WHAT IS AERODYNAMICS?

- "A branch of dynamics that deals with the motion of air and other gaseous fluids, and with the forces acting on bodies in motion relative to such fluids."

… Webster's Dictionary

LET'S DISCUSS…

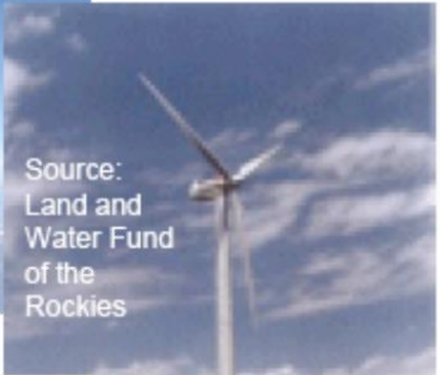
- What does "**Aerodynamics**" mean to you?

- In what other areas or products besides airplanes does aerodynamics matter?

AERODYNAMICS MATTERS...


Source: Boeing


Source: Lexus


Source: Land and Water Fund of the Rockies


Source: lancearmstrong.com


Source: Gold Racing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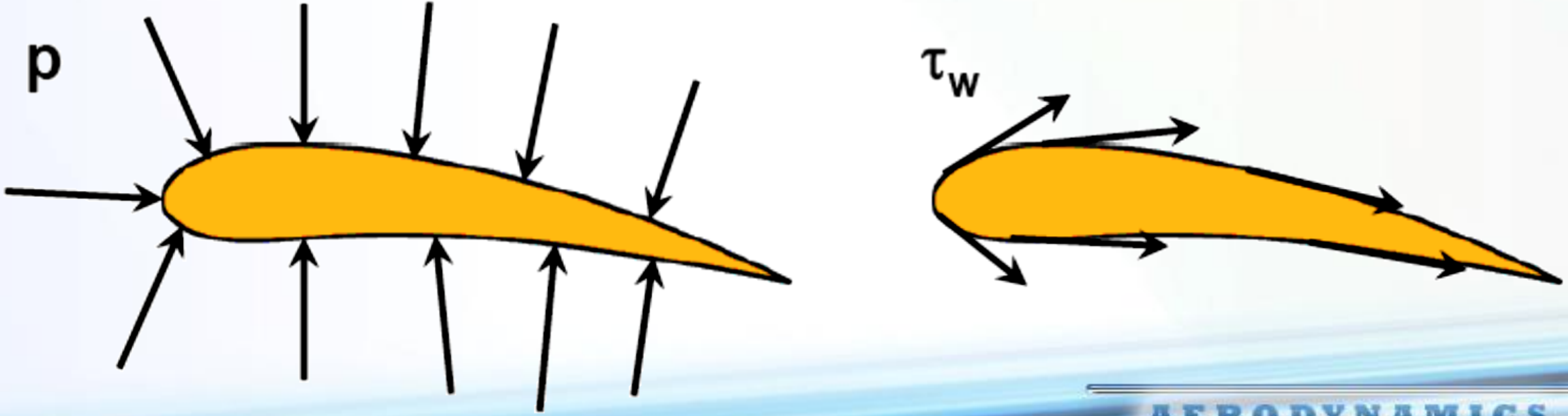
SOURCE OF AERODYNAMIC FORCES…

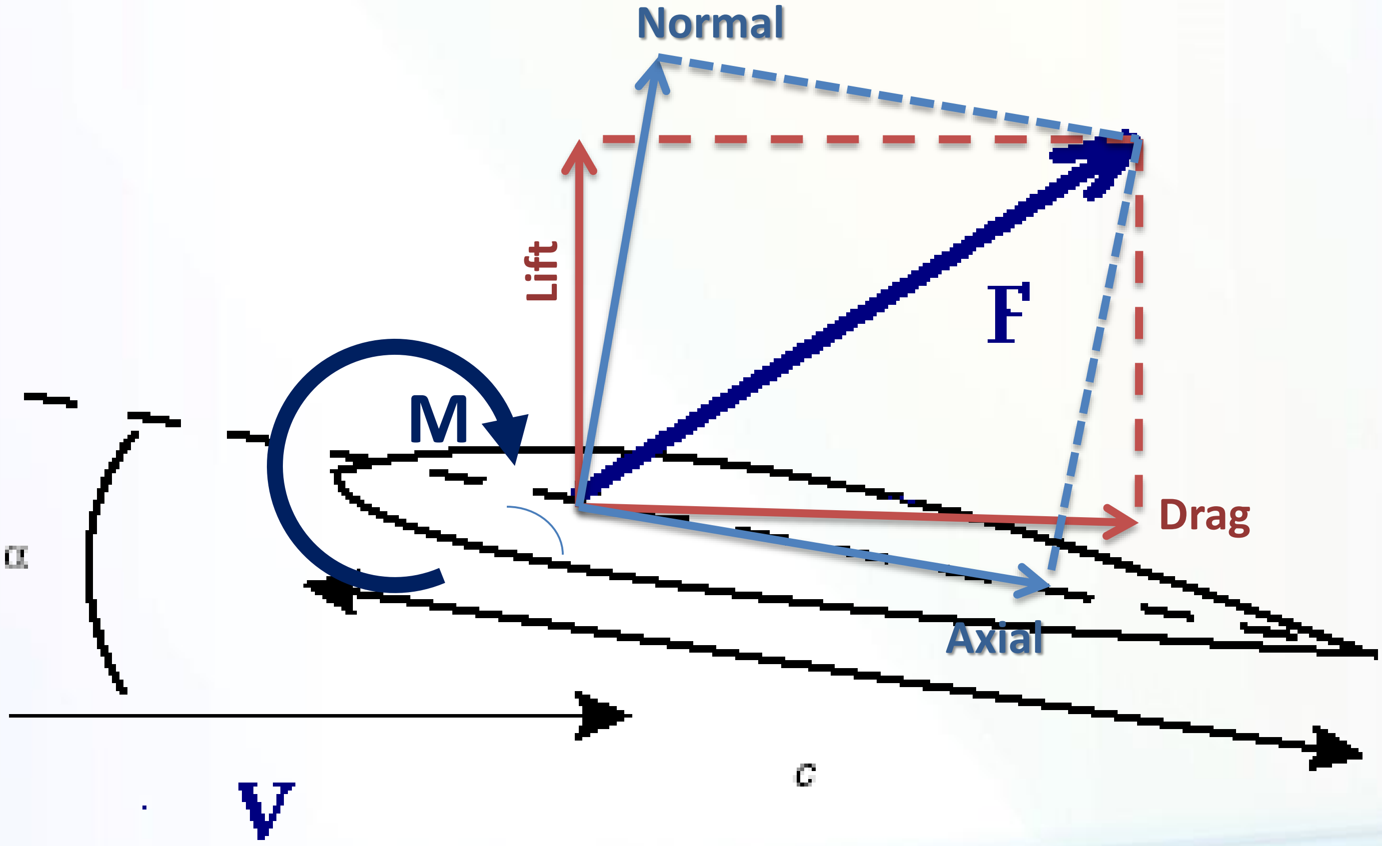
- “Theoretical and experimental aerodynamicists labor to calculate and measure flow fields of many types”

- … Because “ the aerodynamic forces exerted by the airflow on the surface of an airplane, missile, etc., stems from only two simple natural sources:
  - **Pressure distribution** over the surface (normal to surface)
  - **Shear stress (friction)** over the surface (tangential to surface)



SOURCE OF AERODYNAMIC FORCES…

"LIFT" AND "DRAG" COMPONENTS OF AERODYNAMIC FORCE



AERODYNAMICS…

- Deals with calculations of Forces and Moments due to body-air relative movement for all range of speeds. From very low speed to several times more than speed of sound

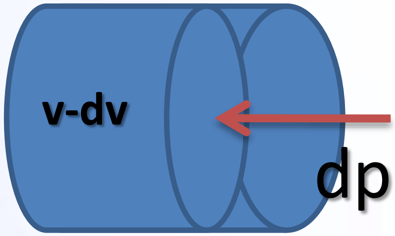
AERODYNAMICS CLASSIFICATION-1…

- Low speed (Incompressible)
- Subsonic
- Transonic
- Supersonic
- Hypersonic

Classification is based on:
Flow Compressibility

DEFINITION OF COMPRESSIBLE FLOW…

- Compressible flow is routinely defined as variable density flow.



$$\tau = -\frac{1}{v}\frac{\partial v}{\partial p} \begin{cases} \tau_T = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T \\ \tau_s = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_s \end{cases}$$

$$\tau_T = 5\times10^{-10} \quad m^2/N \, , \qquad \text{for water at 1atm}$$

$$\tau_T = 1\times10^{-5} \quad m^2/N \, , \qquad \text{for air at 1atm}$$

DEFINITION OF COMPRESSIBLE FLOW…

$$\rho = \frac{1}{v} \quad \Rightarrow \quad \tau = \frac{1}{\rho}\frac{d\rho}{dp} \quad \Rightarrow \quad d\rho = \rho\tau dp$$

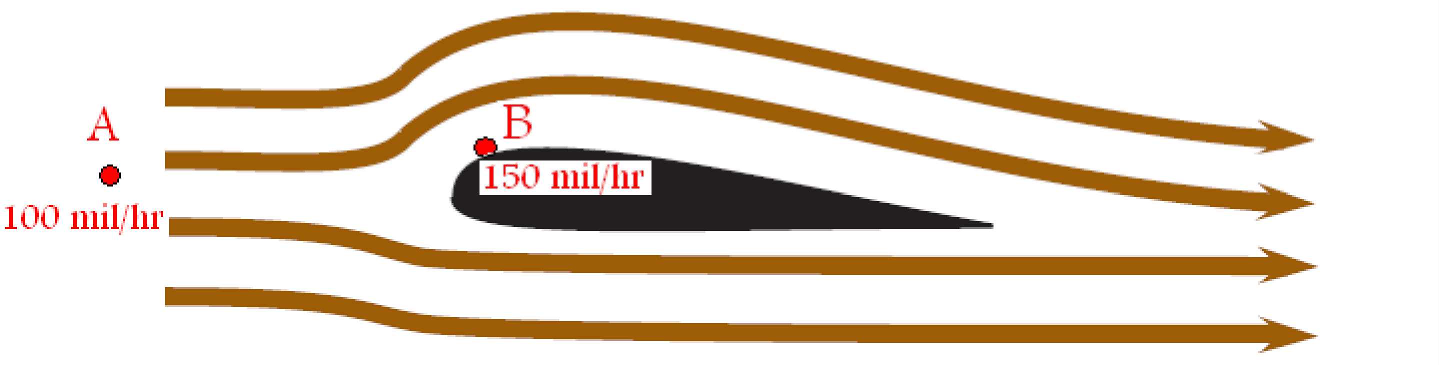
- For the flow of gases with their attendant large values of Compressibility, moderate to strong pressure gradients lead to substantial changes in the density.

- At the same time, such pressure gradients create large velocity changes in the gas.

- Such flows are defined as compressible flows, where density is a variable.

EXAMPLE

- Consider the low-speed flow of air over an airplane wing at standard conditions:



$$p_1 - p_2 = \frac{1}{2}\rho\left(V_2^{\,2} - V_1^{\,2}\right) = \frac{1}{2}(0.002377)\left(220^2 - 147^2\right) = 31.8 \quad lb/ft^2$$

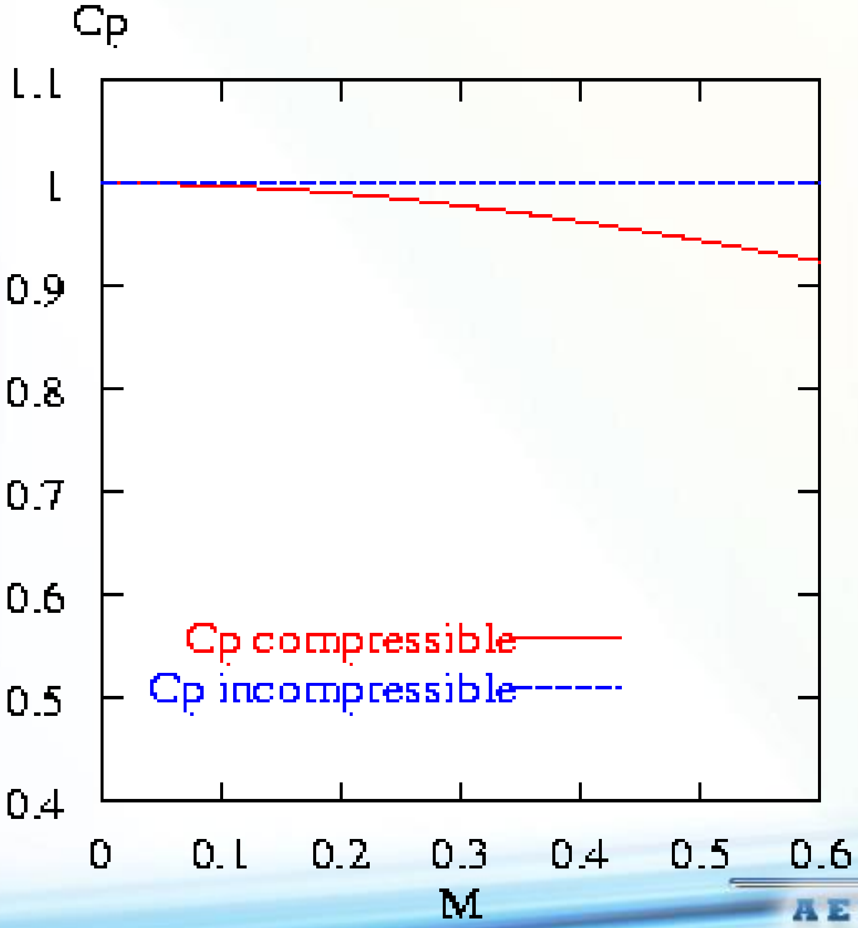
$$\frac{p_1 - p_2}{p_1} = \frac{31.8}{2116} = 0.015$$

- the percentage change in pressure is **1.5%**

EXTENDED CONSTANT-DENSITY FLOWS

- Mach number is defined as the ratio of the local flow velocity to the speed of sound:

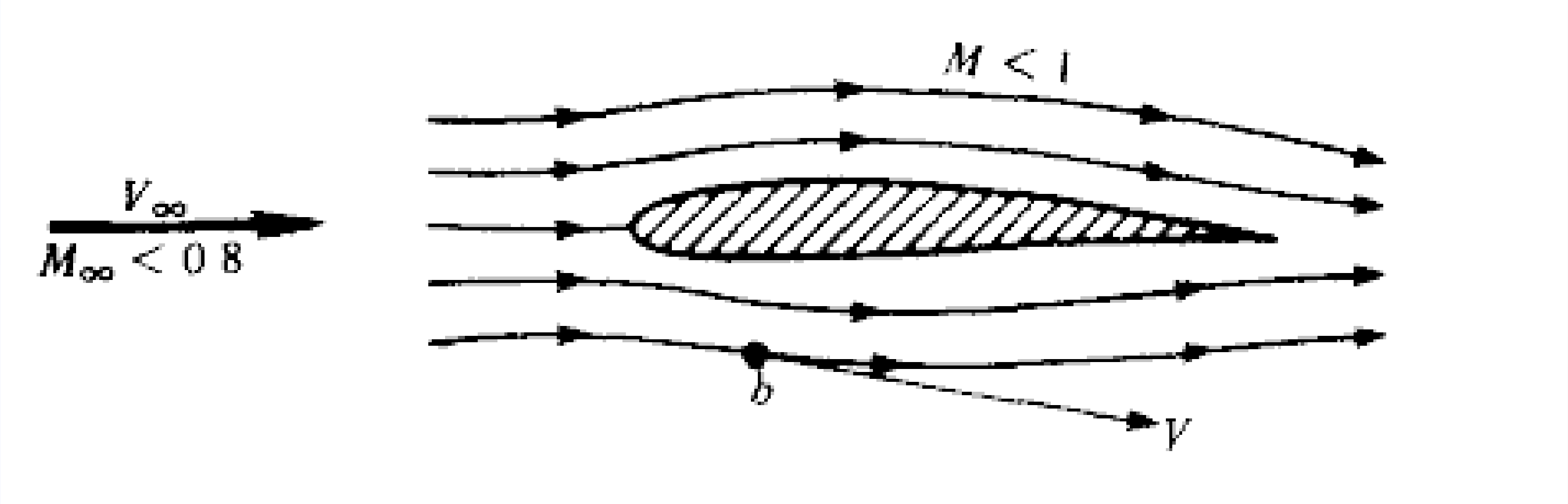
$$M = \frac{V}{a}$$



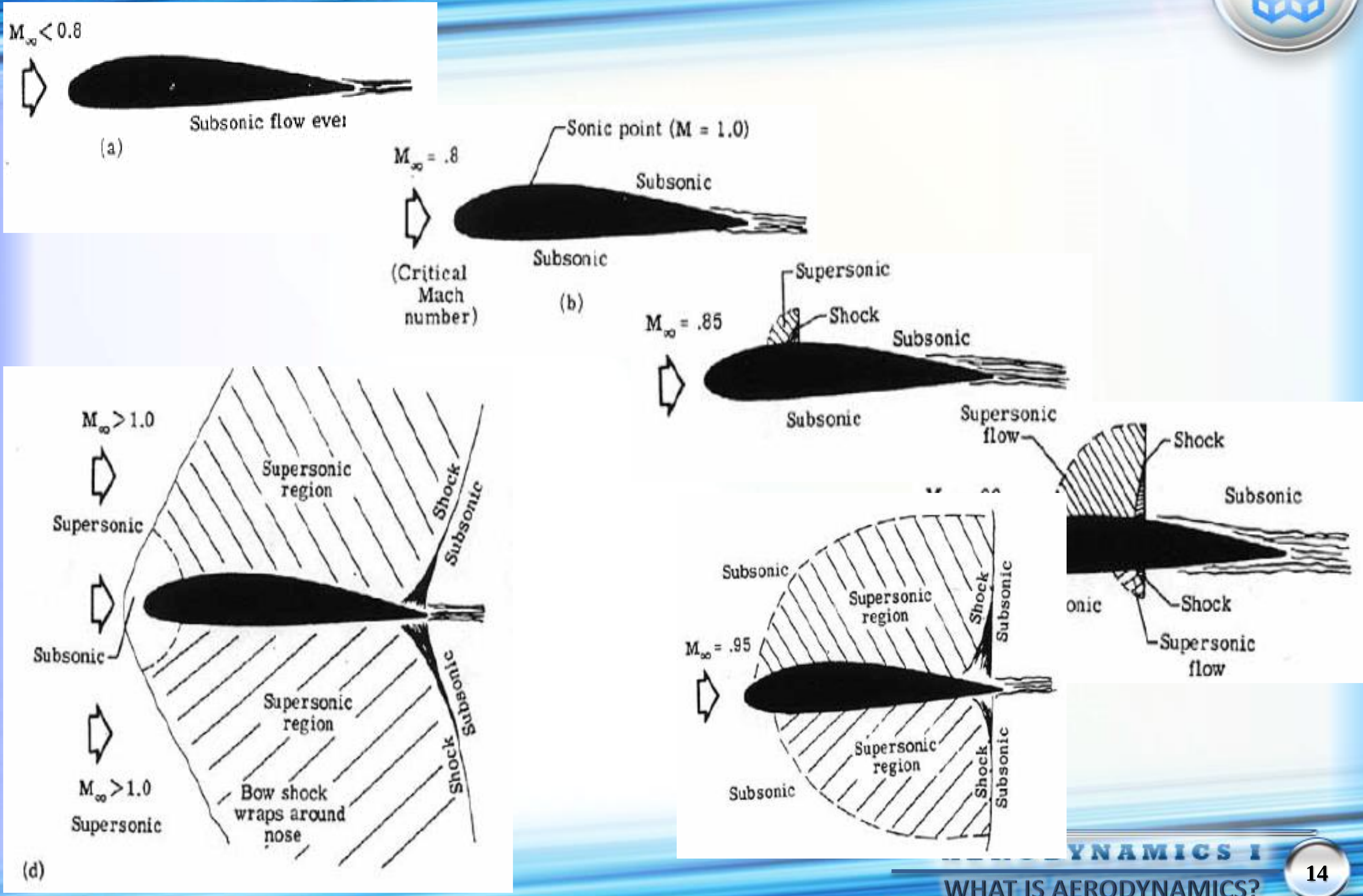
INCOMPRESSIBLE - SUBSONIC FLOWS

$M_\infty < 0.3$ Incompressible Flow

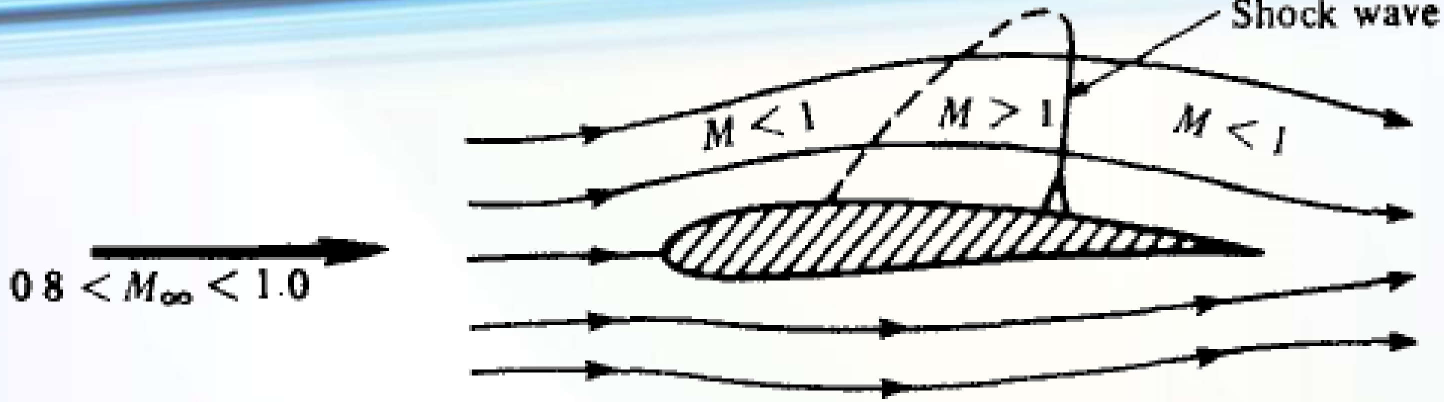
$0.3 < M_\infty < 0.8$ Subsonic Flow

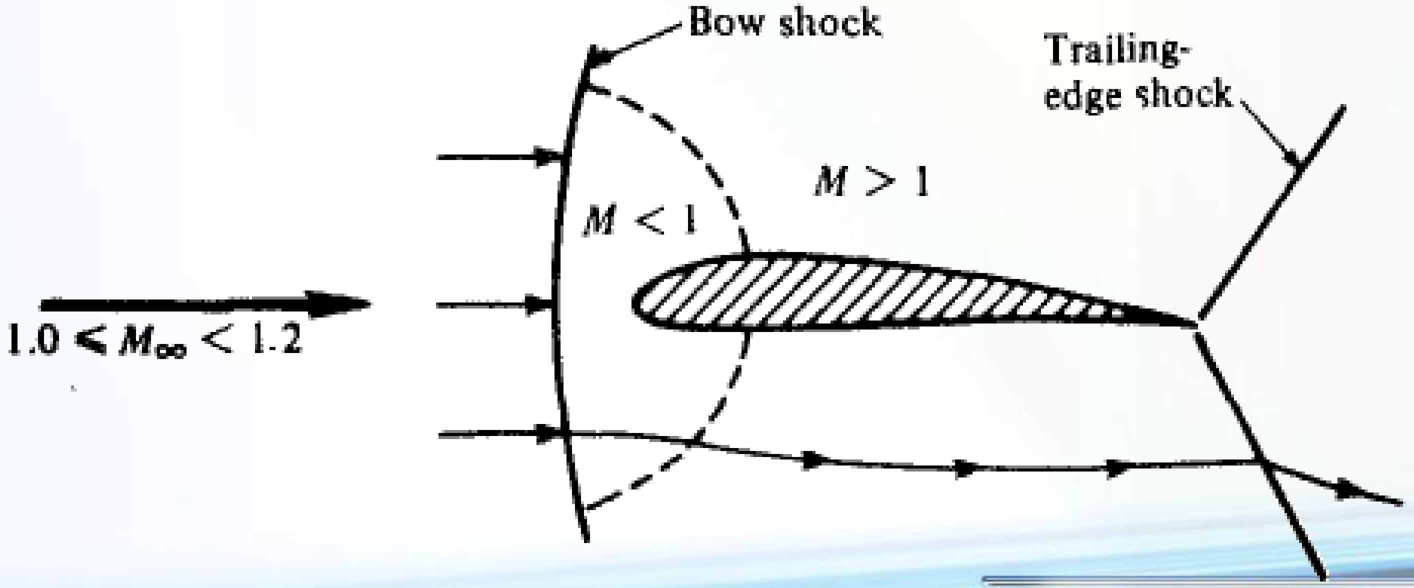


TRANSONIC FLOW



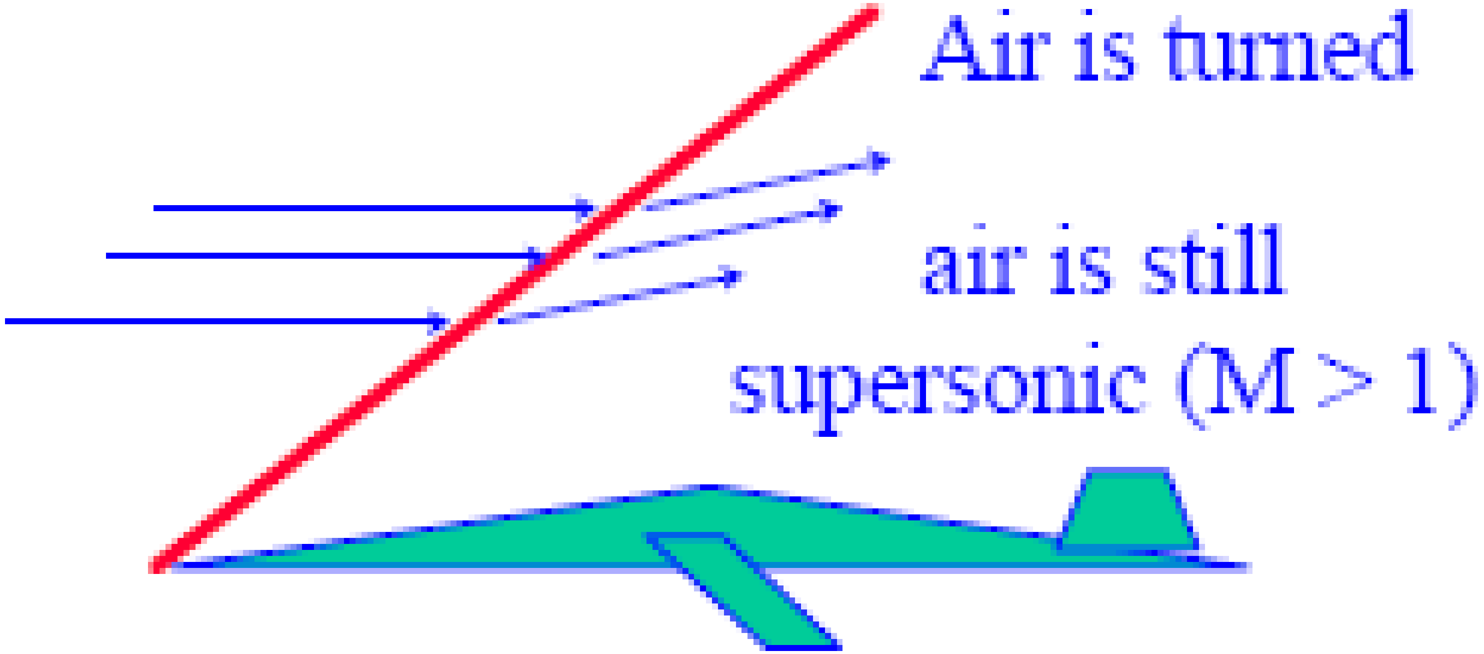
TRANSONIC FLOW





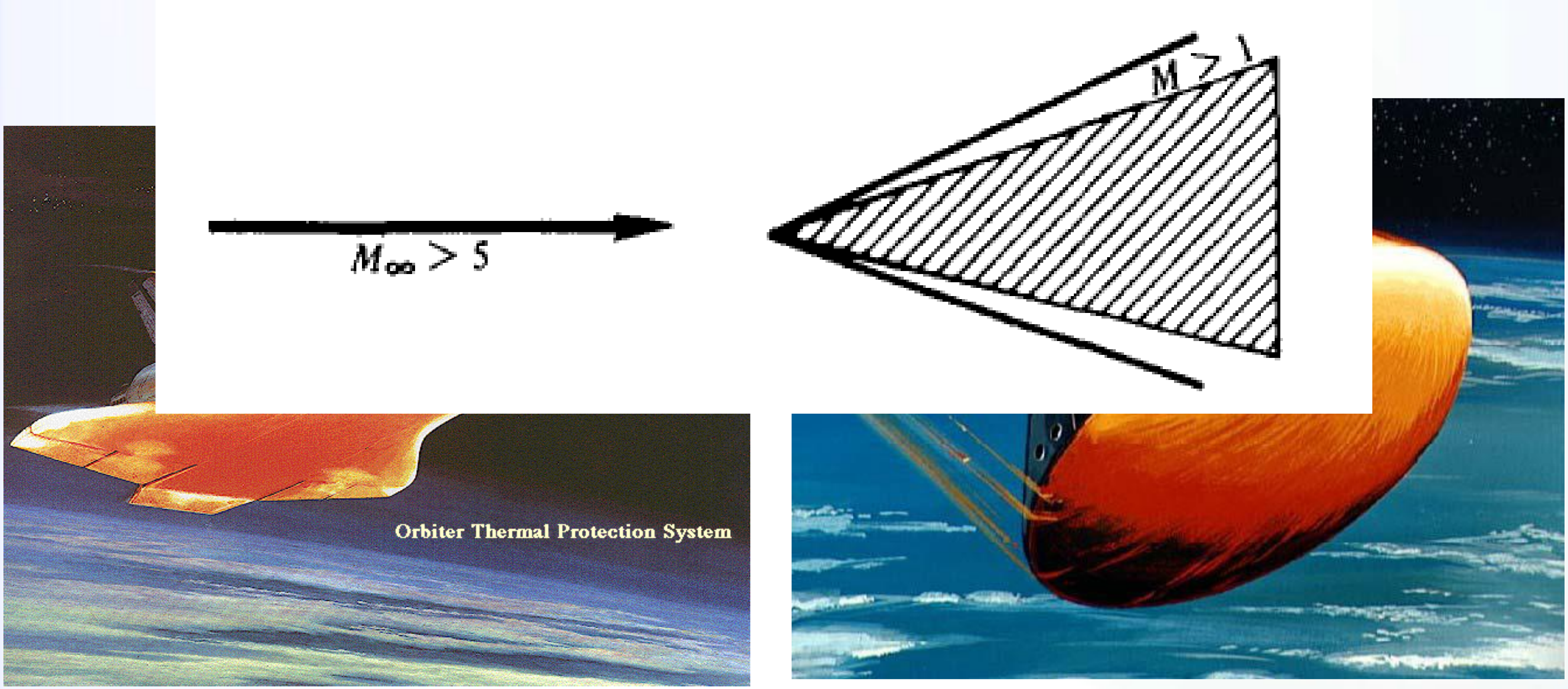
SUPERSONIC FLOW

$M_{\infty} > 1.2$

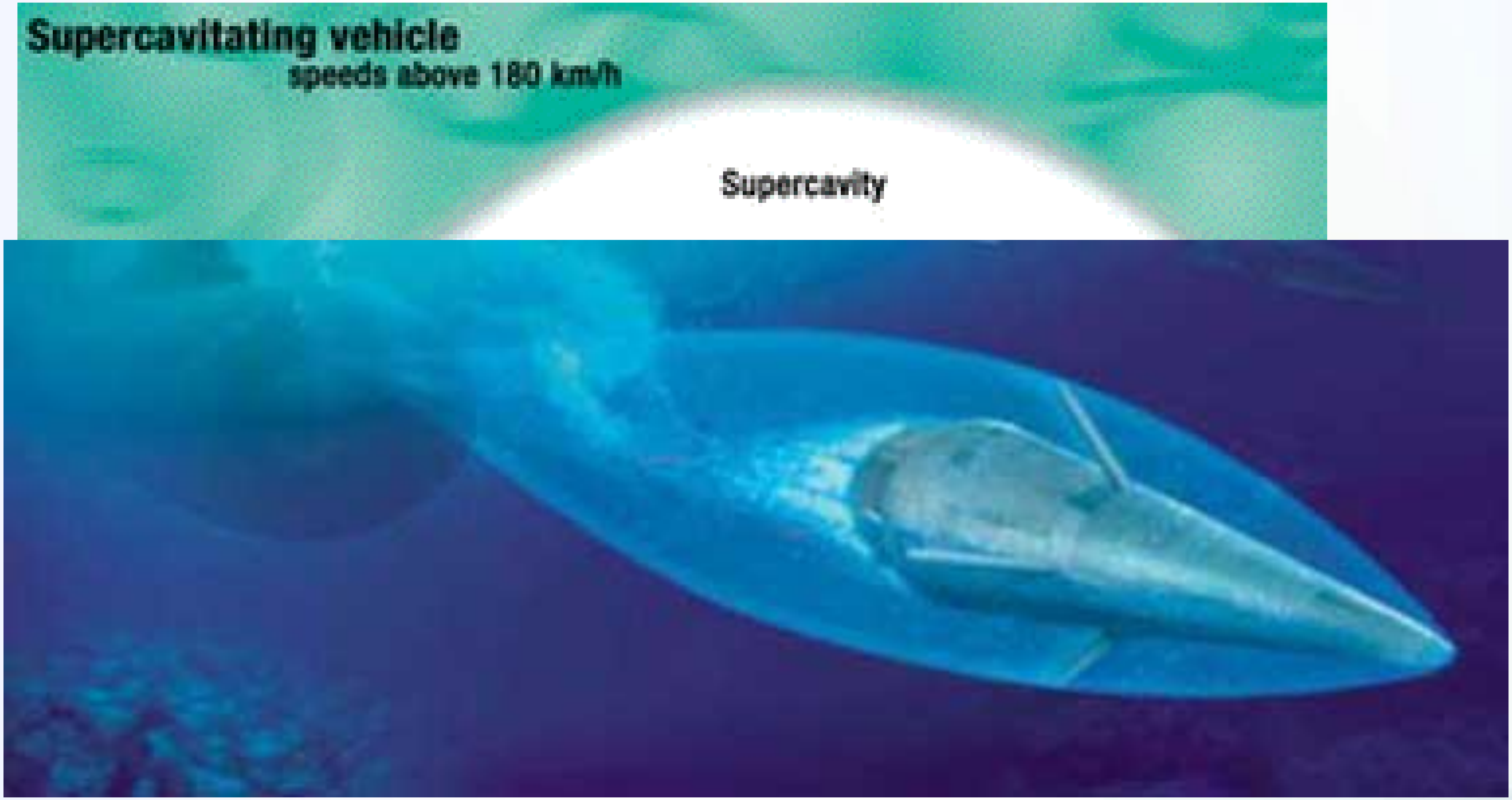


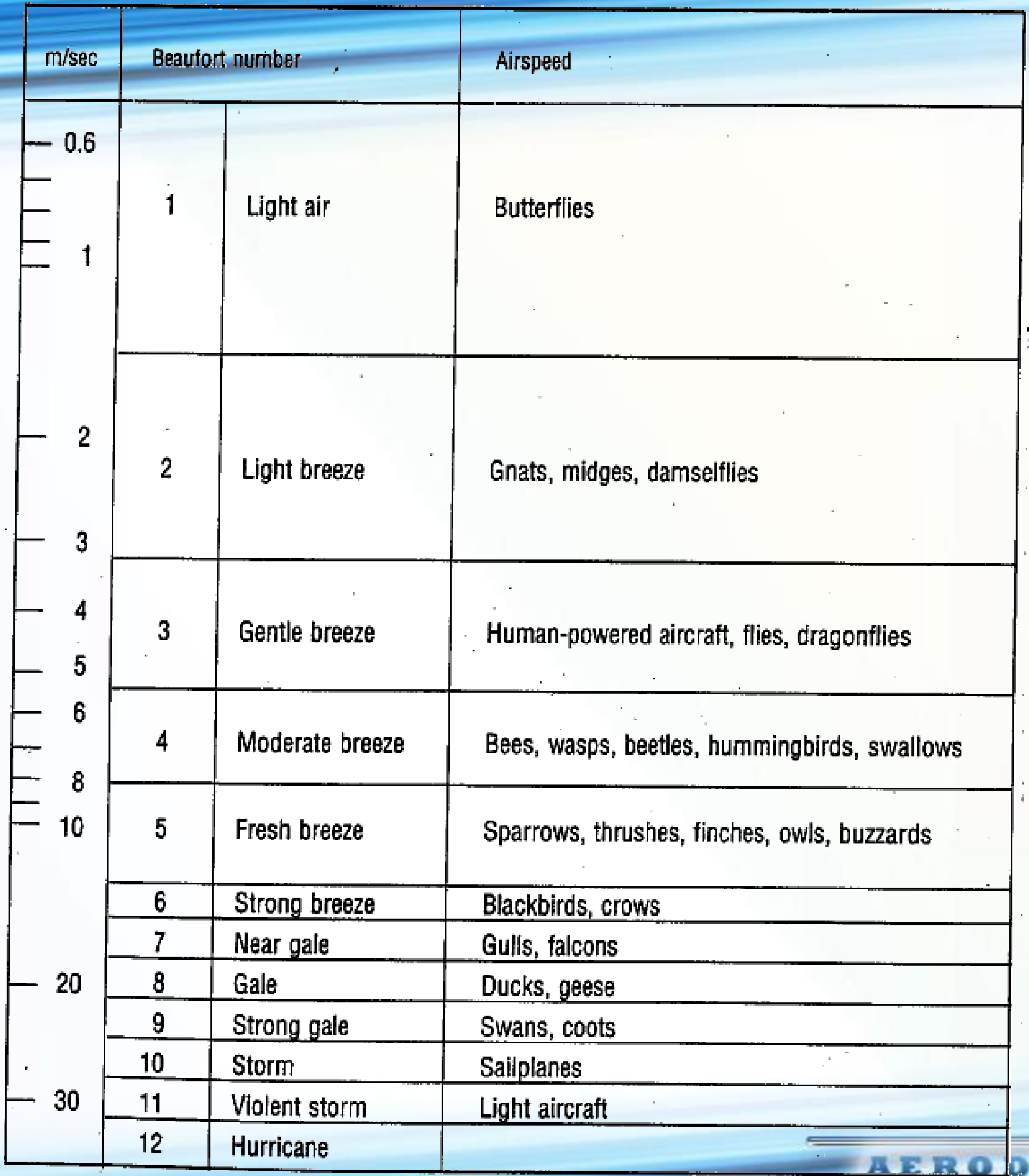
HYPERSONIC FLOW

$M_\infty > 5$



SUPERCAVITATION





m/sec	Beaufort number		Airspeed
0.6 1	1	Light air	Butterflies
2 3	2	Light breeze	Gnats, midges, damselflies
4 5	3	Gentle breeze	Human-powered aircraft, flies, dragonflies
6 8	4	Moderate breeze	Bees, wasps, beetles, hummingbirds, swallows
10	5	Fresh breeze	Sparrows, thrushes, finches, owls, buzzards
	6	Strong breeze	Blackbirds, crows
	7	Near gale	Gulls, falcons
20	8	Gale	Ducks, geese
	9	Strong gale	Swans, coots
	10	Storm	Sailplanes
30	11	Violent storm	Light aircraft
	12	Hurricane	

FLOW CLASSIFICATION - 2

- Inviscid flow:
  - Rotational
  - Irrotational

- Viscous flow:
  - Laminar
  - Turbulent

FLUID MOTION

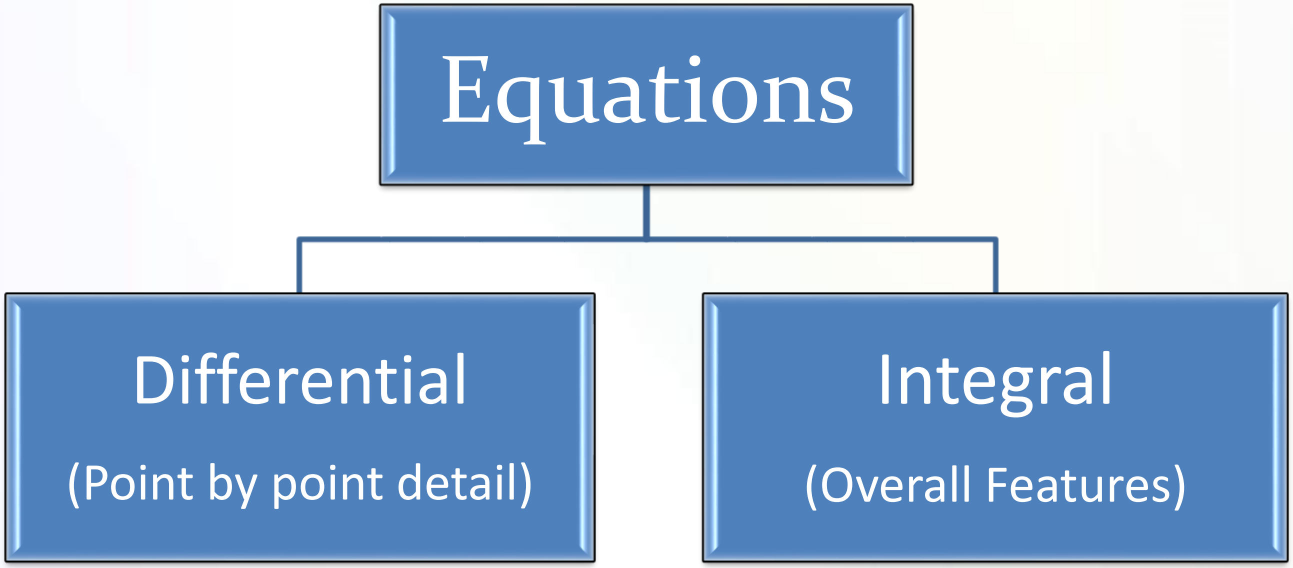
- The motion of the fluid is controlled by:
  - Governing Equations
  - Boundary Conditions

GOVERNING EQUATIONS - CONSERVATION LAWS

- The governing equations are given by conservation laws:

Conservation of mass	Continuity
Conservation of momentum	Newton's 2nd Law, F=ma
Conservation of Energy	1st law of thermodynamics

FORMS OF GOVERNING EQUATIONS



Equations

Differential

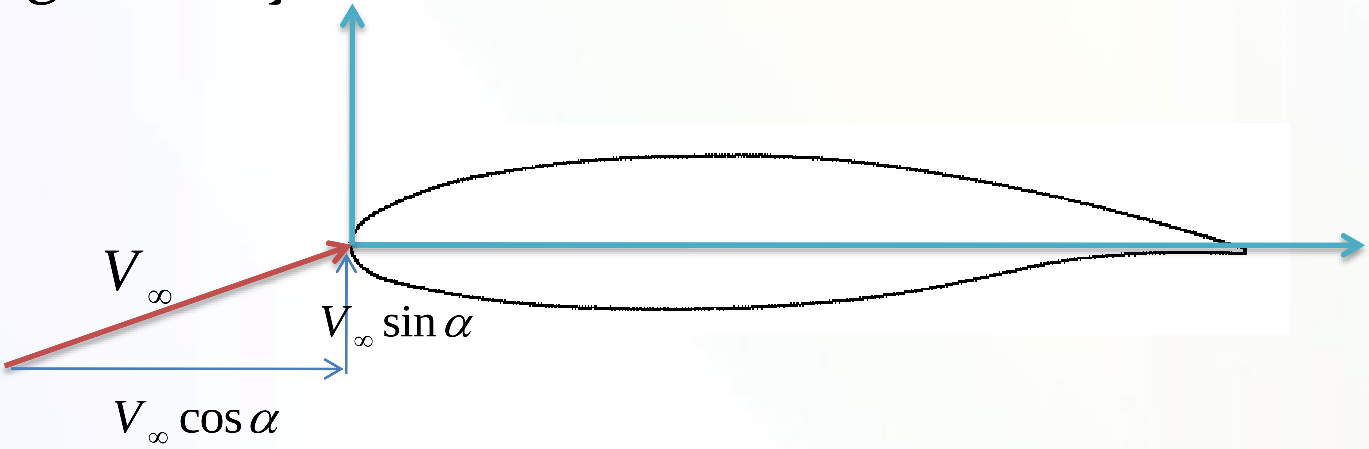
(Point by point detail)

Integral

(Overall Features)

COORDINATE SYSTEMS

- **Cartesian coordinates**: Are normally used to describe vehicle geometry.



- **Cylindrical coordinates**
- **Spherical coordinates**
- **General non-orthogonal curvilinear coordinates**

VARIABLES AND UNKNOWN QUANTITIES

- In general Cartesian coordinates, the independent variables are: x, y, z and t.

- We want to know the velocity components (u, v, w) and the fluid properties (p, ρ, T).

- These six unknowns require six equations:
  - Continuity Equation: 1 Equation
  - Momentum Equations: 3 Equations
  - Energy Equation: 1 Equation
  - Equation of State: 1 Equation

MOTIVATIONS

- We want to find the flow field velocity (u, v, w), pressure (p) and temperature (T) distribution.

- We need to develop a mathematical model of the fluid motion suitable for use in numerical calculations.

- The mathematical model is based on the conservations laws and the fluid properties.

DESCRIPTION OF FLUID MOTION

- ***Lagrangian:***
  - Each fluid particle is traced as it moves around the body.
  - This method corresponds to the conventional concept of Newton's 2nd law

- ***Eulerian:***
  - We look at the entire space around the body as a field, and determine flow properties at various points in the field while the fluid stream past.
  - We consider the distribution of velocity and pressure throughout the field, and ignore the motion of individual fluid particles.

THE CONTINUITY EQUATION

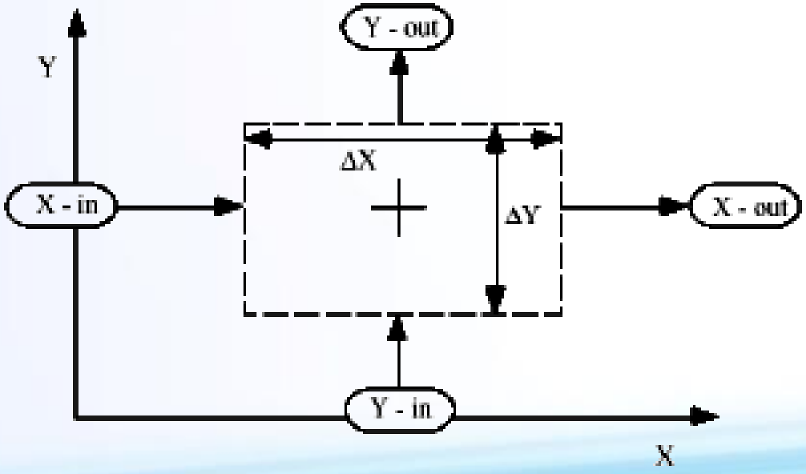
- The statement of Conservation of Mass is in the words simply:

Net outflow of mass through the surface surrounding the volume

=

Time rate of decrease of mass within the volume

[Mass can be neither created or destroyed]



$$[X-out]-[X-in]+[Y-out]-[Y-in]$$
$$= change\ of\ mass\ (decrease)$$
$$= \frac{\partial\rho}{\partial t}\Delta X\Delta Y$$

THE CONTINUITY EQUATION

- The differential form:

2-D $$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}=0$$

3-D $$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}=0$$

Vector form $$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{V})=0$$

- The integral form:

$$\frac{\partial}{\partial t}\iiint\limits_{CV}\rho d\forall+\iint\limits_{CS}\rho\vec{V}.d\vec{s}=0$$

CONSERATION OF MOMENTUM

- *Newton's 2nd law:* The time rate of change of momentum of a body, equals the net force exerted on it.

- For a fixed mass, this is the famous equation:

$$\vec{F} = m\vec{a} = m\frac{D\vec{V}}{Dt}$$

[Force = Time rate of change of momentum]

CONSERATION OF MOMENTUM

- ***Substantial Derivative:***
  - We need to apply Newton's law to a moving fluid element from our fixed coordinate system.
  - Consider any fluid property, $Q(\vec{r},t)$

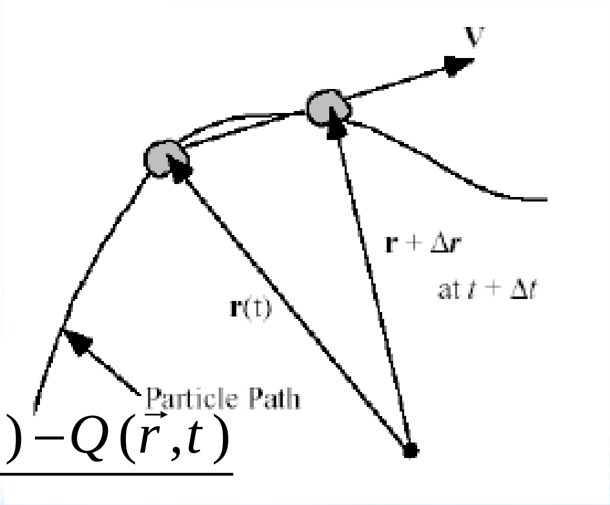
- The change in position of the particle between $\vec{r}$ at t, and $\vec{r}+\Delta\vec{r}$ at $t+\Delta t$ is:

$$\Delta Q = Q(\vec{r}+\Delta s, t+\Delta t) - Q(\vec{r},t)$$

$$= Q(\vec{r}+\vec{V}\Delta t, t+\Delta t) - Q(\vec{r},t)$$

- The rate of change of Q is:

$$\frac{DQ}{Dt} = \lim_{\Delta t\to 0}\frac{\Delta Q}{\Delta t} = \lim_{\Delta t\to 0}\frac{Q(\vec{r}+\vec{V}\Delta t, t+\Delta t) - Q(\vec{r},t)}{\Delta t}$$



CONSERATION OF MOMENTUM

$$\lim_{\Delta t\to 0}\frac{\Delta Q}{\Delta t} = \underbrace{\frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial s}V}_{\text{Substantial derivative}}$$

Local time derivative, *local derivative*

Variation with change of position, *convective derivative*

Substantial derivative

- The second term has the unknown velocity V multiplying a term containing the unknown *Q*. This is important!

- The convective derivative introduces a fundamental nonlinearity into the system.

CONSERATION OF MOMENTUM

- In coordinates, $\vec{V} = \{u, v, w\}$, and the substantial derivative becomes:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

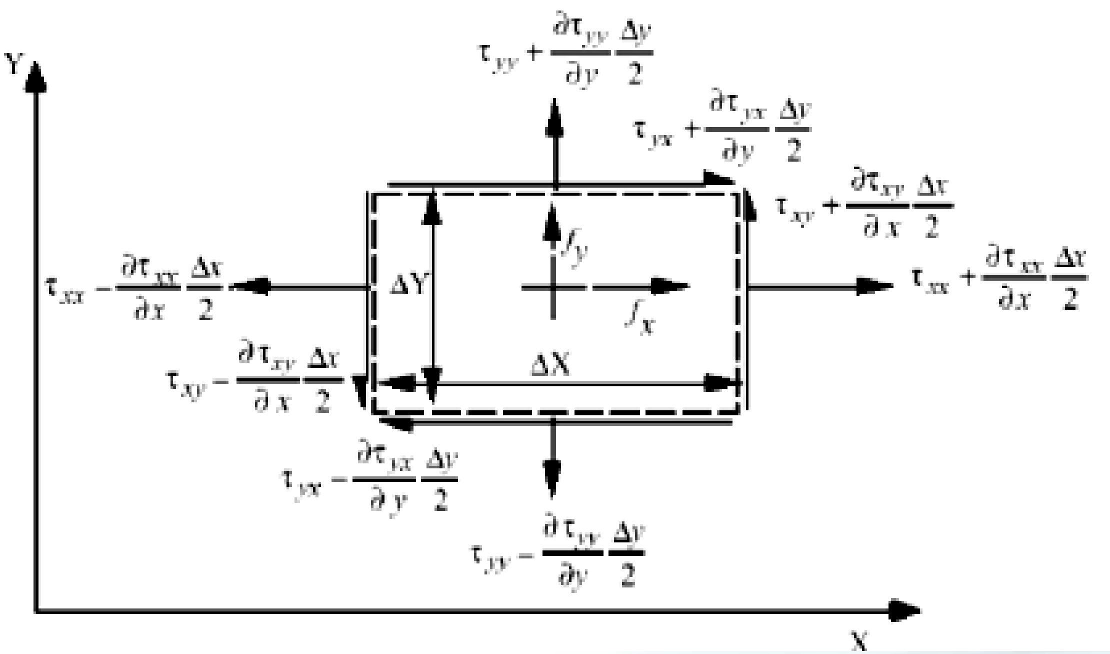
$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

CONSERATION OF MOMENTUM

Sources of the force exerted on the fluid element:

- Body forces
  - Gravitational forces
  - Electromagnetic force
  - …
- Surface forces
  - Pressure
  - Shear stress



CONSERATION OF MOMENTUM

- The net force in the x-direction is found to be:

$$\rho \Delta x\, \Delta y f_x + \frac{\partial}{\partial x}(\tau_{xx})\Delta x\, \Delta y + \frac{\partial}{\partial y}(\tau_{yx})\Delta x\, \Delta y$$

- Using the *Substantial Derivative* and the definition of the mass, $m = \rho \Delta x\, \Delta y\, \Delta z$, and considering the *x* component, $F_x = ma_x$ in three dimensional case, we have:

$$\rho \Delta x\, \Delta y\, \Delta z\, \frac{Du}{Dt} = \rho \Delta x\, \Delta y\, \Delta z f_x + \frac{\partial}{\partial x}(\tau_{xx})\Delta x\, \Delta y\, \Delta z + \frac{\partial}{\partial y}(\tau_{yx})\Delta x\, \Delta y\, \Delta z$$

$$+ \frac{\partial}{\partial z}(\tau_{zx})\Delta x\, \Delta y\, \Delta z$$

CONSERATION OF MOMENTUM

- General conservation of momentum relations:
  - Differential form:

$$\rho\frac{Du}{Dt}=\rho f_x+\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{yx}}{\partial z}$$

$$\rho\frac{Dv}{Dt}=\rho f_y+\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}$$

$$\rho\frac{Dw}{Dt}=\rho f_z+\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}$$

  - Integral Form:

$$\frac{\partial}{\partial t}\iiint_{CV}\rho\vec{V}d\forall+\iint_{CS}(\rho\vec{V}.d\vec{s})\vec{V}=-\iint_{CS}pd\vec{s}+\iiint_{CV}\rho\vec{f}d\forall+\vec{F}_{Viscous}$$

CONSERATION OF MOMENTUM

- Relations between stress and μ based on the assumptions

$$\tau_{xx} = -p - \frac{2}{3}\mu\nabla\cdot\mathbf{V} + 2\mu\frac{\partial u}{\partial x}$$

$$\tau_{yy} = -p - \frac{2}{3}\mu\nabla\cdot\mathbf{V} + 2\mu\frac{\partial v}{\partial y}$$

$$\tau_{zz} = -p - \frac{2}{3}\mu\nabla\cdot\mathbf{V} + 2\mu\frac{\partial w}{\partial x}$$

and

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right)$$

THE CLASSIC NAVIER-STOKES EQUATIONS

- Written in the standard aerodynamics form neglecting the body force.

$$\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}-\frac{2}{3}\mu\nabla\cdot\mathbf{V}\right)+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]$$

$$\rho\frac{Dv}{Dt}=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left(2\mu\frac{\partial v}{\partial y}-\frac{2}{3}\mu\nabla\cdot\mathbf{V}\right)+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]$$

$$\rho\frac{Dw}{Dt}=-\frac{\partial p}{\partial w}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left(2\mu\frac{\partial w}{\partial z}-\frac{2}{3}\mu\nabla\cdot\mathbf{V}\right)$$

- These equations are:
  - Non-linear *(recall that superposition of solutions is not allowed).*
  - Highly coupled.
  - Long!

EULER EQUATIONS

- When the viscous terms are small and thus ignored, the flow is termed **inviscid**.

- The resulting equations are known as the **Euler Equations.**

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial p}{\rho\partial x}=0$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{\partial p}{\rho\partial y}=0$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{\partial p}{\rho\partial z}=0$$

EULER EQUATIONS

- Euler Equations in cylindrical coordinate system:

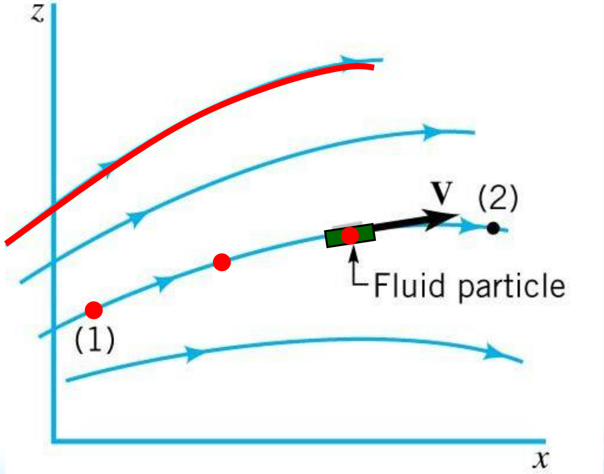
$$\rho a_r = \rho\left(\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}\right) = \rho g_r - \frac{\partial p}{\partial r}$$

$$\rho a_\theta = \rho\left(\frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}\right) = \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$\rho a_z = \rho\left(\frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z}$$

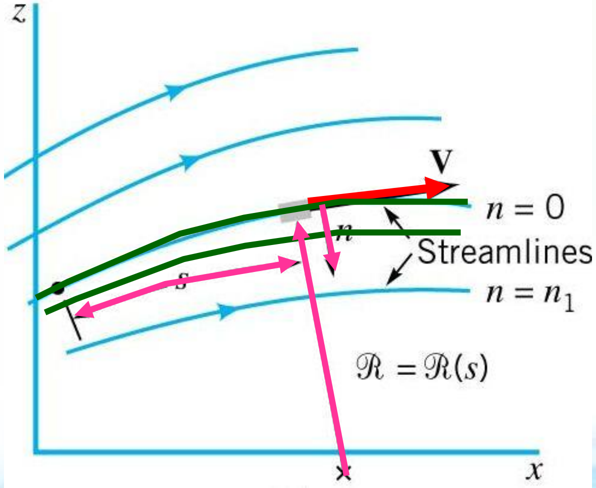
NEWTON’S SECOND LAW: FLUID DYNAMICS

- We describe the motion of each particle with a velocity vector: **V**
- Particles follow specific paths base on the velocity of the particle.
- Location of particle is based on its initial position at an initial time, and its velocity along the path.
- If the flow is a steady flow, each successive particle will follow the same path.



NEWTON'S SECOND LAW: STEADY FLOW

- For Steady Flow, each particle slides along its path, and the velocity vector is every tangent to the path.
- The lines that the velocity vectors are tangent to are called ***streamlines***.
- We can introduce streamline coordinate, s(t) along the streamline and n, normal to the streamline.
- Then $\Re(s)$ is the radius of curvature of the streamline.



NEWTON'S SECOND LAW: STEADY FLOW

$$V = V(s) \longrightarrow V = ds/dt$$

$$\mathbf{a} = d\mathbf{V}/dt$$

- For 2-D Flows, there are two acceleration components:
  - s-direction by chain rule:

$$a_s = dV/dt = (\partial V/\partial s)(ds/dt) = (\partial V/\partial s)V$$

  - Normal direction (n) is the centrifugal acceleration:

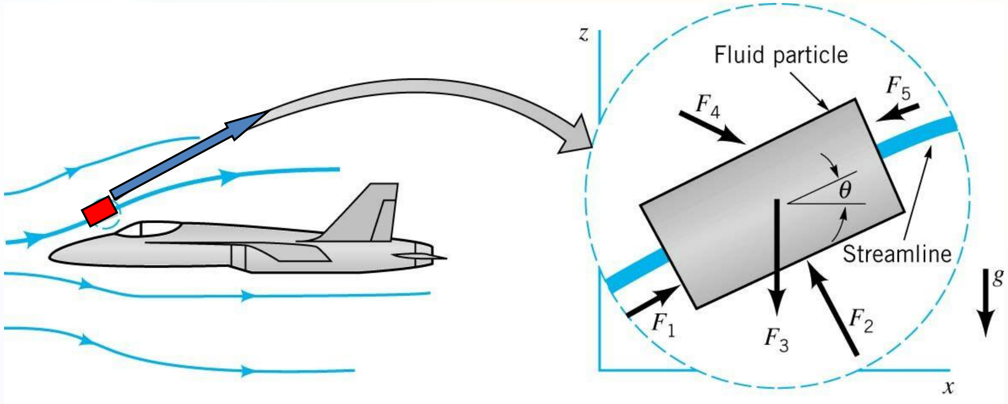
$$a_n = \frac{V^2}{\mathcal{R}}$$

- In general there is acceleration along the streamline:

$$\partial V/\partial s \neq 0$$

- There is also acceleration normal to the streamline: $\mathcal{R} \neq \infty$
- However, to produce an acceleration there must be a force!

NEWTON'S SECOND LAW: STEADY FLOW F.B.D.



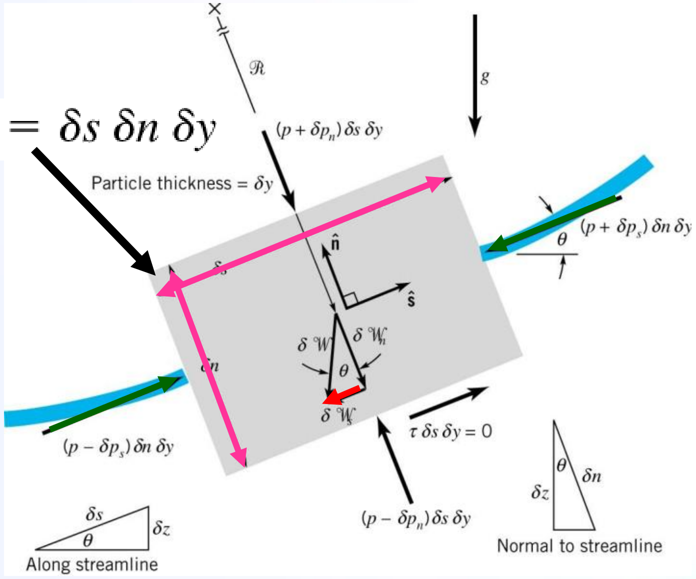
- Remove, the fluid particle from its surroundings.
- Draw the F.B.D. of the flow.
- Assume pressure forces and gravity forces are important.
- Neglect surface tension and viscous forces.

NEWTON’S SECOND LAW: ALONG A STREAMLINE

- Use Streamline coordinates, our element is ds x dn x dy, and the unit vectors are **n** and **s**, and apply Newton’s Second Law in the Streamline Direction.
- Streamline, **F** = m**a**:

$$\sum \delta F_s = \delta m\, a_s = \delta m\, V \frac{\partial V}{\partial s} = \rho\, \delta \forall\, V \frac{\partial V}{\partial s}$$

$$\delta \forall = \delta s\, \delta n\, \delta y$$



- Gravity Forces:

$$\delta \mathcal{W}_s = -\delta \mathcal{W} \sin\theta = -\gamma\, \delta \forall \sin\theta$$

- Pressure Forces (Taylor Series):

$$\delta F_{ps} = (p - \delta p_s)\, \delta n\, \delta y - (p + \delta p_s)\, \delta n\, \delta y = -2\, \delta p_s\, \delta n\, \delta y$$

$$= -\frac{\partial p}{\partial s} \delta s\, \delta n\, \delta y = -\frac{\partial p}{\partial s} \delta \forall$$

$\delta p_s \approx \frac{\partial p}{\partial s} \frac{\delta s}{2}$ arises since pressures vary in a fluid. P is the pressure at the center of the element.

Shear Forces: Neglected, Inviscid!

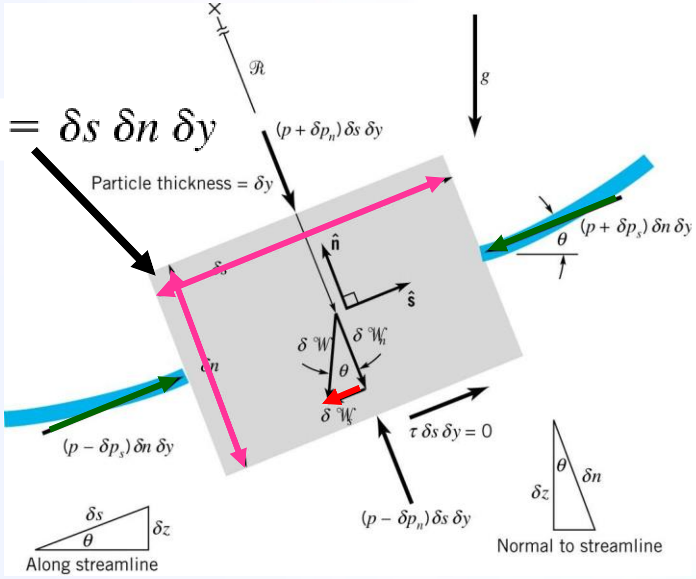
NEWTON'S SECOND LAW: ALONG A STREAMLINE

- Then $\rho\,\delta\forall\,V\frac{\partial V}{\partial s}$ = $\sum\delta F_s = \delta\mathcal{W}_s + \delta F_{ps} = \left(-\gamma\sin\theta - \frac{\partial p}{\partial s}\right)\delta\forall$

Divide out volume, recall $a_s = V\frac{\partial V}{\partial s}$

$$-\gamma\sin\theta - \frac{\partial p}{\partial s} = \rho V\frac{\partial V}{\partial s} = \rho a_s$$

$\delta\forall = \delta s\,\delta n\,\delta y$



- The change of fluid particle speed is accomplished by the appropriate combination of pressure gradient and particle weight along the streamline.
- In a static fluid the R.H.S is zero, and pressure and gravity balance. In a dynamic fluid, the pressure and gravity are unbalanced causing fluid flow.
- In a dynamic fluid, the pressure and gravity are unbalanced causing fluid flow.

NEWTON'S SECOND LAW: ALONG A STREAMLINE

$$-\gamma \sin\theta - \frac{\partial p}{\partial s} = \rho V \frac{\partial V}{\partial s} = \rho a_s$$

- Note, we can rewrite terms in the above equation:

$$\sin\theta = dz/ds$$

$$V\, dV/ds = \tfrac{1}{2} d(V^2)/ds$$

$$dp = (\partial p/\partial s)\, ds + (\partial p/\partial n)\, \cancel{dn} = (\partial p/\partial s)\, ds$$

0 = constant along a streamline

- Then

$$-\gamma \frac{dz}{ds} - \frac{dp}{ds} = \frac{1}{2} \rho \frac{d(V^2)}{ds}$$

- Simplifying,

$$dp + \frac{1}{2} \rho d(V^2) + \gamma\, dz = 0 \qquad \text{(along a streamline)}$$

NEWTON'S SECOND LAW: ALONG A STREAMLINE

- Integrate,

$$\int \frac{dp}{\rho} + \frac{1}{2}V^2 + gz = C \qquad \text{(along a streamline)}$$

- In general, we can not integrate the pressure term because density can vary with temperature and pressure; however, for now we assume constant density.

$$\boxed{p + \tfrac{1}{2}\rho V^2 + \gamma z = \text{constant along streamline}}$$

Celebrated Bernoulli's Equation

- Assumptions:
  I. Viscous effects are assumed negligible *(inviscid)*.
  II. The flow is assumed steady.
  III. The flow is assume incompressible.
  IV. The equation is applicable along a streamline

* We can apply along a streamline in planar and non-planar flows!

NEWTON’S SECOND LAW: ALONG A STREAMLINE

$$-\frac{1}{\rho}\frac{\partial p}{\partial s} - g\frac{\partial z}{\partial s} = V\frac{\partial V}{\partial s}$$

Integrating along S

$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Incompressible flow

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

BERNOULLI'S EQUATION

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

1. **Steady Flow**
2. **No Friction**
3. **Flow Along a Streamline**
4. **Incompressible Flow**

STATIC, STAGNATION AND DYNAMIC PRESSURES

$$p_0 = p + \frac{1}{2}\rho V^2$$

Stagnation Pressure

Static Pressure

Dynamic Pressure

$$q = \frac{1}{2}\rho V^2$$

FLUID KINEMATICS

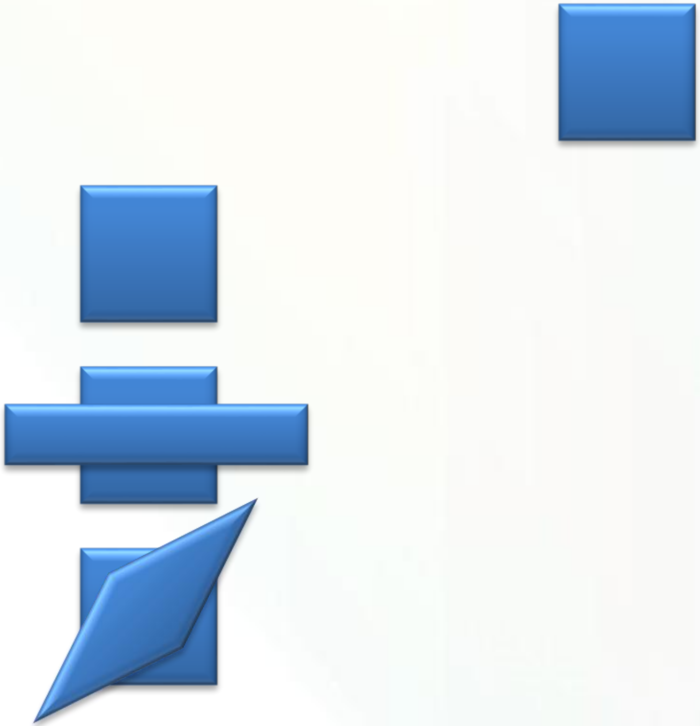
- Motion of a rigid body:
  - Translation: all points in the body, move in parallel straight lines.

  - Rotation: all points in the body move in circular paths about the axis of rotation.

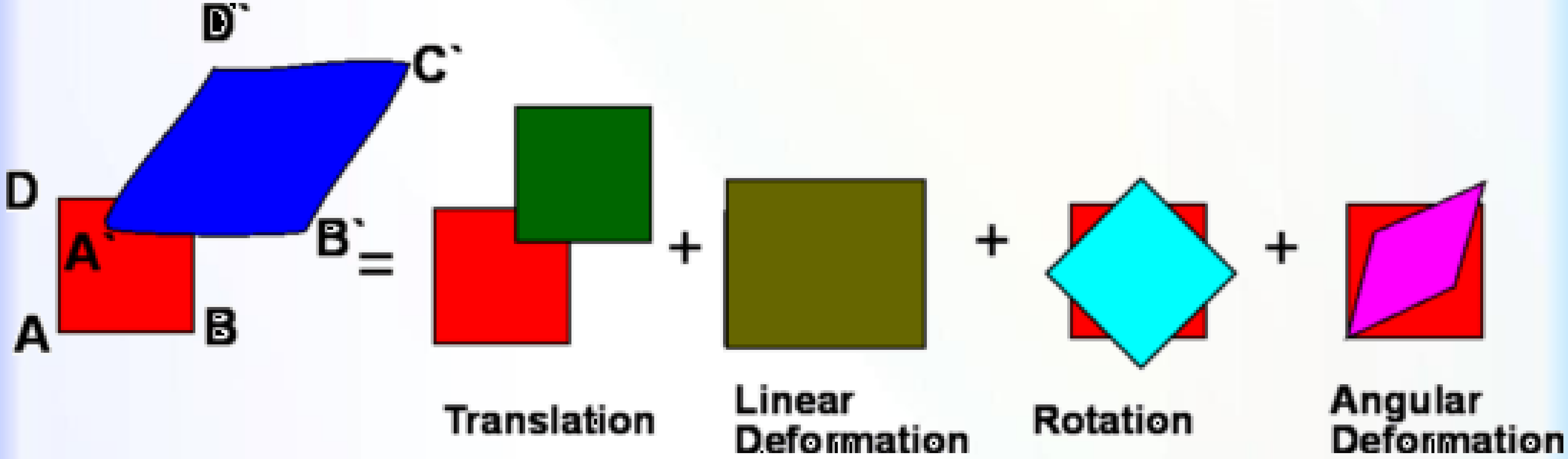
  - General motion

FLUID KINEMATICS

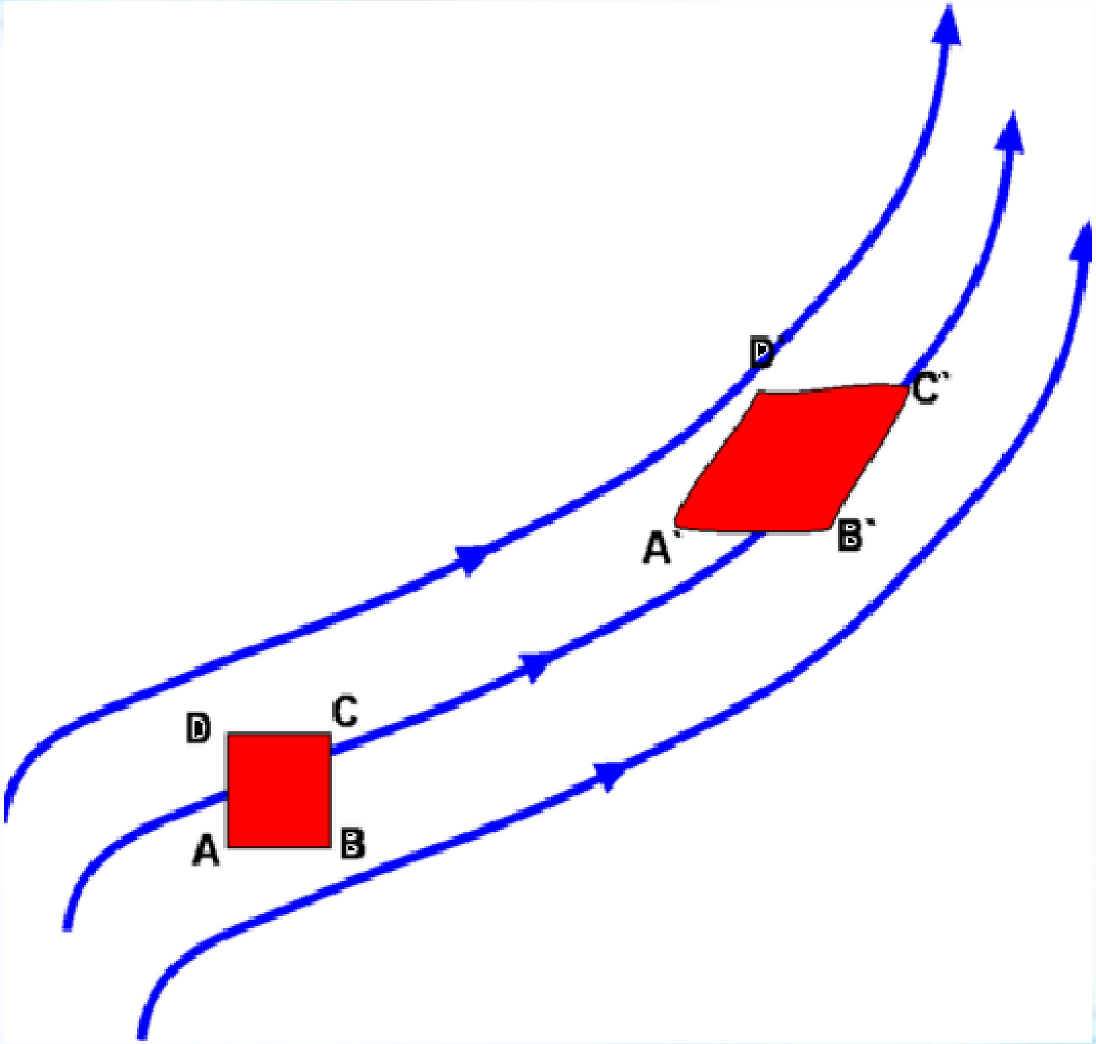
- We can decompose the motion of an infinitesimal fluid particle, into four components:
    - Translation
    - Rotation
    - Linear deformation *(Linear strain)*
    - Angular deformation *(Shear strain)*



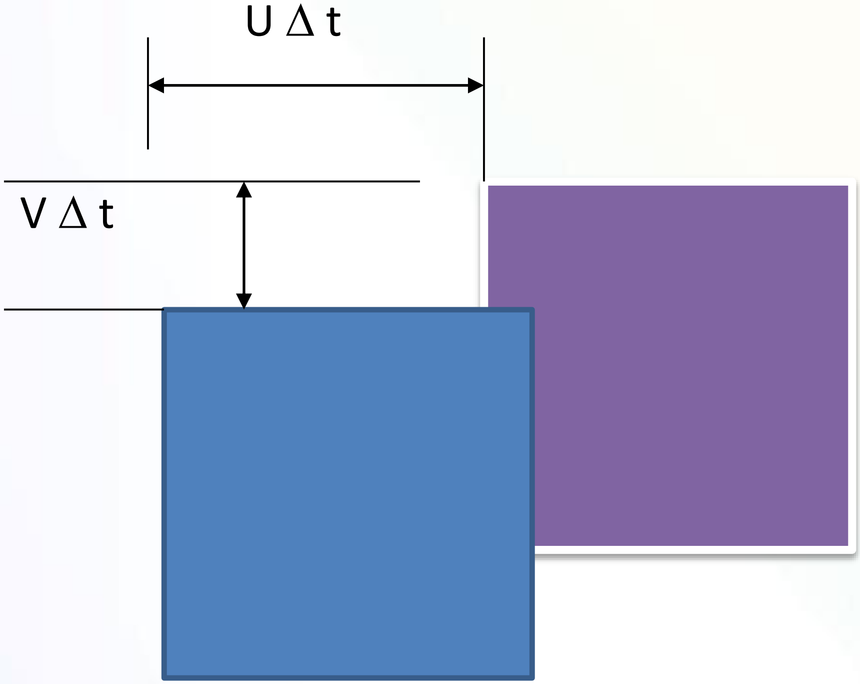
FLUID KINEMATICS



FLUID KINEMATICS

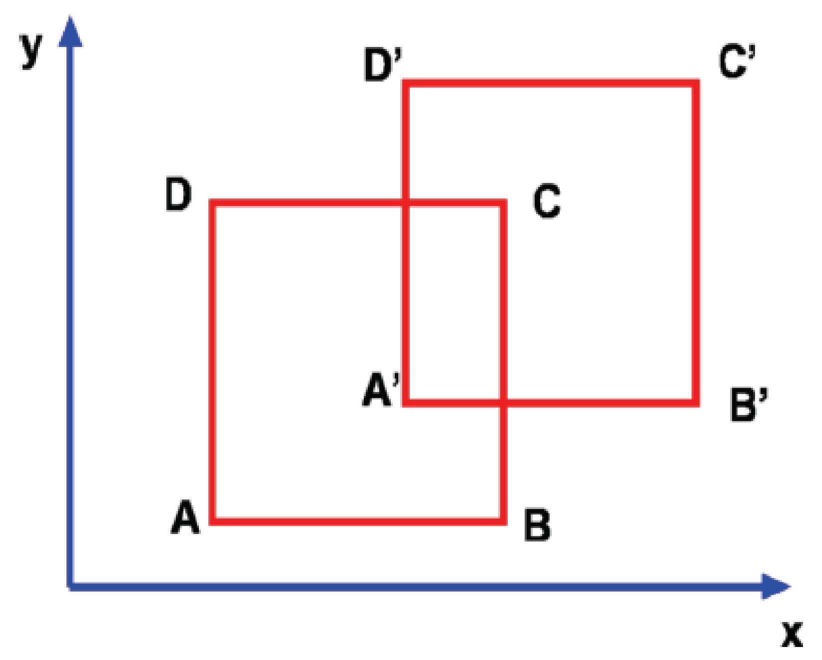


TRANSLATION



No Shear Stress

TRANSLATION



$$\vec{V}_p(t) = \vec{V}(x,y,z,t) \Rightarrow \vec{V}_p(t+dt) = \vec{V}(x+dx, y+dy, z+dz, t+dt)$$

$$d\vec{V}_p = \frac{\partial \vec{V}}{\partial x} dx_p + \frac{\partial \vec{V}}{\partial y} dy_p + \frac{\partial \vec{V}}{\partial z} dz_p + \frac{\partial \vec{V}}{\partial t} dt$$

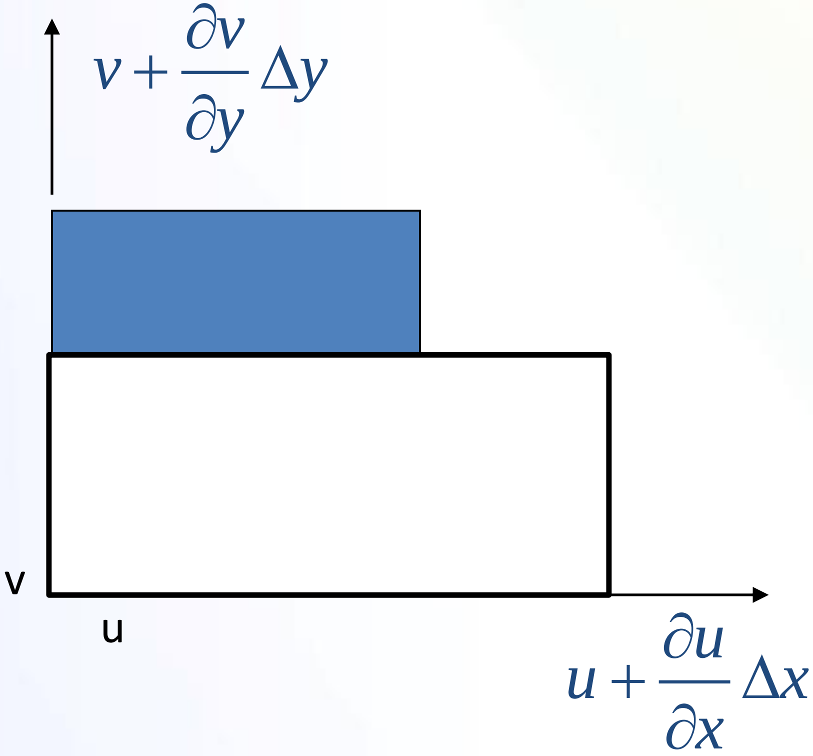
TRANSLATION

$$\frac{d\overrightarrow{V}_p}{\partial t} = \frac{\partial \overrightarrow{V}}{\partial x}\frac{dx_p}{dt} + \frac{\partial \overrightarrow{V}}{\partial y}\frac{dy_p}{dt} + \frac{\partial \overrightarrow{V}}{\partial z}\frac{dz_p}{dt} + \frac{\partial \overrightarrow{V}}{\partial t}$$

Acceleration of fluid element

$$\boxed{\vec{a}_p = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}$$

LINEAR DEFORMATION



$$\frac{\Delta\forall}{\Delta t}=\Delta x\Delta y\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

$$\nabla\vec{V}=\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

Incompressible flow

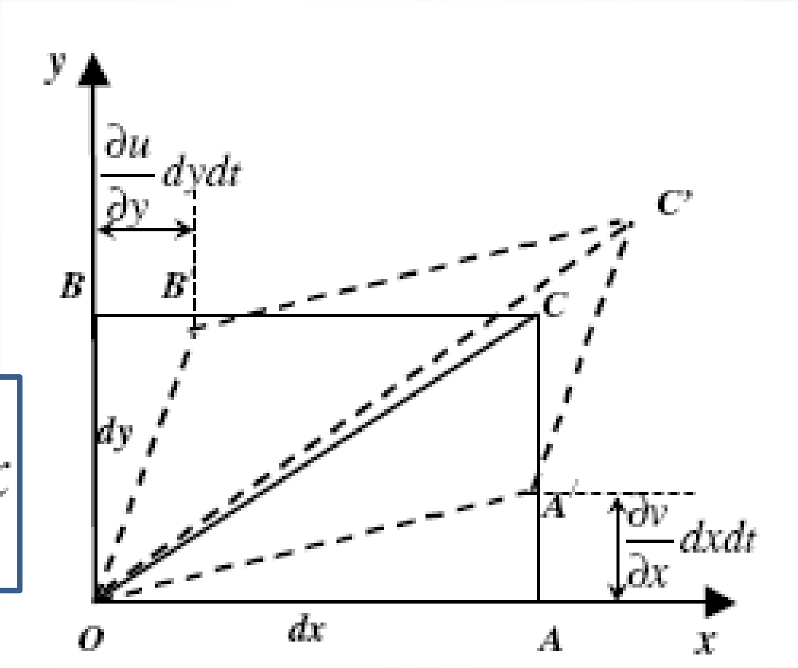
$$\boxed{\Delta\vec{V}=0}$$

LINEAR DEFORMATION

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \vec{\nabla} \cdot \vec{V} = 0$$

ANGULAR DEFORMATION



$$\varepsilon_{xy} = \frac{\partial u}{\partial y} dy\, dt / dy + \frac{\partial v}{\partial x} dx\, dt / dx$$

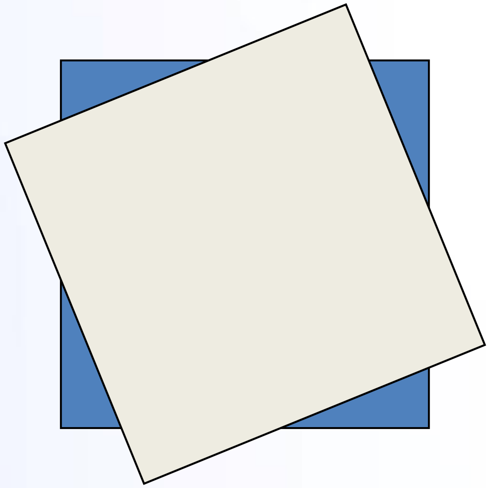
Similarly

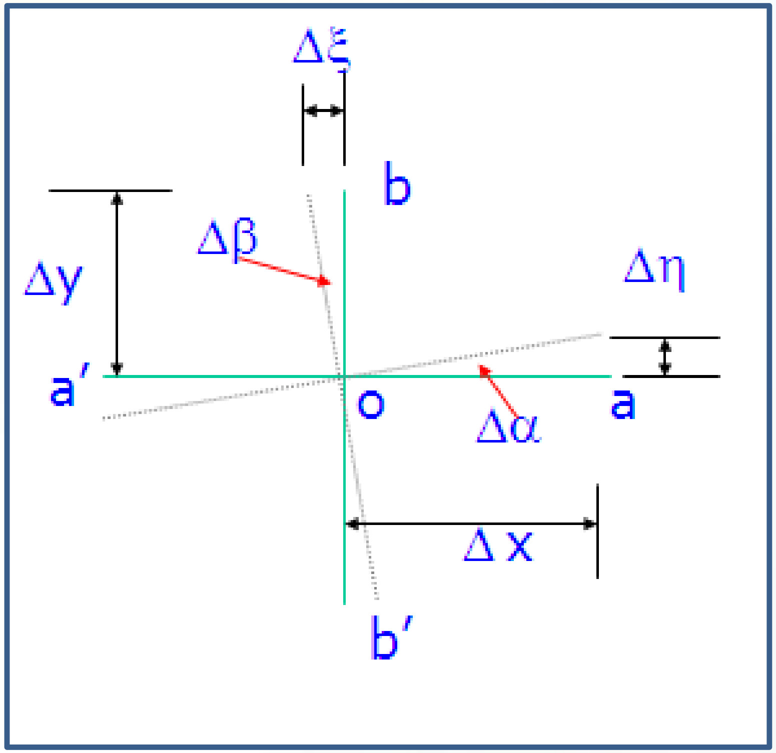
$$\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & \dfrac{\partial v}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & \dfrac{\partial w}{\partial z} \end{pmatrix}$$

ROTATION





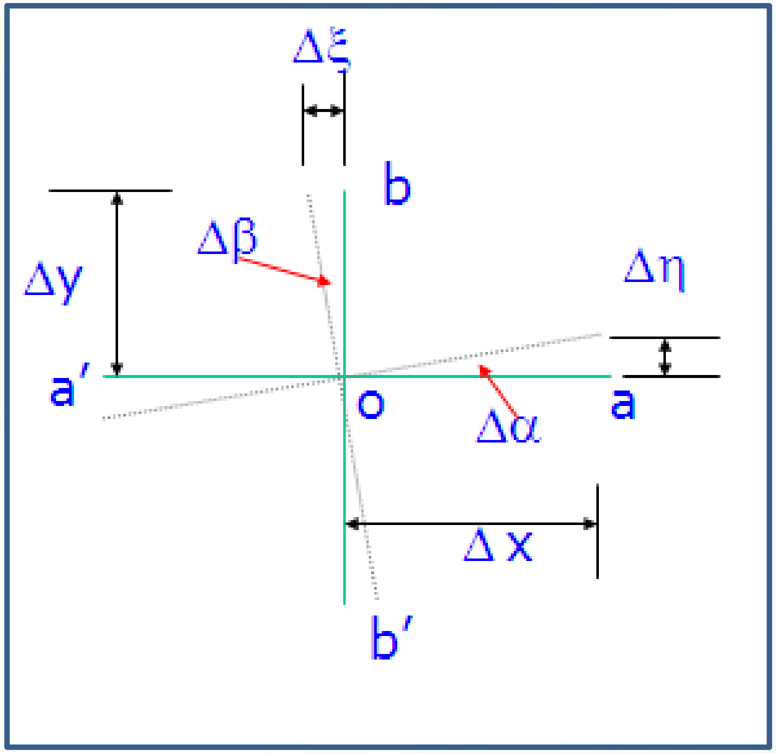
ROTATION

$$\omega_{oa} = \lim_{\Delta t \to 0} \frac{\Delta \alpha}{\Delta t} = \ldots$$

$$= \frac{\partial v}{\partial x}$$

$$\omega_{ob} = \lim_{\Delta t \to 0} \frac{\Delta \beta}{\Delta t} = \ldots$$

$$= -\frac{\partial u}{\partial y}$$



ROTATION

$$\omega_z = \frac{1}{2}\left(\omega_{oa} + \omega_{ob}\right) \longrightarrow \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

In three-dimensional space: $\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$

$$\vec{\omega} = \frac{1}{2}\vec{\nabla} \times \vec{V}$$

$$\vec{\omega} = \frac{1}{2}\mathrm{curl}\vec{V}$$

[Vorticity = Curl of velocity]

ROTATIONAL AND IRROTATIONAL FLOWS

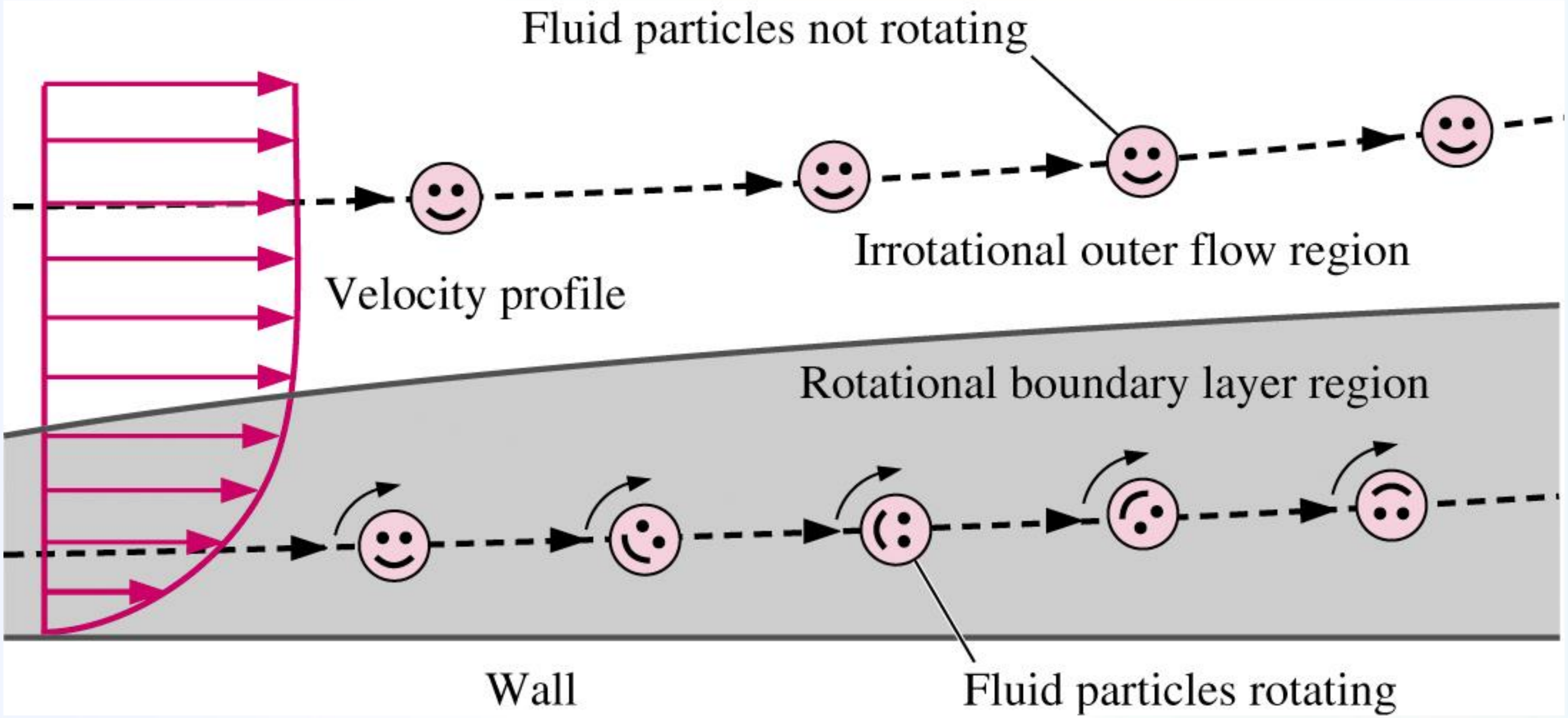
Rotational flow:

$\vec{\nabla} \times \vec{V} \neq 0$ at every point. The fluid elements have a finite angular velocity.

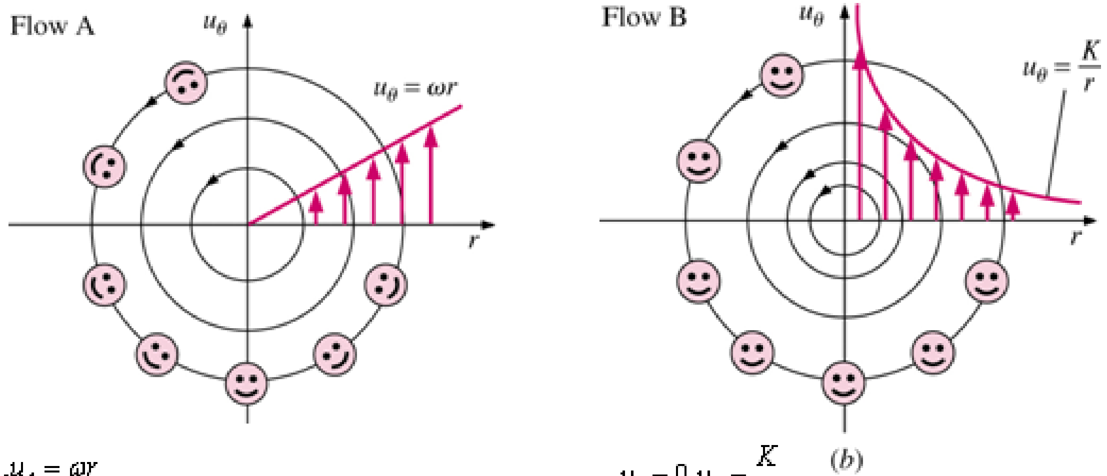
Irrotational flow:

$\vec{\nabla} \times \vec{V} = 0$ at every point. The fluid elements have no angular velocity *(pure translation).*

ROTATIONAL AND IRROTATIONAL FLOWS



ROTATIONAL AND IRROTATIONAL FLOWS



$u_r = 0, u_\theta = \omega r$

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z = \frac{1}{r}\left(\frac{\partial(\omega r^2)}{\partial r} - 0\right)\vec{e}_z = 2\omega\vec{e}_z$$

$u_r = 0, u_\theta = \frac{K}{r}$

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z = \frac{1}{r}\left(\frac{\partial(K)}{\partial r} - 0\right)\vec{e}_z = 0\vec{e}_z$$

Flow **A** is rotational
Flow **B** is irrotational

ROTATIONAL AND IRROTATIONAL FLOWS

BERNOULLI'S EQUATION FOR IRROTATIONAL FLOWS

$$\omega_x = \omega_y = \omega_z = 0$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0$$

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$

Euler Equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + f_x$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + f_y$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + f_z$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + f_x$$

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + f_y$$

$$u\frac{\partial u}{\partial z} + v\frac{\partial v}{\partial z} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + f_z$$

BERNOULLI'S EQUATION FOR IRROTATIONAL FLOWS

$$u\frac{\partial u}{\partial x}dx + v\frac{\partial v}{\partial x}dx + w\frac{\partial w}{\partial x}dx = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx + f_x dx \Bigg|$$

$$u\frac{\partial u}{\partial y}dy + v\frac{\partial v}{\partial y}dy + w\frac{\partial w}{\partial y}dy = -\frac{1}{\rho}\frac{\partial p}{\partial y}dy + f_y dy \Bigg|$$

$$u\frac{\partial u}{\partial z}dz + v\frac{\partial v}{\partial z}dz + w\frac{\partial w}{\partial z}dz = -\frac{1}{\rho}\frac{\partial p}{\partial z}dz + f_z dz \Bigg|$$

$$f_z = -g$$

$$\frac{\partial}{\partial x}\left(\frac{u^2+v^2+w^2}{2}\right) = \frac{\partial}{\partial x}\left(\frac{V^2}{2}\right)$$

Adding above equations:

$$\frac{\partial}{\partial x}\left(\frac{V^2}{2}\right)dx + \frac{\partial}{\partial y}\left(\frac{V^2}{2}\right)dy + \frac{\partial}{\partial z}\left(\frac{V^2}{2}\right)dz = d\left(\frac{V^2}{2}\right)$$

BERNOULLI'S EQUATION FOR IRROTATIONAL FLOWS

$$d\left(\frac{V^2}{2}\right) = -\frac{1}{\rho}dp - gdz$$

Integrating

$$gz + \int \frac{dp}{\rho} + \frac{V^2}{2} = C$$

For $\rho = const.$ *(Incompressible flow):*

$$\boxed{z + \frac{p}{\rho g} + \frac{V^2}{2g} = C}$$

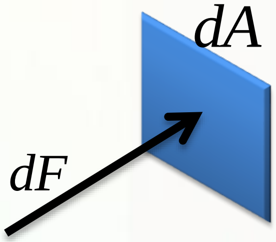
AERODYNAMIC VARIABLES

- The most frequent used terms in aerodynamics are:
  - Pressure
  - Density
  - Temperature
  - Velocity
  - Viscosity

AERODYNAMIC VARIABLES

- **Pressure**:
  - Pressure can be defined at any point in a fluid, whether liquid or gas.
  - Pressure is the normal force per unit area exerted on a surface due to the time rate of change of momentum of the gas molecules impacting on that surface.

$$p = \lim_{dA \to 0}\left(\frac{dF}{dA}\right)$$

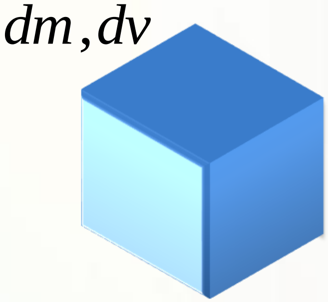


  - Pressure is defined at a *point* in the fluid (or solid). Pressure is a *point property*.
  - Dimension: [M/T²L], [FL/T] (T: Time)

AERODYNAMIC VARIABLES

- **Density**:
  - Density is defined as the “mass per unit volume”. It’s the mass of the fluid contained in an incremental volume surrounding the point.

$$\rho = \lim_{dv \to 0}\left(\frac{dm}{dv}\right)$$



  - In a fluid, density may vary from point to point. Density is a *point property*.
  - Dimension: $[M/L^3]$, $[FT^2/L^4]$ (T: Time)

AERODYNAMIC VARIABLES

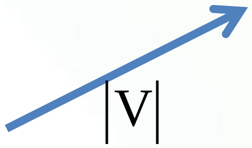
- **Temperature**:
  - Temperature is directly proportional to the average kinetic energy of the molecules of the fluid.

$$KE = \frac{3}{2} kT$$

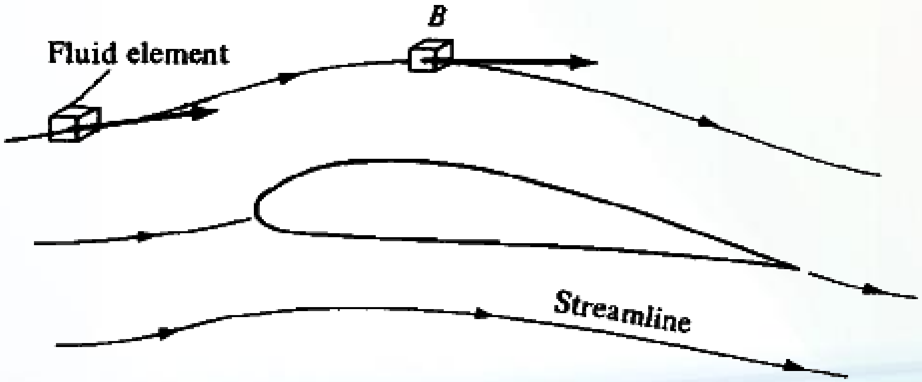
- KE: mean molecular kinetic energy
- k: Boltzmann constant

AERODYNAMIC VARIABLES

- **Velocity**:
  - Flow velocity is a vector quantity; it has both magnitude and direction.



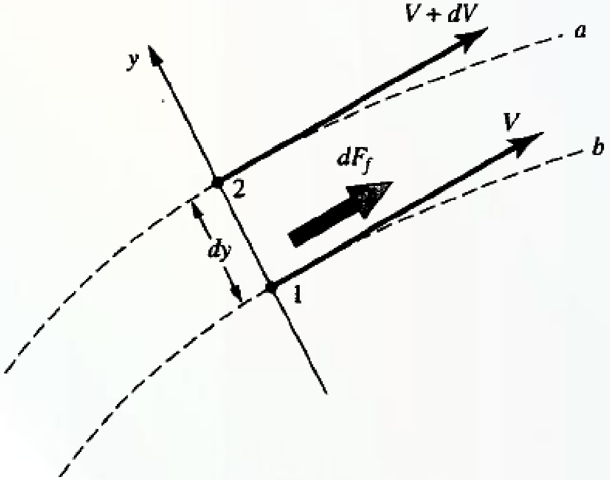
  - The velocity of a flowing fluid at any fixed point *B*, is the velocity of an infinitesimally small fluid element as it sweeps through *B*.



AERODYNAMIC VARIABLES

- **Viscosity**:
  - Viscosity of a fluid is regarded as its tendency to resist sliding between layers.
  - In a Newtonian fluid, the shearing stress is proportional to the rate of shearing deformation. The constant of proportionality is called the coefficient of viscosity μ.

$$\tau = \mu \frac{dV}{dy}$$



  - Viscosity of a fluid relates to the transport of momentum in the direction of the velocity gradient (but opposite in sense. Viscosity is a *transport property*.

AERODYNAMIC VARIABLES

- **Viscosity**:
  - The coefficient of viscosity depends on the composition of the fluid, its temperature and its pressure.
  - Sutherland's formula can be used to derive the dynamic viscosity of an ideal gas as a function of the temperature:

$$\mu = \mu_0 \frac{T_0 + C}{T + C} \left(\frac{T}{T_0}\right)^{3/2}$$

where:
μ = dynamic viscosity in (Pa·s) at input temperature T
μ_0 = reference viscosity in (Pa·s) at reference temperature T_0
T = input temperature in kelvin
T_0 = reference temperature in kelvin
C = Sutherland's constant for the gaseous material in question
Valid for temperatures between $0 < T < 555$ K with an error due to pressure less than 10% below 3.45 MPa

AERODYNAMIC VARIABLES

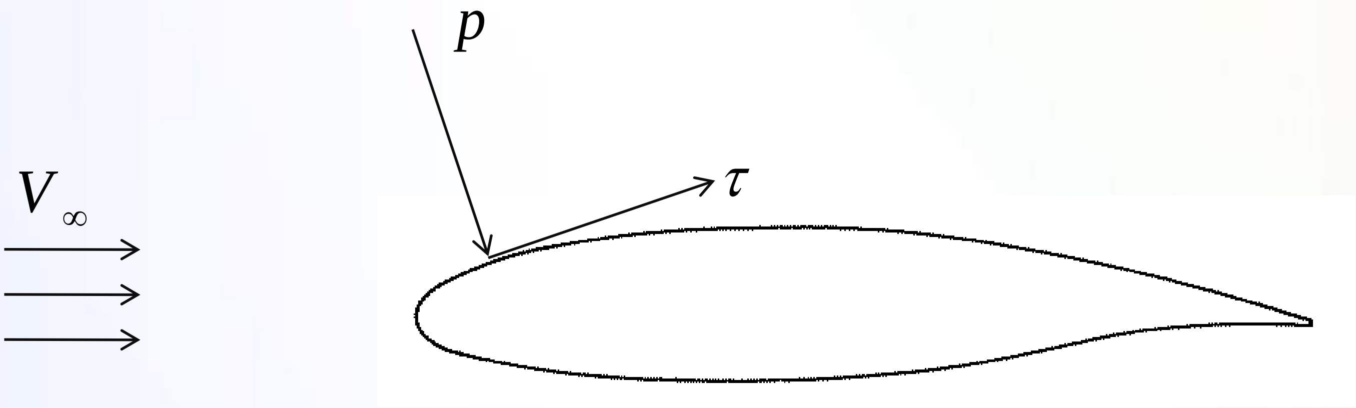
- Viscosity:
  - The coefficient of viscosity depends on the composition of the fluid, its temperature and its pressure.
  - Sutherland's formula can be used to derive the dynamic viscosity of an ideal gas as a function of the temperature:

$$\mu = \mu_0 \frac{T_0 + C}{T + C} \left(\frac{T}{T_0} \right)^{3/2}$$

Gas	C [K]	T_o [K]	μ_o [10^{-6} Pa s]
air	120	291.15	18.27
nitrogen	111	300.55	17.81
oxygen	127	292.25	20.18
carbon dioxide	240	293.15	14.8
carbon monoxide	118	288.15	17.2
hydrogen	72	293.85	8.76
ammonia	370	293.15	9.82
sulfur dioxide	416	293.65	12.54
helium	79.4	273	19

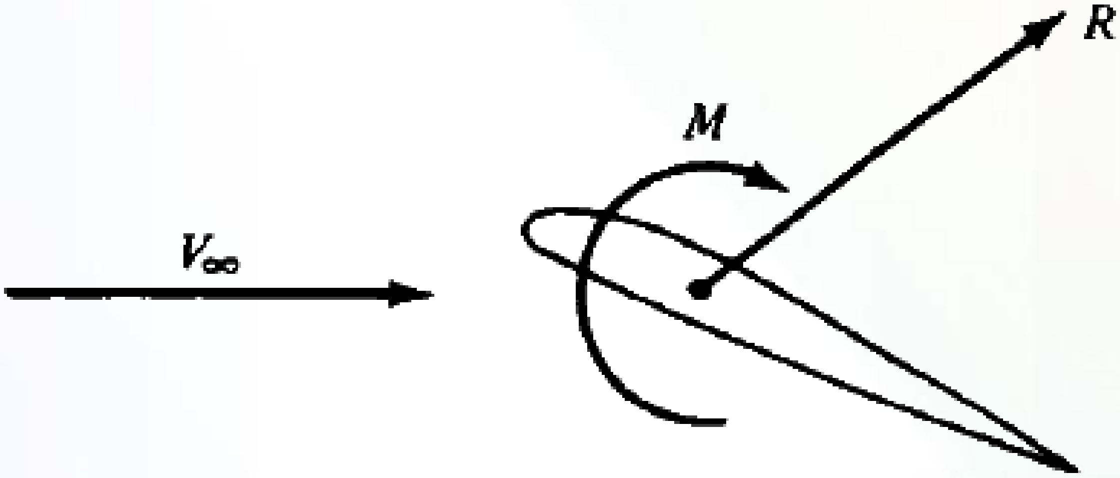
AERODYNAMIC FORCES AND MOMENTS

- Sources of aerodynamic forces and moments:
  - Pressure distribution *(Normal to the surface)*
  - Shear stress distribution *(Tangential to the surface)*



AERODYNAMIC FORCES AND MOMENTS

- The net effect of pressure and shear stress distribution, integrated over the body surface is:
  - Aerodynamic force: *R*
  - Aerodynamic moment: *M*



AERODYNAMIC FORCES AND MOMENTS

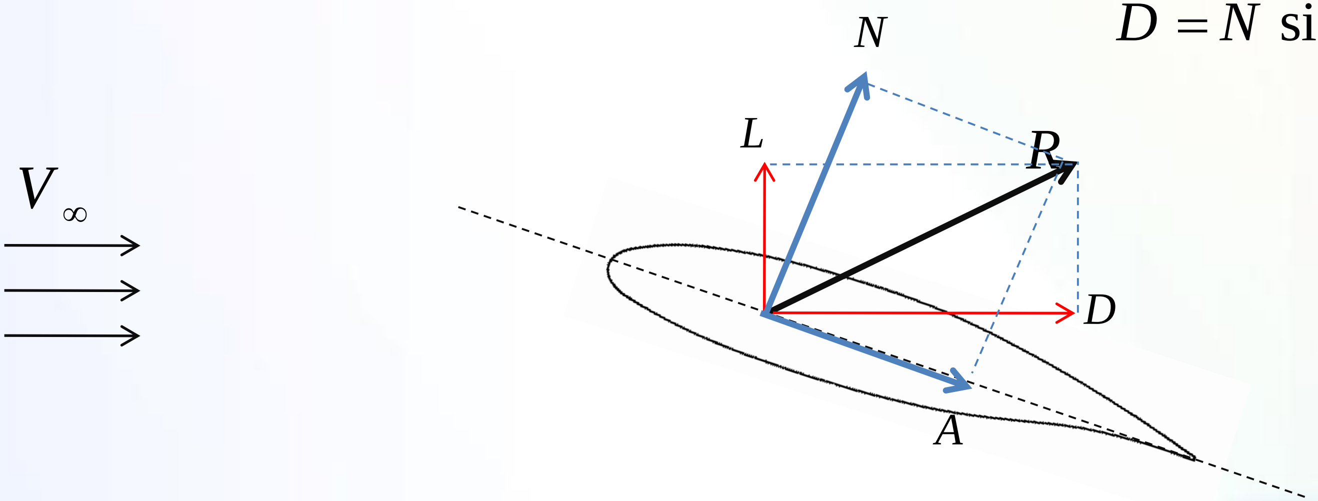
- Components of aerodynamic force (*R*):

1. *L:* Lift *(perpendicular to freestream velocity)*
   D: Drag *(parallel to freestram velocity)*

2. *N:* Normal force *(perpendicular to chord)*
   A: Axial force *(parallel to chord)*

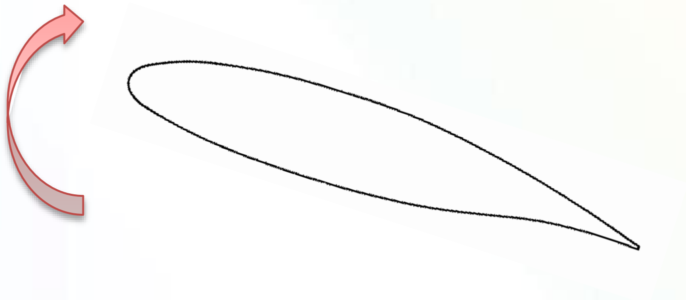
$$L = N \cos\alpha - A \sin\alpha$$

$$D = N \sin\alpha + A \cos\alpha$$

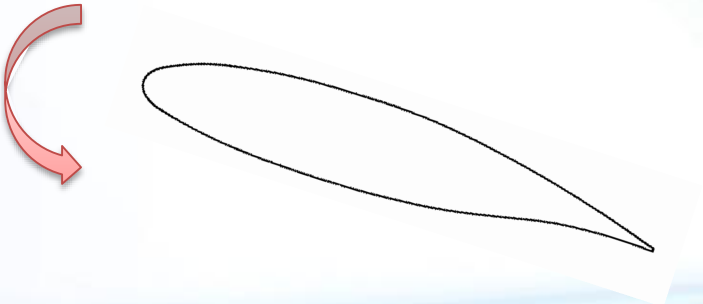


SIGN CONVENTIOM FOR AERODYNAMIC MOMENT

- By convention:
  - Positive moments tend to increase the angle of attack:



  - Negative moments tend to decrease the angle of attack:



AERODYNAMIC COEFFICIENTS

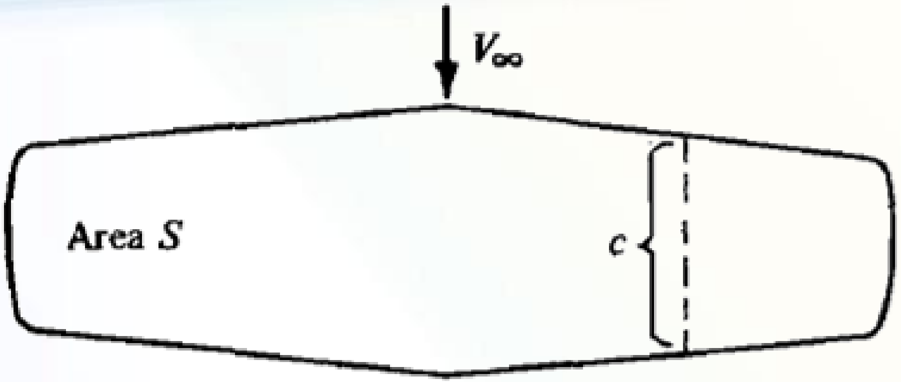
- The dimensionless force and moment coefficients:

Lift coefficient	$C_L = \frac{L}{q_\infty S}$
Drag coefficient	$C_D = \frac{D}{q_\infty S}$
Normal force coefficient	$C_N = \frac{N}{q_\infty S}$
Axial force coefficient	$C_A = \frac{A}{q_\infty S}$
Moment coefficient	$C_M = \frac{M}{q_\infty S l}$

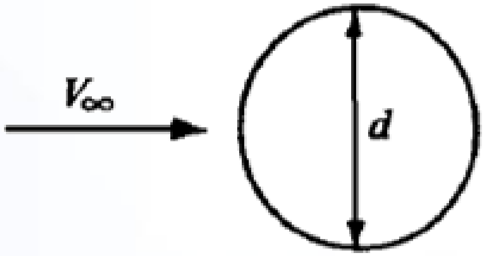
Where:

- q is called the freestream dynamic pressure: $q_\infty = \frac{1}{2}\rho_\infty v_\infty^2$
- l: reference length
- S: reference area

AERODYNAMIC COEFFICIENTS



S = planform area
$l = c$ = chord length



S = cross-sectional area = $\frac{\pi d^2}{4}$
$l = d$ = diameter

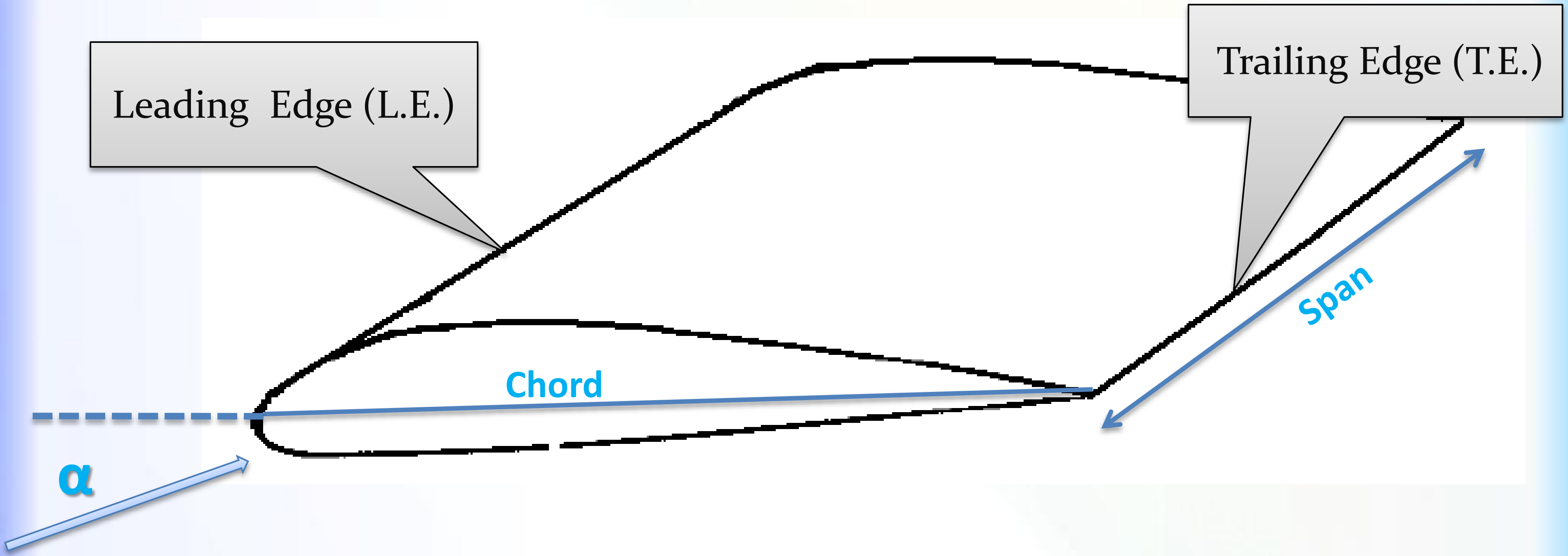
AERODYNAMIC COEFFICIENTS

- Two additional dimensionless quantities:

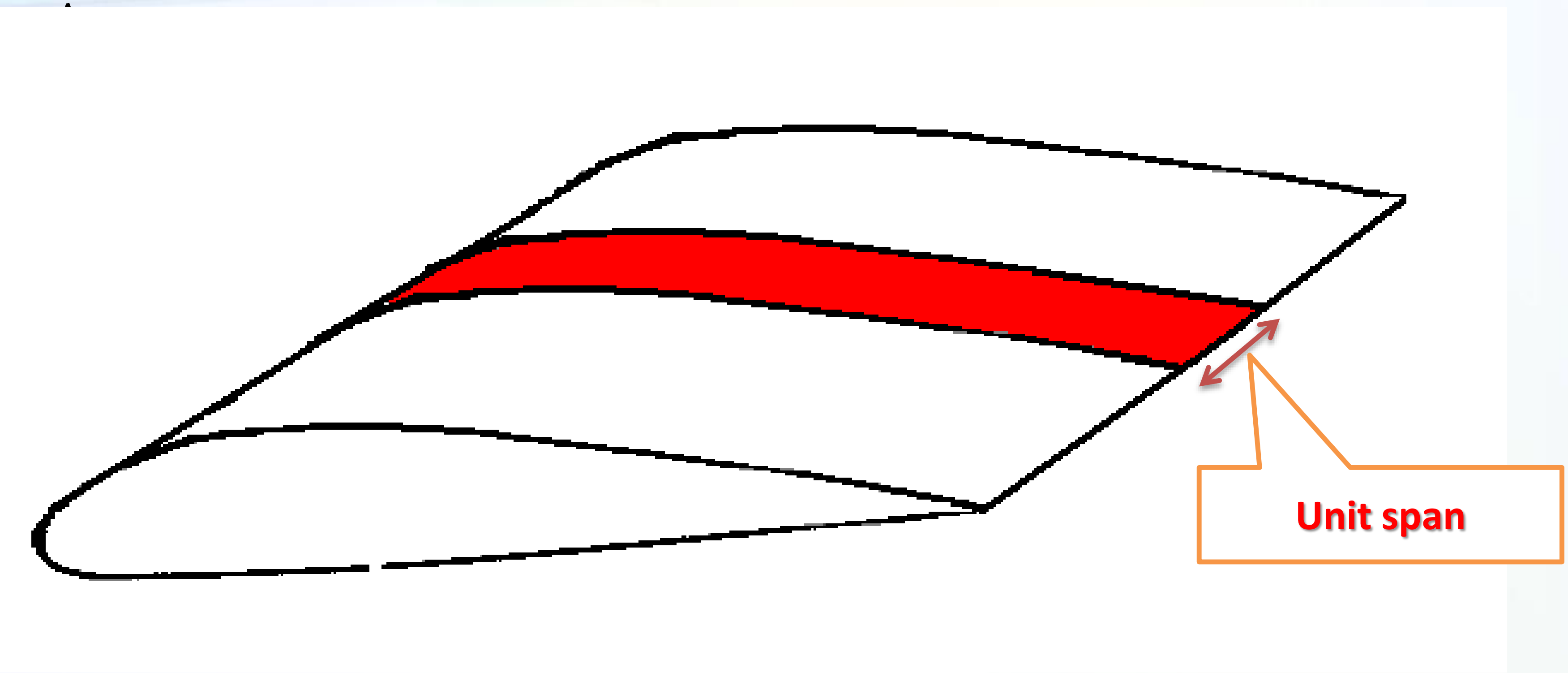
Pressure coefficient $C_p = \dfrac{p - p_\infty}{q_\infty}$

Skin friction coefficient $C_f = \dfrac{\tau}{q_\infty}$

WING PARAMETERS



TWO-DIMENSIONAL BODIES

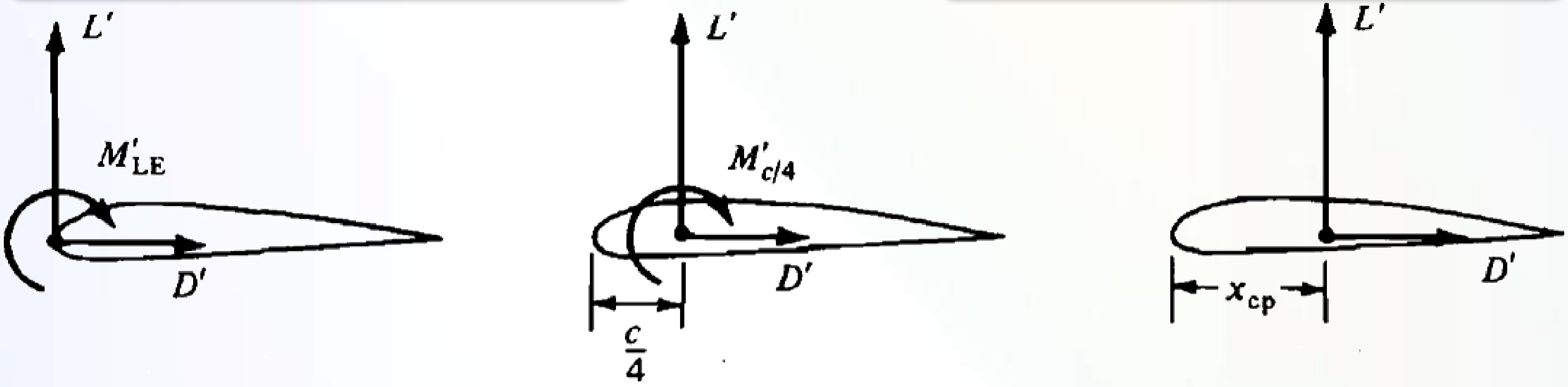


$$S = c \times 1 \Rightarrow \quad c_l \equiv \frac{L'}{q_\infty c} \qquad c_d \equiv \frac{D'}{q_\infty c} \qquad c_m \equiv \frac{M'}{q_\infty c^2}$$

CENTER OF PRESSURE

- Center of pressure is a point about which the aerodynamic moment is zero.

Center of Pressure ⟺ $M = 0$



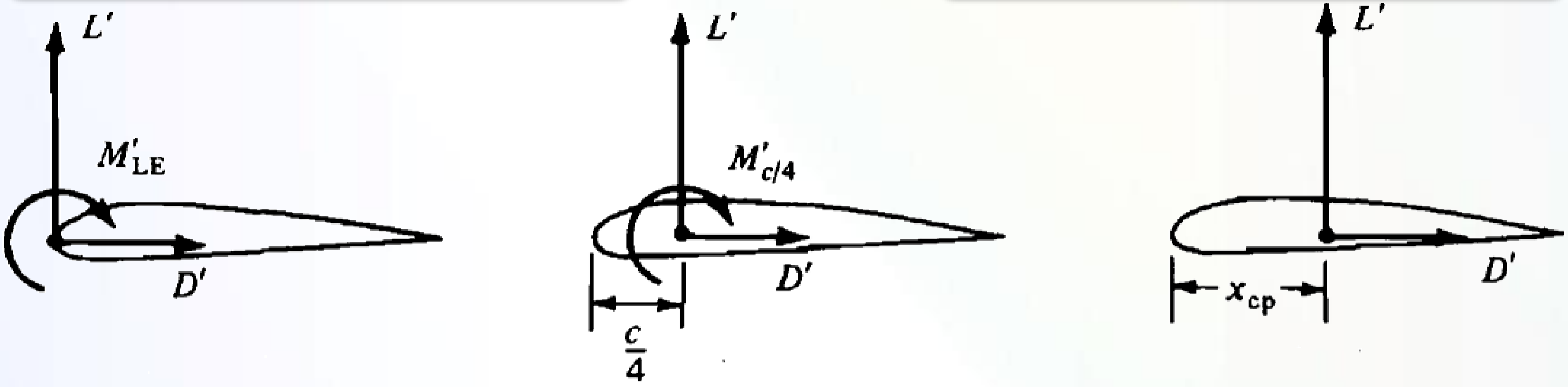
$$x_{cp} = -\frac{M'_{LE}}{N'}$$

- For small angles of attack: $L' \approx N'$ ⟹ $x_{cp} = -\frac{M'_{LE}}{L'}$

CENTER OF PRESSURE

- Center of pressure is a point about which the aerodynamic moment is zero.

Center of Pressure ⟷ $M = 0$



$$M'_{LE} = -\frac{c}{4}L' + M'_{c/4} = -x_{cp}L'$$

CENTER OF PRESSURE - EXAMPLE

In low-speed, incompressible flow, the following experimental data are obtained for an airfoil section at an angle of attack of 4°:

$c_{l} =$ 0.85 and $c_{m,\,c/4}$ = -0.09.

Calculate the location of the center of pressure.

$$x_{cp} = \frac{c}{4} - \frac{M'_{c/4}}{L'}$$

$$\frac{x_{cp}}{c} = \frac{1}{4} - \frac{(M_{c/4}/q_{\infty}c^{2})}{(L'/q_{\infty}c)} = \frac{1}{4} - \frac{c_{m,c/4}}{c_{l}}$$

$$= \frac{1}{4} - \frac{(-0.09)}{0.85} = 0.356$$

DIMENSIONAL ANALYSIS:
THE BUCKINGHAM PI THEOREM

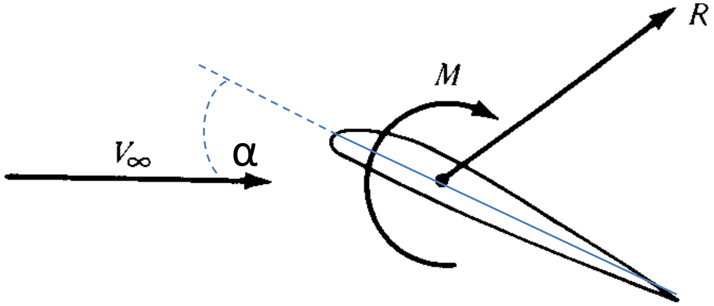
Question:
What physical quantities determine the variation of Aerodynamic forces and moments?

The answer can be found from the powerful method of

dimensional analysis

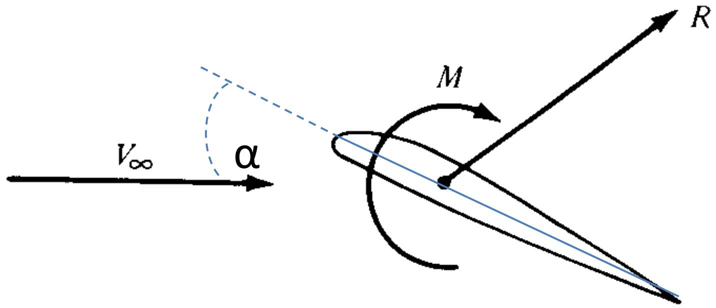
DIMENSIONAL ANALYSIS: THE BUCKINGHAM PI THEOREM

On a physical, intuitive basis, we expect the aerodynamic force to depend on:



1. Freestream velocity, V_∞
2. Freestream density, ρ_∞ .
3. Viscosity of the fluid, μ_∞ .
4. The size of the body, represented by some chosen reference length. Reference length is the chord length c.
5. The compressibility of the fluid. Compressibility is related to the speed of sound, a. Therefore, let us represent the influence of compressibility on aerodynamic forces and moments by the free stream speed of sound, a_∞.

DIMENSIONAL ANALYSIS: THE BUCKINGHAM PI THEOREM



$$R = f\,(\rho_\infty, V_\infty, c, \mu_\infty, a_\infty)$$

DIMENSIONAL ANALYSIS: THE BUCKINGHAM PI THEOREM

- The object of dimensional analysis is to group several variables together to form a new variable that is nondimensional.

- Dimensional analysis is based on the obvious fact that an equation dealing the real physical world, each term must have the same dimensions:

$$\psi + \eta + \zeta = \phi$$

The above equation can be made dimensionless by dividing by any one of the terms, say, ϕ

$$\frac{\psi}{\phi} + \frac{\eta}{\phi} + \frac{\zeta}{\phi} = 1$$

DIMENSIONAL ANALYSIS: THE BUCKINGHAM PI THEOREM

Let ***K*** equal the number of fundamental dimensions required to describe the physical variables. *(In mechanics, all physical variables can be expressed in terms of the dimensions of mass, length, and time; hence, K = 3.)*

Let **P_1, P_2,. . . , P_N** represent **N** physical variables in the physical relation

$$f_1(P_1, P_2, \ldots, P_N) = 0$$

Then, the physical relation may be reexpressed as a relation of *(N – K)* dimensionless products (called Π **products**),

$$f_2(\Pi_1, \Pi_2, \ldots, \Pi_N) = 0$$

DIMENSIONAL ANALYSIS: THE BUCKINGHAM PI THEOREM

Each Π product is a dimensionless product of a set of K physical variables plus one other physical variable. Let P_1, P_2, …, P_K be the selected set of K physical variables. Then

$$\Pi_1 = f_3(P_1, P_2, \ldots, P_K, P_{K+1})$$

$$\Pi_2 = f_4(P_1, P_2, \ldots, P_K, P_{K+2})$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Pi_{N-K} = f_5(P_1, P_2, \ldots, P_K, P_N)$$

The choice of repeating variable, should be such that:

- They include all the K dimensions used in problem.
- The dependent variable should appear in only one of the Π products.

DIMENSIONAL ANALYSIS: THE BUCKINGHAM PI THEOREM

$$R = f\left(\rho_\infty, V_\infty, c, \mu_\infty, a_\infty\right)$$

$$\Downarrow$$

$$g\left(R, \rho_\infty, V_\infty, c, \mu_\infty, a_\infty\right) = 0$$

m = dimensions of mass
l = dimension of length
t = dimension of time

→ $K = 3$

DIMENSIONAL ANALYSIS: THE BUCKINGHAM PI THEOREM

- Physical variables and their dimensions:

$$[R] = mlt^{-2}$$

$$[\rho_\infty] = ml^{-3}$$

$$[V_\infty] = lt^{-1}$$

$$[c] = l$$

$$[\mu_\infty] = ml^{-1}t^{-1}$$

$$[a_\infty] = lt^{-1}$$

N = 6
K = 3

DIMENSIONAL ANALYSIS:
THE BUCKINGHAM PI THEOREM

$g(R, \rho_\infty, V_\infty, c, \mu_\infty, a_\infty) = 0$ can be reexpressed in terms of N – K =3 dimensionless Π products

$$f_2(\Pi_1, \Pi_2, \Pi_3) = 0$$

The Π products are:

$$\Pi_1 = f_3(\rho_\infty, V_\infty, c, R)$$
$$\Pi_2 = f_4(\rho_\infty, V_\infty, c, \mu_\infty)$$
$$\Pi_3 = f_5(\rho_\infty, V_\infty, c, a_\infty)$$

DIMENSIONAL ANALYSIS:
THE BUCKINGHAM PI THEOREM

$$\Pi_1 = f_3(\rho_\infty, V_\infty, c, R)$$

$$\Pi_1 = \rho_\infty^d V_\infty^b c^e R$$

$$[\Pi_1] = (ml^{-3})^d (lt^{-1})^b (l)^e (mlt^{-2})$$

$$d + 1 = 0$$

$$-3d + b + e + 1 = 0$$

$$-b - 2 = 0$$

↓

$d = -1$, $b = -2$, and $e = -2$

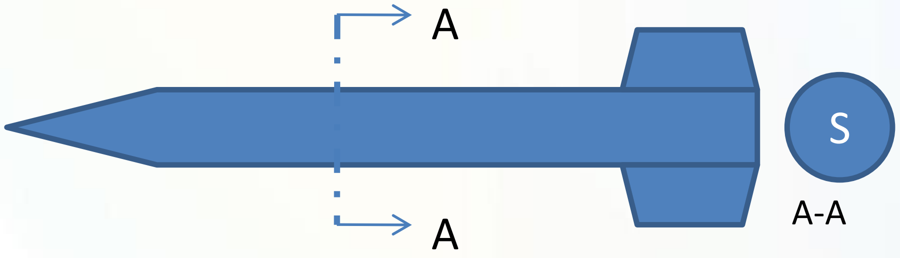
DIMENSIONAL ANALYSIS: THE BUCKINGHAM PI THEOREM

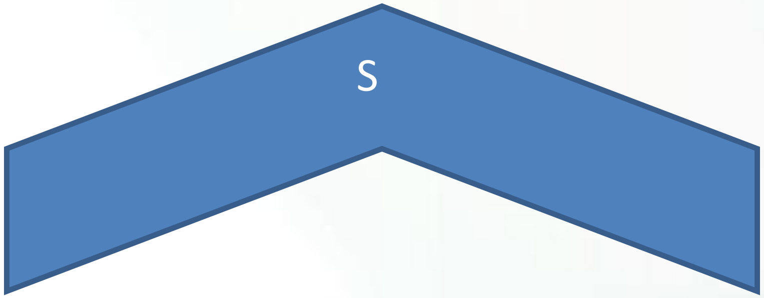
$$\Pi_1 = R\,\rho_\infty^{-1} V_\infty^{-2} c^{-2}$$

$$= \frac{R}{\rho_\infty V_\infty^{2} c^{2}}$$

Force Coefficient

$$C_R = \frac{R}{\frac{1}{2}\rho_\infty V_\infty^{2} S}$$





DIMENSIONAL ANALYSIS: THE BUCKINGHAM PI THEOREM

$$\Pi_2 = \rho_\infty V_\infty^h c^i \mu^j$$

$$[\Pi_2] = (ml^{-3})(lt^{-1})^h(l)^i(ml^{-1}t^{-1})^j$$

For m: $1 + j = 0$

For l: $-3 + h + i - j = 0$

For t: $-h - j = 0$

$$\boxed{j = -1,\ h = 1,\ \text{and}\ i = 1}$$

$$\boxed{\Pi_2 = \frac{\rho_\infty V_\infty c}{\mu_\infty}}$$

$$\boxed{Re = \frac{\rho_\infty V_\infty c}{\mu}}$$

Reynolds Number

DIMENSIONAL ANALYSIS: THE BUCKINGHAM PI THEOREM

$$\Pi_3 = V_\infty \rho_\infty^k c^r a_\infty^s$$

$$[\Pi_3] = (lt^{-1})(ml^{-3})^k(l)^r(lt^{-1})^s$$

For m: $k = 0$

For l: $1 - 3k + r + s = 0$

For t: $-1 - s = 0$

$k = 0$, $s = -1$, and $r = 0$

$$\Pi_3 = \frac{V_\infty}{a_\infty}$$

$$M = \frac{V_\infty}{a_\infty}$$

Mach Number

DIMENSIONAL ANALYSIS: THE BUCKINGHAM PI THEOREM

$$f_2\left(\frac{R}{\frac{1}{2}\rho_\infty V_\infty^2 S}, \frac{\rho_\infty V_\infty c}{\mu_\infty}, \frac{V_\infty}{a_\infty}\right) = 0$$

$$f_2(C_R, \mathrm{Re}, M_\infty) = 0$$

$$C_R = f_6(Re\,, M_\infty)$$

$$C_L = f_7(Re\,, M_\infty)$$

$$C_D = f_8(Re\,, M_\infty)$$

$$C_M = \frac{M}{\frac{1}{2}\rho_\infty V_\infty^2 Sc}$$

$$C_M = f_9(Re\,, M_\infty)$$

DIMENSIONAL ANALYSIS:
THE BUCKINGHAM PI THEOREM

If α is allowed to vary, then:

C_L, C_D, and C_M will in general depend on the value of α.

$$\boxed{C_L = f_7(Re\,, M_{\infty,} \alpha)}$$

$$\boxed{C_D = f_8(Re\,, M_{\infty,} \alpha)}$$

$$\boxed{C_M = f_9(Re\,, M_{\infty,} \alpha)}$$

FLOW SIMILARITY

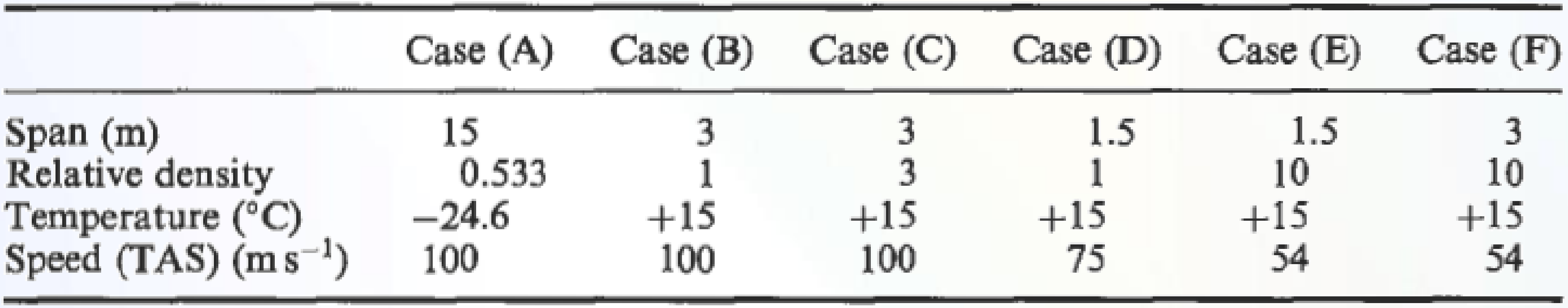
By definition, different flows are ***dynamically similar* *if*:**

1. The bodies and any other solid boundaries are geometrically similar for both flows.

2. The similarity parameters are the same for both flows.

FLOW SIMILARITY - EXAMPLE

An aircraft and some scale models of it are tested under various conditions: given below. Which cases are dynamically similar to the aircraft in flight, given as case (A)?



	Case (A)	Case (B)	Case (C)	Case (D)	Case (E)	Case (F)
Span (m)	15	3	3	1.5	1.5	3
Relative density	0.533	1	3	1	10	10
Temperature (°C)	−24.6	+15	+15	+15	+15	+15
Speed (TAS) ($\mathrm{m\,s^{-1}}$)	100	100	100	75	54	54

Case (A) represents the full-size aircraft at 6000 m. The other cases represent models under test in various types of wind-tunnel

FLOW SIMILARITY – EXAMPLE (CONT.)

The **Reynolds number** $\rho VD/\mu$ *may be calculated for each case*

(Viscosity from Sutherland's formula $\mu = \mu_0 \dfrac{T_0 + C}{T + C}\left(\dfrac{T}{T_0}\right)^{3/2}$*)*

These are found to be:

Case (A)	$Re = 5.52 \times 10^7$	Case (D)	$Re = 7.75 \times 10^6$
Case (B)	$Re = 1.84 \times 10^7$	Case (E)	$Re = 5.55 \times 10^7$
Case (C)	$Re = 5.56 \times 10^7$	Case (F)	$Re = 1.11 \times 10^8$

Cases (A), (C) and (E) are dynamically similar.

BERNOULLI'S EQUATION

- The Bernoulli's equation is a powerful and useful equation that relates pressure changes to velocity and elevation changes along a streamline.

$$\frac{p}{\rho}+\frac{V^{2}}{2}+gz=C$$

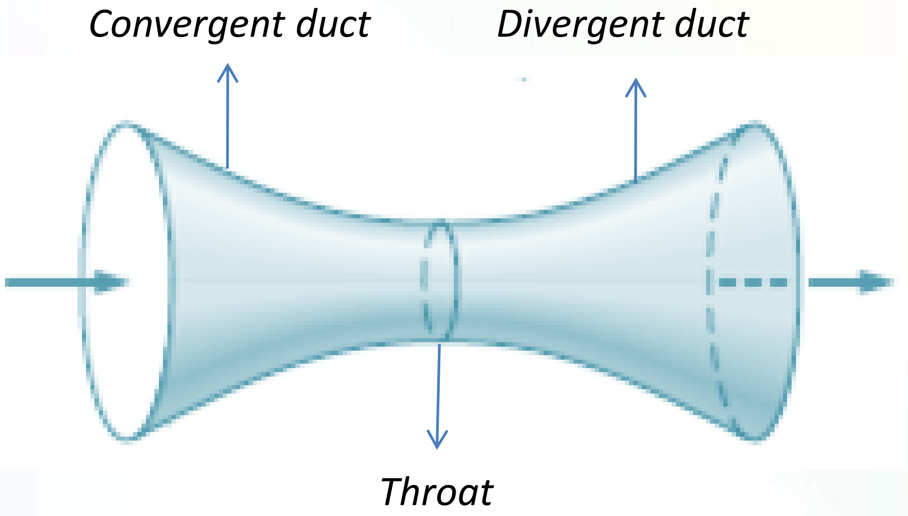
- The Bernoulli's equation gives correct results when applied to flow situations where the following four restrictions are reasonable:
  - Steady flow
  - Incompressible flow
  - Inviscid flow
  - Flow along a streamline *(In general, the Bernoulli's constant [C] has different values along different stramlines)*

BERNOULLI'S EQUATION APPLICATIONS

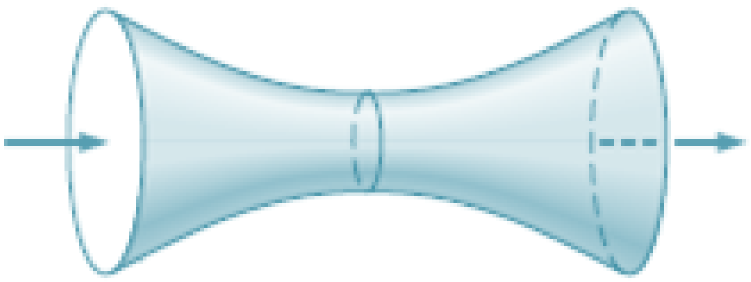
- Bernoulli's equation is applicable to the following two devices:
  - Venturi: *Flowmeter, low-speed wind tunnel, Airspeed measurement*
  - Pitot-tube: *Airspeed measurement*

BERNOULLI'S EQUATION APPLICATIONS: VENTURI

- Venturi is a convergent-divergent duct. It's a device that finds many applications in engineering.



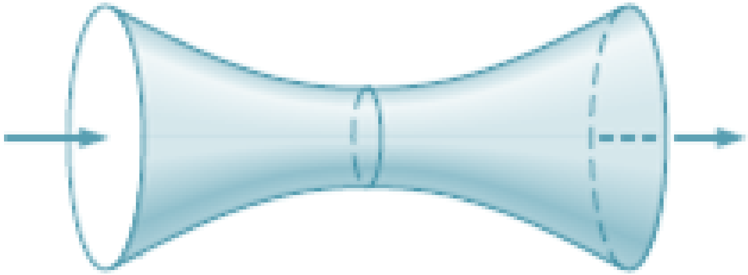
BERNOULLI'S EQUATION APPLICATIONS: VENTURI



- In general, venturi is a three-dimensional duct with elliptical or rectangular cross section which vary from one location to another.

$$A = A(x)$$

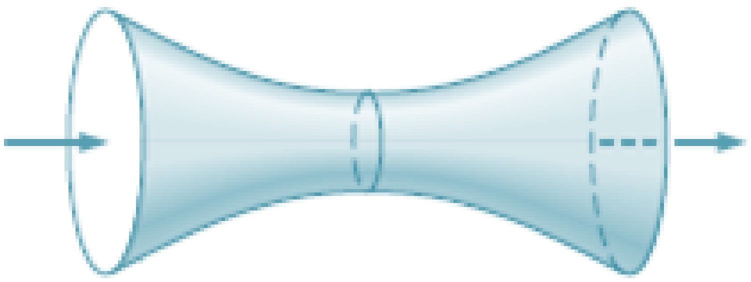
BERNOULLI'S EQUATION APPLICATIONS: VENTURI



- For moderate variation of area, it is reasonable to assume that the flowfield properties (velocity, pressure,…) are uniform across any cross section, and vary only in direction of flow.

$$\underbrace{A = A(x) \qquad V = V(x) \qquad p = p(x)}_{\textit{Quasi-one-dimensional flow}}$$

BERNOULLI'S EQUATION APPLICATIONS: VENTURI



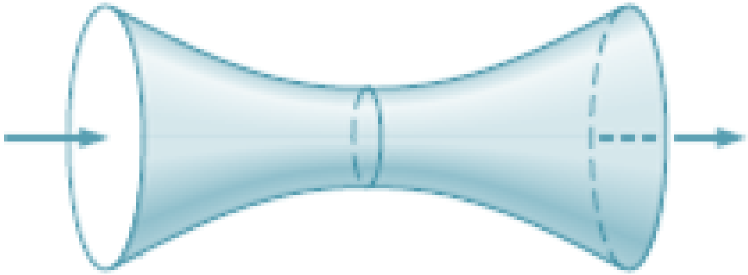
- For steady flow through the venturi, continuity equation gives:

$$\rho VA = const. \longrightarrow$$ the mass flow through the duct is constant.

- For incompressible flow:

$$VA = Q = const.$$

BERNOULLI'S EQUATION APPLICATIONS: VENTURI



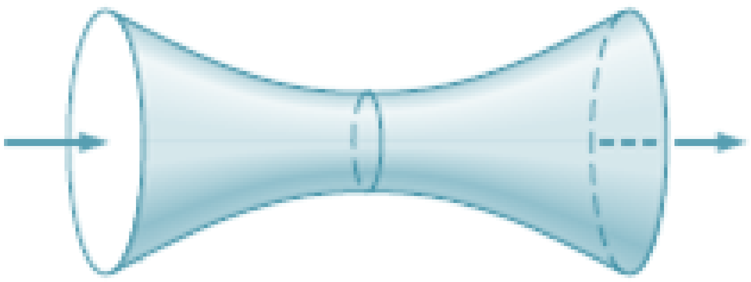
- For a given variation of area A(x):

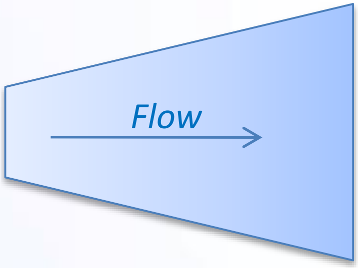
$$V(x) = \frac{Q}{A(x)}$$

- Using Bernoulli's equation:

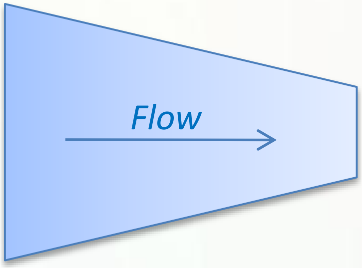
$$p(x) + \frac{\rho [V(x)]^2}{2} = const.$$

BERNOULLI'S EQUATION APPLICATIONS: VENTURI





Velocity decreases (Continuity)
Pressure increases (Bernoulli)

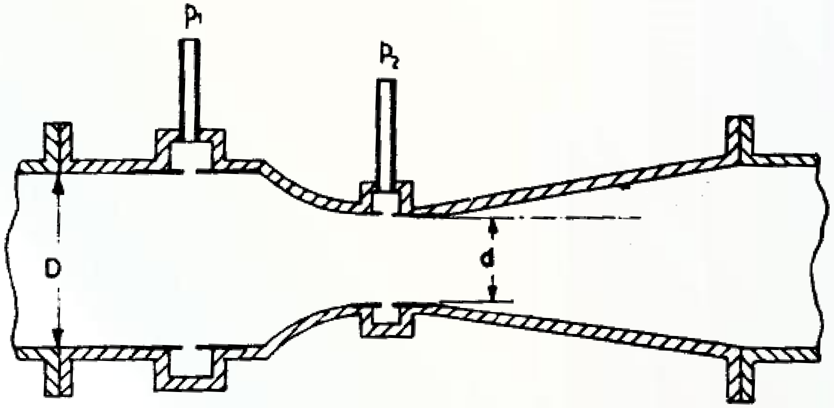


Velocity increases (Continuity)
Pressure decreases (Bernoulli)

BERNOULLI'S EQUATION APPLICATIONS: VENTURI

Venturi applications: Speed measurement

- Venturi can be used to measure airspeed.



- For a venturi *(with a given inlet [station 1] to throat [station 2] area ratio)* and known pressure difference p_1-p_2, the inlet velocity can be obtained from the combination of continuity and Bernoulli's equation:

$$V_1 = \sqrt{\frac{2(p_1 - p_2)}{\rho[(A_1 / A_2)^2 - 1]}}$$

BERNOULLI'S EQUATION APPLICATIONS: VENTURI

Venturi applications: Wind tunnel

- Another application of venturi is the low-speed wind tunnel.

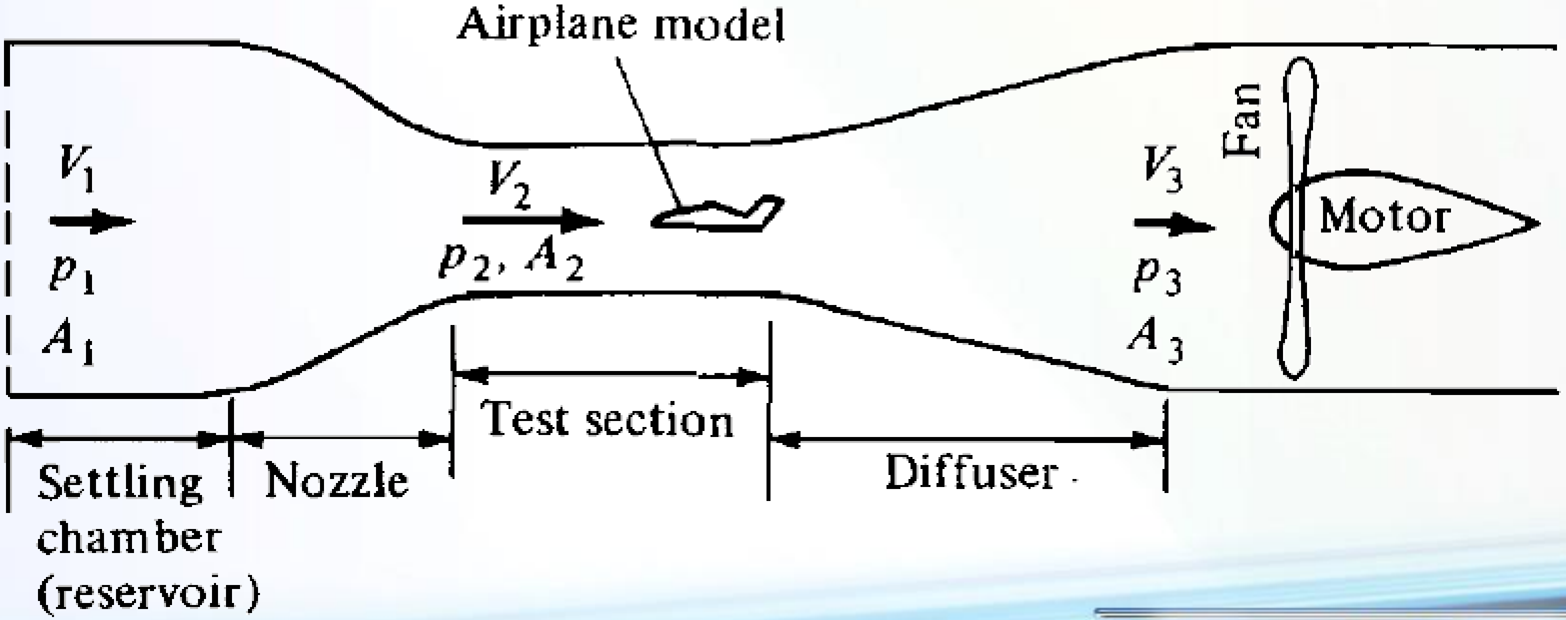
- A low-speed wind tunnel is a large venturi, where the airflow is driven by a fan connected to some type of motor drive.

BERNOULLI'S EQUATION APPLICATIONS: VENTURI

Venturi applications: Wind tunnel

- There are two general types of low-speed wind tunnels:

  1. Open-circuit tunnel

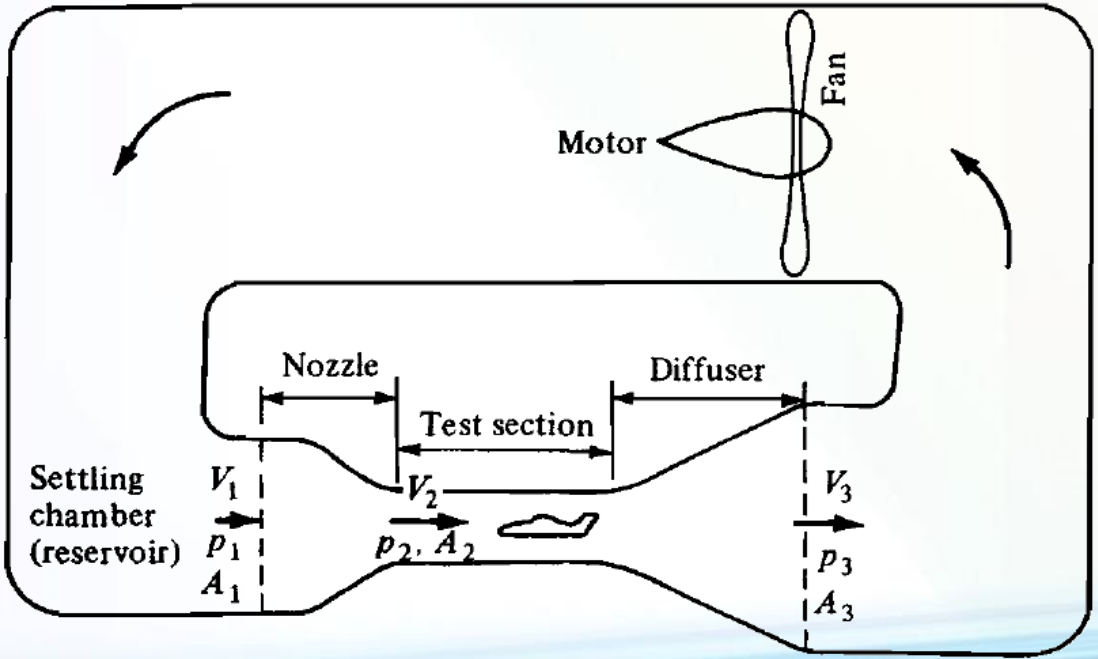


BERNOULLI'S EQUATION APPLICATIONS: VENTURI

Venturi applications: Wind tunnel

- There are two general types of low-speed wind tunnels:

  2. Closed-circuit tunnel



BERNOULLI'S EQUATION APPLICATIONS: VENTURI

Venturi applications: Wind tunnel

- The air velocity in the test section of a low-speed wind tunnel (with fixed area ratio A_2/A_1), is obtained from the combination of continuity and Bernoulli's equation:

$$V_2 = \sqrt{\frac{2(p_1 - p_2)}{\rho[1-(A_2/A_1)^2]}}$$

- In low-speed wind tunnels, a method of measuring the pressure difference P1-P2, is by means of manometers.

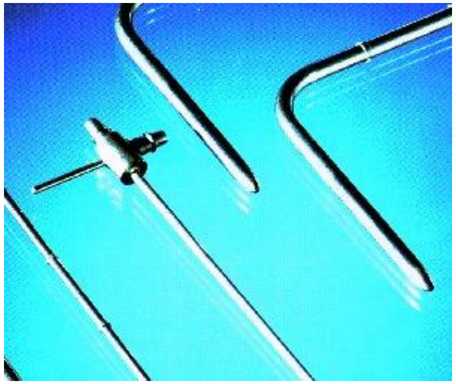
BERNOULLI'S EQUATION APPLICATIONS: PITOT TUBE

- Pitot tube is one of the most common and frequently used instruments in any modern aerodynamic laboratory.
- Pitot tube is the most common device for measuring flight velocities of airplanes.

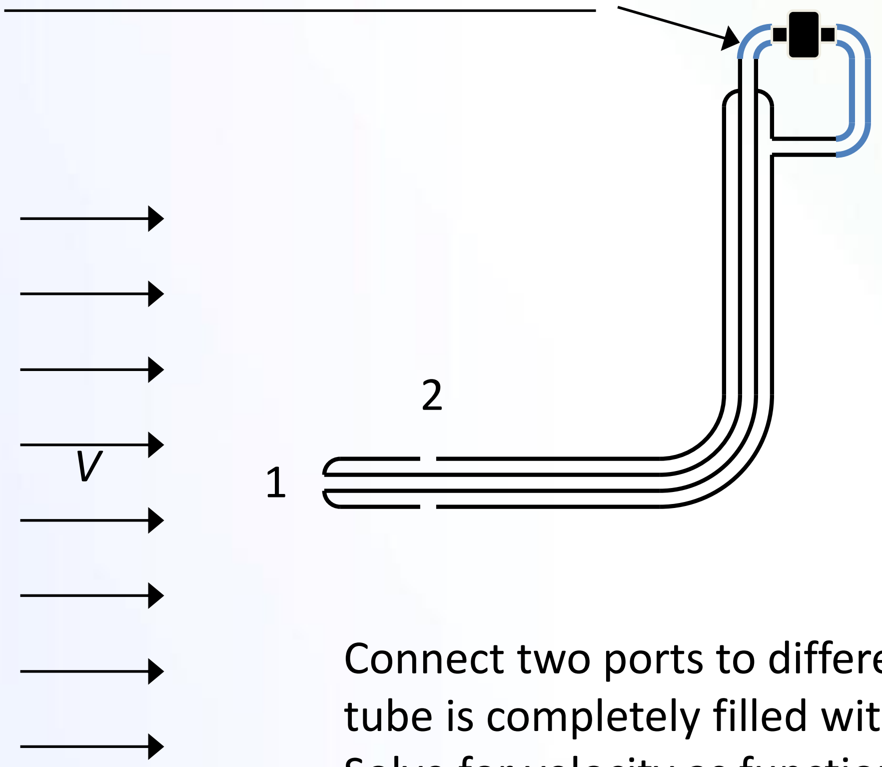
- Can connect a differential pressure transducer to directly measure $V^2/2g$.
- Can be used to measure the flow of water in pipelines.

BERNOULLI'S EQUATION APPLICATIONS: PITOT TUBE

Point measurement!



BERNOULLI'S EQUATION APPLICATIONS: PITOT TUBE



Stagnation pressure tap

Static pressure tap

$$\frac{p_1}{\gamma} + \cancel{z_1} + \cancel{\frac{V_1^2}{2g}} = \frac{p_2}{\gamma} + \cancel{z_2} + \frac{V_2^2}{2g}$$

$V_1 = \underline{0}$

$z_1 = z_2$

$$V = \sqrt{\frac{2}{\rho} \quad p_1 - p_2}$$

Connect two ports to differential pressure transducer. Make sure Pitot tube is completely filled with the fluid that is being measured.

Solve for velocity as function of pressure difference

FUNDUMENTALS OF INVISCID, INCOMPRESSIBLE FLOW

$$\cancel{\frac{\partial \rho}{\partial t}}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\cancel{\frac{\partial \rho w}{\partial z}}=0 \Rightarrow \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

INCOMPRESSIBLE **2-D**

Irrotational

$$\Rightarrow \frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0$$

VELOCITY POTENTIAL

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0$$

Velocity Potential is defined as: $u=\frac{\partial \phi}{\partial x}, v=\frac{\partial \phi}{\partial y}$

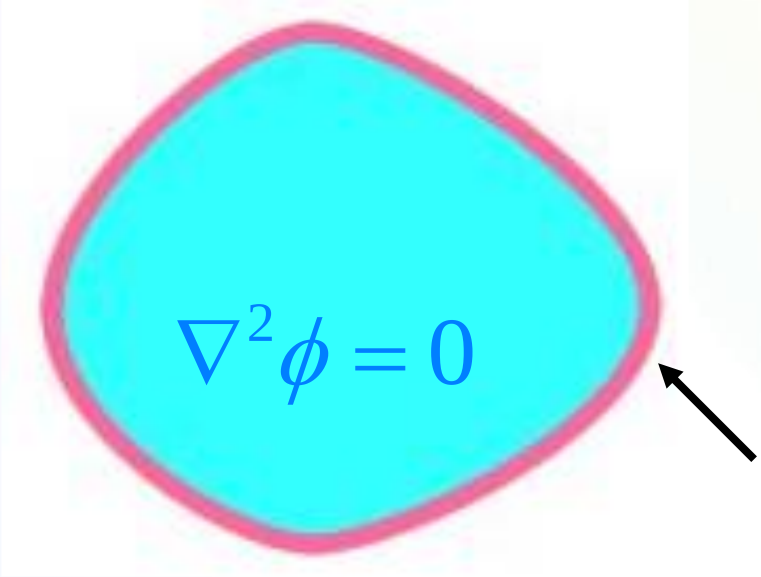
1- ϕ automatically satisfies the Irrotationality condition.

2- If it has to meet the continuity requirement it has to obey,

$$\nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

VELOCITY POTENTIAL

Thus the problem is reduced to that of finding ϕ.

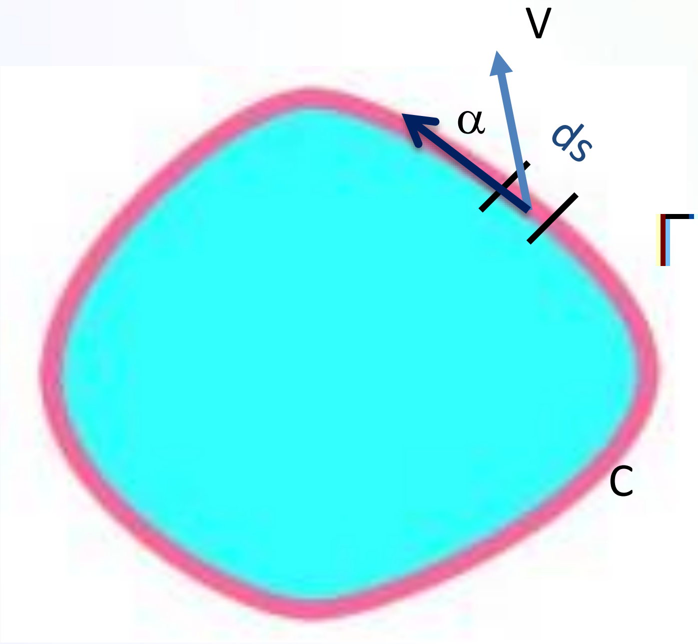


Boundary Condition
On the boundary

$$\phi = c \text{ or } \frac{\partial\phi}{\partial \mathrm{n}} = d$$

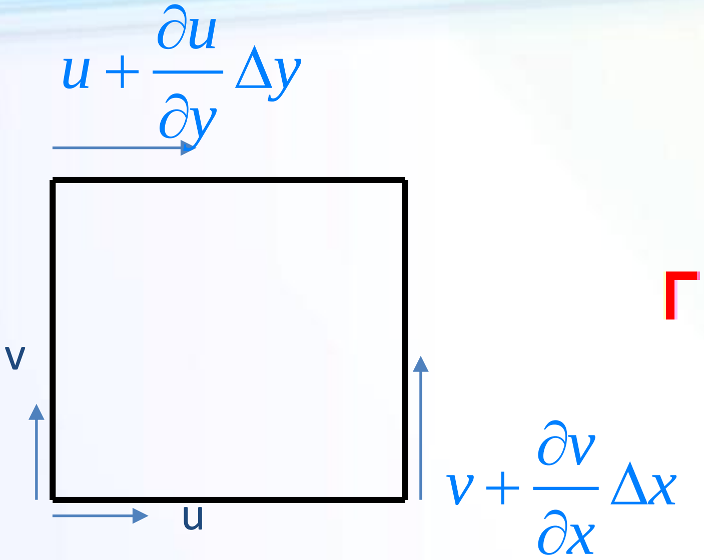
CIRCULATION

Circulation, Γ is defined as the line integral of tangential velocity component around a closed curve in the flow.



$$\Gamma = \oint_c \vec{V} \cdot \overrightarrow{ds} = \oint_c V \cos \propto ds$$

CIRCULATION



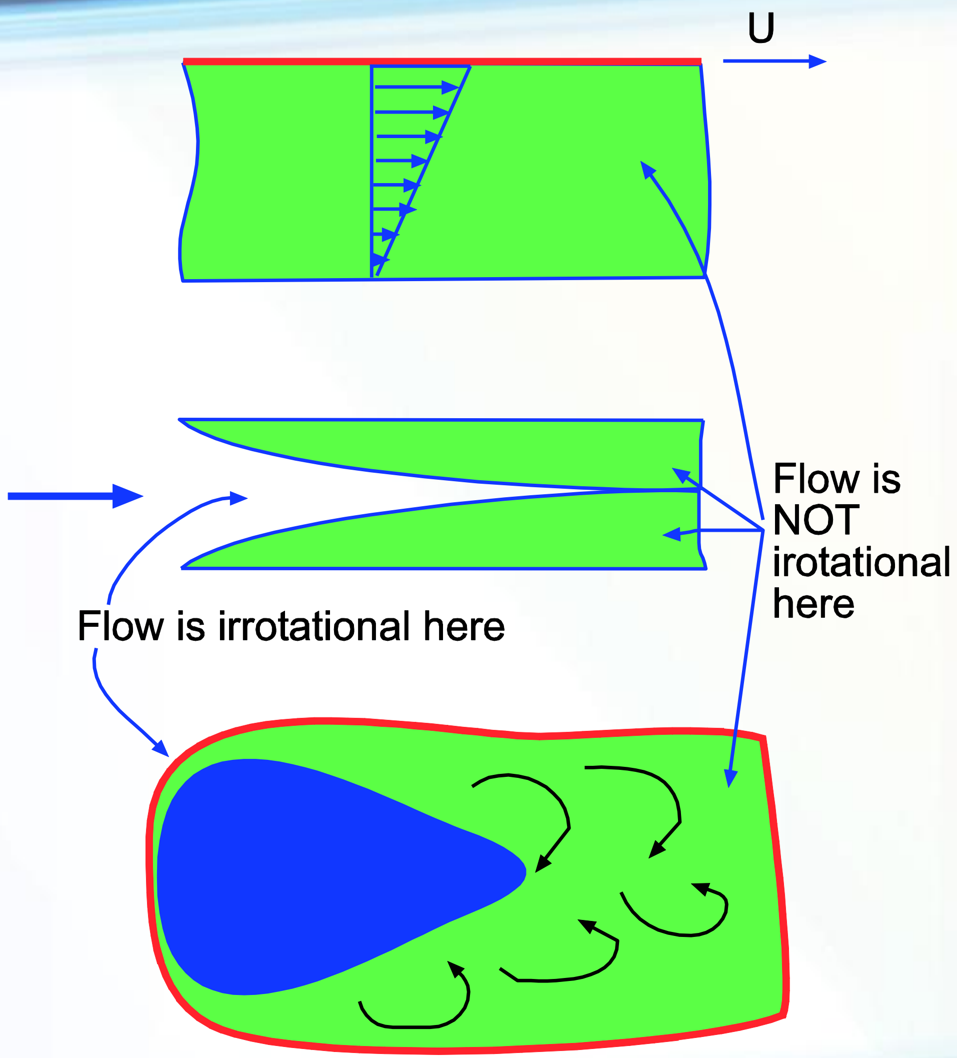
$$\Gamma = \oint_c \vec{V}\cdot\vec{ds} = \iint_A \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\Delta x\Delta y$$

Kelvin-Stokes theorem:

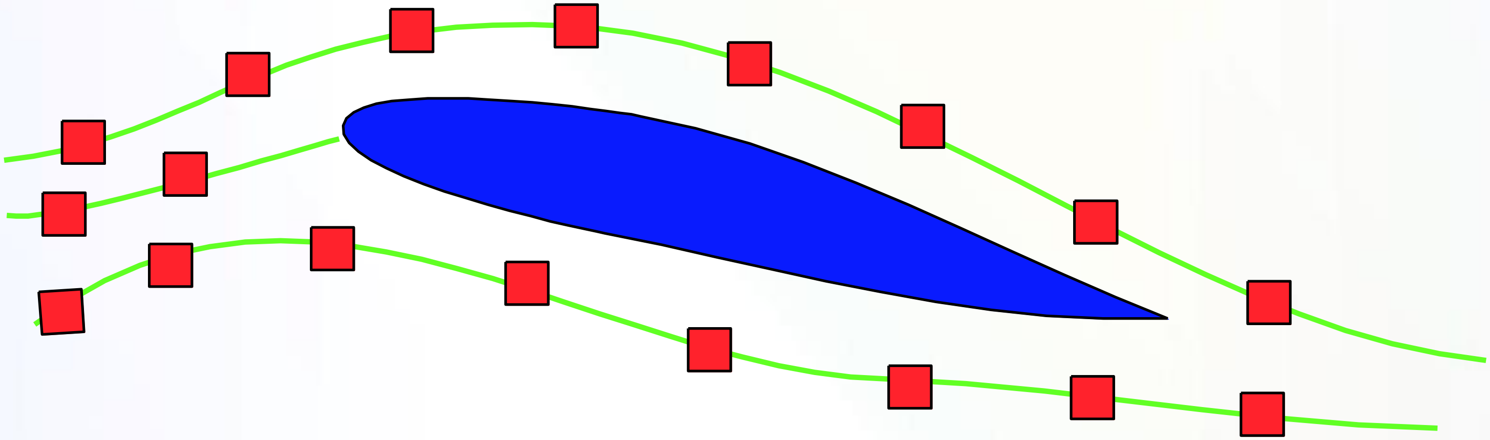
$$\Gamma = \oint_C \mathbf{V}\cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{V})\cdot d\mathbf{S}$$

Thus for an irrotational flow circulation around any closed contour is zero.

CIRCULATION



CIRCULATION



STREAM FUNCTION, ψ

Stream Function, ψ is defined such that

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}$$

ψ = constant, denotes a streamline.

STREAM FUNCTION, ψ

For irrotationality, we have

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

PROPERTIES OF STREAMLINES

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy$$

$$d\psi = 0 \Rightarrow 0 = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy \Rightarrow -\frac{\overset{-v}{\cancel{\dfrac{\partial\psi}{\partial x}}}}{\underset{u}{\cancel{\dfrac{\partial\psi}{\partial y}}}} = \frac{dy}{dx}$$

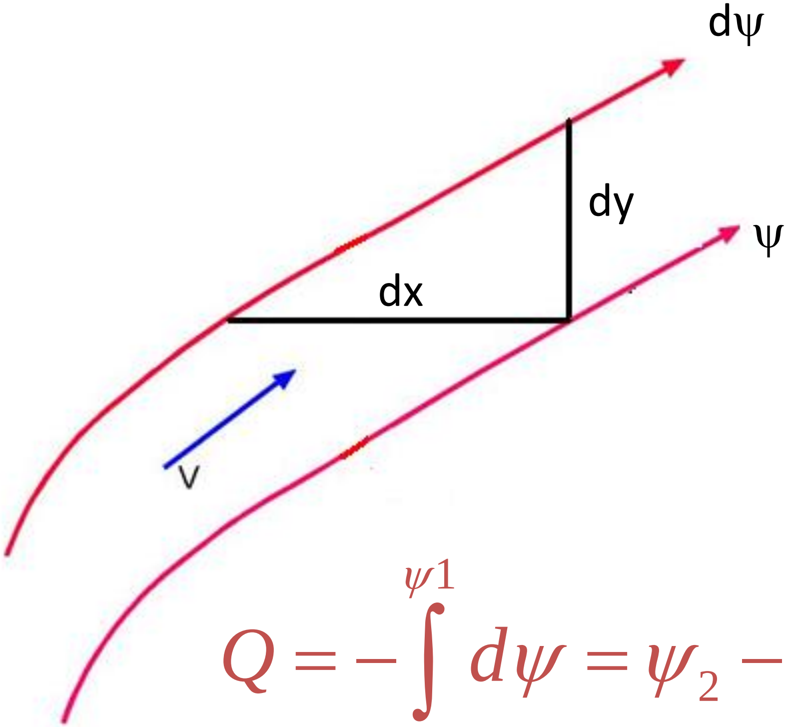
$$\Rightarrow \boxed{\frac{v}{u} = \frac{dy}{dx}}$$

PROPERTIES OF STREAMLINES

Difference dψ between successive streamlines is proportional to volumetric flow rate.

$$-vdx + udy = dQ$$

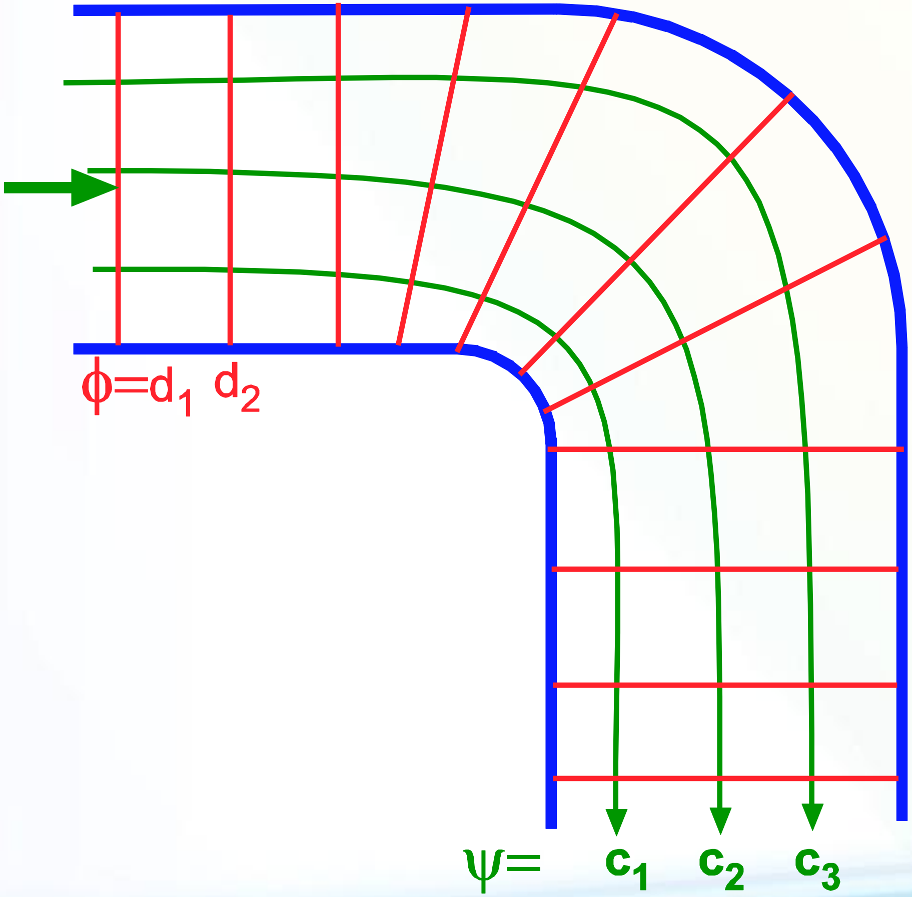
$$\underbrace{\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy}_{d\psi = dQ} = dQ$$



$$Q = -\int_{\psi 2}^{\psi 1} d\psi = \psi_2 - \psi_1$$

PROPERTIES OF STREAMLINES

Streamlines and velocity potential lines are normal to each other.



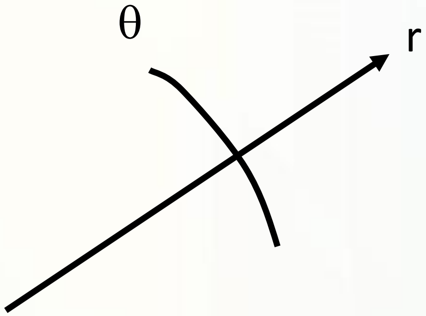
CYLINDRICAL COORDINATE SYSTEM

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, v_\theta = -\frac{\partial \psi}{\partial r}$$

$$v_r = \frac{\partial \phi}{\partial r}, v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$\nabla \phi = \frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$



CAUCHY-REIMANN EQUATIONS

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y};\quad \frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}$$

$$\frac{\partial\phi}{\partial r}=\frac{1}{r}\frac{\partial\psi}{\partial\theta};\quad \frac{1}{r}\frac{\partial\phi}{\partial\theta}=-\frac{\partial\psi}{\partial r}$$

ELEMENTARY PLANE FLOWS

Elemantary plane flows:

✓ Uniform flow

✓ Source / Sink flow

✓ Doublet Flow

✓ Vortex flow

ELEMENTARY PLANE FLOWS

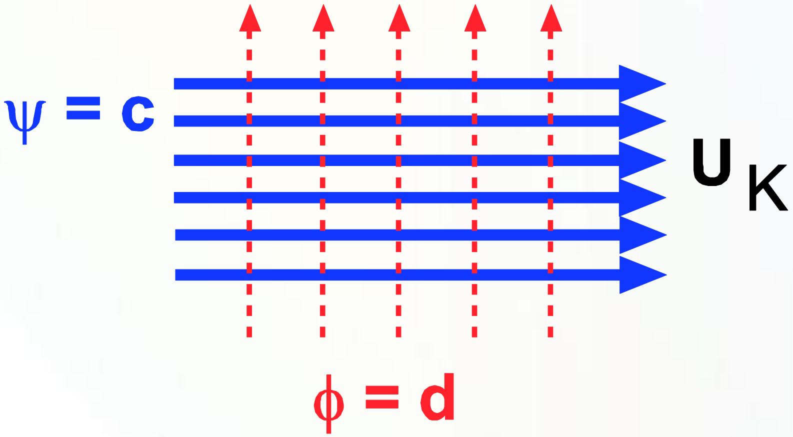
- ✓ Uniform flow

$$\psi = U_\infty y$$

$$\phi = U_\infty x$$

$$\psi = (U_\infty \cos\alpha) y - (U_\infty \sin\alpha) x$$

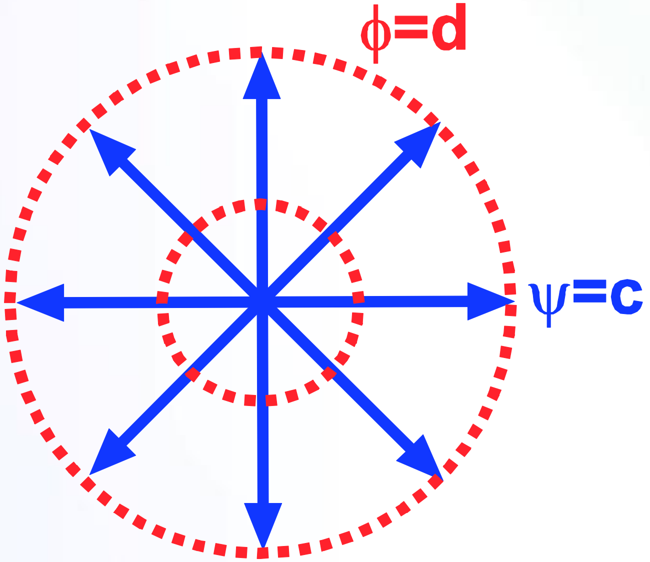
$$\phi = (U_\infty \cos\alpha) x + (U_\infty \sin\alpha) y$$



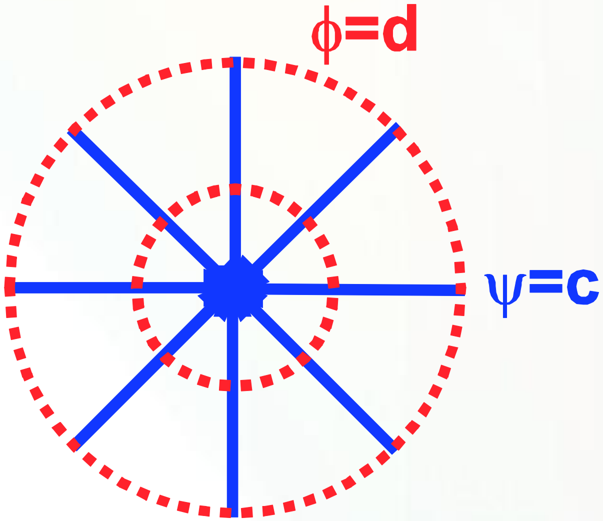
ELEMENTARY PLANE FLOWS

- ✓ Source/Sink flow

$$\phi = \frac{q}{2\pi} \ln r \qquad \psi = \frac{q}{2\pi} \theta$$



Source



Sink

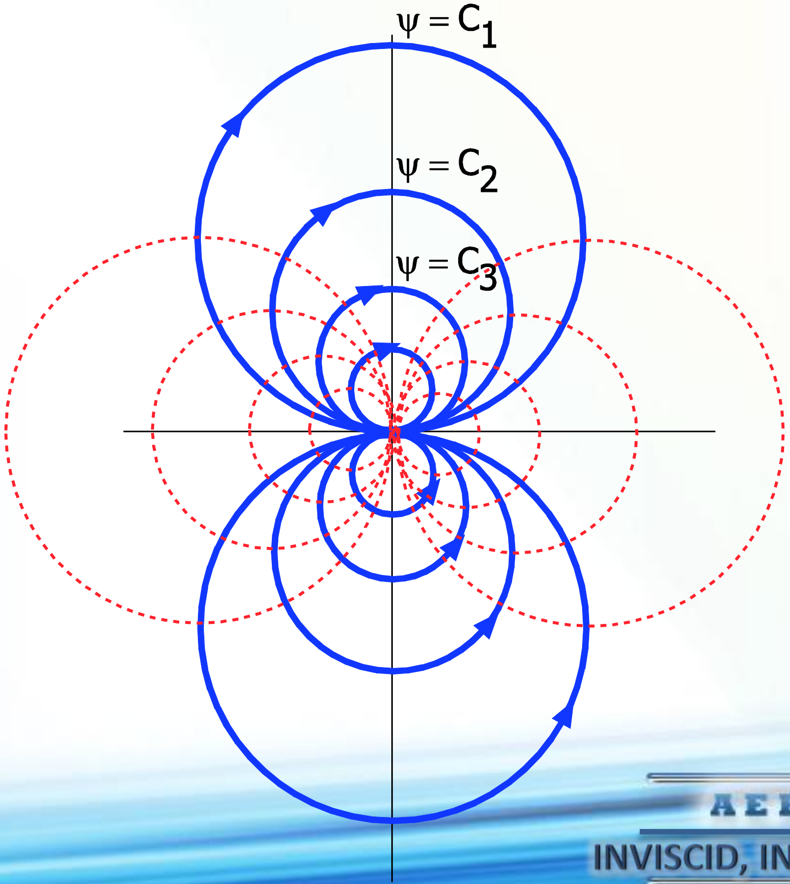
SUPERPOSITION OF ELEMENTARY PLANE FLOWS

✓ Doublet

Source and Sink approach each other ie., a→o, But qa/π is constant or finite.

$$\phi = \frac{\Lambda}{r}\cos\theta$$

$$\psi = -\frac{\Lambda}{r}\sin\theta$$

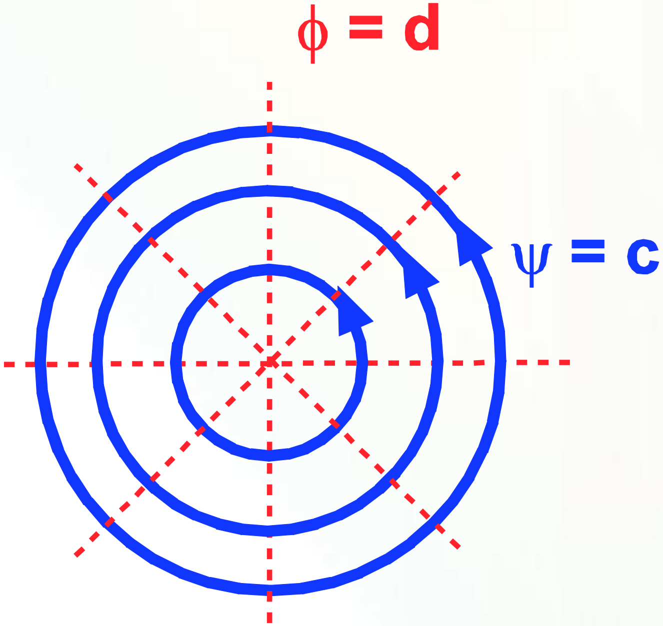


ELEMENTARY PLANE FLOWS

- ✓ Vortex flow

$$\psi = \frac{k}{2\pi} \ln r$$

$$\phi = \frac{k}{2\pi} \theta$$



SUPERPOSITION OF ELEMENTARY PLANE FLOWS

Laplace Equation is linear. So if ϕ_1 and ϕ_2 are two solutions, $\phi_3 = \phi_1 \pm \phi_2$ is also a solution.

Simple flows are superposed to calculate more complex flows.

NOTE: A solid wall is also a streamline. This helps us locate solid boundaries.

POTENTIAL FLOW: PLANE POTENTIAL FLOWS

Laplace's equation is a second-order *linear* Partial Differential Equation. The fact that the Laplace's equation is linear is particularly important, because linear superposition of solutions is allowed:

$\phi_3 = \phi_1 + \phi_2$ where $\phi_1(x, y, z)$ and $\phi_2(x, y, z)$ are solutions of Laplace's equation

For simplicity, we consider 2D (planar) flows:

Cartesian: $u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y}$

Cylindrical: $v_r = \frac{\partial \phi}{\partial r} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$

We note that the stream functions also exist for 2D planar flows:

Cartesian: $u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$

Cylindrical: $v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r}$

POTENTIAL FLOW: PLANE POTENTIAL FLOWS

For irrotational, planar flow: $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$

Now substitute the stream function: $\dfrac{\partial}{\partial y}\left(\dfrac{\partial \psi}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(-\dfrac{\partial \psi}{\partial x}\right)$

Then, $\dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2} = 0 \Longrightarrow$ Laplace’s Equation

For plane, irrotational flow, we use either the potential or the stream function, which both must satisfy Laplace’s equations in two dimensions.

Lines of constant Ψ are streamlines: $\left.\dfrac{dy}{dx}\right|_{\text{along } \psi = \text{constant}} = \dfrac{v}{u}$

Now, the change of ϕ from one point *(x, y)* to a nearby point *(x + dx, y + dy)* is:

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = u\,dx + v\,dy$$

Along lines of constant ϕ we have dϕ =0,

$$\cancelto{0}{d\phi} = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = u\,dx + v\,dy \Longrightarrow \left.\frac{dy}{dx}\right|_{\text{along } \phi = \text{constant}} = -\frac{u}{v}$$

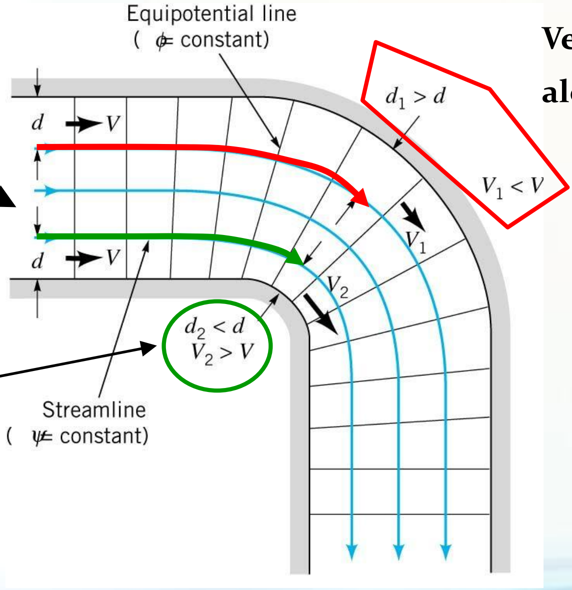
POTENTIAL FLOW: PLANE POTENTIAL FLOWS

Lines of constant ϕ are called Equipotential lines.

The Equipotential lines are orthogonal to lines of constant Ψ (streamlines) where they intersect.

The flow net consists of a family of streamlines and equipotential lines.

The combination of streamlines and equipotential lines are used to visualize a graphical flow situation.



The velocity is inversely proportional to the spacing between streamlines.

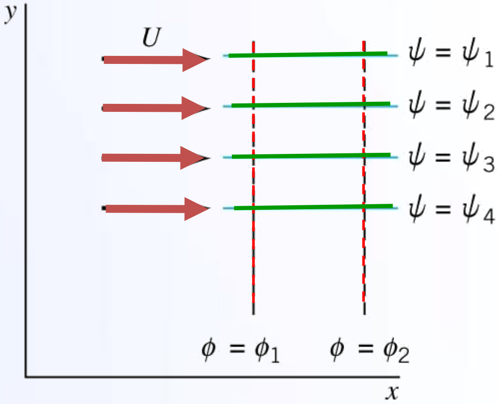
Velocity decreases along this streamline.

Velocity increases along this streamline.

POTENTIAL FLOW: UNIFORM FLOW

The simplest plane potential flow is a uniform flow in which the streamlines are all parallel to each other.

Consider a uniform flow in the x-direction:



Integrate the two equations:

$$\frac{\partial \phi}{\partial x} = U \longrightarrow \phi = Ux + f(y) + C$$

$$\frac{\partial \phi}{\partial y} = 0 \longrightarrow \phi = f(x) + C$$

Matching the solution $\phi = Ux + C$

C is an arbitrary constant, can be set to zero: $\boxed{\phi = Ux}$

Now for the stream function solution:

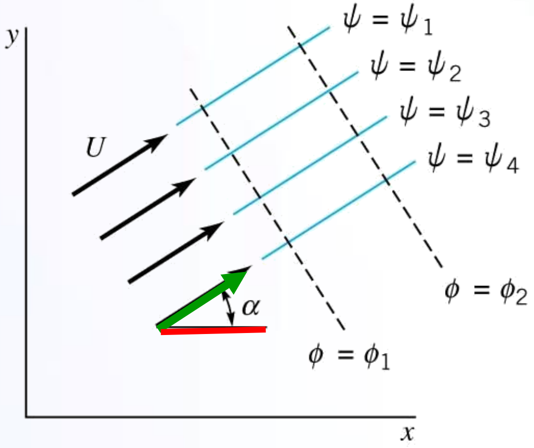
$$\frac{\partial \psi}{\partial y} = U$$

$$\frac{\partial \psi}{\partial x} = 0$$

Integrating the two equations similar to above. $\longrightarrow$ $\boxed{\psi = Uy}$

POTENTIAL FLOW: UNIFORM FLOW

For Uniform Flow in an Arbitrary direction, α :

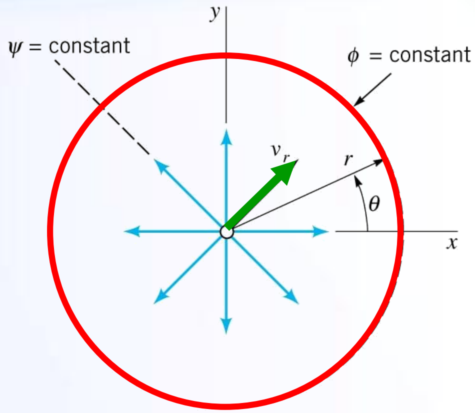


$$\boxed{\phi = U(x \cos \alpha + y \sin \alpha)}$$

$$\boxed{\psi = U(y \cos \alpha - x \sin \alpha)}$$

POTENTIAL FLOW: SOURCE/SINK FLOW

Source Flow:



Source/Sink Flow is a purely radial flow.

Fluid is flowing radially from a line through the origin perpendicular to the x-y plane.

Let *m* be the volume rate emanating from the line (per unit length).

Then, to satisfy mass conservation:

$$(2\pi r)v_r = m \Rightarrow v_r = \frac{m}{2\pi r}$$

Since the flow is purely radial: $v_\theta = 0$

Now, the velocity potential can be obtained:

$$v_r = \frac{\partial \phi}{\partial r} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \Rightarrow \frac{\partial \phi}{\partial r} = \frac{m}{2\pi r} \qquad \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0$$

$$\downarrow \qquad \downarrow$$

$$v_r = \frac{m}{2\pi r} \qquad 0$$

Integrate $\Rightarrow$ $\boxed{\phi = \frac{m}{2\pi}\ln r}$

m is the strength of the source or sink!

If m is positive, the flow is radially outward, source flow.
If m is negative, the flow is radially inward, sink flow.

This potential flow does not exist at r = 0, the origin, because it is not a “real” flow, but can approximate flows.

POTENTIAL FLOW: SOURCE/SINK FLOW

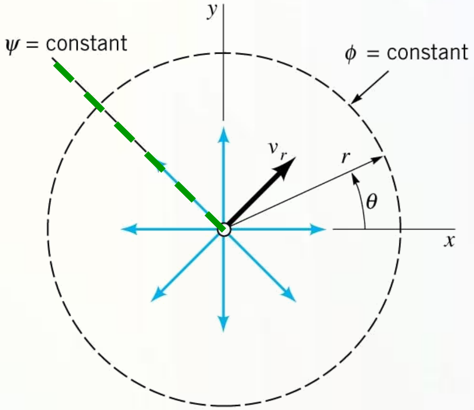
Now, obtain the stream function for the flow:

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r} \qquad \Rightarrow \qquad \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{m}{2\pi r} \qquad \frac{\partial \psi}{\partial r} = 0$$

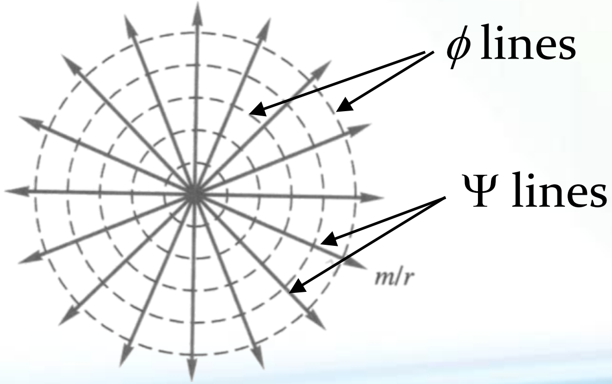
$$\downarrow \qquad\qquad\qquad \downarrow$$

$$v_r = \frac{m}{2\pi r} \qquad\qquad 0$$

Then, integrate to obtain the solution: $\boxed{\psi = \frac{m}{2\pi}\theta}$

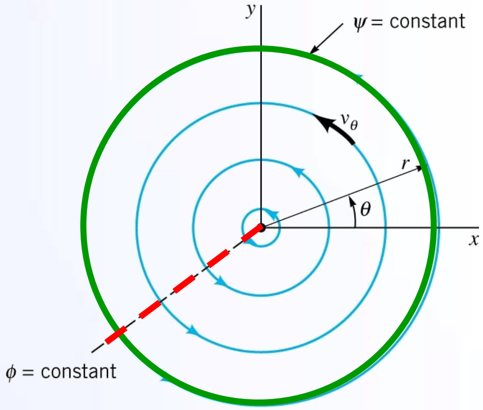


The streamlines are radial lines and the equipotential lines are concentric circles centered about the origin:



POTENTIAL FLOW: VORTEX FLOW

In vortex flow, the streamlines are concentric circles, and the equipotential lines are radial lines.



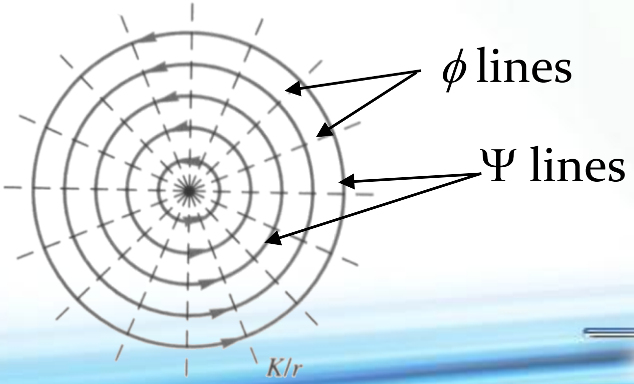
Solution: $\phi = K\theta$ $\psi = -K \ln r$

where K is a constant.

The sign of K determines whether the flow rotates clockwise or counterclockwise.

In this case: $v_r = 0$, $v_\theta = \dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} = -\dfrac{\partial\psi}{\partial r} = \dfrac{K}{r}$

The tangential velocity varies inversely with the distance from the origin. At the origin it encounters a singularity becoming infinite.

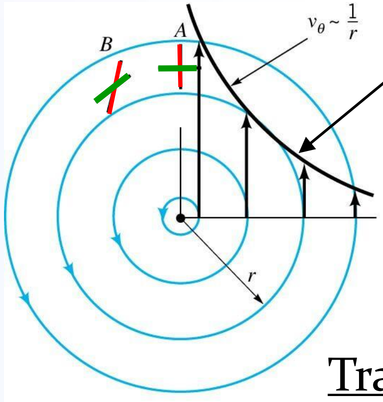


POTENTIAL FLOW: VORTEX FLOW

How can a vortex flow be irrotational?

Rotation refers to the orientation of a fluid element and not the path followed by the element.

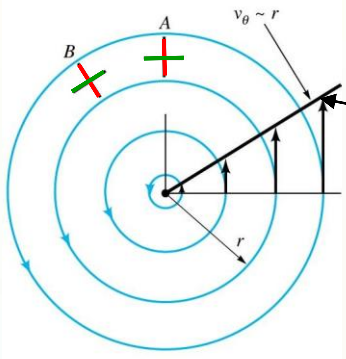
Irrotational Flow: Free Vortex



Velocity increases inward.

i.e., water draining from a bathtub

Rotational Flow: Forced Vortex



Velocity increases outward.

i.e., a rotating tank filled with fluid

<u>Traveling from A to B, consider two sticks</u>

Irrotational Flow:

Initially, sticks aligned, one in the flow direction, and the other perpendicular to the flow.

As they move from A to B the perpendicular-aligned stick rotates clockwise, while the flow-aligned stick rotates counter clockwise.

The average angular velocities cancel each other, thus, the flow is irrotational.

Rotational Flow: Rigid Body Rotation

Initially, sticks aligned, one in the flow direction, and the other perpendicular to the flow.

As they move from A to B they sticks move in a rigid body motion, and thus the flow is rotational.

POTENTIAL FLOW: VORTEX FLOW

A combined vortex flow is one in which there is a forced vortex at the core, and a free vortex outside the core.

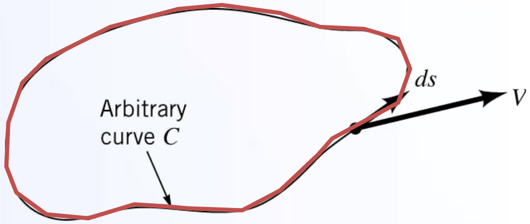
$$v_\theta = \omega r \qquad r \le r_0$$

$$v_\theta = \frac{K}{r} \qquad r > r_0$$

A Hurricane is approximately a combined vortex

Circulation is a quantity associated with vortex flow. It is defined as the line integral of the tangential component of the velocity taken around a closed curve in the flow field.

$$\Gamma = -\int_C V \cdot ds$$



$$\mathbf{V} = \nabla\phi \longrightarrow \mathbf{V} \cdot d\mathbf{s} = \nabla\phi \cdot d\mathbf{s} = d\phi$$

$$\Gamma = \oint_C d\phi = 0 \longrightarrow$$

For irrotational flow the circulation is generally zero.

POTENTIAL FLOW: VORTEX FLOW

However, if there are singularities in the flow, the circulation is not zero if the closed curve includes the singularity.

For the free vortex: $v_\theta = \frac{K}{r}$

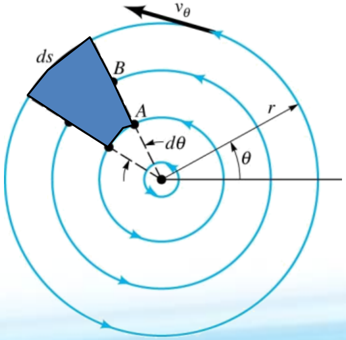
$$\Gamma = \int_0^{2\pi} -\frac{K}{r}(rd\theta) = -2\pi K$$

The circulation is non-zero and constant for the free vortex: $K = -\Gamma/2\pi$

The velocity potential and the stream function can be rewritten in terms of the circulation:

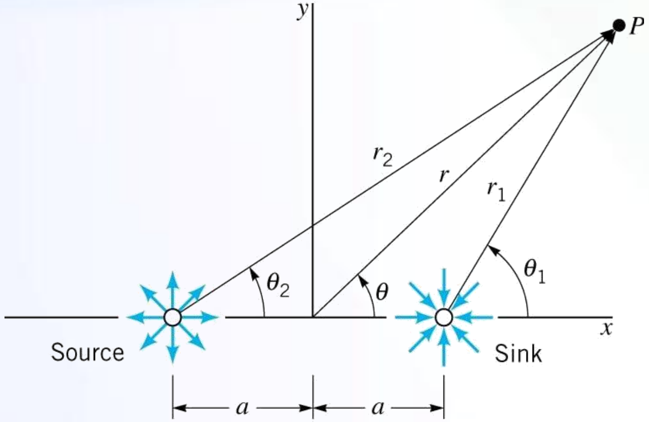
$$\phi = -\frac{\Gamma}{2\pi}\theta \qquad \psi = \frac{\Gamma}{2\pi}\ln r$$

An example in which the closed surface circulation will be zero:



POTENTIAL FLOW: DOUBLET FLOW

Combination of a Equal Source and Sink Pair:



$$\psi = -\frac{m}{2\pi}(\theta_1 - \theta_2)$$

Rearrange and take tangent,

$$\tan\left(-\frac{2\pi\psi}{m}\right) = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

Note, the following: $\tan\theta_1 = \frac{r\sin\theta}{r\cos\theta - a}$ and $\tan\theta_2 = \frac{r\sin\theta}{r\cos\theta + a}$

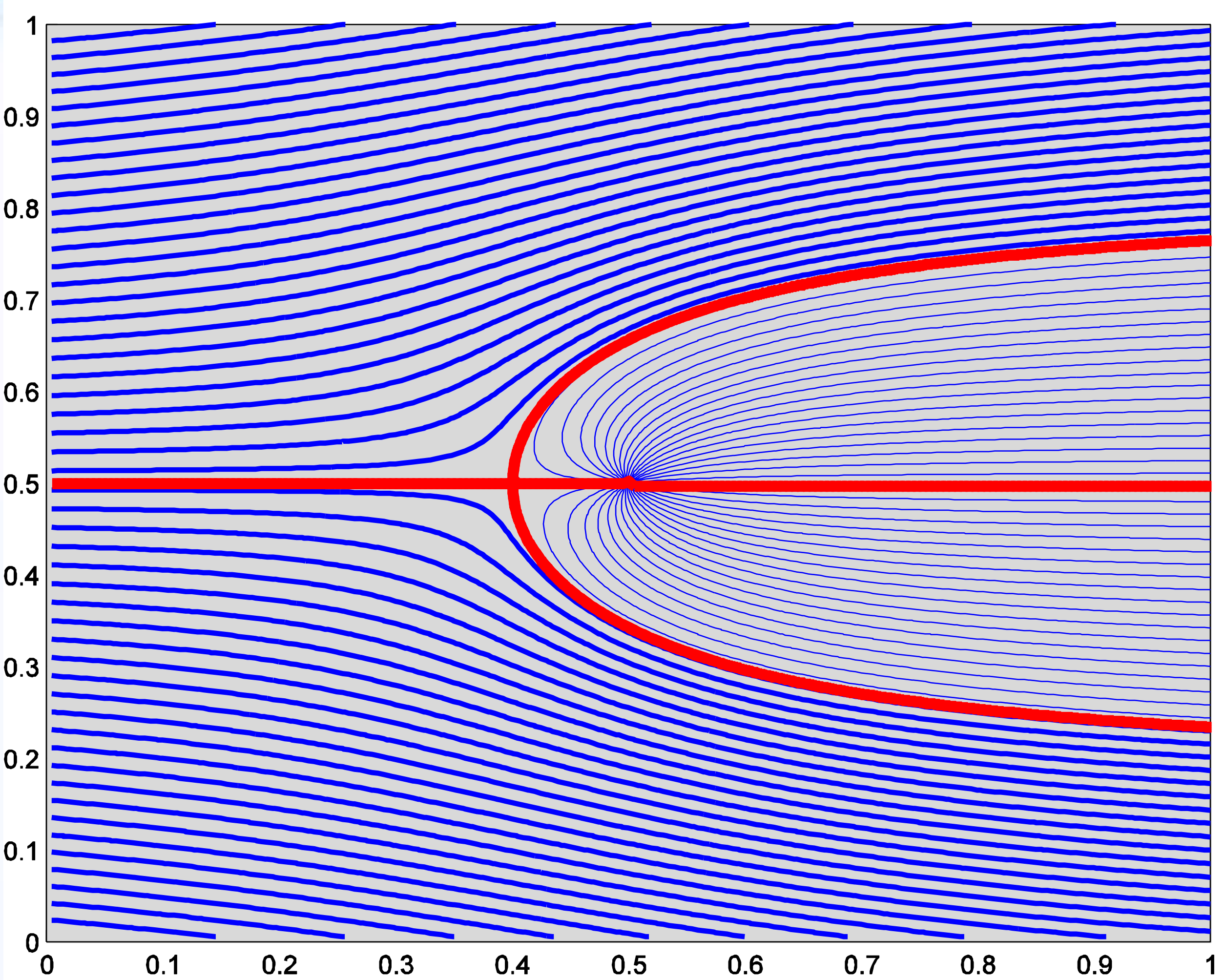
Substituting the above expressions, $\tan\left(-\frac{2\pi\psi}{m}\right) = \frac{2ar\sin\theta}{r^2 - a^2}$

Then, $\psi = -\frac{m}{2\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{r^2 - a^2}\right)$

If a is small, then tangent of angle is approximated by the angle:

$$\psi = -\frac{m}{2\pi}\frac{2ar\sin\theta}{r^2 - a^2} = -\frac{mar\sin\theta}{\pi(r^2 - a^2)}$$

POTENTIAL FLOW: DOUBLET FLOW



POTENTIAL FLOW: DOUBLET FLOW

$$\psi = -\frac{mar \sin \theta}{\pi(r^2 - a^2)}$$

Now, we obtain the doublet flow by letting the source and sink approach one another, and letting the strength increase.

$a \rightarrow 0$

$m \rightarrow \infty$

ma/π is then constant.

$r/(r^2 - a^2) \rightarrow 1/r$

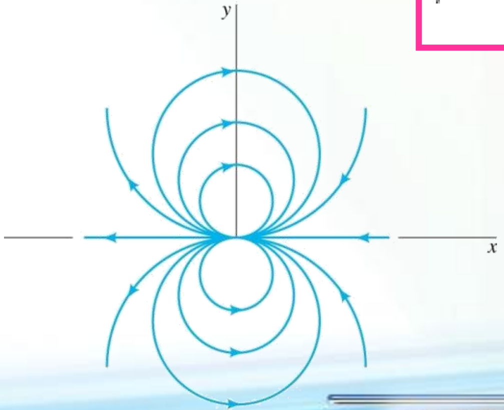
$$\boxed{\psi = -\frac{K \sin \theta}{r}}$$

K is the strength of the doublet, and is equal to ma/π.

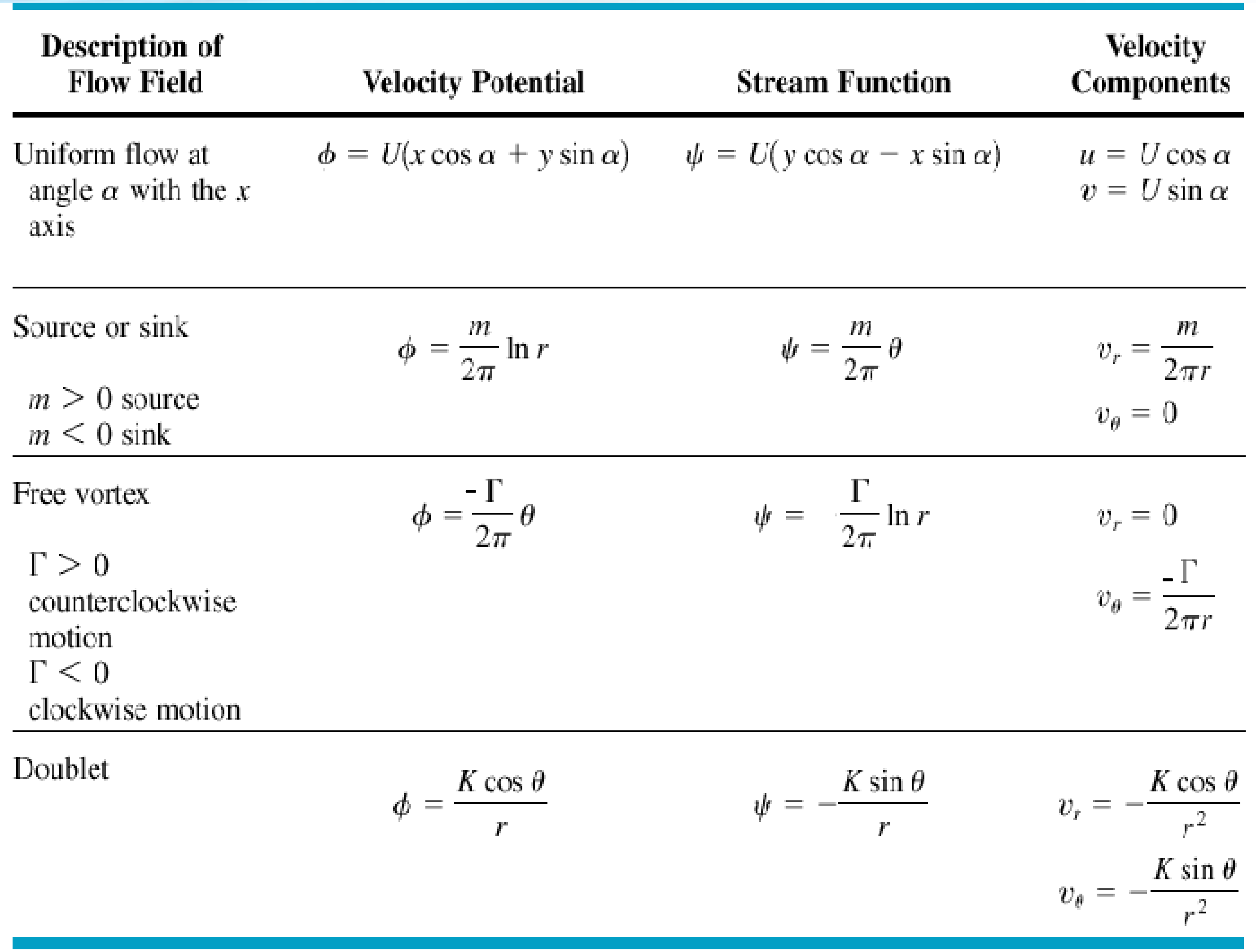
The corresponding velocity potential then is the following:

$$\boxed{\phi = \frac{K \cos \theta}{r}}$$

Streamlines of a Doublet:



POTENTIAL FLOW: SUMMARY OF BASIC FLOWS



Description of Flow Field	Velocity Potential	Stream Function	Velocity Components
Uniform flow at angle α with the x axis	$\phi = U(x \cos \alpha + y \sin \alpha)$	$\psi = U(y \cos \alpha - x \sin \alpha)$	$u = U \cos \alpha$ $v = U \sin \alpha$
Source or sink $m > 0$ source $m < 0$ sink	$\phi = \frac{m}{2\pi} \ln r$	$\psi = \frac{m}{2\pi} \theta$	$v_r = \frac{m}{2\pi r}$ $v_\theta = 0$
Free vortex $\Gamma > 0$ counterclockwise motion $\Gamma < 0$ clockwise motion	$\phi = \frac{-\Gamma}{2\pi} \theta$	$\psi = \frac{\Gamma}{2\pi} \ln r$	$v_r = 0$ $v_\theta = \frac{-\Gamma}{2\pi r}$
Doublet	$\phi = \frac{K \cos \theta}{r}$	$\psi = -\frac{K \sin \theta}{r}$	$v_r = -\frac{K \cos \theta}{r^2}$ $v_\theta = -\frac{K \sin \theta}{r^2}$

POTENTIAL FLOW: SUPERPOSITION OF BASIC FLOWS

Because Potential Flows are governed by linear partial differential equations, the solutions can be combined in superposition.

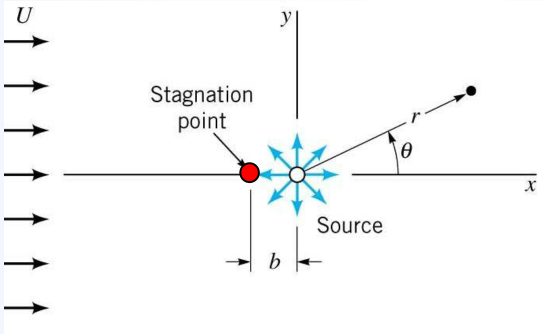
Any streamline in an inviscid flow acts as solid boundary, such that there is no flow through the boundary or streamline.

Thus, some of the basic velocity potentials or stream functions can be combined to yield a streamline that represents a particular body shape.

The superposition representing a body can lead to describing the flow around the body in detail.

SUPERPOSITION OF POTENTIAL FLOWS: RANKINE HALF-BODY

The Rankine Half-Body is a combination of a source and a uniform flow.



Stream Function (cylindrical coordinates):

$$\psi = \psi_{\text{uniform flow}} + \psi_{\text{source}}$$

$$= Ur\sin\theta + \frac{m}{2\pi}\theta$$

Potential Function (cylindrical coordinates):

$$\phi = Ur\cos\theta + \frac{m}{2\pi}\ln r$$

There will be a stagnation point, somewhere along the negative x-axis where the source and uniform flow cancel ($\theta = \pi$):

Evaluate the radial velocity: $v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$

For the source: $v_r = \frac{m}{2\pi r}$

For the uniform flow: $v_r = U\cos\theta$

For $\theta = \pi$, $v_r = U$

Then for a stagnation point, at some $r = -b$, $\theta = \pi$:

$$v_r = -\frac{m}{2\pi} \quad \text{and} \quad U = \frac{m}{2\pi b} \Longrightarrow b = \frac{m}{2\pi U}$$

SUPERPOSITION OF POTENTIAL FLOWS: RANKINE HALF-BODY

Now, the stagnation streamline can be defined by evaluating ψ at r = b, and θ = π.

$$\psi_{\text{stagnation}} = \frac{m}{2}$$

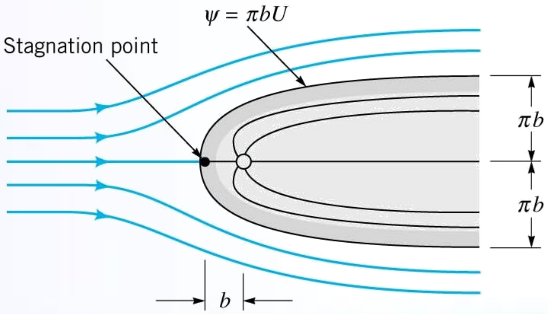
Now, we note that m/2 = πbU, so following this constant streamline gives the outline of the body:

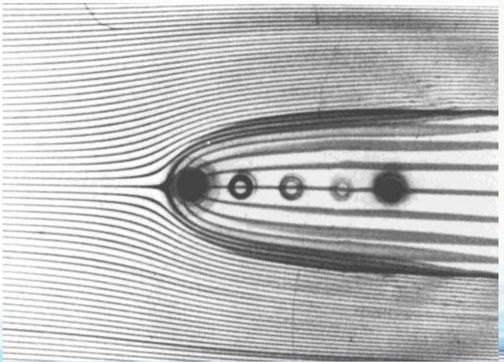
$$\psi = \psi_{\text{uniform flow}} + \psi_{\text{source}} \Longrightarrow \pi b U = U r \sin\theta + b U \theta$$

Then, $r = \dfrac{b(\pi - \theta)}{\sin\theta}$ describes the half-body outline.

So, the source and uniform flows can be used to describe an aerodynamic body.

The other streamlines can be obtained by setting y constant and plotting:





SUPERPOSITION OF POTENTIAL FLOWS: RANKINE HALF-BODY

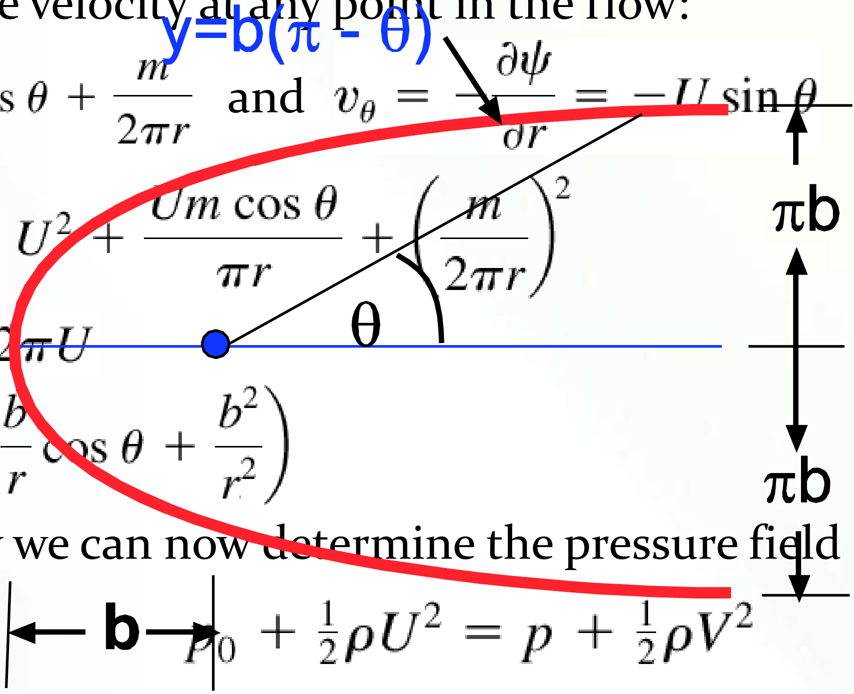
The width of the half-body:

$$y = b(\pi - \theta)$$

$$\theta \to 0 \text{ or } \theta \to 2\pi \Longrightarrow \pm b\pi$$

Total width then, $2\pi b$

The magnitude of the velocity at any point in the flow:



$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\cos\theta + \frac{m}{2\pi r} \quad \text{and} \quad v_\theta = -\frac{\partial\psi}{\partial r} = -U\sin\theta$$

$$V^2 = v_r^2 + v_\theta^2 = U^2 + \frac{Um\cos\theta}{\pi r} + \left(\frac{m}{2\pi r}\right)^2$$

Noting, $b = m/2\pi U$

$$V^2 = U^2\left(1 + 2\frac{b}{r}\cos\theta + \frac{b^2}{r^2}\right)$$

Knowing, the velocity we can now determine the pressure field using the Bernoulli Equation:

$$p_0 + \frac{1}{2}\rho U^2 = p + \frac{1}{2}\rho V^2$$

p_o and U are at a point far away from the body and are known.

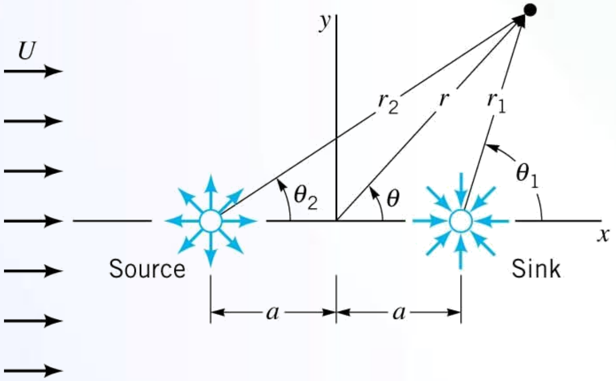
SUPERPOSITION OF POTENTIAL FLOWS: RANKINE HALF-BODY

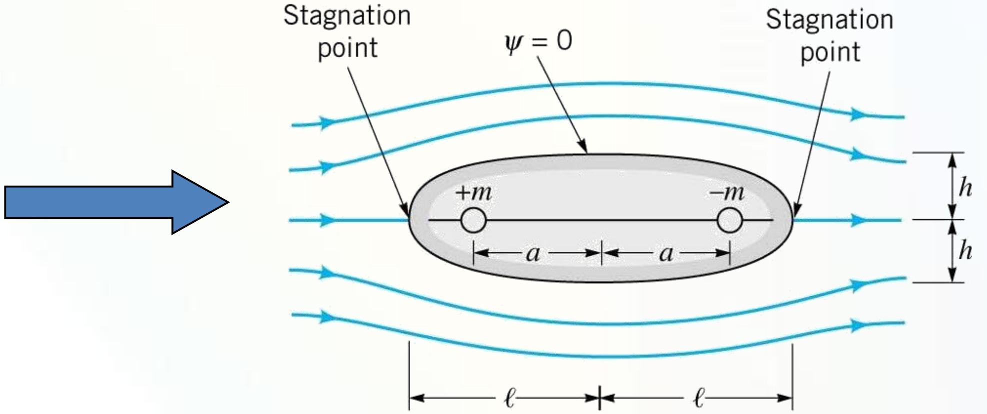
Notes on this type of flow:

- Provides useful information about the flow in the front part of streamlined body.
- A practical example is a bridge pier or a strut placed in a uniform stream
- In a potential flow the tangent velocity is not zero at a boundary, it “slips”
- The flow slips due to a lack of viscosity (an approximation result).
- At the boundary, the flow is not properly represented for a “real” flow.
- Outside the boundary layer, the flow is a reasonable representation.
- The pressure at the boundary is reasonably approximated with potential flow.
- The boundary layer is to thin to cause much pressure variation.

SUPERPOSITION OF POTENTIAL FLOWS: RANKINE OVAL

Rankine Ovals are the combination a source, a sink and a uniform flow, producing a closed body.





Some equations describing the flow:

Potential and Stream Function

$$\psi = Ur\sin\theta - \frac{m}{2\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{r^2 - a^2}\right)$$

$$\psi = Uy - \frac{m}{2\pi}\tan^{-1}\left(\frac{2ay}{x^2 + y^2 - a^2}\right)$$

$$\phi = Ur\cos\theta - \frac{m}{2\pi}(\ln r_1 - \ln r_2)$$

The body half-length

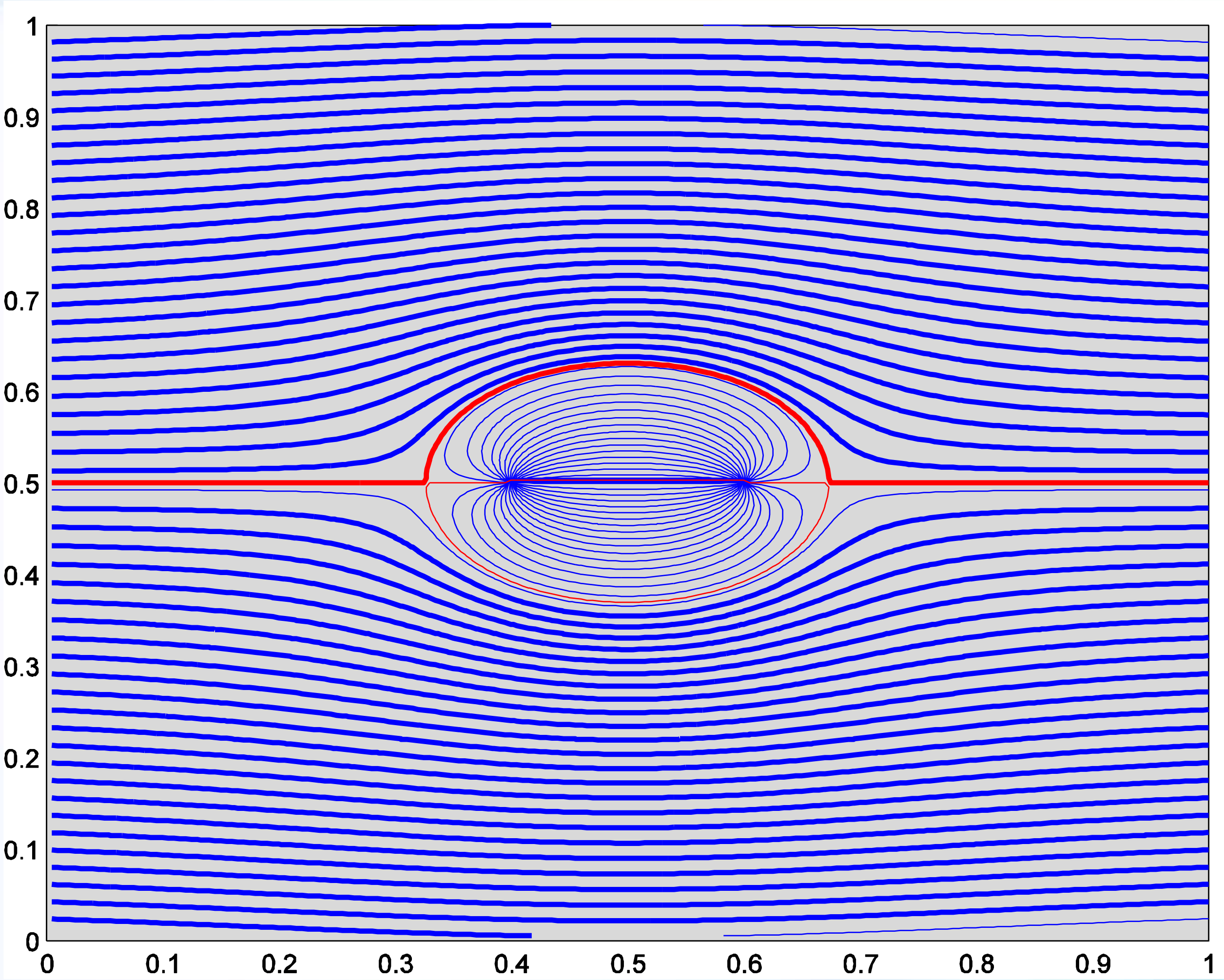
$$\ell = \left(\frac{ma}{\pi U} + a^2\right)^{1/2}$$

The body half-width

$$h = \frac{h^2 - a^2}{2a}\tan\frac{2\pi Uh}{m}$$

“Iterative”

SUPERPOSITION OF POTENTIAL FLOWS: RANKINE OVAL

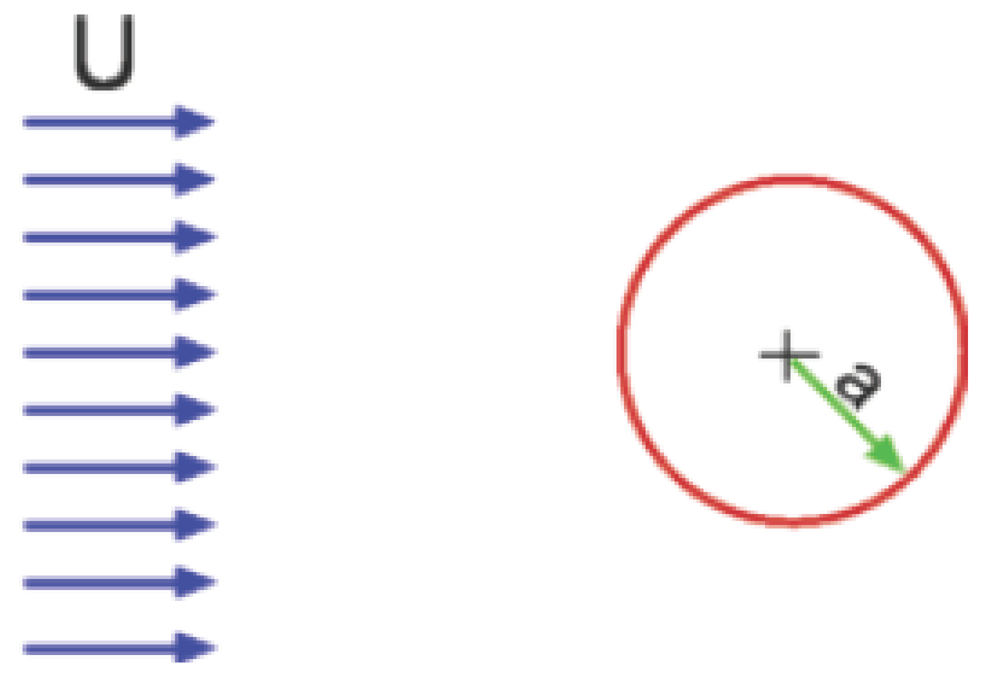


SUPERPOSITION OF POTENTIAL FLOWS: RANKINE OVAL

Notes on this type of flow:

- Provides useful information about the flow about a streamlined body.
- At the boundary, the flow is not properly represented for a “real” flow.
- Outside the boundary layer, the flow is a reasonable representation.
- The pressure at the boundary is reasonably approximated with potential flow.
- Only the pressure on the front of the body is accurate though.
- Pressure outside the boundary is reasonably approximated.

SUPERPOSITION OF POTENTIAL FLOWS: FLOW AROUND A CIRCULAR CYLINDER



SUPERPOSITION OF POTENTIAL FLOWS: FLOW AROUND A CIRCULAR CYLINDER

Combines a uniform flow and a doublet flow:

$$\psi = Ur\sin\theta - \frac{K\sin\theta}{r} \quad \text{and} \quad \phi = Ur\cos\theta + \frac{K\cos\theta}{r}$$

Then require that the stream function is constant for r = a, where a is the radius of the circular cylinder:

$$\psi = \left(U - \frac{K}{r^2}\right) r\sin\theta \quad \psi = 0 \text{ for } r = a \Rightarrow U - \frac{K}{a^2} = 0 \Rightarrow K = Ua^2$$

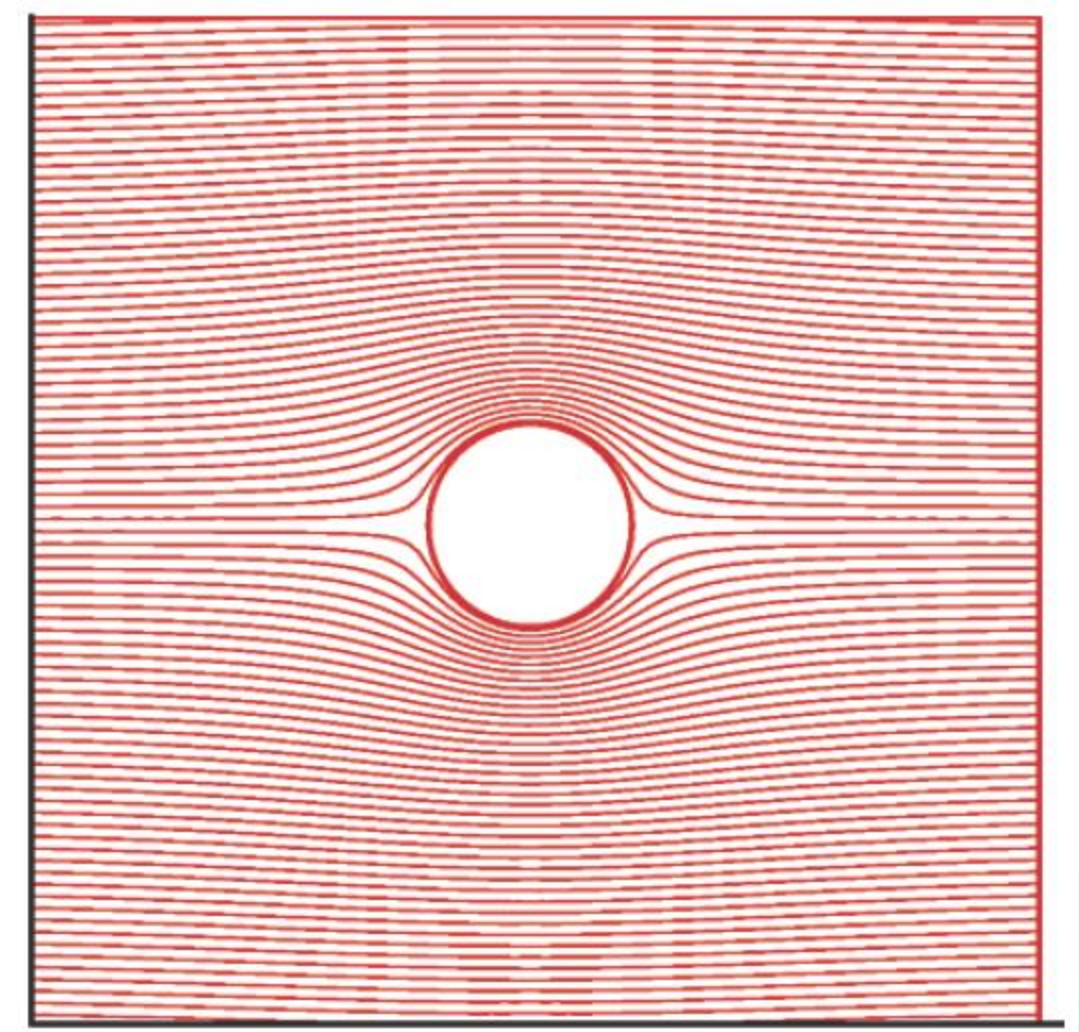
Then, $\psi = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta$ and $\phi = Ur\left(1 + \frac{a^2}{r^2}\right)\cos\theta$

Then the velocity components are:

$$v_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

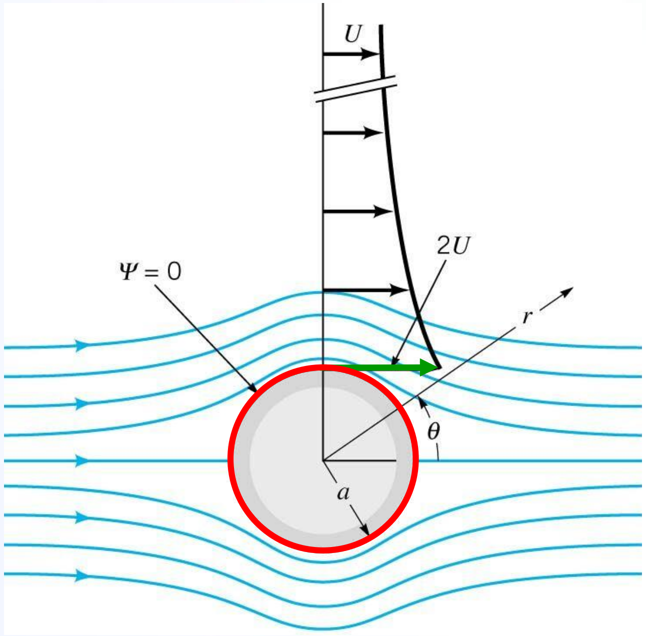
SUPERPOSITION OF POTENTIAL FLOWS: FLOW AROUND A CIRCULAR CYLINDER

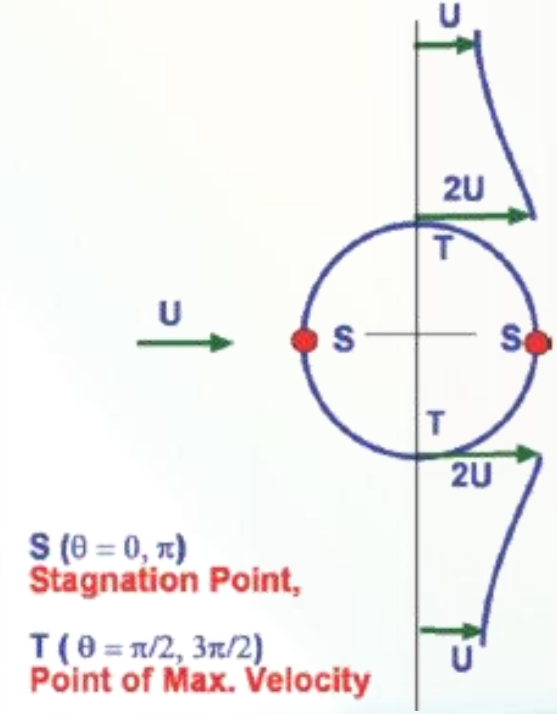


SUPERPOSITION OF POTENTIAL FLOWS: FLOW AROUND A CIRCULAR CYLINDER

At the surface of the cylinder (r = a): $v_{\theta s} = -2U\sin\theta$

The maximum velocity occurs at the top and bottom of the cylinder, of magnitude 2U.





SUPERPOSITION OF POTENTIAL FLOWS: FLOW AROUND A CIRCULAR CYLINDER

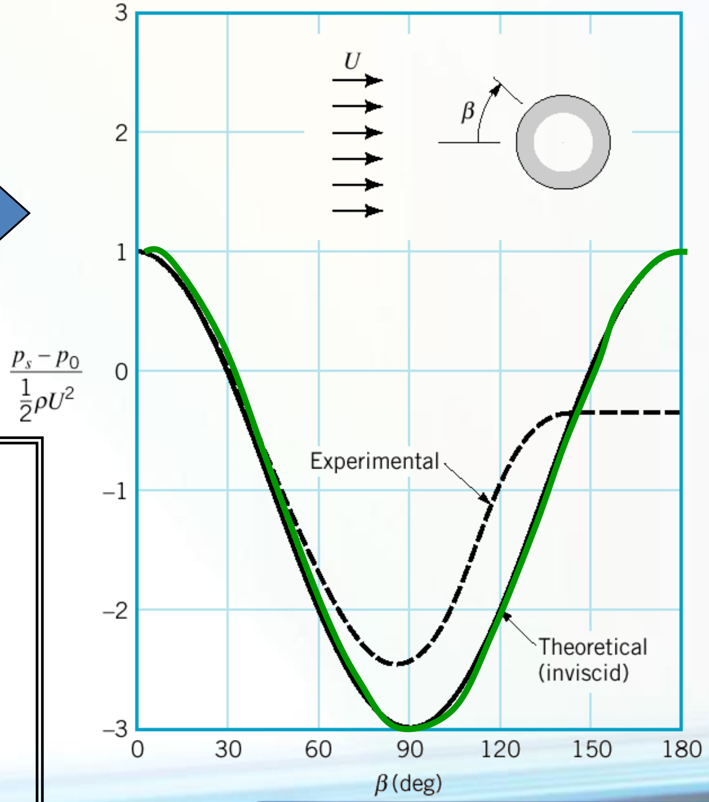
Pressure distribution on a circular cylinder found with the Bernoulli's equation

$$p_0 + \tfrac{1}{2}\rho U^2 = p_s + \tfrac{1}{2}\rho v_{\theta s}^2$$

Then substituting for the surface velocity: $v_{\theta s} = -2U \sin\theta$

$$\boxed{p_s = p_0 + \tfrac{1}{2}\rho U^2(1 - 4\sin^2\theta)}$$

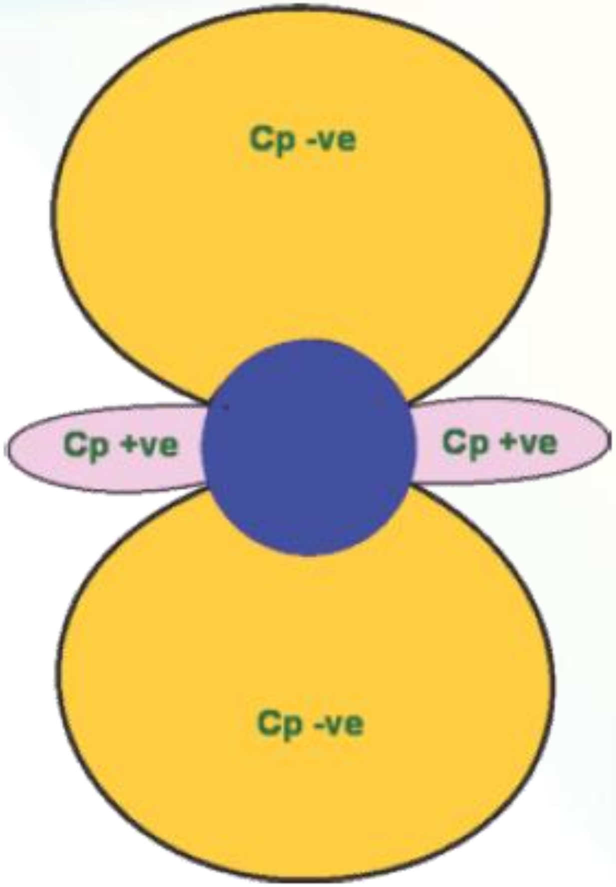
$$Cp = \frac{p_s - p_0}{\frac{1}{2}\rho U^2}$$



Theoretical and experimental results agree well on the front of the cylinder.

Flow separation on the back-half in the real flow due to viscous effects causes differences between the theory and experiment.

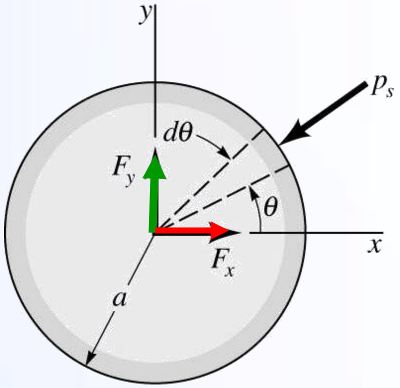
SUPERPOSITION OF POTENTIAL FLOWS: FLOW AROUND A CIRCULAR CYLINDER



Cp distribution for flow past a circular cylinder plotted around the cylinder.

SUPERPOSITION OF POTENTIAL FLOWS: FLOW AROUND A CIRCULAR CYLINDER

The resultant force per unit force acting on the cylinder can be determined by integrating the pressure over the surface (equate to lift and drag).



$$F_x = -\int_0^{2\pi} p_s \cos\theta \, a \, d\theta \qquad \text{(Drag)}$$

$$F_y = -\int_0^{2\pi} p_s \sin\theta \, a \, d\theta \qquad \text{(Lift)}$$

Jean le Rond d'Alembert (1717-1783)

Substituting, $p_s = p_0 + \frac{1}{2}\rho U^2(1 - 4\sin^2\theta)$

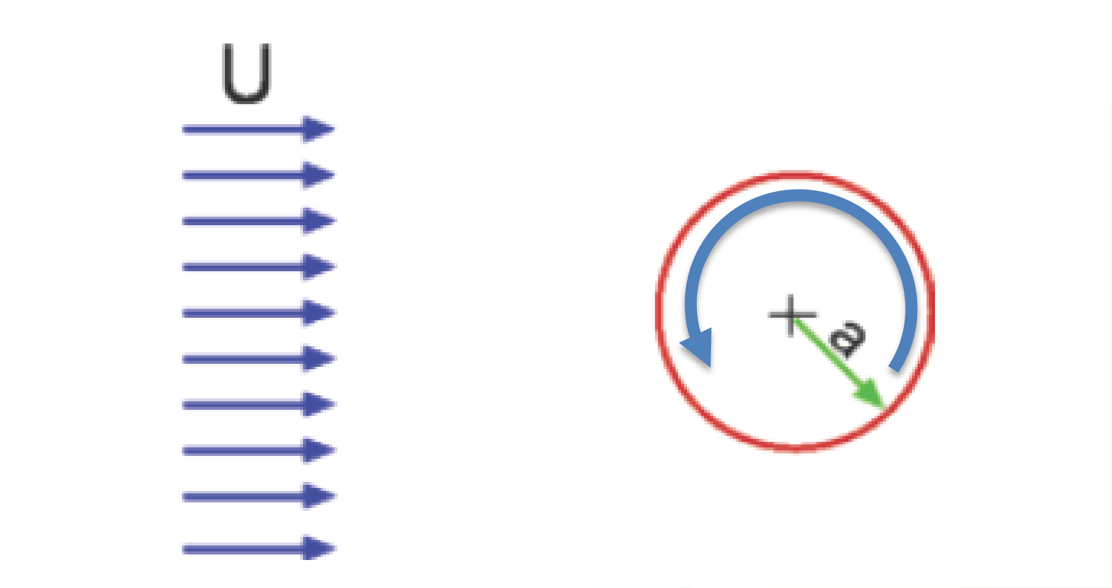
Evaluating the integrals: $F_x = 0 \text{ and } F_y = 0$

Both drag and lift are predicted to be zero on fixed cylinder in a uniform flow?

Mathematically, this makes sense since the pressure distribution is symmetric about cylinder, ahowever, in practice/experiment we see substantial drag on a circular cylinder (d'Alembert's Paradox, 1717-1783).

Viscosity in real flows is the Culprit Again!

SUPERPOSITION OF POTENTIAL FLOWS:
FLOW AROUND A LIFTING CYLINDER



$$\underbrace{\text{Uniform Flow + Doublet}}_{\text{Circular Cylinder}} + \text{Vortex}$$

SUPERPOSITION OF POTENTIAL FLOWS: FLOW AROUND A LIFTING CYLINDER

$$\psi = U_\infty r\left(1 - \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi}\ln r$$

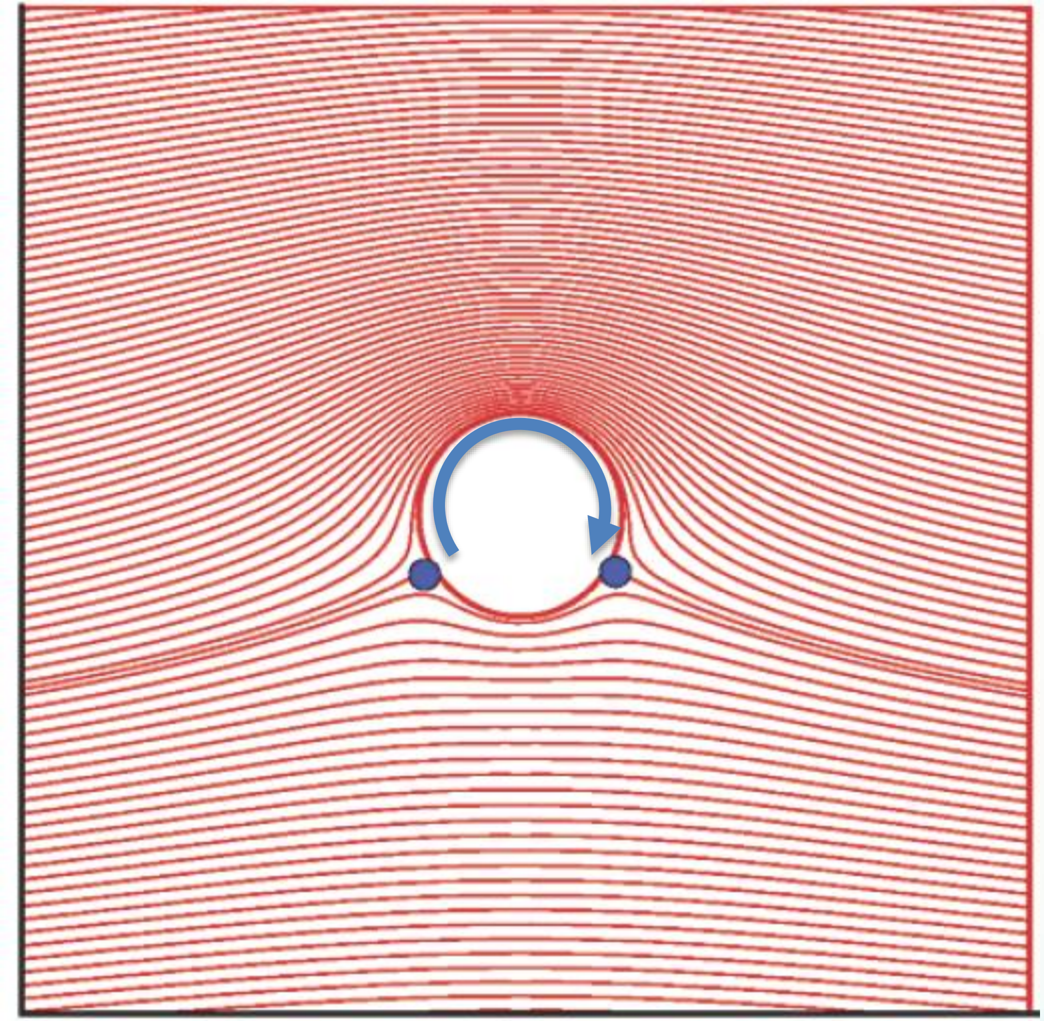
$$\phi = U_\infty r\left(1 + \frac{a^2}{r^2}\right)\cos\theta - \frac{\Gamma}{2\pi}\theta$$

Consequently the velocity components will be,

$$v_r = U_\infty\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = -U_\infty\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r}$$

SUPERPOSITION OF POTENTIAL FLOWS: FLOW AROUND A LIFTING CYLINDER



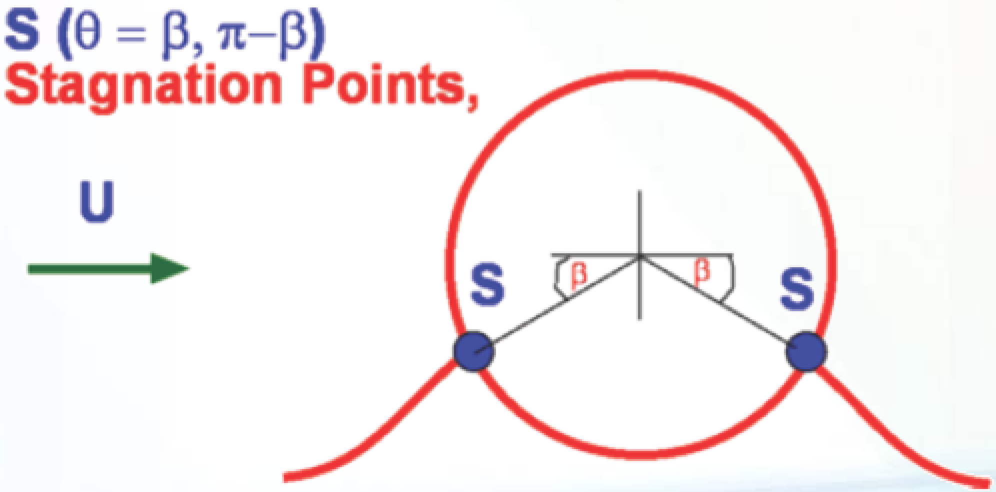
Flow past a Lifting Cylinder

SUPERPOSITION OF POTENTIAL FLOWS:
FLOW AROUND A LIFTING CYLINDER

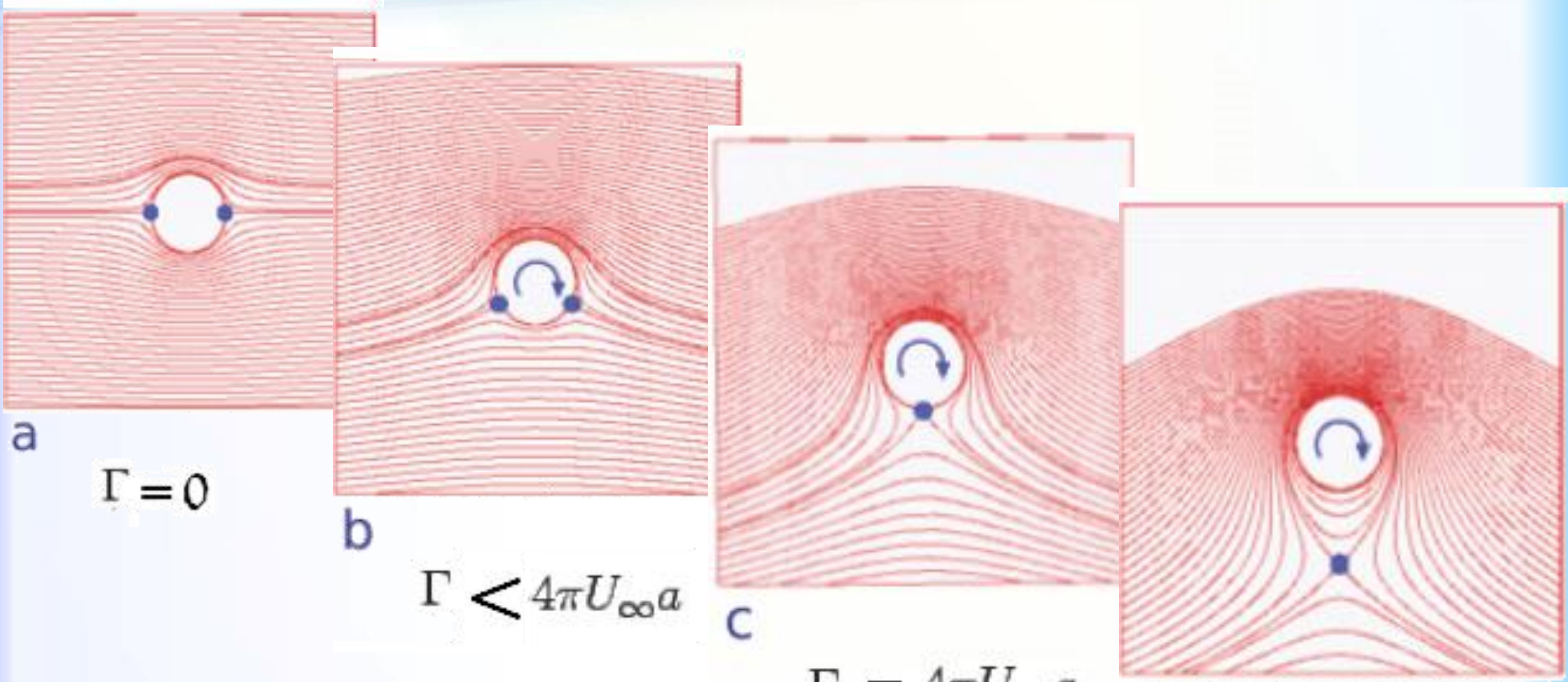
At r = a, the radial velocity is still zero allowing us to consider the same circular cylinder as the "body".

$$v_{\theta_s} = -2\,U_\infty \sin\theta - \frac{\Gamma}{2\pi a}$$

The stagnation points: $v\theta_s = 0$ $\Rightarrow$ $\sin\beta = \frac{-\,\Gamma}{4\,\pi\,U_\infty\,a}$



SUPERPOSITION OF POTENTIAL FLOWS: FLOW AROUND A LIFTING CYLINDER



a

$\Gamma = 0$

b

$\Gamma < 4\pi U_\infty a$

c

$\Gamma = 4\pi U_\infty a$

d

$\Gamma > 4\pi U_\infty a$

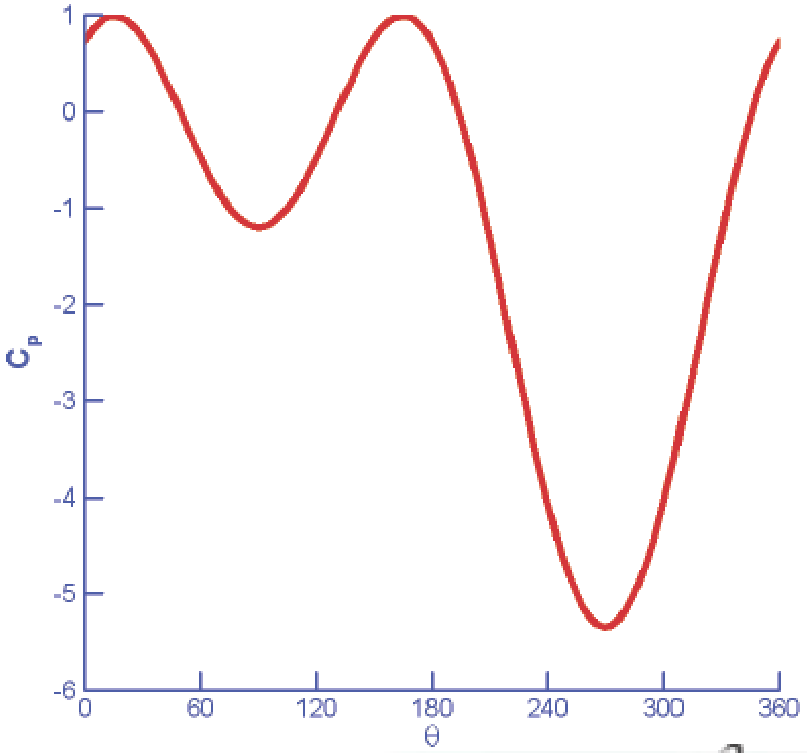
SUPERPOSITION OF POTENTIAL FLOWS: FLOW AROUND A LIFTING CYLINDER

Surface Pressure Distribution and Lift

$$p_s = p_\infty + \frac{1}{2}\rho U_\infty^2 - 2\, U_\infty^2 (\sin\theta - \sin\beta)^2$$

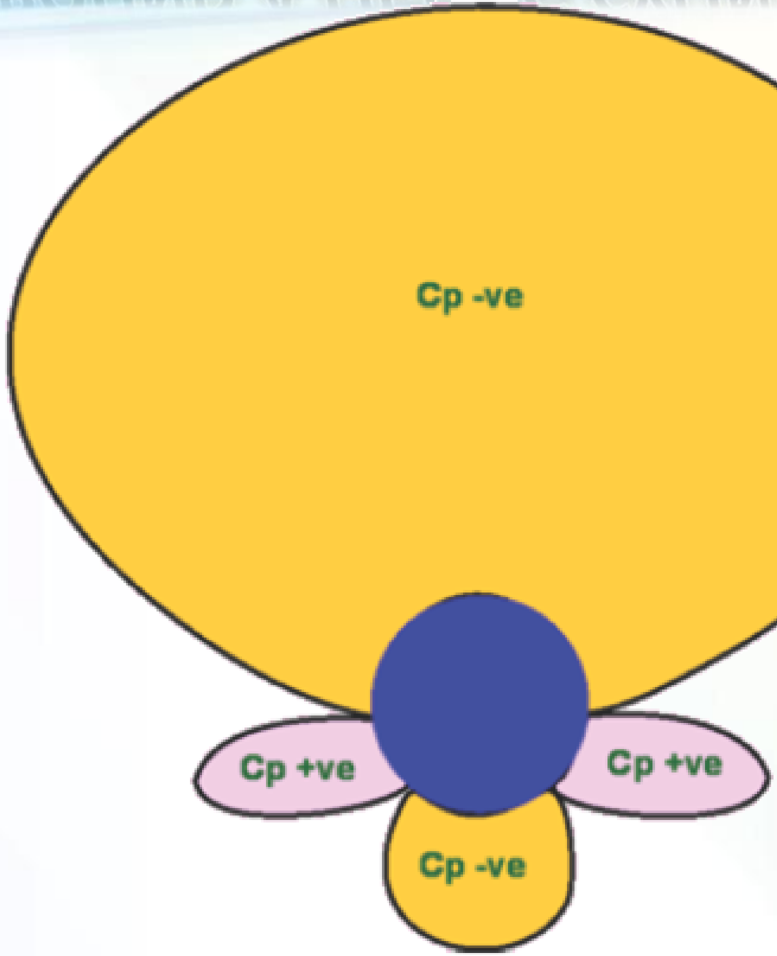
$$C_p = 1 - \left(\frac{v_\theta}{U_\infty}\right)^2$$

$$= 1 - 4\,(\sin\theta - \sin\beta)^2$$



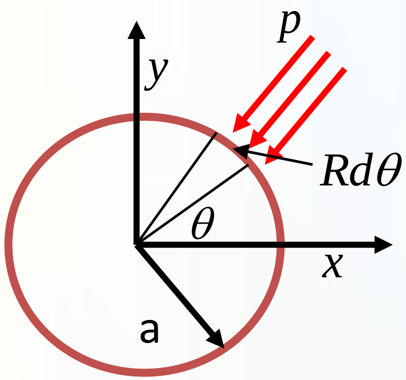
Cp distribution for a lifting cylinder, β =-15⁰.

SUPERPOSITION OF POTENTIAL FLOWS: FLOW AROUND A LIFTING CYLINDER



Cp distribution for a lifting cylinder plottd around the cylinder. $\beta = -15^0$

SUPERPOSITION OF POTENTIAL FLOWS: FLOW AROUND A LIFTING CYLINDER



$$p_s = p_\infty + \frac{1}{2}\rho U_\infty^2\left(1-4\left(\sin\theta - \sin\beta\right)^2\right)$$

$$p_s = p_\infty + \frac{1}{2}\rho U_\infty^2\left(1-4\sin^2\theta - 4\sin^2\beta + 8\sin\theta\sin\beta\right)$$

$$L = -\int_0^{2\pi} a\sin\theta\left[p_\infty + \frac{1}{2}\rho U_\infty^2\left(1-4\sin^2\theta - 4\sin^2\beta + 8\sin\theta\sin\beta\right)\right]d\theta$$

$$L = -\int_0^{2\pi} a\left[p_\infty + \frac{1}{2}\rho U_\infty^2\left(\sin\theta-4\sin^3\theta - 4\sin^2\beta\sin\theta + 8\sin^2\theta\sin\beta\right)\right]d\theta$$

SUPERPOSITION OF POTENTIAL FLOWS: FLOW AROUND A LIFTING CYLINDER

$$L = -\int_0^{2\pi} \cancel{a p_\infty \sin\theta \, d\theta} + \frac{1}{2}\rho U_\infty^2 a \int_0^{2\pi} \cancel{4 \sin^3\theta \, d\theta}$$

$$+ \frac{1}{2}\rho U_\infty^2 a \int_0^{2\pi} \cancel{\sin^2\beta \sin\theta \, d\theta} - \frac{1}{2}\rho U_\infty^2 a \int_0^{2\pi} 8 \sin^2\theta \sin\beta \, d\theta$$

$$\sin\beta = \frac{-\Gamma}{4\pi U_\infty a}$$

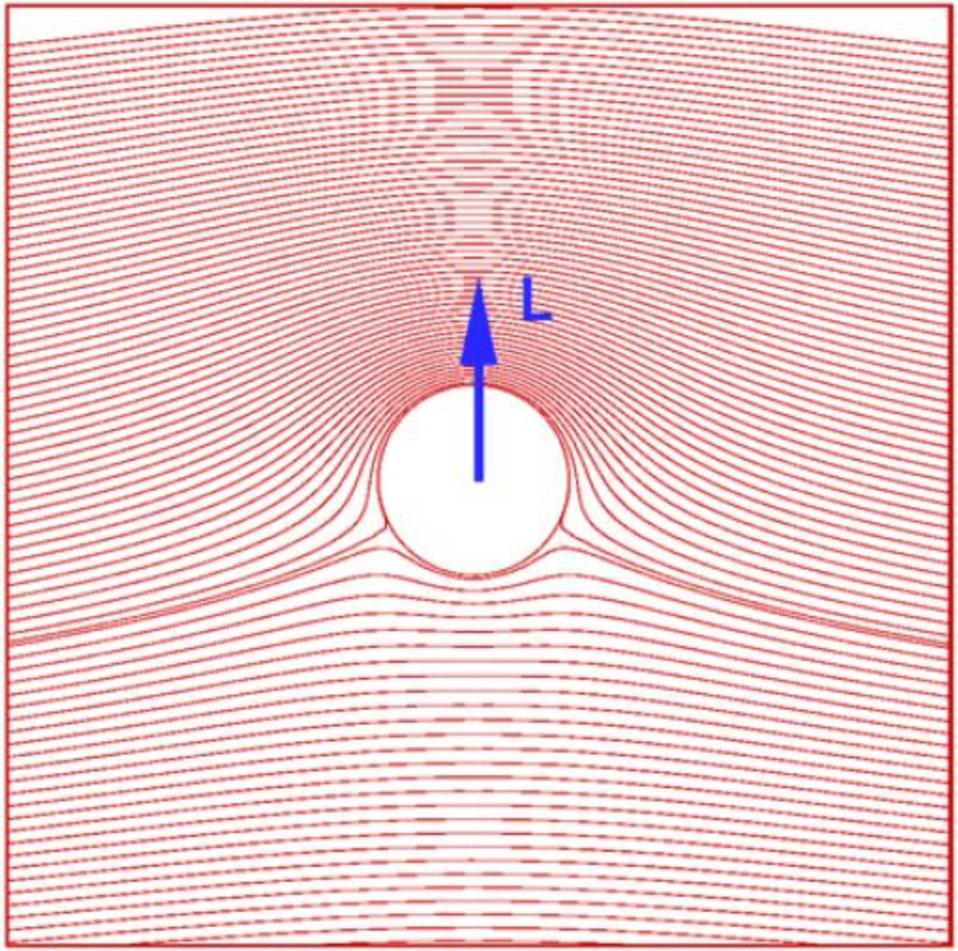
$$L = -\frac{1}{2}\rho U_\infty^2 a \int_0^{2\pi} 8 \sin^2\theta \sin\beta$$

$$= -4\rho U_\infty^2 \sin\beta \left[\frac{\theta}{2} - \frac{\sin^2\theta}{4}\right]_0^{2\pi}$$

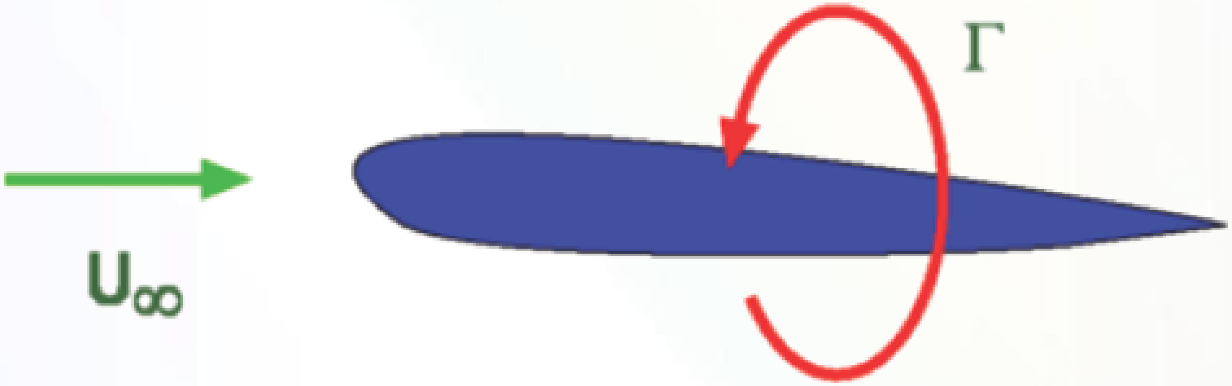
$$= -4\pi\rho U_\infty^2 a \sin\beta$$

$$\boxed{L = \rho U_\infty \Gamma}$$

SUPERPOSITION OF POTENTIAL FLOWS: FLOW AROUND A LIFTING CYLINDER

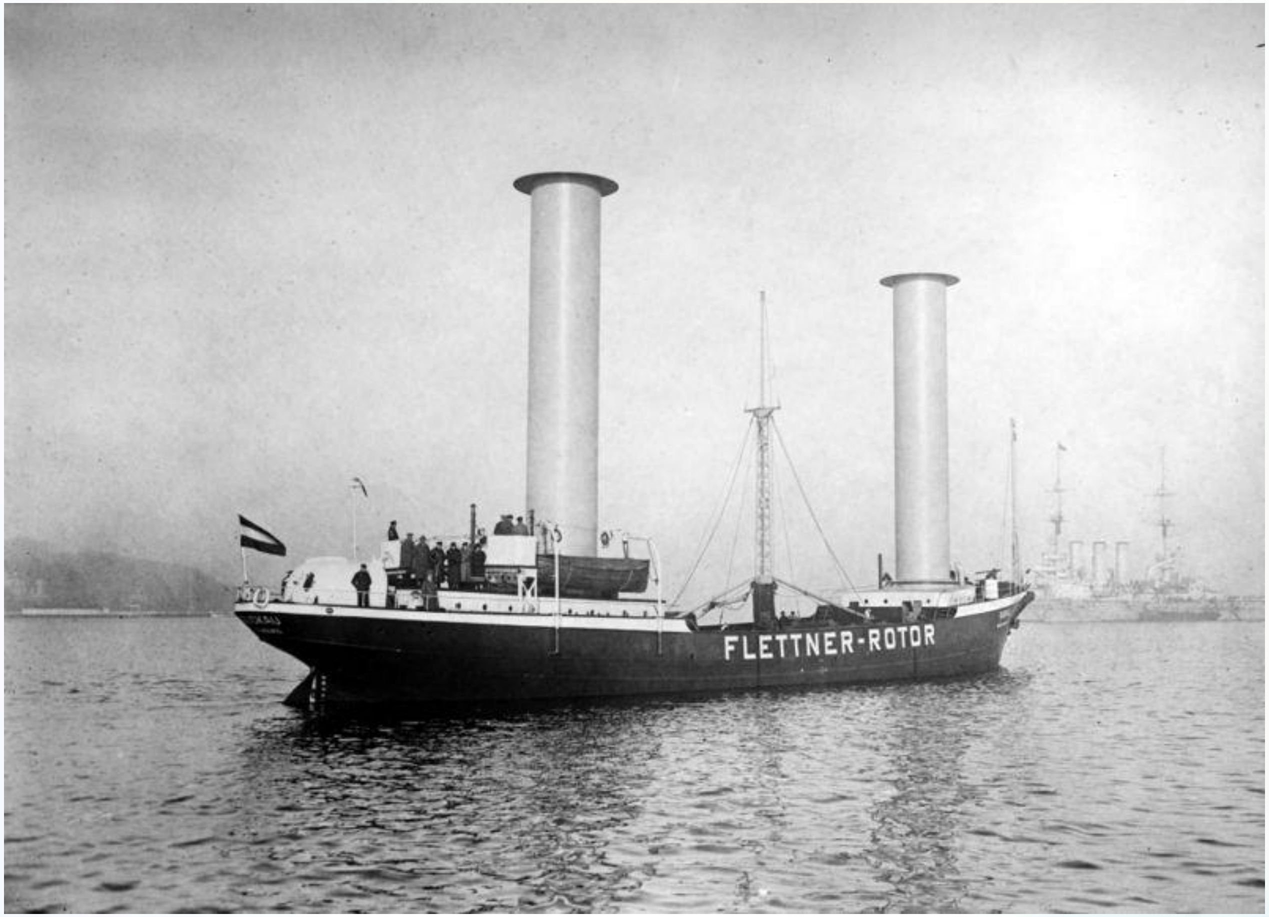


KUTTA-JOUKOWSKY THEOREM

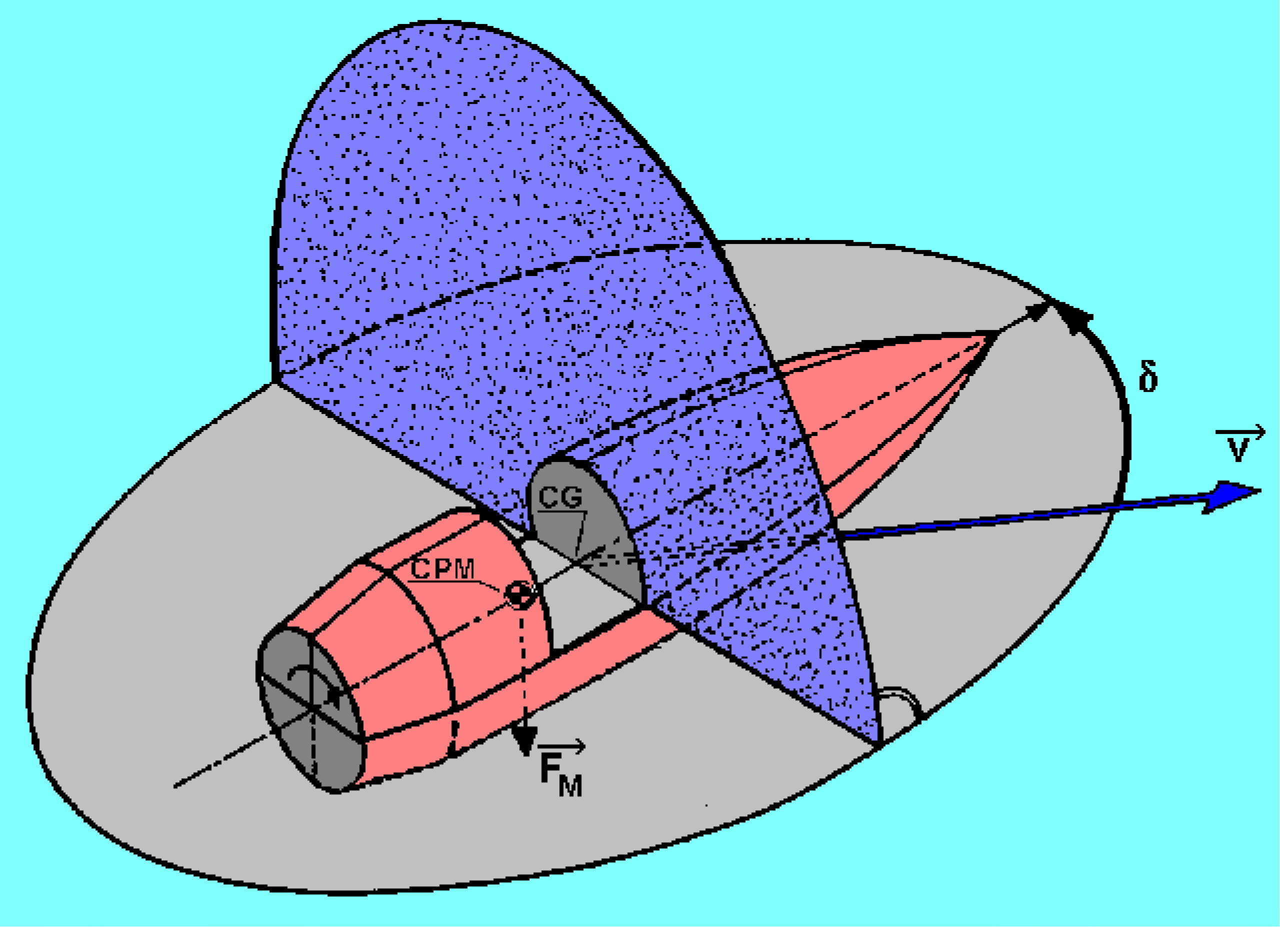


$$L = \rho U_\infty \Gamma$$

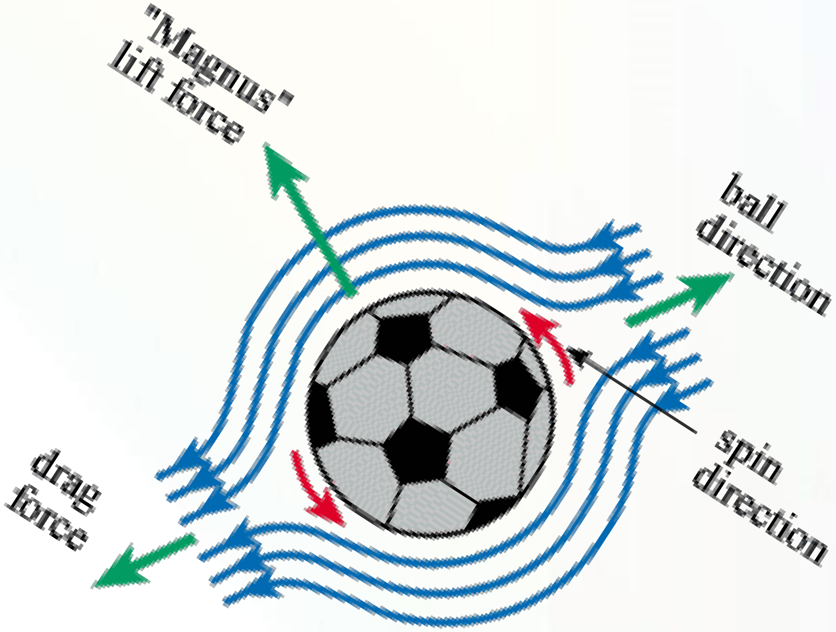
Flettner’s Ship



MAGNUS EFFECTS



Bend it like Beckham



Dynamic lift

Beckham, Applied Physicist

Distance 25 m

Initial $v = 25$ m/s

Flight time 1s

Spin at 10 rev/s

Lift force ~ 4 N

Ball mass ~ 400 g

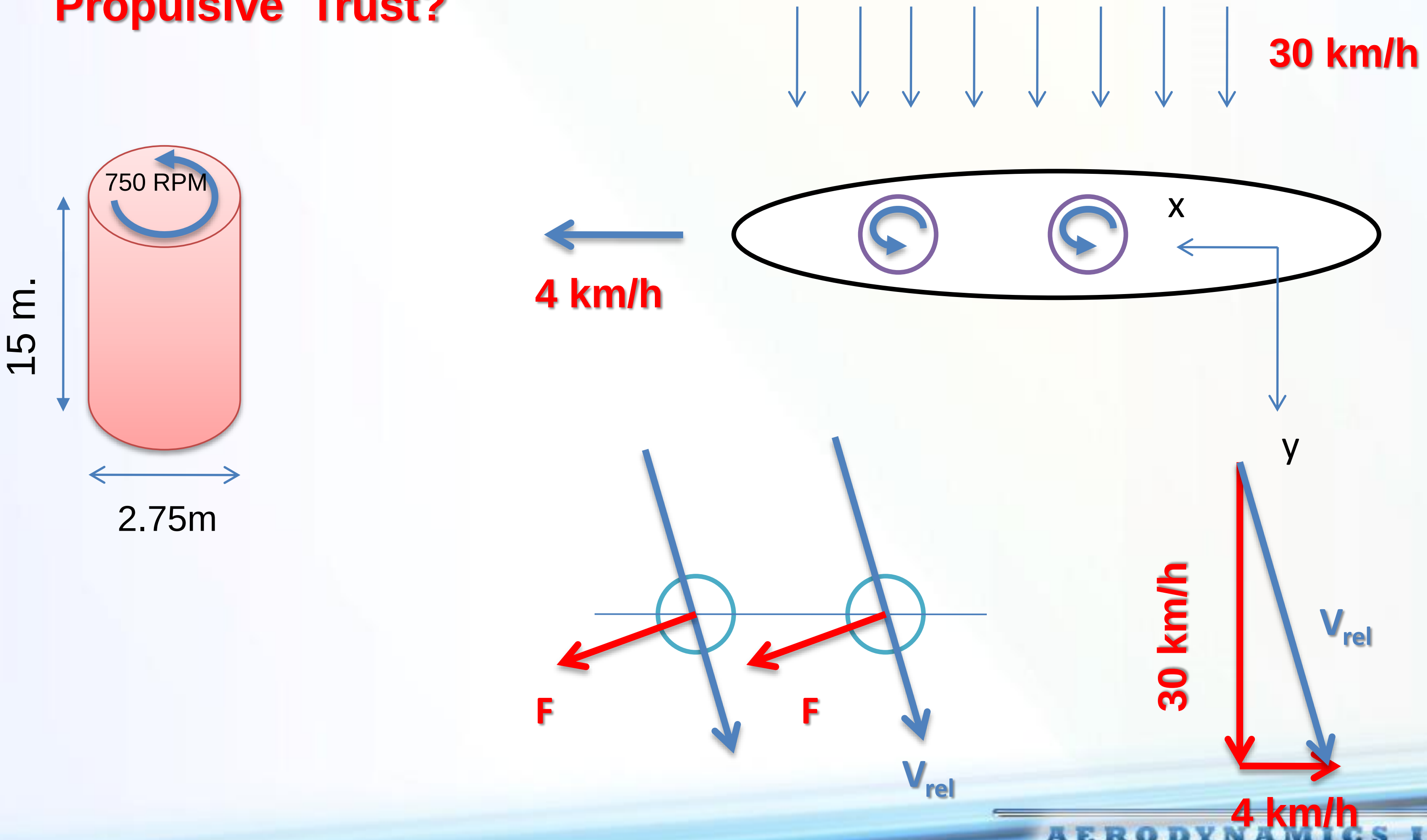
$a = 10$ m/s^2

A swing of 5 m!

MAGNUS EFFECTS - EXAMPLE

Flettner's Ship with the following conditions

Propulsive Trust?



MAGNUS EFFECTS - EXAMPLE

$$V_{rel} = 30j - 4i$$

$$V_{rel} = \sqrt{30^2 + 4^2} = 30.27\frac{km}{h} = 8.41\, m/s$$

$$\Gamma = (\omega r)(2\pi r) = \left[(750)\frac{2\pi}{60}\right](1.375)^2 2\pi = 933\, m^2/s$$

$$\mathrm{F} = \rho \mathrm{V}_{\mathrm{rel}}\Gamma = (1.229)(8.41)(933) = 9643\, N/m$$

$$F_T = 2(9643)(15) = 289\, kN$$

$$(\mathrm{F_T})_{\mathrm{Prop}} = \mathrm{F_T}\cos\alpha = 289\frac{30}{(30^2 + 4^2)^{\frac{1}{2}}}$$

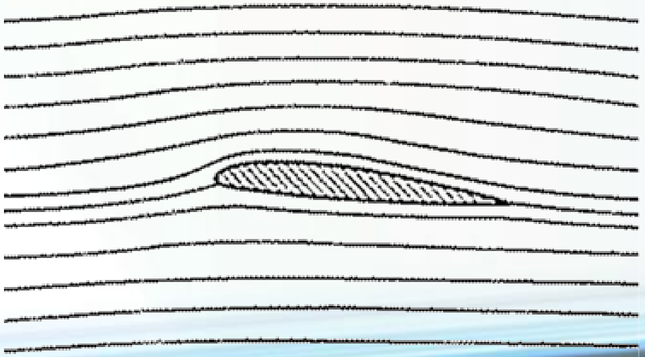
$$= 287\ \mathrm{kN}$$

NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD

We added our elementary flows in certain ways and discovered that the dividing streamlines turned out to fit the shapes of special bodies *(semi-infinite body, Rankine oval and both the nonlifting and the lifting flows over a circular cylinder)*

This ***indirect*** method of starting with a given combination of elementary flows and seeing what body shape comes out of it can hardly be used in a practical sense for bodies of arbitrary shape.

Do we know in advance the correct combination of elementary flows to synthesize the flow over an airfoil?



The answer is NO.

NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD

In order to determine the flow over a specified body, we want a **direct** method.

In direct methods, we specify the shape of an arbitrary body and solve for the distribution of singularities which, in combination with a uniform stream, produce the flow over the given body.

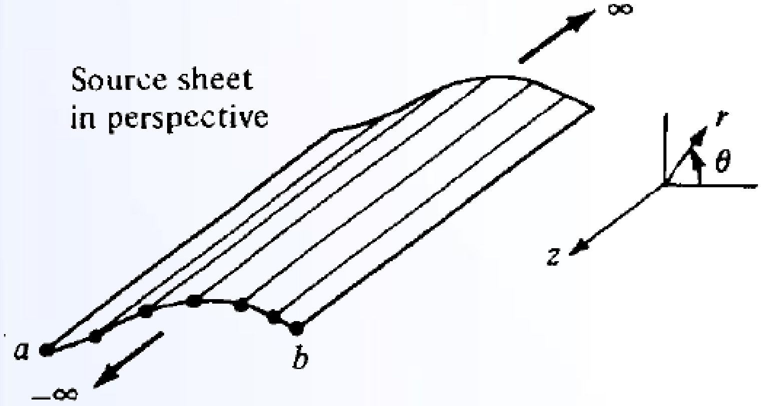
We consider a numerical method appropriate for solution on a computer. The technique is called the *Source Panel Method* and is limited to nonlifting flows over arbitrary bodies.

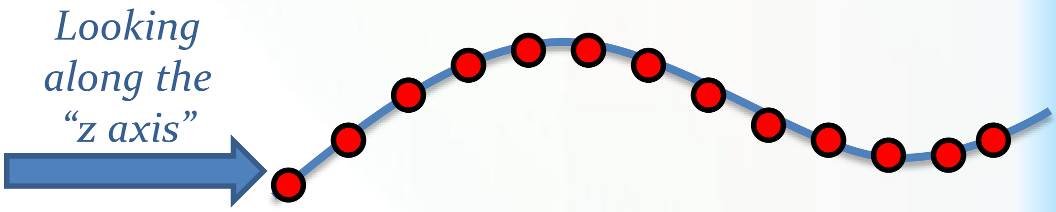
NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD

Let us extend the concept of a source or sink.

Imagine that we have an infinite number of line sources side by side, where the strength of each line source is infinitesimally small.

These side-by-side line sources form a ***source sheet***.



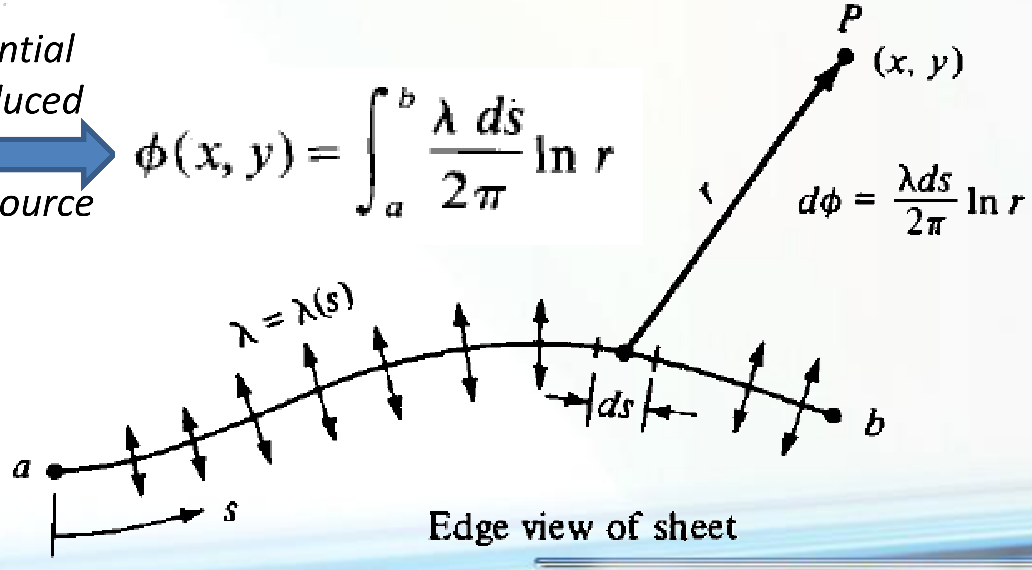

Looking along the “z axis”

NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD

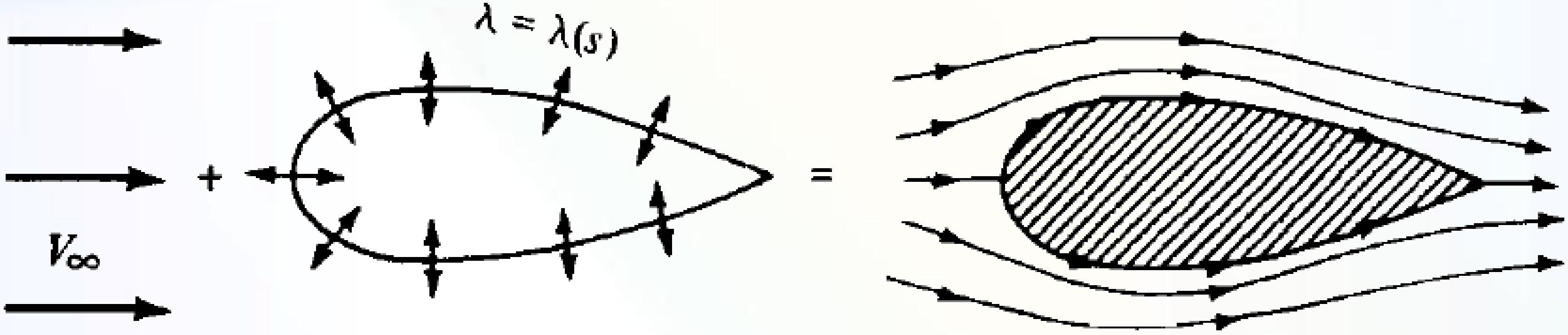
Define $\lambda = \lambda(s)$ to be the source strength per unit length along s. Therefore, the strength of an infinitesimal portion ds of the sheet is λds.

The small section of the source sheet of strength λds, induces an infinitesimally small potential $d\phi$ at point P:

$$d\phi = \frac{\lambda\, ds}{2\pi} \ln r \quad \xrightarrow[\textit{by the entire source sheet}]{\textit{Velocity potential at point P, induced}} \quad \phi(x, y) = \int_a^b \frac{\lambda\, ds}{2\pi} \ln r$$



NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD



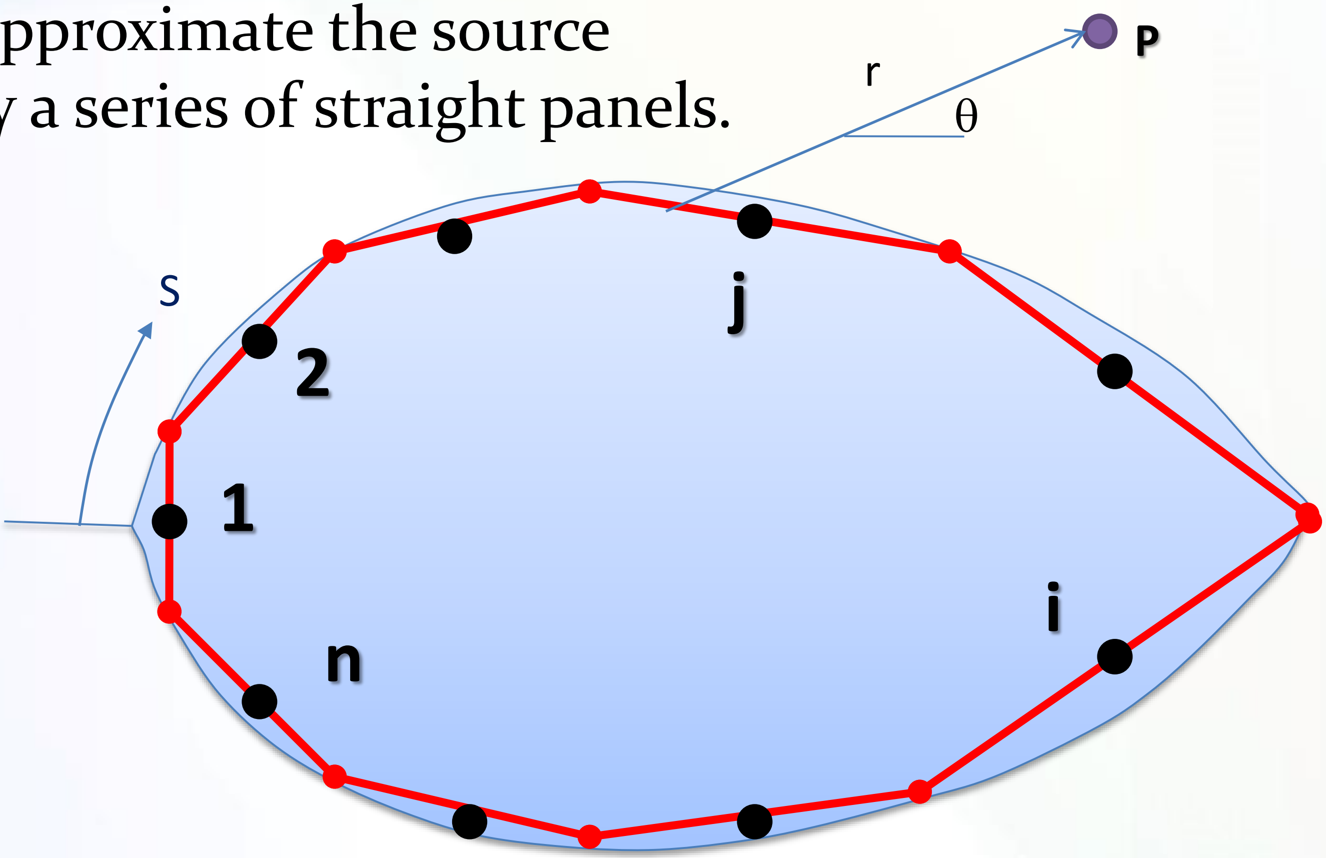
Uniform flow + **Source sheet on surface Flow over the body** = **Flow over the body**

Our problem is one of finding the appropriate $\lambda(s)$.

The solution of this problem is carried out numerically.

NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD

Let us approximate the source sheet by a series of straight panels.



● Control Points

● Boolean points

NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD

$$\phi(x, y) = \int_a^b \frac{\lambda\, ds}{2\pi} \ln r$$

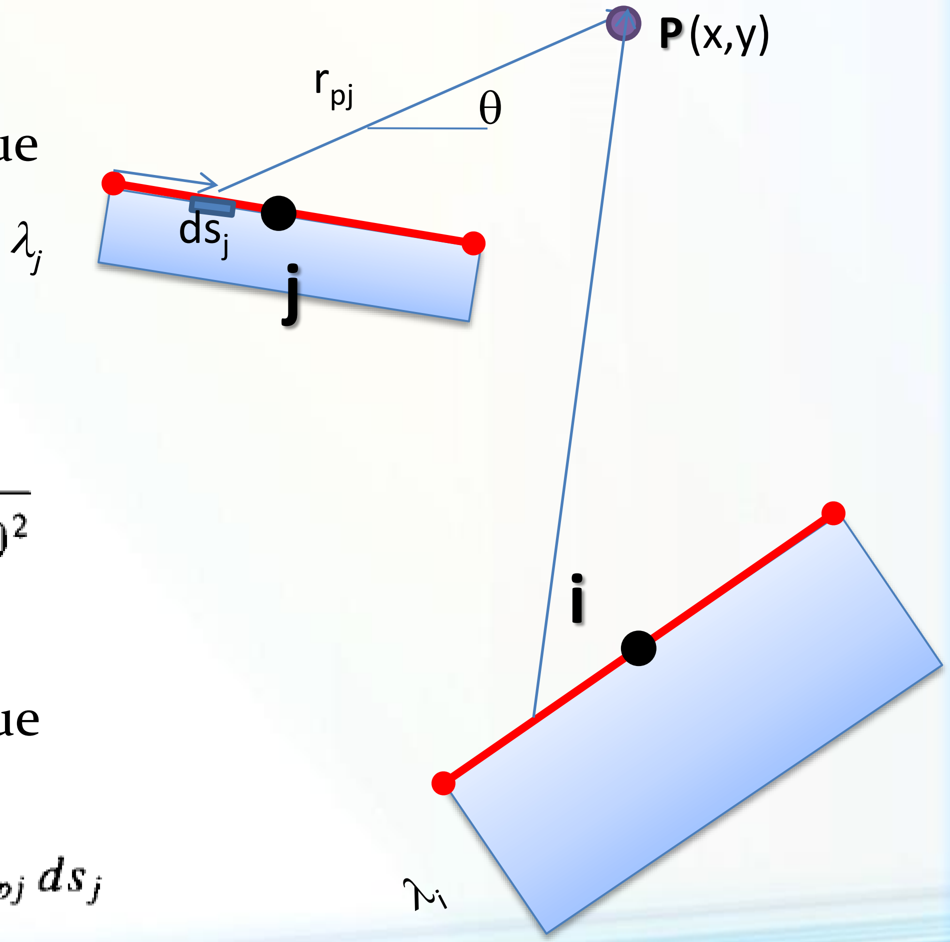
The velocity potential induced at *P* due to the *j*th panel is:

$$\Delta\phi_j = \frac{\lambda_j}{2\pi} \int_j \ln r_{pj}\, ds_j$$

Where: $r_{pj} = \sqrt{(x - x_j)^2 + (y - y_j)^2}$

The velocity potential induced at *P* due to *all* the panels:

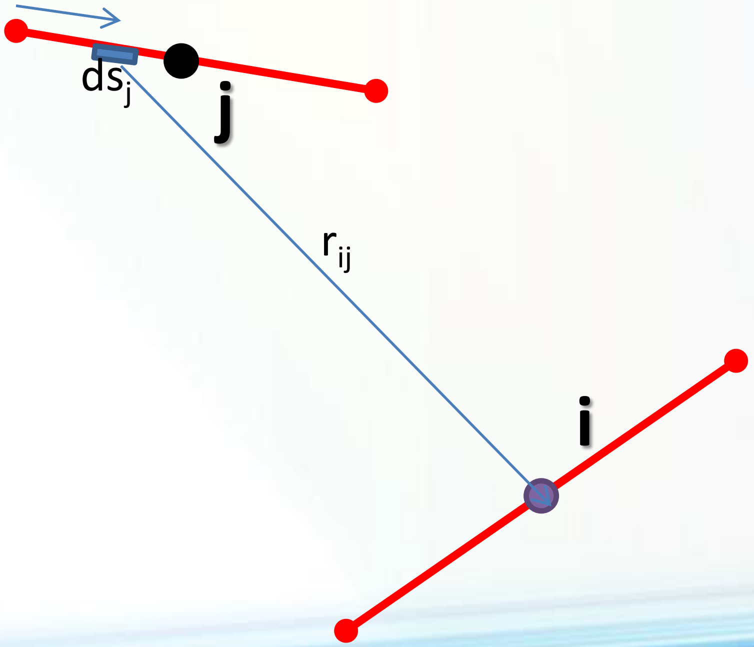
$$\phi(P) = \sum_{j=1}^{n} \Delta\phi_j = \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \ln r_{pj}\, ds_j$$



NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD

$$\phi(x_i, y_i) = \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \ln r_{ij}\, ds_j$$

$$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$



NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD

The boundary condition at solid walls states that:

$$V_{\infty,n} + V_n = 0$$

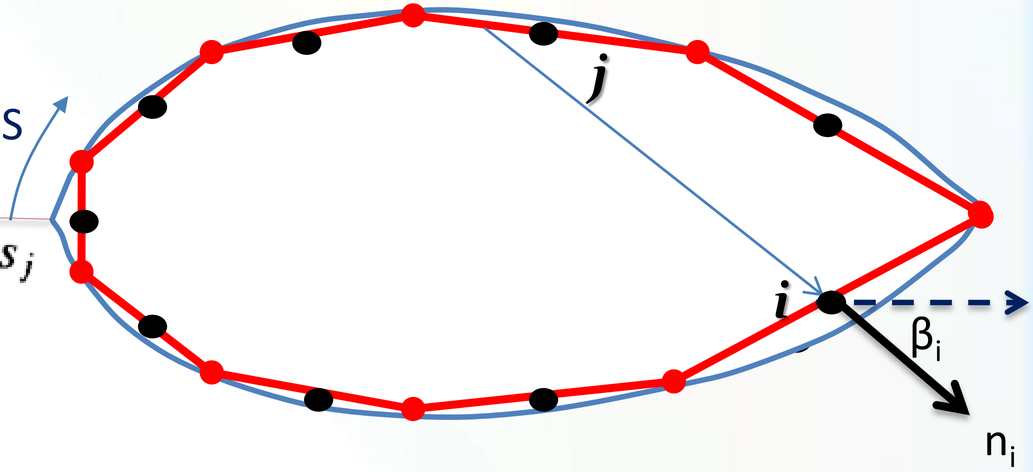
$$\phi(x_i, y_i) = \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \ln r_{ij}\, ds_j$$

Where:

$$V_{\infty,n} = \mathbf{V}_\infty \cdot \mathbf{n}_i = V_\infty \cos \beta_i$$

$$V_n = \frac{\partial}{\partial n_i}[\phi(x_i, y_i)]$$

$$= \frac{\lambda_i}{2} + \sum_{\substack{j=1 \\ (j \neq i)}}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial n_i}(\ln r_{ij})\, ds_j$$



$$V_{\infty,n} + V_n = 0 \quad \Longrightarrow \quad \frac{\lambda_i}{2} + \sum_{\substack{j=1 \\ (j \neq i)}}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial n_i}(\ln r_{ij})\, ds_j + V_\infty \cos \beta_i = 0$$

NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD

$$\frac{\lambda_i}{2} + \sum_{\substack{j=1 \\ (j \neq 1)}}^{n} \frac{\lambda_j}{2\pi} \overset{I_{i,j}}{\boxed{\int_j \frac{\partial}{\partial n_i}(\ln r_{ij})}}\, ds_j + V_\infty \cos\beta_i = 0 \Longrightarrow \frac{\lambda_i}{2} + \sum_{\substack{j=1 \\ (j \neq 1)}}^{n} \frac{\lambda_j}{2\pi} I_{i,j} + V_\infty \cos\beta_i = 0$$

The integral $I_{i,j}$ is evaluated at the jth control point and the integral is taken over the complete jth panel:

$$I_{i,j} = \int_j \frac{\partial}{\partial n_i}(\ln r_{ij})\, ds_j \qquad r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

$$\frac{\partial}{\partial n_i}(\ln r_{ij}) = \frac{1}{r_{ij}} \frac{\partial r_{ij}}{\partial n_i} = \frac{1}{r_{ij}} \frac{1}{2} [(x_i - x_j)^2 + (y_i - y_j)^2]^{-1/2} \left[2(x_i - x_j) \frac{dx_i}{dn_i} + 2(y_i - y_j) \frac{dy_i}{dn_i} \right]$$

$$\frac{\partial}{\partial n_i}(\ln r_{ij}) = \frac{(x_i - x_j)\cos\beta_i + (y_i - y_j)\sin\beta_i}{(x_i - x_j)^2 + (y_i - y_j)^2}$$

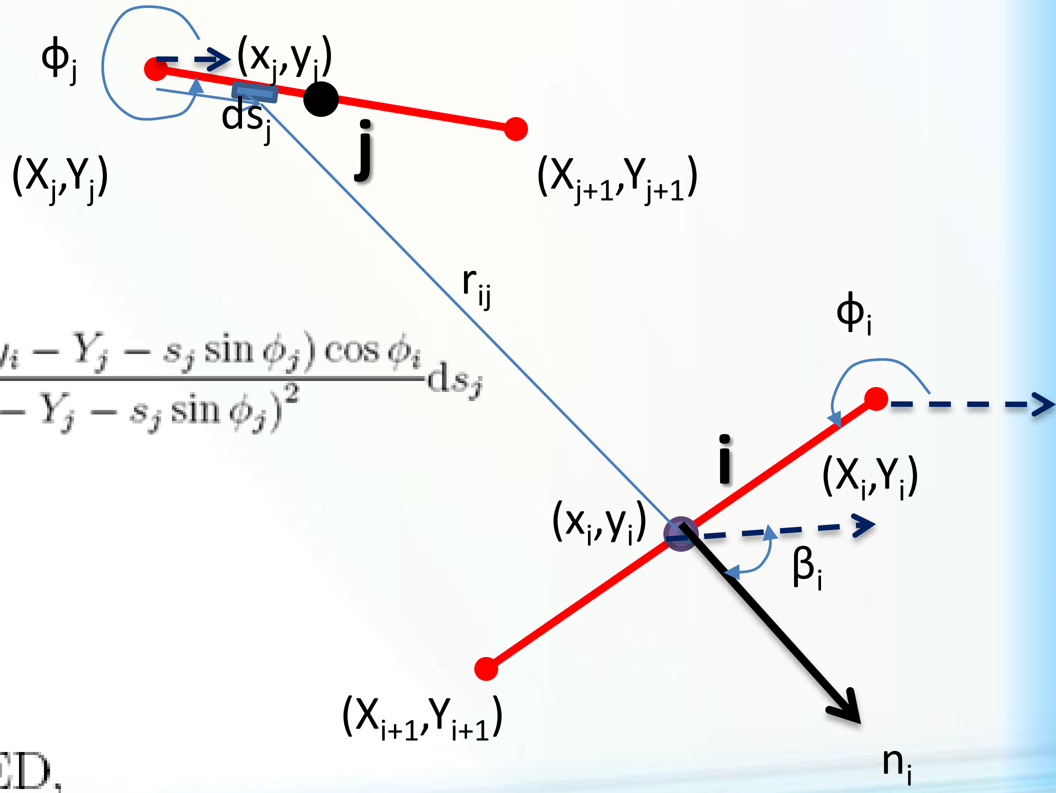
NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD

$$\beta_i = \Phi_i + \frac{\pi}{2} \Rightarrow \cos\beta_i = -\sin\Phi_i \qquad \sin\beta_i = \cos\Phi_i$$

$$x_i = \frac{X_i + X_{i+1}}{2} \qquad y_i = \frac{Y_i + Y_{i+1}}{2}$$

$$x_j = X_j + s_j \cos\phi_j \qquad y_j = Y_j + s_j \sin\phi_j$$

$$S_j = \sqrt{(X_{j+1} - X_j)^2 + (Y_{j+1} - Y_j)^2}$$



$$I_{ij} = \int_0^{S_j} \frac{(x_i - X_j - s_j\cos\phi_j)(-\sin\phi_i) + (y_i - Y_j - s_j\sin\phi_j)\cos\phi_i}{(x_i - X_j - s_j\cos\phi_j)^2 + (y_i - Y_j - s_j\sin\phi_j)^2} ds_j$$

Now, considering that

s_j is a VARIABLE and

$x_i,\ y_i,\ X_i,\ Y_i,\ S_j,\ \phi_i,\ \phi_j$ are FIXED,

NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD

So that $I_{i,j}$ becomes:

$$I_{ij} = \int_0^{S_j} \frac{C s_j + D}{s_j^2 + 2A s_j + B} \mathrm{d}s_j$$

Where

$$A = -(x_i - X_j)\cos\phi_j - (y_i - Y_j)\sin\phi_j$$

$$B = (x_i - X_j)^2 + (y_i - Y_j)^2$$

$$C = \sin(\phi_i - \phi_j) = \sin\phi_i \cos\phi_j - \cos\phi_i \sin\phi_j$$

$$D = (y_i - Y_j)\cos\phi_i - (x_i - X_j)\sin\phi_i$$

$$E = \sqrt{B - A^2} = (x_i - X_j)\sin\phi_j - (y_i - Y_j)\cos\phi_j$$

We obtain an expression for $I_{i,j}$ from any table of integrals:

$$I_{ij} = \frac{C}{2} \ln\left(\frac{S_j^2 + 2AS_j + B}{B}\right) + \frac{D - AC}{E}\left(\arctan\frac{S_j + A}{E} - \arctan\frac{A}{E}\right)$$

NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD

$$\frac{\lambda_i}{2} + \sum_{\substack{j=1 \\ (j \neq 1)}}^{n} \frac{\lambda_j}{2\pi} I_{i,j} + V_\infty \cos \beta_i = 0$$

With known values of $I_{i,j}$'s, this is a linear algebraic equation with n unknowns $\lambda_1, \lambda_2, \ldots, \lambda_n$.

This equation represents the flow boundary condition evaluated at the control point of the *i*th panel.

If we apply this equation to the control point of all the panels, the results will be a system of n linear algebraic equations with n unknowns ($\lambda_1, \lambda_2, \ldots, \lambda_n$)

The values of λ_j's should obey the relation: $\sum_{j=1}^{n} \lambda_j S_j = 0$

NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD

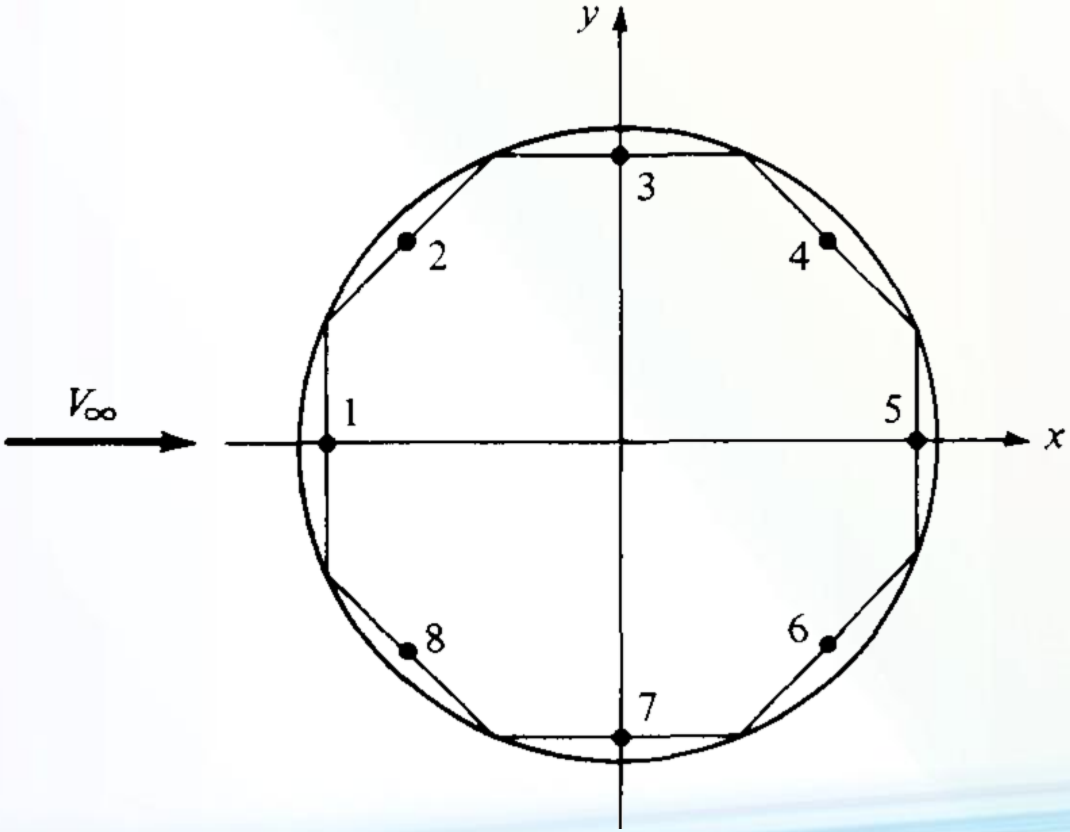
The total surface velocity at the *i*th control point is the sum of the contribution from the freestream and from the source panels:

$$V_i = V_{\infty,s} + V_s = V_\infty \sin\beta_i + \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \boxed{\int_j \frac{\partial}{\partial s}(\ln r_{ij})\, ds_j}$$

$$\frac{D-AC}{2E}\ln\left(\frac{S_j^2+2AS_j+B}{B}\right) - C\left(\arctan\frac{S_j+A}{E} - \arctan\frac{A}{E}\right)$$

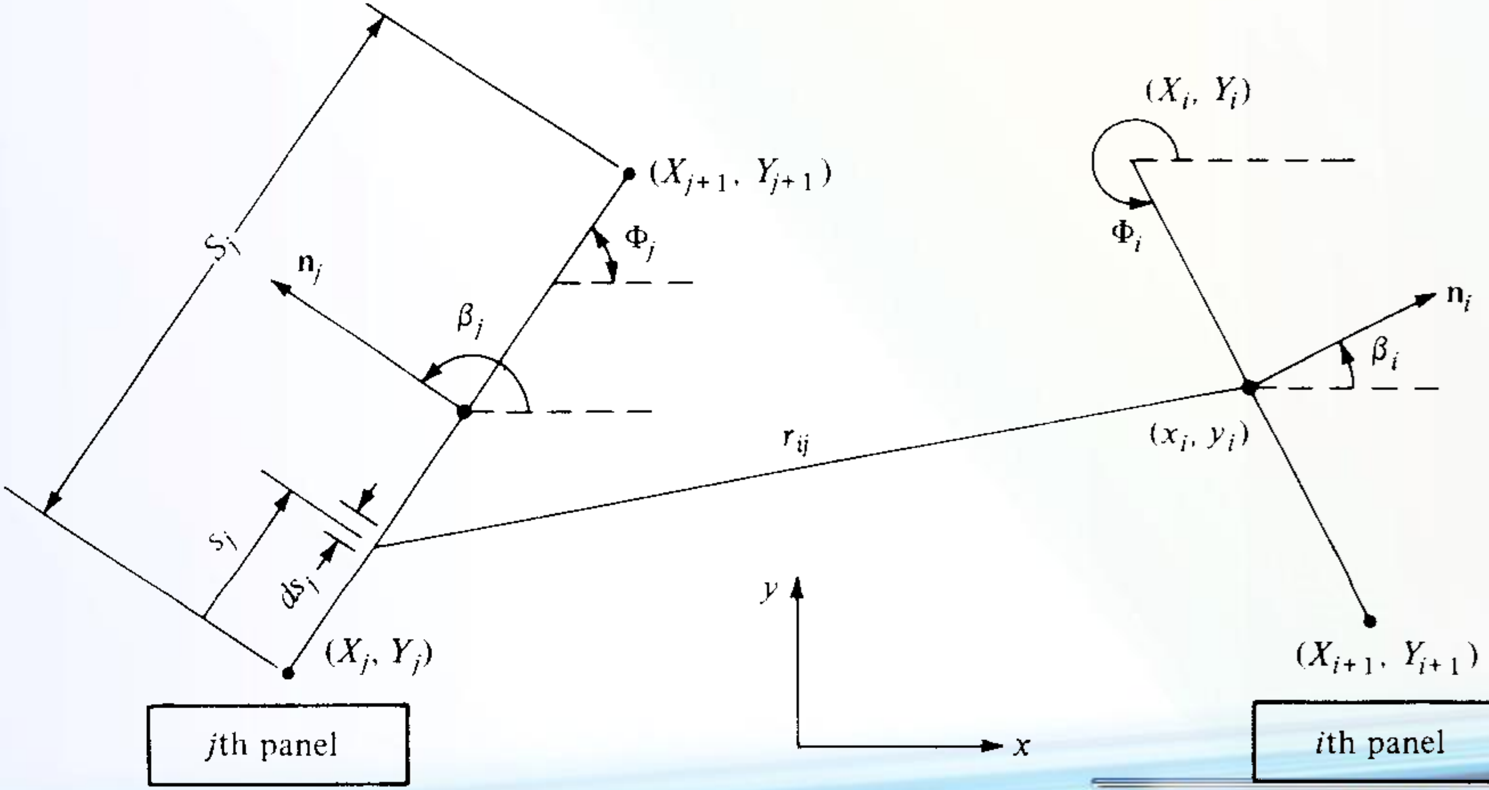
NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD

EXAMPLE: Calculate the pressure coefficient distribution around a circular cylinder using the source panel technique.



NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD

$$I_{i,j} = \int_j \frac{\partial}{\partial n_i} (\ln r_{ij})\, ds_j$$



NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD

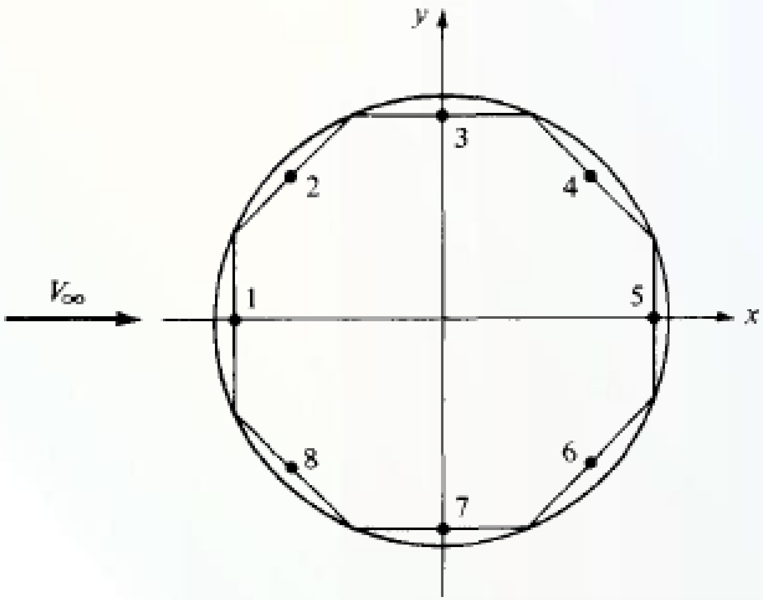
$$I_{i,j} = \frac{C}{2}\ln\left(\frac{S_j^2 + 2AS_j + B}{B}\right) + \frac{D - AC}{E}\left(\tan^{-1}\frac{S_j + A}{E} - \tan^{-1}\frac{A}{E}\right)$$

$\boxed{I_{4,2}.}$

$X_j = -0.9239$ $X_{j+1} = -0.3827$ $Y_j = 0.3827$

$Y_{j+1} = 0.9239$ $\Phi_i = 315°$ $\Phi_j = 45°$

$x_i = 0.6533$ $y_i = 0.6533$



$A = -1.3065$ $B = 2.5607$ $C = -1$ $D = 1.3065$

$S_j = 0.7654$ $E = 0.9239$

$\boxed{I_{4,2} = 0.4018}$

NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD

Similarly, $I_{4,3} = 0.3528$, $I_{4,5} = 0.3528$, $I_{4,6} = 0.4018$, $I_{4,7} = 0.4074$, and $I_{4,8} = 0.4084$.

$$0.4074\lambda_1 + 0.4018\lambda_2 + 0.3528\lambda_3 + \pi\lambda_4 + 0.3528\lambda_5 + 0.4018\lambda_6 + 0.4074\lambda_7 + 0.4084\lambda_8 = -0.7071\,2\pi V_\infty$$

$\lambda_1/2\pi V_\infty = 0.3765$

$\lambda_2/2\pi V_\infty = 0.2662$

$\lambda_3/2\pi V_\infty = 0$

$\lambda_4/2\pi V_\infty = -0.2662$

$\lambda_5/2\pi V_\infty = -0.3765$

$\lambda_6/2\pi V_\infty = -0.2662$

$\lambda_7/2\pi V_\infty = 0$

$\lambda_8/2\pi V_\infty = 0.2662$

$$\sum_{j=1}^{n} \lambda_j = 0$$

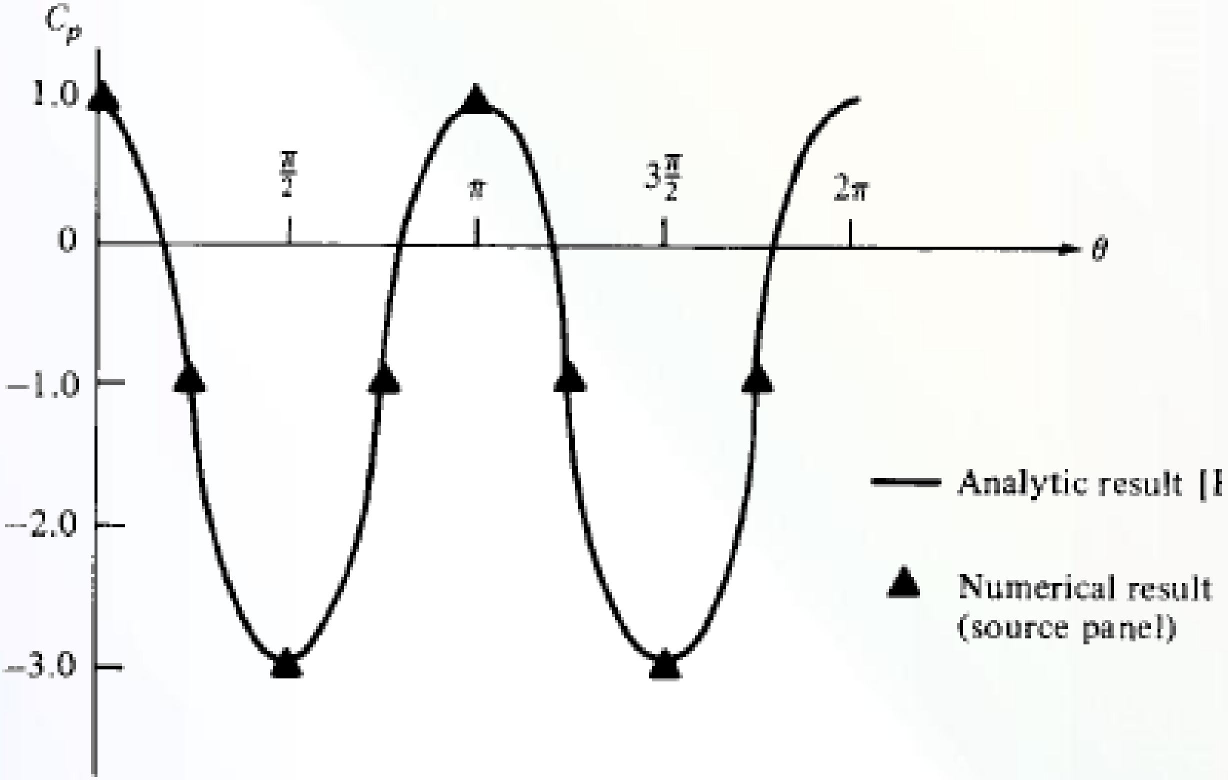
NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD

$$V_i = V_{\infty,s} + V_s = V_\infty \sin \beta_i + \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \boxed{\int_j \frac{\partial}{\partial s} (\ln r_{ij})} \, ds_j$$

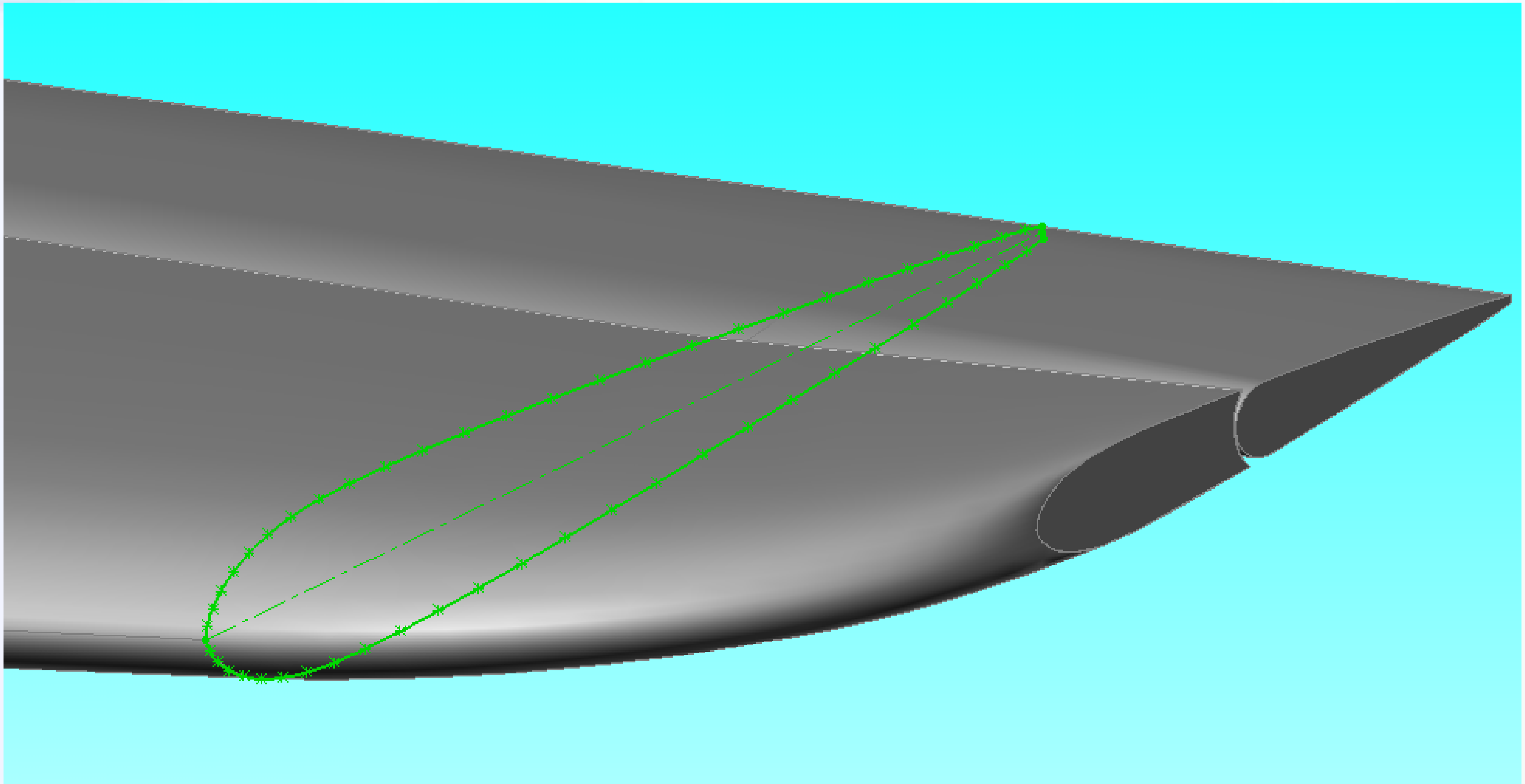
$$\int_j \frac{\partial}{\partial s} (\ln r_{ij}) \, ds_j = \frac{D - AC}{2E} \ln \frac{S_j^2 + 2AS_j + B}{B} - C\left(\tan^{-1} \frac{S_j + A}{E} - \tan^{-1} \frac{A}{E}\right)$$

$$\boxed{C_{p,i} = 1 - \left(\frac{V_i}{V_\infty}\right)^2}$$

NONLIFTING FLOWS OVER ARBITRARY BODIES: THE NUMERICAL SOURCE PANEL METHOD

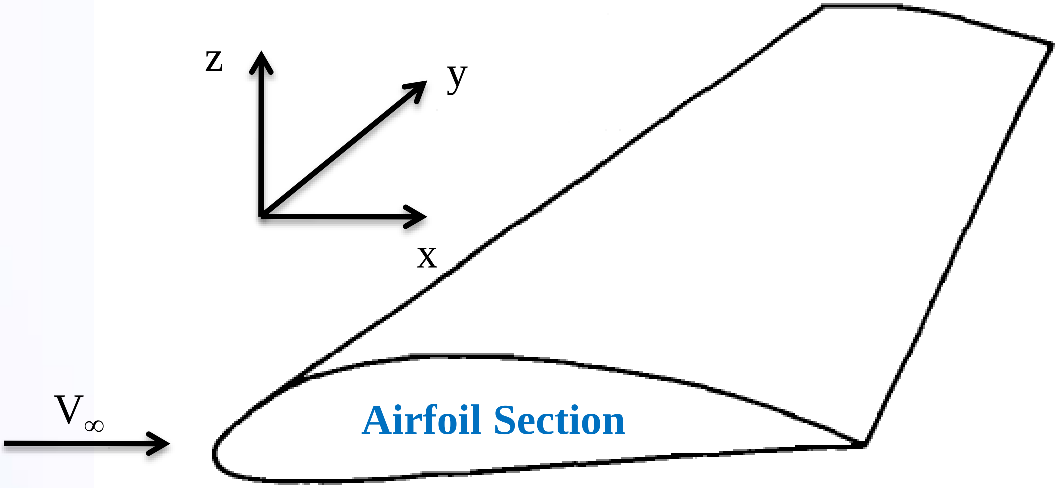


INCOMPRESSIBLE FLOW OVER AIRFOILS



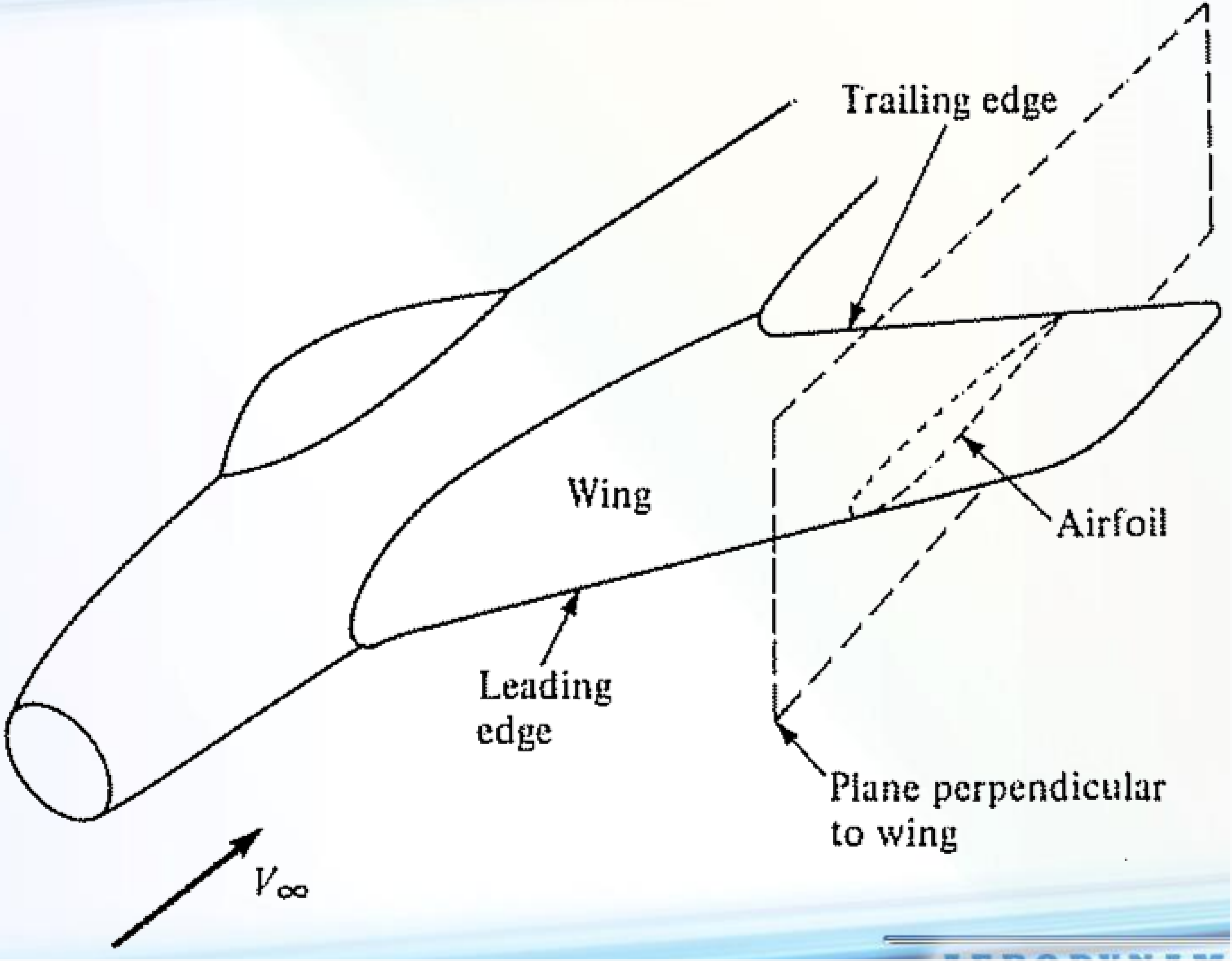
WHAT IS AN AIRFOIL?

- Consider a wing as shown in the figure.
  - The wing extends in the y direction
  - The freestream velocity is parallel to *xz* plane

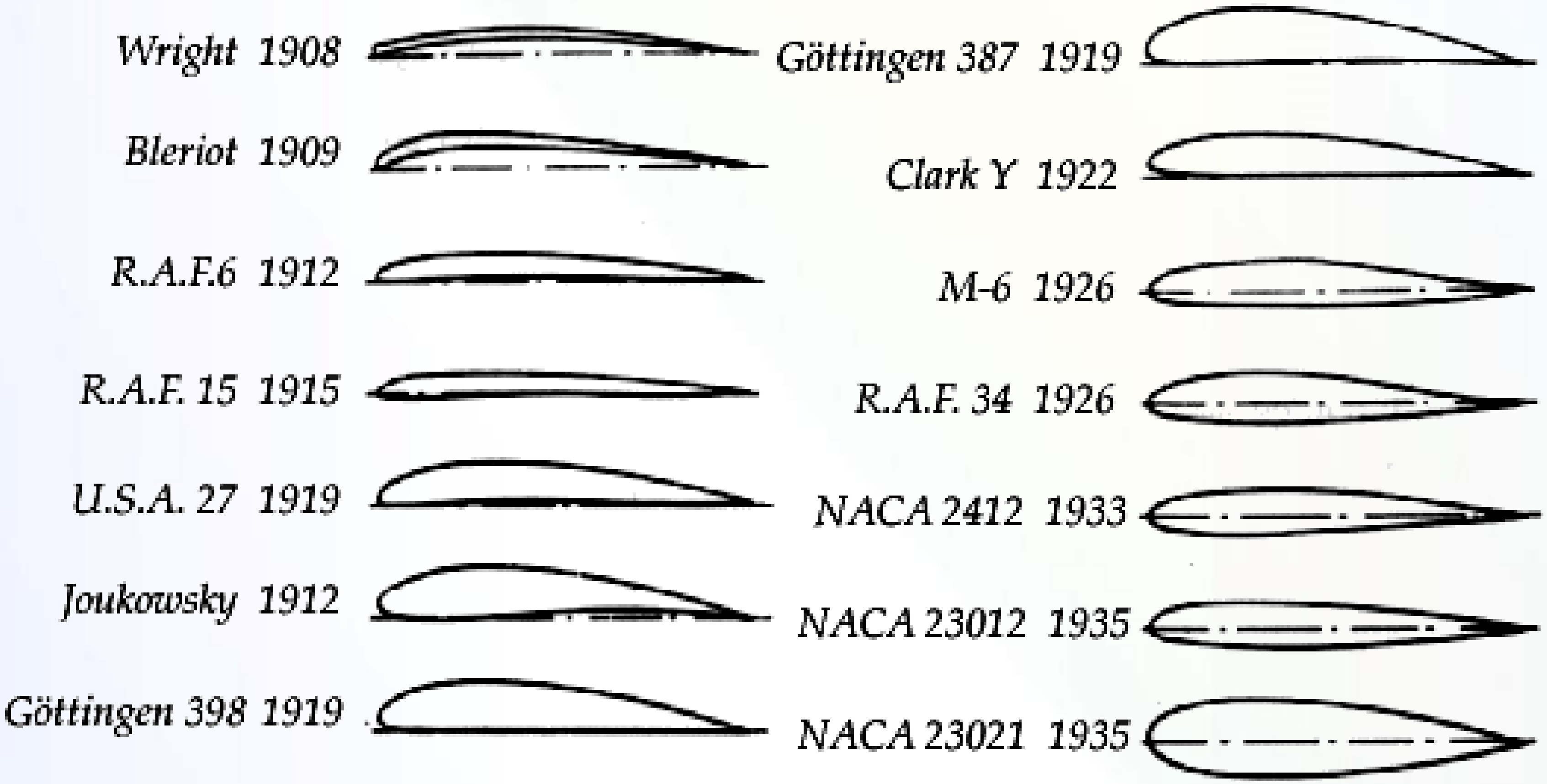


- Any section of the wing cut by a plane parallel to the *xz* plane is called an ***airfoil***.

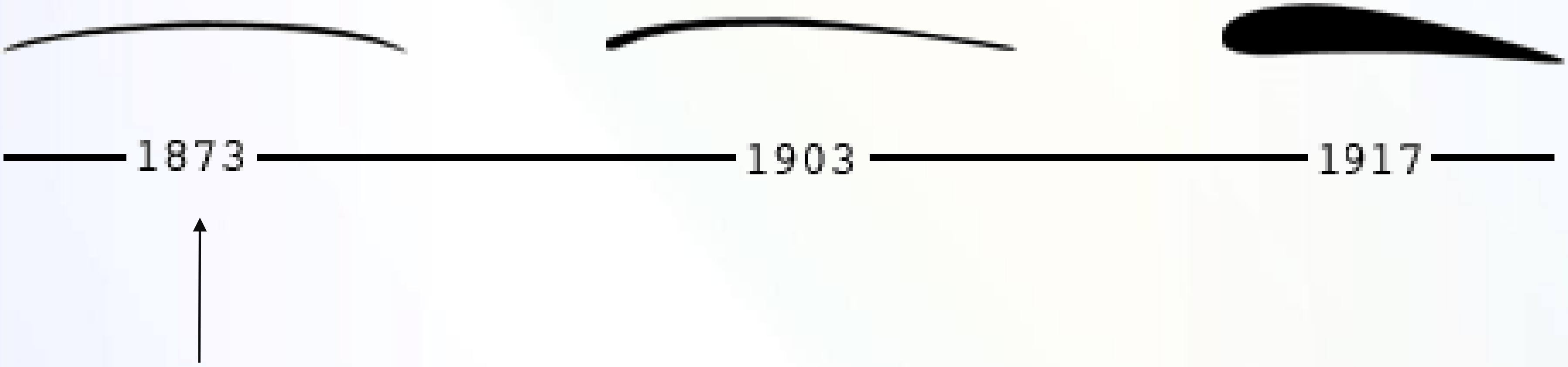
WHAT IS AN AIRFOIL?



EVOLUTION OF AIRFOILS

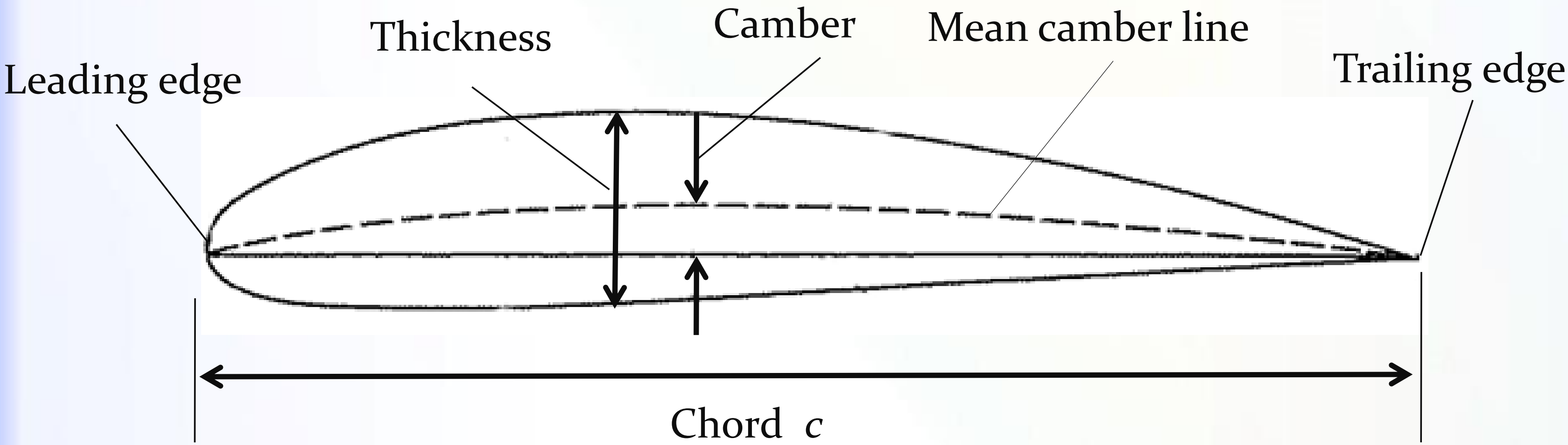


EVOLUTION OF AIRFOILS



Early Designs - Designers mistakenly believed that these airfoils with sharp leading edges will have low drag. In practice, they stalled quickly, and generated considerable drag.

AIRFOIL NOMENCLATURE



AIRFOIL GEOMETRY

Airfoil geometry is often characterized by a few parameters such as:

- Maximum thickness
- Maximum camber
- Position of max thickness
- Position of max camber
- Nose radius.

One can generate a reasonable airfoil section given these parameters.

NACA AIRFOILS

The **NACA** identified different airfoil shapes with a logical numbering system.

The primary reference volume for all the NACA subsonic airfoil studies remains:

Abbott, I.H., and Von Doenhoff, A.E., *“Theory of Wing Sections”, Dover, 1959.*

THE NACA FOUR-DIGIT AIRFOIL

- The first family of NACA airfoils, developed in the 1930s, was the "four-digit" series.
- The numbering system for these airfoils is defined by:

NACA MPXX

Where:

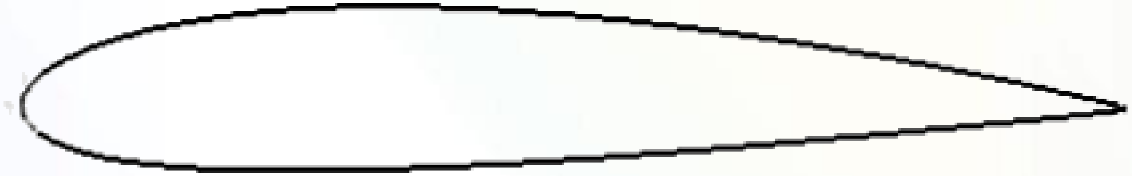
M is the maximum camber in *hundredths of chord*.

P is the location of the maximum camber in *tenths of the chord*.

XX is the maximum thickness, *t/c, in percent chord.*

THE NACA FOUR-DIGIT AIRFOIL - EXAMPLE

NACA 2415



The maximum camber is 0.02c

Maximum camber is located at 0.4c from the leading edge.

The maximum thickness is 0.15c

THE NACA FIVE-DIGIT AIRFOIL

- This airfoil is an extension of the 4 digit series. The numbering system for these airfoils is defined by:

NACA LMMXX

Where:

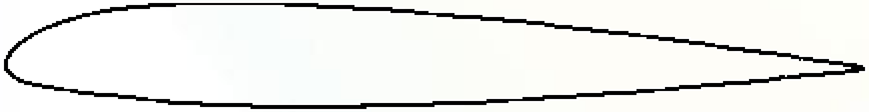
L: is the amount of camber; the design lift coefficient is 3L/2, in tenths

MM: the location of maximum camber along the chord from the leading edge is MM/2, in hundredths of the chord

XX: is the maximum thickness, t/c, in percent chord.

THE NACA FIVE-DIGIT AIRFOIL - EXAMPLE

NACA 23012



12% thick airfoil,

The design lift coefficient is 0.3,

The position of max camber is located at x/c = 0.15,

The “standard” 5 digit foil camber line is used.

THE NACA 6-SERIES LAMINAR FLOW AIRFOIL

- One of the most widely used family of NACA airfoils is the "6-series" laminar flow airfoils, developed during World War II.

AB,C-DEE

Where:

A: Is the series designation.

B: Location of minimum pressure in tenth of chord from the leading edge *(for the basic symmetric thickness distribution at zero lift)*

C: The range of lift coefficient in tenth above and below the design lift coefficient in which favourable pressure gradients exist on both surfaces

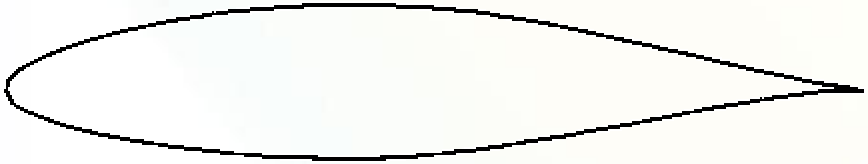
D: The design lift coefficient in tenth

EE: the maximum thickness in hundredths of chord

After the six-series sections, airfoil design became much more specialized for the particular application.

THE NACA 6-SERIES LAMINAR FLOW AIRFOIL - EXAMPLE

NACA 65,3-218



6 is the series designation.

The maximum pressure occurs at 0.5c for the basic symmetric thickness distribution at zero lift.

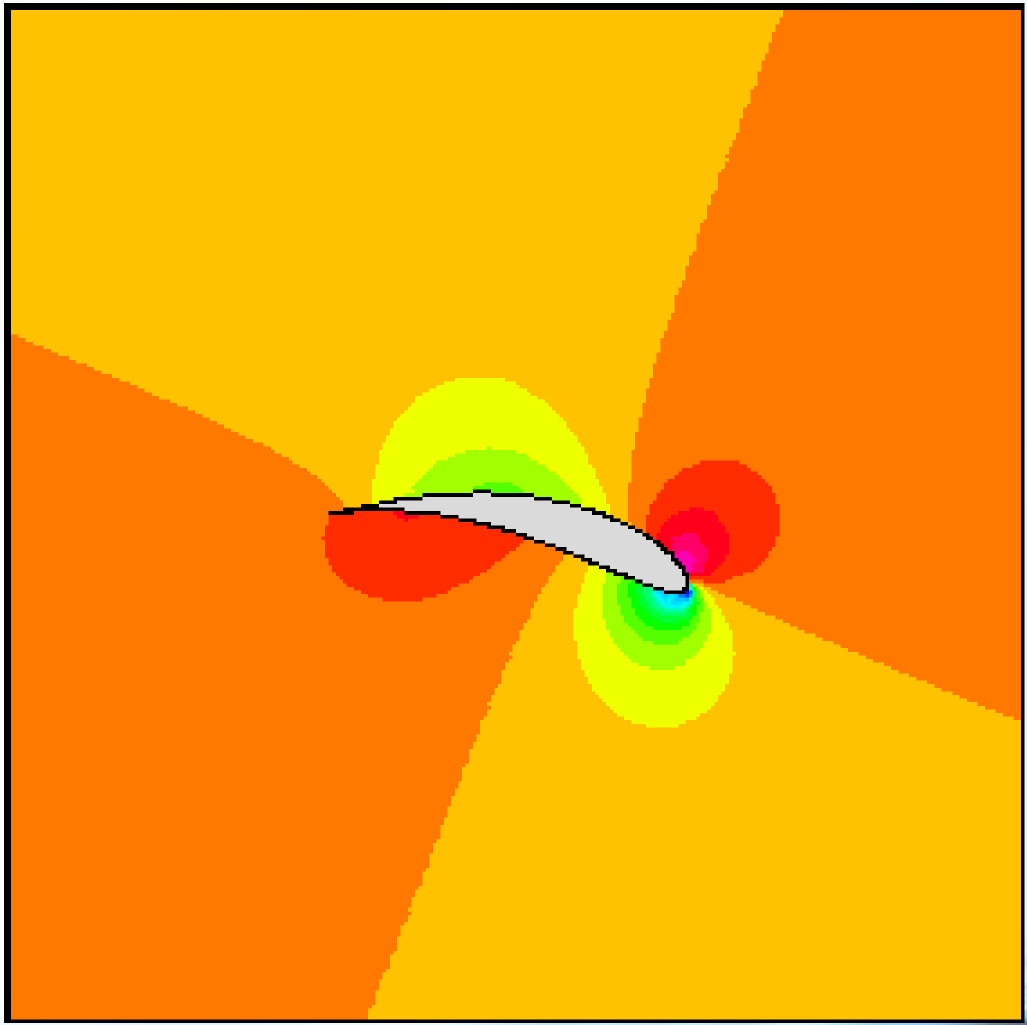
The range of lift coefficient above and below the design lift coefficient in which favourable pressure gradients exist on both surfaces is 0.3

The design lift coefficient is 0.2.

The airfoil is 18 percent thick.

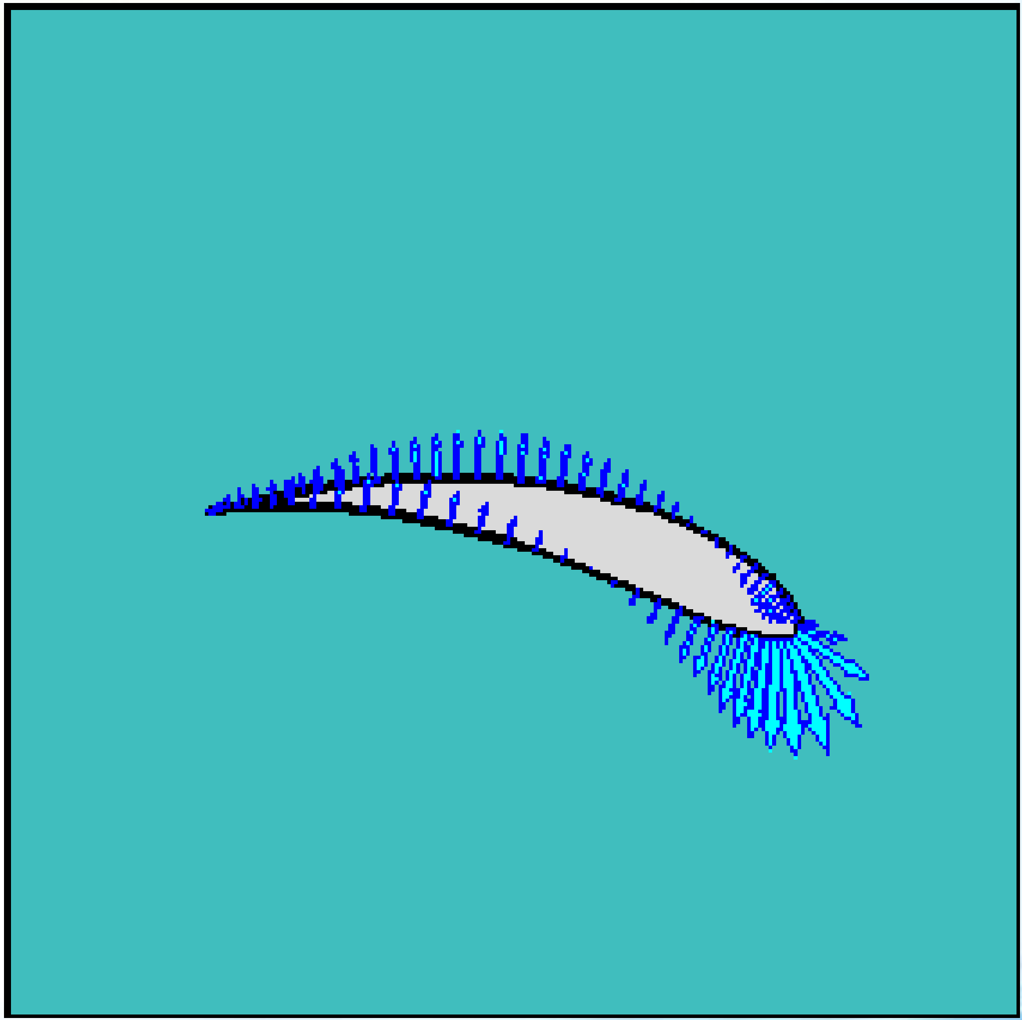
LIFT GENERATION

- Generation of lift by an airfoil is due to the imbalance of pressure distribution over top and bottom surfaces.



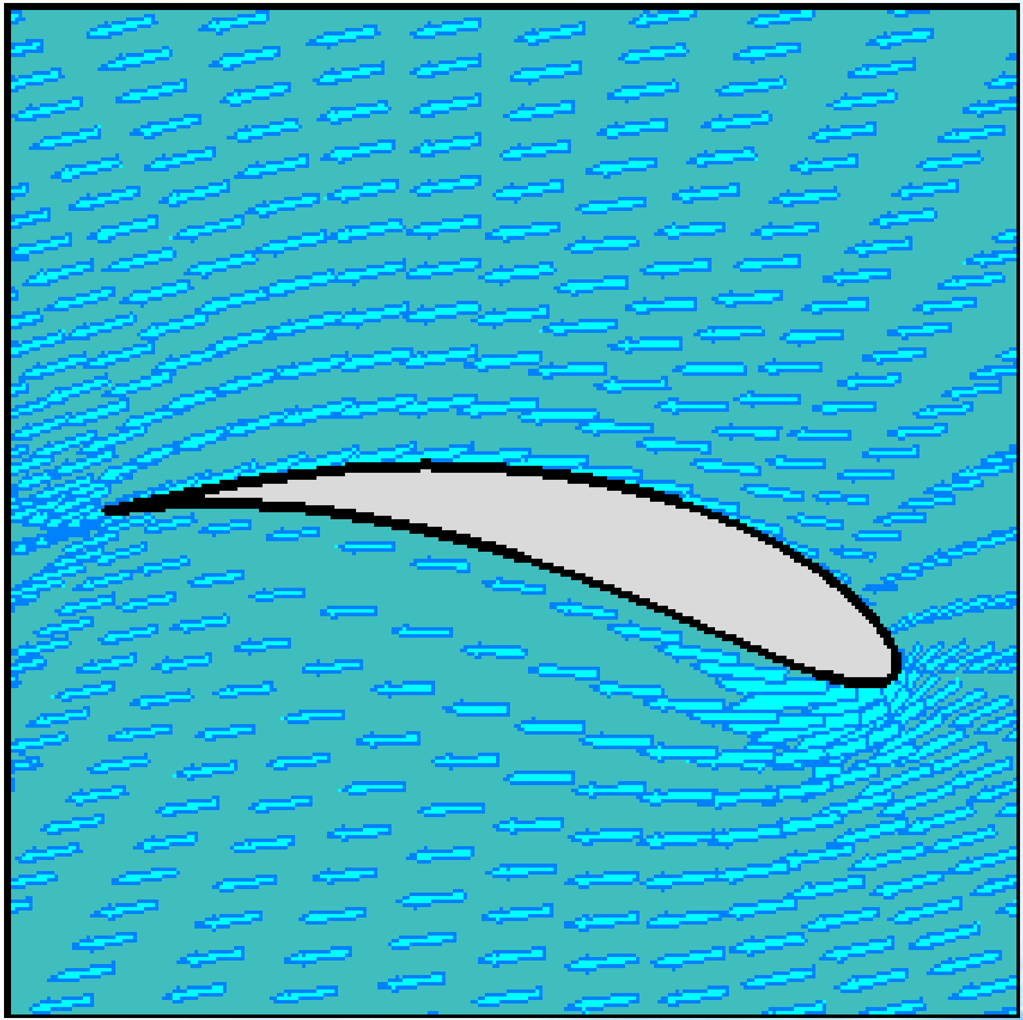
LIFT GENERATION

- If pressure on top is lower than pressure on bottom surface, lift is generated.



LIFT GENERATION

- Flow velocity over the top of airfoil is faster than over bottom surface.

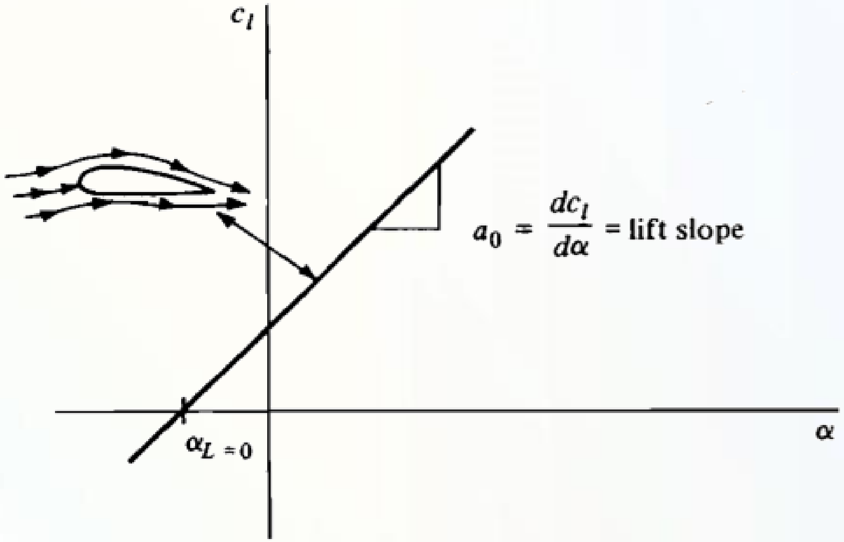


VARIATION OF LIFT WITH ANGLE OF ATTACK

- The lift coefficient of an airfoil changes as the Angle-of-Attack changes.

VARIATION OF LIFT WITH ANGLE OF ATTACK
LOW-TO-MODERATE ANGLES OF ATTACK

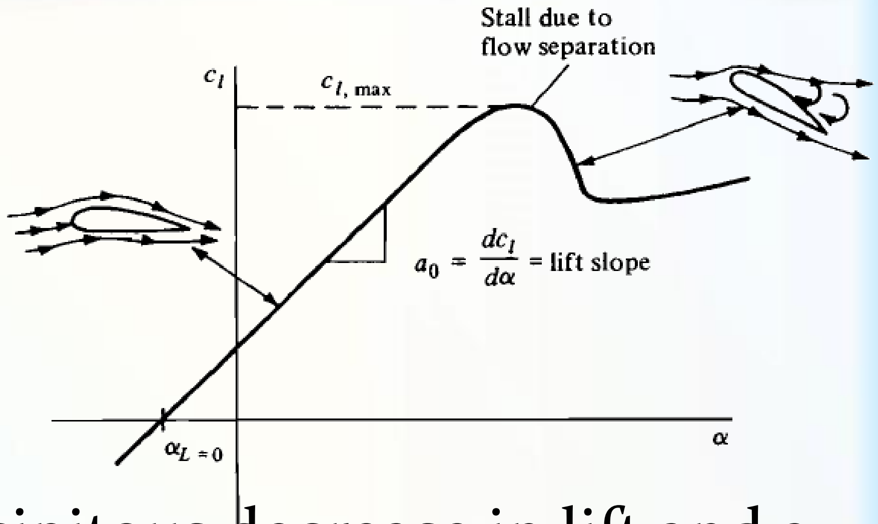
- At low-to-moderate angles of attack, c_l varies linearly with α.



- The slope of this straight line is called the *lift slope*.

- In this region, the flow moves smoothly over the airfoil and is attached over most of the surface.

VARIATION OF LIFT WITH ANGLE OF ATTACK
HIGHT ANGLES OF ATTACK



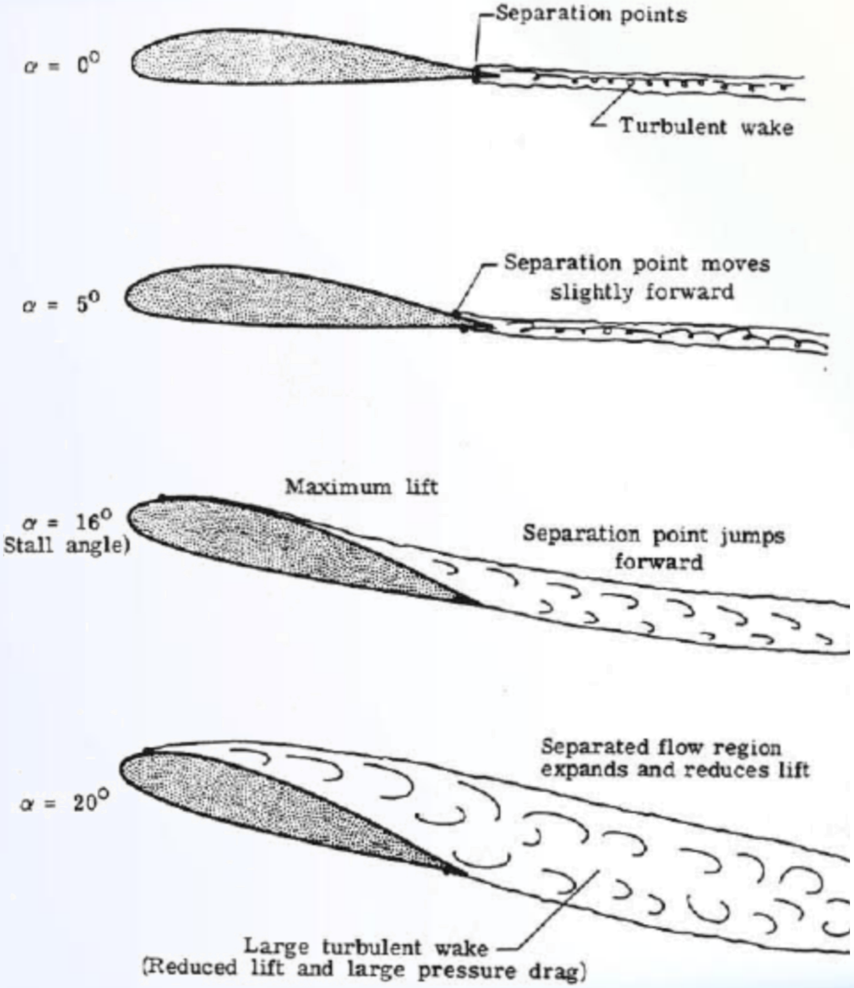
- As α becomes large, the flow tends to separate from the top surface of the airfoil.

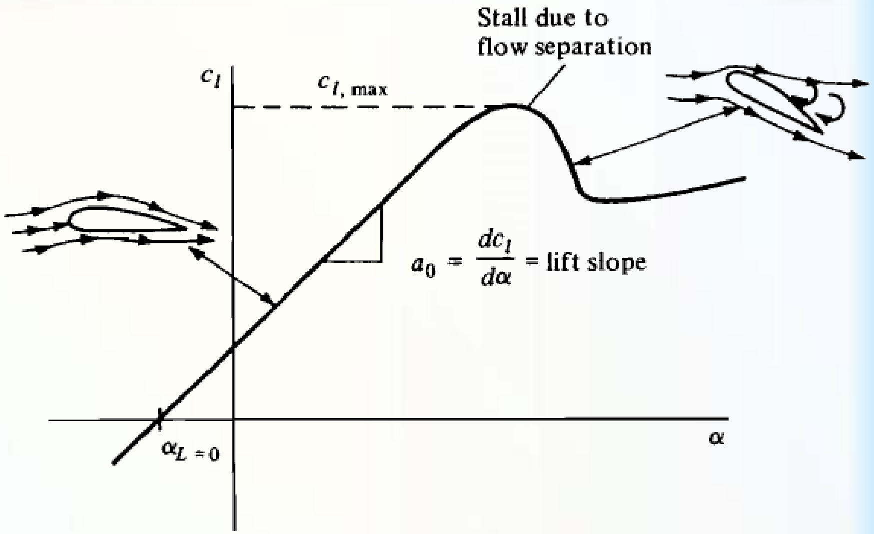
- The consequance of this separated flow at high α is a precipitous decrease in lift and a large increase in drag.

- Under such conditions, the airfoil is said to be *stalled*.

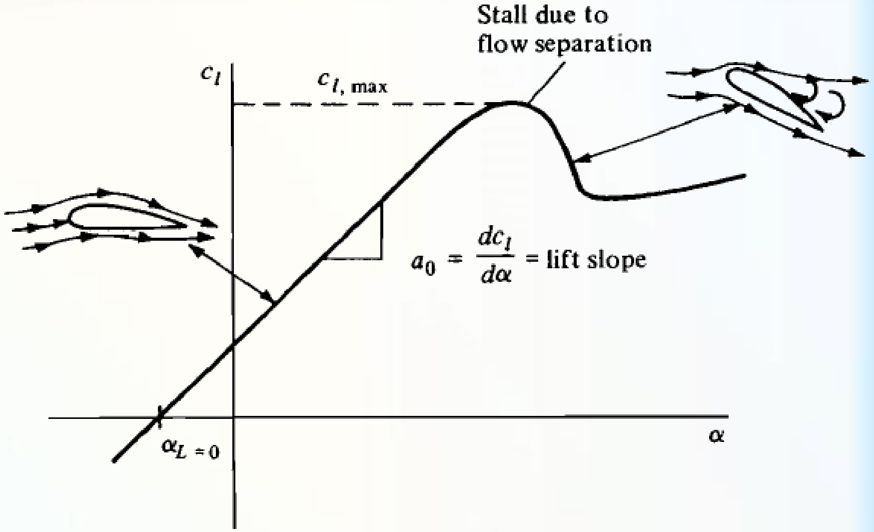
- The maximum value of c_l, which occurs just prior to the stall, is denoted by $c_{l,max}$.

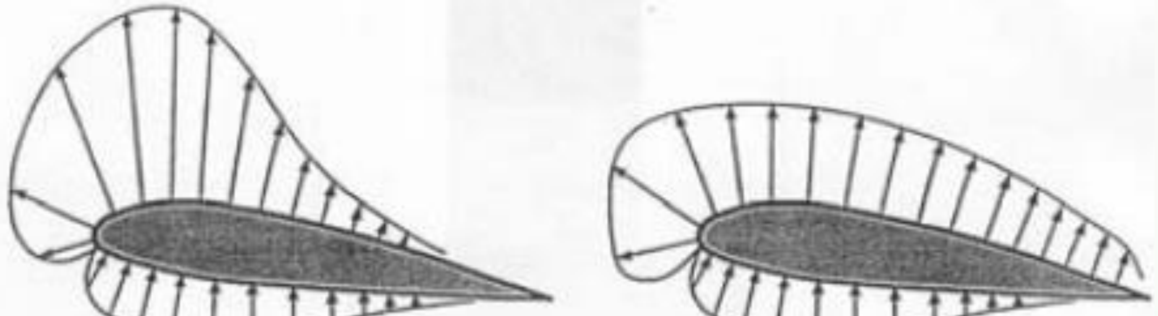
VARIATION OF LIFT WITH ANGLE OF ATTACK





VARIATION OF LIFT WITH ANGLE OF ATTACK

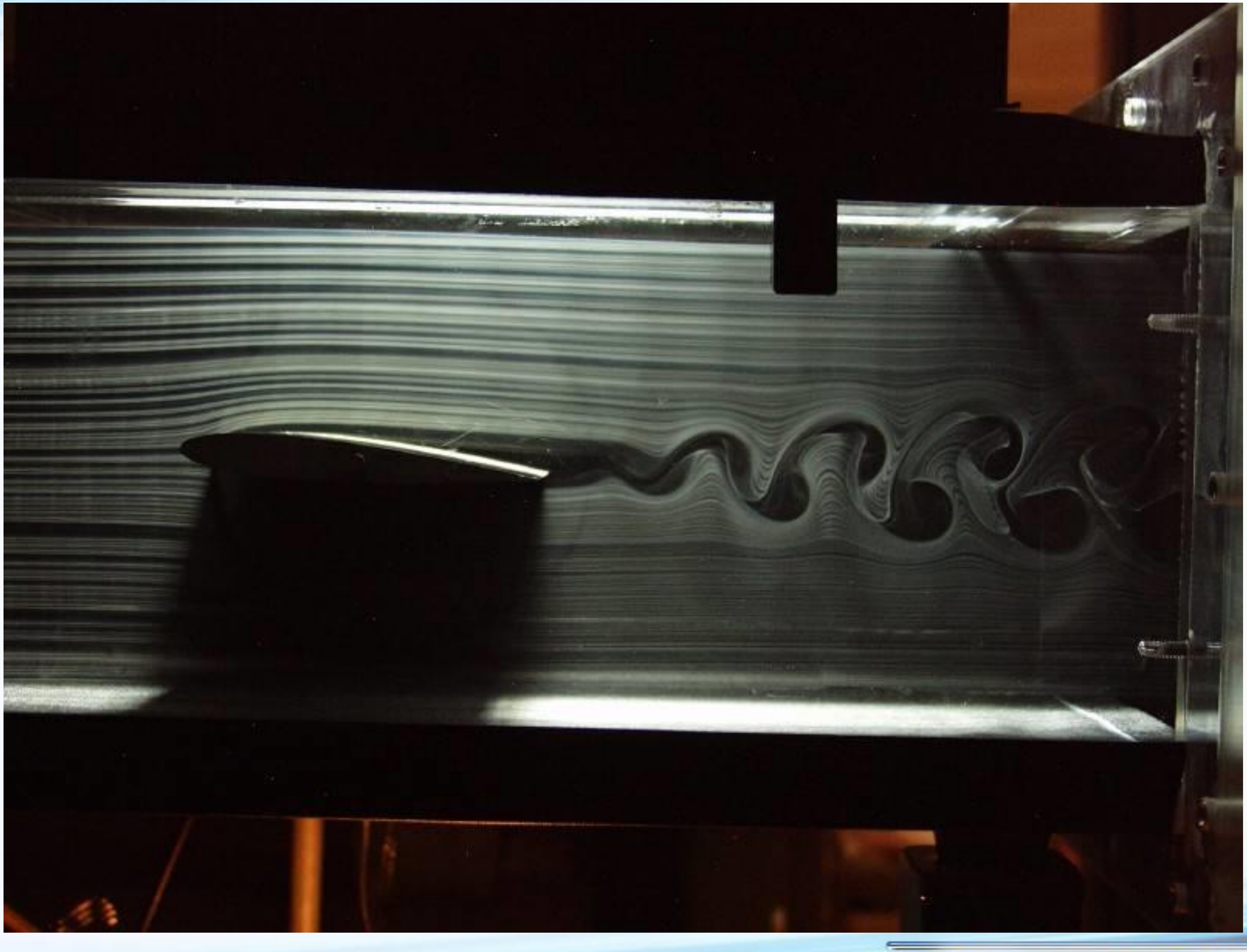




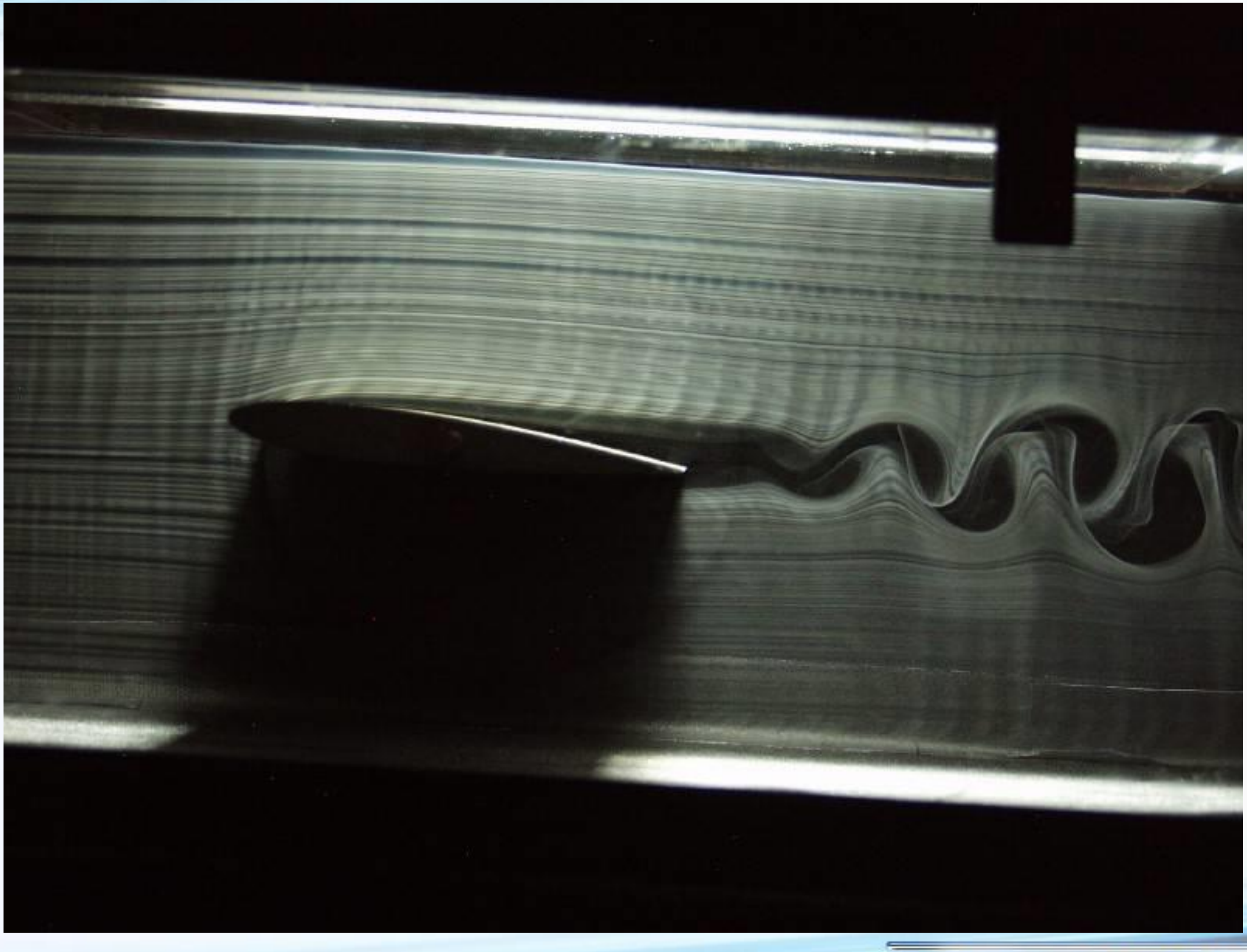
Just Before Stalling

Just After Stalling

VARIATION OF LIFT WITH ANGLE OF ATTACK (α=2 deg)

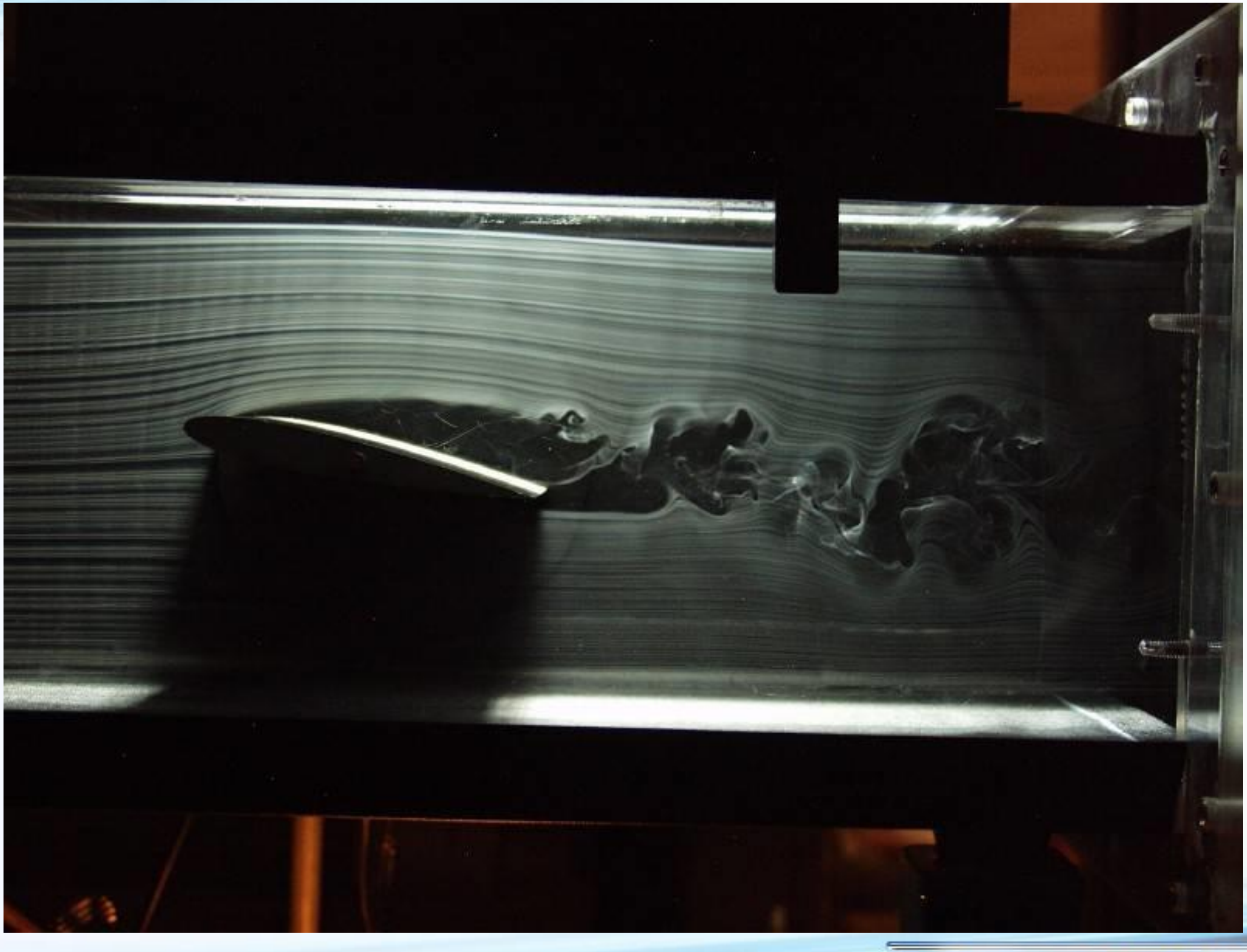


VARIATION OF LIFT WITH ANGLE OF ATTACK (α=3 deg)

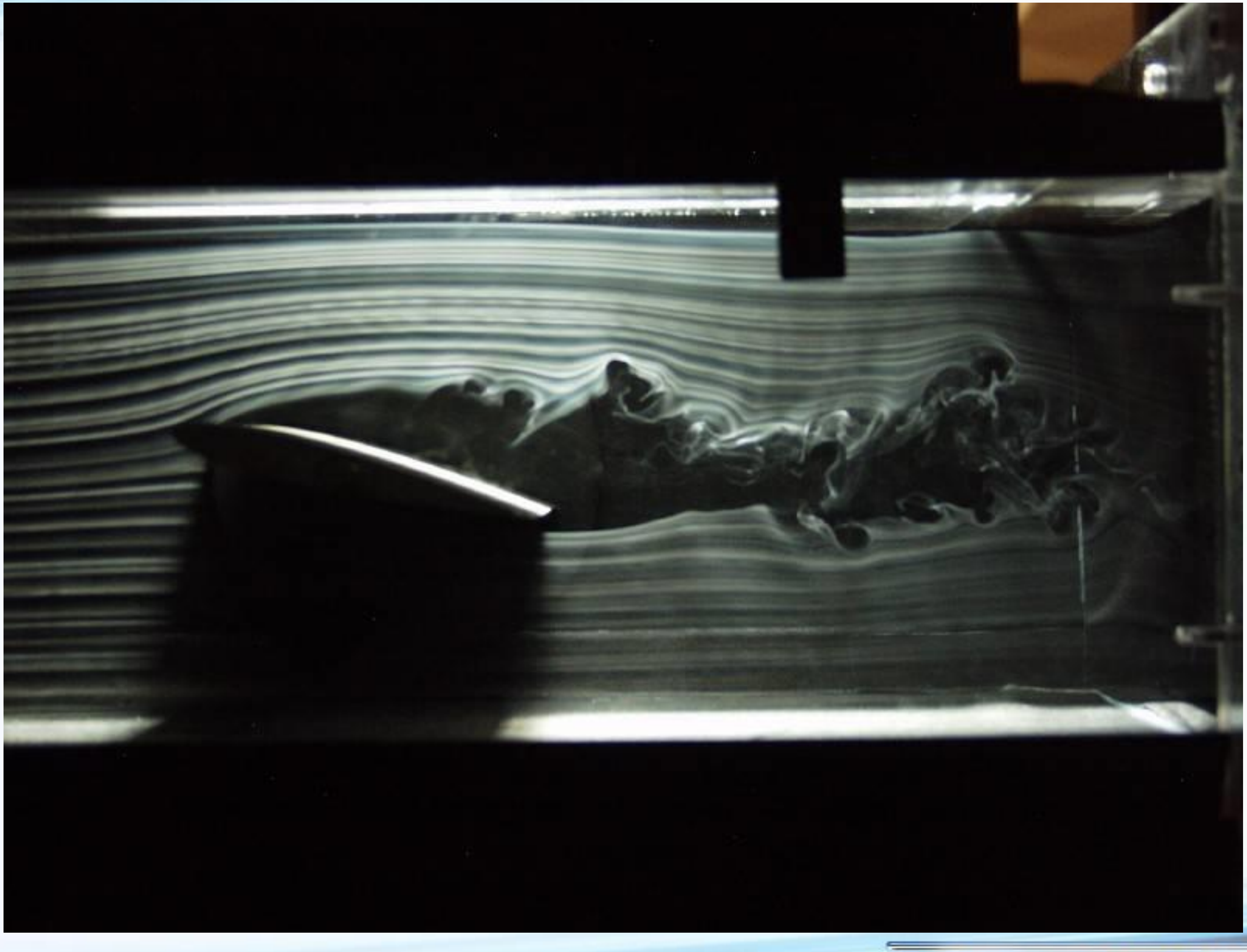


VARIATION OF LIFT WITH ANGLE OF ATTACK
(α=6 deg)

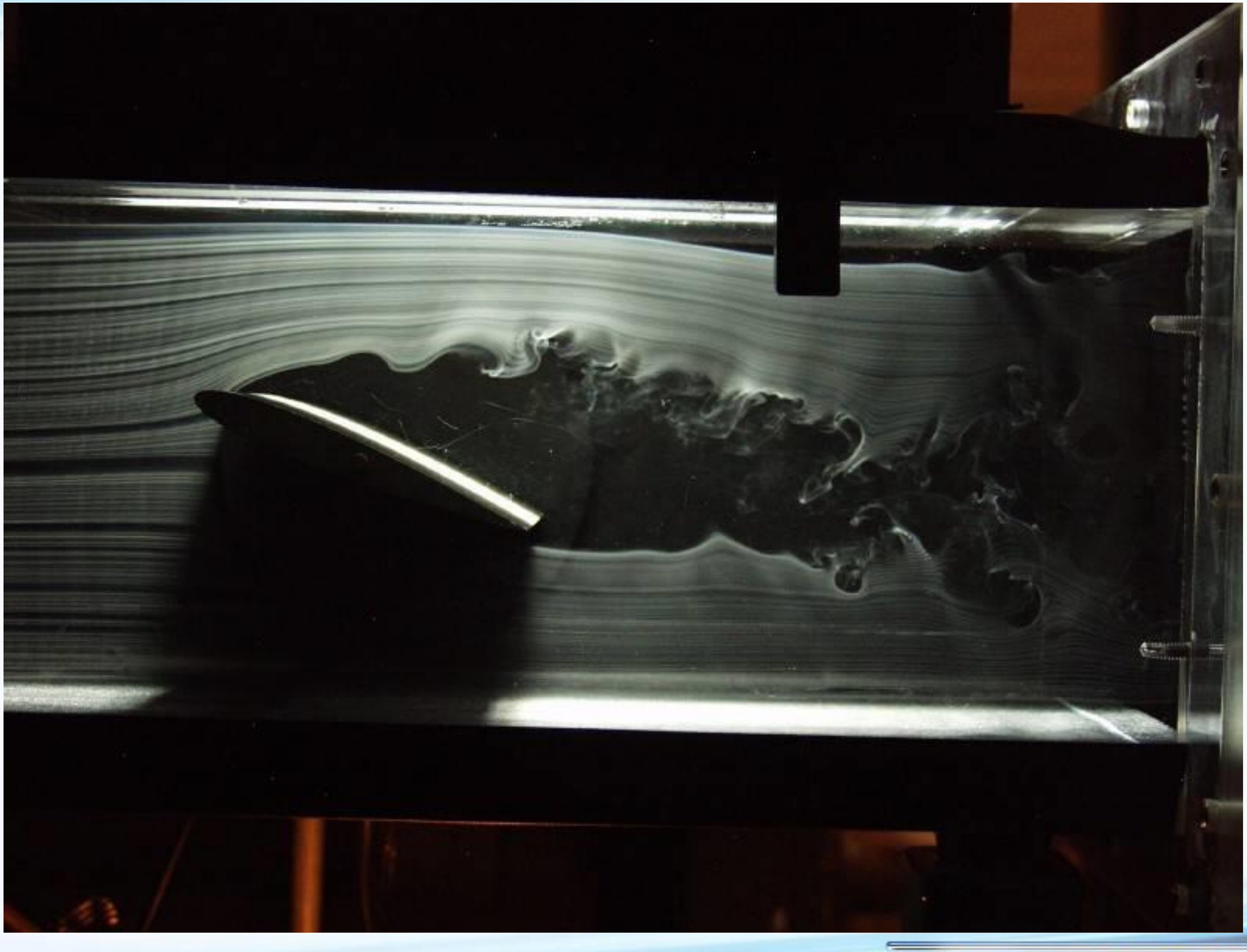
VARIATION OF LIFT WITH ANGLE OF ATTACK (α=9 deg)



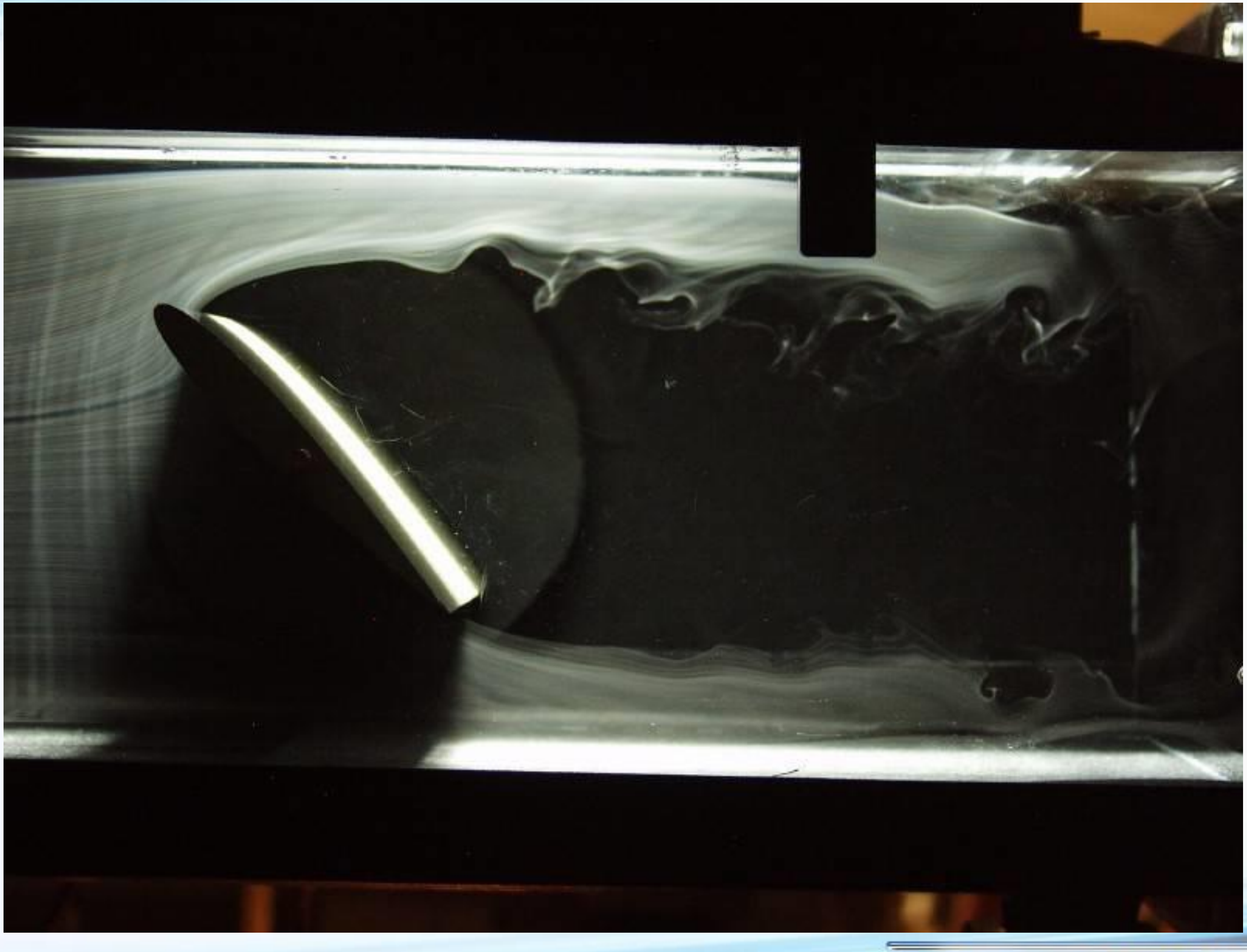
VARIATION OF LIFT WITH ANGLE OF ATTACK
(α=12 deg)



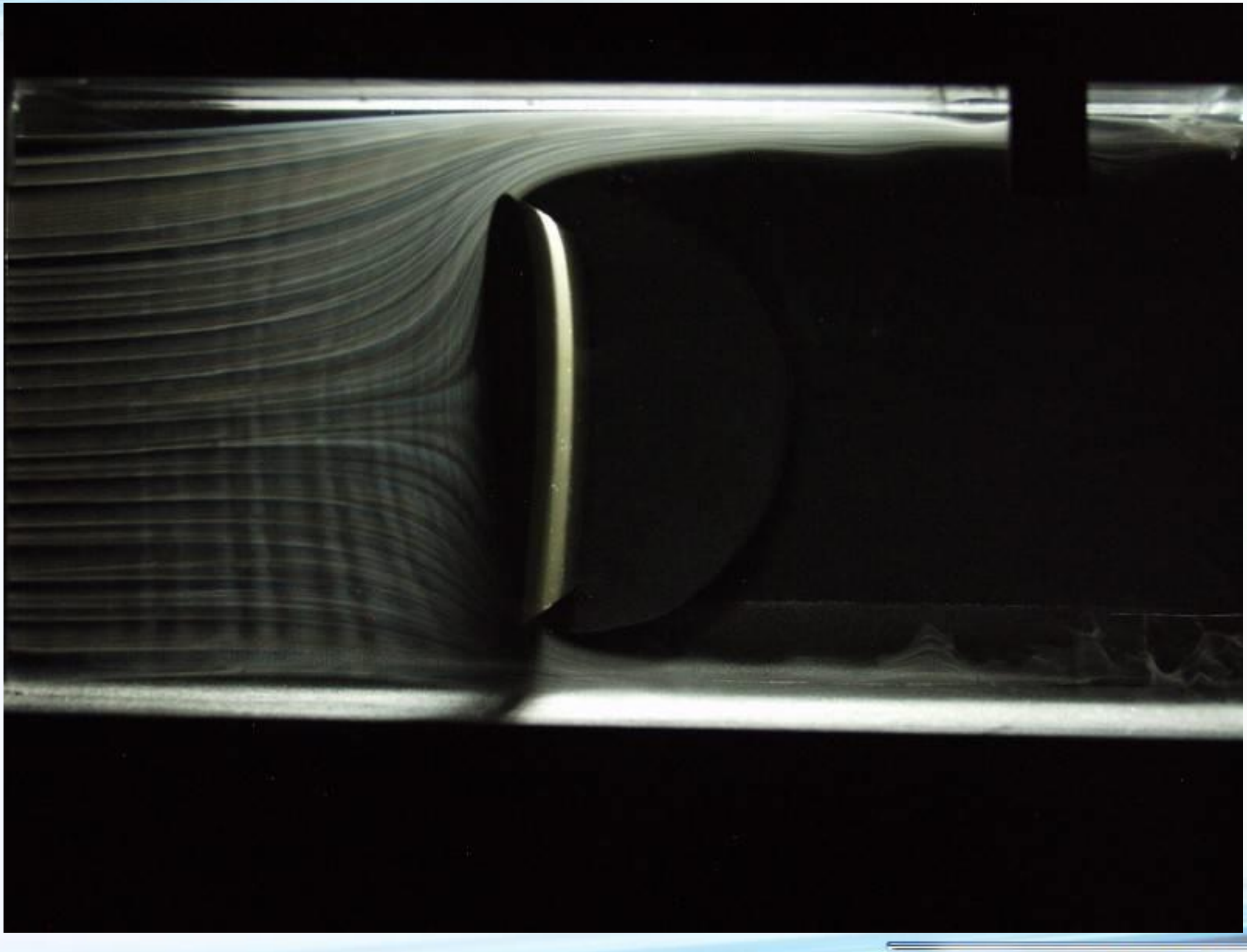
VARIATION OF LIFT WITH ANGLE OF ATTACK
(α=20 deg)



VARIATION OF LIFT WITH ANGLE OF ATTACK (α=60 deg)

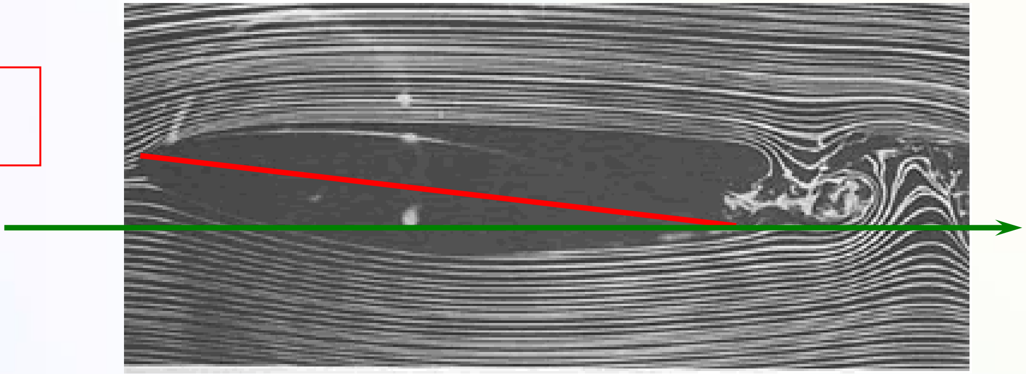


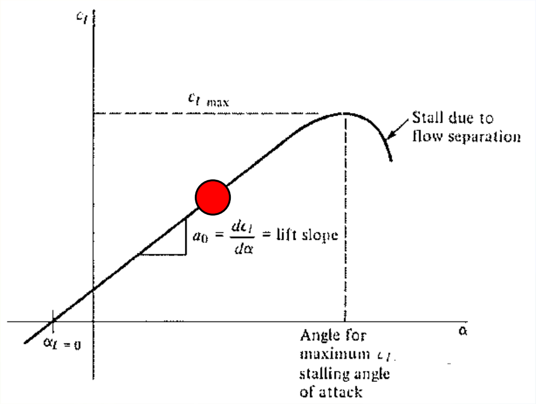
VARIATION OF LIFT WITH ANGLE OF ATTACK (α=90 deg)



VARIATION OF LIFT WITH ANGLE OF ATTACK (α=90 deg)

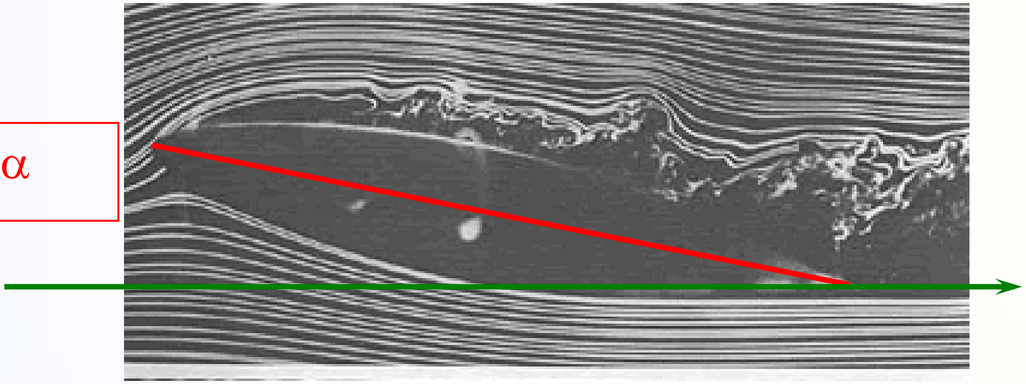
Low α

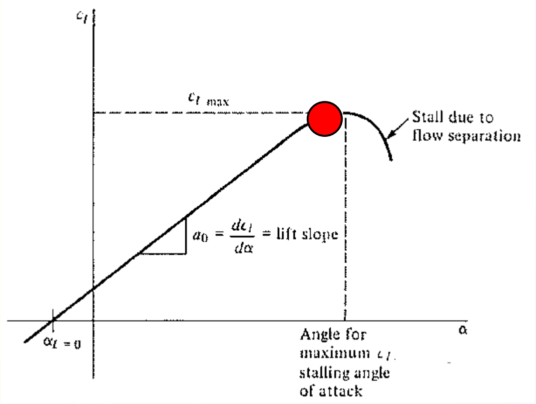




VARIATION OF LIFT WITH ANGLE OF ATTACK
(α=90 deg)

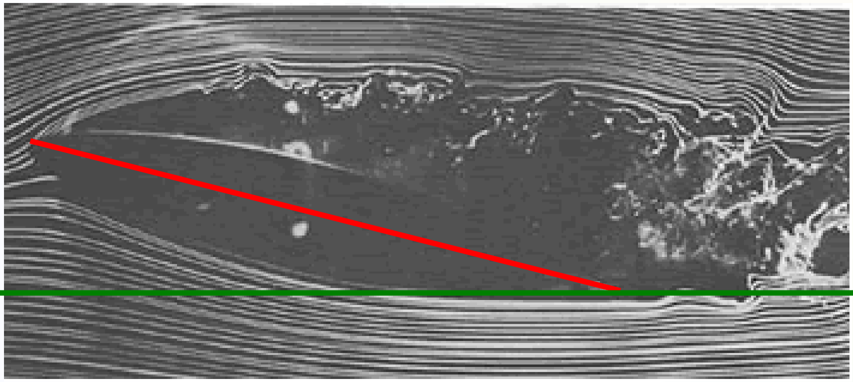
Moderate α

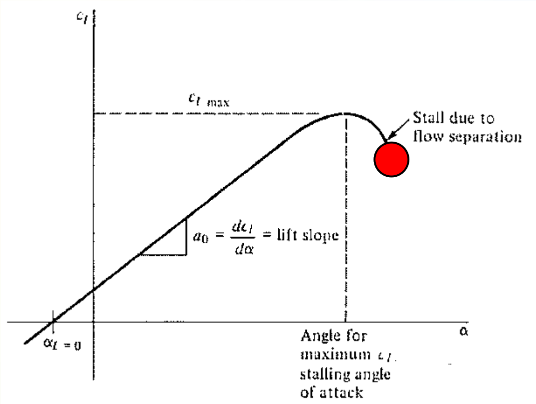




VARIATION OF LIFT WITH ANGLE OF ATTACK
(α=90 deg)

High α

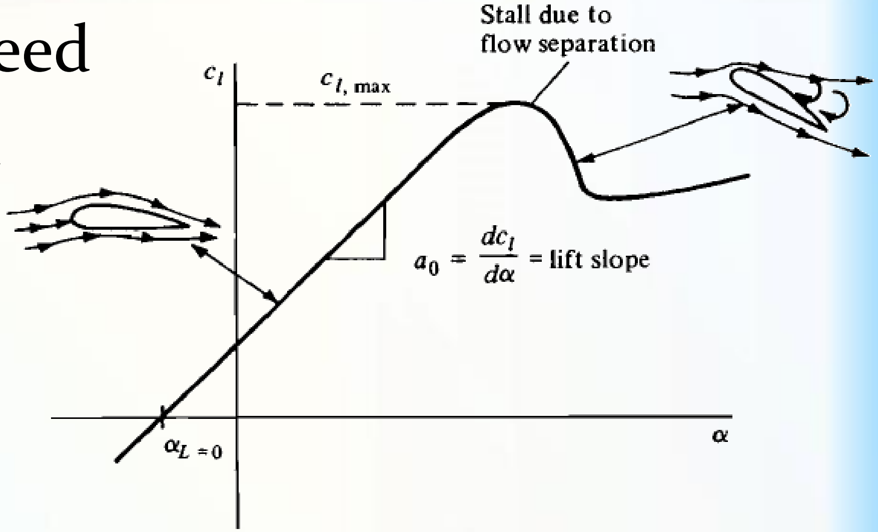




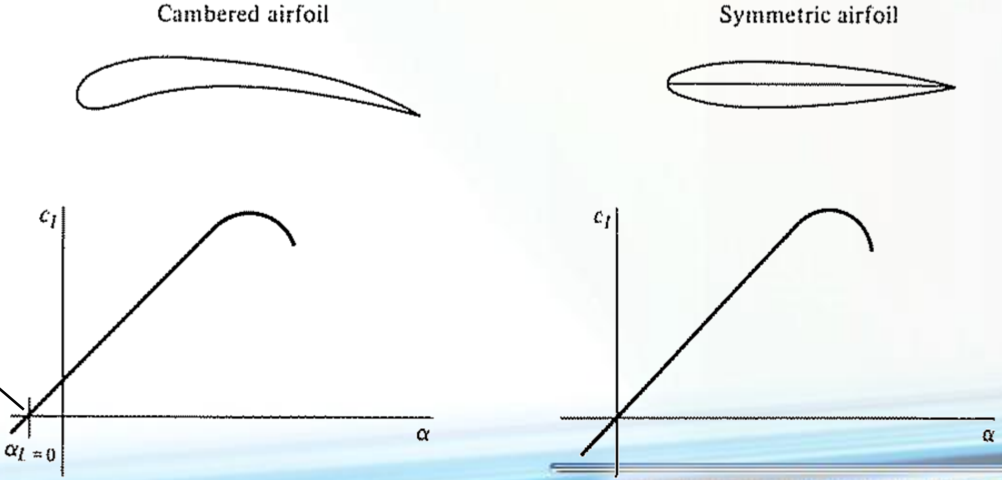
VARIATION OF LIFT WITH ANGLE OF ATTACK

- $c_{l,max}$ determines the stalling speed of an airplane. The higher is $c_{l,max}$ the lower is the stalling speed.

- The value of α when lift equals zero is called the "*zero-lift angle of attack*" ($\alpha_{L=0}$).



At negative α airfoil will have zero lift


Cambered airfoil

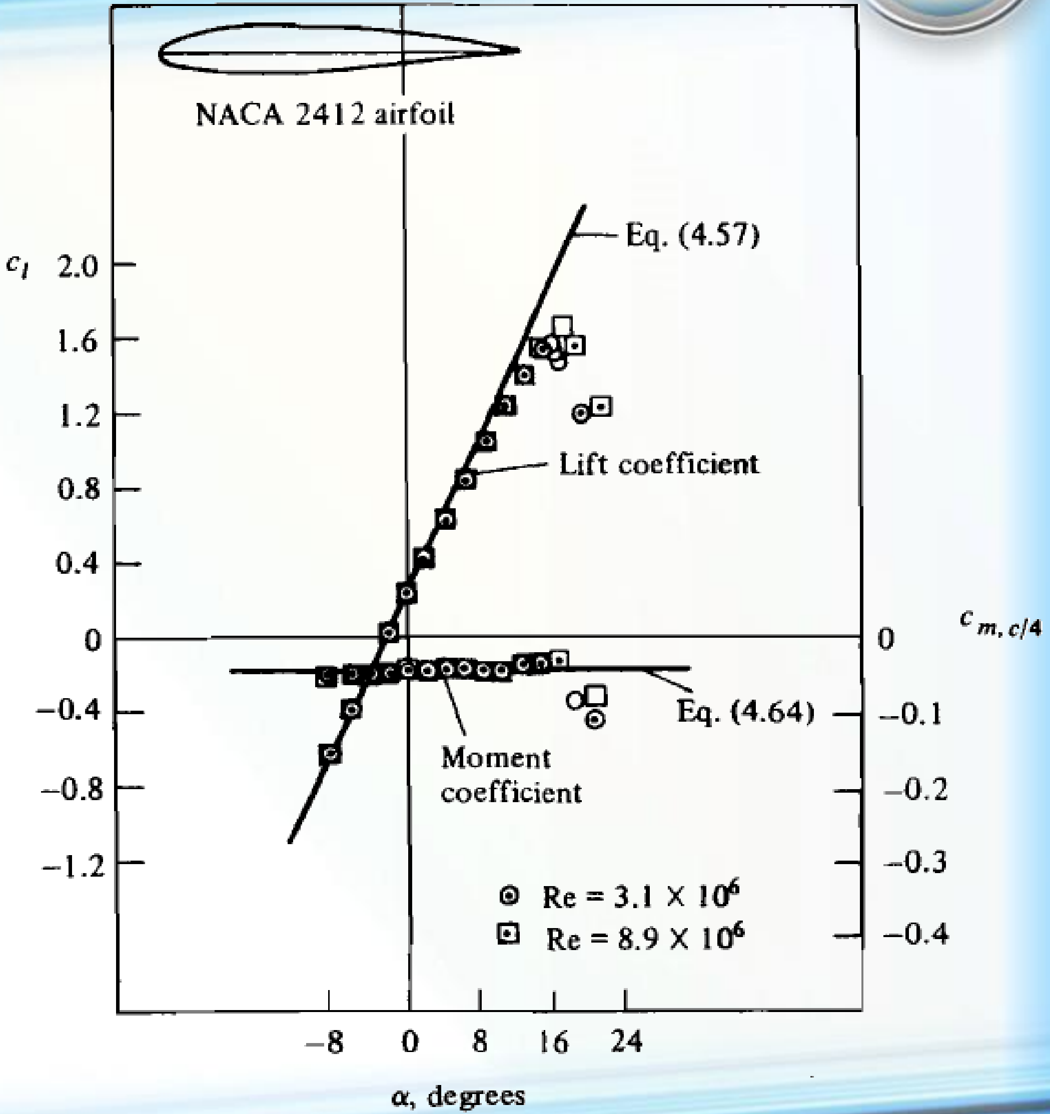
Symmetric airfoil

EFFECT OF REYNOLD NUMBER

- The lift slope is not influenced by *Re*.

- $c_{l,max}$ is dependent upon *Re*.

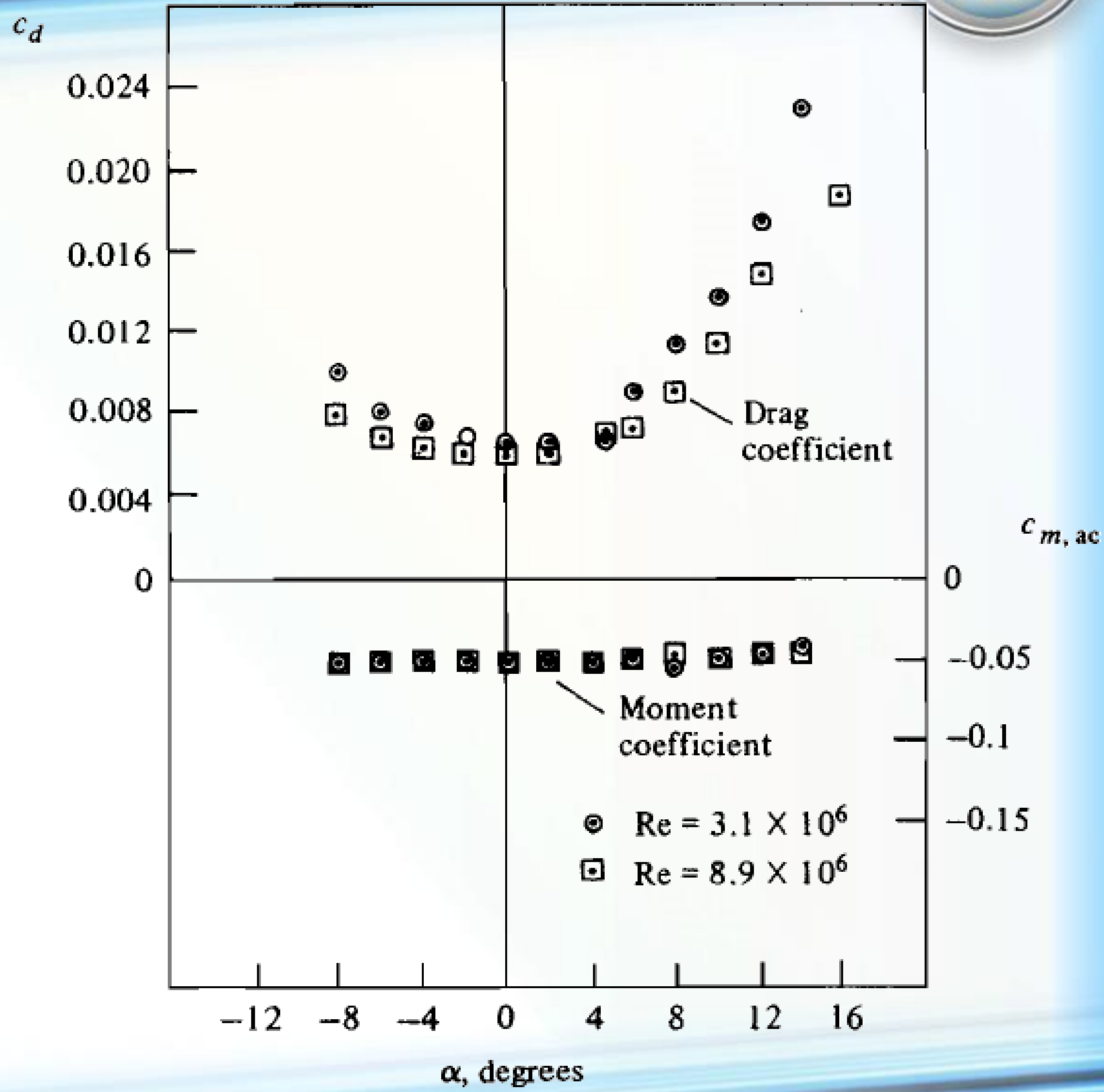
- The moment coefficient is insensitive to *Re* except At large α.



EFFECT OF REYNOLD NUMBER

- The sum of *skin friction drag* and *pressure drag* yields the *profile drag*.

Profile drag coefficient is sensitive to *Re*.



EXAMPLE

Consider an NACA 2412 airfoil with a chord of 0.64 m in an airstream at standard sea level conditions. The freestream velocity is 70 m/s. The lift per unit span is 1254 N/m. Calculate the angle of attack and the drag per unit span.

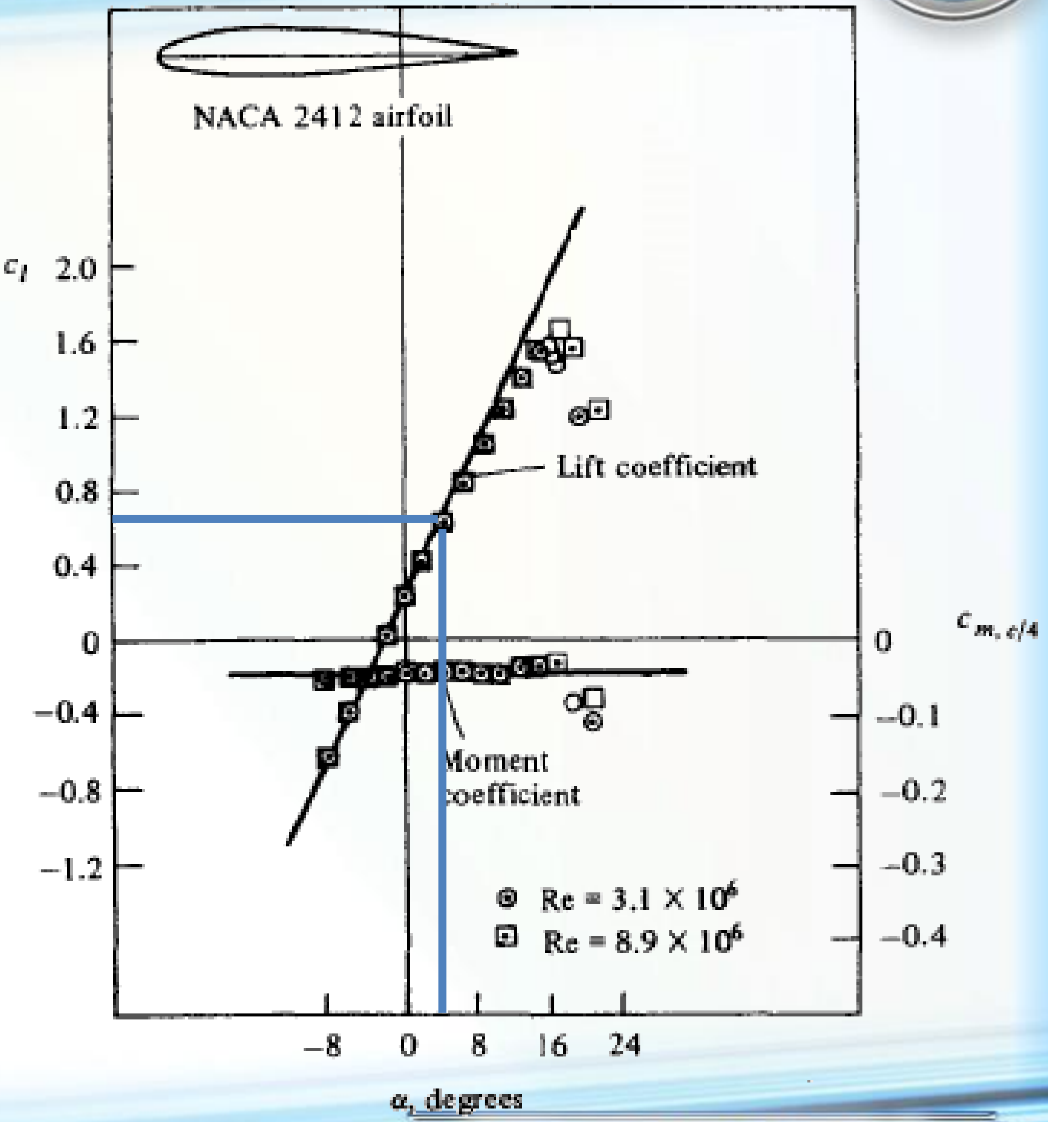
At standard sea level, $\rho = 1.23\ \text{kg/m}^3$:

$$q_\infty = \tfrac{1}{2}\rho_\infty V_\infty^2 = \tfrac{1}{2}(1.23)(70)^2 = 3013.5\ \text{N/m}^2$$

$$c_l = \frac{L'}{q_\infty S} = \frac{L'}{q_\infty c(1)} = \frac{1254}{3013.5(0.64)} = 0.65$$

EXAMPLE (CONT.)

$c_l = 0.65$, we obtain $\boxed{\alpha = 4^\circ}$.



EXAMPLE (CONT.)

To obtain the drag per unit span, we must use the data diagram.
Since $c_d = f(Re)$, let us calculate Re. At standard sea level, $\mu = 1.789 \text{ x } 1\,0^{-5}$ kg/(m.s). Hence,

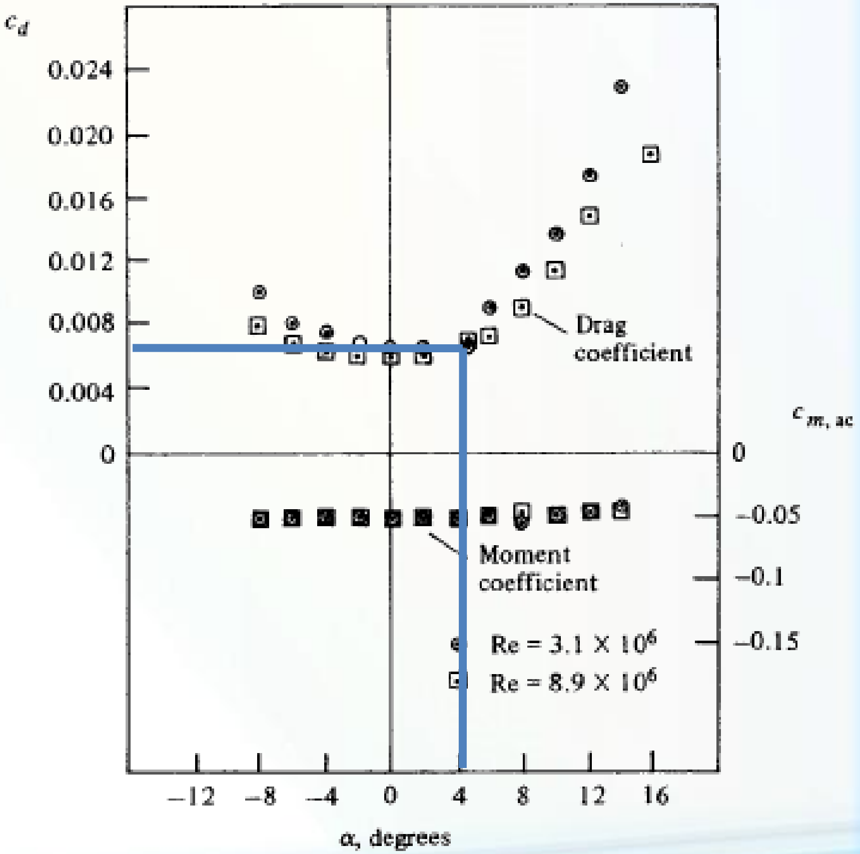
$$\mathrm{Re} = \frac{\rho_\infty V_\infty c}{\mu_\infty} = \frac{1.23(70)(0.64)}{1.789 \times 10^{-5}} = 3.08 \times 10^6$$

Therefore, using the data for $\mathrm{Re} = 3.1 \times 10^6$

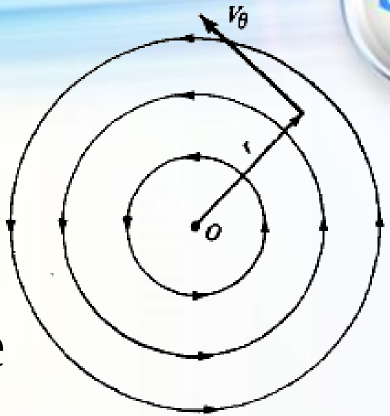
we find $c_d = 0.0068$. Thus,

$$D' = q_\infty S c_d = q_\infty c(1) c_d$$

$$= 3013.5(0.64)(0.0068) = \boxed{13.1 \text{ N/m}}$$



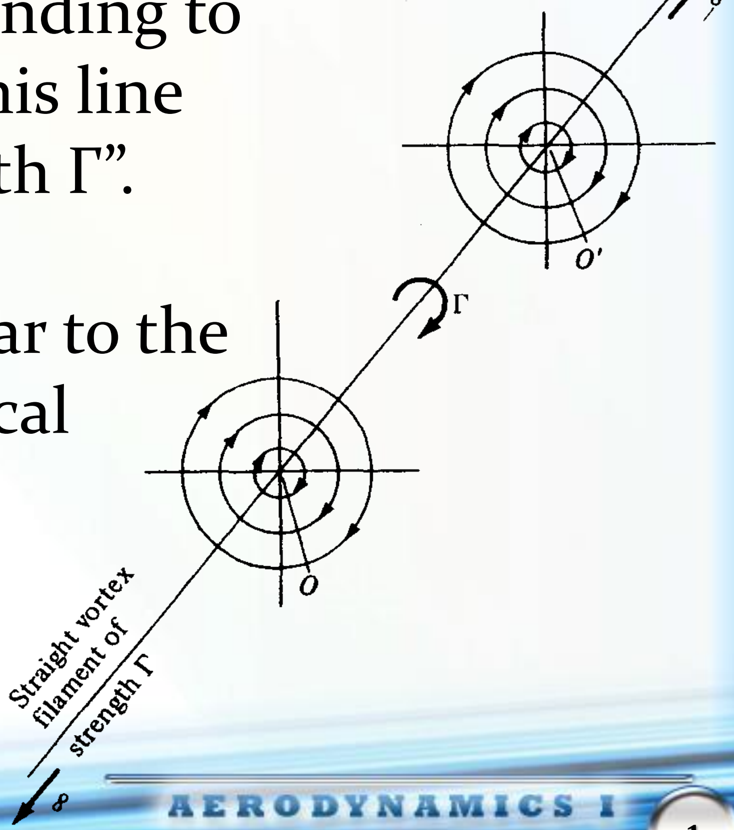
THE VORTEX FILAMENT



- Let us expand the concept a point vortex.

- Imagine a straight line perpendicular to the page, going through point O, and extending to infinity both out and into the page. This line is a straight “vortex filament of strength Γ”.

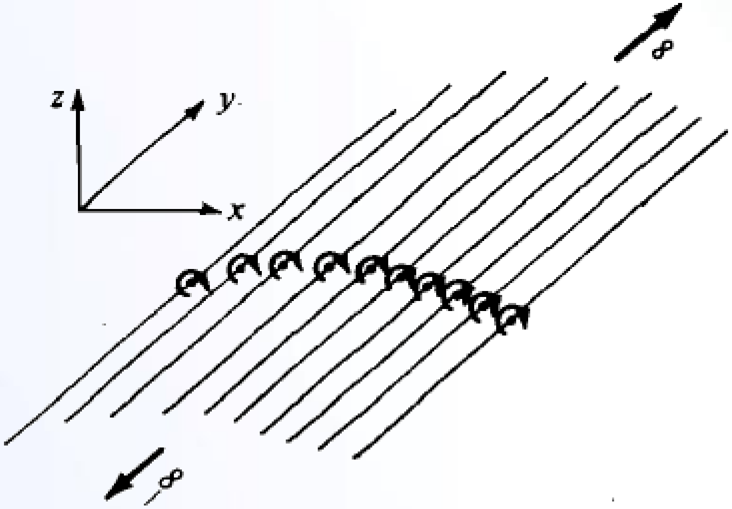
- The flows in the planes perpendicular to the vortex filament at O and O’ are identical to each other and are identical to the flow induced by a point vortex of strength Γ.



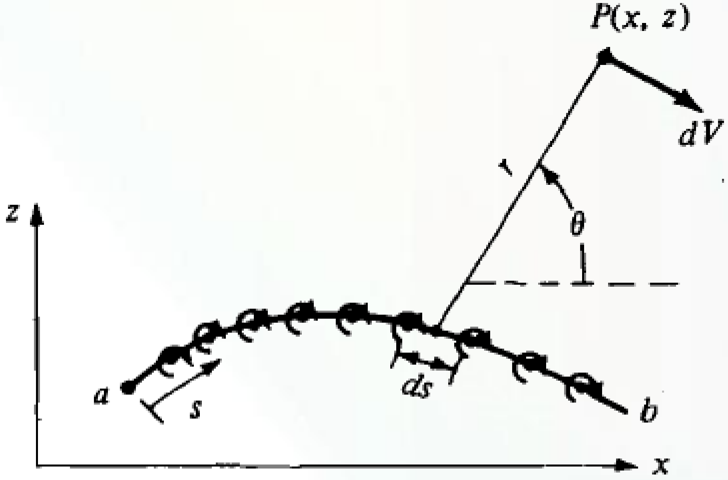
THE VORTEX SHEET

- Imagine an infinite number of straight vortex filaments side by side, where the strength of each filament is infinitesimally small.

- These side by side vortex filaments form a vortex sheet.

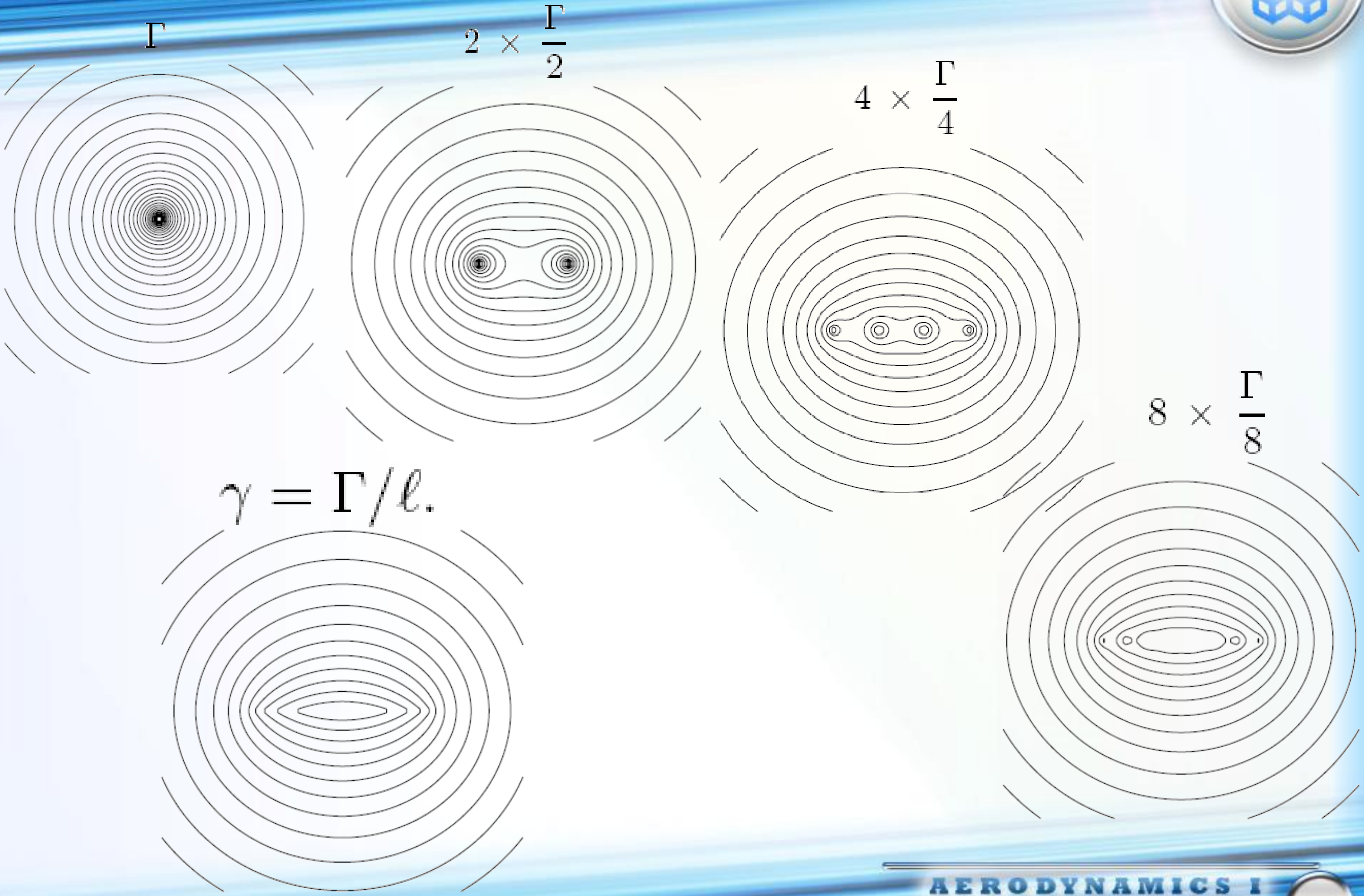


Vortex sheet in perspective



Edge view of sheet

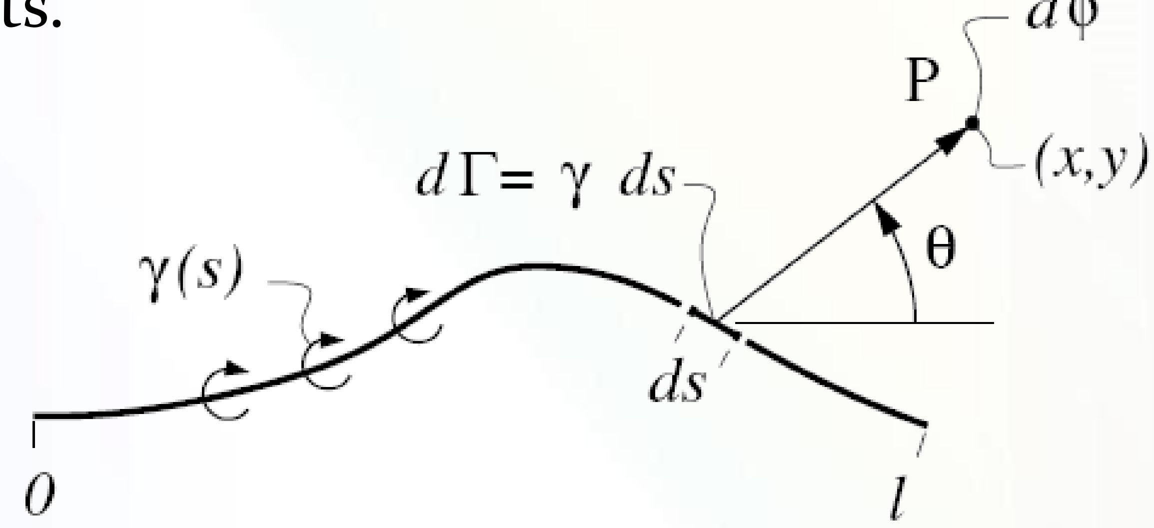
THE VORTEX SHEET



$$\gamma = \Gamma / \ell.$$

PROPERTIES OF VORTEX SHEETS

- The analysis of the vortex sheet closely follows that of the source sheets.



$$\phi(x,y) = -\int_0^{\ell} \frac{\gamma}{2\pi}\theta\, ds$$

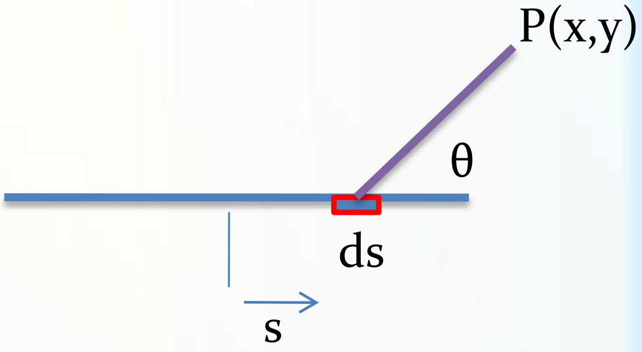
$$\Gamma = \int_0^{l} \gamma ds$$

PROPERTIES OF VORTEX SHEETS

- For a straight vortex sheet extending from (-l/2,0) to (l/2,0), with a constant strength γ, the potential and the velocity components at point P are given by:

$$\phi(x,y) = -\int_0^{\ell} \frac{\gamma}{2\pi}\,\theta\, ds$$

$$\phi(x,y) = \frac{\gamma}{2\pi}\int_{-\ell/2}^{\ell/2} -\arctan\frac{y}{x-s}\; ds$$

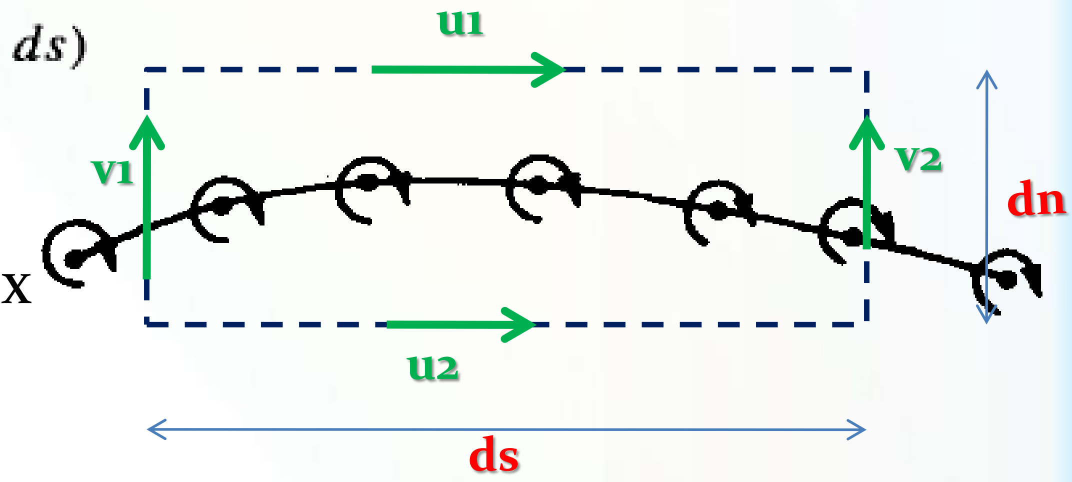


$$u(x,y) = \frac{\partial\phi}{\partial x} = \frac{\gamma}{2\pi}\int_{-\ell/2}^{\ell/2}\frac{\partial}{\partial x}\left[-\arctan\frac{y}{x-s}\right] ds = \frac{\gamma}{2\pi}\int_{-\ell/2}^{\ell/2}\frac{y}{(x-s)^2+y^2}\, ds$$

$$v(x,y) = \frac{\partial\phi}{\partial y} = \frac{\gamma}{2\pi}\int_{-\ell/2}^{\ell/2}\frac{\partial}{\partial y}\left[-\arctan\frac{y}{x-s}\right] ds = \frac{\gamma}{2\pi}\int_{-\ell/2}^{\ell/2}\frac{-x}{(x-s)^2+y^2}\, ds$$

THE VORTEX SHEET

- Consider a rectangular path enclosing a section of a vortex sheet of length ds. The circulation around the path is:



$$\Gamma = -(v_2\,dn - u_1\,ds - v_1\,dn + u_2\,ds)$$

$$\Gamma = (u_1 - u_2)\,ds + (v_1 - v_2)\,dn$$

- The strenght of the vortex sheet contained inside the path is: $\Gamma = \gamma\,ds$

$$\gamma\,ds = (u_1 - u_2)\,ds + (v_1 - v_2)\,dn$$

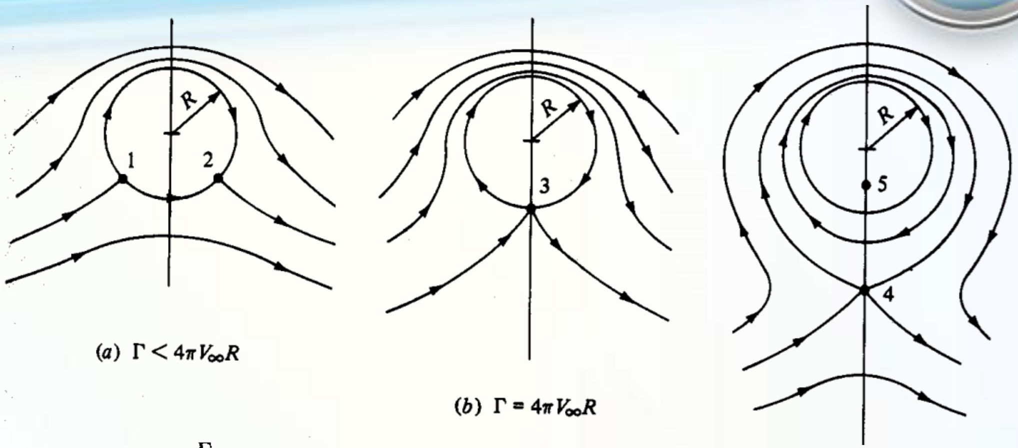
Let dn ⟶ 0

$$\gamma\,ds = (u_1 - u_2)\,ds$$

$$\boxed{\gamma = u_1 - u_2}$$

THE KUTTA CONDITION

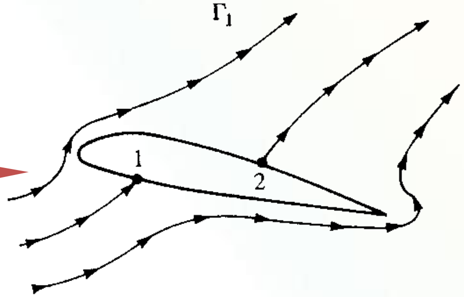
Potential flow with lift is not unique!
(Circulation Γ may have any value)

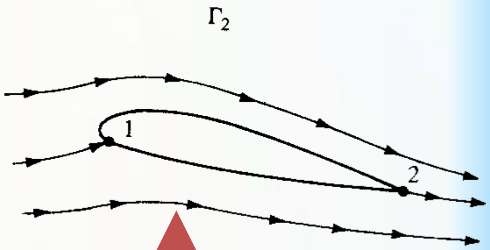


(a) $\Gamma < 4\pi V_\infty R$

(b) $\Gamma = 4\pi V_\infty R$

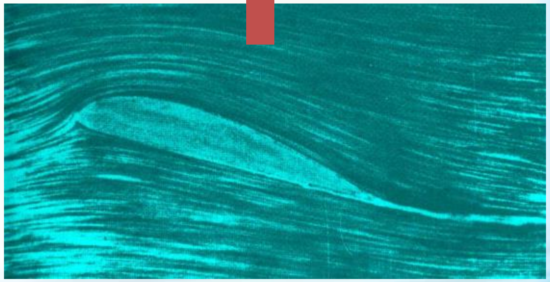
The same happens for the flow around an airfoil



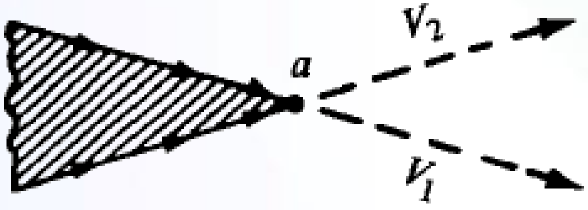


Which flow occurs in reality?
The flow that leaves smoothly at the trailing edge

(The “Kutta condition”)



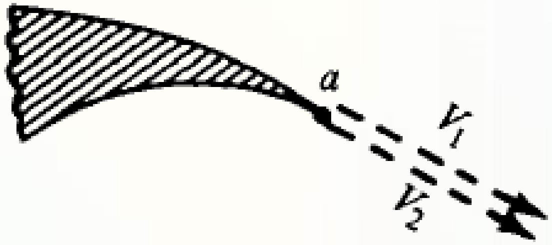
IMPLEMENTATION OF THE KUTTA CONDITION



Finite angle

At point a: $V_1 = V_2 = 0$

(Point a is a stagnation point)



Cusp

At point a: $V_1 = V_2 \neq 0$

$$p_a + \tfrac{1}{2}\rho V_1^2 = p_a + \tfrac{1}{2}\rho V_2^2$$

$$V_1 = V_2$$

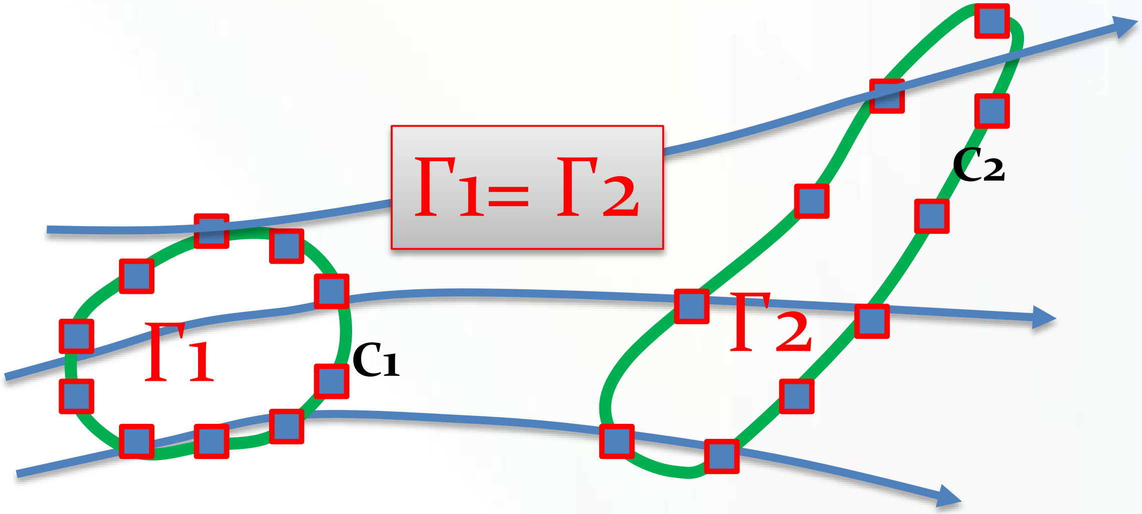
the Kutta condition expressed in terms of the strength of the vortex sheet is:

$$\gamma(\mathrm{TE}) = \gamma(a) = V_1 - V_2 \quad \longrightarrow \quad \gamma(\mathrm{TE}) = 0$$

KELVIN'S CIRCULATION THEOREM AND THE STARTING VORTEX

Question: *How does nature generate this circulation?*

$$\boxed{\frac{D\Gamma}{Dt} = 0}$$



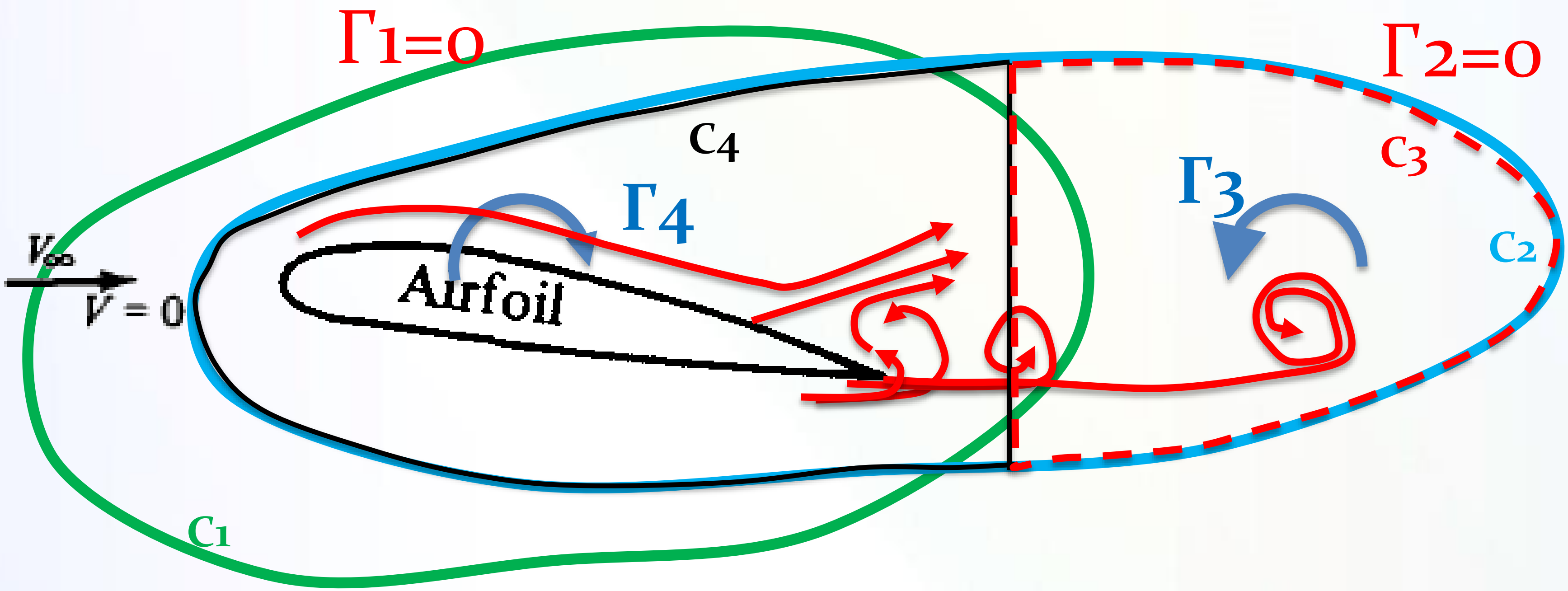
time t1

$$\Gamma_1 = \int_{C_1} V \, .ds$$

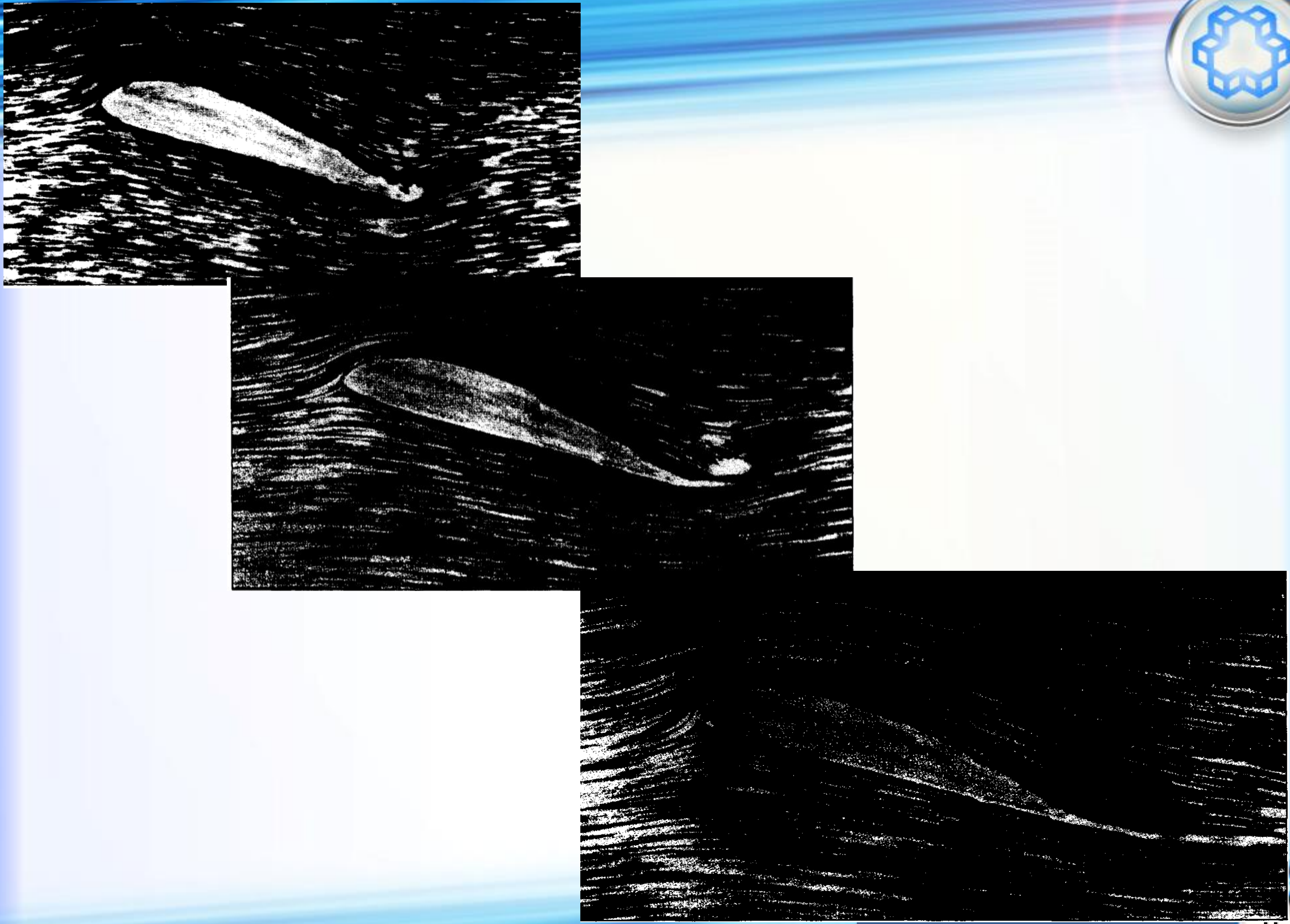
time t2

$$\Gamma_2 = \int_{C_2} V \, .ds$$

STARTING VORTEX



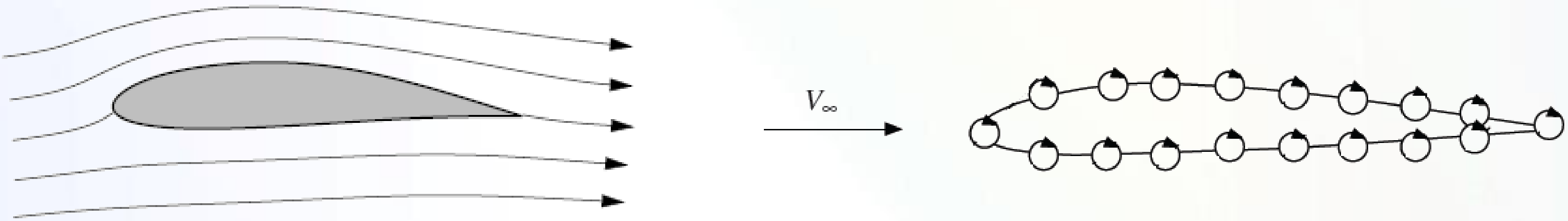
$$\Gamma 2=\Gamma 3+\Gamma 4=0 \Rightarrow \Gamma 4=-\Gamma 3$$



AIRFOIL APPROXIMATION

o Consider an airfoil of arbitrary shape and thickness in a free stream with velocity V∞

o Replace the airfoil surface with a vortex sheet of variable strength γ(i).



o Calculate the variation of γ as a function of s such that the induced velocity field from the vortex sheet when added to the uniform velocity of magnitude will make the vortex sheet (hence the airfoil surface) a streamline of the flow.

$$\Gamma = \int \gamma \, ds \quad \Rightarrow \quad L' = \rho_\infty V_\infty \Gamma$$

AIRFOIL APPROXIMATION

- No general analytical solution for $\gamma = \gamma(s)$ exists for an airfoil of arbitrary shape and thickness. Rather, the strength of the vortex sheet must be found numerically

foundation of the vortex panel method

Analytical solution?

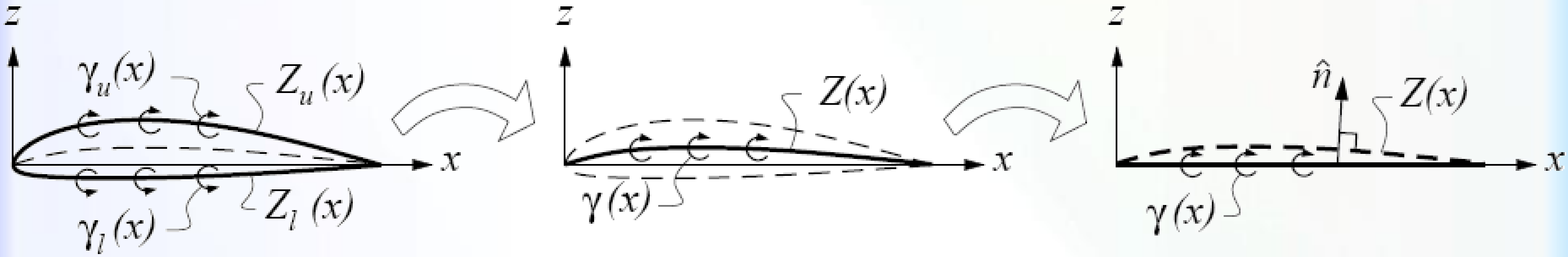
Thin airfoil approximation

THIN AIRFOIL APPROXIMATION

1) The airfoil is assumed to be thin, with small maximum camber and thickness relative to the chord, and is assumed to operate at a small angle of attack, $\alpha \ll 1$.

2) The upper and lower vortex sheets are superimposed together into a single vortex sheet $\gamma = \gamma u + \gamma \ell$, which is placed on the x axis rather than on the curved mean camber line $Z = (Zu + Z\ell)/2$.

3) The flow-tangency condition $V \cdot n = 0$ is applied on the x-axis at $z = 0$, rather than on the camber line at $z = Z$. But the normal vector *n* is normal to the actual camber line shape, as shown in the figure.

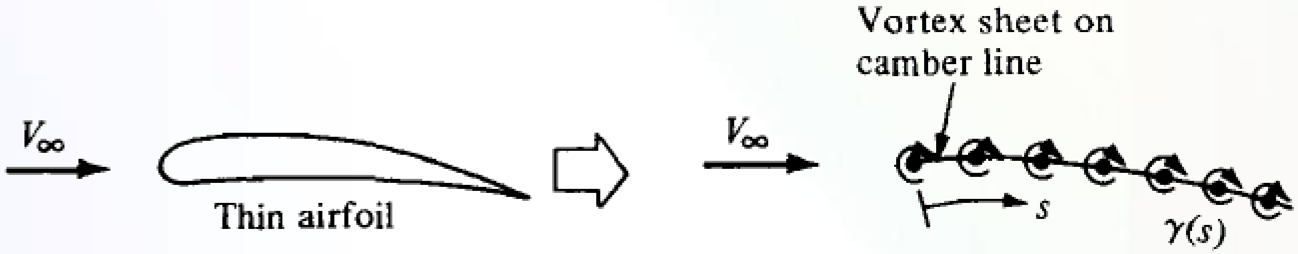


CLASSICAL THIN AIRFOIL THEORY

Thin airfoils can be simulated by a vortex sheet placed along the camber line.

Our purpose is to calculate the variation of *γ(s)* such that:

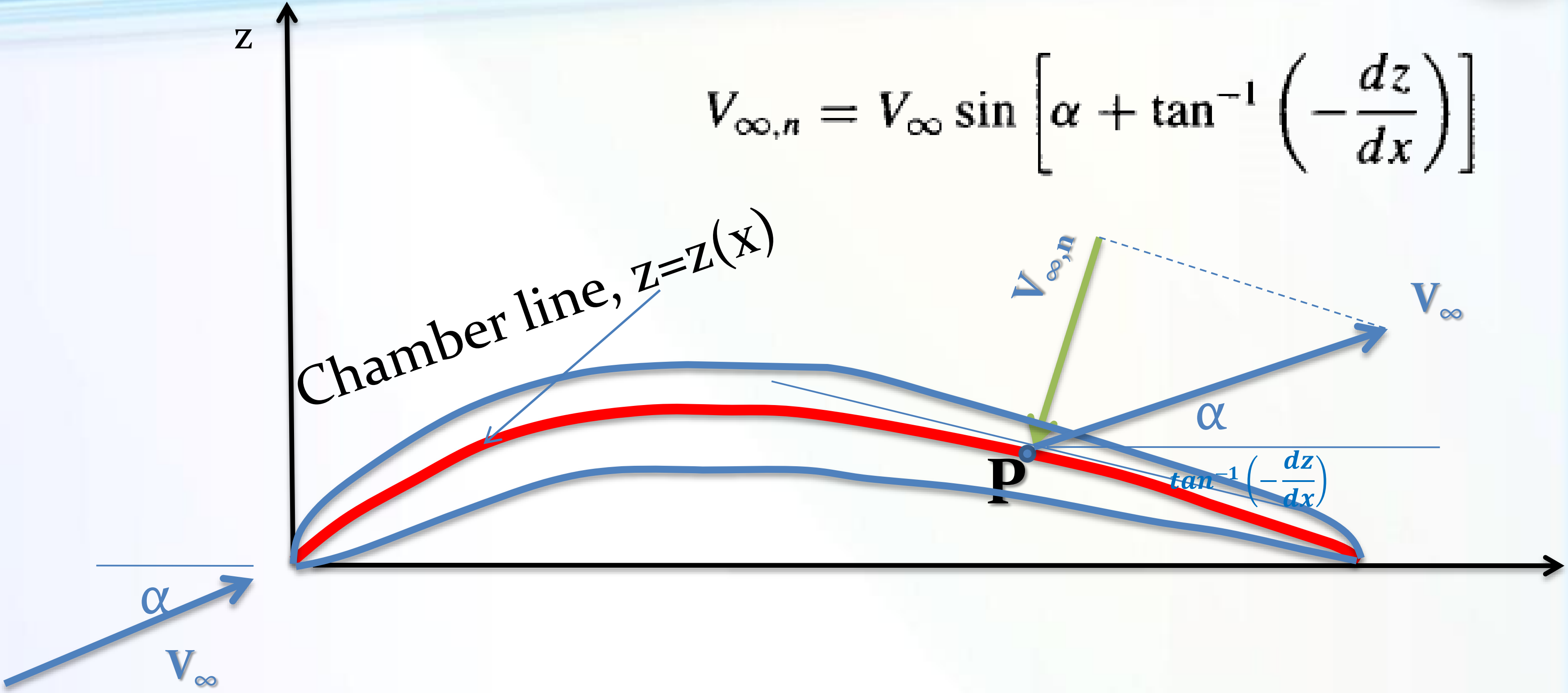
1) The camber line becomes a streamline of the flow
2) The Kutta condition is satisfied *(γ(TE)=0)*.



Once we have found the particular γ(s) that satisfies above conditions, then the total circulation Γ around the airfoil is found by integrating γ(s) from the leading edge to the trailing edge.

$$L = \rho V_\infty \Gamma$$

CLASSICAL THIN AIRFOIL THEORY



$$V_{\infty,n} = V_\infty \sin\left[\alpha + \tan^{-1}\left(-\frac{dz}{dx}\right)\right]$$

For the camber line to be a streamline: $V_{\infty,n} + w'(s) = 0$

CLASSICAL THIN AIRFOIL THEORY

$$V_{\infty,n} = V_\infty \sin\left[\alpha + \tan^{-1}\left(-\frac{dz}{dx}\right)\right]$$

$$\tan^{-1}\left(-\frac{dz}{dx}\right) \simeq -\frac{dz}{dx}$$

$$V_{\infty,n} = V_\infty \sin\left[\alpha - \frac{dz}{dx}\right]$$

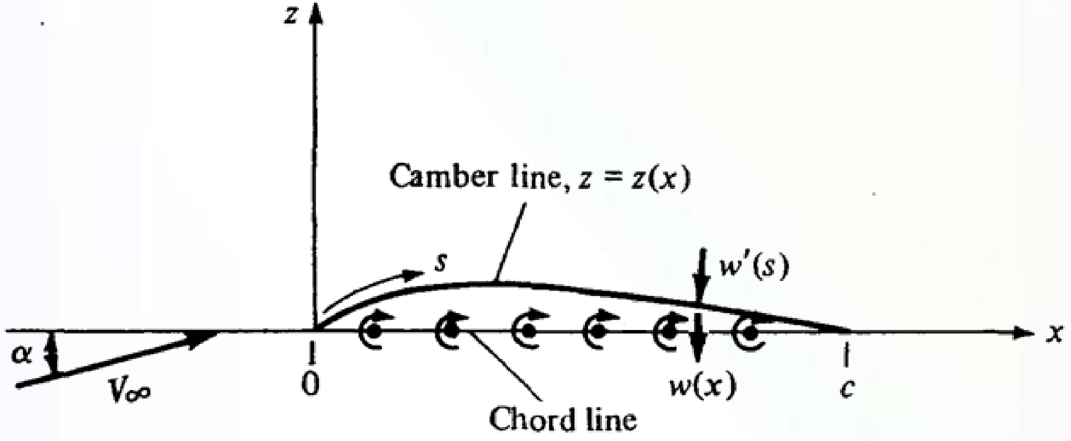
$$\sin\left[\alpha - \frac{dz}{dx}\right] \simeq \left[\alpha - \frac{dz}{dx}\right]$$

$$\boxed{V_{\infty,n} = V_\infty\left(\alpha - \frac{dz}{dx}\right)}$$

CLASSICAL THIN AIRFOIL THEORY

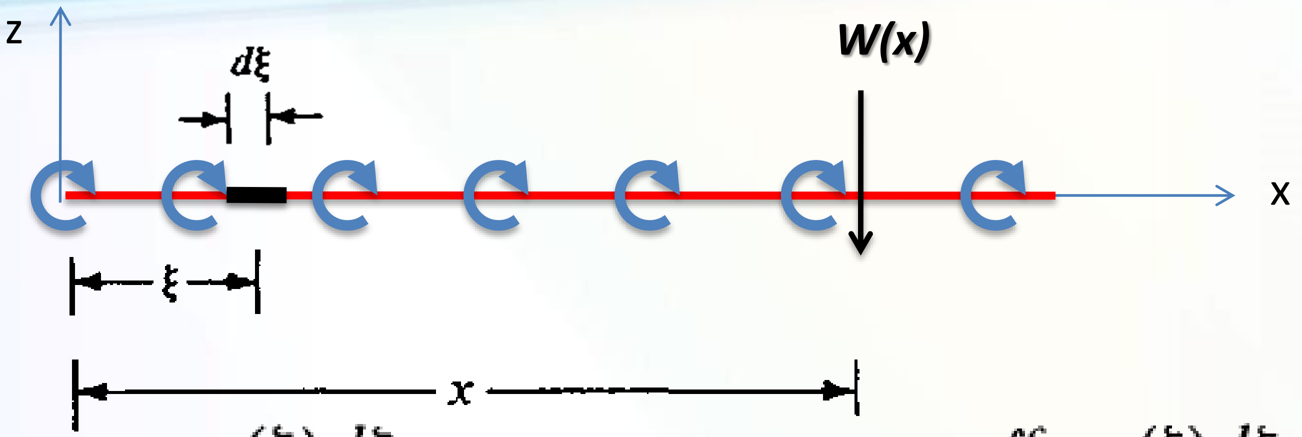
$$\boxed{V_{\infty,n} + w'(s) = 0}$$

Let w(x) denote the component of velocity normal to the chord line induced by the vortex sheet,



If the airfoil is thin, the camber line is close to the chord line, and it is consistent with thin airfoil theory to make the approximation that $w'(s) \approx w(x)$

CLASSICAL THIN AIRFOIL THEORY



$$dw = -\frac{\gamma(\xi)\,d\xi}{2\pi(x-\xi)} \Rightarrow w(x) = -\int_0^c \frac{\gamma(\xi)\,d\xi}{2\pi(x-\xi)}$$

$$V_{\infty,n} + w'(s) = 0 \Rightarrow V_\infty\left(\alpha - \frac{dz}{dx}\right) - \int_0^c \frac{\gamma(\xi)\,d\xi}{2\pi(x-\xi)} = 0$$

$$\boxed{\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\,d\xi}{x-\xi} = V_\infty\left(\alpha - \frac{dz}{dx}\right)}$$

Fundamental equation of thin airfoil theory

CLASSICAL THIN AIRFOIL THEORY
THE SAYMMETRICAL AIRFOIL

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\,d\xi}{x-\xi} = V_\infty\left(\alpha - \frac{dz}{dx}\right)$$

The central problem of thin airfoil theory is to solve the above equation for γ(ξ), subject to the Kutta condition, namely, $\gamma(c) = 0$.

Special Case: A symmetric airfoil has no camber; the camber line is coincident with the chord line. For this case:

$$dz/dx = 0$$

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\,d\xi}{x-\xi} = V_\infty\alpha$$

CLASSICAL THIN AIRFOIL THEORY
THE SAYMMETRICAL AIRFOIL

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\,d\xi}{x-\xi} = V_\infty\alpha$$

$$\xi = \frac{c}{2}(1-\cos\theta)$$

$$x = \frac{c}{2}(1-\cos\theta_0)$$

$$d\xi = \frac{c}{2}\sin\theta\,d\theta$$

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\,d\theta}{\cos\theta-\cos\theta_0} = V_\infty\alpha$$

$$\boxed{\gamma(\theta) = 2\alpha V_\infty\frac{(1+\cos\theta)}{\sin\theta}}$$

VERIFICATION OF THE SOLUTION

Is $\gamma(\theta) = 2\alpha V_\infty \frac{(1+\cos\theta)}{\sin\theta}$ the solution of $\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\, d\theta}{\cos\theta - \cos\theta_0} = V_\infty\alpha$?

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta}{(\cos\theta-\cos\theta_0)}d\theta = \frac{1}{2\pi}\int_0^\pi \overbrace{2\alpha V_\infty \frac{1+\cos\theta}{\sin\theta}}\frac{\sin\theta}{(\cos\theta-\cos\theta_0)}d\theta$$

$$= V_\infty\alpha\frac{1}{\pi}\int_0^\pi \underbrace{\frac{1+\cos\theta}{(\cos\theta-\cos\theta_0)}d\theta}$$

$$= V_\infty\alpha\frac{1}{\pi}\left[\pi(0+1)\right]$$

$$= V_\infty\alpha$$

Standard integrals:
(n=0,1,2...)

$$\boxed{\int_0^\pi \frac{\cos n\theta}{(\cos\theta-\cos\theta_0)}d\theta = \pi\frac{\sin n\theta_0}{\sin\theta_0}}$$

VERIFICATION OF THE SOLUTION

$$\gamma(\theta) = 2\alpha V_\infty \frac{(1 + \cos\theta)}{\sin\theta}$$

Note that at the trailing edge, where $\theta=\pi$, the above equation yields:

$$\gamma(\pi) = 2\alpha V_\infty \frac{0}{0}$$

However, using L'Hospital's rule on Equation $\gamma(\theta)$

$$\gamma(\pi) = 2\alpha V_\infty \frac{-\sin\pi}{\cos\pi} = 0$$

Thus, the equation also satisfies the Kutta condition.

THIN SYMMETRICAL AIRFOILS
LIFT SLOPE

$$\Gamma = \int_0^c \gamma(\xi)\, d\xi$$

$$\Gamma = \frac{c}{2}\int_0^\pi \gamma(\theta)\sin\theta\, d\theta$$

$$\Gamma = \alpha c V_\infty \int_0^\pi (1+\cos\theta)\, d\theta = \pi\alpha c V_\infty$$

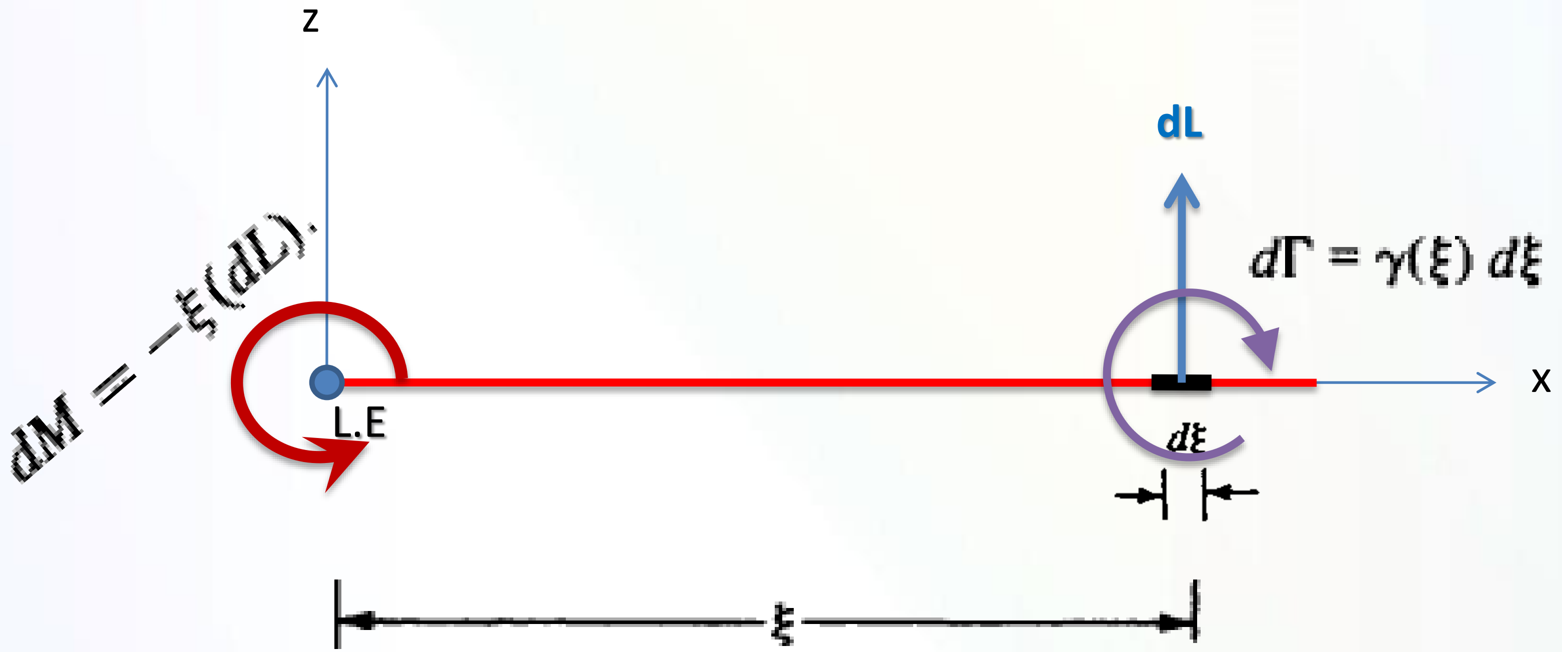
$$L' = \rho_\infty V_\infty \Gamma = \pi\alpha c \rho_\infty V_\infty^2$$

$$c_l = \frac{L'}{q_\infty s} \quad \Rightarrow \quad c_l = 2\pi\alpha \quad \Rightarrow \quad \text{Lift slope} = \frac{dc_l}{d\alpha} = 2\pi$$

$$S = c(1)$$

THIN SYMMETRICAL AIRFOILS
MOMENT ABOUT THE LEADING EDGE

The moment about the leading edge can be calculated as follows:



$$M'_{\mathrm{LE}} = -\int_0^c \xi(dL) = -\rho_\infty V_\infty \int_0^c \xi\gamma(\xi)\, d\xi$$

THIN SYMMETRICAL AIRFOILS
MOMENT ABOUT THE LEADING EDGE

$$M'_{LE} = -\int_0^c \xi\,(dL) = -\rho_\infty V_\infty \int_0^c \xi\gamma(\xi)\,d\xi$$

$$= -\rho_\infty V_\infty \int_0^\pi \left(\frac{c}{2}(1-\cos\theta) \right)\left(2\alpha V_\infty \frac{1+\cos\theta}{\sin\theta} \right)\left(\frac{c}{2}\sin\theta\,d\theta \right)$$

$$= -\frac{\rho_\infty V_\infty{}^2 \alpha c^2}{2} \int_0^\pi (1-\cos\theta)(1+\cos\theta) d\theta = -\frac{1}{2}\rho_\infty V_\infty{}^2 \alpha c^2 \frac{\pi}{2}$$

$$\boxed{c_{m,LE} = \frac{M'_{LE}}{\frac{1}{2}\rho_\infty V_\infty{}^2 c^2.(1)} = -\frac{\pi\alpha}{2}}$$

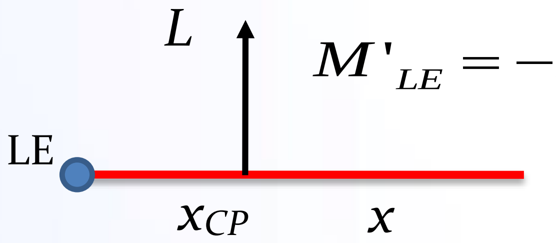
THIN SYMMETRICAL AIRFOILS - THE CENTER OF PRESSURE AND THE AERODYNAMIC CENTER

Lift coefficient:

$$c_l = \frac{L'}{\frac{1}{2}\rho V_\infty^{\ 2} c.(1)} = 2\pi\alpha$$

Moment coefficient about leading edge:

$$c_{m,LE} = \frac{M'_{LE}}{\frac{1}{2}\rho V_\infty^{\ 2} c^2.(1)} = -\frac{\pi\alpha}{2}$$



$$M'_{LE} = -L' x_{CP}$$

Center of pressure:

$$\boxed{\frac{x_{CP}}{c} = -\frac{c_{m,LE}}{c_l} = \frac{1}{4}}$$

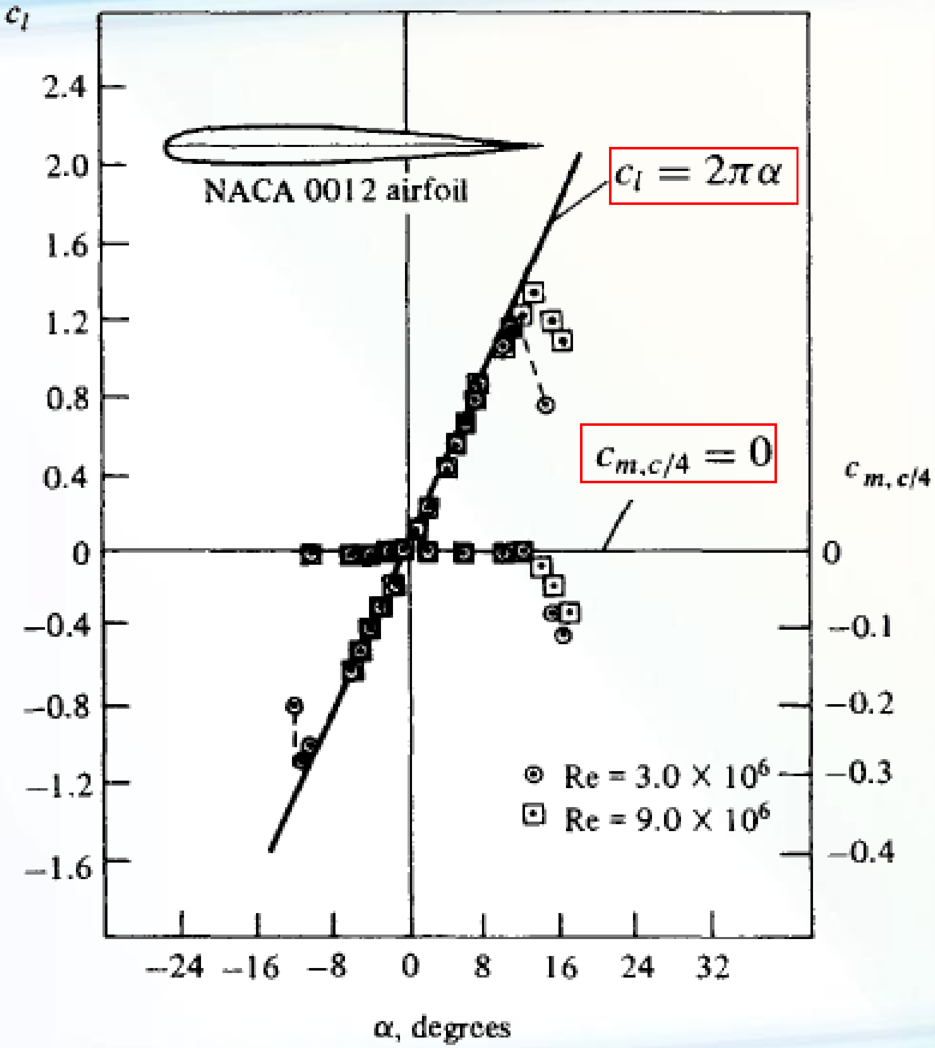
$$c_{m,x} = c_l \frac{x - x_{CP}}{c} = c_{m,LE} + c_l \frac{x}{c}$$

Moment coefficient about quarter-chord point:

$$\boxed{c_{m,c/4} = 0}$$

quarter-chord point is also the aerodynamic center: $c_{m,c/4}$ is independent of α!

THIN SYMMETRICAL AIRFOILS



THE SYMMETRICAL AIRFOIL: SUMMARY

Vorticity distribution (=lift distribution)

$$\boxed{\gamma(\theta) = 2\alpha V_\infty \frac{1+\cos\theta}{\sin\theta} = 2\alpha V_\infty \sqrt{\frac{c-x}{x}}}$$

Lift coefficient:

$$\boxed{c_l = \frac{L'}{\frac{1}{2}\rho V_\infty^{\;2} c.(1)} = 2\pi\alpha}$$

Lift slope:

$$\boxed{\frac{dc_l}{d\alpha} = 2\pi}$$

Moment coefficient
about quarter-chord point:

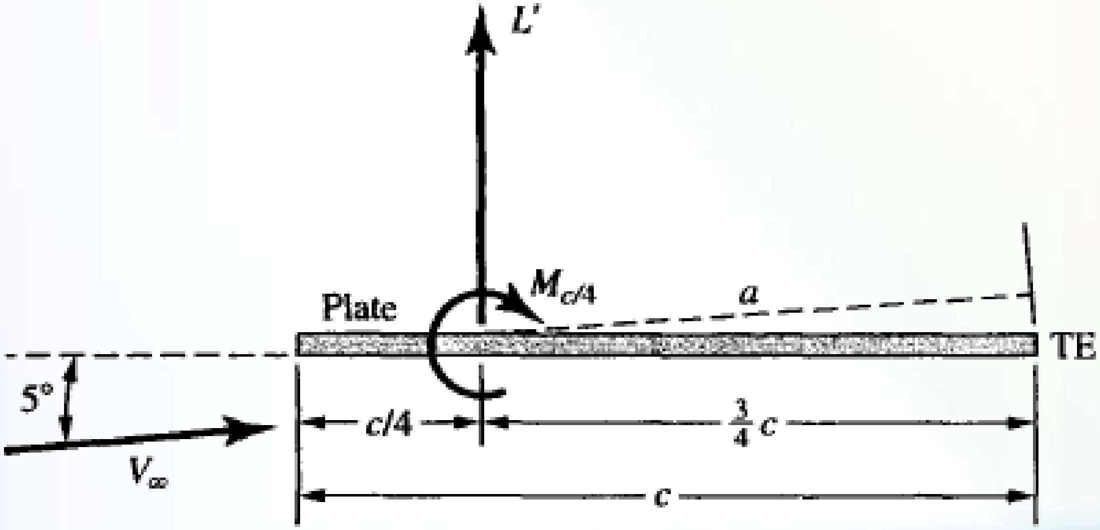
$$\boxed{c_{m,c/4} = c_{m,LE} + \frac{c_l}{4} = 0}$$

quarter-chord point is both the <u>center of pressure</u>: ($c_{m,c/4} = 0$)
and the <u>aerodynamic center</u>: ($c_{m,c/4}$ is independent of α)

THIN AIRFOILS – EXAMPLE

Consider a thin flat plate at 5 deg. angle of attack.
Calculate the:

(a) Lift coefficient,
(b) Moment coefficient about the leading edge,
(c) Moment coefficient about the quarter chord point,
(d) Moment coefficient about the trailing edge.



THIN AIRFOILS – EXAMPLE (CONT.)

$$c_\ell = 2\pi\alpha \qquad \alpha = \frac{5}{57.3} = 0.0873\ \text{rad} \qquad c_\ell = 2\pi(0.0873) = \boxed{0.5485}$$

$$c_{m,\ell e} = -\frac{c_\ell}{4} = -\frac{0.5485}{4} = \boxed{-0.137}$$

$$c_{m,c/4} = \boxed{0}$$

$$a = \left(\tfrac{3}{4}c\right)\cos\alpha = \left(\tfrac{3}{4}c\right)\cos 5^\circ \qquad \cos\alpha \approx 1 \qquad a = \left(\tfrac{3}{4}c\right)$$

$$M'_{\text{te}} = \left(\frac{3}{4}c\right)L' + M'_{c/4}$$

$$c_{m,\text{te}} = \frac{M'_{\text{te}}}{q_\infty c^2} = \left(\frac{3}{4}c\right)\frac{L'}{q_\infty c^2} + \frac{M'_{c/4}}{q_\infty c^2}$$

$$c_{m,\text{te}} = \frac{3}{4}c_\ell + \cancel{c_{m,c/4}} \Rightarrow c_{m,\text{te}} = \frac{3}{4}c_\ell \Rightarrow c_{m,\text{te}} = \frac{3}{4}(0.5485) = \boxed{0.411}$$

THE CAMBERED AIRFOIL

For a cambered airfoil, dz/dx is finite.

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\,d\xi}{x-\xi} = V_\infty\left(\alpha - \frac{dz}{dx}\right) \Longrightarrow \frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\,d\theta}{\cos\theta - \cos\theta_0} = V_\infty\left(\alpha - \frac{dz}{dx}\right)$$

The solution for this more general problem can be written as a Fourier series:

$$\gamma(\theta) = 2V_\infty\left(\underbrace{A_0\frac{1+\cos\theta}{\sin\theta}}_{\substack{\text{“Basic solution”}\\ \text{for the symmetrical}\\ \text{airfoil: } A_o = \alpha}} + \underbrace{\sum_{n=1}^{\infty} A_n \sin n\theta}_{\text{Additional terms}}\right)$$

The coefficients A_n (n=0,1,2,…) depend on the shape of the camber line z(x).

The coefficient A_o depends also on α.

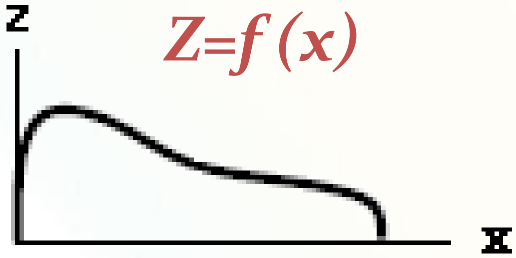
THE CAMBERED AIRFOIL

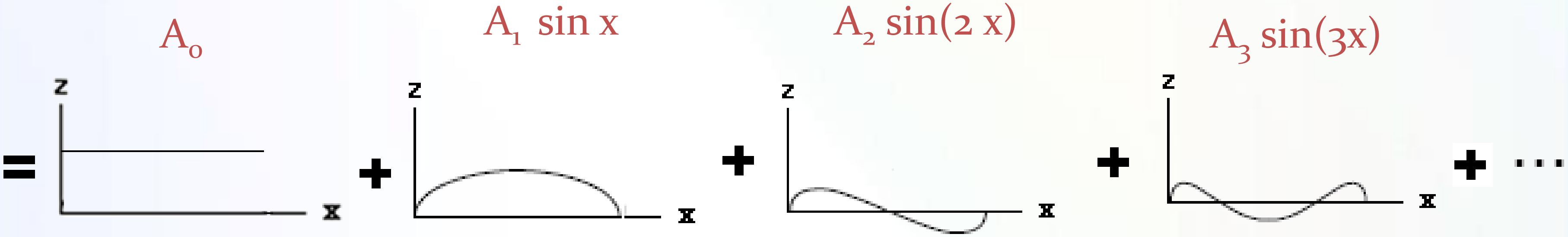
The coefficients A_0 and *A„(n = 1, 2, 3 , . . .)* in the above equation must be specific values in order that the camber line be a streamline of the flow.

$$\frac{1}{2\pi}\int_0^{\pi}\frac{\gamma(\theta)\sin\theta\,d\theta}{\cos\theta-\cos\theta_0}=V_\infty\left(\alpha-\frac{dz}{dx}\right)\qquad\gamma(\theta)=2V_\infty\left(A_0\frac{1+\cos\theta}{\sin\theta}+\sum_{n=1}^{\infty}A_n\sin n\theta\right)$$

$$\frac{1}{\pi}\int_0^{\pi}\frac{A_0(1+\cos\theta)\,d\theta}{\cos\theta-\cos\theta_0}+\frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{\pi}\frac{A_n\sin n\theta\sin\theta\,d\theta}{\cos\theta-\cos\theta_0}=\alpha-\underbrace{\frac{dz}{dx}}_{f(x)}$$

THE CAMBERED AIRFOIL





$$A_0 - \sum_{n=1}^{\infty} A_n \cos n\theta_0 = \alpha - \frac{dz}{dx} \Rightarrow \frac{dz}{dx} = (\alpha - A_0) + \sum_{n=1}^{\infty} A_n \cos n\theta_0$$

THE CAMBERED AIRFOIL

In general, the Fourier cosine series representation of a function $f(\theta)$ over an interval $0< \theta <\pi$ is given by:

$$f(\theta) = B_0 + \sum_{n=1}^{\infty} B_n \cos n\theta$$

$$B_0 = \frac{1}{\pi}\int_0^{\pi} f(\theta)\, d\theta \qquad\qquad B_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta\, d\theta$$

$$\frac{dz}{dx} = (\alpha - A_0) + \sum_{n=1}^{\infty} A_n \cos n\theta_0$$

$$\alpha - A_0 = \frac{1}{\pi}\int_0^{\pi} \frac{dz}{dx}\, d\theta_0 \quad \Longrightarrow \quad \boxed{A_0 = \alpha - \frac{1}{\pi}\int_0^{\pi} \frac{dz}{dx}\, d\theta_0}$$

$$\boxed{A_n = \frac{2}{\pi}\int_0^{\pi} \frac{dz}{dx}\cos n\theta_0\, d\theta_0}$$

THE CAMBERED AIRFOIL – AERODYNAMIC COEFFICIENTS

The total circulation due to the entire vortex sheet from the leading edge to the trailing edge is:

$$\Gamma = \int_0^c \gamma(\xi)\,d\xi = \frac{c}{2}\int_0^\pi \gamma(\theta)\sin\theta\,d\theta \qquad \gamma(\theta) = 2V_\infty\left(A_0\frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right)$$

$$\Gamma = cV_\infty\left[A_0 \underbrace{\int_0^\pi (1+\cos\theta)\,d\theta}_{\pi} + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^\pi \sin n\theta \sin\theta\,d\theta}_{\begin{cases}\pi/2 & \text{for } n=1\\ 0 & \text{for } n\neq 1\end{cases}}\right]$$

$$\Gamma = cV_\infty\left(\pi A_0 + \frac{\pi}{2}A_1\right)$$

$$L' = \rho_\infty V_\infty \Gamma = \rho_\infty V_\infty^2 c\left(\pi A_0 + \frac{\pi}{2}A_1\right)$$

THE CAMBERED AIRFOIL – AERODYNAMIC COEFFICIENTS

$$c_l = \frac{L'}{\frac{1}{2}\rho_\infty V_\infty^2 c(1)} = \pi(2A_0 + A_1)$$

$$\boxed{c_l = 2\pi\left[\alpha + \frac{1}{\pi}\int_0^\pi \frac{dz}{dx}(\cos\theta_0 - 1)\,d\theta_0\right]}$$

$$\boxed{\text{Lift slope} \equiv \frac{dc_l}{d\alpha} = 2\pi}$$

$$c_l = \frac{dc_l}{d\alpha}(\alpha - \alpha_{L=0})$$

$$c_l = 2\pi(\alpha - \alpha_{L=0})$$

$$\boxed{\alpha_{L=0} = -\frac{1}{\pi}\int_0^\pi \frac{dz}{dx}(\cos\theta_0 - 1)\,d\theta_0}$$

THE CAMBERED AIRFOIL – AERODYNAMIC COEFFICIENTS

$$M'_{\text{LE}} = -\int_0^c \xi(dL) = -\rho_\infty V_\infty \int_0^c \xi\gamma(\xi)\, d\xi$$

$$c_{m,\text{le}} = -\frac{\pi}{2}\left(A_0 + A_1 - \frac{A_2}{2}\right)$$

$$c_{m,\text{le}} = -\left[\frac{c_l}{4} + \frac{\pi}{4}(A_1 - A_2)\right] \qquad c_{m,c/4} = \frac{\pi}{4}(A_2 - A_1)$$

THE CAMBERED AIRFOIL – CENTER OF PRESSURE

$$x_{cp} = -\frac{M'_{LE}}{L'} = -\frac{c_{m,le}c}{c_l} \qquad x_{cp} = \frac{c}{4}\left[1 + \frac{\pi}{c_l}(A_1 - A_2)\right]$$

As the lift approaches zero, x_{cp} moves toward infinity; that is, it leaves the airfoil. For this reason, the center of pressure is not always a convenient point at which to draw the force system on an airfoil.

the force-and-moment system on an airfoil is more conveniently considered at the *aerodynamic center*.

THE CAMBERED AIRFOIL – EXAMPLE

Consider an NACA 23012 airfoil. The mean camber line for this airfoil is given by

$$\frac{z}{c} = 2.6595\left[\left(\frac{x}{c}\right)^3 - 0.6075\left(\frac{x}{c}\right)^2 + 0.1147\left(\frac{x}{c}\right)\right] \qquad \text{for } 0 \le \frac{x}{c} \le 0.2025$$

$$\frac{z}{c} = 0.02208\left(1 - \frac{x}{c}\right) \qquad \text{for } 0.2025 \le \frac{x}{c} \le 1.0$$

Calculate:

a) the angle of attack at zero lift,
b) the lift coefficient when $\alpha = 4°$,
c) the moment coefficient about c/4
d) the location of the center of pressure in terms of x_{cp}/c, when $\alpha = 4°$.

THE CAMBERED AIRFOIL – EXAMPLE (CONT.)

We will need dz/dx. From the given shape of the mean camber line, this is

$$\frac{dz}{dx} = 2.6595\left[3\left(\frac{x}{c}\right)^2 - 1.215\left(\frac{x}{c}\right) + 0.1147\right] \qquad \text{for } 0 \le \frac{x}{c} \le 0.2025$$

and
$$\frac{dz}{dx} = -0.02208 \qquad \text{for } 0.2025 \le \frac{x}{c} \le 1.0$$

$$x = (c/2)(1 - \cos\theta)$$

$$\frac{dz}{dx} = 2.6595\left[\frac{3}{4}(1 - 2\cos\theta + \cos^2\theta) - 0.6075(1 - \cos\theta) + 0.1147\right]$$

or
$$= 0.6840 - 2.3736\cos\theta + 1.995\cos^2\theta \qquad \text{for } 0 \le \theta \le 0.9335 \text{ rad}$$

and
$$= -0.02208 \qquad \text{for } 0.9335 \le \theta \le \pi$$

$$\alpha_{L=0} = -\frac{1}{\pi}\int_0^\pi \frac{dz}{dx}(\cos\theta - 1)\, d\theta$$

THE CAMBERED AIRFOIL – EXAMPLE (CONT.)

$$\alpha_{L=0} = -\frac{1}{\pi}\int_0^{0.9335}(-0.6840 + 3.0576\cos\theta - 4.3686\cos^2\theta + 1.995\cos^3\theta)\,d\theta$$

$$-\frac{1}{\pi}\int_{0.9335}^{\pi}(0.02208 - 0.02208\cos\theta)\,d\theta$$

$$\int \cos\theta\, d\theta = \sin\theta$$

$$\int \cos^2\theta\, d\theta = \tfrac{1}{2}\sin\theta\cos\theta + \tfrac{1}{2}\theta$$

$$\int \cos^3\theta\, d\theta = \tfrac{1}{3}\sin\theta(\cos^2\theta + 2)$$

➡

$$\alpha_{L=0} = -\frac{1}{\pi}[-2.8683\theta + 3.0576\sin\theta - 2.1843\sin\theta\cos\theta$$

$$+0.665\sin\theta(\cos^2\theta + 2)]_0^{0.9335}$$

$$-\frac{1}{\pi}[0.02208\theta - 0.02208\sin\theta]_{0.9335}^{\pi}$$

$$\alpha_{L=0} = -\frac{1}{\pi}(-0.0065 + 0.0665) = -0.0191 \text{ rad}$$

$$\boxed{\alpha_{L=0} = -1.09^\circ}$$

THE CAMBERED AIRFOIL – EXAMPLE (CONT.)

(b)

$$\alpha = 4^\circ = 0.0698 \text{ rad}$$

$$c_l = 2\pi(\alpha - \alpha_{L=0}) = 2\pi(0.0698 + 0.0191) = \boxed{0.559}$$

(c)

$$c_{m,c/4} = \frac{\pi}{4}(A_2 - A_1)$$

$$\begin{aligned}
A_1 &= \frac{2}{\pi}\int_0^\pi \frac{dz}{dx}\cos\theta\, d\theta \\
&= \frac{2}{\pi}\int_0^{0.9335} (0.6840\cos\theta - 2.3736\cos^2\theta + 1.995\cos^3\theta)\, d\theta \\
&\quad + \frac{2}{\pi}\int_{0.9335}^{\pi} (-0.02208\cos\theta)\, d\theta \\
&= \frac{2}{\pi}[0.6840\sin\theta - 1.1868\sin\theta\cos\theta - 1.1868\theta + 0.665\sin\theta(\cos^2\theta + 2)]_0^{0.9335} \\
&\quad + \frac{2}{\pi}[-0.02208\sin\theta]_{0.09335}^{\pi} \\
&= \frac{2}{\pi}(0.1322 + 0.0177) = 0.0954
\end{aligned}$$

THE CAMBERED AIRFOIL – EXAMPLE (CONT.)

$$A_2 = \frac{2}{\pi}\int_0^{\pi}\frac{dz}{dx}\cos 2\theta\, d\theta = \frac{2}{\pi}\int_0^{\pi}\frac{dz}{dx}(2\cos^2\theta - 1)\, d\theta$$

$$= \frac{2}{\pi}\int_0^{0.9335}(-0.6840 + 2.3736\cos\theta - 0.627\cos^2\theta$$

$$- 4.747\cos^3\theta + 3.99\cos^4\theta)\, d\theta$$

$$+ \frac{2}{\pi}\int_{0.9335}^{\pi}(0.02208 - 0.0446\cos^2\theta)\, d\theta$$

$$\boxed{\int \cos^4\theta\, d\theta = \tfrac{1}{4}\cos^3\theta\sin\theta + \tfrac{3}{8}(\sin\theta\cos\theta + \theta)}$$

$$A_2 = \frac{2}{\pi}\left\{-0.6840\,\theta + 2.3736\sin\theta - 0.628\left(\frac{1}{2}\right)(\sin\theta\cos\theta + \theta)\right.$$

$$\left. - 4.747\left(\frac{1}{3}\right)\sin\theta(\cos^2\theta + 2) + 3.99\left[\frac{1}{4}\cos^3\sin\theta + \frac{3}{8}(\sin\theta\cos\theta + \theta)\right]\right\}_0^{0.9335}$$

$$+ \frac{2}{\pi}\left[0.02208\theta - 0.0446\left(\frac{1}{2}\right)(\sin\theta\cos\theta + \theta)\right]_{0.9335}^{\pi}$$

$$= \frac{2}{\pi}(0.11384 + 0.01056) = 0.0792$$

THE CAMBERED AIRFOIL – EXAMPLE (CONT.)

$$c_{m,c/4} = \frac{\pi}{4}(A_2 - A_1) = \frac{\pi}{4}(0.0792 - 0.0954)$$

$$\boxed{c_{m,c/4} = -0.0127}$$

(d)

$$x_{\text{cp}} = \frac{c}{4}\left[1 + \frac{\pi}{c_l}(A_1 - A_2)\right]$$

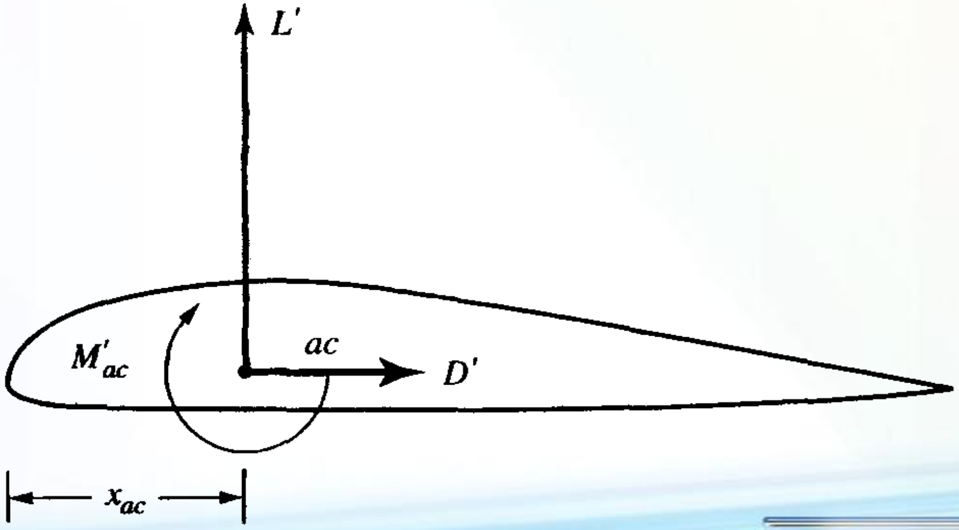
$$\frac{x_{\text{cp}}}{c} = \frac{1}{4}\left[1 + \frac{\pi}{0.559}(0.0954 - 0.0792)\right] = 0.273$$

	Experiment	Thin airfoil
$\alpha_{L=0}$	-1.09°	-1.1°
c_l (at $\alpha = 4^\circ$)	0.559	0.55
$c_{m,c/4}$	-0.0127	-0.01

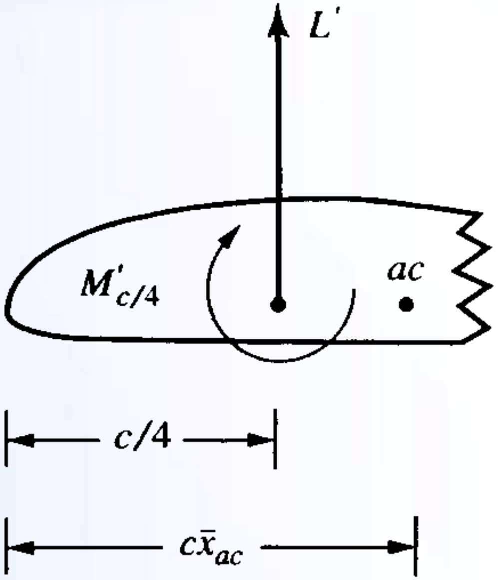
THE AERODYNAMIC CENTER

The *aerodynamic center* is a point on a body about which the aerodynamically generated moment is *independent of angle of attack.*

For most conventional airfoils, the aerodynamic center is close to, but not necessarily exactly at, the quarter-chord point.



THE AERODYNAMIC CENTER



$$M'_{ac} = L'(c\bar{x}_{ac} - c/4) + M'_{c/4}$$

$$\frac{M'_{ac}}{q_\infty Sc} = \frac{L'}{q_\infty S}(\bar{x}_{ac} - 0.25) + \frac{M'_{c/4}}{q_\infty Sc}$$

$$c_{m,ac} = c_l(\bar{x}_{ac} - 0.25) + c_{m,c/4}$$

$$\frac{dc_{m,ac}}{d\alpha} = \frac{dc_l}{d\alpha}(\bar{x}_{ac} - 0.25) + \frac{dc_{m,c/4}}{d\alpha}$$

$$0 = \underbrace{\frac{dc_l}{d\alpha}}_{a_o}(\bar{x}_{ac} - 0.25) + \underbrace{\frac{dc_{m,c/4}}{d\alpha}}_{m_o}$$

THE AERODYNAMIC CENTER

$$0 = a_0(\bar{x}_{ac} - 0.25) + m_0$$

$$\boxed{\bar{x}_{ac} = -\frac{m_0}{a_0} + 0.25}$$

The equation proves that, for a body with linear lift and moment curves, that is, where a_o and m_o are fixed values, the aerodynamic center exists as a fixed point on the airfoil.

THE AERODYNAMIC CENTER - EXAMPLE

Consider the NACA 23012 airfoil.
Where is it's aerodynamic center?

$\alpha_{L=0}$	$-1.1°$
c_l (at $\alpha = 4°$)	0.55
$c_{m,c/4}$ (at $\alpha = 4°$)	-0.005

$$a_0 = \frac{0.55 - 0}{4 - (-1.1)} = 0.1078 \text{ per degree}$$

$$m_0 = \frac{-0.005 - (-0.0125)}{4 - (-4)} = 9.375 \times 10^{-4} \text{ per degree}$$

$$\bar{x}_{ac} = -\frac{m_0}{a_0} + 0.25$$

$$= -\frac{9.375 \times 10^{-4}}{0.1078} + 0.25$$

$$= \boxed{0.241}$$