appear as in Fig. 16-27($b$).

**16.11.** A plane element in a body is subject to a normal compressive stress in the $x$-direction of 12,000 lb/in² as well as a shearing stress of 4000 lb/in², as shown in Fig. 16-28. ($a$) Determine the normal and shearing stress intensities on a plane inclined at an angle of 30° to the normal stress. ($b$) Determine the maximum and minimum values of the normal stress that may exist on

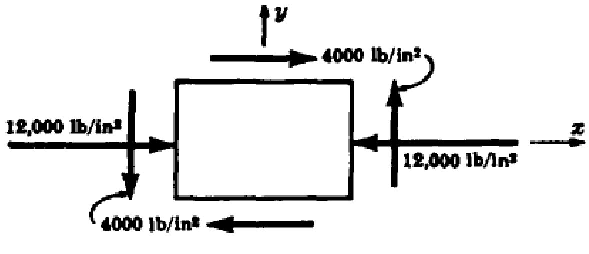



Fig. 16-28

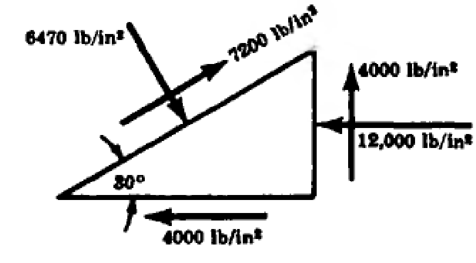



Fig. 16-29

inclined planes and the direction of these stresses. (*c*) Find the magnitude and direction of the maximum shearing stress that may exist on an inclined plane.

(*a*) By the sign convention for normal and shearing stresses adopted in Problem 16.7, we have here $\sigma_x = -12{,}000\ \text{lb/in}^2$, $\tau_{xy} = -4000\ \text{lb/in}^2$. From (*1*) of Problem 16.7, the normal stress on the 30° plane is

$$\sigma = -12{,}000/2 - (-12{,}000/2)\cos 60^\circ - 4000\sin 60^\circ = -6470\ \text{lb/in}^2$$

From (*2*) of Problem 16.7, the shearing stress on the 30° plane is

$$\tau = \tfrac{1}{2}(-12{,}000)\sin 60^\circ - 4000\cos 60^\circ = -7200\ \text{lb/in}^2$$

The positive directions of the normal and shearing stresses on an inclined plane were illustrated in Fig. 16-16. By this sign convention the stresses on the 30° plane appear as in Fig. 16-29.

(*b*) The values of the principal stresses were given by (*5*) and (*6*) of Problem 16.7. From (*5*),

$$\sigma_{max} = -12{,}000/2 + \sqrt{(-12{,}000/2)^2 + (-4000)^2} = 1220\ \text{lb/in}^2$$

From (*6*),

$$\sigma_{min} = -12{,}000/2 - \sqrt{(-12{,}000/2)^2 + (-4000)^2} = -13{,}220\ \text{lb/in}^2$$

The tensile principal stress is usually referred to as the maximum, even though its absolute value is smaller than that of the compressive stress.

The directions of the planes on which these principal stresses occur are given by (*3*) of Problem 16.7 to be

$$\tan 2\theta_p = -\frac{\tau_{xy}}{\frac{1}{2}\sigma_x} = -\frac{-4000}{-12{,}000/2} = -2/3$$

The angles defined by $2\theta_p$ lie in the second and fourth quadrants since the tangent is negative. Hence $2\theta_p = 146°20'$ and $2\theta'_p = 326°20'$. Thus the principal planes are defined by $\theta_p = 73°10'$ and $\theta'_p = 163°10'$. If $\theta_p = 73°10'$, together with the given values of $\sigma_x$ and $\tau_{xy}$, is now substituted in (*1*) of Problem 16.7, we find

$$\begin{aligned}\sigma &= \tfrac{1}{2}\sigma_x - \tfrac{1}{2}\sigma_x\cos 2\theta + \tau_{xy}\sin 2\theta \\ &= -12{,}000/2 - (-12{,}000/2)\cos 146°20' - 4000\sin 146°20' = -13{,}220\ \text{lb/in}^2\end{aligned}$$

Thus the principal stress of $-13{,}220\ \text{lb/in}^2$ occurs on the principal plane oriented at 73°10′ to the *x*-axis. The principal stresses are shown in Fig. 16-30. The shearing stresses on these principal planes are zero.

(*c*) The value of the maximum shearing stress is found from (*8*) of Problem 16.7 to be

$$\tau_{\substack{max\\min}} = \pm\sqrt{(\tfrac{1}{2}\sigma_x)^2 + (\tau_{xy})^2} = \pm\sqrt{(-12{,}000/2)^2 + (-4000)^2} = \pm 7220\ \text{lb/in}^2$$

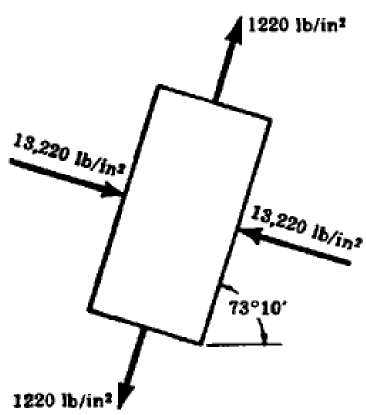



Fig. 16-30

The directions of the planes on which these shearing stresses occur was found in (7) of Problem 16.7 to be

$$\tan 2\theta_s = \frac{\frac{1}{2}\sigma_x}{\tau_{xy}} = \frac{-12{,}000/2}{-4000} = \frac{3}{2}$$

Thus $2\theta_s = 56°20'$ and $2\theta_s' = 236°20'$; or $\theta_s = 28°10'$ and $\theta_s' = 118°10'$. From (2) of Problem 16.7, the shearing stress on any plane inclined at an angle $\theta$ with the $x$-axis is

$$\tau = \tfrac{1}{2}\sigma_x \sin 2\theta + \tau_{xy}\cos 2\theta = \tfrac{1}{2}(-12{,}000)\sin 56°20' - 4000\cos 56°20' = -7220 \text{ lb/in}^2$$

Thus the shearing stress on the 28°10′ plane is negative. The positive sense of shearing stress was shown in Fig. 16-16.

The normal stresses on the planes of maximum shearing stress were found in (9) of Problem 16.7 to be

$$\sigma = \tfrac{1}{2}\sigma_x = -12{,}000/2 = -6000 \text{ lb/in}^2$$

This normal stress acts on each of the planes of maximum shearing stress, as shown in Fig. 16-31.

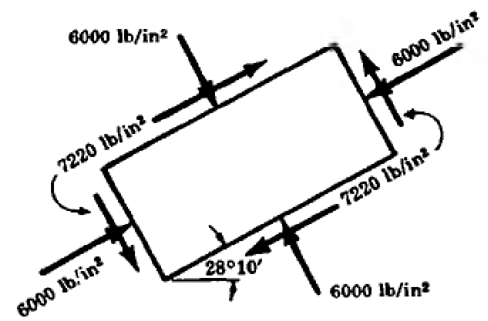



Fig. 16-31

**16.12.** A plane element is subject to the stresses shown in Fig. 16-32. Using Mohr's circle, determine (*a*) the principal stresses and their directions and (*b*) the maximum shearing stresses and the directions of the planes on which they occur.

The procedure for the construction of Mohr's circle was outlined in Problem 16.8. Following the instructions there, the shearing stresses on the vertical faces of the above element are negative, those on the horizontal faces are positive. Thus the stress condition of $\sigma_x = -12{,}000$ lb/in², $\tau_{xy} = -4000$ lb/in² existing on the vertical faces of the element plots as point $b$ in Fig. 16-33. The stress condition of $\tau_{xy} = 4000$ lb/in², together with a zero normal stress on the horizontal faces, plots as point $d$. Line $\overline{bd}$ is drawn, its midpoint $c$ is located, and a circle of radius $\overline{cb} = \overline{cd}$ is drawn with $c$ as a center. This is Mohr's circle. The endpoints of the diameter $\overline{bd}$ represent the stress conditions existing in the element if it has the original orientation shown in Fig. 16-32.

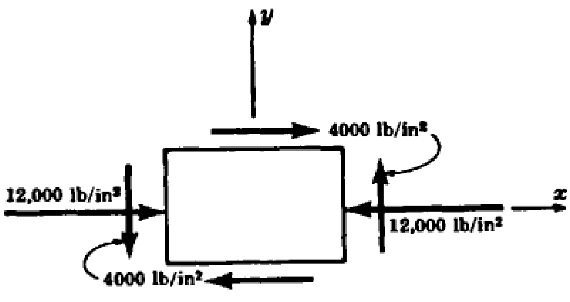



Fig. 16-32

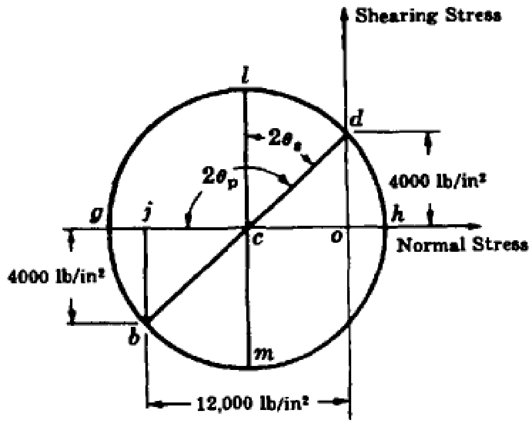



Fig. 16-33

(a) The principal stresses are represented by points $g$ and $h$ (Fig. 16-33), as demonstrated in Problem 16.8. They may be determined either by direct measurement from the above diagram or by realizing that the coordinate of $c$ is $-6000$, and that $\overline{cd} = \sqrt{(6000)^2 + (4000)^2} = 7220$. Thus the minimum principal stress is

$$\sigma_{min} = \overline{og} = +(\overline{oc} + \overline{cg}) = -6000 - 7220 = -13{,}220 \text{ lb/in}^2$$

The maximum principal stress is

$$\sigma_{max} = \overline{oh} = \overline{ch} - \overline{co} = 7220 - 6000 = 1220 \text{ lb/in}^2$$

The angle $2\theta_p$ designated above is given by $\tan 2\theta_p = -4000/6000 = -2/3$ since $\tan(180° - \theta) = -\tan\theta$. Hence $2\theta_p = 146°20'$, and $\theta_p = 73°10'$. This value could of course have been obtained by direct measurement of angle $dcg$ in Mohr's circle. Thus the principal stress of $-13{,}220$ lb/in² represented by point $g$ acts on a plane oriented $73°10'$ from the original $x$-axis. The principal stresses thus appear as in Fig. 16-34. It is evident from Mohr's circle that the shearing stresses on these planes are zero, since points $g$ and $h$ lie on the horizontal axis of Mohr's circle.

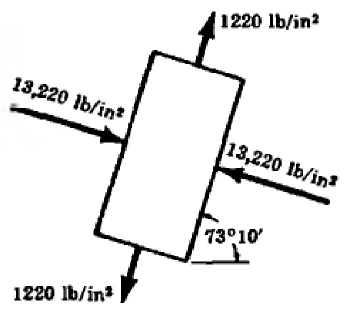



Fig. 16-34

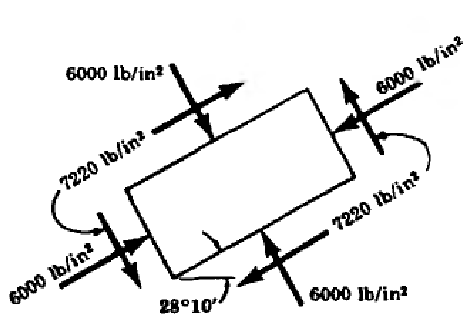



Fig. 16-35

(b) The maximum shearing stress is represented by $\overline{cl}$ in Mohr's circle. This radius has already been found to be equal to 7220 lb/in². The angle $2\theta_s$ may be found either by direct measurement from Mohr's circle or simply by subtracting 90° from the above value of $2\theta_p$. This leads to $\theta_s = 28°10'$. The shearing stress represented by point $l$ is positive; hence on this $28°10'$ plane the shearing stress tends to rotate the element in a clockwise direction. Also, from Mohr's circle the abscissa of point $l$ is $-6000$ lb/in² and this represents the normal stress occurring on the planes of maximum shearing stresses, as shown in Fig. 16-35.

**16.13.** Consider a plane element removed from a stressed elastic member. In general such an element will be subject to normal stresses in each of two perpendicular directions, as well as shearing stresses. Let these stresses be denoted by $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ and have the positive directions shown in Fig. 16-36. (*a*) Determine the magnitudes of the normal and shearing stresses on a plane inclined at an angle $\theta$ to the $x$-axis. (*b*) Also determine the maximum and minimum values of the normal stress that may exist on inclined planes and the directions of these stresses. (*c*) Finally, find the magnitude and direction of the maximum shearing stress that may exist on an inclined plane.

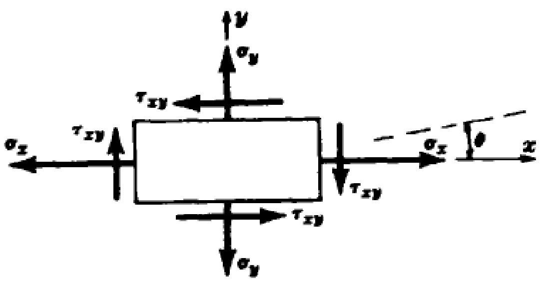


**Fig. 16-36**

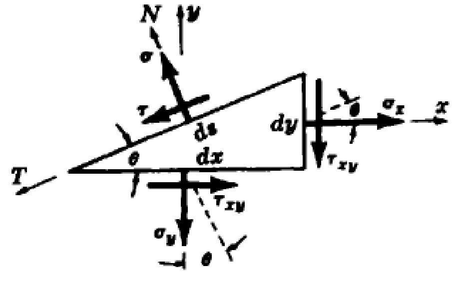


**Fig. 16-37**

(*a*) Evidently the desired stresses acting on the inclined planes are internal quantities with respect to the element shown in Fig. 16-36. Following the usual procedure of introducing a cutting plane so as to render the desired quantities external to the new section, we cut the originally rectangular element along the plane inclined at the angle $\theta$ to the $x$-axis and thus obtain the triangular element shown in Fig. 16-37. Since we have removed half of the material in the rectangular element, we must replace it by the effect that it exerted upon the remaining lower triangle shown and this effect in general consists of both normal and shearing forces acting along the inclined plane. We shall designate the magnitudes of the normal and shearing stresses corresponding to these forces by $\sigma$ and $\tau$, respectively. Thus our problem reduces to finding the unknown stresses $\sigma$ and $\tau$ in terms of the known stresses $\sigma_x$, $\sigma_y$, and $\tau_{xy}$. Chapter 17 illustrates the manner of determination of the stresses $\sigma_x$, $\sigma_y$, and $\tau_{xy}$. It is to be carefully noted that the free-body diagram, Fig. 16-37, indicates stresses acting on the various faces of the element, and not forces. Each of these stresses is assumed to be uniformly distributed over the area on which it acts.

We shall introduce the $N$- and $T$-axes normal and tangential to the inclined plane as shown. Let $t$ denote the thickness of the element perpendicular to the plane of the page. Let us begin, by summing forces in the $N$-direction. For equilibrium we have

$$\Sigma F_N = \sigma t\,ds - \sigma_x t\,dy \sin\theta - \tau_{xy} t\,dy \cos\theta - \sigma_y t\,dx \cos\theta - \tau_{xy} t\,dx \sin\theta = 0$$

Substituting $dy = ds \sin\theta$, $dx = ds\cos\theta$ in the equilibrium equation,

$$\sigma\,ds = \sigma_x\,ds \sin^2\theta + \sigma_y\,ds\cos^2\theta + 2\tau_{xy}\,ds \sin\theta\cos\theta$$

Introducing the identities $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, $\sin 2\theta = 2\sin\theta\cos\theta$, we find

$$\sigma = \tfrac{1}{2}\sigma_x(1-\cos 2\theta) + \tfrac{1}{2}\sigma_y(1+\cos 2\theta) + \tau_{xy}\sin 2\theta$$

or

$$\sigma = \tfrac{1}{2}(\sigma_x+\sigma_y) - \tfrac{1}{2}(\sigma_x-\sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (1)$$

Thus the normal stress $\sigma$ on any plane inclined at an angle $\theta$ with the $x$-axis is known as a function of $\sigma_x$, $\sigma_y$, $\tau_{xy}$, and $\theta$.

Next, summing forces acting on the element in the $T$-direction, we find

$$\Sigma F_T = \tau t\,ds - \sigma_x t\,dy\cos\theta + \tau_{xy} t\,dy \sin\theta - \tau_{xy} t\,dx\cos\theta + \sigma_y t\,dx\sin\theta = 0$$

Substituting for $dx$ and $dy$ as before, we get

$$\tau\,ds = \sigma_x\,ds\sin\theta\cos\theta - \tau_{xy}\,ds\sin^2\theta + \tau_{xy}\,ds\cos^2\theta - \sigma_y\,ds\sin\theta\cos\theta$$

Introducing the previous identities and the relation $\cos 2\theta = \cos^2\theta - \sin^2\theta$, this last equation becomes

$$\tau = \tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (2)$$

Thus the shearing stress $\tau$ on any plane inclined at an angle $\theta$ with the $x$-axis is known as a function of $\sigma_x$, $\sigma_y$, $\tau_{xy}$, and $\theta$.

(*b*) To determine the maximum value that the normal stress $\sigma$ may assume as the angle $\theta$ varies, we shall differentiate Eq. (*1*) with respect to $\theta$ and set this derivative equal to zero. Thus

$$\frac{d\sigma}{d\theta} = (\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

Hence the values of $\theta$ leading to maximum and minimum values of the normal stress are given by

$$\tan 2\theta_p = -\frac{\tau_{xy}}{\tfrac{1}{2}(\sigma_x - \sigma_y)} \qquad (3)$$

The planes defined by the angles $\theta_p$ are called *principal planes*. The normal stresses that exist on these planes are designated as *principal stresses*. They are the maximum and minimum values that the normal stress may assume in the element under consideration. The values of the principal stresses may easily be found by considering the graphical interpretation of (*3*) given in Fig. 16-38. Evidently the tangent of either of the angles designated as $2\theta_p$ has the value given in (*3*). Thus there are two solutions of (*3*), and consequently two values of $2\theta_p$ (differing by 180°) and also two values of $\theta_p$ (differing by 90°). It is to be noted that Fig. 16-38 bears no direct relationship to the triangular element whose free-body diagram was given in Fig. 16-37.

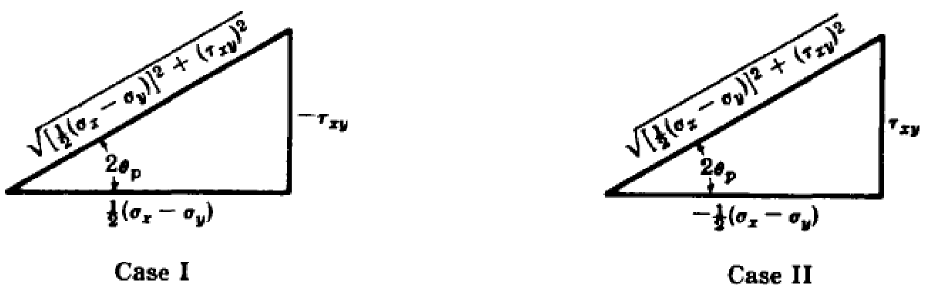



**Fig. 16-38**

The values of $\sin 2\theta_p$ and $\cos 2\theta_p$ as found from the above two diagrams may now be substituted in (*1*) to yield the maximum and minimum values of the normal stresses. Observing that

$$\sin 2\theta_p = \frac{\mp\tau_{xy}}{\sqrt{[\frac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2}} \qquad \cos 2\theta_p = \frac{\pm\frac{1}{2}(\sigma_x - \sigma_y)}{\sqrt{[\frac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2}}$$

where the upper signs pertain to Case I and the lower to Case II, we obtain from (*1*)

$$\sigma = \tfrac{1}{2}(\sigma_x + \sigma_y) \pm \sqrt{[\tfrac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2} \qquad (4)$$

The maximum normal stress is

$$\sigma_{max} = \tfrac{1}{2}(\sigma_x + \sigma_y) + \sqrt{[\tfrac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2} \qquad (5)$$

The minimum normal stress is

$$\sigma_{min} = \tfrac{1}{2}(\sigma_x + \sigma_y) - \sqrt{[\tfrac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2} \qquad (6)$$

The stresses given by (*5*) and (*6*) are the principal stresses and they occur on the principal planes defined by (*3*). By substituting one of the values of $\theta_p$ from (*3*) into Eq. (*1*), one may readily determine which of the two principal stresses is acting on that plane. The other principal stress naturally acts on the other principal plane.

By substituting the values of the angle $2\theta_p$ as given by (3) or by Fig. 16-38 into (2), it is readily seen that the shearing stresses $\tau$ on the principal planes are zero.

(c) To determine the maximum value that the shearing stress $\tau$ may assume as the angle $\theta$ varies, we shall differentiate Eq. (2) with respect to $\theta$ and set this derivative equal to zero. Thus

$$\frac{d\tau}{d\theta} = (\sigma_x - \sigma_y)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

The values of $\theta$ leading to the maximum values of the shearing stress are thus

$$\tan 2\theta_s = \frac{\frac{1}{2}(\sigma_x - \sigma_y)}{\tau_{xy}} \tag{7}$$

The planes defined by the two solutions to this equation are the planes of maximum shearing stress.

Again, a graphical interpretation of (7) is convenient. The two values of the angle $2\theta_s$ satisfying this equation may be represented as in Fig. 16-39. From these diagrams we have

$$\sin 2\theta_s = \frac{\pm\frac{1}{2}(\sigma_x - \sigma_y)}{\sqrt{[\frac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2}} \qquad \cos 2\theta_s = \frac{\pm\tau_{xy}}{\sqrt{[\frac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2}}$$

where the upper (positive) sign refers to Case I and the lower (negative) sign applies to Case II. Substituting these values in (2) we find

$$\tau_{\substack{\max \\ \min}} = \pm\sqrt{[\frac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2} \tag{8}$$

Here the positive sign represents the maximum shearing stress, the negative sign the minimum shearing stress.


Fig. 16-39

If we compare (3) and (7), it is evident that the angles $2\theta_p$ and $2\theta_s$ differ by 90°, since the tangents of these angles are the negative reciprocals of one another. Hence the planes defined by the angles $\theta_p$ and $\theta_s$ differ by 45°; that is, the planes of maximum shearing stress are oriented 45° from the planes of maximum normal stress.

It is also of interest to determine the normal stresses on the planes of maximum shearing stress. These planes are defined by (7). If we now substitute the values of $\sin 2\theta_s$ and $\cos 2\theta_s$ in Eq. (1) for normal stress, we find

$$\sigma = \frac{1}{2}(\sigma_x + \sigma_y) \tag{9}$$

Thus on each of the planes of maximum shearing stress is a normal stress of magnitude $\frac{1}{2}(\sigma_x + \sigma_y)$.

**16.14.** Discuss a graphical representation of the analysis presented in Problem 16.13.

For given values of $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ we proceed this way:

1. Introduce a rectangular coordinate system in which normal stresses are represented along the horizontal axis and shearing stresses along the vertical axis. The scales used on these two axes must be equal.

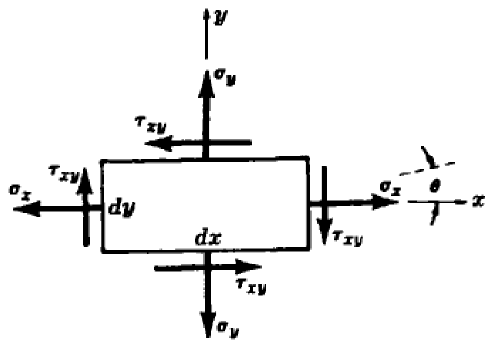



**Fig. 16-40**

2. With reference to the original rectangular element considered in Problem 16.13 and reproduced in Fig. 16-40, we shall introduce the sign convention that shearing stresses are positive if they tend to rotate the element clockwise, and negative if they tend to rotate it counterclockwise. Here the shearing stresses on the vertical faces are positive, those on the horizontal faces are negative. Also, tensile normal stresses are considered to be positive, compressive stresses negative.
3. We first locate point $b$ by laying out $\sigma_x$ and $\tau_{xy}$ to their given values. The shear stress $\tau_{xy}$ on the vertical faces on which $\sigma_x$ acts is positive; hence this value is plotted as positive in Fig. 16-41.

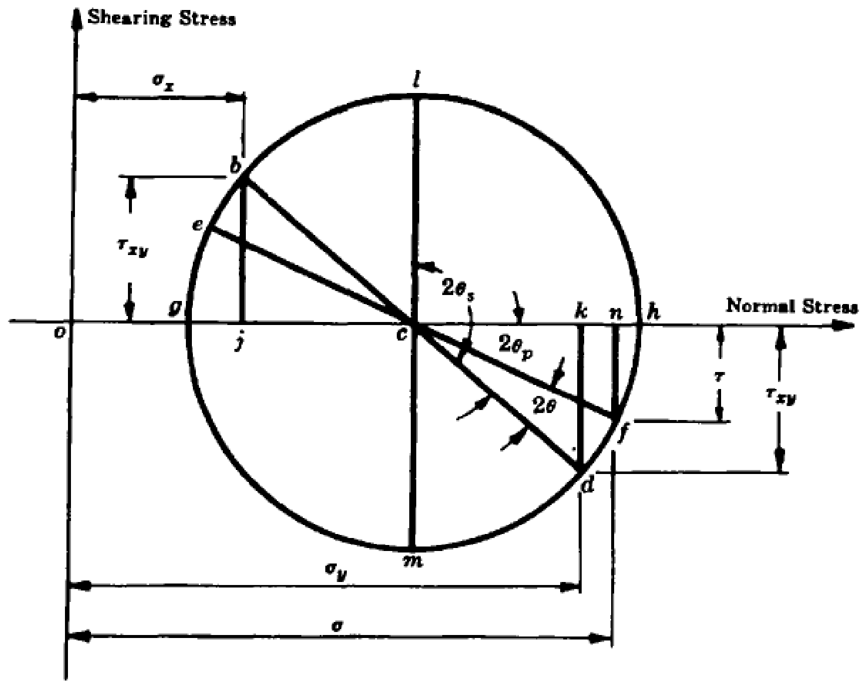



**Fig. 16-41**

4. We next locate point $d$ in a similar manner by laying off $\sigma_y$ and $\tau_{xy}$ to their given values. Figure 16-41 is drawn on the assumption that $\sigma_y > \sigma_x$ although the treatment presented here holds if $\sigma_y < \sigma_x$. The shear stress $\tau_{xy}$ on the horizontal faces on which $\sigma_y$ acts is negative; hence this value is plotted below the reference axis.
5. Next, we draw line $\overline{bd}$, locate midpoint $c$, and draw a circle having its center at $c$ and radius equal to $\overline{cb}$. This is known as Mohr's circle.

We shall first show that the points $g$ and $h$ along the horizontal diameter of the circle represent the principal stresses. To do this we note that the point $c$ lies at a distance $\frac{1}{2}(\sigma_x + \sigma_y)$ from the origin of the coordinate system. Also, the line segment $\overline{jk}$ is of length $\sigma_y - \sigma_x$; hence $\overline{ck}$ is of length $\frac{1}{2}(\sigma_y - \sigma_x)$. From the right triangle relationship we have

$$(\overline{cd})^2 = (\overline{ck})^2 + (\overline{kd})^2 \qquad \text{or} \qquad \overline{cd} = \sqrt{[\tfrac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2}$$

Also, $\overline{cg} = \overline{ch} = \overline{cd}$. Hence the $x$-coordinate of point $h$ is $\overline{oc} + \overline{ch}$ or

$$\tfrac{1}{2}(\sigma_x + \sigma_y) + \sqrt{[\tfrac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2}$$

But this expression is exactly the maximum principal stress, as found in (5) of Problem 16.13. Likewise the $x$-coordinate of point $g$ is $\overline{oc} - \overline{gc}$ or

$$\tfrac{1}{2}(\sigma_x + \sigma_y) - \sqrt{[\tfrac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2}$$

and this expression is exactly the minimum principal stress, as found in (6) of Problem 16.13. Consequently the points $g$ and $h$ represent the principal stresses existing in the original element. We see that the tangent of $\angle kcd = \overline{dk}/\overline{ck} = \tau_{xy}/\frac{1}{2}(\sigma_y - \sigma_x)$. But from (3) of Problem 16.13 we had

$$\tan 2\theta_p = -\frac{\tau_{xy}}{\frac{1}{2}(\sigma_x - \sigma_y)}$$

and by comparison of these two relations we see that $\angle kcd = 2\theta_p$; that is, a counterclockwise rotation from the diameter $\overline{bd}$ (corresponding to the stresses in the $x$- and $y$-directions) leads us to the diameter $\overline{gh}$, representing the principal planes, on which the principal stresses occur. The principal planes lie at an angle $\theta_p$ from the $x$-direction.

Thus Mohr's circle is a convenient device for finding the principal stresses, since one can merely establish the circle for a given set of stresses $\sigma_x$, $\sigma_y$, $\tau_{xy}$, then measure $\overline{og}$ and $\overline{oh}$. These abscissas represent the principal stresses to the same scale used in plotting $\sigma_x$, $\sigma_y$, $\tau_{xy}$.

It is now apparent that the radius of Mohr's circle,

$$\overline{cd} = \sqrt{[\tfrac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2}$$

corresponds to the maximum shearing stress as found in (8) of Problem 16.13. Actually, the shearing stress on any plane is represented by the ordinate to Mohr's circle; hence we should consider the radial lines $\overline{cl}$ and $\overline{cm}$ as representing the maximum shearing stress. The angle $dcl$ is evidently $2\theta_s$ and hence it is apparent that the double angle between the planes of maximum normal stress and the planes of maximum shearing stress ($\angle kcl$) is 90°; hence the planes of maximum shearing stress are oriented 45° from the planes of maximum normal stress.

Evidently the endpoints of the diameter $\overline{bd}$ represent the stresses acting in the original $x$- and $y$-directions. We shall now demonstrate that the endpoints of any other diameter such as $\overline{ef}$ (at an angle $2\theta$ with $\overline{bd}$) represent the stresses on a plane inclined at an angle $\theta$ to the $x$-axis. To do this we note that the abscissa of point $f$ is given by

$$\begin{aligned}\sigma = \overline{oc} + \overline{cn} &= \tfrac{1}{2}(\sigma_x + \sigma_y) + \overline{cf}\cos(2\theta_p - 2\theta)\\ &= \tfrac{1}{2}(\sigma_x + \sigma_y) + \overline{cf}(\cos 2\theta_p \cos 2\theta + \sin 2\theta_p \sin 2\theta)\\ &= \tfrac{1}{2}(\sigma_x + \sigma_y) + \sqrt{[\tfrac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2}\,(\cos 2\theta_p \cos 2\theta + \sin 2\theta_p \sin 2\theta)\end{aligned}$$

But from an inspection of triangle $ckd$ in Mohr's circle it is evident that

$$\sin 2\theta_p = \frac{\tau_{xy}}{\sqrt{[\frac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2}} \qquad \cos 2\theta_p = \frac{\frac{1}{2}(\sigma_y - \sigma_x)}{\sqrt{[\frac{1}{2}(\sigma_x - \sigma_y)]^2 + (\tau_{xy})^2}} \tag{1}$$

Substituting the values of $\tau_{xy}$ and $\frac{1}{2}(\sigma_y - \sigma_x)$ from these last two equations into the previous equation, we find

$$\sigma = \tfrac{1}{2}(\sigma_x + \sigma_y) - \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

But this is exactly the normal stress on a plane inclined at an angle $\theta$ to the $x$-axis as derived in (1) of Problem 16.13.