فصل چهارم

تابع فازی

Fuzzy Function

۶-۶- توابع فازی روی مجموعه های فازی

یک تابع فازی[۹] حالت عمومی شده مفهوم تابع کلاسیک است . تـابع کلاسـیک f یـک تبـدیل از دامنه D به فضای S که برد نامیده می شود است. ($f(D) \subseteq S$)

جنبه های مختلفی از مفهوم تابع کلاسیک می تواند جهت رسیدن به یک تـابع فـازی، بـه صـورت فازی در نظر گرفته شود. بنابراین سطوح مختلفی از فازی بودن یک تابع کلاسیک می تواند مطـرح شود که به شرح ذیل است :

- **الف)** تصویر کردن[۱۰] دامنه به برد می تواند کلاسیک باشد . اگر دامنـه یـک مجموعـه فازی باشد آن گاه یک تصویر کلاسیک از یک دامنه فـازی[۱۱]، یـک مجموعـه فـازی کـه برد[۱۲] تابع است را تولید می کند. در این حالت تابع کلاسیک است امـا دامنـه و بـرد آن فازی است.
- **ب)** تابع تصویر به صورت فازی مطرح شود که به این حالت تابع فازی گفته می شود.
- **ج)** توابع کلاسیک می توانند خواص فازی داشته باشند یـا ایـن کـه توسـط یکسـری محدودیت های فازی محدود شوند.

6.1.1 Function with Fuzzy Constraint

Definition (Function with fuzzy constraint) Let X and Y be crisp sets, and f be a crisp function. A and B are fuzzy sets defined on universal sets X and Y respectively. Then the function satisfying the condition $\mu_A(x) \le \mu_B(f(x))$ is called a function with constraints on fuzzy domain A and fuzzy range B.

Example 6.2 Consider two fuzzy sets,

$$A = \{(1, 0.5), (2, 0.8)\}, \quad B = \{(2, 0.7), (4, 0.9)\}$$

and a function

$$y = f(x) = 2x, \quad \text{for} \quad x \in A, \quad y \in B.$$

We see the function f satisfies the condition, $\mu_A(x) \le \mu_B(y)$.

6.1.2 Propagation of Fuzziness by Crisp Function

Definition (Fuzzy extension function) Fuzzy extension function propagatetes the ambiguity of independent variables to dependent variables. when f is a crisp function from X to Y, the fuzzy extension function f defines the image $f(\widetilde{X})$ of fuzzy set $\widetilde{X}$. That is, the extension principle is applied(see section 3.4).

$$\mu_{f(\widetilde{x})}(y) = \begin{cases} \max\limits_{x \in f^{-1}(y)} \mu_{\widetilde{x}}(x), & if \quad f^{-1}(y) \ne \phi \\ 0, & if \quad f^{-1}(y) = \phi \end{cases}$$

where, $f^{-1}(y)$ is inverse image of y.

In this section, we use the sign ~ for the emphasis of fuzzy variable.

Example 6.4 There is a crisp function,

$$f(x) = 3\widetilde{x} + 1$$

where its domain is $A = \{(0, 0.9), (1, 0.8), (2, 0.7), (3, 0.6), (4, 0.5)\}$
and its range is $B = [\,0, 20\,]$

The independent variables have ambiguity and the fuzziness is propagated to the crisp set B. Then, we can obtain a fuzzy set B' in B

$$B' = \{(1,0.9), (4, 0.8), (7, 0.7), (10, 0.6), (13, 0.5)\}.$$

6.1.3 Fuzzifying Function of Crisp Variable

Fuzzifying function of crisp variable is a function which produces image of crisp domain in a fuzzy set.

Definition (Single fuzzifying fuction) Fuzzifying function from X to Y is the mapping of X in fuzzy power set $\tilde{P}(Y)$.

$$\tilde{f} : X \rightarrow \tilde{P}(Y) \qquad \square$$

That is to say, the fuzzifying function is a mapping from domain to fuzzy set of range. Fuzzifying function and the fuzzy relation coincides with each other in the mathematical manner. So to speak, fuzzifying function can be interpreted as fuzzy relation R defined as following:

$$\forall (x, y) \in X \times Y$$

$$\mu_{\tilde{f}(x)}(y) = \mu_R(x, y)$$

Example 6.5 Consider two crisp sets $A = \{2, 3, 4\}$ and $B = \{2, 3, 4, 6, 8, 9, 12\}$

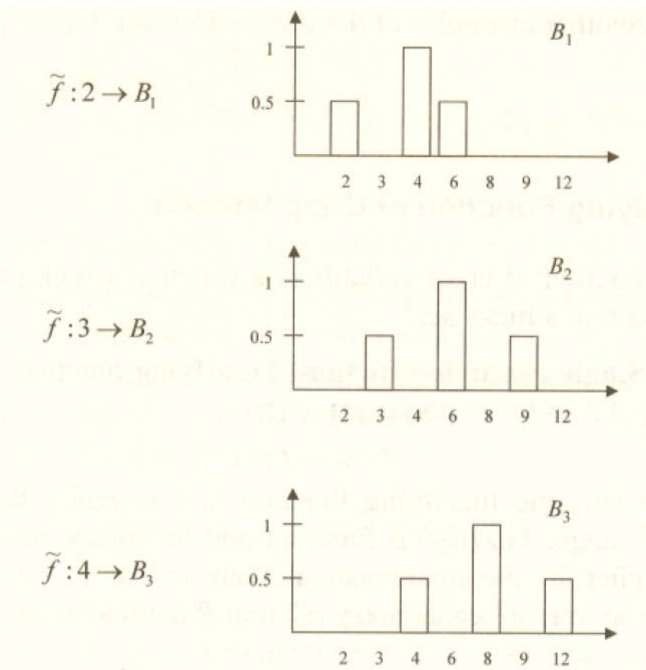
A fuzzifying function $\tilde{f}$ maps the elemets in A to power set $\tilde{P}(B)$ in the following manner.

$$\tilde{f}(2) = B_{1,} \quad \tilde{f}(3) = B_{2,} \quad \tilde{f}(4) = B_3$$

$$\text{where} \quad \tilde{P}(B) = \{B_1, B_2, B_3\}$$

$B_1 = \{(2, 0.5), (4, 1), (6. 0.5)\}$ $B_2 = \{(3. 0.5), (6,1), (9, 0.5),$ $B_3 = \{(4, 0.5), (8, 1), (12, 0.5)\}$

If we look at the mapping in detail, we can see the relationship as shown in (Fig 6.1)



Definition (Fuzzy bunch of functions) Fuzzy bunch of crisp functions from X to Y is defined with fuzzy set of crisp function f_i ($i = 1, \ldots, n$) and it is denoted as

$$\tilde{f} = \{(f_i, \mu_{\tilde{f}}(f_i)) \mid f_i : X \to Y, i \in \aleph\}$$

$$f_i = f(x), \quad \forall x \in X \qquad \square$$

This function produces fuzzy set as its outcome.

Example 6.6 In the case of crisp sets f_1, f_2 and f_3, the bunch will be, for example,

$X = \{1, 2, 3\}$

$$\tilde{f} = \{(f_1, 0.4), (f_2, 0.7), (f_3, 0.5)\}$$

$$f_1(x) = x, \quad f_2(x) = x^2, \quad f_3(x) = -x + 1$$

By f_1, we get $\tilde{f} = \{(1, 0.4), (2, 0.4), (3, 0.4)\}$

By f_2, $\tilde{f}_2 = \{(1, 0.7), (4, 0.7), (9, 0.7)\}$

By f_3, $\tilde{f}_3 = \{(0, 0.5), (-1, 0.5), (-2, 0.5)\}$

then, we can summarize the outputs as follows :

$\tilde{f}(1) = \{(1, 0.4), (1, 0.7), (0, 0.5)\} = \{(0, 0.5), (1, 0.7)\}$

$\tilde{f}(2) = \{(2, 0.4), (4, 0.7), (-1, 0.5)\} = \{(-1, 0.5), (2, 0.4), (4, 0.7)\}$

$\tilde{f}(3) = \{(3, 0.4), (9, 0.7), (-2, 0.5)\} = \{(-2, 0.5), (3, 0.4), (9, 0.7)\}$

6.2.1 Maximizing and Minimizing Set

Definition (Maximizing set) Let f be the function having real values in X and the highest and the lowest value of f be *sup(f)* and *inf(f)* respectively. At this time, the maximizing set M is defined as a fuzzy set.

$$\forall x \in X, \quad \mu_M(x) = \frac{f(x) - \inf(f)}{\sup(f) - \inf(f)} \quad \square$$

That is, the maximizing set M is a fuzzy set and defined by the possibility of x to make the maximum value sup(*f*). The possibility of x to be in the range of M is defined from the relative normalized position in the interval [inf(*f*), sup(*f*)]. Here the interval [inf(*f*), sup(*f*)] denotes the possible range of $f(x)$ to have some values. Minimizing set of f is defined as the maximizing set of $-f$.

Example 6.8 Let's have a look at $f(x)$ of (Fig 6.3.) The interval of values is as follows :

$$[\inf(f), \sup(f)] = [10, 20],\ 1 \le x \le 10$$

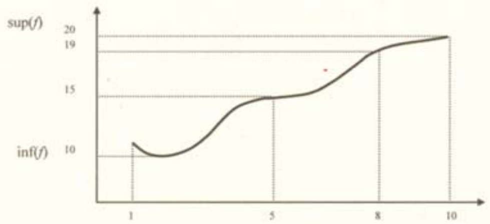
and when $x = 5$, $f(x) = 15$. Then the possibility of $x = 5$ to be in the maximizing set M is calculated as follows :

$$\mu_M(5) = (15 - 10) / (20 - 10) = 5 / 10 = 0.5$$

also if when $x = 8$, $f(x) = 19$,

$$\mu_M(8) = (19 - 10) / (20 - 10) = 9 / 10 = 0.9$$

$\mu_M(x)$ denotes the possibility of x to make maximum value of f. Here, we might say that two independent variables $x = 5$ and $x = 8$ make the maximum value of $f(x)$ =20 with the possibilities 0.5 and 0.9, respectively $\square$



$1 \le x \le 10$

$10 \le f(x) \le 20$

6.2.2 Maximum Value of Crisp Function

(1) Crisp Domain

Assume x_0 is the independent variable which makes function f be the maximum value in crisp domain D. We might utilize the maximizing set M to find the value x_0 .That is, x_0 shall be the element that enables $\mu_M(x)$ to be the maximum value.

$$\mu_M(x_0) = \max_{x \in D} \mu_M(x)$$

$\mu_M(x)$ is the membership function of maximizing set. At this time, maximum value of f will be $f(x_0)$. $\mu_M(x_0)$ can be written as in the following, denoting domain D as a crisp set.

$$\begin{aligned} \mu_M(x_0) &= \max_{x \in D} \mu_M(x) \\ &= \max_{x \in x} \min[\mu_M(x), \mu_D(x)] \end{aligned}$$

Example 6.10 There is a function (F.g. 6.5) and its domain.

$$f(x) = \cos x, \quad x \in D = [0, \ 2\pi\ ]$$

$$\mu_M(x) = \frac{\cos x - \inf(\cos x)}{\sup(\cos x) - \inf(\cos x)} = \frac{\cos x - (-1)}{1-(-1)} = \frac{1}{2}\cos x + \frac{1}{2}$$

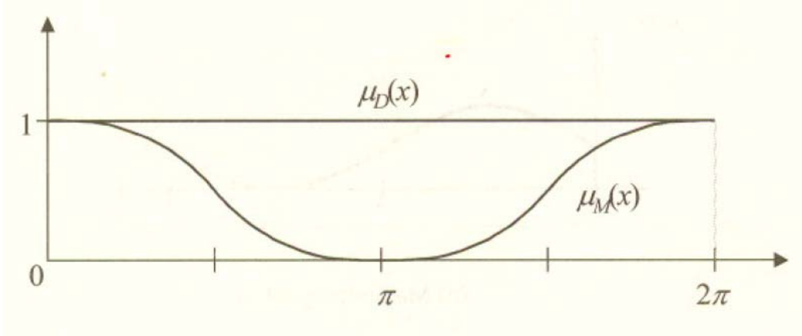
$$\begin{aligned} \mu_D(x) &= 1 \text{ for } 0 \le x \le 2\pi, \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Maximum value $f(x_0)$ is obtained at x_0 where

$$\begin{aligned} \mu_M(x_0) &= Max\ Min[\mu_M(x), \mu_D(x)] \\ &= \underset{0 \le x \le 2\pi}{Max}\ \mu_M(x), \\ &= 1 \quad \text{when} \quad x_0 = 0 \text{ and } 2\pi \end{aligned}$$

Therefore, the maximum value

$f(x_0) = 1$ is obtained when $x_0 = 0$ and 2π □



Maximum Value of Crisp Function

2- Fuzzy Domain

Now getting the maximum value $f(x_0)$ when domain is expressed in fuzzy set. To make f be the maximum value by x_0, following two conditions should be met.

- Set $\mu_M(x)$ as maximum
- Set $\mu_D(x)$ as maximum

For arbitrary element x_1 corresponding to the maximum f, it is necessary to satisfy the above two conditions on $\mu_M(x)$ and $\mu_D(x)$. The possibility of x_1 to make the maxim value of f is determined by the minimum of $\mu_M(x_1)$ and $\mu_D(x_1)$, that is,

$$\text{Min}[\mu_M(x_1), \mu_D(x_1)].$$

Therefore, the point x_0 which enables the function f to be the maximum is defined as follows.

$$\underset{x \in X}{MaxMin}[\mu_M(x), \mu_D(x)] = \mu(x_0)$$

At this time, the maximum value is $f(x_0)$. Here $\mu_M(x)$ is membership function of maximizing set and $\mu_D(x)$ is that of fuzzy domain(Fig 6.6). Comparing x_0 with x_1 in the figure, x_1 enables f to be maximum rather than x_0.

$$f(x_1) > f(x_0) \text{ or } \mu_M(x_1) > \mu_M(x_0)$$

but since $\mu_D(x_1)$ is very much smaller than $\mu_D(x_0)$, $f(x_0)$ is selected as the maximum value.

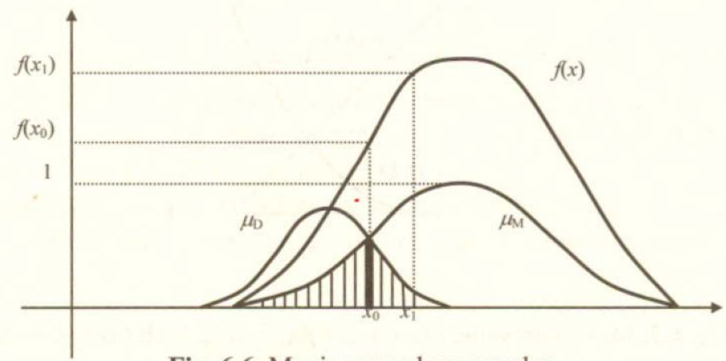


Fig. 6.6. Maximum value as scalar

Example 6.11 There are a function and a fuzzy domain(Fig 6.7).

$$f(x) = -x + 2,\ x \in D$$

$$\mu_D(x) = x^2 \quad \text{for} \quad 0 \le x \le 1$$
$$= 0 \quad \text{otherwise}$$

We can get the maximizing function.

$$\mu_M(x) = \frac{-x+2-1}{2-1} = -x+1$$

From the following equation,

$$\mu_f(x_0) = \text{Max Min}[\mu_M(x),\ \mu_D(x)]$$

the point x_0 is obtained when

$$\mu_M(x) = \mu_D(x) \quad \text{for} \quad 0 \le x \le 1$$

$$-x + 1 = x^2, \quad x \cong 0.6$$

Therefore, we have the maximum value $f(x_0) = 1.4$ when $x_0 = 0.6$

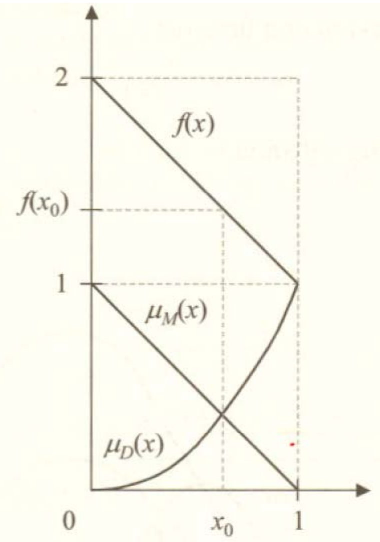


Fig. 6.7. Maximum value of example $f(x) = -x+2$ with fuzzy domain

Example 6.12 We have a crisp function f and its fuzzy domain D. Let's find the maximum value of f with D.

$$f(x) = \cos x, \quad x \in D$$

$$\mu_D(x) = Min\ [1, \frac{x}{\pi}] \quad \text{for } 0 \leq x \leq 2\pi$$

$$= 0 \qquad \text{otherwise}$$

$$\mu_M(x) = \frac{1}{2}\cos x + \frac{1}{2}$$

In (Fig 6.8.)

Max Min $[\mu_M(x), \mu_D(x)]$ is obtained when $x_0 = 2\pi$

then $f(x_0) = 1$

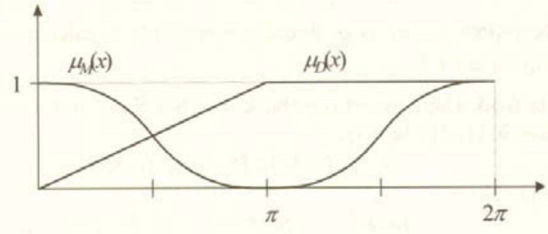


Fig. 6.8. Maximum value $f(x) = \cos(x)$ with fuzzy domain

Integration and Differenciation of Fuzzy Function

Integration

(1) Integration of fuzzifying function in crisp interval

Definition (Integration of fuzzying function) In non-fuzzy interval $[a, b] \in \Re$, let the fuzzifying function have fuzzy value $\tilde{f}(x)$ for $x \in [a, b]$. Integration $\tilde{I}(a,b)$ of the fuzzifying function in $[a, b]$ is defined as follows:

$$\tilde{I}(a,b) = \{(\int_a^b f_\alpha^-(x)dx + \int_a^b f_\alpha^+(x)dx,\ \alpha) \mid \alpha \in [0,1]\}$$

Here f_α^+ and f_α^- are α-cut functions of $\tilde{f}(x)$. Note that the plus sign(+) in the above formula is to express enumeration in fuzzy set but not addition. Therefore, the total integration is obtained by aggregating integrations of each α-cut function.

If we apply the α-cut operation to the fuzzifying function, we can get f_α^+ or f_α^- which are α-cut functions. We can calculate the integration of each function :

$$\tilde{I}_\alpha^- = \int_a^b f_\alpha^-(x)dx \text{ and } \tilde{I}_\alpha^- = \int_a^b f_\alpha^+(x)dx.$$

Now we can say that the possibility of $\tilde{I}_\alpha^-$ or $\tilde{I}_\alpha^+$ to be a member of total integration $\tilde{I}(a,b)$ is α. Recall the principle in calculating the fuzzy cardinality in sec 1.5.5.

Example 6.13 There is a fuzzy bunch of functions and we want to get integration in [1, 2](Fig 6.9).

$$\tilde{f} = \{(f_1, 0.4), (f_2, 0.7), (f_3, 0.4)\}$$

$X = [1, 2]$

$$f_1(x) = x, \quad f_2(x) = x^2, \quad f_3(x) = x + 1$$

i) Integration at $\alpha = 0.7$,

$f = f_2(x) = x^2$

$$I_\alpha(1,2) = \int_1^2 x^2 dx = \frac{1}{3}x^3]_1^2 = \frac{7}{3}$$

The integration result is $\frac{7}{3}$ with possibility 0.7

Therefore , $\tilde{I}_{0.7}(1,2) = \{(\frac{7}{3}, 0.7)\}$

ii) $\alpha = 0.4$, there are two functions

$$f^+ = f_1(x) = x$$
$$f^- = f_3(x) = x+1$$

$$I_\alpha^+(1, 2) = \int_1^2 x\ dx = \frac{1}{2}x^2]_1^2 = \frac{3}{2}$$

$$I_\alpha^-(1, 2) = \int_1^2 (x+1)dx = \frac{1}{2}x^2 + x]_1^2 = \frac{5}{2}$$

The integration results are $\frac{3}{2}$ with possibility 0.4 and $\frac{5}{2}$ with 0.4.

then, $\tilde{I}_{0.4}(1, 2) = \{(\frac{3}{2}, 0.4), (\frac{5}{2}, 0.4)\}$

Finally, we have the total integration.

$$\tilde{I}(1, 2) = \{(\frac{7}{3}, 0.7), (\frac{3}{2}, 0.4), (\frac{5}{2}, 0.4)\}$$ □

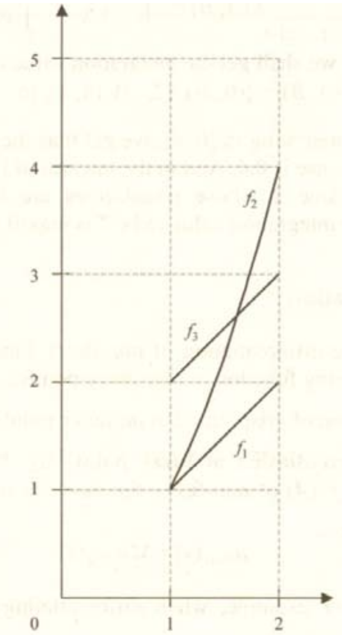


Fig. 6.9. Integration of fuzzy function with crisp interval

(2) Integration crisp function in fuzzy interval

In this part, we shall deal with the integration of non-fuzzy function in fuzzy interval [*A*, *B*] of which the boundaries are determined by two fuzzy sets *A* and *B*.(Fig 6.10)

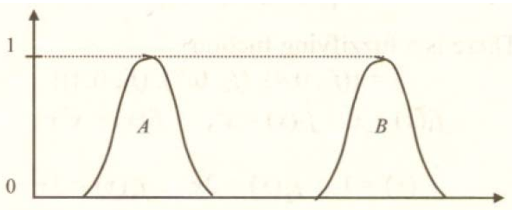


Fig. 6.10. Fuzzy interval

Definition (Integration in fuzzy interval) Integration *I*(*A*, *B*) of non-fuzzy function *f* in fuzzy interval [*A*, *B*] is defined as,

$$\mu_{I(a,b)}(z) = \underset{x,y}{Max} \quad Min[\mu_A(x), \mu_B(x)]$$

$$z = \int_x^y f(u)du$$

Example 6.14 Following shows the integration of function $f(x) = 2$ in fuzzy interval $[A, B]$.

$$A = \{(4, .8), (5, 1), (6, .4)\}$$
$$B = \{(6, .7), (7, 1), (8\ .2)\}$$
$$f(x) = 2, x \in [4, 8]$$

$$\tilde{I}(A, B) = \int_A^B f(x)dx = \int_A^B 2dx$$

Like (Table 7.1.), we shall get the integration value I(A, B).

$$\tilde{I}(A, B) = \{(0, .4), (2, .7), (4, 1), (6, .8), (8, .2)\}$$

For instance, integrating in [6, 6], we get 0 as the integration value. The possibility of this case is 0.4. And in the interval of [5, 6] and [6, 7], we get the integration value 2 whose possibilities are 0.7 and 0.4 . So the possibility for the integration value to be 2 is max[0.7, 0.4] = 0.7. □

Table 6.1. Fuzzy Integration

[a, b]	$\int_a^b 2\,dx$	min[$\mu_A(a), \mu_B(b)$]
[4, 6]	4	.7
[4, 7]	6	.8
[4, 8]	8	.2
[5, 6]	2	.7
[5, 7]	4	1.0
[5, 8]	6	.2
[6, 6]	0	.4
[6, 7]	2	.4
[6, 8]	4	.2

Differentiation

(1) Differentiation of crisp function on fuzzy points

Definition (Differentiation at fuzzy point) By the extension principle, differentiation $f'(A)$ of non-fuzzy function f at fuzzy point or fuzzy set A is defined as

$$\mu_{f'(A)}(y) = \underset{f(x)=y}{Max}\, \mu_A(x) \quad \square$$

Example 6.15 For example, when differentiating function $f(x) = x^3$ at fuzzy point A,

$$A = \{(-1, 0.4), (0, 1), (1, 0.6)\}$$

from $f'(x) = 3x^2$,

$$f'(A) = \{(3, 0.4), (0, 1), (3, 0.6)\}$$
$$= \{(0, 1), (3, 0.6)\} \quad \square$$

Example 6.16 There is a fuzzifying fuction

$$\tilde{f} = \{(f_1, 0.4), (f_2, 0.7), (f_3, 0.4)\}$$
$$f_1(x) = x, \quad f_2(x) = x^2, \quad f_3(x) = x^3+1$$

First, we have

$$f_1'(x) = 1 \quad f_2'(x) = 2x, \quad f_3'(x) = 3x^2$$
$$f_1'(0.5) = 1 \quad \text{when} \quad \alpha = 0.4$$
$$f_2'(0.5) = 1 \quad \text{when} \quad \alpha = 0.7$$
$$f_3'(0.5) = 0.75 \quad \text{when} \quad \alpha = 0.4$$
$$\frac{d\tilde{f}}{dx}(x_0) = \{(1, 0.4), (1, 0.7), (0.75, 0.4)\}$$
$$= \{(1, 0.7), (0.75, 0.4)\} \quad \square$$