Aerodynamics-II

KN Toosi University of Technology

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Lecture Hours: Sundays &Tuesdays 13:00-15:00

Office Hours: Sundays &Tuesdays 15-16:30 or by appointment

Teaching Assistant: Mr. Mehdi Talebi

Textbooks:

1- John Anderson, “Fundamentals of Aerodynamics” 4th Edition, McGraw-Hill, 2005.

2-John Bertin, “Aerodynamics for Engineers”, 4th Edition, Prentice Hall.

3- E. L. Houghton, P. W. Carpenter “ Aerodynamics for engineering students” 5th edition, McGraw-Hill, 2003.

4-A. M. Kuethe, C. Y. Chow, “Foundations of aerodynamics” , John Wiley, 5th edition, 1998

Class policy

Lecture:

1- Attendance is not mandatory but is recommended.

2-On-time arrival is expected. Late arrival and leaving the class before ending the lecture will be punished .

3- Sending and receiving SMS or any item that can disturb the lecture *is prohibited .*

Complementary Problems:

10 sets of problems from the texts and other sources will be emailed and will be discussed in the teaching assistant classes.

Grading policy:

Midterm-I	15%
Midterm-II	15%
Midterm-III	15%
Seminar(or project)	10%
TA class	5%
Final Exam	40%

Weeks	Date	Items	
1		Syllabus, policies and introductions	
		Elements of compressible flows and a review of thermodynamics	
2		Governing equations of inviscid , compressible flows	
		Governing equations -continue	
3		Normal shock waves and related topics	
		Normal shock waves –cont.	
4		Normal shock waves –cont.	
		Oblique shock waves	
5		Oblique shock waves-cont.	
		Oblique shock waves-cont.	Midterm-I

Week s	Date	Items
6		Shock interactions and reflections
		Shock interactions and reflections
7		Prandtl-Meyer expansion waves
		Prandtl-Meyer expansion waves-cont.
8		Supersonic airfoils
		Supersonic airfoils-cont
9		Compressible flow through converging-diverging ducts
		Compressible flow through converging-diverging ducts-cont.
10		Compressible flow through converging-diverging ducts-cont.
		Supersonic Wind tunnel Midterm-II

Weeks	Date	Items
11		Compressible flow in pipes with friction
		Compressible flow in pipes with friction-cont.
12		Compressible flow in pipes with heat transfer
		Compressible flow in pipes with heat transfer-cont.
13		Full potential equation
		Linear theory in compressible subsonic flows
14		Linear theory in compressible subsonic flows-cont.
		Linear theory in compressible supersonic
15		Linear theory in compressible supersonic-cont.
		Introduction to hypersonic flows Midterm-III

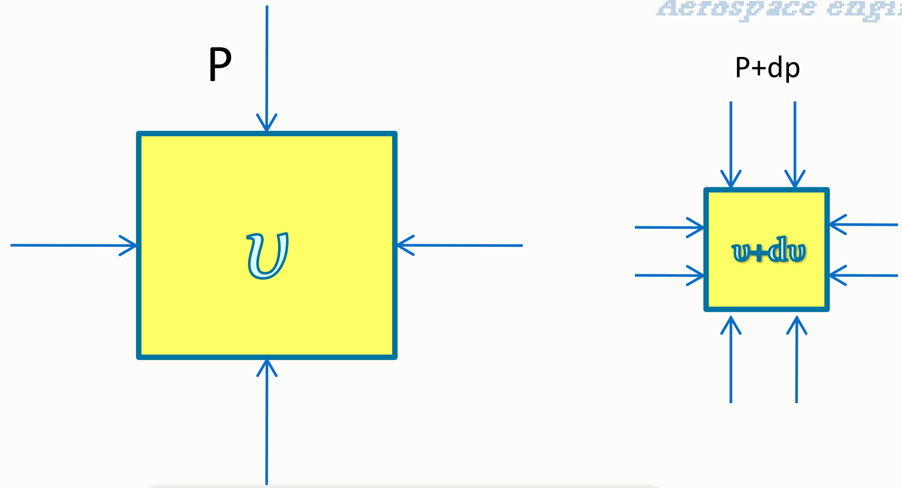
Aerodynamics classification

- Low speed (Incompressible)
- Subsonic
- Transonic
- Supersonic
- Hypersonic

Classification is base on flow compressibility

DEFINITION OF COMPRESSIBLE FLOW

- Compressible Fluid

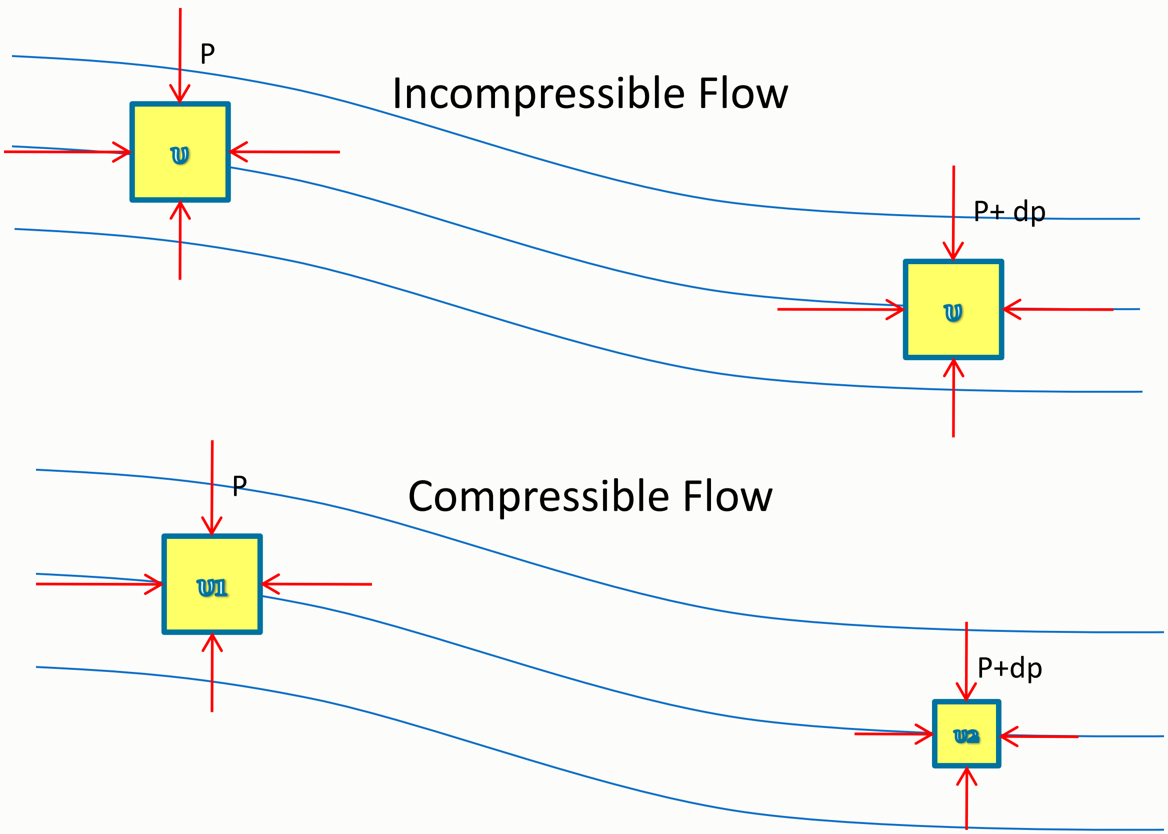


$$\tau = -\frac{1}{v}\frac{dv}{dP} = \frac{1}{\rho}\frac{d\rho}{dP} \Rightarrow \tau_T = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T \Rightarrow \tau_s = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_s$$

$\tau_t = 5\times 10^{-10}\ m^2/N$ for water at 1 Atm.

$\tau_t = 1\times 10^{-5}\ m^2/N$ for air at 1 Atm.

Compressible flow is routinely defined as variable density flow.



In compressible flows
Two thermodynamic properties vary

All thermodynamics properties vary

Thermodynamic properties are so important in compressible (high speed) Aerodynamics

A brief review on thermodynamics

Perfect gas:

A perfect gas is one whose individual molecules interact only via direct collisions, with no other intermolecular forces present.

$$p = \rho R T$$

For air, $R = 287$ J/kg-K$^{\circ}$

Perfect gas

Calorically Perfect

Thermally Perfect

Internal Energy and Enthalpy

consider a finite volume of gas consisting of a large number of molecules. The sum of the energies of all the molecules in this volume is defined as the internal energy (e) of the gas.

A related quantity is the specific enthalpy, denoted by h, and related to the other variables by

$$h = e + pv$$

The units of e and h are (velocity)^2, or m^2/s^2 in SI units.

For a calorically perfect gas:

$$e = c_v T$$

$$h = c_p T$$

$$h - e = pv = (c_p - c_v)T$$

$$pv = RT$$

$$c_p - c_v = R$$

$$\gamma \equiv c_p/c_v$$

For air:

$$\gamma = 1.4$$

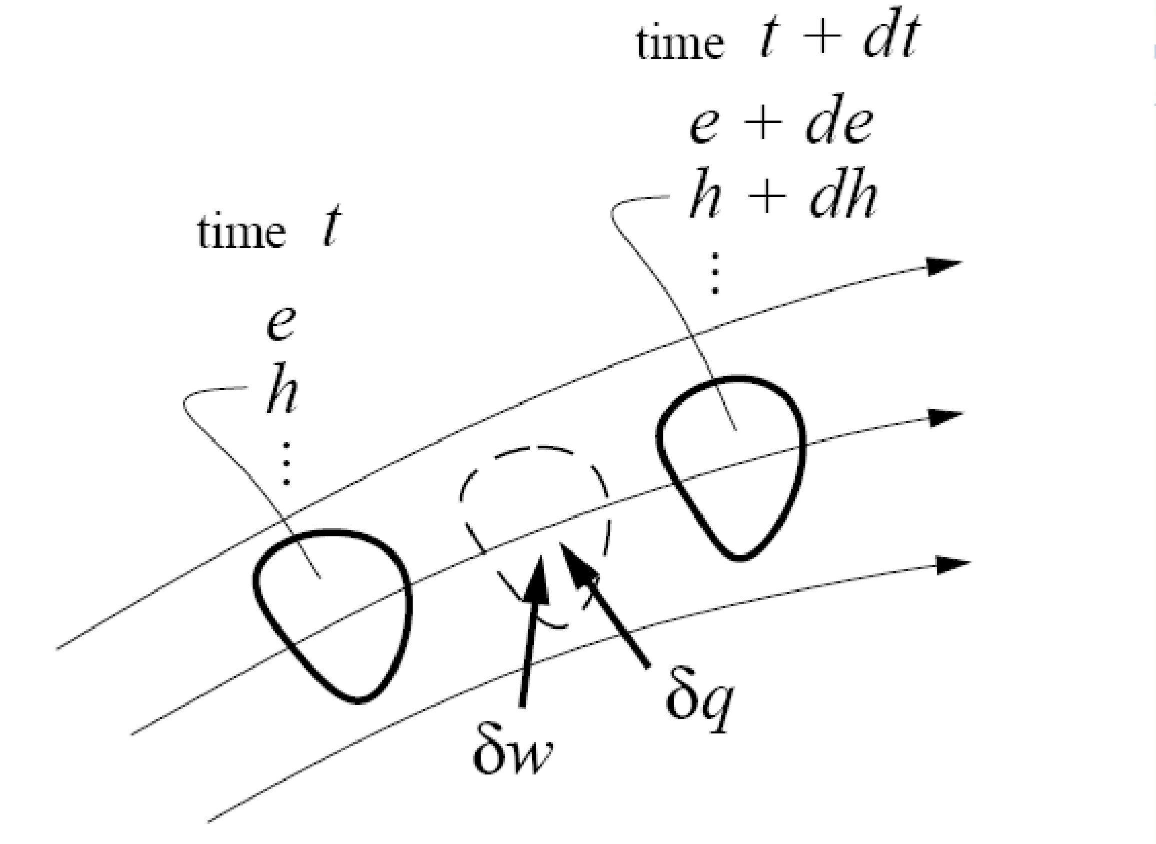
$$\frac{1}{\gamma - 1} = 2.5$$

$$\frac{\gamma}{\gamma - 1} = 3.5$$

$$c_v = \frac{1}{\gamma - 1} R$$

$$c_p = \frac{\gamma}{\gamma - 1} R$$

First Law of Thermodynamics

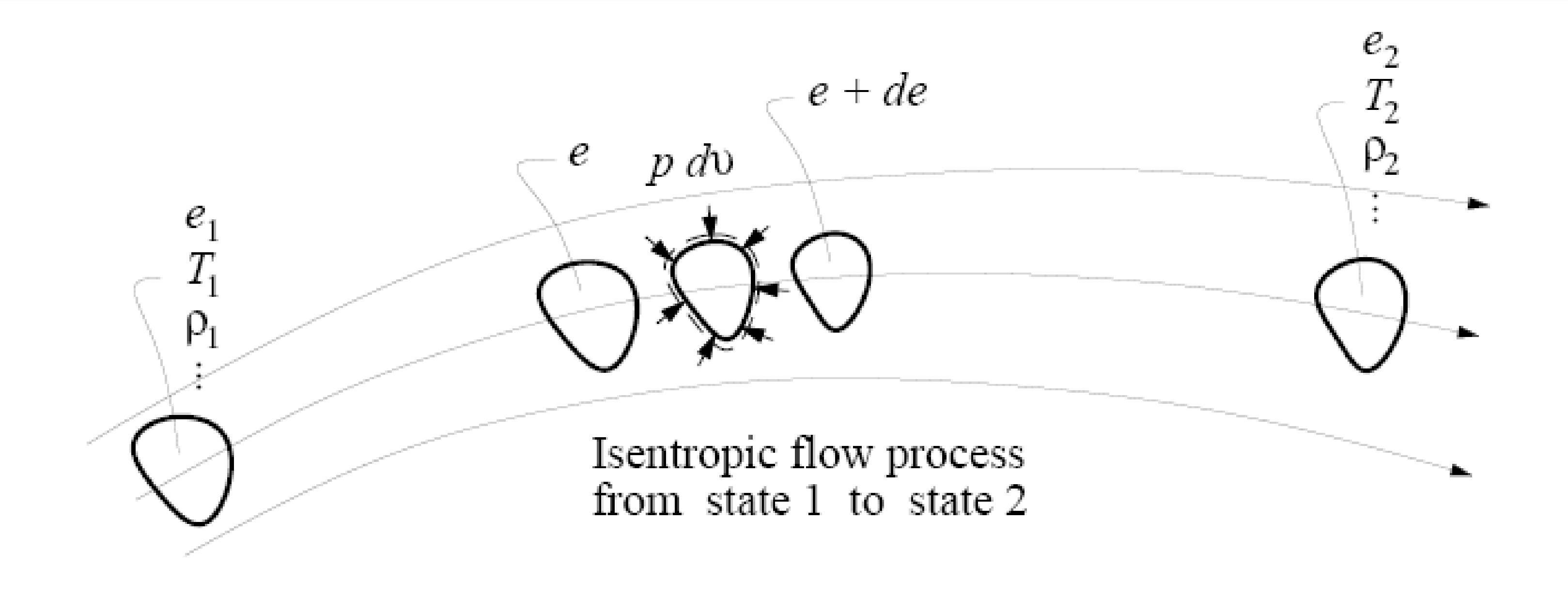


$$\delta q + \delta w = de$$

1. Adiabatic process, where no heat is transferred,

2. Reversible process, no dissipation occurs, implying that work must be only via volumetric compression $dw = -p\,dv$

3. Isentropic process, which is both adiabatic and reversible, implying $-p\,dv = de$



Entropy and the Second Law of Thermodynamics

Consider a block of ice in contact with a red-hot plate of steel.

Experience tells us that the ice will warm up (and probably melt) and the steel plate will cool down.

The first law allows that the ice may get cooler and the steel plate hotter—just as long as energy is conserved during the process.

To ascertain the proper direction of a process, let us define a new state variable, the entropy, as follows:

$$ds = \frac{\delta q_{\mathrm{rev}}}{T}$$

$$\boxed{ds = \frac{\delta q}{T} + ds_{\mathrm{irrev}}}$$

δq is the actual amount of heat added to the system during **an actual irreversible process**

$$\boxed{ds_{\text{irrev}} \geq 0} \Rightarrow \boxed{ds \geq \frac{\delta q}{T}}$$

The above equations are forms of the *second law of thermodynamics.*

The second law tells us in what direction a process will take place.

The practical calculation of entropy is carried out as follows.

$$\delta q - p\,dv = de$$

$$ds = \frac{\delta q_{\text{rev}}}{T}$$

$$T\,ds - p\,dv = de \Rightarrow \boxed{T\,ds = de + p\,dv}$$

$$dh = de + p\,dv + v\,dp$$

$$T\,ds = de + p\,dv$$

$$\boxed{T\,ds = dh - v\,dp}$$

$$de = c_v\,dT \text{ and } dh = c_p\,dT$$

$$ds = c_v \frac{dT}{T} + \frac{p\,dv}{T}$$

$$ds = c_p \frac{dT}{T} - \frac{v\,dp}{T}$$

$$pv = RT$$

$$v/T = R/p$$

$$ds = c_p \frac{dT}{T} - R \frac{dp}{p}$$

$$s_2 - s_1 = \int_{T_1}^{T_2} c_p \frac{dT}{T} - \int_{p_1}^{p_2} R \frac{dp}{p}$$

$$\boxed{s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}}$$

$$\boxed{s_2 - s_1 = c_v \ln \frac{T_2}{T_1} + R \ln \frac{v_2}{v_1}}$$

For an adiabatic process, *$\delta q = 0$. Also, for a* reversible process, *$ds_{irr} = 0$.*
Thus, for an adiabatic, reversible process, $ds = 0$

$$0 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

$$\ln \frac{p_2}{p_1} = \frac{c_p}{R} \ln \frac{T_2}{T_1}$$

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{c_p/R}$$

$$\frac{c_p}{R} = \frac{\gamma}{\gamma - 1}$$

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}$$

$$0 = c_v \ln \frac{T_2}{T_1} + R \ln \frac{v_2}{v_1}$$

$$\ln \frac{v_2}{v_1} = -\frac{c_v}{R} \ln \frac{T_2}{T_1}$$

$$\frac{v_2}{v_1} = \left(\frac{T_2}{T_1}\right)^{-c_v/R}$$

$$\frac{c_v}{R} = \frac{1}{\gamma - 1}$$

$$\frac{v_2}{v_1} = \left(\frac{T_2}{T_1}\right)^{-1/(\gamma-1)}$$

$$\boxed{\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}}$$

Isentropic relations

$$-p\,dv = de \;\Rightarrow\; dv = d\left(\frac{1}{\rho}\right) = -\frac{d\rho}{\rho^2} \;\Rightarrow\; \frac{\rho_2}{\rho_1} = \frac{p_2}{p_1}\frac{T_1}{T_2}$$

$$de = c_v\,dT = \frac{1}{\gamma-1}R\,dT \;\Rightarrow\; p\frac{d\rho}{\rho^2} = \frac{1}{\gamma-1}R\,dT \;\Rightarrow$$

$$\frac{d\rho}{\rho} = \frac{1}{\gamma-1}\frac{\rho R}{p}dT \;\Rightarrow\; \frac{d\rho}{\rho} = \frac{1}{\gamma-1}\frac{dT}{T} \;\xrightarrow{\text{Integration}}$$

$$\ln\rho = \frac{1}{\gamma-1}\ln T + \text{const.} \;\Rightarrow\; \frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{1/(\gamma-1)} \;\Rightarrow$$

$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1}\frac{T_1}{T_2} \;\Rightarrow\; \frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)} \qquad \frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$$

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}$$

Why is this equation so important?

The answers rest on the fact that a large number of practical compressible flow problems can be assumed to be isentropic contrary to what you might initially think.

For example, consider the flow over an airfoil or through a rocket engine. In the regions adjacent to the airfoil surface and the rocket nozzle walls, a boundary layer is formed wherein the dissipative mechanisms of viscosity, thermal conduction, and diffusion are strong. Hence, the entropy increases within these boundary layers.

However, consider the fluid elements moving outside the boundary layer. Here, the dissipative effects of viscosity, etc., are very small and can be neglected. Moreover, no heat is being transferred to or from the fluid element (i.e., we are not heating the fluid element with a Bunsen burner or cooling it in a refrigerator); thus, the flow outside the boundary layer is adiabatic.

EXAMPLE

Consider a Boeing 747 flying at a standard altitude of 36,000 ft. The pressure at a point on the wing is 400 lb/ft2 . Assuming isentropic flow over the wing, calculate the temperature at this point.

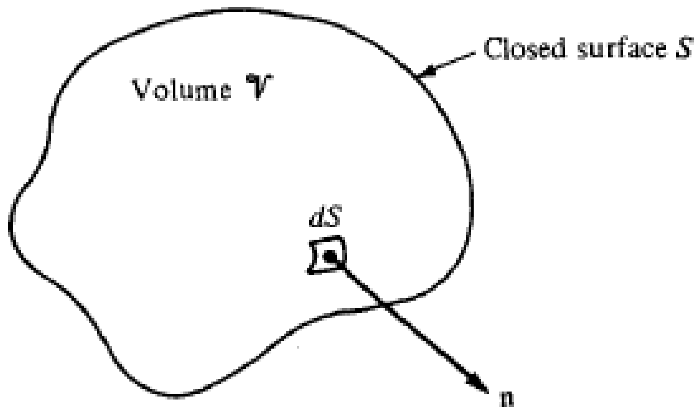
at a standard altitude of 36,000 ft, $p_\infty = 476\ \text{lb/ft}^2$ and $T_\infty = 391\ ^\circ\text{R}$.

$$\frac{p}{p_\infty} = \left(\frac{T}{T_\infty}\right)^{\gamma/(\gamma-1)} \Rightarrow T = T_\infty \left(\frac{p}{p_\infty}\right)^{(\gamma-1)/\gamma} = 391 \left(\frac{400}{476}\right)^{0.4/1.4} = \boxed{372\ ^\circ\text{R}}$$

GOVERNING EQUATIONS FOR INVISCID, COMPRESSIBLE FLOW

	Incompressible flow	Compressible flow
variables:	$\vec{V}$, p	$\vec{V}$, p , ρ , T
equations:	mass , $\overrightarrow{\text{momentum}}$	mass , $\overrightarrow{\text{momentum}}$, energy , state

Laplace's equation and
Bernoulli's equation,



$$\iiint_{\mathcal{V}} \rho \, d\mathcal{V} = \text{volume integral of a scalar } \rho \text{ over the volume } \mathcal{V} \text{ (the result is a scalar)}$$

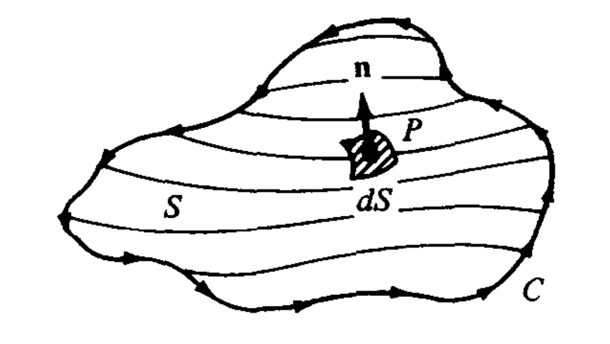
Let A be a vector field in space.
The volume integral over the volume V of the quantity A is written as:

$$\iiint_{\mathcal{V}} \mathbf{A} \, d\mathcal{V} = \text{volume integral of a vector } \mathbf{A} \text{ over the volume } \mathcal{V} \text{ (the result is a vector)}$$

Relations Between Line, Surface, and Volume Integrals

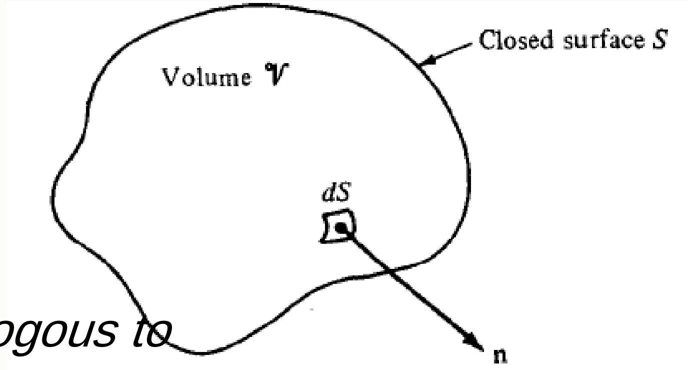
The line integral of A over *C is related to the* surface integral of A over S by *Stokes' theorem:*

$$\oint_C \mathbf{A} \cdot \mathbf{ds} = \iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{dS}$$



The surface and volume integrals of the vector field A are related through the *divergence theorem:*

$$\oiint_S \mathbf{A} \cdot \mathbf{dS} = \iiint_{\mathcal{V}} (\nabla \cdot \mathbf{A})\, d\mathcal{V}$$

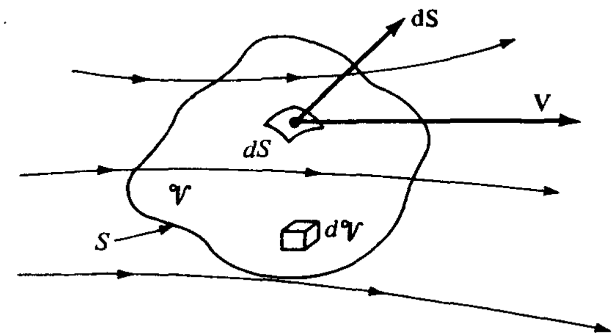


If *p represents a scalar field, a vector relationship analogous to above Equation* is given by the *gradient theorem:*

$$\oiint_S p\, \mathbf{dS} = \iiint_{\mathcal{V}} \nabla p\, d\mathcal{V}$$

CONTINUITY EQUATION

$$\boxed{\frac{\partial}{\partial t}\oiiint_{\mathcal{V}} \rho\, d\mathcal{V} + \oiint_{S} \rho \mathbf{V}\cdot \mathbf{dS} = 0}$$



FIX C.V.

Divergence

$$\oiiint_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\mathcal{V} + \oiint_{S} \rho \mathbf{V}\cdot \mathbf{dS} = 0$$

$$\oiint_{S} (\rho \mathbf{V})\cdot \mathbf{dS} = \oiiint_{\mathcal{V}} \nabla\cdot(\rho \mathbf{V})\, d\mathcal{V}$$

$$\oiiint_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\mathcal{V} + \oiiint_{\mathcal{V}} \nabla\cdot(\rho \mathbf{V})\, d\mathcal{V} = 0$$

$$\oiiint_{\mathcal{V}} \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \mathbf{V})\right] d\mathcal{V} = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \mathbf{V}) = 0}$$

MOMENTUM EQUATION

$$\mathbf{F} = m\mathbf{a}$$

A more general form

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{V})$$

Physical principle: *Force = time rate of change of momentum*

$$\mathbf{F} = \iiint_{\mathcal{V}} \rho \mathbf{f}\, d\mathcal{V} - \oiint_{S} p\, \mathbf{dS} + \mathbf{F}_{\text{viscous}}$$

$$\frac{d}{dt}(m\mathbf{V}) = \oiint_{S} (\rho \mathbf{V} \cdot \mathbf{dS})\mathbf{V} + \frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho \mathbf{V}\, d\mathcal{V}$$

$$\frac{d}{dt}(m\mathbf{V}) = \mathbf{F}$$

$$\boxed{\frac{\partial}{\partial t}\oiiint_{\mathcal{V}} \rho\mathbf{V}\, d\mathcal{V} + \oiint_{S} (\rho\mathbf{V}\cdot\mathbf{dS})\mathbf{V} = -\oiint_{S} p\,\mathbf{dS} + \oiiint_{\mathcal{V}} \rho\mathbf{f}\, d\mathcal{V} + \mathbf{F}_{\text{viscous}}}$$

Apply the gradient theorem:

$$-\oiint_{S} p\,\mathbf{dS} = -\oiiint_{\mathcal{V}} \nabla p\, d\mathcal{V}$$

$$\oiiint_{\mathcal{V}} \frac{\partial(\rho\mathbf{V})}{\partial t} d\mathcal{V} + \oiint_{S} (\rho\mathbf{V}\cdot\mathbf{dS})\mathbf{V} = -\oiiint_{\mathcal{V}} \nabla p\, d\mathcal{V} + \oiiint_{\mathcal{V}} \rho\mathbf{f}\, d\mathcal{V} + \mathbf{F}_{\text{viscous}}$$

$$\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$$

The *x component of momentum equation*

$$\iiint_{\mathcal{V}} \frac{\partial(\rho u)}{\partial t}\, d\mathcal{V} + \oint\!\!\oint_S (\rho \mathbf{V} \cdot \mathbf{dS}) u = -\iiint_{\mathbf{V}} \frac{\partial p}{\partial x}\, d\mathcal{V} + \iiint_{\mathcal{V}} \rho f_x \, d\mathcal{V} + (F_x)_{\text{viscous}}$$

Apply the divergence theorem: $\oint\!\!\oint_S (\rho \mathbf{V} \cdot \mathbf{dS}) u = \oint\!\!\oint_S (\rho u \mathbf{V}) \cdot \mathbf{dS} = \iiint_{\mathcal{V}} \nabla \cdot (\rho u \mathbf{V})\, d\mathcal{V}$

$$\iiint_{\mathcal{V}} \left[\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \mathbf{V}) + \frac{\partial p}{\partial x} - \rho f_x - (\mathcal{F}_x)_{\text{viscous}} \right] d\mathcal{V} = 0$$

$$\boxed{\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \mathbf{V}) = -\frac{\partial p}{\partial x} + \rho f_x + (\mathcal{F}_x)_{\text{viscous}}}$$

$$\boxed{\frac{\partial(\rho v)}{\partial t} + \nabla \cdot (\rho v \mathbf{V}) = -\frac{\partial p}{\partial y} + \rho f_y + (\mathcal{F}_y)_{\text{viscous}}}$$

$$\boxed{\frac{\partial(\rho w)}{\partial t} + \nabla \cdot (\rho w \mathbf{V}) = -\frac{\partial p}{\partial z} + \rho f_z + (\mathcal{F}_z)_{\text{viscous}}}$$

ENERGY EQUATION

$$\delta q + \delta w = de$$

Let us apply the first law to the fluid flowing through the fixed control volume:

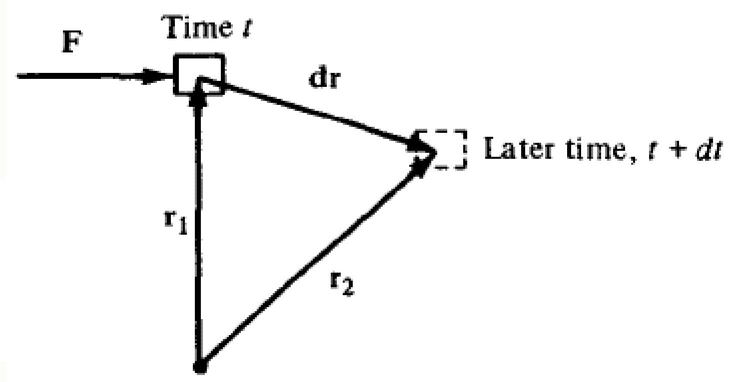
B_1 = rate of heat added to fluid inside control volume from surroundings
B_2 = rate of work done on fluid inside control volume
B_3 = rate of change of energy of fluid as it flows through control volume

$$B_1 + B_2 = B_3$$

$$\text{Rate of volumetric heating} = \oiiint_{\mathcal{V}} \dot{q}\rho \, d\mathcal{V}$$

$$B_1 = \oiiint\limits_{\mathcal{V}} \dot{q}\rho \, d\mathcal{V} + \dot{Q}_{\text{viscous}}$$

$$\text{Rate of doing work on moving body} = \mathbf{F} \cdot \mathbf{V}$$



$$\begin{array}{r}\text{Rate of work done on fluid inside} \\ \mathcal{V} \text{ due to pressure force on } S\end{array} = -\oiint\limits_{S} (p \, \mathbf{dS}) \cdot \mathbf{V}$$

$$\begin{array}{r}\text{Rate of work done on fluid} \\ \text{inside } \mathcal{V} \text{ due to body forces}\end{array} = \oiiint\limits_{\mathcal{V}} (\rho \mathbf{f} \, d\mathcal{V}) \cdot \mathbf{V}$$

$$B_2 = -\oiint\limits_{S} p\mathbf{V} \cdot \mathbf{dS} + \oiiint\limits_{\mathcal{V}} \rho(\mathbf{f} \cdot \mathbf{V}) \, d\mathcal{V} + \dot{W}_{\text{viscous}}$$

$$\begin{array}{r}\text{Net rate of flow of total}\\ \text{energy across control surface}\end{array} = \iint_S (\rho \mathbf{V} \cdot \mathbf{dS}) \left(e + \frac{V^2}{2}\right)$$

$$\iiint_{\mathcal{V}} \rho \left(e + \frac{V^2}{2}\right) d\mathcal{V}$$

The total energy contained in the elemental volume *dV is* ***ρ(e + V²/2) dV:***

$$\iiint_{\mathcal{V}} \rho \left(e + \frac{V^2}{2}\right) d\mathcal{V}$$

$$\begin{array}{r}\text{Time rate of change of total energy}\\ \text{inside } \mathcal{V} \text{ due to transient variations}\\ \text{of flow-field variables}\end{array} = \frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho \left(e + \frac{V^2}{2}\right) d\mathcal{V}$$

$$B_3 = \frac{\partial}{\partial t} \oiiint_{\mathcal{V}} \rho \left(e + \frac{V^2}{2} \right) d\mathcal{V} + \oiint_S (\rho \mathbf{V} \cdot \mathbf{dS}) \left(e + \frac{V^2}{2} \right)$$

$$\boxed{\begin{aligned} &\oiiint_{\mathcal{V}} \dot{q} \rho \, d\mathcal{V} + \dot{Q}_{\text{viscous}} - \oiint_S p \mathbf{V} \cdot \mathbf{dS} + \oiiint_{\mathcal{V}} \rho (\mathbf{f} \cdot \mathbf{V}) \, d\mathcal{V} + \dot{W}_{\text{viscous}} \\ &\quad = \frac{\partial}{\partial t} \oiiint_{\mathcal{V}} \rho \left(e + \frac{V^2}{2} \right) d\mathcal{V} + \oiint_S \rho \left(e + \frac{V^2}{2} \right) \mathbf{V} \cdot \mathbf{dS} \end{aligned}}$$

Applying the divergence theorem

$$\boxed{\begin{aligned} \frac{\partial}{\partial t} \left[\rho \left(e + \frac{V^2}{2} \right) \right] + \nabla \cdot \left[\rho \left(e + \frac{V^2}{2} \right) \mathbf{V} \right] &= \rho \dot{q} - \nabla \cdot (p \mathbf{V}) + \rho (\mathbf{f} \cdot \mathbf{V}) \\ &\quad + \dot{Q}'_{\text{viscous}} + \dot{W}'_{\text{viscous}} \end{aligned}}$$

If the flow is steady ($\partial/\partial t = 0$),

inviscid ($\dot{Q}_{\text{viscous}} = 0$ and $W_{\text{viscous}} = 0$)

adiabatic (no heat addition, $\dot{q} = 0$),

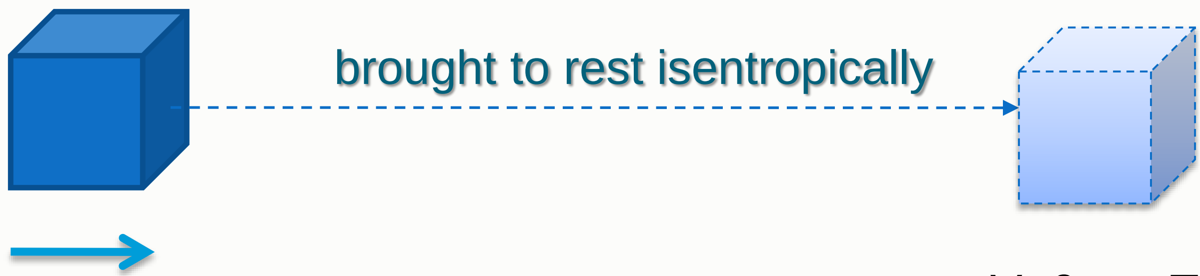
without body forces ($\mathbf{f} = 0$)

$$\boxed{\oiint_S \rho\left(e + \frac{V^2}{2}\right)\mathbf{V}\cdot\mathbf{dS} = -\oiint_S p\mathbf{V}\cdot\mathbf{dS}}$$

$$\boxed{\nabla\cdot\left[\rho\left(e + \frac{V^2}{2}\right)\mathbf{V}\right] = -\nabla\cdot(p\mathbf{V})}$$

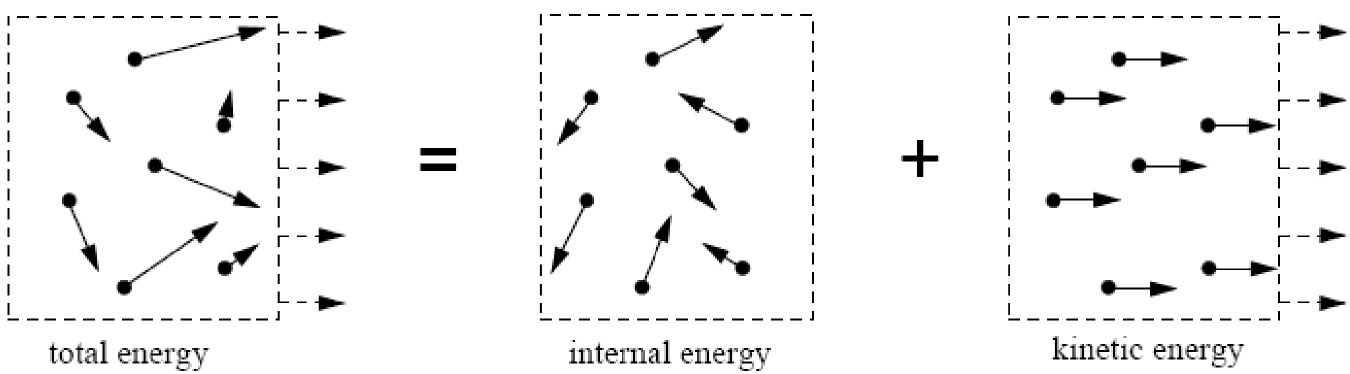
Static and stagnation (Total) properties

Consider a fluid element passing through a given point in a flow where the local pressure, temperature, density, Mach number, and velocity are *p, T, ρ, M, and* V, respectively



V, p, T, ρ

$V=0, p_0, T_0, \rho_0$



$$e_o = e + \frac{1}{2}V^2$$

$$h_o = h + \frac{1}{2}V^2 = e + \frac{p}{\rho} + \frac{1}{2}V^2 = e_o + \frac{p}{\rho}$$

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$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} = 0$$

$$\rho \frac{D(p/\rho)}{Dt} = \rho \frac{\rho Dp/Dt - p D\rho/Dt}{\rho^2} = \frac{Dp}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt}$$

$$\rho \frac{D(p/\rho)}{Dt} = \frac{Dp}{Dt} + p\nabla \cdot \mathbf{V} = \frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p + p \nabla \cdot \mathbf{V}$$

$$\nabla \cdot p\mathbf{V} \equiv p\nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla p$$

$$\rho \frac{D(e + V^2/2)}{Dt} = -\nabla \cdot p\mathbf{V}$$

$$\rho \frac{D}{Dt}\left(\overbrace{e + \frac{p}{\rho}}^{=h} + \frac{V^2}{2}\right) = -\cancel{p\nabla \cdot \mathbf{V}} - \cancel{\mathbf{V} \cdot \nabla p} + \frac{\partial p}{\partial t} + \cancel{\mathbf{V} \cdot \nabla p} + \cancel{p\nabla \cdot \mathbf{V}}$$

$$\rho \frac{D(h + V^2/2)}{Dt} = \frac{\partial p}{\partial t}$$

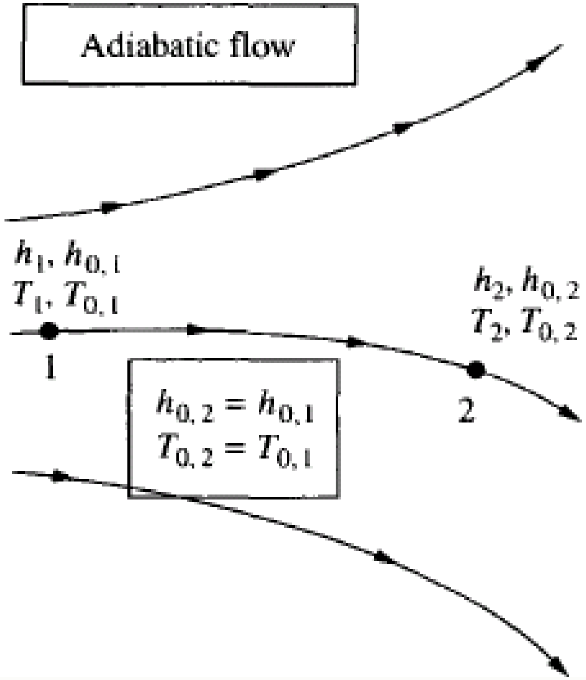
If the flow is steady, $\partial p/\partial t = 0$

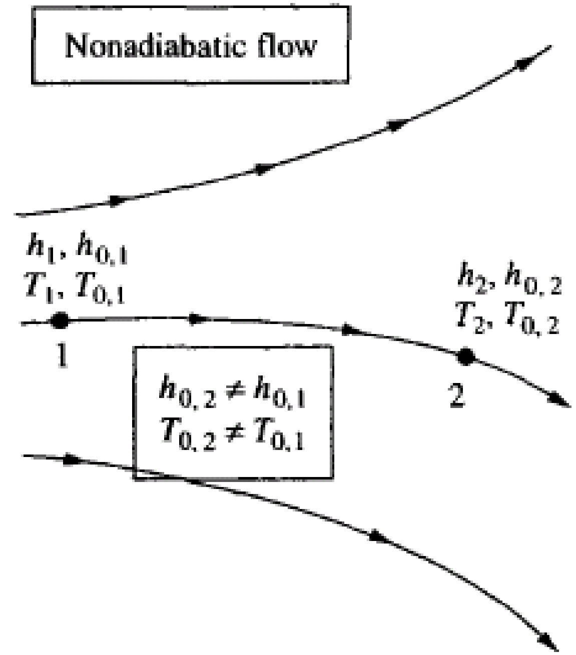
$$\rho \frac{D(h + V^2/2)}{Dt} = 0 \quad \Rightarrow \quad \boxed{h + \frac{V^2}{2} = \text{const}}$$

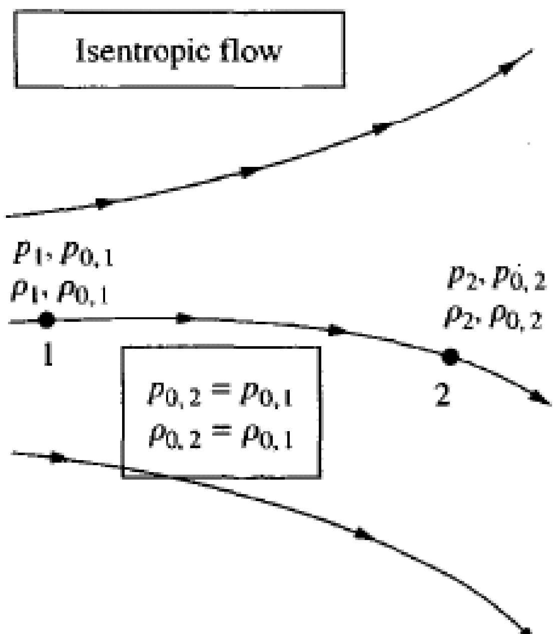
$$\boxed{h + \frac{V^2}{2} = h_0} \quad \Rightarrow \quad \boxed{h_0 = \text{const}}$$

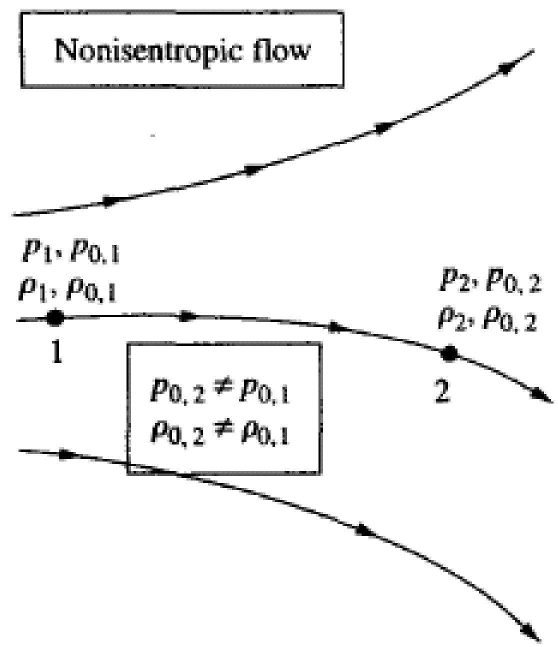
For a calorically perfect gas, $h_0 = c_p T_0$. $\Rightarrow$ $\boxed{T_0 = \text{const}}$

In an adiabatic flow Stagnation(Total) temperature remains constant


Adiabatic flow
$h_1, h_{0,1}$
$T_1, T_{0,1}$
1
$h_2, h_{0,2}$
$T_2, T_{0,2}$
2
$h_{0,2} = h_{0,1}$
$T_{0,2} = T_{0,1}$


Nonadiabatic flow
$h_1, h_{0,1}$
$T_1, T_{0,1}$
1
$h_2, h_{0,2}$
$T_2, T_{0,2}$
2
$h_{0,2} \neq h_{0,1}$
$T_{0,2} \neq T_{0,1}$


Isentropic flow
$p_1, p_{0,1}$
$\rho_1, \rho_{0,1}$
1
$p_2, p_{0,2}$
$\rho_2, \rho_{0,2}$
2
$p_{0,2} = p_{0,1}$
$\rho_{0,2} = \rho_{0,1}$


Nonisentropic flow
$p_1, p_{0,1}$
$\rho_1, \rho_{0,1}$
1
$p_2, p_{0,2}$
$\rho_2, \rho_{0,2}$
2
$p_{0,2} \neq p_{0,1}$
$\rho_{0,2} \neq \rho_{0,1}$

EXAMPLE

At a point in an airflow the pressure, temperature, and velocity are 1 atm, 320 K, and 1000 m/s. Calculate the total temperature and total pressure at this point.

$$h + \frac{V^2}{2} = h_0 \quad \Rightarrow \quad c_p T + \frac{V_2}{2} = c_p T_0$$

$$h = c_p T$$

$$c_p = \frac{\gamma R}{\gamma - 1}, \quad \rightarrow \quad T_0 = T + \frac{V^2}{2c_p} = T + \left(\frac{\gamma - 1}{2\gamma R}\right) V^2$$

$$T_0 = 320 + \left[\frac{0.4}{2(1.4)(287)}\right](1000)^2 = 320 + 497.8$$

$$T_0 = \boxed{817.8\ \text{K}}$$

By definition, the total pressure is the pressure that would exist if the flow at th point were slowed isentropically to zero velocity.

Hence, we can use the isentropic relations in to relate total to static conditions.

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

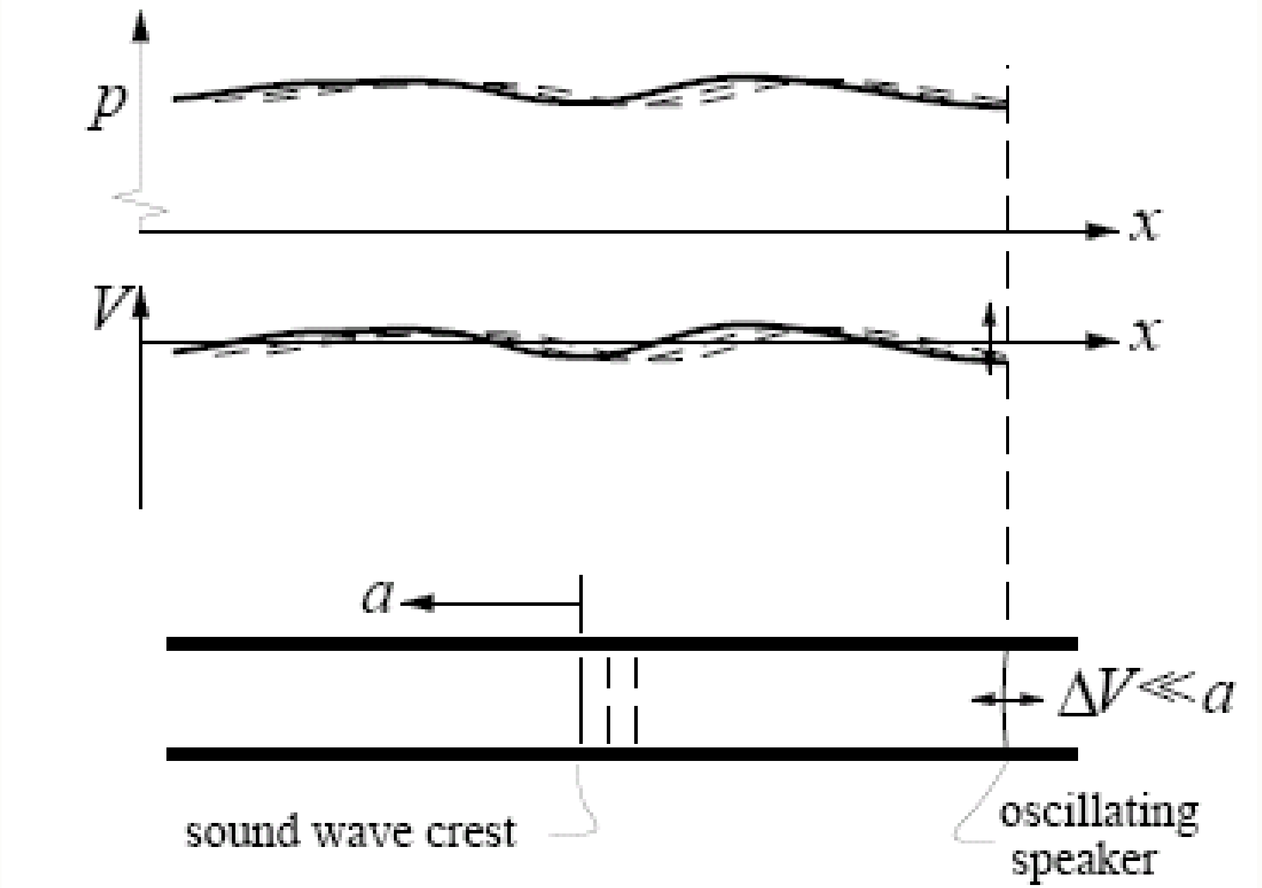
$$p_0 = p\left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}} = (1 \text{ atm})\left(\frac{817.8}{320}\right)^{\frac{1.4}{0.4}}$$

$$P_0 = \boxed{26.7 \text{ atm}}$$

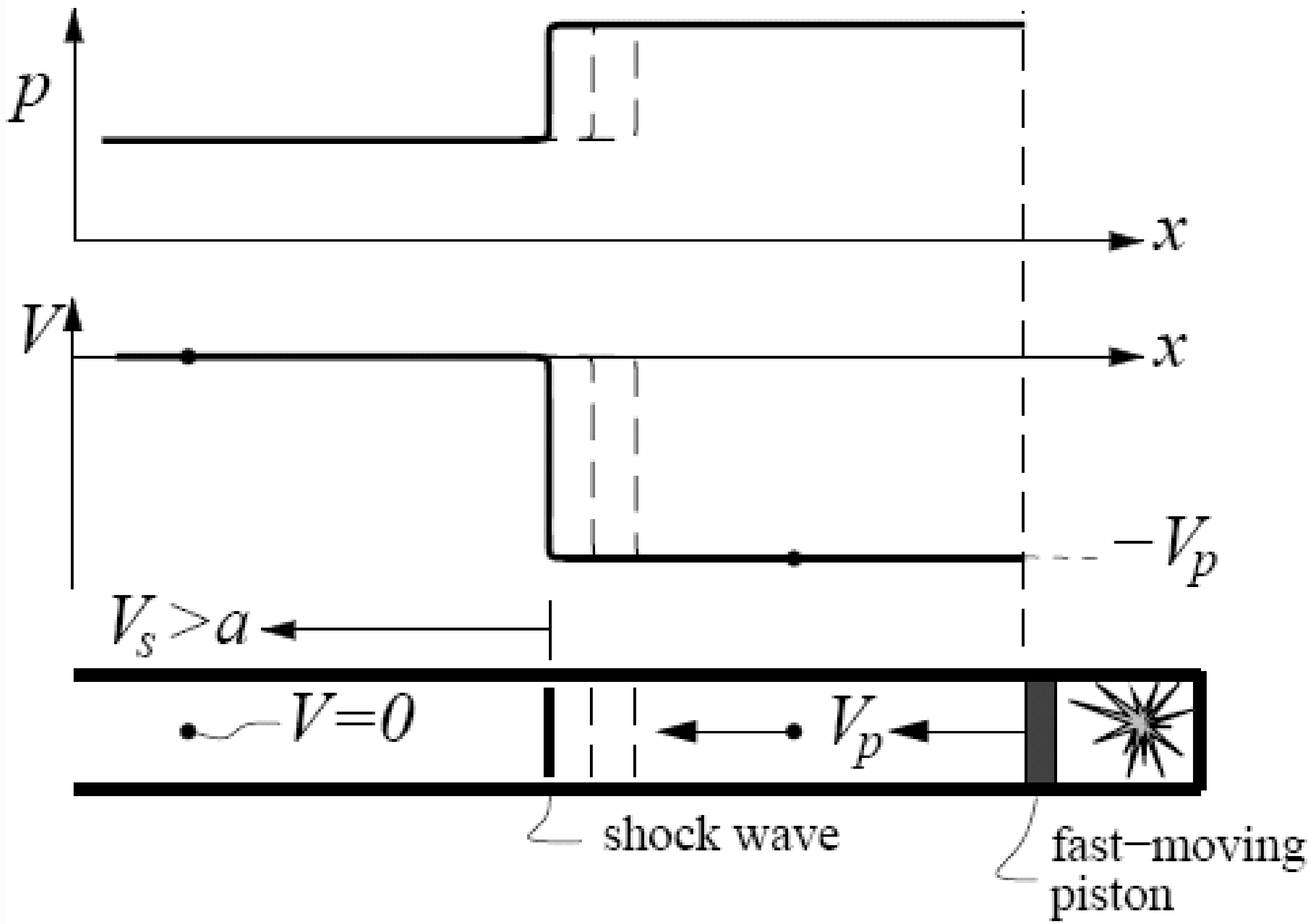
Introduction to Shock Waves

Compressibility of a fluid allows the existence of waves, which are variations in , p, and h (or temperature T), which self propagate through the fluid at some speed.

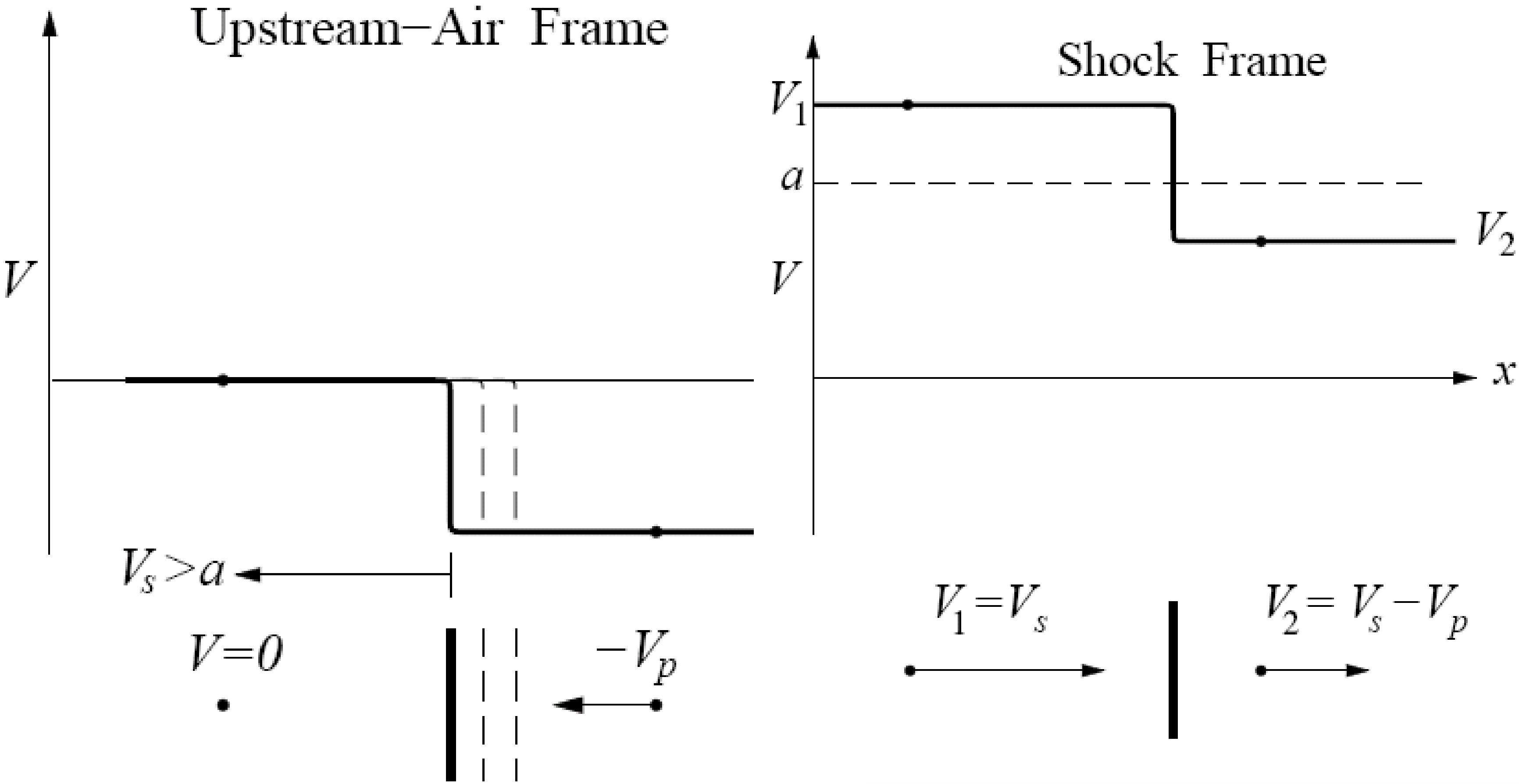
Ordinary sound consists of very small variations which move at the speed of sound a

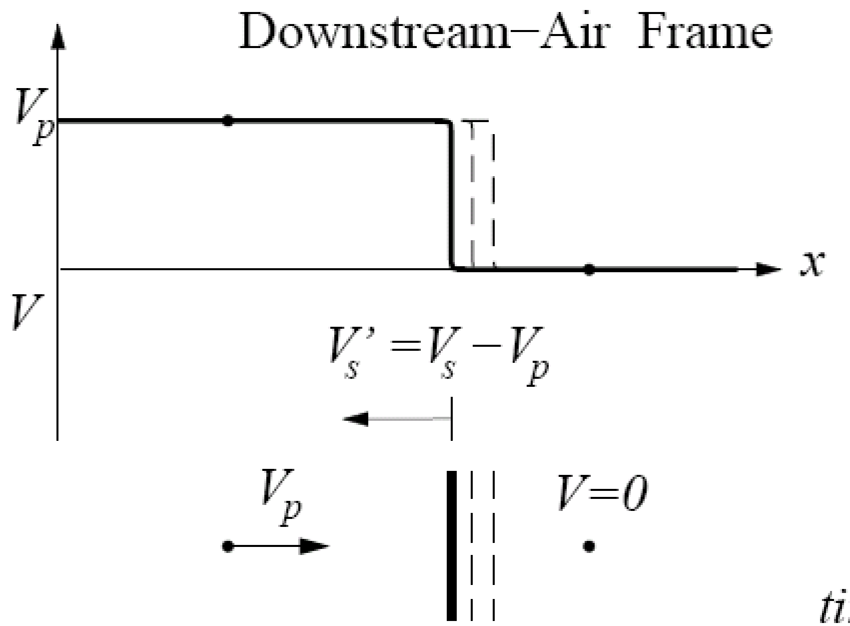


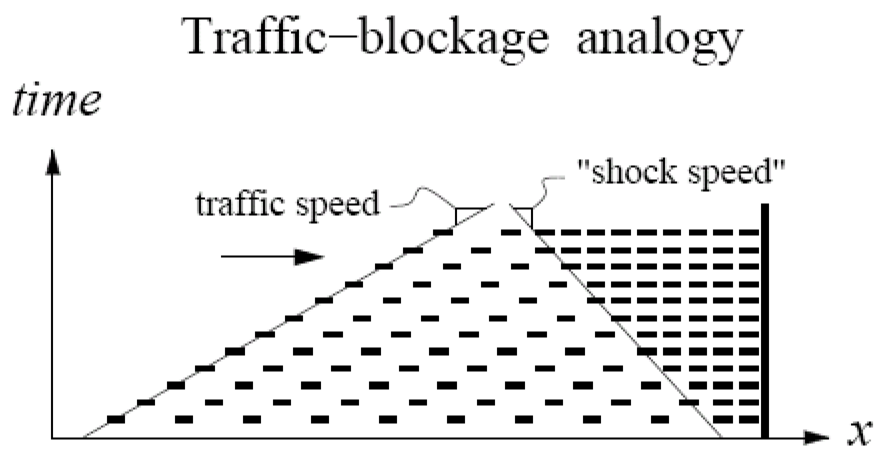
A shock wave has a finite variation in flow quantities and moves at a larger speed $V_s > a$



Frames (Upstream, Shock, Downstream)







This situation is closely analogous to how a traffic blockage propagates backward against the oncoming traffic.

Dissipation in Shock

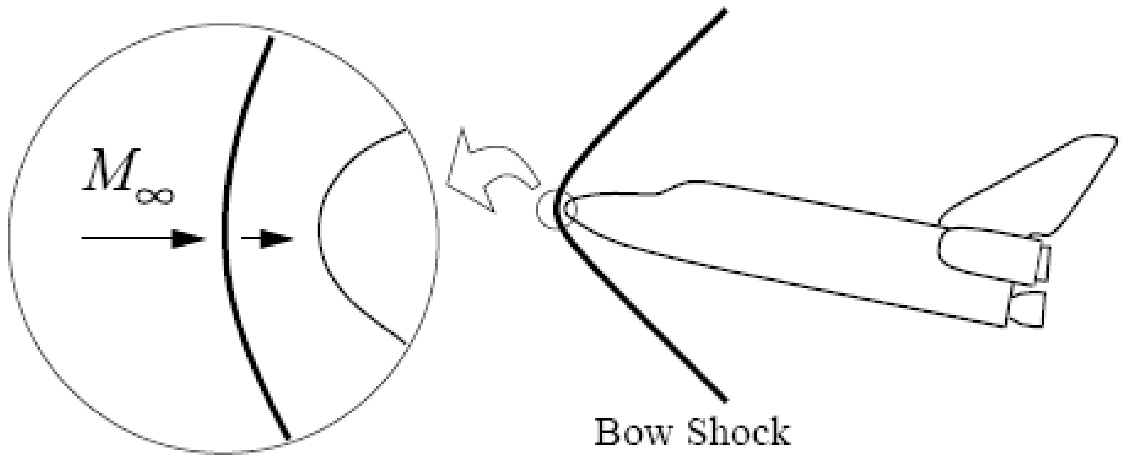
The flow passing through a shock wave undergoes an adiabatic process, since there is no heat being supplied (there's nothing there to provide heat!).

But because a shock wave is typically very thin less than 1 micron at sea level — there are strong viscous forces acting on the fluid passing through it, so the process is irreversible. Therefore, the stagnation quantities have the following relations across a shock wave:

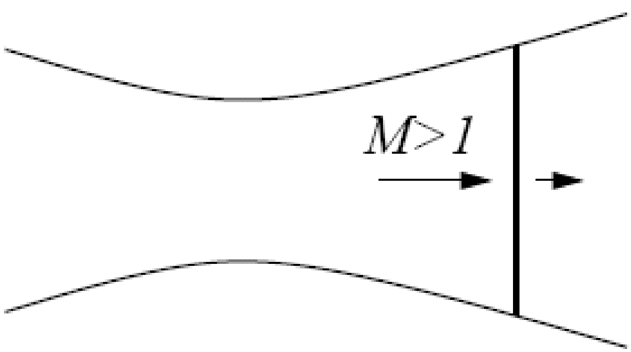
$$\boxed{\begin{aligned} h_{o_1} &= h_{o_2} \\ \rho_{o_1} &> \rho_{o_2} \\ p_{o_1} &> p_{o_2} \end{aligned}}$$

Normal Shock Waves

A normal shock wave appears in many types of supersonic flows.



Bow Shock



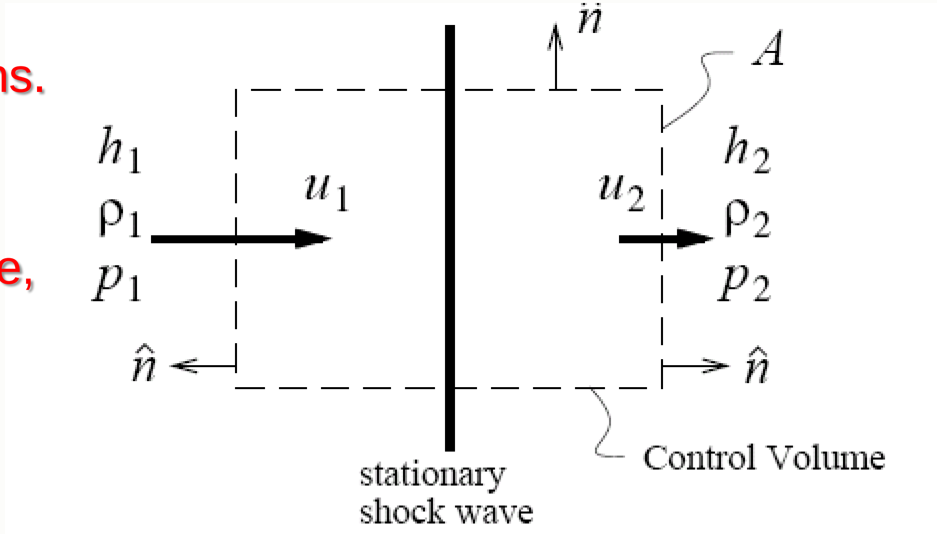
Nozzle Shock

Shock jump relations

We examine the flow in the frame in which the shock is stationary.

A control volume is defined straddling the shock. The flow in the shock has the following properties:

1. Flow is steady, so ∂()/∂t = 0 in all equations.
2. Flow is adiabatic, so $q^{\cdot} = 0$.
3. Body forces such as gravity are negligible,



The shock wave is a thin region of highly viscous flow.
The flow through the shock is adiabatic but nonsentropic

Mass continuity

$$\oiint \rho \vec{V} \cdot \hat{n}\, dA = 0$$

$$-\rho_1 u_1 A + \rho_2 u_2 A = 0$$

$$\boxed{\rho_1 u_1 = \rho_2 u_2}$$

x-Momentum

$$\oiint \rho \vec{V} \cdot \hat{n}\, u\, dA + \oiint p \hat{n} \cdot \hat{\imath}\, dA = 0$$

$$-\rho_1 u_1^2 A + \rho_2 u_2^2 A - p_1 A + p_2 A = 0$$

$$\boxed{\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2}$$

Energy

$$\oiint \rho \vec{V} \cdot \hat{n}\, h_o\, dA = 0 \rightarrow -\rho_1 u_1 h_{o_1} A + \rho_2 u_2 h_{o_2} A = 0$$

$$\rightarrow h_{o_1} = h_{o_2} \rightarrow \boxed{h_1 + \frac{1}{2} u_1^2 = h_2 + \frac{1}{2} u_2^2}$$

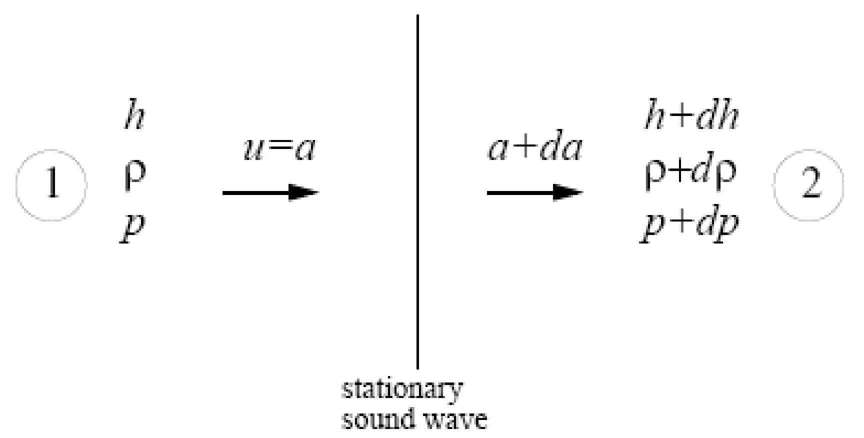
Equation of State

$$\boxed{p_2 = \frac{\gamma - 1}{\gamma} \rho_2 h_2}$$

Speed of Sound

we first consider an infinitesimally weak shock wave, also known as a sound wave. Because the velocity gradients and hence the viscous action is small, the flow process through the wave is isentropic.

The objective here is to determine this a in terms of the other variables by the governing applying equations



The mass equation

$$\rho a \;=\; (\rho + d\rho)(a + da) \;=\; \rho a \;+\; a\,d\rho \;+\; \rho\,da$$

$$\boxed{da \;=\; -\frac{a}{\rho}\,d\rho}$$

x-momentum

$$\rho a^2 \;+\; p \;=\; (\rho + d\rho)(a + da)^2 \;+\; (p + dp) \;=\; \rho a^2 \;+\; a^2\,d\rho \;+\; 2a\rho\,da \;+\; p \;+\; dp$$

$$\boxed{0 \;=\; 2a\rho\,da \;+\; a^2\,d\rho \;+\; dp}$$

Combining the mass and momentum equations

$$0 = 2a\rho\left(-\frac{a}{\rho}\,d\rho\right) + a^2\,d\rho + dp \longrightarrow 0 = -a^2\,d\rho + dp$$

$$\boxed{a^2 = \frac{dp}{d\rho}}$$

We could now relate p and ρ and thus get dp/dρ using the energy and state equations and the above equation. But an algebraically simpler approach is to use one of the isentropic relations instead, which are valid for this weak wave. The simplest relation for this purpose is

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} \longrightarrow \frac{p}{\rho^{\gamma}} = \text{const} = c \longrightarrow p = c\rho^{\gamma}$$

$$\frac{dp}{d\rho} = c\gamma\rho^{\gamma-1} = \left(\frac{p}{\rho^{\gamma}}\right)\gamma\rho^{\gamma-1} = \frac{\gamma p}{\rho} \longrightarrow \frac{dp}{p} = \gamma\,\frac{d\rho}{\rho}$$

$$\frac{dp}{d\rho} = \gamma \frac{p}{\rho} = \gamma RT$$

$$a^2 = \frac{dp}{d\rho}$$

$$\boxed{a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma RT}}$$

it clearly states that the *speed of sound in a calorically perfect gas is a function of temperature only.*

Recall the definition of compressibility

$$\tau_s = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_s$$

$$v = 1/\rho \rightarrow dv = -d\rho/\rho^2$$

$$\tau_s = -\rho\left[-\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial p}\right)_s\right]$$

$$= \frac{1}{\rho(\partial p/\partial \rho)_s}$$

$$\tau_s = \frac{1}{\rho a^2} \rightarrow a = \sqrt{\frac{1}{\rho \tau_s}}$$

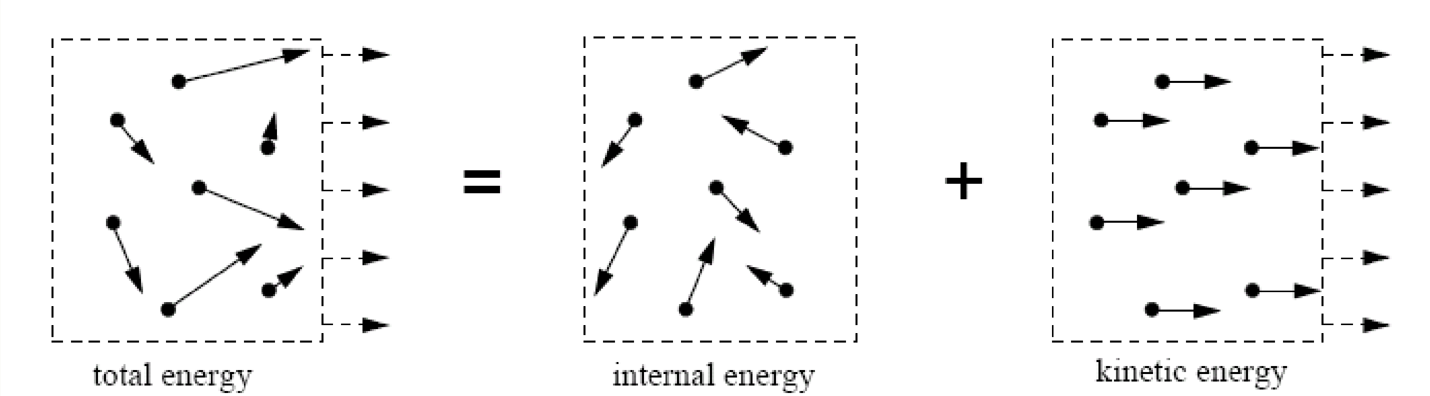
case of an incompressible fluid $\tau_s = 0$

$\rightarrow$ Mach number M = V/a, is zero.

In regard to additional physical meaning of the Mach number, consider a fluid element moving along a streamline. The kinetic and internal energies per unit mass are *$V^2/2$ and e, respectively.*

$$\frac{V^2/2}{e} = \frac{V^2/2}{c_v T} = \frac{V^2/2}{RT/(\gamma - 1)} = \frac{(\gamma/2)V^2}{a^2/(\gamma - 1)} = \frac{\gamma(\gamma - 1)}{2} M^2$$

In other words, the Mach number is a measure of the directed motion of the gas compared with the random thermal motion of the molecules.



EXAMPLE

Consider an airplane flying at a velocity of 250 m/s. Calculate its Mach number if it is flying at a standard altitude of

(a) sea level, ***(b) 5 km,*** ***(c) 10 km.***

for the standard atmosphere, at sea level, $T_\infty = 288$ K.

$$a_\infty = \sqrt{\gamma RT} = \sqrt{(1.4)(287)(288)} = 340.2 \text{ m/s} \rightarrow M_\infty = \frac{V_\infty}{a_\infty} = \frac{250}{340.2} = \boxed{0.735}$$

(b) At 5 km, $\rightarrow$ $T_\infty = 255.7$

$$a_\infty = \sqrt{(1.4)(287)(255.7)} = 320.5 \text{ m/s} \rightarrow M_\infty = \frac{V_\infty}{a_\infty} = \frac{250}{320.2} = \boxed{0.78}$$

(c) 10 km $\rightarrow$ $T_\infty = 223.3.$

$$a_\infty = \sqrt{(1.4)(287)(223.3)} = 299.5 \text{ m/s} \rightarrow M_\infty = \frac{V_\infty}{a_\infty} = \frac{250}{299.5} = \boxed{0.835}$$

EXAMPLE

Calculate the ratio of kinetic energy to internal energy at a point in an airflow where the Mach number is:
(a) *M = 2,* **(b) *M = 20.***

$$(a)\ \frac{V^2/2}{e} = \frac{\gamma(\gamma-1)}{2}M^2 = \frac{(1.4)(0.4)}{2}(2)^2 = \boxed{1.12}$$

$$(b)\ \frac{V^2/2}{e} = \frac{\gamma(\gamma-1)}{2}M^2 = \frac{(1.4)(0.4)}{2}(20)^2 = \boxed{112}$$

Examining these two results, we see that at Mach 2, the kinetic energy and internal energy are about the same, whereas at the large hypersonic Mach number of 20, the kinetic energy is more than a hundred times larger than the internal energy. This is one characteristic of hypersonic flows—high ratios of kinetic to internal energy.

SPECIAL FORMS OF THE ENERGY EQUATION

we elaborate upon the energy equation for adiabatic flow,

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$$

one-dimensional flow,

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

$$h = c_p T \rightarrow c_p T_1 + \frac{u_1^2}{2} = c_p T_2 + \frac{u_2^2}{2} \rightarrow \frac{\gamma R T_1}{\gamma - 1} + \frac{u_1^2}{2} = \frac{\gamma R T_2}{\gamma - 1} + \frac{u_2^2}{2}$$

$$\frac{a_1^2}{\gamma - 1} + \frac{u_1^2}{2} = \frac{a_2^2}{\gamma - 1} + \frac{u_2^2}{2}$$

If we consider point to be a stagnation point,

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{a_0^2}{\gamma - 1}$$

sonic flow, where $u = a^*$

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{a^{*2}}{\gamma - 1} + \frac{a^{*2}}{2} \longrightarrow \frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{\gamma + 1}{2(\gamma - 1)} a^{*2}$$

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$$c_p T + \frac{u^2}{2} = c_p T_0$$

in a steady, adiabatic, inviscid flow,

$$c_p T_1 + \frac{u_1^2}{2} = c_p T_2 + \frac{u_2^2}{2} = c_p T_0 = \text{const}$$

$$\frac{T_0}{T} = 1 + \frac{u^2}{2c_p T} = 1 + \frac{u^2}{2\gamma RT/(\gamma - 1)} = 1 + \frac{u^2}{2a^2/(\gamma - 1)}$$

$$= 1 + \frac{\gamma - 1}{2}\left(\frac{u}{a}\right)^2$$

$$\boxed{\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2}$$

Total pressure p_0 *and total density* ρ_o,
These definitions involve an *isentropic compression of the flow to* zero velocity.

$$\frac{p_0}{p} = \left(\frac{\rho_0}{\rho}\right)^{\gamma} = \left(\frac{T_0}{T}\right)^{\gamma/(\gamma-1)}$$

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)}$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{1/(\gamma-1)}$$

The above equations are very important;
they should be branded on your mind.

Consider a point in a general flow where the velocity is exactly sonic (i.e., where *M = 1*).

$$\frac{T^*}{T_0} = \frac{2}{\gamma + 1}$$

$$\frac{p^*}{p_0} = \left(\frac{2}{\gamma + 1}\right)^{\gamma/(\gamma-1)}$$

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma + 1}\right)^{1/(\gamma-1)}$$

For $\gamma = 1.4$, these ratios are

$$\frac{T^*}{T_0} = 0.833 \qquad \frac{p^*}{p_0} = 0.528 \qquad \frac{\rho^*}{\rho_0} = 0.634$$

EXAMPLE

Consider a point in an airflow where the local Mach number, static pressure, and static temperature are 3.5, 0.3 atm, and 180 K, respectively. Calculate the local values of *po, To,*

M = 3.5, →

$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)}$$

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

→

$$p_0/p = 76.27$$

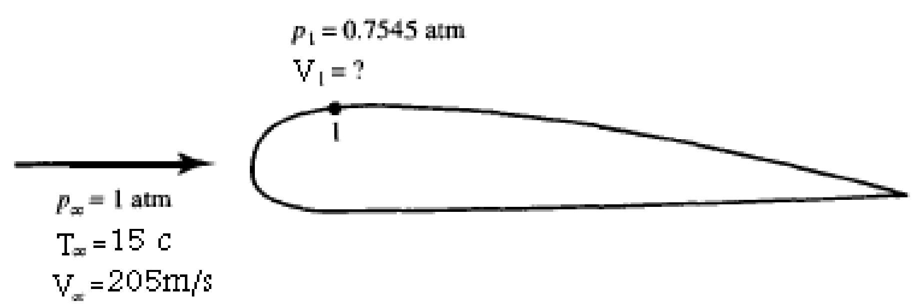
$$T_0/T = 3.45.$$

$$p_0 = \left(\frac{p_0}{p}\right) p = 76.27(0.3\ \text{atm}) = \boxed{22.9\ \text{atm}}$$

$$T_0 = \frac{T_0}{T} T = 3.45(180) = \boxed{621\ \text{K}}$$

EXAMPLE

calculation of the velocity at a point on an airfoil when we were given the pressure at that point and the freestream velocity and pressure and temperature is required. Calculate the velocity using a) incompressible , b) compressible assumptions for the air flow



a) incompressible

At standard sea level conditions, $\rho_\infty = 1.23$ kg/m^3 and $p_\infty = 1.01 \times 10^5$ N/m^2. Hence,

$$p_\infty + \tfrac{1}{2}\rho V_\infty^2 = p + \tfrac{1}{2}\rho V^2 \longrightarrow V = \sqrt{\frac{2(p_\infty - p)}{\rho} + V_\infty^2}$$

$$= \sqrt{\frac{2(1.01 - 0.7545) \times 10^5}{1.23} + (205)^2} = \boxed{289.1 \text{ m/s}}$$

b) compressible

T_∞=15 c=288 K ⟶ a_∞=340.2 m/s ⟶ $M_\infty = V_\infty / a_\infty$=340.2 m/s=0.6

$$p_{0,\infty} = \frac{p_{0,\infty}}{p_\infty} p_\infty = (1.276)(1) = 1.276 \text{ atm}$$

Recall that for an isentropic flow, the total pressure is constant throughout the flow.

$$p_{0,1} = p_{0,\infty} = 1.276 \text{ atm}$$

$$\frac{p_{0,1}}{p_1} = \frac{1.276}{0.7545} = 1.691 \longrightarrow M_1 = 0.9$$

The flow is isentropic, $\frac{p_1}{p_\infty} = \left(\frac{T_1}{T_\infty}\right)^{\gamma/(\gamma-1)} \longrightarrow T_1 = T_\infty \left(\frac{p_1}{p_\infty}\right)^{(\gamma-1)/\gamma} \longrightarrow$ T1=266 K

$a_1 = \sqrt{\gamma R T_1} \longrightarrow a_1 =$ 327 m/s $\longrightarrow V_1 = M_1 a_1 \longrightarrow \boxed{\text{V1=294.3 m/s}}$

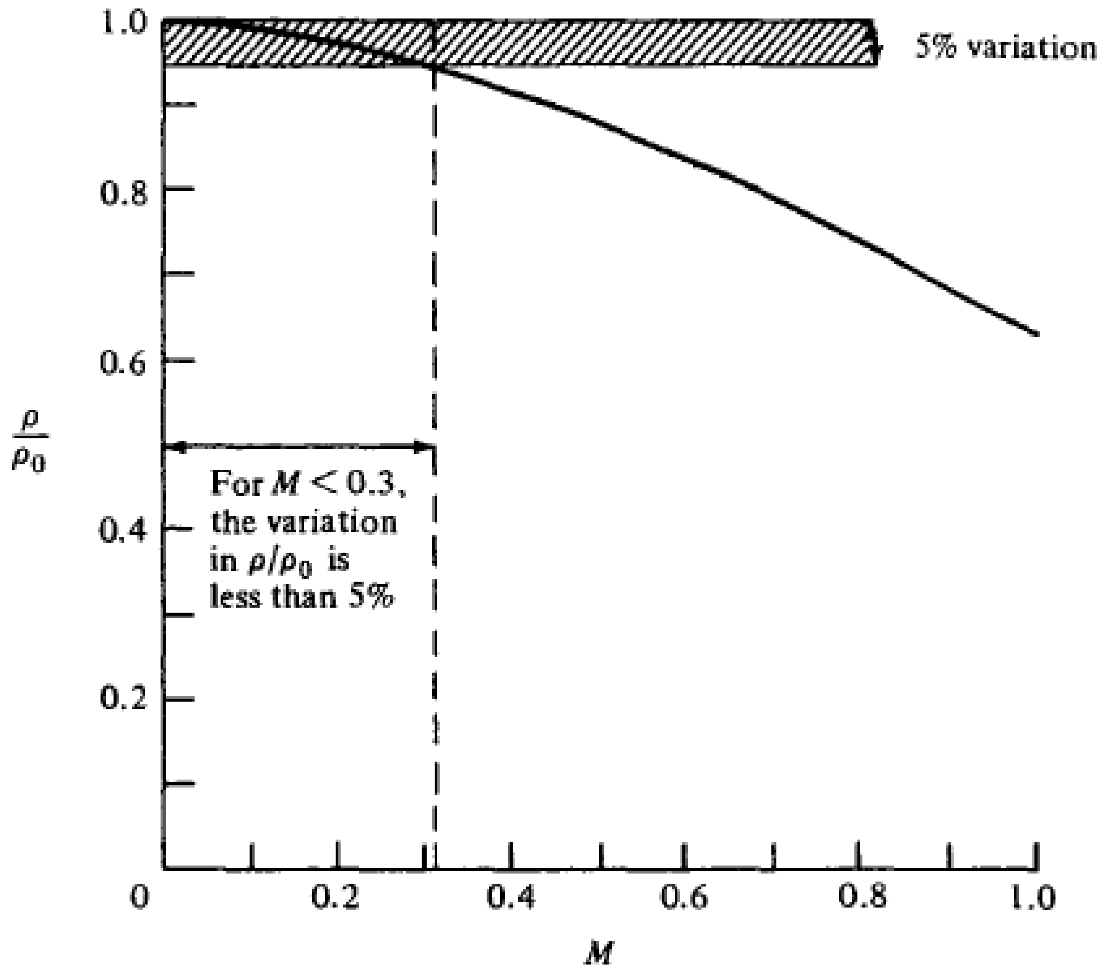
WHEN IS A FLOW COMPRESSIBLE?

There is no specific answer to this question;

We have stated several times in the preceding chapters the rule of thumb that a flow can be reasonably assumed to be incompressible when M < 0.3, whereas it should be considered compressible when M > 0.3.
There is nothing magic about the value 0.3, but it is a convenient dividing line. We are now in a position to add substance to this rule of thumb.

Consider a fluid element initially at rest, say, an element of the air around you. The density of this gas at rest is p 0 . Let us now accelerate this fluid element isentropically to some velocity V and Mach number *M:*

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{1/(\gamma-1)}$$



To obtain additional insight into the significance of Figure:

How the ratio ρ/ρo affects the change in pressure associated with a given change in velocity?

Euler's equation:

$$dp = -\rho V\,dV \quad \rightarrow \quad \frac{dp}{p} = -\frac{\rho}{p}V^2\frac{dV}{V}$$

If we now assume that the density is constant, say, equal to ρo

$$\left(\frac{dp}{p}\right)_0 = -\frac{\rho_0}{p}V^2\frac{dV}{V}$$

$$\frac{dp}{p} = -\frac{\rho}{p}V^2\frac{dV}{V}$$

$$\frac{dp/p}{(dp/p)_0} = \frac{\rho}{\rho_0}$$

consider the flow of air through a nozzle starting in the reservoir at nearly zero velocity and standard sea level values of *po = 2116 lb/ft2* and *To = 510°R, and expanding to a velocity of 350 ft/s at the nozzle exit. The* pressure at the nozzle exit will be calculated assuming first incompressible flow and then compressible flow.

Incompressible flow: From Bernoulli's equation,

$$p = p_0 - \tfrac{1}{2}\rho V^2 = 2116 - \tfrac{1}{2}(0.002377)(350)^2 = \boxed{1970 \text{ lb/ft}^2}$$

Compressible flow: From the energy equation, Equation (8.30), with c_p = 6006[(ft) (lb)/slug°R] for air,

$$T = T_0 - \frac{V^2}{2c_p} = 519 - \frac{(350)^2}{2(6006)} = 508.8°\text{R}$$

From Equation (7.32),

$$\frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{\gamma/(\gamma-1)} = \left(\frac{508.8}{519}\right)^{3.5} = 0.9329$$

$$p = 0.9329 p_0 = 0.9329(2116) = \boxed{1974 \text{ lb/ft}^2}$$

Note that the two results are almost the same, with the compressible value of pressure only 0.2 percent higher than the incompressible value.

Also, note that the Mach number at the exit is 0.317

On the other hand, if this flow were to continue to expand to a velocity of 900 ft/s, a repeat of the above calculation yields the following results for the static pressure at the end of the expansion:

Incompressible (Bernoulli's equation): $p = 1153\ \text{lb/ft}^2$

Compressible: $p = 1300\ \text{lb/ft}^2$

Here, the difference between the two sets of results is considerable—a 13 percent difference. In this case, the Mach number at the end of the expansion is 0.86.

CALCULATION OF NORMAL SHOCK-WAVE PROPERTIES

$p/\rho = a^2/\gamma$

$$\boxed{\begin{aligned} \rho_1 u_1 &= \rho_2 u_2 \\ \rho_1 u_1^2 + p_1 &= \rho_2 u_2^2 + p_2 \end{aligned}}$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

$$p_2 = \frac{\gamma - 1}{\gamma}\rho_2 h_2$$

$$u_1 - u_2 = \frac{1}{\gamma}\left(\frac{a_2^2}{u_2} - \frac{a_1^2}{u_1}\right)$$

$$h_1 + \frac{1}{2}u_1^2 \equiv h_o = h_2 + \frac{1}{2}u_2^2$$

$$\begin{aligned} a_1^2 &= (\gamma-1)h_1 = (\gamma-1)\left(h_o - \frac{1}{2}u_1^2\right) \\ a_2^2 &= (\gamma-1)h_2 = (\gamma-1)\left(h_o - \frac{1}{2}u_2^2\right) \end{aligned}$$

$$u_1 - u_2 = \frac{\gamma-1}{\gamma}\left(\frac{h_o}{u_2} - \frac{h_o}{u_1} + \frac{1}{2}(u_1 - u_2)\right)$$

$$u_1 - u_2 = \frac{\gamma-1}{\gamma}\left(\frac{h_o}{u_2} - \frac{h_o}{u_1} + \frac{1}{2}(u_1-u_2)\right)$$

Dividing by $u_1 - u_2$

$$1 = \frac{\gamma-1}{\gamma}\left(\frac{h_o}{u_1 u_2} + \frac{1}{2}\right)$$

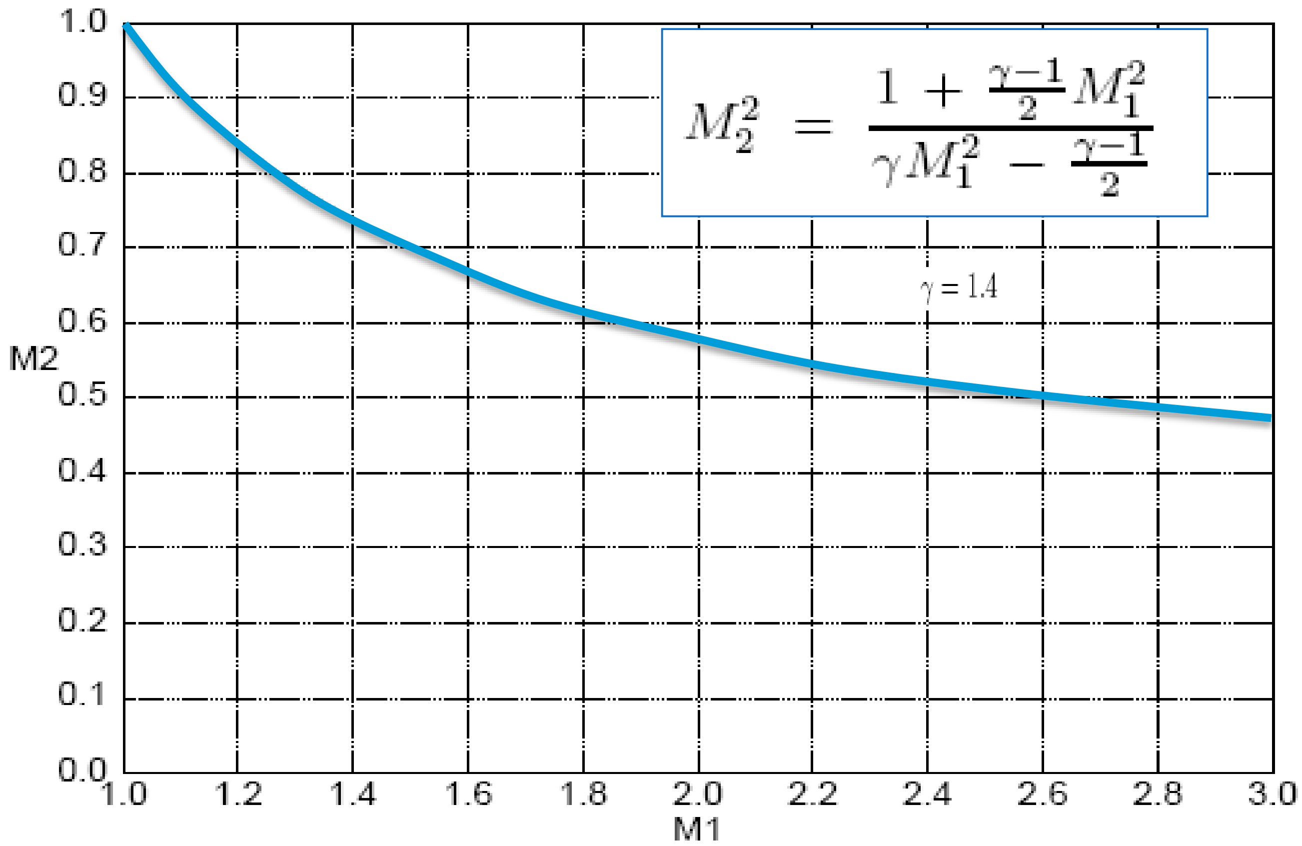
$$\frac{(\gamma-1)h_o}{u_1 u_2} = \frac{\gamma+1}{2}$$

$$\frac{(\gamma-1)^2 h_o^2}{u_1^2 u_2^2} = \left(\frac{\gamma+1}{2}\right)^2$$

Since $h_o = h_{o_1} = h_{o_2}$

$$(\gamma-1)^2 h_o^2 = (\gamma-1)h_{o_1}\,(\gamma-1)h_{o_2} = a_1^2\left(1+\frac{\gamma-1}{2}M_1^2\right)\, a_2^2\left(1+\frac{\gamma-1}{2}M_2^2\right)$$

$$\boxed{M_2^2 = \frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2 - \frac{\gamma-1}{2}}}$$



The $M_1 \to 1^+$, $M_2 \to 1^-$ limit corresponds to infinitesimal shock, or a sound wave.

$M_2(M_1)$ function is not shown for $M_1 < 1$, since this would correspond to an "expansion shock" which is physically impossible based on irreversibility considerations.

$$\lim_{M_1 \to \infty} M_2 = \sqrt{\frac{\gamma - 1}{2\gamma}} = 0.378$$

Static jump relations

From the mass equation

$$1 = \frac{\gamma-1}{\gamma}\left(\frac{h_o}{u_1 u_2} + \frac{1}{2}\right)$$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{u_1^2}{u_1 u_2}$$

$$u_1^2 = M_1^2 a_1^2 = M_1^2 \frac{(\gamma-1)h_o}{1+\frac{\gamma-1}{2}M_1^2}$$

$$\frac{1}{u_1 u_2} = \frac{1}{(\gamma-1)h_o}\frac{\gamma+1}{2}$$

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}}$$

The combination of the momentum equation and mass equation gives:

$$p_2 - p_1 = \rho_1 u_1^2 - \rho_2 u_2^2 = \rho_1 u_1^2 \left(1 - \frac{u_2}{u_1}\right) = \rho_1 u_1^2 \left(1 - \frac{\rho_1}{\rho_2}\right)$$

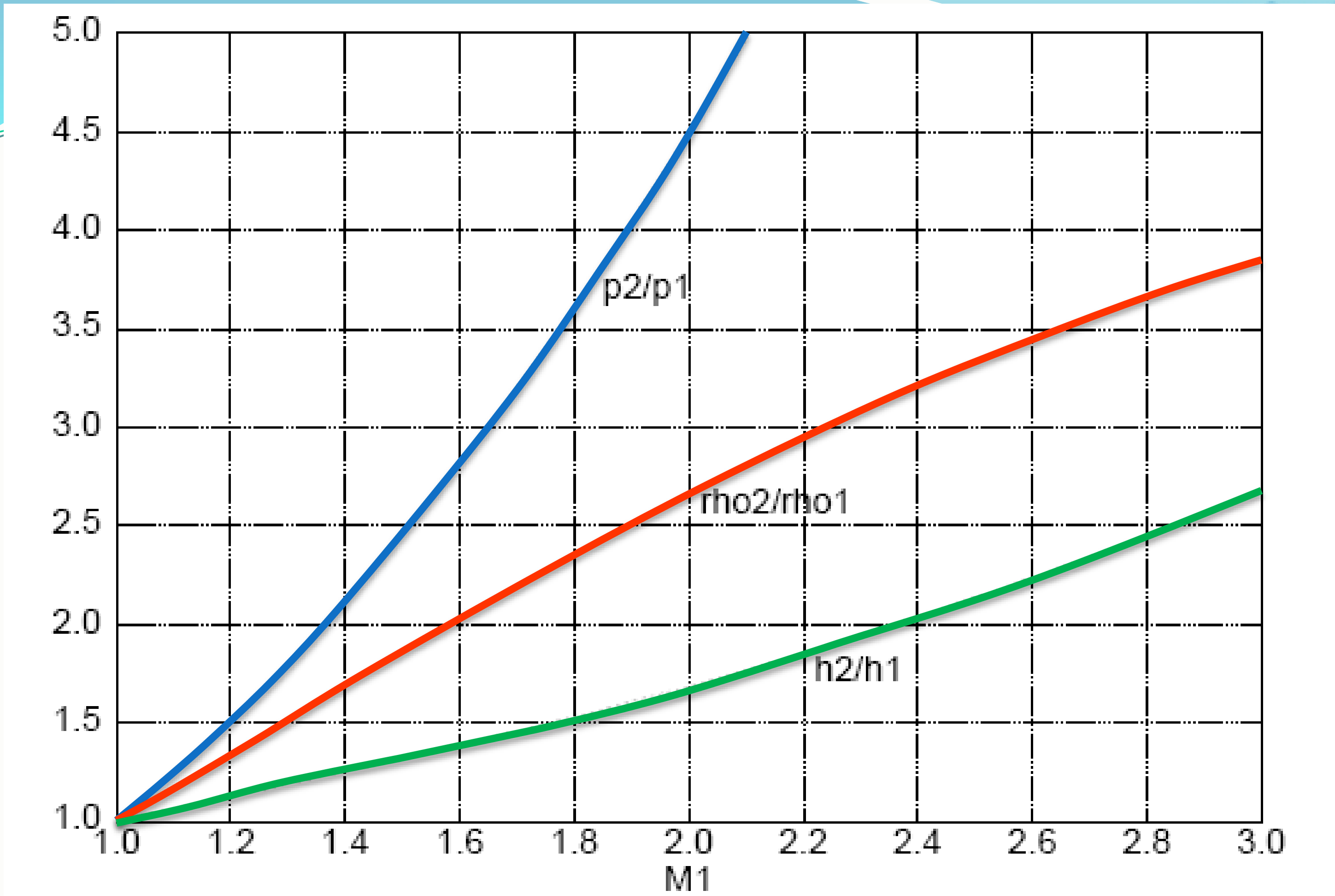
$$\rho u^2 = \gamma p M^2$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{2 + (\gamma-1)M_1^2}$$

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2}$$

$$\boxed{\frac{T_2}{T_1} = \frac{h_2}{h_1} = \left[1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right)\right]\frac{2 + (\gamma-1)M_1^2}{(\gamma+1)M_1^2}}$$

$$\boxed{\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right)}$$



$$\lim_{M_1 \to \infty} \frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{\gamma - 1} = 6$$

$$\lim_{M_1 \to \infty} \frac{p_2}{p_1} = \infty \qquad \lim_{M_1 \to \infty} \frac{T_2}{T_1} = \infty$$

Recall that the second law of thermodynamics determines the *direction which a given process* can take.

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

$$s_2 - s_1 = c_p \ln \left\{ \left[1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1) \right] \frac{2 + (\gamma - 1)M_1^2}{(\gamma+1)M_1^2} \right\} - R \ln \left[1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1) \right]$$

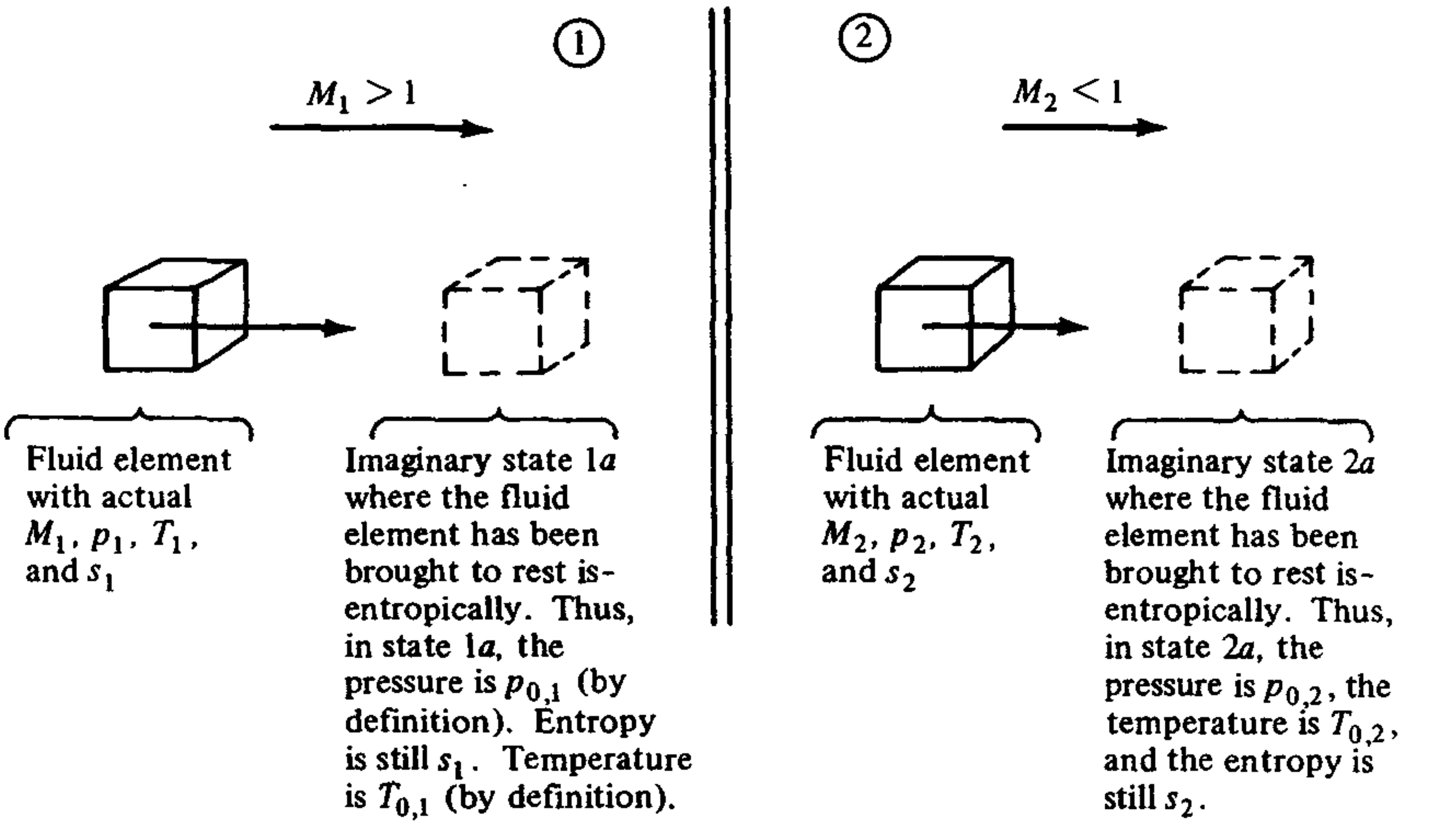
we see that the entropy change s2 — s1 across the shock is a function of M1 only. The second law dictates that:

$$s_2 - s_1 \geq 0$$

if $M_1 = 1$, $s_2 = s_1$ if $M_1 > 1$, then $s_2 - s_1 > 0$.

if $M_1 < 1$ $s_2 - s_1 < 0$.

What happens to total conditions across a shock wave?



$$c_pT_1 + \frac{u_1^2}{2} = c_pT_2 + \frac{u_2^2}{2}$$

$$c_pT_0 = c_pT + \frac{u^2}{2}$$

$$c_pT_{0,1} = c_pT_{0,2}$$

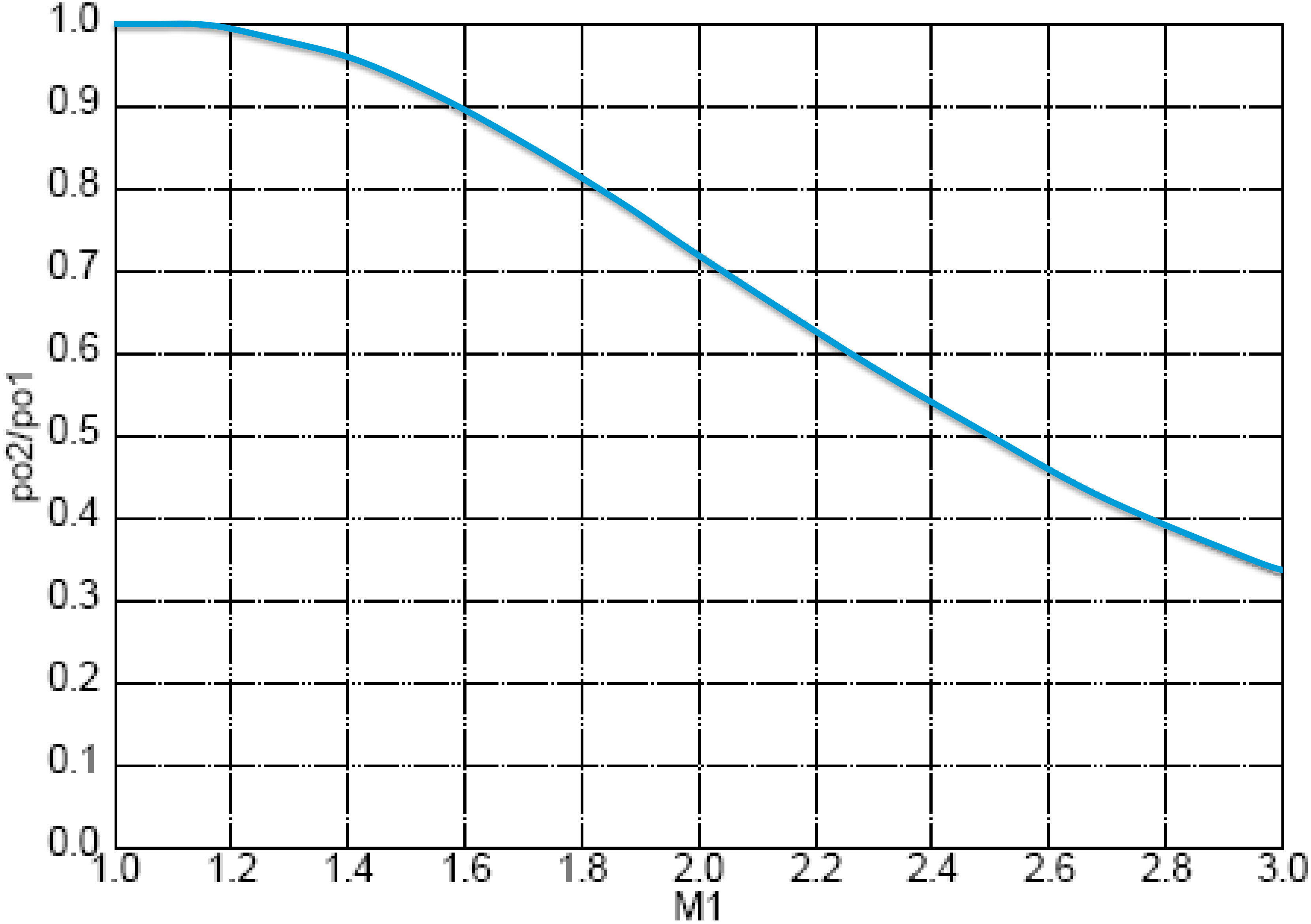
$$\boxed{T_{0,1} = T_{0,2}}$$

$$s_{2a} - s_{1a} = c_p \ln \frac{T_{2a}}{T_{1a}} - R \ln \frac{p_{2a}}{p_{1a}}$$

$$T_{0,2} = T_{0,1}$$

$$s_2 - s_1 = -R \ln \frac{p_{0,2}}{p_{0,1}}$$

$$\boxed{\frac{p_{0,2}}{p_{0,1}} = e^{-(s_2-s_1)/R}}$$


po2/po1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
M1

EXAMPLE

Consider a normal shock wave in air where the upstream flow properties are u1= 680 m/s, T1= 288 K, and p1= 1 atm.
Calculate the velocity, temperature, and pressure downstream of the shock.

$$a_1 = \sqrt{\gamma R T_1} = \sqrt{1.4(287)(288)} = 340 \text{ m/s}$$

$$M_1 = \frac{u_1}{a_1} = \frac{680}{340} = 2$$

$$p_2/p_1 = 4.5, \; T_2/T_1 = 1.687, \; M_2 = 0.5774,$$

$$p_2 = \frac{p_2}{p_1} p_1 = 4.5(1 \text{ atm}) = \boxed{4.5 \text{ atm}}$$

$$T_2 = \frac{T_2}{T_1} T_1 = 1.687(288) = \boxed{486 \text{ K}}$$

$$a_2 = \sqrt{\gamma R T_2} = \sqrt{1.4(287)(486)} = 442 \text{ m/s}$$

$$u_2 = M_2 a_2 = 0.5774(442) = \boxed{255 \text{ m/s}}$$

EXAMPLE

Consider a normal shock wave in a supersonic airstream where the pressure upstream of the shock is 1 atm. Calculate the loss of total pressure across the shock wave when the upstream Mach number is

(a) M1 = 2, (b) M1 = 4.

Compare these two results and comment on their implication.

(a) The upstream total pressure is obtained from

$$p_{0,1} = \left(\frac{p_{0,1}}{p_1}\right) p_1$$

$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)}$$

$$p_{0,1}/p_1 = 7.824$$

$$\frac{p01}{p02} = \left(\frac{p01}{p1}\right)\left(\frac{p1}{p2}\right)\left(\frac{p2}{p02}\right) = \frac{\left(\frac{p01}{p1}\right)}{\left(\frac{p02}{p2}\right)\left(\frac{p2}{p1}\right)} = \frac{\left(1+\frac{\gamma-1}{2}M_1^2\right)^{\frac{\gamma}{\gamma-1}}}{\left(1+\frac{\gamma-1}{2}M_2^2\right)^{\frac{\gamma}{\gamma-1}}\left(1+\frac{2\gamma}{\gamma+1}\left(M_1^2-1\right)\right)}$$

$$= \frac{\left(1+\frac{\gamma-1}{2}M_1^2\right)^{\frac{\gamma}{\gamma-1}}}{\left(1+\frac{\gamma-1}{2}\left(\frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2-\frac{\gamma-1}{2}}\right)\right)^{\frac{\gamma}{\gamma-1}}\left(1+\frac{2\gamma}{\gamma+1}\left(M_1^2-1\right)\right)}$$

$$p_{0,2} = \left(\frac{p_{0,2}}{p_{0,1}}\right) p_{0,1}$$

$$p_{0,2} = (0.7209)(7.824) = 5.64 \text{ atm}$$

the loss of total pressure

$$p_{0,1} - p_{0,2} = 7.824 - 5.64 = \boxed{2.184 \text{ atm}}$$

(b) M1 = 4.

$$p_{0,1} = \left(\frac{p_{0,1}}{p_1}\right) p_1 = (151.8)(1 \text{ atm}) = 151.8 \text{ atm}$$

$$p_{0,2} = \left(\frac{p_{0,2}}{p_{0,1}}\right) p_{0,1} = (0.1388)(151.8) = 21.07 \text{ atm}$$

$$\rightarrow \quad p_{0,1} - p_{0,2} = 151.8 - 21.07 = \boxed{130.7 \text{ atm}}$$

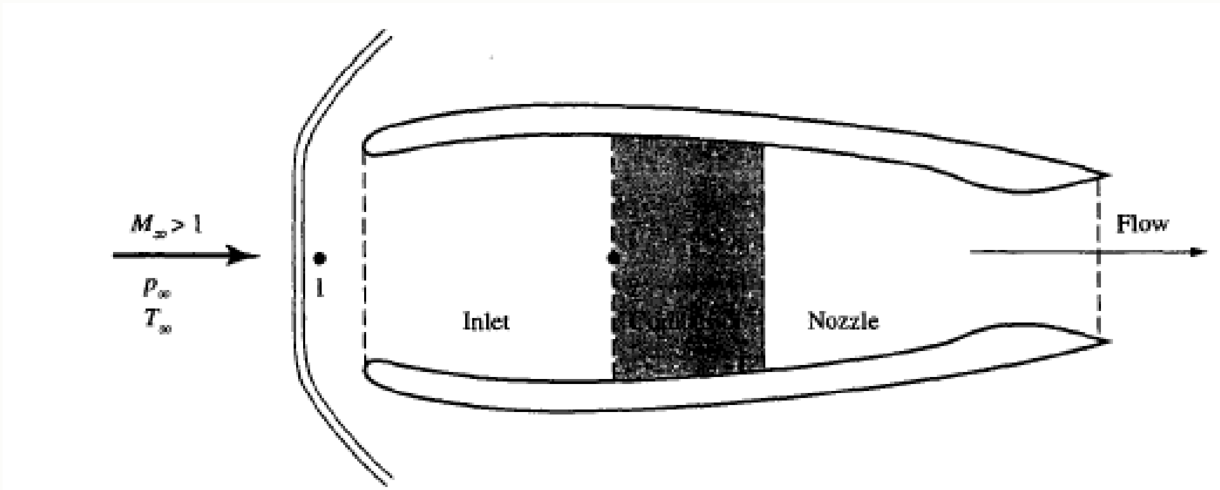
In any flow, total pressure is a precious commodity.

Any loss of total pressure reduces the flow's ability to do useful work. Losses of total pressure reduce the performance of any flow device, and cost money. **We will see this time-and-time-again in subsequent chapters. In this example, we see that for a normal shock at Mach 2, the loss of total pressure was 2.184 atm, whereas simply by doubling the Mach number to 4, the loss of total pressure was a whopping 130.7 atm.**

The moral to this story is that, if you are going to suffer a normal shock wave in a flow, everything else being equal, you want the normal shock to occur at the lowest possible upstream Mach number.

EXAMPLE

A ramjet engine is an air-breathing propulsion device with essentially no rotating machinery (no rotating compressor blades, turbine, etc.).



The ramjet is flying at Mach 2 at a standard altitude of 10 km, where the air pressure and temperature are 2.65 x 104 N/m2 and 223.3 K, respectively. Calculate the air temperature and pressure at point 2 when the Mach number at that point is 0.2.

$$M_\infty = 2 \longrightarrow p_{0,\infty} = \left(\frac{p_{0,\infty}}{p_\infty}\right) p_\infty = (7.824)(2.65 \times 10^4) = 2.07 \times 10^5 \text{ N/m}^2$$

$$T_{0,\infty} = \left(\frac{T_{0,\infty}}{T_\infty}\right) T_\infty = (1.8)(223.3) = 401.9 \text{ K}$$

$$p_{0,1} = \left(\frac{p_{0,1}}{p_{0,\infty}}\right) p_{0,\infty} = (0.7209)(2.07 \times 10^5) = 1.49 \times 10^5 \text{ N/m}^2$$

$$T_{0,1} = T_{0,\infty} = 401.9 \text{ K}$$

$$M_2 = 0.2$$

$$p_{0,2} = 1.49 \times 10^5 \text{ N/m}^2$$

$$T_{0,2} = 401.9 \text{ K.}$$

$$p_2 = \left(\frac{p_2}{p_{0,2}}\right)(p_{02}) = \frac{1.49 \times 10^5}{1.028} = \boxed{1.45 \times 10^5 \text{ N/m}^2}$$

$$T_2 = \left(\frac{T_2}{T_{0,2}}\right)(T_{02}) = \frac{401.9}{1.008} = \boxed{399 \text{ K}}$$

Air pressures and temperatures on the order of 1.42 atm and 399 K entering the combustor are very tolerable conditions for low-speed subsonic combustion.

EXAMPLE

Repeat Example 8.10, except for a freestream Mach number = 10. Assume that the ramjet has been redesigned so that the Mach number at point 2 remains equal to 0.2.

for *M =10,* →

$$p_{0,\infty} = \left(\frac{p_{0,\infty}}{p_\infty}\right) p_\infty = (0.4244 \times 10^5)(2.65 \times 10^4) = 1.125 \times 10^9 \text{ N/m}^2$$

$$T_{0,\infty} = \left(\frac{T_{0,\infty}}{T_\infty}\right) T_\infty = (21)(223.3) = 4690 \text{ K}$$

At point 1,

$$p_{0,1} = \left(\frac{p_{0,1}}{p_{0,\infty}}\right)(p_{0,\infty}) = (0.3045 \times 10^{-2})(1.125 \times 10^9) = 3.43 \times 10^6 \text{ N/m}^2$$

$$T_{0,1} = T_{0,\infty} = 4690 \text{ K}$$

$$p_{0,2}/p_2 = 1.028$$

$$T_{0,2}/T_2 = 1.008$$

$$M_2 = 0.2$$

$$p_{0,2} = p_{0,1} = 3.43 \times 10^6 \text{ N/m}^2$$

$$T_{0,2} = T_{0,1} = 4690 \text{ K}$$

$$p_2 = \left(\frac{p_2}{p_{0,2}}\right)(p_{0,2}) = \frac{3.43 \times 10^6}{1.028} = \boxed{3.34 \times 10^6 \text{ N/m}^2}$$

$$T_2 = \left(\frac{T_2}{T_{0,2}}\right)(T_{0,2}) = \frac{4690}{1.008} = \boxed{4653 \text{ K}}$$

In atmospheres,

$$p_2 = \frac{3.34 \times 10^6}{1.02 \times 10^5} = \boxed{32.7 \text{ atm}}$$

Compared to the rather benign conditions at point 2 existing for the case treated in previous example, in the present example the air entering the combustor is at a pressure and temperature of 32.7 atm and 4653 K—both extremely severe conditions.

The temperature is so hot that the fuel injected into the combustor will decompose rather than burn, with little or no thrust being produced. Moreover, the pressure is so high that the structural design of the combustor would have to be extremely heavy, assuming in the first place that some special heat-resistant material could be found that could handle the high temperature.

In short, a conventional ramjet, where the flow is slowed down to a l ow subsonic Mach number before entering the combustor, will not work at high, hypersonic Mach numbers.

The solution to this problem is not to slow the flow inside the engine to low subsonic speeds, but rather to slow it only to a lower but still supersonic speed. In this manner, the temperature and pressure increase inside the engine will be smaller and can be made tolerable.

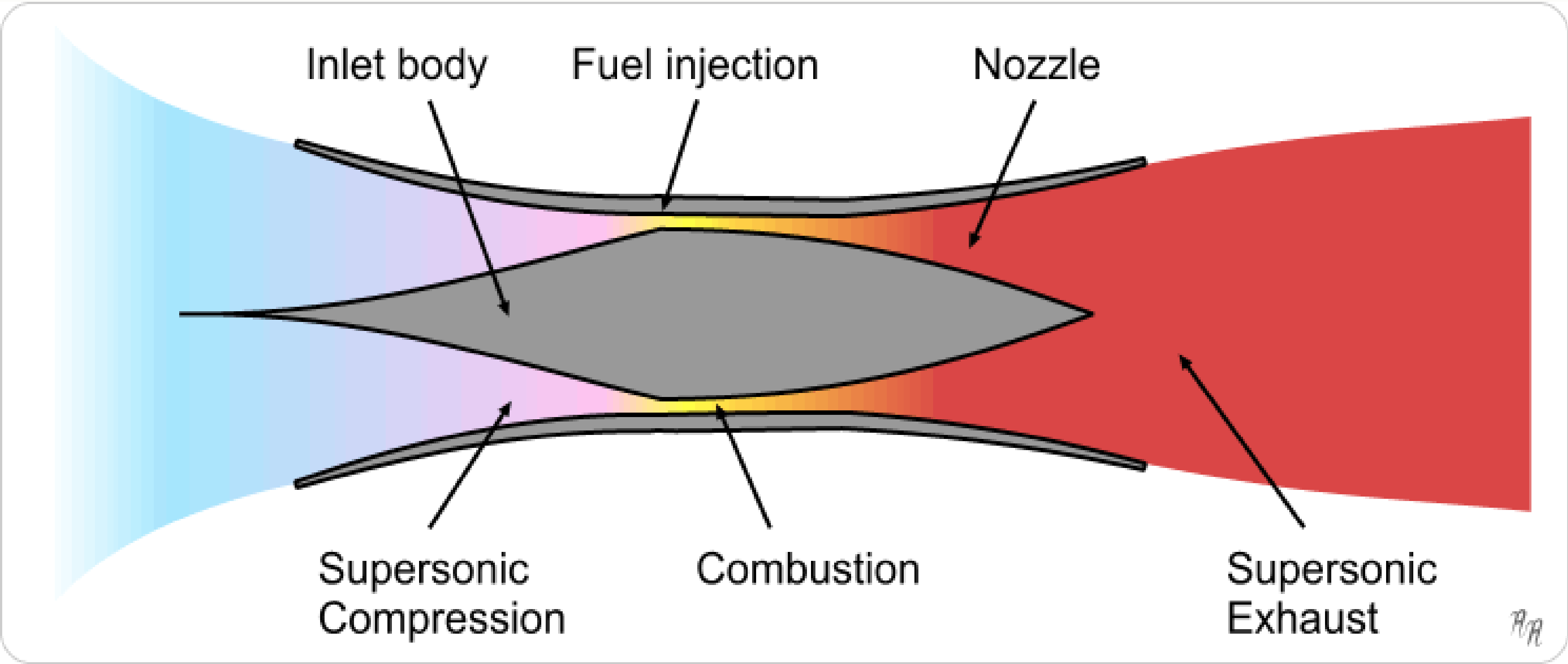
In such a ramjet, the entire flowpath through the engine remains at supersonic speed, including inside the combustor. This necessitates the injection and mixing of the fuel in a supersonic stream—a challenging technical problem. This type of ramjet, where the flow is supersonic throughout, is called a supersonic combustion ramjet—SCRAMjet

Ramjet

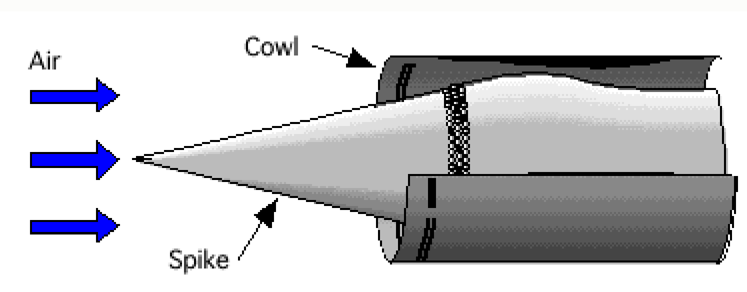
a stovepipe jet, is a form of jet engine at contains no major moving parts

scramjet

A scramjet (supersonic combustion ramjet) is a variation of a ramjet with the key difference being that the flow in the combustor is supersonic. At higher speeds it is necessary to combust supersonically to maximize the efficiency of the combustion process.



Supersonic Combustion Ram Jet SCRAMJET Engines



In this engine, compressors are not used.

Shock waves in front of the aircraft and inside the inlet
slow down the flow and increase the pressure.

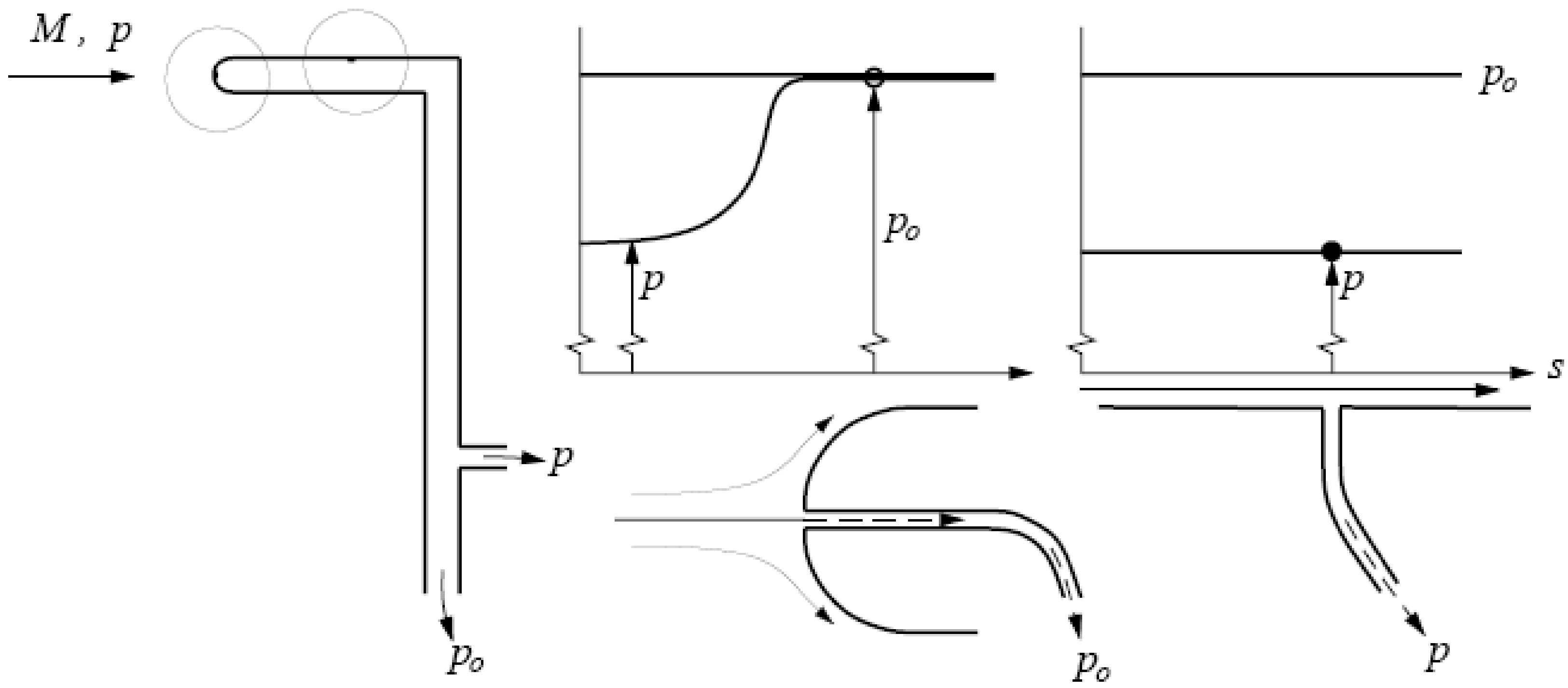
The flow inside the entire engine, including the compressor,
is supersonic.

An aerospace plane will use a SCRAMJET engine.

Subsonic Compressible Flow

$$\frac{p_o}{p} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)}$$

$$M^2 = \frac{2}{\gamma-1}\left[\left(\frac{p_o}{p}\right)^{(\gamma-1)/\gamma} - 1\right]$$

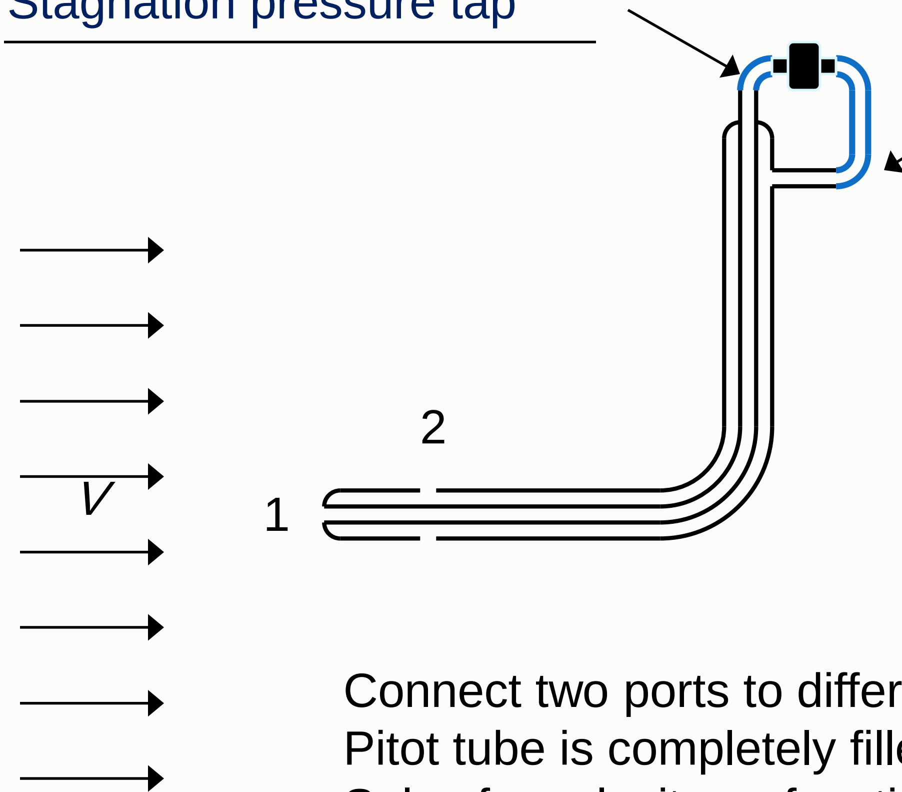


MEASUREMENT OF VELOCITY IN A COMPRESSIBLE FLOW

Pitot Tube

Stagnation pressure tap

Static pressure tap



$$\frac{p_1}{\gamma} + \cancel{z_1} + \cancel{\frac{V_1^2}{2g}} = \frac{p_2}{\gamma} + \cancel{z_2} + \frac{V_2^2}{2g}$$

$V_1 = 0$

$z_1 = z_2$

$$V = \sqrt{\frac{2}{\rho}\ p_1 - p_2}$$

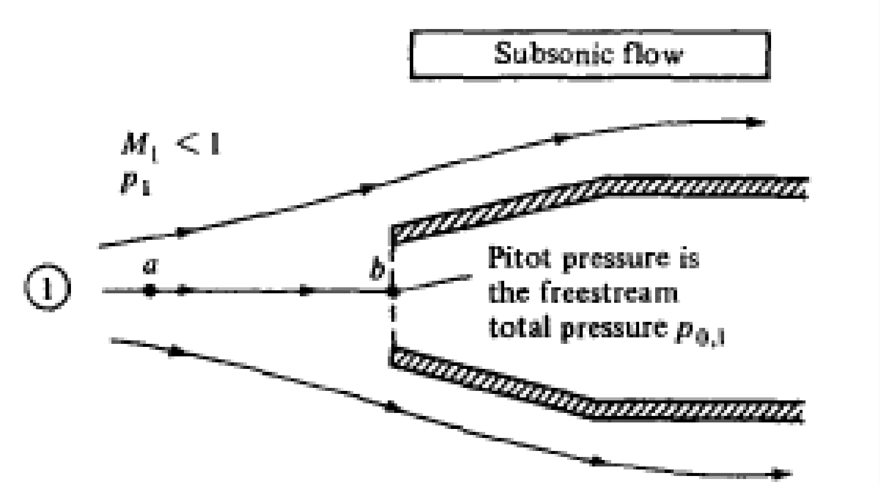
Connect two ports to differential pressure transducer. Make sure Pitot tube is completely filled with the fluid that is being measured. Solve for velocity as function of pressure difference

$$\frac{p_{0,1}}{p_1} = \left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{\gamma/(\gamma-1)}$$

$$\boxed{M_1^2 = \frac{2}{\gamma - 1}\left[\left(\frac{p_{0,1}}{p_1}\right)^{(\gamma-1)/\gamma} - 1\right]}$$

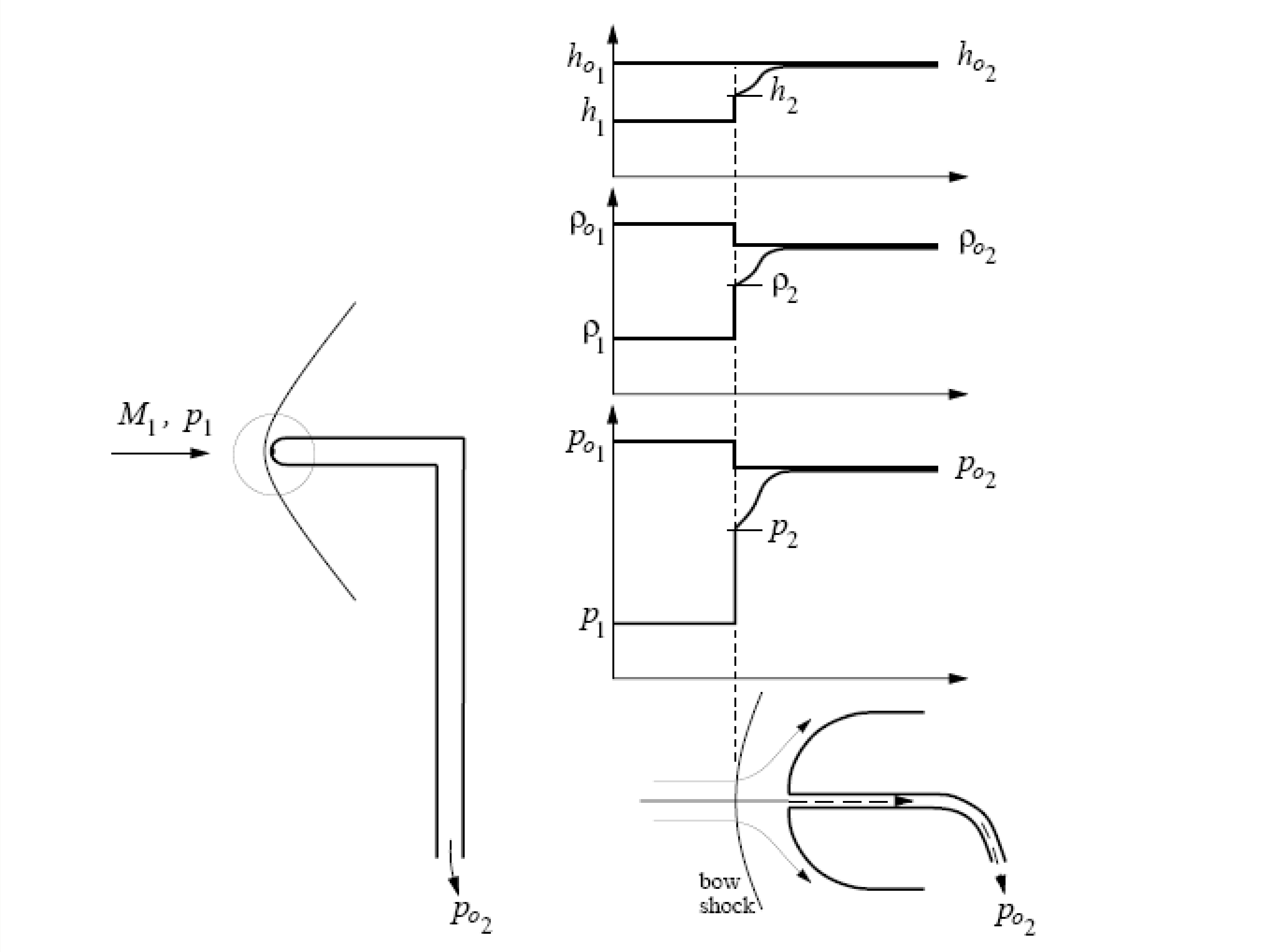
$$M_1 = u_1/a_1.$$

$$u_1^2 = \frac{2a_1^2}{\gamma - 1}\left[\left(\frac{p_{0,1}}{p_1}\right)^{(\gamma-1)/\gamma} - 1\right]$$



Unlike incompressible flow, a knowledge of Po,1 and p1 is not sufficient to obtain u1 we also need the freestream speed of sound, a1.

Supersonic Flow

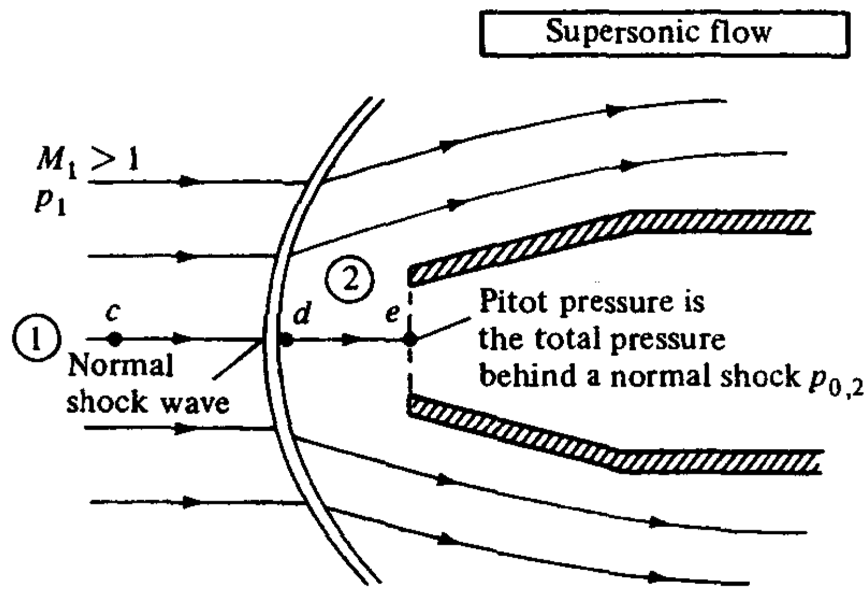


$$\frac{p_{0,2}}{p_1} = \frac{p_{0,2}}{p_2}\frac{p_2}{p_1}$$

$$\frac{p_{0,2}}{p_2} = \left(1 + \frac{\gamma - 1}{2}M_2^2\right)^{\gamma/(\gamma-1)}$$

$$M_2^2 = \frac{1 + [(\gamma - 1)/2]M_1^2}{\gamma M_1^2 - (\gamma - 1)/2}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)$$



$$\boxed{\frac{p_{0,2}}{p_1} = \left(\frac{(\gamma + 1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma - 1)}\right)^{\gamma/(\gamma-1)} \frac{1 - \gamma + 2\gamma M_1^2}{\gamma + 1}}$$

Rayleigh Pitot tube formula.

EXAMPLE

A Pitot tube is inserted into an airflow where the static pressure is 1 atm. Calculate the flow Mach number when the Pitot tube measures (a) 1.276 atm, (b) 2.714 atm, (c) 12.06 atm.

First, we must assess whether the flow is subsonic or supersonic. At Mach 1, the Pitot tube would measure $p_0 = p/0.528 = 1.893p$. Hence, when $p_0 < 1.893$ atm, the flow is subsonic, and when $p_0 > 1.893$ atm, the flow is supersonic.

(*a*) Pitot tube measurement = 1.276 atm.

The flow is supersonic.

$$M_1^2 = \frac{2}{\gamma - 1}\left[\left(\frac{p_{0,1}}{p_1}\right)^{(\gamma-1)/\gamma} - 1\right]$$

$\boxed{M = 0.6}$

(*b*) Pitot tube measurement = 2.714 atm.

The flow is supersonic.

$$\frac{p_{0,2}}{p_1} = \left(\frac{(\gamma+1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma-1)}\right)^{\gamma/(\gamma-1)} \frac{1-\gamma+2\gamma M_1^2}{\gamma+1}$$

$\boxed{M_1 = 1.3}$

(*c*) Pitot tube measurement = 12.06 atm.

The flow is supersonic.

$$\frac{p_{0,2}}{p_1} = \left(\frac{(\gamma+1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma-1)}\right)^{\gamma/(\gamma-1)} \frac{1-\gamma+2\gamma M_1^2}{\gamma+1}$$

$\boxed{M_1 = 3.0}$

EXAMPLE

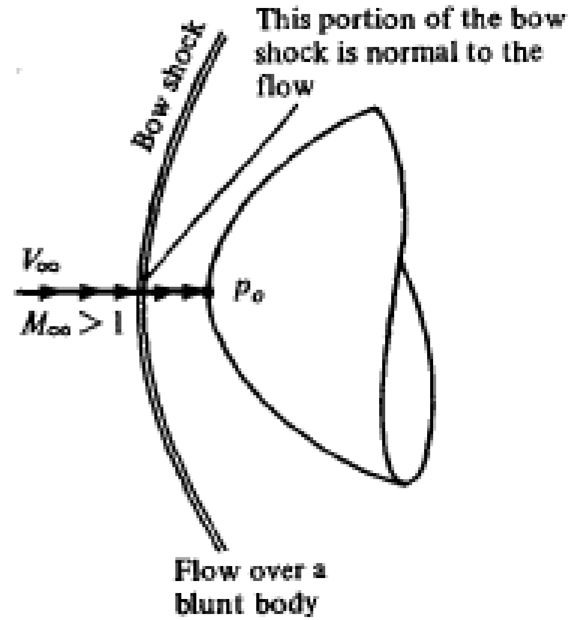
Consider a hypersonic missile flying at Mach 8 at an altitude of 20,000 ft, where the pressure is 973.3 lb/ft2 . The nose of the missle is blunt. Calculate the pressure at the stagnation point on the nose.

For Mach 8, po,1/p1= 82.87.

$$p_s = p_{0,2} = \left(\frac{p_{0,2}}{p_1}\right)(p_1) = 82.87(973.3) = \boxed{8.07 \times 10^4 \text{ lb/ft}^2}$$

Since 1 atm = 2116 lb/ft^2,

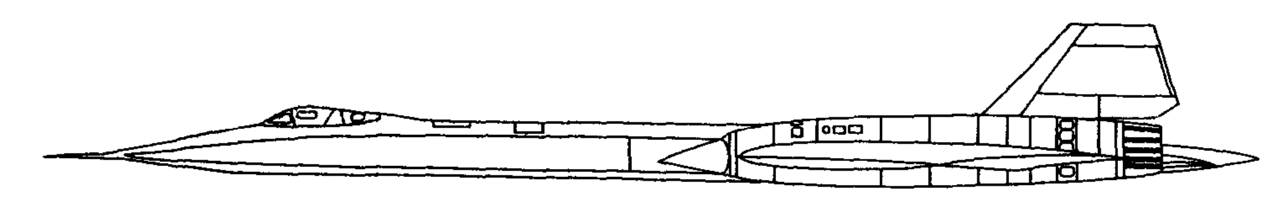
$$p_s = \frac{8.07 \times 10^4}{2116} = \boxed{38.1 \text{ atm}}$$



Note that the pressure at the nose of the missile is quite high—38.1 atm. This is typical of hypersonic flight at low altitude.

EXAMPLE

Consider the Lockheed SR-71 Blackbird flying at a standard altitude of 25 km. The pressure measured by a Pitot tube on this airplane is 3.88 x 10^4 N/m^2. Calculate the velocity of the airplane.



At an altitude of 25 km, p = 2.5273x10^3 N / m^2 and T = 216.66 K.

$$\frac{p_{0,1}}{p_1} = \frac{3.88 \times 10^4}{2.5273 \times 10^3} = 15.35$$

$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)}$$

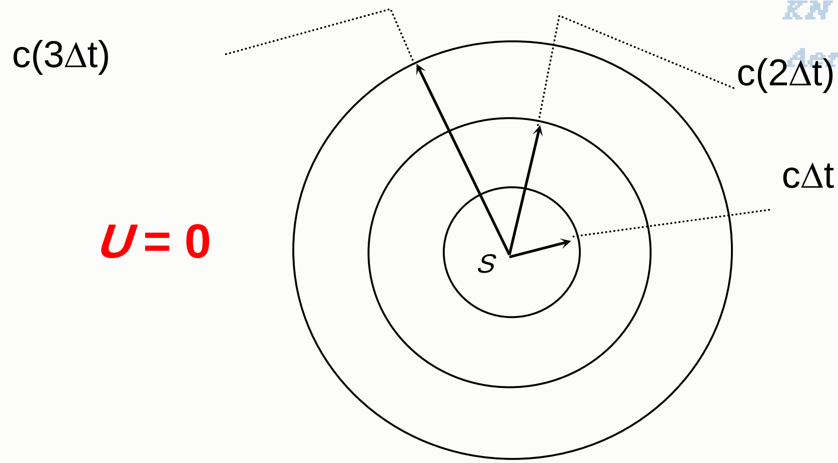
$$M_1 = 3.4$$

$$V_1 = M_1 a_1 = (3.4)(295) = \boxed{1003 \text{ m/s}}$$

Oblique Shock and Expansion Waves

KN Toosi University of Technology
Aerospace engineering Faculty

SOUND WAVE PROPAGATION (pressure disturbance)

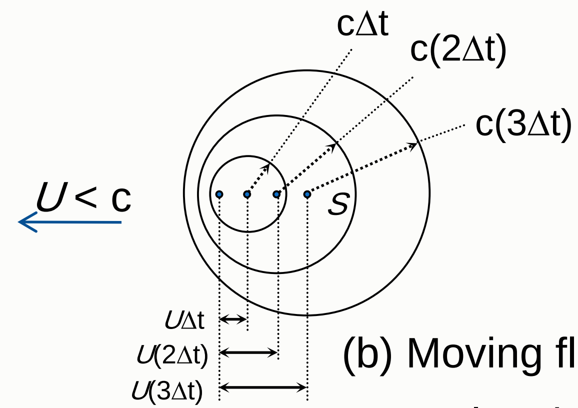


(a) Stationary fluid

radial propagation only

Pressure disturbance occurring at an interval of every Δt

S is the disturbance source



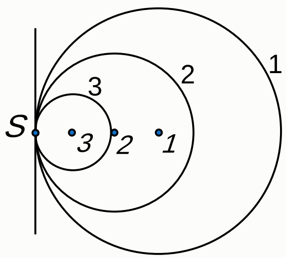
(b) Moving fluid

Doppler shift

radial + axial propagation

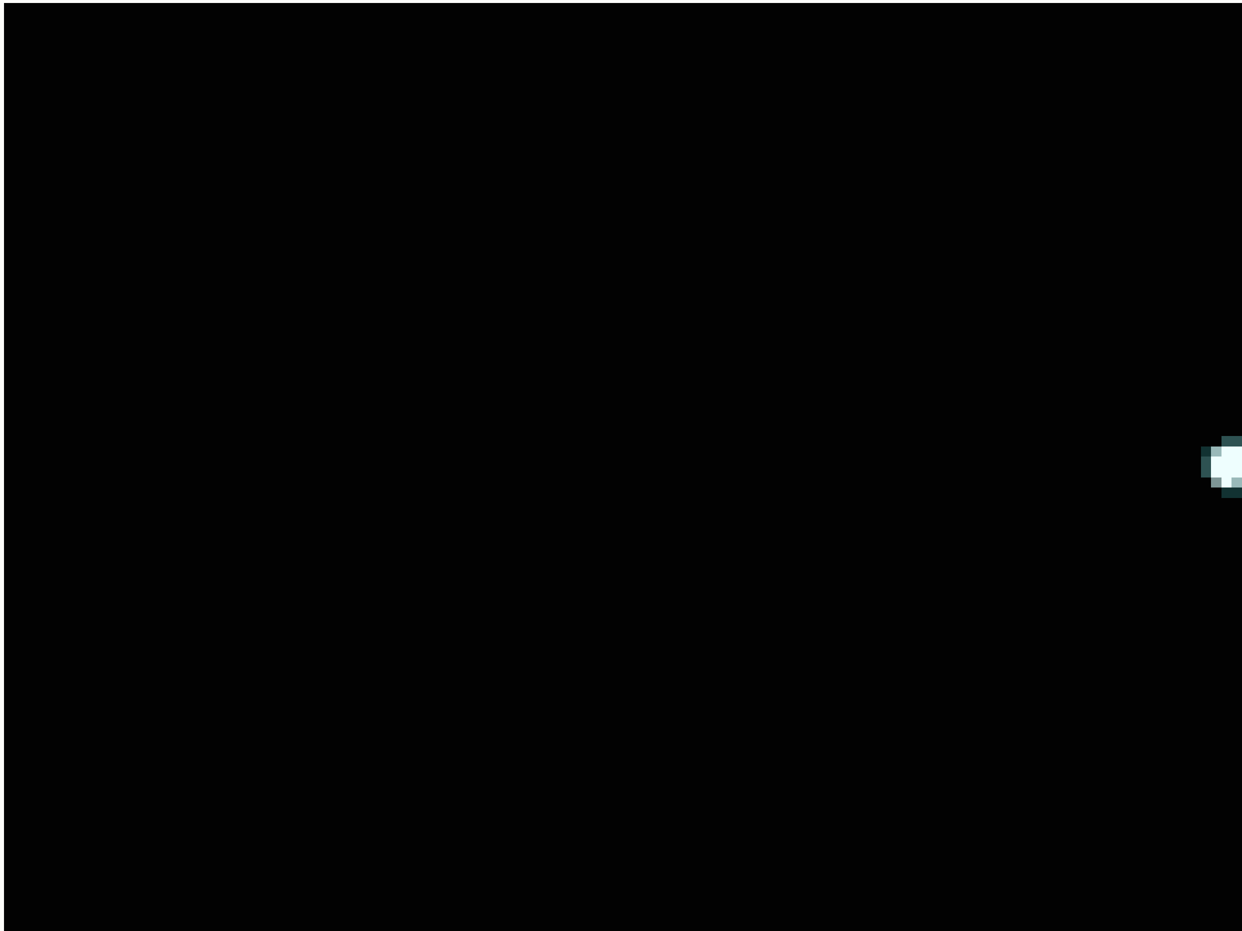
Ma < 1 subsonic

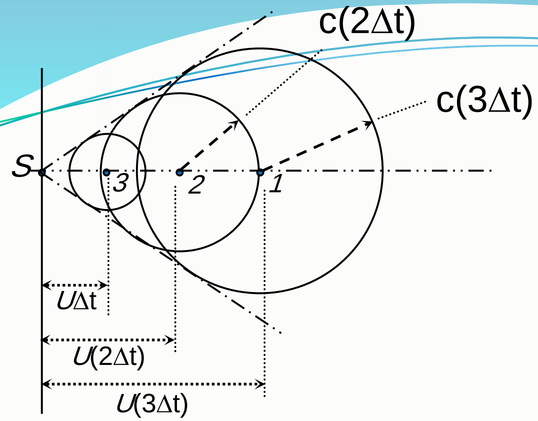
Case (c): $U = c$



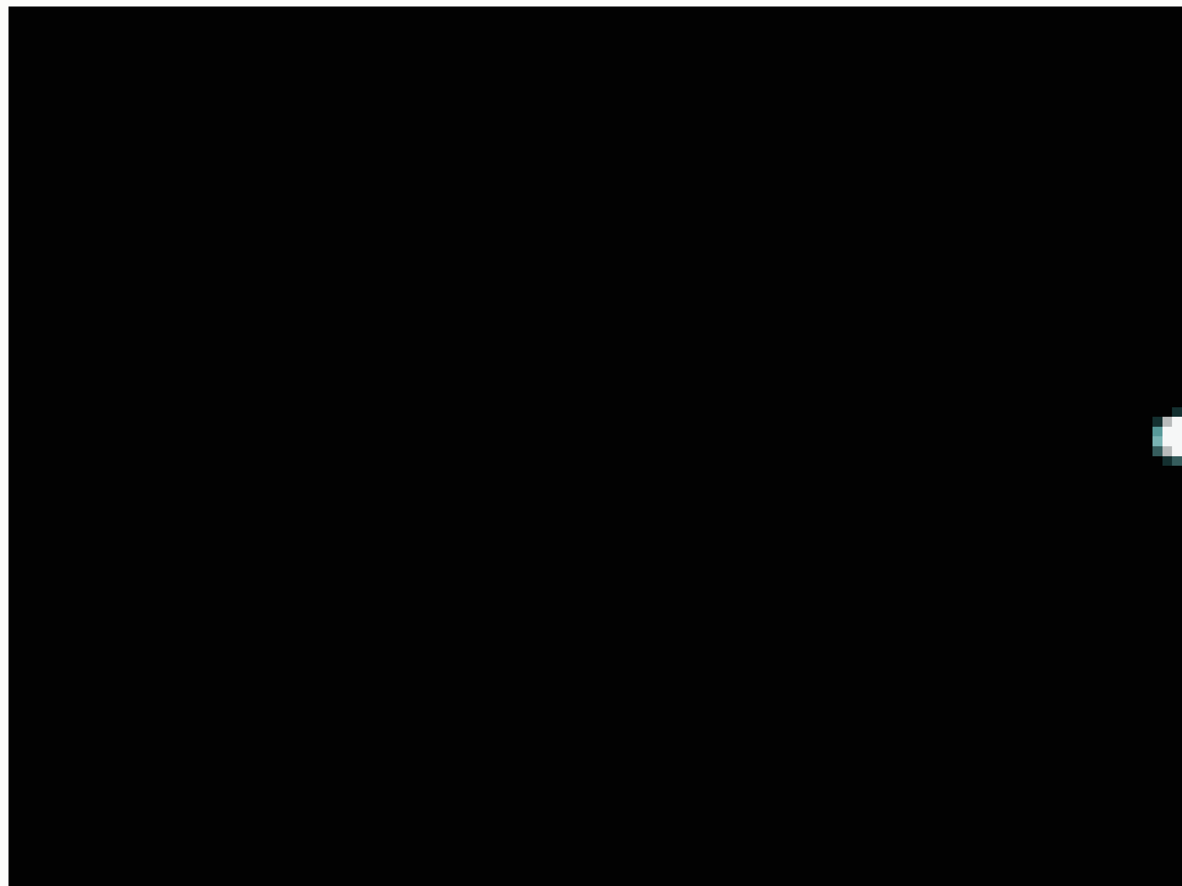
$Ma = 1$ sonic

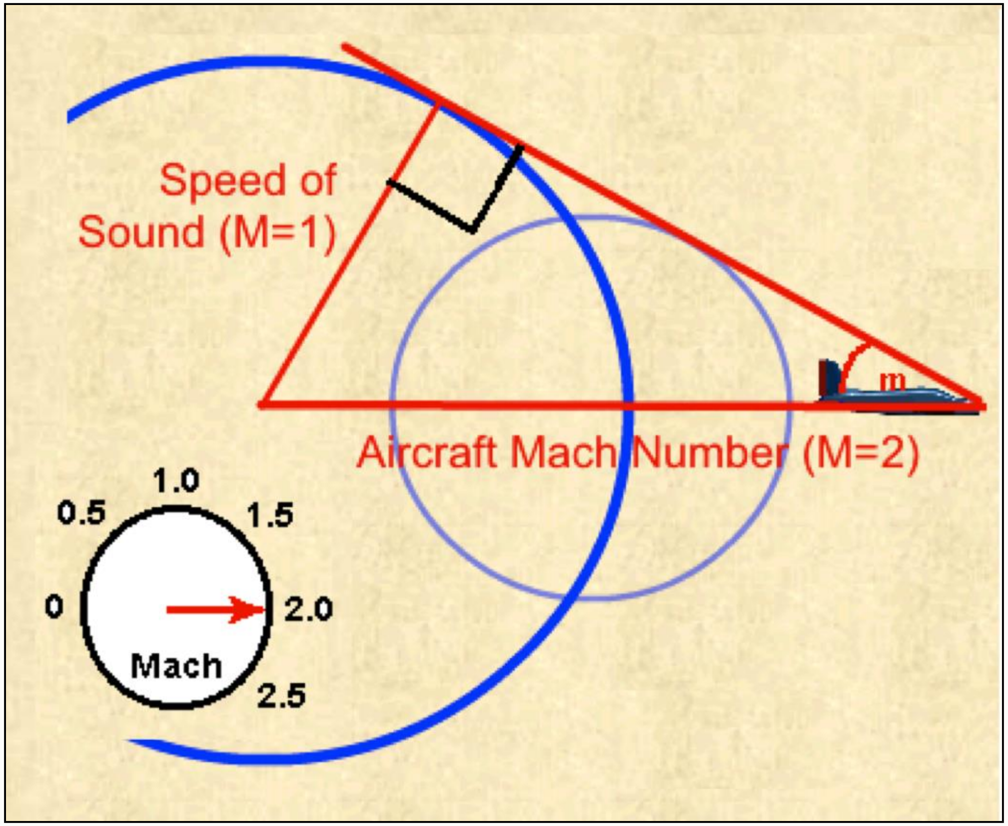
all wavefronts touch source S





Case (d): $U > c$





$$\sin\mu = \frac{at}{Vt} = \frac{a}{V} = \frac{1}{M}$$

$$\mu = \sin^{-1}\frac{1}{M}$$

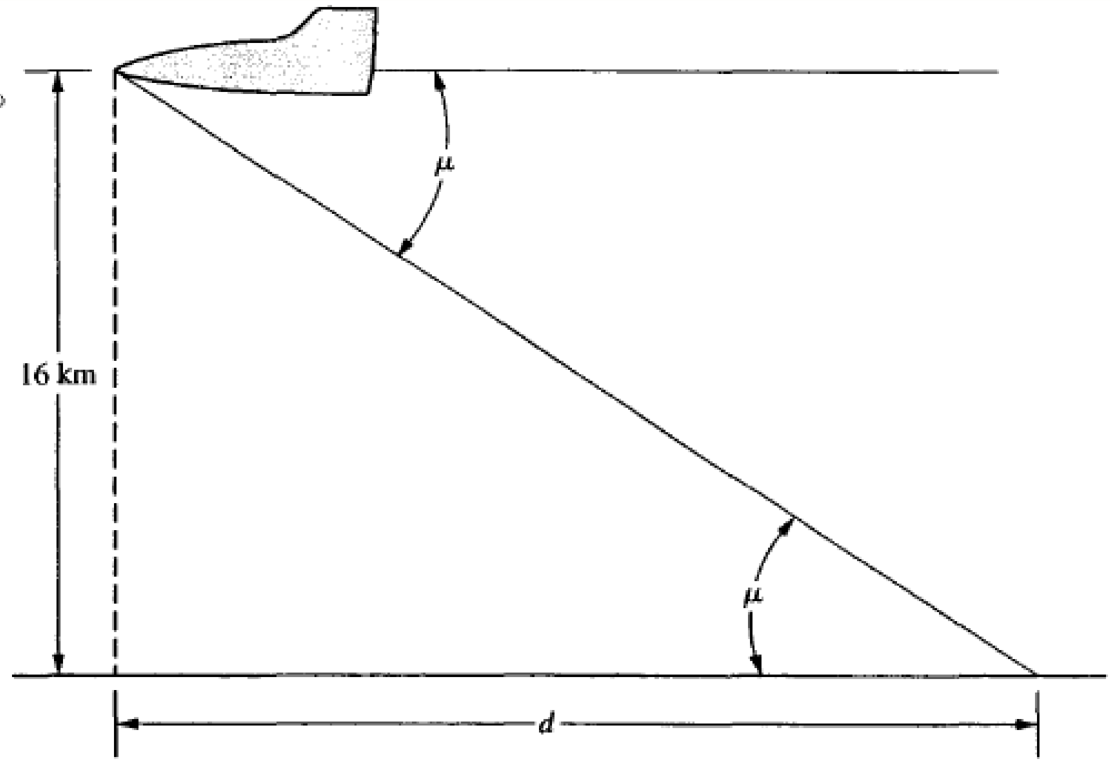
$$\mu = \frac{1}{\sin(M)}$$

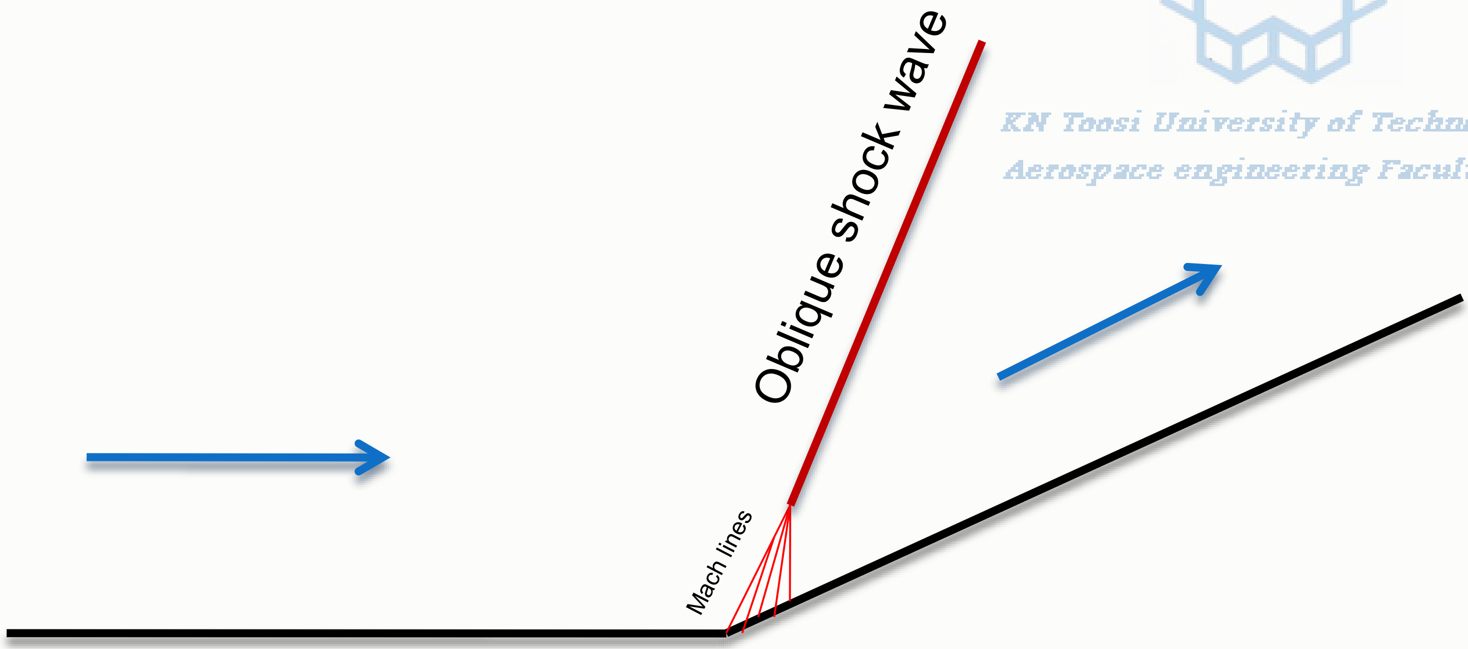
A supersonic airplane is flying at Mach 2 at an altitude of 16 km. Assume the shock wave pattern from the airplane (see Figure 9.1) quickly coalesces into a Mach wave that intersects the ground behind the airplane, causing a "sonic boom" to be heard by a bystander on the ground. At the instant the sonic boom is heard, how far ahead of the bystander is the airplane?

$$\mu = \sin^{-1}\left(\frac{1}{M}\right) = \sin^{-1}\left(\frac{1}{2}\right) = 30^\circ$$

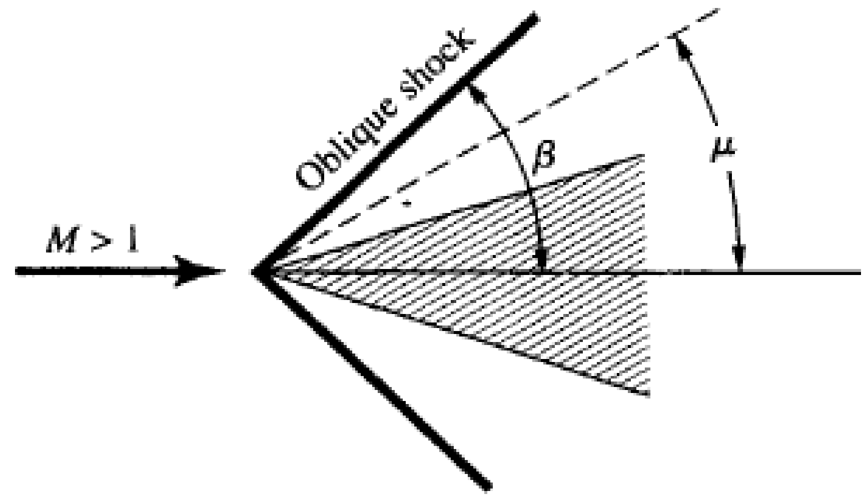
$$\tan \mu = \frac{16 \text{ km}}{d}$$

$$d = \frac{16 \text{ km}}{\tan \mu} = \frac{16}{0.577} = \boxed{27.7 \text{ km}}$$

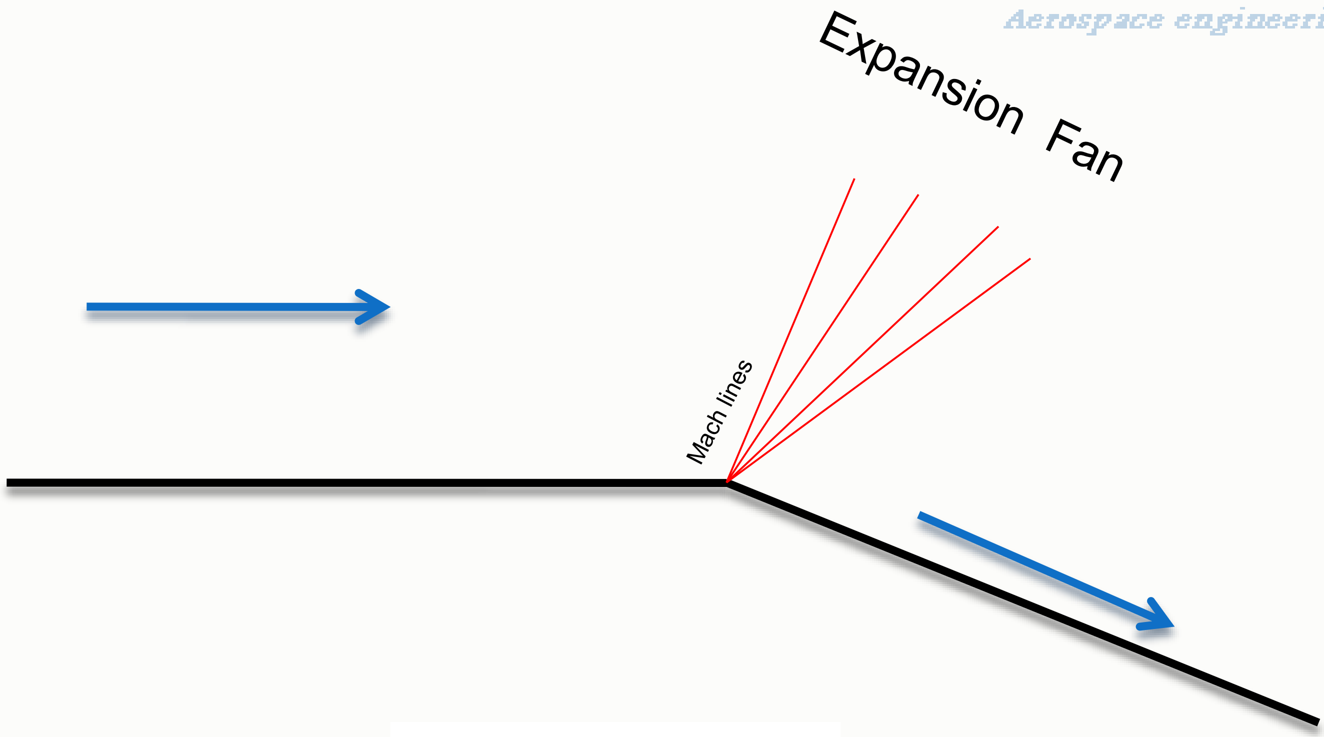




(a) Concave corner

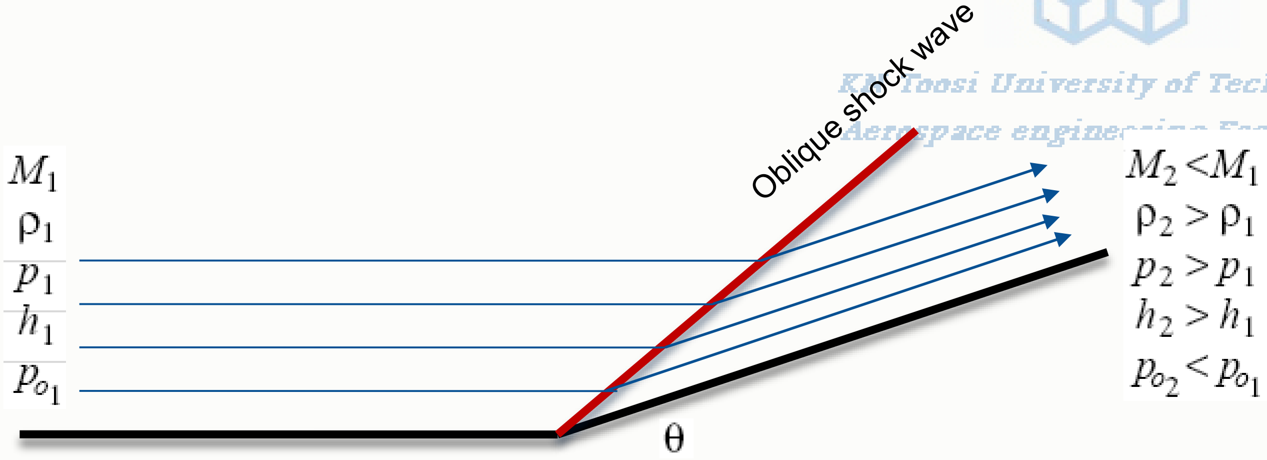


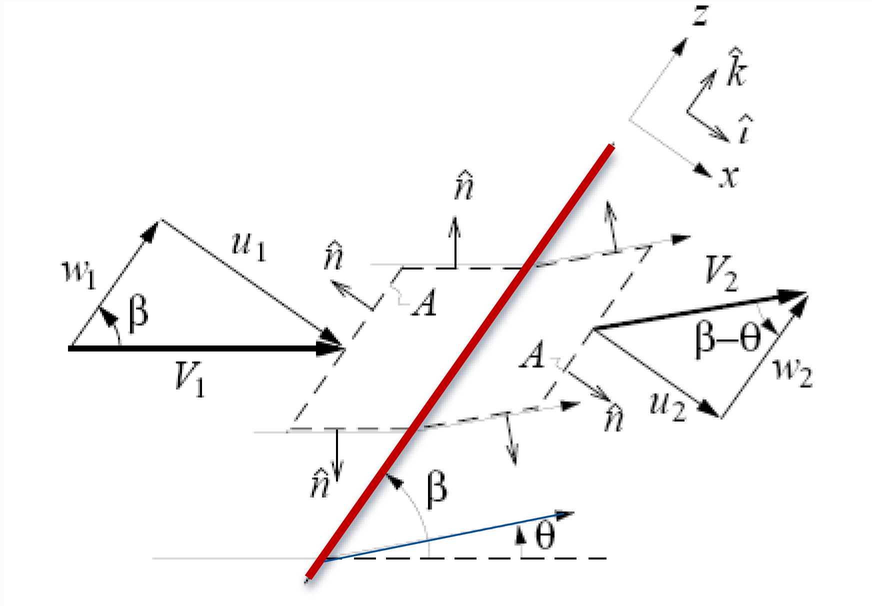
Relation between the oblique shock-wave angle and the Mach angle.



(b) Convex corner

OBLIQUE SHOCK RELATIONS



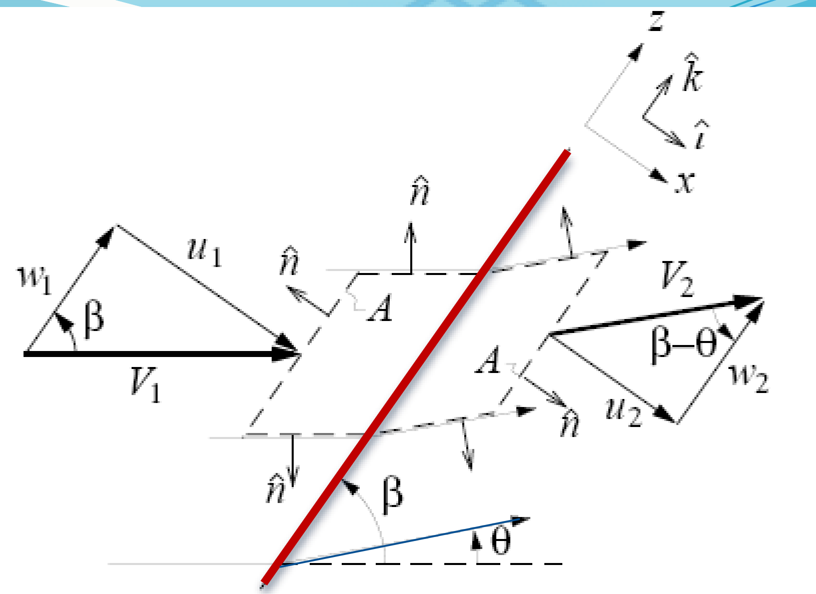


Mass continuity

$$\oiint \rho \vec{V} \cdot \hat{n} \, dA = 0$$

$$-\rho_1 u_1 A + \rho_2 u_2 A = 0$$

$$\rho_1 u_1 = \rho_2 u_2$$



x-Momentum

$$\oiint \rho \vec{V} \cdot \hat{n} \, u \, dA + \oiint p \hat{n} \cdot \hat{\imath} \, dA = 0$$

$$-\rho_1 u_1^2 A + \rho_2 u_2^2 A - p_1 A + p_2 A = 0$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

z-Momentum

$$\oiint \rho \vec{V} \cdot \hat{n} \, w \, dA + \oiint p \hat{n} \cdot \hat{k} \, dA = 0$$

$$-\rho_1 u_1 A \, w_1 + \rho_2 u_2 A \, w_2 = 0$$

$$w_1 = w_2$$

Energy

$$\oiint \rho \vec{V} \cdot \hat{n} \, h_o \, dA = 0$$

$$-\rho_1 u_1 h_{o_1} A + \rho_2 u_2 h_{o_2} A = 0$$

$$h_{o_1} = h_{o_2}$$

$$h_1 + \frac{1}{2}\left(u_1^2 + w_1^2\right) = h_2 + \frac{1}{2}\left(u_2^2 + w_2^2\right)$$

$$h_1 + \frac{1}{2} u_1^2 = h_2 + \frac{1}{2} u_2^2$$

Equation of State

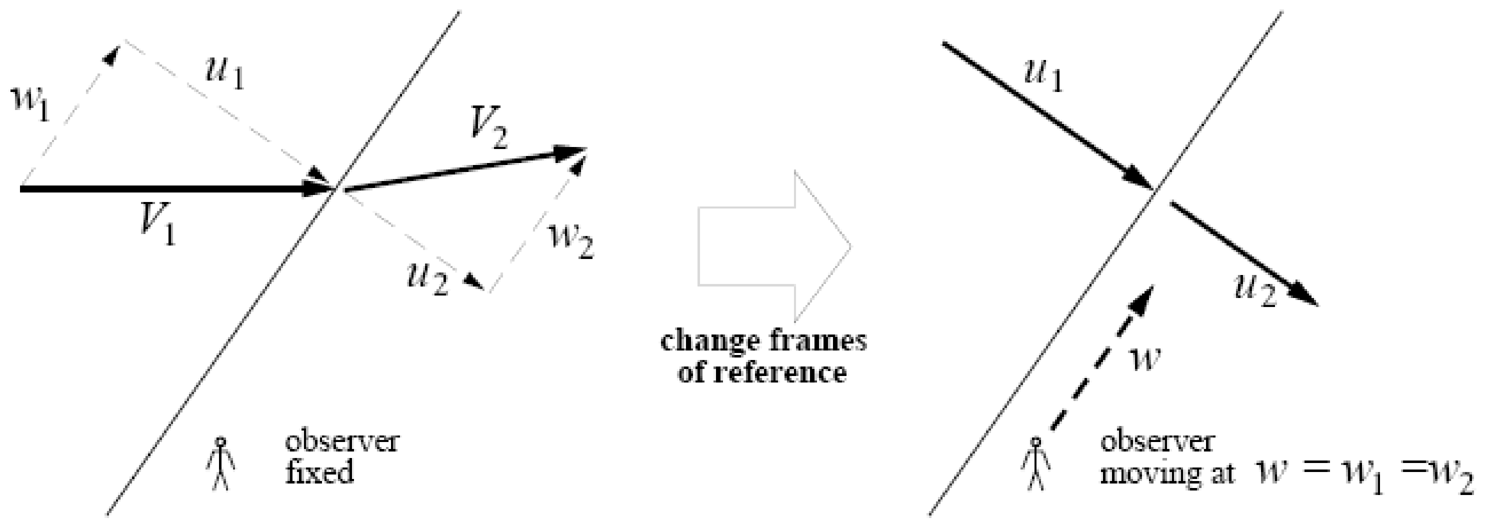
$$p_2 = \frac{\gamma - 1}{\gamma} \rho_2 h_2$$

Oblique/normal shock equivalence

It is apparent that equations mass, x-momentum , energy, state equations are in fact identical to the normal-shock equations derived earlier. The one addition z-momentum equation simply states that the tangential velocity component doesn't change across a shock.

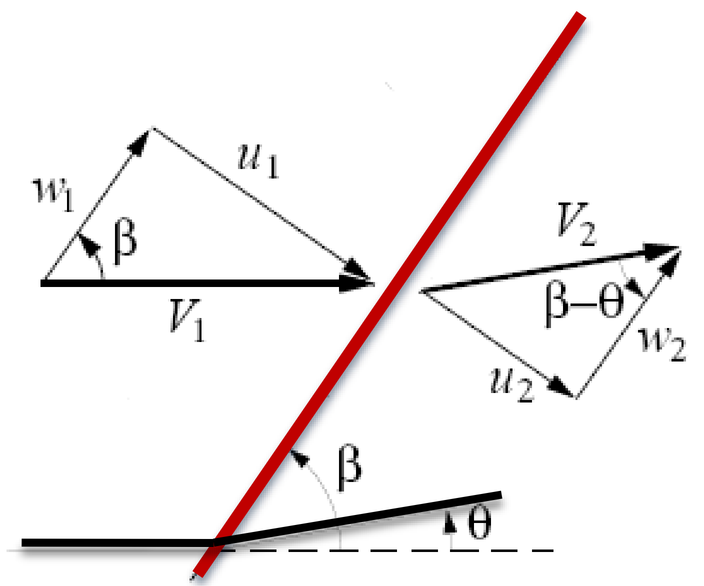
The effective equivalence between an oblique and a normal shock allows re-use of the already derived normal shock jump relations.

We only need to construct the necessary transformation from one frame to the other.



First we define the normal Mach number components seen by the moving observer.

$$M_{n_1} \equiv \frac{u_1}{a_1} = \frac{V_1 \sin\beta}{a_1} = M_1 \sin\beta$$



$$M_{n_2}^2 = \frac{1 + \frac{\gamma-1}{2} M_{n_1}^2}{\gamma M_{n_1}^2 - \frac{\gamma-1}{2}}$$

$$M_2 = \frac{M_{n2}}{\sin(\beta - \theta)}$$

The static property ratios are likewise obtained using the previous normal-shock relations.

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$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_{n_1}^2}{2 + (\gamma-1)M_{n_1}^2}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_{n_1}^2 - 1\right)$$

$$\frac{h_2}{h_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2}$$

$$\frac{p_{o_2}}{p_{o_1}} = \frac{p_2}{p_1}\left(\frac{h_1}{h_2}\right)^{\gamma/(\gamma-1)}$$

To allow application of the above relations, we still require the wave angle β.

$$w_1 = w_2, \quad \Rightarrow \quad \frac{\tan(\beta-\theta)}{\tan\beta} = \frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} = \frac{(\gamma+1)M_1^2\sin^2\beta}{2+(\gamma-1)M_1^2\sin^2\beta}$$

Solving this for θ gives

$$\boxed{\tan\theta = \frac{2}{\tan\beta}\,\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma+\cos 2\beta)+2}}$$

$$\beta(\theta, M_1).$$

Oblique-shock analysis: Summary

Starting from the known upstream Mach number M1 and the flow deflection angle (body surface angle) θ, the oblique-shock analysis proceeds as follows.

$$\boxed{\theta\,,\ M_1}$$

$$\tan\theta = \frac{2}{\tan\beta}\,\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma+\cos 2\beta) + 2}$$

$$\boxed{\beta}$$

$$M_{n,1} = M_1 \sin\beta$$

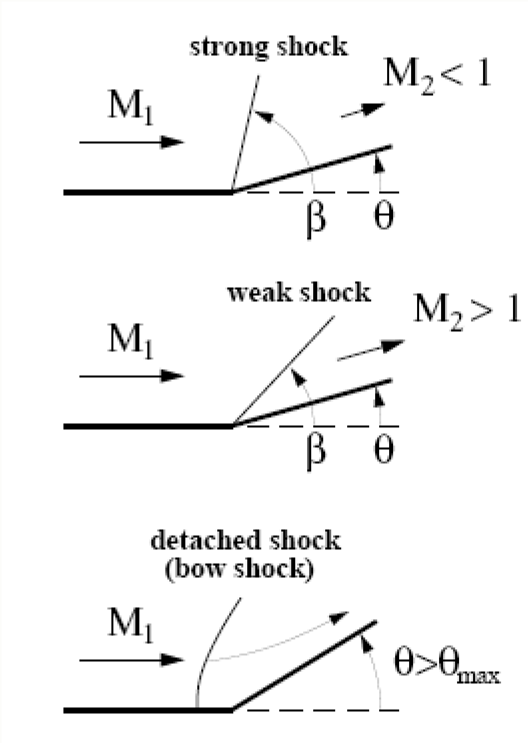
$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_{n_1}^2}{2 + (\gamma-1)M_{n_1}^2}$$

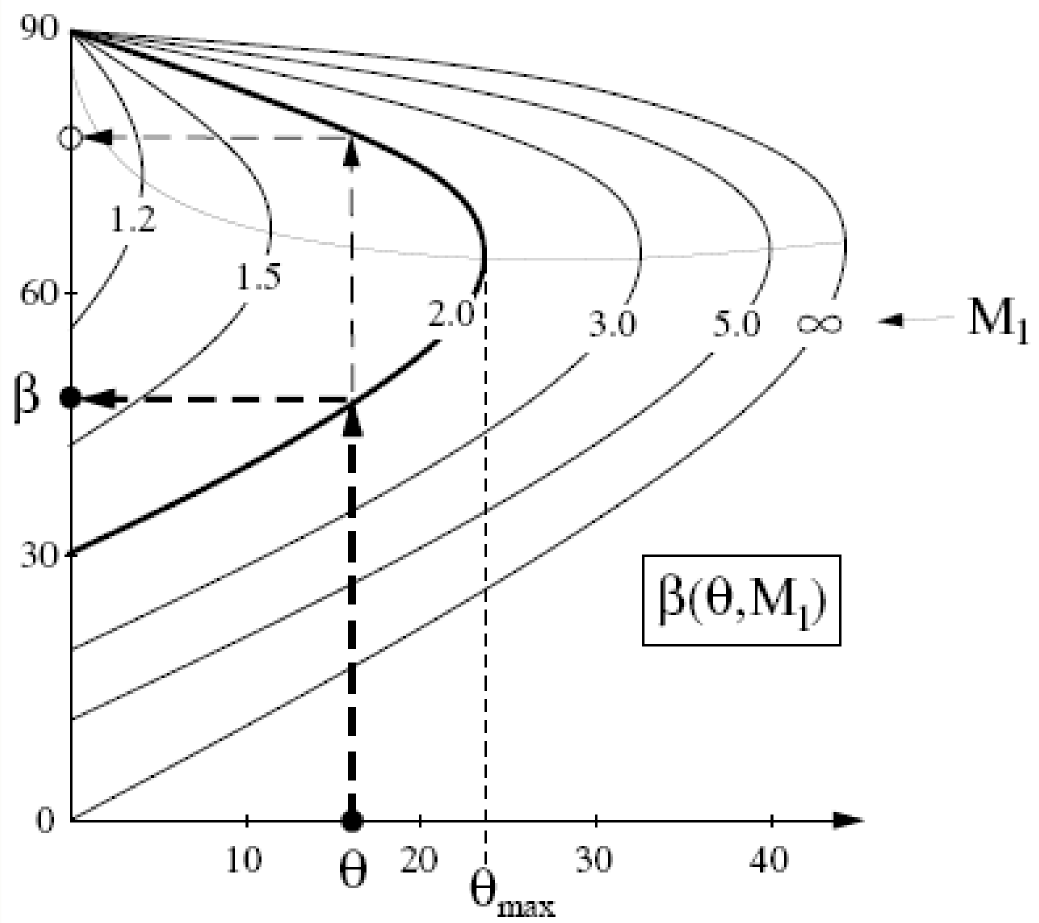
$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_{n_1}^2 - 1\right)$$

$$\frac{h_2}{h_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2}$$

$$\frac{p_{o_2}}{p_{o_1}} = \frac{p_2}{p_1}\left(\frac{h_1}{h_2}\right)^{\gamma/(\gamma-1)}$$

$$\tan\theta = \frac{2}{\tan\beta}\,\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}$$





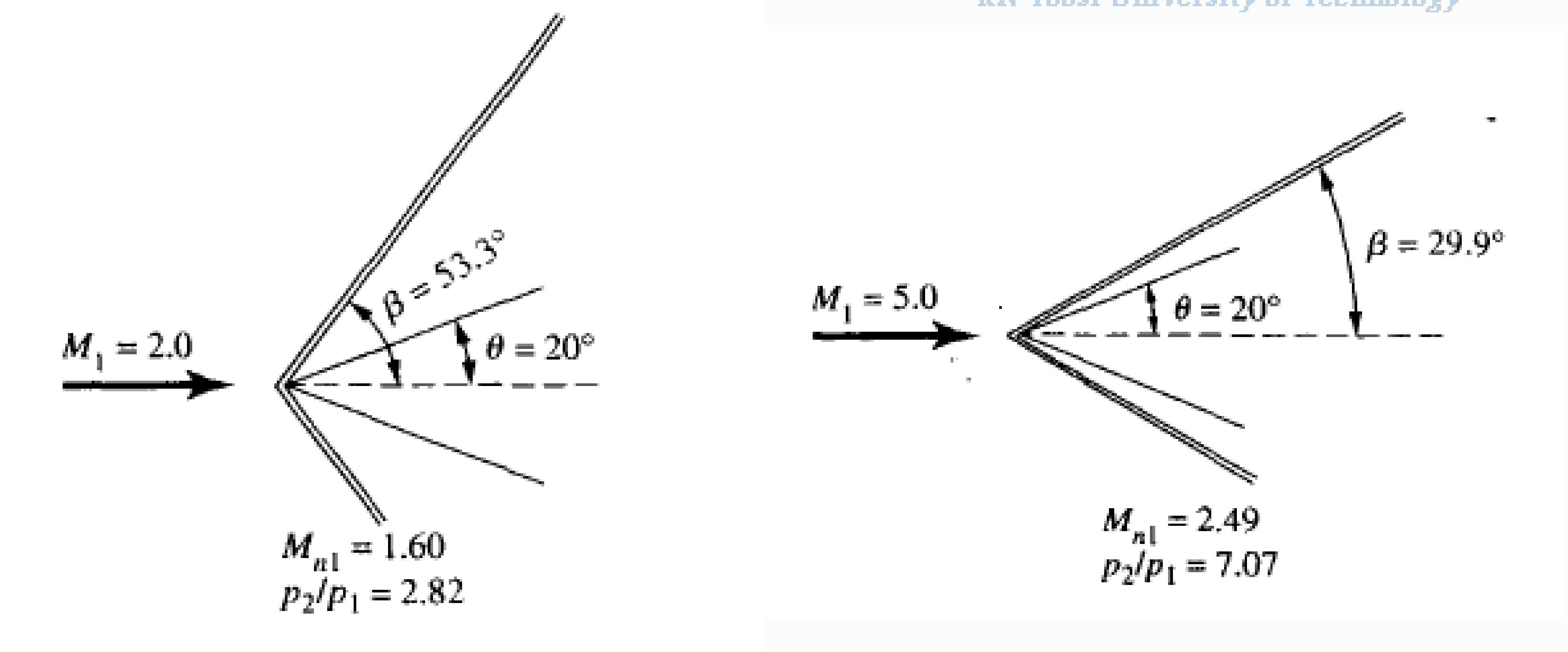
1-There is a maximum turning angle θ_{max} for any given upstream Mach number $M1$. If the wall angle exceeds this, or $\theta > \theta_{max}$, no oblique shock is possible. Instead, a detached shock forms ahead of the concave corner. Such a detached shock is in fact the same as a bow shock discussed earlier.

1. If $\theta < \theta_{max}$, two distinct oblique shocks with two different β angles are physically possible. The smaller β case is called a weak shock, and is the one most likely to occur in a typical supersonic flow. The larger β case is called a strong shock, and is unlikely to form over a straight-wall wedge. The strong shock has a subsonic flow behind it.

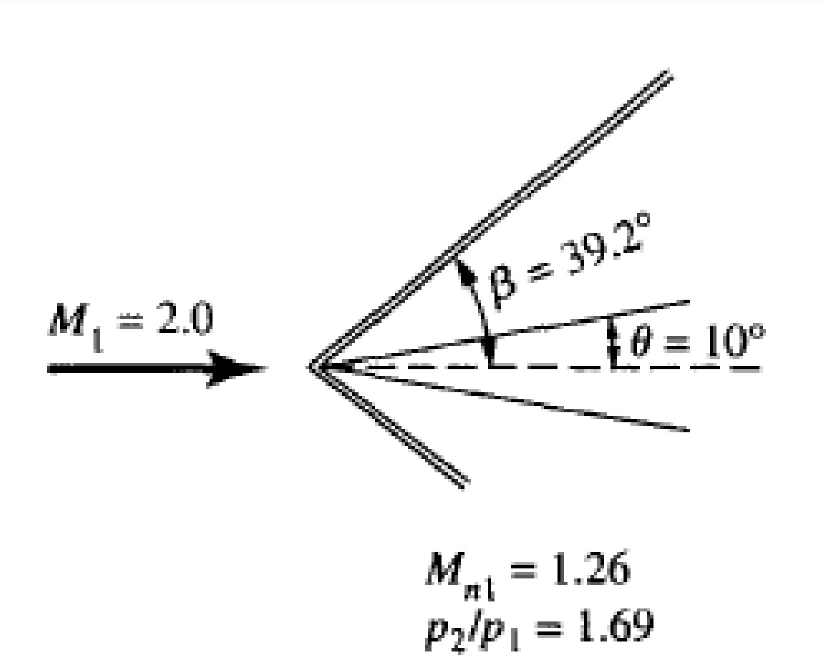
2. The strong-shock case in the limit $\theta \rightarrow 0$ and $\beta \rightarrow 90^\circ$, in the upper-left corner of the oblique shock chart, corresponds to the normal-shock case.

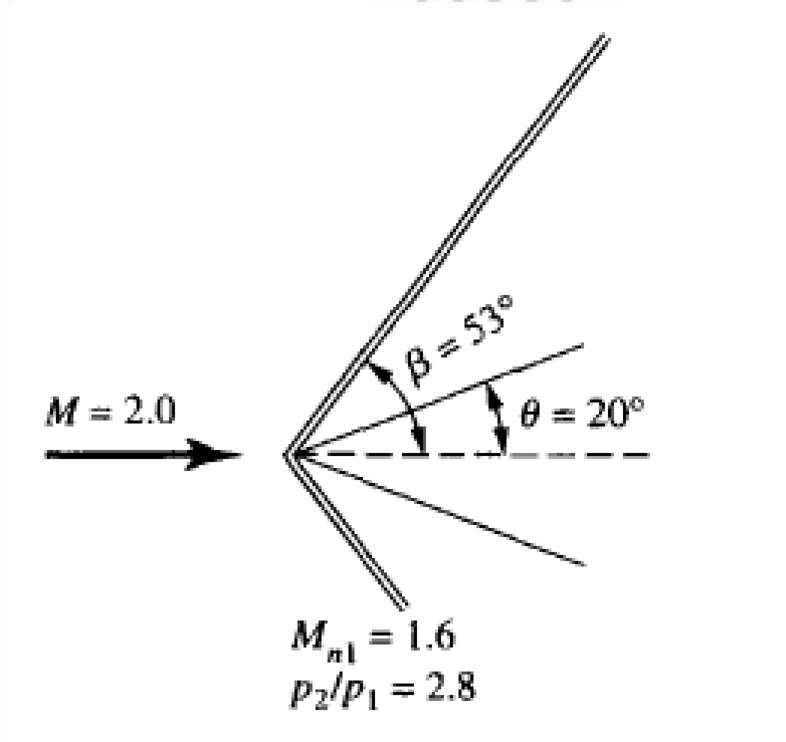
4- The Weak-shock case in the limit $\theta \rightarrow 0$ and $\beta \rightarrow \mu$, in the upper-left corner of the oblique shock chart, corresponds to the normal-shock case.

Effects of increasing the upstream Mach number.



Effect of increasing the deflection angle





Example

Consider a supersonic flow with M = 2, p = 1 atm, and T = 288 K. This flow is deflected at a compression corner through 20°. Calculate M, p, T, po, and T0 behind the resulting oblique shock wave.

for M1 = 2 and *θ = 20°* ⇒ β = 53.4°

Mn,1 = M1 sin β = 2 sin 53.4° = 1.606.

$$M_{n_2}^2 = \frac{1 + \frac{\gamma-1}{2} M_{n_1}^2}{\gamma M_{n_1}^2 - \frac{\gamma-1}{2}}$$

Mn, 2 = 0.6684

$$\frac{p_2}{p_1} = 2.82 \qquad \frac{T_2}{T_1} = 1.388 \qquad \frac{p_{0,2}}{p_{0,1}} = 0.8952$$

$$M_2 = \frac{M_{n,2}}{\sin(\beta - \theta)} = \frac{0.6684}{\sin(53.4 - 20)} = \boxed{1.21}$$

$$p_2 = \frac{p_2}{p_1} p_1 = 2.82(1 \text{ atm}) = \boxed{2.82 \text{ atm}}$$

$$T_2 = \frac{T_2}{T_1} T_1 = 1.388(288) = \boxed{399.7 \text{ K}}$$

For M1 = 2, $p_{0,1}/p_1 = 7.824$ and $T_{0,1}/T_1 = 1.8$

$$p_{0,2} = \frac{p_{0,2}}{p_{0,1}} \frac{p_{0,1}}{p_1} p_1 = 0.8952(7.824)(1 \text{ atm}) = \boxed{7.00 \text{ atm}}$$

$$T_{0,2} = T_{0,1} = \frac{T_{0,1}}{T_1} T_1 = 1.8(288) = \boxed{518.4 \text{ K}}$$

Example

Consider an oblique shock wave with a wave angle of 30°. The upstream flow Mach number is 2.4. Calculate the deflection angle of the flow, the pressure and temperature ratios across the shock wave, and the Mach number behind the wave.

$$M_1 = 2.4 \text{ and } \beta = 30^\circ$$

$$\tan\theta = \frac{2}{\tan\beta}\,\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}$$

$$\boxed{\theta = 6.5^\circ}$$

$$M_{n,1} = M_1 \sin\beta = 2.4 \sin 30^\circ = 1.2$$

$$\frac{p_2}{p_1} = \boxed{1.513}$$

$$\frac{T_2}{T_1} = \boxed{1.128}$$

$$M_{n,2} = 0.8422$$

$$M_2 = \frac{M_{n,2}}{\sin(\beta - \theta)} = \frac{0.8422}{\sin(30 - 6.5)} = \boxed{2.11}$$

Two aspects are illustrated by this example:

1. This is a fairly weak shock wave only a 51 percent increase in pressure across the wave. Indeed this case is close to that of a Mach wave, where $\mu = \sin^{-1}(1/M) = \sin^{-1}(1/2.4) = 24.6°$. *The* shock-wave angle of 30° is not much larger than μ; the deflection angle of 6.5° is also small—consistent with the relative weakness of the shock wave.

2. Only two properties need to be specified in order to define uniquely a given oblique shock wave. In this example, *M_1 and θ were those two properties.* In pervious example, the specified *M_1 and θ were the two properties. Once any* two properties about the oblique shock are specified, the shock is uniquely defined. This is analogous to the case of a normal shock wave studied in Previous chapter. There, we proved that all the changes across a normal shock wave were uniquely defined by specifying only *one property, such as M_1.*

Example

Consider an oblique shock wave with *β = 35° and a pressure ratio p2/p1= 3. Calculate* the upstream Mach number.

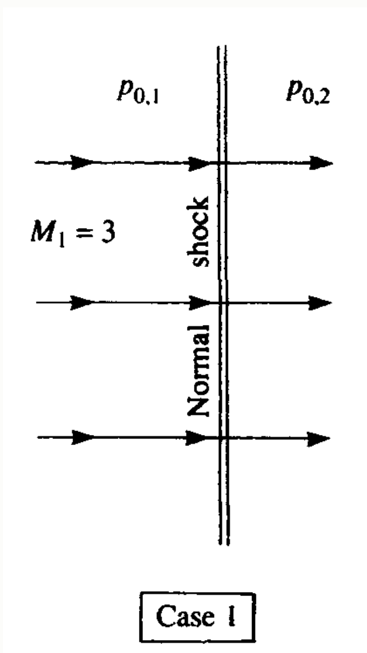
$p_2/p_1 = 3,$ $\Rightarrow$ $M_{n,1} = 1.64$

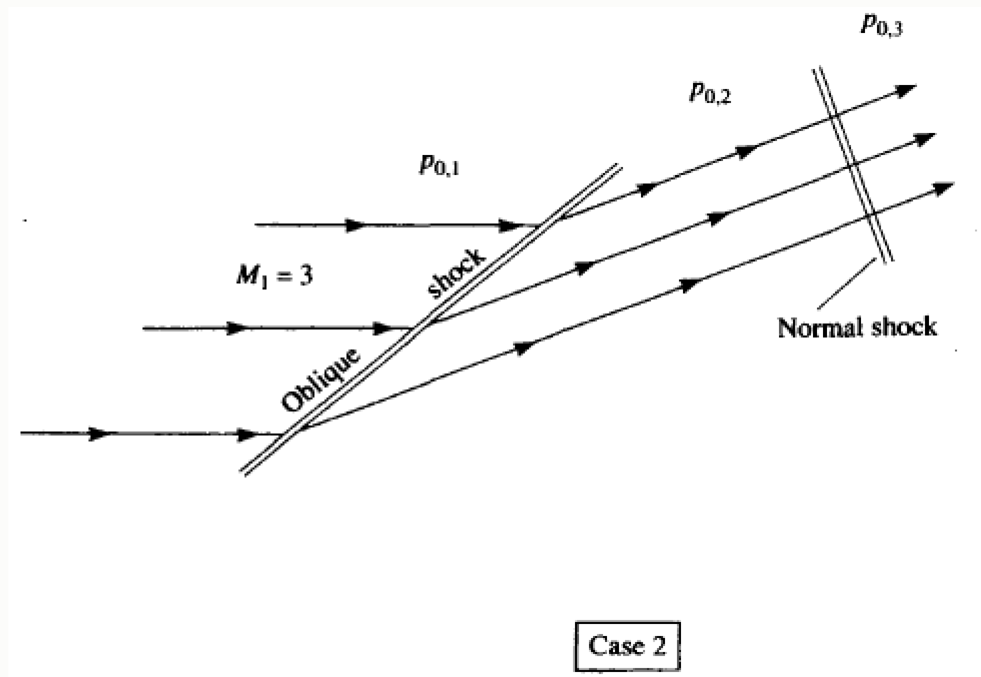
$$M_{n,1} = M_1 \sin\beta$$

$$M_1 = \frac{M_{n,1}}{\sin\beta} = \frac{1.66}{\sin 35^\circ} = \boxed{2.86}$$

Example

Consider a Mach 3 flow. It is desired to slow this flow to a subsonic speed. Consider two separate ways of achieving this: (1) the Mach 3 flow is slowed by passing directly through a normal shock wave; (2) the Mach 3 flow first passes through an oblique shock with a 40° wave angle, and then subsequently through a normal shock. These two cases are sketched in Figure. Calculate the ratio of the final total pressure values for the two cases, that is, the total pressure behind the normal shock for case 2 divided by the total pressure behind the normal shock for case 1. Comment on the significance of the result.





For case 1, at *M = 3,* $\Rightarrow$ $$\left(\frac{p_{0_2}}{p_{0_1}}\right)_{\text{case 1}} = 0.3283$$

For case 2, we have *Mn1 = M1sin β= 3 sin40° = 1.93.*

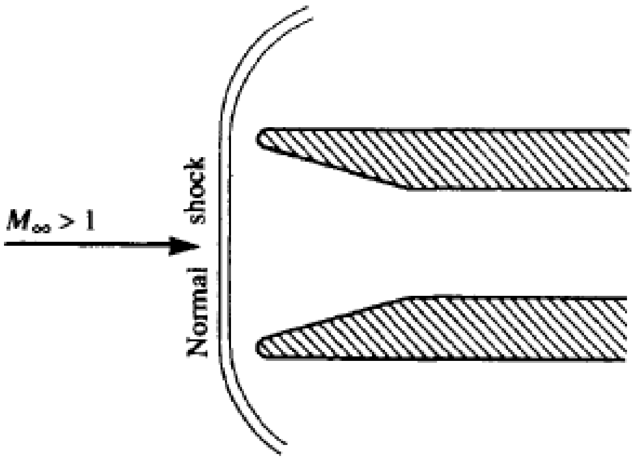
$\Rightarrow$ $$\frac{p_{0_2}}{p_{0_1}} = 0.7535 \quad \text{and} \quad M_{n,2} = 0.588$$

for $M_1 = 3$ and $\beta = 40°$ $\Rightarrow$ $\theta = 22°$

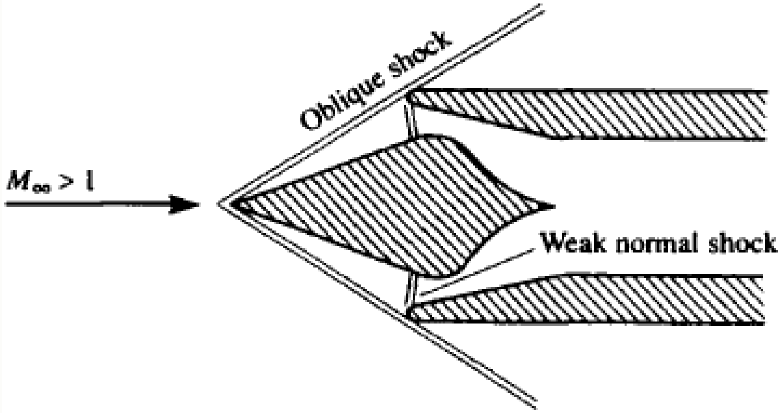
$$M_2 = \frac{M_{n,2}}{\sin(\beta - \theta)} = \frac{0.588}{\sin(40 - 22)} = 1.90$$ $\Rightarrow$ $p_{0_3}/p_{0_2} = 0.7674.$

$$\left(\frac{p_{0_3}}{p_{0_1}}\right)_{\text{case 2}} = \left(\frac{p_{0_2}}{p_{0_1}}\right)\left(\frac{p_{0_3}}{p_{0_2}}\right) = (0.7535)(0.7674) = 0.578$$

$$\left(\frac{p_{0_3}}{p_{0_1}}\right)_{\text{case 2}} \bigg/ \left(\frac{p_{0_2}}{p_{0_1}}\right)_{\text{case 1}} = \frac{0.578}{0.3283} = \boxed{1.76}$$

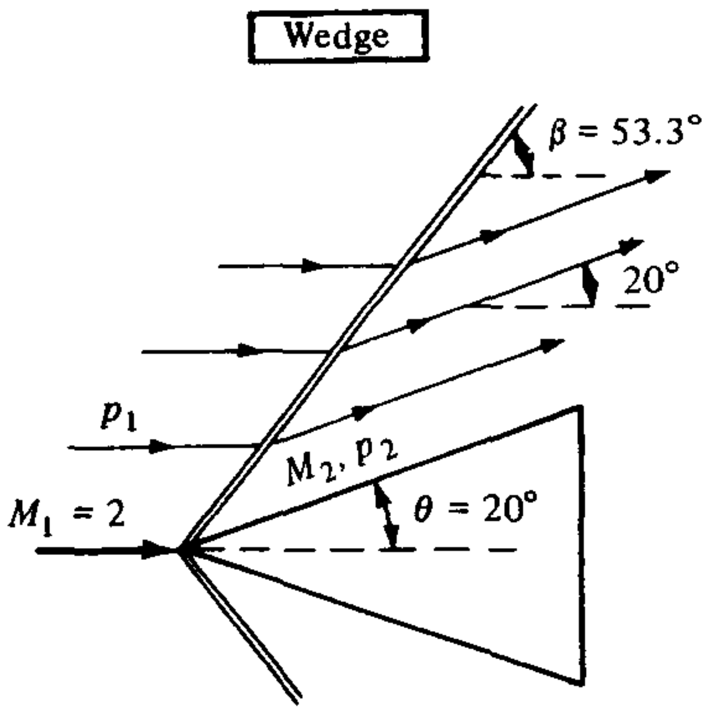


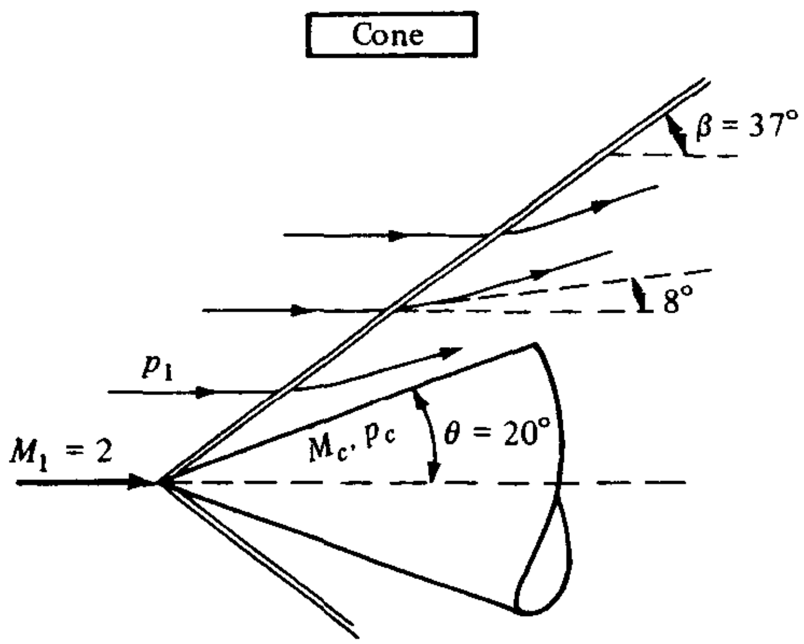
(*a*) Normal shock inlet



(*b*) Oblique shock inlet

SUPERSONIC FLOW OVER WEDGES AND CONES





The main differences between the supersonic flow over a cone and wedge, both with the same body angle, are that

(1) the shock wave on the cone is weaker,

(2) the cone surface pressure is less,

(3) the streamlines above the cone surface are curved

Example

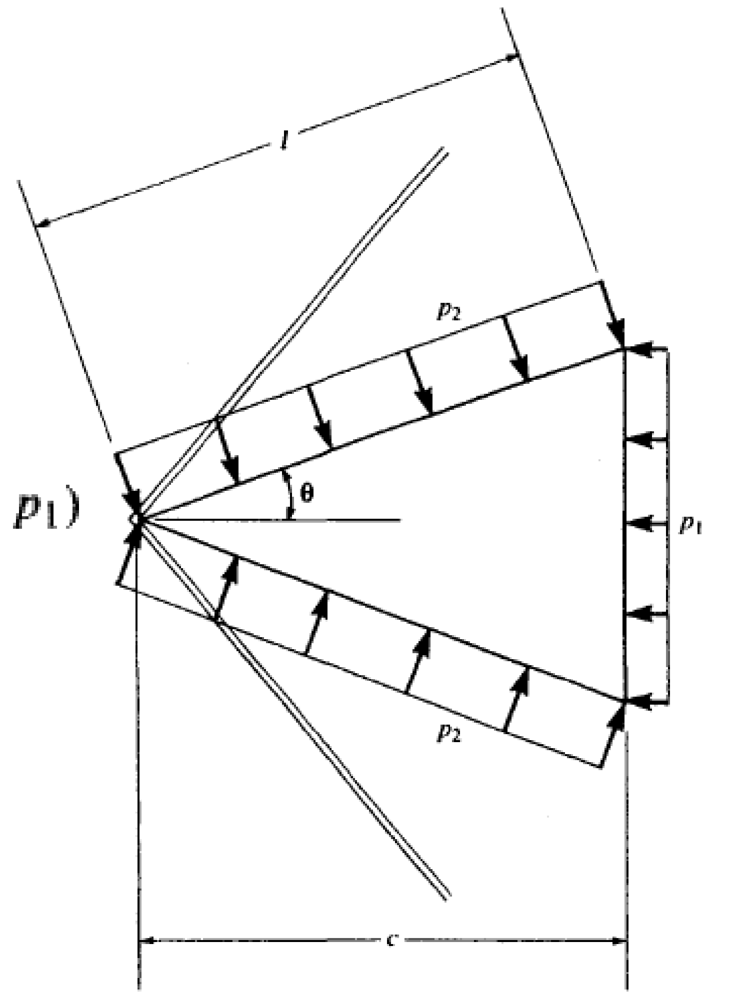
Consider a wedge with a 15° half angle in a Mach 5 flow, as sketched in the figure. Calculate the drag coefficient for this wedge. (Assume that the pressure over the base is equal to freestream static pressure, as shown in the figure)

$$c_d = \frac{D'}{q_1 S} = \frac{D'}{q_1 c(1)} = \frac{D'}{q_1 c}$$

$$D' = 2p_2 l \sin\theta - 2p_1 l \sin\theta = (2l \sin\theta)(p_2 - p_1)$$

$$D' = 2p_2 l \sin\theta - 2p_1 l \sin\theta = (2l \sin\theta)(p_2 - p_1)$$

$$l = \frac{c}{\cos\theta}$$



$$D' = (2c \tan\theta)(p_2 - p_1)$$

$$c_d = (2 \tan\theta)\left(\frac{p_2 - p_1}{q_1}\right)$$

$$q_1 \equiv \frac{1}{2}\rho_1 V_1^2 = \frac{1}{2}\rho_1 \frac{\gamma p_1}{\gamma p_1} V_1^2 = \frac{\gamma p_1}{2a_1^2} V_1^2 = \frac{\gamma}{2} p_1 M_1^2$$

$$c_d = (2\tan\theta)\left(\frac{p_2 - p_1}{(\gamma/2)p_1 M_1^2}\right) = \frac{4\tan\theta}{\gamma M_1^2}\left(\frac{p_2}{p_1} - 1\right)$$

$$c_d = (2\tan\theta)\left(\frac{p_2 - p_1}{(\gamma/2)p_1 M_1^2}\right) = \frac{4\tan\theta}{\gamma M_1^2}\left(\frac{p_2}{p_1} - 1\right)$$

for $M_1 = 5$ and $\theta = 15°$

$$\beta = 24.2°$$

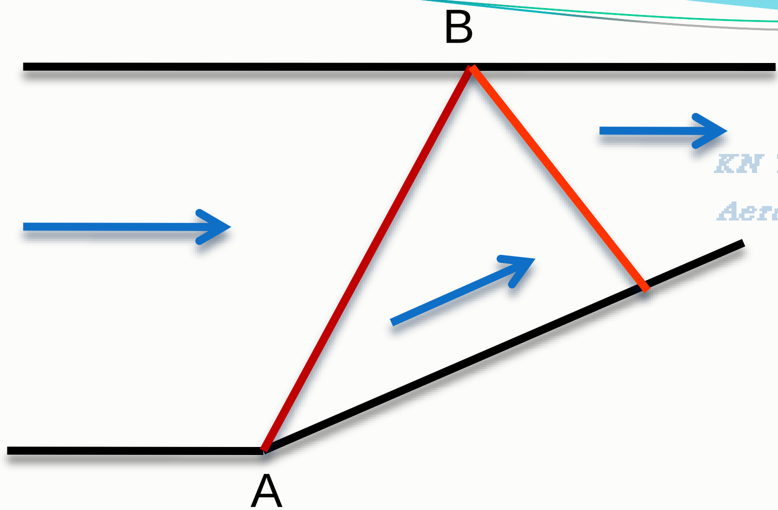
$$M_{n,1} = M_1 \sin\beta = 5\sin(24.2°) = 2.05$$

$$\frac{p_2}{p_1} = 4.736$$

$$c_d = \frac{4\tan\theta}{\gamma M_1^2}\left(\frac{p_2}{p_1} - 1\right) = \frac{4\tan 15°}{(1.4)(5)^2}(4.736 - 1) = \boxed{0.114}$$

The drag is finite for this case. In a supersonic or hypersonic inviscid flow over a two-dimensional body, the drag is always finite. D'Alembert's paradox does not hold for freestream Mach numbers such that shock waves appear in t he flow.

SHOCK INTERACTIONS AND REFLECTIONS



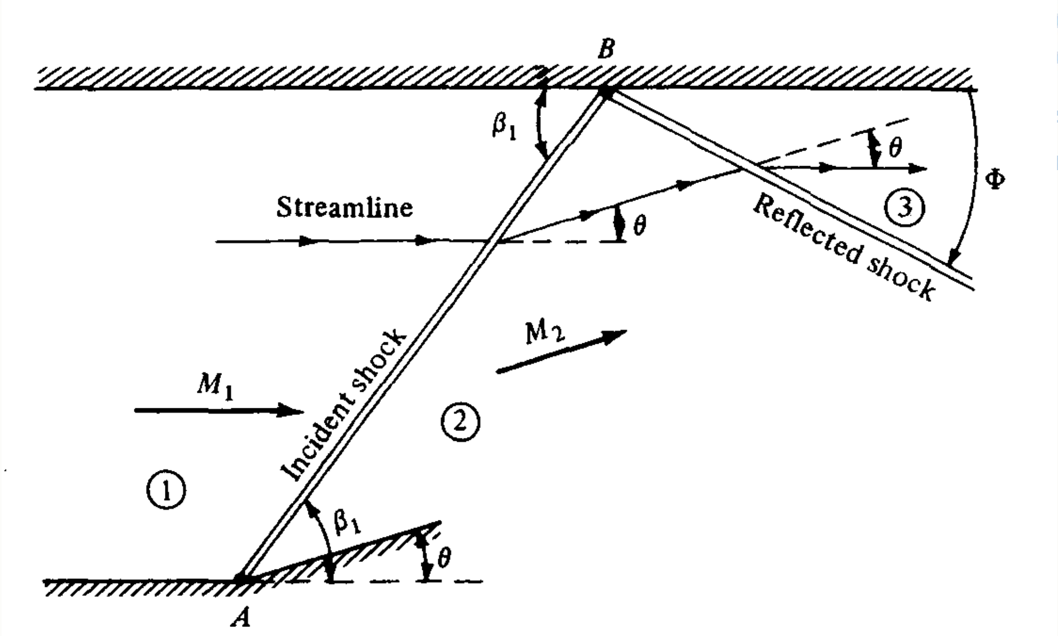
consider an oblique shock wave generated by a concave corner,

Assume that a straight, horizontal wall is present above the corner,

The shock wave generated at point A, called the incident shock wave, impinges on the upper wall at point B.

Question: ***Does the shock wave simply disappear at point B? If not,*** **what happens to it?**

SHOCK INTERACTIONS AND REFLECTIONS

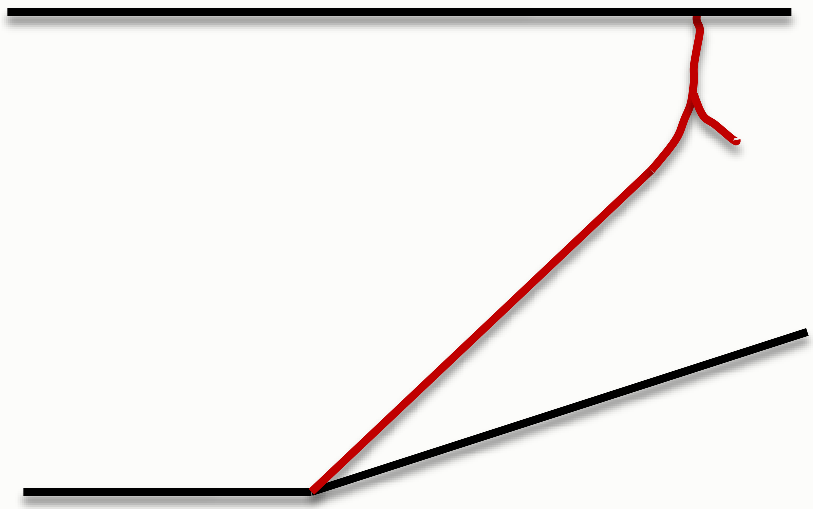


the flow must be tangent everywhere along the upper wall; if the flow in region 2 were to continue unchanged, it would run into the wall and have no place to go. Hence, the flow in region 2 must eventually be bent downward through the angle θ in order to maintain a flow tangent to the upper wall. Nature accomplishes this downward deflection via a second shock wave originating at the impingement point *B 9. This second shock is called the* ***reflected shock wave.***

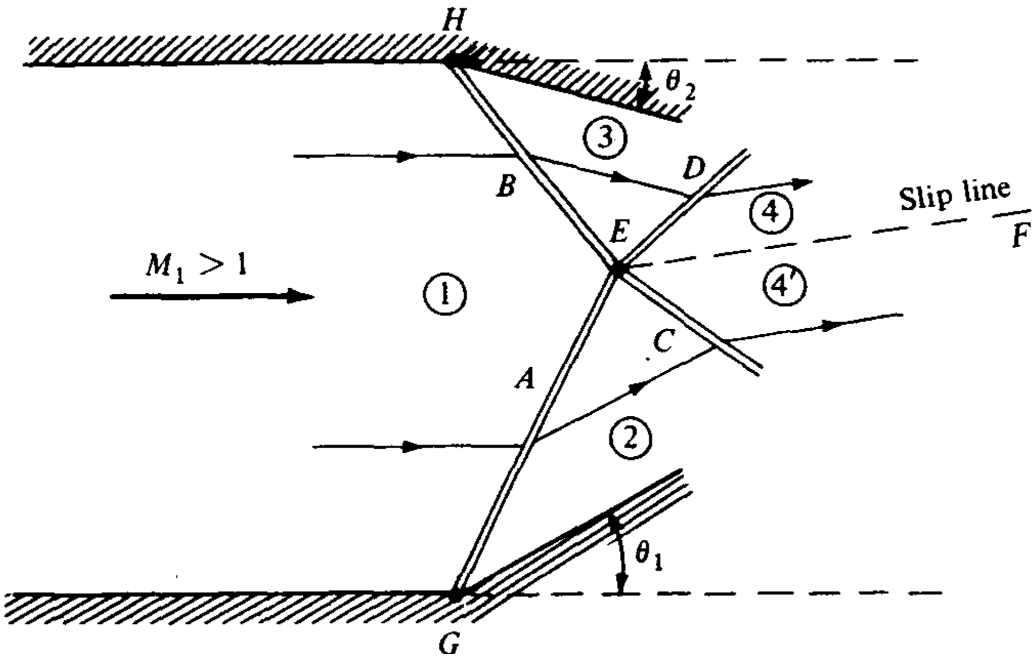
Assume that *M1 is only slightly* above the minimum Mach number necessary for a straight, attached shock wave at the given deflection angle *θ.*

we know that the Mach number decreases across a shock . *This* decrease may be enough such that *M2 is not above the minimum Mach number for* the required deflection *θ through the reflected shock.*

In such a case, our oblique shock theory does not allow a solution for a straight reflected shock wave.

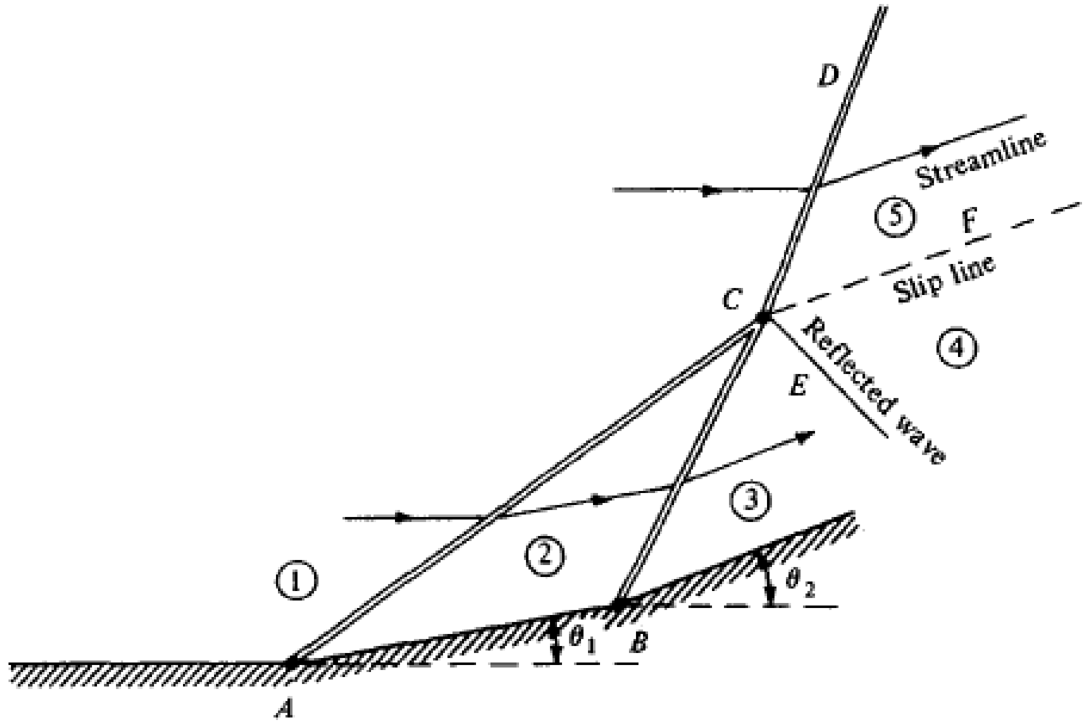


Another type of shock interaction is shown in figure.



At the intersection, wave A is refracted and continues as wave D. Similarly, wave B is refracted and continues as wave C. The flow behind the refracted shock D is denoted by region 4; the flow behind the refracted shock C is denoted by region 4'.

Across the slip line, the pressures are constant (i.e., $p_4 = p_{4'}$) , and the direction (but not necessarily the magnitude) of velocity is the same, namely, parallel to the slip line. All other properties in regions 4 and 4' are different, most notably the entropy $s_4 \neq s_{4'}$).

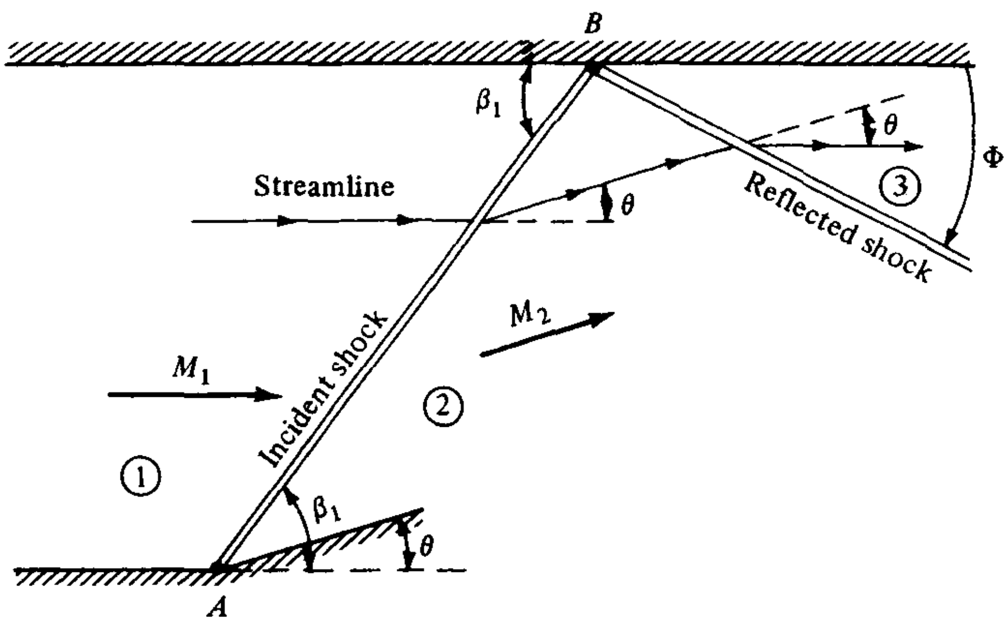


The intersection occurs at point *C, at which the two shocks merge*

and propagate as the stronger shock CD, usually along with a weak reflected wave CE. This reflected wave is necessary to adjust the flow so that the velocities in regions 4 and 5 are in the same direction. Again, a slip line CF trails downstream of the intersection point.

EXAMPLE

Consider an oblique shock wave generated by a compression corner with a 10° deflection angle. The Mach number of the flow ahead of the corner is 3.6; the flow pressure and temperature are standard sea level conditions. The oblique shock wave subsequently impinges on a straight wall opposite the compression comer. Calculate the angle of the reflected shock wave Φ relative to the straight wall. Also, obtain the pressure, temperature, and Mach number behind the reflected wave.



$$M_1 = 3.6 \text{ and } \theta = 10^\circ \xrightarrow{\theta\text{-}\beta\text{-}M \text{ diagram}} \beta_1 = 24^\circ$$

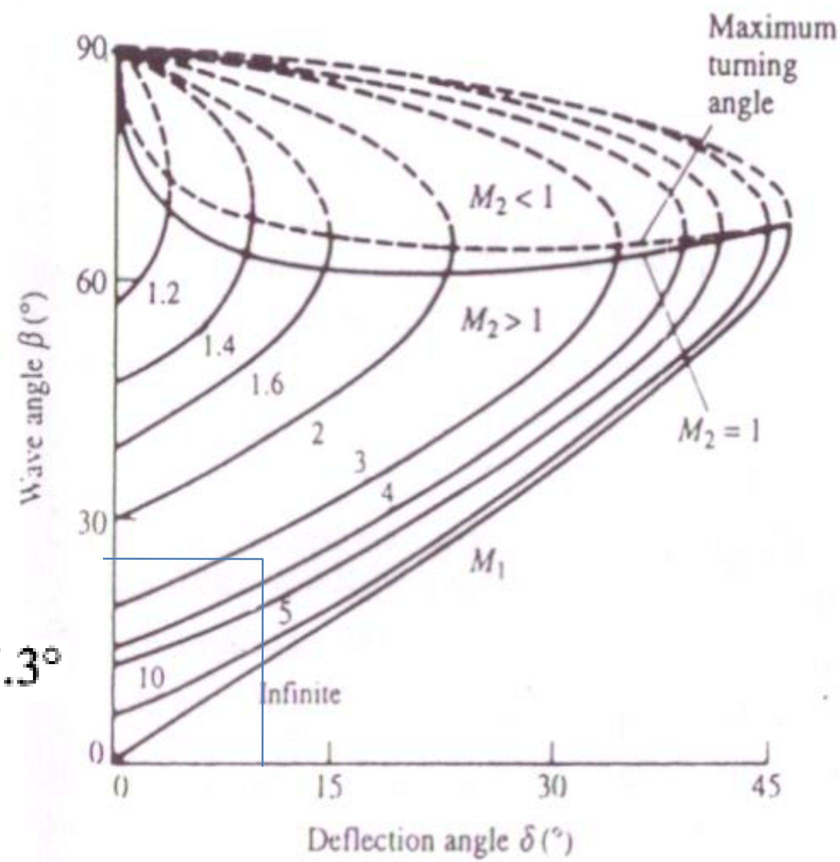
$$M_{n,1} = M_1 \sin\beta_1 = 3.6 \sin 24^\circ = 1.464$$

$$\Downarrow$$

$$M_{n,2} = 0.7157 \qquad \frac{p_2}{p_1} = 2.32 \qquad \frac{T_2}{T_1} = 1.294$$

$$M_2 = \frac{M_{n,2}}{\sin(\beta - \theta)} = \frac{0.7157}{\sin(24 - 10)} = 2.96$$

$$M_2 = 2.96 \text{ and } \theta = 10^\circ \xrightarrow{\theta\text{-}\beta\text{-}M \text{ diagram}} \beta_2 = 27.3^\circ$$



$$\Phi = \beta_2 - \theta = 27.3 - 10 = \boxed{17.3^\circ}$$

Also, the normal component of the upstream Mach number relative to the reflected shock is

$$M_2 \sin \beta_2 = (2.96) \sin 27.3^\circ = 1.358.$$

$$\frac{p_3}{p_2} = 1.991 \qquad \frac{T_3}{T_2} = 1.229 \qquad M_{n,3} = 0.7572$$

$$M_3 = \frac{M_{n,3}}{\sin(\beta_2 - \theta)} = \frac{0.7572}{\sin(27.3 - 10)} = \boxed{2.55}$$

For standard sea level conditions, $p_1 = 2116\ \mathrm{lb/ft^3}$ and $T_1 = 519^\circ\mathrm{R}$. Thus,

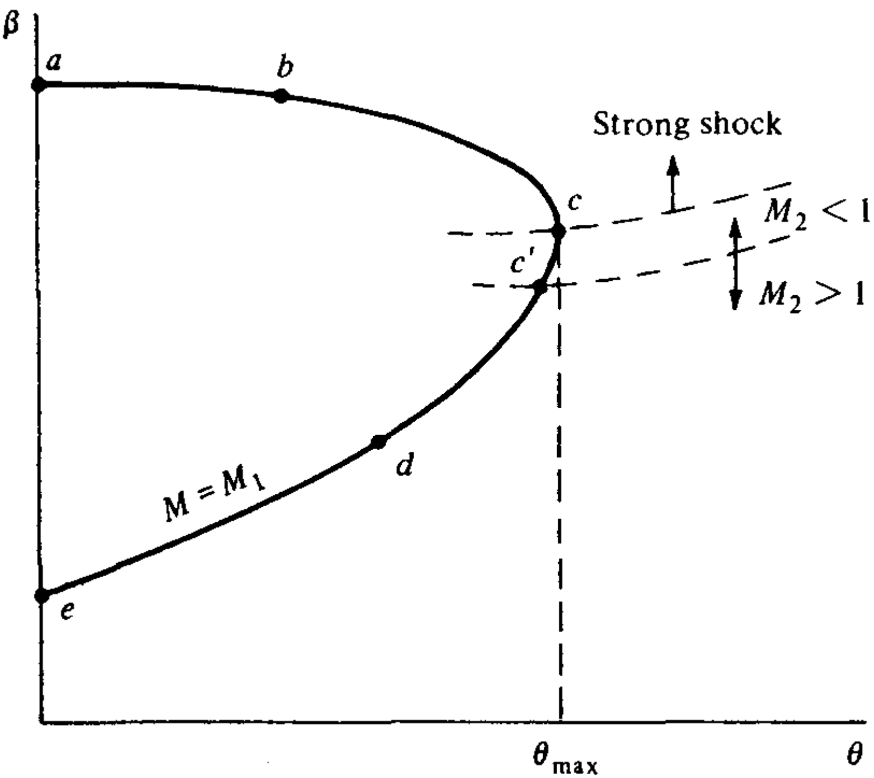
$$p_3 = \frac{p_3}{p_2}\frac{p_2}{p_1}p_1 = (1.991)(2.32)(2116) = \boxed{9774\ \mathrm{lb/ft^3}}$$

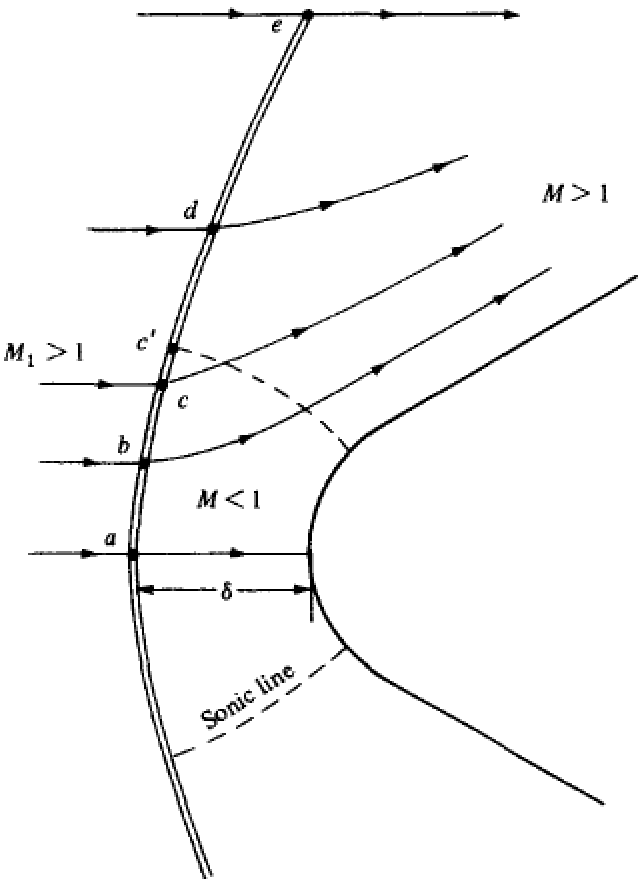
$$T_3 = \frac{T_3}{T_2}\frac{T_2}{T_1}T_1 = (1.229)(1.294)(519) = \boxed{825^\circ\mathrm{R}}$$

Note that the reflected shock is weaker than the incident shock,

DETACHED SHOCK WAVE IN FRONT OF A BLUNT BODY

The curved bow shock which stands in front of a blunt body in a supersonic flow is sketched in Figure

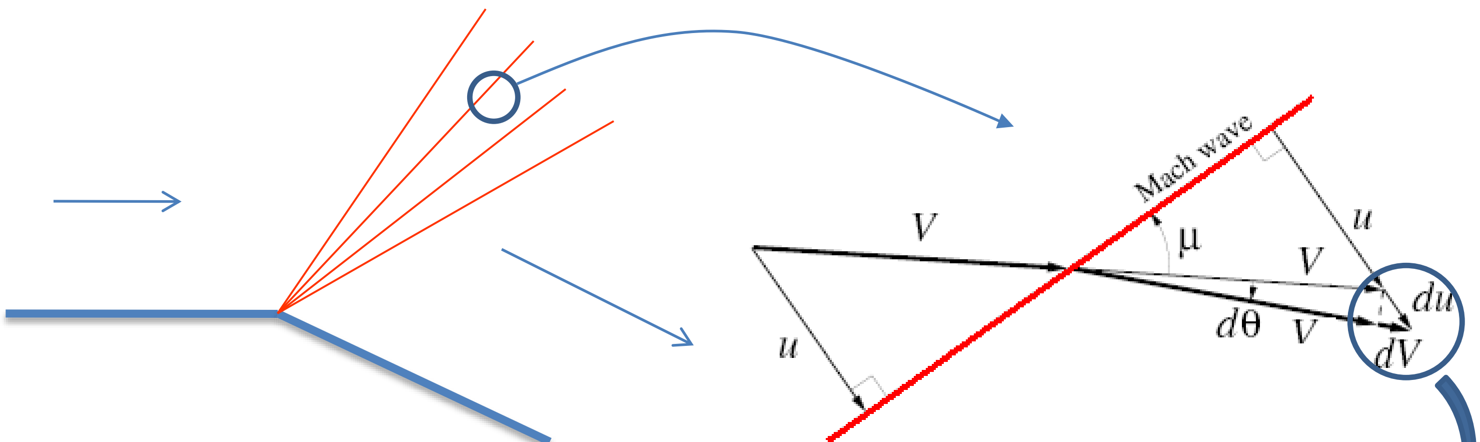


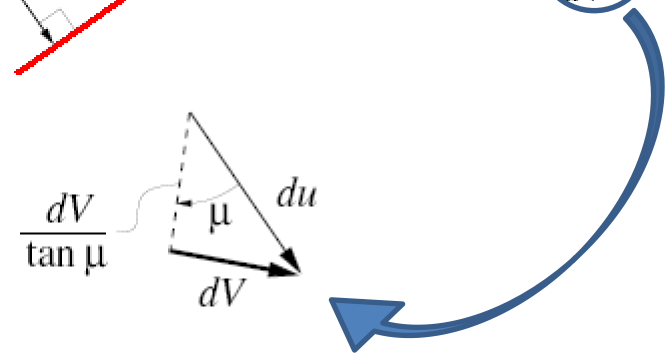


δ is defined as the *shock detachment distance.*

The solution of this flow field is not trivial. Indeed, the supersonic blunt-body problem was a major focus for supersonic aerodynamicists during the 1950s and 1960s, spurred by the need to understand the high-speed flow over blunt-nosed missiles and reentry bodies.

PRANDTL-MEYER EXPANSION WAVES





From the u-V and du-dV velocity triangles:

$$d\theta = \frac{dV}{\tan\mu}\frac{1}{V}$$

$$\frac{1}{\tan\mu} = \frac{\cos\mu}{\sin\mu} = \frac{\sqrt{1-\sin^2\mu}}{\sin\mu} \xrightarrow{\sin\mu = 1/M} \frac{\sqrt{1-\sin^2\mu}}{\sin\mu} = \frac{\sqrt{1-1/M^2}}{1/M}$$

$$\frac{\sqrt{1-1/M^2}}{1/M} = \sqrt{M^2-1}$$

$$d\theta = \frac{dV}{\tan\mu}\frac{1}{V}$$

$$d\theta = \sqrt{M^2-1}\,\frac{dV}{V}$$

Prandtl-Meyer Function

$$V = M\,a = M\,a_o \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-1/2}$$

$$\ln V = \ln M + \ln a_o - \frac{1}{2} \ln\left(1 + \frac{\gamma - 1}{2} M^2\right)$$

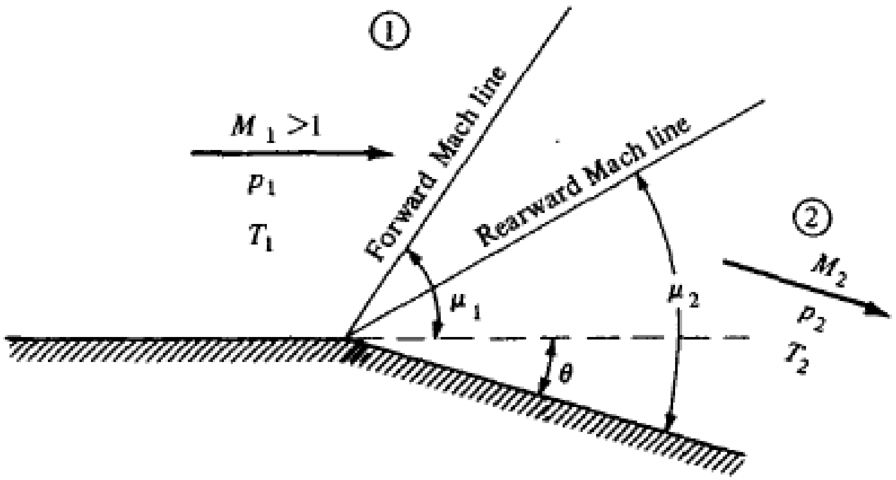
$$\frac{dV}{V} = \frac{dM}{M} - \frac{1}{2} \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-1} \frac{\gamma - 1}{2} 2M\,dM$$

$$\frac{dV}{V} = \frac{1}{1 + \frac{\gamma-1}{2} M^2} \frac{dM}{M}$$

$$d\theta = \sqrt{M^2 - 1}\,\frac{dV}{V}$$

$$d\theta = \frac{\sqrt{M^2 - 1}}{1 + \frac{\gamma-1}{2} M^2} \frac{dM}{M}$$

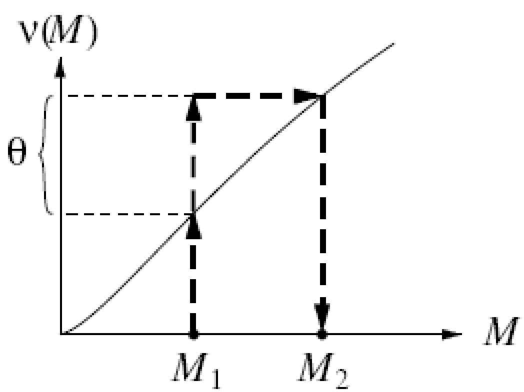
$$\int_{\theta_1}^{\theta_2} d\theta \;=\; \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\,\frac{dM}{M}$$

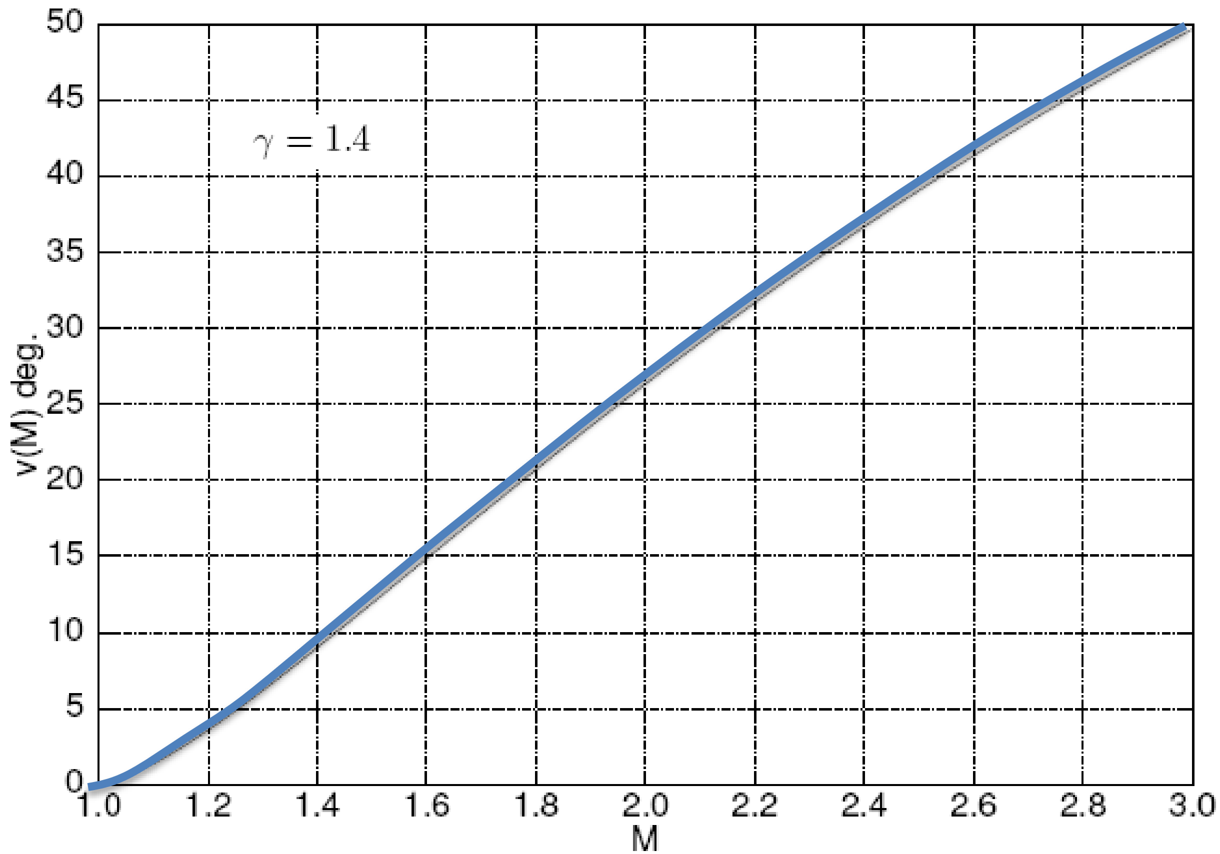


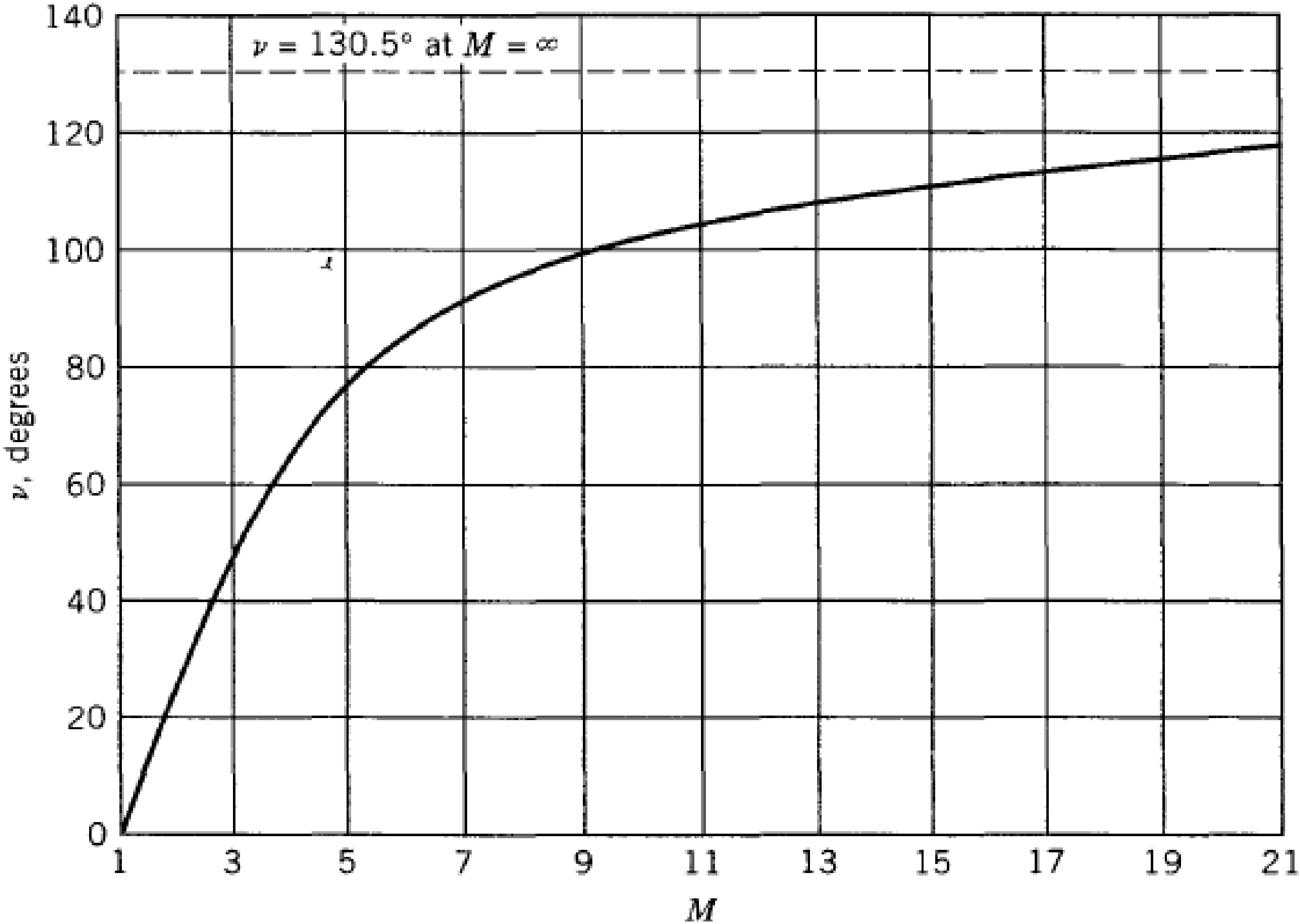
$$\overbrace{\theta_2-\theta_1}^{\theta} \;=\; \nu(M_2) \;-\; \nu(M_1)$$

where $$\nu(M) \;\equiv\; \sqrt{\frac{\gamma+1}{\gamma-1}}\,\arctan\sqrt{\frac{\gamma-1}{\gamma+1}\,(M^2-1)} \;-\; \arctan\sqrt{M^2-1}$$

ν (M) is called the Prandtl-Meyer function,

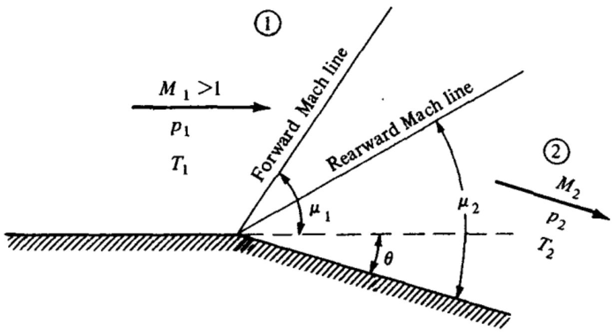

ν(M)
θ
M
M1
M2


γ = 1.4
ν(M) deg.
M
0
5
10
15
20
25
30
35
40
45
50
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0



$\nu = 130.5°$ at $M = \infty$
ν, degrees
0
20
40
60
80
100
120
140
1
3
5
7
9
11
13
15
17
19
21
M

How do the above results solve the problem stated in Figure



1. For the given M1, *obtain* ν(M1)

2. Calculate ν(M2) from using θ + ν(M1)= v(M2)

3. Obtain M2 from Appendix C corresponding to the value of ν(M2) or using prandtl-Meyer function

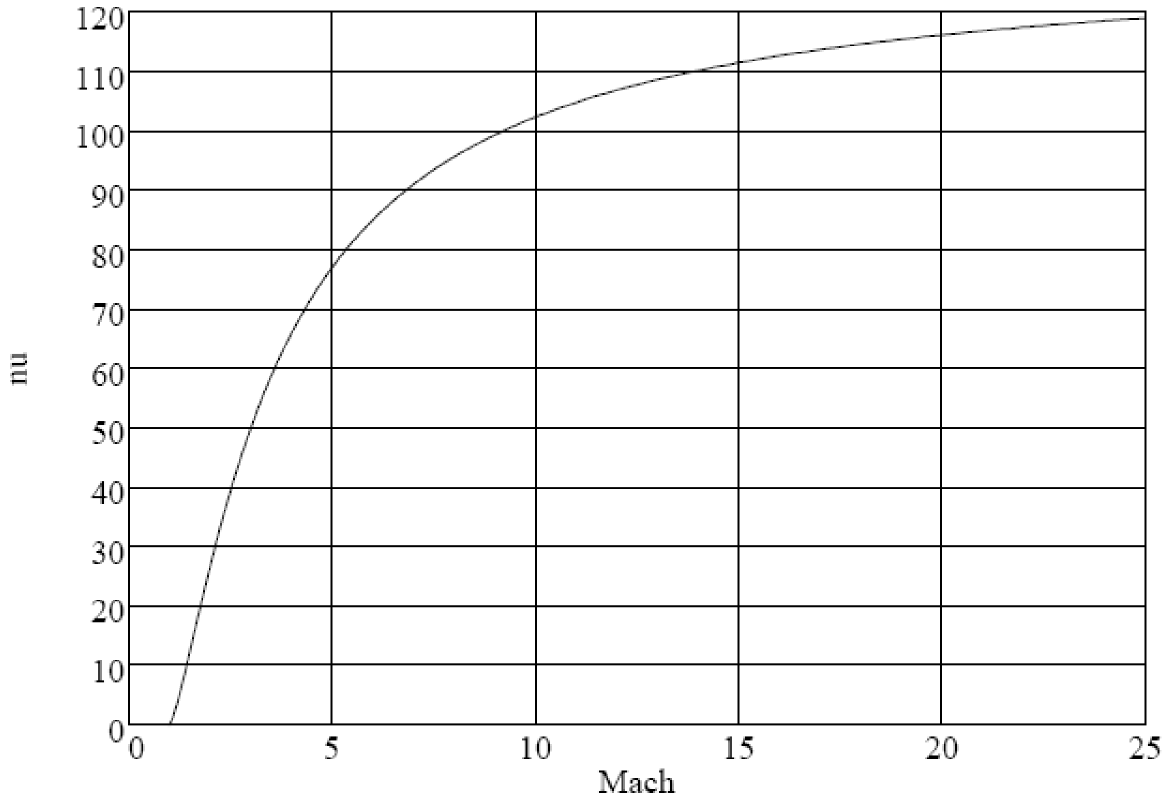
$$\boxed{\nu = \left(\frac{\gamma+1}{\gamma-1}\right)^{1/2} \tan^{-1}\left[\frac{\gamma-1}{\gamma+1}(M^2-1)\right]^{1/2} - \tan^{-1}(M^2-1)^{1/2}}$$

4. The expansion wave is isentropic; hence, po and T0 are constant through the wave.

$$\frac{T_2}{T_1} = \frac{T_2/T_{0,2}}{T_1/T_{0,1}} = \frac{1+[(\gamma-1)/2]M_1^2}{1+[(\gamma-1)/2]M_2^2} \qquad \frac{p_2}{p_1} = \frac{p_2/p_0}{p_1/p_0} = \left(\frac{1+[(\gamma-1)/2]M_1^2}{1+[(\gamma-1)/2]M_2^2}\right)^{\gamma/(\gamma-1)}$$

How do you compute the inverse of the Prandtl-Meyer Function?

$$\nu = \frac{1}{\lambda}\tan^{-1}(\lambda\beta) - \tan^{-1}(\beta) \qquad \lambda = \sqrt{\frac{\gamma - 1}{\gamma + 1}} \qquad \beta = \sqrt{M^2 - 1}.$$



In the usual approach for calculation, you compute the value of ν for the upstream Mach number and then add the angle thru which the flow is turned. The downstream Mach number is that which corresponds to the downstream ν. But, there is no easy way to compute this.

Since the equation is relatively simple, most students of compressible flow theory try to use algebraic and trigonometric manipulations to obtain an expression for Mach as a function of ν. Alas, it does not seem possible and we are left with numerical procedures.

Given a sufficiently dense table of ν vs. Mach, one can do a reverse table lookup.

This is probably the method of choice for students who are doing an off-line calculation.

In simple shock-expansion theory. If one is developing a computing procedure this approach can still be used by building a large table of ν versus Mach and then doing interpolation as Mach versus ν.

There is a certain lack of elegance to this approach and people would like a nice neat function that simply returned Mach for a given ν.

In September 1975, I.M. Hall published a paper in the Aeronautical Journal in which he described an rational polynomial function that approximated the inverse Prandtl-Meyer function with accuracy sufficient for most engineering purposes.

The function is simply a cubic polynomial divided by a quadratic.

$$M = \frac{1 + Ay + By^2 + Cy^3}{1 + Dy + Ey^2}$$

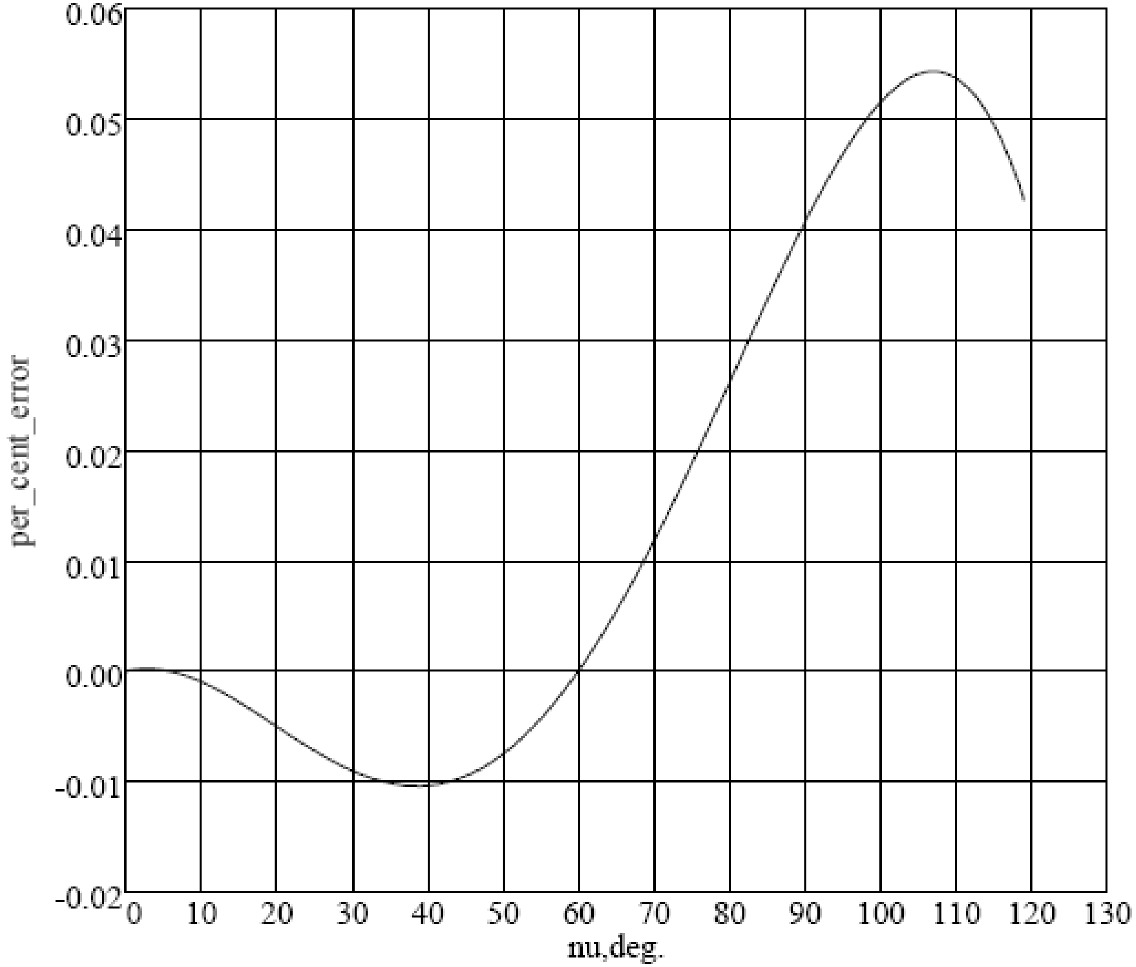
$$y = \left(\frac{\nu}{\nu_\infty}\right)^{2/3}$$

$$\nu_\infty = \frac{\pi}{2}(\sqrt{6} - 1)$$

A	1.3604
B	0.0962
C	-0.5127
D	-0.6722
E	-0.3278

All of the constants shown here are for γ=1.4. The original paper by Hall describes the calculation for other specific heats.

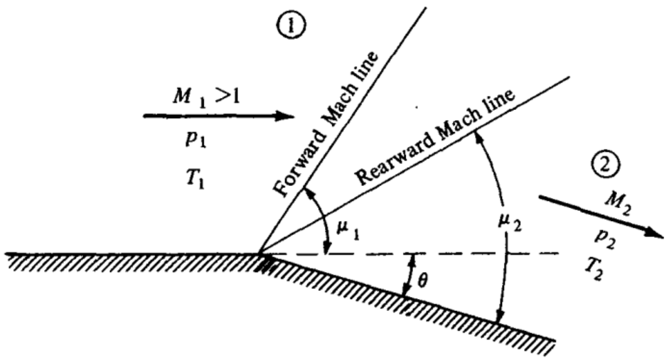
The accuracy of the Hall approximation is quite good with the maximum error much less that one-tenth of one percent.



EXAMPLE

A supersonic flow with *M1= 1.5, p1=1 atm, and T1 = 288 K is expanded* around a sharp comer) through a deflection angle of 15°. Calculate *M2, p2, T2,* P0,2, T0,2 and the angles that the forward and rearward Mach lines make with respect to the upstream flow direction.

$$\nu = \left(\frac{\gamma+1}{\gamma-1}\right)^{1/2} \tan^{-1}\left[\frac{\gamma-1}{\gamma+1}(M^2-1)\right]^{1/2} - \tan^{-1}(M^2-1)^{1/2}$$



$M_1 = 1.5 \Rightarrow \nu_1 = 11.91°$

$\nu_2 = \nu_1 + \theta = 11.91 + 15 = 26.91° \Rightarrow M_2 = 2.0$

$M_1 = 1.5, \Rightarrow p_{0,1}/p_1 = 3.671$ and $T_{0,1}/T_1 = 1.45.$

$M_2 = 2.0, \Rightarrow p_{0,2}/p_2 = 7.824$ and $T_{0,2}/T_2 = 1.8.$

Since the flow is isentropic, $T_{0,2} = T_{0,1}$ and $p_{0,2} = p_{0,1}$

$$p_2 = \frac{p_2}{p_{0,2}} \frac{p_{0,2}}{p_{0,1}} \frac{p_{0,1}}{p_1} p_1 = \frac{1}{7.824}(1)(3.671)(1 \text{ atm}) = \boxed{0.469 \text{ atm}}$$

$$T_2 = \frac{T_2}{T_{0,2}} \frac{T_{0,2}}{T_{0,1}} \frac{T_{0,1}}{T_1} T_1 = \frac{1}{1.8}(1)(1.45)(288) = \boxed{232 \text{ K}}$$

$$p_{0,2} = p_{0,1} = \frac{p_{0,1}}{p_1} p_1 = 3.671(1 \text{ atm}) = \boxed{3.671 \text{ atm}}$$

$$T_{0,2} = T_{0,1} = \frac{T_{0,1}}{T_1} T_1 = 1.45(288) = \boxed{417.6 \text{ K}}$$

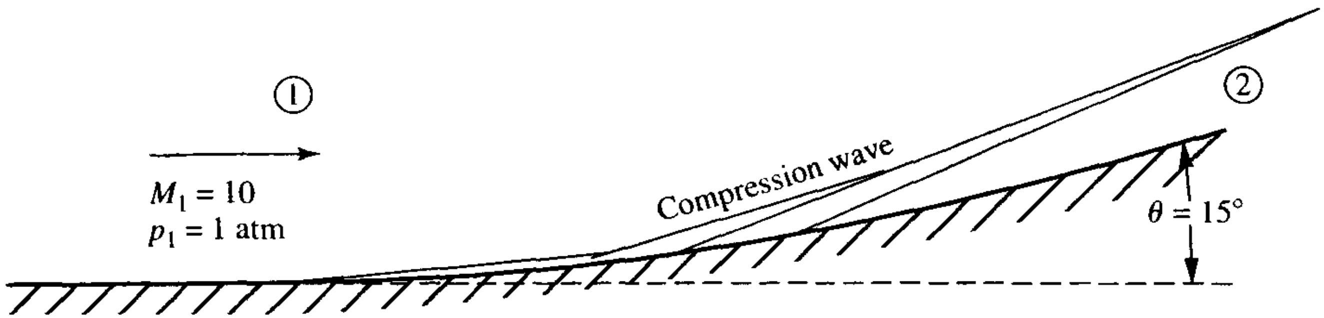
$$\text{Angle of forward Mach line} = \mu_1 = \boxed{41.81^\circ}$$

$$\text{Angle of rearward Mach line} = \mu_2 - \theta = 30 - 15 = \boxed{15^\circ}$$

EXAMPLE

An isentropic compression wave is one of the possible compression mechanisms in SCRAMjets.

Consider the isentropic compression surface sketched in Figure. The Mach number and pressure upstream of the wave are M1 = 10 and *p1= 1atm, respectively. The flow is turned through a total angle* of 15°. Calculate the Mach number and pressure in region 2 behind the compression wave. Compare the results with those of a sharp corner.



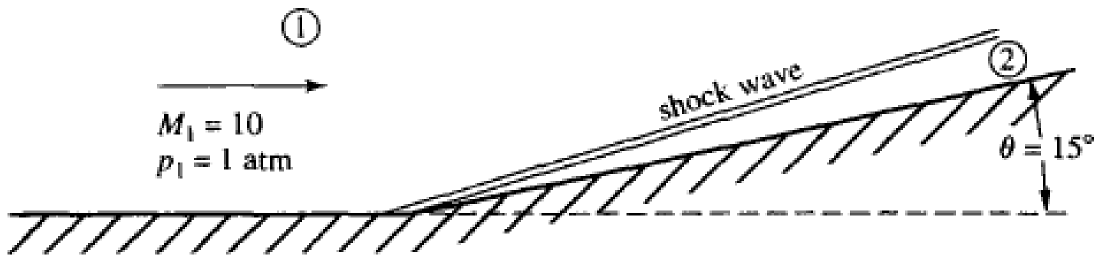
(*a*) Isentropic compression corner

$M_1 = 10 \rightarrow \nu_1 = 102.3° \rightarrow \nu_2 = \nu_1 - \theta = 102.3 - 15 = 87.3° \rightarrow \boxed{M_2 = 6.4}$

for $M_1 = 10$, $p_{0,1}/p_1 = 0.4244 \times 10^5$ and for $M_2 = 6.4$, $p_{0,2}/p_2 = 0.2355 \times 10^4$

$$p_{0,2} = p_{0,1} \rightarrow p_2 = \left(\frac{p_2}{p_{0,2}}\right)\left(\frac{p_{0,2}}{p_{0,1}}\right)\left(\frac{p_{0,1}}{p_1}\right)p_1 = \left(\frac{1}{0.2355 \times 10^4}\right)(1)(0.4244 \times 10^5)(1)$$

$$= \boxed{18.02 \text{ atm}}$$



(b) Shock compression corner

$M_1 = 10$ and $\theta = 15°$ → $\beta = 20°$ → $M_{n,1} = M_1 \sin\beta = (10)\sin 20° = 34.2$

$M_{n,1} = 3.42,$ → $p_2/p_1 = 13.32,\ p_{0,2}/p_{0,1} = 0.2322,$ and $M_{n,2} = 0.4552.$

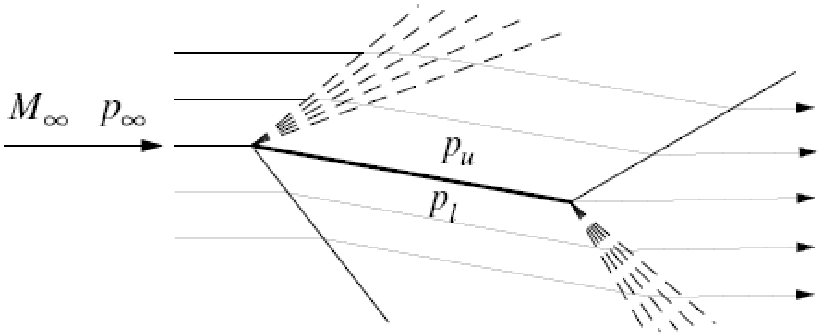
$$M_2 = \frac{M_{n,2}}{\sin(\beta - \theta)} = \frac{0.4552}{\sin(20 - 15)} = \boxed{5.22}$$

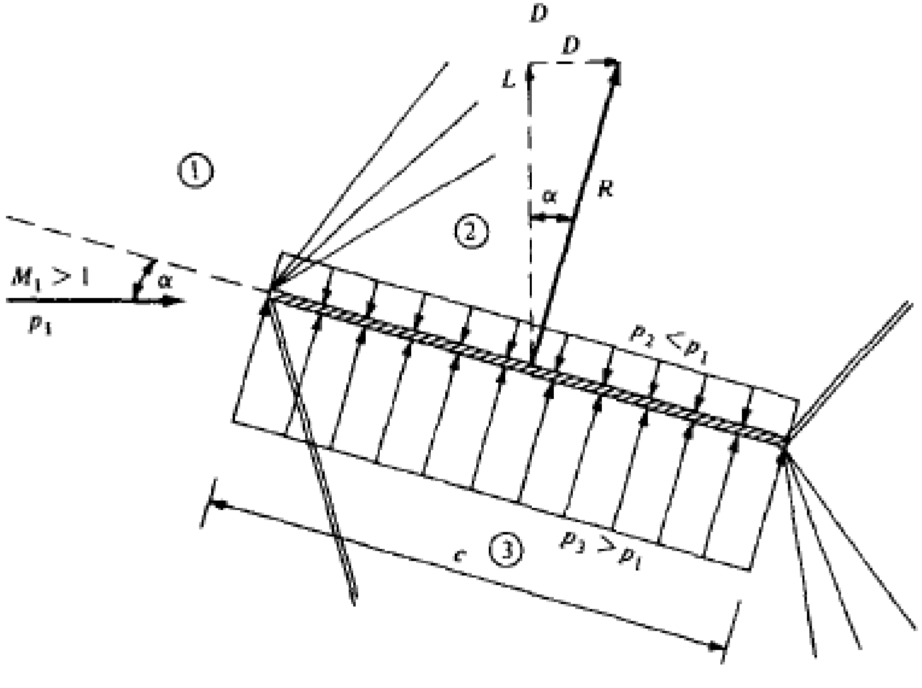
$$p_2 = (p_2/p_1)p_1 = 13.32(1) = \boxed{13.32 \text{ atm}}$$

The inefficiency of the shock wave is measured by the loss of total pressure across the shock; total pressure drops by about 77 percent across the shock. This emphasizes why designers of supersonic and hypersonic inlets would love to have the compression process carried out via isentropic compression waves.

it is a point design for the given upstream Mach number.

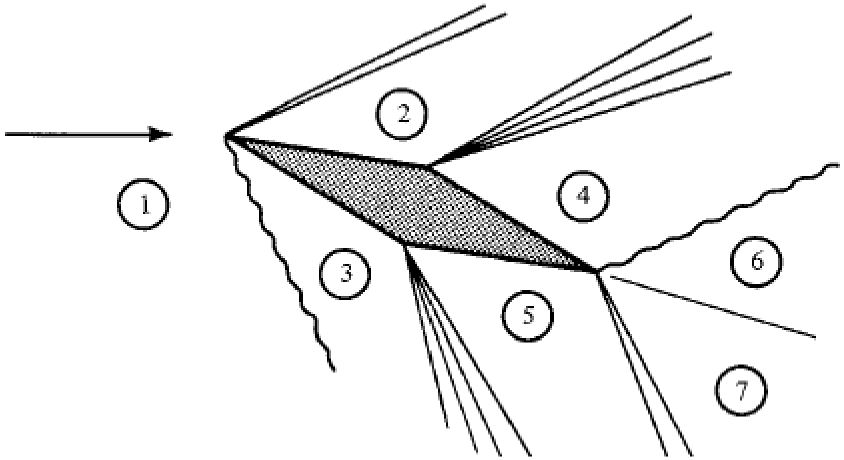
Shock-Expansion Theory

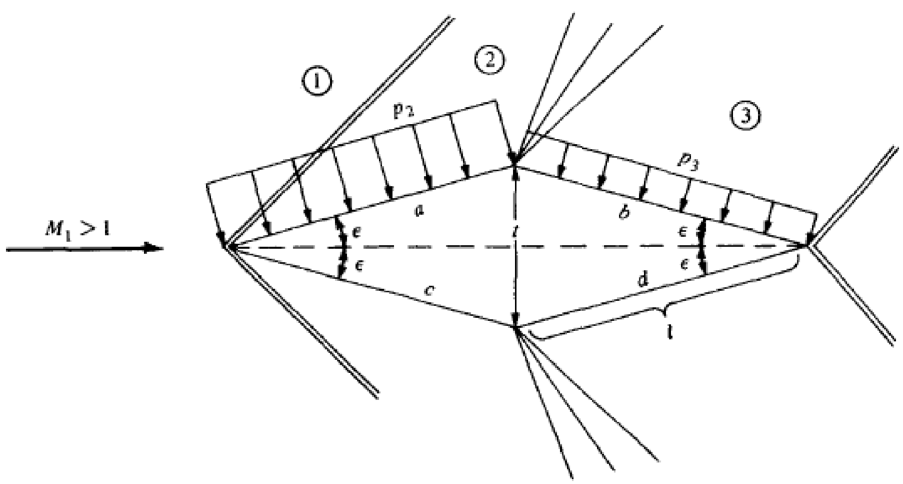


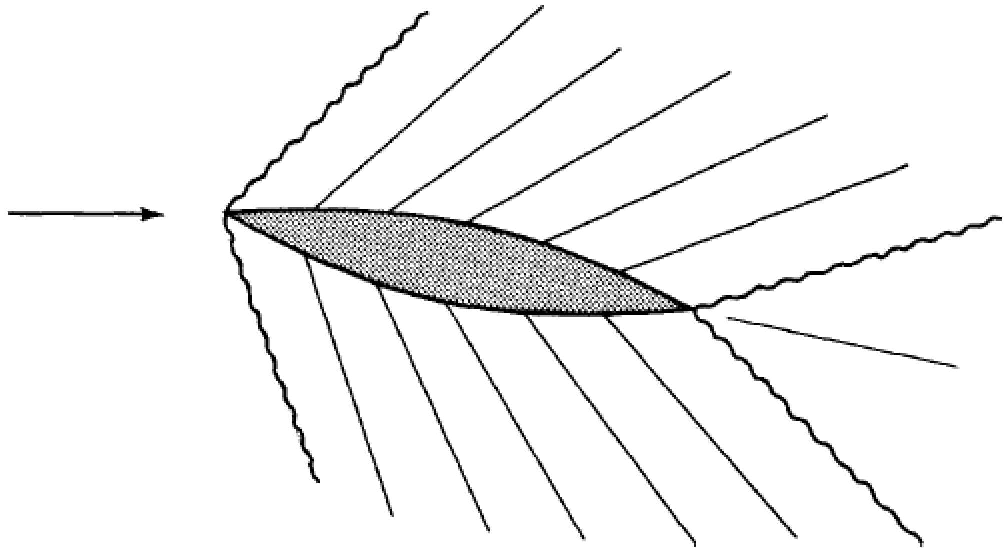


The diamond-shape airfoil

example of the application of shock-expansion theory,



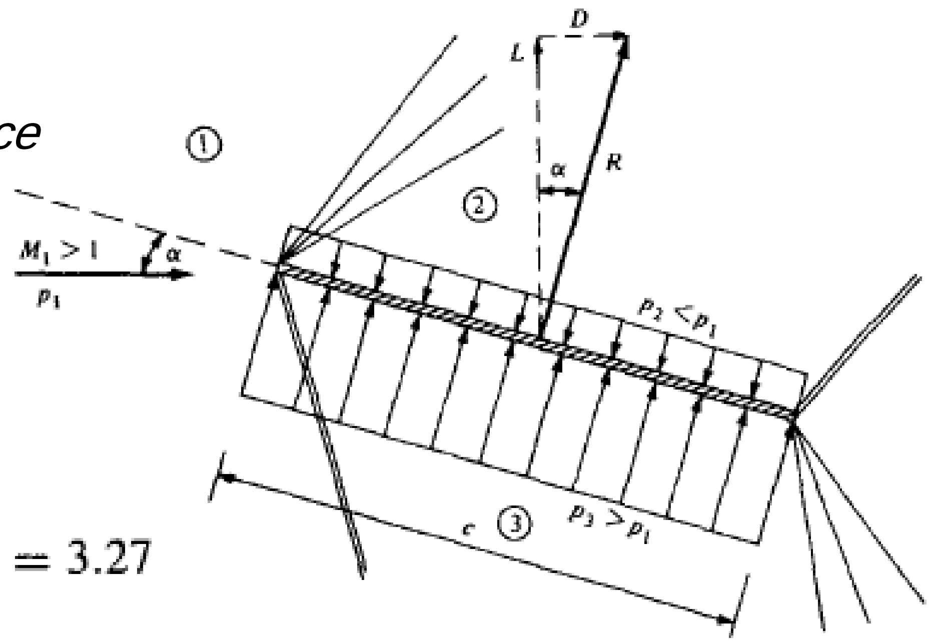




Biconvex airfoil at low angle of attack

EXAMPLE

Calculate the lift and drag coefficiens for a flat plate at a 5° angle of attack in a Mach 3 flow.



First, calculate *p2/p1 on the upper surface*

$$\nu_2 = \nu_1 + \theta$$

where $\theta = \alpha$.

$$\text{for } M_1 = 3,\ \nu_1 = 49.76^\circ$$

$$\nu_2 = 49.76^\circ + 5^\circ = 54.76^\circ \longrightarrow M_2 = 3.27$$

$$\text{for } M_1 = 3,\ p_{0_1}/p_1 = 36.73;\ \text{for } M_2 = 3.27,\ p_{0_2}/p_2 = 55.$$

$$\text{Since } p_{0_1} = p_{0_2}, \longrightarrow \frac{p_2}{p_1} = \frac{p_{0_1}}{p_1} \Big/ \frac{p_{0_2}}{p_2} = \frac{36.73}{55} = 0.668$$

Next, calculate p3/p1 on the bottom surface.

From the θ-β-M diagram $\longrightarrow$ $M_1 = 3$ and $\theta = 5°$, $\beta = 23.1°$

$$M_{n,1} = M_1 \sin\beta = 3\sin 23.1° = 1.177 \longrightarrow p_3/p_1 = 1.458$$

$$L' = (p_3 - p_2)c\cos\alpha \longrightarrow c_l = \frac{L'}{q_1 S} = \frac{L'}{(\gamma/2)p_1 M_1^2 c} = \frac{2}{\gamma M_1^2}\left(\frac{p_3}{p_1} - \frac{p_2}{p_1}\right)\cos\alpha$$

$$= \frac{2}{(1.4)(3)^2}(1.458 - 0.668)\cos 5° = \boxed{0.125}$$

$$D' = (p_3 - p_2)c\sin\alpha \longrightarrow c_d = \frac{D'}{q_1 S} = \frac{2}{\gamma M_1^2}\left(\frac{p_3}{p_1} - \frac{p_2}{p_1}\right)\sin\alpha$$

$$= \frac{2}{(1.4)(3^2)}(1.458 - 0.668)\sin 5° = \boxed{0.011}$$

Note that to calculate these coefficients we did not need to know the freestream pressure, density, or velocity. All we needed to know was:

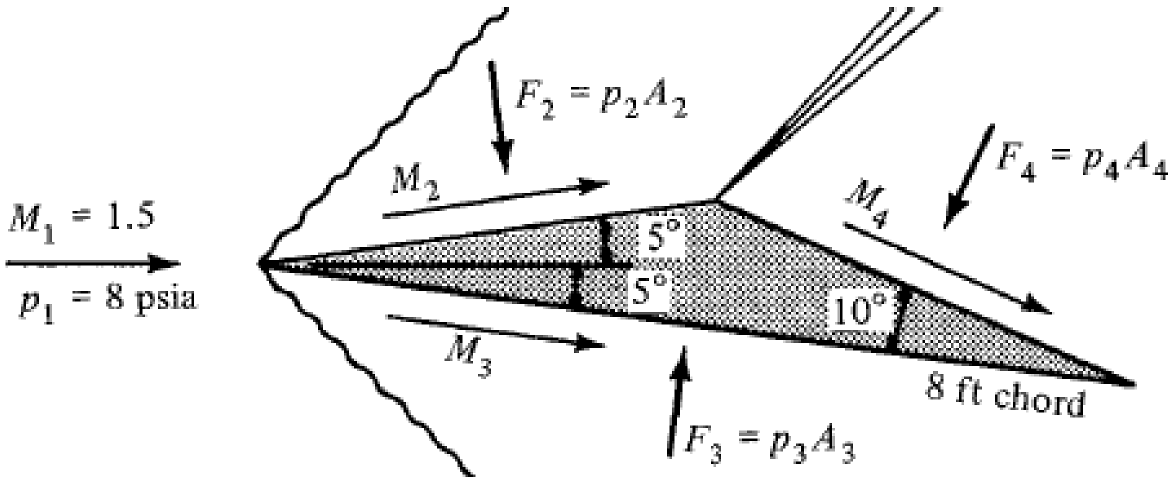
1. The shape of the body

2. The angle of attack

3. The freestream Mach number

EXAMPLE

It has been suggested that a supersonic airfoil be designed as an isosceles triangle with 10° equal angles and an 8-ft chord. When operating at a 5° angle of attack the air flow appears as shown in Figure.
Find the pressures on the various surfaces and the lift and drag forces when flying at *M = 1.5 through air with a pressure of 8 psia.*



From the θ-β-M diagram → $M1 = 1.5$ and $\theta = 5°$, $\beta = 48°$:

$$M1n = M1 \sin\beta = 1.5 (\sin 48°) = 1.115$$

M2n = 0.900 and p2/p1 = 1.2838

The Prandtl–Meyer expansion turns the flow by 20°:

v4 = v2 + 20 = 6.7213 + 20 = 26.7213 $\longrightarrow$ *M4 = 2.012*

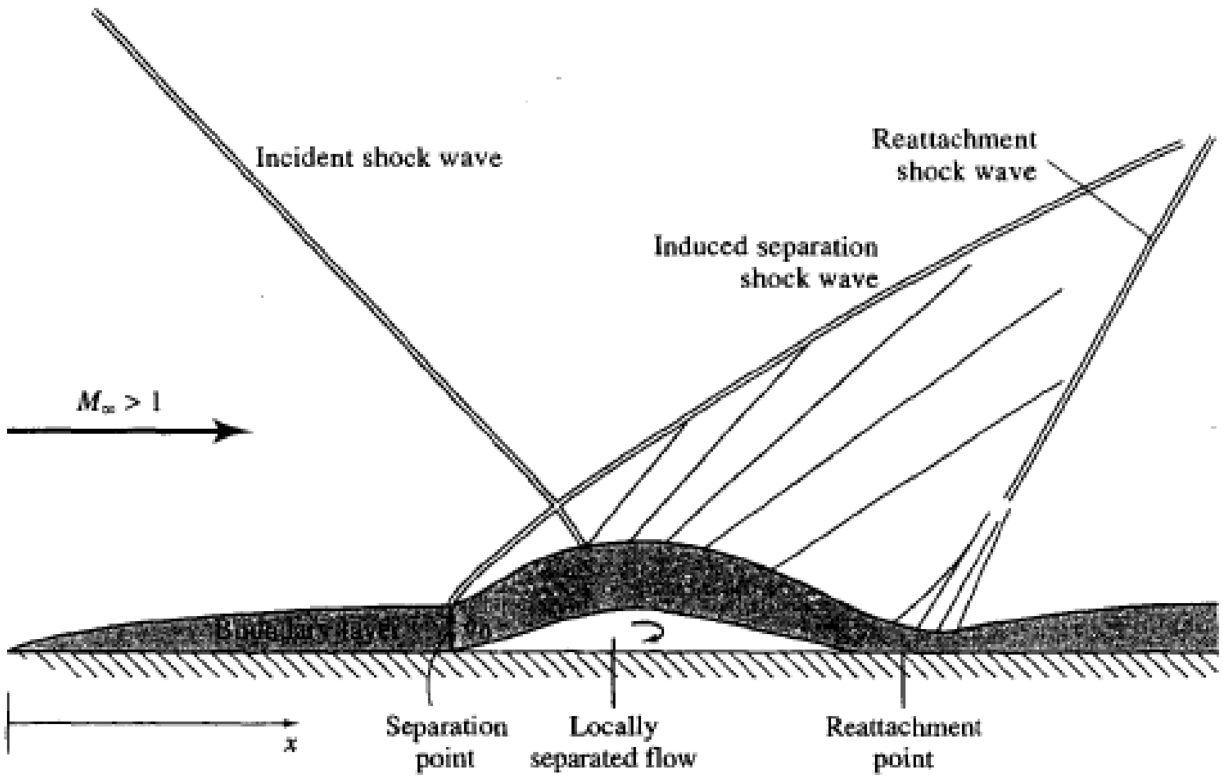
Note that conditions in region 3 are identical with region 2.We now find the pressures. The lift force (perpendicular to the free stream) will be

$$L = F_3 \cos 5^\circ - F_2 \cos 5^\circ - F_4 \cos 15^\circ$$

The lift per unit span will be 3728 lbf.

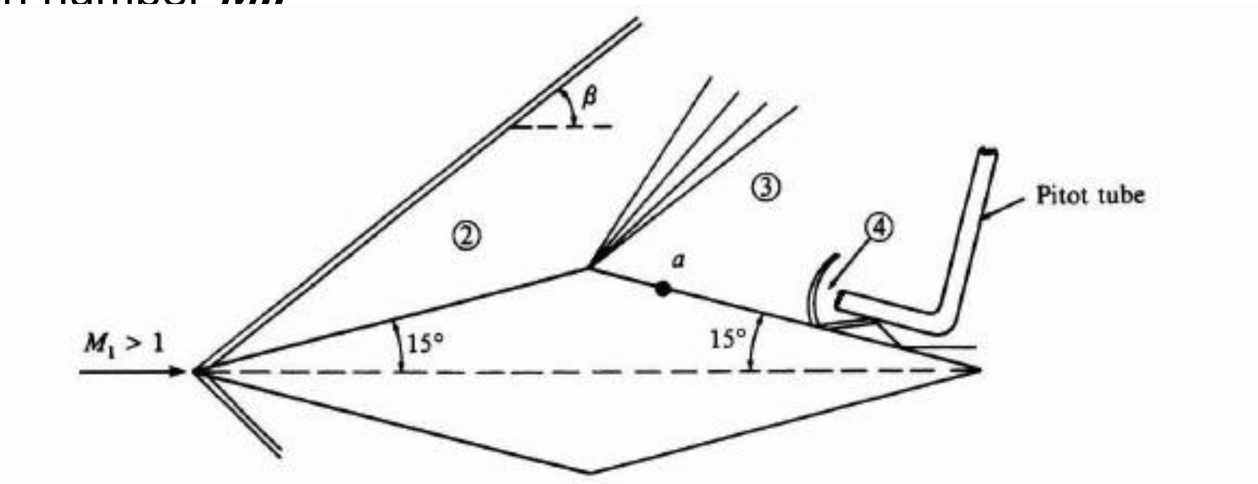
VISCOUS FLOW:
SHOCK-WAVE/BOUNDARY-LAYER INTERACTION

Shock waves and boundary layers do not mix.
Bad things can happen when a shock wave impinges on a boundary layer.



Example

Consider the arrangement shown in the following Fig.. A 15 Deg. half-angle diamond wedge airfoil is in a supersonic flow at zero angle of attack. A Pitot tube is inserted into the flow at the location shown in the Fig.. The pressure measured by the Pitot tube is 2.596 atm. At point a on the backface, the pressure is 0.1 atm. Calculate the free-stream Mach number ***Ml.***



The pressure at point a is the static pressure in region 3.

$$\frac{p_{o4}}{p_3} = \frac{2.596}{0.1} = 25.96$$

, for $p_{o4}/p_3 = 25.96$: $\longrightarrow$ $M_3 = 4.45$

for $M_3 = 4.45$ $\longrightarrow$ $\nu_3 = 71.27^\circ$

$\longrightarrow$ $\nu_2 = \nu_3 - \theta = 71.27 - 30 = 41.27^\circ$

$\nu_2 = 41.27^\circ$ $\longrightarrow$ $M_2 = 2.6.$

$\longrightarrow$ $M_{n_2} = M_2 \sin(\beta - \theta) = 2.6 \sin(\beta - 15^\circ)$

$$M_{n_2} = 0.588.$$

Assume $M_1 = 4$. Then $\beta = 27^\circ$, $M_{n_1} = M_1 \sin\beta = 4\sin 27^\circ = 1.816$.

$$\longrightarrow \quad M_{n_2} = 0.588.$$

Putting these results into **Eq:** $M_{n_2} = M_2 \sin(\beta - \theta) = 2.6\sin(\beta - 15^\circ)$

$$0.588 \stackrel{?}{=} 2.6\sin 10.5^\circ = 0.47$$

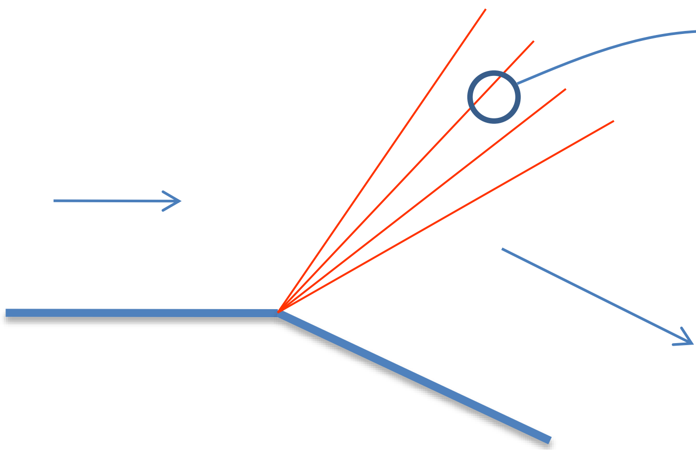
This does not check. We are going in the wrong direction

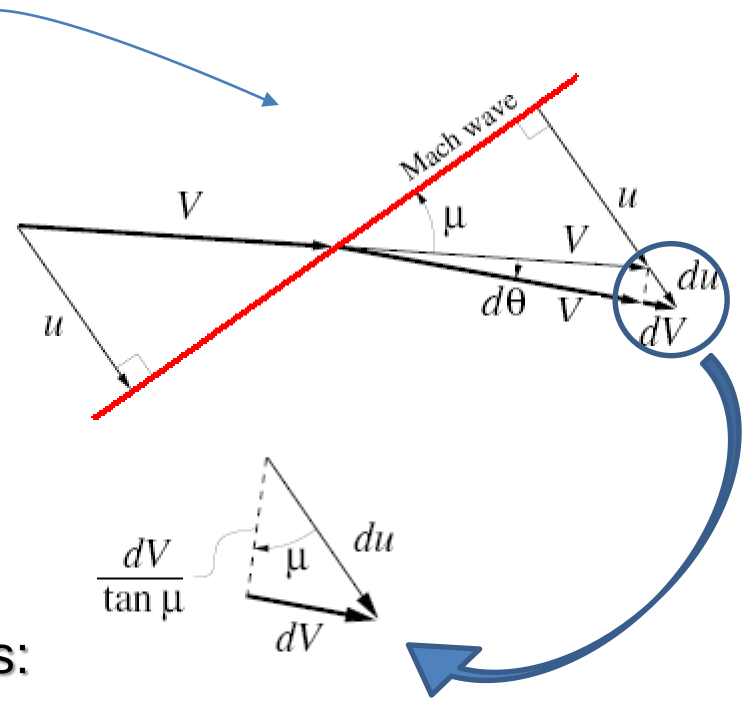
Assume $M_1 = 3.5$. Then $\beta = 29.2^\circ$, $M_{n_1} = 3.5\sin 29.2^\circ = 1.71$

$$\longrightarrow \quad M_{n_2} = 0.638.$$

$$0.588 \stackrel{?}{=} 2.6\sin 10.5^\circ = 0.47$$

PRANDTL-MEYER EXPANSION WAVES





From the u-V and du-dV velocity triangles:

$$d\theta = \frac{dV}{\tan\mu}\frac{1}{V}$$

$$\frac{1}{\tan\mu} = \frac{\cos\mu}{\sin\mu} = \frac{\sqrt{1-\sin^2\mu}}{\sin\mu} \xrightarrow{\sin\mu = 1/M} \frac{\sqrt{1-\sin^2\mu}}{\sin\mu} = \frac{\sqrt{1-1/M^2}}{1/M}$$

$$\frac{\sqrt{1-1/M^2}}{1/M} = \sqrt{M^2-1}$$

$$d\theta = \frac{dV}{\tan\mu}\frac{1}{V}$$

$$d\theta = \sqrt{M^2-1}\,\frac{dV}{V}$$

Prandtl-Meyer Function

$$V = M\,a = M\,a_o\left(1 + \frac{\gamma-1}{2}M^2\right)^{-1/2}$$

$$\ln V = \ln M + \ln a_o - \frac{1}{2}\ln\left(1 + \frac{\gamma-1}{2}M^2\right)$$

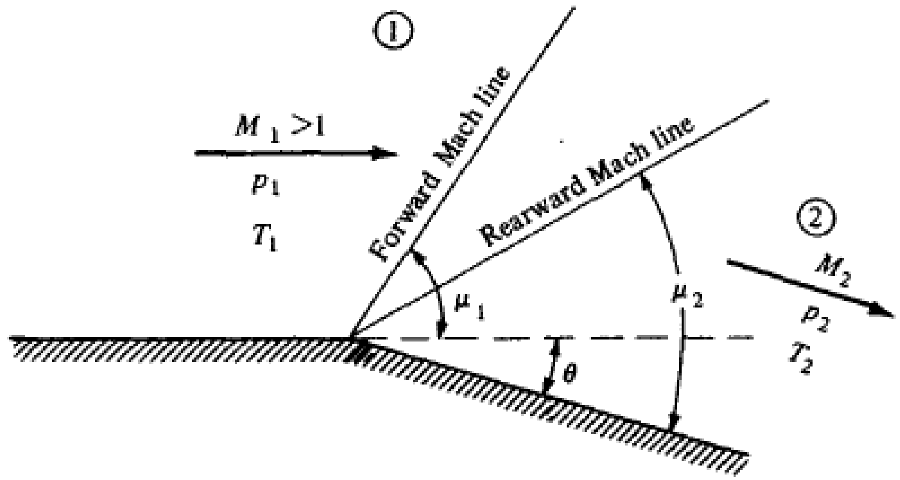
$$\frac{dV}{V} = \frac{dM}{M} - \frac{1}{2}\left(1 + \frac{\gamma-1}{2}M^2\right)^{-1}\frac{\gamma-1}{2}\,2M\,dM$$

$$\frac{dV}{V} = \frac{1}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$

$$d\theta = \sqrt{M^2-1}\,\frac{dV}{V}$$

$$d\theta = \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$

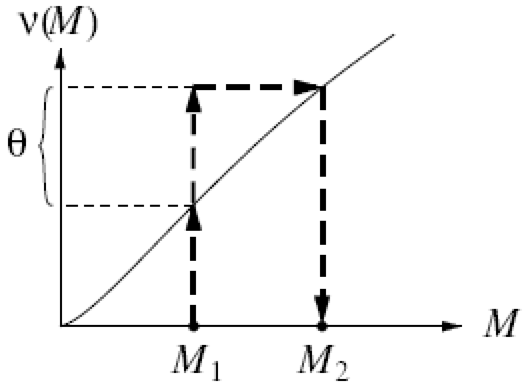
$$\int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$



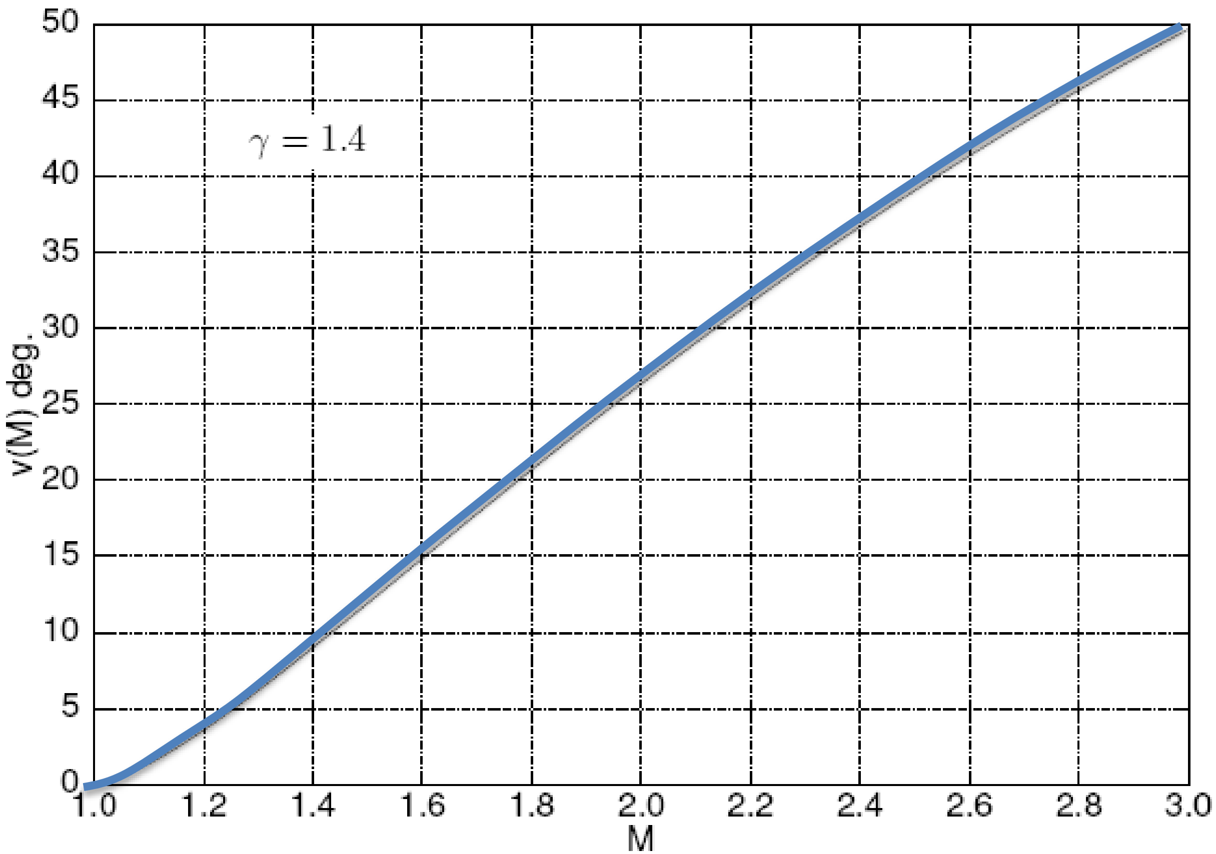
$$\overbrace{\theta_2 - \theta_1}^{\theta} = \nu(M_2) - \nu(M_1)$$

where $$\nu(M) \equiv \sqrt{\frac{\gamma+1}{\gamma-1}}\arctan\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \arctan\sqrt{M^2-1}$$

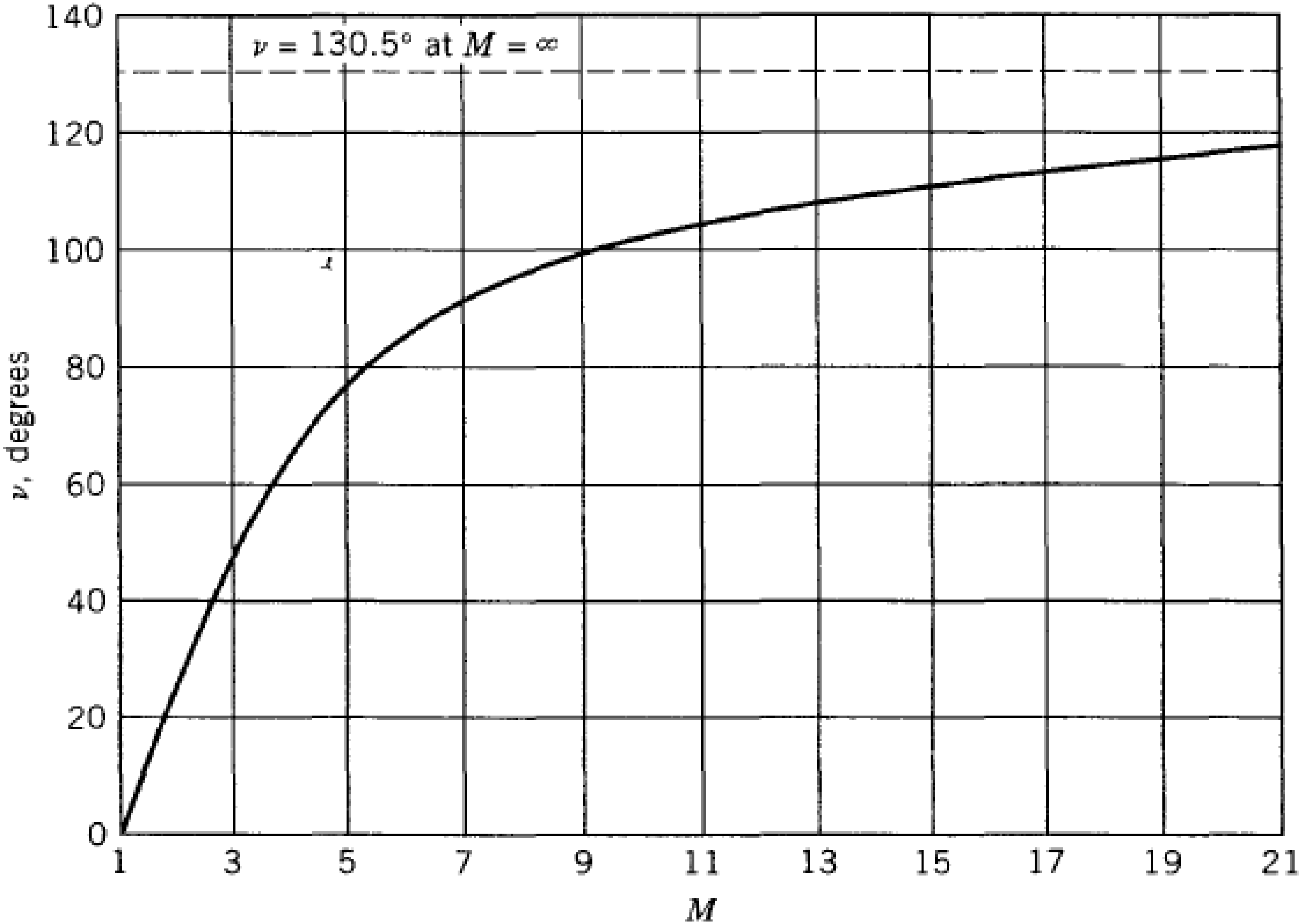
ν (M) is called the Prandtl-Meyer function,



ν(M)
θ
M
M₁
M₂

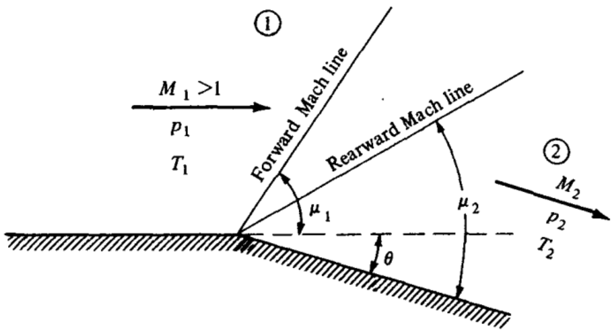


γ = 1.4
ν(M) deg.
M



ν = 130.5° at M = ∞
ν, degrees
0
20
40
60
80
100
120
140
1
3
5
7
9
11
13
15
17
19
21
M

How do the above results solve the problem stated in Figure



1. For the given M1, *obtain* ν(M1)

2. Calculate ν(M2) from using $\theta + \nu$(M1)= v(M2)

3. Obtain M2 from Appendix C corresponding to the value of ν(M2) or using prandtl-Meyer function

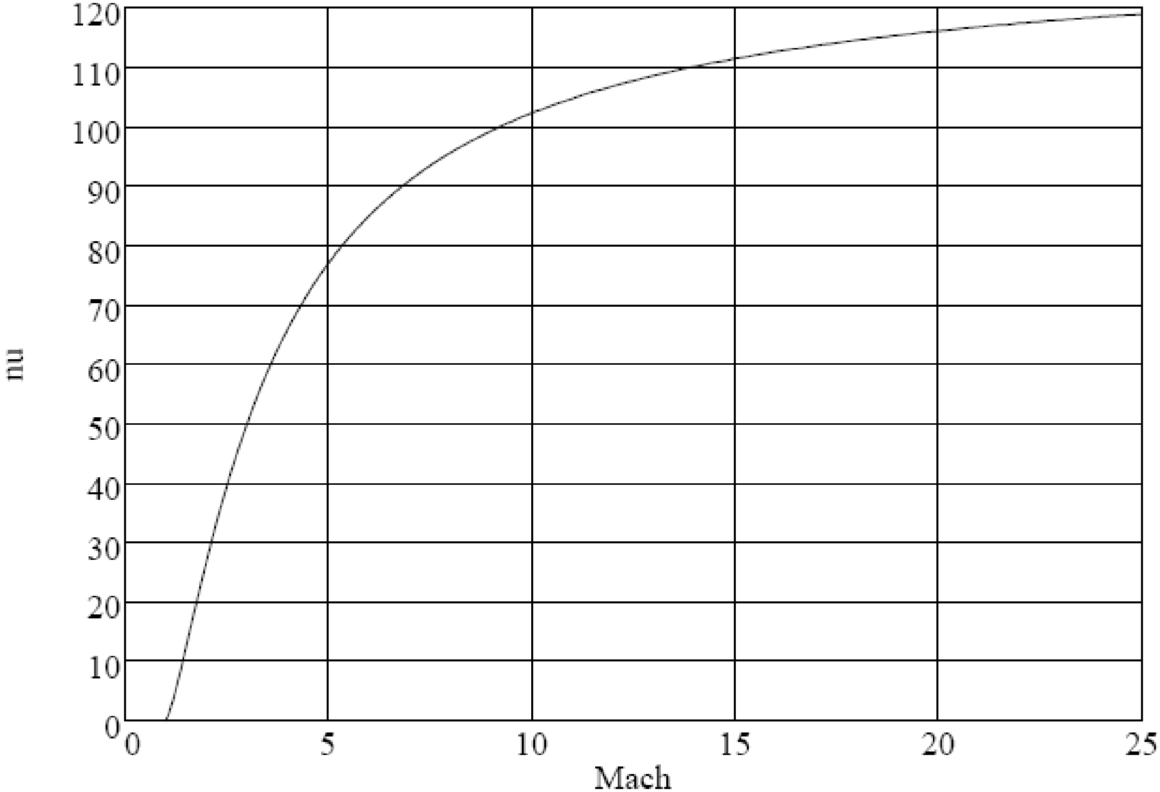
$$\boxed{\nu = \left(\frac{\gamma+1}{\gamma-1}\right)^{1/2} \tan^{-1}\left[\frac{\gamma-1}{\gamma+1}(M^2-1)\right]^{1/2} - \tan^{-1}(M^2-1)^{1/2}}$$

4. The expansion wave is isentropic; hence, po and T0 are constant through the wave.

$$\frac{T_2}{T_1} = \frac{T_2/T_{0,2}}{T_1/T_{0,1}} = \frac{1+[(\gamma-1)/2]M_1^2}{1+[(\gamma-1)/2]M_2^2} \qquad \frac{p_2}{p_1} = \frac{p_2/p_0}{p_1/p_0} = \left(\frac{1+[(\gamma-1)/2]M_1^2}{1+[(\gamma-1)/2]M_2^2}\right)^{\gamma/(\gamma-1)}$$

How do you compute the inverse of the Prandtl-Meyer Function?

$$\nu = \frac{1}{\lambda}\tan^{-1}(\lambda\beta) - \tan^{-1}(\beta) \qquad ; \lambda = \sqrt{\frac{\gamma-1}{\gamma+1}} \qquad \beta = \sqrt{M^2-1}.$$



In the usual approach for calculation, you compute the value of ν for the upstream Mach number and then add the angle thru which the flow is turned. The downstream Mach number is that which corresponds to the downstream ν. But, there is no easy way to compute this.

Since the equation is relatively simple, most students of compressible flow theory try to use algebraic and trigonometric manipulations to obtain an expression for Mach as a function of ν. Alas, it does not seem possible and we are left with numerical procedures.

Given a sufficiently dense table of ν vs. Mach, one can do a reverse table lookup.

This is probably the method of choice for students who are doing an off-line calculation.

In simple shock-expansion theory. If one is developing a computing procedure this approach can still be used by building a large table of ν versus Mach and then doing interpolation as Mach versus ν.

There is a certain lack of elegance to this approach and people would like a nice neat function that simply returned Mach for a given ν.

In September 1975, I.M. Hall published a paper in the Aeronautical Journal in which he described an rational polynomial function that approximated the inverse Prandtl-Meyer function with accuracy sufficient for most engineering purposes.

The function is simply a cubic polynomial divided by a quadratic.

$$M = \frac{1 + Ay + By^2 + Cy^3}{1 + Dy + Ey^2}$$

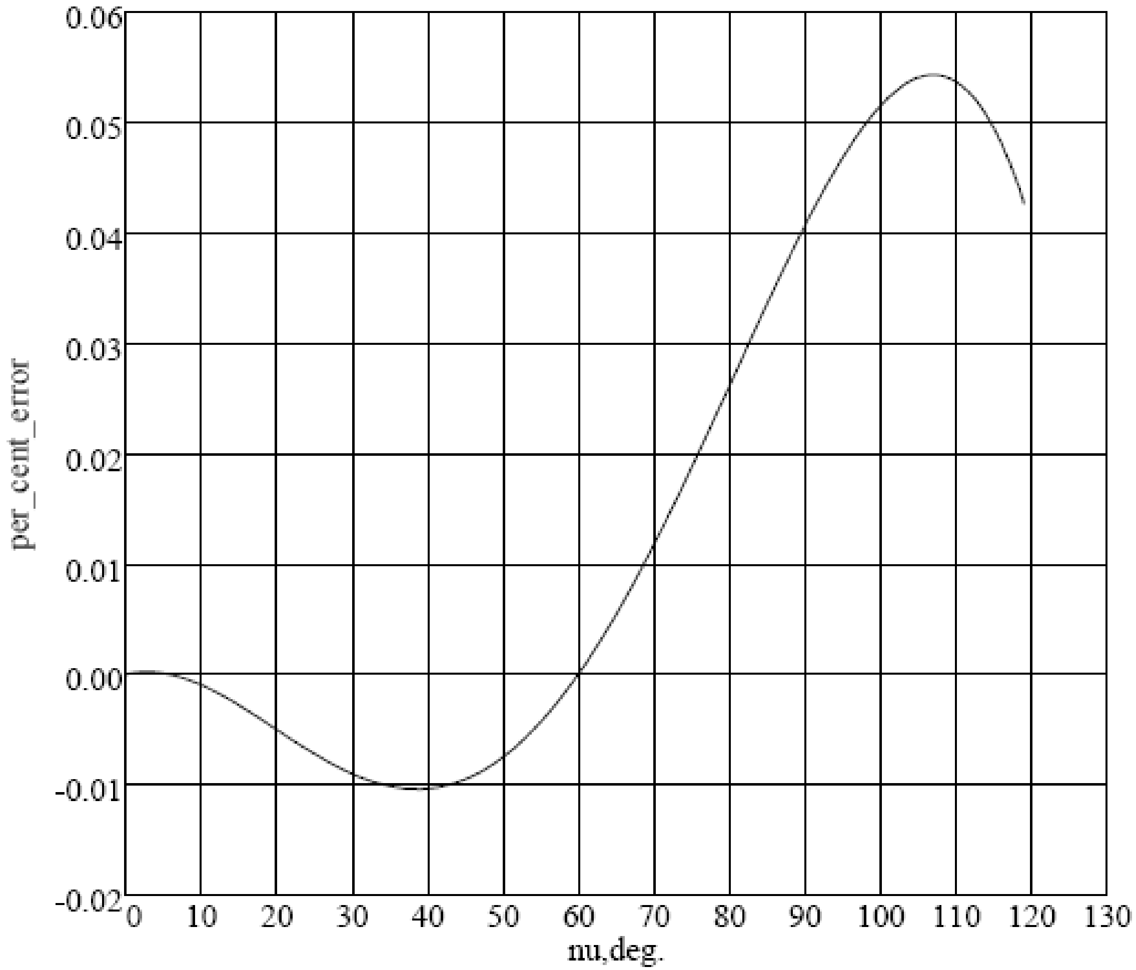
$$y = \left(\frac{\nu}{\nu_\infty}\right)^{2/3}$$

$$\nu_\infty = \frac{\pi}{2}(\sqrt{6} - 1)$$

A	1.3604
B	0.0962
C	-0.5127
D	-0.6722
E	-0.3278

All of the constants shown here are for γ=1.4. The original paper by Hall describes the calculation for other specific heats.

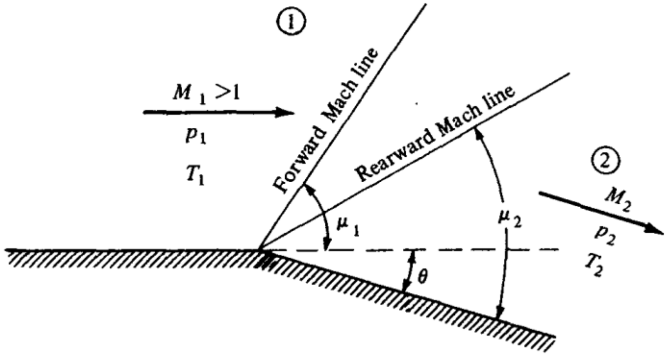
The accuracy of the Hall approximation is quite good with the maximum error much less that one-tenth of one percent.



EXAMPLE

A supersonic flow with *M1= 1.5, p1=1 atm, and T1 = 288 K is expanded* around a sharp comer) through a deflection angle of 15°. Calculate *M2, p2, T2,* P0,2, T0,2 and the angles that the forward and rearward Mach lines make with respect to the upstream flow direction.

$$\boxed{\nu = \left(\frac{\gamma+1}{\gamma-1}\right)^{1/2} \tan^{-1}\left[\frac{\gamma-1}{\gamma+1}(M^2-1)\right]^{1/2} - \tan^{-1}(M^2-1)^{1/2}}$$



$M_1 = 1.5 \Rightarrow \nu_1 = 11.91^\circ$

$\nu_2 = \nu_1 + \theta = 11.91 + 15 = 26.91^\circ \Rightarrow M_2 = 2.0$

$M_1 = 1.5, \Rightarrow p_{0,1}/p_1 = 3.671$ and $T_{0,1}/T_1 = 1.45.$

$M_2 = 2.0, \Rightarrow p_{0,2}/p_2 = 7.824$ and $T_{0,2}/T_2 = 1.8.$

Since the flow is isentropic, $T_{0,2} = T_{0,1}$ and $p_{0,2} = p_{0,1}$

$$p_2 = \frac{p_2}{p_{0,2}} \frac{p_{0,2}}{p_{0,1}} \frac{p_{0,1}}{p_1} p_1 = \frac{1}{7.824}(1)(3.671)(1\ \text{atm}) = \boxed{0.469\ \text{atm}}$$

$$T_2 = \frac{T_2}{T_{0,2}} \frac{T_{0,2}}{T_{0,1}} \frac{T_{0,1}}{T_1} T_1 = \frac{1}{1.8}(1)(1.45)(288) = \boxed{232\ \text{K}}$$

$$p_{0,2} = p_{0,1} = \frac{p_{0,1}}{p_1} p_1 = 3.671(1\ \text{atm}) = \boxed{3.671\ \text{atm}}$$

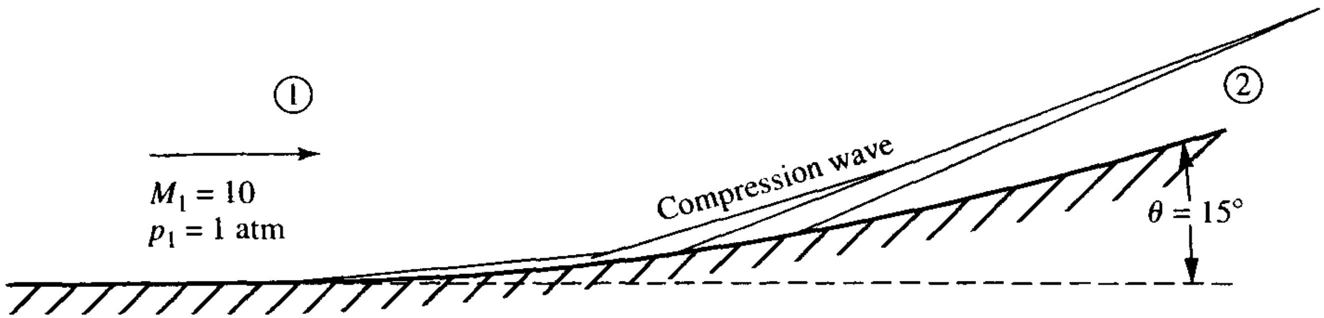
$$T_{0,2} = T_{0,1} = \frac{T_{0,1}}{T_1} T_1 = 1.45(288) = \boxed{417.6\ \text{K}}$$

$$\text{Angle of forward Mach line} = \mu_1 = \boxed{41.81^\circ}$$

$$\text{Angle of rearward Mach line} = \mu_2 - \theta = 30 - 15 = \boxed{15^\circ}$$

EXAMPLE

An isentropic compression wave is one of the possible compression mechanisms in SCRAMjets.
Consider the isentropic compression surface sketched in Figure. The Mach number and pressure upstream of the wave are M1 = 10 and *p1= 1atm, respectively. The flow is turned through a total angle* of 15°. Calculate the Mach number and pressure in region 2 behind the compression wave. Compare the results with those of a sharp corner.



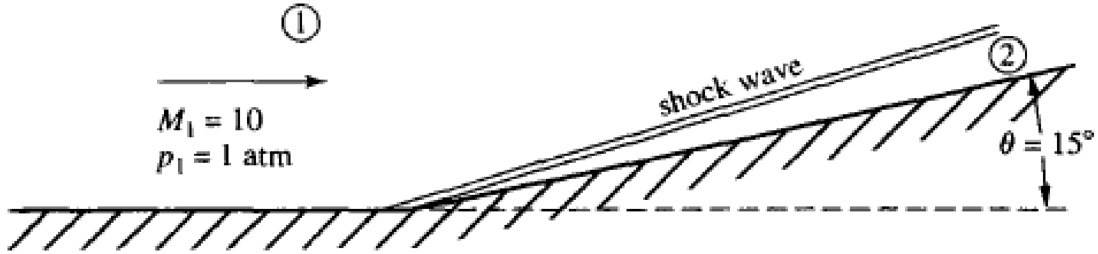
(*a*) Isentropic compression corner

$$M_1 = 10 \longrightarrow \nu_1 = 102.3° \longrightarrow \nu_2 = \nu_1 - \theta = 102.3 - 15 = 87.3° \longrightarrow \boxed{M_2 = 6.4}$$

for $M_1 = 10$, $p_{0,1}/p_1 = 0.4244 \times 10^5$ and for $M_2 = 6.4$, $p_{0,2}/p_2 = 0.2355 \times 10^4$

$$p_{0,2} = p_{0,1} \longrightarrow p_2 = \left(\frac{p_2}{p_{0,2}}\right)\left(\frac{p_{0,2}}{p_{0,1}}\right)\left(\frac{p_{0,1}}{p_1}\right)p_1 = \left(\frac{1}{0.2355 \times 10^4}\right)(1)(0.4244 \times 10^5)(1)$$

$$= \boxed{18.02 \text{ atm}}$$



(b) Shock compression corner

$M_1 = 10$ and $\theta = 15°$ $\rightarrow$ $\beta = 20°$ $\rightarrow$ $M_{n,1} = M_1 \sin\beta = (10)\sin 20° = 34.2$

$M_{n,1} = 3.42,$ $\rightarrow$ $p_2/p_1 = 13.32,\ p_{0,2}/p_{0,1} = 0.2322,$ and $M_{n,2} = 0.4552.$

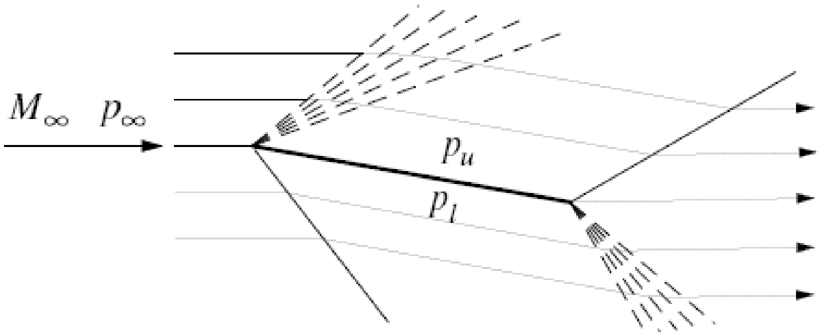
$$M_2 = \frac{M_{n,2}}{\sin(\beta - \theta)} = \frac{0.4552}{\sin(20 - 15)} = \boxed{5.22}$$

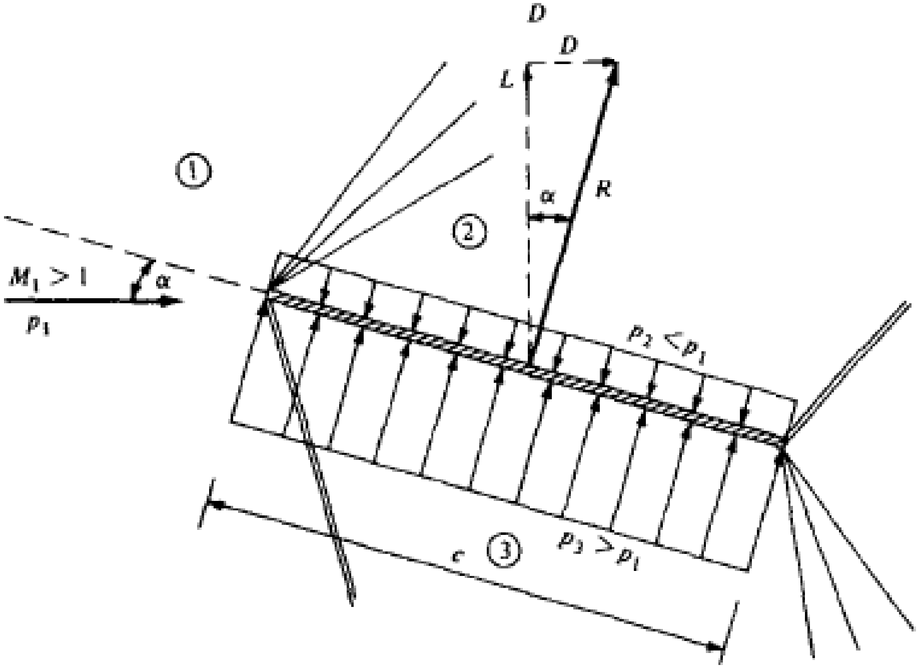
$$p_2 = (p_2/p_1)p_1 = 13.32(1) = \boxed{13.32 \text{ atm}}$$

The inefficiency of the shock wave is measured by the loss of total pressure across the shock; total pressure drops by about 77 percent across the shock. This emphasizes why designers of supersonic and hypersonic inlets would love to have the compression process carried out via isentropic compression waves.

it is a point design for the given upstream Mach number.

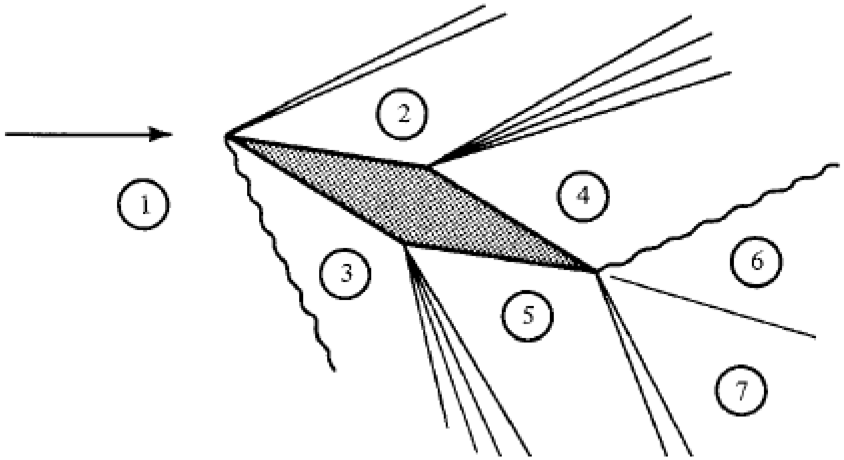
Shock-Expansion Theory

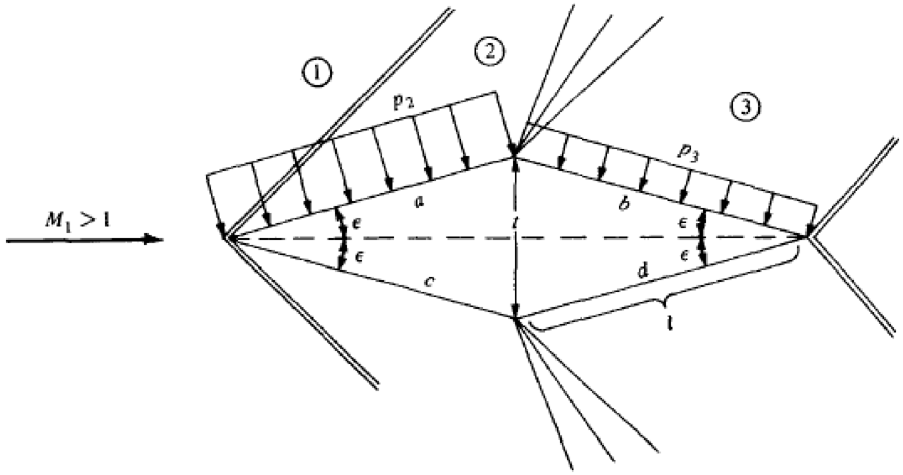


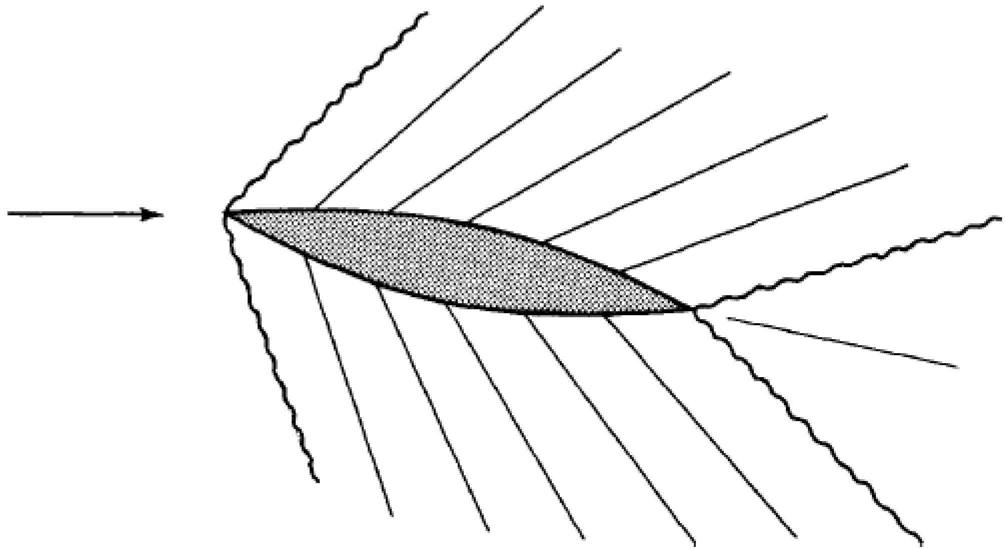


The diamond-shape airfoil

example of the application of shock-expansion theory,



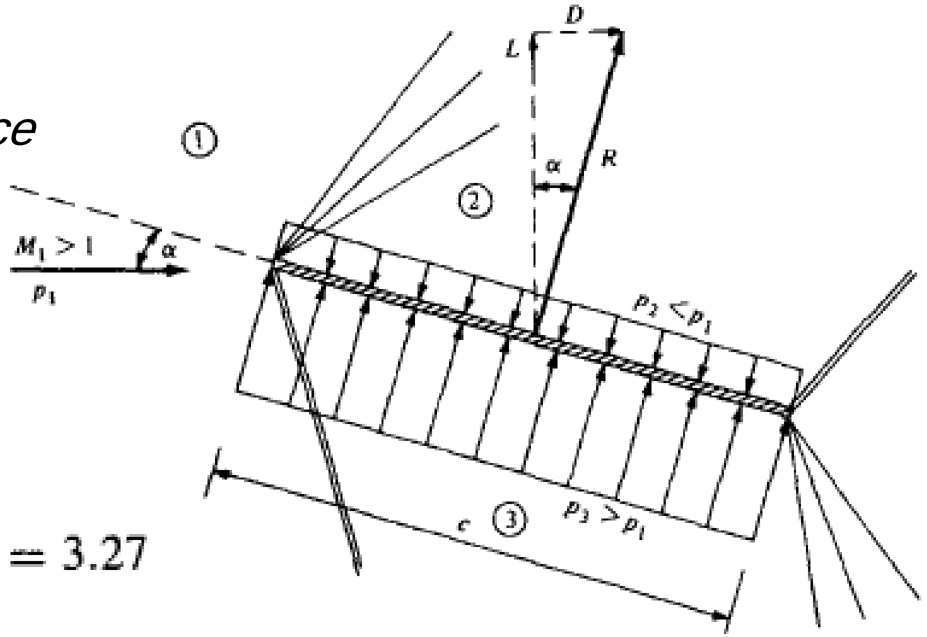




Biconvex airfoil at low angle of attack

EXAMPLE

Calculate the lift and drag coefficiens for a flat plate at a 5° angle of attack in a Mach 3 flow.



First, calculate *p2/p1 on the upper surface*

$$\nu_2 = \nu_1 + \theta$$

where $\theta = \alpha$.

$$\text{for } M_1 = 3,\ \nu_1 = 49.76^\circ$$

$$\nu_2 = 49.76^\circ + 5^\circ = 54.76^\circ \longrightarrow M_2 = 3.27$$

for $M_1 = 3$, $p_{0_1}/p_1 = 36.73$; for $M_2 = 3.27$, $p_{0_2}/p_2 = 55$.

Since $p_{0_1} = p_{0_2}$, $\longrightarrow$ $$\frac{p_2}{p_1} = \frac{p_{0_1}}{p_1} \Big/ \frac{p_{0_2}}{p_2} = \frac{36.73}{55} = 0.668$$

Next, calculate p3/p1 on the bottom surface.

From the θ-β-M diagram $\longrightarrow$ $M_1 = 3$ and $\theta = 5°$, $\beta = 23.1°$

$$M_{n,1} = M_1 \sin\beta = 3 \sin 23.1° = 1.177 \longrightarrow p_3/p_1 = 1.458$$

$$L' = (p_3 - p_2)c\cos\alpha \longrightarrow c_l = \frac{L'}{q_1 S} = \frac{L'}{(\gamma/2)p_1 M_1^2 c} = \frac{2}{\gamma M_1^2}\left(\frac{p_3}{p_1} - \frac{p_2}{p_1}\right)\cos\alpha$$

$$= \frac{2}{(1.4)(3)^2}(1.458 - 0.668)\cos 5° = \boxed{0.125}$$

$$D' = (p_3 - p_2)c\sin\alpha \longrightarrow c_d = \frac{D'}{q_1 S} = \frac{2}{\gamma M_1^2}\left(\frac{p_3}{p_1} - \frac{p_2}{p_1}\right)\sin\alpha$$

$$= \frac{2}{(1.4)(3^2)}(1.458 - 0.668)\sin 5° = \boxed{0.011}$$

Note that to calculate these coefficients we did not need to know the freestream pressure, density, or velocity. All we needed to know was:

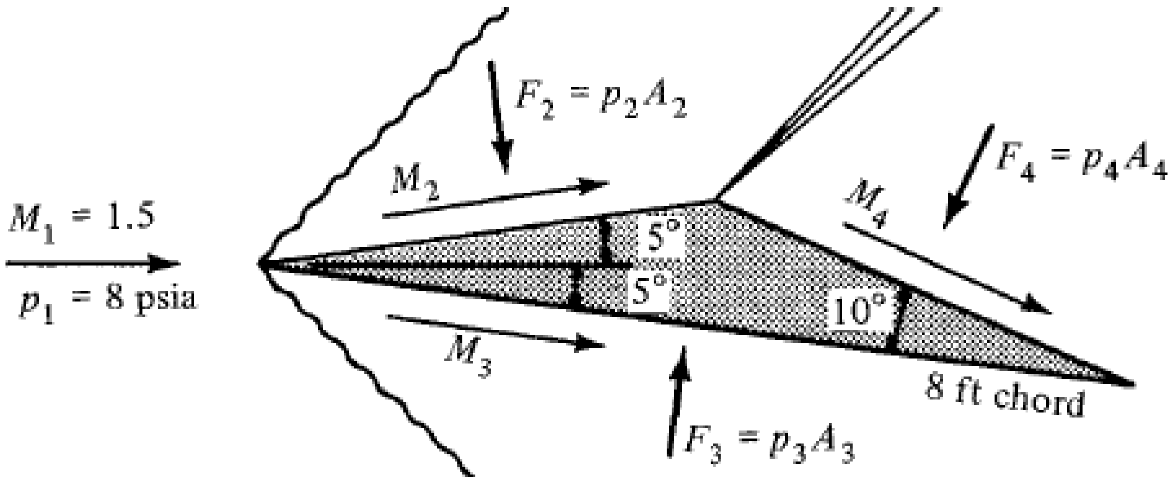
1. The shape of the body

2. The angle of attack

3. The freestream Mach number

EXAMPLE

It has been suggested that a supersonic airfoil be designed as an isosceles triangle with 10° equal angles and an 8-ft chord. When operating at a 5° angle of attack the air flow appears as shown in Figure.
Find the pressures on the various surfaces and the lift and drag forces when flying at *M = 1.5 through air with a pressure of 8 psia.*



From the θ-β-M diagram → $M1 = 1.5$ and $\theta = 5°$, $\beta = 48°$:

$$M1n = M1 \sin\beta = 1.5 (\sin 48°) = 1.115$$

$M2n = 0.900$ and $p2/p1 = 1.2838$

The Prandtl–Meyer expansion turns the flow by 20°:

$v4 = v2 + 20 = 6.7213 + 20 = 26.7213 \longrightarrow M4 = 2.012$

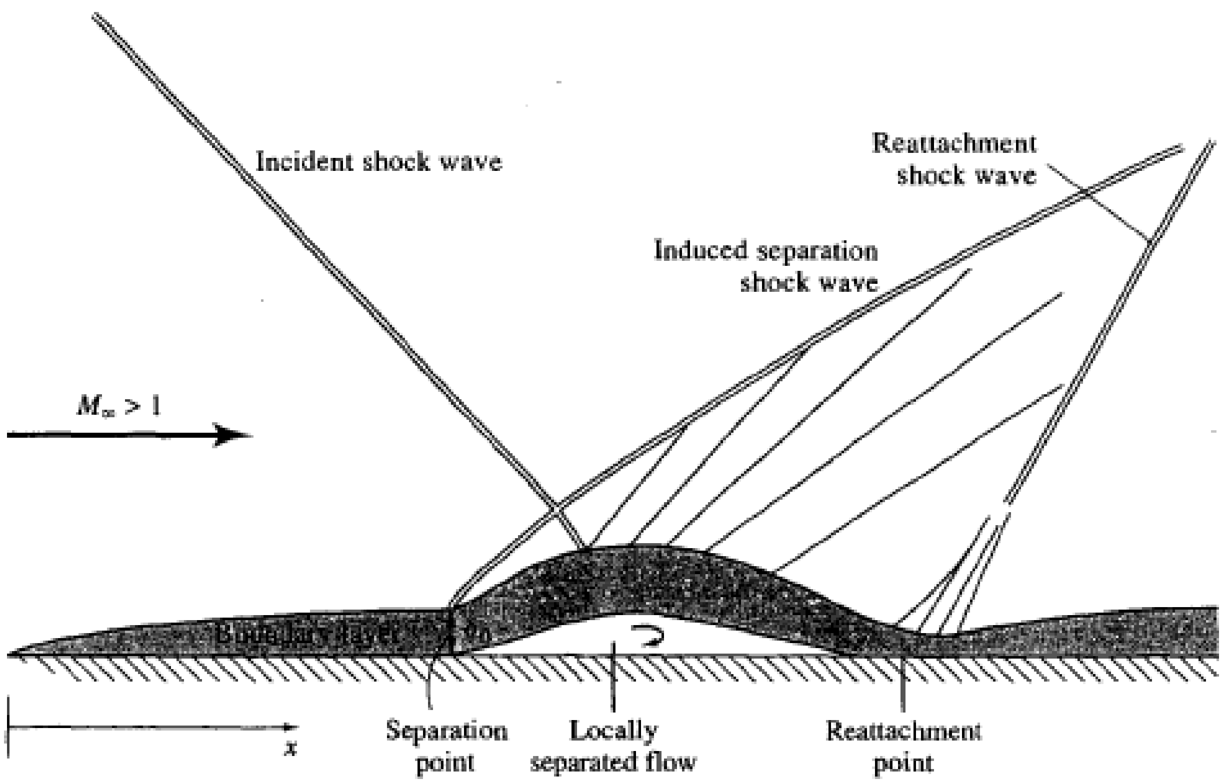
Note that conditions in region 3 are identical with region 2.We now find the pressures. The lift force (perpendicular to the free stream) will be

$$L = F_3 \cos 5^\circ - F_2 \cos 5^\circ - F_4 \cos 15^\circ$$

The lift per unit span will be 3728 lbf.

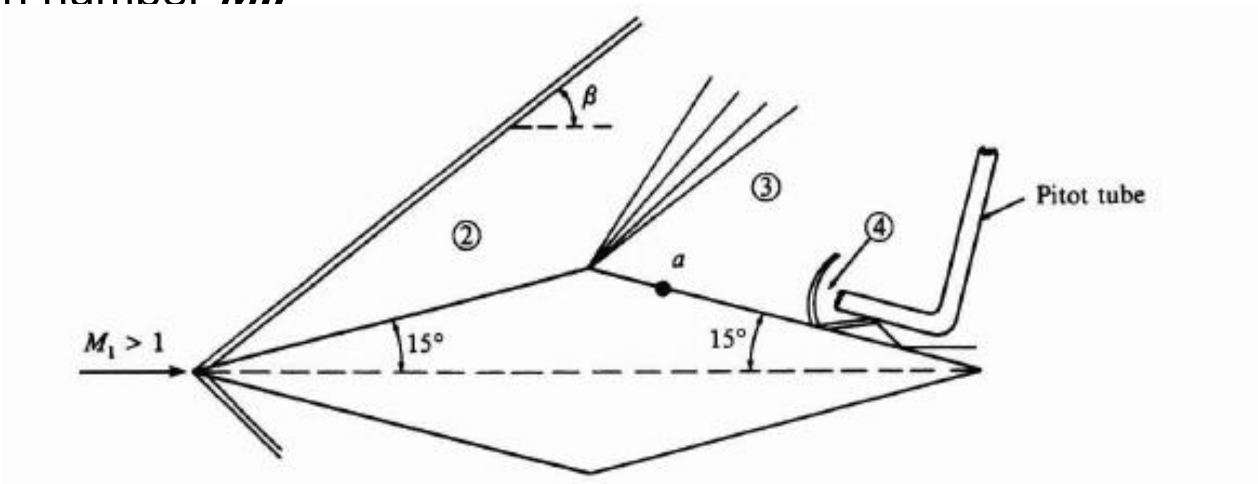
VISCOUS FLOW:
SHOCK-WAVE/BOUNDARY-LAYER INTERACTION

Shock waves and boundary layers do not mix.
Bad things can happen when a shock wave impinges on a boundary layer.



Example

Consider the arrangement shown in the following Fig.. A 15 Deg. half-angle diamond wedge airfoil is in a supersonic flow at zero angle of attack. A Pitot tube is inserted into the flow at the location shown in the Fig.. The pressure measured by the Pitot tube is 2.596 atm. At point a on the backface, the pressure is 0.1 atm. Calculate the free-stream Mach number ***MI.***



The pressure at point a is the static pressure in region 3.

$$\frac{p_{o4}}{p_3} = \frac{2.596}{0.1} = 25.96$$

, for $p_{o4}/p_3 = 25.96$: $\longrightarrow$ $M_3 = 4.45$

for $M_3 = 4.45$ $\longrightarrow$ $\nu_3 = 71.27°$

$\longrightarrow$ $\nu_2 = \nu_3 - \theta = 71.27 - 30 = 41.27°$

$\nu_2 = 41.27°$ $\longrightarrow$ $M_2 = 2.6.$

$\longrightarrow$ $M_{n_2} = M_2 \sin(\beta - \theta) = 2.6 \sin(\beta - 15°)$

$$M_{n_2} = 0.588.$$

Assume $M_1 = 4$. Then $\beta = 27^\circ$, $M_{n_1} = M_1 \sin\beta = 4 \sin 27^\circ = 1.816$.

$$\longrightarrow \quad M_{n_2} = 0.588.$$

Putting these results into **Eq:** $M_{n_2} = M_2 \sin(\beta - \theta) = 2.6 \sin(\beta - 15^\circ)$

$$0.588 \stackrel{?}{=} 2.6 \sin 10.5^\circ = 0.47$$

This does not check. We are going in the wrong direction

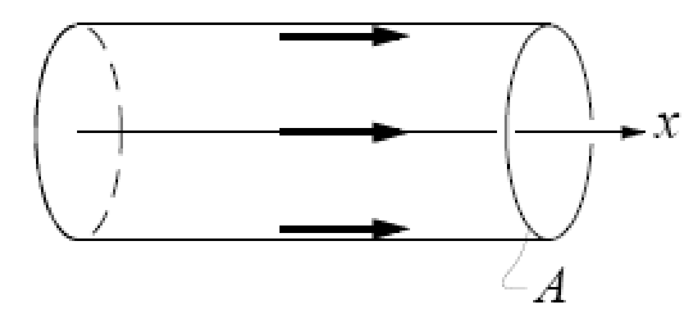
Assume $M_1 = 3.5$. Then $\beta = 29.2^\circ$, $M_{n_1} = 3.5 \sin 29.2^\circ = 1.71$

$$\longrightarrow \quad M_{n_2} = 0.638.$$

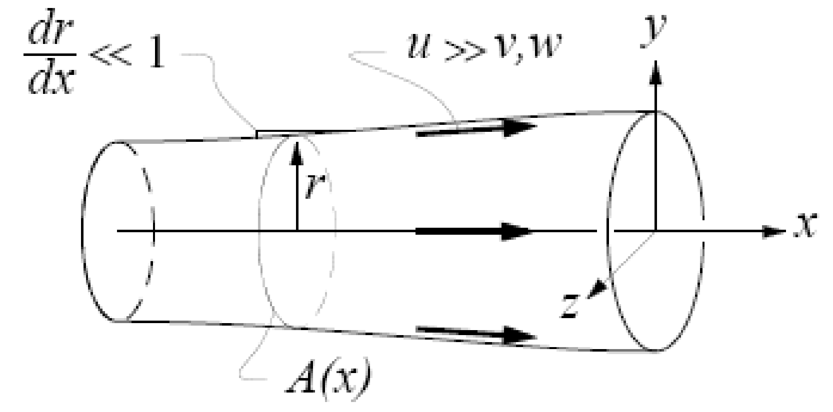
$$0.588 \stackrel{?}{=} 2.6 \sin 10.5^\circ = 0.47$$

Compressible Flow Through Nozzles, Diffusers, and Wind Tunnels

Compressible Channel Flow



1−D Flow



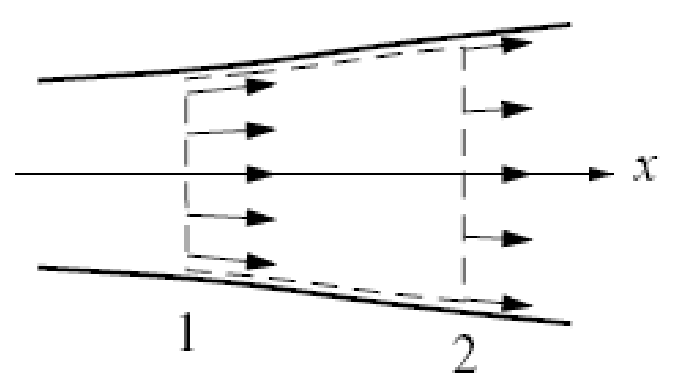
Quasi−1−D Flow

A quasi-one-dimensional flow is one in which all variables vary primarily along one direction, say x. A flow in a duct with slowly-varying area A(x) is the case of interest here. In practice this means that the slope of the duct walls is small. Also, the x-velocity component u dominates the y and z-components v and w.

Governing equations

Application of the integral mass continuity equation to a segment of the duct bounded by any two x locations gives

$$
\begin{aligned}
\oiint \rho \vec{V} \cdot \hat{n}\, dA &= 0 \\
-\rho_1 u_1 \iint_1 dA + \rho_2 u_2 \iint_2 dA &= 0 \\
-\rho_1 u_1 A_1 + \rho_2 u_2 A_2 &= 0
\end{aligned}
$$



Since stations 1 or 2 can be placed at any arbitrary location x, we can define the duct mass flow which is constant all along the duct, and relates the density, velocity, and area.

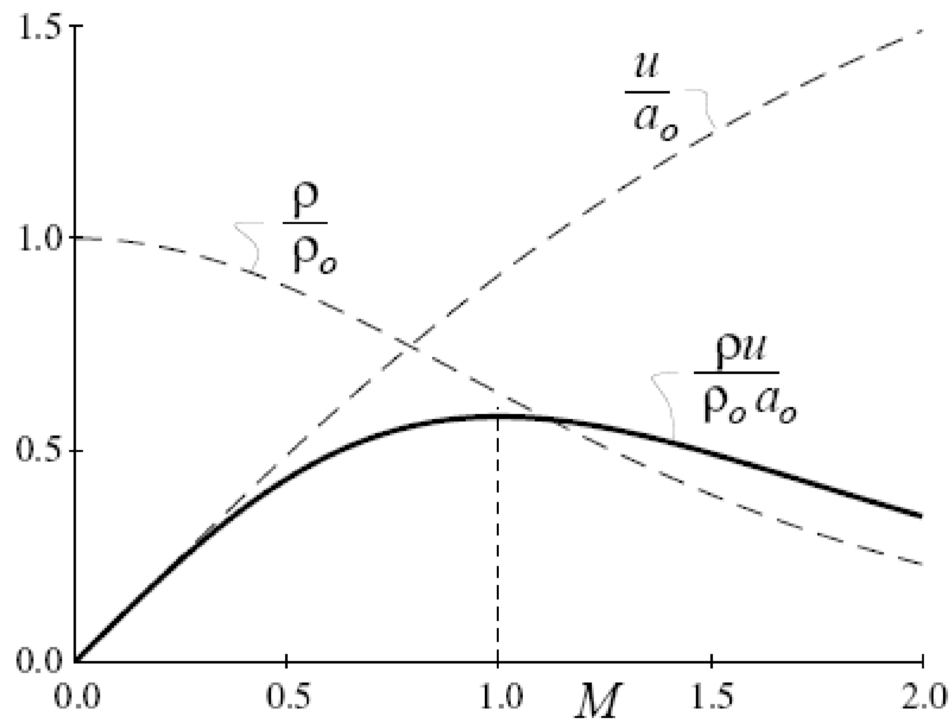
$$\rho(x)\, u(x)\, A(x) \equiv \dot{m} = \text{constant}$$

If we assume that the flow in the duct is isentropic, at least piecewise-isentropic between shocks,

$$\frac{\rho}{\rho_o} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{1}{\gamma - 1}}$$

$$\frac{u}{a_o} = \frac{M\, a}{a_o} = M \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{1}{2}}$$

$$\frac{\rho u}{\rho_o a_o} = M \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{\gamma + 1}{2(\gamma - 1)}}$$



The significance of ρu is that it represents the inverse of the duct area, or $A \sim \dfrac{1}{\rho u}$

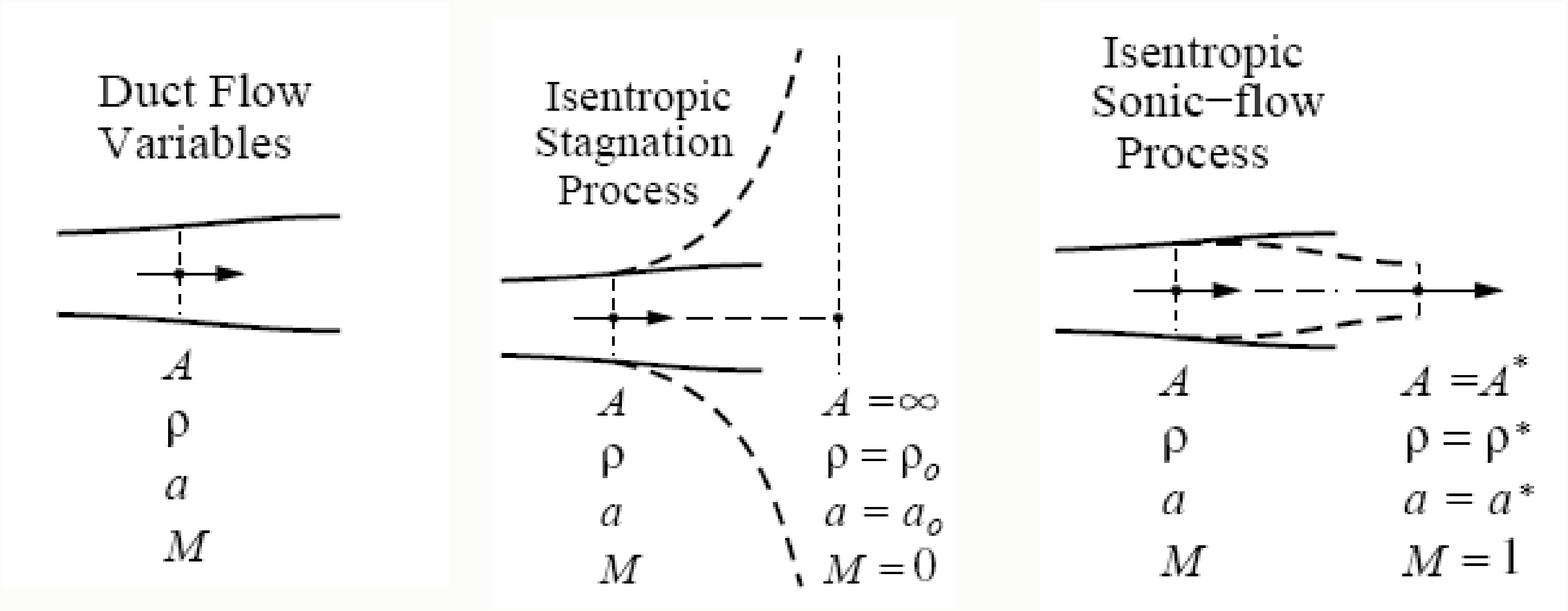
It is evident that the maximum possible mass flux occurs at a location where locally M = 1.

$$\frac{d}{dM}\left(\frac{\rho u}{\rho_o a_o}\right) = \left(1 - M^2\right)\left(1 + \frac{\gamma-1}{2}M^2\right)^{-\frac{\gamma-3}{2(\gamma-1)}} = 0 \longrightarrow M = 1$$

Sonic conditions

The stagnation conditions ρ_o and a_o were used to normalize the various quantities.

For compressible duct flows, it is very convenient to also define sonic conditions which can serve as alternative normalizing quantities.



The advantage of the sonic-flow process is that it produces a well-defined sonic throat area A* while for the stagnation process A tends to infinity, and cannot be used for normalization.

The ratios between the stagnation and sonic conditions are readily obtained from the usual isentropic relations, with M = 1 plugged in.

Numerical values are also given for γ = 1.4

$$\frac{\rho^*}{\rho_o} = \left(1 + \frac{\gamma - 1}{2}\right)^{-\frac{1}{\gamma-1}} = 0.6339$$

$$\frac{a^*}{a_o} = \left(1 + \frac{\gamma - 1}{2}\right)^{-\frac{1}{2}} = 0.9129$$

$$\frac{p^*}{p_o} = \left(1 + \frac{\gamma - 1}{2}\right)^{-\frac{\gamma}{\gamma-1}} = 0.5283$$

$$\dot{m} = \rho u A = \rho^* u^* A^*$$

note that $u^* = a^*$ since $M = 1$ at the sonic throat.

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{a^*}{u} = \frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho}\frac{a^*}{a_o}\frac{a_o}{u}$$

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{a^*}{u} = \frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho}\frac{a^*}{a_o}\frac{a_o}{u}$$

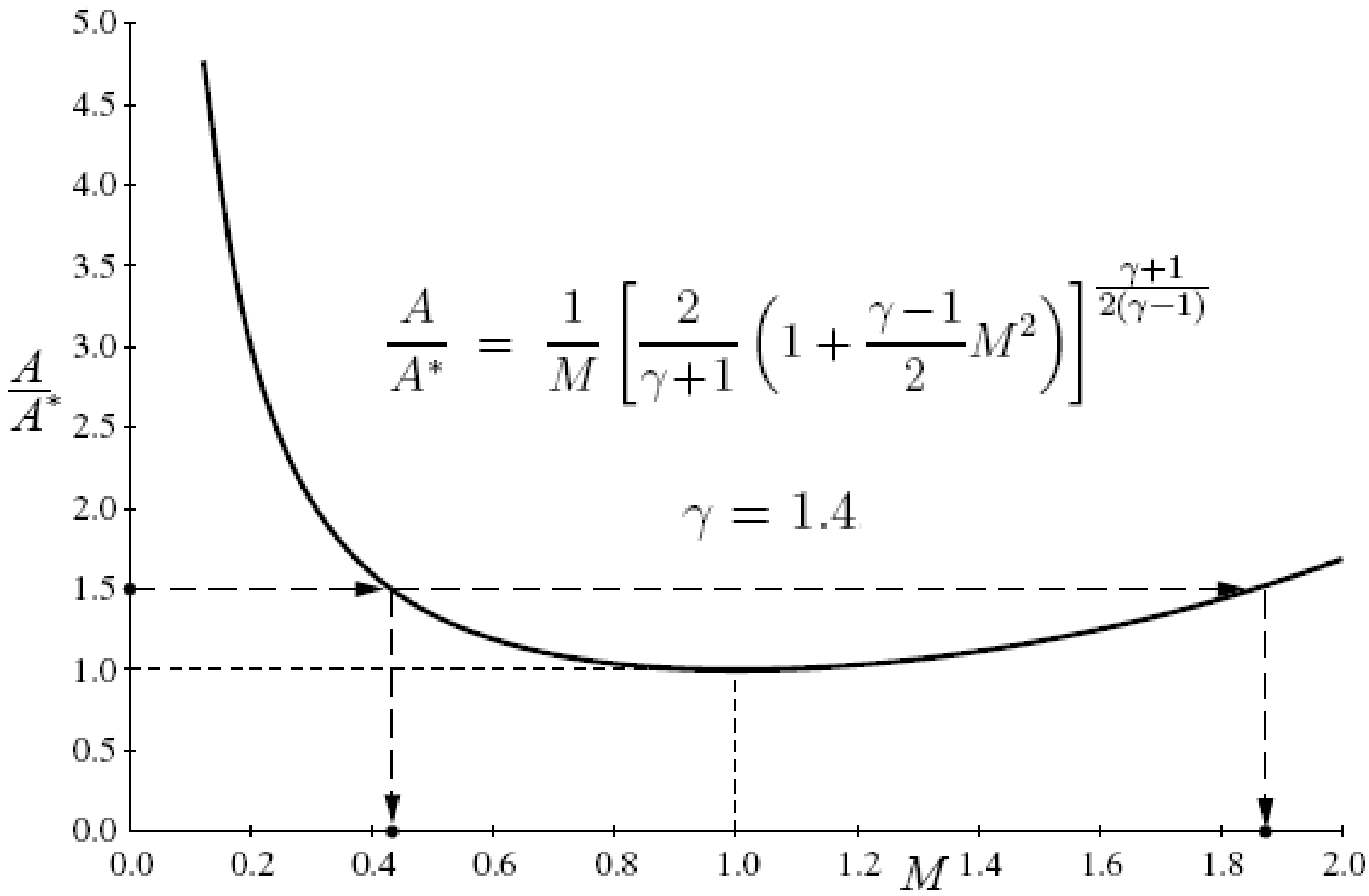
$$\frac{\rho^*}{\rho_o} = \left(1+\frac{\gamma-1}{2}\right)^{-\frac{1}{\gamma-1}}$$

$$\frac{a^*}{a_o} = \left(1+\frac{\gamma-1}{2}\right)^{-\frac{1}{2}}$$

$$\frac{\rho u}{\rho_o a_o} = M\left(1+\frac{\gamma-1}{2}M^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$$

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

This is the area-Mach relation



If the duct geometry A(x) is given, and A is defined from the known duct mass flow and stagnation quantities, then M(x) can be determined using the graphical technique shown in the figure,

EXAMPLE

Consider the isentropic supersonic flow through a convergent-divergent nozzle with an exit-to-throat area ratio of 10.25. The reservoir pressure and temperature are 5 atm and 600°R, respectively. Calculate M, p, and T at the nozzle exit.

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

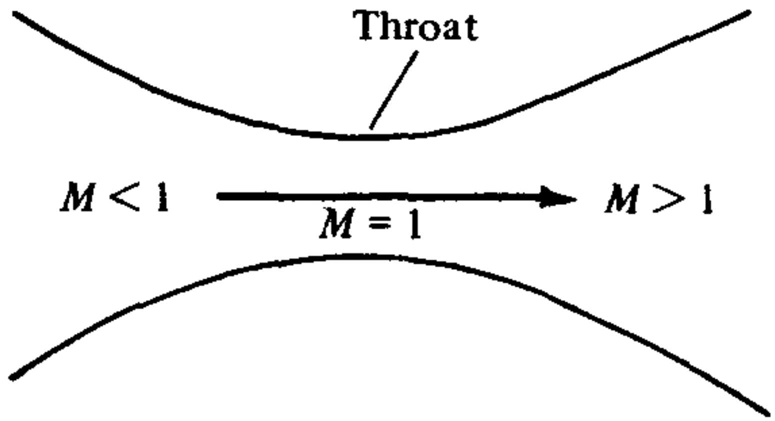
For *Ae/A* = 10.25* → $M_e = \boxed{3.95}$

$$\frac{p_e}{p_0} = \frac{1}{142} \quad \text{and} \quad \frac{T_e}{T_0} = \frac{1}{4.12}$$

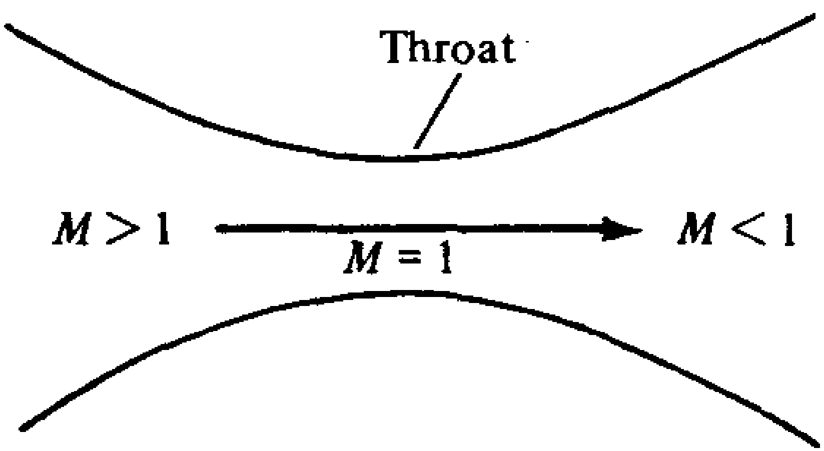
→

$$p_e = 0.007 p_0 = 0.007(5) = \boxed{0.035 \text{ atm}}$$

$$T_e = 0.2427 T_0 = 0.2427(600) = \boxed{145.6^\circ\text{R}}$$



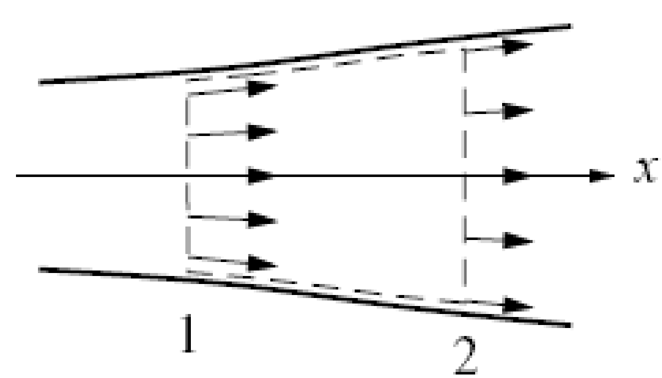
Supersonic nozzle



Supersonic diffuser

The momentum equation, Equation

$$\oint\!\!\!\oint_S (\rho \mathbf{V} \cdot \mathbf{dS})\mathbf{V} = -\oint\!\!\!\oint_S p\, \mathbf{dS}$$

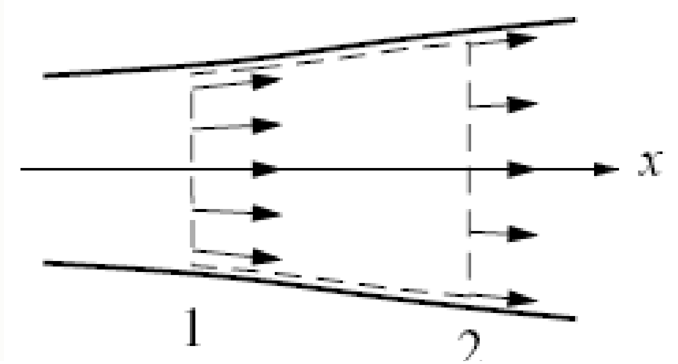


$$\oint\!\!\!\oint_S (\rho \mathbf{V} \cdot \mathbf{dS})u = -\oint\!\!\!\oint_S (p\, dS)_x$$

$$-\rho_1 u_1^2 A_1 + \rho_2 u_2^2 A_2 = -(-p_1 A_1 + p_2 A_2) + \int_{A_1}^{A_2} p\, dA$$

$$\boxed{p_1 A_1 + \rho_1 u_1^2 A_1 + \int_{A_1}^{A_2} p\, dA = p_2 A_2 + \rho_2 u_2^2 A_2}$$

$$\oiint_S \rho\left(e+\frac{V^2}{2}\right)\mathbf{V}\cdot\mathbf{dS} = -\oiint_S p\mathbf{V}\cdot\mathbf{dS}$$



$$\rho_1\left(e_1+\frac{u_1^2}{2}\right)(-u_1A_1)+\rho_2\left(e_2+\frac{u_2^2}{2}\right)(u_2A_2) = -(-p_1u_1A_1+p_2u_2A_2)$$

$$p_1u_1A_1+\rho_1u_1A_1\left(e_1+\frac{u_1^2}{2}\right) = p_2u_2A_2+\rho_2u_2A_2\left(e_2+\frac{u_2^2}{2}\right)$$

$$\frac{p_1}{\rho_1}+e_1+\frac{u_1^2}{2} = \frac{p_2}{\rho_2}+e_2+\frac{u_2^2}{2}$$

$h = e + pv = e + p/\rho$

$$h_1+\frac{u_1^2}{2} = h_2+\frac{u_2^2}{2}$$

$$\boxed{h_0 = \text{const}}$$

$$\boxed{\rho_1 u_1 A_1 = \rho_2 u_2 A_2}$$

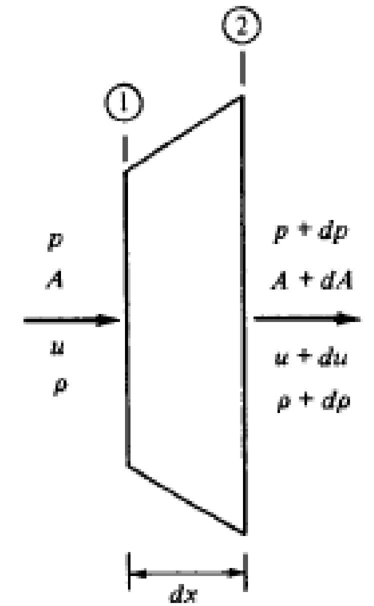
$$\boxed{p_1 A_1 + \rho_1 u_1^2 A_1 + \int_{A_1}^{A_2} p\, dA = p_2 A_2 + \rho_2 u_2^2 A_2}$$

$$\boxed{h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}}$$

$$p_2 = \rho_2 R T_2$$
$$h_2 = c_p T_2$$

Five equations for the five unknowns P2, u2, p2, T2, and h2. We could, in principle, solve these equations directly for the unknown flow quantities at station 2

$$\rho(x)\,u(x)\,A(x) \;\equiv\; \dot{m} \;=\; \text{constant}$$



$$d(\rho u A) = 0$$

$$p_1 A_1 + \rho_1 u_1^2 A_1 + \int_{A_1}^{A_2} p\, dA = p_2 A_2 + \rho_2 u_2^2 A_2$$

$$pA + \rho u^2 A + p\, dA = (p + dp)(A + dA) + (\rho + d\rho)(u + du)^2 (A + dA)$$

$$A\, dp + A u^2\, d\rho + \rho u^2\, dA + 2\rho u A\, du = 0$$

$$\rho u^2\, dA + \rho u A\, du + A u^2\, d\rho = 0$$

$$dp = -\rho u\, du$$

$$dp = -\rho u\, du$$

$$\frac{dp}{\rho} = \frac{dp}{d\rho}\frac{d\rho}{\rho} = -u\, du$$

$$\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s$$

$$\frac{dp}{d\rho} = a^2$$

$$a^2 \frac{d\rho}{\rho} = -u\, du$$

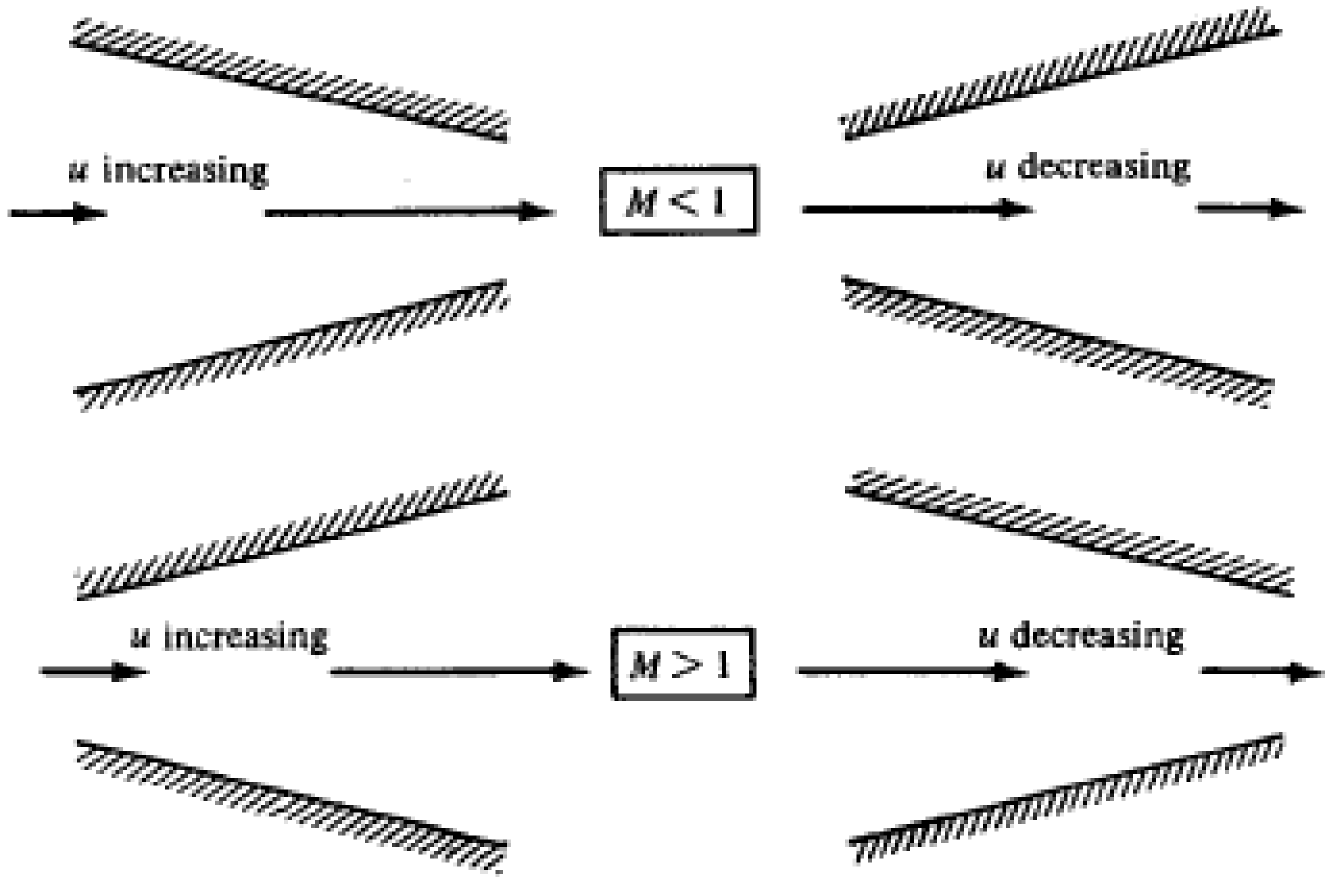
$$\frac{d\rho}{\rho} = -\frac{u\, du}{a^2} = -\frac{u^2}{a^2}\frac{du}{u} = -M^2\frac{du}{u}$$

$$d(\rho u A) = 0$$

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

$$\boxed{\frac{dA}{A} = (M^2 - 1)\frac{du}{u}}$$

$$\boxed{\frac{dA}{A} = (M^2 - 1)\frac{du}{u}}$$



Nozzle Flows

Consider a duct with a throat, connected at its inlet to a very large still air reservoir with total pressure and enthalpy p_r, h_r.

The duct exit is now subjected to an adjustable exit static pressure p_e

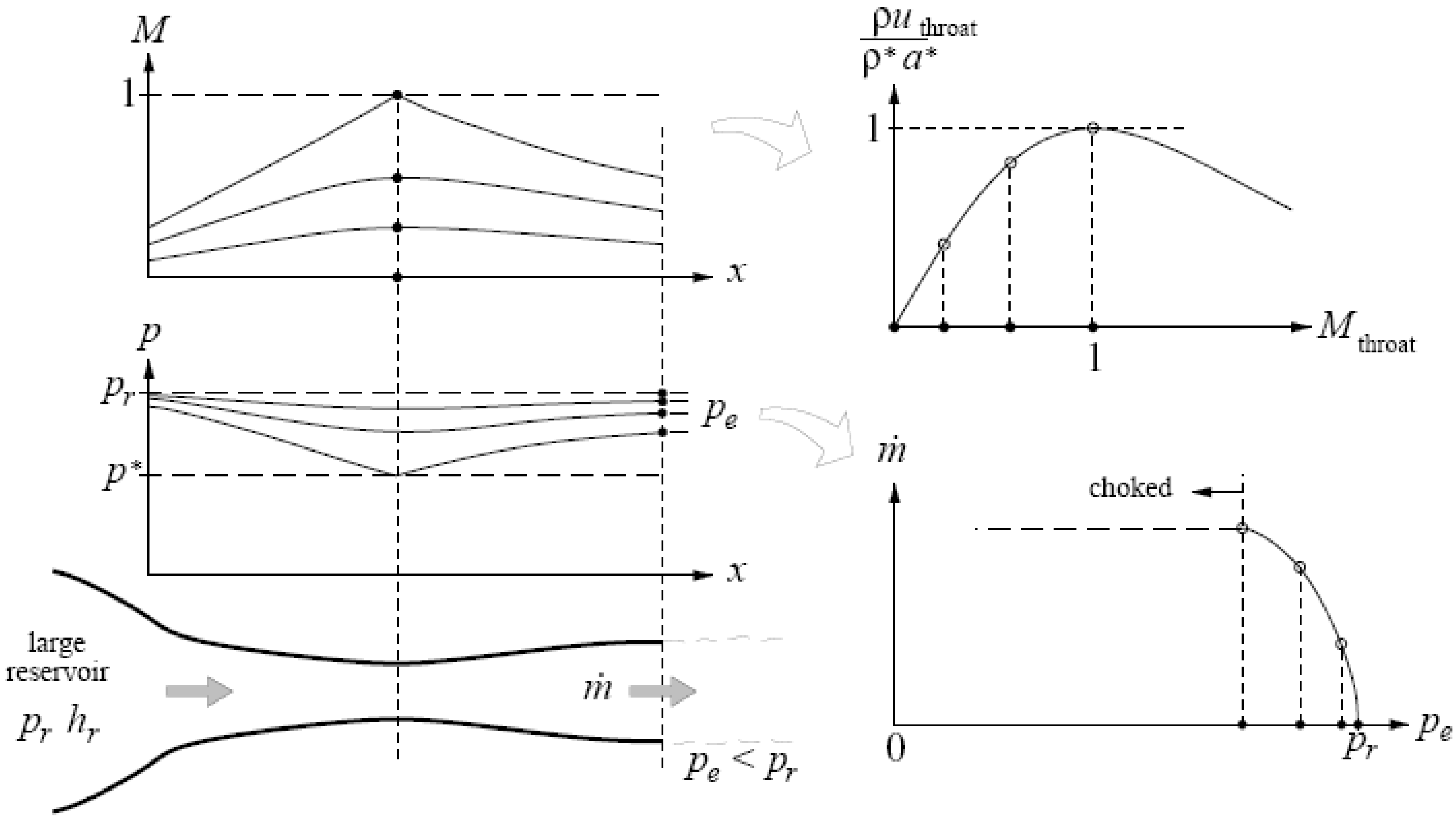
$$p_o = p_r$$

$$a_o^2 = (\gamma-1)h_o = (\gamma-1)h_r$$

$$\rho_o = \frac{\gamma p_o}{(\gamma-1)h_o} = \frac{\gamma p_r}{(\gamma-1)h_r}$$

$$M_e^2 = \frac{2}{\gamma-1}\left[(p_o/p_e)^{\frac{\gamma-1}{\gamma}} - 1\right]$$

$$\dot{m} = \rho_e u_e A_e = \frac{\gamma p_o}{\sqrt{(\gamma-1)h_o}} M_e \left(1 + \frac{\gamma-1}{2}M_e^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} A_e$$



M
1
x
p
p_r
p_e
p^*
x
large reservoir
$p_r\ h_r$
$\dot{m}$
$p_e < p_r$
$\frac{\rho u}{\rho^* a^*}$ throat
1
M_{throat}
1
$\dot{m}$
choked
0
p_r
p_e

Mass flux **ρu** at the throat reaching its maximum possible value **ρ* a***, which is given by:

$$\rho^* a^* = \rho_o a_o \frac{\rho^*}{\rho_o}\frac{a^*}{a_o} = \frac{\gamma p_r}{\sqrt{(\gamma-1)h_r}}\left(1+\frac{\gamma-1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$$

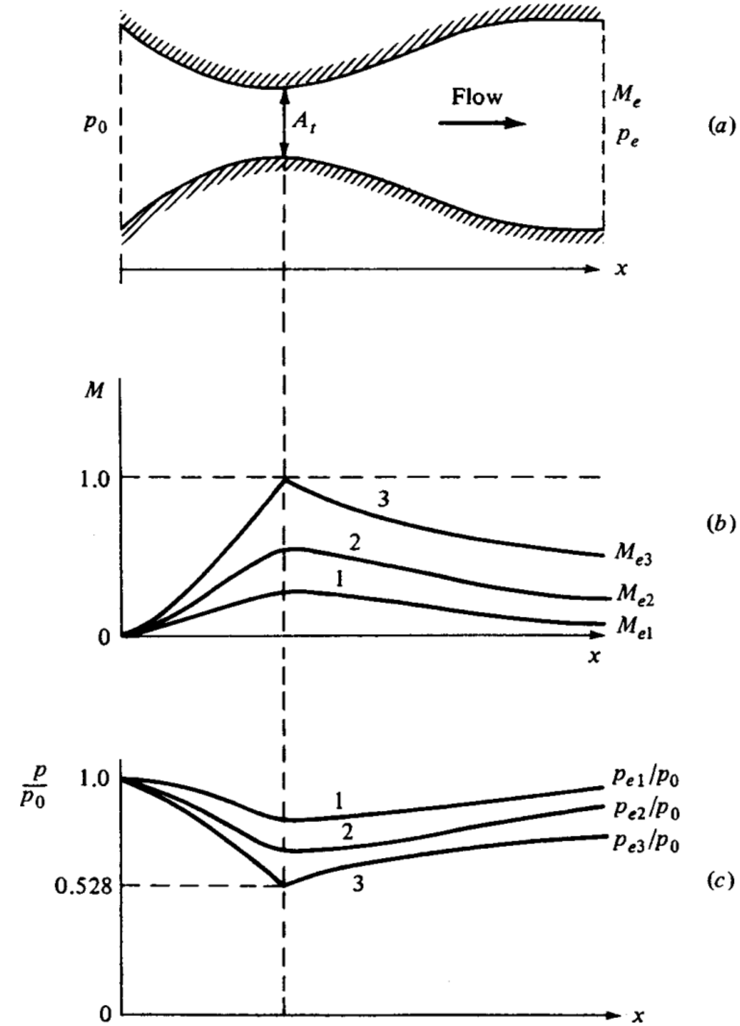
Therefore, the only way to change the mass flow of a choked duct is to change the reservoir's total properties **p_r** and/or **h_r**.

$$\dot{m} = \frac{p_0 A^*}{\sqrt{T_0}}\sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}}$$

Question:

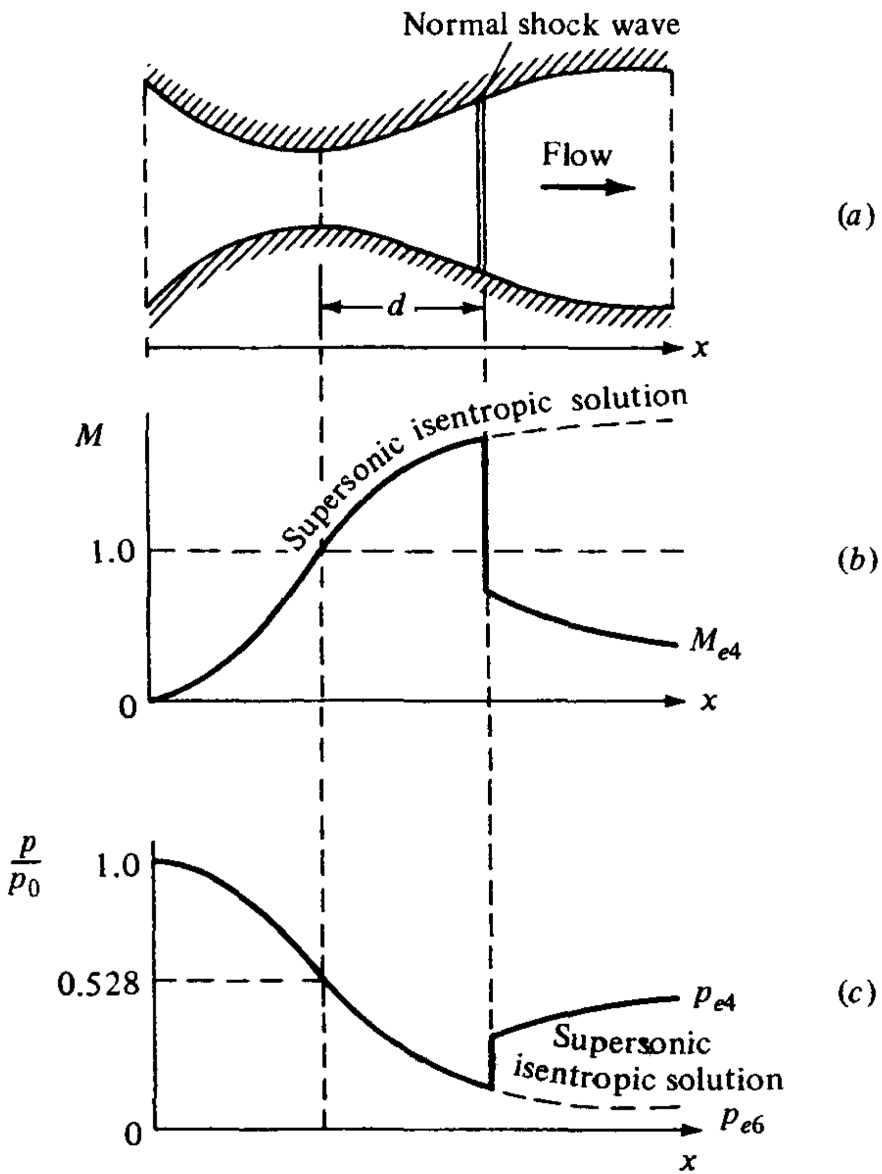
What happens in the duct when pe is reduced below pe3 that produces chock condition In the convergent?

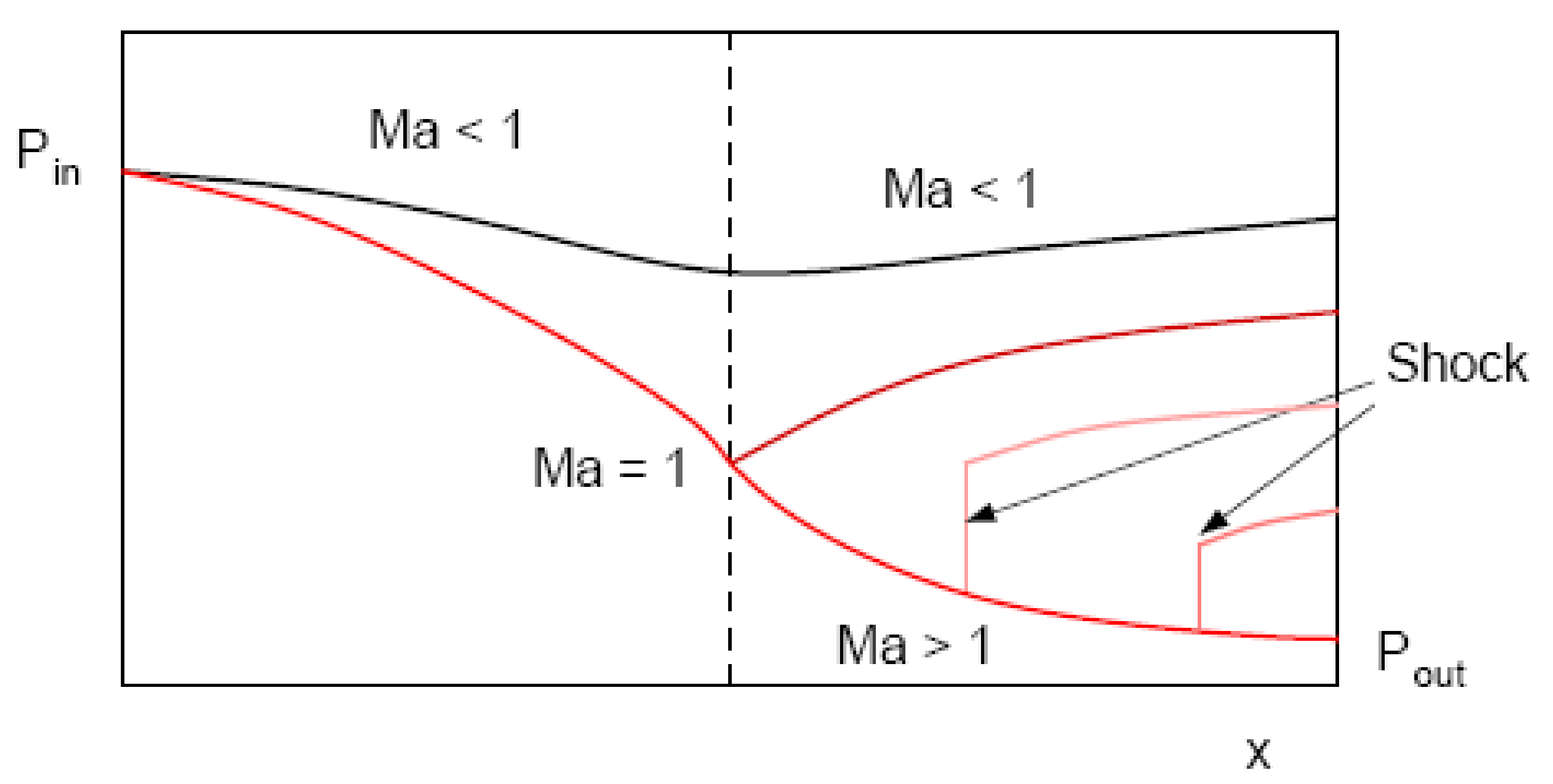
A lot happens in the divergent section of the duct. As the exit pressure is reduced below p e 3 , a region of supersonic flow appears downstream of the throat. However, the exit pressure is too high to allow an isentropic supersonic flow throughout the entire divergent section.



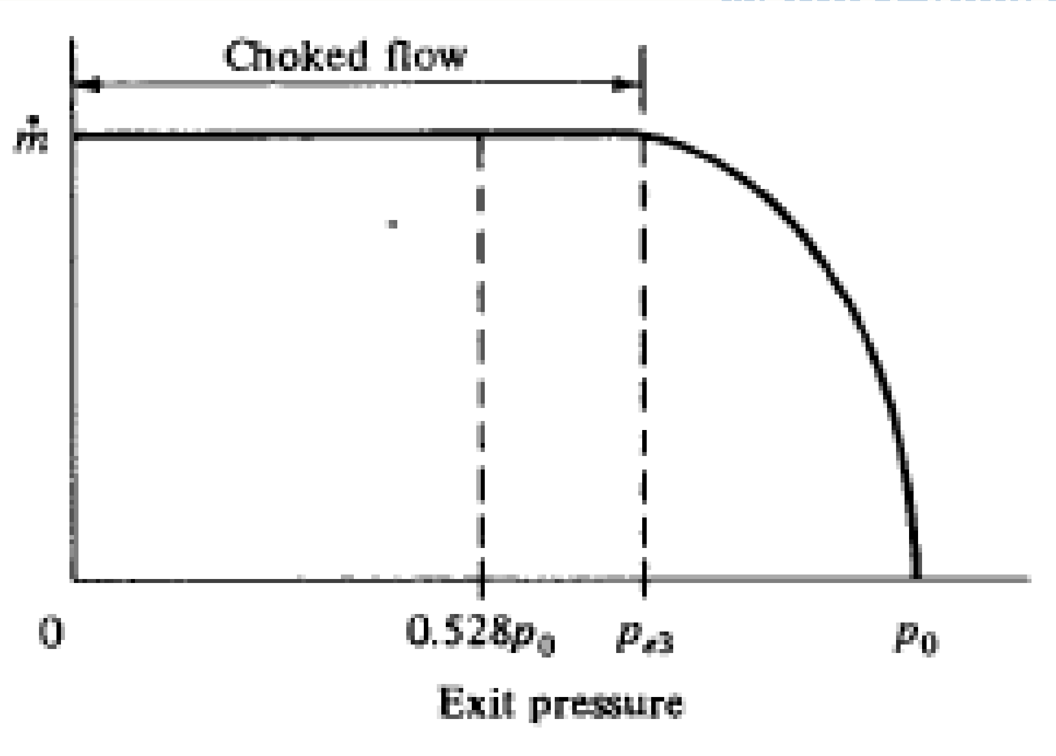
Instead,

For *pe* less than pe3 but substantially higher than the fully isentropic value a normal shock wave is formed downstream of the throat.


Normal shock wave
Flow
d
x
(a)
M
Supersonic isentropic solution
1.0
0
M_{e4}
x
(b)
$\frac{p}{p_0}$
1.0
0.528
0
p_{e4}
Supersonic isentropic solution
p_{e6}
x
(c)



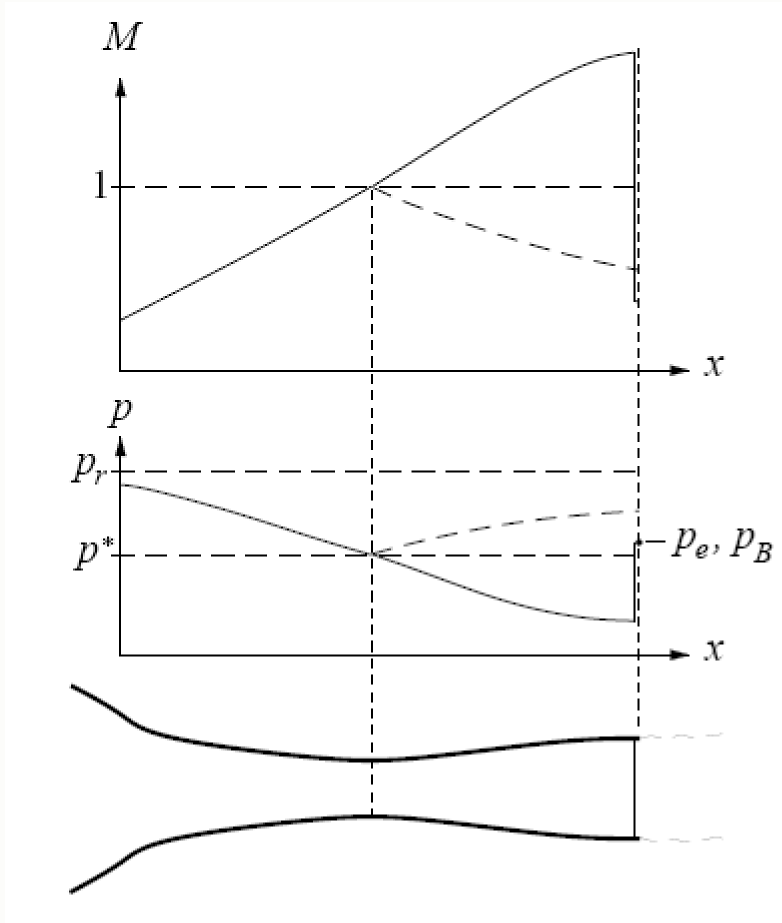
P in
Ma < 1
Ma < 1
Shock
Ma = 1
Ma > 1
P out
x


Choked flow
ṁ
0
0.528p₀
pₑ₃
p₀
Exit pressure

Supersonic-exit flows

With sufficiently low back pressure, the shock can be moved back to nearly the exit plane.

If the back pressure is reduced further, below the sonic pressure p* the exit flow becomes supersonic, leading to three possible types of exit flow.

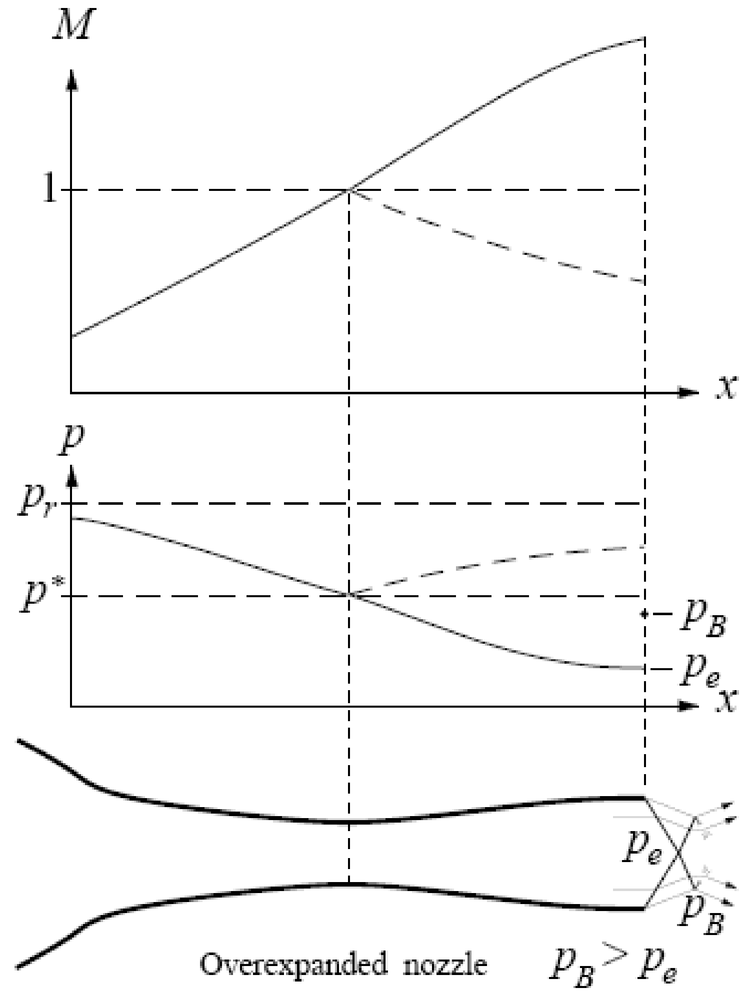


1-Overexpanded nozzle flow

In this case, $p_B < p^*$

the exit flow is supersonic, but $p_B > p_e$, so the flow must adjust to a higher pressure.

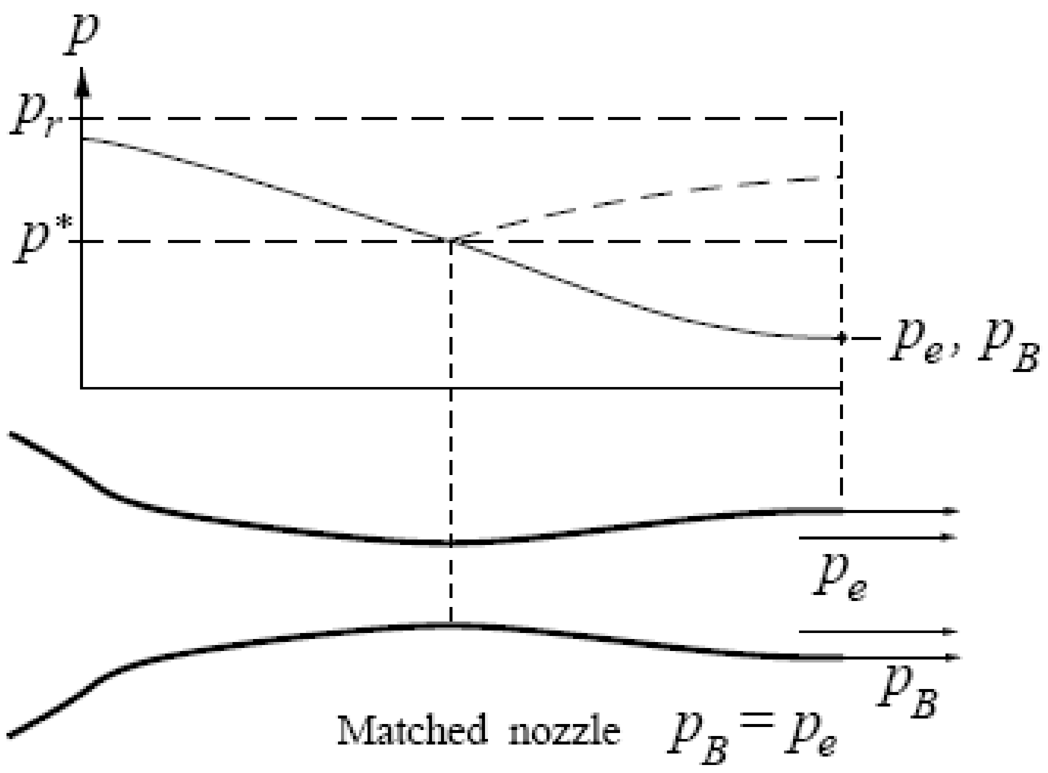
The streamline at the edge of the jet behaves much like a solid wall, whose turning angle adjusts itself so that the post-shock pressure is equal to p_B.



2- Matched nozzle flow.

In this case, the back pressure is reduced further until $p_B = p_e$.

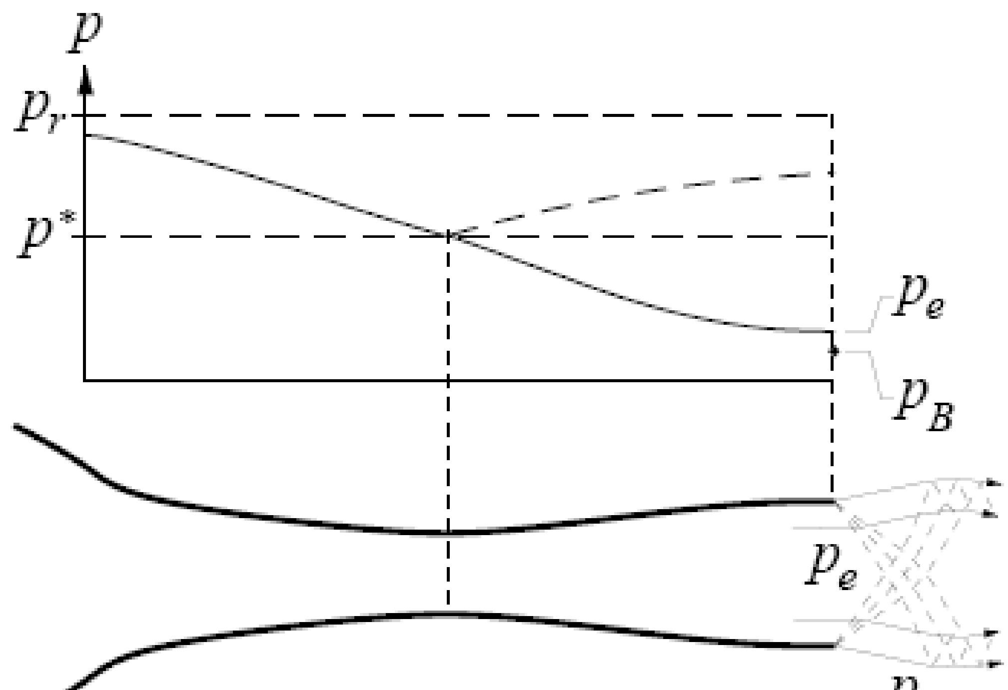
The duct nozzle flow comes out at the same pressure as the surrounding air, and hence no turning takes place.



3-Underexpanded nozzle flow.

In this case, the back pressure is reduced below the isentropic exit pressure, so that p_B < pe.

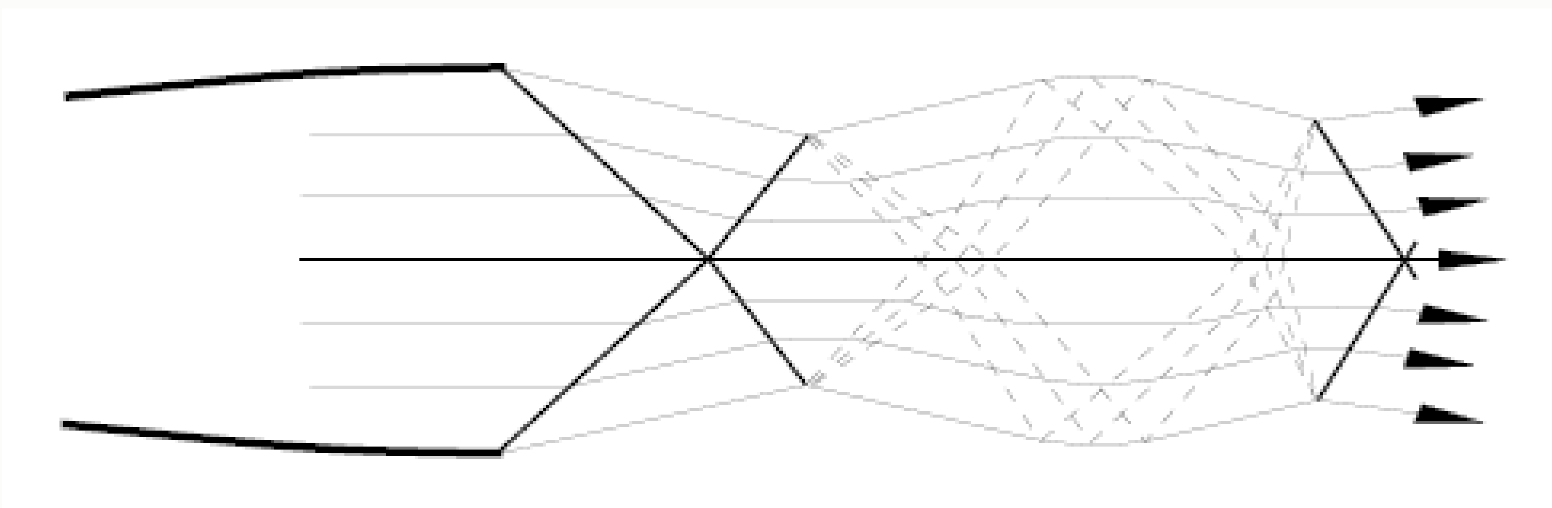
The duct nozzle flow must now expand to reach p_B, which is done through expansion fans attached to the duct nozzle edges



Underexpanded nozzle $p_B < p_e$

Jet shock diamonds

In the underexpanded and overexpanded nozzle flows, each initial oblique shock or expansion fan impinges on the opposite edge of the jet, turning the flow away or towards the centerline. The shock or expansion fan reflects off the edge, and propagates back to the other side, repeating the cycle until the jet dissipates though mixing. These flow patterns are known as shock diamonds, which are often visible in the exhaust of rocket or jet engines.



Determination of Choked Nozzle Flows

$$\dot{m} = \rho_e u_e A_e = \frac{\gamma p_o}{\sqrt{(\gamma-1)h_o}} M_e \left(1 + \frac{\gamma-1}{2} M_e^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} A_e$$

$$\downarrow$$

$$\dot{m} = \rho^* a^* A_t = \frac{\gamma p_r}{\sqrt{(\gamma-1)h_r}} \left(1 + \frac{\gamma-1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} A_t \qquad \text{(choked)}$$

$$\dot{m} = \frac{\gamma p_e}{\sqrt{(\gamma-1)h_r}} M_e \left(1 + \frac{\gamma-1}{2} M_e^2\right)^{1/2} A_e \qquad \text{(choked)}$$

$$\downarrow$$

$$M_e^2 \left(1 + \frac{\gamma-1}{2} M_e^2\right) = \left(\frac{p_r}{p_e} \frac{A_t}{A_e}\right)^2 \left(1 + \frac{\gamma-1}{2}\right)^{-\frac{\gamma+1}{\gamma-1}} \qquad \text{(choked)}$$

$$M_e^2\left(1+\frac{\gamma-1}{2}M_e^2\right) = \left(\frac{p_r}{p_e}\frac{A_t}{A_e}\right)^2\left(1+\frac{\gamma-1}{2}\right)^{-\frac{\gamma+1}{\gamma-1}} \qquad \text{(choked)}$$

$$p_{oe} = p_e\left(1+\frac{\gamma-1}{2}M_e^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{p_{oe}}{p_r} = \left(\frac{p_{o_2}}{p_{o_1}}\right)_{\text{shock}} = f(M_1)$$

where f(M1) is the shock total pressure ratio function, also available in tabulated form above equation .
Therefore implicitly determines M1 just in front of the shock, which together with the universal flow area function A/A* = f(M) determines the nozzle area at the shock.

EXAMPLE

Consider the isentropic flow through a convergent-divergent nozzle with an exit-to-throat area ratio of 2. The reservoir pressure and temperature are 1 atm and 288 K, respectively.
Calculate the Mach number, pressure, and temperature at both the throat and the exit for the cases where (a) the flow is supersonic at the exit and *(b) the flow is subsonic* throughout the entire nozzle except at the throat, where *M = 1.*

*(a) **At the throat, the flow is sonic. Hence,***

$$M_t = \boxed{1.0}$$

$$p_t = p^* = \frac{p^*}{p_0} p_0 = 0.528(1 \text{ atm}) = \boxed{0.528 \text{ atm}}$$

$$T_t = T^* = \frac{T^*}{T_0} = 0.833(288) = \boxed{240 \text{ K}}$$

At the exit, the flow is supersonic. $A_e/A^* = 2,$ $M_e = \boxed{2.2}$

$$p_e = \frac{p_e}{p_0} p_0 = \frac{1}{10.69}(1\text{ atm}) = \boxed{0.0935\text{ atm}}$$

$$T_e = \frac{T_e}{T_0} T_0 = \frac{1}{1.968}(288) = \boxed{146\text{ K}}$$

(b) At the throat, the flow is still sonic. Hence, from above, Mt = 1.0, pt = 0.528 atm, and Tt = 240 K.

However, at all other locations in the nozzle, the flow is subsonic. At the exit, where Ae/A* = 2 ,

$M_e = \boxed{0.3}$

$$p_e = \frac{p_e}{p_0} p_0 = \frac{1}{1.064}(1\text{ atm}) = \boxed{0.94\text{ atm}}$$

$$T_e = \frac{T_e}{T_0} T_0 = \frac{1}{1.018}(288) = \boxed{282.9\text{ K}}$$

EXAMPLE

For the nozzle in previous example, assume the exit pressure is 0.973 atm. Calculate the Mach numbers at the throat and the exit.

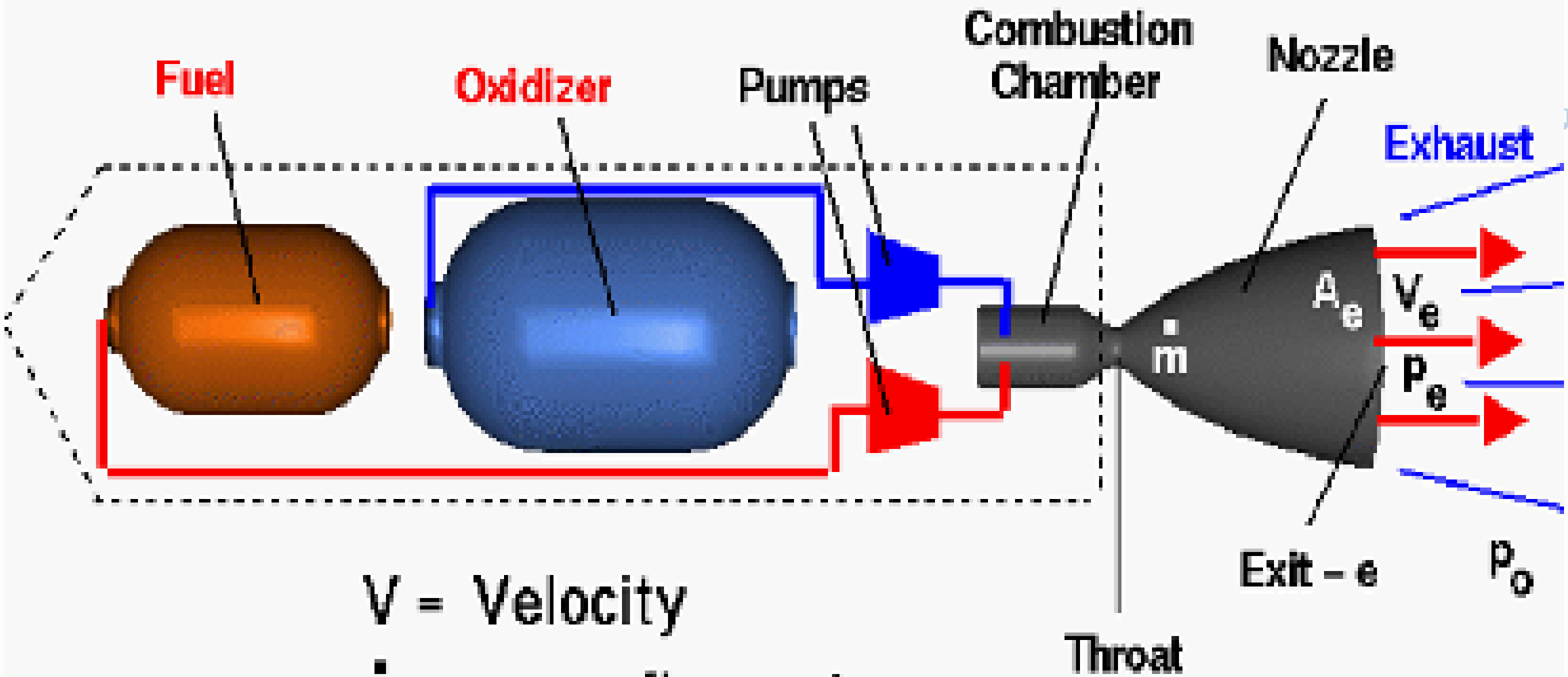
In the previous example, we saw that if pe = 0.94 atm, the flow is sonic at the throat, but subsonic. in this case, the flow is subsonic throughout the nozzle, including at the throat. For this case, A* takes on a reference value, and the actual geometric throat area is denoted by At.

$$\frac{p_0}{p_e} = \frac{1}{0.973} = 1.028$$

$$M_e = \boxed{0.2} \quad \text{and} \quad \frac{A_e}{A^*} = 2.964$$

$$\frac{A_t}{A^*} = \frac{A_t}{A_e}\frac{A_e}{A^*} = 0.5(2.964) = 1.482$$

From the subsonic portion of Appendix A, for At/A* = 1.482, we have:



V = Velocity

$\dot{m}$ = mass flow rate

p = pressure

Thrust = $F = \dot{m}\, V_e + (p_e - p_a)\, A_e$

EXAMPLE

Consider a rocket engine. Liquid hydrogen and oxygen are burned in the combustion chamber producing a combustion gas pressure and temperature of 30 atm and 3500 K, respectively. The area of the rocket nozzle throat is 0.4 m2 . The area of the exit is designed so that the exit pressure exactly equals the ambient atmospheric pressure at a standard altitude of 20 km. Assume an isentropic flow through the rocket engine nozzle with an effective value of the ratio of specific heats y = 1.22, and a constant value of the specific gas constant *R = 520 J/(kg)(K).*

(a) Calculate the thrust of the rocket engine.

(b) Calculate the area of the nozzle exit.

we first need to obtain the value of mass flow, *and* exit velocity, *ue.*

The mass flow is constant through the nozzle and is equal to m˙ = puA evaluated at any location in the nozzle.

A convenient location to evaluate m˙ is at the

$$\dot{m} = \rho^* u^* A^*$$

To obtain ρ^*, we need $\rho_0 = p_0/RT_0$. Noting that (1 atm) $= 1.01 \times 10^5$ N/m^2,

$$\rho_0 = \frac{(30)(1.01 \times 10^5)}{(520)(3500)} = 1.665 \text{ kg/m}^3$$

$$\frac{\rho*}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} = \left(\frac{2}{1.22+1}\right)^{\frac{1}{1.22-1}} = \left(\frac{2}{2.22}\right)^{4.545} = 0.622$$

$$\rho^* = 0.622\rho_0 = 0.622(1.665) = 1.036 \text{ kg/m}^3$$

At the throat, the flow velocity is equal to the local speed of sound, $u^* = a^*$

$$\frac{T^*}{T_0} = \frac{2}{\gamma + 1} = \frac{2}{2.22} = 0.901 \quad \Rightarrow \quad T^* = 0.901 T_0 = 0.901(3500) = 3154\,K$$

$$a^* = \sqrt{\gamma R T^*} = \sqrt{(1.22)(520)(3154)} = 1415 \text{ m/s}$$

$$\dot{m} = \rho^* u^* A^* = (1.036)(1415)(0.4) = 586.4 \text{ kg/s}$$

Next, we need to obtain the exit velocity *ue*.

$$\frac{p_0}{p_e} = \left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{\frac{\gamma}{\gamma - 1}}$$

where, from the statement of the problem, pe is equal to the ambient pressure at a standard altitude of 20 km. From Appendix D, at 20 km, poo = 5.5293 x 103 N/m2 .

$$p_e = p_\infty = 5529 N/m^2$$

$$1 + \frac{\gamma - 1}{2} M_e^2 = \left(\frac{p_0}{p_e}\right)^{\frac{\gamma-1}{\gamma}} = \left[\frac{(30)(1.01 \times 10^5)}{5529}\right]^{\frac{0.22}{1.22}} = (548)^{0.18} = 3.111$$

$$\frac{\gamma - 1}{2} M_e^2 = 2.111 \Rightarrow M_e^2 = (2.111)\left(\frac{2}{0.22}\right) = 19.19$$

$$M_e = 4.38$$

$$\frac{T_0}{T_e} = 1 + \frac{\gamma - 1}{2} M_e^2 = 3.111 \Rightarrow T_e = \frac{T_0}{3.111} = \frac{3500}{3.111} = 1125 \text{ K}$$

$$a_e = \sqrt{\gamma R T_e} = \sqrt{(1.22)(520)(1125)} = 844.8 \text{ m/s}$$

$$u_e = M_e a_e = (4.38)(844.8) = 3700 \text{ m/s}$$

Intermediate check: We can check this value of 3700 m/s for ue by directly using the energy equation,

$$c_p T_0 = c_p T_e + \frac{u_e^2}{2}$$

$$c_p = \frac{\gamma R}{\gamma - 1} = \frac{(1.22)(520)}{0.22} = 2883.6 \frac{\text{J}}{\text{kg} \cdot \text{K}}$$

$$u_e^2 = 2c_p(T_0 - T_e) = 2(2883.6)(3500 - 1125) = 1.3697 \times 10^7$$

$$u_e = 3700 \text{ m/s}$$

$$T = \dot{m} u_e = (586.4)(3700) = \boxed{2.17 \times 10^6 \text{ N}}$$

$$\left(\frac{A_e}{A^*}\right)^2 = \frac{1}{M_e^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_e^2\right)\right]^{\frac{\gamma+1}{\gamma-1}}$$

$$\frac{\gamma+1}{\gamma-1} = \frac{2.22}{0.22} = 10.1$$

$$\frac{\gamma}{\gamma+1} = \frac{2}{2.22} = 0.9$$

$$1+\frac{\gamma-1}{2}M_e^2 = 3.111$$

$$M_e = 4.38$$

$$\left(\frac{A_e}{A^*}\right)^2 = \frac{1}{(4.38)^2}[(0.9)(3.111)]^{10.1} = 1710.8$$

$$\frac{A_e}{A^*} = 41.36$$

$$A_e = (41.36)A^* = (41.36)(0.4) = \boxed{16.5\ \mathrm{m}^2}$$

EXAMPLE

Calculate the mass flow through the rocket engine described in the previous example using the closed-form analytical expression given:

$$\dot{m} = \frac{p_0 A^*}{\sqrt{T_0}} \sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}}$$

We have *po = 30 atm, T0 = 3500 K, A* = 0.4 m2 , R =*520 J/(kg)(K), and γ = 1.22.

$$p_0 = 30 \text{ atm} = (30)(1.01 \times 10^5) = 3.03 \times 10^6 \text{ N/m}^2$$

$$\gamma / R = 1.22/510 = 2.346 \times 10^{-3}$$

$$\frac{2}{\gamma+1} = \frac{2}{2.22} = 0.9$$

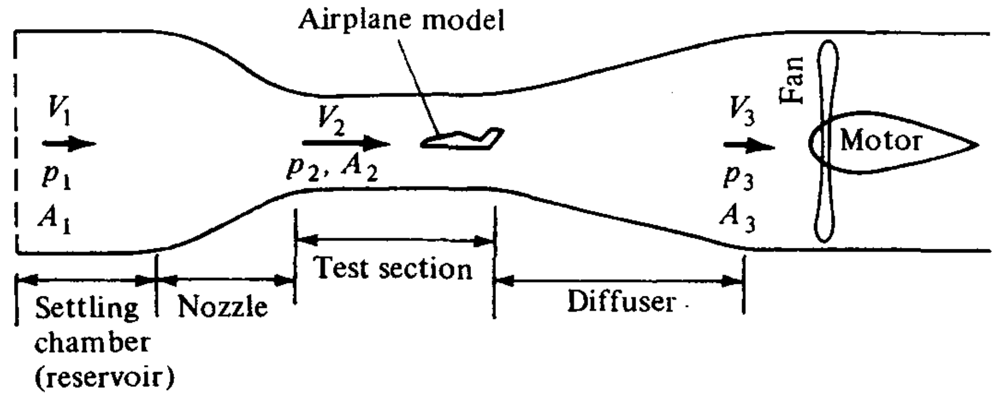
$$\frac{\gamma+1}{\gamma-1} = \frac{2.22}{0.22} = 10.09$$

$$\dot{m} = \frac{(3.03 \times 10^6)(0.4)}{\sqrt{3500}} \sqrt{(2.346 \times 10^{-3})(0.9)^{10.09}} = \boxed{583.2 \text{ kg/s}}$$

This result, obtained from a single equation, compares well with the value of 586.4 kg/s obtained from a sequence of calculations that is subject to a larger cumulative roundoff error

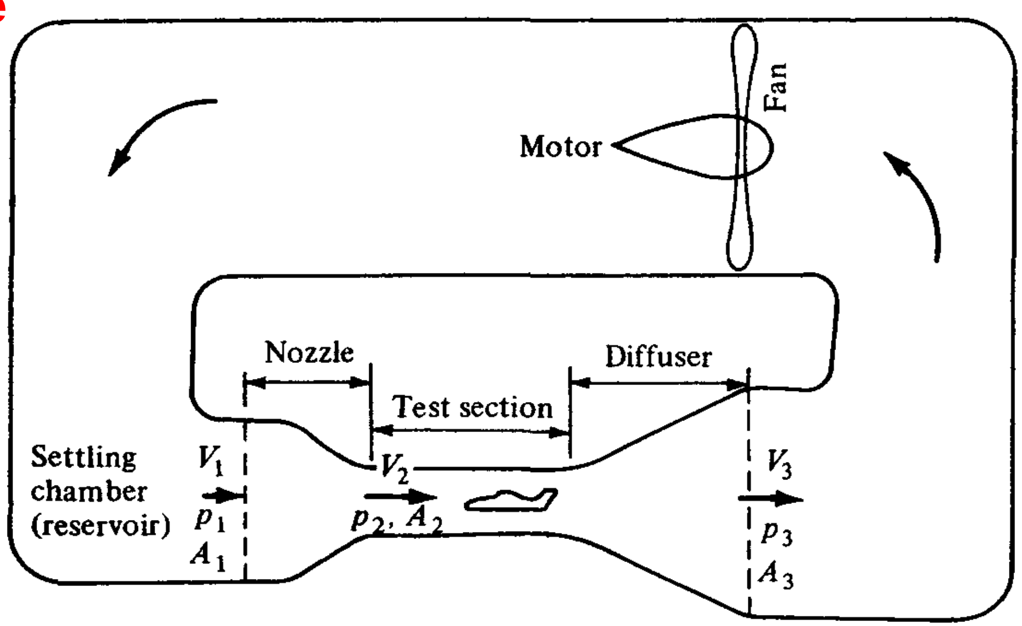
DIFFUSERS

In general,
we can define a diffuser as any duct designed to slow an incoming gas flow to lower velocity at the exit of the diffuser. The incoming flow can be subsonic, or it can be supersonic, as discussed in the present section.
or **it can be supersonic, that will be discussed**



(a) Open-circuit tunnel

However, the shape of the diffuser is drastically different, depending on whether the incoming flow is subsonic or supersonic.



(b) Closed-circuit tunnel

The total pressure of a flowing gas is a measure of the capacity of the flow to perform useful work. Let us consider two examples:

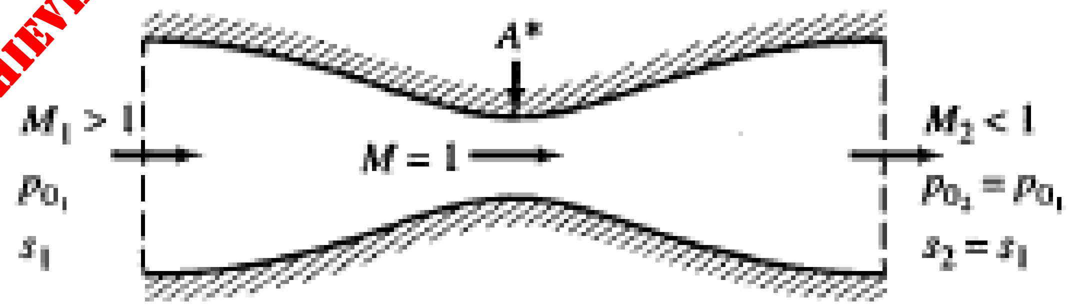
1. **A pressure vessel containing stagnant air at 10 atm**

2. **A supersonic flow at $M = 2.16$ *and* $p = 1$ *atm***

In case 1, the air velocity is zero; hence, $p_o = p = 10$ *atm. Now, imagine that* we want to use air to drive a piston in a piston-cylinder arrangement, where useful work is performed by the piston being displaced through a distance. The air is ducted into the cylinder from a large manifold, in the same vein as the reciprocating internal combustion engine in our automobile. In case, 1, the pressure vessel can act as the manifold; hence, the pressure on the piston is 10 atm, and a certain amount of useful work is performed, say, W_1. *However, in case 2, the supersonic* flow must be slowed to a low velocity before we can readily feed it into the manifold. If this slowing process can be achieved without loss of total pressure, then the pressure in the manifold in this case is also 10 atm (assuming $V = 0$), and the same amount of useful work W_1 *is performed. On the other hand, assume* that in slowing down the supersonic stream, a loss of 3 atm takes place in the total pressure. Then the pressure in the manifold is only 7 atm, with the consequent generation of useful work W_2, *which is less than in the first case; that is,* $W_2 < W_1$.

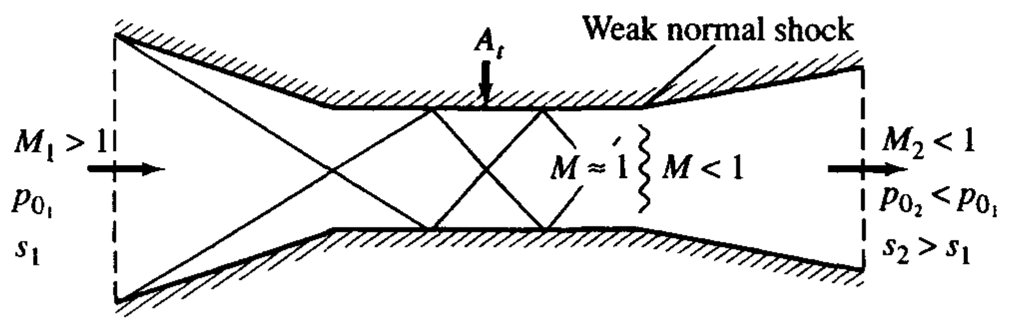
A diffuser is a duct designed to slow an incoming gas flow to lower velocity at the exit of the diffuser with as small a loss in total pressure as possible.

IT CAN NEVER BE ACHIEVED.



(*a*) Ideal (isentropic) supersonic diffuser

The art of diffuser design is to obtain as small a total pressure loss as possible, that is, to design the convergent, divergent, and constant-area throat sections so that Po2/Po1 is as close to unity as possible

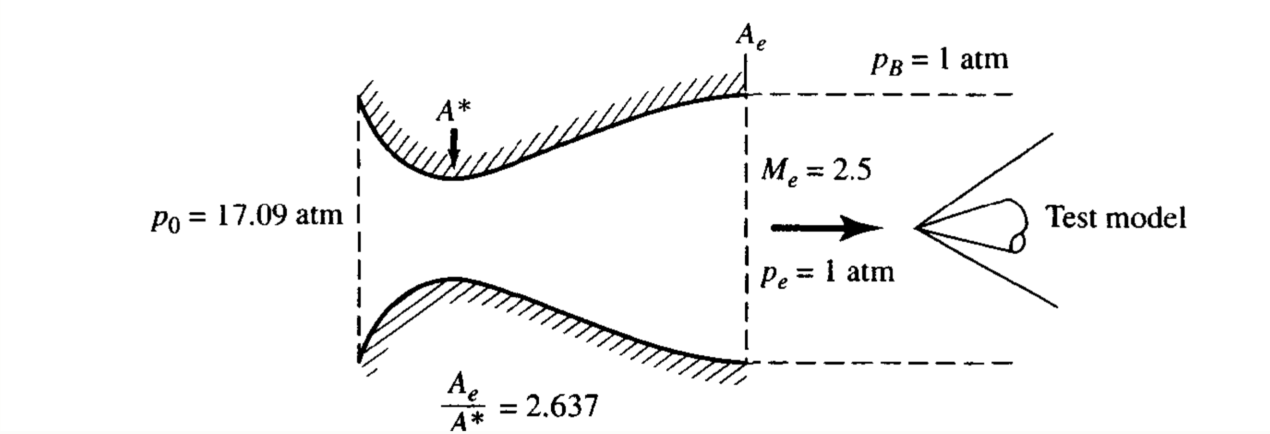


(*b*) Actual supersonic diffuser

SUPERSONIC WIND TUNNELS

Imagine that you want to create a Mach 2.5 uniform flow in a laboratory for the purpose of testing a model of a supersonic vehicle, say, a cone.

How do you do it?

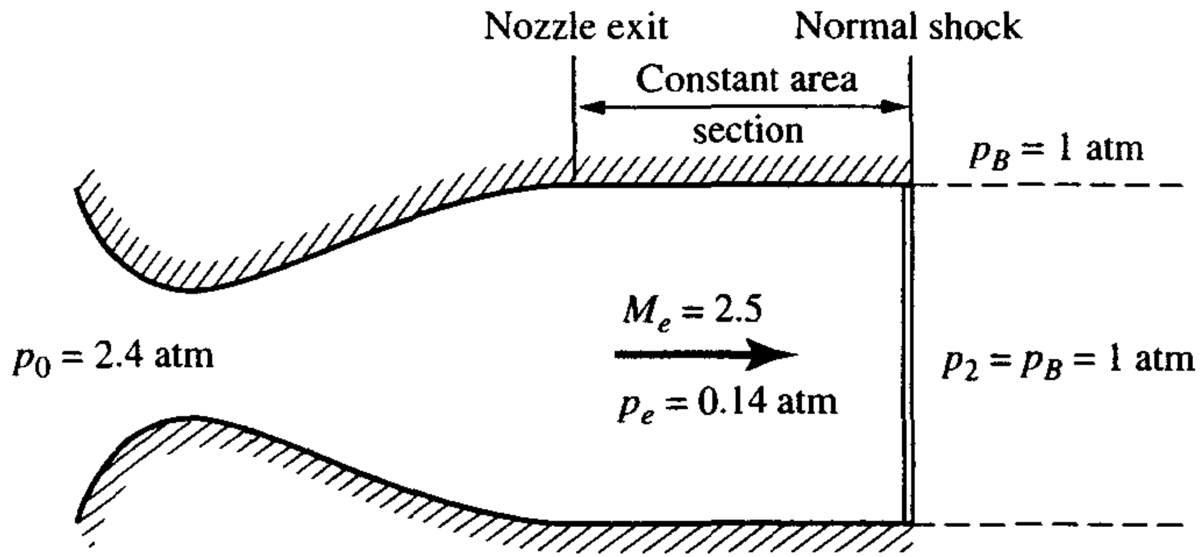


Here, the Mach 2.5 flow passes into the surroundings as a "free jet." The test model is placed in the flow downstream of the nozzle exit.

Question:

can you accomplish your objective in a more efficient way, at less cost?

Instead of the free jet, imagine that you have a long constant-area section downstream of the nozzle exit, with a normal shock wave standing at the end of the constant-area section



The static pressure ratio across the normal shock is p2/pe = 7.125.

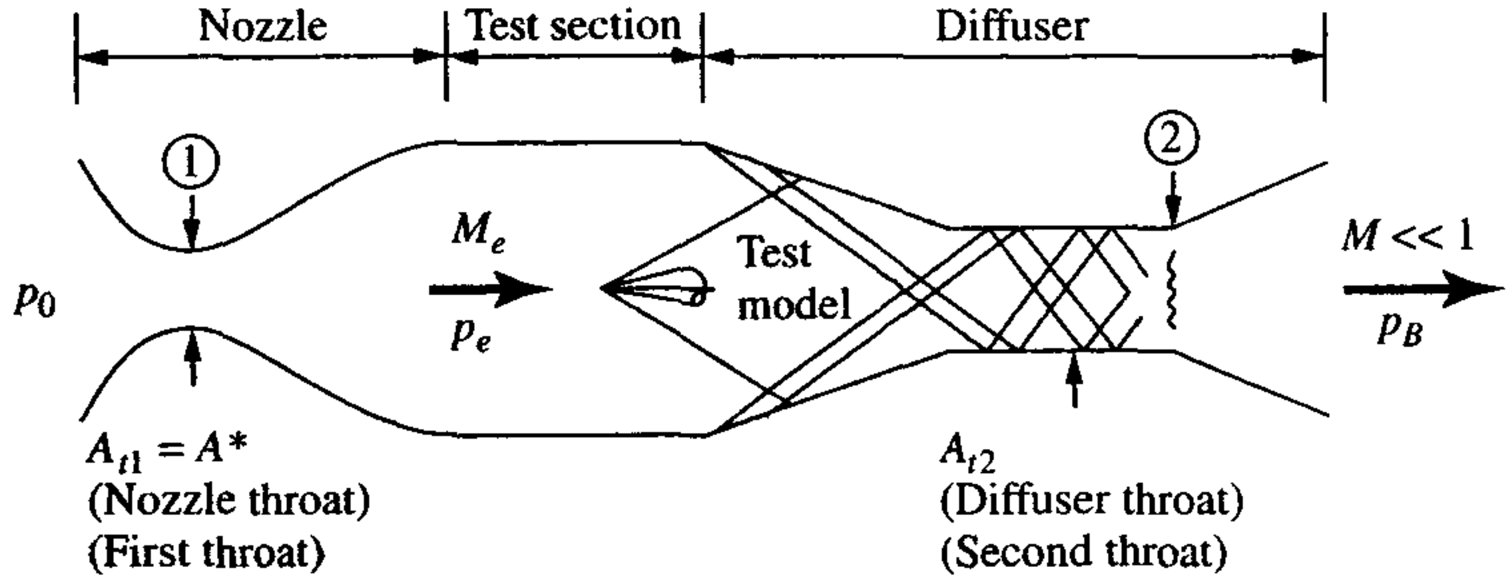
The normal shock wave is acting as a diffuser, slowing the air originally at Mach 2.5 to the subsonic value of Mach 0.513 immediately behind the shock. **Hence, by the addition of this "diffuser," we can more efficiently produce our uniform Mach 2.5 flow.**

1. A normal shock is the strongest possible shock, hence creating the largest total pressure loss. If we could replace the normal shock with a weaker shock, the total pressure loss would be less, and the required reservoir pressure po would be less than 2.4 atm.

2. It is extremely difficult to hold a normal shock wave stationary at the duct exit; in real life, flow unsteadiness and instabilities would cause the shock to move somewhere else and to fluctuate constantly in position. Thus, we could never be certain about the quality of the flow in the constant-area duct.

3. As soon as a test model is introduced into the constant-area section, the oblique waves from the model would propagate downstream, causing the flow to become two- or three-dimensional. The normal shock could not exist in such a flow.

Hence, let us replace the normal shock with the oblique shock diffuser shown in Figure



The main source of total pressure loss in a supersonic wind tunnel is the diffuser.

$$\dot{m}_1 = \dot{m}_2 \Rightarrow \rho_1^* a_1^* A_{t,1} = \rho_2 u_2 A_{t,2}$$

KN Toosi, $A_{t,2} \neq A_{t,1}$.

Aerospace engineering Faculty

Question: How does $A_{t,2}$ differ from $A_{t,1}$?

$$\frac{A_{t,2}}{A_{t,1}} = \frac{\rho_1^* a_1^*}{\rho_2^* a_2^*}$$

Hence, the flow throughout the wind tunnel is adiabatic,

$$\Rightarrow a_1^* = a_2^* \Rightarrow \frac{A_{t,2}}{A_{t,1}} = \frac{\rho_1^*}{\rho_2^*} \Rightarrow \frac{\rho_1^*}{\rho_2^*} = \frac{p_1^*/RT_1^*}{p_2^*/RT_2^*} = \frac{p_1^*}{p_2^*}$$

$$\Rightarrow \frac{A_{t,2}}{A_{t,1}} = \frac{p_1^*}{p_2^*}$$

$$p_1^* = p_{0,1}\left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)}$$

$$p_2^* = p_{0,2}\left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)}$$

$$\Rightarrow \quad \frac{A_{t,2}}{A_{t,1}} = \frac{p_{0,1}}{p_{0,2}}$$

$$p_{0,2} < p_{0,1} \quad \Rightarrow \quad A_{t,2} > A_{t,1}.$$

Only in the case of an ideal isentropic diffuser, where p0 = constant,

$$\Rightarrow \quad A_{t,2} = A_{t,1}$$

EXAMPLE

For the preliminary design of a Mach 2 supersonic wind tunnel, calculate the ratio of the diffuser throat area to the nozzle throat area.

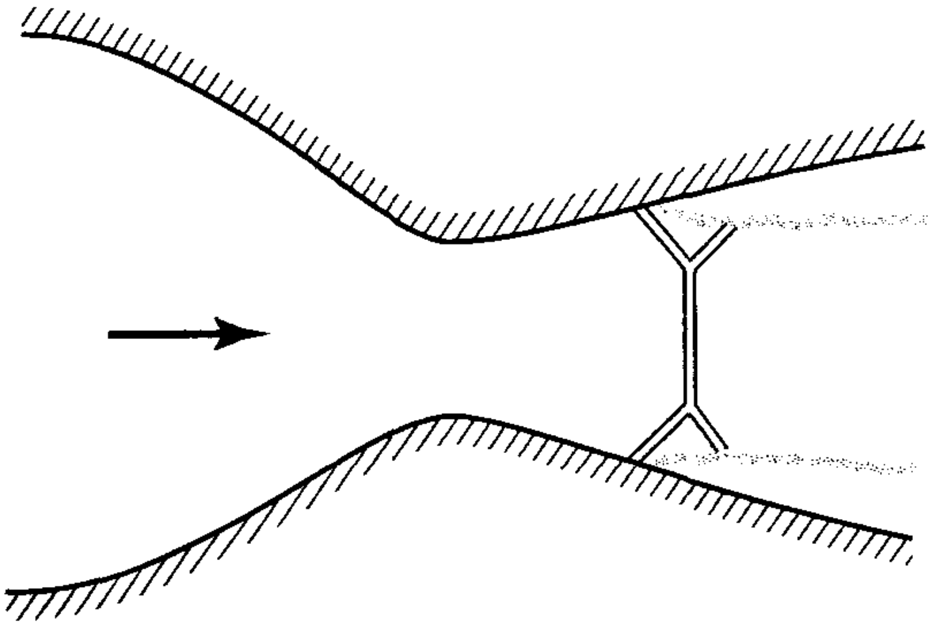
Assuming a normal shock wave at the entrance of the diffuser (for starting),

M=2 ⇒ $p_{0,2}/p_{0,1} = 0.7209$

$$\frac{A_{t,2}}{A_{t,1}} = \frac{p_{0,1}}{p_{0,2}} = \frac{1}{0.7209} = \boxed{1.387}$$

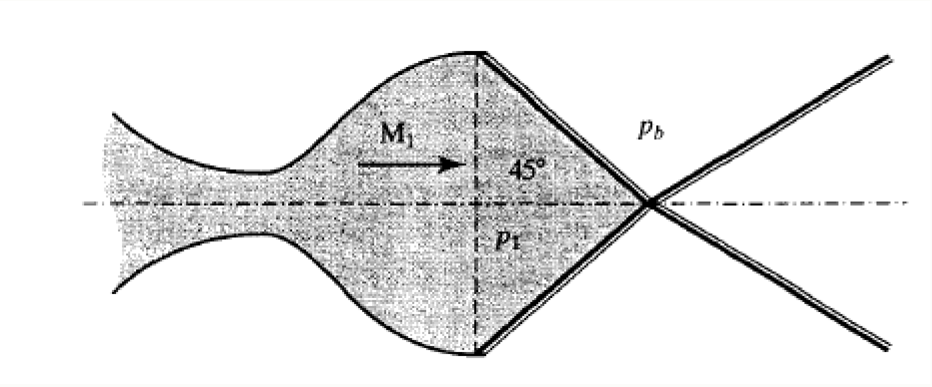
VISCOUS FLOW: SHOCK-WAVE/BOUNDARY-LAYER INTERACTION INSIDE NOZZLES

The adverse pressure gradient across the shock causes the boundary layer to separate from the nozzle wall. A lambda-type shock pattern occurs at the two feet of the shock near the wall, and the core of the nozzle flow, now separated from the wall, flows downstream at almost constant area.



EXAMPLE

A supersonic nozzle is designed to operate at Mach 2.0. Under a certain operating condition, however, an oblique shock making a 45° angle with the flow direction is observed at the nozzle exit plane. What percent of increase in stagnation pressure would be necessary to eliminate this shock and maintain supersonic flow at the nozzle exit?



For *M1 = 2.0,* $\longrightarrow$ *p1/po1 = 0.1278.*

The component of M1 normal to the oblique wave is M1sin 45° = 1.4142

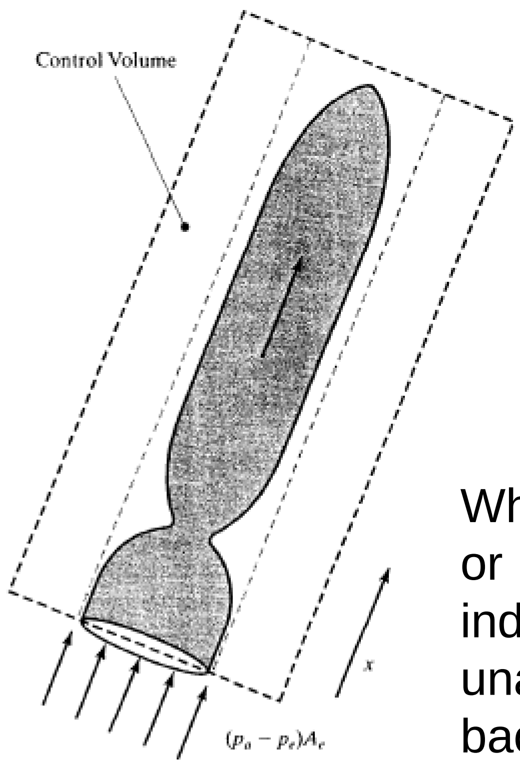
From the normal shock relations, $\longrightarrow$ *Pb/P1= 2.1667.*

pb/po1= (pb/p1)(Pl/Po1) = (2.1667)(0.1278) = 0.2769

po1=(1/0.2769)Pb = (3.6114)Pb

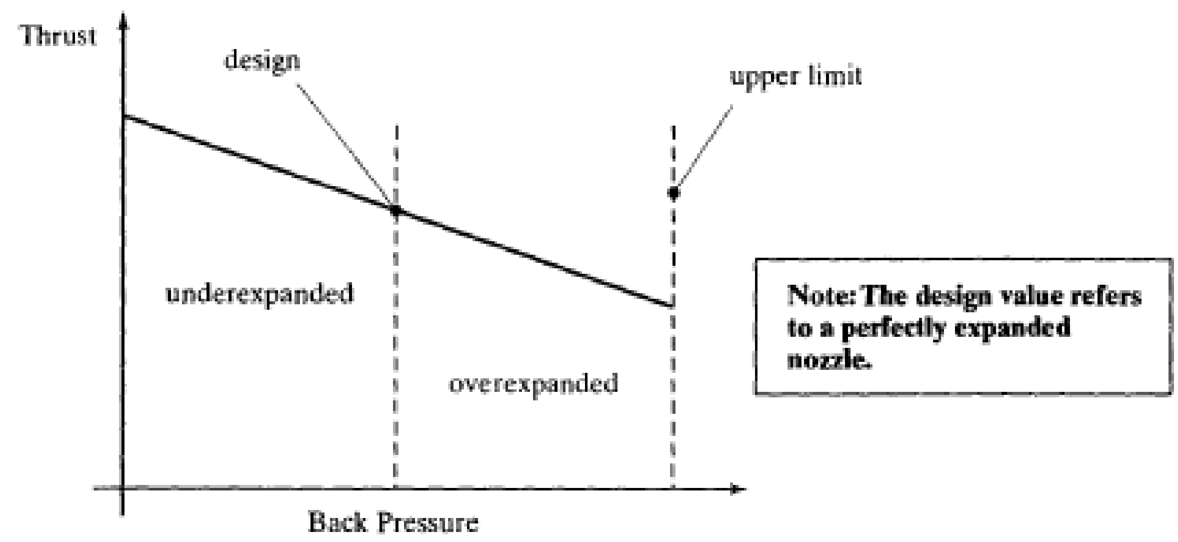
On the other hand, for supersonic exit flow with no shocks (i.e., Match nozzle)
po1= (l/0.1278)pb = 7.8247Pb

Thus, an increase of:
[(7.8247 - 3.6114)/3.6114]100 = 116.7 percent
in stagnation pressure is required.

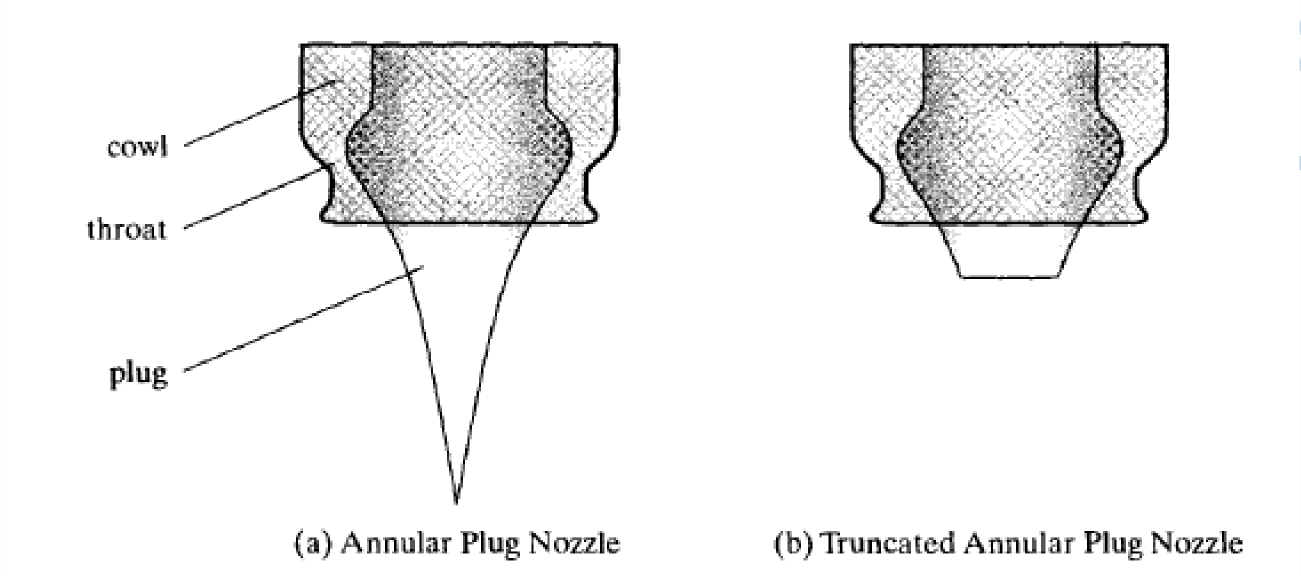


$$\mathbf{T} = (p_e - p_a)A_e + \dot{m}V_e$$

When a supersonic nozzle is operating in the under- or overexpanded regimes, with flow in the nozzle independent of back pressure, the exit velocity is unaffected by back pressure. Thus, over this range of back pressures,the above shows that larger thrusts are developed in the underexpanded case *(Pe > Pa)* and smaller thrusts in the overexpanded case (Pe < Pa)·

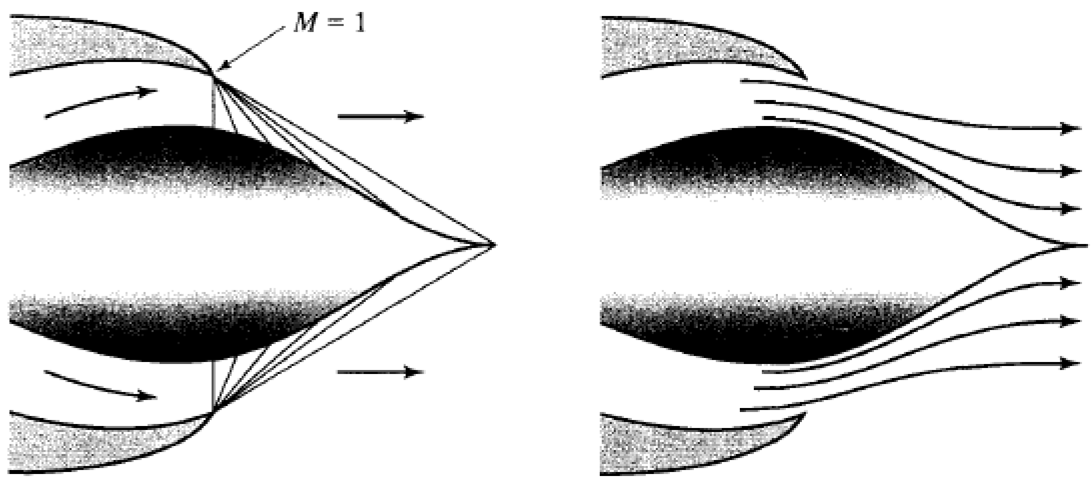


For back pressures greater than the upper limit indicated, a normal shock appears in the diverging portion of the nozzle, the exit velocity becomes subsonic, and this analysis no longer applies.



The plug nozzle was studied in the 1950s and '60s and reconsidered for use on the RLV X-33 in the 1990s.
This device is intended to allow the flow to be directed or controlled by the ambient pressure (since ambient pressure varies with altitude, this mechanism is termed altitude adaptation)
rather than by the nozzle walls.
In this nozzle, the supersonic flow is not confined within solid walls, but is exposed to the ambient pressure.

Plug nozzle operation at the design pressure ratio is depicted in Figure

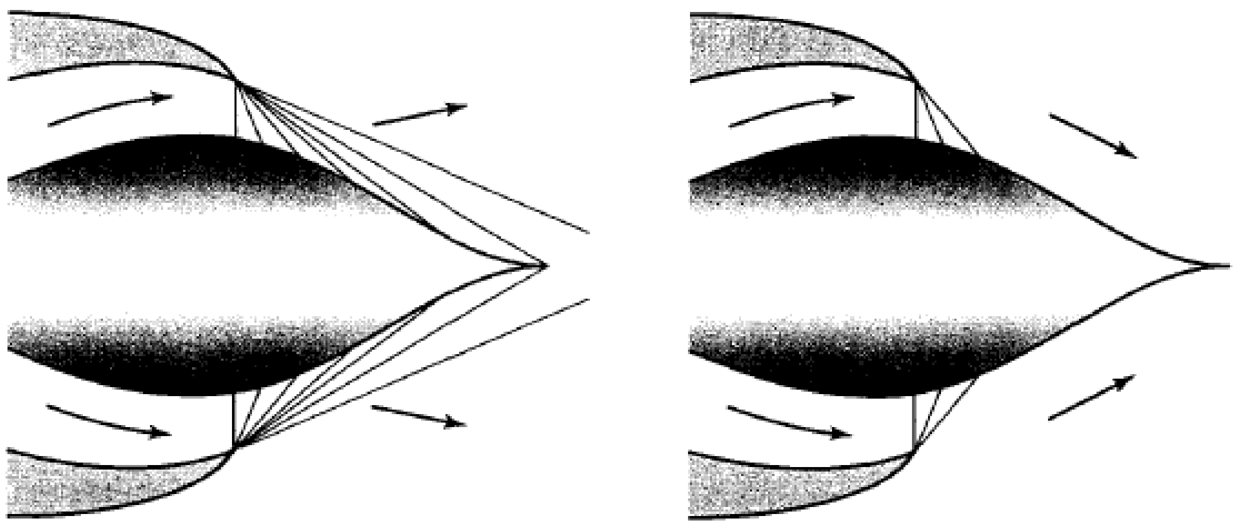


(a) Wave pattern for design (b) Streamlines for design

The annular flow first expands internally up to Mach 1 at the throat. The remainder of the expansion to the back pressure occurs with the flow exposed to ambient pressure. Since the throat pressure is considerably higher than the back pressure, a Prandtl-Meyer expansion fan is attached to the throat cowling as shown.

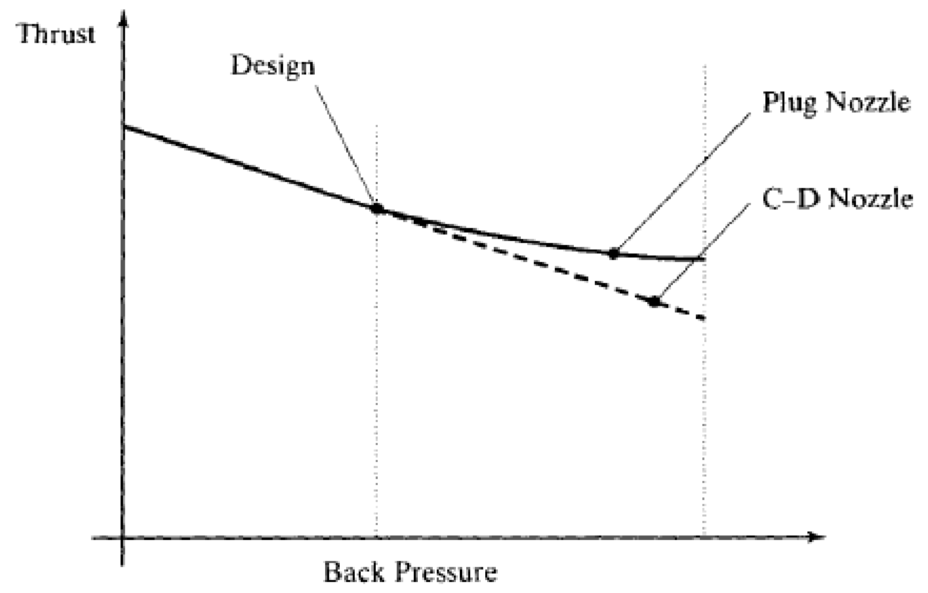
The plug is designed so that, at the design pressure ratio, the final expansion wave intersects the plug apex.

To produce a maximum axial thrust, it is necessary for the exit flow to have an axial direction. Therefore, the flow at the throat cowling must be directed toward the axis so that the turning produced by the expansion fan will yield axial flow at the plug apex.



(a) Underexpanded (b) Overexpanded

The expansion along the plug is controlled by the back pressure, whereas the converging-diverging nozzle expansion is controlled by nozzle geometry.



Comparison of Thrust and Back Pressure for Plug and C-D Nozzles

EXAMPLE

Performance computations comparing overexpanded plug and C-D nozzles

A rocket nozzle is designed to operate with a ratio of chamber pressure to ambient pressure pc/pa of 50. Compare the performance of a plug nozzle with that of a converging diverging nozzle for two cases where the nozzle is operating overexpanded:

pc/pa = 40 and pc/pa = 20.

Make the comparison on the basis of thrust coefficient CT = T/(Pc At),
where T is the thrust and At is the area of throat. Assume that γ=1.4 and in both cases
Neglect the effect of non axial exit velocity components.

For the design case:

From $p_e/p_o = p_a/p_c = 1/50 = 0.02$,

and since in the design case the flow is isentropic, we can determine the Mach number at the exit:

$M_e = 3.2077 \longrightarrow T_e/T_o = T_e/T_c = 0.3270.$

Now, from the definition of the thrust coefficient,

$$C_T = \frac{(\dot{m}_t V_e)}{p_c A_t} = \frac{(\rho_t A_t V_t) V_e}{p_c A_t} = \left(\frac{p_t}{RT_t}\right)\frac{V_t V_e}{p_c} = \left(\frac{p_t}{p_c}\right)\left(\frac{p_c}{RT_o}\right)\left(\frac{T_o}{T_t}\right)\frac{(M_t a_t)(M_e a_e)}{p_c}$$

Because the nozzle is choked, *Mt = 1, and therefore,*

$$\frac{p_t}{p_c} = \left(\frac{\gamma + 1}{2}\right)^{\gamma/(1-\gamma)} = 0.5283$$

$$\frac{T_t}{T_c} = \frac{2}{\gamma + 1} = 0.8333$$

$$C_T = \left[\frac{0.5283p_c}{R(0.8333T_c)}\right]\left[\frac{\sqrt{(1.4)(R)(0.8333)T_c}}{p_c}\right](3.2077)\sqrt{(1.4)(R)(0.3270)T_c} = 1.4862$$

For the converging-diverging nozzle operating off design:

$$C_T = (C_T)_{\text{design}} + \frac{A_e(p_e - p_a)}{A_t p_c} = 1.4862 + \frac{A_e}{A_t}\left(\frac{p_e}{p_c} - \frac{p_a}{p_c}\right)$$

where at $M_e = 3.2077$, $A_e/A_t = A_e/A^* = 5.1584$. So for $p_c/p_a = 40$,

$$C_T = 1.4862 + 5.1584\left(\frac{1}{50} - \frac{1}{40}\right) = 1.4604$$

For $p_c/p_a = 20$,

$$C_T = 1.4862 + 5.1584(0.02 - 0.05) = 1.3314$$

For the plug nozzle:

Flow in the plug nozzle does not continue to expand below ambient pressure, so there is no pressure term in the expression for thrust.

Now, at $\frac{p_c}{p_a} = 40, \qquad M_e = 3.0570, \qquad \frac{T_e}{T_c} = 0.3485$, and

$$C_T = \frac{\dot{m}_t V_e}{p_c A_t} = \left[\frac{0.5283 p_c}{R(0.8333T_c)}\right]\left[\frac{A_t\sqrt{1.4(R)0.8333T_c}}{p_c A_t}\right]3.0570\sqrt{1.4R(0.3485T_c)} = 1.4622$$

whereas for $p_c/p_a = 20$, $M_e = 2.6015$, $\frac{T_e}{T_c} = 0.4249$, and $C_T = 1.3740$

The plug nozzle is marginally superior to the C-D nozzle near the design point when operating in the overexpanded regime; however, the gap widens as the chamber-to-ambientpressure ratio decreases.

Adiabatic Duct Flow with Friction

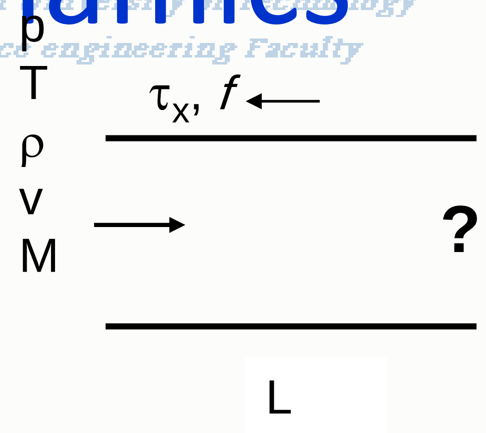
✓**Area changes**
Friction,
heat transfer

Are the most important factors affecting the properties in a flow system. Up to this point we have considered only one of these factors, that of variations in area.

- Friction must be included for flow through long ducts, especially if the cross-sectional area is small.

- Here, we study compressible flow with significant wall friction, but negligible heat transfer in ducts of constant cross section.

Fanno Flow - Thermodynamics



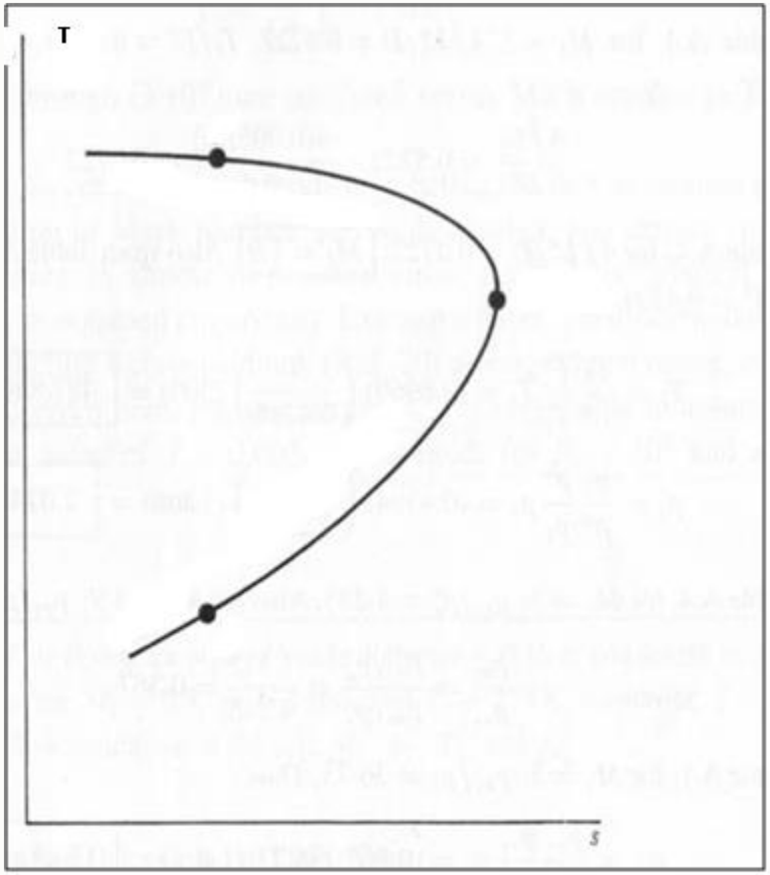
- Steady, 1-d, constant area, adiabatic flow with no external work but *with friction*
- Conserved quantities
  - since adiabatic, no work: **h_o=constant**
  - since A=const: mass flux=**ρv=constant**

  – combining: $h_o = h + (\rho v)^2/2\rho = \text{constant}$

- On h-s diagram, can draw **Fanno Line**
  – line connecting points with same h_o and ρv

Fanno line

$$s - s_1 = C_v \ln \frac{T}{T_1} - R \ln \frac{\rho}{\rho_1} = C_v \ln \frac{T}{T_1} + R \ln \frac{V}{V_1}$$



$$V = \sqrt{2Cp\left(T_0 - T\right)}$$

$$\boxed{h_0 = h + \frac{V^2}{2}}$$

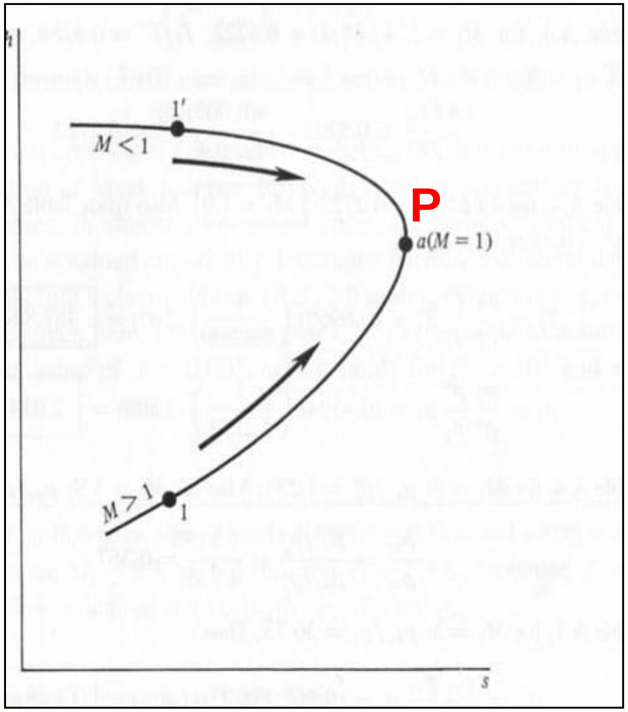
$$\frac{s - s_1}{C_v} = \ln \frac{T}{T_1} + \frac{\gamma - 1}{2} \ln \frac{T_0 - T}{T_0 - T_1}$$

$$= \ln T + \frac{\gamma - 1}{2} \ln\left(T_0 - T\right) + const$$

At point P

$$ds/dT = 0 \;\Rightarrow\; \frac{1}{T} - \frac{r-1}{2(T_0 - T)} = 0$$

$$C_p(T_0 - T) = \frac{V^2}{2} \;\Rightarrow\; \frac{1}{T} = \frac{\gamma - 1}{2.\dfrac{V^2}{2}\dfrac{(\gamma-1)}{\gamma R}} = \frac{\gamma R}{V^2} \;\Rightarrow\; V^2 = \gamma R T = a^2$$



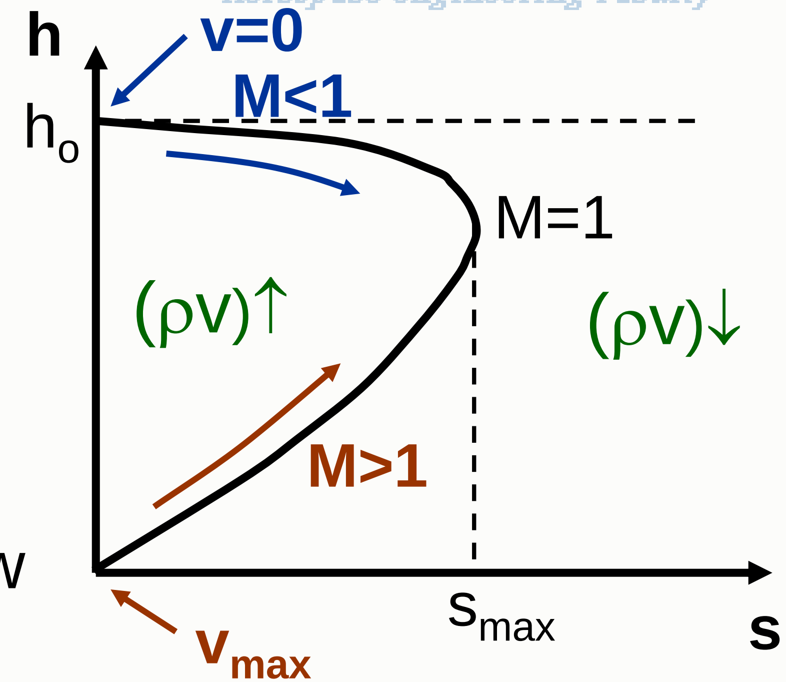
$$\therefore M = 1$$

T high $\rightarrow$ V low $\rightarrow$ above P , M < 1

T low $\rightarrow$ V high $\rightarrow$ below P , M > 1

Fanno Line

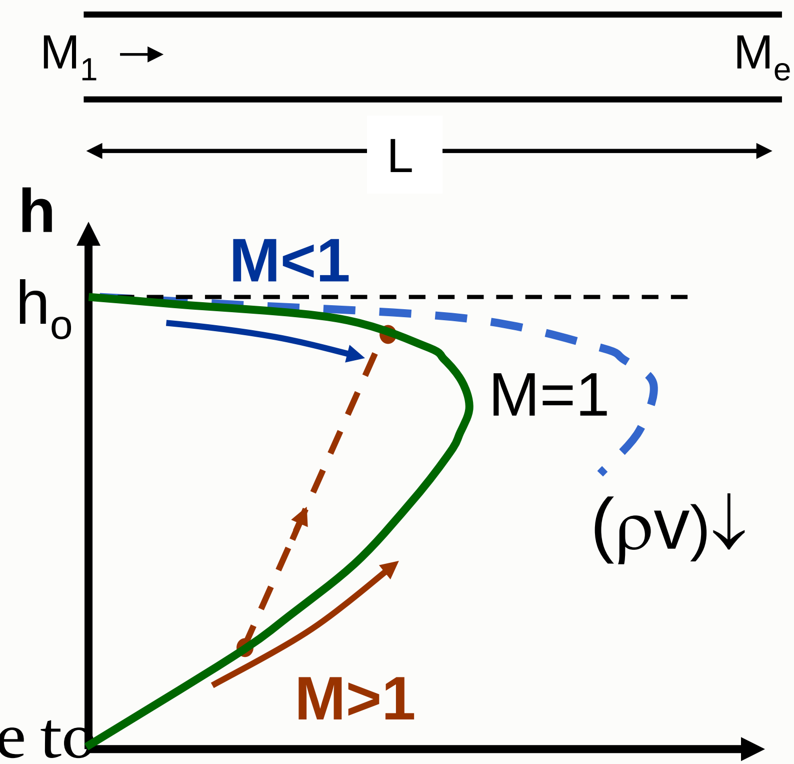
- Velocity change (due to friction) associated with entropy change

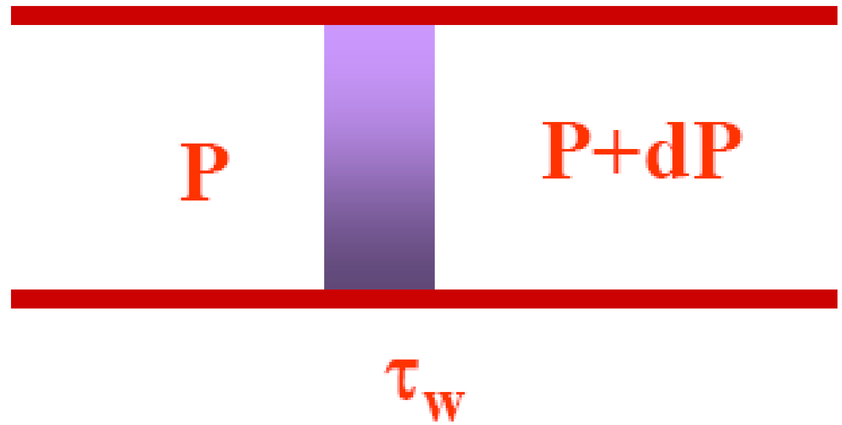


- Friction can only increase entropy
  – can only approach M=1
  – friction alone can not allow flow to transition between sub/supersonic
- Two solutions given (ρv,h_o,s): subsonic & supersonic
  – change mass flux: new Fanno line

Fanno Line - Choking

- Total friction experienced by flow increases with length of "flow", e.g., duct length, L
- For long enough duct, M_e=1 ($L=L_{max}$)
- What happens if $L>L_{max}$
  - flow already "choked"
  - **subsonic flow**: must move to different Fanno line (– – –), i.e., lower mass flux
  - **supersonic flow**: get a shock (– – –)



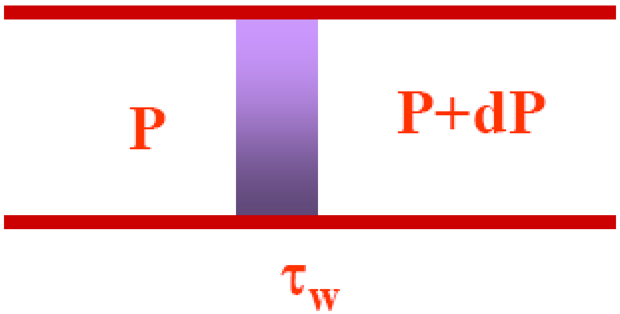


Energy Equation $\xrightarrow{1}$ $\boxed{C_P dT + V dV = 0}$

Continuity Equation

$\boxed{\rho V = \text{Const.}}$ $\xrightarrow{2}$ $\boxed{\dfrac{d\rho}{\rho} + \dfrac{dV}{V} = 0}$

Momentum Equation



$$PA - (P + dP)A - \tau_w D dx = \rho V A (V + dV - V)$$

3

$$dP + 4\tau_w \pi \frac{dx}{D} + \rho V dV = 0$$

$$\tau_w = \frac{f}{4}\frac{\rho V^2}{2}$$

Equation of State

$$P = \rho R T$$

4

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

Mach Number

$$M^2 = \frac{V^2}{kRT}$$

5

$$\frac{2dM}{M} = \frac{2dV}{V} - \frac{dT}{T}$$

1
$$C_p dT + VdV = 0$$

2
$$\frac{d\rho}{\rho} + \frac{dV}{V} = 0$$

3
$$dP + 4\tau_w \pi \frac{dx}{D} + \rho V dV = 0$$

4
$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

5
$$\frac{2dM}{M} = \frac{2dV}{V} - \frac{dT}{T}$$

5 Equations for 5 Unknowns

$$\frac{dM}{M}, \frac{dV}{V}, \frac{dT}{T}, \frac{dP}{P}, \frac{d\rho}{\rho}$$

Fanno Line – Mach Equations

$$\frac{dM^2}{M^2} = \frac{\gamma M^2\left(1+\frac{\gamma-1}{2}M^2\right)}{1-M^2}\frac{fdx}{D}$$

(X.6)

$$\frac{dT}{T} = \frac{dh}{h} = \frac{-\gamma(\gamma-1)M^4}{2\left(1-M^2\right)}\frac{fdx}{D}$$

(X.8)

$$\frac{dp}{p} = \frac{-\gamma M^2\left[1+(\gamma-1)M^2\right]}{2\left(1-M^2\right)}\frac{fdx}{D}$$

(X.7)

$$\frac{d\rho}{\rho} = -\frac{dv}{v} = \frac{-\gamma M^2}{2\left(1-M^2\right)}\frac{fdx}{D}$$

(X.9)

$$\frac{ds}{R} = -\frac{dp_o}{p_o} = \frac{\gamma M^2}{2}\frac{fdx}{D}$$

(X.10)

- can write each as only f(M)
- p_o loss due to entropy rise

Property Variations

- Look at signs of previous equations to see how properties changed by friction as we move along flow
  - $(1\text{-}M^2)$ term makes $M<1$ different than $M>1$

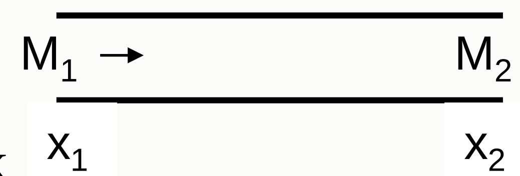
	M<1	M>1
s	↑	↑
p_o	↓	↓
M	↑	↓
h,T	↓	↑
p	↓	↑
ρ	↓	↑
v	↑	↓

- Friction increases s, $\Rightarrow p_o$ drop
- Friction drives $M \rightarrow 1$
- h_o, T_o const: h,T opposite to M
- p, ρ same as T (like isen. flow)
- ρv=const: v opposite of ρ

A Solution Method

- Need to integrate (X.6-10) to find how properties change along length of flow ($f$dx/D)
  - can integrate or use tables of integrated values

- Mach number variation



$$\int_{M_1^2}^{M_2^2} \frac{\left(1-M^2\right)dM^2}{\gamma M^4\left(1+\frac{\gamma-1}{2}M^2\right)} = \int_0^L \frac{f\left(\mathrm{Re},\mathrm{surface}\right)dx}{D}$$

function of Reynolds number (v) and surface roughness

$$\int_{M_1^2}^{1} \frac{\left(1-M^2\right)dM^2}{\gamma M^4\left(1+\frac{\gamma-1}{2}M^2\right)} = \frac{\bar{f}\,L_{max}}{D}$$

$$\Rightarrow \frac{\bar{f}\,L_{max}}{D} = f\,(M)\ \text{only}$$

1) use avg. f

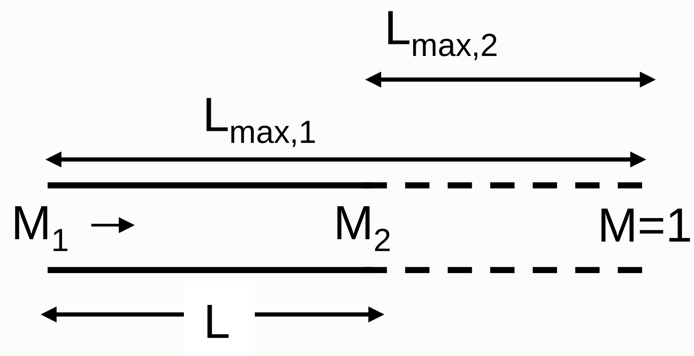
2) to **tabularize solution**, use reference condition: $M_2=1$, $L_2=L_{max}$

Use of Tables

- To get change in M, use change in $f\,L_{max}/D$ (like using A/A^*)

$$\frac{fL}{D} = \int_{M_1^2}^{M_2^2} \left\{ \frac{\left(1-M^2\right)}{\gamma M^4 \left(1+\frac{\gamma-1}{2}M^2\right)} \right\} dM^2$$

$$= \int_{M_1^2}^{1} \{-\} dM^2 - \int_{M_2^2}^{1} \{-\} dM^2$$



(X.11)

$$\frac{fL}{D} = \left.\frac{fL_{max}}{D}\right)_{M_1} - \left.\frac{fL_{max}}{D}\right)_{M_2}$$

so if you know $f\,L/D$ and M_1, 1) look up $f\,L_{max}/D$ at M_1
2) calculate $f\,L_{max}/D$ at M_2
3) look up corresponding M_2

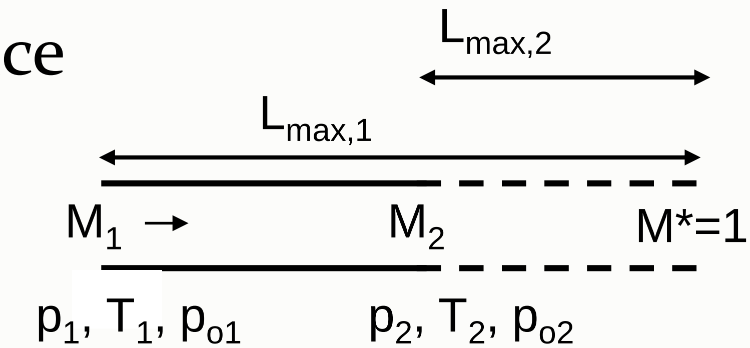
- Find values in Appendix E in John

Mach Number-Distance Relationship

$$\frac{\bar{f}L_{max}}{D}=\frac{1-M^2}{\gamma M^2}+\frac{\gamma+1}{2\gamma}\ln\frac{(\gamma+1)M^2}{2(1+\frac{\gamma-1}{2}M^2)}$$

TD Property Changes

- To get changes in T, p, p_o, … can also use M=1 condition as reference condition (*)



- Integrate (X.7-10), e.g.,

$$\int_{p_1}^{p_2} \frac{dp}{p} = \int_{M_1}^{M_2} -\frac{1}{2} \frac{1+(\gamma-1)M^2}{1+\frac{\gamma-1}{2}M^2} \frac{dM^2}{M^2}$$

$$\frac{p_2}{p_1} = \left[\frac{M_1^2\left(1+\frac{\gamma-1}{2}M_1^2\right)}{M_2^2\left(1+\frac{\gamma-1}{2}M_2^2\right)} \right]^{1/2} \Rightarrow \frac{p}{p^*} = \frac{1}{M}\sqrt{\frac{\frac{\gamma+1}{2}}{1+\frac{\gamma-1}{2}M^2}}$$

Fanno Flow Property Changes

- Summarize results in terms of reference conditions

$$\frac{T}{T^*} = \frac{(\gamma+1)/2}{1+\frac{\gamma-1}{2}M^2}$$ **(X.12)**

$$\frac{p_o}{p_o^*} = \frac{1}{M}\left(\frac{T}{T^*}\right)^{\frac{\gamma+1}{2(1-\gamma)}}$$ **(X.14)**

$$\frac{p}{p^*} = \frac{1}{M}\sqrt{\frac{T}{T^*}}$$ **(X.13)**

$$\frac{v}{v^*} = \frac{\rho^*}{\rho} = M\sqrt{\frac{T}{T^*}}$$ **(X.15)**

- In terms of initial and final properties

$$\frac{T_2}{T_1} = \frac{\left(1+\frac{\gamma-1}{2}M_1^2\right)}{\left(1+\frac{\gamma-1}{2}M_2^2\right)}$$

(X.16) (T_o=const)

$$\frac{p_2}{p_1} = \frac{M_1}{M_2}\sqrt{\frac{T_2}{T_1}}$$

(X.17)

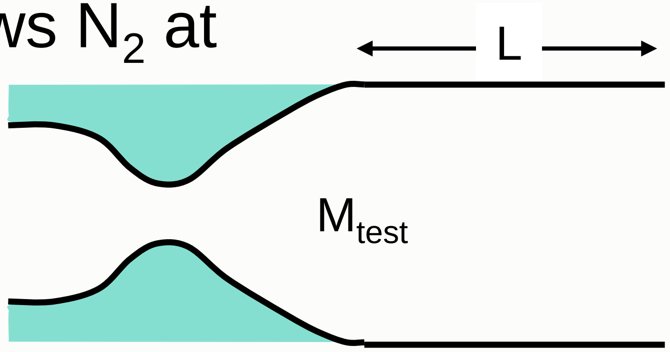
$$\frac{p_{o2}}{p_{o1}} = \frac{M_1}{M_2}\left(\frac{T_2}{T_1}\right)^{\frac{\gamma+1}{2(1-\gamma)}}$$

(X.18)

$$\frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} = \frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}}$$ **(X.19)**

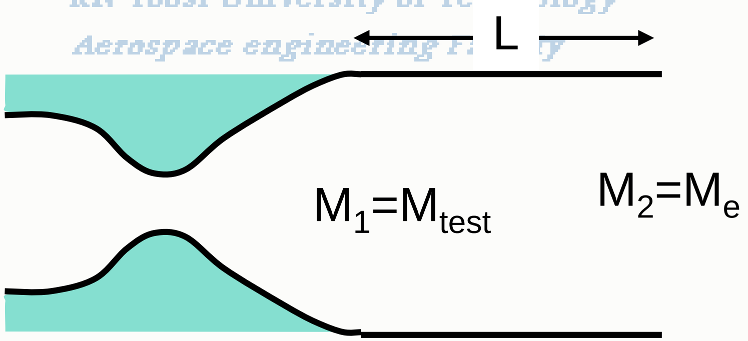
Example

- **Given:** Exit of supersonic nozzle connected to straight walled test section. Test section flows N_2 at M_{test}=3.0, T_o=290. K, p_o=500. kPa, L=1m, D=10 cm, f=0.005



- **Find:**
  - M, T, p at end of test section
  - $p_{o,exit}/p_{o,inlet}$
  - L_{max} for test section
- **Assume:** N_2 is tpg/cpg, γ=1.4, steady, adiabatic, no work

Solution



- **Analysis:**
  - **M_e**

    (X.11) $$\frac{fL}{D} = \left.\frac{fL_{max}}{D}\right)_{3.0} - \left.\frac{fL_{max}}{D}\right)_{M_e}$$

$$\left.\frac{fL_{max}}{D}\right)_{M_e} = 0.5222 - \frac{0.005(100)}{10} = 0.4722$$

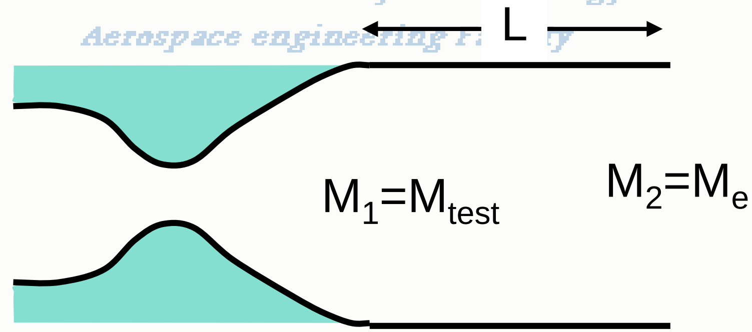
(Appendix E) $$M_e = 2.70$$

another solution is M=0.605, but since started M>1, can't be subsonic

  - **T**

    (T_o const) $$T_2 = T_1 \frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2} = \frac{T_o}{1+\frac{\gamma-1}{2}M_2^2} = 118\text{K}$$

Solution (con't)



– **p**

$$p_2 = p_1 \frac{M_1}{M_2}\sqrt{\frac{T_2}{T_1}} \quad \text{(X.17)}$$

$$p_1 = p_{o1}\left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{-\gamma/\gamma-1}$$

$$= \frac{500\text{kPa}}{2.8^{3.5}} = 13.6\text{kPa}$$

$$\frac{T_2}{T_1} = \frac{1 + ((\gamma-1)/2)M_1^2}{1 + ((\gamma-1)/2)M_2^2} = 1.14$$

$$p_2 = 13.6\text{kPa}\,\frac{3.0}{2.7}\sqrt{1.14} = 16.1\text{kPa}$$

– $\mathbf{p_{o,e}/p_{o,test}}$

$$\text{(X.18)} \quad \frac{p_{o2}}{p_{o1}} = \frac{M_1}{M_2}\left(\frac{T_2}{T_1}\right)^{\frac{\gamma+1}{2(1-\gamma)}} = \frac{3.0}{2.7}(1.14)^{-3} = 0.75$$

25% loss in stagnation pressure due to friction

Solution (con't)

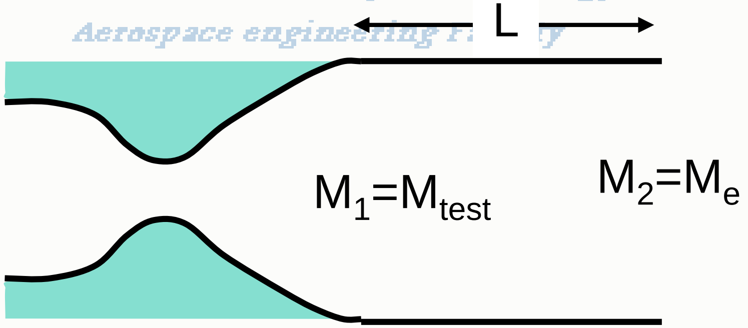
– L_{max}

$$L_{max} = \left.\frac{fL_{max}}{D}\right)_{M_{test}} \frac{D}{f}$$

$$= 0.5222 \frac{0.1\text{m}}{0.005}$$

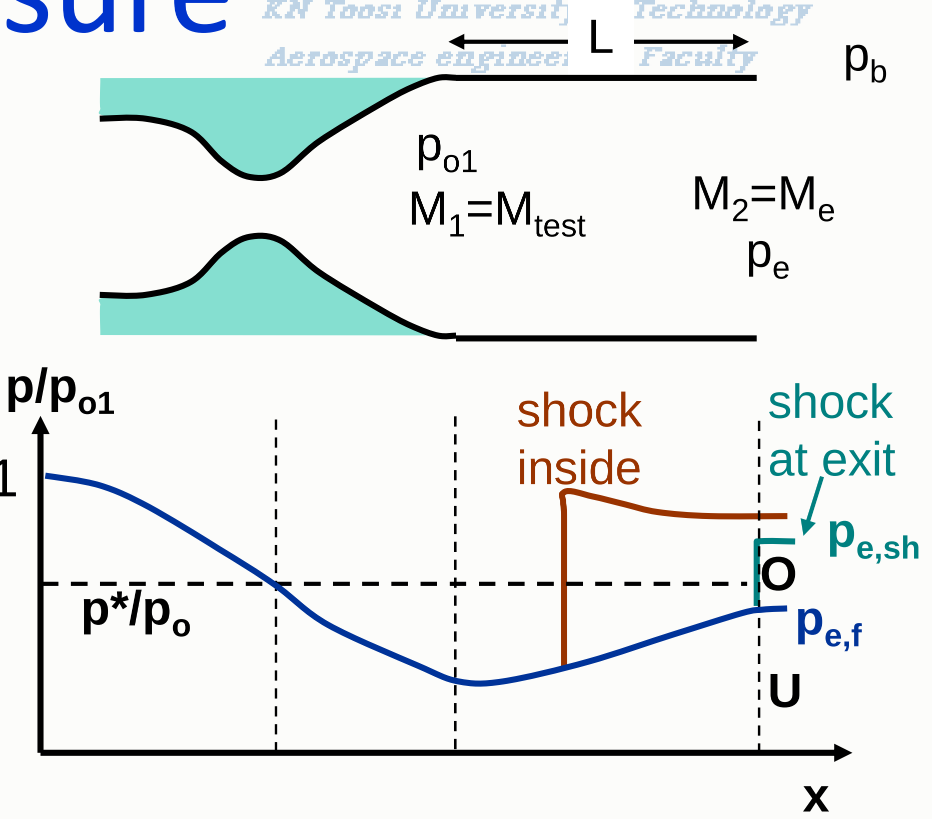
$$= \boxed{10.4\text{m}}$$

10 m long section would have M=1 at exit



$L<L_{max}$, Back Pressure

- Last problem (supersonic duct), what would happen if calculated exit pressure ($p_{e,f}$) did not match actual back pressure (p_b)

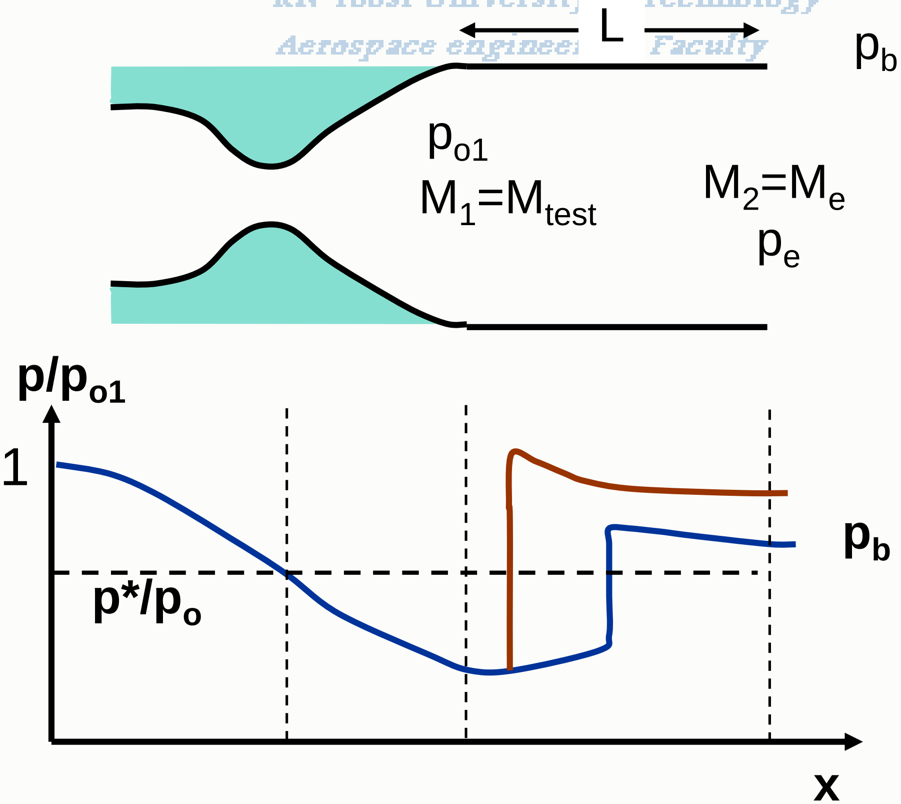


- $p_b<p_{e,f}$: expansion outside duct (underexpanded)
- $p_{e,f} < p_b<p_{e,sh}$: oblique shocks outside duct (overexpanded)
- $p_{e,sh}<p_b$: shocks inside duct (until shock reaches ~throat)

$L>L_{max}$, Back Pressure

- Can't have flow transition to subsonic with pure Fanno flow
  ⇒shock in duct

- Shock location determined by back pressure
  - raise p_b
  - shock moves upstream until shock reaches M=1 location in nozzle

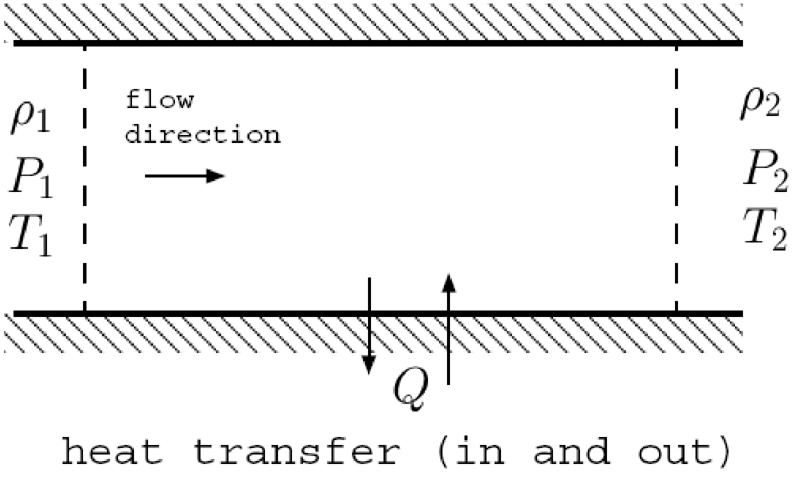


Rayleigh Flow

Rayleigh flow is model describing a frictionless flow with heat transfer through a pipe of constant cross sectional area.

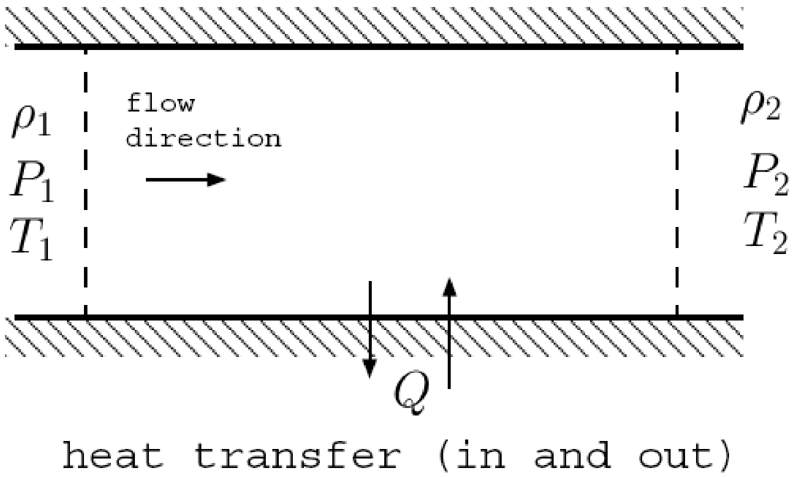
In practice Rayleigh flow isn't a really good model for the real situation. Yet, Rayleigh flow is practical and useful concept in a obtaining trends and limits such as the density and pressure change due to external cooling or heating.

the heat transfer can be in two directions not like the friction (there is no negative friction).



Continuity

$$\dot{m} = \rho A V = \text{ const}$$



$$\rho V = \text{ const}$$

$$\boxed{\rho V = G = \text{ const}}$$

Energy

$$h_{t1} + q = h_{t2} + w_s$$

$$\boxed{h_{t1} + q = h_{t2}}$$

This is the first major flow category for which the total enthalpy has *not* been constant.

Momentum

$$p_1 A - p_2 A = \rho A V \left(V_2 - V_1\right)$$

$$p_1 - p_2 = \rho V \left(V_2 - V_1\right) = G \left(V_2 - V_1\right)$$

$$p_1 + G V_1 = p_2 + G V_2$$

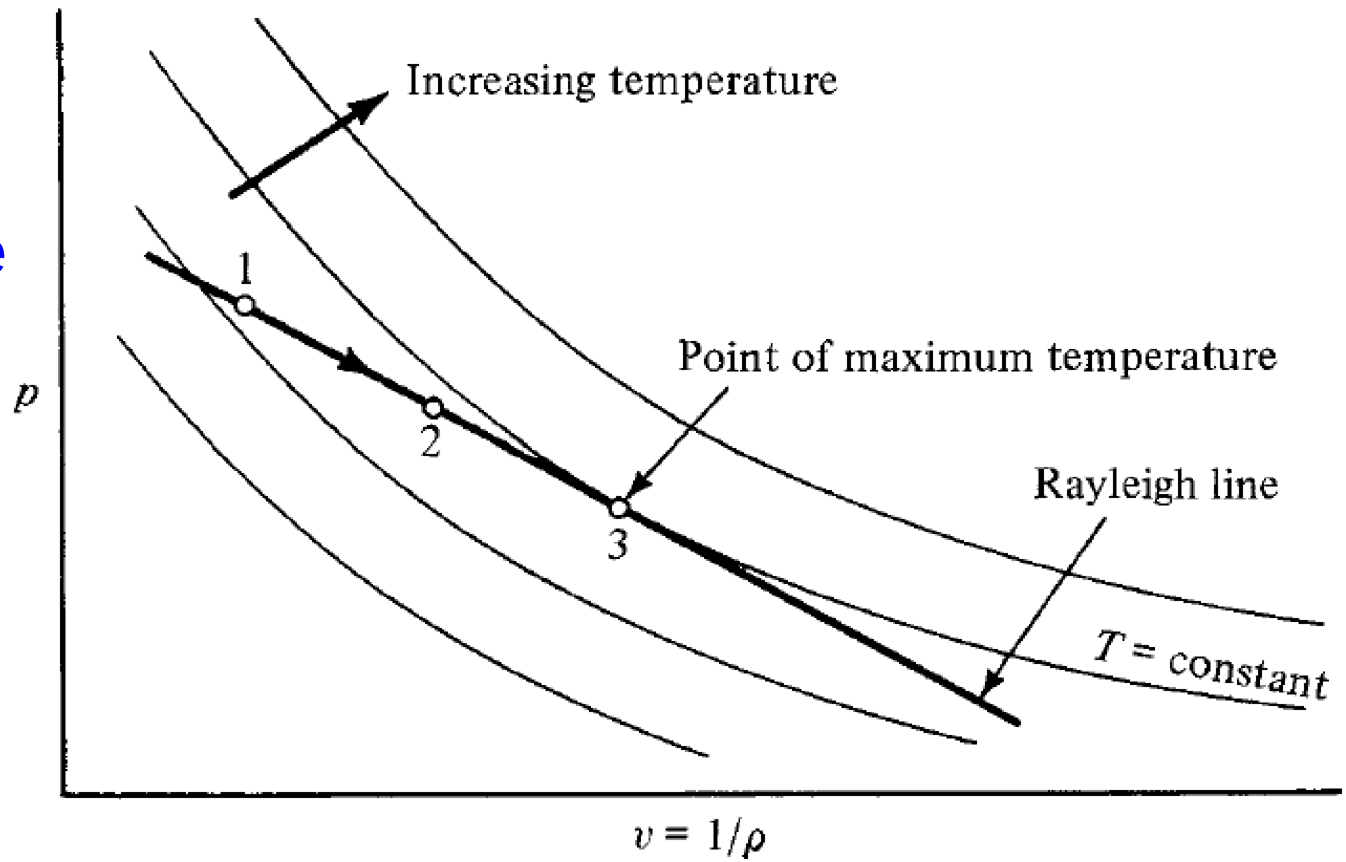
$$\boxed{p + \rho V^2 = \text{const}}$$

$$p + \frac{G^2}{\rho} = \text{const}$$

$$p + \frac{G^2}{\rho} = \text{const}$$

lines of constant temperature: $pv = \text{const}$

point 3 is reached where the temperature is a maximum. Is this a limiting point of some sort?



To answer these questions, we must turn elsewhere

$$ds_e = \frac{\delta q}{T}$$

For a T = constant line, $pv = RT = \text{ const}$

$$p\,dv + v\,dp = 0$$

$$\boxed{\frac{dp}{dv} = -\frac{p}{v}}$$

For an S = constant line, $pv^{\gamma} = \text{ const}$

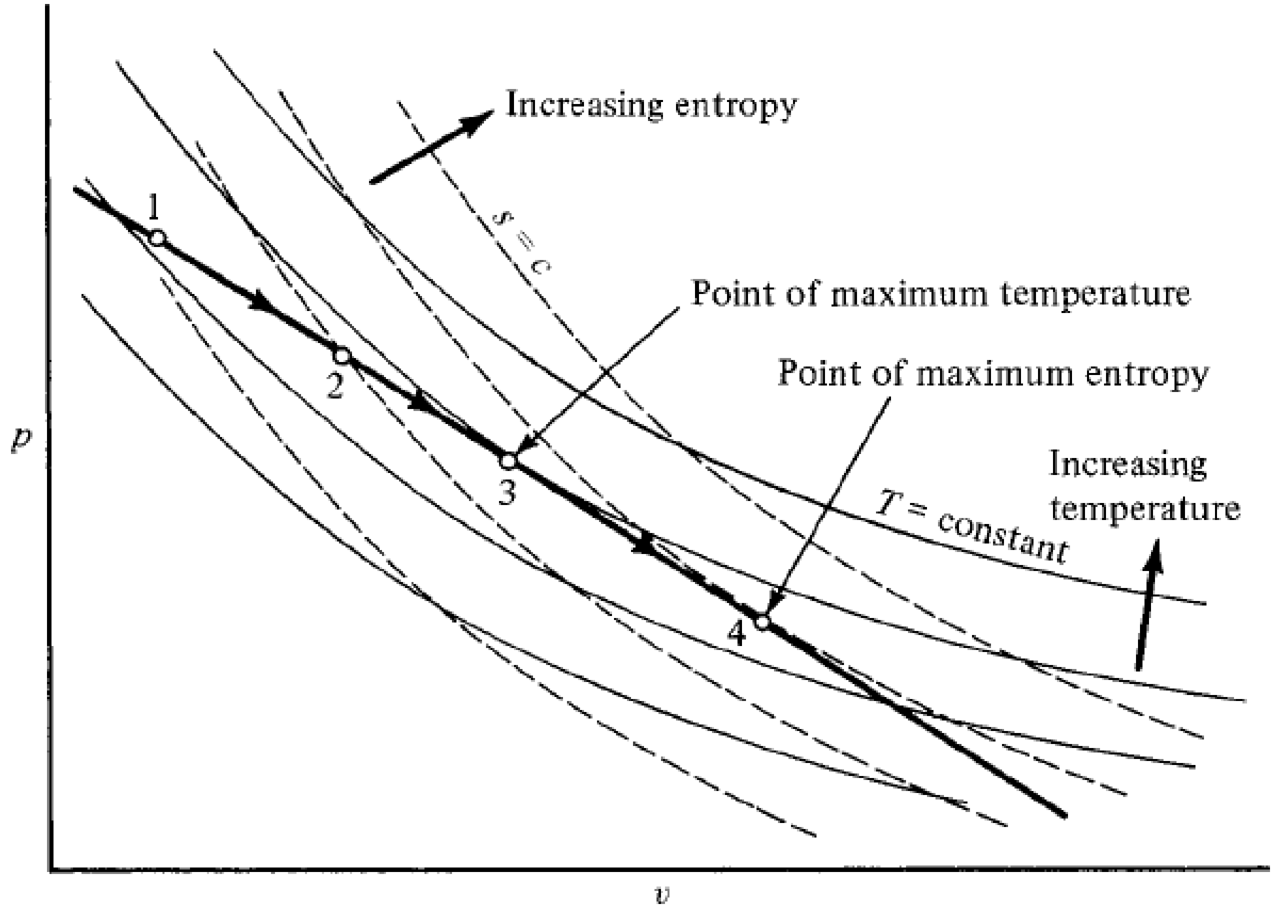
$$v^{\gamma}\,dp + p\gamma\,v^{\gamma-1}\,dv = 0$$

$$\boxed{\frac{dp}{dv} = -\gamma\,\frac{p}{v}}$$

We now see that not only can we reach the point of maximum temperature, but more heat can be added to take us beyond this point.

From point 3 to 4, we add heat to the system and its temperature decreases.

the effects of heat addition are normally thought of as causing the fluid density to decrease. This requires the velocity to increase since *ρV = constant by continuity.*



This velocity increase automatically

Some of the heat that is added to the system is converted into this increase in kinetic energy of the fluid,

Noting that kinetic energy is proportional to the square of velocity, we realize that as higher velocities are reached, the addition of more heat is accompanied by much greater increases in kinetic energy. Eventually, we reach a point where *all of the heat* energy added is required for the kinetic energy increase.

Let's discuss the h(T)-s diagram

$$ds = c_p \frac{dT}{T} - R\frac{dP}{P}$$

$$s - s^* = c_p \ln \frac{T}{T^*} - R \ln \frac{P}{P^*}$$

$$\rho = \frac{p}{RT}$$

$$V^2 = M^2 a^2 = M^2 \gamma RT$$

$$p + \rho V^2 = \text{const}$$

$$p(1 + \gamma M^2) = \text{const}$$

$$\boxed{\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}}$$

$$M^2 = \frac{1 + \gamma}{\gamma} \frac{P^*}{P} - \frac{1}{\gamma}$$

Equation of state

$$\frac{T_2}{T_1} = \frac{P_2}{P_1} \frac{\rho_1}{\rho_2}$$

$$\rho = \frac{\dot{m}}{AV}$$

$$V = M\sqrt{\gamma RT}$$

$$\frac{\sqrt{T}}{PM} = \text{Const.}$$

$$p(1 + \gamma M^2) = \text{const}$$

$$\frac{T}{\dfrac{M^2}{(1 + \gamma M^2)^2}} = \text{Const.}$$

$$\frac{T_1(1 + \gamma M_1^2)^2}{M_1^2} = \frac{T_2(1 + \gamma M_2^2)^2}{M_2^2}$$

$$\frac{T}{T^*} = \frac{(1 + \gamma)^2 M^2}{(1 + \gamma M^2)^2}$$

$$\frac{P}{P^*} = \frac{1+\gamma}{1+\gamma M^2}$$

$$\frac{T}{T^*} = \frac{(1+\gamma)^2 M^2}{(1+\gamma M^2)^2}$$

$$\frac{P}{P^*} = \frac{1}{M}\sqrt{\frac{T}{T^*}}$$

$$M^2 = \frac{1+\gamma}{\gamma}\frac{P^*}{P} - \frac{1}{\gamma}$$

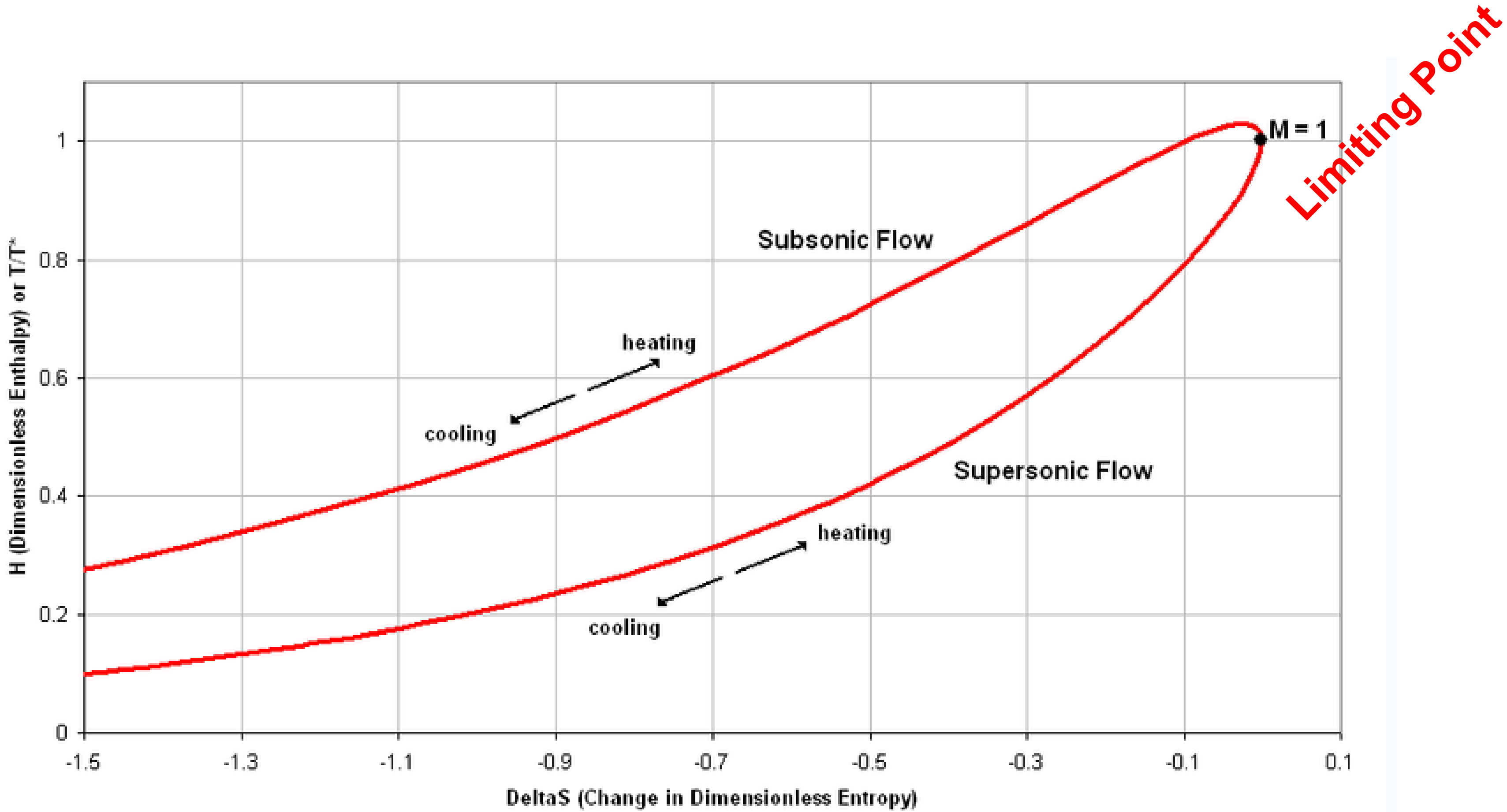
$$\frac{P}{P^*}\sqrt{\frac{(1+\gamma)P^*/P - 1}{\gamma}} = \sqrt{\frac{T}{T^*}}$$

$$\frac{\gamma T}{T^*} = \left(\frac{P}{P^*}\right)^2\left[(1+\gamma)\frac{P^*}{P} - 1\right]$$

$$\frac{P}{P^*} = \frac{1+\gamma}{2} \pm \frac{\sqrt{(1+\gamma)^2 - 4\gamma(T/T^*)}}{2}$$

$$s - s^* = c_p \ln \frac{T}{T^*} - R \ln \frac{P}{P^*}$$

$$\frac{s - s^*}{c_p} = \ln \frac{T}{T^*} - \frac{\gamma - 1}{\gamma} \ln \left[\frac{(\gamma + 1) \pm \sqrt{(\gamma + 1)^2 - 4\gamma(T/T^*)}}{2} \right]$$



For heat addition, the entropy must increase and the flow moves to the right.

Limiting Point

$$p + \frac{G^2}{\rho} = \text{const}$$

$$dp + G^2 \left(-\frac{d\rho}{\rho^2} \right) = 0$$

$$\frac{dp}{d\rho} = \frac{G^2}{\rho^2} = V^2$$

$$V^2 = \frac{dp}{d\rho}$$

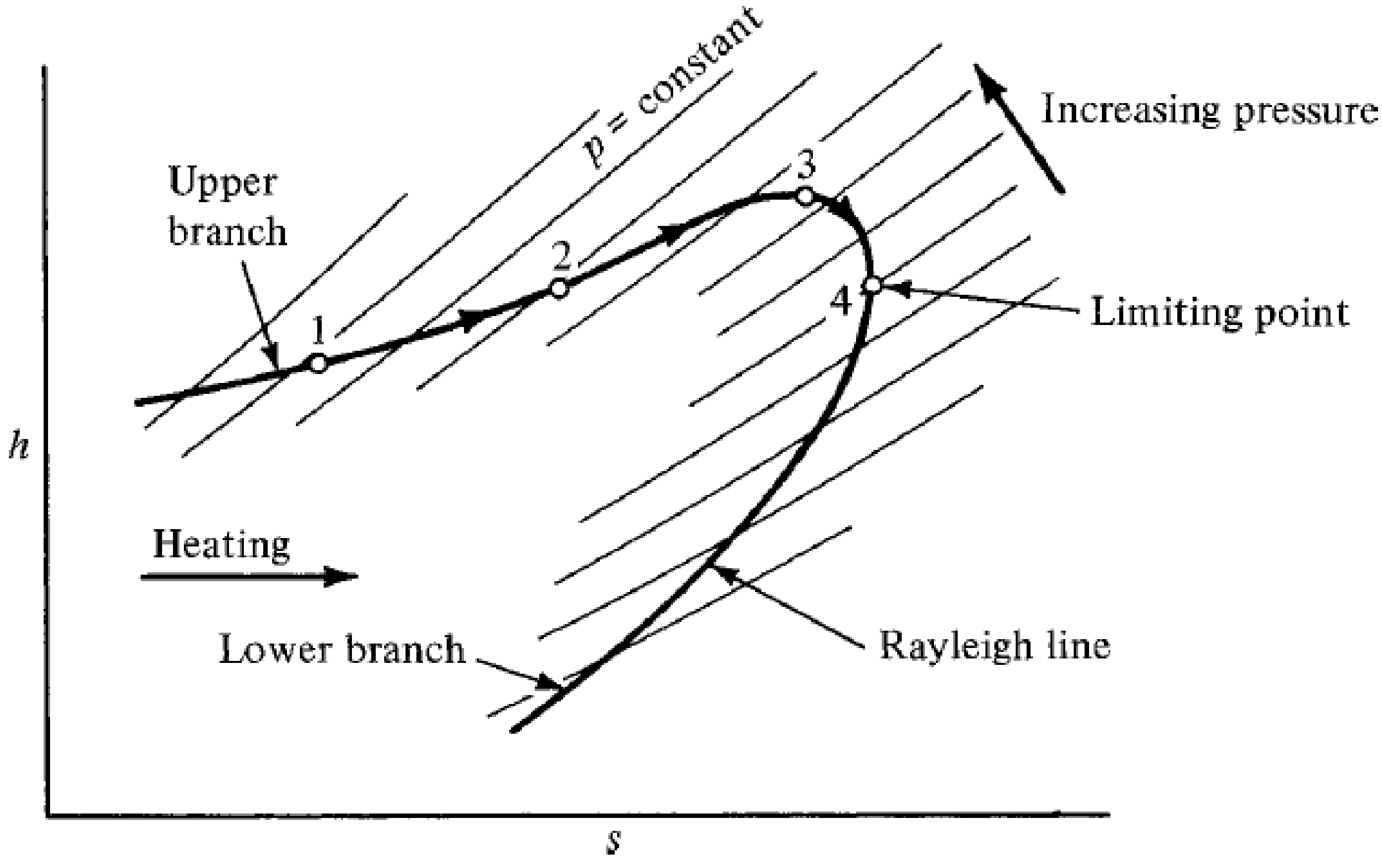
Is valid *anyplace along the Rayleigh line. Now for a differential movement at* the limit point of maximum entropy, *ds = 0 or s = const.*

$$V^2 = \left(\frac{\partial p}{\partial \rho} \right)_{s=c}$$

(at the limit point)

This is immediately recognized as sonic velocity.

We have been discussing a *familiar heating process along the upper branch.* *What* about the lower branch?



Another interesting fact can be shown to be true at the limit point.

$$dp = V^2 \, d\rho$$

$$T\,ds = dh - \frac{dp}{\rho}$$

$$M = 1, ds = 0$$

$$0 = dh + V\,dV$$

$$dh_t = 0$$

(at the limit point)

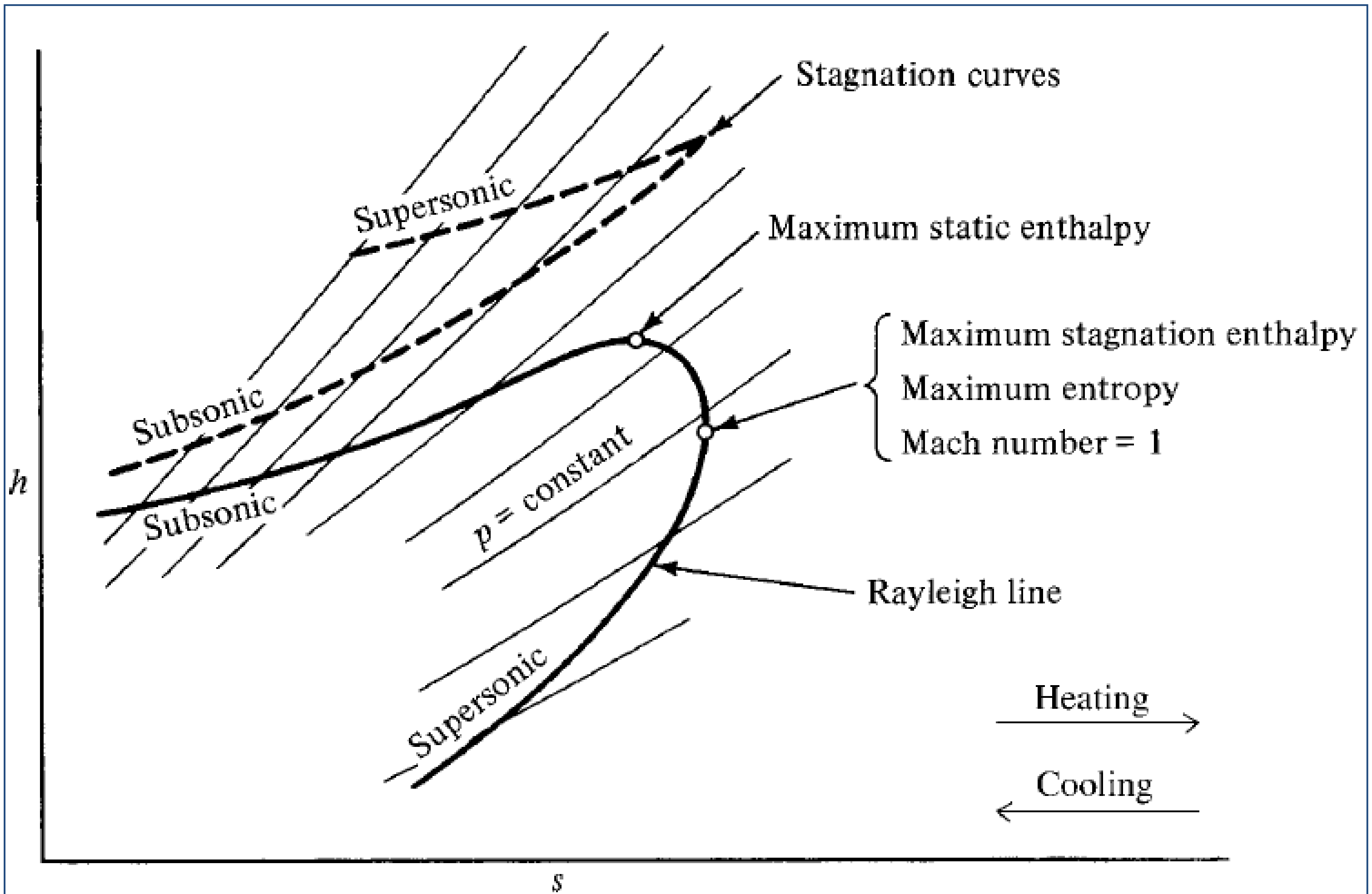
$$h_t = h + \frac{V^2}{2}$$

$$dh_t = dh + V\,dV$$

The stagnation enthalpy increases as long as heat can be added. At the point of maximum entropy, no more heat can be added and thus *ht must be a maximum at this location.*

$$\delta q = \cancel{\delta w_s} + dh_t \qquad \delta q = dh_t$$

$$dh_t = T\,ds_e = T\,ds \qquad \boxed{\frac{dh_t}{ds} = T}$$



Stagnation Conditions

$$T_t = T\left(1 + \frac{\gamma - 1}{2}M^2\right) \Rightarrow \frac{T_{t2}}{T_{t1}} = \frac{T_2}{T_1}\left(\frac{1 + [(\gamma - 1)/2]M_2^{\ 2}}{1 + [(\gamma - 1)/2]M_1^{\ 2}}\right)$$

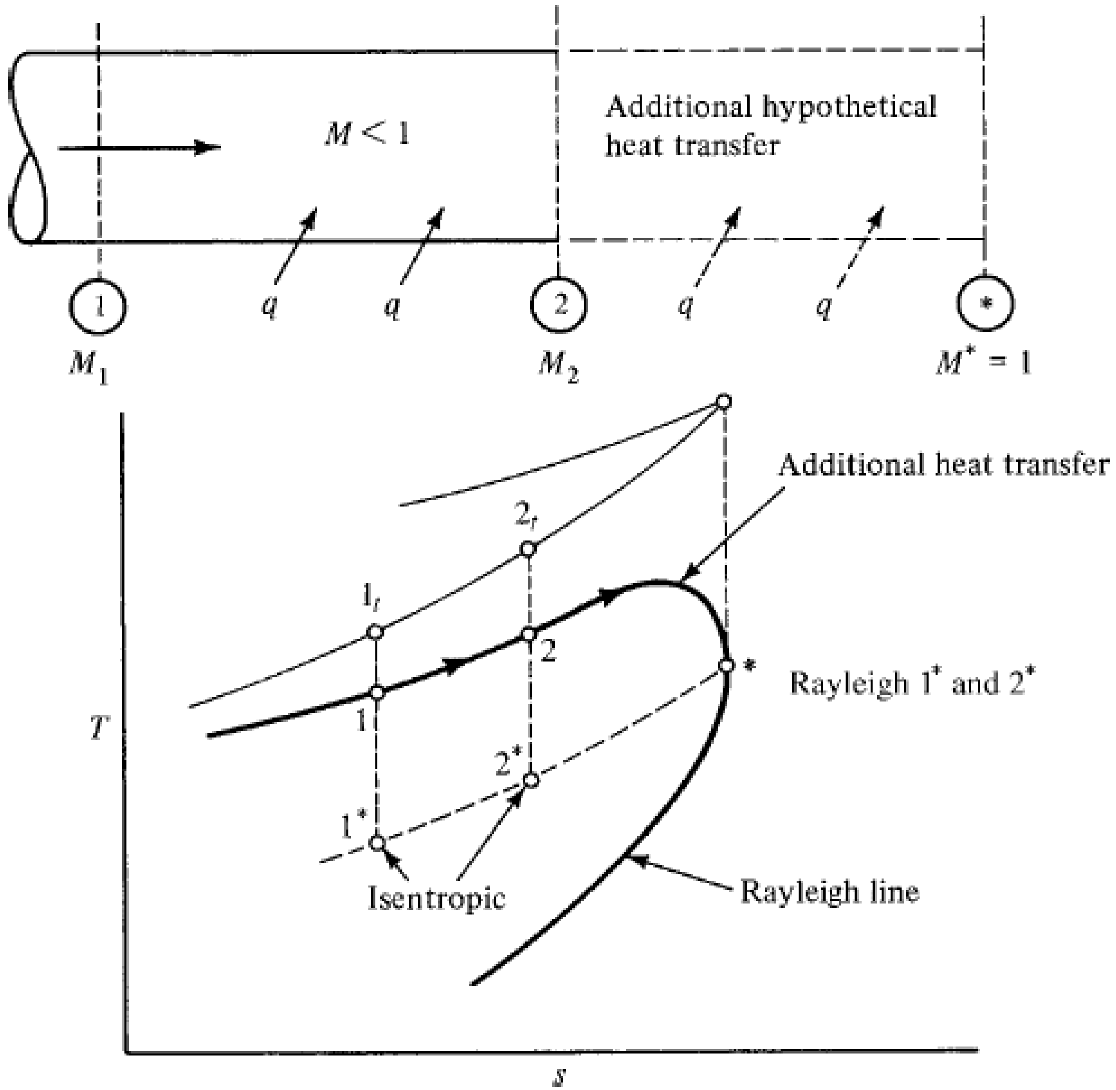
$$\boxed{\frac{T_{t2}}{T_{t1}} = \left(\frac{1 + \gamma M_1^{\ 2}}{1 + \gamma M_2^{\ 2}}\right)^2 \frac{M_2^{\ 2}}{M_1^{\ 2}}\left(\frac{1 + [(\gamma - 1)/2]M_2^{\ 2}}{1 + [(\gamma - 1)/2]M_1^{\ 2}}\right)}$$

$$p_t = p\left(1 + \frac{\gamma - 1}{2}M^2\right)^{\gamma/(\gamma-1)} \Rightarrow \frac{p_{t2}}{p_{t1}} = \frac{p_2}{p_1}\left(\frac{1 + [(\gamma - 1)/2]M_2^{\ 2}}{1 + [(\gamma - 1)/2]M_1^{\ 2}}\right)^{\gamma/(\gamma-1)}$$

$$\boxed{\frac{p_{t2}}{p_{t1}} = \frac{1 + \gamma M_1^{\ 2}}{1 + \gamma M_2^{\ 2}}\left(\frac{1 + [(\gamma - 1)/2]M_2^{\ 2}}{1 + [(\gamma - 1)/2]M_1^{\ 2}}\right)^{\gamma/(\gamma-1)}}$$

REFERENCE STATE ANDTHE RAYLEIGH TABLE

We introduce still another $*$ reference state defined as before, in that the Mach number of unity must be reached by some particular process.



$$\frac{p_2}{p_1} = \frac{1 + \gamma {M_1}^2}{1 + \gamma {M_2}^2}$$

$p_2 \Rightarrow p \qquad M_2 \Rightarrow M$ (any value)

$p_1 \Rightarrow p^* \qquad M_1 \Rightarrow 1$

$$\frac{p}{p^*} = \frac{1 + \gamma}{1 + \gamma M^2} = f(M, \gamma)$$

$$\frac{T}{T^*} = \frac{M^2(1 + \gamma)^2}{(1 + \gamma M^2)^2} = f(M, \gamma)$$

$$\frac{\rho}{\rho^*} = \frac{1 + \gamma M^2}{(1 + \gamma) M^2} = f(M, \gamma)$$

$$\frac{T_t}{T_t^*} = \frac{2(1+\gamma)M^2}{(1+\gamma M^2)^2}\left(1+\frac{\gamma-1}{2}M^2\right) = f(M,\gamma)$$

$$\frac{p_t}{p_t^*} = \frac{1+\gamma}{1+\gamma M^2}\left(\frac{1+[(\gamma-1)/2]M^2}{(\gamma+1)/2}\right)^{\gamma/(\gamma-1)} = f(M,\gamma)$$

Values for the functions represented in equations are listed in the Rayleigh table

APPLICATIONS

The procedure for solving Rayleigh flow problems is quite similar to the approach used for Fanno flow except that the tie between the two locations in Rayleigh flow is determined by heat transfer considerations rather than by duct friction. The recommended steps are, therefore, as follows

1. **Sketch the physical situation (including the hypothetical $*$ reference point).**

1. **2. Label sections where conditions are known or desired.**

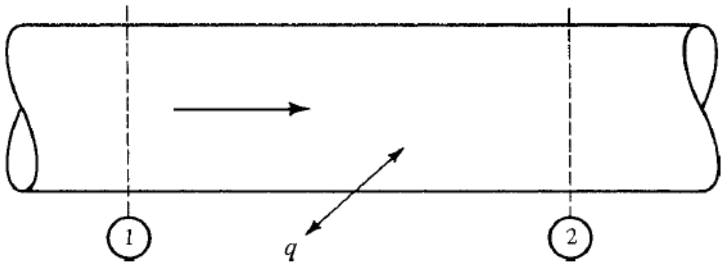
3. List all given information with units.

4. Determine the unknown Mach number.

5. Calculate the additional properties desired.

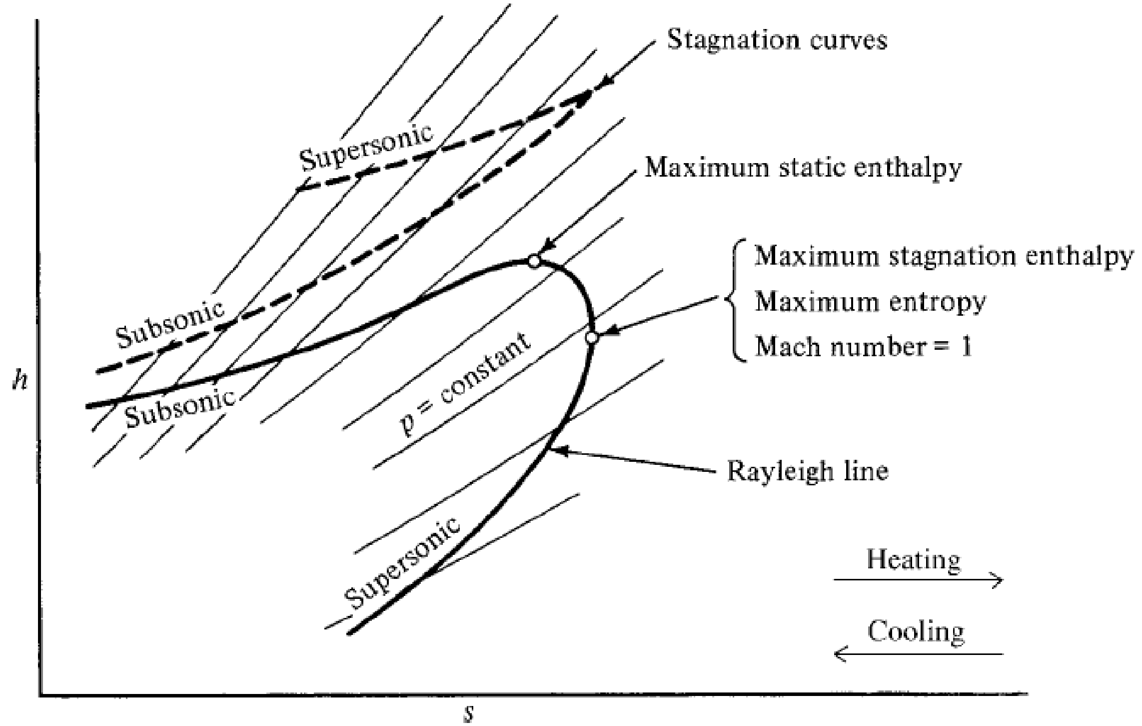
Example

For Figure, given *M1 = 1.5, p1 = 10 psia, and M2 = 3.0, find p2 and* the direction of heat transfer.



$$p_2 = \frac{p_2}{p^*}\frac{p^*}{p_1}p_1 = (0.1765)\left(\frac{1}{0.5783}\right)(10) = 3.05 \text{ psia}$$

The flow is getting more supersonic, or moving away from the $*$ reference point.



Example

Given *M2 = 0.93, Tt2 = 300°C, and Tt1 = 100°C, find M1 and p2/p1.*
To determine conditions at section 1 we must establish the ratio

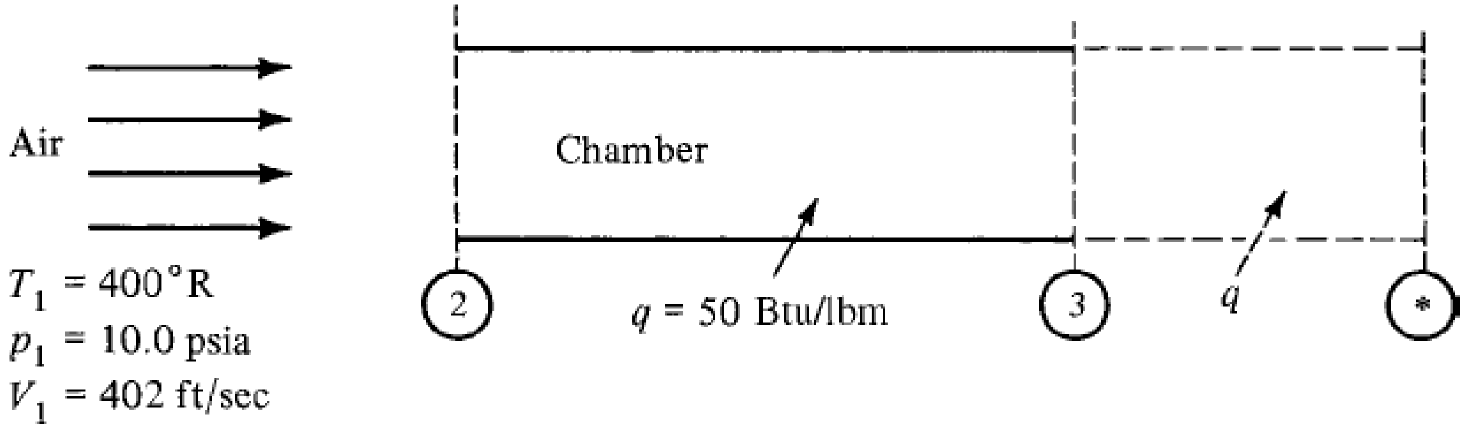
$$\frac{T_{t1}}{T_t{}^*} = \frac{T_{t1}}{T_{t2}}\frac{T_{t2}}{T_t{}^*} = \left(\frac{273+100}{273+300}\right)(0.9963) = 0.6486$$

$T_t / T_t{}^* = 0.6486$ ⇨ $M_1 = 0.472$

⇨ $$\frac{p_2}{p_1} = \frac{p_2}{p^*}\frac{p^*}{p_1} = (1.0856)\left(\frac{1}{1.8294}\right) = 0.593$$

Example

A constant-area combustion chamber is supplied air at 400°R and 10.0 psia (Below figure). The air stream has a velocity of 402 ft/sec. Determine the exit conditions if 50 Btu/lbm is added in the combustion process and the chamber handles the maximum amount of air possible.



$$T_2 = T_1 = 400°\text{R} \qquad p_2 = p_1 = 10.0 \text{ psia} \qquad V_2 = V_1 = 402 \text{ ft/sec}$$

$$a_2 = \sqrt{\gamma g_c R T_2} = [(1.4)(32.2)(53.3)(400)]^{1/2} = 980 \text{ ft/sec}$$

$$M_2 = \frac{V_2}{a_2} = \frac{402}{980} = 0.410$$

$$T_{t2} = \frac{T_{t2}}{T_2} T_2 = \left(\frac{1}{0.9675} \right) (400) = 413°\text{R}$$

$$M_2 = 0.41 \quad \Rightarrow \quad \frac{T_{t2}}{T_t{}^*} = 0.5465 \qquad \frac{T_2}{T^*} = 0.6345 \qquad \frac{p_2}{p^*} = 1.9428$$

$$\Delta T_t = \frac{q}{c_p} = \frac{50}{0.24} = 208°\text{R} \quad \Rightarrow \quad T_{t3} = T_{t2} + \Delta T_t = 413 + 208 = 621°\text{R}$$

$$\Rightarrow \quad \frac{T_{t3}}{T_t{}^*} = \frac{T_{t3}}{T_{t2}} \frac{T_{t2}}{T_t{}^*} = \left(\frac{621}{413} \right) (0.5465) = 0.8217$$

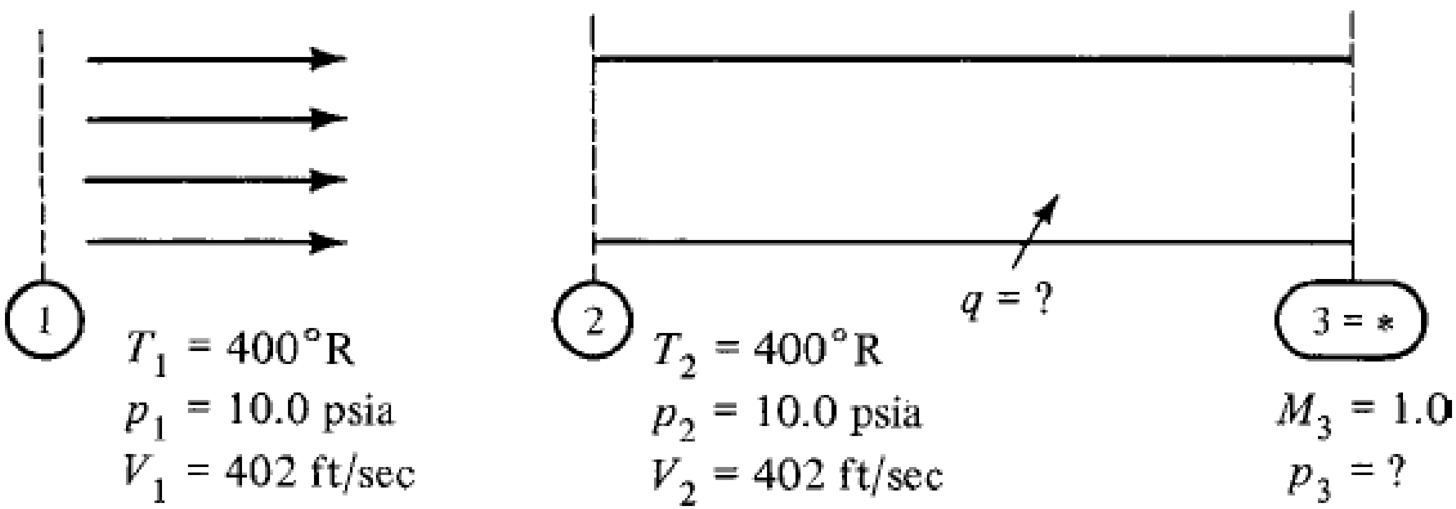
Rayleigh table

$$\Rightarrow \quad M_3 = 0.603 \qquad \frac{T_3}{T^*} = 0.9196 \qquad \frac{p_3}{p^*} = 1.5904$$

$$p_3 = \frac{p_3}{p^*}\frac{p^*}{p_2}p_2 = (1.5904)\left(\frac{1}{1.9428}\right)(10.0) = 8.19 \text{ psia}$$

$$T_3 = \frac{T_3}{T^*}\frac{T^*}{T_2}T_2 = (0.9196)\left(\frac{1}{0.6345}\right)(400) = 580°\text{R}$$

How much more heat (fuel) could be added without changing conditions at the entrance to the duct?



$M_2 = 0.41$ and $T_{t2} = 413°\text{R}$.

$$T_{t3} = T_t{}^* = \frac{T_t{}^*}{T_{t2}} T_{t2} = \left(\frac{1}{0.5465}\right)(413) = 756°\text{R}$$

$$p_3 = p^* = \frac{p^*}{p_2} p_2 = \left(\frac{1}{1.9428}\right)(10.0) = 5.15 \text{ psia}$$

$$q = c_p\,\Delta T_t = (0.24)(756 - 413) = 82.3 \text{ Btu/lbm}$$

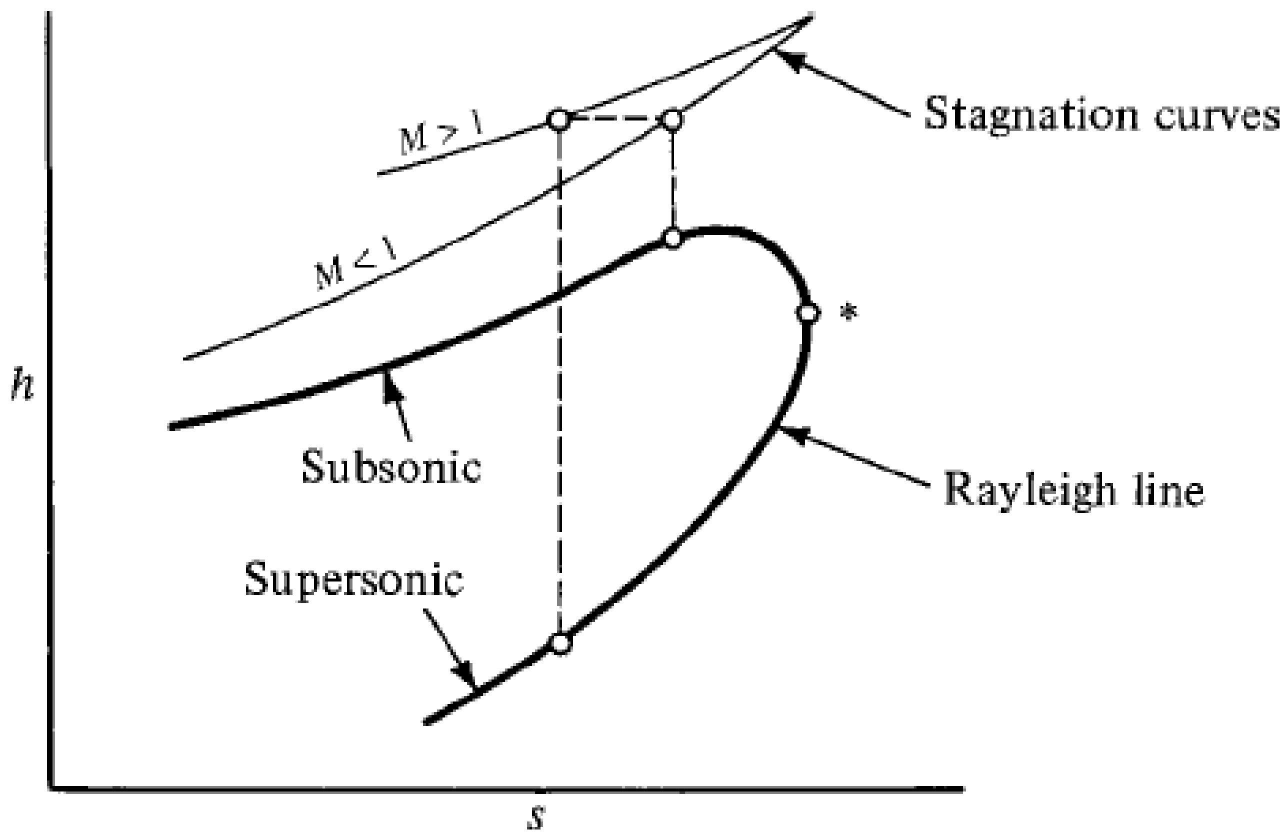
CORRELATION WITH SHOCKS

Some similarities between Rayleigh flow and normal shocks

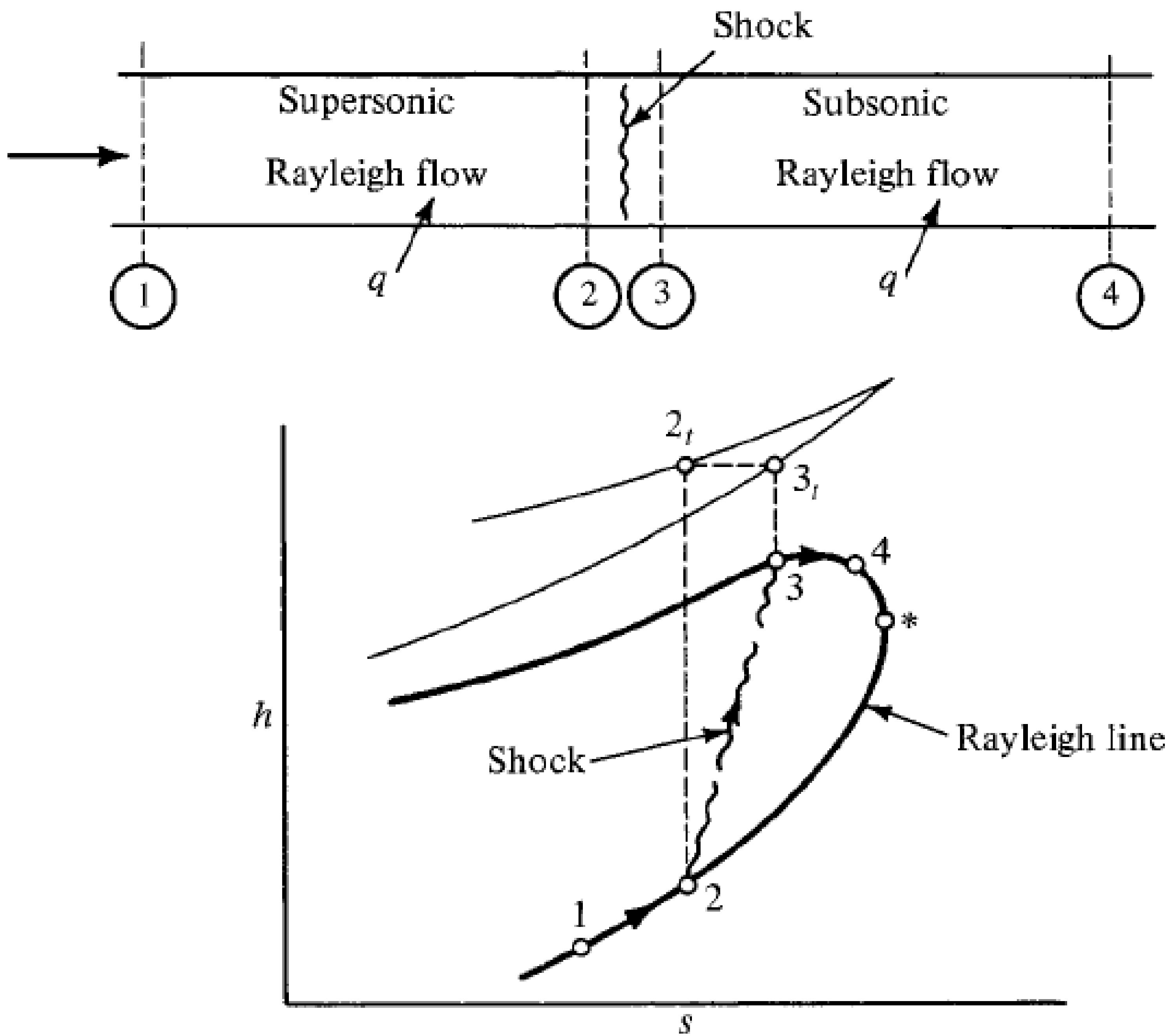
1. The end points before and after a normal shock represent states with the same mass flow per unit area, the same impulse function, and the same stagnation enthalpy.

2. A Rayleigh line represents states with the same mass flow per unit area and the same impulse function. All points on a Rayleigh line do not have the same stagnation enthalpy because of the heat transfer involved. To move along a Rayleigh line requires this heat transfer.

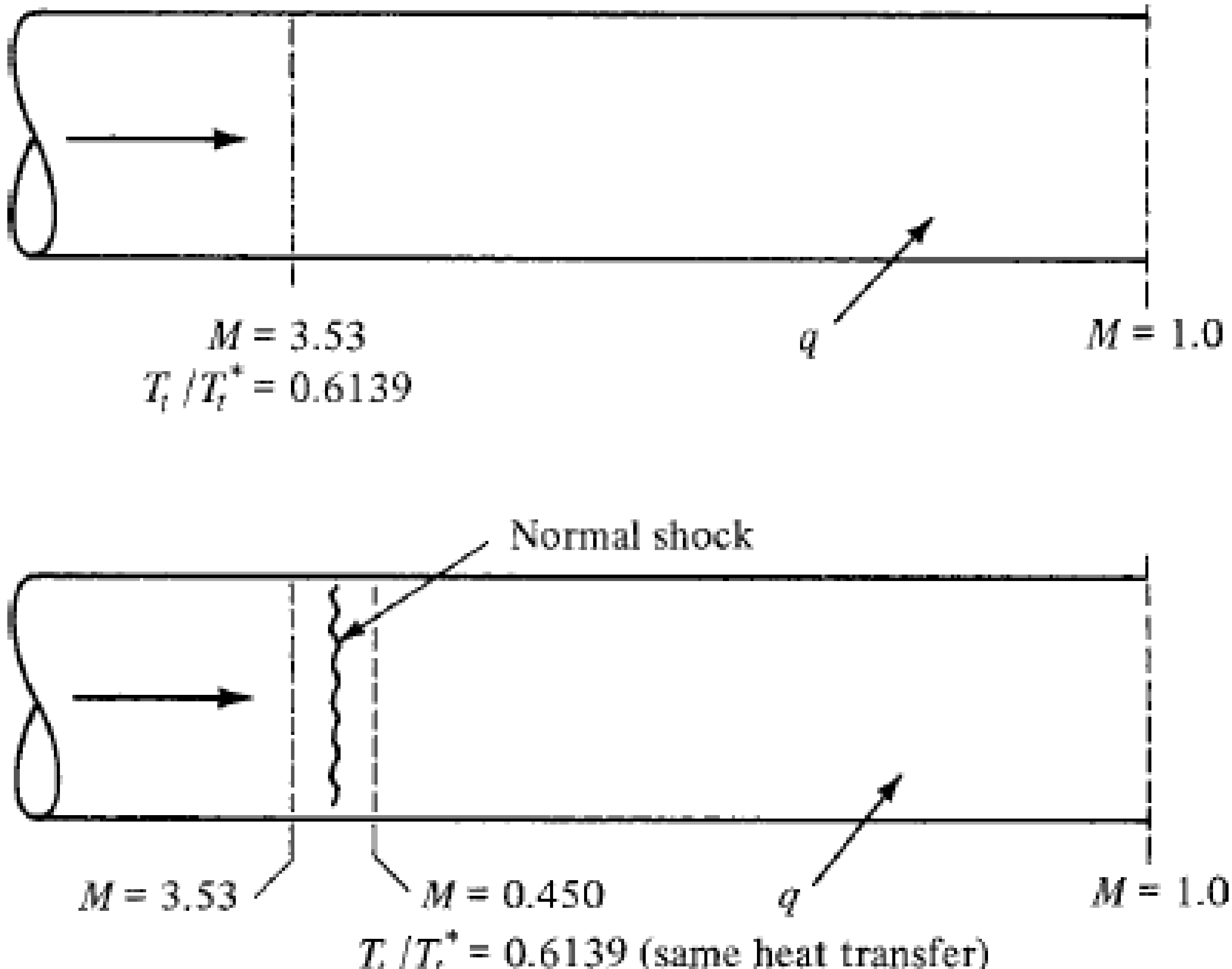
Shock Wave Equations	Rayleigh Flow Equations
$\rho_1 V_1 = \rho_2 V_2$	$\rho V = G = \text{const}$
$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$	$h_{t1} + q = h_{t2}$
$p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2$	$p + \rho V^2 = \text{const}$



For every point on the supersonic branch of the Rayleigh line there is a corresponding point on the subsonic branch with the same stagnation enthalpy. Thus these two points satisfy all three conditions for the end points of a normal shock and could be connected by such a shock.



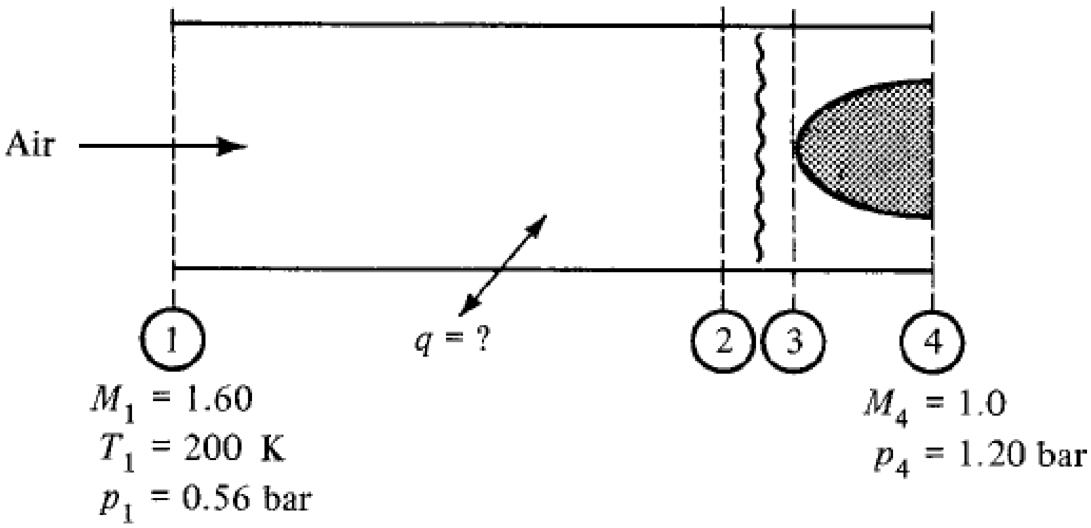
Shock
Supersonic
Rayleigh flow
Subsonic
Rayleigh flow
q
q
1
2
3
4
2_t
3_t
4
3
*
h
Shock
Rayleigh line
1
2
s



$M = 3.53$
$T_t / T_t^* = 0.6139$
q
$M = 1.0$
Normal shock
$M = 3.53$
$M = 0.450$
q
$M = 1.0$
$T_t / T_t^* = 0.6139$ (same heat transfer)

Example

Air enters a constant-area duct with a Mach number of 1.6, a temperature of 200 K, and a pressure of 0.56 bar . After some heat transfer a normal shock occurs, whereupon the area is reduced as shown. At the exit the Mach number is found to be 1.0 and the pressure is 1.20 bar. Compute the amount and direction of heat transfer.



The flow from 3 to 4 is isentropic; thus:

$$p_{t3} = p_{t4} = \frac{p_{t4}}{p_4} p_4 = \left(\frac{1}{0.5283}\right)(1.20) = 2.2714 \text{ bar}$$

From the Rayleigh table we find M3 = 0.481 and from the shock table, M2 = 2.906.

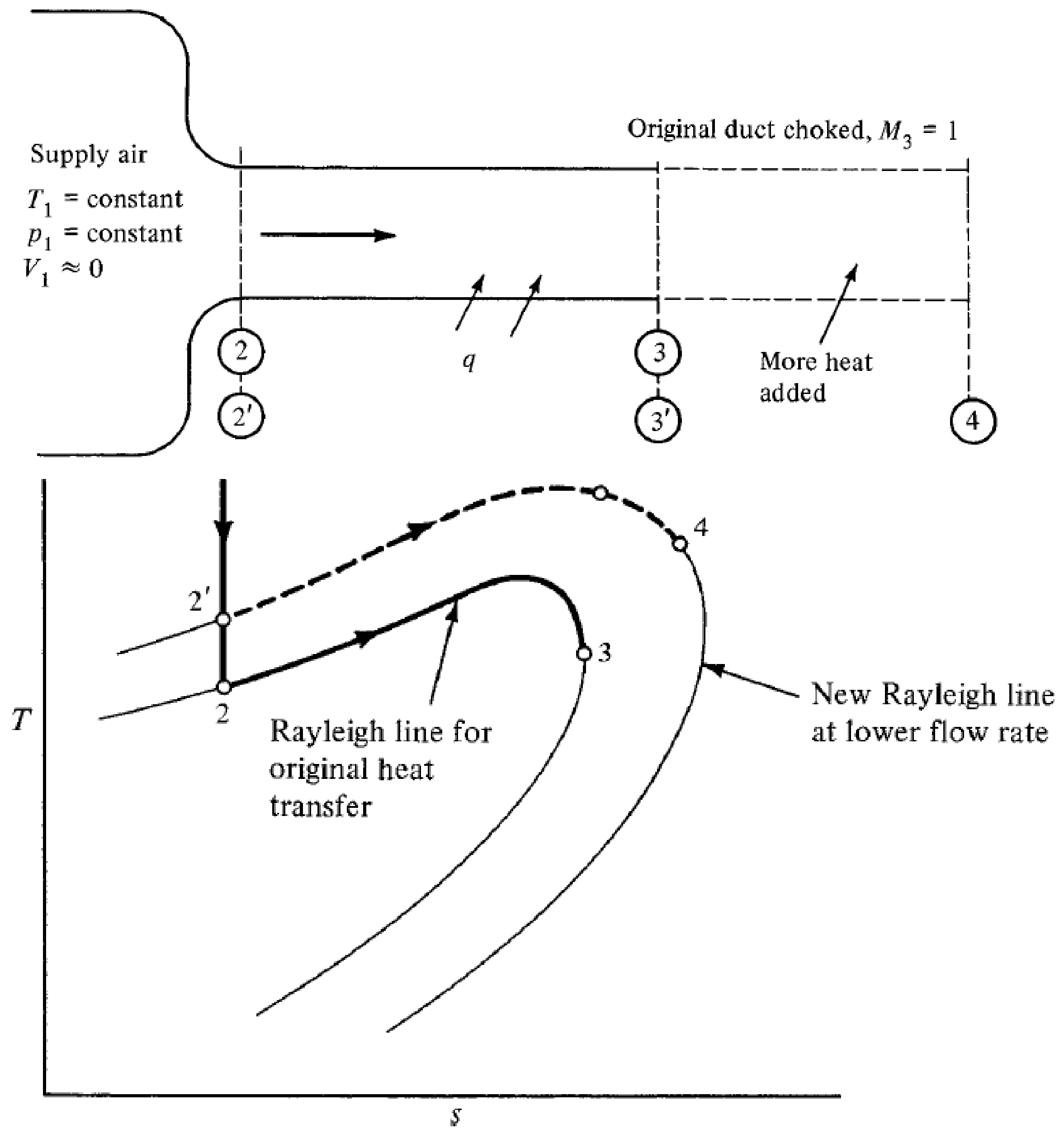
$$T_{t1} = \frac{T_{t1}}{T_1} T_1 = \left(\frac{1}{0.6614}\right)(200) = 302 \text{ K}$$

$$T_{t2} = \frac{T_{t2}}{T_t{}^*} \frac{T_t{}^*}{T_{t1}} T_{t1} = (0.6629)\left(\frac{1}{0.8842}\right)(302) = 226 \text{ K}$$

$$q = c_p(T_{t2} - T_{t1}) = (1000)(226 - 302) = -7.6 \times 10^4 \text{ J/kg}$$

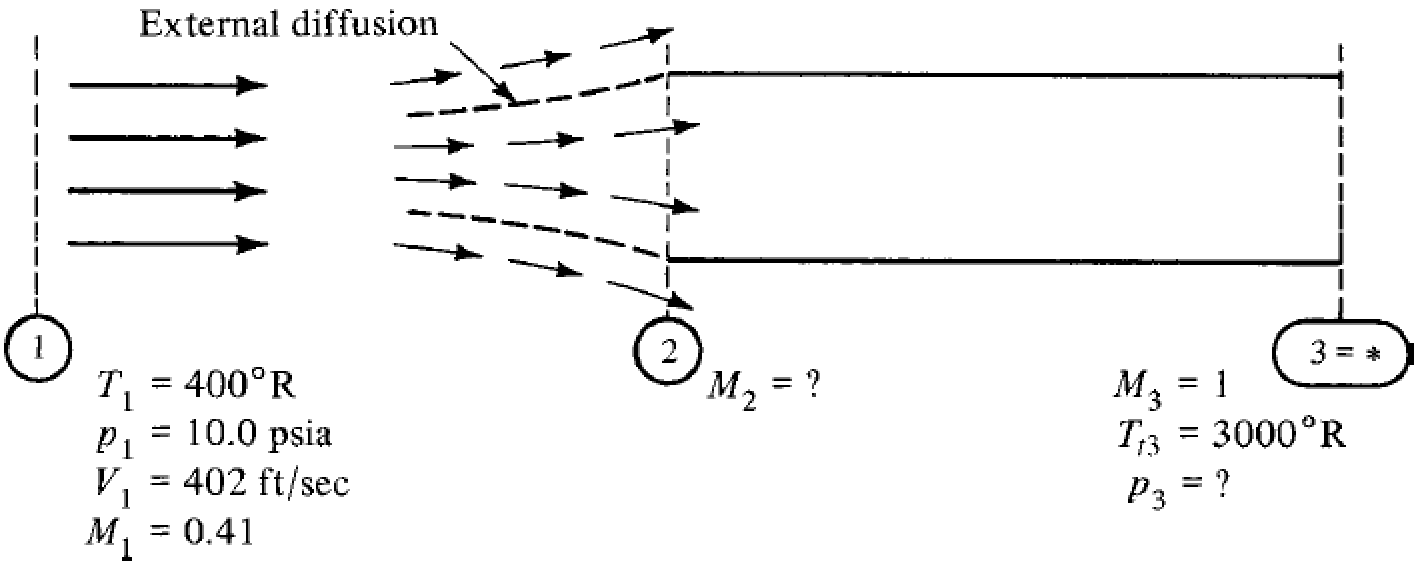
THERMAL CHOKING

Once sufficient heat has been added, we reach Mach 1 at the end of the duct.
The T –s diagram for this is shown as path 1–2–3. This is called thermal choking



Example

let us add sufficient fuel to raise the outlet stagnation temperature to 3000°R. Assume that the receiver pressure is very low so that sonic velocity still exists at the exit. The additional entropy generated by the extra fuel can only be accommodated by moving to a new Rayleigh line at a decreased flow rate which lowers the inlet Mach number. If the chamber is fed by the same air stream some spillage must occur at the entrance. We would like to know the Mach number at the inlet and the pressure at the exit.



Since it is isentropic from the free stream to the inlet:

$$T_{t2} = T_{t1} = 413°\text{R}$$

since $M_3 = 1$, we know that $T_{t3} = T_t{}^*$.

$$\frac{T_{t2}}{T_t{}^*} = \frac{T_{t2}}{T_{t3}}\frac{T_{t3}}{T_t{}^*} = \left(\frac{413}{3000}\right)(1) = 0.1377$$

from the Rayleigh table, $M_2 = 0.176$ and $p_2/p^* = 2.3002$.

$$p_2 = \frac{p_2}{p_{t2}}\frac{p_{t2}}{p_{t1}}\frac{p_{t1}}{p_1}p_1 = (0.9786)(1)\left(\frac{1}{0.8907}\right)(10.0) = 10.99 \text{ psia}$$

$$p_3 = \frac{p_3}{p^*}\frac{p^*}{p_2}p_2 = (1)\left(\frac{1}{2.3002}\right)(10.99) = 4.78 \text{ psia}$$

Suppose that in the previous example we were unable to lower the receiver pressure to 4.78 psia. Assume that as fuel was added to raise the stagnation temperature to 3000°R, the pressure in the receiver was maintained at its previous value of 5.15 psia.

This would lower the flow rate even further as we move to another Rayleigh line with a lower mass velocity, *and t his time the exit velocity would not be quite sonic.*

***Although* both *M2 and M3 are unknown, two pieces of information are given at the exit. Two* simultaneous equations could be written, but it is easier to use tables and a trial and-error solution.**

The important thing to remember is that once a *subsonic flow is thermally choked, the addition of more heat causes the flow rate to decrease. Just how* much it decreases and whether or not the exit remains sonic depends on the pressure that exists after the exit.

Shock Wave Equations

$$\rho_1 V_1 = \rho_2 V_2$$

$$h_1 + \frac{V_1^{\,2}}{2} = h_2 + \frac{V_2^{\,2}}{2}$$

$$p_1 + \rho_1 V_1^{\,2} = p_2 + \rho_2 V_2^{\,2}$$

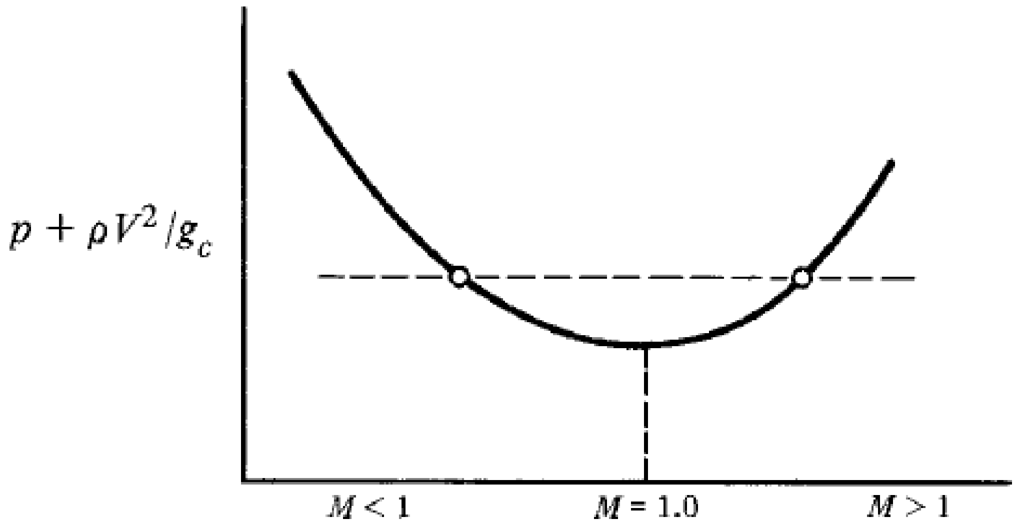
Fanno Equations

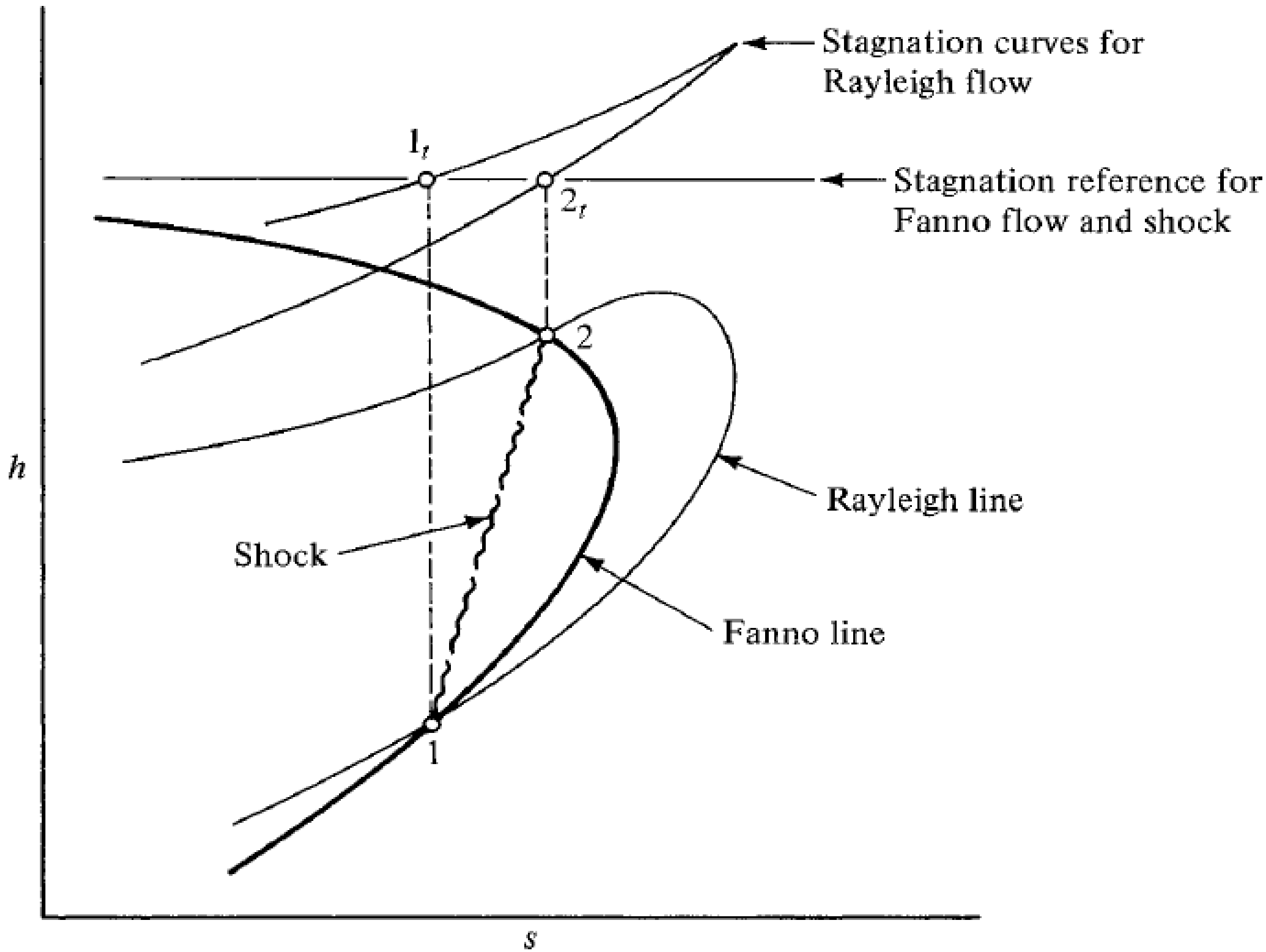
$$\rho_1 V_1 = \rho_2 V_2$$

$$h_1 + \frac{V_1^{\,2}}{2} = h_2 + \frac{V_2^{\,2}}{2}$$

$$\left(p_1 + \rho_1 V_1^{\,2}\right) - \frac{F_f}{A} = p_2 + \rho_2 V_2^{\,2}$$

Variation of $p + \rho V^2/g_c$ in Fanno flow.




Stagnation curves for Rayleigh flow
1_t
2_t
Stagnation reference for Fanno flow and shock
2
h
Rayleigh line
Shock
Fanno line
1
s

FLOW WITH FRICTION

- *Area changes*, *Friction* and *Heat transfer* are the most important factors affecting the properties in a flow system.

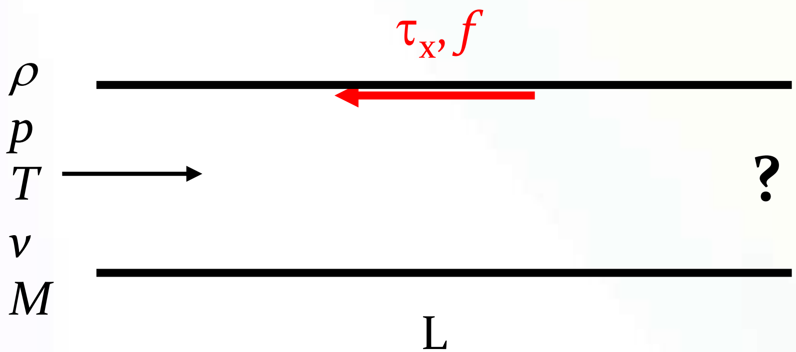
- Up to this point we have considered only one of these factors, that of variations in area. In a real flow situation, however, frictional forces are present and may have a decisive effect on the resultant flow characteristics.

- Friction must be included for flow through long ducts, especially if the cross-sectional area is small.

- Here, we study *compressible flow* with significant wall *friction*, but negligible heat transfer in ducts of constant cross section.

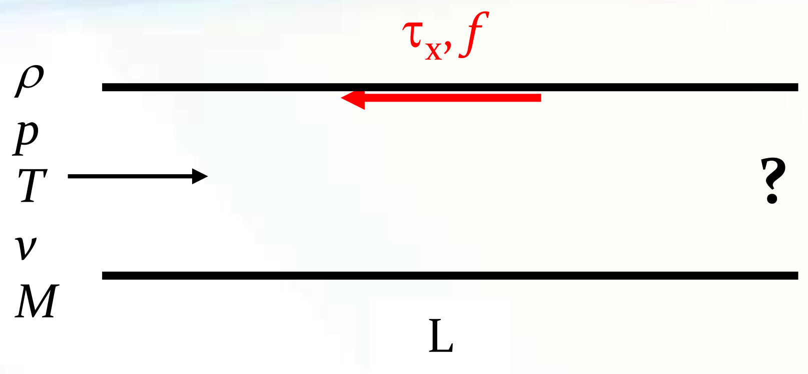
FANNO LINE FLOW

- Consider *one-dimensional*, *steady*, *adiabatic* flow with *no external work* but with *friction* of a perfect gas with constant specific heats through a *constant-area* channel.



FANNO LINE FLOW

FANNO LINE FLOW



- Conserved quantities:
  - ❖ Since adiabatic, no work: $\boldsymbol{h_o}$ **= constant**
  - ❖ Since A = const: mass flux = $\boldsymbol{\rho v}$ **= constant**

$$h + \frac{V^2}{2} = \text{constant} = h_o$$

$$\rho V = \text{constant}$$

$h_o = h + (\rho v)^2/2\rho = \text{constant}$

FANNO LINE FLOW

- On *h-s* diagram, we can draw *Fanno Line* – The line connecting points with same h_o and ρv.

$$T\,ds = dh - \frac{dp}{\rho} = du - \frac{p}{\rho^2}d\rho \Rightarrow ds = \frac{du}{T} - R\frac{d\rho}{\rho}$$

- Assuming constant specific heats, with state 1 a reference state in the flow, above Equation may be integrated to produce

$$s - s_1 = c_v \ln\frac{T}{T_1} - R\ln\frac{\rho}{\rho_1}$$

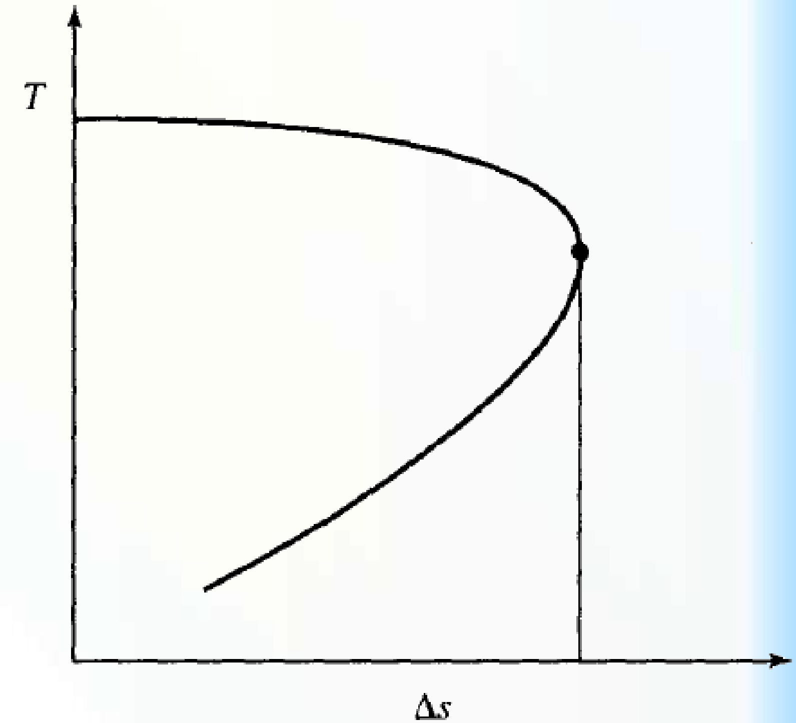
$$\rho V = \text{constant}$$

$$s - s_1 = c_v \ln\frac{T}{T_1} + R\ln\frac{V}{V_1}$$

FANNO LINE FLOW

- From the energy equation, we have

$$V = \sqrt{2(h_o - h)} = \sqrt{2c_p(T_o - T)}$$



So that

$$s - s_1 = c_v \ln\frac{T}{T_1} + R\ln\frac{V}{V_1}$$

$$\frac{s - s_1}{c_v} = \ln\left(\frac{T}{T_1}\right) + \frac{\gamma - 1}{2}\ln\left(\frac{T_o - T}{T_o - T_1}\right)$$

$$= \frac{1}{\gamma}\ln T + \frac{\gamma - 1}{2\gamma}\ln(T_o - T) + \text{constant}$$

FANNO LINE FLOW

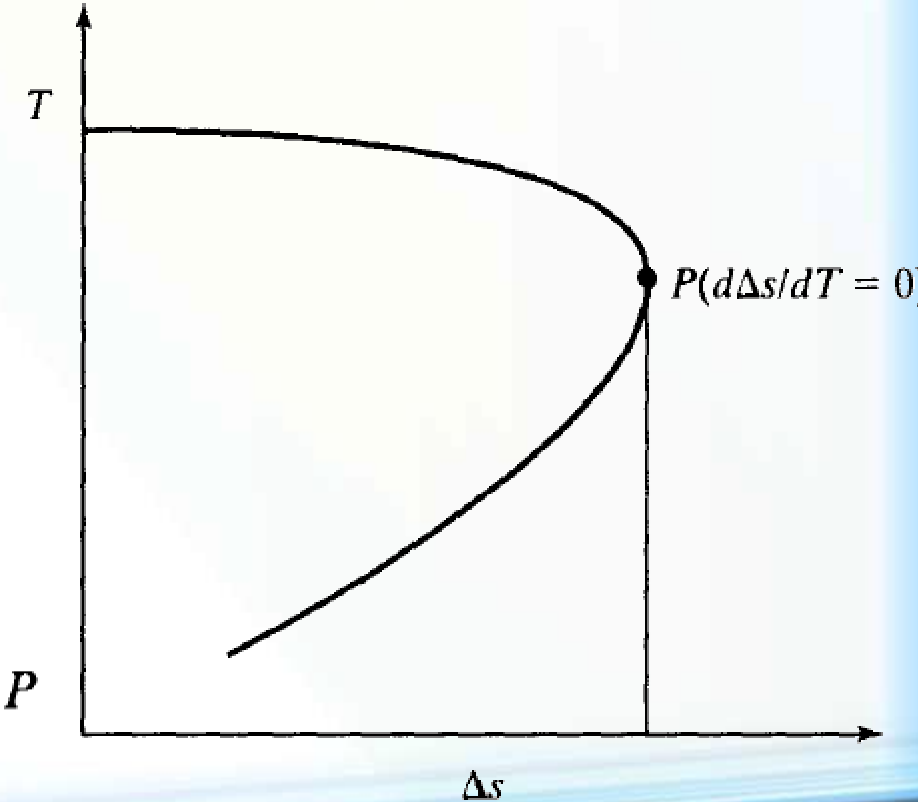
- The Fanno line represents the locus of states that can be obtained under the assumptions of Fanno flow for a fixed mass flow and total enthalpy.

- Consider the point of tangency P, where $d\Delta s/dT = 0$.

$$\frac{s - s_1}{c_v} = \ln\left(\frac{T}{T_1}\right) + \frac{\gamma - 1}{2}\ln\left(\frac{T_o - T}{T_o - T_1}\right)$$

Differentiating

$$\frac{1}{c_v}\frac{d\Delta s}{dT} = \frac{1}{T} - \frac{\gamma - 1}{2(T_o - T)} = 0 \quad \text{at point } P$$



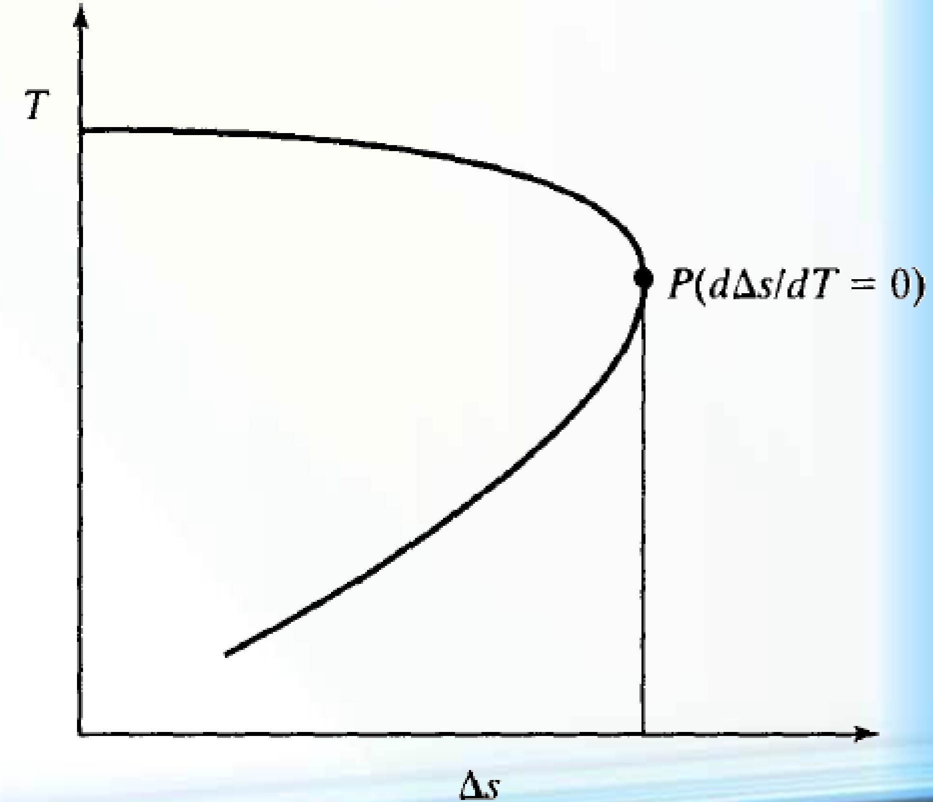
FANNO LINE FLOW

$$\frac{1}{c_v}\frac{d\Delta s}{dT} = \frac{1}{T} - \frac{\gamma - 1}{2(T_o - T)} = 0$$

$$c_p(T_o - T) = \frac{V^2}{2}$$

$$V^2 = c_p(\gamma - 1)T = \gamma RT$$

$$V^2 = a^2 \quad \text{for} \quad ds = 0$$

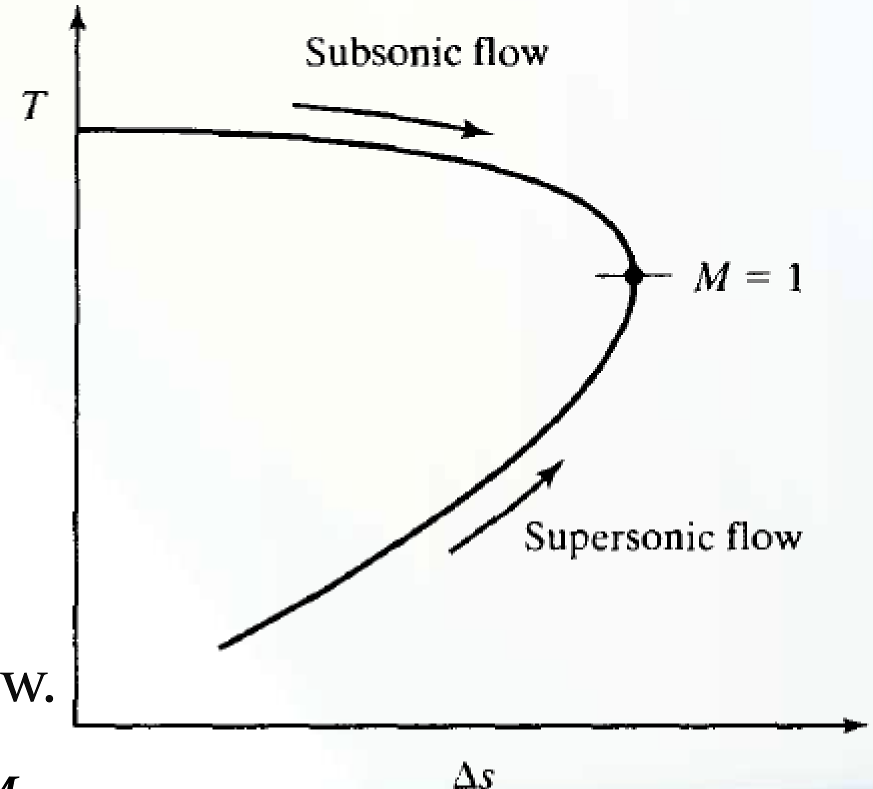


- So that $M = 1$ at point P.

FANNO LINE FLOW

- According to the energy equation, higher velocities are associated with lower enthalpies or temperatures.

  - The section of the Fanno line on *T-s* coordinates that lies *above* point *P* corresponds to *subsonic* flow.
  - The section of the Fanno line on *T-s* coordinates that lies *below* point *P* corresponds to *supersonic* flow.



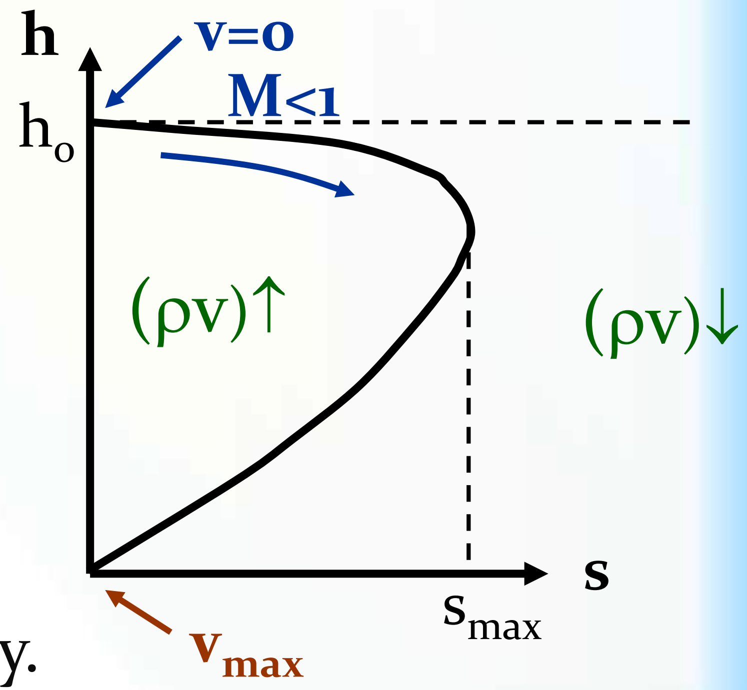
T high $\rightarrow$ V low $\rightarrow$ above P , $M < 1$

T low $\rightarrow$ V high $\rightarrow$ below P , $M > 1$

SUBSONIC FLOW WITH FRICTION

- Consider subsonic adiabatic flow in a constant-area tube. The flow is irreversible because of friction, so for this adiabatic case, $ds > 0$. In other words, the entropy increases in the flow direction.

- Returning to the *T-s* diagram, we see that for a given mass flow, the state of the fluid continually moves to the right, corresponding to an entropy rise. Thus,

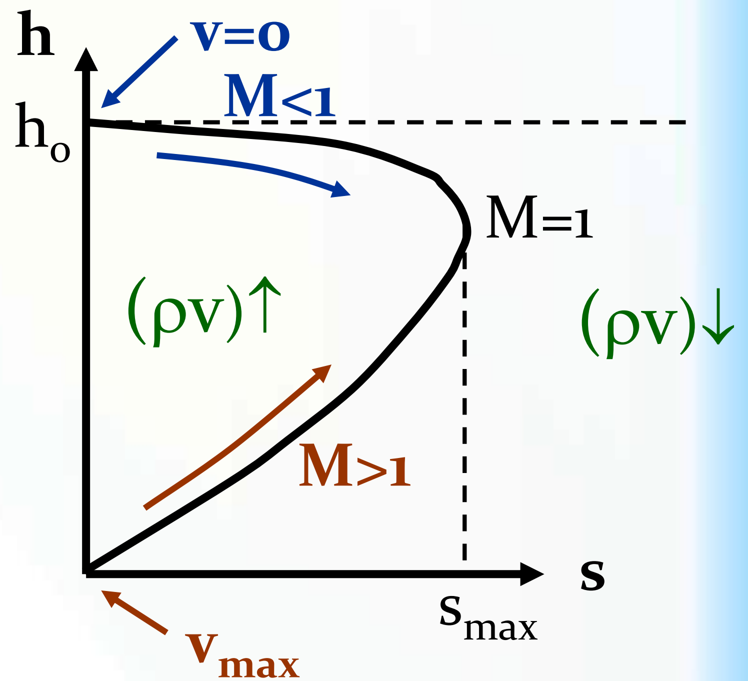


  ❖ For *subsonic* flow with friction, the Mach number *increases* to unity.

SUPERSONIC FLOW WITH FRICTION

○ Consider subsonic adiabatic flow in a constant-area tube. The flow is irreversible because of friction, so for this adiabatic case, $ds > 0$. In other words, the entropy increases in the flow direction.

○ Returning to the *T-s* diagram, we see that for a given mass flow, the state of the fluid continually moves to the right, corresponding to an entropy rise. Thus,



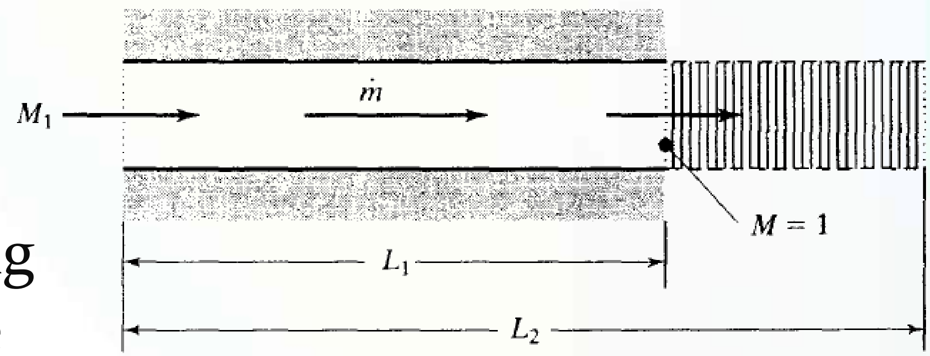
- ❖ For *supersonic* flow, the entropy must again increase, so the flow Mach number here *decreases* to unity.

Friction alone can not allow flow to transition between sub/supersonic flows.

FANNO LINE CHOCKED FLOW (SUBSONIC)

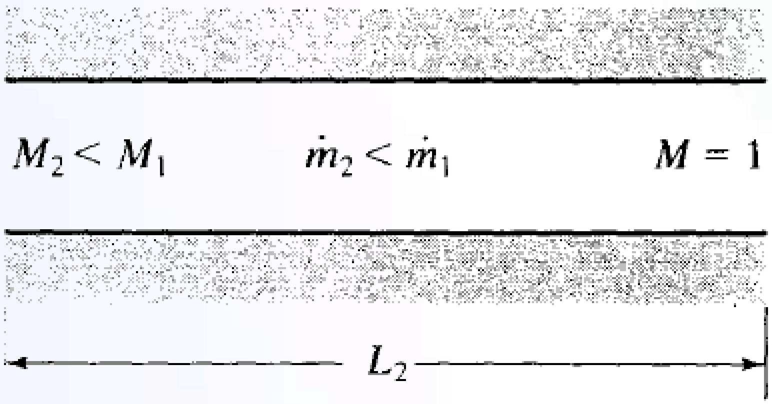
- Suppose now that the duct is long enough for a flow initially subsonic to reach Mach 1 and that an additional length is added.

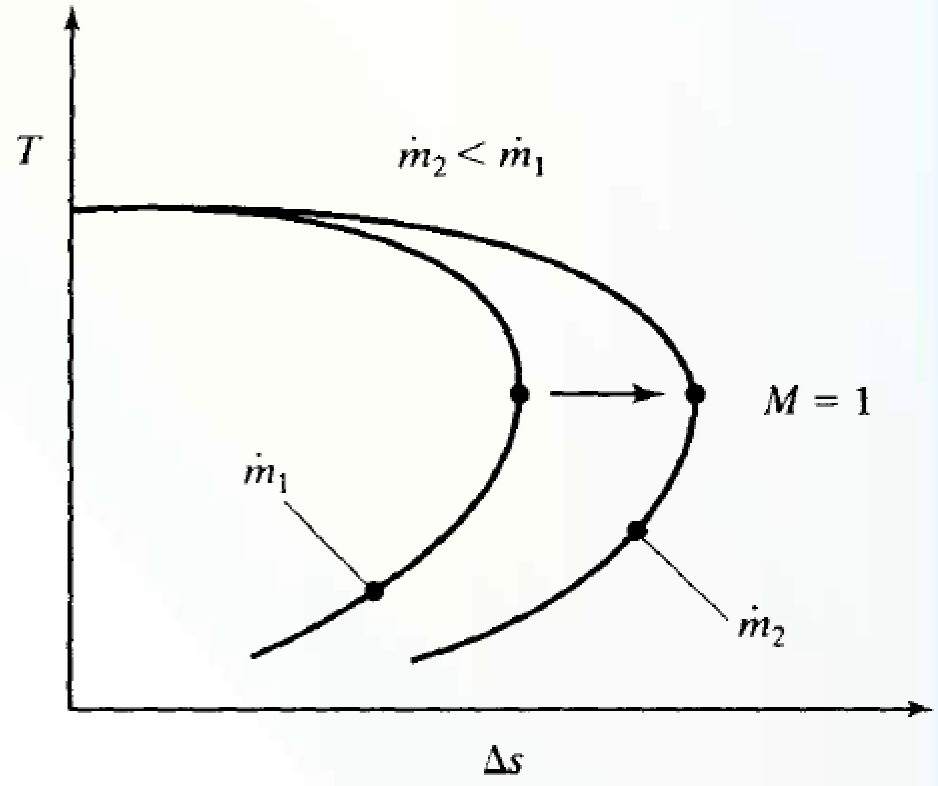
- The flow Mach number for the given mass flow cannot exceed unity without decreasing the entropy. However, from the second law of thermodynamics, this is impossible for an adiabatic flow.



- The additional length brings about a reduction in mass flow; the flow jumps to another Fanno line. Essentially, the duct is choked due to friction.

FANNO LINE CHOCKED FLOW (SUBSONIC)

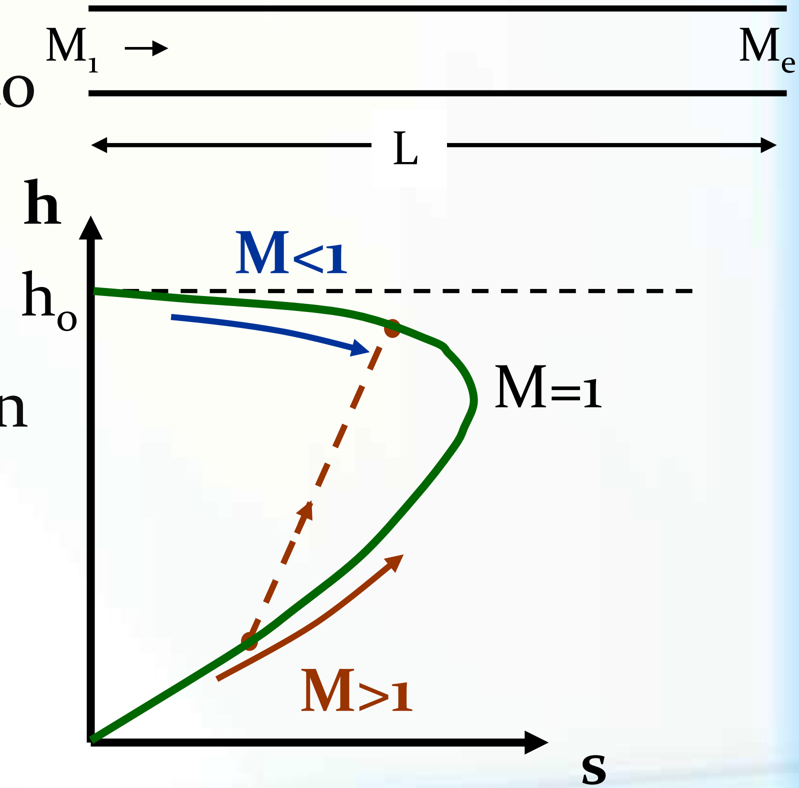




FANNO LINE CHOCKED FLOW (SUPERSONIC)

- Suppose the inlet flow is supersonic and the duct length is made greater than the L_{max} required to produce Mach 1.

- With the supersonic flow unable to "sense" changes in duct length occurring ahead of it, the flow adjusts to the additional length by means of a normal shock rather than a flow reduction.

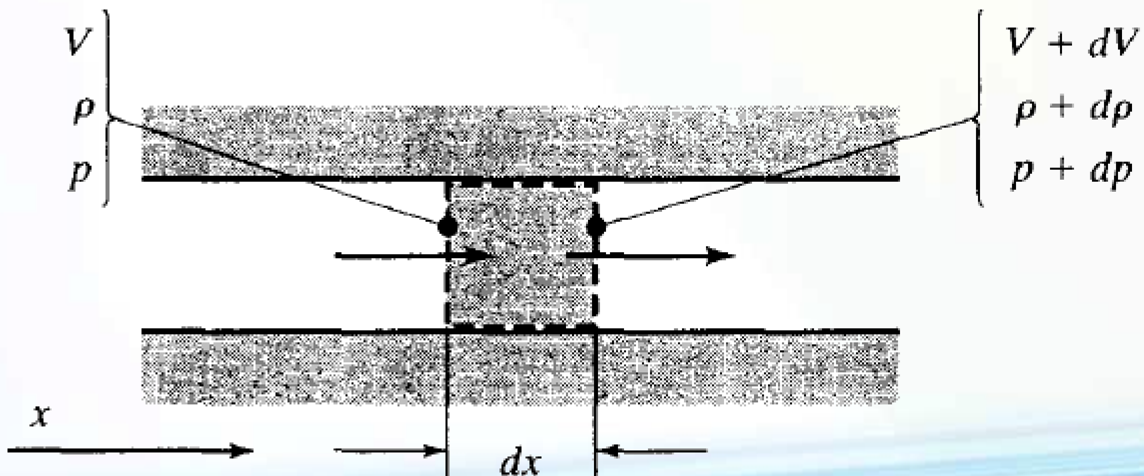


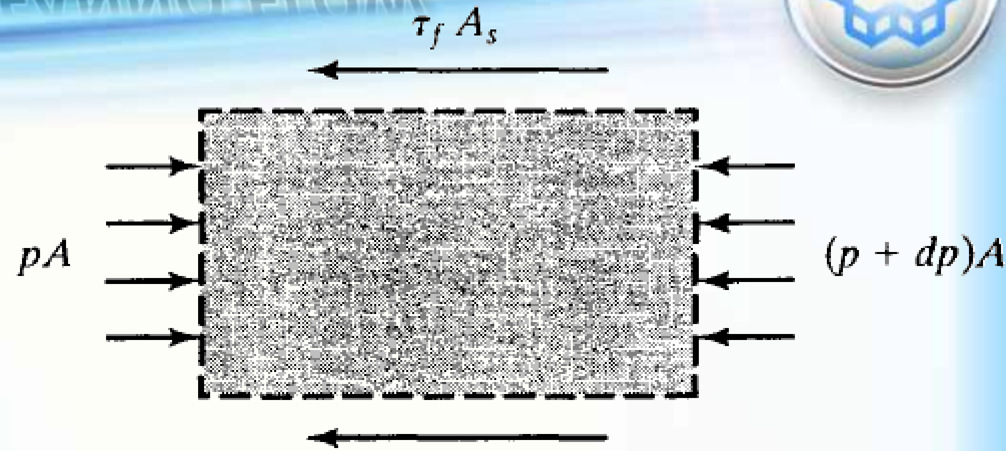
- The location of the shock in the duct is determined by the back pressure imposed on the duct.

WORKING RELATIONS FOR FANNO FLOW

- From practical considerations, it is necessary to determine the change in properties with actual duct length. This requires the use of the momentum equation, with a term accounting for the frictional forces acting on the control volume.

- Select a control volume as shown in Figure





- Applying the momentum equation for steady flow, we get

$$\sum F_x = \iint_{\text{c.s.}} V_x(\rho \mathbf{V} \cdot d\mathbf{A})$$

$$pA - (p + dp)A - \tau_f A_s = (\rho A V)(V + dV) - (\rho A V)V$$

$$-A dp - \tau_f A_s = \rho A V dV$$

Define the hydraulic diameter as $D_h = 4\dfrac{\text{Cross-Sectional Area}}{\text{Wetted Perimeter}} = \dfrac{4A}{P}$

For a circular duct $D_h = \dfrac{4(\pi D^2/4)}{\pi D} = D$

For a square duct $D_h = \dfrac{4S^2}{4S} = S$ $\quad S$ = the width of each side

WORKING RELATIONS FOR FANNO FLOW

WORKING RELATIONS FOR FANNO FLOW

- Since the area over which friction acts, A_s, is equal to the perimeter of the duct times the incremental length dx, it may be replaced using the hydraulic diameter to obtain

$$-A dp - \tau_f (dx) \frac{4A}{D_h} = \rho A V dV$$

- define a friction coefficient $f \equiv 4\tau_f / (\tfrac{1}{2}\rho V^2)$

f is dependent on the flow Reynolds number and the relative wall roughness ε/D_h. However, since $\mathrm{Re} = \rho V D/\mu = 4\dot{m}/(\pi D \mu)$ for Fanno flow, the flow rate and the diameter are constant, and assuming constant dynamic viscosity, we see that the Reynolds number is also constant and therefore does not affect the friction coefficient.

WORKING RELATIONS FOR FANNO FLOW

Some texts define a friction coefficient:

$$\text{“}f\text{”} = \tau_f / \left(\tfrac{1}{2}\rho V^2\right) = C_f$$

That is, they use the skin friction coefficient as the friction coefficient.
The friction coefficient in those texts is referred to as the *Fanning friction factor.*
The friction coefficient used here is sometimes referred to as the *Darcy friction factor.*
The relation between the two friction coefficients is $f = 4''f''$.

WORKING RELATIONS FOR FANNO FLOW

$$-A\,dp - \tau_f(dx)\frac{4A}{D_h} = \rho AV\,dV$$

$$f \equiv 4\tau_f/(\tfrac{1}{2}\rho V^2)$$

$$-dp - \frac{1}{2}\rho V^2 f\frac{dx}{D} = \rho V\,dV$$

- It is desirable to integrate above Equation to obtain, for example, an expression for Mach number and pressure change over a given duct length. First we divide the foregoing equation by *p*:

$$\frac{dp}{p} + \frac{1}{2}\gamma M^2 f\frac{dx}{D} + \gamma M^2\frac{dV}{V} = 0$$

- To obtain an expression for *M* in terms of *x*, *dV*/*V* and *dp*/*p* must be replaced in this Equation.

WORKING RELATIONS FOR FANNO FLOW

- From continuity, ρV = constant

$$\rho V = \frac{p}{RT}\sqrt{\gamma RT}\frac{V}{\sqrt{\gamma RT}} = \sqrt{\frac{\gamma}{R}}\left(\frac{pM}{\sqrt{T}}\right) = \text{constant}$$

- Taking the logarithm of this expression and then differentiating produces

$$\frac{dp}{p} = -\frac{dM}{M} + \frac{1}{2}\frac{dT}{T} = 0$$

- Moreover, from the definition of the Mach number, we have

$$\frac{dV}{V} = \frac{dM}{M} + \frac{1}{2}\frac{dT}{T}$$

WORKING RELATIONS FOR FANNO FLOW

$$\frac{dp}{p} + \frac{1}{2}\gamma M^2 f \frac{dx}{D} + \gamma M^2 \frac{dV}{V} = 0$$

$$\frac{dp}{p} = -\frac{dM}{M} + \frac{1}{2}\frac{dT}{T} = 0$$

$$\frac{dV}{V} = \frac{dM}{M} + \frac{1}{2}\frac{dT}{T}$$

$$\left(-\frac{dM}{M} + \frac{1}{2}\frac{dT}{T}\right) + \frac{1}{2}\gamma M^2\left(f\frac{dx}{D}\right) + \gamma M^2\left(\frac{dM}{M} + \frac{1}{2}\frac{dT}{T}\right) = 0$$

or

$$\left(f\frac{dx}{D}\right) = -\frac{2}{\gamma M^2}\left(-\frac{dM}{M} + \frac{1}{2}\frac{dT}{T}\right) - 2\left(\frac{dM}{M} + \frac{1}{2}\frac{dT}{T}\right)$$

$$= \frac{2}{\gamma}\frac{dM}{M^3} - \frac{dM^2}{M^2} - \frac{dT}{T} - \frac{1}{\gamma M^2}\frac{dT}{T}$$

WORKING RELATIONS FOR FANNO FLOW

$$\left(f\frac{dx}{D}\right) = -\frac{2}{\gamma M^2}\left(-\frac{dM}{M} + \frac{1}{2}\frac{dT}{T}\right) - 2\left(\frac{dM}{M} + \frac{1}{2}\frac{dT}{T}\right)$$

$$= \frac{2}{\gamma}\frac{dM}{M^3} - \frac{dM^2}{M^2} - \frac{dT}{T} - \frac{1}{\gamma M^2}\frac{dT}{T}$$

But, for this adiabatic flow, To = constant, so

$$T\left(1 + \frac{\gamma - 1}{2}M^2\right) = \text{constant} \xrightarrow{\text{Logarithm and differentiating}} \frac{dT}{T} + \frac{d\left(1 + \frac{\gamma-1}{2}M^2\right)}{1 + \frac{\gamma-1}{2}M^2} = 0$$

$$f\frac{dx}{D} = \frac{2}{\gamma}\frac{dM}{M^3} - \frac{dM^2}{M^2} + \frac{d\left(1 + \frac{\gamma-1}{2}M^2\right)}{1 + \frac{\gamma-1}{2}M^2} + \left(\frac{1}{\gamma M^2}\right)\frac{d\left(1 + \frac{\gamma-1}{2}M^2\right)}{1 + \frac{\gamma-1}{2}M^2}$$

WORKING RELATIONS FOR FANNO FLOW

$$\frac{1}{(\gamma M^2)\left(1+\frac{\gamma-1}{2}M^2\right)} = \frac{c_1}{(\gamma M^2)} + \frac{c_2}{\left(1+\frac{\gamma-1}{2}M^2\right)}$$

Expanding the right-hand side and comparing the numerator of both sides reveals that

$$c_1 = 1 \qquad\qquad c_2 = -(\gamma - 1)/(2\gamma)$$

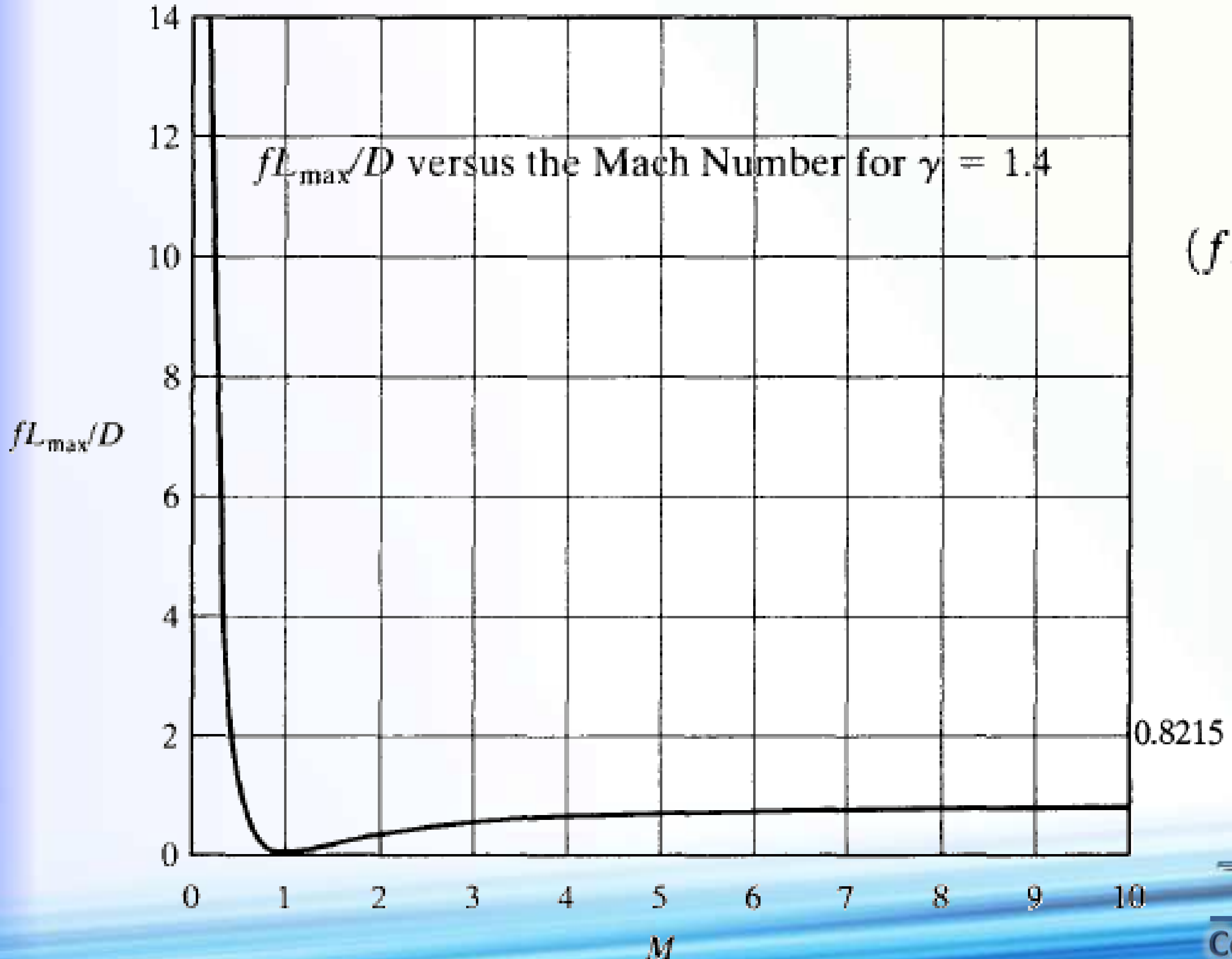
$$\left(\frac{1}{\gamma M^2}\right)\frac{d\left(1+\frac{\gamma-1}{2}M^2\right)}{1+\frac{\gamma-1}{2}M^2} = \frac{\gamma-1}{2\gamma}\frac{d(M^2)}{M^2} - \frac{\gamma-1}{2\gamma}\frac{d\left(1+\frac{\gamma-1}{2}M^2\right)}{1+\frac{\gamma-1}{2}M^2}$$

$$f\frac{dx}{D} = \left(\frac{\gamma+1}{2\gamma}\right)\frac{d\left(1+\frac{\gamma-1}{2}M^2\right)}{1+\frac{\gamma-1}{2}M^2} + \frac{2}{\gamma}\frac{dM}{M^3} - \left(\frac{\gamma+1}{2\gamma}\right)\frac{d(M^2)}{M^2}$$

This equation may be integrated term by term to determine M as a function of duct Length. It is convenient for Fanno flow (frictional, constant-area, adiabatic flow) to choose, as a reference point, $M = 1$ and $L = L_{max}$. For the lower limit of integration, select $x = 0$ at $M = M$.

WORKING RELATIONS FOR FANNO FLOW

$$\frac{f L_{max}}{D} = \left(\frac{\gamma+1}{2\gamma}\right)\ln\left(\frac{\frac{\gamma+1}{2}}{1+\frac{\gamma-1}{2}M^2}\right) - \frac{1}{\gamma}\left(1-\frac{1}{M^2}\right) - \left(\frac{\gamma+1}{2\gamma}\right)\ln\left(\frac{1}{M^2}\right)$$



$$M \to \infty$$

$$(f L_{max}/D) \to [(\gamma + 1)/(2\gamma)]$$

WORKING RELATIONS FOR FANNO FLOW

If it is required to find p versus M or (fdx/D), dV/V can be eliminated from

$$\frac{dp}{p} + \frac{1}{2}\gamma M^2 f\frac{dx}{D} + \gamma M^2 \frac{dV}{V} = 0$$

$$\frac{dV}{V} = \frac{dM}{M} + \frac{1}{2}\frac{dT}{T} \longrightarrow \frac{dT}{T} + \frac{d\left(1 + \frac{\gamma - 1}{2}M^2\right)}{1 + \frac{\gamma - 1}{2}M^2} = 0$$

$$\frac{dp}{p} + \frac{1}{2}\gamma M^2 f\frac{dx}{D} + \gamma M^2\frac{dM}{M} - \frac{1}{2}\frac{\gamma M^2(\gamma - 1)M^2}{1 + \frac{\gamma - 1}{2}M^2}\frac{dM}{M} = 0$$

$$f\frac{dx}{D} = \left[\frac{(1 - M^2)}{1 + \frac{\gamma - 1}{2}M^2}\right]\left(\frac{2}{\gamma M^2}\right)\frac{dM}{M}$$

WORKING RELATIONS FOR FANNO FLOW

$$\frac{dp}{p} + \frac{1 - M^2}{1 + \frac{\gamma - 1}{2} M^2} \frac{dM}{M} + \gamma M^2 \frac{dM}{M} - \frac{1}{2} \frac{\gamma M^2 (\gamma - 1) M^2}{1 + \frac{\gamma - 1}{2} M^2} \frac{dM}{M} = 0$$

$$\frac{dp}{p} = -\left[\frac{1 + (\gamma - 1) M^2}{1 + \frac{\gamma - 1}{2} M^2} \right] \frac{dM}{M}$$

$$\int_p^{p^*} \frac{dp}{p} = -\int_M^1 \left[\frac{1 + \frac{\gamma - 1}{2} M^2}{1 + \frac{\gamma - 1}{2} M^2} \right] \frac{dM}{M} - \int_M^1 \left[\frac{\frac{\gamma - 1}{2} M^2}{1 + \frac{\gamma - 1}{2} M^2} \right] \frac{dM}{M}$$

$$\ln\left(\frac{p}{p^*} \right) = \ln\left(\frac{1}{M} \right) + \frac{1}{2} \ln\left[\frac{\gamma + 1}{2 + (\gamma - 1) M^2} \right] = \ln\left\{ \left(\frac{1}{M} \right) \left[\frac{\gamma + 1}{2 + (\gamma - 1) M^2} \right]^{1/2} \right\}$$

WORKING RELATIONS FOR FANNO FLOW

$$\frac{p}{p^*} = \left(\frac{1}{M}\right)\left[\frac{\gamma + 1}{2 + (\gamma - 1)M^2}\right]^{1/2}$$

The above equation can be found by an alternative approach that has the advantage of not requiring integration.

Since we are dealing with adiabatic flow of a perfect gas,

$$\frac{T_1}{T_2} = \left(\frac{T_o}{T_2}\right)\left(\frac{T_1}{T_o}\right) = \frac{1 + \dfrac{\gamma - 1}{2}M_2^2}{1 + \dfrac{\gamma - 1}{2}M_1^2} = \frac{2 + (\gamma - 1)M_2^2}{2 + (\gamma - 1)M_1^2}$$

$$\rho_1 V_1 = \rho_2 V_2 \qquad \rho = p/(RT) \qquad V = Ma = M\sqrt{\gamma RT}$$

$$\frac{p_1}{p_2} = \frac{M_2}{M_1}\sqrt{\frac{T_1}{T_2}} \qquad \frac{p_1}{p_2} = \frac{M_2}{M_1}\left[\frac{2 + (\gamma - 1)M_2^2}{2 + (\gamma - 1)M_1^2}\right]^{1/2}$$

The density ratio follows immediately from the perfect-gas relation:

$$\frac{\rho_1}{\rho_2} = \frac{p_1}{RT_1}\frac{RT_2}{p_2} = \frac{p_1}{p_2}\frac{T_2}{T_1} = \frac{M_2}{M_1}\left[\frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2}\right]^{1/2} = \frac{V_2}{V_1}$$

WORKING RELATIONS FOR FANNO FLOW

Because of the irreversibilities involved with Fanno flow, the total pressure will always decrease in the direction of flow, which is not the case for isentropic flow.

$$\frac{p_{o1}}{p_{o2}} = \frac{p_{o1}}{p_1}\frac{p_1}{p_2}\frac{p_2}{p_{o2}} = \left(1 + \frac{\gamma - 1}{2}M_1^2\right)^{\frac{\gamma}{\gamma-1}}\frac{M_2}{M_1}\left[\frac{2 + (\gamma - 1)M_2^2}{2 + (\gamma - 1)M_1^2}\right]^{1/2}\left(1 + \frac{\gamma - 1}{2}M_2^2\right)^{-\frac{\gamma}{\gamma-1}}$$

$$\frac{p_{o1}}{p_{o2}} = \frac{M_2}{M_1}\left[\frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2}\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

Because the flow is also one dimensional and adiabatic, the equations for the conservation of mass and energy, but not momentum, for flow through a normal shockwave are the same as that for Fanno flow.

$$\frac{s_2 - s_1}{R} = -\ln\left(\frac{p_{o2}}{p_{o1}}\right)$$

WORKING RELATIONS FOR FANNO FLOW

For computational purposes, the reference state where the Mach number reaches unity is used, and, as mentioned previously, the symbol for this state is *.

$$\frac{T}{T^*} = \frac{2 + (\gamma - 1)1^2}{2 + (\gamma - 1)M^2} = \frac{(\gamma + 1)}{2 + (\gamma - 1)M^2}$$

$$\frac{p}{p^*} = \left(\frac{1}{M}\right)\left[\frac{\gamma + 1}{2 + (\gamma - 1)M^2}\right]^{1/2}$$

$$\frac{\rho}{\rho^*} = \frac{V^*}{V} = \frac{1}{M}\left[\frac{2 + (\gamma - 1)M^2}{(\gamma + 1)}\right]^{1/2}$$

$$\frac{p_o}{p_o^*} = \frac{1}{M}\left[\frac{2 + (\gamma - 1)M^2}{(\gamma + 1)}\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

PROPERTY VARIATIONS

- Look at signs of previous equations to see how properties changed by friction as we move along flow
  - (1-M^2) term makes M<1 different than M>1

	M<1	M>1
s	↑	↑
p_o	↓	↓
M	↑	↓
h,T	↓	↑
p	↓	↑
ρ	↓	↑
v	↑	↓

- Friction increases $s \Rightarrow p_o$ drop
- Friction drives $M \rightarrow 1$
- h_o, T_o = const: h, T opposite to M
- p, ρ same as T (like isen. flow)
- ρv=const: v opposite of ρ

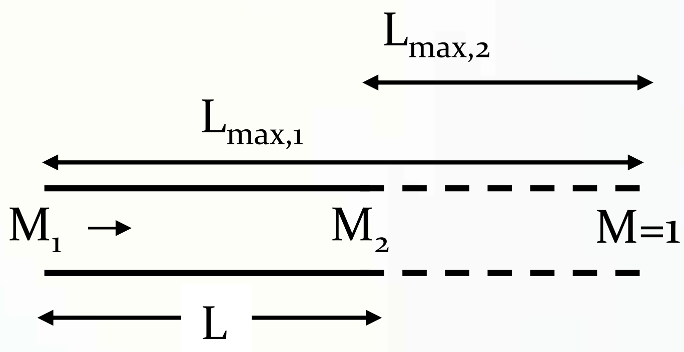
USE OF TABLES

- To get change in M, use change in $f\,L_{max}/D$ (like using A/A^*)

$$\frac{fL}{D}=\int_{M_1}^{M_2}\frac{\left(1-M^2\right)}{\left(1+\frac{\gamma-1}{2}M^2\right)}\left(\frac{2}{\gamma M^2}\right)\frac{dM}{M}$$

$$=\int_{M_1}^{1}\left\{-\right\}dM-\int_{M_2}^{1}\left\{-\right\}dM$$

$$\boxed{\frac{fL}{D}=\left.\frac{fL_{\max}}{D}\right)_{M_1}-\left.\frac{fL_{\max}}{D}\right)_{M_2}}$$



- If you know $f\,L/D$ and M_1,
  1. Look up $f\,L_{max}/D$ at M_1
  2. Calculate $f\,L_{max}/D$ at M_2
  3. Look up corresponding M_2

Find values in Appendix F in "Gas Dynamics", James E. John

USE OF TABLES

TABLE F.1 Fanno Line Flow ($\gamma = 1.4$)

M	T/T^*	p/p^*	p_o/p_o^*	ρ/ρ^*	fL_{max}/D
0.00	1.2000	∞	∞	∞	∞
0.02	1.1999	54.7701	28.9421	45.6454	1778.4499
0.04	1.1996	27.3817	14.4815	22.8254	440.3522
0.06	1.1991	18.2508	9.6659	15.2200	193.0311
0.08	1.1985				
0.10	1.1976				
0.12	1.1966				
0.14	1.1953				
0.16	1.1939				
0.18	1.1923				
0.20	1.1905				
0.22	1.1885				
0.24	1.1863				
0.26	1.1840				
0.28	1.1815				
0.30	1.1788				
0.32	1.1750				

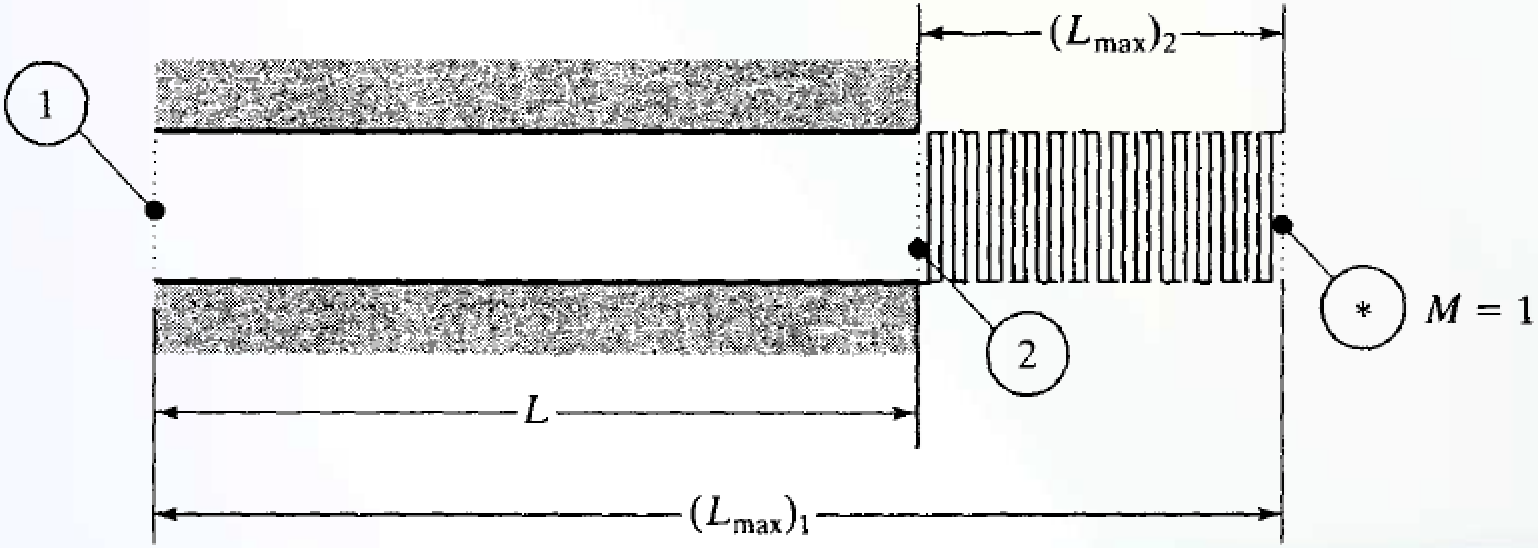
TABLE F.1 *(Continued)*

M	T/T^*	p/p^*	p_o/p_o^*	ρ/ρ^*	fL_{max}/D
2.10	0.6376	0.3802	1.8369	0.5963	0.3339
2.12	0.6320	0.3750	1.8690	0.5934	0.3394
2.14	0.6263	0.3698	1.9018	0.5905	0.3449
2.16	0.6208	0.3648	1.9354	0.5876	0.3503
2.18	0.6152	0.3598	1.9698	0.5848	0.3556
2.20	0.6098	0.3549	2.0050	0.5821	0.3609
2.22	0.6043	0.3502	2.0409	0.5794	0.3661
2.24	0.5989	0.3455	2.0777	0.5768	0.3712
2.26	0.5936	0.3409	2.1153	0.5743	0.3763
2.28	0.5883	0.3364	2.1538	0.5718	0.3813
2.30	0.5831	0.3320	2.1931	0.5694	0.3862
2.32	0.5779	0.3277	2.2333	0.5670	0.3911
2.34	0.5728	0.3234	2.2744	0.5647	0.3959
2.36	0.5677	0.3193	2.3164	0.5624	0.4006

EXAMPLE

Flow enters a constant-area, insulated duct with Mach number of 0.60, static pressure of 150 kPa, and static temperature of 300 K. Assume a duct length of 45 cm, a duct diameter of 3 cm, and a friction coefficient of 0.02.

Determine the *Mach number*, *static pressure*, and *static temperature* at the duct outlet. Assume that $\gamma = 1.4$.



EXAMPLE (CONT.)

$$M_1 = 0.60, \quad \xrightarrow[\text{tabulated in Appendix}]{\frac{fL_{\max}}{D} = \left(\frac{\gamma+1}{2\gamma}\right)\ln\left(\frac{\frac{\gamma+1}{2}}{1+\frac{\gamma-1}{2}M^2}\right) - \frac{1}{\gamma}\left(1-\frac{1}{M^2}\right) - \left(\frac{\gamma+1}{2\gamma}\right)\ln\left(\frac{1}{M^2}\right)} \quad (fL_{\max}/D)_1 = 0.4908,$$

$$fL/D = 0.30. \qquad (L_{\max})_2 = (L_{\max})_1 - L$$

$$\left(\frac{fL_{\max}}{D}\right)_2 = \left(\frac{fL_{\max}}{D}\right)_1 - \frac{fL}{D} = 0.4908 - 0.30 = 0.1908$$

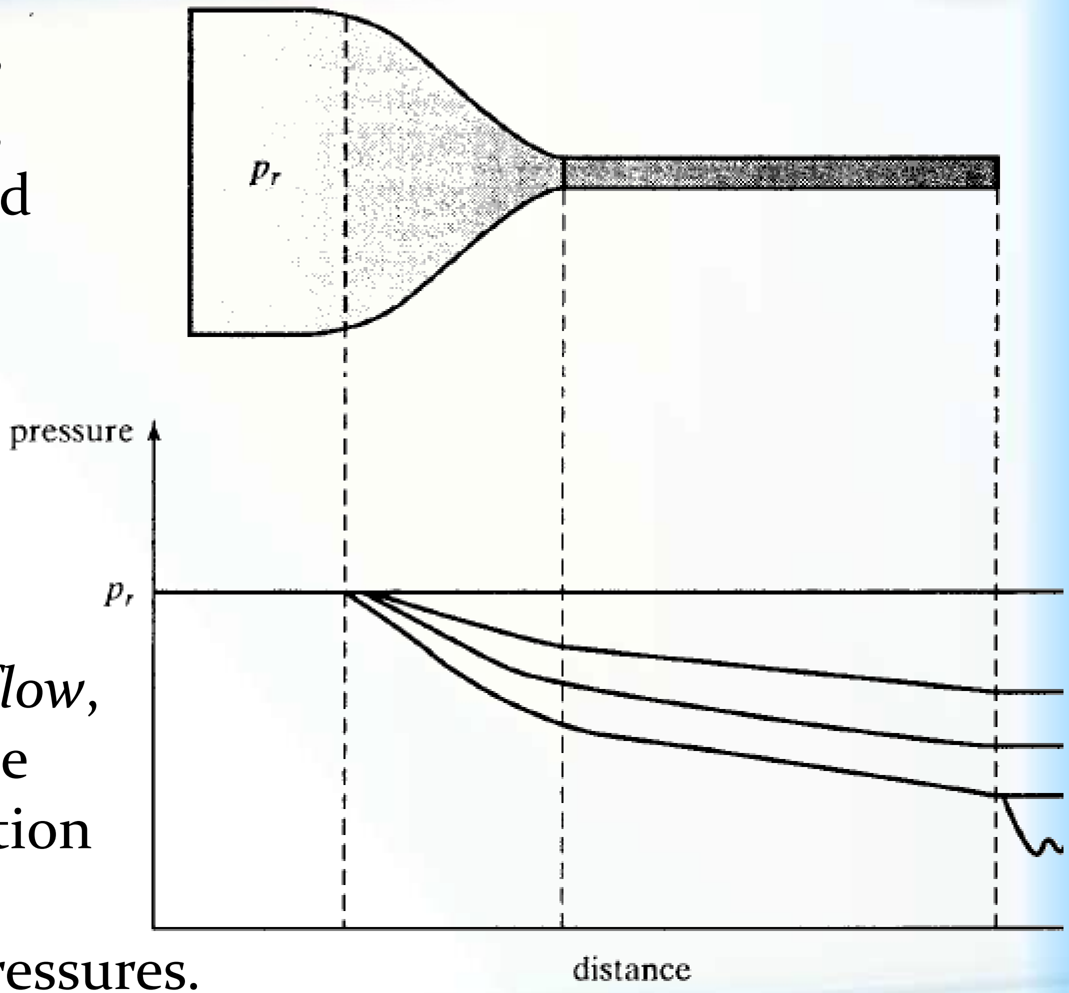
$$\xrightarrow{\text{tabulated in Appendix}} \quad M_2 = 0.7093$$

$$\frac{p_2}{p_1} = \frac{(p_2/p^*)}{(p_1/p^*)} = \frac{1.4722}{1.7634} = 0.8349 \qquad p_2 = (0.8349)150 = 125.2350 \text{ kPa}$$

$$\frac{T_2}{T_1} = \frac{(T_2/T^*)}{(T_1/T^*)} = \frac{1.0903}{1.1194} = 0.9740 \qquad T_2 = (0.9740)300 = 292.2012 \text{ K}$$

FLOW THROUGH A CONVERGING NOZZLE AND CONSTANT-AREA DUCT IN SERIES

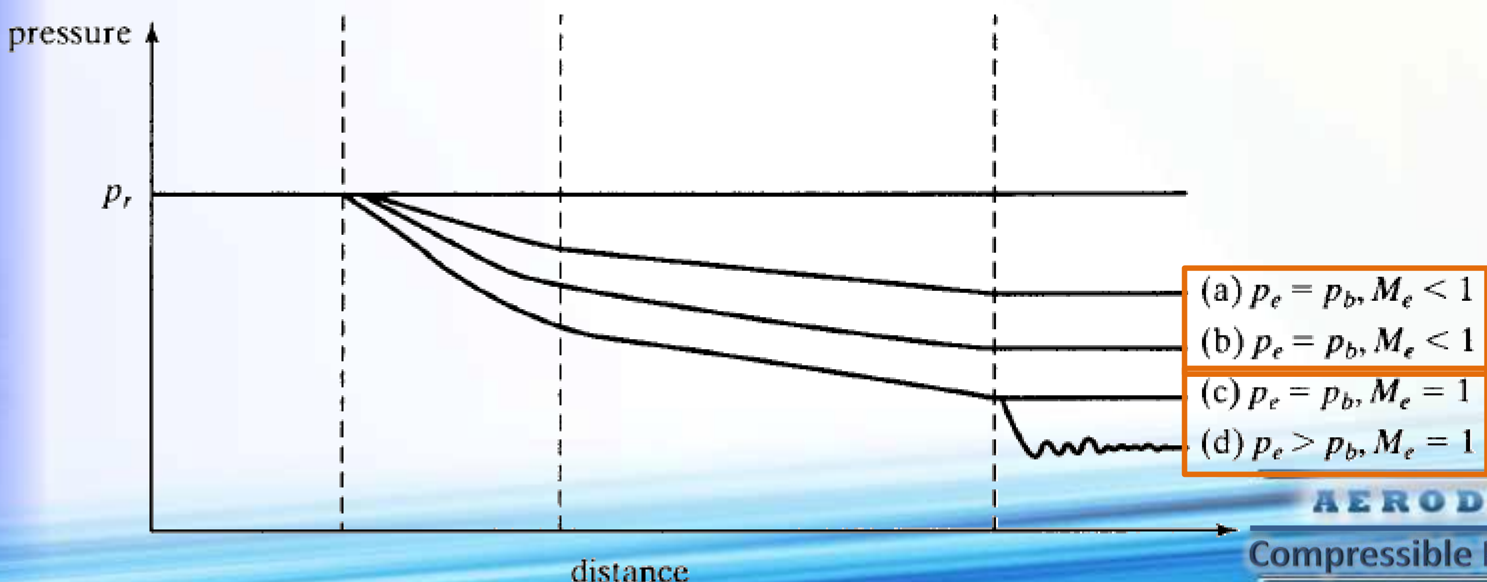
- Very often, a situation occurs where a duct is fed by a nozzle, with the duct back pressure and nozzle stagnation pressure the known quantities. Consider a duct supplied by a converging nozzle, with flow provided by a reservoir at pressure p_r.



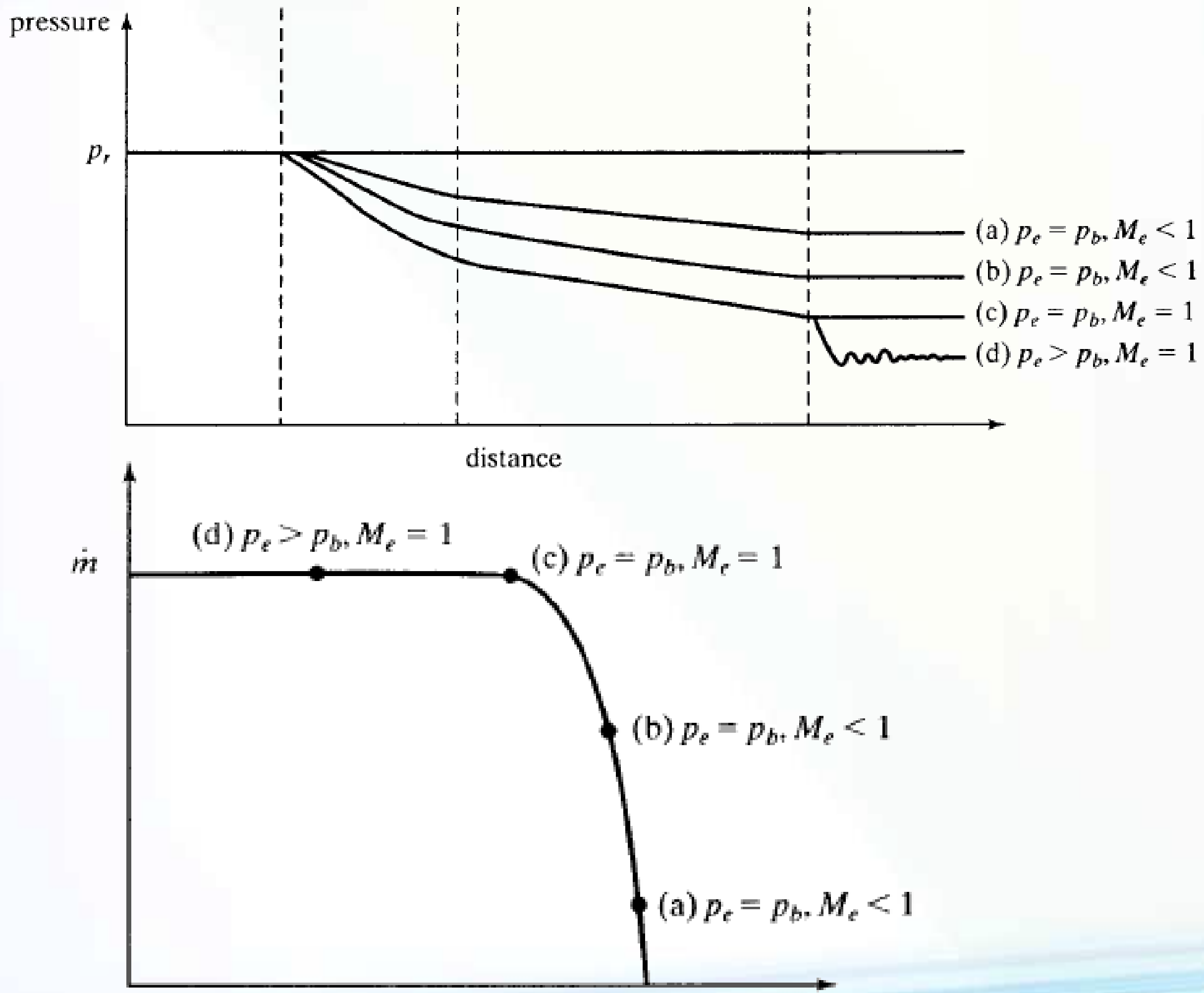
- Assuming *isentropic nozzle flow*, with *Fanno flow in the duct*, the system static pressure distribution p versus the distance x can be determined for various back pressures. (p_r is maintained constant.)

FLOW THROUGH A CONVERGING NOZZLE AND CONSTANT-AREA DUCT IN SERIES

- As p_b is lowered below p_r curves (a) and (b) in are obtained, with pressure decreasing in both nozzle and duct.
- Eventually, when the back pressure is decreased to that of curve (c), Mach number 1 occurs at the duct exit. Further decreases in back pressure cannot be "sensed" by the reservoir; for all back pressures below that of curve (c) [e.g., the underexpanded flow at curve (d)], the mass flow rate remains the same as that of curve (c).



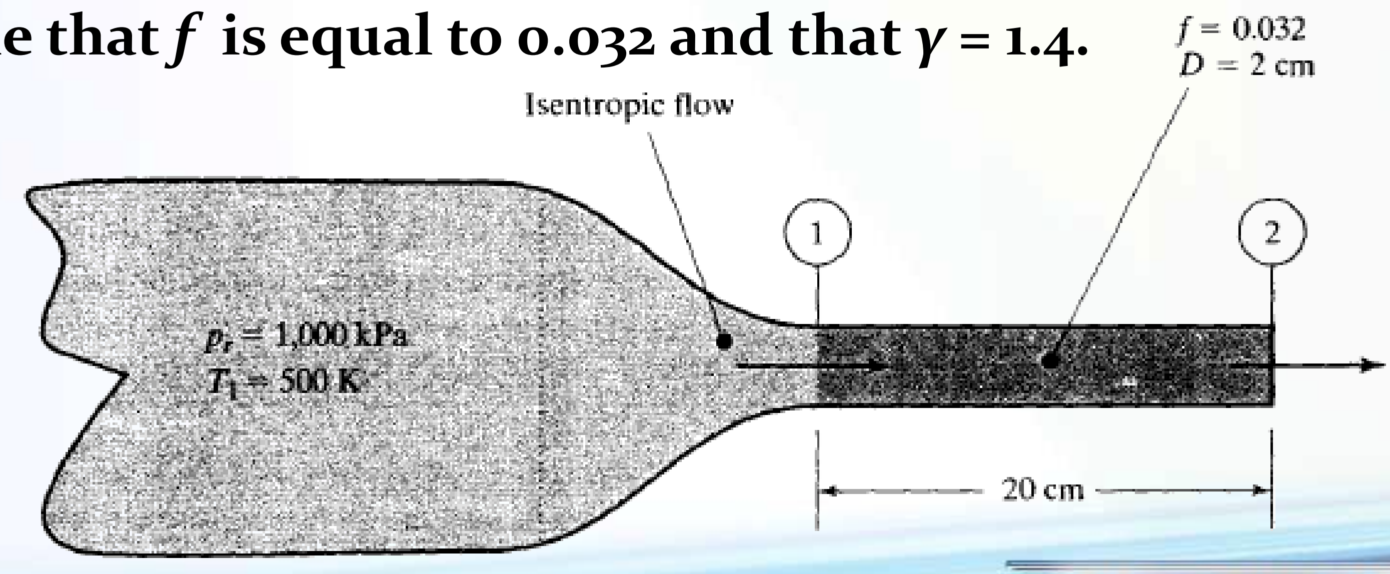
FLOW THROUGH A CONVERGING NOZZLE AND CONSTANT-AREA DUCT IN SERIES



EXAMPLE

A constant-area duct that is 20 cm in length by 2 cm in diameter is connected to a reservoir through a converging nozzle, as shown in Figure. For a reservoir pressure and temperature of 1 MPa and 500 K, determine the maximum air flow rate in kilograms per second through the system and the range of back pressures over which this flow is realized.

Repeat these calculations for a converging nozzle with no duct. Assume that f is equal to 0.032 and that $\gamma = 1.4$.



EXAMPLE (CONT.)

For maximum mass flow through the nozzle-duct system, M_2 is equal to unity. For this condition, the actual fL/D of the duct becomes equal to $(f L_{max}/D)_1 = 0.32.$

$\longrightarrow$ $M_1 = 0.6517.$ $\longrightarrow$ $(p/p_o)_1 = 0.7518$ and $(T/T_o)_1 = 0.9217.$

$$p_1 = (0.7518)1.000 = 751.8 \text{ kPa and } T_1 = (0.9217)500 = 460.85 \text{ K}.$$

$$\dot{m}_{max} = \left(\frac{p}{RT}\right)_1 A_1 M_1 \sqrt{\gamma R T_1}$$

$$= \left[\frac{(751.8)}{(0.287)(460.85)}\right]\left[\frac{\pi}{4}(4 \times 10^{-4})\right]\left[(0.6517)\sqrt{(1.4)(287)(460.85)}\right]$$

$$= (5.6841 \text{ kg/m}^3)(0.00031416 \text{ m}^2)(280.4352 \text{ m/s})$$

$$= 0.5008 \text{ kg/s}$$

EXAMPLE (CONT.)

$$p_2/p_1 = p^*/p_1 = 1/1.6138, \text{ or } p^* = 751.8/1.6138 = 465.8570 \text{ kPa},$$

so the system is choked over the range of back pressures from **0** to **465.8570 kPa.**

If the duct were to be removed, choking would occur with Mach 1 at the nozzle exit. For this condition $p_1 = (0.5283)(1{,}000) = 528.3 \text{ kPa}$

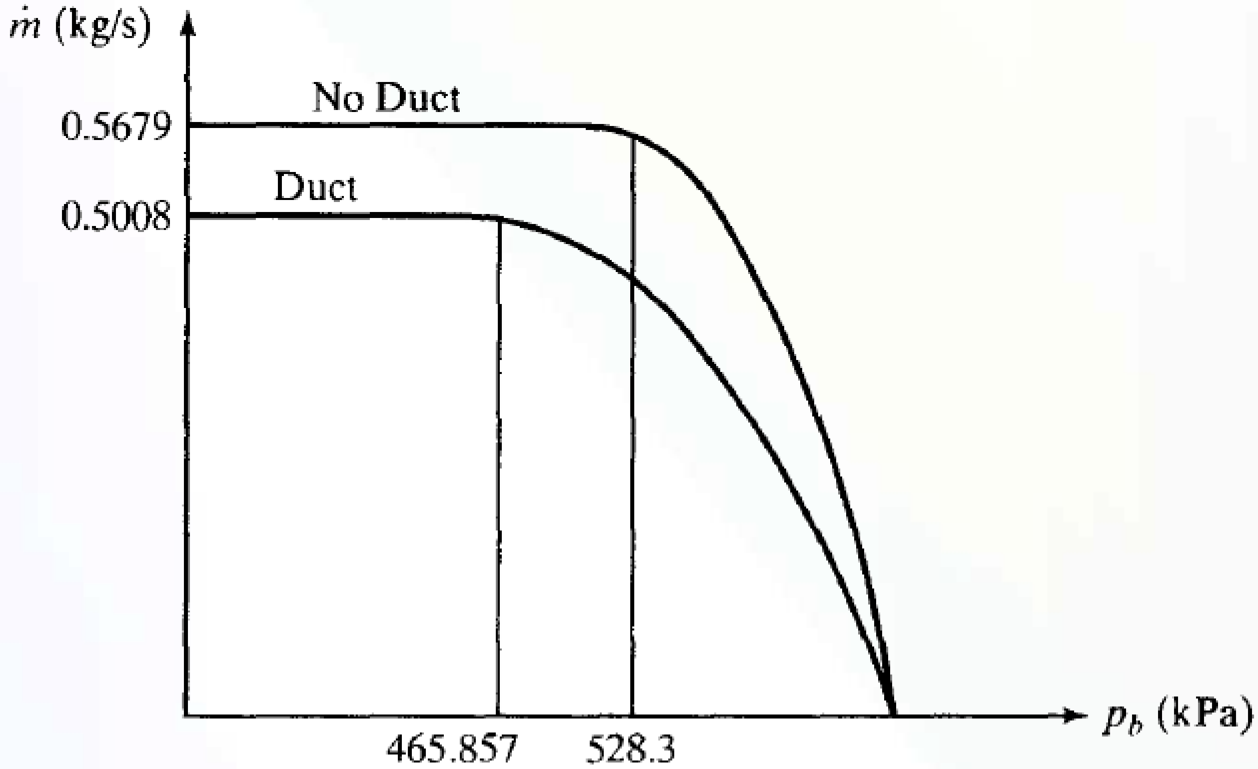
$$T_1 = 0.8333(500 \text{ K}) = 416.65 \text{ K},$$

So the maximum mass flow rate is:

$$(4.4180)(0.00031416)(409.1576) = 0.5679 \text{ kg/s}.$$

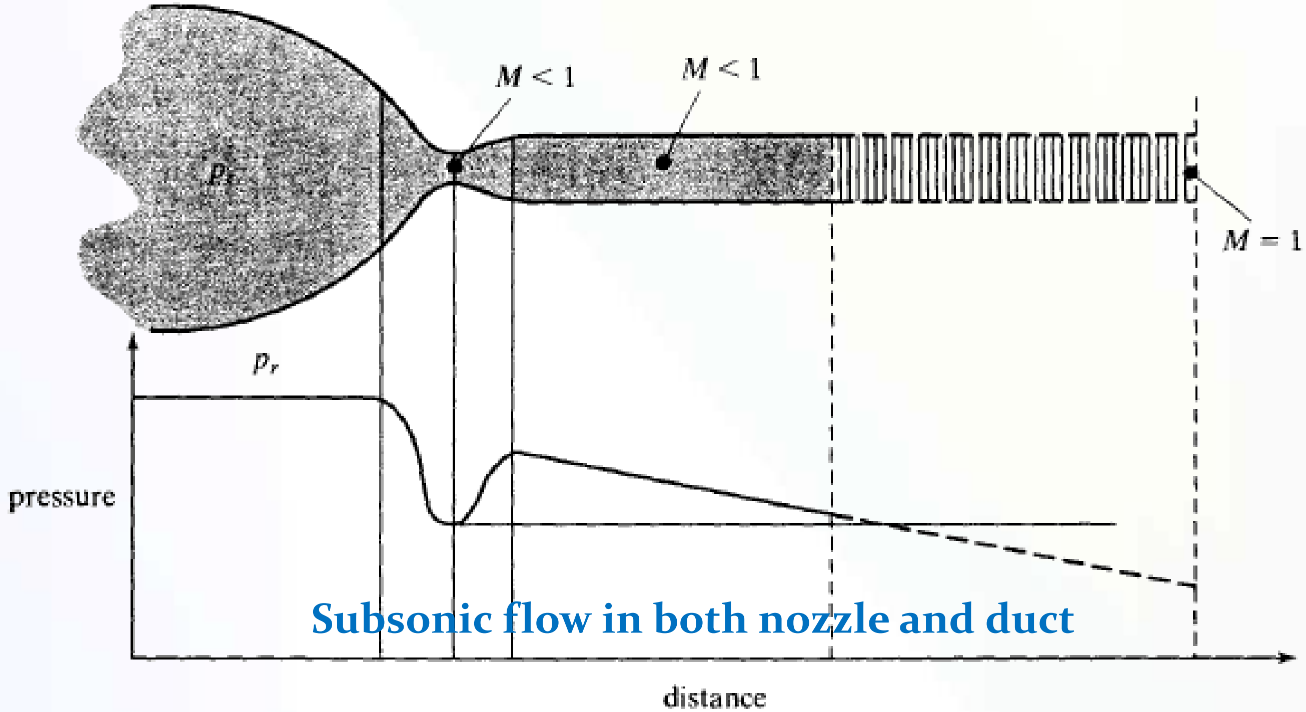
For this case, the system is choked over the back-pressure range from **0** to **528.3 kPa**.

EXAMPLE (CONT.)



FLOW THROUGH A C-D NOZZLE AND CONSTANT-AREA DUCT IN SERIES

When a duct is connected to a reservoir through a converging-diverging nozzle, the situation becomes somewhat more complex.



Subsonic flow in both nozzle and duct

If the duct is long enough (see the dashed curve in the figure), the system reaches Mach 1 first at the duct exit; in this case, the nozzle is not choked. Once Mach 1 is reached, no further increase in mass flow rate can occur by reduction of the system back pressure.

FLOW THROUGH A C-D NOZZLE AND CONSTANT-AREA DUCT IN SERIES

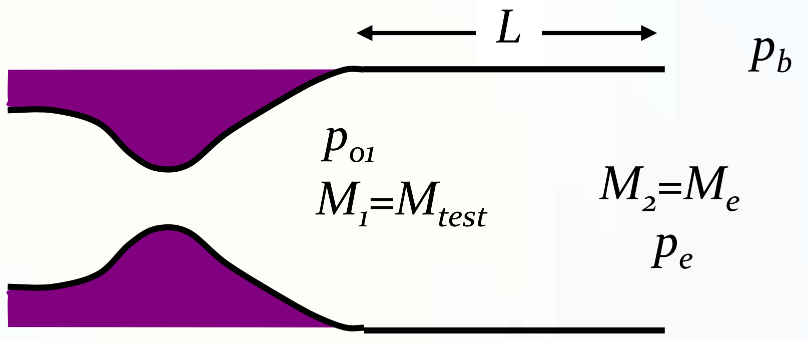
- With supersonic flow at the nozzle exit, there is the possibility of shocks in the duct. Note, however, that once the back pressure is just low enough to produce Mach 1 at the nozzle throat, the system is choked, with no further increase in mass flow possible.

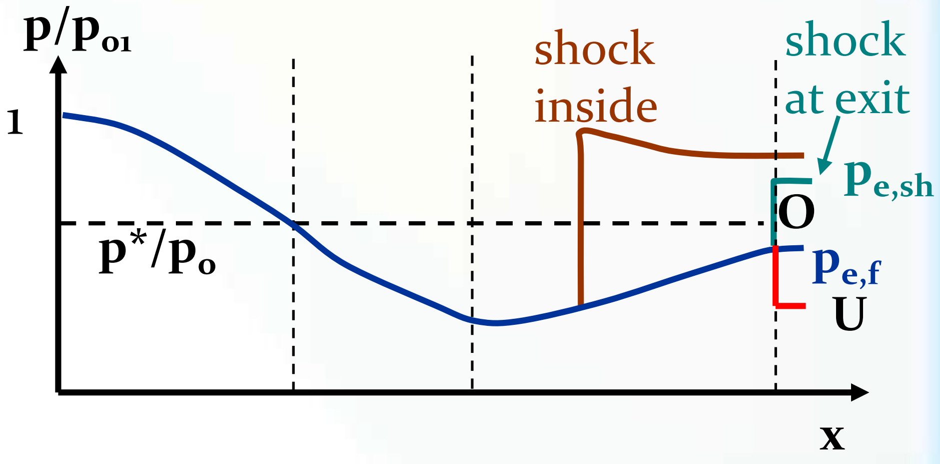
- Unlike the case previously discussed, here, once the throat velocity reaches the velocity of sound, the mass flow rate is unaffected by duct length; that is, the conditions at the throat remain fixed, and therefore $\dot{m} = \rho_t A_t V_t$. Now the system is choked by the nozzle, not the duct.

- Let us consider the flow pattern obtained with supersonic flow at the duct inlet in the following two cases:
  1. $L < L_{max}$
  2. $L > L_{max}$

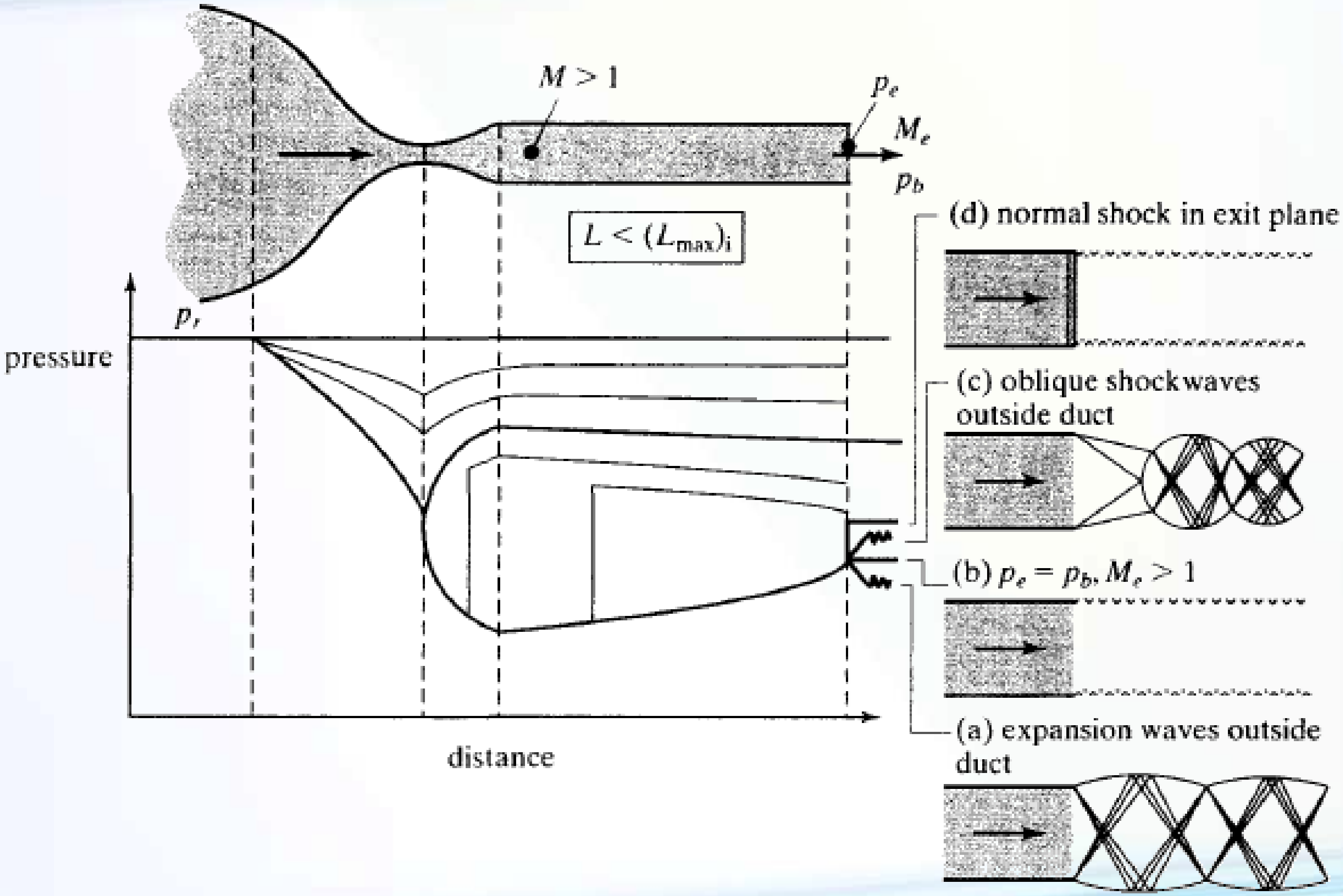
C-D NOZZLE AND CONSTANT-AREA DUCT SUPERSONIC DUCT INLET / $L < L_{MAX}$

- $p_b < p_{e,f}$: Expansion outside duct (underexpanded). the exit Mach number must be either supersonic or unity.
- $p_{e,f} < p_b < p_{e,sh}$: oblique shocks outside duct (overexpanded)
- $p_{e,sh} < p_b$: Shock waves inside duct. For a high-enough back pressure, the shock moves into the nozzle, thus eliminating supersonic flow in the duct.





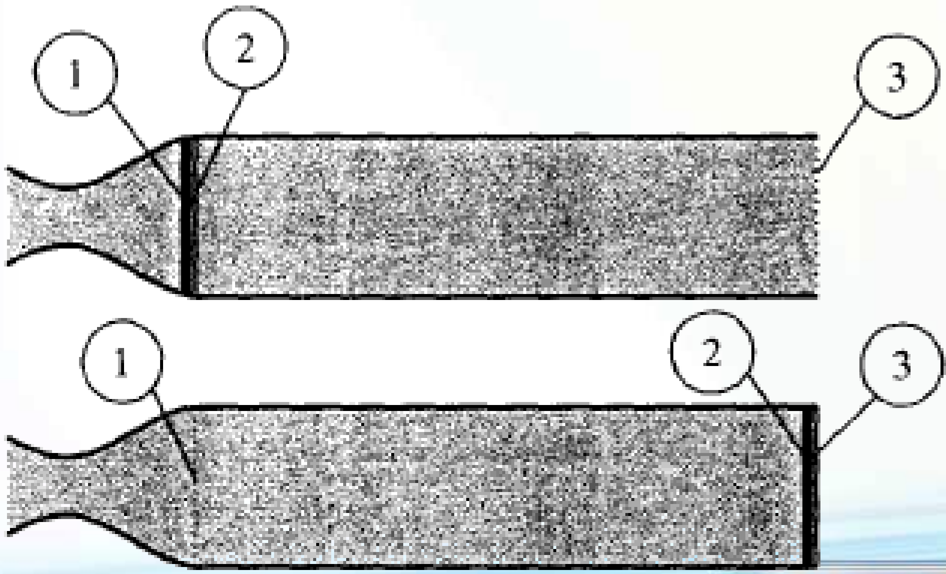
C-D NOZZLE AND CONSTANT-AREA DUCT
SUPERSONIC DUCT INLET / $L < L_{MAX}$



EXAMPLE

A converging-diverging nozzle with area ratio of 2 to 1 is supplied by a reservoir containing air at 500 kPa. The nozzle exhausts into a constant-area duct of length-to-diameter ratio of 10 and friction coefficient f of 0.02.

Determine the range of system back pressures over which a normal shock appears in the duct. Assume isentropic flow in the nozzle and Fanno flow in the duct. Assume that γ =1.4.



EXAMPLE (CONT.)

$$(A/A^*)_1 = 2.0, \longrightarrow M_1 = 2.1972. \qquad (fL_{max}/D)_1 = 0.3602.$$

$$fL/D = 0.20, \longrightarrow L < (L_{max})_1$$

Shock at the duct inlet

$$M_1 = 2.1972. \longrightarrow M_2 = 0.5474. \longrightarrow (fL_{max}/D)_2 = 0.7429.$$

$$\left(\frac{fL_{max}}{D}\right)_3 = \left(\frac{fL_{max}}{D}\right)_2 - \frac{fL}{D} = 0.7429 - 0.20 = 0.5427 \longrightarrow M_3 = 0.5874.$$

$$p_b = p_3 = \left(\frac{p_3}{p^*}\right)\left(\frac{p^*}{p_2}\right)\left(\frac{p_2}{p_1}\right)\left(\frac{p_1}{p_{o1}}\right)\left(\frac{p_{o1}}{p_r}\right)p_r$$

$$= (1.8071)\left(\frac{1}{1.9438}\right)(5.4656)(0.09393)(1)500 = 238.5961\ \text{kPa}$$

EXAMPLE (CONT.)

Shock at the duct exit

$M_1 = 2.1972. \longrightarrow (fL_{\max}/D)_1 = 0.3602.$

$$\left(\frac{fL_{\max}}{D}\right)_2 = \left(\frac{fL_{\max}}{D}\right)_1 - \frac{fL}{D} = 0.3602 - 0.20 = 0.1602$$

$M_2 = 1.5663. \longrightarrow$ From the normal-shock relations: $p_3/p_2 = 2.6955.$

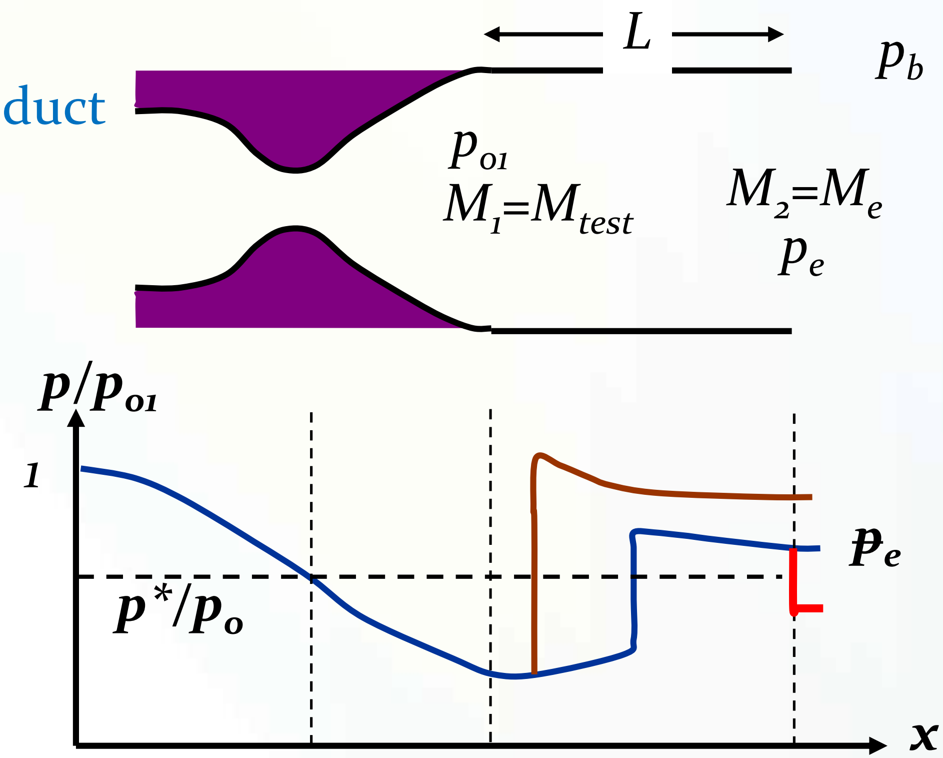
$$p_b = p_3 = \left(\frac{p_3}{p_2}\right)\left(\frac{p_2}{p^*}\right)\left(\frac{p^*}{p_1}\right)\left(\frac{p_1}{p_{o1}}\right)\left(\frac{p_{o1}}{p_r}\right)p_r$$

$$= (2.6955)\left(\frac{0.5728}{0.3556}\right)(0.09393)(1)500 = 203.9177 \text{ kPa}$$

$$\boxed{203.918 \text{ kPa} < p_b < 238.596 \text{ kPa}}$$

C-D NOZZLE AND CONSTANT-AREA DUCT SUPERSONIC DUCT INLET / L > L$_{MAX}$

- Supersonic flow + ($L > L_{max}$)
  $\Rightarrow$ a normal shock occurs in the duct

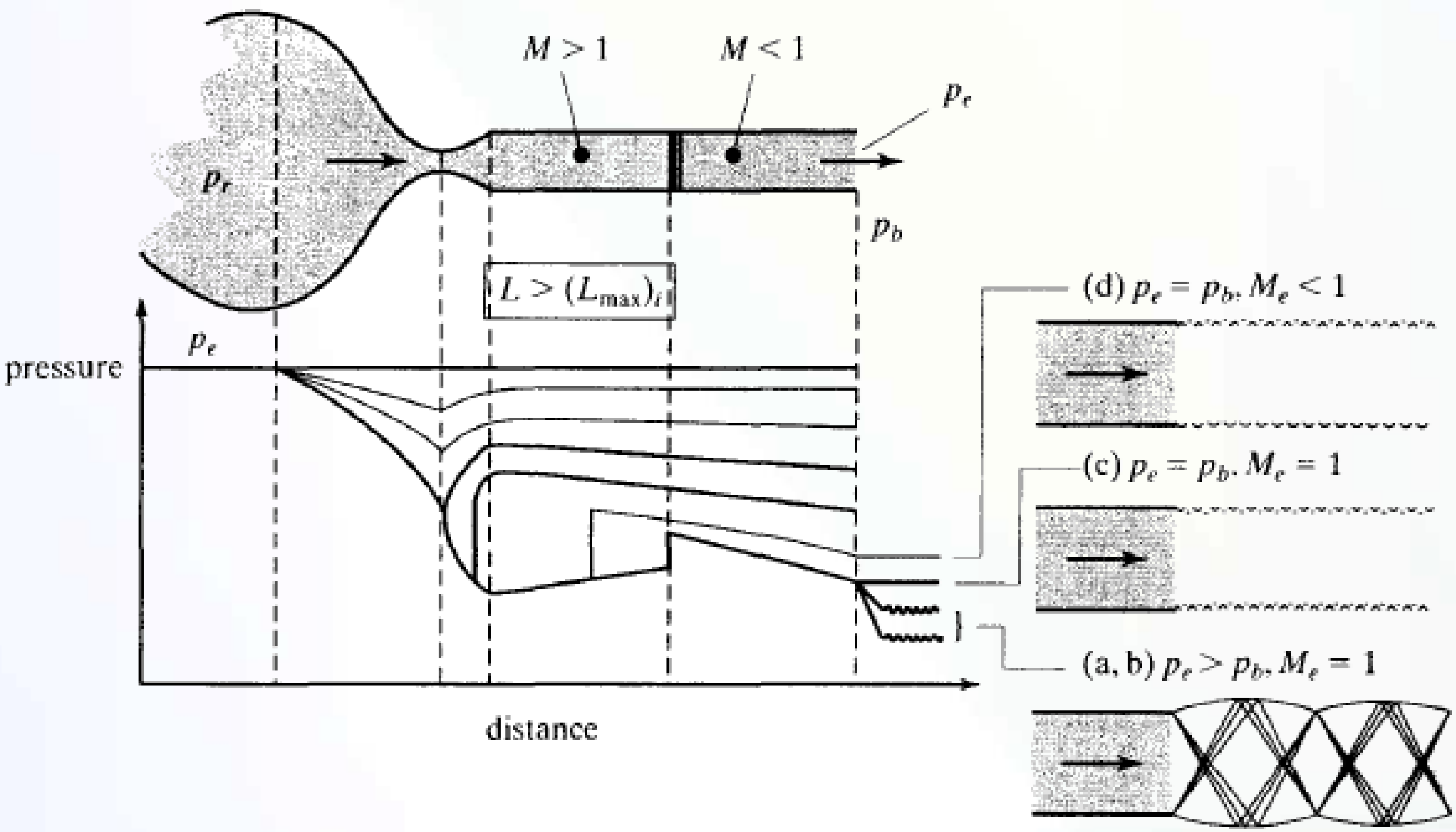


(Shock location is determined by back pressure)

- ○ $p_b < p_e$: Expansion waves must occur at the duct exit (with the exit-plane Mach number equal to unity).
- ○ $p_b > p_e$: The normal shock moves upstream toward the duct inlet, with the exit Mach number subsonic and the back pressure equal to the exitplane pressure.

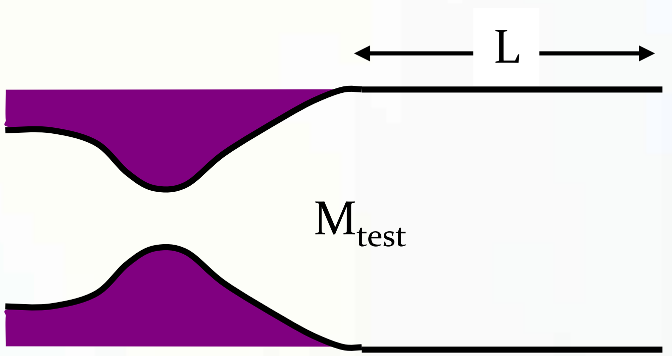
For a high-enough back pressure, the shock moves into the nozzle, thus eliminating supersonic flow in the duct.

C-D NOZZLE AND CONSTANT-AREA DUCT SUPERSONIC DUCT INLET / L > L_{MAX}



EXAMPLE

- **Given:** Exit of supersonic nozzle connected to straight walled test section. Test section flows N_2 at $M_{test} = 3.0$, $T_o = 290.$ K, p_o=500. kPa,
- L = 1m, D = 10 cm, f = 0.005



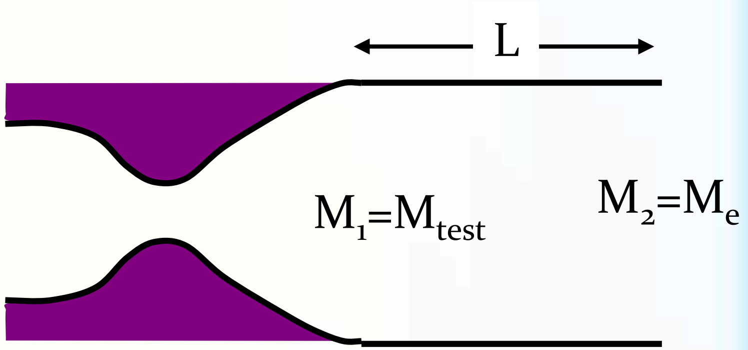
- **Find:**
  - M, T, p at end of test section
  - $p_{o,exit}/p_{o,inlet}$
  - L_{max} for test section

- **Assume:** N_2 is tpg/cpg, $\gamma = 1.4$, steady, adiabatic, no work

EXAMPLE (CONT.)

- **Analysis:**

❖ M_e



$$\frac{fL}{D} = \left.\frac{fL_{\max}}{D}\right)_{3.0} - \left.\frac{fL_{\max}}{D}\right)_{M_e}$$

$$\left.\frac{fL_{\max}}{D}\right)_{M_e} = 0.5222 - \frac{0.005(1)}{0.1} = 0.4722$$

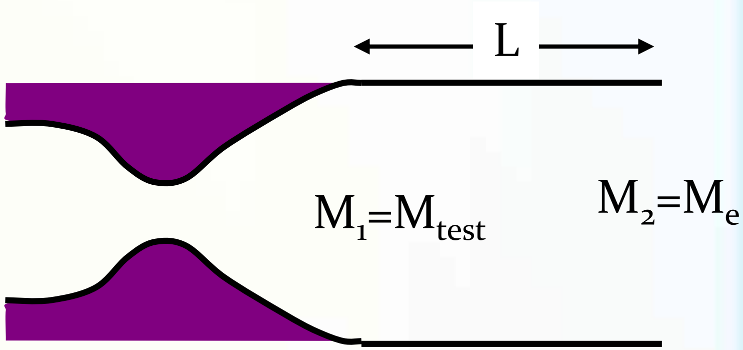
(Appendix) $\boxed{M_e = 2.70}$

another solution is M=0.605, but since started M>1, can't be subsonic

❖ T

(T_o const)

$$T_2 = T_1 \frac{1 + \frac{\gamma - 1}{2} M_1^2}{1 + \frac{\gamma - 1}{2} M_2^2} = \frac{T_o}{1 + \frac{\gamma - 1}{2} M_2^2} = \boxed{118\,\mathrm{K}}$$



❖ *p*

$$p_2 = p_1 \frac{M_1}{M_2}\sqrt{\frac{T_2}{T_1}}$$

$$p_1 = p_{o1}\left(1+\frac{\gamma-1}{2}M_1^2\right)^{-\gamma/\gamma-1}$$

$$= \frac{500\,\text{kPa}}{2.8^{3.5}} = 13.6\,\text{kPa}$$

$$\frac{T_2}{T_1} = \frac{1+((\gamma-1)/2)M_1^2}{1+((\gamma-1)/2)M_2^2} = 1.14$$

$$p_2 = 13.6\frac{3.0}{2.7}\sqrt{1.14} = 16.1\,\text{kPa}$$

❖ $p_{o,e}/p_{o,test}$

$$\frac{p_{o2}}{p_{o1}} = \frac{M_1}{M_2}\left(\frac{T_2}{T_1}\right)^{\frac{\gamma+1}{2(1-\gamma)}} = \frac{3.0}{2.7}(1.14)^{-3} = 0.75$$

25% loss in stagnation pressure due to friction

EXAMPLE (CONT.)

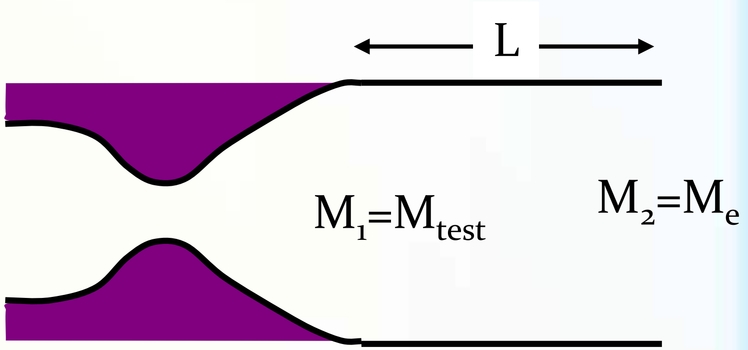
❖ L_{max}

$$L_{\max} = \left.\frac{fL_{\max}}{D}\right)_{M_{test}} \frac{D}{f}$$

$$= 0.5222\,\frac{0.1m}{0.005}$$

$$= \boxed{10.4m}$$

10.4 m long section
would have M=1 at exit



Compressible flows

Differential Approach

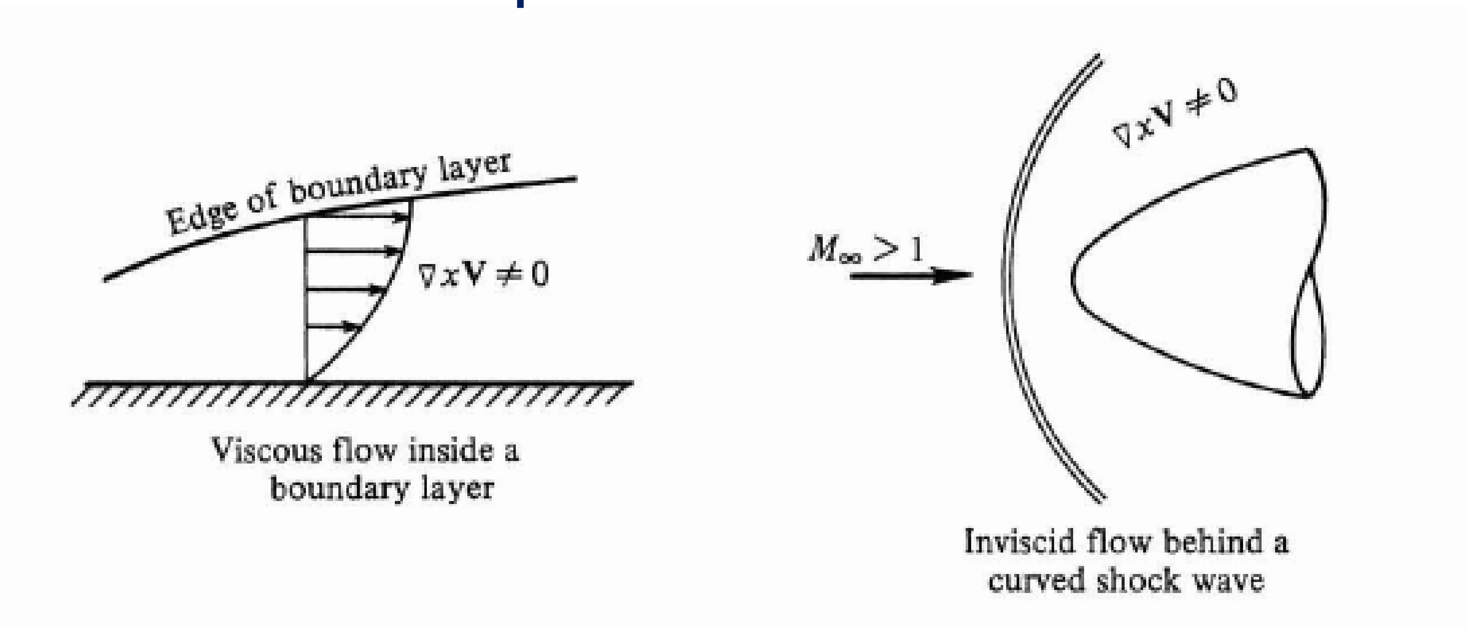
Velocity Potential Equation

IRROTATIONAL FLOW

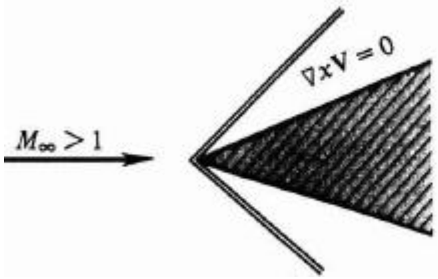
Vorticity is twice the angular velocity of a fluid element $\nabla \times \mathbf{V} = 2\boldsymbol{\omega}$

A flow where $\nabla \times \mathbf{V} \neq 0$ throughout is called a Rotational flow

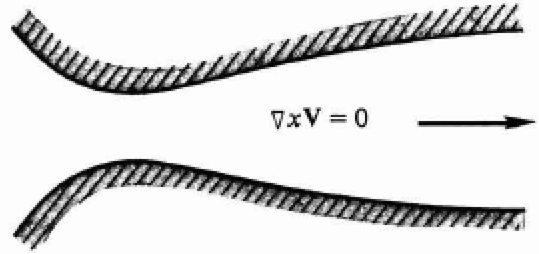
Examples of rotational flows.



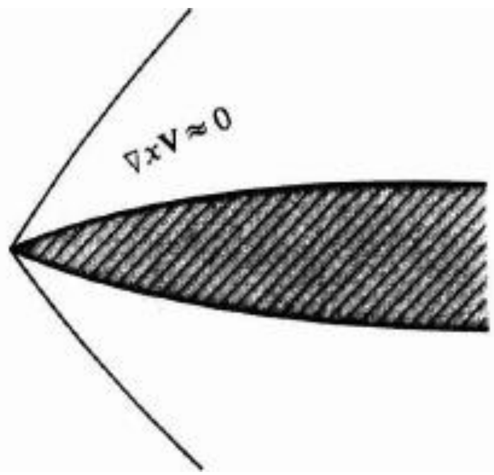
Examples of irrotational flows.



Flowfield over a sharp wedge or cone



Two-dimensional or axisymmetric nozzle flows



Flowfield behind the shock wave on a *slender, sharpnosed* body is almost irrotational. For analysis, we usually assume $\nabla x \mathbf{V} = 0$ for this case.

Consider an irrotational flow in more detail

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - \mathbf{j}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

$$\boxed{\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \qquad \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z} \qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}}$$

irrotationality conditions

Consider Euler's equation without body forces.

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x}$$

$$\Rightarrow \quad -\frac{\partial p}{\partial x}\,dx = \rho u \frac{\partial u}{\partial x}\,dx + \rho v \frac{\partial u}{\partial y}\,dx + \rho w \frac{\partial u}{\partial z}\,dx$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$$

$$\Rightarrow \quad -\frac{\partial p}{\partial x}\,dx = \rho u \frac{\partial u}{\partial x}\,dx + \rho v \frac{\partial v}{\partial x}\,dx + \rho w \frac{\partial w}{\partial x}\,dx$$

$$-\frac{\partial p}{\partial x}\,dx = \frac{1}{2}\rho \frac{\partial u^2}{\partial x}\,dx + \frac{1}{2}\rho \frac{\partial v^2}{\partial x}\,dx + \frac{1}{2}\rho \frac{\partial w^2}{\partial x}\,dx$$

Similarly

$$-\frac{\partial p}{\partial x}\,dx = \frac{1}{2}\rho\frac{\partial u^2}{\partial x}\,dx + \frac{1}{2}\rho\frac{\partial v^2}{\partial x}\,dx + \frac{1}{2}\rho\frac{\partial w^2}{\partial x}\,dx$$

$$-\frac{\partial p}{\partial y}\,dy = \frac{1}{2}\rho\frac{\partial u^2}{\partial y}\,dy + \frac{1}{2}\rho\frac{\partial v^2}{\partial y}\,dy + \frac{1}{2}\rho\frac{\partial w^2}{\partial y}\,dy$$

$$-\frac{\partial p}{\partial z}\,dz = \frac{1}{2}\rho\frac{\partial u^2}{\partial z}\,dz + \frac{1}{2}\rho\frac{\partial v^2}{\partial z}\,dz + \frac{1}{2}\rho\frac{\partial w^2}{\partial z}\,dz$$

$$-\left(\frac{\partial p}{\partial x}\,dx + \frac{\partial p}{\partial y}\,dy + \frac{\partial p}{\partial z}\,dz\right) = \frac{1}{2}\rho\frac{\partial V^2}{\partial x}\,dx + \frac{1}{2}\rho\frac{\partial V^2}{\partial y}\,dy + \frac{1}{2}\rho\frac{\partial V^2}{\partial z}\,dz$$

where $V^2 = u^2 + v^2 + w^2$

$$-\left(\frac{\partial p}{\partial x}\,dx + \frac{\partial p}{\partial y}\,dy + \frac{\partial p}{\partial z}\,dz\right) = \frac{1}{2}\rho\frac{\partial V^2}{\partial x}\,dx + \frac{1}{2}\rho\frac{\partial V^2}{\partial y}\,dy + \frac{1}{2}\rho\frac{\partial V^2}{\partial z}\,dz$$

$$-dp = \tfrac{1}{2}\rho\,d(V^2)$$

The velocity potential equation

Consider a vector $\mathbf{A}$. If $\nabla \times \mathbf{A} = 0$ everywhere, then $\mathbf{A}$ can always be expressed as $\nabla\zeta$, where ζ is a scalar function. This stems directly from the vector identity, curl (grad) $\equiv 0$. Hence,

$$\nabla \times \nabla\zeta = 0$$

where ζ is any scalar function.

For irrotational flow, $\nabla \times \mathbf{V} = 0$

Hence, we can define a scalar function $\Phi = \Phi(x, y, z)$

$$\mathbf{V} \equiv \nabla\Phi$$

where Φ, is called the velocity potential.

$$\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$$

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\mathbf{i} + \frac{\partial\Phi}{\partial y}\mathbf{j} + \frac{\partial\Phi}{\partial z}\mathbf{k}$$

$$u = \frac{\partial\Phi}{\partial x} \quad v = \frac{\partial\Phi}{\partial y} \quad w = \frac{\partial\Phi}{\partial z}$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

$$\frac{\partial}{\partial x}\rho\mathbf{\Phi}_x + \frac{\partial}{\partial y}\rho\mathbf{\Phi}_y + \frac{\partial}{\partial z}\rho\mathbf{\Phi}_z = 0$$

$$\rho(\mathbf{\Phi}_{xx} + \mathbf{\Phi}_{yy} + \mathbf{\Phi}_{zz}) + \mathbf{\Phi}_x\frac{\partial\rho}{\partial x} + \mathbf{\Phi}_y\frac{\partial\rho}{\partial y} + \mathbf{\Phi}_z\frac{\partial\rho}{\partial z} = 0$$

we eliminate ρ

$$dp = -\rho V\,dV = -\frac{\rho}{2}\,d(V^2) = -\frac{\rho}{2}\,d(u^2 + v^2 + w^2)$$

$$\boxed{dp = -\rho\,d\left(\frac{\mathbf{\Phi}_x^2 + \mathbf{\Phi}_y^2 + \mathbf{\Phi}_z^2}{2}\right)}$$

Recalling that the flow is isentropic, $\Rightarrow$ $\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s = a^2$ $\Rightarrow$ $\boxed{d\rho = \frac{dp}{a^2}}$

$$d\rho = -\frac{\rho}{a^2}\,d\left(\frac{\mathbf{\Phi}_x^2 + \mathbf{\Phi}_y^2 + \mathbf{\Phi}_z^2}{2}\right)$$

$$\rho(\mathbf{\Phi}_{xx} + \mathbf{\Phi}_{yy} + \mathbf{\Phi}_{zz}) + \mathbf{\Phi}_x \boxed{\frac{\partial \rho}{\partial x}} + \mathbf{\Phi}_y \boxed{\frac{\partial \rho}{\partial y}} + \mathbf{\Phi}_z \boxed{\frac{\partial \rho}{\partial z}} = 0$$

$$d\rho = -\frac{\rho}{a^2}\, d\left(\frac{\mathbf{\Phi}_x^2 + \mathbf{\Phi}_y^2 + \mathbf{\Phi}_z^2}{2}\right)$$

$$\frac{\partial \rho}{\partial x} = -\frac{\rho}{a^2}\frac{\partial}{\partial x}\left(\frac{\mathbf{\Phi}_x^2 + \mathbf{\Phi}_y^2 + \mathbf{\Phi}_z^2}{2}\right)$$

$$\frac{\partial \rho}{\partial x} = -\frac{\rho}{a^2}\left(\mathbf{\Phi}_x\mathbf{\Phi}_{xx} + \mathbf{\Phi}_y\mathbf{\Phi}_{yx} + \mathbf{\Phi}_z\mathbf{\Phi}_{zx}\right)$$

$$\frac{\partial \rho}{\partial y} = -\frac{\rho}{a^2}(\mathbf{\Phi}_x\mathbf{\Phi}_{xy} + \mathbf{\Phi}_y\mathbf{\Phi}_{yy} + \mathbf{\Phi}_z\mathbf{\Phi}_{zy})$$

$$\frac{\partial \rho}{\partial z} = -\frac{\rho}{a^2}(\mathbf{\Phi}_x\mathbf{\Phi}_{xz} + \mathbf{\Phi}_y\mathbf{\Phi}_{yz} + \mathbf{\Phi}_z\mathbf{\Phi}_{zz})$$

$$\left(1-\frac{\Phi_x^2}{a^2}\right)\Phi_{xx}+\left(1-\frac{\Phi_y^2}{a^2}\right)\Phi_{yy}+\left(1-\frac{\Phi_z^2}{a^2}\right)\Phi_{zz}$$
$$-\frac{2\Phi_x\Phi_y}{a^2}\Phi_{xy}-\frac{2\Phi_x\Phi_z}{a^2}\Phi_{xz}-\frac{2\Phi_y\Phi_z}{a^2}\Phi_{yz}=0$$

velocity potential equation.

From the energy equation $h_o = \text{const}$

$$c_pT+\frac{V^2}{2}=c_pT_o \qquad \frac{\gamma RT}{\gamma-1}+\frac{V^2}{2}=\frac{\gamma RT_o}{\gamma-1} \qquad \frac{a^2}{\gamma-1}+\frac{V^2}{2}=\frac{a_o^2}{\gamma-1}$$

$$a^2=a_o^2-\frac{\gamma-1}{2}V^2=a_o^2-\frac{\gamma-1}{2}(u^2+v^2+w^2)$$

$$\boxed{a^2=a_o^2-\frac{\gamma-1}{2}\left(\Phi_x^2+\Phi_y^2+\Phi_z^2\right)}$$

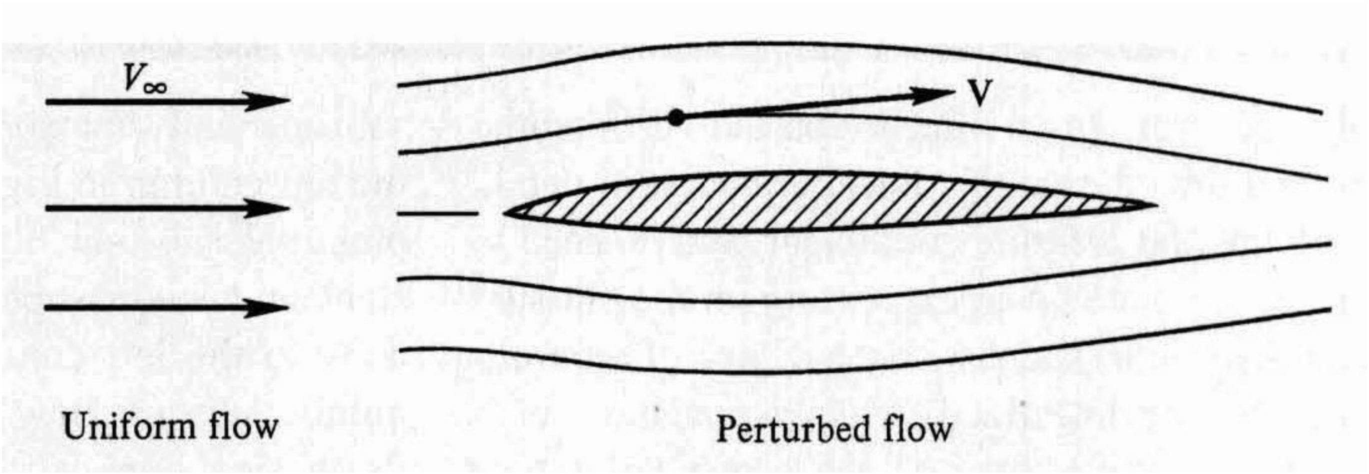
Note that the velocity potential equation is a *nonlinear partial differential* equation. It applies to any irrotational, isentropic flow: subsonic, transonic, supersonic, or hypersonic

It also applies to incompressible flow, where $a \longrightarrow \infty$, *hence* yielding the familiar Laplace's equation,

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0$$

There is no general closed-form solution to the velocity potential equation,

Linearized Flow



$$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k},$$

u', v', and w' are the *perturbation velocities* in the x, y, and z directions

$$V_x = V_\infty + u'$$
$$V_y = v'$$
$$V_z = w'$$

In terms of the velocity potential,

$$\nabla\Phi = \mathbf{V} = (V_\infty + u')\mathbf{i} + v'\mathbf{j} + w'\mathbf{k}$$

$$V_x = V_\infty + u' = \frac{\partial \Phi}{\partial x} = V_\infty + \frac{\partial \phi}{\partial x}$$

$$V_y = v' = \frac{\partial \Phi}{\partial y} = \frac{\partial \phi}{\partial y}$$

$$V_z = w' = \frac{\partial \Phi}{\partial z} = \frac{\partial \phi}{\partial z}$$

$$\Phi_{xx} = \frac{\partial^2 \phi}{\partial x^2}$$

$$\Phi_{yy} = \frac{\partial^2 \phi}{\partial y^2}$$

$$\Phi_{zz} = \frac{\partial^2 \phi}{\partial z^2}$$

Inserting in velocity potential equation:

$$\left[a^2 - \left(V_\infty + \frac{\partial \phi}{\partial x}\right)^2\right]\frac{\partial^2 \phi}{\partial x^2} + \left[a^2 - \left(\frac{\partial \phi}{\partial y}\right)^2\right]\frac{\partial^2 \phi}{\partial y^2} + \left[a^2 - \left(\frac{\partial \phi}{\partial z}\right)^2\right]\frac{\partial^2 \phi}{\partial z^2}$$
$$- 2\left(V_\infty + \frac{\partial \phi}{\partial x}\right)\frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial x\,\partial y} - 2\left(V_\infty + \frac{\partial \phi}{\partial x}\right)\frac{\partial \phi}{\partial z}\frac{\partial^2 \phi}{\partial x\,\partial z} - 2\frac{\partial \phi}{\partial y}\frac{\partial \phi}{\partial z}\frac{\partial^2 \phi}{\partial y\,\partial z} = 0$$

The above equation is called the perturbation-velocity potential equation

Since the total enthalpy is constant throughout the flow,

$$h_\infty + \frac{V_\infty^2}{2} = h + \frac{V^2}{2} = h + \frac{(V_\infty + u')^2 + v'^2 + w'^2}{2}$$

$$\frac{a_\infty^2}{\gamma - 1} + \frac{V_\infty^2}{2} = \frac{a^2}{\gamma - 1} + \frac{(V_\infty + u')^2 + v'^2 + w'^2}{2}$$

$$a^2 = a_\infty^2 - \frac{\gamma - 1}{2}(2u'V_\infty + u'^2 + v'^2 + w'^2)$$

$$
\begin{aligned}
&(1-M_\infty^2)\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\\
&\quad= M_\infty^2\left[(\gamma+1)\frac{u'}{V_\infty}+\left(\frac{\gamma+1}{2}\right)\frac{u'^2}{V_\infty^2}+\left(\frac{\gamma-1}{2}\right)\left(\frac{v'^2+w'^2}{V_\infty^2}\right)\right]\frac{\partial u'}{\partial x}\\
&\qquad+M_\infty^2\left[(\gamma-1)\frac{u'}{V_\infty}+\left(\frac{\gamma+1}{2}\right)\frac{v'^2}{V_\infty^2}+\left(\frac{\gamma-1}{2}\right)\left(\frac{w'^2+u'^2}{V_\infty^2}\right)\right]\frac{\partial v'}{\partial y}\\
&\qquad+M_\infty^2\left[(\gamma-1)\frac{u'}{V_\infty}+\left(\frac{\gamma+1}{2}\right)\frac{w'^2}{V_\infty^2}+\left(\frac{\gamma-1}{2}\right)\left(\frac{u'^2+v'^2}{V_\infty^2}\right)\right]\frac{\partial w'}{\partial z}\\
&\qquad+M_\infty^2\left[\frac{v'}{V_\infty}\left(1+\frac{u'}{V_\infty}\right)\left(\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}\right)+\frac{w'}{V_\infty}\left(1+\frac{u'}{V_\infty}\right)\left(\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x}\right)\right.\\
&\qquad\qquad\left.+\frac{u'w'}{V_\infty^2}\left(\frac{\partial w'}{\partial y}+\frac{\partial v'}{\partial z}\right)\right]
\end{aligned}
$$

We now specialize to the case of *small perturbations,* i.e., we assume the u', v', and w' are small compared to V_∞:

$$\frac{u'}{V_\infty},\ \frac{v'}{V_\infty},\ \text{and}\ \frac{w'}{V_\infty} \ll 1 \qquad \left(\frac{u'}{V_\infty}\right)^2,\ \left(\frac{v'}{V_\infty}\right)^2,\ \text{and}\ \left(\frac{w'}{V_\infty}\right)^2 \lll 1$$

1. For $0 \le M_\infty \le 0.8$ and for $M_\infty \ge 1.2$, the magnitude of

$$M_\infty^2 \left[(\gamma + 1)\frac{u'}{V_\infty} + \cdots\right] \frac{\partial u'}{\partial x}$$

is small in comparison to the magnitude of

$$(1 - M_\infty^2)\,\frac{\partial u'}{\partial x}$$

Thus, *ignore* the former term.

2. For $M_\infty \le 5$ (approximately),

$$M_\infty^2 \left[(\gamma - 1) \frac{u'}{V_\infty} + \cdots \right] \frac{\partial v'}{\partial y}$$

is small in comparison to $\partial v'/\partial y$,

$$M_\infty^2 \left[(\gamma - 1) \frac{u'}{V_\infty} + \cdots \right] \frac{\partial w'}{\partial z}$$

is small in comparison to $\partial w'/\partial z$, and

$$M_\infty^2 \left[\frac{v'}{V_\infty} \left(1 + \frac{u'}{V_\infty} \right) \left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} \right) + \cdots \right] \approx 0$$

$$\boxed{(1 - M_\infty^2) \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0}$$

$$\boxed{\left(1 - M_\infty^2\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0}$$

1. The perturbations must be *small*.
2. From item 1 in the list above, we see that *transonic flow* ($0.8 \leq M_\infty \leq 1.2$) is excluded.
3. From item 2 in that same list we see that *hypersonic flow* ($M_\infty \geq 5$) is excluded.

LINEARIZED PRESSURE COEFFICIENT

$$C_p \equiv \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2}$$

$$\frac{1}{2}\rho_\infty V_\infty^2 = \frac{1}{2}\frac{\gamma p_\infty}{\gamma p_\infty}\rho_\infty V_\infty^2 = \frac{\gamma}{2} p_\infty \frac{V_\infty^2}{a_\infty^2} = \frac{\gamma}{2} p_\infty M_\infty^2$$

$$C_p = \frac{p - p_\infty}{(\gamma/2) p_\infty M_\infty^2} = \frac{p_\infty (p/p_\infty - 1)}{(\gamma/2) p_\infty M_\infty^2}$$

$$\boxed{C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)}$$

$$h + \frac{V^2}{2} = h_\infty + \frac{V_\infty^2}{2}$$

$$T + \frac{V^2}{2c_p} = T_\infty + \frac{V_\infty^2}{2c_p}$$

$$T - T_\infty = \frac{V_\infty^2 - V^2}{2c_p} = \frac{V_\infty^2 - V^2}{2\gamma R/(\gamma - 1)}$$

$$\frac{T}{T_\infty} - 1 = \frac{\gamma - 1}{2} \frac{V_\infty^2 - V^2}{\gamma R T_\infty} = \frac{\gamma - 1}{2} \frac{V_\infty^2 - V^2}{a_\infty^2}$$

$$V^2 = (V_\infty + u')^2 + v'^2 + w'^2$$

$$\frac{T}{T_\infty} = 1 - \frac{\gamma - 1}{2a_\infty^2}(2u'V_\infty + u'^2 + v'^2 + w'^2)$$

$$\frac{T}{T_\infty} = 1 - \frac{\gamma - 1}{2a_\infty^2}(2u'V_\infty + u'^2 + v'^2 + w'^2)$$

$$p/p_\infty = (T/T_\infty)^{\gamma/(\gamma-1)}$$

$$\frac{p}{p_\infty} = \left[1 - \frac{\gamma - 1}{2a_\infty^2}(2u'V_\infty + u'^2 + v'^2 + w'^2)\right]^{\gamma/(\gamma-1)}$$

$$u'/V_\infty \ll 1: u'^2/V_\infty,\ v'^2/V_\infty^2, \text{ and } w'^2/V_\infty^2 \lll 1$$

$$\frac{p}{p_\infty} = (1 - \varepsilon)^{\gamma/(\gamma-1)}$$

Where *ε is small.*

From the binomial expansion, neglecting higher-order terms,

$$\frac{p}{p_\infty} = 1 - \frac{\gamma}{\gamma - 1}\varepsilon + \cdots$$

$$\frac{p}{p_\infty} = 1 - \frac{\gamma}{2} M_\infty^2 \left(\frac{2u'}{V_\infty} + \frac{u'^2 + v'^2 + w'^2}{V_\infty^2} \right) + \cdots$$

$$C_p = \frac{2}{\gamma M_\infty^2} \left(\frac{p}{p_\infty} - 1 \right)$$

$$C_p = \frac{2}{\gamma M_\infty^2} \left[1 - \frac{\gamma}{2} M_\infty^2 \left(\frac{2u'}{V_\infty} + \frac{u'^2 + v'^2 + w'^2}{V_\infty^2} \right) + \cdots - 1 \right]$$
$$= -\frac{2u'}{V_\infty} - \frac{u'^2 + v'^2 + w'^2}{V_\infty^2} + \cdots$$

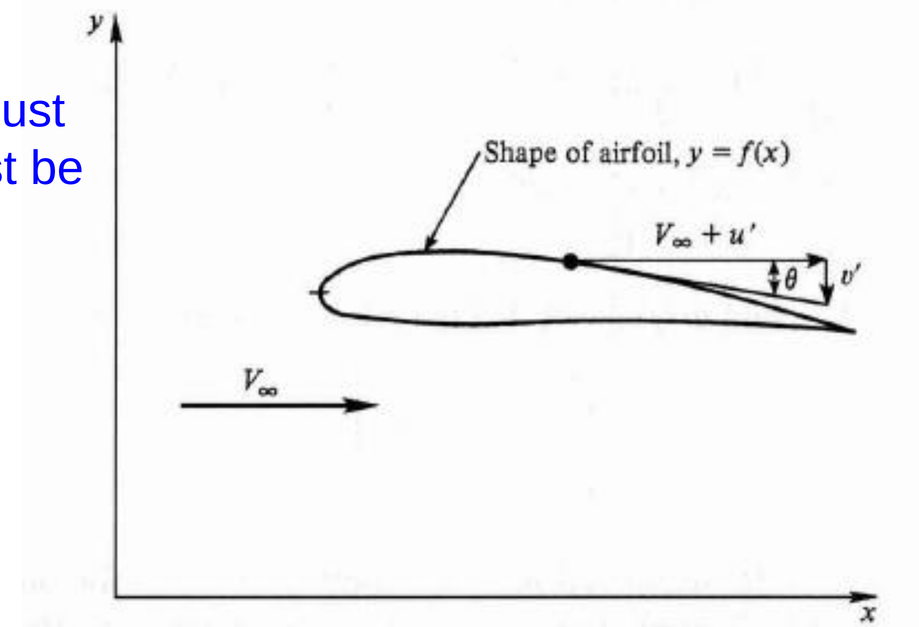
Since u'^2/V_∞^2, v'^2/V_∞^2, and $w'^2/V_\infty^2 \lll 1$,

$$\boxed{C_p = -\frac{2u'}{V_\infty}}$$

***linearized pressure coefficient*,**

LINEARIZED SUBSONIC FLOW

Consider the compressible subsonic flow over a thin airfoil at small angle of attack (hence small perturbations),



The usual inviscid flow boundary condition must hold at the surface, i.e., the flow velocity must be tangent to the surface.

$$\frac{df}{dx} = \frac{v'}{V_\infty + u'} = \tan\theta$$

For small perturbations, $u' \ll V_\infty$, and $\tan\theta \approx \theta$

$$\frac{df}{dx} = \frac{v'}{V_\infty} = \theta$$

Since $v' = \partial\phi/\partial y$

$$\frac{\partial\phi}{\partial y} = V_\infty \frac{df}{dx}$$

The subsonic compressible flow over the airfoil is governed by the linearized perturbation-velocity potential equation .
For two-dimensional flow, this becomes

$$\beta^2 \phi_{xx} + \phi_{yy} = 0 \quad \text{where } \beta \equiv \sqrt{1 - M_\infty^2}.$$

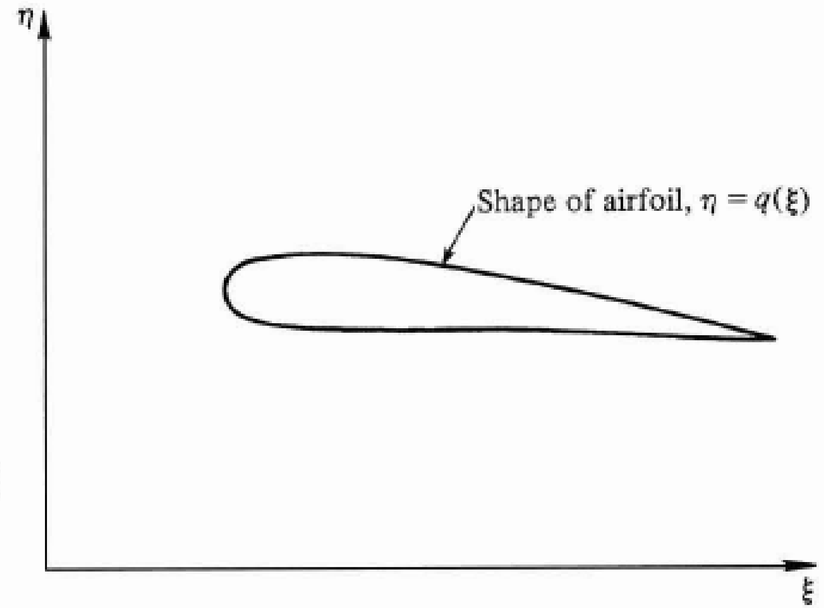
It can be transformed to a familiar incompressible form by considering a transformed coordinate system (ξ, η)

$$\xi = x$$

$$\eta = \beta y$$

$$\bar{\phi}(\xi, \eta) = \beta \phi(x, y)$$

$$\frac{\partial \xi}{\partial x} = 1 \qquad \frac{\partial \xi}{\partial y} = 0 \qquad \frac{\partial \eta}{\partial x} = 0 \qquad \frac{\partial \eta}{\partial y} = \beta$$



$$\phi_x = \frac{\partial \phi}{\partial x} = \frac{1}{\beta}\frac{\partial \bar{\phi}}{\partial x} = \frac{1}{\beta}\left[\frac{\partial \bar{\phi}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \bar{\phi}}{\partial \eta}\frac{\partial \eta}{\partial x}\right] = \frac{1}{\beta}\frac{\partial \bar{\phi}}{\partial \xi} = \frac{\bar{\phi}_\xi}{\beta}$$

$$\phi_{xx} = \frac{1}{\beta}\bar{\phi}_{\xi\xi}$$

$$\phi_y = \frac{\partial \phi}{\partial y} = \frac{1}{\beta}\frac{\partial \bar{\phi}}{\partial y} = \frac{1}{\beta}\left[\frac{\partial \bar{\phi}}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \bar{\phi}}{\partial \eta}\frac{\partial \eta}{\partial y}\right] = \frac{\partial \bar{\phi}}{\partial \eta} = \bar{\phi}_\eta$$

$$\phi_{yy} = \bar{\phi}_{\eta\eta}\beta$$

$$\beta^2\phi_{xx} + \phi_{yy} = 0$$

$$\beta^2\left(\frac{1}{\beta}\bar{\phi}_{\xi\xi}\right) + \beta\bar{\phi}_{\eta\eta} = 0$$

$$\boxed{\bar{\phi}_{\xi\xi} + \bar{\phi}_{\eta\eta} = 0}$$

$$y = f(x) \Rightarrow \eta = q(\xi)$$

Boundary condition in (x, y)

$$V_\infty \frac{df}{dx} = \frac{\partial \phi}{\partial y} = \frac{1}{\beta}\frac{\partial \bar{\phi}}{\partial y} = \frac{\partial \bar{\phi}}{\partial \eta}$$

Boundary condition in (ξ, η)

$$V_\infty \frac{dq}{d\xi} = \frac{\partial \bar{\phi}}{\partial \eta}$$

$$\frac{df}{dx} = \frac{dq}{d\xi}$$

it demonstrates that the shape of the airfoil in (x, y) and (ξ,η) space is the same

Pressure coefficient

$$C_p = -\frac{2u'}{V_\infty} = -\frac{2}{V_\infty}\frac{\partial \phi}{\partial x} = -\frac{2}{V_\infty}\frac{1}{\beta}\frac{\partial \bar{\phi}}{\partial x} = -\frac{2}{V_\infty}\frac{1}{\beta}\frac{\partial \bar{\phi}}{\partial \xi}$$

Denoting the incompressible perturbation velocity in the ξ direction by $\bar{u}$, where $\bar{u} = \partial\bar{\phi}/\partial\xi$

$C_p = \frac{1}{\beta}\left(-\frac{2\bar{u}}{V_\infty}\right)$ Since (ξ, η) space corresponds to incompressible flow $-\frac{2\bar{u}}{V_\infty} = C_{p_o}$

where C_{p_o} is the *incompressible* pressure coefficient.

$$\boxed{C_p = \frac{C_{p_o}}{\sqrt{1 - M_\infty^2}}}$$

This is called the **Prandtl-Glauert rule**; it is a similarity rule which relates incompressible flow over a given two-dimensional profile to subsonic compressible flow over the same profile

$$C_L = \frac{L}{\frac{1}{2}\rho_\infty V_\infty^2 S}$$

$$C_M = \frac{M}{\frac{1}{2}\rho_\infty V_\infty^2 S l}$$

$$\boxed{C_L = \frac{C_{L_o}}{\sqrt{1 - M_\infty^2}}}$$

$$\boxed{C_M = \frac{C_{M_o}}{\sqrt{1 - M_\infty^2}}}$$

These are also called the Prandtl-Glauert rule

They are exceptionally practical aerodynamic formulas for the approximate compressibility correction to low-speed lift and moments on slender two-dimensional aerodynamic shapes.

An important effect of compressibility on subsonic flowfields can be seen by noting that

$$u' = \frac{\partial \phi}{\partial x} = \frac{1}{\beta}\frac{\partial \bar{\phi}}{\partial x} = \frac{1}{\beta}\frac{\partial \bar{\phi}}{\partial \xi} = \frac{\bar{u}}{\beta} = \frac{\bar{u}}{\sqrt{1 - M_\infty^2}}$$

IMPROVED COMPRESSIBILITY CORRECTIONS

The importance of accurate compressibility corrections reached new highs during the rapid increase in airplane speeds spurred by World War II.
Efforts were made to improve upon the Prandtl-Glauert rule discussed prevously the more popular formulas are given below.

In an effort to obtain an improved compressibility correction, Laitone applied locally in the flow, i.e.,

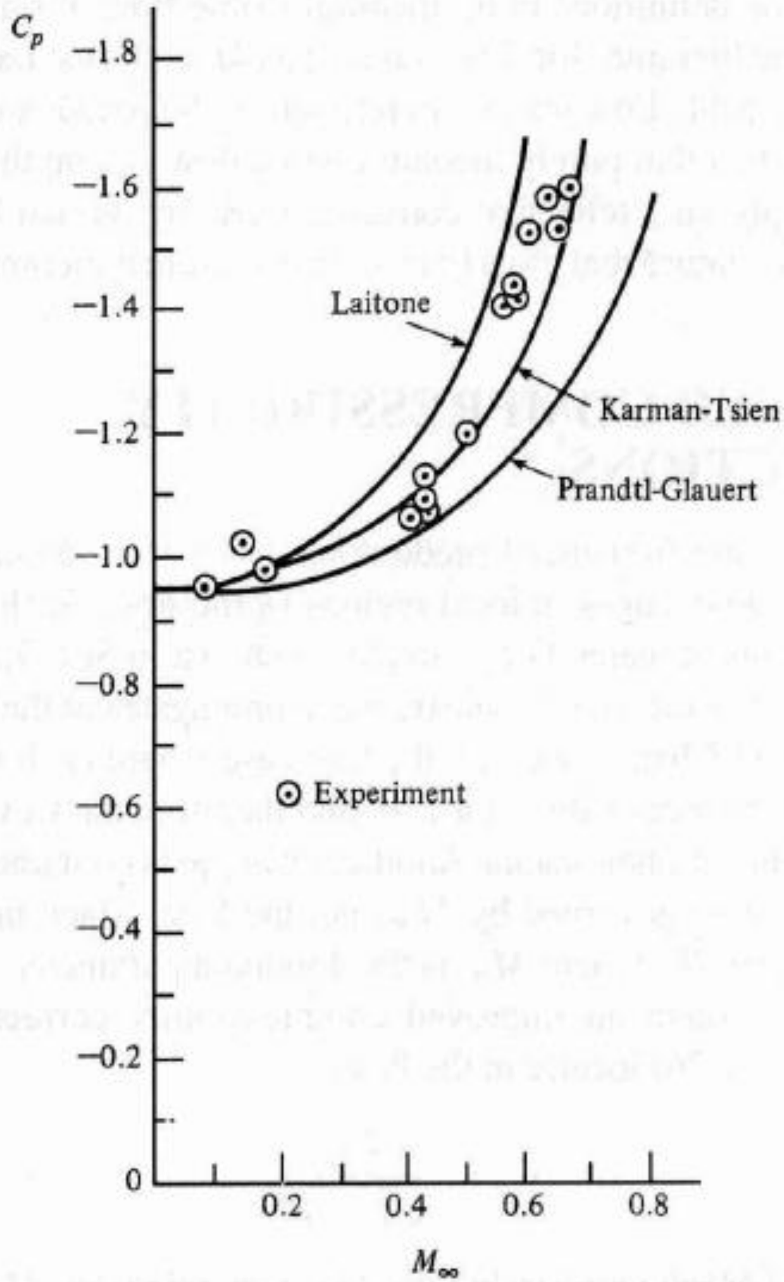
$$C_p = \frac{C_{p_o}}{\sqrt{1 - M^2}}$$

where M is the local Mach number. In turn, M can be related to M_∞ and the pressure coefficient through the isentropic flow relations.
The resulting compressibility correction is:

$$C_p = \frac{C_{p_o}}{\sqrt{1 - M_\infty^2} + \left[M_\infty^2 \left(1 + \frac{\gamma - 1}{2} M_\infty^2 \right) \Big/ 2\sqrt{1 - M_\infty^2} \right] C_{p_o}}$$

Another compressibility correction that has been adopted widely is that due to von Karman and Tsien

$$C_p = \frac{C_{p_o}}{\sqrt{1 - M_\infty^2} + \left(\frac{M_\infty^2}{1 + \sqrt{1 - M_\infty^2}} \right) \frac{C_{p_o}}{2}}$$



Comparison of several compressibility corrections with experiment for an NACA 4412 airfoil at an angle of attack α = 1◦53'.

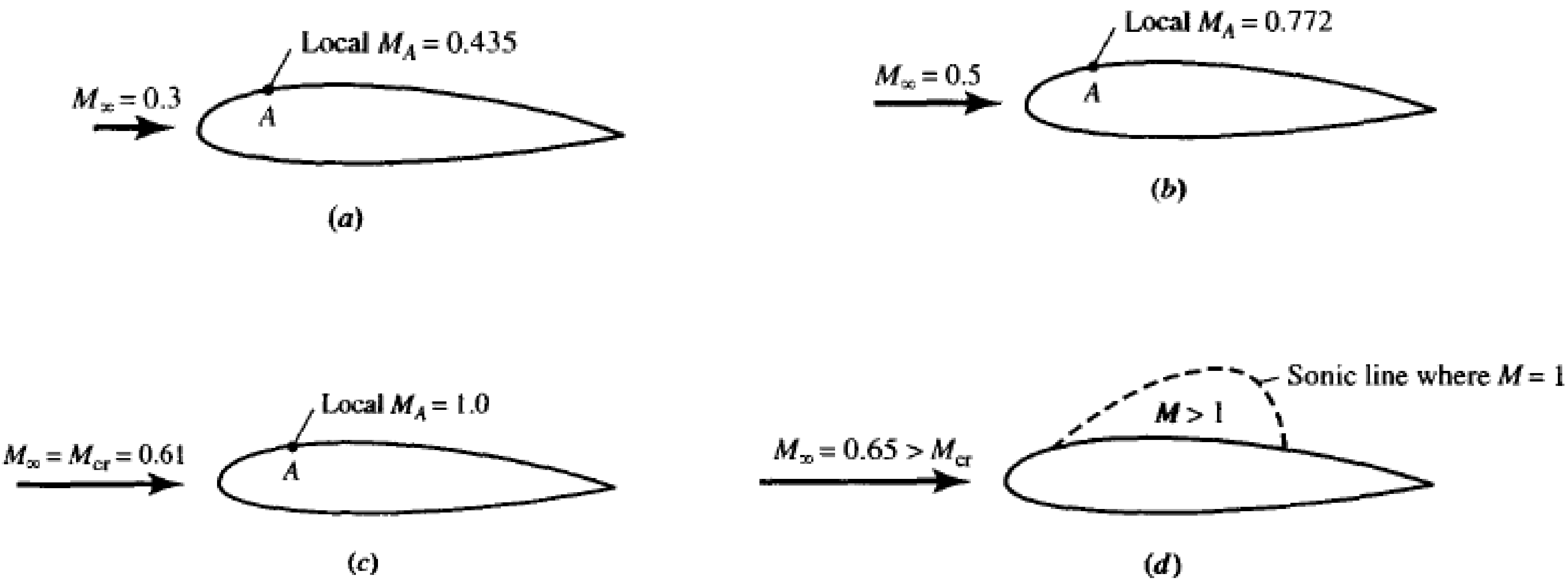
Note that the Prandtl-Glauert rule, although the simplest to apply, under predicts the experimental values, whereas the mproved compressibility corrections are clearly more accurate.

This is because both the Laitone and Karman-Tsien rules bring in the nonlinear aspects of the flow.

CRITICAL MACH NUMBER

We have now finished our discussion of linearized flow and the associated compressibility corrections. such linearized theory does not apply to the transonic flow regime

$$0.8 \le M_\infty \le 1.2.$$



Linearized flow will fail even for M∞ lower than 0.8

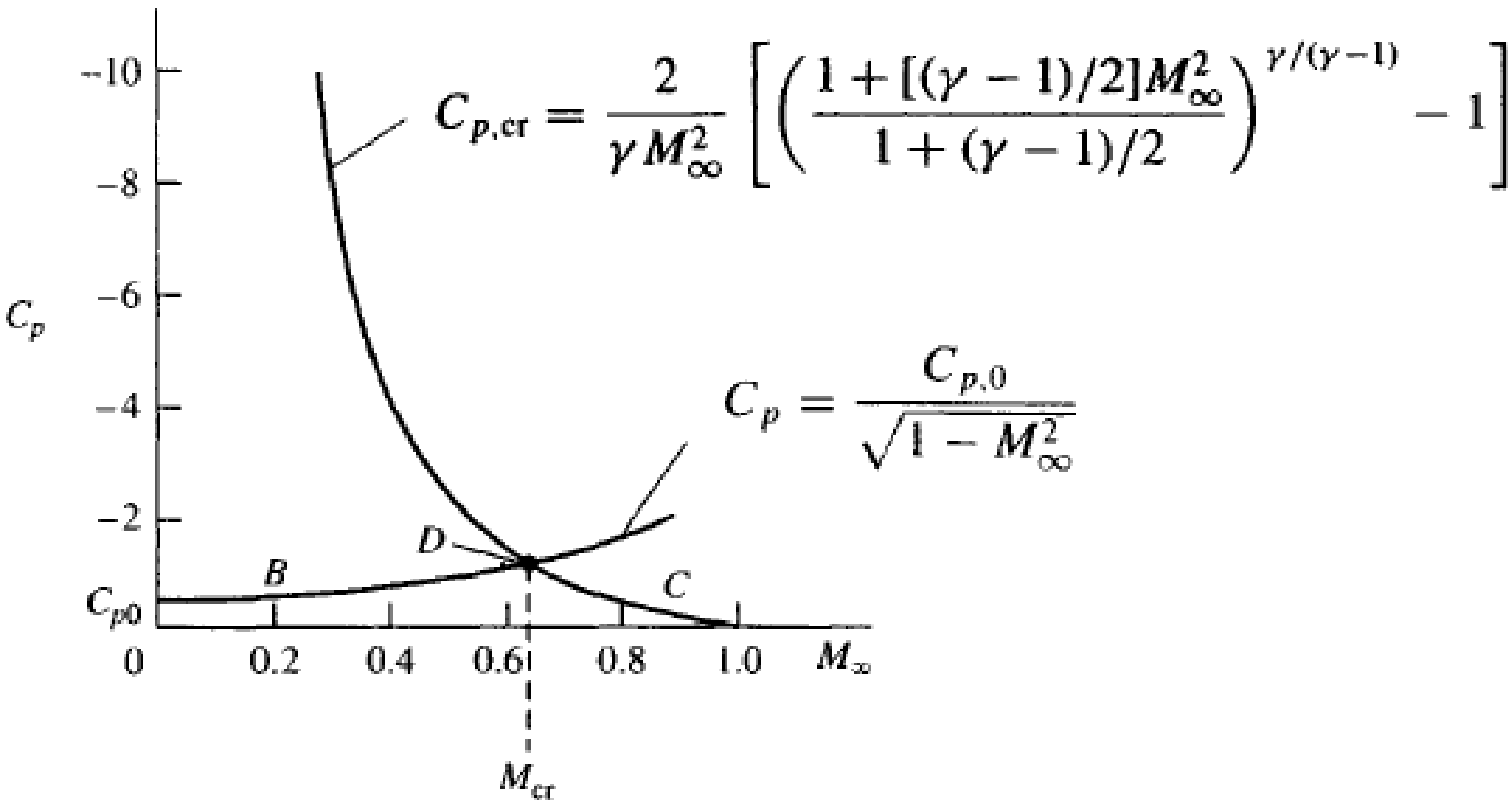
Let p_∞ and p_A represent the static pressures in the freestream and at point A

For isentropic flow,

$$\frac{p_A}{p_\infty} = \frac{p_A/p_0}{p_\infty/p_0} = \left(\frac{1+[(\gamma-1)/2]M_\infty^2}{1+[(\gamma-1)/2]M_A^2}\right)^{\gamma/(\gamma-1)}$$

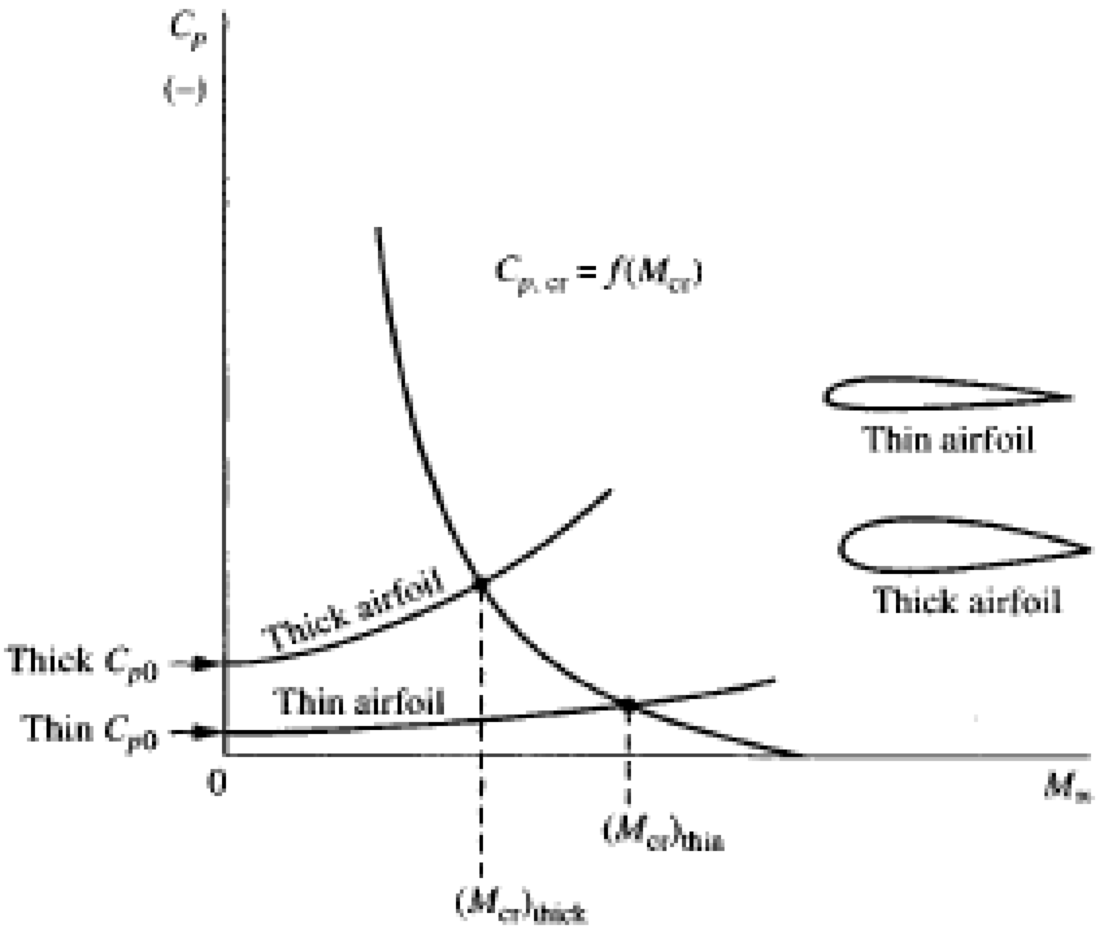
$$C_{p,A} = \frac{2}{\gamma M_\infty^2}\left(\frac{p_A}{p_\infty} - 1\right)$$

$$C_{p,A} = \frac{2}{\gamma M_\infty^2}\left[\left(\frac{1+[(\gamma-1)/2]M_\infty^2}{1+[(\gamma-1)/2]M_A^2}\right)^{\gamma/(\gamma-1)} - 1\right]$$



$C_{p,\mathrm{cr}} = \frac{2}{\gamma M_\infty^2}\left[\left(\frac{1+[(\gamma-1)/2]M_\infty^2}{1+(\gamma-1)/2}\right)^{\gamma/(\gamma-1)} - 1\right]$
$C_p = \frac{C_{p,0}}{\sqrt{1-M_\infty^2}}$
C_p
−10
−8
−6
−4
−2
C_{p0}
0
0.2
0.4
0.6
0.8
1.0
M_∞
B
C
D
M_{cr}

For high-speed airplanes, it is desirable to have M_{cr} as high as possible. Hence, modern high-speed subsonic airplanes are usually designed with relatively thin airfoils.



Effect of airfoil thickness on critical Mach number.

Example

In this, we illustrate the estimation of the critical Mach number for an airfoil using

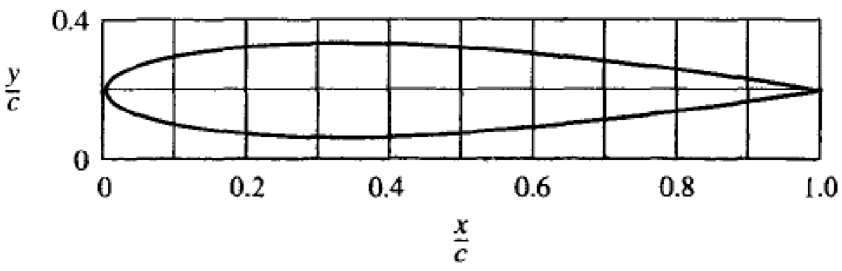
(a) ***the graphical solution discussed in this section, and***

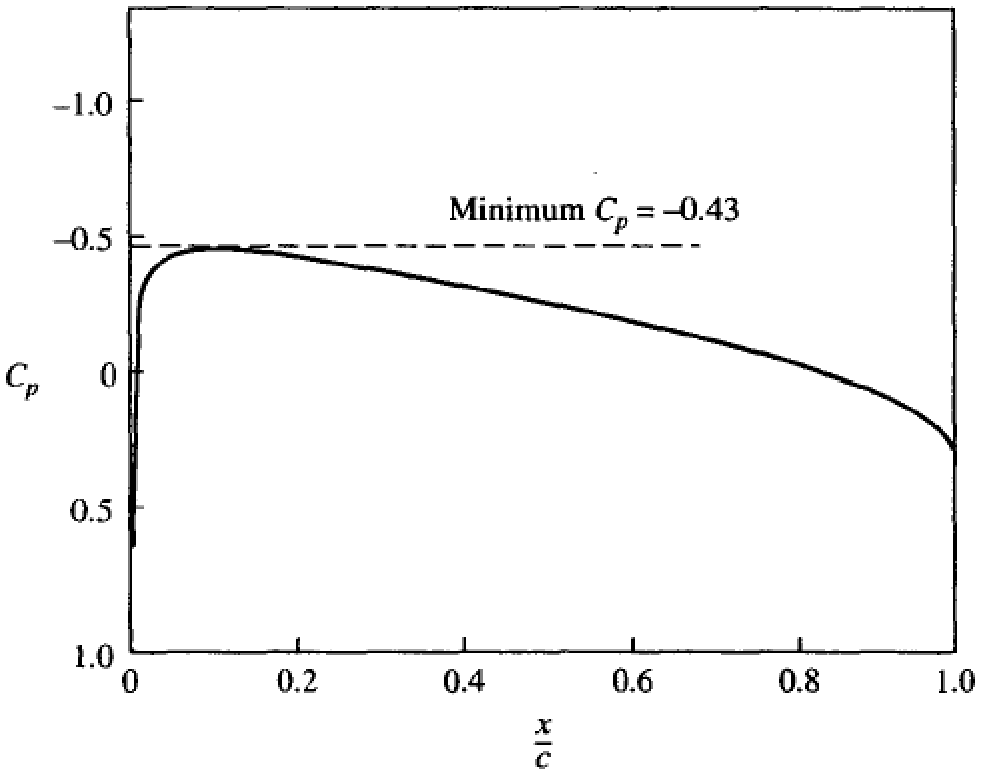
(b) an analytical solution using

a closed-form equation

Consider the NACA 0012 airfoil at zero angle of attack

The pressure coefficient distribution over this airfoil, measured in a wind tunnel at low speed, is given. From this information, estimate the critical Mach number of the NACA 0012 airfoil at zero angle of attack.





(a) Graphical Solution. ***First, let us accurately plot the curve of Cp_{Cr} versus M_{cr} from*** **Equation**

$$C_{p,\mathrm{cr}} = \frac{2}{\gamma M_{\mathrm{cr}}^2}\left[\left(\frac{1+[(\gamma-1)/2]M_{\mathrm{cr}}^2}{1+(\gamma-1)/2}\right)^{\gamma/(\gamma-1)} - 1\right]$$

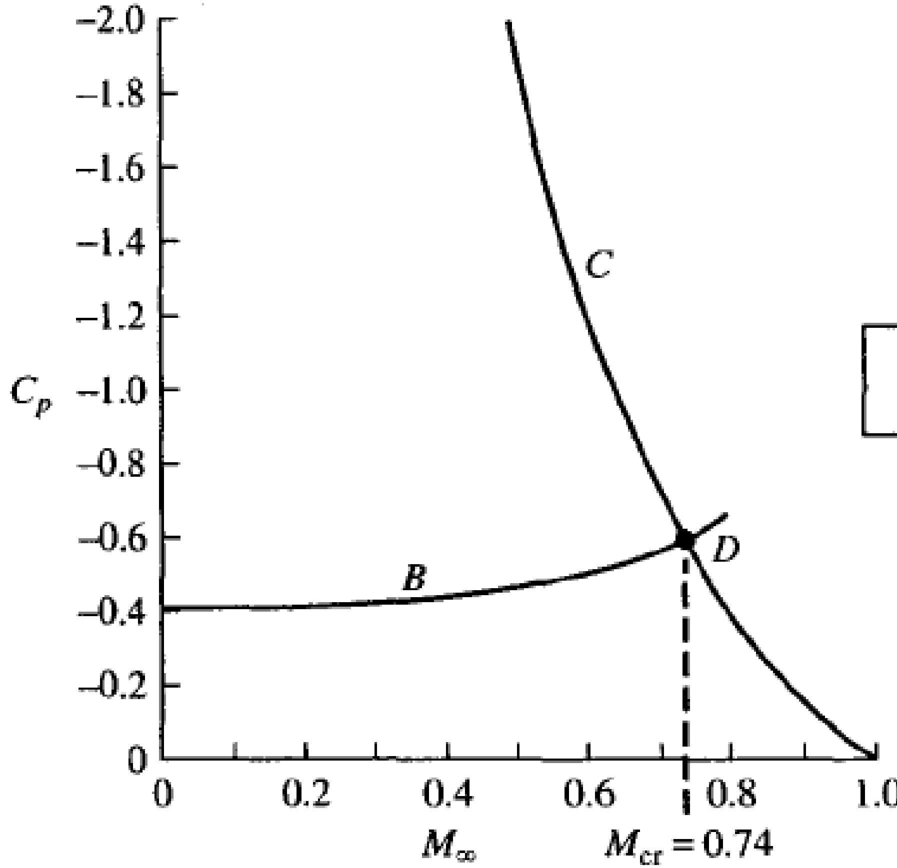
M_∞	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$C_{p,\mathrm{cr}}$	−3.66	−2.13	−1.29	−0.779	−0.435	−0.188	0

From the pressure coefficient distribution given in the figure.
The minimum value of Cp on the surface is -0.43.

$$(C_{p,0})_{\min} = -0.43.$$

$$(C_p)_{\min} = \frac{(C_{p,0})_{\min}}{\sqrt{1-M_\infty^2}} = \frac{-0.43}{\sqrt{1-M_\infty^2}}$$

M_∞	0	0.2	0.4	0.6	0.8
$(C_p)_{\min}$	−0.43	−0.439	−0.469	−0.538	−0.717



$M_{cr} = 0.74$

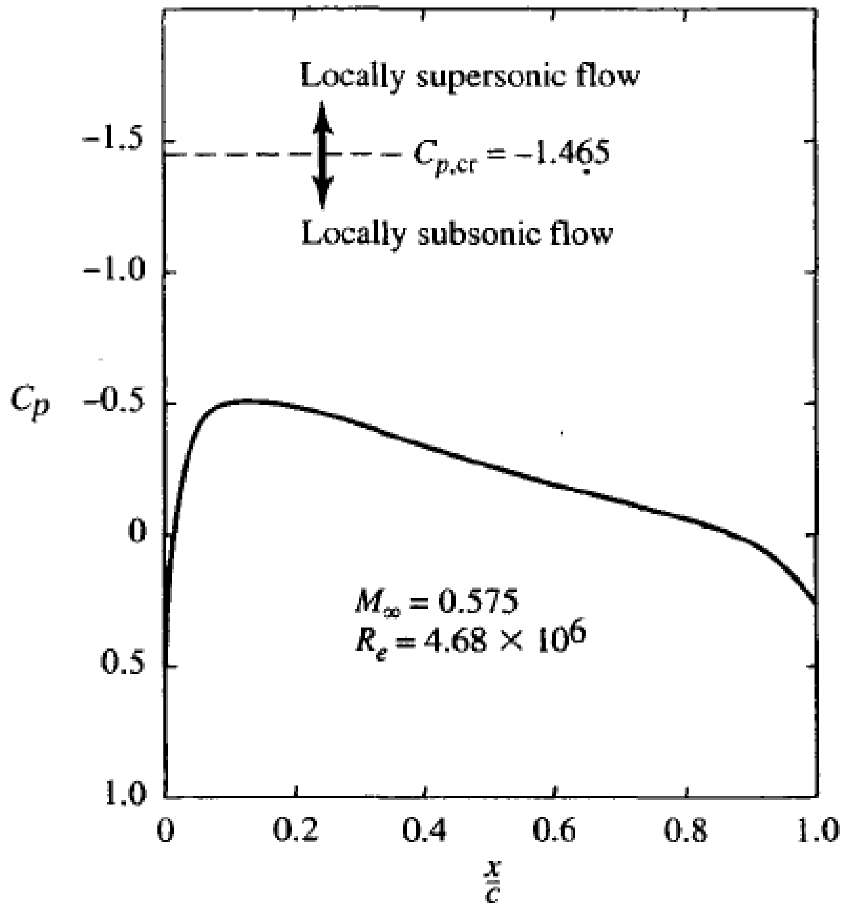
(b) Analytical Solution.

$$(C_p)_{\min} = \frac{-0.43}{\sqrt{1 - M_\infty^2}} \Rightarrow \frac{-0.43}{\sqrt{1 - M_{cr}^2}} = \frac{2}{\gamma M_{cr}^2}\left[\left(\frac{1 + [(\gamma - 1)/2]M_{cr}^2}{1 + (\gamma - 1)/2}\right)^{\gamma/\gamma - 1} - 1\right]$$

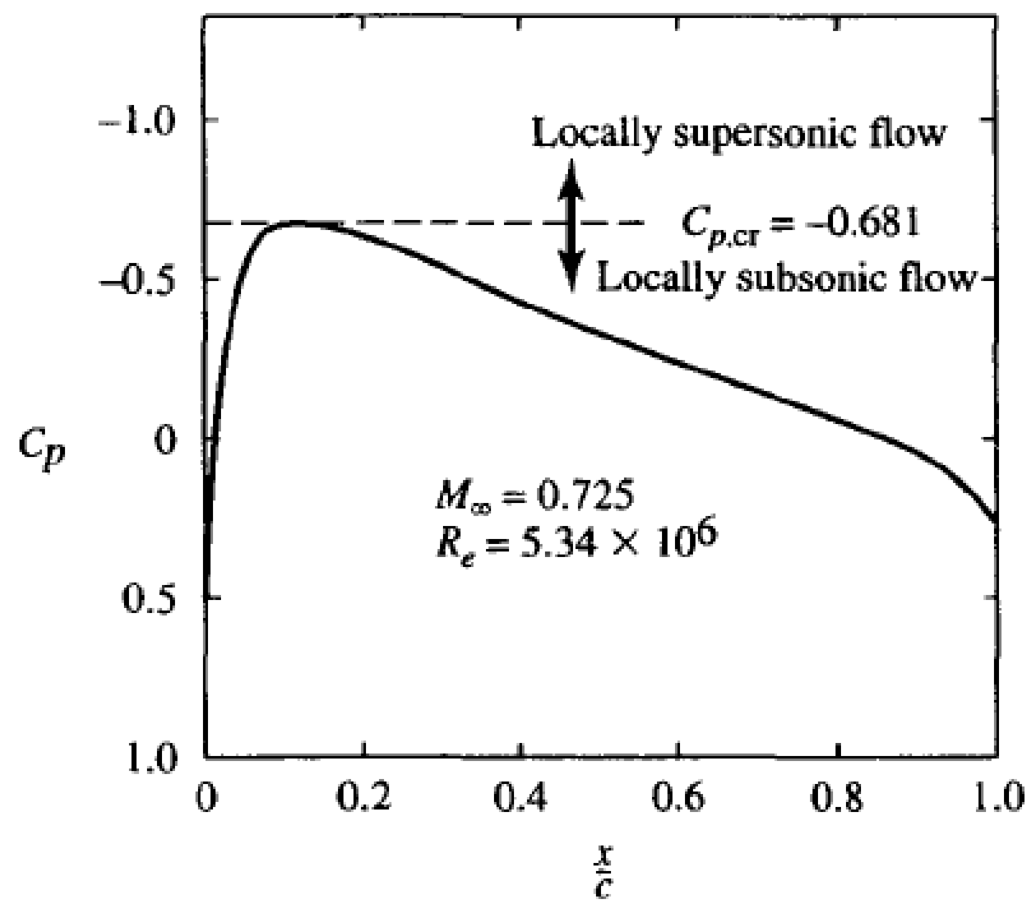
$$\Rightarrow \boxed{M_{cr} = 0.7371}$$

***Question:* How accurate is the estimate of the critical Mach number in this example?**

Wind tunnel measurements of the surface pressure distributions for this airfoil at zero angle of attack



Locally supersonic flow
$C_{p,\text{cr}} = -1.465$
Locally subsonic flow
C_p
$M_\infty = 0.575$
$R_e = 4.68 \times 10^6$
$\frac{x}{c}$

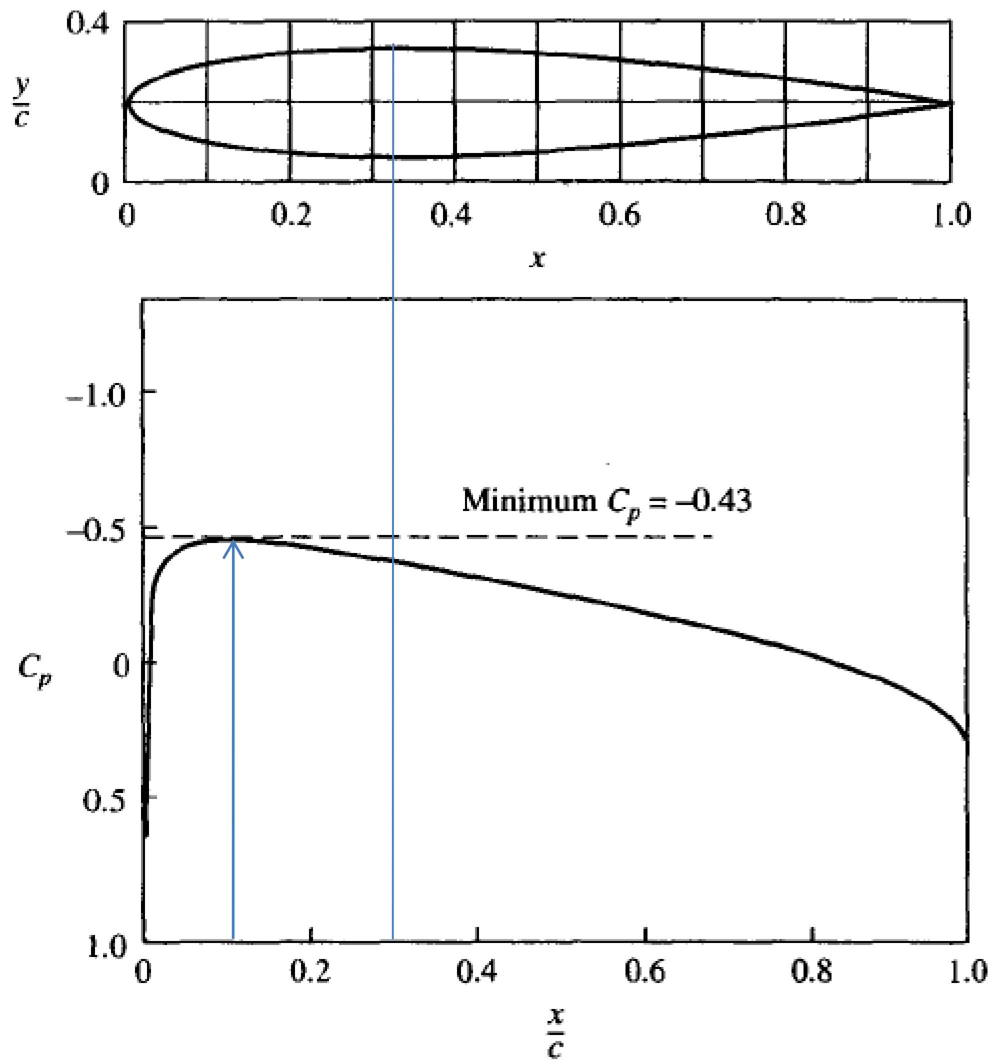


Locally supersonic flow
$C_{p,\text{cr}} = -0.681$
Locally subsonic flow
C_p
$M_\infty = 0.725$
$R_e = 5.34 \times 10^6$
$\frac{x}{c}$

A Comment on the Location of Minimum Pressure (Maximum Velocity)

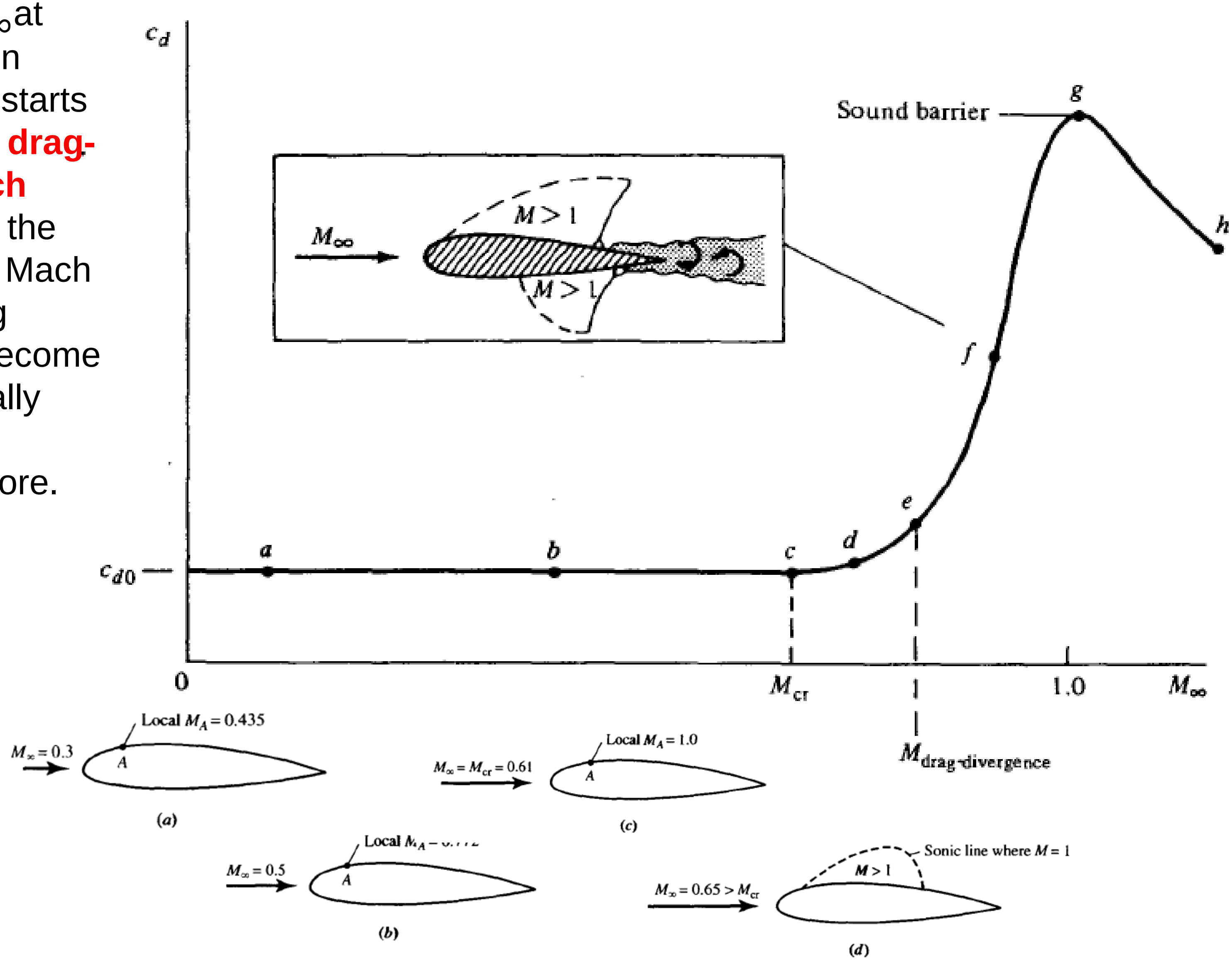
The point of minimum pressure (hence maximum velocity) does *not correspond to the location of maximum thickness of the airfoil.*

Nature places the maximum velocity at a point which satisfies the physics of the whole flow field, not just what is happening in a local region of the flow.
The point of maximum velocity is dictated by the complete shape of the airfoil, not just by the shape in a local region.



DRAG-DIVERGENCE MACH NUMBER: THE SOUND BARRIER

The value of M_∞ at which this sudden increase in drag starts is defined as the **drag-divergence Mach number**. Beyond the drag-divergence Mach number, the drag coefficient can become very large, typically increasing by a factor of 10 or more.



Design for higher speed aircraft •

The question as to whether one may delay the drag divergence Mach number to a value closer to 1 is a fascinating subject of novel aerodynamic designs:

– Use of thin airfoils

– Low-aspect-ratio wing

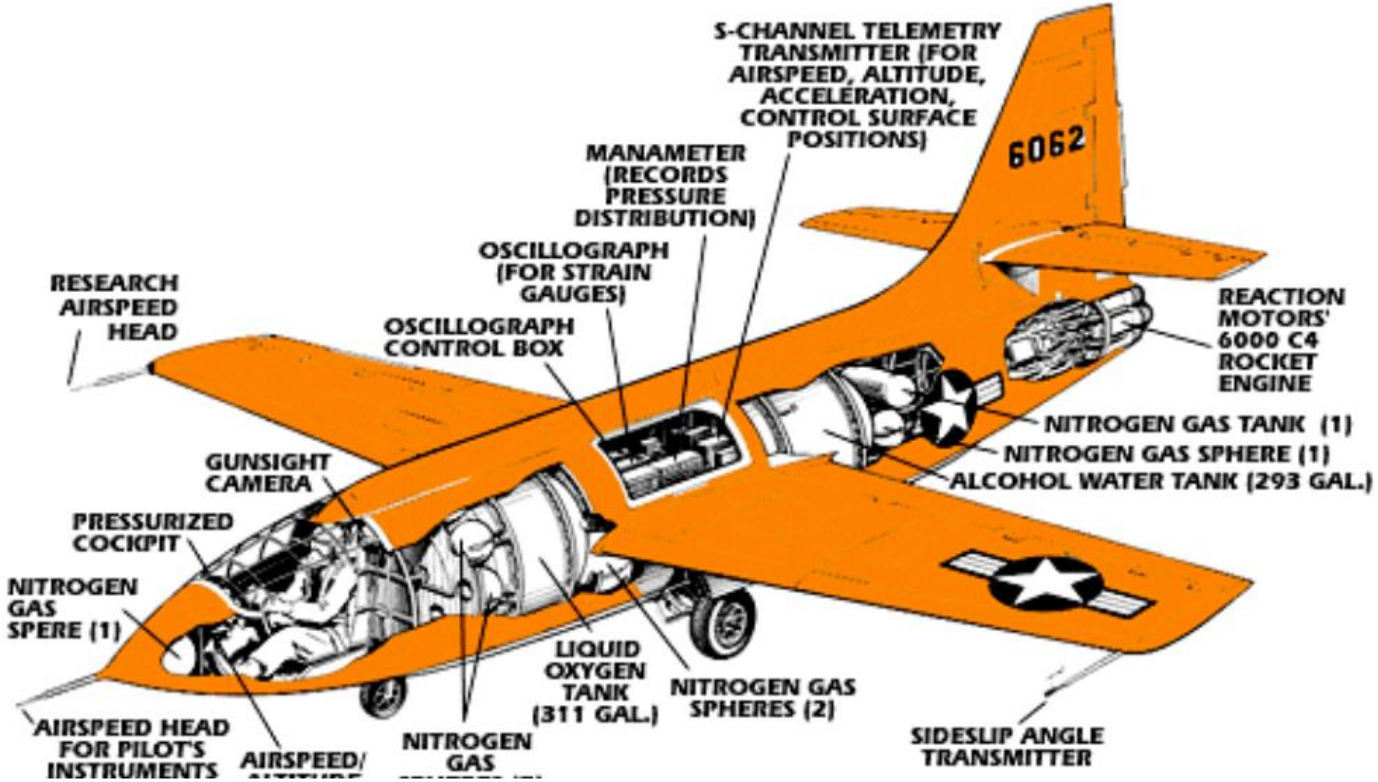
– Use of sweep of the wing forward or back

– Removal of boundary layer and vortex generators

– Supercritical technology

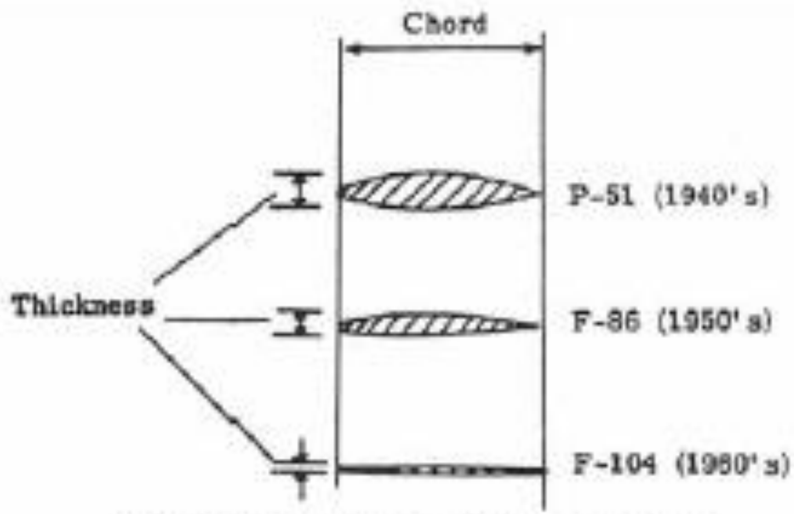
– Area-rule technology

Bell-x1



Airfoil with
10% Thickness

Use of thin airfoils



(a) Changes in airfoil sections.

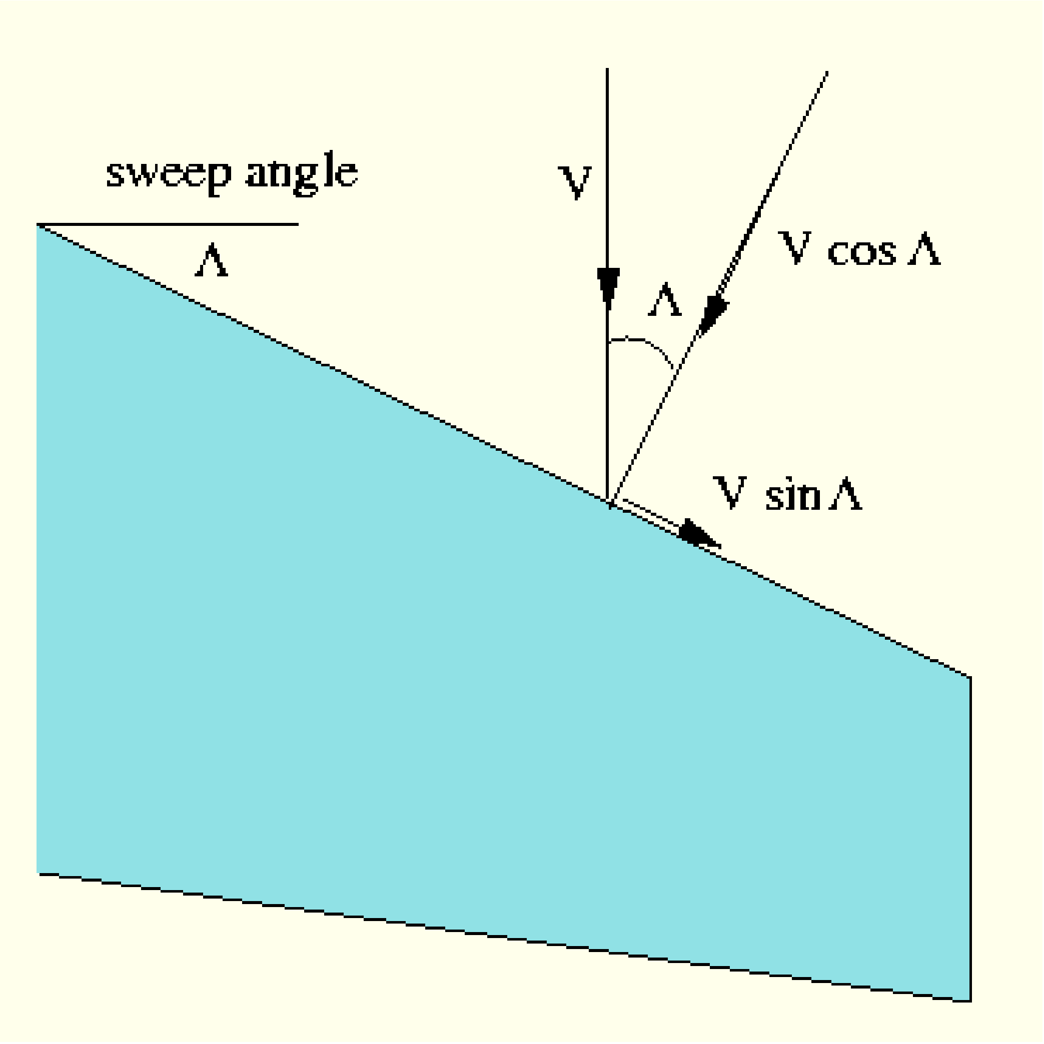
(b) F-104G airplane.

F-104: thin airfoils and Low-aspect-ratio wing

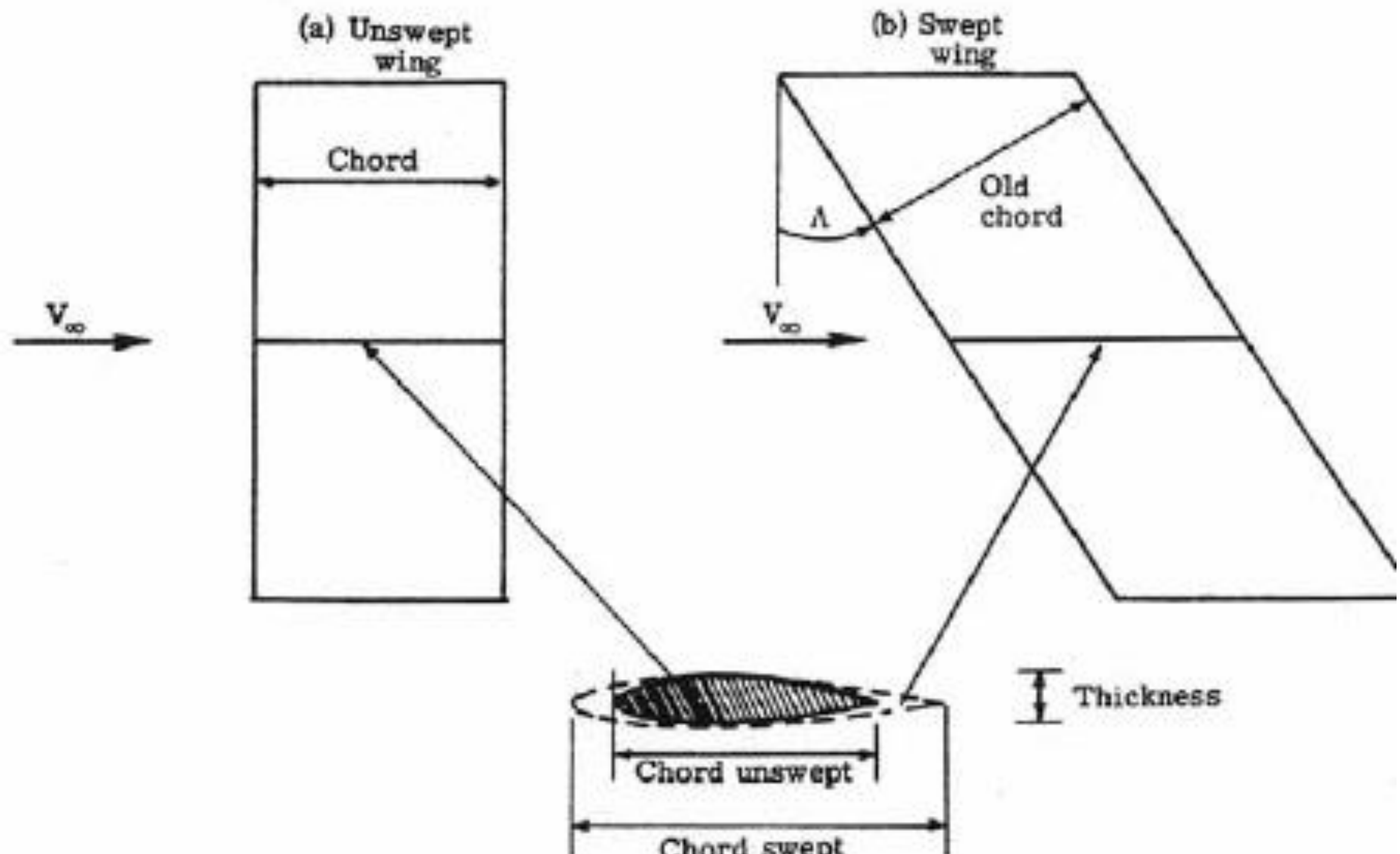
- The F-104 was designed to achieve the minimum possible wave drag but was penalized with low subsonic lift.

- As a result, the landing speed of this airplane was particularly high and landing mishaps were common among untrained pilots.

Swept-Back Wings

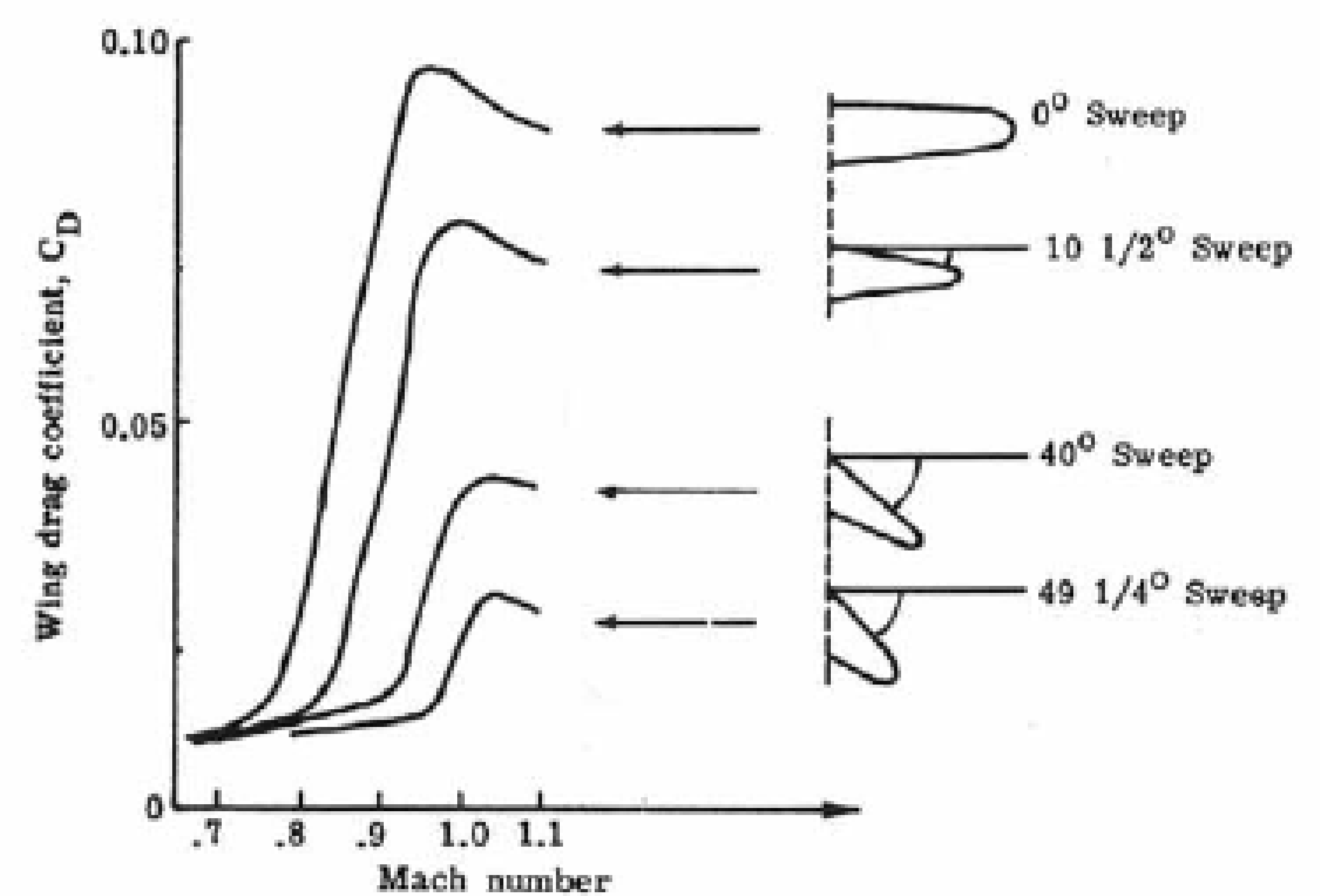


Sweep reduces effective thickness-chord ratio



Use of sweep of the wing

Effects of sweep on wing transonic drag coefficient

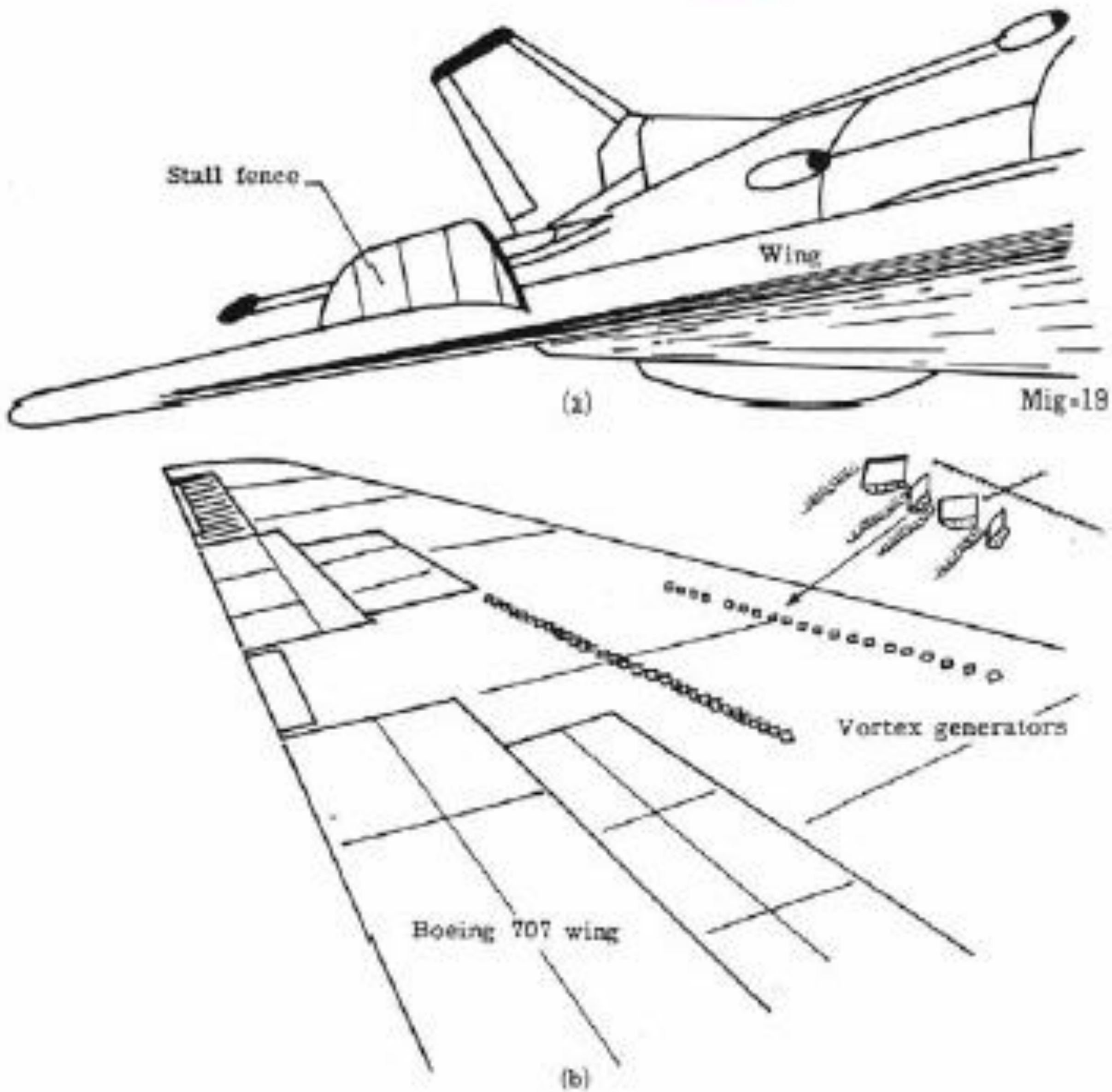


X-29

Removal of boundary layer and vortex generators

- A major disadvantage of swept wings is that there is a spanwise flow along the wing, and the boundary layer will thicken toward the tips for sweepback and toward the roots for sweep forward.
- The spanwise flow may be reduced by the use of stall fences and vortex generators.
- Wing twist is another possible solution to this spanwise flow condition.

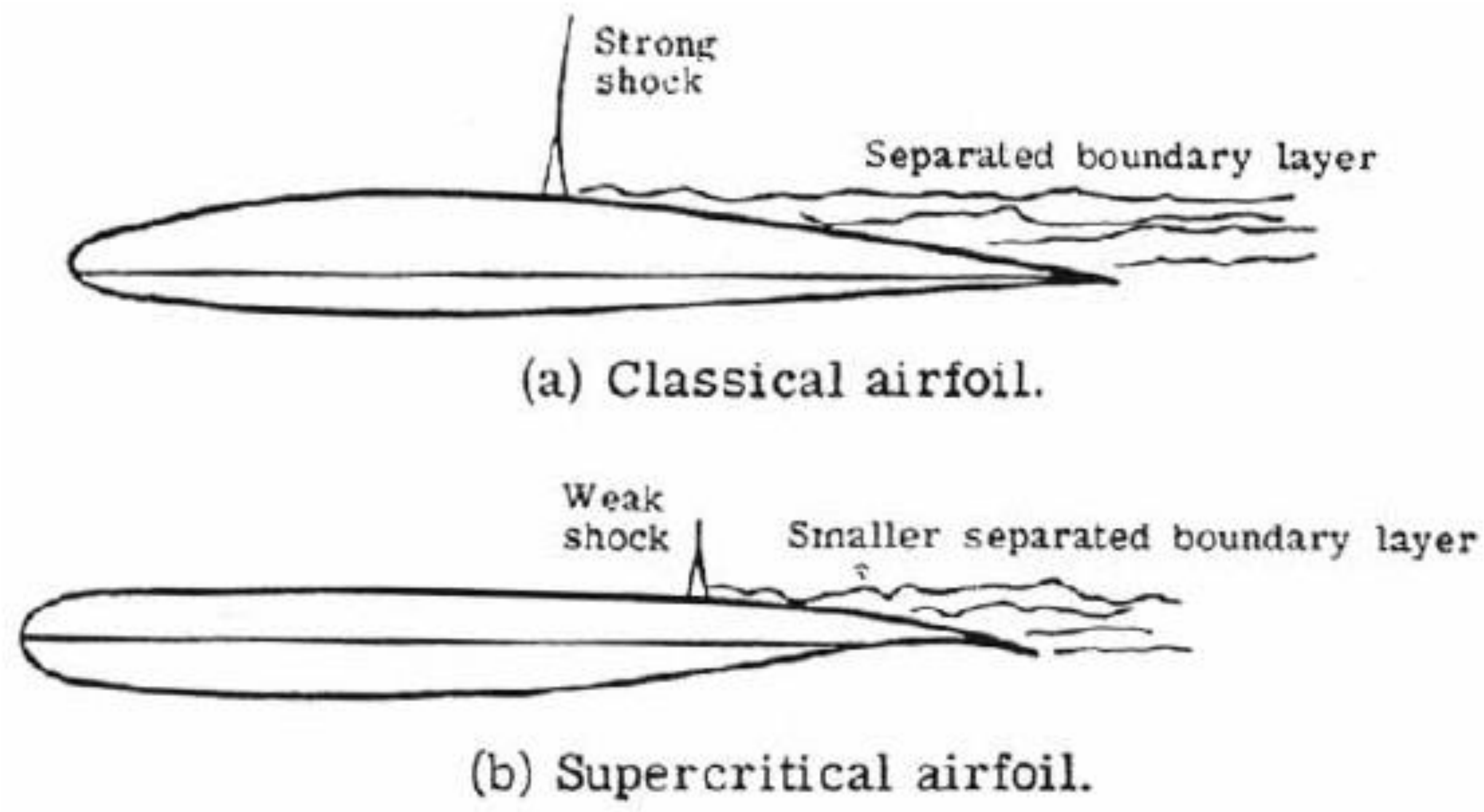
Stall fences and vortex generators

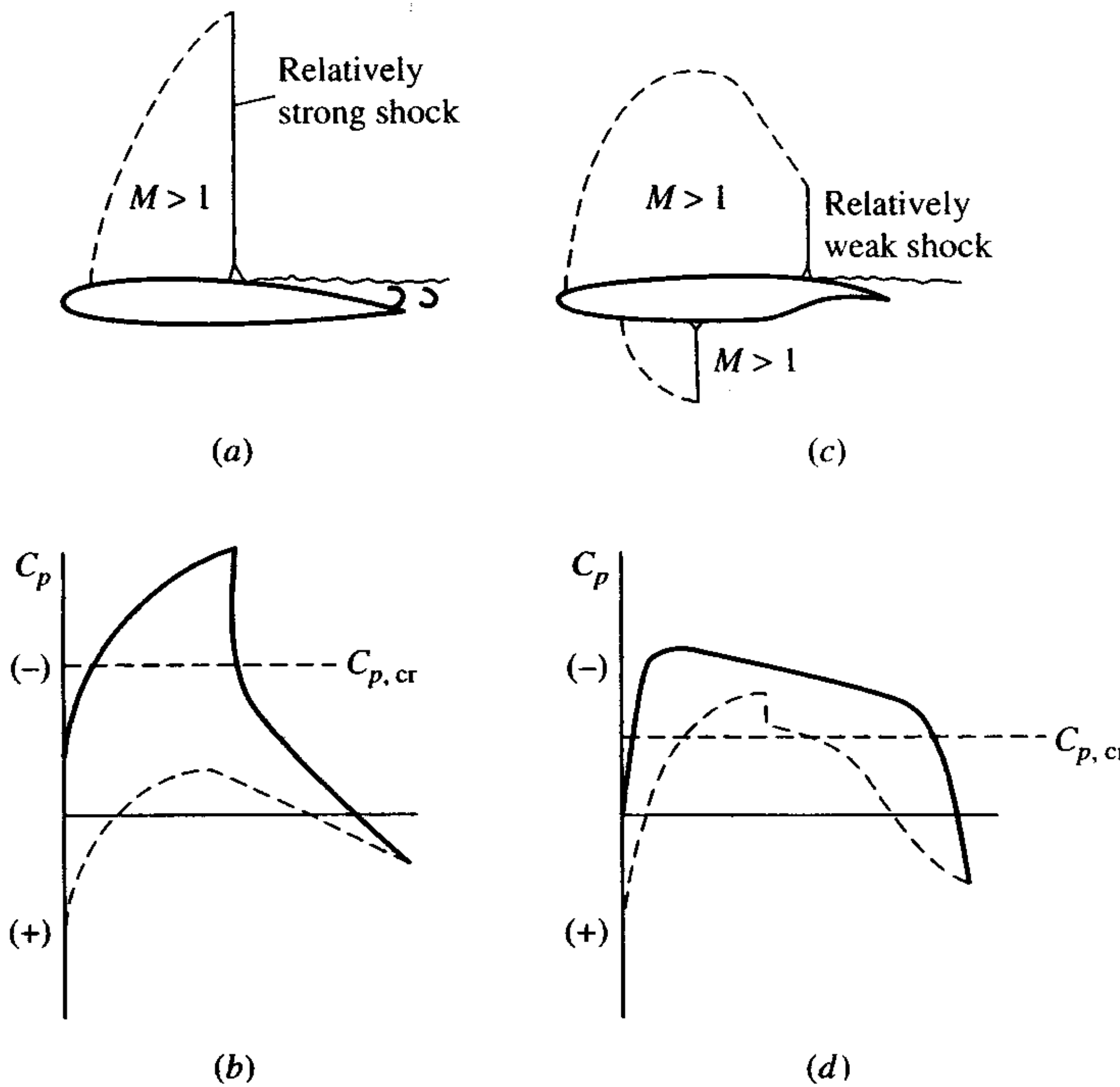


Supercritical Airfoils

- Developed by Dr. Whitcomb of the NASA
- The airfoil has a flattened upper surface which delays the formation and strength of the shocks to a point closer to the trailing edge.
  - The flattened upper surface exhibits a reduction of lift, to counteract this, the supercritical airfoil has increased camber at the trailing edge
- Shock- induced separation is greatly decreased.
- The critical Mach number is delayed even up to 0.99.
- This delay represents a major increase in commercial airplane performance.

The classical airfoil and the supercritical airfoil operating near the Mach 1 region



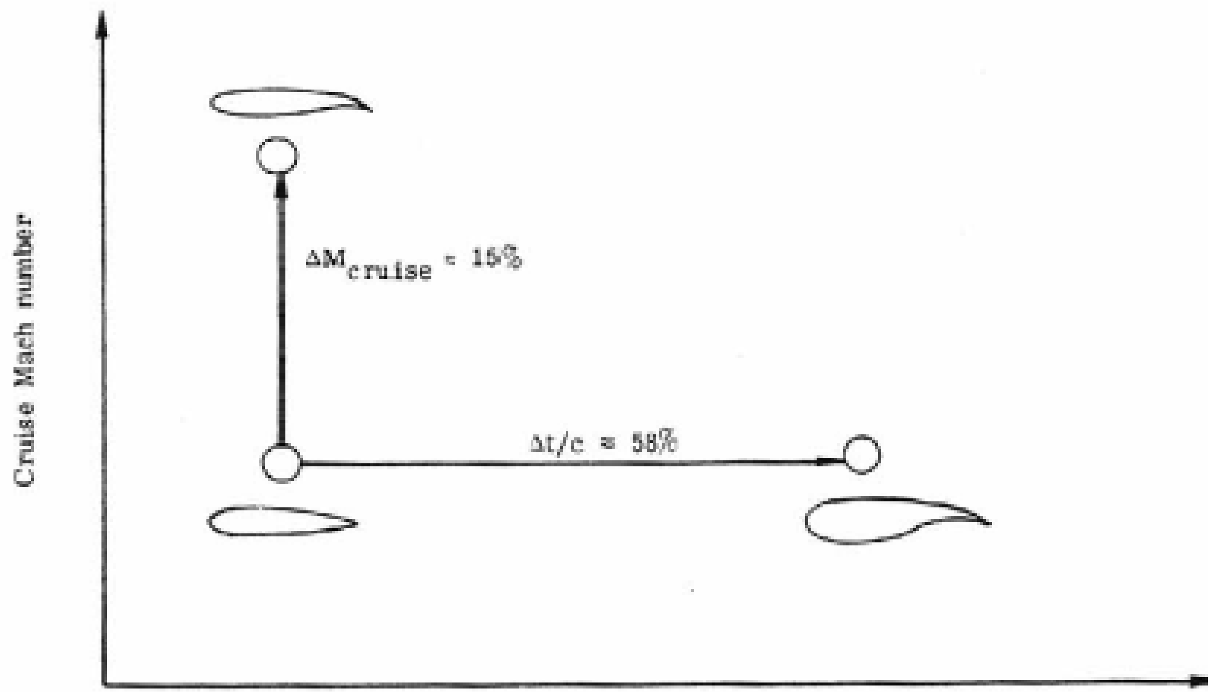


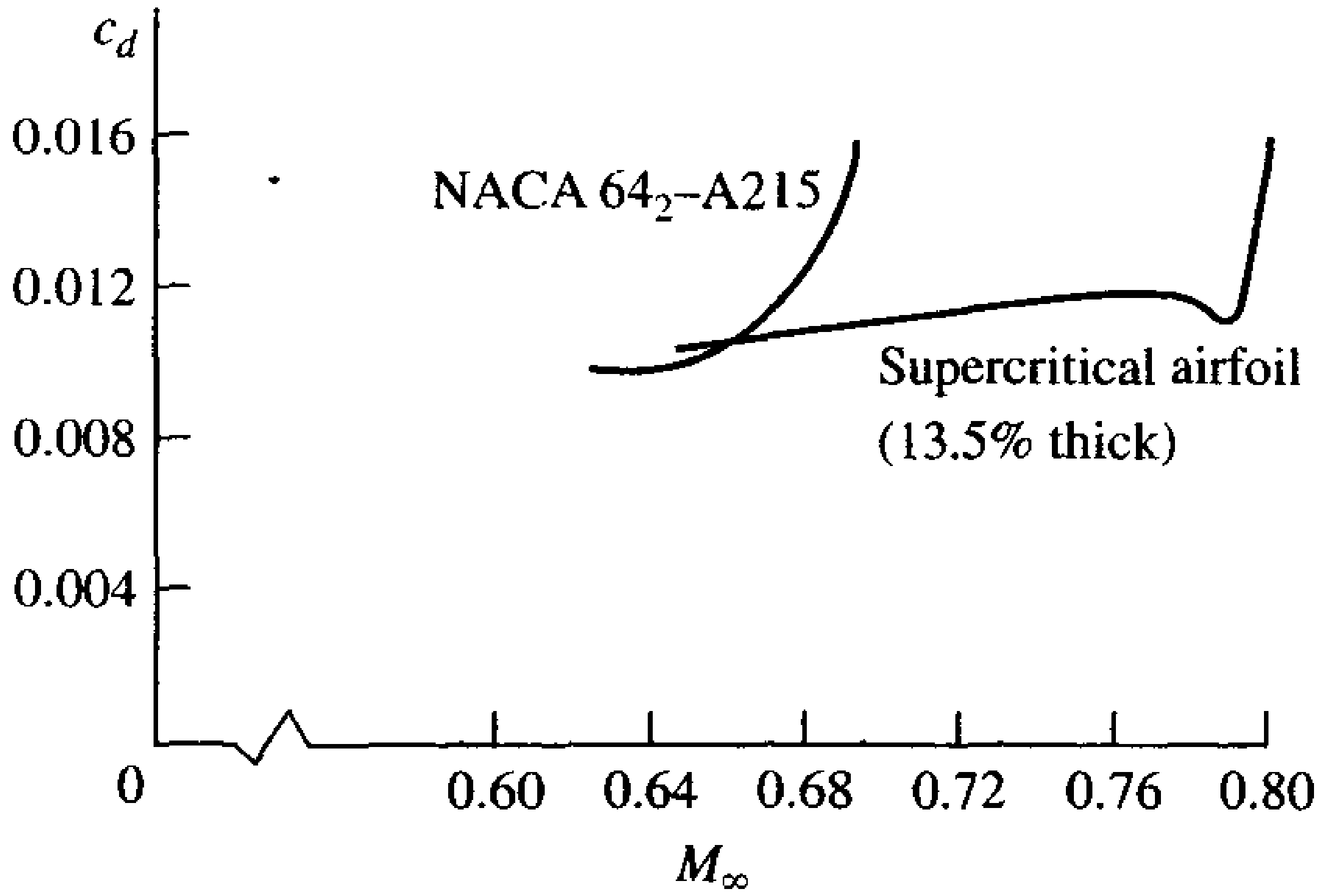
NACA 64_2–A215 airfoil
$M_\infty = 0.69$

Supercritical airfoil (13.5% thick)
$M_\infty = 0.79$

Supercritical Airfoils: Benefits

- The airfoil permits high subsonic cruise near Mach 1 before the transonic drag rise
- At lower drag divergence Mach numbers, the supercritical airfoil permits a thicker wing section to be used without a drag penalty This airfoil reduces structural weight and permits higher lift at lower speeds.

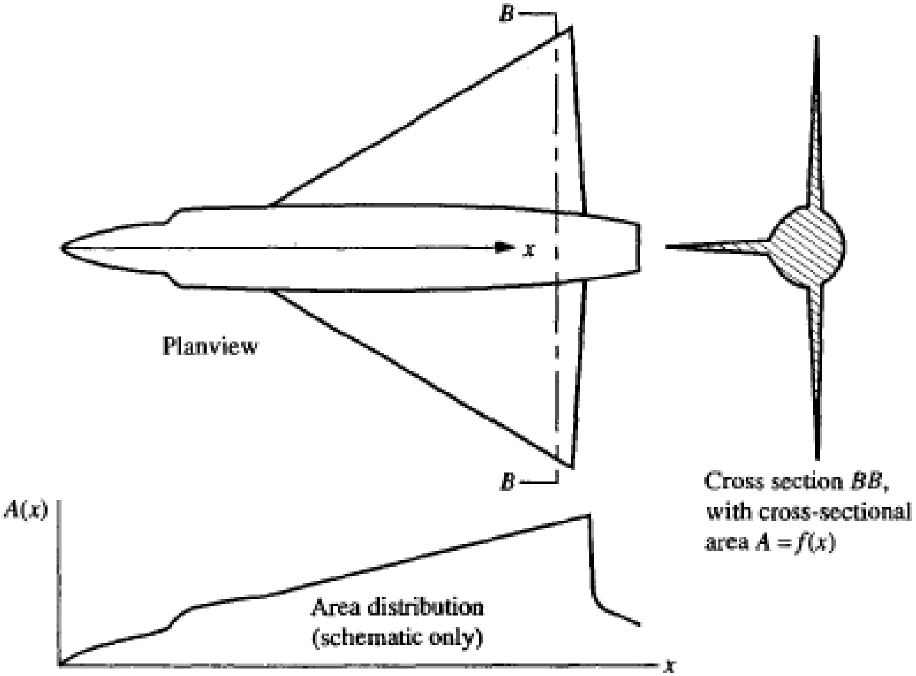


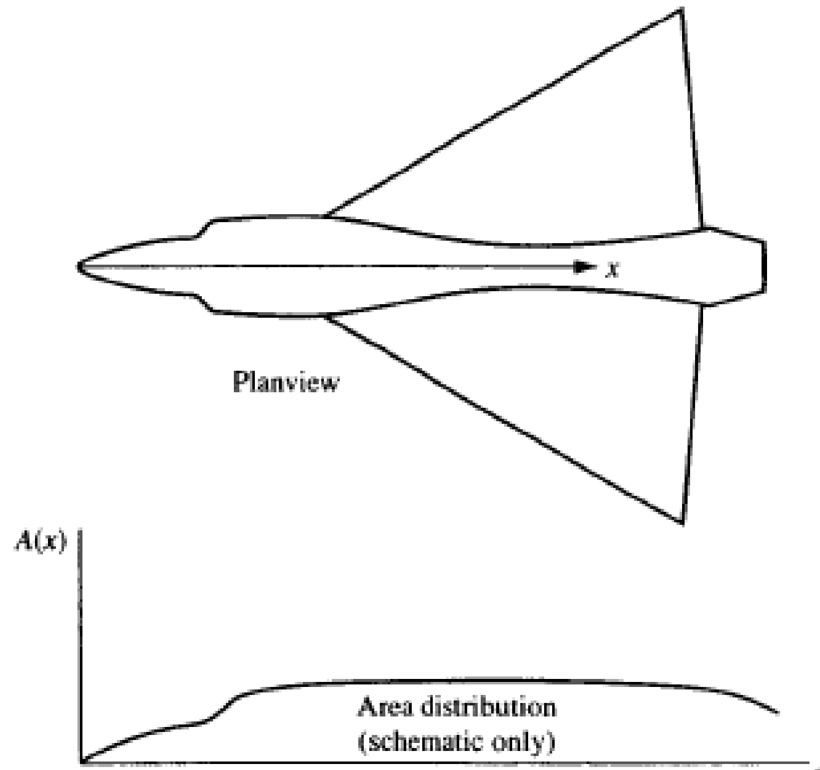


c_d
0.016
0.012
0.008
0.004
0
NACA 64_2–A215
Supercritical airfoil
(13.5% thick)
0.60
0.64
0.68
0.72
0.76
0.80
M_∞

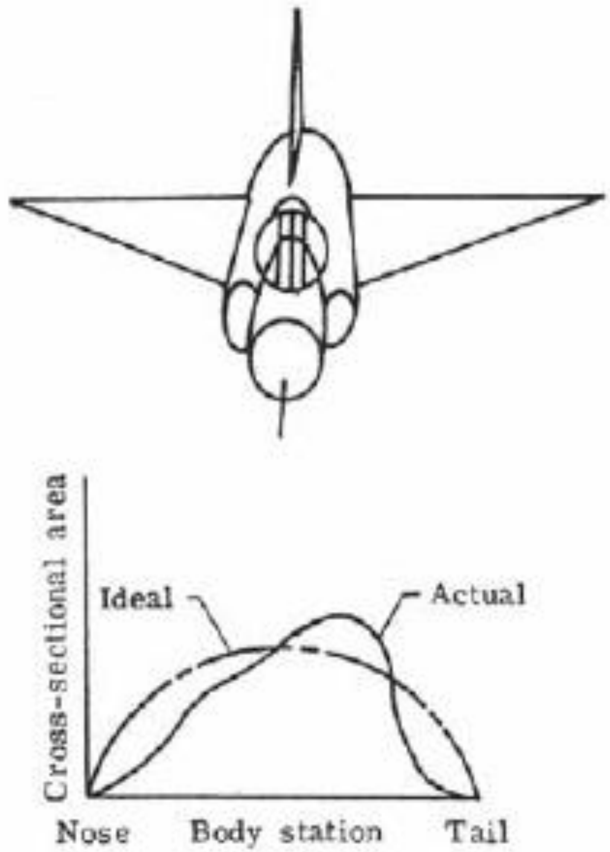
THE AREA RULE

For almost a century, it was well known by ballisticians that the speed of a supersonic bullet or artillery shell with a cross-sectional area was higher than projectiles with abrupt or smooth *variation of* discontinuous area distributions. In the mid-1950s, an aeronautical engineer at the NACA Langley Aeronautical Laboratory, Richard T. Whitcomb, put this knowledge to work on the problem of transonic flight of airplanes.

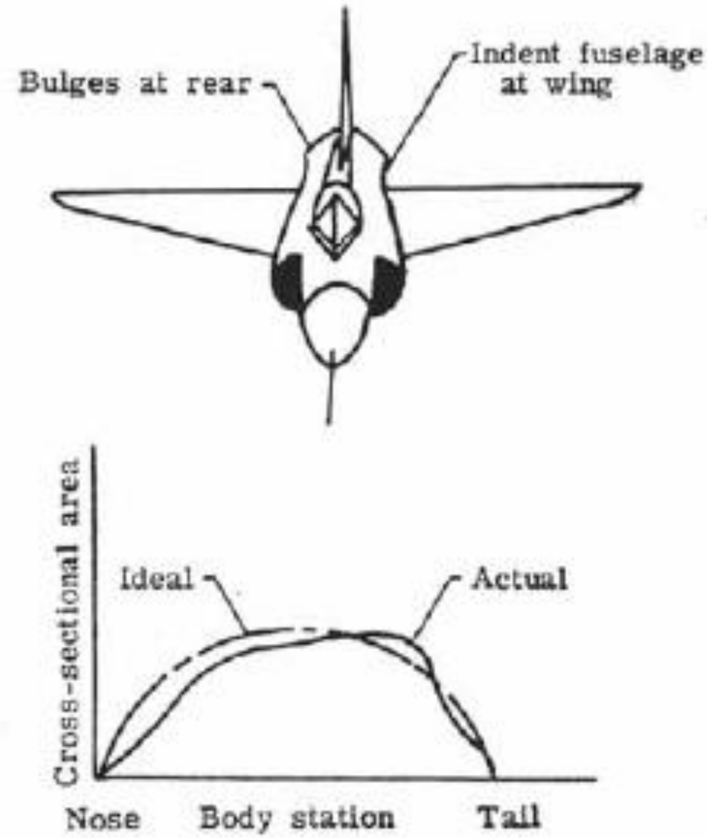




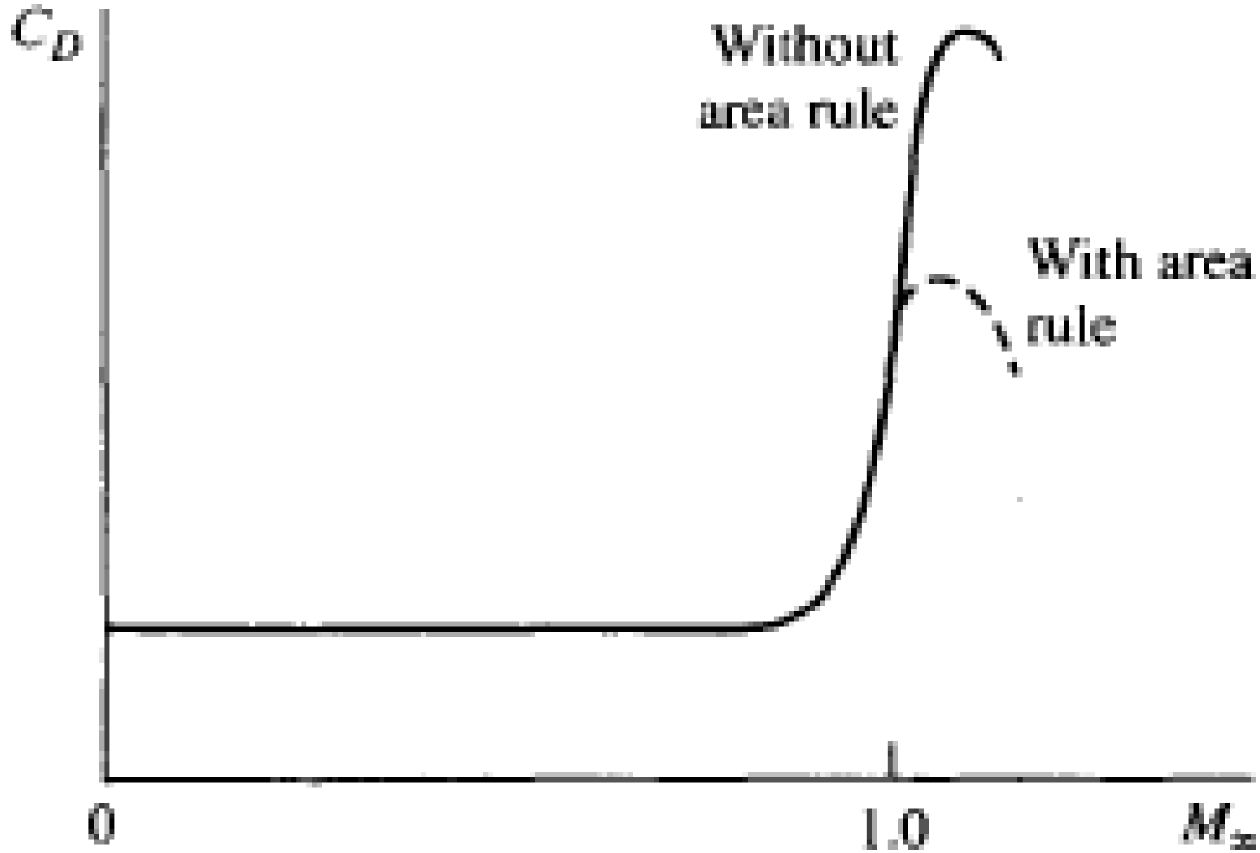
Area Rule



(a) YF-102A before area ruling.

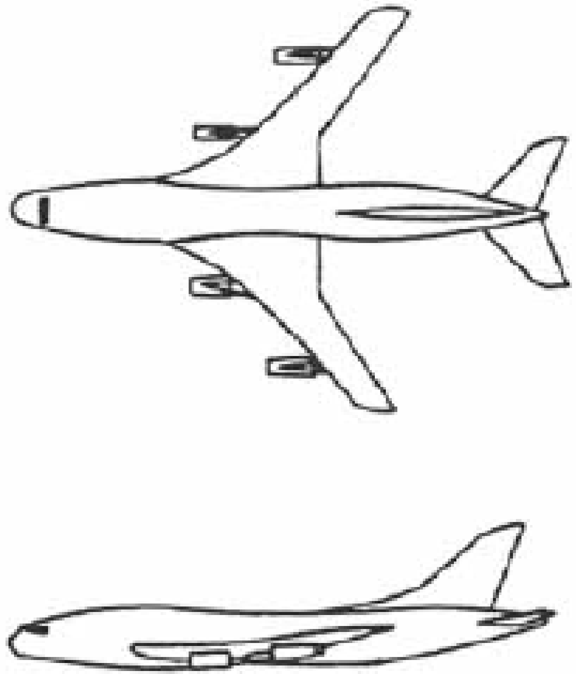


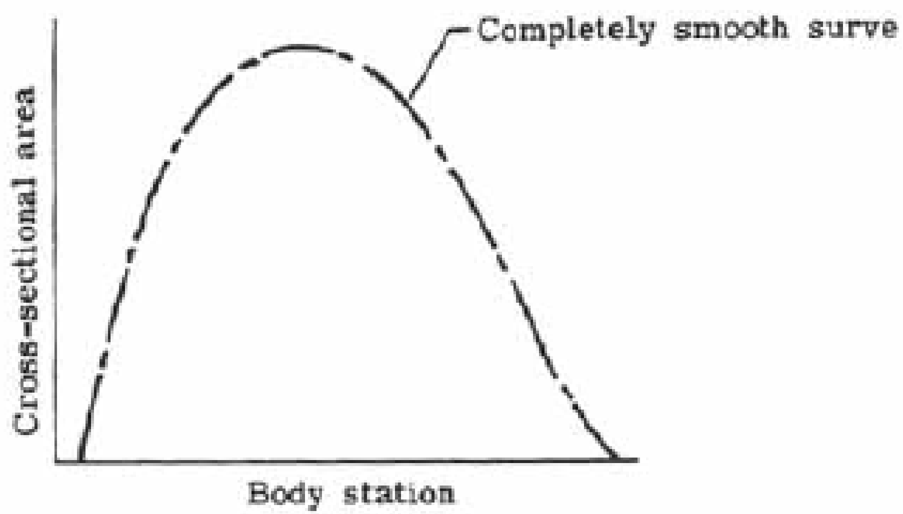
(b) F-102A after area ruling.



C_D
Without area rule
With area rule
0
1.0
M_∞

- Recently, the area-rule concept has been applied to design a near-sonic transport capable of cruising at Mach numbers around 0.99.
- In addition to area ruling, a supercritical wing is used.
- The shocks and drag divergence are delayed to a near-sonic Mach number.






787
BOEING

Linearized Flovvs

So far, We have dealt almost exclusively with one-dimensional flow.

Exceptions were the treatment of oblique shocks and Prandtl-Meyer flow in yet even these cases could be handled componentwise as equivalent one dimensional flows.

One-dimensional analysis has been shown to be useful for obtaining good engineering approximations to a wide variety of flow problems. However, such an analysis is necessarily an approximation; no real flow exists that is truly one dimensional. Furthermore, many problems cannot even be approached with a one dimensional analysis

For example, the one-dimensional equations are inadequate for the design of the contour of such a nozzle (A versus x). Likewise, the flow over a cambered supersonic wing cannot be predicted on the basis of a simple one-dimensional theory

For these, and a great many other, practical cases, we must develop the equations of motion for multidimensional gas dynamics and find means for solving these equations subject to prescribed boundary conditions.

A Review of the Governing Equations

- The Continuity Equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

- The N-S equations (Conservation of Momentum)

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu \nabla \cdot \mathbf{V}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left(2\mu \frac{\partial v}{\partial y} - \frac{2}{3}\mu \nabla \cdot \mathbf{V}\right) + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right]$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial w} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left(2\mu \frac{\partial w}{\partial z} - \frac{2}{3}\mu \nabla \cdot \mathbf{V}\right)$$

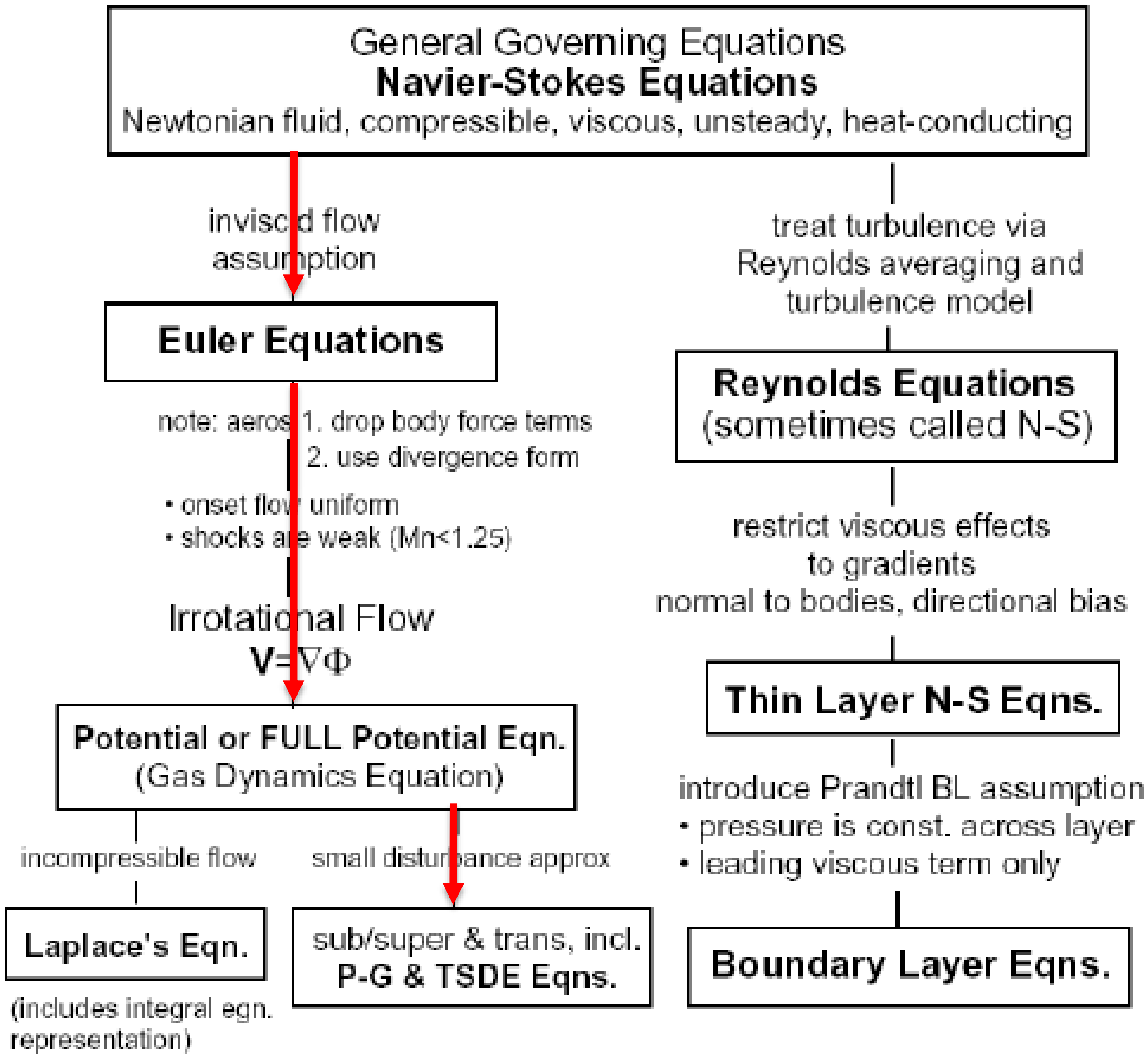
- The Energy Equation

$$\rho \frac{Dh}{Dt} - \frac{Dp}{Dt} = \nabla \cdot (k \nabla T) \qquad + \qquad \Phi$$

$$\mathbf{\Phi} = \mu \left\{ 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 \right.$$
$$\left. + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 \right\}$$

Summary on Governing Equations

The connection between various flow field models

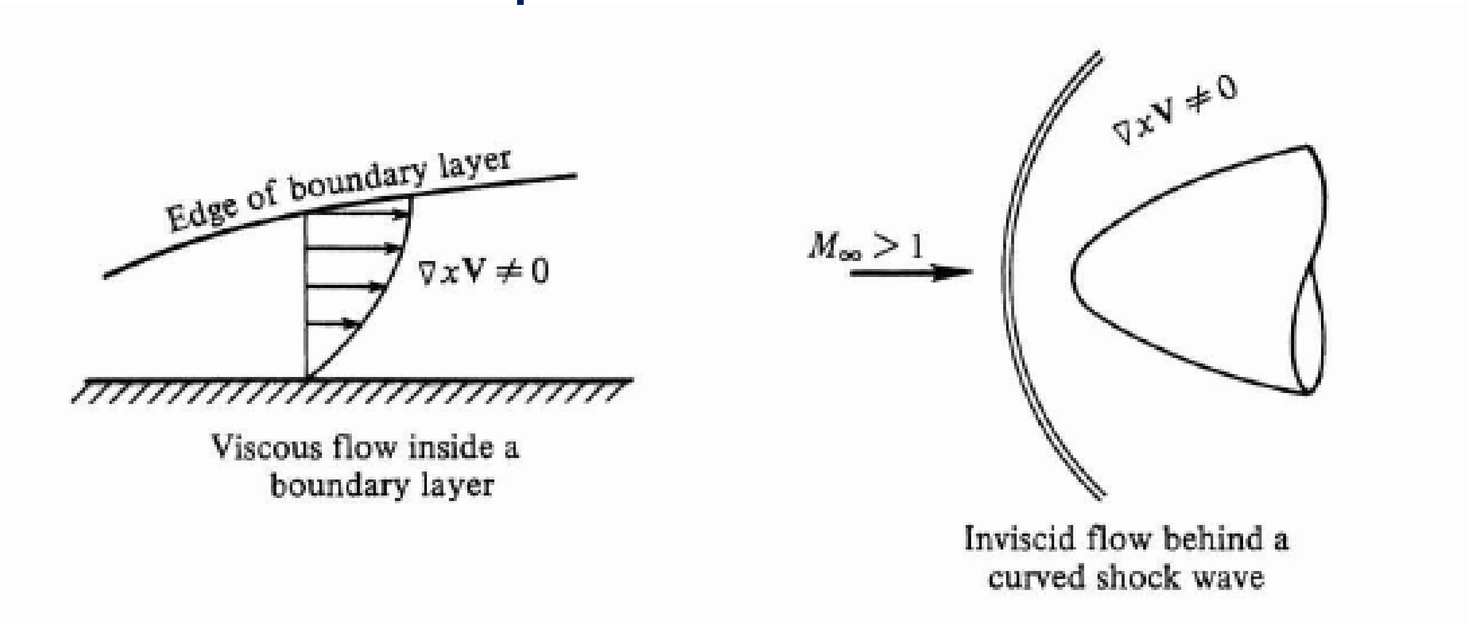


IRROTATIONAL FLOW

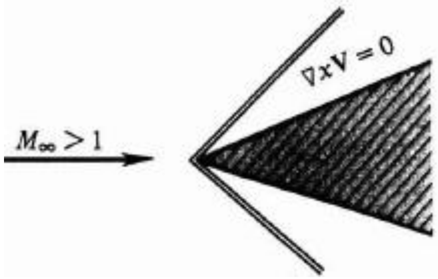
Vorticity is twice the angular velocity of a fluid element $\nabla \times \mathbf{V} = 2\boldsymbol{\omega}$

A flow where $\nabla \times \mathbf{V} \neq 0$ throughout is called a Rotational flow

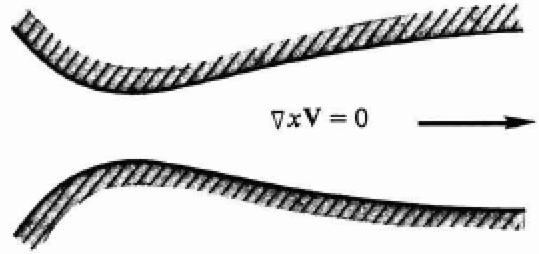
Examples of rotational flows.



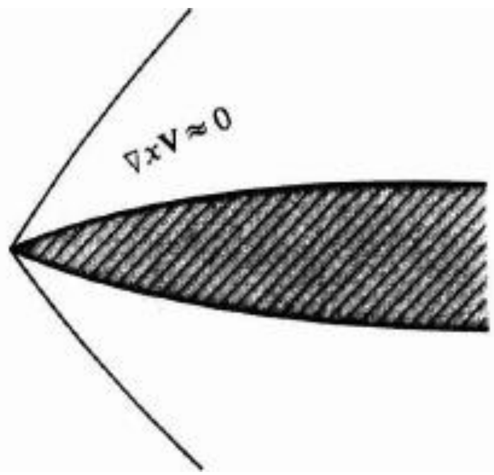
Examples of irrotational flows.



Flowfield over a sharp wedge or cone



Two-dimensional or axisymmetric nozzle flows



Flowfield behind the shock wave on a *slender, sharpnosed* body is almost irrotational. For analysis, we usually assume $\nabla x \mathbf{V} = 0$ for this case.

Consider an irrotational flow in more detail

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - \mathbf{j}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

$$\boxed{\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \qquad \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z} \qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}}$$

irrotationality conditions

Consider Euler's equation without body forces.

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x}$$

$$\Rightarrow \quad -\frac{\partial p}{\partial x}\, dx = \rho u \frac{\partial u}{\partial x}\, dx + \rho v \frac{\partial u}{\partial y}\, dx + \rho w \frac{\partial u}{\partial z}\, dx$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$$

$$\Rightarrow \quad -\frac{\partial p}{\partial x}\, dx = \rho u \frac{\partial u}{\partial x}\, dx + \rho v \frac{\partial v}{\partial x}\, dx + \rho w \frac{\partial w}{\partial x}\, dx$$

$$-\frac{\partial p}{\partial x}\, dx = \frac{1}{2}\rho \frac{\partial u^2}{\partial x}\, dx + \frac{1}{2}\rho \frac{\partial v^2}{\partial x}\, dx + \frac{1}{2}\rho \frac{\partial w^2}{\partial x}\, dx$$

Similarly

$$-\frac{\partial p}{\partial x}\,dx = \frac{1}{2}\rho\frac{\partial u^2}{\partial x}\,dx + \frac{1}{2}\rho\frac{\partial v^2}{\partial x}\,dx + \frac{1}{2}\rho\frac{\partial w^2}{\partial x}\,dx$$

$$-\frac{\partial p}{\partial y}\,dy = \frac{1}{2}\rho\frac{\partial u^2}{\partial y}\,dy + \frac{1}{2}\rho\frac{\partial v^2}{\partial y}\,dy + \frac{1}{2}\rho\frac{\partial w^2}{\partial y}\,dy$$

$$-\frac{\partial p}{\partial z}\,dz = \frac{1}{2}\rho\frac{\partial u^2}{\partial z}\,dz + \frac{1}{2}\rho\frac{\partial v^2}{\partial z}\,dz + \frac{1}{2}\rho\frac{\partial w^2}{\partial z}\,dz$$

$$-\left(\frac{\partial p}{\partial x}\,dx + \frac{\partial p}{\partial y}\,dy + \frac{\partial p}{\partial z}\,dz\right) = \frac{1}{2}\rho\frac{\partial V^2}{\partial x}\,dx + \frac{1}{2}\rho\frac{\partial V^2}{\partial y}\,dy + \frac{1}{2}\rho\frac{\partial V^2}{\partial z}\,dz$$

where $V^2 = u^2 + v^2 + w^2$

$$-\left(\frac{\partial p}{\partial x}\,dx + \frac{\partial p}{\partial y}\,dy + \frac{\partial p}{\partial z}\,dz\right) = \frac{1}{2}\rho\frac{\partial V^2}{\partial x}\,dx + \frac{1}{2}\rho\frac{\partial V^2}{\partial y}\,dy + \frac{1}{2}\rho\frac{\partial V^2}{\partial z}\,dz$$

$$-dp = \tfrac{1}{2}\rho\,d(V^2)$$

The velocity potential equation

Consider a vector **A**. If $\nabla \times \mathbf{A} = 0$ everywhere, then **A** can always be expressed as $\nabla\zeta$, where ζ is a scalar function. This stems directly from the vector identity, curl (grad) $\equiv 0$. Hence,

$$\nabla \times \nabla\zeta = 0$$

where ζ is any scalar function.

For irrotational flow, $\nabla \times \mathbf{V} = 0$

Hence, we can define a scalar function $\Phi = \Phi(x, y, z)$

$$\mathbf{V} \equiv \nabla\Phi$$

where Φ, is called the velocity potential.

$$\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$$

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\mathbf{i} + \frac{\partial\Phi}{\partial y}\mathbf{j} + \frac{\partial\Phi}{\partial z}\mathbf{k}$$

$$u = \frac{\partial\Phi}{\partial x} \quad v = \frac{\partial\Phi}{\partial y} \quad w = \frac{\partial\Phi}{\partial z}$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

$$\frac{\partial}{\partial x}\rho\mathbf{\Phi}_x + \frac{\partial}{\partial y}\rho\mathbf{\Phi}_y + \frac{\partial}{\partial z}\rho\mathbf{\Phi}_z = 0$$

$$\rho(\mathbf{\Phi}_{xx} + \mathbf{\Phi}_{yy} + \mathbf{\Phi}_{zz}) + \mathbf{\Phi}_x\frac{\partial\rho}{\partial x} + \mathbf{\Phi}_y\frac{\partial\rho}{\partial y} + \mathbf{\Phi}_z\frac{\partial\rho}{\partial z} = 0$$

we eliminate ρ

$$dp = -\rho V\,dV = -\frac{\rho}{2}\,d(V^2) = -\frac{\rho}{2}\,d(u^2 + v^2 + w^2)$$

$$\boxed{dp = -\rho\,d\left(\frac{\boldsymbol{\Phi}_x^2 + \boldsymbol{\Phi}_y^2 + \boldsymbol{\Phi}_z^2}{2}\right)}$$

Recalling that the flow is isentropic, ⇨ $\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s = a^2$ ⇨ $\boxed{d\rho = \frac{dp}{a^2}}$

$$d\rho = -\frac{\rho}{a^2}\,d\left(\frac{\boldsymbol{\Phi}_x^2 + \boldsymbol{\Phi}_y^2 + \boldsymbol{\Phi}_z^2}{2}\right)$$

$$\rho(\mathbf{\Phi}_{xx} + \mathbf{\Phi}_{yy} + \mathbf{\Phi}_{zz}) + \mathbf{\Phi}_x \boxed{\frac{\partial \rho}{\partial x}} + \mathbf{\Phi}_y \boxed{\frac{\partial \rho}{\partial y}} + \mathbf{\Phi}_z \boxed{\frac{\partial \rho}{\partial z}} = 0$$

$$d\rho = -\frac{\rho}{a^2}\, d\left(\frac{\mathbf{\Phi}_x^2 + \mathbf{\Phi}_y^2 + \mathbf{\Phi}_z^2}{2}\right)$$

$$\frac{\partial \rho}{\partial x} = -\frac{\rho}{a^2}\frac{\partial}{\partial x}\left(\frac{\mathbf{\Phi}_x^2 + \mathbf{\Phi}_y^2 + \mathbf{\Phi}_z^2}{2}\right)$$

$$\frac{\partial \rho}{\partial x} = -\frac{\rho}{a^2}(\mathbf{\Phi}_x\mathbf{\Phi}_{xx} + \mathbf{\Phi}_y\mathbf{\Phi}_{yx} + \mathbf{\Phi}_z\mathbf{\Phi}_{zx})$$

$$\frac{\partial \rho}{\partial y} = -\frac{\rho}{a^2}(\mathbf{\Phi}_x\mathbf{\Phi}_{xy} + \mathbf{\Phi}_y\mathbf{\Phi}_{yy} + \mathbf{\Phi}_z\mathbf{\Phi}_{zy})$$

$$\frac{\partial \rho}{\partial z} = -\frac{\rho}{a^2}(\mathbf{\Phi}_x\mathbf{\Phi}_{xz} + \mathbf{\Phi}_y\mathbf{\Phi}_{yz} + \mathbf{\Phi}_z\mathbf{\Phi}_{zz})$$

$$\left(1-\frac{\Phi_x^2}{a^2}\right)\Phi_{xx}+\left(1-\frac{\Phi_y^2}{a^2}\right)\Phi_{yy}+\left(1-\frac{\Phi_z^2}{a^2}\right)\Phi_{zz}$$
$$-\frac{2\Phi_x\Phi_y}{a^2}\Phi_{xy}-\frac{2\Phi_x\Phi_z}{a^2}\Phi_{xz}-\frac{2\Phi_y\Phi_z}{a^2}\Phi_{yz}=0$$

velocity potential equation.

From the energy equation $h_o = \text{const}$

$$c_pT+\frac{V^2}{2}=c_pT_o \qquad \frac{\gamma RT}{\gamma-1}+\frac{V^2}{2}=\frac{\gamma RT_o}{\gamma-1} \qquad \frac{a^2}{\gamma-1}+\frac{V^2}{2}=\frac{a_o^2}{\gamma-1}$$

$$a^2=a_o^2-\frac{\gamma-1}{2}V^2=a_o^2-\frac{\gamma-1}{2}(u^2+v^2+w^2)$$

$$\boxed{a^2=a_o^2-\frac{\gamma-1}{2}\left(\Phi_x^2+\Phi_y^2+\Phi_z^2\right)}$$

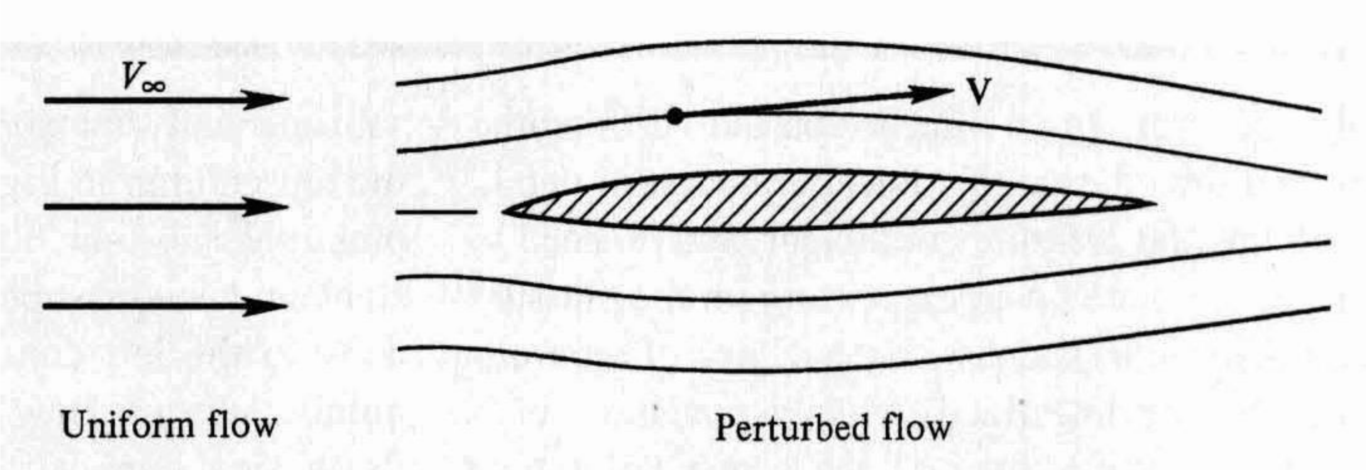
Note that the velocity potential equation is a *nonlinear partial differential* equation. It applies to any irrotational, isentropic flow: subsonic, transonic, supersonic, or hypersonic

It also applies to incompressible flow, where $a \longrightarrow \infty$, ***hence*** yielding the familiar Laplace's equation,

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0$$

There is no general closed-form solution to the velocity potential equation,

Linearized Flow



$$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k},$$

u', v', and w' are the *perturbation velocities* in the x, y, and z directions

$$V_x = V_\infty + u'$$

$$V_y = v'$$

$$V_z = w'$$

In terms of the velocity potential,

$$\nabla\Phi = \mathbf{V} = (V_\infty + u')\mathbf{i} + v'\mathbf{j} + w'\mathbf{k}$$

$$V_x = V_\infty + u' = \frac{\partial \Phi}{\partial x} = V_\infty + \frac{\partial \phi}{\partial x}$$

$$V_y = v' = \frac{\partial \Phi}{\partial y} = \frac{\partial \phi}{\partial y}$$

$$V_z = w' = \frac{\partial \Phi}{\partial z} = \frac{\partial \phi}{\partial z}$$

$$\Phi_{xx} = \frac{\partial^2 \phi}{\partial x^2}$$

$$\Phi_{yy} = \frac{\partial^2 \phi}{\partial y^2}$$

$$\Phi_{zz} = \frac{\partial^2 \phi}{\partial z^2}$$

Inserting in velocity potential equation:

$$\left[a^2 - \left(V_\infty + \frac{\partial \phi}{\partial x}\right)^2\right] \frac{\partial^2 \phi}{\partial x^2} + \left[a^2 - \left(\frac{\partial \phi}{\partial y}\right)^2\right] \frac{\partial^2 \phi}{\partial y^2} + \left[a^2 - \left(\frac{\partial \phi}{\partial z}\right)^2\right] \frac{\partial^2 \phi}{\partial z^2}$$

$$- 2\left(V_\infty + \frac{\partial \phi}{\partial x}\right) \frac{\partial \phi}{\partial y} \frac{\partial^2 \phi}{\partial x \, \partial y} - 2\left(V_\infty + \frac{\partial \phi}{\partial x}\right) \frac{\partial \phi}{\partial z} \frac{\partial^2 \phi}{\partial x \, \partial z} - 2 \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial z} \frac{\partial^2 \phi}{\partial y \, \partial z} = 0$$

The above equation is called the perturbation-velocity potential equation

Since the total enthalpy is constant throughout the flow,

$$h_\infty + \frac{V_\infty^2}{2} = h + \frac{V^2}{2} = h + \frac{(V_\infty + u')^2 + v'^2 + w'^2}{2}$$

$$\frac{a_\infty^2}{\gamma - 1} + \frac{V_\infty^2}{2} = \frac{a^2}{\gamma - 1} + \frac{(V_\infty + u')^2 + v'^2 + w'^2}{2}$$

$$a^2 = a_\infty^2 - \frac{\gamma - 1}{2}(2u'V_\infty + u'^2 + v'^2 + w'^2)$$

$$
\begin{aligned}
&(1 - M_\infty^2)\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \\
&\quad = M_\infty^2\left[(\gamma+1)\frac{u'}{V_\infty} + \left(\frac{\gamma+1}{2}\right)\frac{u'^2}{V_\infty^2} + \left(\frac{\gamma-1}{2}\right)\left(\frac{v'^2 + w'^2}{V_\infty^2}\right)\right]\frac{\partial u'}{\partial x} \\
&\qquad + M_\infty^2\left[(\gamma-1)\frac{u'}{V_\infty} + \left(\frac{\gamma+1}{2}\right)\frac{v'^2}{V_\infty^2} + \left(\frac{\gamma-1}{2}\right)\left(\frac{w'^2 + u'^2}{V_\infty^2}\right)\right]\frac{\partial v'}{\partial y} \\
&\qquad + M_\infty^2\left[(\gamma-1)\frac{u'}{V_\infty} + \left(\frac{\gamma+1}{2}\right)\frac{w'^2}{V_\infty^2} + \left(\frac{\gamma-1}{2}\right)\left(\frac{u'^2 + v'^2}{V_\infty^2}\right)\right]\frac{\partial w'}{\partial z} \\
&\qquad + M_\infty^2\left[\frac{v'}{V_\infty}\left(1 + \frac{u'}{V_\infty}\right)\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right) + \frac{w'}{V_\infty}\left(1 + \frac{u'}{V_\infty}\right)\left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}\right)\right. \\
&\qquad\qquad \left. + \frac{u'w'}{V_\infty^2}\left(\frac{\partial w'}{\partial y} + \frac{\partial v'}{\partial z}\right)\right]
\end{aligned}
$$

We now specialize to the case of *small perturbations,* i.e., we assume the u', v', and w' are small compared to V_∞:

$$\frac{u'}{V_\infty}, \frac{v'}{V_\infty}, \text{ and } \frac{w'}{V_\infty} \ll 1 \qquad \left(\frac{u'}{V_\infty}\right)^2, \left(\frac{v'}{V_\infty}\right)^2, \text{ and } \left(\frac{w'}{V_\infty}\right)^2 \lll 1$$

1. For $0 \leq M_\infty \leq 0.8$ and for $M_\infty \geq 1.2$, the magnitude of

$$M_\infty^2 \left[(\gamma + 1)\frac{u'}{V_\infty} + \cdots\right] \frac{\partial u'}{\partial x}$$

is small in comparison to the magnitude of

$$(1 - M_\infty^2)\,\frac{\partial u'}{\partial x}$$

Thus, *ignore* the former term.

2. For $M_\infty \leq 5$ (approximately),

$$M_\infty^2\left[(\gamma-1)\frac{u'}{V_\infty}+\cdots\right]\frac{\partial v'}{\partial y}$$

is small in comparison to $\partial v'/\partial y$,

$$M_\infty^2\left[(\gamma-1)\frac{u'}{V_\infty}+\cdots\right]\frac{\partial w'}{\partial z}$$

is small in comparison to $\partial w'/\partial z$, and

$$M_\infty^2\left[\frac{v'}{V_\infty}\left(1+\frac{u'}{V_\infty}\right)\left(\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}\right)+\cdots\right]\approx 0$$

$$\boxed{\left(1-M_\infty^2\right)\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0}$$

$$\boxed{\left(1 - M_\infty^2\right)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0}$$

1. The perturbations must be *small*.

2. From item 1 in the list above, we see that *transonic flow* ($0.8 \leq M_\infty \leq 1.2$) is excluded.

3. From item 2 in that same list we see that *hypersonic flow* ($M_\infty \geq 5$) is excluded.

LINEARIZED PRESSURE COEFFICIENT

$$C_p \equiv \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2}$$

$$\frac{1}{2}\rho_\infty V_\infty^2 = \frac{1}{2}\frac{\gamma p_\infty}{\gamma p_\infty}\rho_\infty V_\infty^2 = \frac{\gamma}{2} p_\infty \frac{V_\infty^2}{a_\infty^2} = \frac{\gamma}{2} p_\infty M_\infty^2$$

$$C_p = \frac{p - p_\infty}{(\gamma/2) p_\infty M_\infty^2} = \frac{p_\infty (p/p_\infty - 1)}{(\gamma/2) p_\infty M_\infty^2}$$

$$\boxed{C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)}$$

$$h + \frac{V^2}{2} = h_\infty + \frac{V_\infty^2}{2}$$

$$T + \frac{V^2}{2c_p} = T_\infty + \frac{V_\infty^2}{2c_p}$$

$$T - T_\infty = \frac{V_\infty^2 - V^2}{2c_p} = \frac{V_\infty^2 - V^2}{2\gamma R/(\gamma - 1)}$$

$$\frac{T}{T_\infty} - 1 = \frac{\gamma - 1}{2}\frac{V_\infty^2 - V^2}{\gamma R T_\infty} = \frac{\gamma - 1}{2}\frac{V_\infty^2 - V^2}{a_\infty^2}$$

$$V^2 = (V_\infty + u')^2 + v'^2 + w'^2$$

$$\frac{T}{T_\infty} = 1 - \frac{\gamma - 1}{2a_\infty^2}(2u'V_\infty + u'^2 + v'^2 + w'^2)$$

$$\frac{T}{T_\infty} = 1 - \frac{\gamma - 1}{2a_\infty^2}(2u'V_\infty + u'^2 + v'^2 + w'^2)$$

$$p/p_\infty = (T/T_\infty)^{\gamma/(\gamma-1)}$$

$$\frac{p}{p_\infty} = \left[1 - \frac{\gamma - 1}{2a_\infty^2}(2u'V_\infty + u'^2 + v'^2 + w'^2)\right]^{\gamma/(\gamma-1)}$$

$$u'/V_\infty \ll 1 \text{: } u'^2/V_\infty,\ v'^2/V_\infty^2, \text{ and } w'^2/V_\infty^2 \lll 1$$

$$\frac{p}{p_\infty} = (1 - \varepsilon)^{\gamma/(\gamma-1)}$$

Where *ε is small.*

From the binomial expansion, neglecting higher-order terms,

$$\frac{p}{p_\infty} = 1 - \frac{\gamma}{\gamma - 1}\varepsilon + \cdots$$

$$\frac{p}{p_\infty} = 1 - \frac{\gamma}{2} M_\infty^2 \left(\frac{2u'}{V_\infty} + \frac{u'^2 + v'^2 + w'^2}{V_\infty^2} \right) + \cdots$$

$$C_p = \frac{2}{\gamma M_\infty^2} \left(\frac{p}{p_\infty} - 1 \right)$$

$$\begin{aligned} C_p &= \frac{2}{\gamma M_\infty^2} \left[1 - \frac{\gamma}{2} M_\infty^2 \left(\frac{2u'}{V_\infty} + \frac{u'^2 + v'^2 + w'^2}{V_\infty^2} \right) + \cdots - 1 \right] \\ &= -\frac{2u'}{V_\infty} - \frac{u'^2 + v'^2 + w'^2}{V_\infty^2} + \cdots \end{aligned}$$

Since u'^2/V_∞^2, v'^2/V_∞^2, and $w'^2/V_\infty^2 \lll 1$,

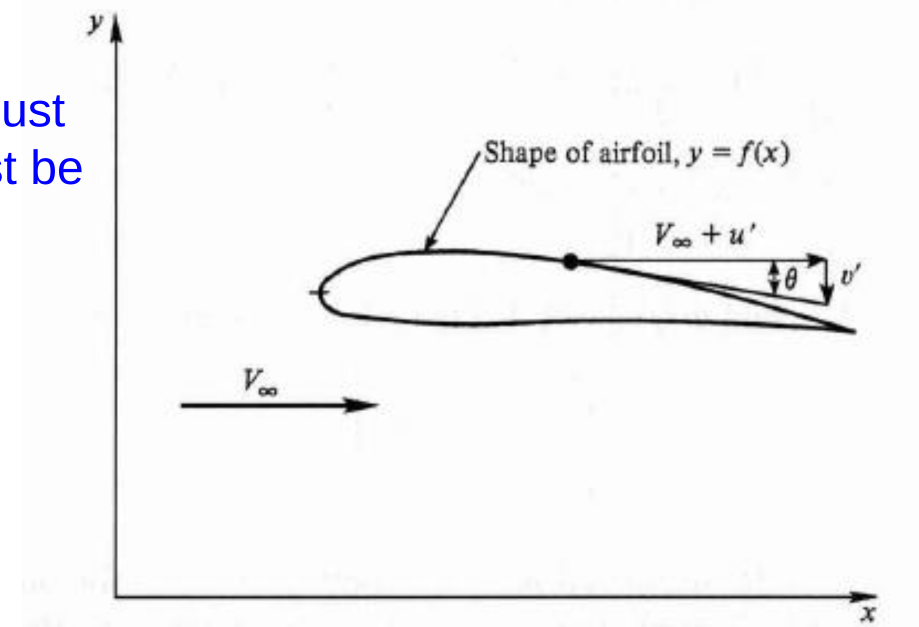
$$\boxed{C_p = -\frac{2u'}{V_\infty}}$$

***linearized pressure coefficient*,**

LINEARIZED SUBSONIC FLOW

Consider the compressible subsonic flow over a thin airfoil at small angle of attack (hence small perturbations),

The usual inviscid flow boundary condition must hold at the surface, i.e., the flow velocity must be tangent to the surface.



$$\frac{df}{dx} = \frac{v'}{V_\infty + u'} = \tan\theta$$

For small perturbations, $u' \ll V_\infty$, and $\tan\theta \approx \theta$

$$\frac{df}{dx} = \frac{v'}{V_\infty} = \theta$$

Since $v' = \partial\phi/\partial y$

$$\frac{\partial\phi}{\partial y} = V_\infty \frac{df}{dx}$$

SIMILARITY LAWS FOR SUBSONIC FLOW

For small-perturbation, linearized, compressible flow

$$(1 - M_\infty^2)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

For incompressible, two-dimensional, steady potential flow,

$$\frac{\partial^2 \phi_i}{\partial x_i^2} + \frac{\partial^2 \phi_i}{\partial y_i^2} = 0$$

Solutions to Laplace's equation are available for a wide variety of boundary conditions. Thus, it would seem logical to try to transform the linearized, compressible potential equation into the incompressible potential equation, so as to utilize available incompressible flow solutions for problems in compressible flow.

Consider a thin body in the (x, y) plane immersed in a uniform compressible flow U_∞ of Mach number M_∞. We shall transform this flow to the incompressible plane (x_i, y_i). Let

$$x_i = k_1 x$$

$$y_i = k_2 y$$

$$\phi_i = k_3 \phi$$

$$U_{\infty_i} = k_4 U_\infty$$

where k_1, k_2, k_3, and k_4 are scale factors of the transformation.

$$\boxed{(1 - M_\infty^2)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0}$$

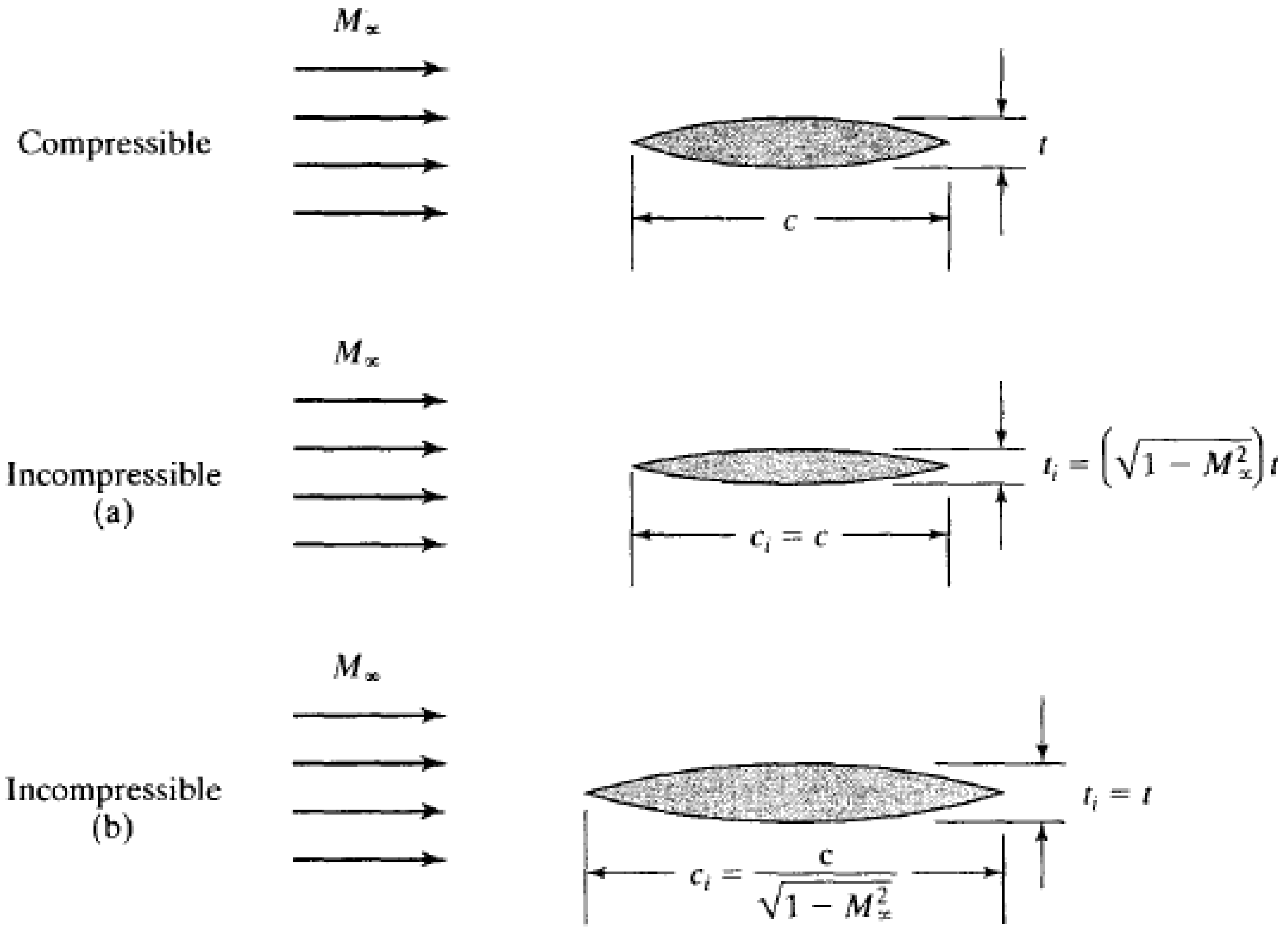
$$k_1^2 \frac{(1 - M_\infty^2)}{k_3} \frac{\partial^2 \phi_i}{\partial x_i^2} + \frac{k_2^2}{k_3} \frac{\partial^2 \phi_i}{\partial y_i^2} = 0$$

In order to transform

$$\frac{k_1}{k_2} = \frac{1}{\sqrt{1 - M_\infty^2}}$$

For example, consider compressible flow over a thin airfoil of chord c and thickness *t.*

$$\frac{t_i}{c_i} = \frac{k_2 t}{k_1 c} = \frac{t}{c}\sqrt{1 - M_\infty^2}$$



for small perturbation flow:

$$\left(\frac{dy}{dx}\right)_b = \frac{v_p}{U_\infty}$$

$$v_p = \partial\phi_p/\partial y.$$

$$\frac{k_1}{k_2}\left(\frac{dy_i}{dx_i}\right)_b = \frac{k_4}{U_\infty}\frac{k_2}{k_3}\frac{\partial\phi_{p_i}}{\partial y_i}$$

To satisfy the boundary conditions at the body surface in the incompressible plane, it is necessary that:

$$\frac{k_1}{k_2} = \frac{k_4 k_2}{k_3}$$

since

$$\frac{k_1}{k_2} = \frac{1}{\sqrt{1 - M_\infty^2}}$$

$$\frac{k_2 k_4}{k_3} = \frac{1}{\sqrt{1 - M_\infty^2}}$$

It is important also to compare the pressure coefficients in the two planes.

For compressible flow, $$C_p = -2u_p/U_\infty$$

For incompressible flow $$p + \tfrac{1}{2}\rho V^2 = \text{constant}$$

By definition $$C_{p_i} = \frac{(p - p_\infty)_i}{\frac{1}{2}\rho U_{\infty i}^2}$$

$$p_{\infty_i} + \frac{1}{2}\rho U_{\infty_i}^2 = p_i + \frac{1}{2}\rho V_i^2 = p_i + \frac{1}{2}\rho\left[(u_{p_i} + U_{\infty_i})^2 + v_{p_i}^2\right]$$

$$(p - p_\infty)_i = -\frac{1}{2}\rho U_{\infty_i}^2\left[2\frac{u_{p_i}}{U_{\infty_i}} + \left(\frac{u_{p_i}}{U_{\infty_i}}\right)^2 + \left(\frac{v_{p_i}}{U_{\infty_i}}\right)^2\right]$$

Dropping smaller terms, we obtain

$$C_{p_i} = -\frac{2u_{p_i}}{U_{\infty_i}}$$

For the compressible plane,

$$\boxed{\frac{1}{\sqrt{1 - M_\infty^2}}}$$

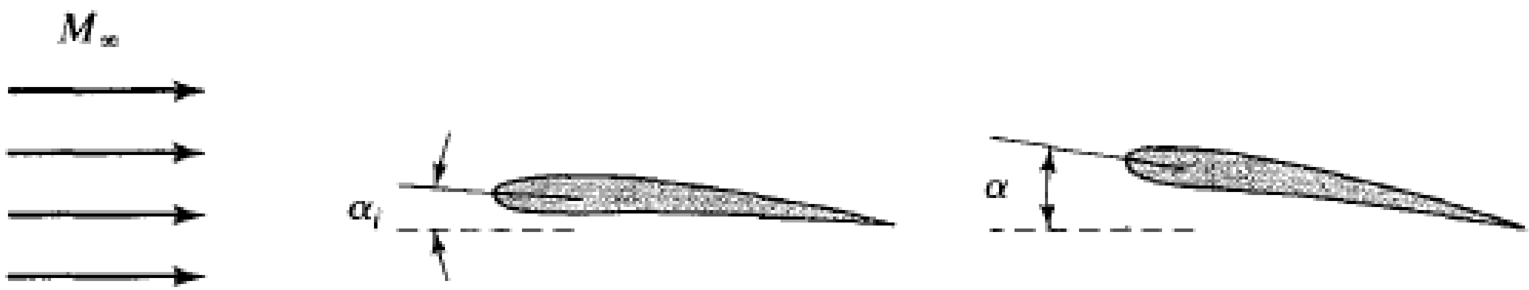
$$C_p = -\frac{2u_p}{U_\infty} = -\frac{2\dfrac{\partial\phi_p}{\partial x}}{U_\infty} = -\frac{2\dfrac{k_1 k_4}{k_3}\dfrac{\partial\phi_{pi}}{\partial x_i}}{U_{\infty i}} = -\frac{k_1}{k_2}\frac{k_2 k_4}{k_3}\frac{2u_{pi}}{U_{\infty i}}$$

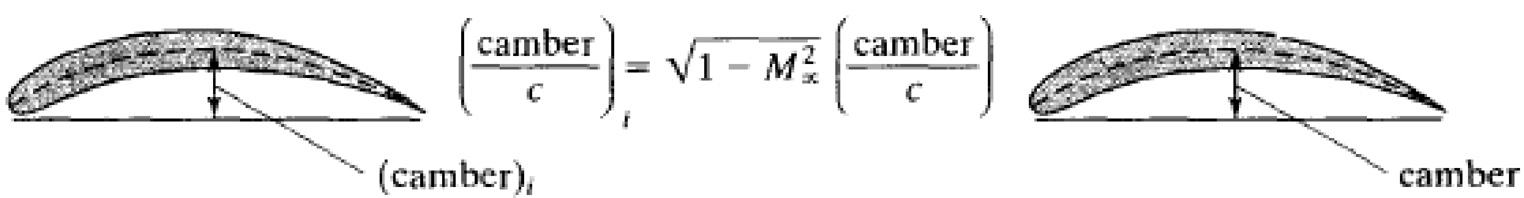
$$\boxed{C_p = \frac{C_{pi}}{1 - M_\infty^2}}$$

The same relationship holds for other airfoil characteristics involving a ratio of the Y dimension to the x dimension.

$$\alpha_i = \alpha\sqrt{1 - M_\infty^2}$$

$$\frac{(\text{camber}/c)_i}{(\text{camber}/c)} = \sqrt{1 - M_\infty^2}$$





- - - - Mean line (equidistant between upper and lower surfaces)

$$L = \int_0^c p_l \, dx - \int_0^c p_u \, dx$$

$$C_L \equiv \frac{L}{\frac{1}{2}\rho U_\infty^2 c} = \frac{L}{\frac{1}{2}\rho U_\infty^2 c} = \int_0^c C_{p_l} \frac{dx}{c} - \int_0^c C_{p_u} \frac{dx}{C}$$

Thus, the lift coefficients for the airfoils in the compressible and incompressible planes are in the same ratio as the pressure coefficient:

$$C_{Li} = C_L(1 - M_\infty^2)$$

The similarity relations are called the Goethert rules.

It must be emphasized that these rules are valid only for thin airfoils with small angles of attack, small camber ratios, and so on.

Example **Application of Goethert similarity rules for subsonic flow past a two-dimensional airfoil**

A two-dimensional airfoil has a thickness ratio (maximum thickness to chord) of 0.04 and a camber ratio of 0.015. When tested in a low-speed wind tunnel (incompressible flow, M_∞ = 0) at an angle of attack of 3°, the lift coefficient C_L is measured to be 0.6. It is desired to determine the performance of a similar airfoil at M_∞ = 0.5. Using the Goethert rules, determine the geometrical characteristics of the related airfoil in compressible flow at the given M_∞, and determine the lift coefficient.

Solution:

$$\sqrt{1 - M_\infty^2} = \sqrt{1 - (0.5)^2} = 0.8660$$

$$\left(\frac{t}{c}\right)_{M_\infty=0.5} = \left(\frac{t}{c}\right)_i \frac{1}{\sqrt{1 - M_\infty^2}} = \left(\frac{t}{c}\right)_i \frac{1}{0.8660} = (0.04)(1.1547) = 0.0462$$

$$\left(\frac{\text{camber}}{c}\right)_{M_\infty=0.5} = \left(\frac{\text{camber}}{c}\right)_i \frac{1}{\sqrt{1 - M_\infty^2}} = (0.015)(1.1547) = 0.0173$$

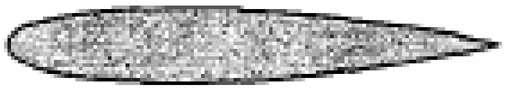
$$\alpha_{M_\infty=0.5} = \alpha_i \frac{1}{\sqrt{1 - M_\infty^2}} = (3)(1.1547) = 3.4641°$$

$$C_{L_{M_\infty=0.5}} = \frac{C_{Li}}{1 - M_\infty^2} = \frac{0.6}{0.75} = 0.80$$

Although the Goethert rules have been shown to possess some application, it would seem far more useful to have a comparison between performance of the same airfoil in compressible and incompressible flows

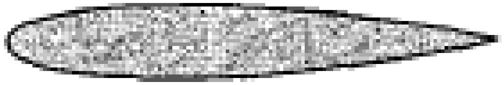
Let us consider two airfoils in the incompressible plane, related to a third foil in the compressible plane. The first incompressible airfoil.

Compressible $\qquad \dfrac{t}{c}$



Incompressible (iA) $\qquad \left(\dfrac{t}{c}\right)_{iA} = \left(\dfrac{t}{c}\right)\sqrt{1 - M_\infty^2}$

Incompressible (iB) $\qquad \left(\dfrac{t}{c}\right)_{iB} = \left(\dfrac{t}{c}\right)$



we need a relation between the pressure coefficients for the incompressible airfoils.

It can be shown that for thin bodies in incompressible, two-dimensional potential flow, related by $(y/c)_i = Kf(x/c)_i$

With f being the same function for all bodies,

$$\frac{\left(\frac{y}{c}\right)_{iB}}{\left(\frac{y}{c}\right)_{iA}} = \frac{K_{iB}}{K_{iA}} = \frac{1}{\sqrt{1 - M_\infty^2}}$$

which means that:

$$C_p = \frac{C_{pi}}{1 - M_\infty^2} \longrightarrow$$

$$\frac{C_{piA}}{C_{piB}} = \frac{K_{iA}}{K_{iB}} = \sqrt{1 - M_\infty^2}$$

$$C_{piA} = C_p(1 - M_\infty^2)$$

$$C_{piB} = C_p\sqrt{1 - M_\infty^2}$$

$$C_{piA} = C_p(1 - M_\infty^2)$$

$$C_{piB} = C_p\sqrt{1 - M_\infty^2}$$

$$\boxed{C_p = \frac{C_{pi}}{\sqrt{1 - M_\infty^2}}}$$

This expression relates the pressure coefficient on a body immersed in two-dimensional compressible flow of Mach number M_∞ to the pressure coefficient on the same body immersed in an incompressible flow.

Let us now attempt to determine the shape of a third incompressible airfoil call it C-that has the same pressure coefficient at corresponding points as the compressible foil

$$C_{piC} = C_p \longrightarrow \frac{C_{piC}}{C_{piB}} = \frac{1}{1 - M_\infty^2} \longrightarrow \frac{C_{piC}}{C_{piB}} = \frac{\left(\frac{y}{c}\right)_{iC}}{\left(\frac{y}{c}\right)_{iB}} = \frac{\left(\frac{t}{c}\right)_{iC}}{\left(\frac{t}{c}\right)_{iB}}$$

Therefore, for $C_{piC} = C_{p}$

$$\left(\frac{t}{c}\right)_{iC} = \frac{1}{\sqrt{1 - M_\infty^2}}\left(\frac{t}{c}\right)_{iB} = \frac{1}{\sqrt{1 - M_\infty^2}}\frac{t}{c}$$

$$\alpha_{iC} = \frac{1}{\sqrt{1 - M_\infty^2}}\alpha$$

$$\left(\frac{\text{camber}}{c}\right)_{iC} = \frac{1}{\sqrt{1 - M_\infty^2}}\frac{\text{camber}}{c}$$

$$C_p = \frac{C_{pi}}{\sqrt{1 - M_\infty^2}}$$

The similarity laws for subsonic compressible flow, as given by Eqs. are called the Prandtl-Glauert rules.

Example **Application of the Prandtl-Glauert similarity rule for subsonic Dowpast a two-dimensional airfoil**

The two-dimensional airfoil of the previous example, when tested in a low-speed wind tunnel at an angle of attack of 3°, is found to have a lift coefficient CL of 0.6. Determine the lift coefficient of the same airfoil at M∞ = 0.50.

Solution

From the Prandtl-Glauert similarity rule,

$$C_{L_{M_\infty=0}} = C_{Li} = C_L\sqrt{1 - M_\infty^2} \longrightarrow C_L = \frac{0.6}{\sqrt{1-(0.5)^2}} = \frac{0.6}{0.8660} = 0.6928$$

we compared the lift coefficients of two different airfoils, one in incompressible flow and the other, of different dimensions, in compressible flow

IMPROVED COMPRESSIBILITY CORRECTIONS

The importance of accurate compressibility corrections reached new highs during the rapid increase in airplane speeds spurred by World War II.
Efforts were made to improve upon the Prandtl-Glauert rule discussed prevously the more popular formulas are given below.

In an effort to obtain an improved compressibility correction, Laitone applied locally in the flow, i.e.,

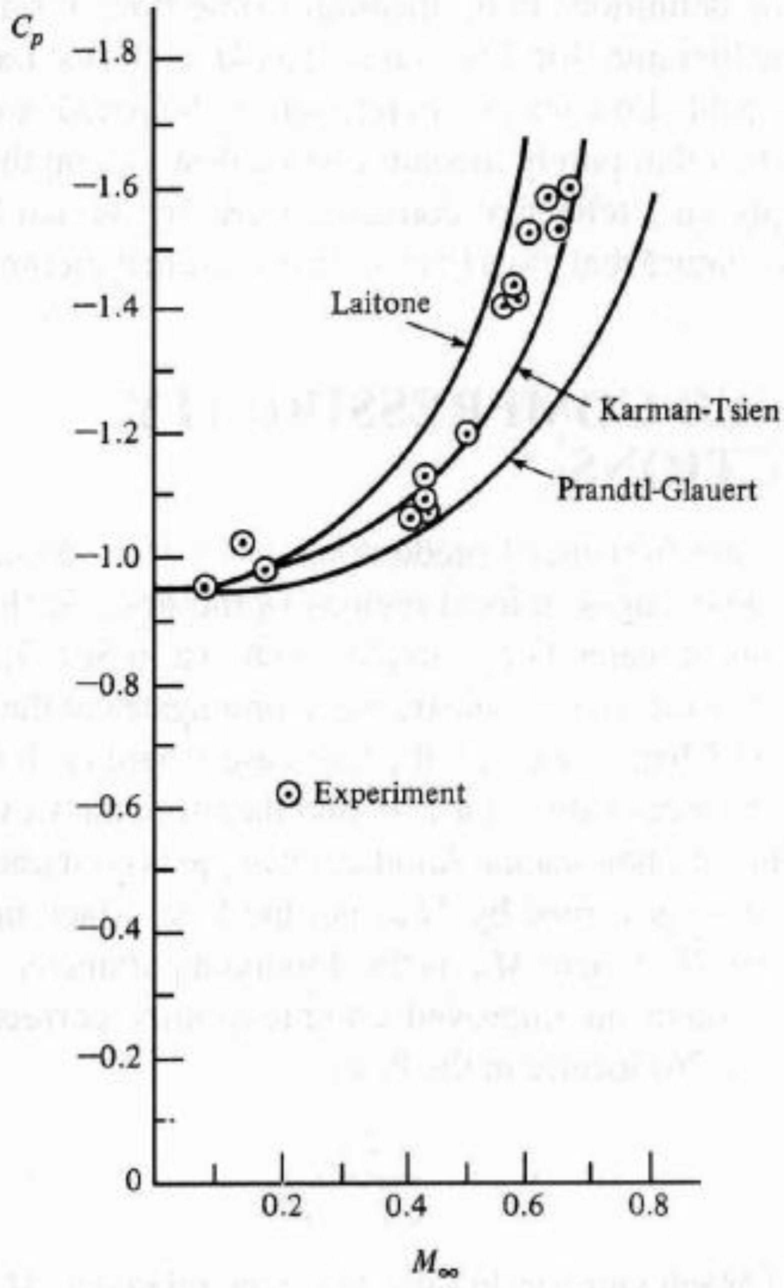
$$C_p = \frac{C_{p_o}}{\sqrt{1 - M^2}}$$

where M is the local Mach number. In turn, M can be related to M_∞ and the pressure coefficient through the isentropic flow relations.
The resulting compressibility correction is:

$$C_p = \frac{C_{p_o}}{\sqrt{1 - M_\infty^2} + \left[M_\infty^2 \left(1 + \frac{\gamma - 1}{2} M_\infty^2\right) \Big/ 2\sqrt{1 - M_\infty^2}\right] C_{p_o}}$$

Another compressibility correction that has been adopted widely is that due to von Karman and Tsien

$$C_p = \frac{C_{p_o}}{\sqrt{1 - M_\infty^2} + \left(\frac{M_\infty^2}{1 + \sqrt{1 - M_\infty^2}}\right)\frac{C_{p_o}}{2}}$$



Comparison of several compressibility corrections with experiment for an NACA 4412 airfoil at an angle of attack α = 1°53'.

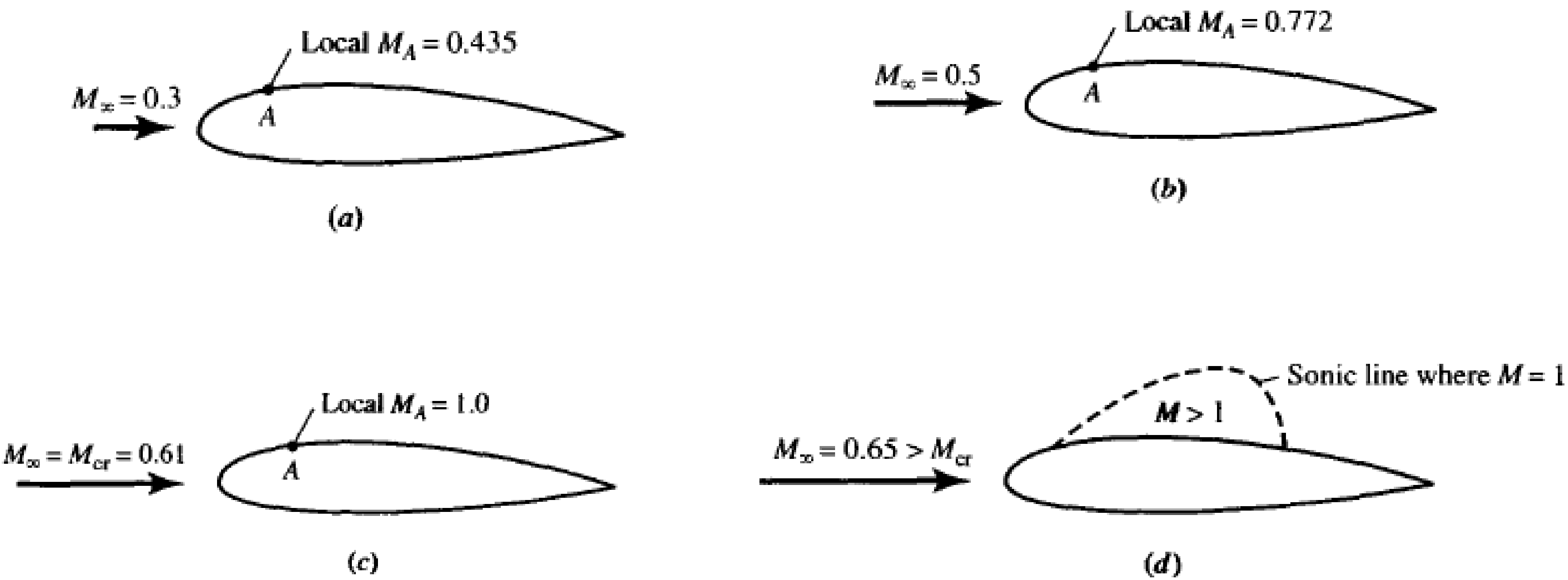
Note that the Prandtl-Glauert rule, although the simplest to apply, under predicts the experimental values, whereas the mproved compressibility corrections are clearly more accurate.

This is because both the Laitone and Karman-Tsien rules bring in the nonlinear aspects of the flow.

CRITICAL MACH NUMBER

We have now finished our discussion of linearized flow and the associated compressibility corrections. such linearized theory does not apply to the transonic flow regime

$$0.8 \le M_\infty \le 1.2.$$



Linearized flow will fail even for M∞ lower than 0.8

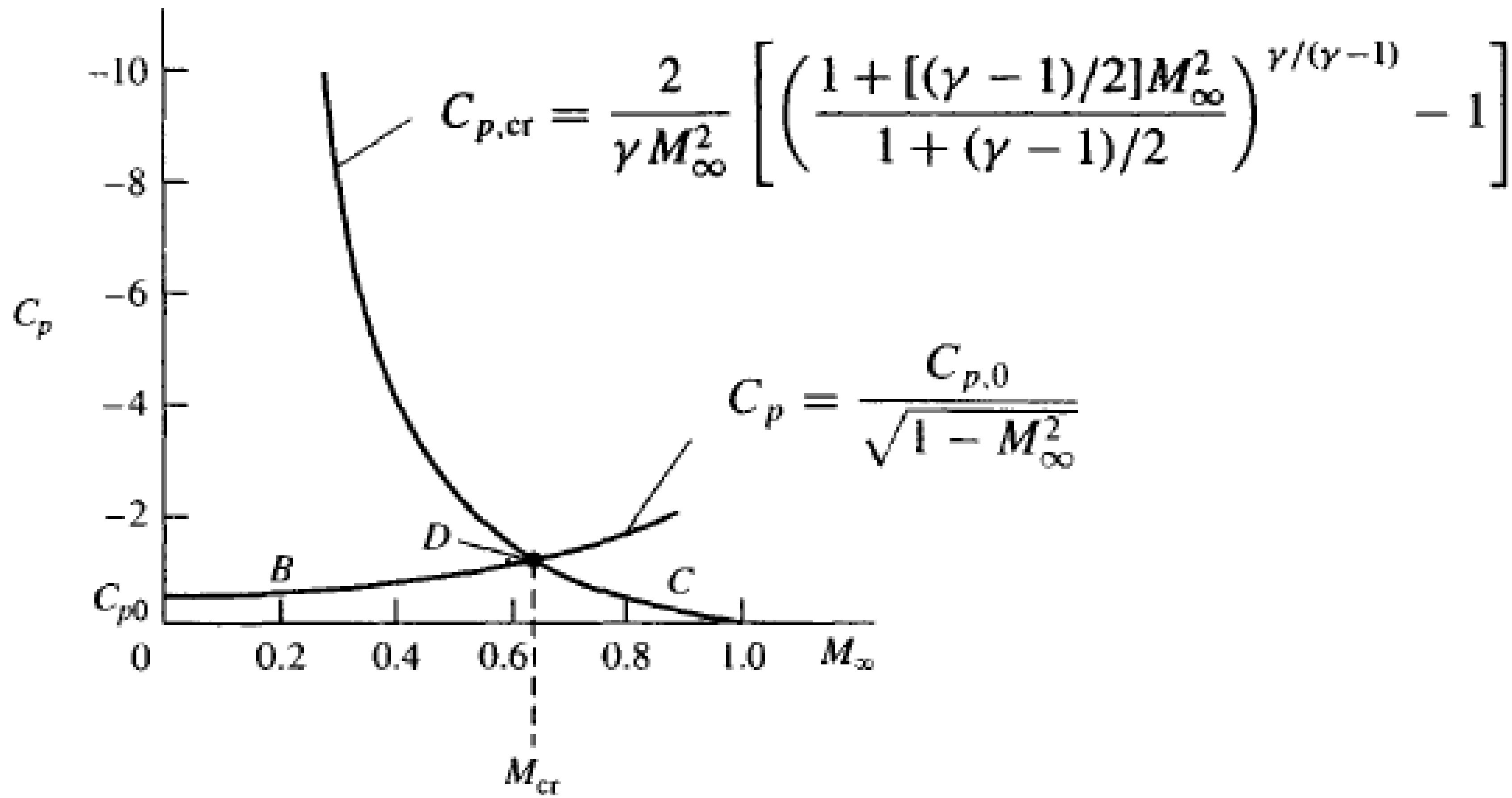
Let p_∞ and p_A represent the static pressures in the freestream and at point A

For isentropic flow,

$$\frac{p_A}{p_\infty} = \frac{p_A/p_0}{p_\infty/p_0} = \left(\frac{1+[(\gamma-1)/2]M_\infty^2}{1+[(\gamma-1)/2]M_A^2}\right)^{\gamma/(\gamma-1)}$$

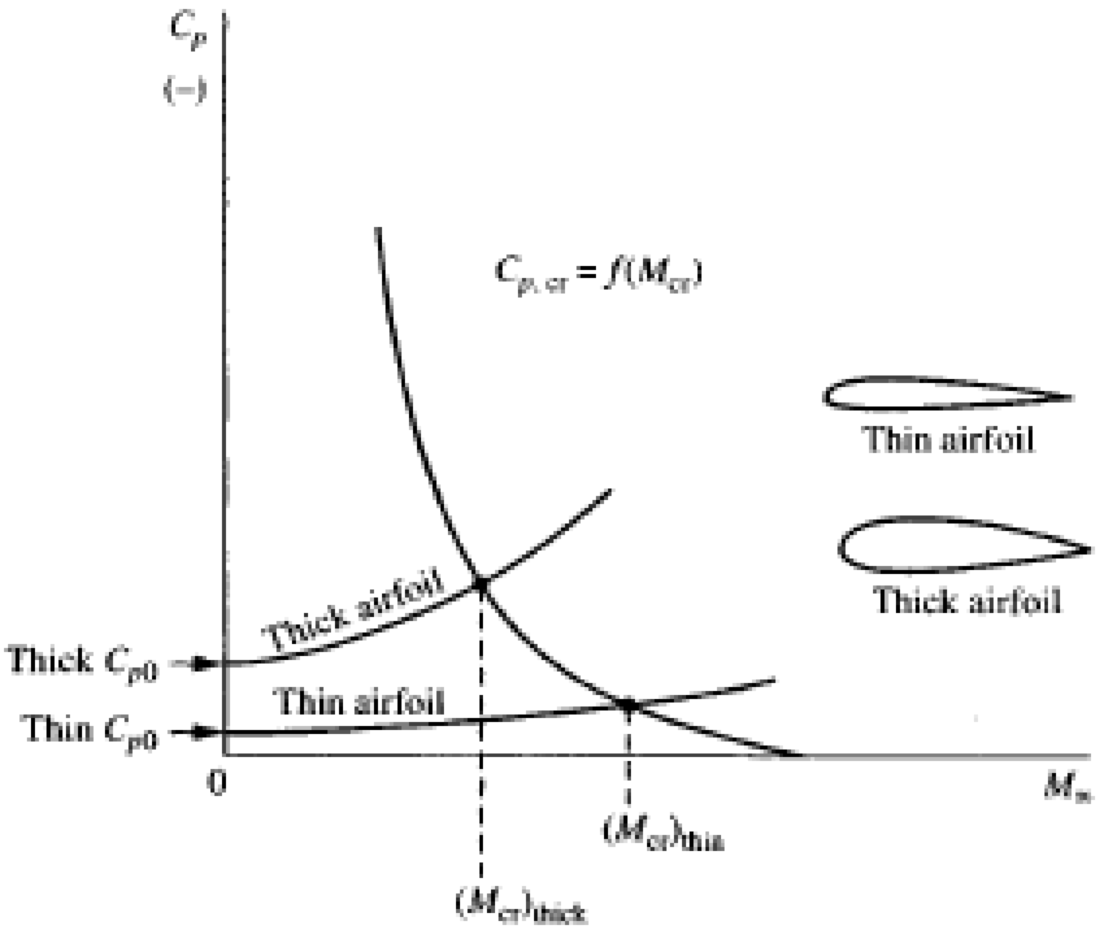
$$C_{p,A} = \frac{2}{\gamma M_\infty^2}\left(\frac{p_A}{p_\infty} - 1\right)$$

$$C_{p,A} = \frac{2}{\gamma M_\infty^2}\left[\left(\frac{1+[(\gamma-1)/2]M_\infty^2}{1+[(\gamma-1)/2]M_A^2}\right)^{\gamma/(\gamma-1)} - 1\right]$$



$C_{p,\mathrm{cr}} = \dfrac{2}{\gamma M_\infty^2}\left[\left(\dfrac{1+[(\gamma-1)/2]M_\infty^2}{1+(\gamma-1)/2}\right)^{\gamma/(\gamma-1)} - 1\right]$
$C_p = \dfrac{C_{p,0}}{\sqrt{1-M_\infty^2}}$
C_p
-10
-8
-6
-4
-2
C_{p0}
0
0.2
0.4
0.6
0.8
1.0
M_∞
B
C
D
M_{cr}

For high-speed airplanes, it is desirable to have M_{cr} as high as possible. Hence, modern high-speed subsonic airplanes are usually designed with relatively thin airfoils.



Effect of airfoil thickness on critical Mach number.

Example

In this, we illustrate the estimation of the critical Mach number for an airfoil using

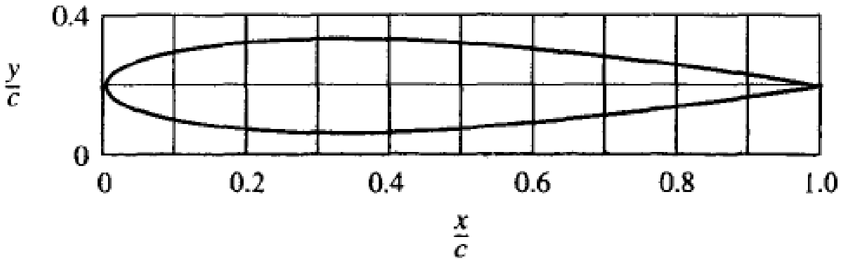
*(a) **the graphical solution discussed in this section, and***

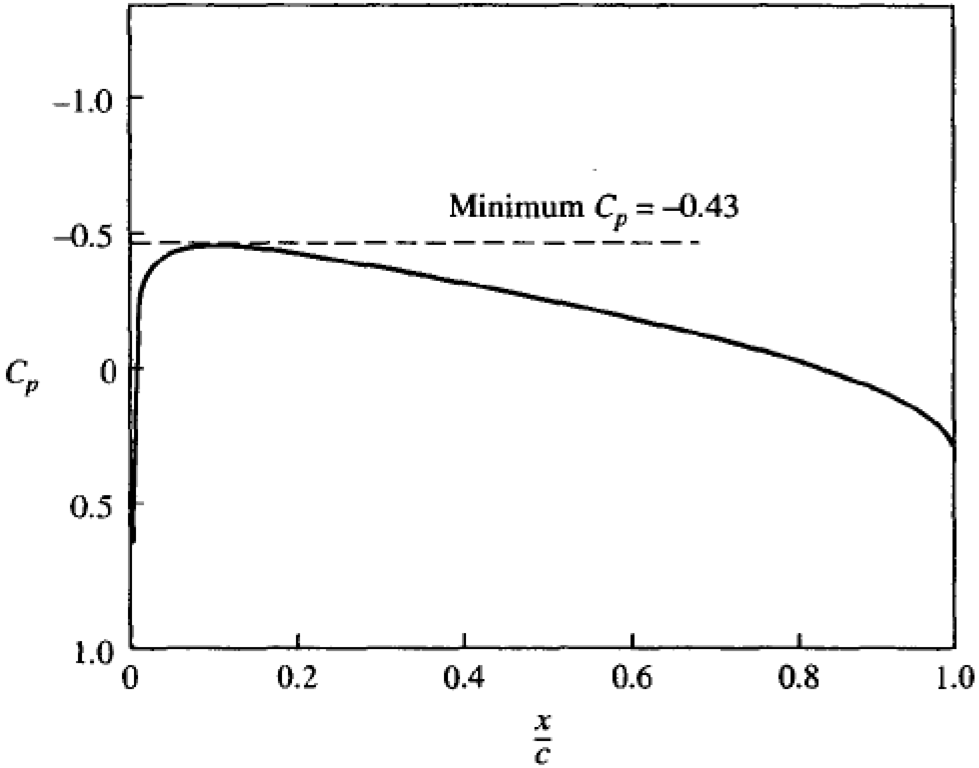
(b) an analytical solution using

a closed-form equation

Consider the NACA 0012 airfoil at zero angle of attack

The pressure coefficient distribution over this airfoil, measured in a wind tunnel at low speed, is given. From this information, estimate the critical Mach number of the NACA 0012 airfoil at zero angle of attack.





(a) Graphical Solution. ***First, let us accurately plot the curve of Cp_{Cr} versus M_{cr} from*** **Equation**

$$C_{p,\text{cr}} = \frac{2}{\gamma M_{\text{cr}}^2}\left[\left(\frac{1+[(\gamma-1)/2]M_{\text{cr}}^2}{1+(\gamma-1)/2}\right)^{\gamma/(\gamma-1)} - 1\right]$$

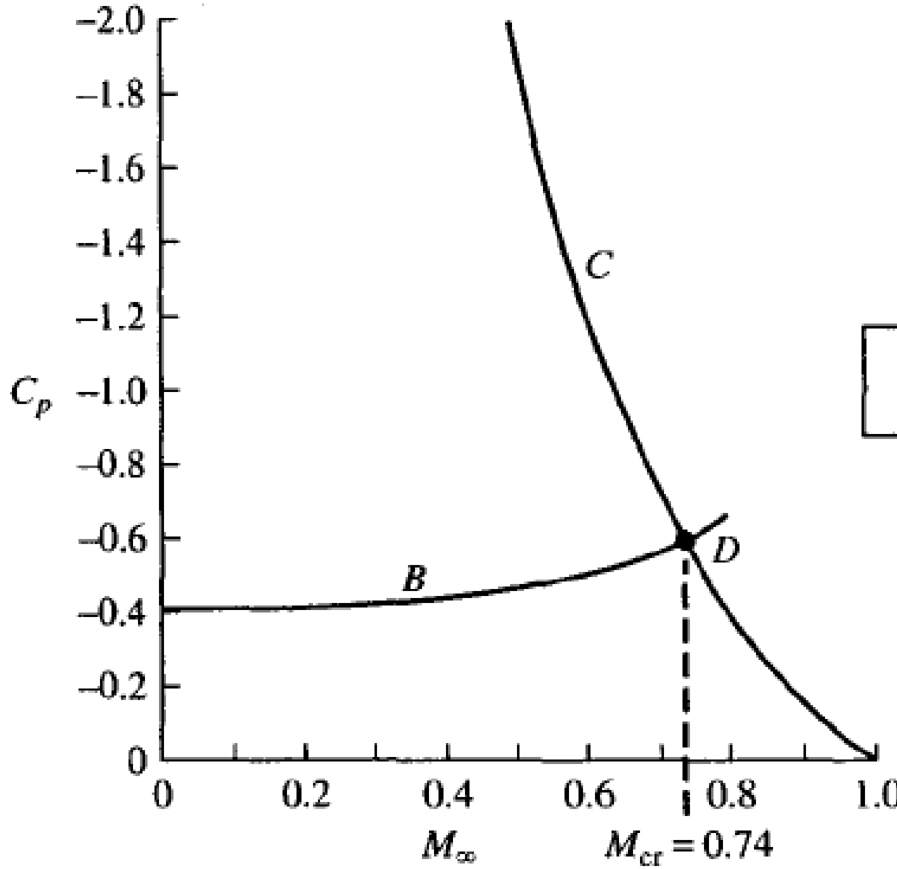
M_∞	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$C_{p,\text{cr}}$	−3.66	−2.13	−1.29	−0.779	−0.435	−0.188	0

From the pressure coefficient distribution given in the figure.
The minimum value of Cp on the surface is -0.43.

$$(C_{p,0})_{\text{min}} = -0.43.$$

$$(C_p)_{\text{min}} = \frac{(C_{p,0})_{\text{min}}}{\sqrt{1-M_\infty^2}} = \frac{-0.43}{\sqrt{1-M_\infty^2}}$$

M_∞	0	0.2	0.4	0.6	0.8
$(C_p)_{\text{min}}$	−0.43	−0.439	−0.469	−0.538	−0.717



$$M_{cr} = 0.74$$

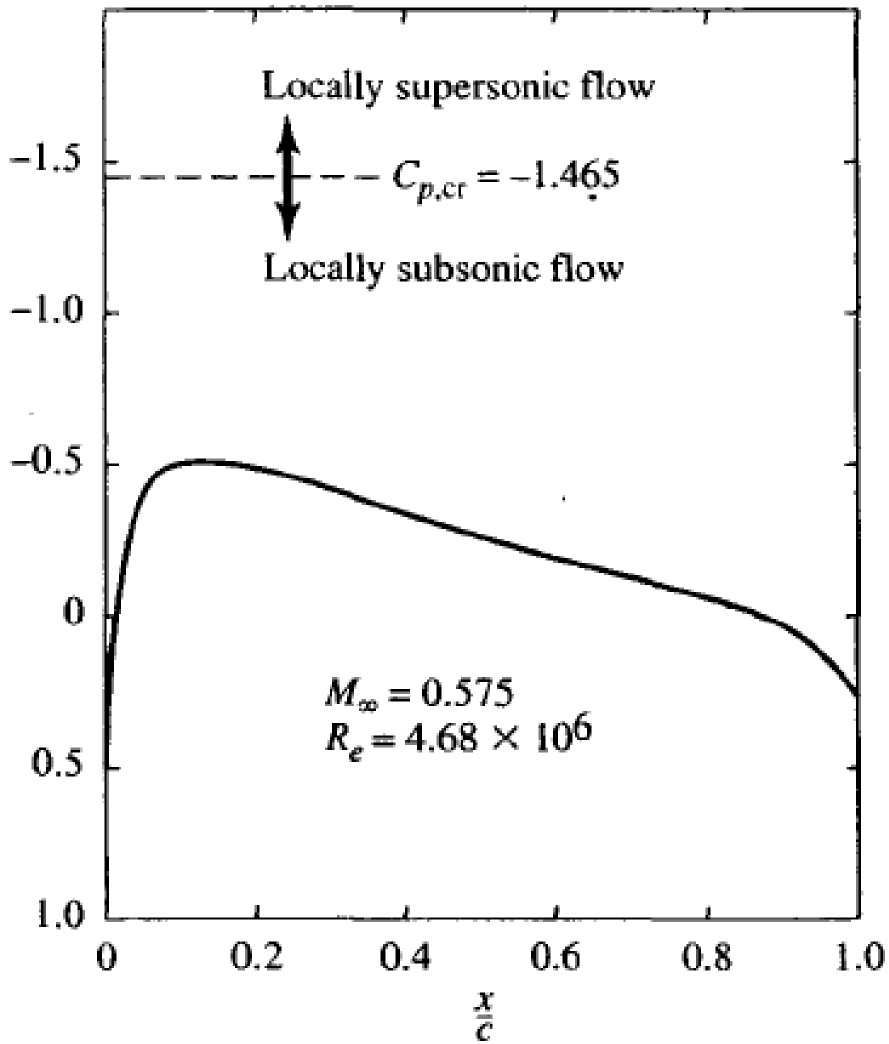
(b) Analytical Solution.

$$(C_p)_{\min} = \frac{-0.43}{\sqrt{1-M_\infty^2}} \Rightarrow \frac{-0.43}{\sqrt{1-M_{cr}^2}} = \frac{2}{\gamma M_{cr}^2}\left[\left(\frac{1+[(\gamma-1)/2]M_{cr}^2}{1+(\gamma-1)/2}\right)^{\gamma/\gamma-1} - 1\right]$$

$$\Rightarrow \boxed{M_{cr} = 0.7371}$$

***Question:* How accurate is the estimate of the critical Mach number in this example?**

Wind tunnel measurements of the surface pressure distributions for this airfoil at zero angle of attack



Locally supersonic flow
$C_{p,cr} = -1.465$
Locally subsonic flow
$M_\infty = 0.575$
$R_e = 4.68 \times 10^6$
-1.5
-1.0
-0.5
0
0.5
1.0
0
0.2
0.4
0.6
0.8
1.0
$\frac{x}{c}$

Linearized perturbation-velocity potential equation

$$\beta^2\phi_{xx} + \phi_{yy} = 0$$ where $\beta = \sqrt{1 - M_\infty^2}$. For subsonic flow

$$\lambda^2\phi_{xx} - \phi_{yy} = 0$$ where $\lambda = \sqrt{M_\infty^2 - 1}$ For supersonic flow,

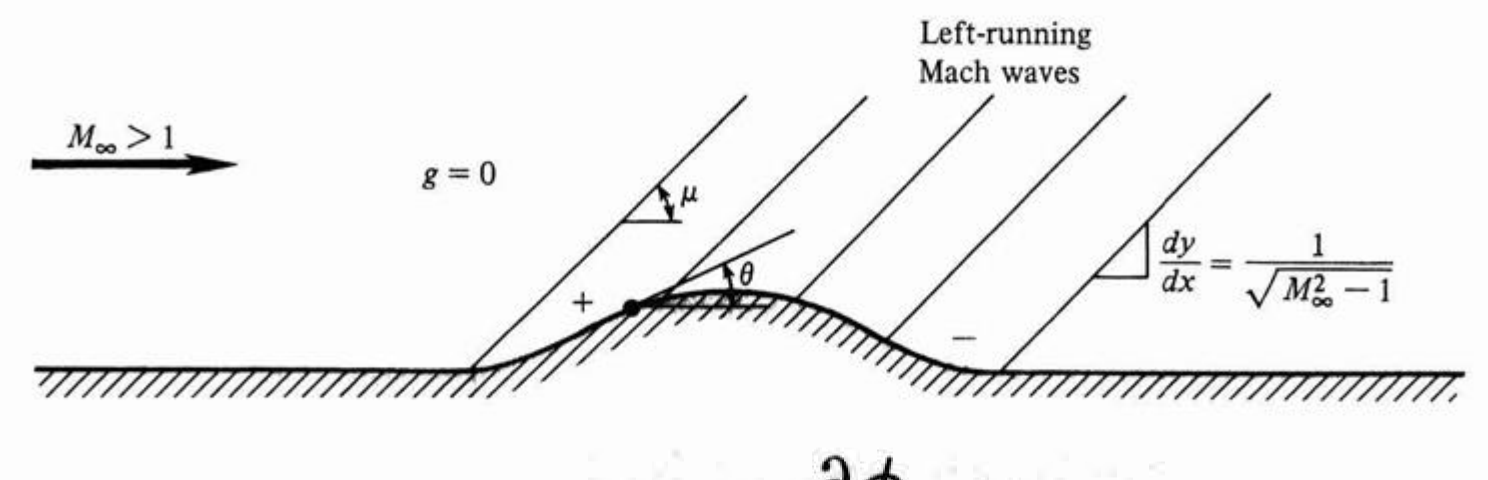
$$\phi = f(x - \lambda y) + g(x + \lambda y)$$

Examining the particular

solution where $g = 0$, and hence $\phi = f(x - \lambda y)$, we see that lines of constant ϕ correspond to $x - \lambda y = \text{const}$, or

$$\frac{dy}{dx} = \frac{1}{\lambda} = \frac{1}{\sqrt{M_\infty^2 - 1}}$$

Consider the supersonic flow over a body or surface which introduces small changes in the flowfield, i.e., flow over a thin airfoil, over a mildly wavy wall, or over a small hump in a surface.



$$\phi = f(x - \lambda y)$$

$$u' = \frac{\partial \phi}{\partial x} = f'$$

$$v' = \frac{\partial \phi}{\partial y} = -\lambda f'$$

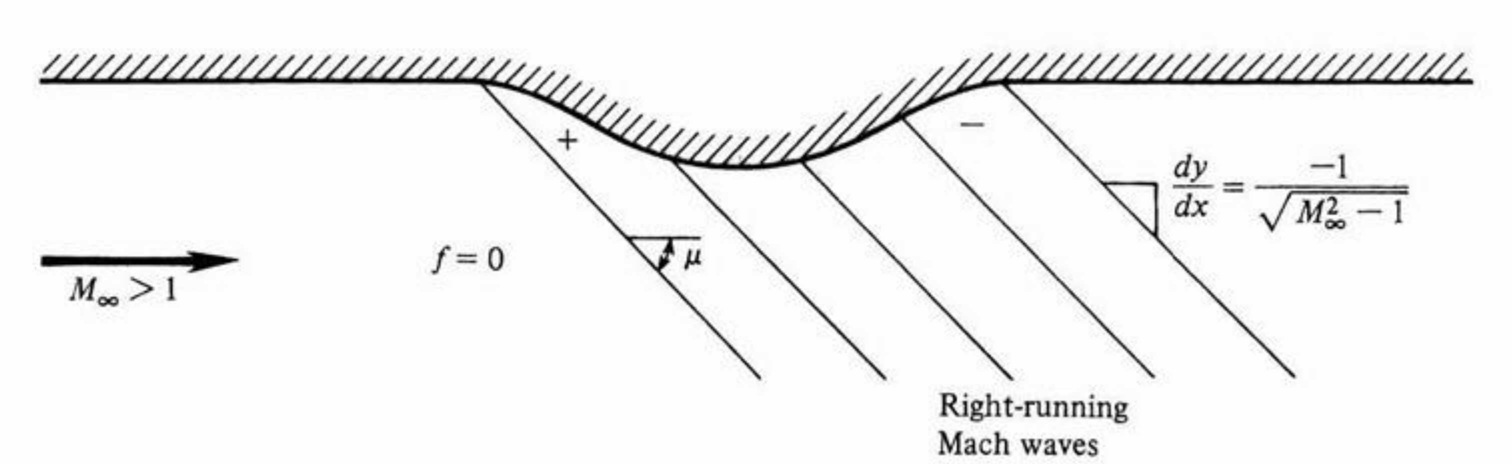
$$u' = -\frac{v'}{\lambda}$$

$$\tan\theta = \frac{dy}{dx} = \frac{v'}{V_\infty + u'}$$

For small perturbations, $u' \ll V_\infty$ and $\tan\theta \approx \theta$.

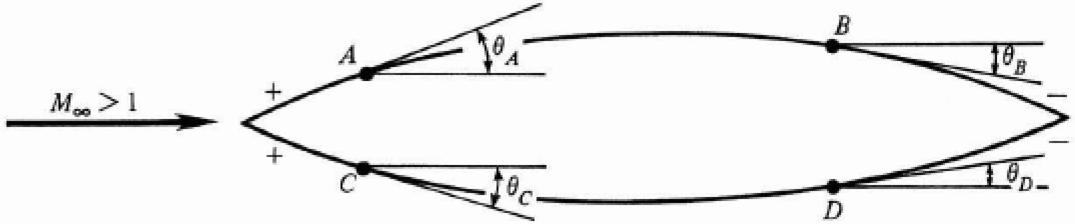
$$v' = V_\infty \theta \quad \xrightarrow{u' = -\frac{v'}{\lambda}} \quad u' = -\frac{V_\infty \theta}{\lambda} \quad \rightarrow \quad C_p = -\frac{2u'}{V_\infty} = \frac{2\theta}{\lambda}$$

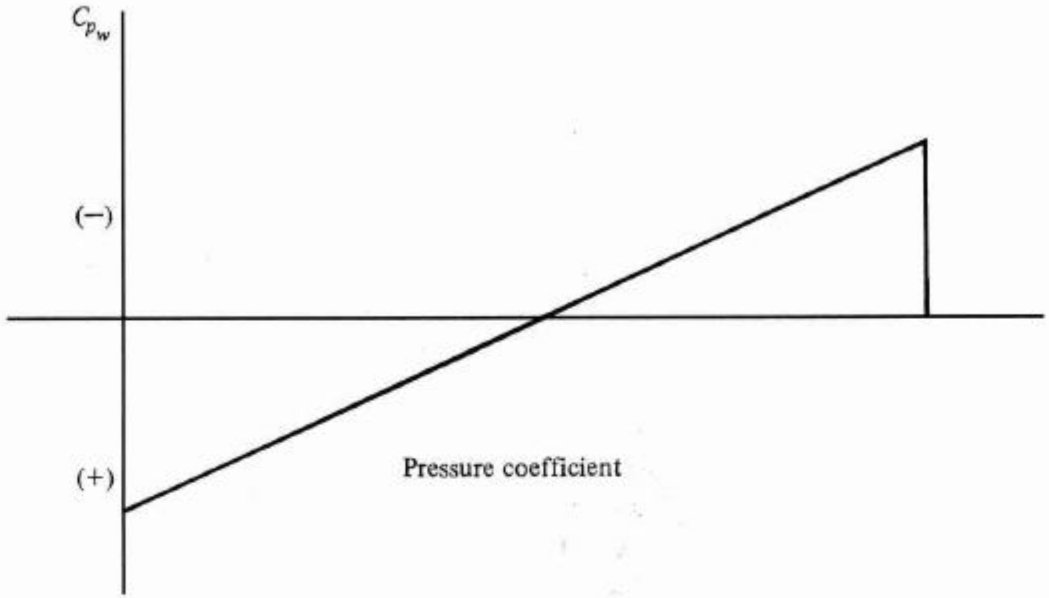
$$\boxed{C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}}$$



$$\phi = f(x - \lambda y) + g(x + \lambda y) \xrightarrow{f = 0} \phi = g(x + \lambda y)$$

$$C_p = \frac{-2\theta}{\sqrt{M_\infty^2 - 1}}$$





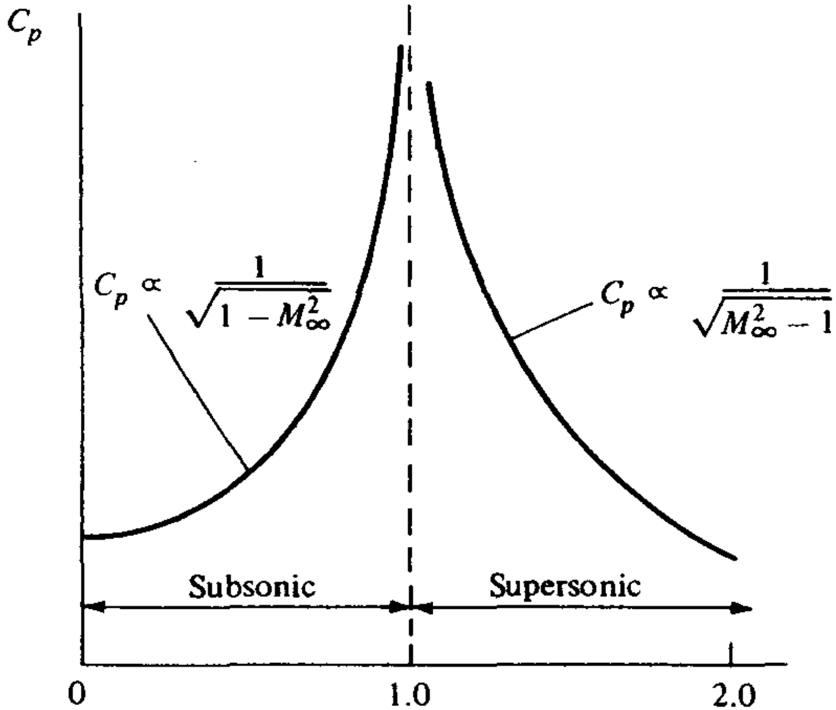
$$C_{pA} = \frac{2\theta_A}{\sqrt{M_\infty^2 - 1}}$$

$$C_{pB} = \frac{2\theta_B}{\sqrt{M_\infty^2 - 1}}$$

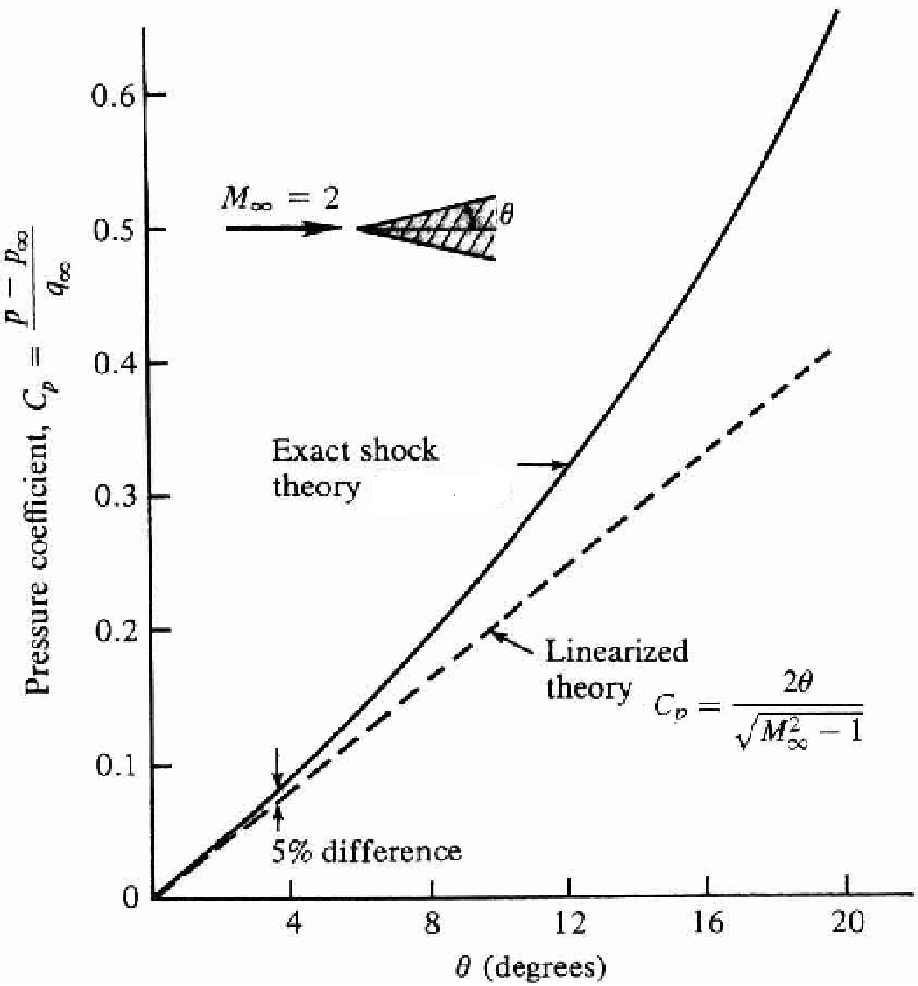
There is no real need to worry about the formal sign
. For any practical application, it is suggested the use of the Eq. along the body with common sense to single out the compression and expansion surfaces on a body.

If the surface is a compression surface, Cp must be positive, no matter whether the surface is on the top or bottom of the body.

Similarly, if the surface is an expansion surface, Cp must be negative.



C_p
$C_p \propto \frac{1}{\sqrt{1 - M_\infty^2}}$
$C_p \propto \frac{1}{\sqrt{M_\infty^2 - 1}}$
Subsonic
Supersonic
0
1.0
2.0



Pressure coefficient, $C_p = \frac{p - p_\infty}{q_\infty}$
0.6
0.5
0.4
0.3
0.2
0.1
0
$M_\infty = 2$
θ
Exact shock theory
Linearized theory
$C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}$
5% difference
4
8
12
16
20
θ (degrees)

$$c_n = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}} \frac{1}{c} \int_0^c dx = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}$$

The axial force coefficient

$$c_a = \frac{1}{c} \int_{\mathrm{LE}}^{\mathrm{TE}} (C_{p,u} - C_{p,l})\, dy$$

the pressures act normal to the surface, and hence there is no component of the pressure force in the x direction. $c_a = 0.$

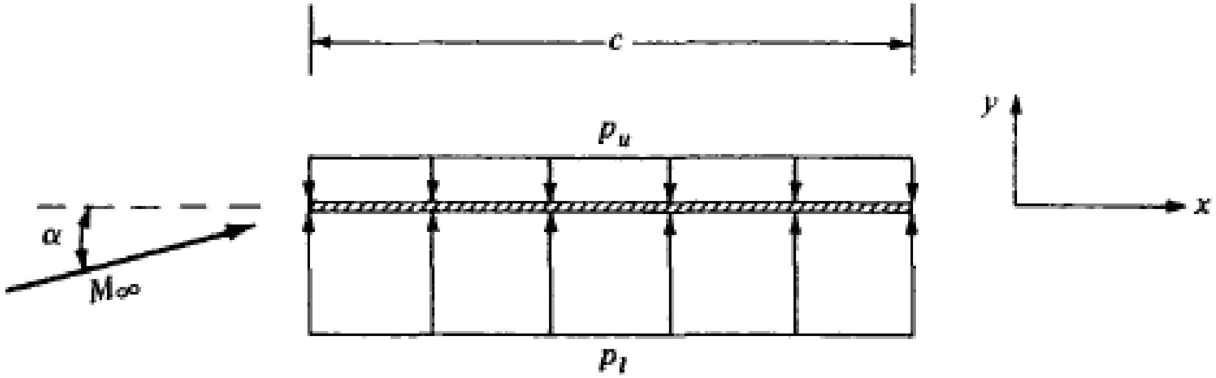
$$c_l = c_n \cos\alpha - c_a \sin\alpha$$

$$c_d = c_n \sin\alpha + c_a \cos\alpha$$

The assumption that a is small $\cos\alpha \approx 1$ and $\sin\alpha \approx \alpha$.

Example

Consider a flat plate at angle of attack in a supersonic flow. Calculate lift and drag coefficient



$$C_{p,l} = \frac{2\alpha}{\sqrt{M_\infty^2 - 1}}$$

$$C_{p,u} = -\frac{2\alpha}{\sqrt{M_\infty^2 - 1}}$$

The normal force coefficient

$$c_n = \frac{1}{c}\int_0^c (C_{p,l} - C_{p,u})\,dx$$

$$c_l = c_n - c_a\alpha$$

$$c_d = c_n\alpha + c_a$$

$$\boxed{c_l = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}}$$

$$\boxed{c_d = \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}}}$$

Within the approximation of linearized theory, cl depends only on a and is independent of the airfoil shape and thickness. However, the same linearized theory gives a wave-drag coefficient in the form of

$$c_d = \frac{4}{\sqrt{M_\infty^2 - 1}}(\alpha^2 + g_c^2 + g_t^2)$$

where g_c and g_t are functions of the airfoil camber and thickness.

COMPRESSIBLE FLOW WITH HEAT TRANSFER

- We have studied the effects of area change and friction on a gas flow. For these cases, flows were assumed to be adiabatic.

- In this session, the effect of heat addition or loss on a one-dimensional gas flow will be investigated. To isolate the effects of heat transfer from the other major factors we assume flow in a *constant-area duct without friction*.

- At first this may seem to be an unrealistic situation, but actually it is a good first approximation to many real problems, as most heat exchangers have constant-area flow passages. It is also a simple and reasonably equivalent process for a constant-area combustion chamber.

- In systems where *high rates of heat transfer* occur, the entropy change caused by the heat transfer is much greater than that caused by friction, or

$$ds_e \gg ds_i$$

Thus

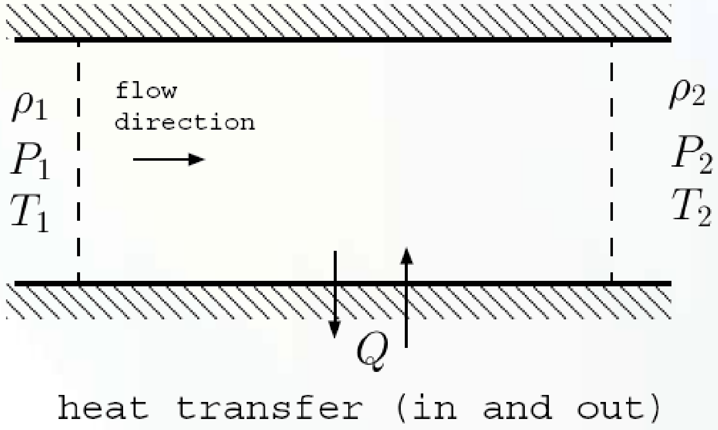
$$ds \approx ds_e$$

and the frictional effects may be neglected.

COMPRESSIBLE FLOW WITH HEAT TRANSFER

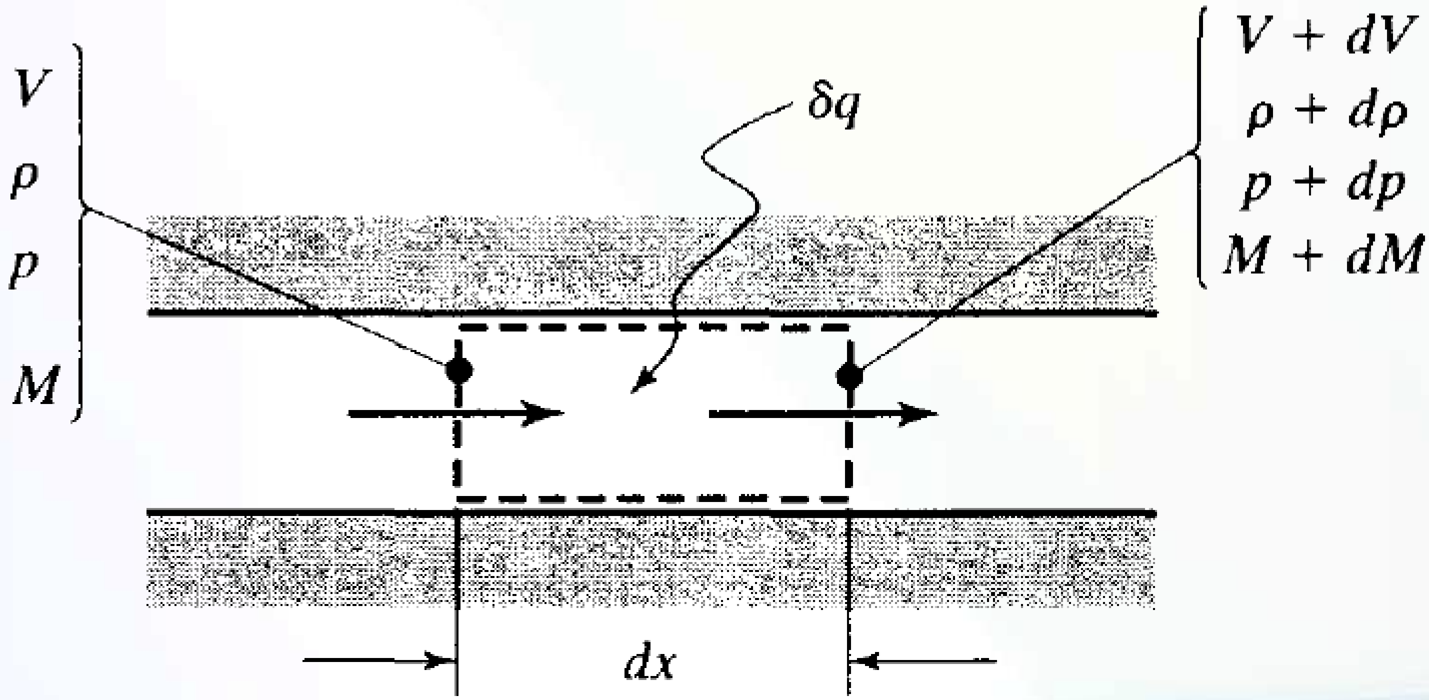
o To isolate the effects of heat transfer we make the following assumptions

- ❖ Perfect gas with constant specific heats
- ❖ Steady flow
- ❖ One-dimensional flow
- ❖ Constant area $dA = 0$
- ❖ Negligible friction $ds_i \approx 0$
- ❖ No shaft work $\delta w_s = 0$
- ❖ Neglect potential $dz = 0$



o We proceed by applying the basic concepts of continuity, energy, and momentum.

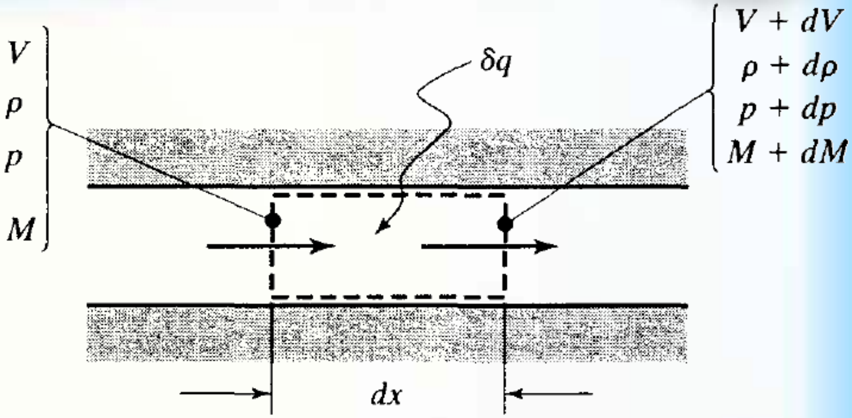
- Consider the control volume shown in Figure, in which a small amount of heat δq, expressed in joules per kilogram (J/kg), is added to the flow.



CONTINUITY EQUATION

- The continuity equation is

$$\rho V = (\rho + d\rho)(V + dV)$$



so that

$$\frac{d\rho}{\rho} + \frac{dV}{V} = 0$$

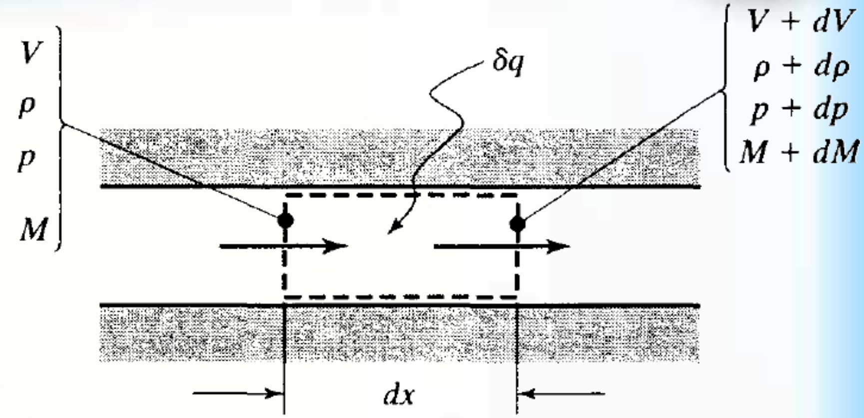
or

$$\rho V = G = \text{const}$$

MOMENTUM EQUATION

- The momentum equation is

$$dp + \rho V\, dV = 0$$



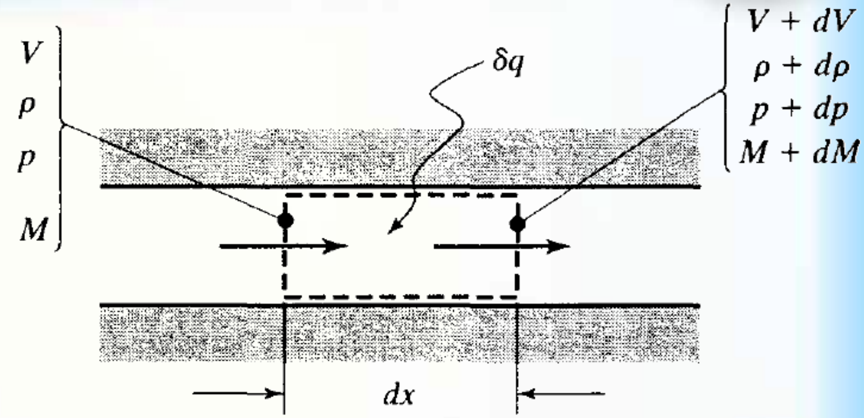
Since the only forces acting on the control volume are pressure forces. With ρV = constant, we can integrate the preceding equation to obtain

$$p + \rho V^2 = \text{constant}$$

$$p + \frac{G^2}{\rho} = \text{const}$$

- The energy equation is

$$\delta q = dh + V\,dV$$



For a perfect gas,

$$dh = c_p\,dT$$

So that

$$\delta q = c_p dT + V\,dV$$

From the definition of stagnation enthalpy and stagnation temperature

$$h_o = c_p T_o = h + \frac{V^2}{2} = c_p T + \frac{V^2}{2} \longrightarrow c_p dT_o = c_p dT + V\,dV$$

$$\delta q = c_p dT_o$$

This is the first major flow category for which the total enthalpy has *not* been constant.

- It is now necessary to express the basic equations in terms of Mach number and arrange the results into a form suitable for application to engineering problems.

- Let's discuss the Rayleigh line on a *p-v* diagram

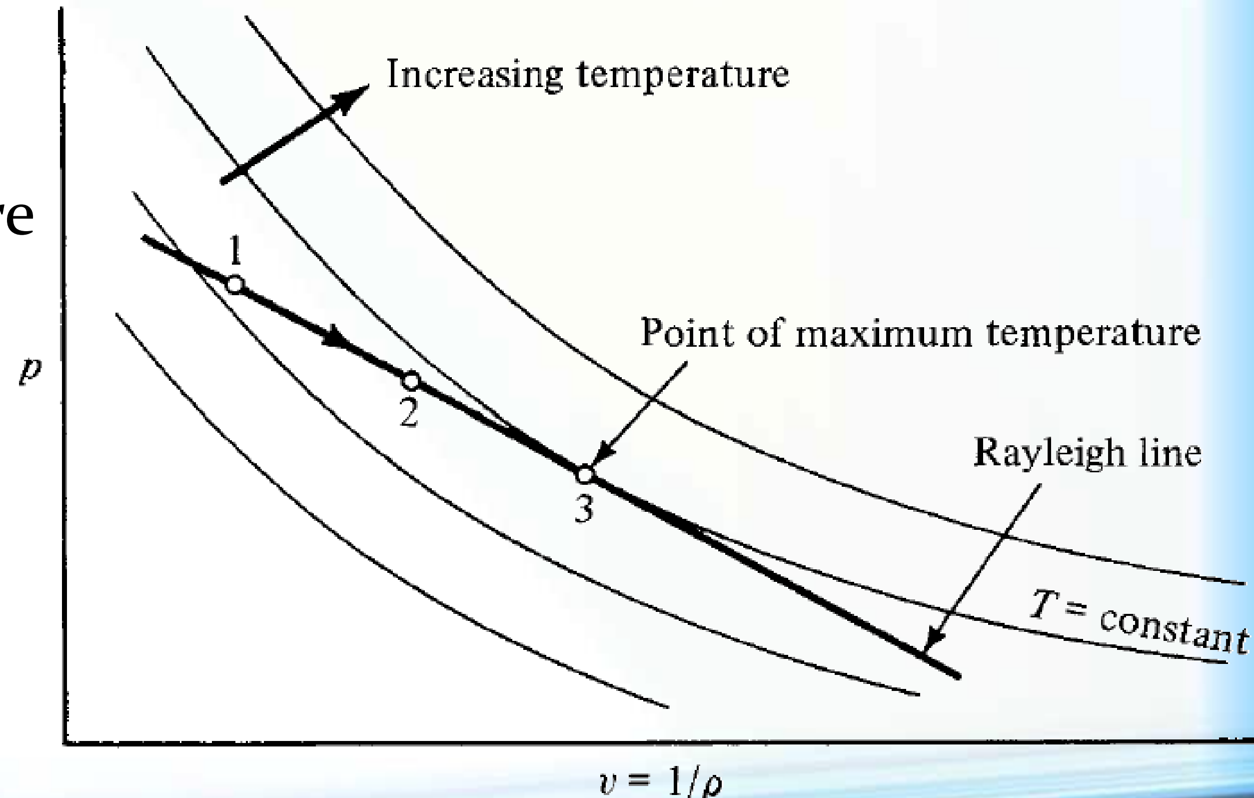
- **Historical Note:** For a constant value of ρV, momentum equation on a *p-v* diagram, is a straight line with a negative slope and was named the Rayleigh line by Aurel Stodola in his classic book *Steam and Gas Turbines*, McGraw-Hill (1927), p. 84. The corresponding flow is thus called *Rayleigh line flow*.

RAYLEIGH LINE FLOW / P-V DIAGRAM

- Rayleigh line: $p + \dfrac{G^2}{\rho} = \text{const}$
- Lines of constant temperature: $pv = \text{const}$

- Point 3 is reached where the temperature is a maximum.

Is this a limiting point of some sort? Have we reached some kind of a choked condition?



RAYLEIGH LINE FLOW / P-V DIAGRAM

To answer these questions, we must turn elsewhere. Recall that the addition of heat causes the entropy of the fluid to increase since

$$ds_e = \frac{\delta q}{T}$$

From our basic assumption of negligible friction, $ds \approx ds_e$

For a *T* = *constant* line, $pv = RT = \text{const}$

$$p\,dv + v\,dp = 0$$

$$\boxed{\frac{dp}{dv} = -\frac{p}{v}}$$

For an *S* = *constant* line, $pv^{\gamma} = \text{const}$

the isentropic line has the greater negative slope

$$v^{\gamma}\,dp + p\gamma v^{\gamma-1}\,dv = 0$$

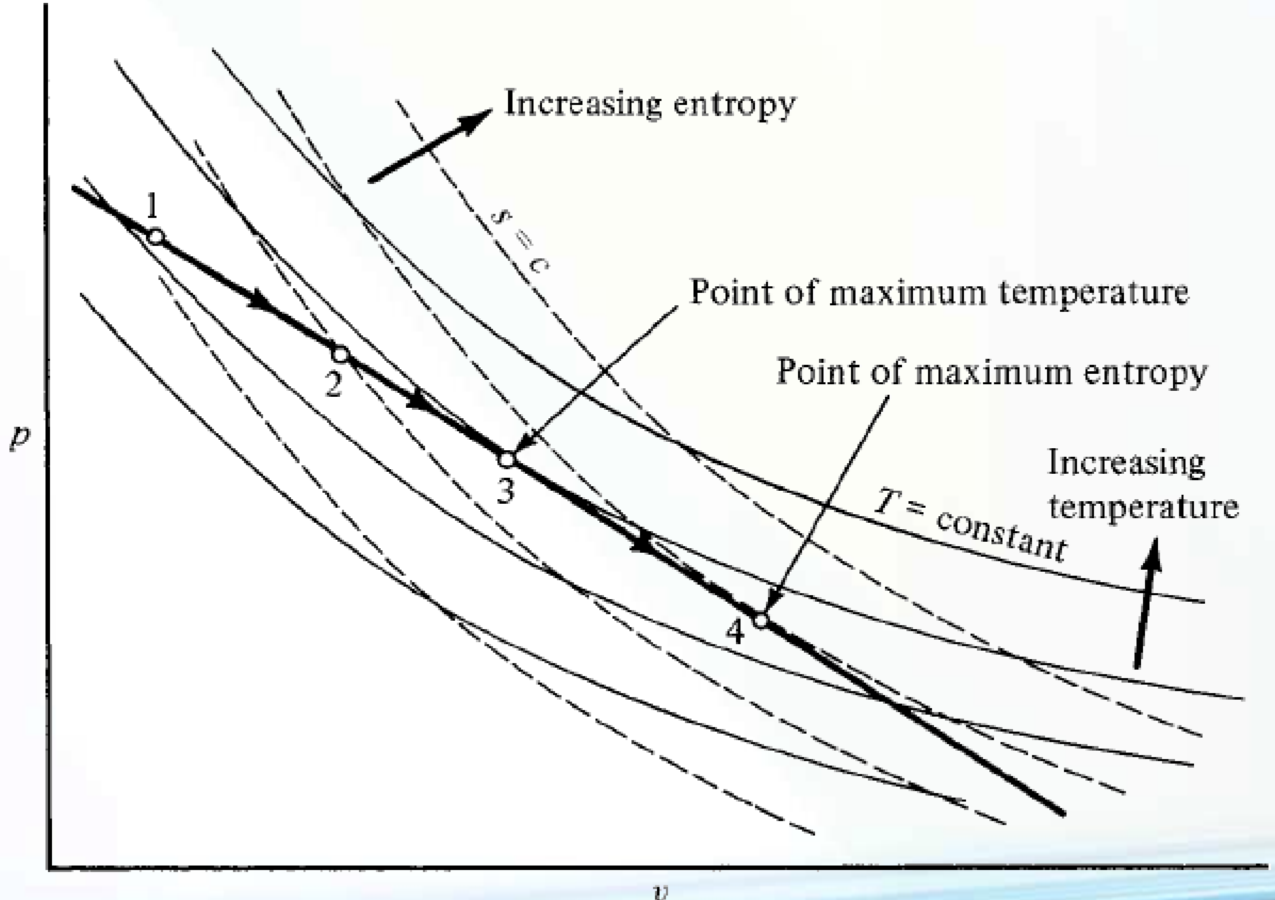
$$\boxed{\frac{dp}{dv} = -\gamma\frac{p}{v}}$$

RAYLEIGH LINE FLOW / P-V DIAGRAM

$$\frac{dp}{dv} = -\frac{p}{v}$$

$$\frac{dp}{dv} = -\gamma\frac{p}{v}$$

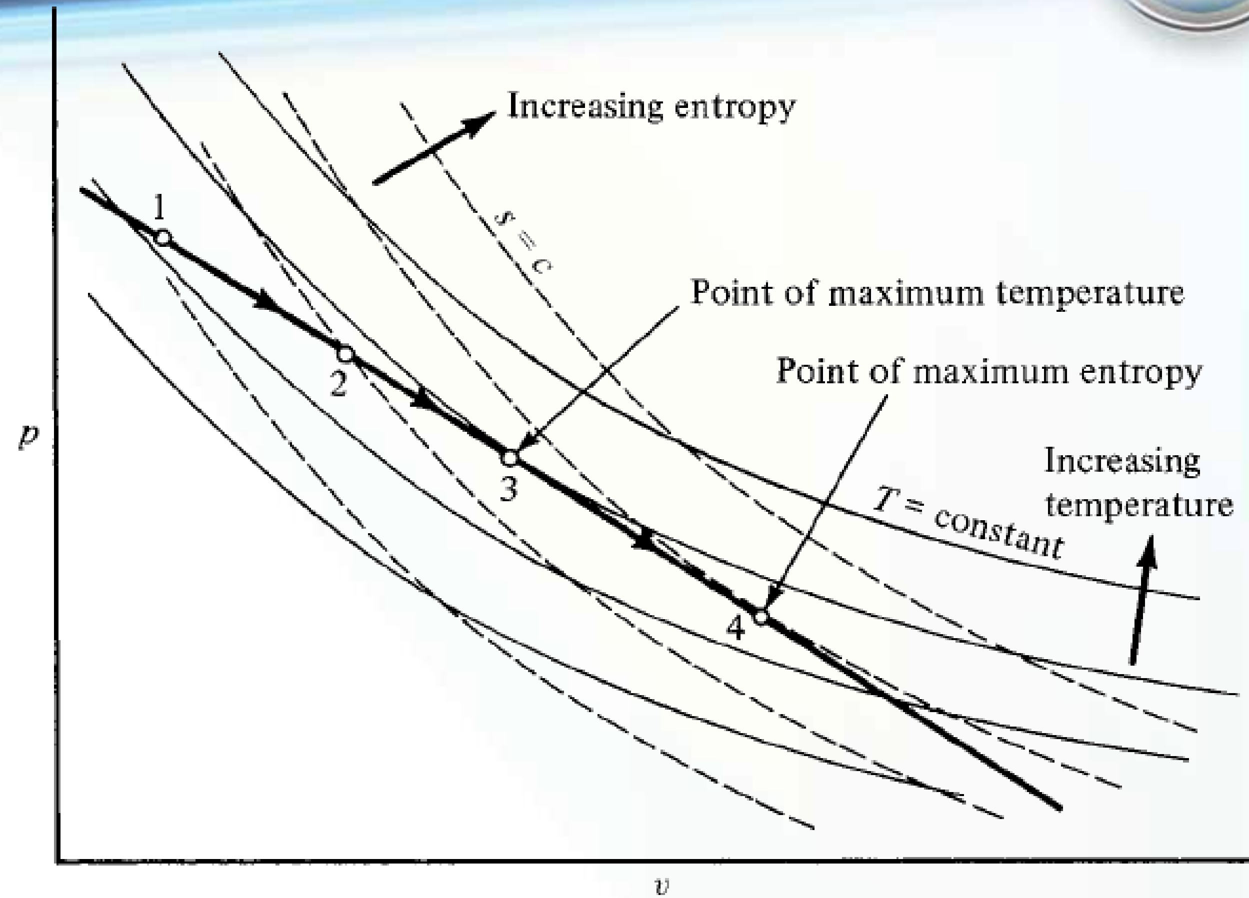
We now see that not only can we reach the point of maximum temperature, but more heat can be added to take us beyond this point.



RAYLEIGH LINE FLOW / P-V DIAGRAM

From point 3 to 4, we add heat to the system and its temperature decreases.

The effects of heat addition are normally thought of as causing the fluid density to decrease. This requires the velocity to increase since ρV = constant by continuity.

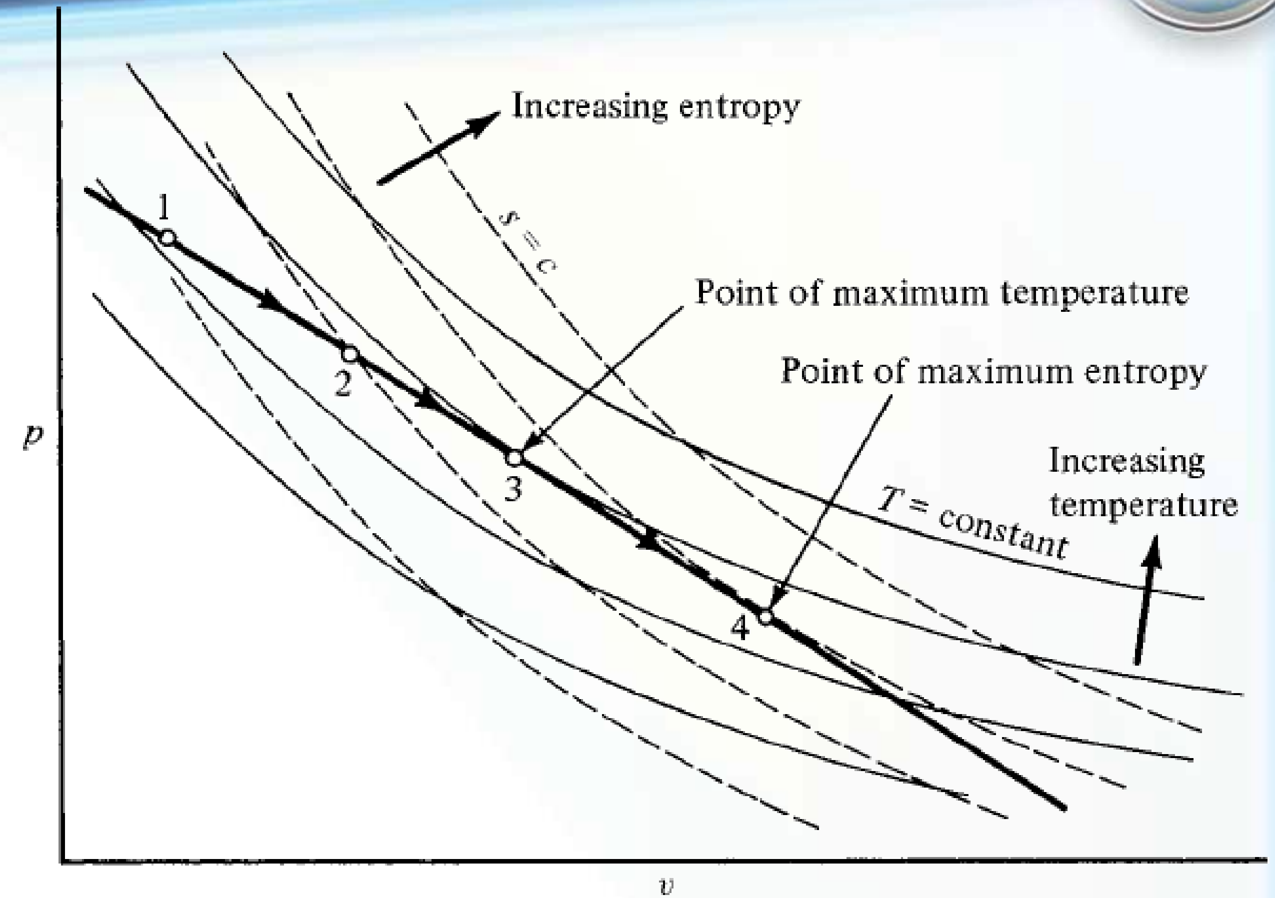


This velocity increase automatically

Some of the heat that is added to the system is converted into this increase in kinetic energy of the fluid.

RAYLEIGH LINE FLOW / P-V DIAGRAM

Noting that kinetic energy is proportional to *the square of velocity*, we realize that as higher velocities are reached, the addition of more heat is accompanied by much greater increases in kinetic energy. Eventually, we reach a point where all of the heat energy added is required for the kinetic energy increase.



RAYLEIGH LINE FLOW / WORKING RELATIONS

$$\rho = \frac{p}{RT}$$

$$V^2 = M^2 a^2 = M^2 \gamma RT$$

$$p + \rho V^2 = \text{const}$$

$$p(1 + \gamma M^2) = \text{const}$$

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}$$

RAYLEIGH LINE FLOW / WORKING RELATIONS

$$p_0 = p\left(1+\frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)} \quad \Rightarrow \quad \frac{p_{02}}{p_{01}} = \frac{p_2}{p_1}\left(\frac{1+[(\gamma-1)/2]M_2^2}{1+[(\gamma-1)/2]M_1^2}\right)^{\gamma/(\gamma-1)}$$

$$\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2} \quad \Rightarrow \quad \frac{p_{02}}{p_{01}} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}\left(\frac{1+[(\gamma-1)/2]M_2^2}{1+[(\gamma-1)/2]M_1^2}\right)^{\gamma/(\gamma-1)}$$

RAYLEIGH LINE FLOW / WORKING RELATIONS

$$T = \frac{p}{\rho R}$$

$$\rho = \frac{\dot{m}}{AV}$$

$$V = M\sqrt{\gamma RT}$$

$$\sqrt{T} = (\text{constant})\,pM$$

$$\sqrt{T} = (\text{constant})\,pM$$

$$p(1+\gamma M^2) = \text{const}$$

$$T = (\text{constant})\frac{M^2}{(1+\gamma M^2)^2}$$

$$\frac{T_1(1+\gamma M_1^2)^2}{M_1^2} = \frac{T_2(1+\gamma M_2^2)^2}{M_2^2}$$

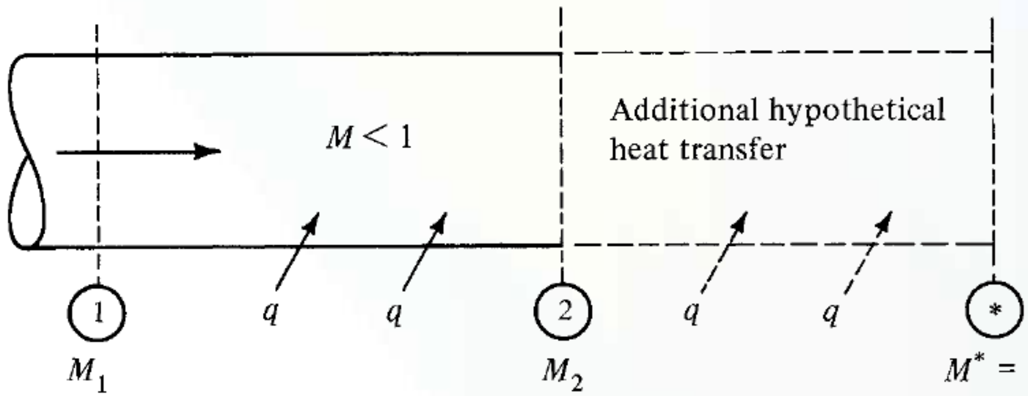
$$\frac{T_2}{T_1} = \frac{M_2^2}{M_1^2}\frac{(1+\gamma M_1^2)^2}{(1+\gamma M_2^2)^2}$$

RAYLEIGH LINE FLOW / WORKING RELATIONS

$$T_0 = T\left(1+\frac{\gamma-1}{2}M^2\right) \Rightarrow \frac{T_{02}}{T_{01}} = \frac{T_2}{T_1}\left(\frac{1+[(\gamma-1)/2]M_2^2}{1+[(\gamma-1)/2]M_1^2}\right)$$

$$\frac{T_2}{T_1} = \frac{M_2^2}{M_1^2}\frac{(1+\gamma M_1^2)^2}{(1+\gamma M_2^2)^2} \Rightarrow \frac{T_{02}}{T_{01}} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2 \frac{M_2^2}{M_1^2}\left(\frac{1+[(\gamma-1)/2]M_2^2}{1+[(\gamma-1)/2]M_1^2}\right)$$

- To facilitate computation, as well as any tabulation of these expressions, let state 2 be a reference state at which Mach number 1 occurs, as was done for both isentropic and Fanno flows. More physical significance will be attached to the Mach 1 state later. Denoting the properties at Mach 1 by (*), we see that



$$\frac{p}{p^*} = \frac{1+\gamma}{1+\gamma M^2}$$

$$\frac{T}{T^*} = \frac{(1+\gamma)^2 M^2}{(1+\gamma M^2)^2}$$

$$\frac{V}{V^*} = \frac{\rho^*}{\rho} = \frac{p^*}{p}\frac{T}{T^*} = \frac{(1+\gamma)M^2}{1+\gamma M^2}$$

RAYLEIGH LINE FLOW / REFERENCE CONDITIONS

RAYLEIGH LINE FLOW / REFERENCE CONDITIONS

- To facilitate computation, as well as any tabulation of these expressions, let state 2 be a reference state at which Mach number 1 occurs, as was done for both isentropic and Fanno flows. More physical significance will be attached to the Mach 1 state later. Denoting the properties at Mach 1 by (*), we see that

$$\frac{T_o}{T_o^*} = \frac{(1+\gamma)^2}{(1+\gamma M^2)^2}\,\frac{M^2}{1}\,\frac{\left(1+\dfrac{\gamma-1}{2}M^2\right)}{\left(1+\dfrac{\gamma-1}{2}\right)} = \frac{(1+\gamma)M^2[2+(\gamma-1)M^2]}{(1+\gamma M^2)^2}$$

$$\frac{p_o}{p_o^*} = \left(\frac{1+\gamma M^2}{1+\gamma}\right)\left(\frac{1+\dfrac{\gamma-1}{2}M^2}{1+\dfrac{\gamma-1}{2}}\right)^{\gamma/(\gamma-1)} = \left(\frac{1+\gamma M^2}{1+\gamma}\right)\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{\gamma/(\gamma-1)}$$

RAYLEIGH LINE FLOW / REFERENCE CONDITIONS

$$\frac{P}{P^*} = \frac{1+\gamma}{1+\gamma M^2}$$

$$\frac{T}{T^*} = \frac{(1+\gamma)^2 M^2}{(1+\gamma M^2)^2}$$

$$\frac{P}{P^*} = \frac{1}{M}\sqrt{\frac{T}{T^*}}$$

$$M^2 = \frac{1+\gamma}{\gamma}\frac{P^*}{P} - \frac{1}{\gamma}$$

$$\frac{P}{P^*}\sqrt{\frac{(1+\gamma)P^*/P - 1}{\gamma}} = \sqrt{\frac{T}{T^*}}$$

$$\frac{\gamma T}{T^*} = \left(\frac{P}{P^*}\right)^2\left[(1+\gamma)\frac{P^*}{P} - 1\right]$$

$$\frac{P}{P^*} = \frac{1+\gamma}{2} \pm \frac{\sqrt{(1+\gamma)^2 - 4\gamma(T/T^*)}}{2}$$

RAYLEIGH LINE FLOW / T-S DIAGRAM

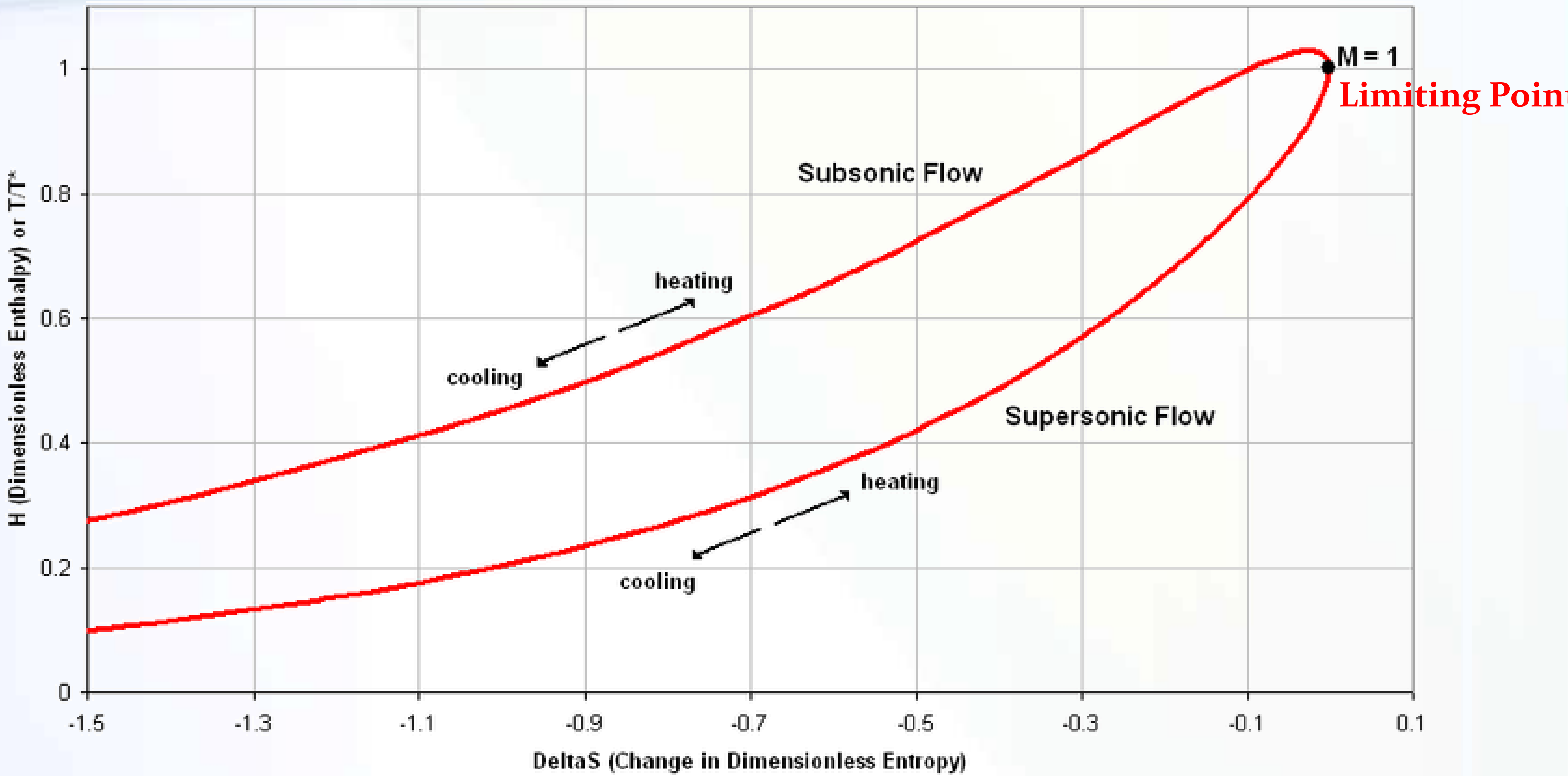
- To gain a better understanding of the effect of heat addition on Mach number, let us determine the locus of states for a given mass flow on a *T-s* diagram; the resultant plot is termed the *Rayleigh line*.

$$s - s^* = c_p \ln \frac{T}{T^*} - R \ln \frac{P}{P^*}$$

$$\frac{P}{P^*} = \frac{1+\gamma}{2} \pm \frac{\sqrt{(1+\gamma)^2 - 4\gamma(T/T^*)}}{2}$$

$$\boxed{\frac{s - s^*}{c_p} = \ln \frac{T}{T^*} - \frac{\gamma - 1}{\gamma} \ln \left[\frac{(\gamma+1) \pm \sqrt{(\gamma+1)^2 - 4\gamma(T/T^*)}}{2}\right]}$$

RAYLEIGH LINE FLOW / T-S DIAGRAM



For heat addition, the entropy must increase and the flow moves to the right.

LIMITING POINT

$$p + \frac{G^2}{\rho} = \text{const} \qquad\qquad dp + G^2\left(-\frac{d\rho}{\rho^2}\right) = 0$$

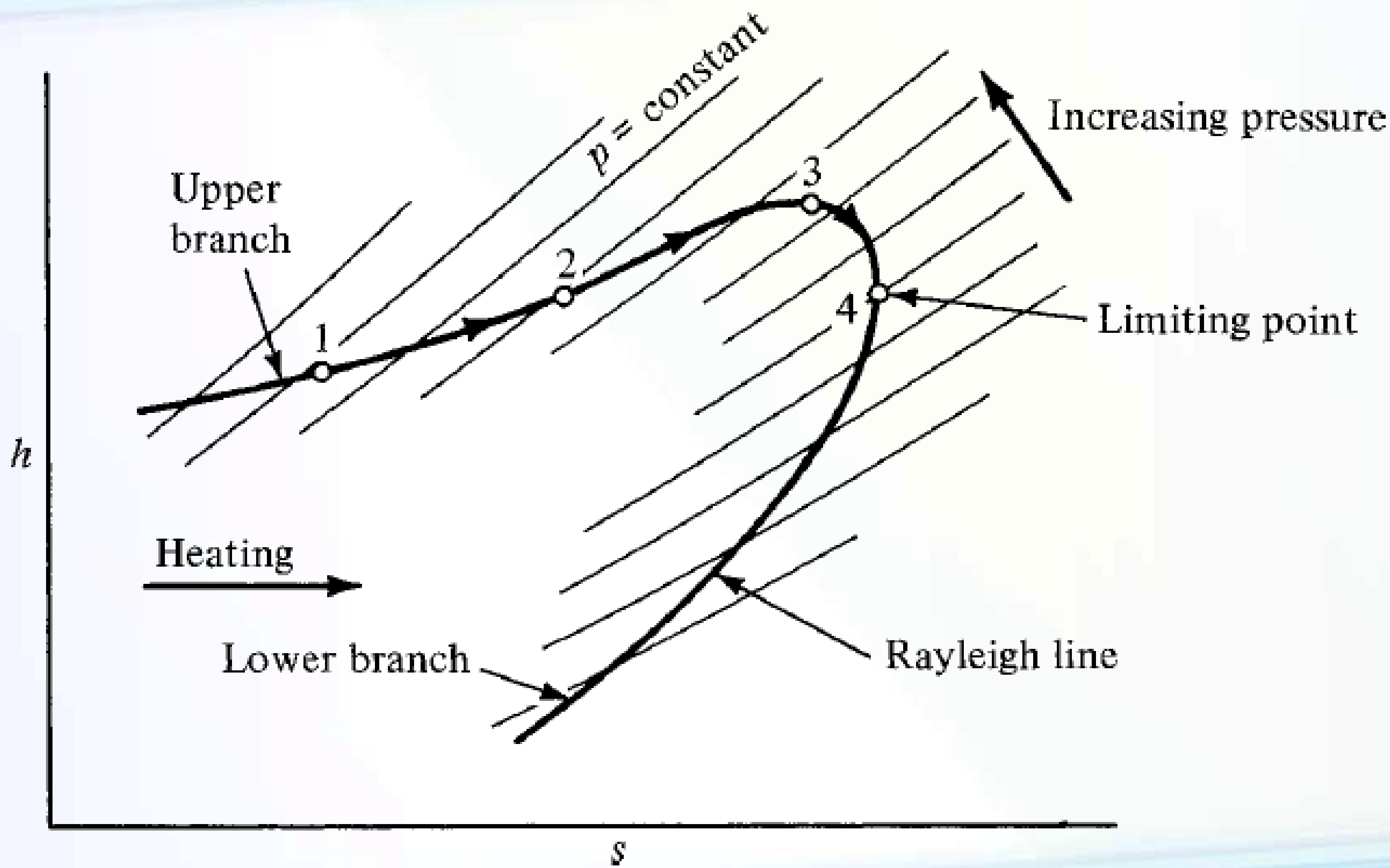
$$\frac{dp}{d\rho} = \frac{G^2}{\rho^2} = V^2 \qquad\qquad V^2 = \frac{dp}{d\rho}$$

This equation is valid *anyplace along the Rayleigh line.* Now for a differential movement at the limit point of maximum entropy, $ds = 0$ or $s = \text{const.}$

$$V^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s=c} \qquad \textit{(at the limit point)}$$

This is immediately recognized as sonic velocity. The upper branch of the Rayleigh line, where property variations appear reasonable, is seen to be a region of subsonic flow and the lower branch is for supersonic flow.

LIMITING POINT



LIMITING POINT

Another interesting fact can be shown to be true at the limit point.

$$dp = V^2\, d\rho$$

$$T\, ds = dh - \frac{dp}{\rho}$$

$$M = 1,\, ds = 0$$

$$0 = dh + V\, dV$$

$$dh_0 = 0$$

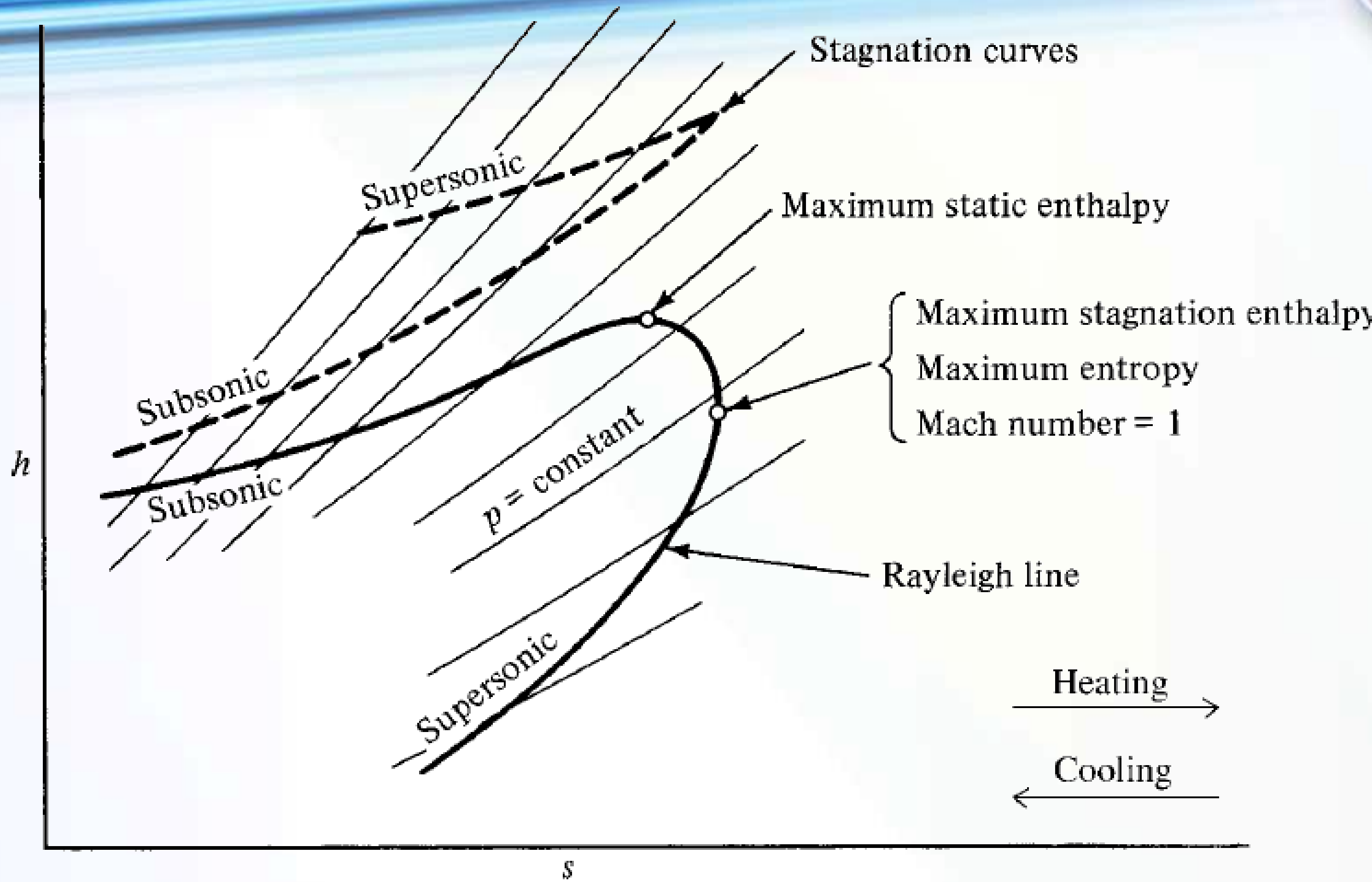
(at the limit point)

$$h_0 = h + \frac{V^2}{2}$$

$$dh_0 = dh + V dV$$

The stagnation enthalpy increases as long as heat can be added. At the point of maximum entropy, no more heat can be added and thus h_o must be a maximum at this location.

STAGTATION CURVE



$$\delta q = \cancel{\delta w_s} + dh_0$$

$$dh_0 = T ds_s = T ds$$

$$\frac{dh_0}{ds} = T$$

APPLICATIONS

The procedure for solving Rayleigh flow problems is quite similar to the approach used for Fanno flow except that the tie between the two locations in Rayleigh flow is determined by heat transfer considerations rather than by duct friction. The recommended steps are, therefore, as follows

1. Sketch the physical situation (including the hypothetical $*$ reference point).

2. Label sections where conditions are known or desired.

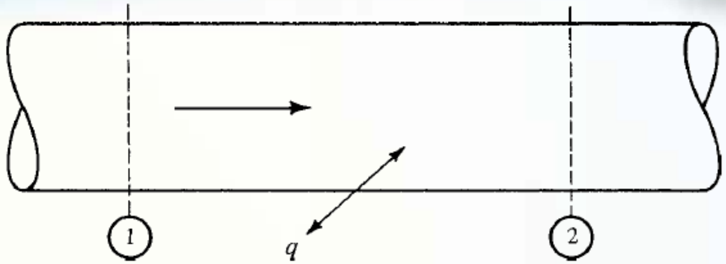
3. List all given information with units.

4. Determine the unknown Mach number.

5. Calculate the additional properties desired.

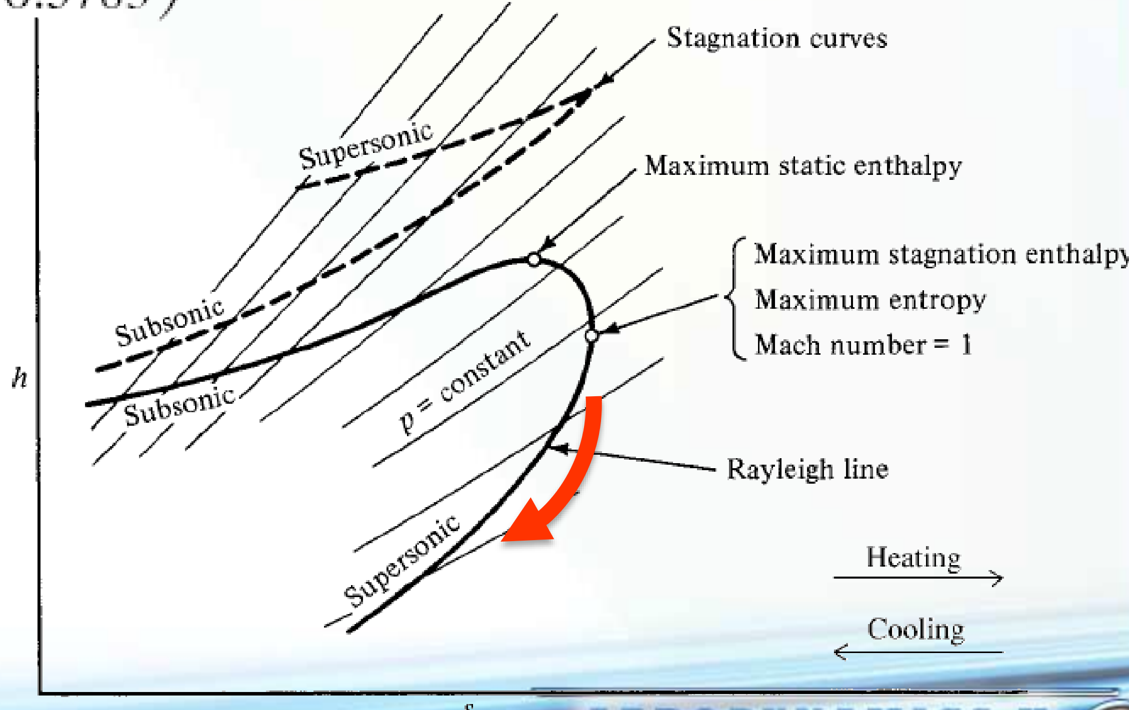
EXAMPLE

For Figure, given $M_1 = 1.5$, $p_1 = 10$ psia, and $M_2 = 3.0$, find p_2 and the direction of heat transfer.



$$p_2 = \frac{p_2}{p^*}\frac{p^*}{p_1}p_1 = (0.1765)\left(\frac{1}{0.5783}\right)(10) = 3.05 \text{ psia}$$

The flow is getting more supersonic, or moving away from the $*$ reference point.



EXAMPLE

Given $M_2 = 0.93$, $T_{t2} = 300°C$, and $T_{t1} = 100°C$, find M_1 and p_2/p_1.

To determine conditions at section 1 we must establish the ratio

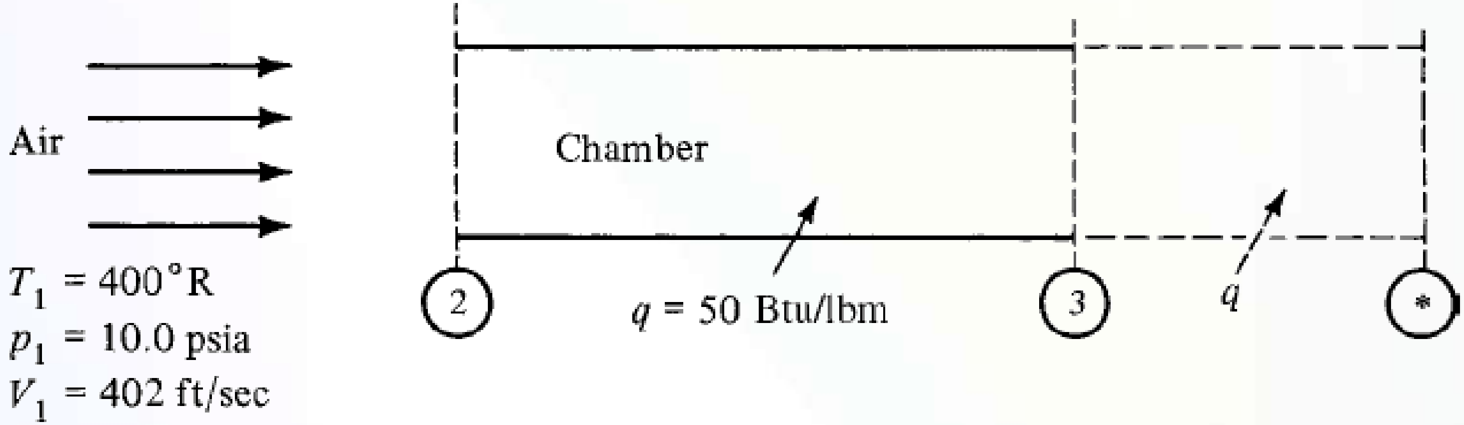
$$\frac{T_{t1}}{T_t{}^*} = \frac{T_{t1}}{T_{t2}}\frac{T_{t2}}{T_t{}^*} = \left(\frac{273+100}{273+300}\right)(0.9963) = 0.6486$$

$$T_t/T_t{}^* = 0.6486 \quad \Rightarrow \quad M_1 = 0.472$$

$$\frac{p_2}{p_1} = \frac{p_2}{p^*}\frac{p^*}{p_1} = (1.0856)\left(\frac{1}{1.8294}\right) = 0.593$$

EXAMPLE

A constant-area combustion chamber is supplied air at 400°R and 10.0 psia. The air stream has a velocity of 402 ft/sec. Determine the exit conditions if 50 Btu/lbm is added in the combustion process and the chamber handles the maximum amount of air possible.



$$T_2 = T_1 = 400°\text{R} \qquad p_2 = p_1 = 10.0 \text{ psia} \qquad V_2 = V_1 = 402 \text{ ft/sec}$$

$$a_2 = \sqrt{\gamma g_c R T_2} = [(1.4)(32.2)(53.3)(400)]^{1/2} = 980 \text{ ft/sec}$$

$$M_2 = \frac{V_2}{a_2} = \frac{402}{980} = 0.410$$

EXAMPLE (CONT.)

$$T_{t2} = \frac{T_{t2}}{T_2} T_2 = \left(\frac{1}{0.9675}\right)(400) = 413°\text{R}$$

$$M_2 = 0.41 \quad \Rightarrow \quad \frac{T_{t2}}{T_t{}^*} = 0.5465 \qquad \frac{T_2}{T^*} = 0.6345 \qquad \frac{p_2}{p^*} = 1.9428$$

$$\Delta T_t = \frac{q}{c_p} = \frac{50}{0.24} = 208°\text{R} \quad \Rightarrow \quad T_{t3} = T_{t2} + \Delta T_t = 413 + 208 = 621°\text{R}$$

$$\Rightarrow \quad \frac{T_{t3}}{T_t{}^*} = \frac{T_{t3}}{T_{t2}} \frac{T_{t2}}{T_t{}^*} = \left(\frac{621}{413}\right)(0.5465) = 0.8217$$

Rayleigh table $\Rightarrow$ $M_3 = 0.603 \qquad \dfrac{T_3}{T^*} = 0.9196 \qquad \dfrac{p_3}{p^*} = 1.5904$

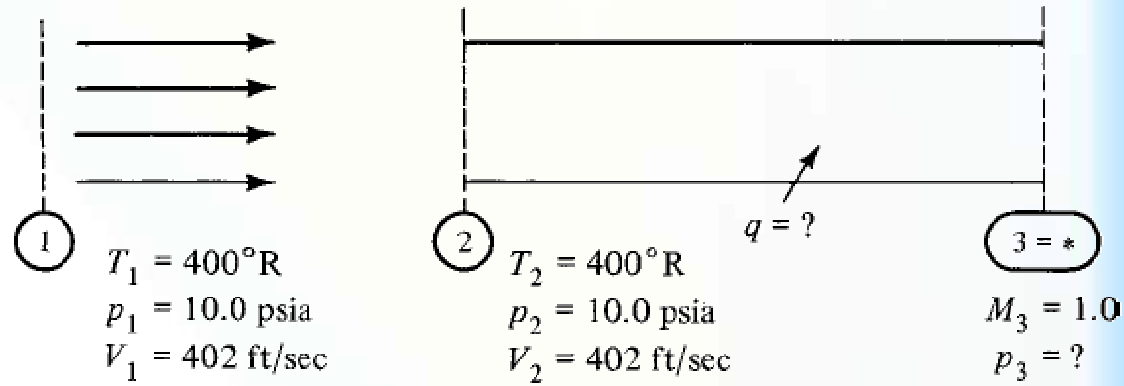
EXAMPLE (CONT.)

$$p_3 = \frac{p_3}{p^*}\frac{p^*}{p_2}p_2 = (1.5904)\left(\frac{1}{1.9428}\right)(10.0) = 8.19 \text{ psia}$$

$$T_3 = \frac{T_3}{T^*}\frac{T^*}{T_2}T_2 = (0.9196)\left(\frac{1}{0.6345}\right)(400) = 580^\circ\text{R}$$

EXAMPLE

In previous example, how much more heat (fuel) could be added without changing conditions at the entrance to the duct?



$M_2 = 0.41$ and $T_{t2} = 413°\text{R}$.

$$T_{t3} = T_t{}^* = \frac{T_t{}^*}{T_{t2}} T_{t2} = \left(\frac{1}{0.5465}\right)(413) = 756°\text{R}$$

$$p_3 = p^* = \frac{p^*}{p_2} p_2 = \left(\frac{1}{1.9428}\right)(10.0) = 5.15 \text{ psia}$$

$$q = c_p \, \Delta T_t = (0.24)(756 - 413) = 82.3 \text{ Btu/lbm}$$

CORRELATION WITH SHOCKS

There are some similarities between Rayleigh flow and normal shocks:

1. The end points before and after a normal shock represent states with the same mass flow per unit area, the same impulse function, and the same stagnation enthalpy.

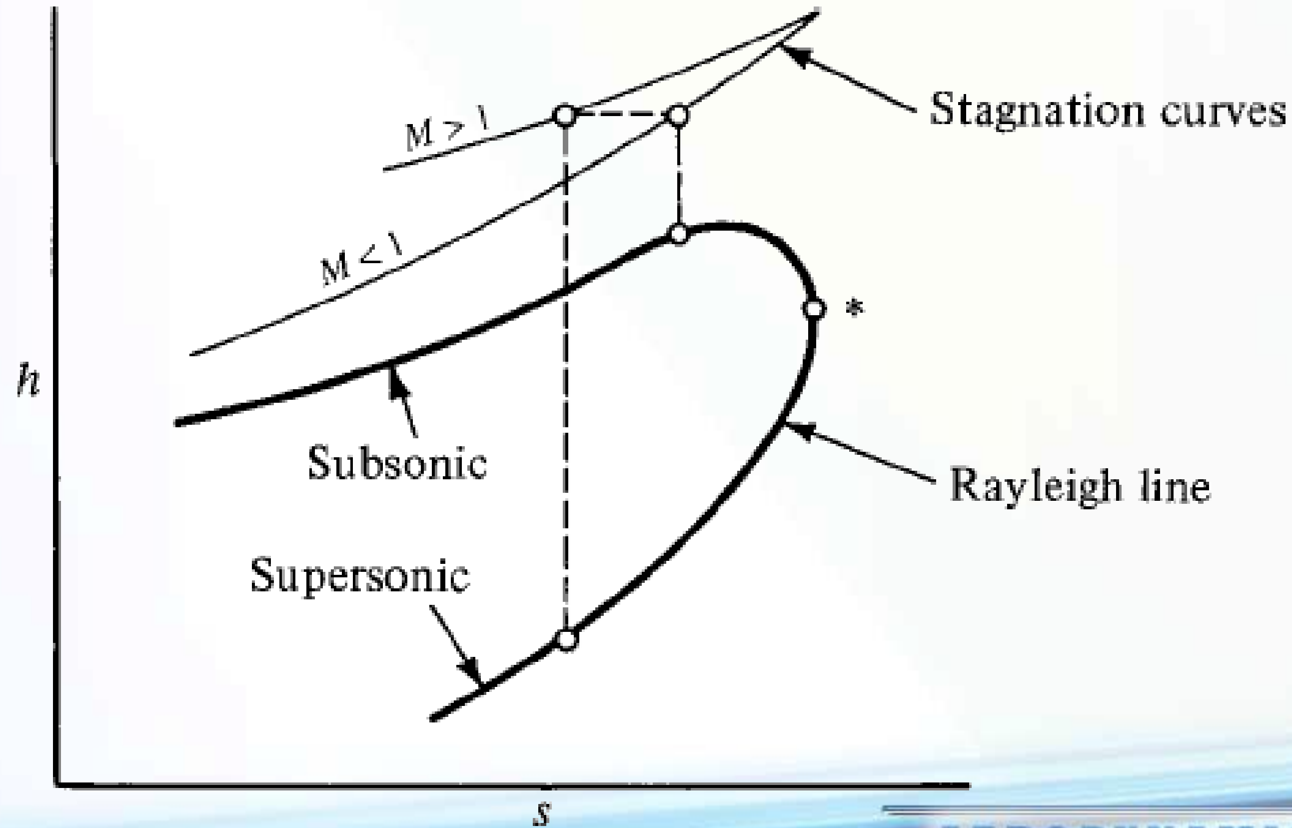
2. A Rayleigh line represents states with the same mass flow per unit area and the same impulse function. All points on a Rayleigh line do not have the same stagnation enthalpy because of the heat transfer involved. To move along a Rayleigh line requires this heat transfer.

CORRELATION WITH SHOCKS

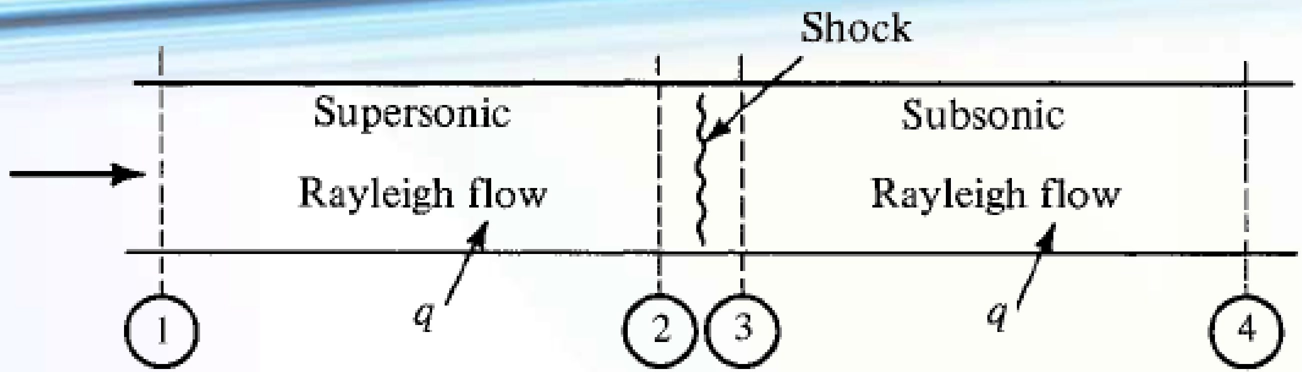
Shock Wave Equations	Rayleigh Flow Equations
$\boxed{\rho_1 V_1 = \rho_2 V_2}$	$\boxed{\rho V = G = \text{const}}$
$\boxed{h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}}$	$\boxed{h_{t1} + q = h_{t2}}$
$\boxed{p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2}$	$\boxed{p + \rho V^2 = \text{const}}$

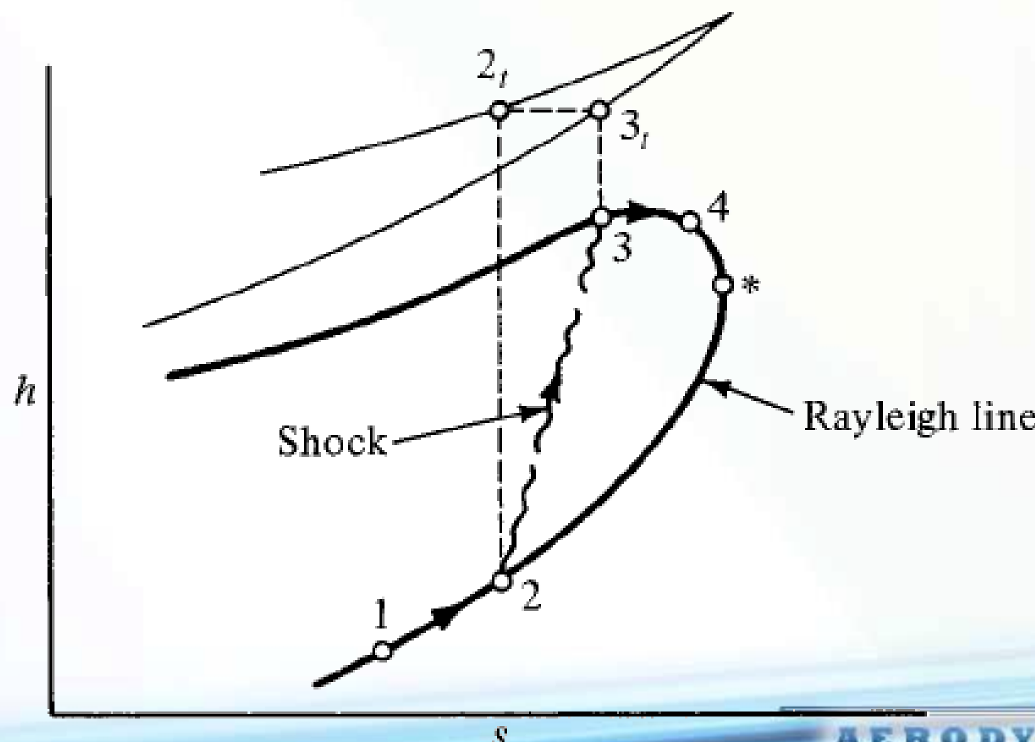
CORRELATION WITH SHOCKS

For every point on the supersonic branch of the Rayleigh line there is a corresponding point on the subsonic branch with the same stagnation enthalpy. Thus these two points satisfy all three conditions for the end points of a normal shock and could be connected by such a shock.



CORRELATION WITH SHOCKS





FANNO FLOW / RAYLEIGH FLOW / NORMAL SHOCK

Shock Wave Equations

$$\boxed{\rho_1 V_1 = \rho_2 V_2}$$

$$\boxed{h_1 + \frac{V_1^{\,2}}{2} = h_2 + \frac{V_2^{\,2}}{2}}$$

$$\boxed{p_1 + \rho_1 V_1^{\,2} = p_2 + \rho_2 V_2^{\,2}}$$

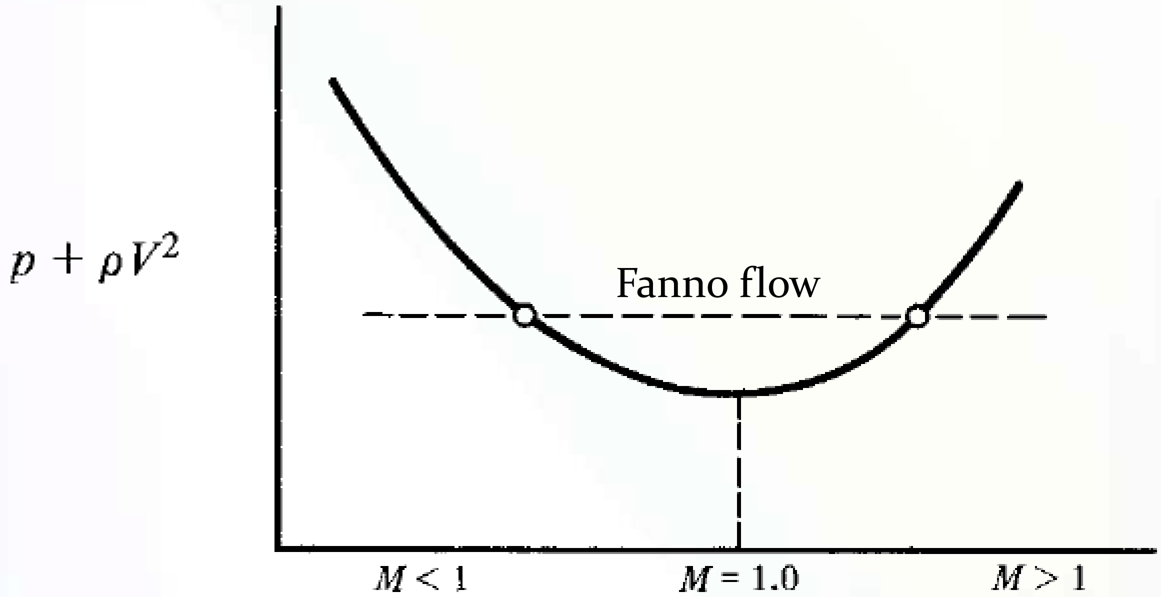
Fanno Equations

$$\boxed{\rho_1 V_1 = \rho_2 V_2}$$

$$\boxed{h_1 + \frac{V_1^{\,2}}{2} = h_2 + \frac{V_2^{\,2}}{2}}$$

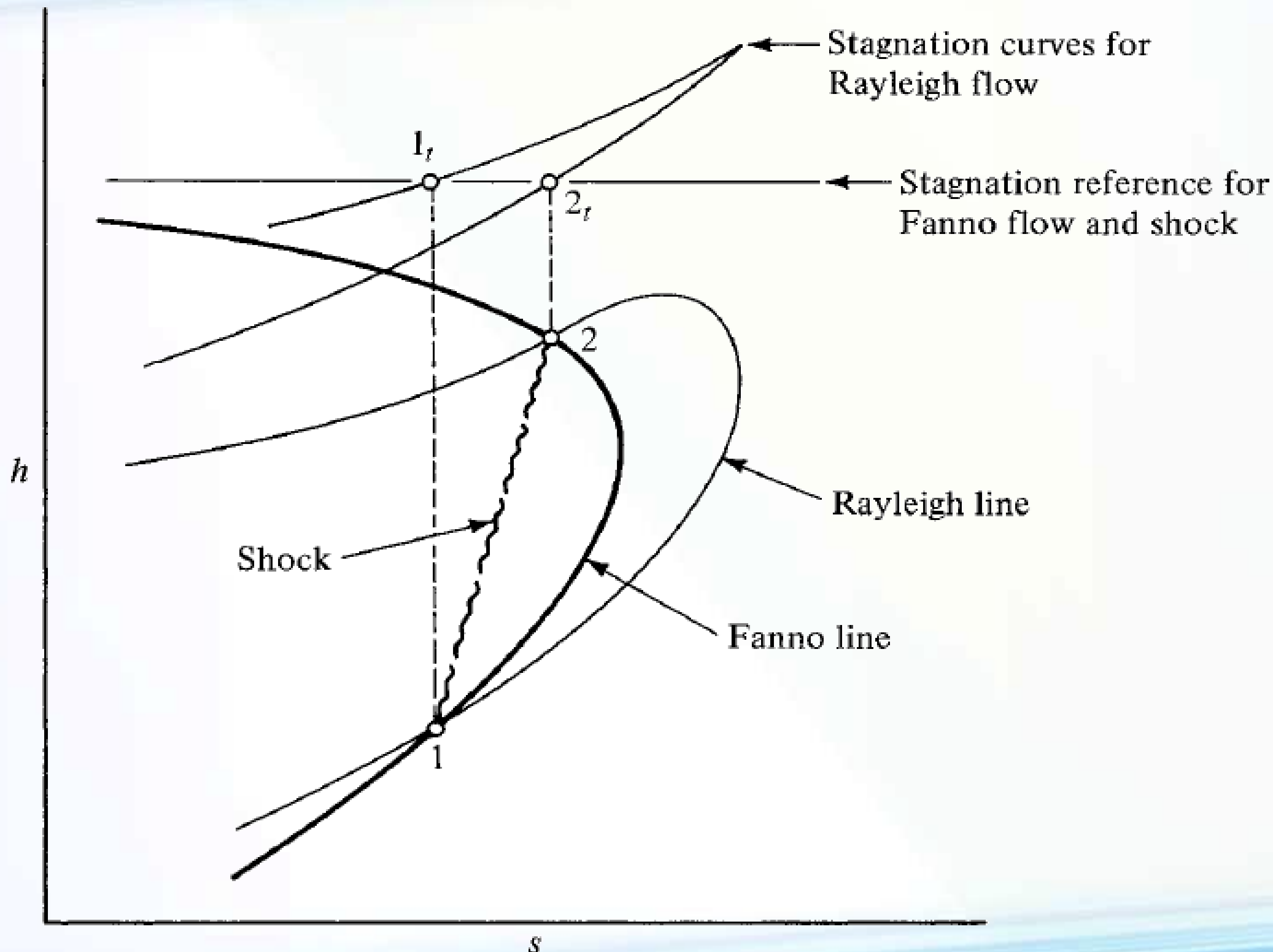
$$\boxed{\left(p_1 + \rho_1 V_1^{\,2}\right) - \frac{F_f}{A} = p_2 + \rho_2 V_2^{\,2}}$$

FANNO FLOW / RAYLEIGH FLOW / NORMAL SHOCK



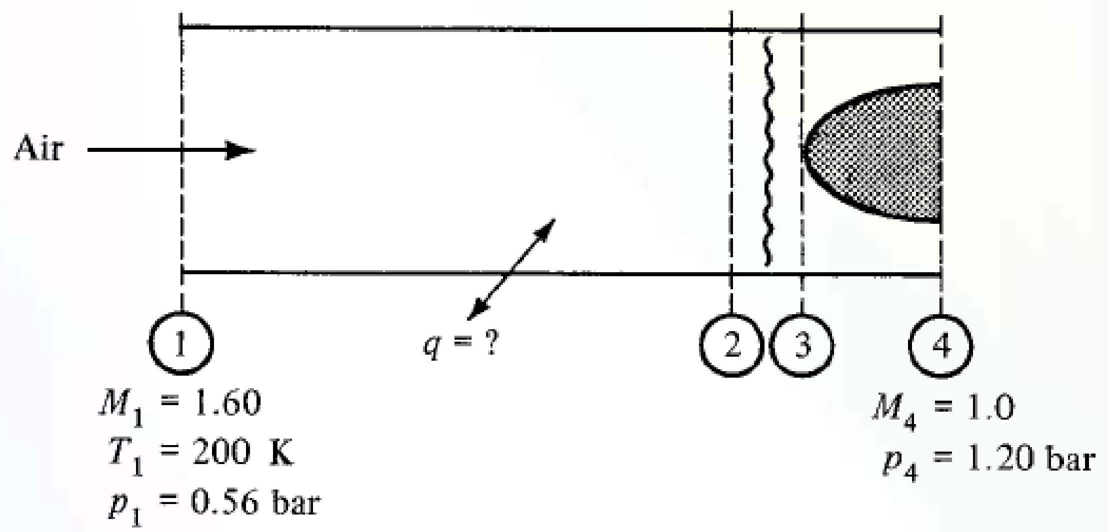
The Fanno continuity and energy equations and the Rayleigh continuity and momentum equations collectively are the same as the continuity, momentum, and energy equations developed for the normal shock. Thus, the locus of states before and after a normal shock appears on a *T*-*s* diagram at the intersection of the Fanno and Rayleigh lines.

FANNO FLOW / RAYLEIGH FLOW / SHOCK WAVE

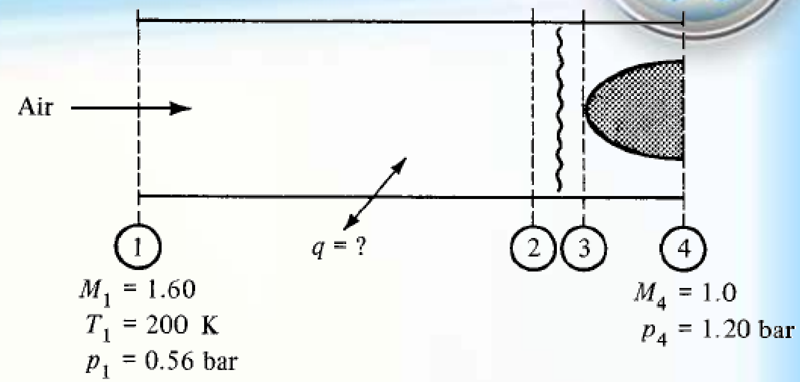


EXAMPLE

Air enters a constant-area duct with a Mach number of 1.6, a temperature of 200 K, and a pressure of 0.56 bar . After some heat transfer a normal shock occurs, whereupon the area is reduced as shown. At the exit the Mach number is found to be 1.0 and the pressure is 1.20 bar. Compute the amount and direction of heat transfer.



EXAMPLE (CONT.)



The flow from 3 to 4 is isentropic; thus:

$$p_{t3} = p_{t4} = \frac{p_{t4}}{p_4} p_4 = \left(\frac{1}{0.5283}\right)(1.20) = 2.2714 \text{ bar}$$

From the Rayleigh table we find $M_3 = 0.481$ and from the shock table, $M_2 = 2.906$.

$$T_{t1} = \frac{T_{t1}}{T_1} T_1 = \left(\frac{1}{0.6614}\right)(200) = 302 \text{ K}$$

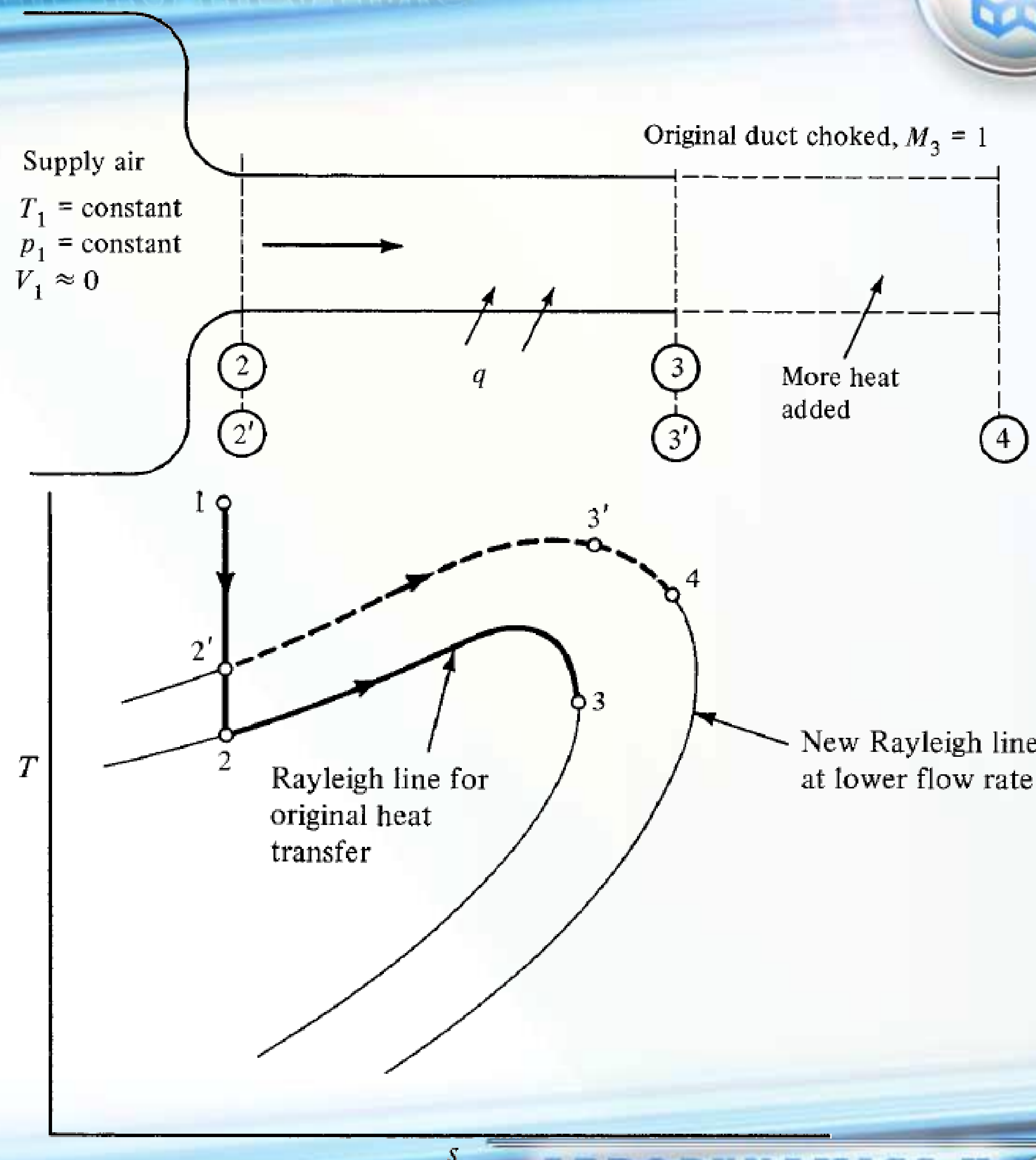
$$T_{t2} = \frac{T_{t2}}{T_t{}^*} \frac{T_t{}^*}{T_{t1}} T_{t1} = (0.6629)\left(\frac{1}{0.8842}\right)(302) = 226 \text{ K}$$

$$q = c_p(T_{t2} - T_{t1}) = (1000)(226 - 302) = -7.6 \times 10^4 \text{ J/kg}$$

THERMAL CHOKING DUE TO HEATING

In Subsonic Rayleigh flow, once sufficient heat has been added, we reach Mach 1 at the end of the duct.
The *T* –*s* diagram for this is shown as path 1–2–3. This is called *thermal choking*.

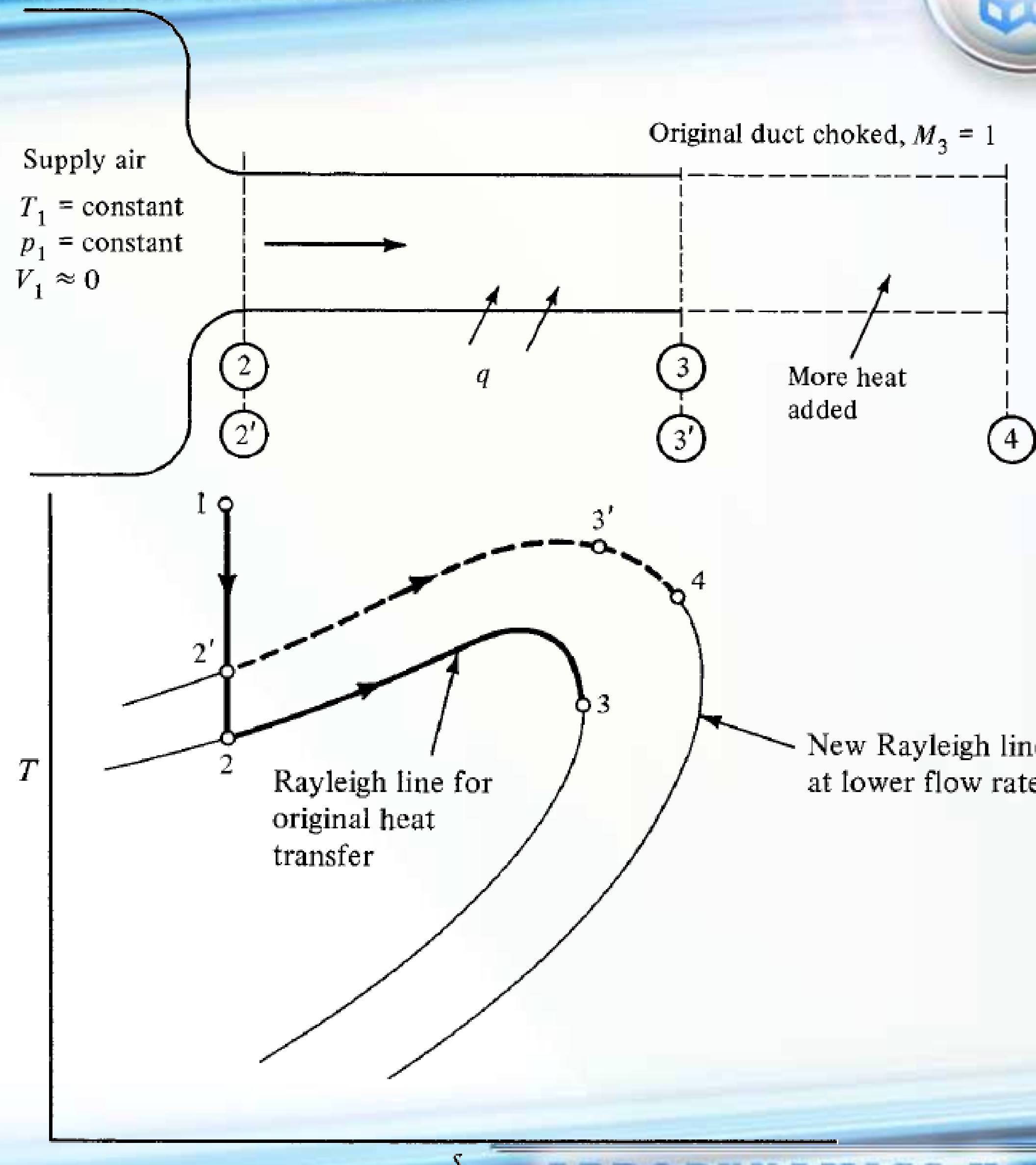
Reduction of the receiver pressure below p_3 would not affect the flow conditions inside the system. However, the addition of more heat will change these conditions.



THERMAL CHOKING DUE TO HEATING

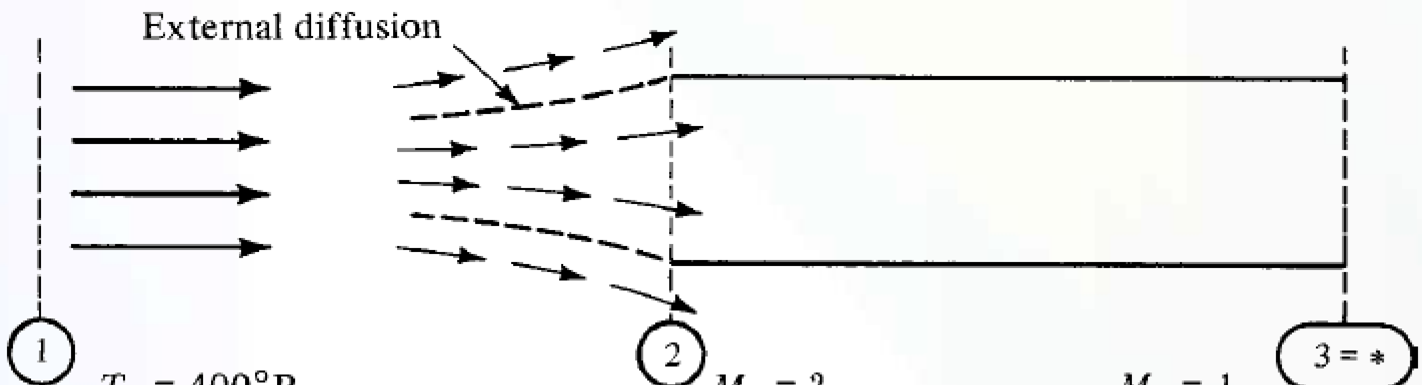
Now suppose that we add more heat to the system.

The only way that the system can reflect the required additional entropy change is to move to a new Rayleigh line at a decreased flow rate. This is shown as path 1–2'–3'– 4 on the *T* –*s* diagram. Whether or not the exit velocity remains sonic depends on how much extra heat is added and on the receiver pressure imposed on the system.



EXAMPLE

Let us add sufficient fuel to the combustion chamber to raise the outlet stagnation temperature to 3000°R. Assume that the receiver pressure is very low so that sonic velocity still exists at the exit. The additional entropy generated by the extra fuel can only be accommodated by moving to a new Rayleigh line at a decreased flow rate which lowers the inlet Mach number. If the chamber is fed by the same air stream some spillage must occur at the entrance. We would like to know the Mach number at the inlet and the pressure at the exit.

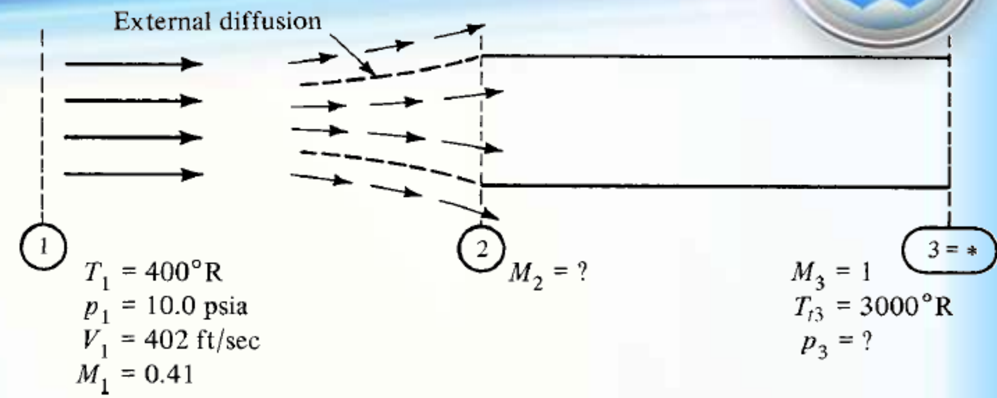


$T_1 = 400°R$
$p_1 = 10.0$ psia
$V_1 = 402$ ft/sec
$M_1 = 0.41$

$M_2 = ?$

$M_3 = 1$
$T_{t3} = 3000°R$
$p_3 = ?$

EXAMPLE (CONT.)



Since it is isentropic from the freestream to the inlet:

$$T_{t2} = T_{t1} = 413°\text{R}$$

since $M_3 = 1$, we know that $T_{t3} = T_t{}^*$.

$$\frac{T_{t2}}{T_t{}^*} = \frac{T_{t2}}{T_{t3}}\frac{T_{t3}}{T_t{}^*} = \left(\frac{413}{3000}\right)(1) = 0.1377$$

from the Rayleigh table, $M_2 = 0.176$ and $p_2/p^* = 2.3002$.

$$p_2 = \frac{p_2}{p_{t2}}\frac{p_{t2}}{p_{t1}}\frac{p_{t1}}{p_1}p_1 = (0.9786)(1)\left(\frac{1}{0.8907}\right)(10.0) = 10.99 \text{ psia}$$

$$p_3 = \frac{p_3}{p^*}\frac{p^*}{p_2}p_2 = (1)\left(\frac{1}{2.3002}\right)(10.99) = 4.78 \text{ psia}$$

EXAMPLE (CONT.)

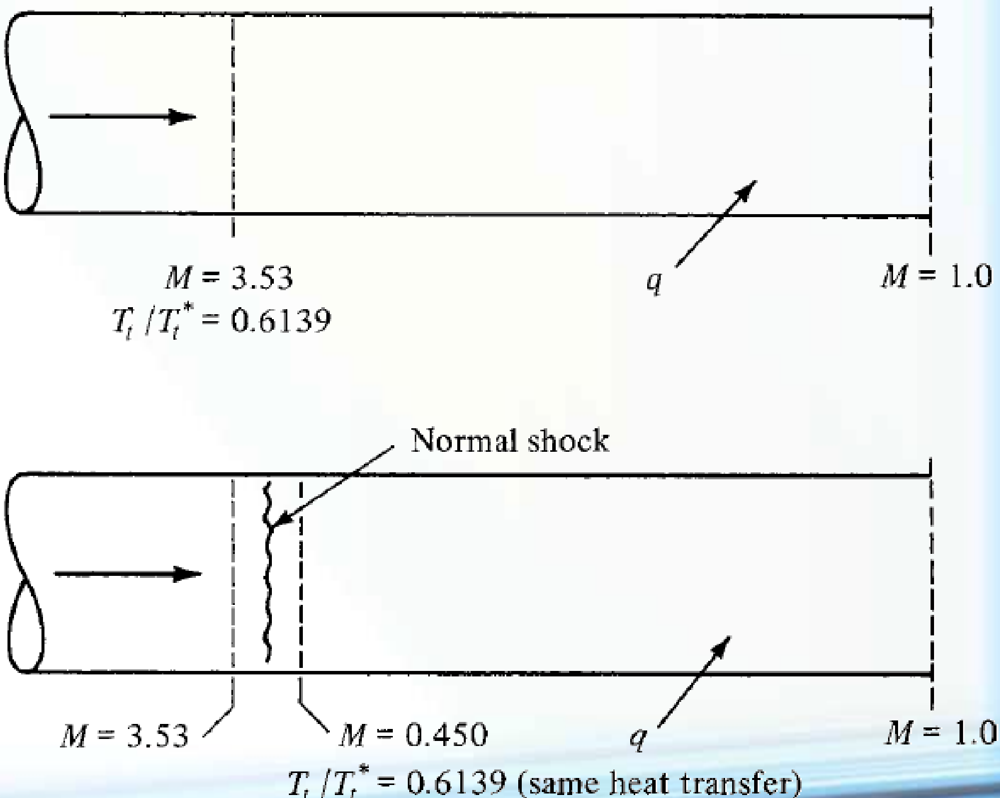
Suppose that in the previous example we were unable to lower the receiver pressure to 4.78 psia. Assume that as fuel was added to raise the stagnation temperature to 3000°R, the pressure in the receiver was maintained at its previous value of 5.15 psia. This would lower the flow rate even further as we move to another Rayleigh line with a lower mass velocity, *and this time the exit velocity would not be quite sonic.*

Although both M_2 and M_3 are unknown, two pieces of information are given at the exit. Two simultaneous equations could be written, but it is easier to use tables and a trial and-error solution.

The important thing to remember is that once a *subsonic flow is thermally choked, the addition of more heat causes the flow rate to decrease.* Just how much it decreases and whether or not the exit remains sonic depends on the pressure that exists after the exit.

EXAMPLE (CONT.)

The parallel between choked Rayleigh and Fanno flow does not quite extend into the supersonic regime. Recall that for choked Fanno flow the addition of more duct introduced a shock in the duct which permitted considerably more friction length to the sonic point. Figure shows a Mach 3.53 flow that has $T_t/T_t^* = 0.6139$. For a given total temperature at this section, the value of T_t/T_t^* is a direct indication of the amount of heat that can be added to the choke point. If a normal shock were to occur at this point, the Mach number after the shock would be 0.450, which also has $T_t/T_t^* = 0.6139$. Thus the heat added after the shock is exactly the same as it would be without the shock.



EXAMPLE

Air (γ = 1.4, R = 0.287 kJ/kg·K, and c_p = 1.004 kJ/kg· K) enters a ramjet combustion chamber with a velocity of 100 m/s and static temperature of 400 K.

- **Determine the maximum amount of heat that can be added in the combustion chamber without reducing the mass-flow rate.**
- **For this q_{max}, find the fuel-air ratio.**
- **If the fuel-air ratio were to be increased by 10 percent, determine the reduction in m for the same inlet stagnation pressure and temperature.**

Assume the heating value of the fuel to be 40 MJ/kg, neglect the fuel flow rate in comparison with the air flow rate, and assume the air to behave as a perfect gas with constant specific heats. Neglect friction.

First, the inlet Mach number is computed as follows:

$$M_1 = \frac{V_1}{a_1} = \frac{100}{\sqrt{(1.4)(287)(400)}} = 0.2494$$

EXAMPLE (CONT.)

From the isentropic flow relations, $(T/T_{o1}) = 0.9877$; thus,

$$T_{o1} = \left(\frac{T_{o1}}{T_1}\right)T_1 = \left(\frac{1}{0.9877}\right)400 = 404.9813 \text{ K}$$

Applying the energy equation to the flow in the combustion chamber, we get

$$\dot{q} = \dot{m}_{\text{air}}c_p(T_{o2} - T_{o1}) = \dot{m}_{\text{fuel}}(\text{HV})$$

$$q_{\max} = \dot{q}_{\max}/\dot{m},\ T_{o2} = T_o^*.$$

From the Rayleigh flow relations, we have

$$T_{o2} = T_o^* = \left(\frac{T_o^*}{T_{o1}}\right)T_{o1} = \frac{404.9813}{0.2558} = 1,583.1950 \text{ K}$$

EXAMPLE (CONT.)

$$\frac{\dot{m}_{\text{fuel}}}{\dot{m}_{\text{air}}} = \frac{c_p(T_{o2} - T_{o1})}{\text{HV}} = \frac{(1.004)(1{,}583.1950 - 404.9813)}{40{,}000} = 0.02957$$

$$q_{\max} = (1.004)(1{,}583.1950 - 404.9813) = 1{,}182.9265 \text{ kJ/kg}$$

If the fuel-air ratio were to be increased by 10 percent,

$$\dot{m}_{\text{fuel}}/\dot{m}_{\text{air}} = (1.10)0.02957 = 0.03253.$$

from the energy equation, we have

$$T_{o2} - T_{o1} = \frac{0.03253(40{,}000)}{1.004} = 1{,}295.8964 \text{ K}$$

$$T_{o2} = T_o^* \longrightarrow T_o^* = 1{,}295.8964 + 404.9813 = 1{,}700.8777 \text{ K}.$$

EXAMPLE (CONT.)

$$\frac{T_{o1}}{T_o^*} = \frac{404.9813}{1{,}700.8777} = 0.2381 \longrightarrow M_1 = 0.2392,$$

$$\longrightarrow p_1/p_{o1} = 0.9610 \text{ and } T_1/T_{o1} = 0.9887. \longrightarrow T_1 = 400.4050\text{ K}.$$

$$M_1 = 0.2494, \longrightarrow p_1/p_{o1} = 0.9577.$$

Thus, the 10-percent increase in fuel-air ratio results in a slight increase of inlet static pressure.

$$\dot{m} = \rho A V = \frac{p}{RT} A\left(M\sqrt{\gamma R T}\right)$$

the resultant effect of increasing fuel flow is a decrease in $\dot{m}$, given by

$$\frac{\dot{m}_{10\%\text{ increase}}}{\dot{m}} = \left(\frac{0.2392}{0.2494}\right)\left(\frac{0.9610}{0.9577}\right)\sqrt{\frac{400}{400.4050}} = 0.9619$$

corresponding to a 3.81 percent decrease of mass flow.

CONVECTION HEAT TRANSFER

The heat exchange at a fluid-solid interface can be described by *Newton's law of cooling*,

$$\dot{q} = \overline{h}A(T_w - T_f)$$

When a fluid is made to flow over the surface, the heat exchange that occurs is termed a *forced convection*.

When the heat exchange takes place as a result of fluid motion brought about by density differences within the fluid, it is termed a *natural convection*.

The mean heat transfer coefficient depends upon the geometry, the flow conditions, and the fluid properties. For convective flows involving air, h ranges approximately from 10 to 500 $W/m^2 \cdot K$. This range is determined from what are usually termed "*low-speed correlations*“.

Low speed is defined in terms of the *Eckert number*

$$E = \frac{V^2}{c_p \Delta T}$$

When this nondimensional number is *small* (low V or high ΔT), the situation is regarded as "*low speed*." As may be easily verified, at speeds approaching *sonic velocity* (300 m/s for air) and for a temperature difference of 100°C, the Eckert number is *close to unity.*

For Eckert numbers on the order of unity ("high-speed flows"), *viscous dissipation* terms within the energy equation become important and have a significant effect on the temperature distribution.

In fact, for these "high-speed flows," the temperature of the surface can actually exceed the freestream static temperature even if the surface is perfectly insulated.

CONVECTION HEAT TRANSFER

This energy exchange is sometimes referred to as *aerodynamic or frictional heating*, and the temperature is called the *adiabatic wall temperature* T_{aw}.

For gases, T_{aw} is relatively close to the stagnation temperature T_o.

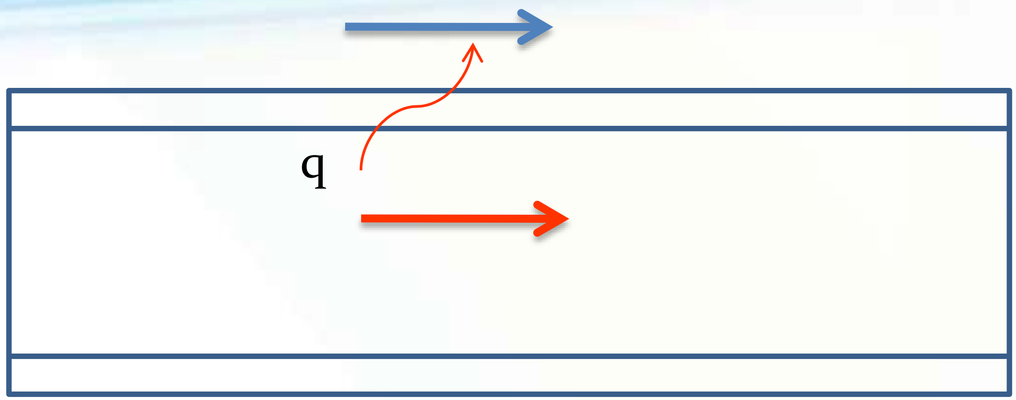
So, for high-speed gas flows, the convective heat transfer may be expressed by

$$\dot{q} = \bar{h}A(T_w - T_o)$$

and the low-speed mean heat transfer coefficient may still be employed.

In ducts, through which gases flow, there can be a heat exchange at the *inside surface* as well as at the external surface.
Moreover, heat must be conducted through the duct wall. To account for all of these heat exchanges, an overall heat transfer coefficient U is often used.

HEAT TRANSFER COEFFICIENT OF PIPES



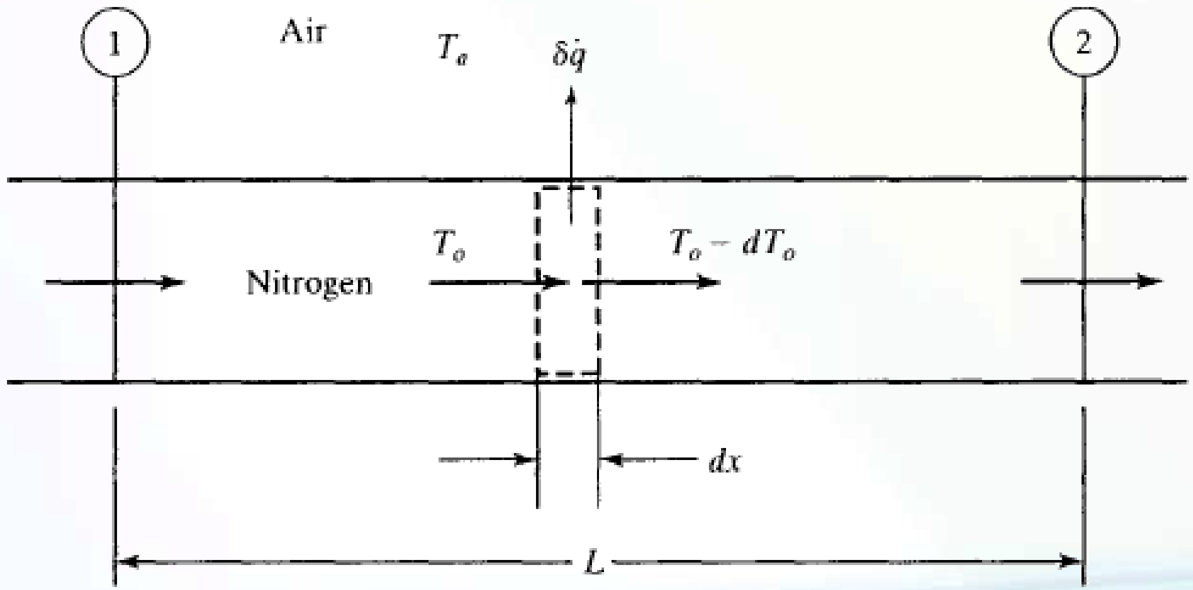
$$U_i = \frac{1}{\frac{1}{\bar{h}_i} + \frac{D_i}{2k}\ln\left(\frac{D_o}{D_i}\right) + \left(\frac{D_i}{D_o}\right)\frac{1}{\bar{h}_o}}$$

When the *wall thickness is small and the thermal conductivity is high*, the overall heat transfer coefficient of the pipe is

$$U = \frac{1}{\frac{1}{\bar{h}_i} + \frac{1}{\bar{h}_o}}$$

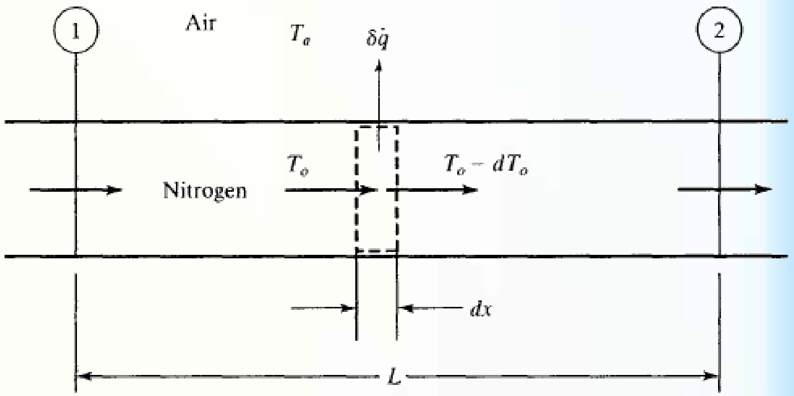
EXAMPLE

Nitrogen (γ = 1.4. R = 0.2968 kJ/kg·K, and c_p = 1.038 kJ/kg·K) enters an uninsulated duct at Mach 2.0, with a stagnation temperature of 1000 K and stagnation pressure of 1.4 MPa. Heat is lost from the nitrogen to the outside ambient air at 20°C,with the mean overall heat transfer coefficient U equal to 60 W/m^2 ·K. The duct's diameter is 5 cm, and its length is 2 m. Determine the outlet stagnation temperature, outlet Mach number, and percent of change of stagnation pressure. Neglect friction.



EXAMPLE (CONT.)

Energy equation:



$$\delta\dot{q} = -\dot{m}c_p dT_o = U\,dA_p(T_o - T_a)$$

$$dA_p = \pi D\,dx$$

$$-\int_1^2 \frac{dT_o}{T_o - T_a} = \int_1^2 \frac{U\pi D}{\dot{m}c_p}dx = \frac{U\pi D}{\dot{m}c_p}\int_1^2 dx$$

$$\ln\frac{T_{o1} - T_a}{T_{o2} - T_a} = \frac{U\pi D}{\dot{m}c_p}L$$

At $M_1 = 2$, from the isentropic relations, $(T/T_o)_1 = 0.5556$ and $(p/p_o)_1 = 0.1278$, so that $T_1 = 555.6$ K and $p_1 = 0.1789$ MPa. Then

$$V_1 = M_1\sqrt{\gamma RT_1} = 2.0\sqrt{1.4(296.8)(555.6)} = 960.9639 \text{ m/s}$$

$$\dot{m} = \frac{p_1}{RT_1}AV_1 = \frac{(178.9 \text{ kN/m}^2)\left(\dfrac{\pi}{4}0.05^2 \text{ m}^2\right)(960.9639 \text{ m/s})}{(0.2968 \text{ kJ/kg}\cdot\text{K})(555.6 \text{ K})} = 2.0470 \text{ kg/s}$$

EXAMPLE (CONT.)

$$\ln\frac{T_{o1}-T_a}{T_{o2}-T_a}=\frac{U\pi D}{\dot{m}c_p}L=\frac{(60\ \text{W/m}^2\cdot\text{K})\pi(0.05\ \text{m})2\ \text{m}}{(2.047\ \text{kg/s})(1.038\ \text{kJ/kg}\cdot\text{K})(1{,}000\ \text{J/kJ})}=0.008871$$

$$\frac{T_{o1}-T_a}{T_{o2}-T_a}=1.0089$$

$$T_{o2}-T_a=\frac{1{,}000-293}{1.0089}=700.7557\ \text{K}\longrightarrow T_{o2}=293+700.7557=993.7557\ \text{K}$$

To find M_2 use the Rayleigh relations, starting from

$$\frac{T_{o2}}{T_o^*}=\frac{T_{o2}}{T_{o1}}\frac{T_{o1}}{T_o^*}=\frac{993.7557}{1{,}000}0.7934=0.7884$$

From this value, we find that $M_2 = 2.0251$.

EXAMPLE (CONT.)

Accordingly, we may further write that

$$p_2 = \frac{p_2}{p^*}\frac{p^*}{p_1}p_1 = \frac{0.3560}{0.3636}178.9 = 175.1606 \text{ kPa}$$

$$p_{o2} = \frac{p_{o2}}{p_2}p_2 = \frac{1}{0.1229}175.1606 = 1{,}425.2288 \text{ kPa}$$

$$\frac{p_{o2} - p_{o1}}{p_{o1}} = \frac{1{,}425.2288 - 1{,}400}{1{,}400}100 = 1.8021\% \text{ increase}$$

Linearized Supersonic Flow

Linearized perturbation-velocity potential equation

$$\beta^2\phi_{xx} + \phi_{yy} = 0$$ where $\beta = \sqrt{1 - M_\infty^2}$. For subsonic flow

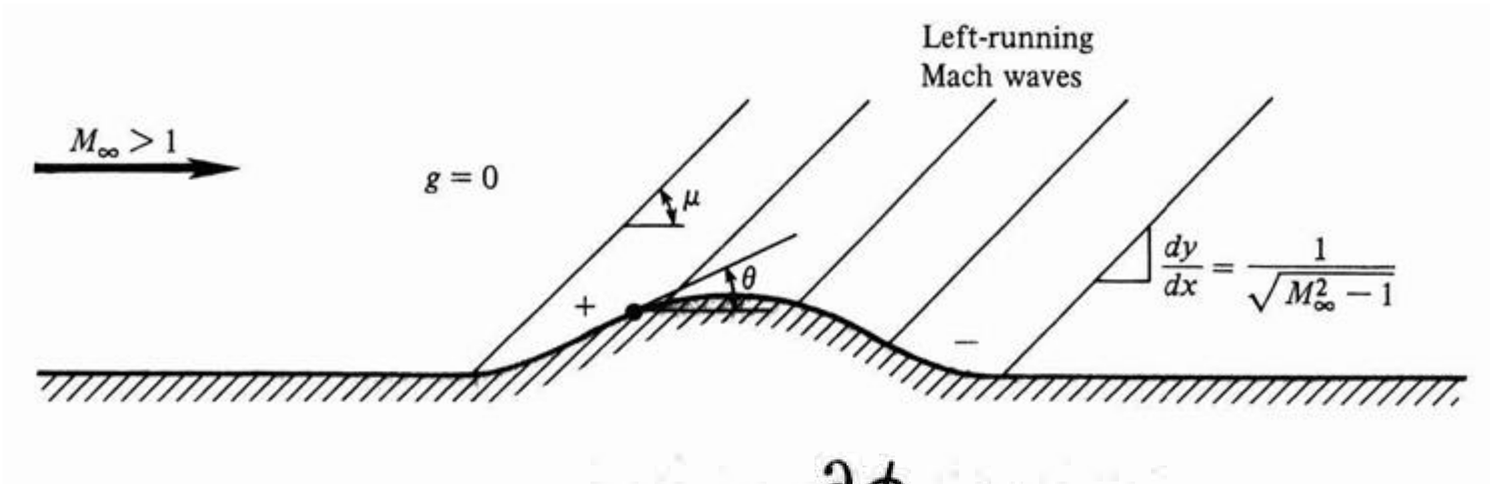
$$\lambda^2\phi_{xx} - \phi_{yy} = 0$$ where $\lambda = \sqrt{M_\infty^2 - 1}$ For supersonic flow,

$$\phi = f(x - \lambda y) + g(x + \lambda y)$$

Examining the particular solution where $g = 0$, and hence $\phi = f(x - \lambda y)$, we see that lines of constant ϕ correspond to $x - \lambda y = \text{const}$, or

$$\frac{dy}{dx} = \frac{1}{\lambda} = \frac{1}{\sqrt{M_\infty^2 - 1}}$$

Consider the supersonic flow over a body or surface which introduces small changes in the flowfield, i.e., flow over a thin airfoil, over a mildly wavy wall, or over a small hump in a surface.



$$\phi = f(x - \lambda y)$$

$$u' = \frac{\partial \phi}{\partial x} = f'$$

$$v' = \frac{\partial \phi}{\partial y} = -\lambda f'$$

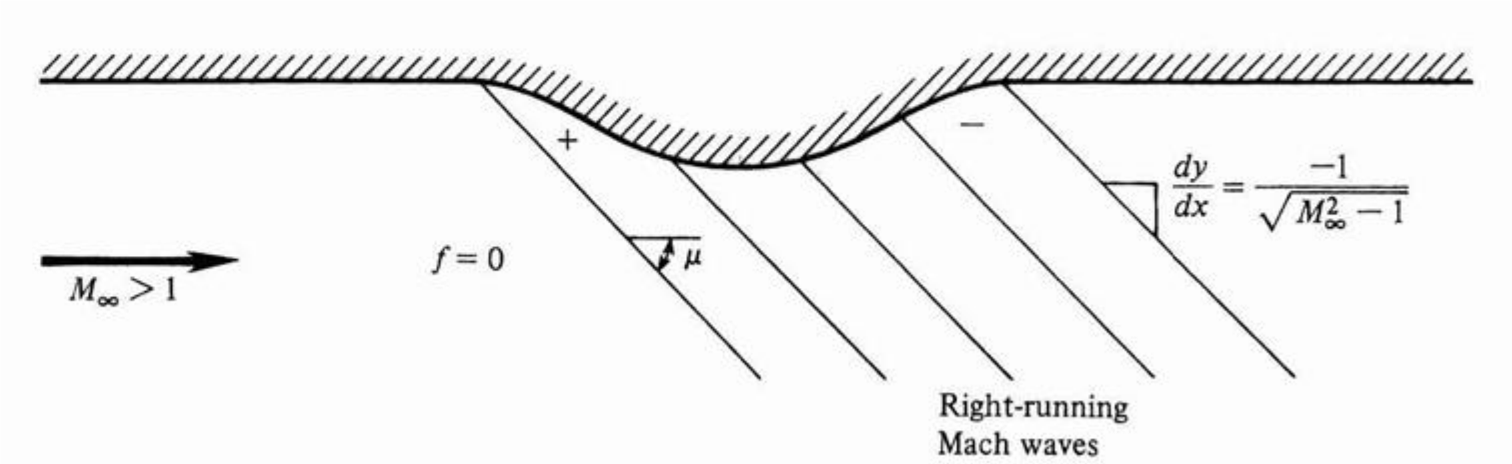
$$u' = -\frac{v'}{\lambda}$$

$$\tan\theta = \frac{dy}{dx} = \frac{v'}{V_\infty + u'}$$

For small perturbations, $u' \ll V_\infty$ and $\tan\theta \approx \theta$.

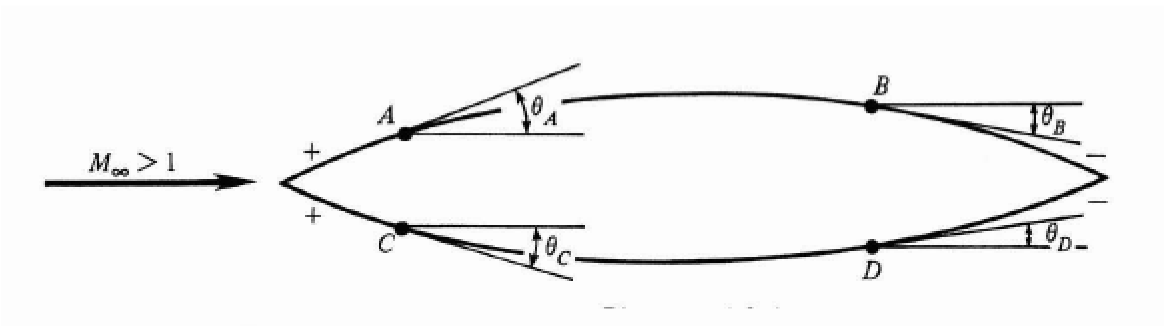
$$v' = V_\infty\theta \xrightarrow{u' = -\frac{v'}{\lambda}} u' = -\frac{V_\infty\theta}{\lambda} \rightarrow C_p = -\frac{2u'}{V_\infty} = \frac{2\theta}{\lambda}$$

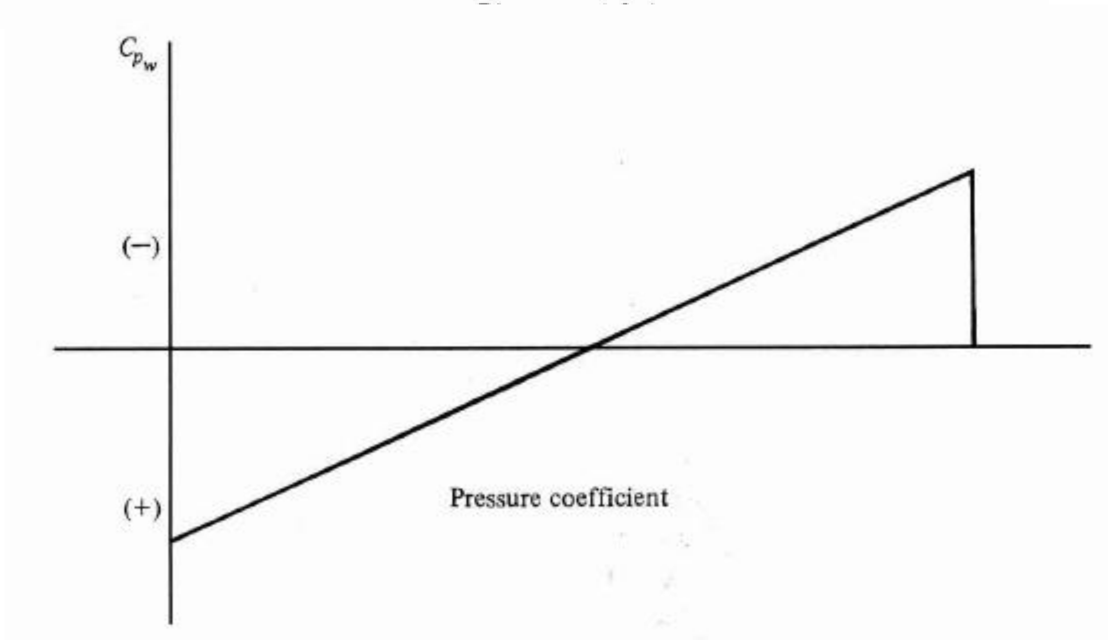
$$\boxed{C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}}$$



$$\phi = f(x - \lambda y) + g(x + \lambda y) \xrightarrow{f = 0} \phi = g(x + \lambda y)$$

$$C_p = \frac{-2\theta}{\sqrt{M_\infty^2 - 1}}$$

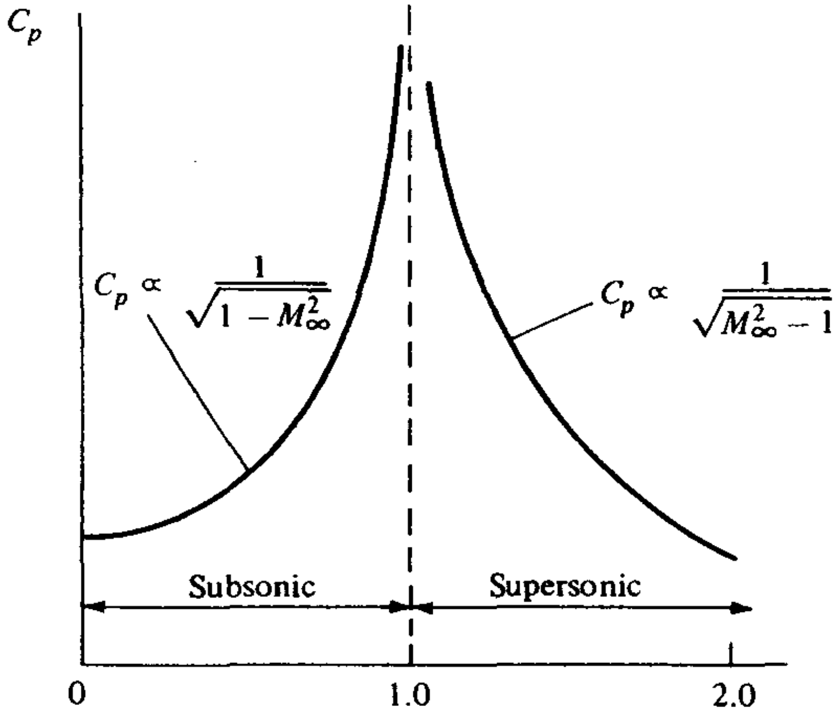


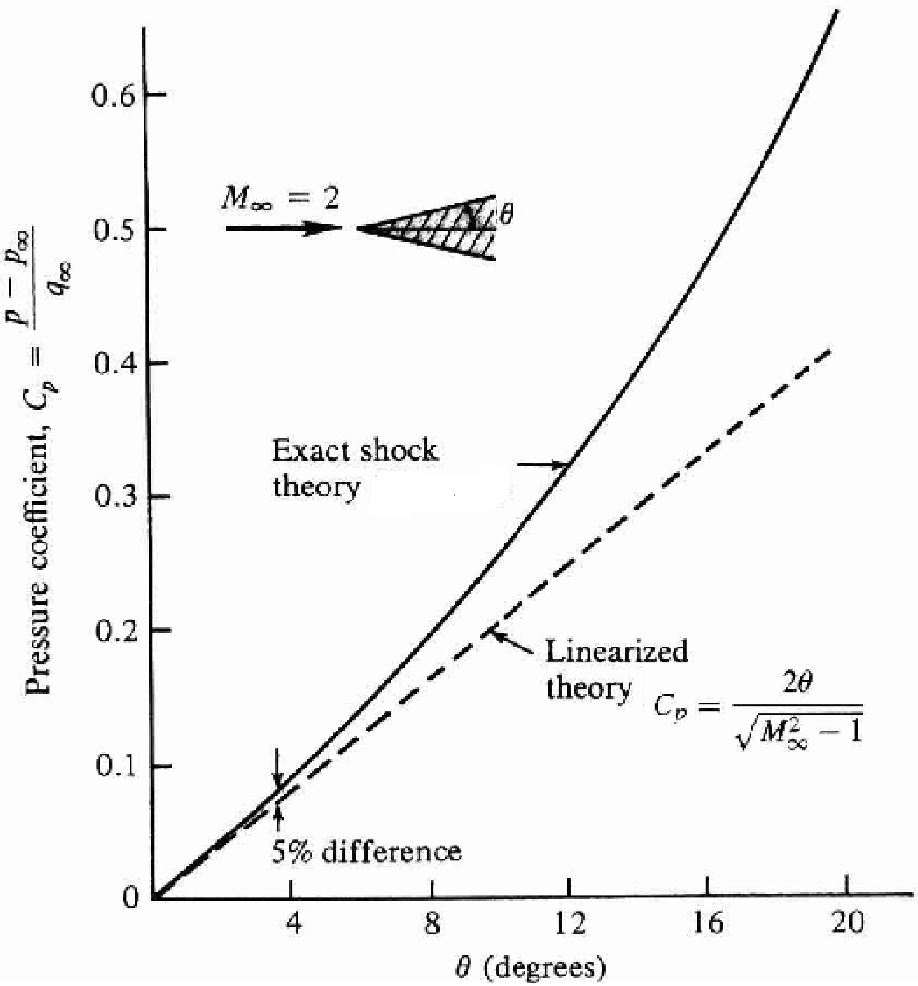


$$C_{pA} = \frac{2\theta_A}{\sqrt{M_\infty^2 - 1}}$$

$$C_{pB} = \frac{2\theta_B}{\sqrt{M_\infty^2 - 1}}$$

There is no real need to worry about the formal sign
. For any practical application, it is suggested the use of the Eq. along the body with common sense to single out the compression and expansion surfaces on a body.
If the surface is a compression surface, Cp must be positive, no matter whether the surface is on the top or bottom of the body.
Similarly, if the surface is an expansion surface, Cp must be negative.


C_p
$C_p \propto \frac{1}{\sqrt{1 - M_\infty^2}}$
$C_p \propto \frac{1}{\sqrt{M_\infty^2 - 1}}$
Subsonic
Supersonic
0
1.0
2.0


Pressure coefficient, $C_p = \frac{p - p_\infty}{q_\infty}$
0.6
0.5
0.4
0.3
0.2
0.1
0
$M_\infty = 2$
θ
Exact shock theory
Linearized theory
$C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}$
5% difference
4
8
12
16
20
θ (degrees)

$$c_n = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}} \frac{1}{c} \int_0^c dx = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}$$

The axial force coefficient

$$c_a = \frac{1}{c} \int_{\mathrm{LE}}^{\mathrm{TE}} (C_{p,u} - C_{p,l})\, dy$$

the pressures act normal to the surface, and hence there is no component of the pressure force in the x direction. $c_a = 0.$

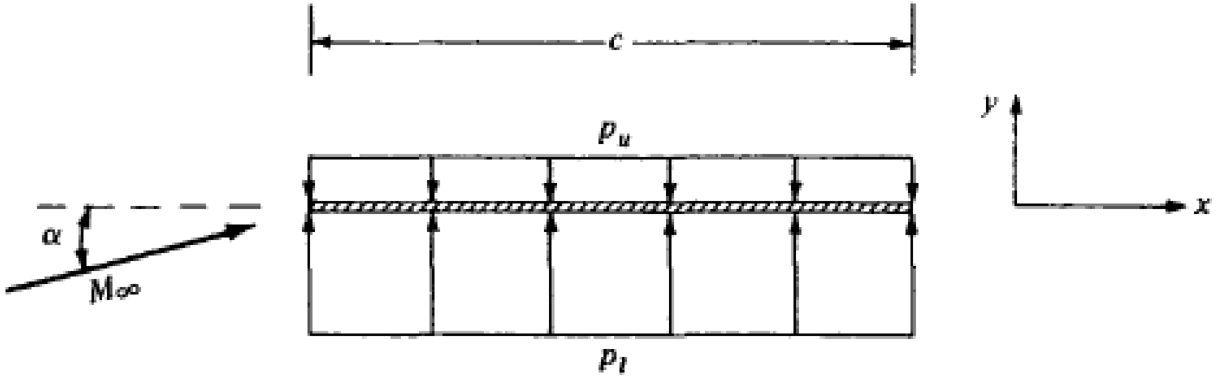
$$c_l = c_n \cos\alpha - c_a \sin\alpha$$

$$c_d = c_n \sin\alpha + c_a \cos\alpha$$

The assumption that a is small $\cos\alpha \approx 1$ and $\sin\alpha \approx \alpha$.

Example

Consider a flat plate at angle of attack in a supersonic flow. Calculate lift and drag coefficient



$$C_{p,l} = \frac{2\alpha}{\sqrt{M_\infty^2 - 1}}$$

$$C_{p,u} = -\frac{2\alpha}{\sqrt{M_\infty^2 - 1}}$$

The normal force coefficient

$$c_n = \frac{1}{c}\int_0^c (C_{p,l} - C_{p,u})\,dx$$

$$c_l = c_n - c_a\alpha$$

$$c_d = c_n\alpha + c_a$$

$$\boxed{c_l = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}}$$

$$\boxed{c_d = \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}}}$$

Within the approximation of linearized theory, cl depends only on a and is independent of the airfoil shape and thickness. However, the same linearized theory gives a wave-drag coefficient in the form of

$$c_d = \frac{4}{\sqrt{M_\infty^2 - 1}}(\alpha^2 + g_c^2 + g_t^2)$$

where g_c and g_t are functions of the airfoil camber and thickness.

Elements of Hypersonic Flow

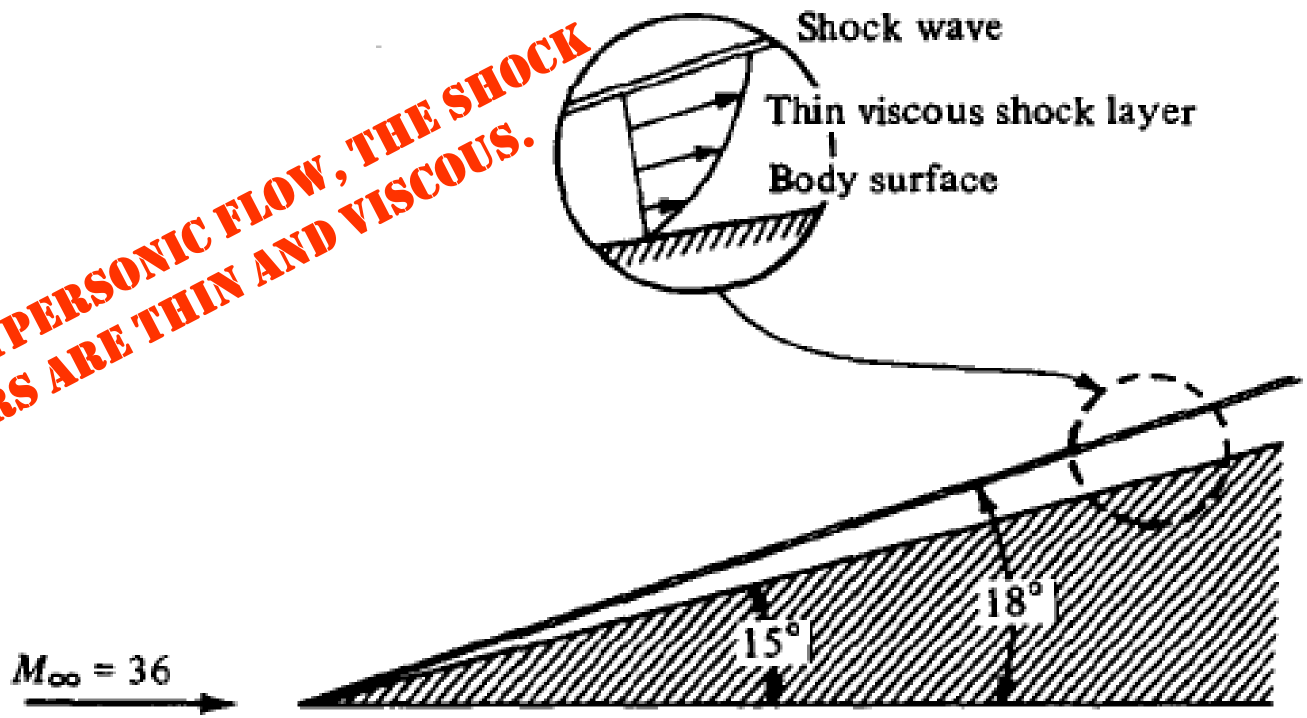
Almost everyone has their own definition of the term hypersonic.
If we were to conduct something like a public opinion poll among those present, and asked everyone to name a Mach number above which the flow of a gas should properly be described as hypersonic there would be a majority of answers round about 5 or 6,
but it would be quite possible for someone to advocate, and defend, numbers as small as 3, or as high as 12.

QUALITATIVE ASPECTS OF HYPERSONIC FLOW

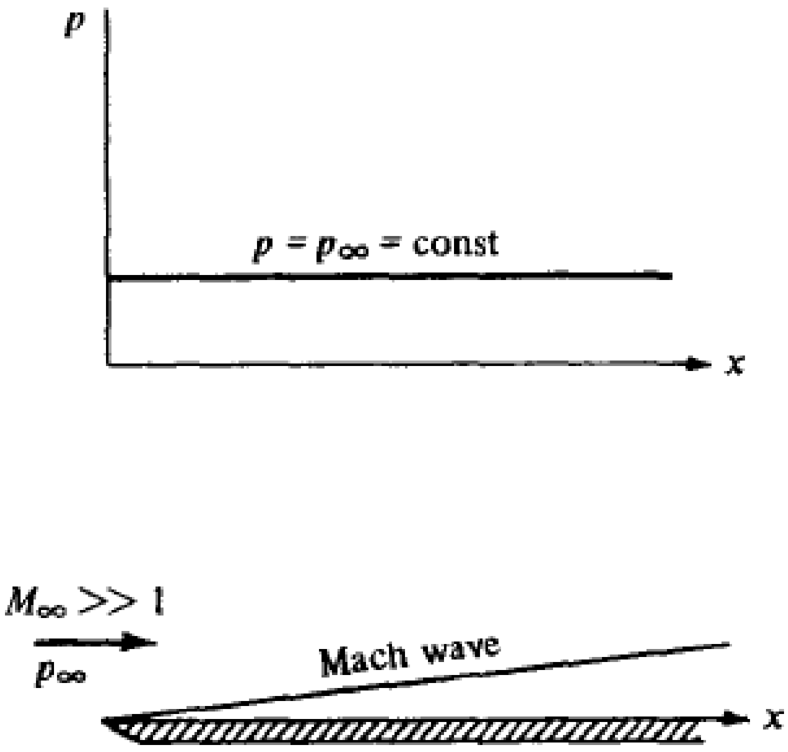
1-Thin shock layer

Consider a 15° half-angle wedge flying at $M_\infty = 36$.

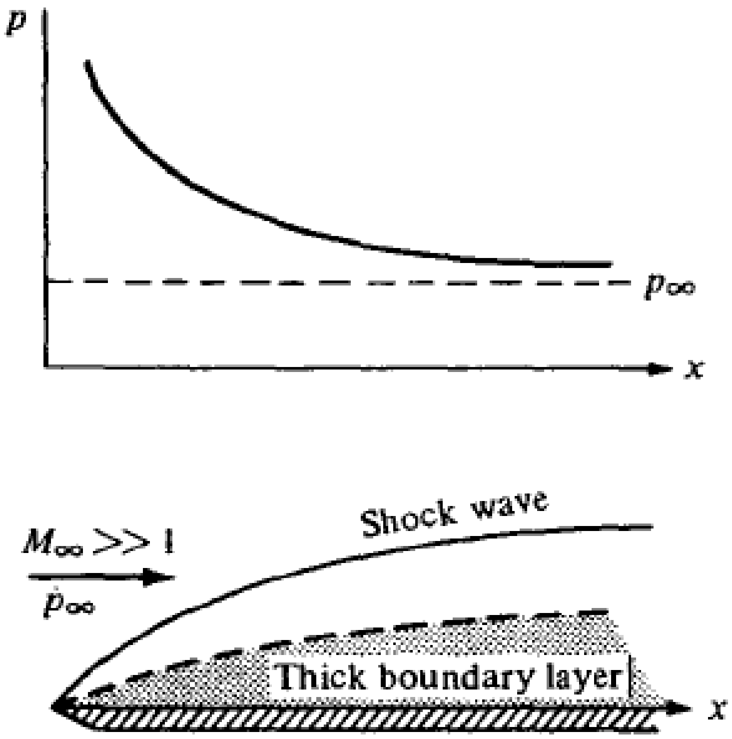
FOR HYPERSONIC FLOW , THE SHOCK LAYERS ARE THIN AND VISCOUS.



2- viscous interaction



(a) No viscous interaction



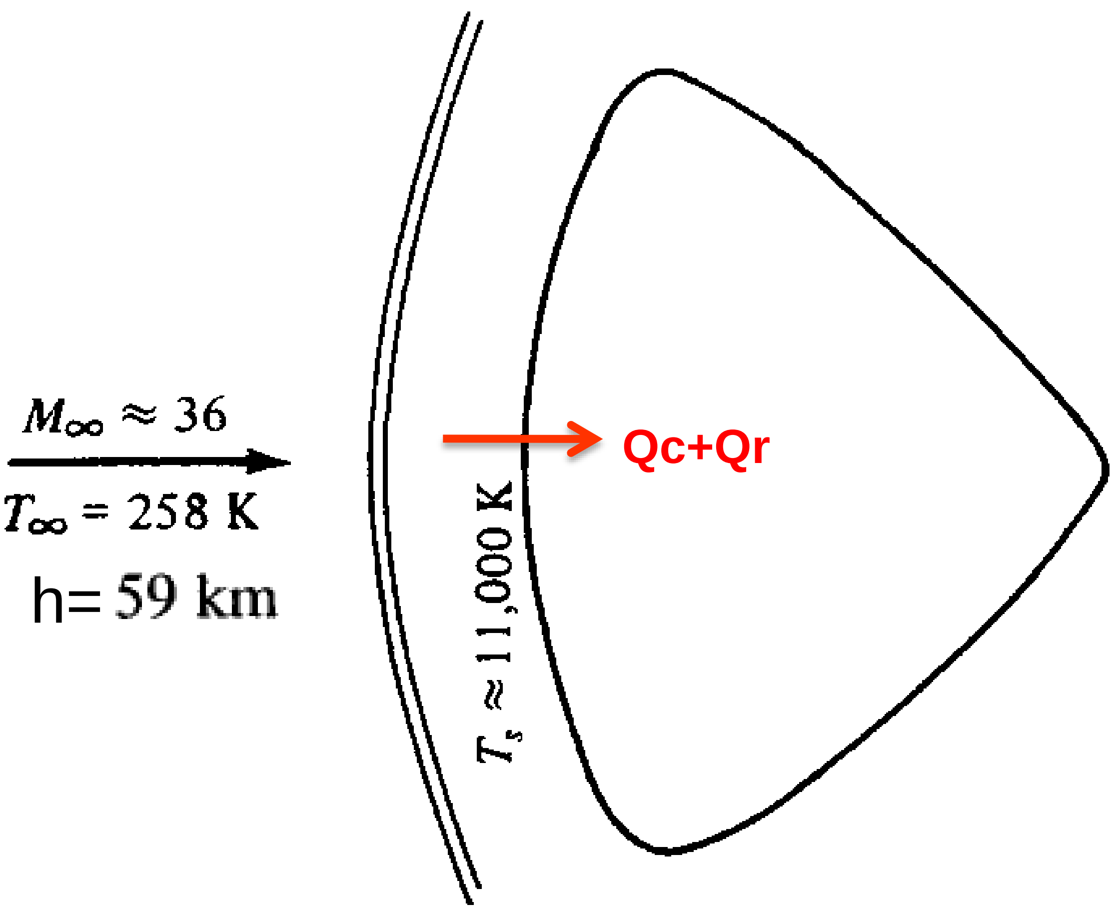
(b) Viscous interaction

3- high temperatures in the shock layer

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

T_0=65000 K

**six times
hotter than
the surface
of the sun!**



Let us examine these high-temperature effects in more detail.

$p = 1$ atm and $T = 288$ K ⇨ 20 percent O_2 and 80 percent N_2 by volume.

increase T to 2000 K ⇨ $O_2 \rightarrow 2O$ $\quad 2000\text{ K} < T < 4000\text{ K}$

increase to 4000 K ⇨ $N_2 \rightarrow 2N$ $\quad 4000\text{ K} < T < 9000\text{ K}$

increase to 9000 K ⇨ $N \rightarrow N^+ + e^-$, $O \rightarrow O^+ + e^-$ $\quad T > 9000\text{ K}$

The presence of these free electrons in the shock layer is responsible for the "communications blackout" experienced over portions of the trajectory of a reentry vehicle.

Associated with the high-temperature shock layers is a large amount of heat transfer to the surface of a hypersonic vehicle

Indeed, for reentry velocities, aerodynamic heating dominates the design of the vehicle,

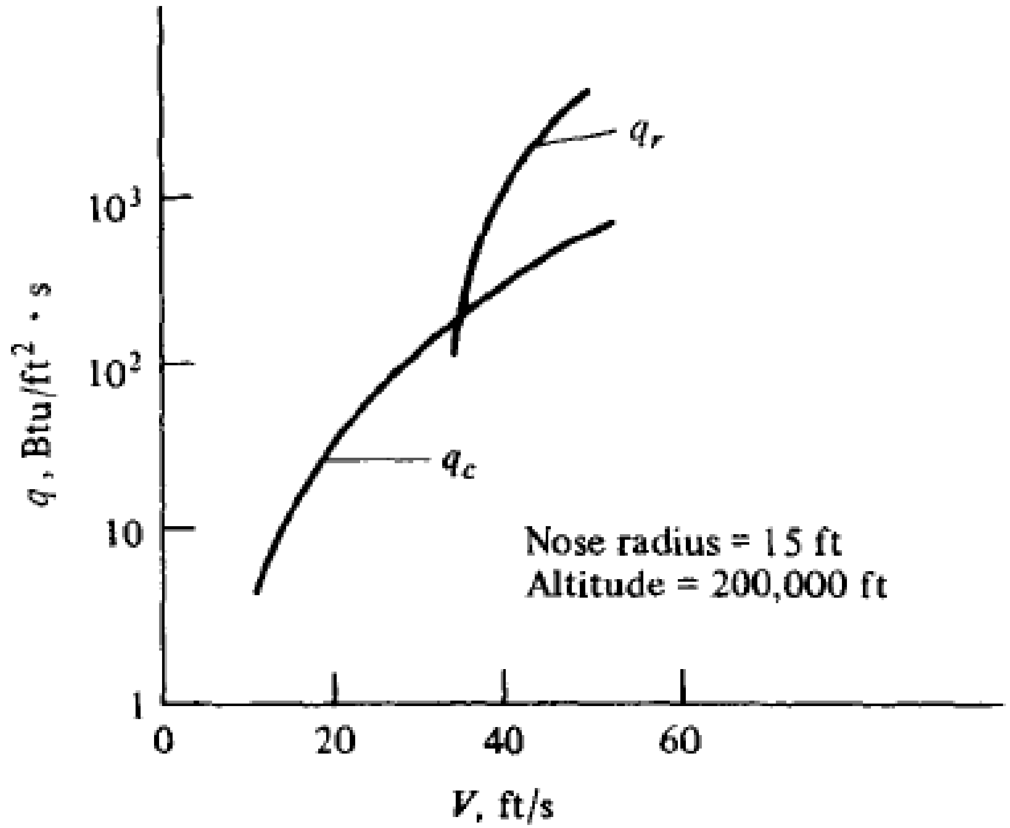
The usual mode of aerodynamic heating is the transfer of energy from the hot shock layer to the surface by means of thermal conduction at the surface;

The gas normal to the surface, then $q_c = -k(\partial T/\partial n)$
is the heat transfer into the surface.

Because ∂T/∂n is a flow-field property generated by the flow of the gas over the body, q_c is called convective heating.

For reentry velocities associated with ICBMs (about 28,000 ft/s), this is the only meaningful mode of heat transfer to the body.

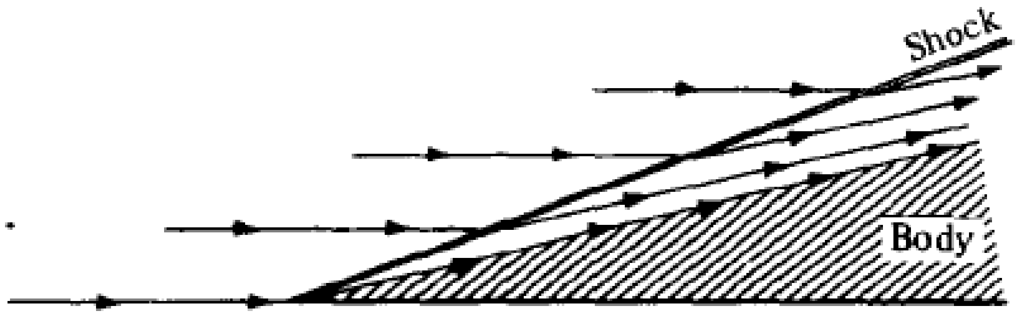
However, at higher velocities, the shock-layer temperature becomes even hotter. From experience, we know that all bodies emit thermal radiation, and from physics you know that blackbody radiation varies as T4;

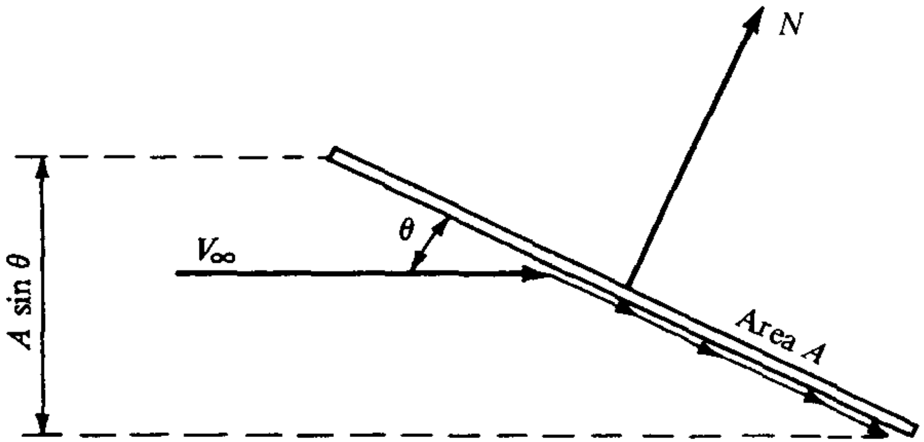


Convective and radiative heating rates of a blunt reentry vehicle as a function of flight velocity.

NEWTONIAN THEORY

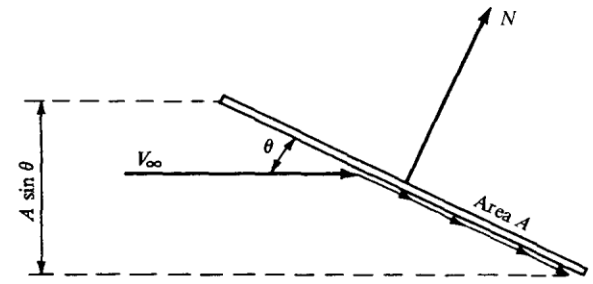
note how close the shock wave lies to the body surface.





Schematic for Newtonian impact theory.

Hence, the time rate of change of momentum is



Mass flow × change in normal component of velocity

$$(\rho_\infty V_\infty A \sin\theta)(V_\infty \sin\theta) = \rho_\infty V_\infty^2 A \sin^2\theta$$

from Newton's second law, $$N = \rho_\infty V_\infty^2 A \sin^2\theta$$

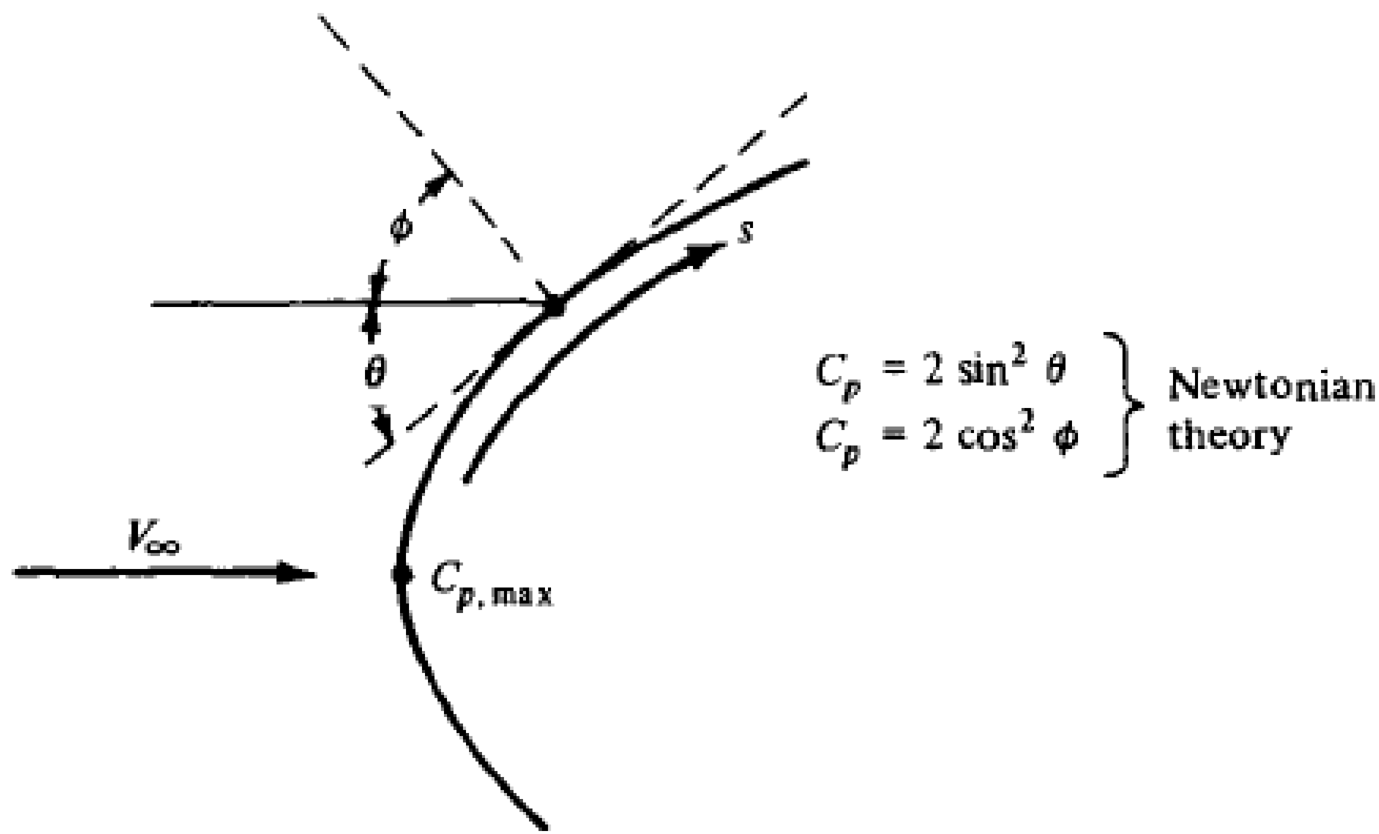
$$\Rightarrow \quad \frac{N}{A} = \rho_\infty V_\infty^2 \sin^2\theta$$

Therefore, when the purely directed motion of the particles in Newton's model results in the normal force per unit area, N/A

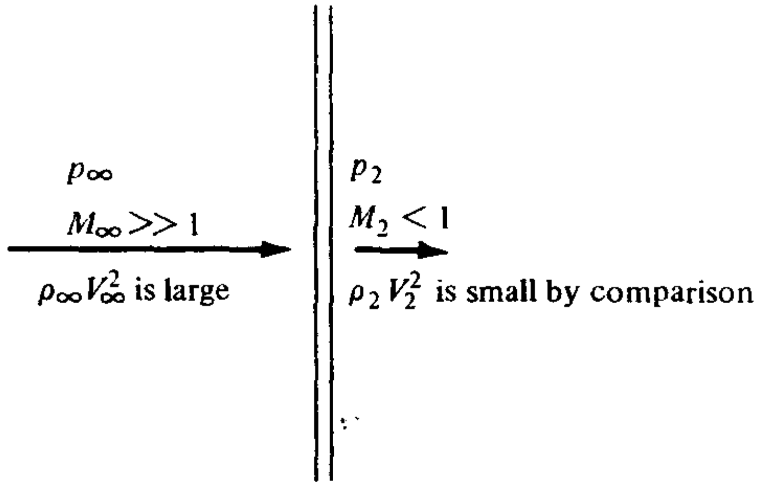
this normal force per unit area must be construed as the pressure difference above P∞, namely, p -p∞ on the surface.

$$p - p_\infty = \rho_\infty V_\infty^2 \sin^2\theta$$

$$\frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 2\sin^2\theta \quad \Rightarrow \quad \boxed{C_p = 2\sin^2\theta}$$



The result that the maximum pressure coefficient approaches 2 at $M_\infty \longrightarrow \infty$ can be obtained independently from the one-dimensional momentum equation,



$$p_\infty + \rho_\infty V_\infty^2 = p_2 + \rho_2 V_2^2$$

Recall that across a normal shock wave the flow velocity decreases, $V_2 < V_\infty$;

$$(\rho_\infty V_\infty^2) \gg (\rho_2 V_2^2)$$

$$M_\infty \to \infty, \quad \Rightarrow \quad p_2 - p_\infty = \rho_\infty V_\infty^2$$

$$C_p = \frac{p_2 - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 2$$

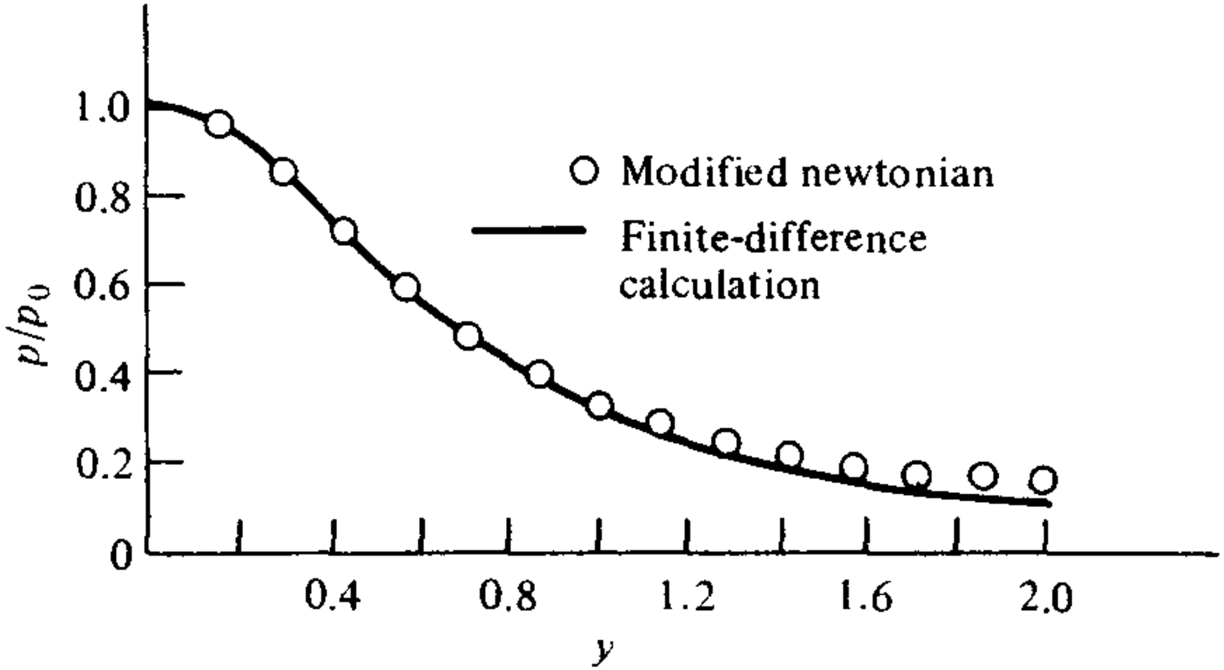
Limitation: M∞ → ∞ & γ=1.4

$C_{p,max}$ for a given M∞ can be readily calculated from normal shock-wave theory.

$$p_{0,2}/p_1 = p_{0,2}/p_\infty \quad \Rightarrow \quad C_{p,\max} = (2/\gamma M_\infty^2)(p_{0,2}/p_\infty - 1).]$$

$$\boxed{C_p = C_{p,\max}\sin^2\theta}$$

Modified Newtonian law.



Surface pressure distribution, paraboloid, $M_\infty = 4$**. Comparison of modified newtonian theory and time-dependent finite-difference calculations.**

THE LIFT AND DRAG OF WINGS AT HYPERSONIC SPEEDS:
Newtonian results for a flat plate at angle of attack

Question: At subsonic speeds, how do the lift coefficient C_L and drag coefficient C_D for a wing vary with angle of attack α?

1. The lift coefficient varies linearly with angle of attack, at least up to the stall;

2.The drag coefficient is given by

$$C_D = c_d + \frac{C_L^2}{\pi e AR}$$

Since C_L is proportional to α, then C_D varies as the square of α.

Question: ***At supersonic speeds, how do C_L and C_D for a wing vary with α?***

1- Lift coefficient varies *linearly with α,*

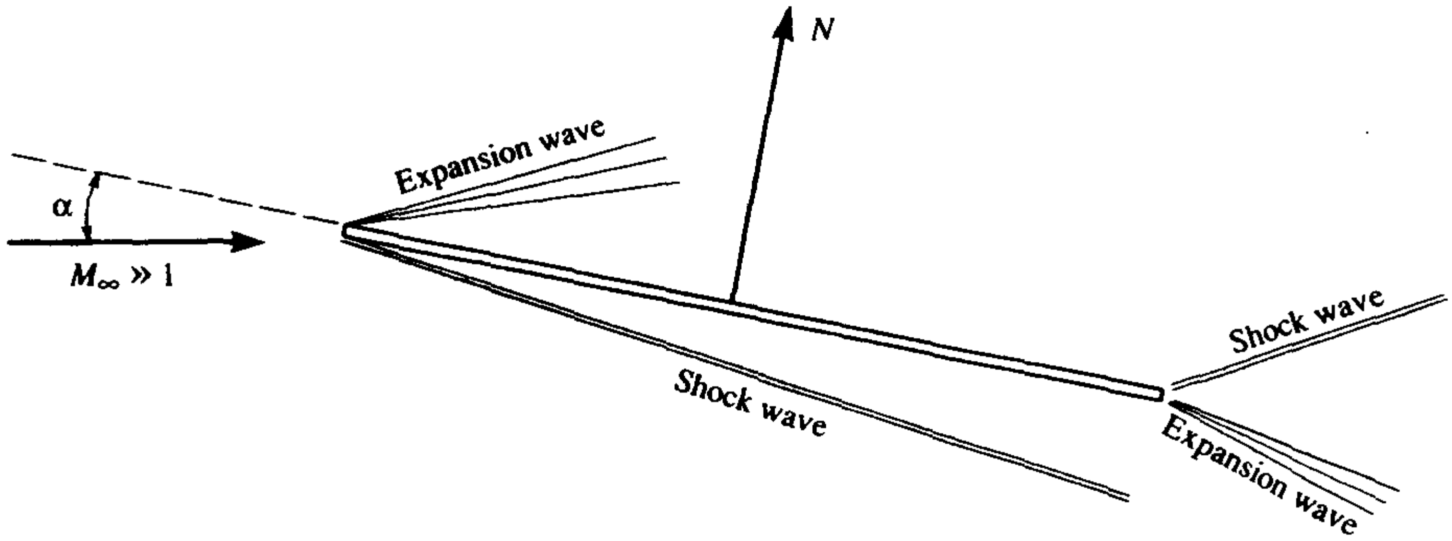
$$c_l = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}$$

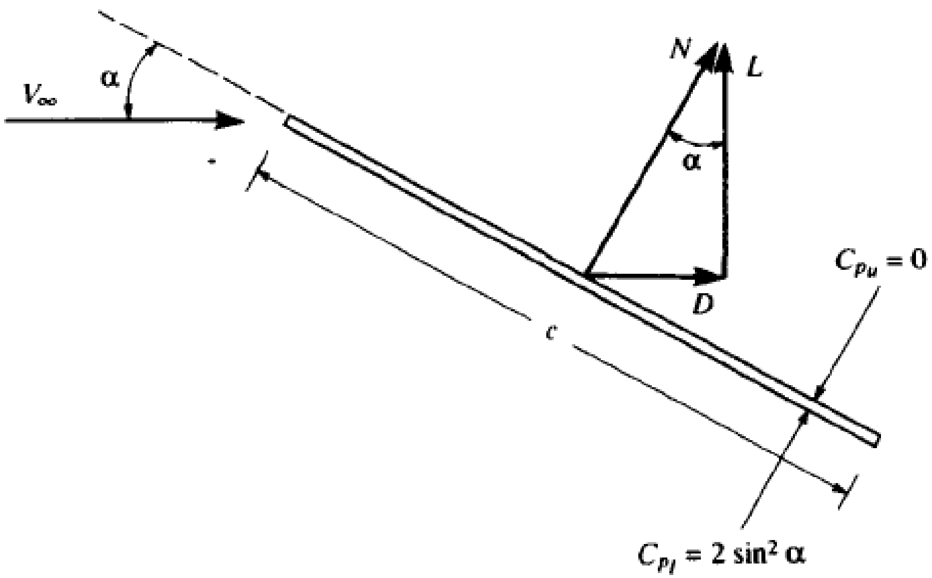
2. Drag coefficient varies as the square ofα,

$$c_d = \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}}$$

The characteristics of a finite wing at supersonic speeds follow essentially the same functional variation with the angle of attack,

Question: **At hypersonic speeds, how do CL and CD for a wing vary with α ?**





$$C_{p,l} = 2\sin^2\alpha \qquad C_{p,u} = 0$$

$$c_n = \frac{1}{c}\int_0^c (C_{p,l} - C_{p,u})\,dx \Rightarrow c_n = \frac{1}{c}(2\sin^2\alpha)c = 2\sin^2\alpha$$

$$\Rightarrow \begin{aligned} c_l &= c_n \cos\alpha \\ c_d &= c_n \sin\alpha \end{aligned} \Rightarrow \frac{L}{D} = \cot\alpha$$

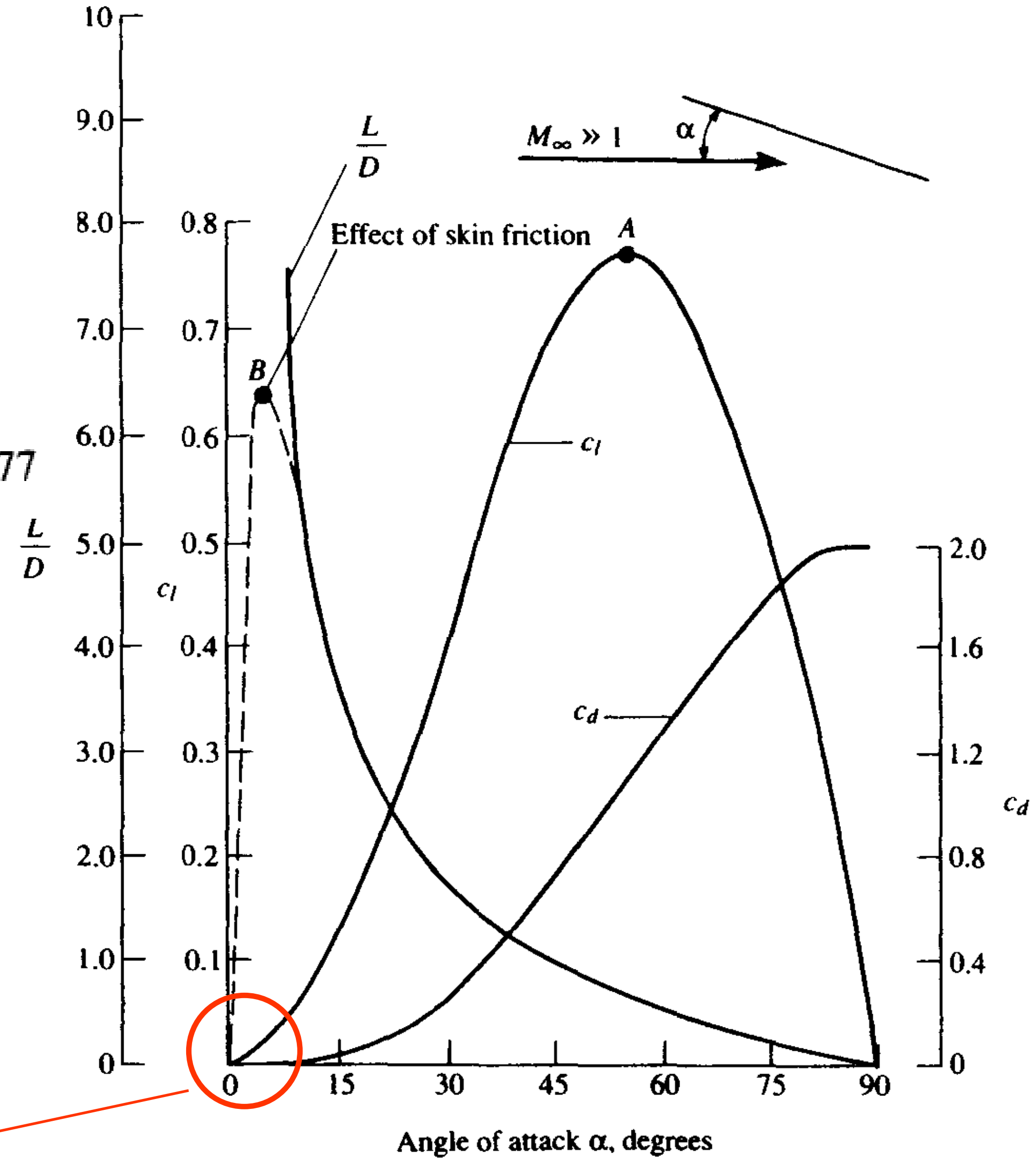
$$\frac{dc_l}{d\alpha} = (2\sin^2 2)(-\sin\alpha) + 4\cos^2\alpha\sin\alpha = 0$$

$$\sin^2\alpha = 2\cos^2\alpha = 2(1-\sin^2\alpha)$$

$$\sin^2\alpha = \tfrac{2}{3}$$

$$\alpha = 54.7^\circ$$

$$c_{l,\max} = 2\sin^2(54.7^\circ)\cos(54.7^\circ) = 0.77$$



Nonlinear even at
small angle of attack

$$c_d = 2\sin^3\alpha + c_{d,0}$$

α is small,

$$c_l = 2\alpha^2$$

$$c_d = 2\alpha^3 + c_{d,0}$$

Let us examine the conditions associated with $(L/D)_{\max}$ more closely.

The value of $(L/D)_{max}$ and the angle of attack at which it occurs are strictly a function of the zero-lift drag coefficient denoted by $cd_{,o}$.

$$c_d = 2\sin^3\alpha + c_{d,0}$$

At small angles of attack

$$c_l = 2\alpha^2$$

$$c_d = 2\alpha^3 + c_{d,0}$$

$$\frac{c_l}{c_d} = \frac{2\alpha^2}{2\alpha^3 + c_{d,0}}$$

$$\frac{d(c_l/c_d)}{d\alpha} = \frac{(2\alpha^3 + c_{d,0})4\alpha - 2\alpha^2(6\alpha^2)}{(2\alpha^3 + c_{d,0})} = 0$$

$$8\alpha^4 + 4\alpha c_{d,0} - 12\alpha^4 = 0$$

$$4\alpha^3 = 4c_{d,0}$$

$$\boxed{\alpha = (c_{d,0})^{1/3}}$$

$$\left(\frac{c_l}{c_d}\right)_{\max} = \frac{2(c_{d,0})^{2/3}}{2c_{d,0} + c_{d,0}} = \frac{2/3}{(c_{d,0})^{1/3}}$$

$$\left(\frac{L}{D}\right)_{\max} = \left(\frac{c_l}{c_d}\right)_{\max} = \boxed{0.67/(c_{d,0})^{1/3}}$$

$$\alpha = (c_{d,0})^{1/3}$$

$$c_d = 2\alpha^3 + c_{d,0}$$

$$\Rightarrow \quad c_d = 2c_{d,0} + c_{d,0} = 3c_{d,0}$$

$$c_d = c_{d,w} + c_{d,0} \quad \Rightarrow \quad \boxed{c_{d,w} = 2c_{d,0}}$$

Accuracy Considerations

Consider an infinitely thin flat plate at an angle of attack of 15° in a Mach 8 flow. Calculate the pressure coefficients on the top and bottom surface, the lift and drag coefficients, and the lift-to-drag ratio using
(a) exact shock-expansion theory,
(b) Newtonian theory. Compare the results.

(a) **the upper surface,** $M_1 = 8$ and $\nu_1 = 95.62°$,

$$\nu_2 = \nu_1 + \theta = 95.62 + 15 = 110.62° \Rightarrow M_2 = 14.32$$

$$M_1 = 8 \Rightarrow p_{0_1}/p_1 = 0.9763 \times 10^4 \qquad p_{0_1} = p_{0_2}$$

$$M_2 = 14.32 \Rightarrow p_{0_2}/p_2 = 0.4808 \times 10^6$$

$$\frac{p_2}{p_1} = \frac{p_{0_1}}{p_1} \Big/ \frac{p_{0_2}}{p_2} = \frac{0.9763 \times 10^4}{0.4808 \times 10^6} = 0.0203$$

$$C_{p_2} = \frac{2}{\gamma M_1^2}\left(\frac{p_2}{p_1} - 1\right) = \frac{2}{(1.4)(8)^2}(0.0203 - 1) = \boxed{-0.0219}$$

on the bottom surface from the oblique shock theory,

from the θ-β-M for $M_1 = 8$ and $\theta = 15°$, $\beta = 21°$:

$$M_{n,1} = M_1 \sin\beta = 8 \sin 21° = 2.87 \qquad p_3/p_1 = 9.443$$

$$C_{p_3} = \frac{2}{\gamma M_1^2}\left(\frac{p_3}{p_2} - 1\right) = \frac{2}{(1.4)(8)^2}(9.443 - 1) = \boxed{0.1885}$$

$$c_n = \frac{1}{c}\int_0^c (C_{p,\ell} - C_{p,u})\,dx = C_{p_3} - C_{p_2} = 0.1885 - (-0.0219) = 0.2104$$

The axial force on the plate is zero,

$$c_\ell = c_n \cos\alpha = 0.2104 \cos 15^\circ = \boxed{0.2032}$$

$$c_d = c_n \sin\alpha = 0.2104 \sin 15^\circ = \boxed{0.0545}$$

$$\frac{L}{D} = \frac{c_\ell}{c_d} = \frac{0.2032}{0.0545} = \boxed{3.73}$$

(b)

$$C_{p_3} = 2\sin^2\alpha = 2\sin^2 15^\circ = \boxed{0.134}$$

$$C_{p_2} = \boxed{0}$$

$$c_\ell = (C_{p_3} - C_{p_2})\cos\alpha = 0.134 \cos 15^\circ = \boxed{0.1294}$$

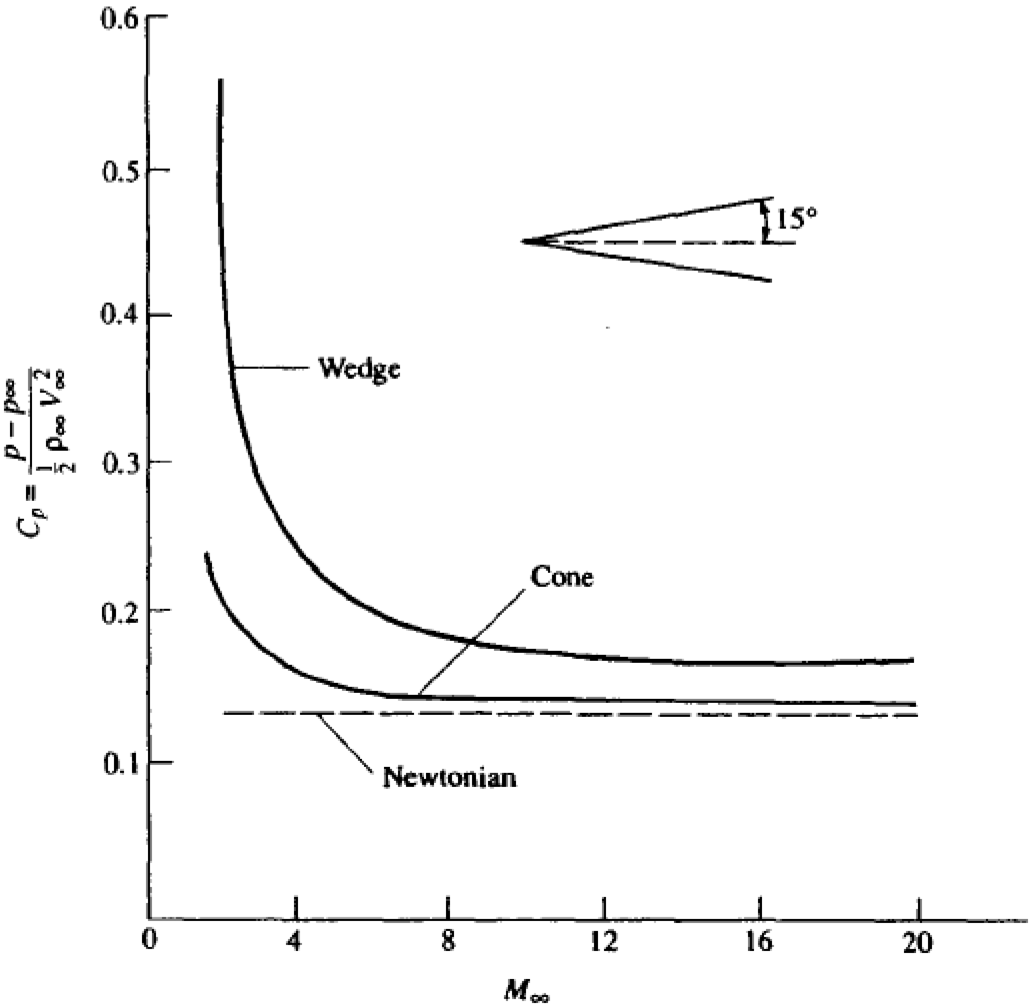
$$c_d = (C_{p_3} - C_{p_2})\sin\alpha = 0.134 \sin 15^\circ = \boxed{0.03468}$$

$$\frac{L}{D} = \frac{c_\ell}{c_d} = \frac{0.1294}{0.3468} = \boxed{3.73}$$

Discussion: **From the above worked example, we see that newtonian theory underpredicts the pressure coefficient on the bottom surface by 29 percent, and of course predicts a value of zero for the pressure coefficient on the upper surface in comparison to -0.0219 from exact theory—an error of 100 percent. Also, newtonian theory underpredicts cl and cd by 36.6 percent. However, the value of L/D from newtonian theory is exactly correct.**

$$\frac{L}{D} = \cot\alpha$$

$$\frac{L}{D} = \cot 15^\circ = 3.73$$



0.6
0.5
0.4
0.3
0.2
0.1
$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2}$
15°
Wedge
Cone
Newtonian
0
4
8
12
16
20
M_∞

HYPERSONIC SHOCK-WAVE RELATIONS AND ANOTHER LOOK AT NEWTONIAN THEORY

Consider the flow through a straight oblique shock wave.

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}\left(M_1^2 \sin^2 \beta - 1\right)$$

$M_1 \to \infty$:

$$\boxed{\frac{p_2}{p_1} = \frac{2\gamma}{\gamma + 1} M_1^2 \sin^2 \beta}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1) M_1^2 \sin^2 \beta}{(\gamma - 1) M_1^2 \sin^2 \beta + 2}$$

$M_1 \to \infty$:

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{\gamma - 1}}$$

$$\frac{T_2}{T_1} = \frac{(p_2/p_1)}{(\rho_2/\rho_1)} \qquad \text{(from the equation of state: } p = \rho RT\text{)}$$

$M_1 \rightarrow \infty$:

$$\boxed{\frac{T_2}{T_1} = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M_1^2 \sin^2\beta}$$

$$\tan\theta = 2\cot\beta\left[\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma+\cos 2\beta)+2}\right]$$

$$\sin\beta \approx \beta$$

$$\cos 2\beta \approx 1$$

$$\tan\theta \approx \sin\theta \approx \theta$$

$$\theta = \frac{2}{\beta}\left[\frac{M_1^2\beta^2 - 1}{M_1^2(\gamma+1)+2}\right]$$

as $M_1 \to \infty$ *and* θ, *hence* β *is small:*

$$\boxed{\frac{\beta}{\theta} = \frac{\gamma + 1}{2}}$$

Note that, for $\gamma = 1.4$,

$$\boxed{\beta = 1.2\theta}$$

It is interesting to observe that, in the hypersonic limit for a slender wedge, the wave angle is only 20 percent larger than the wedge angle—a graphic demonstration of a thin shock layer in hypersonic flow.

$$C_p = \frac{p_2 - p_1}{q_1} \quad \Longrightarrow \quad C_p = \frac{2}{\gamma M_1^2}\left(\frac{p_2}{p_1} - 1\right)$$

an exact relation for *Cp* behind an oblique shock wave

$$C_p = \frac{4}{\gamma + 1}\left(\sin^2\beta - \frac{1}{M_1^2}\right)$$

$M_1 \to \infty$:

$$\boxed{C_p = \left(\frac{4}{\gamma + 1}\right)\sin^2\beta}$$

$\gamma \to 1.0$ ⇒ $C_p \to 2\sin^2\beta$

$M_\infty \to \infty$: ⇒ $\frac{\rho_2}{\rho_\infty} \to \frac{\gamma+1}{\gamma-1}$ ⇒ $\gamma \to 1$ *and* $M_\infty \to \infty$: $\boxed{\frac{\rho_2}{\rho_\infty} \to \infty}$

$\frac{\beta}{\theta} \to \frac{\gamma+1}{2}$ ⇒ *as* $\gamma \to 1$ *and* $M_\infty \to \infty$ *and* θ *and* β *are small:* $\boxed{\beta = \theta}$

$$\boxed{C_p = 2\sin^2\theta}$$

MACH NUMBER INDEPENDENCE

$$C_p = \frac{4}{\gamma + 1}\left[\sin^2\beta - \frac{1}{M_\infty^2}\right]$$