IN THE NAME OF GOD

ORBITAL MECHANICS

CHAPTER 1

DYNAMICS OF POINT MASSES

Student version

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2010

1-DYNAMIC OF POINT MASSES

1.1 INTRODUCTION

⋆This chapter serves as a self-contained reference on:

- The kinematics & dynamics
- Some basic vector operations
- Rotation and concepts which will be used in the following lectures.

1-DYNAMIC OF POINT MASSES

INTROUDUCTION

✶We will review

- The curvilinear motion of particles in three dimensions
- The concepts of force and mass
- The Newton's law of gravitation (Newton's second law of motion)
- The formulas for calculating the time derivatives of moving vectors. (the computation of relative velocity and acceleration).

1-DYNAMIC OF POINT MASSES

1.2 KINEMATICS

✶ To track the motion of particle *P* through space we need a frame of reference consisting of a clock and a cartesian coordinate system.

Clock → track of time *t*

xyz axes → location of the particle

✶ In non-relativistic mechanics, a single "universal" clock serves for all coordinate systems.

Unit of time → [s]

Unit of length → [m] or [km]

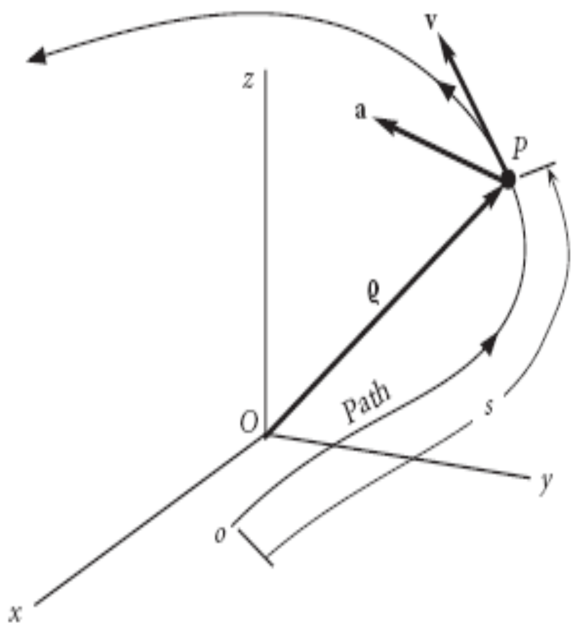
1-DYNAMIC OF POINT MASSES

KINEMATICS

- The position of the particle P at a time t is defined by the position vector $\mathbf{r}(t)$

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$$

$\hat{i}, \hat{j}, \hat{k}$: the unit vectors



- The distance of P from the origin

$$\|\mathbf{r}\| = r = \sqrt{x^2 + y^2 + z^2}$$

$\|r\| = r$:magnitude or length of **r**

- We know

$$r = \sqrt{\mathbf{r} \cdot \mathbf{r}}$$

1-DYNAMIC OF POINT MASSES

KINEMATICS

✶The velocity **v** and acceleration **a** of the particle:

$$\mathbf{v}(t) = \frac{dx(t)}{dt}\hat{\mathbf{i}} + \frac{dy(t)}{dt}\hat{\mathbf{j}} + \frac{dz(t)}{dt}\hat{\mathbf{k}} = v_x(t)\hat{\mathbf{i}} + v_y(t)\hat{\mathbf{j}} + v_z(t)\hat{\mathbf{k}}$$

$$\mathbf{a}(t) = \frac{dv_x(t)}{dt}\hat{\mathbf{i}} + \frac{dv_y(t)}{dt}\hat{\mathbf{j}} + \frac{dv_z(t)}{dt}\hat{\mathbf{k}} = a_x(t)\hat{\mathbf{i}} + a_y(t)\hat{\mathbf{j}} + a_z(t)\hat{\mathbf{k}}$$

✶ The locus of point that a particle occupies as it moves through space is called its path, or trajectory.

The path is a straight line → motion is rectilinear

The path is curved → motion is curvilinear

1-DYNAMIC OF POINT MASSES

KINEMATICS

✶ The velocity vector **v** is tangent to the path:

$$\mathbf{v} = v\hat{\mathbf{u}}_t$$

$\hat{U}_t$:unit vector tangent to the trajectory

v : the magnitude of the velocity **v**.

✶ The distance that P travels along its path in the time interval ds:

$$ds = v\,dt$$

✶ In other words

$$v = \dot{s}$$

✶ Note that

$$v \neq \dot{r}$$

(the magnitude of the derivative of **r** ≠ the derivative of the magnitude of **r**)

1-DYNAMIC OF POINT MASSES

EXAMPLE 1.1

The position vector as a function of time is:

$$\mathbf{r} = (8t^2 + 7t + 6)\hat{\mathbf{i}} + (5t^3 + 4)\hat{\mathbf{j}} + (0.3t^4 + 2t^2 + 1)\hat{\mathbf{k}}\ \text{(m)} \qquad \text{(a)}$$

At t =10s, calculate v and $\dot{r}$,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (16t + 7)\hat{\mathbf{i}} + 15t^2\hat{\mathbf{j}} + (1.2t^3 + 4t)\hat{\mathbf{k}}$$

$$\|\mathbf{v}\| = (1.44t^6 + 234.6t^4 + 272t^2 + 224t + 49)^{\frac{1}{2}}$$

t=10 → v=1953.3 m/s

$$\|\mathbf{r}\| = (0.09t^8 + 26.2t^6 + 68.6t^4 + 152t^3 + 149t^2 + 84t + 53)^{\frac{1}{2}}$$

$$\dot{r} = \frac{dr}{dt} = \frac{0.36t^7 + 78.6t^5 + 137.2t^3 + 228t^2 + 149t + 42}{(0.09t^8 + 26.2t^6 + 68.6t^4 + 152t^3 + 149t^2 + 84t + 53)^{\frac{1}{2}}}$$

t =10 → $\dot{r}$ =1935.5 m/s

1-DYNAMIC OF POINT MASSES

EXAMPLE 1.1

- ✶If **v** is given, then we can find the components of the unit tangent $\hat{U}_t$ in the cartesian coordinate frame of reference

$$\hat{\mathbf{u}}_t = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{v_x}{v}\hat{\mathbf{i}} + \frac{v_y}{v}\hat{\mathbf{j}} + \frac{v_z}{v}\hat{\mathbf{k}} \quad \left(v = \sqrt{v_x^2 + v_y^2 + v_z^2}\right)$$

- ✶The acceleration may be written

$$\mathbf{a} = a_t\hat{\mathbf{u}}_t + a_n\hat{\mathbf{u}}_n$$

- ✶ Where

$$a_t = \dot{v}\ (= \ddot{s}) \qquad a_n = \frac{v^2}{\varrho}$$

1-DYNAMIC OF POINT MASSES

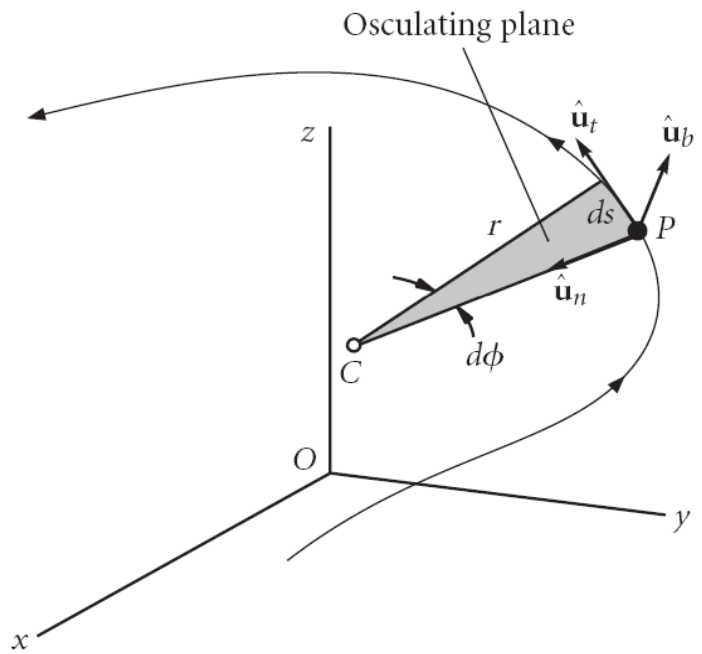
EXAMPLE 1.1

- ϱ : the radius of curvature

- $\hat{U}_t$: the unit tangent

- $\hat{U}_n$: the unit principal

- $r_{c/p}$: the position of C relative to P $r_{c/p} = p\hat{U}_n$

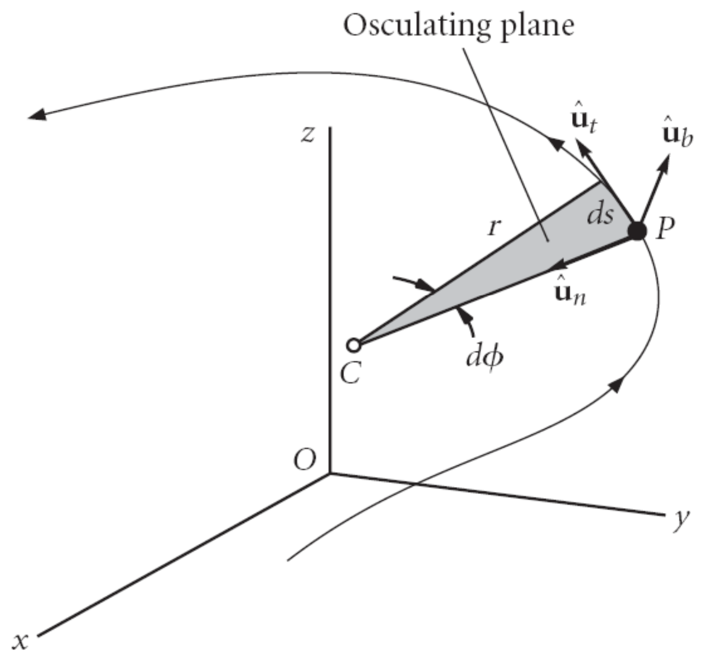


- $\hat{U}_b$: unit normal to the osculating plane $\hat{U}_b = \hat{u}_t \times \hat{u}_n$

1-DYNAMIC OF POINT MASSES

EXAMPLE 1.1

- ✶The center of curvature lies in the osculating plane
- ✶When the particle P moves a distance ds the vector **r** sweep out angle $d\phi$



$$ds = \varrho \, d\phi$$

$$\dot{\phi} = \frac{v}{\varrho}$$

1-DYNAMIC OF POINT MASSES

EXAMPLE 1.2

Relative to a Cartesian coordinate system, the position, velocity and acceleration of a particle relative at a given instant are:

$$\mathbf{r} = 250\hat{\mathbf{i}} + 630\hat{\mathbf{j}} + 430\hat{\mathbf{k}}\ (\text{m})$$

$$\mathbf{v} = 90\hat{\mathbf{i}} + 125\hat{\mathbf{j}} + 170\hat{\mathbf{k}}\ (\text{m/s})$$

$$\mathbf{a} = 16\hat{\mathbf{i}} + 125\hat{\mathbf{j}} + 30\hat{\mathbf{k}}\ (\text{m/s}^2)$$

Find the coordinate of the center of curvature at that instant.

1-DYNAMIC OF POINT MASSES

EXAMPLE 1.2

$$v = \|\mathbf{v}\| = \sqrt{90^2 + 125^2 + 170^2} = 229.4\,\mathrm{m/s}$$

$$\hat{\mathbf{u}}_t = \frac{\mathbf{v}}{v} = \frac{90\hat{\mathbf{i}} + 125\hat{\mathbf{j}} + 170\hat{\mathbf{k}}}{229.4} = 0.3923\hat{\mathbf{i}} + 0.5449\hat{\mathbf{j}} + 0.7411\hat{\mathbf{k}}$$

$$a_t = \mathbf{a} \cdot \hat{\mathbf{u}}_t = (16\hat{\mathbf{i}} + 125\hat{\mathbf{j}} + 30\hat{\mathbf{k}}) \cdot (0.3923\hat{\mathbf{i}} + 0.5449\hat{\mathbf{j}} + 0.7411\hat{\mathbf{k}}) = 96.62\,\mathrm{m/s^2}$$

$$a = \sqrt{16^2 + 125^2 + 30^2} = 129.5\,\mathrm{m/s^2}$$

$\mathbf{a} = a_t\hat{\mathbf{u}}_t + a_n\hat{\mathbf{u}}_n$

$\hat{\mathbf{u}}_t$ and $\hat{\mathbf{u}}_n$ are perpendicular

$\rightarrow$ $a^2 = a_t^2 + a_n^2$

1-DYNAMIC OF POINT MASSES

EXAMPLE 1.2

$$a_n = \sqrt{a^2 - a_t^2} = \sqrt{129.5^2 - 96.62^2} = 86.29 \text{ m/s}^2$$

$$\hat{\mathbf{u}}_n = \frac{1}{a_n}(\mathbf{a} - a_t\hat{\mathbf{u}}_t)$$

$$= \frac{1}{86.29}[(16\hat{\mathbf{i}} + 125\hat{\mathbf{j}} + 30\hat{\mathbf{k}}) - 96.62(0.3923\hat{\mathbf{i}} + 0.5449\hat{\mathbf{j}} + 0.7411\hat{\mathbf{k}})]$$

$$= -0.2539\hat{\mathbf{i}} + 0.8385\hat{\mathbf{j}} - 0.4821\hat{\mathbf{k}}$$

$$\varrho = \frac{v^2}{a_n} = \frac{229.4^2}{86.29} = 609.9 \text{ m}$$

$$\mathbf{r}_C = \mathbf{r} + \mathbf{r}_{C/P}$$

$$= \mathbf{r} + \varrho\hat{\mathbf{u}}_n = 250\hat{\mathbf{i}} + 630\hat{\mathbf{j}} + 430\hat{\mathbf{k}} + 609.9(-0.2539\hat{\mathbf{i}} + 0.8385\hat{\mathbf{j}} - 0.4821\hat{\mathbf{k}})$$

$$= 95.16\hat{\mathbf{i}} + 1141\hat{\mathbf{j}} + 136.0\hat{\mathbf{k}} \text{ (m)}$$

The coordinate of C are:

$$\underline{x = 95.16 \text{ m}} \qquad \underline{y = 1141 \text{ m}} \qquad \underline{z = 136.0 \text{ m}}$$

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ORBITAL MECHANICS

CHAPTER 2

NEWTON'S LAW OF GRAVITATION

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2-NEWTON'S LAW OF GRAVITATION

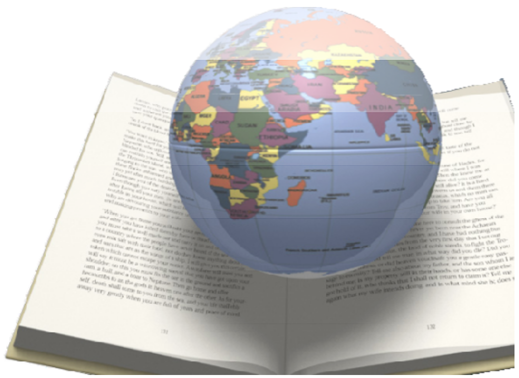
- Two men, Tycho Brache and Johan Kepler, laid the groundwork for Newton's greatest discoveries, 50 years later than his birth. (1592)
- Tycho, was recording accurate data on the position of the planets.
- Kepler by using the Tycho's data found and published his three law of planetary motion (1601- 1619)

KEPLER'S LAWS

__First Law__–The orbit of each planet is an ellipse, with the sun at a focus.

__Second Law__–The line joining the planet to the sun sweeps out equal areas in equal times.

__Third Law__–The square of the period of a planet is proportional to the cube of its mean distance from the sun.



2-NEWTON'S LAW OF GRAVITATION

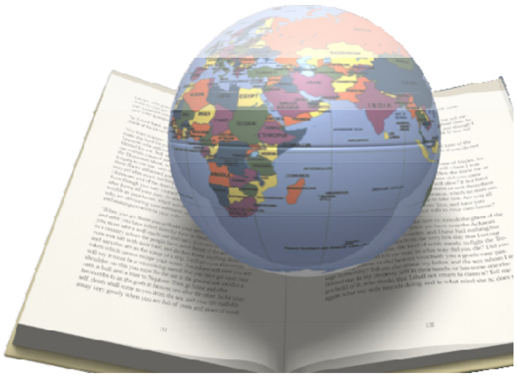
- Still, Kepler's laws were only a description not an explanation of planetary motion.
- The 23- year-old Newton conceived the law of gravitation, the laws of motion and developed the fundamental concepts of differential calculus. (1665- 1666)
- Newton publish his discoveries some 20 years later!, in book "The Mathematical Principles of Natural Philosophy" or more simply " The Principia" (1687)
- In book I of the principle Newton introduces his three laws of motion:

NEWTON'S LAWS

First Law—Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.

Second Law—The rate of change of momentum is proportional to the force impressed and is in the same direction as that force.

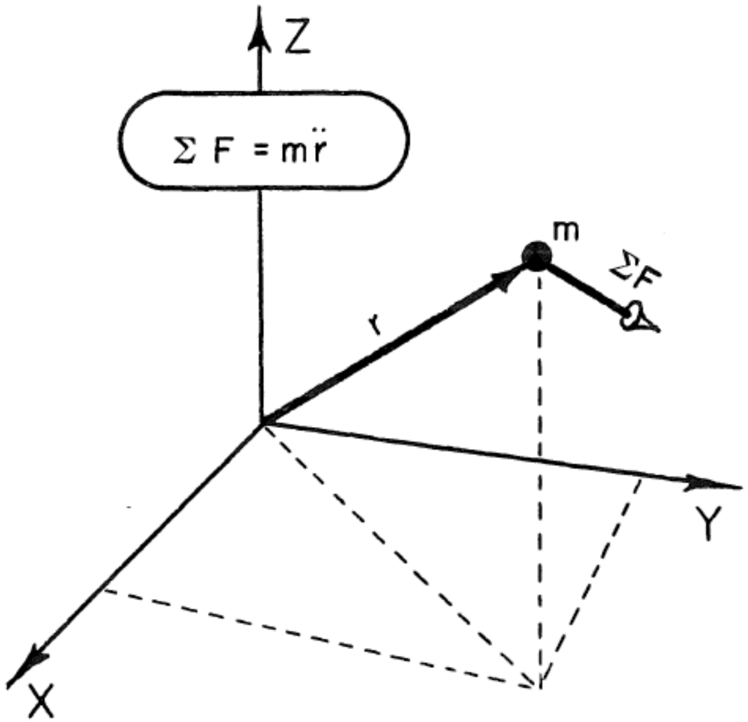
Third Law—To every action there is always opposed an equal reaction.



2-NEWTON'S LAW OF GRAVITATION

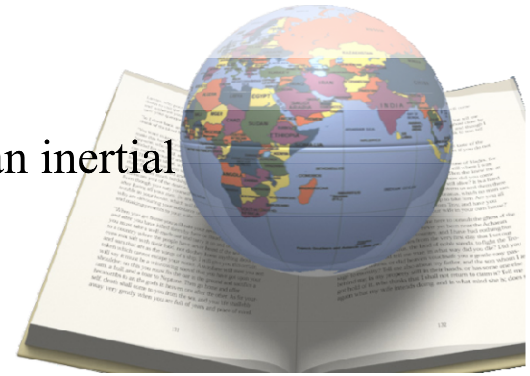
- The second law can be expressed mathematically as follows:

$$\Sigma \mathbf{F} = m\,\ddot{\mathbf{r}} \qquad (1)$$



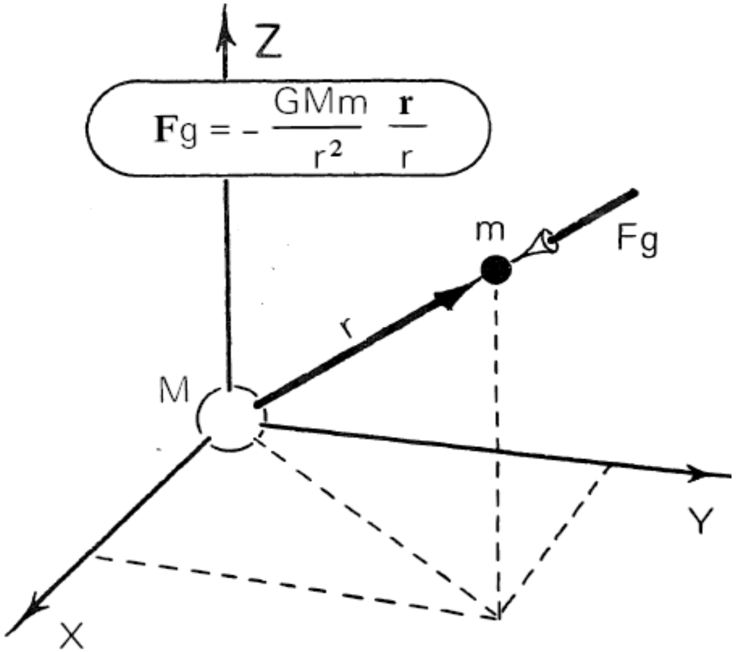
ΣF : The vector sum of all forces acting on the mass

$m\ddot{r}$: The vector acceleration of the mass measured relative to an inertial reference from



2-NEWTON'S LAW OF GRAVITATION

✶Newton formulated the law of gravity by stating that any two bodies attract one another with a force proportional to the product of their masses and inversely proportional to the square of the distance between them:

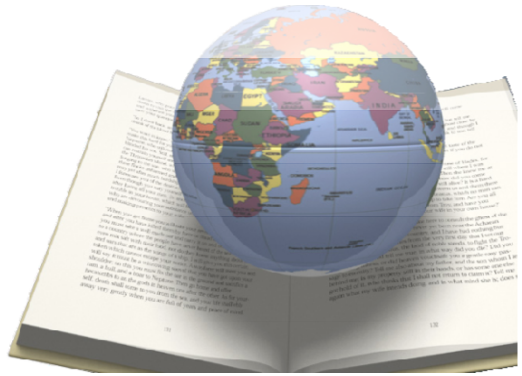


$$\mathbf{F}_g = -\frac{GMm}{r^2}\frac{\mathbf{r}}{r} \qquad (2)$$

F_g :The force on mass m due to mass M

`r :The vector from M to m

$G : 6.6742\times10^{11}\, m^3/kg.s^2$ The universal gravitational constant



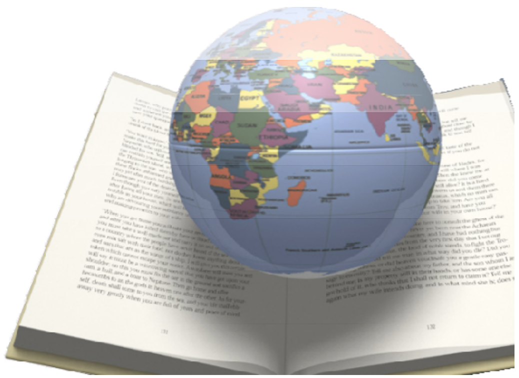
2-NEWTON'S LAW OF GRAVITATION

- ✶ Mass, like length and time is a primitive physical concept
- ✶ It can't be defined in terms of any other physical concept.
- ✶ Mass is simply the quantity of matter.
- ✶ More practically, mass is a measure of the inertia of a body.
- ✶ Inertia in an object's resistance to changing its state of motion.

inertia ↑ ⟶ mass ↑

inertia ↓ ⟶ mass ↓

- ✶ The unit of mass is "Kg"



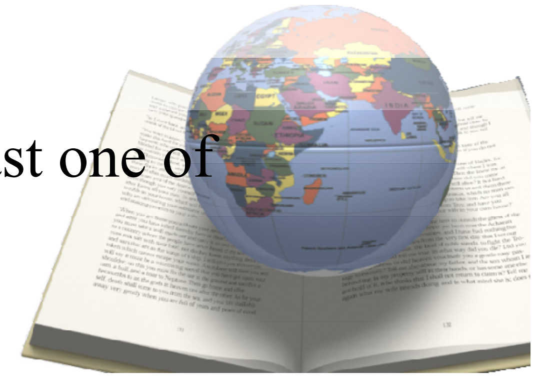
2-NEWTON’S LAW OF GRAVITATION

- ✶ Force is the action of one physical body on another, either through direct contact or through a distance
- ✶ Gravity is an example of force acting through a distance.
- ✶ The gravitational force between two masses m_1 and m_2 having a distance r between their centers is: (Newton’s law of gravity)

$$F_g = G\frac{m_1 m_2}{r^2} \qquad (3)$$

G: universal gravitational constant

- ✶ The force of gravity is too small unless at least one of the masses is extremely big.



2-NEWTON'S LAW OF GRAVITATION

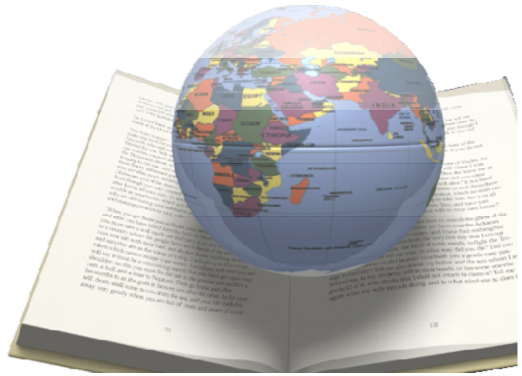
- The force of a large mass (such as the earth) on a mass many orders of magnitude smaller (such as a person) is called weight.
- The weight of the small body is:

$$W = G\frac{Mm}{r^2} = m(\frac{GM}{r^2})$$

$$W = mg \qquad (4)$$

$$g = \frac{GM}{r^2} \qquad (5)$$

$g : (m/s^2)$ acceleration of gravity

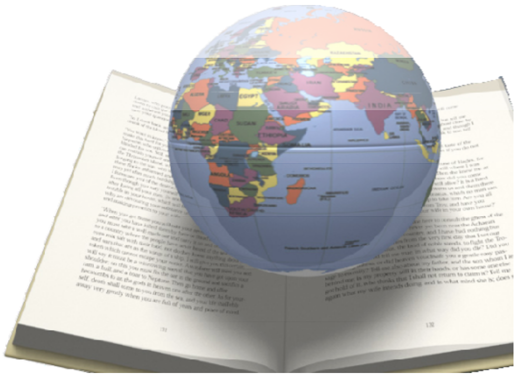


2-NEWTON'S LAW OF GRAVITATION

- If planetary gravity is the only force acting on a body the body is said to be in free fall.
- In free fall, the are no contact forces, so there can be no sense of weight.
- Even though the weight is not zero, a person in free fall experiences weightlessness, or absence of gravity.

$$R_E = 6378km \longrightarrow g_0 = \frac{GM}{R_E^2} \qquad (6)$$

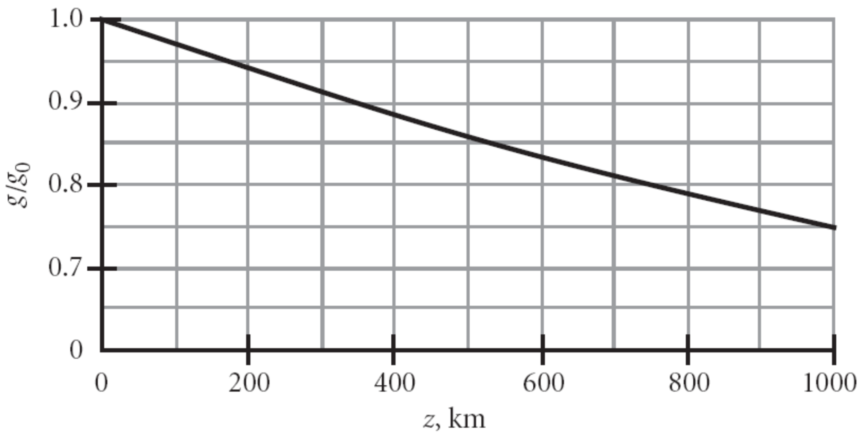
$$g_0 = \frac{GM}{R_E^2} \longrightarrow g_0 = 9.807\, m/s^2$$



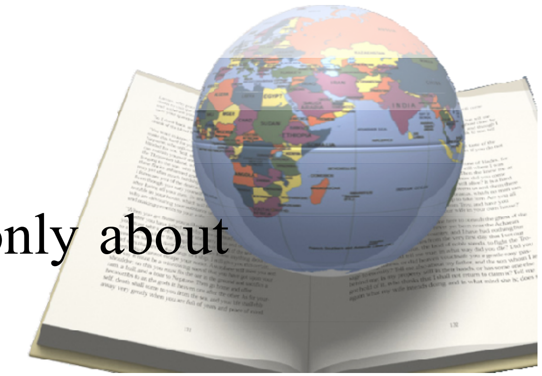
2-NEWTON'S LAW OF GRAVITATION

$$(6),(5) \rightarrow g = g_0 \frac{R_E^2}{(R_E + z)^2} = \frac{g_0}{(1 + z/R_E)^2} \qquad (8)$$

- ✶ Measurement's show that a altitudes on the order of 10 kilometers g is only three-tenths of a percent (%0.3) less than its sea-level value.
- ✶ Thus under ordinary conditions, we ignore the variation of g with altitude.



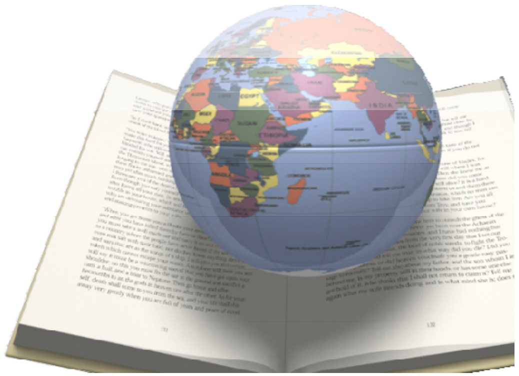
- ✶ At space station altitude (300km), weight is only about 10 percent less than it is on the earth's surface



2-NEWTON’S LAW OF GRAVITATION

EXAMPLE 2.1

- Show that in the absence of an atmosphere, the shape of a low altitude ballistic trajectory is a parabola. Assume the acceleration of gravity g is constant and neglect the earth’s curvature



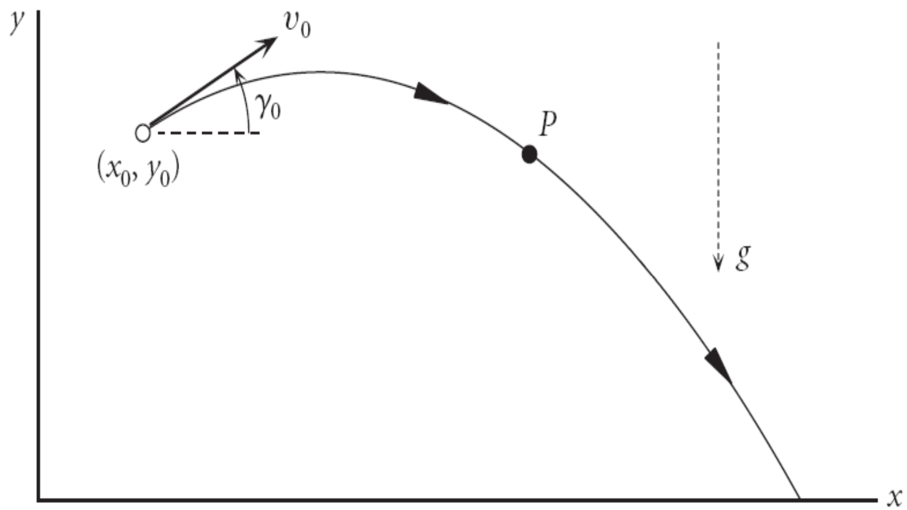
2-NEWTON'S LAW OF GRAVITATION

EXAMPLE 2.1

$t = 0$: Time of launched

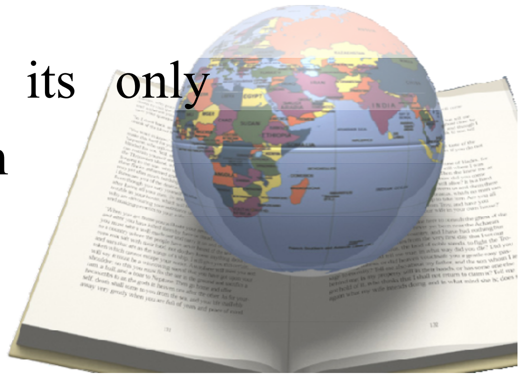
v_0 : speed

γ_0 : Flight path angle



Solution:

Since the projectile is in free fall after launch, its only acceleration is that of gravity in the negative y-direction



$$\ddot{x} = 0$$
$$\ddot{y} = -g$$

2-NEWTON'S LAW OF GRAVITATION

EXAMPLE 2.1

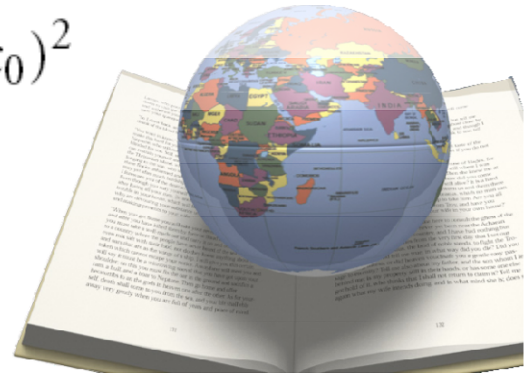
- Integrating with respect to time and applying the initial conditions leads to:

$$x = x_0 + (v_0 \cos \gamma_0)t \qquad \text{(a)}$$

$$y = y_0 + (v_0 \sin \gamma_0)t - \frac{1}{2}gt^2 \qquad \text{(b)}$$

- Solving (a) for t and substituting the result into (b) yields.

$$y = y_0 + (x - x_0)\tan\gamma_0 - \frac{1}{2}\frac{g}{v_0 \cos \gamma_0}(x - x_0)^2$$

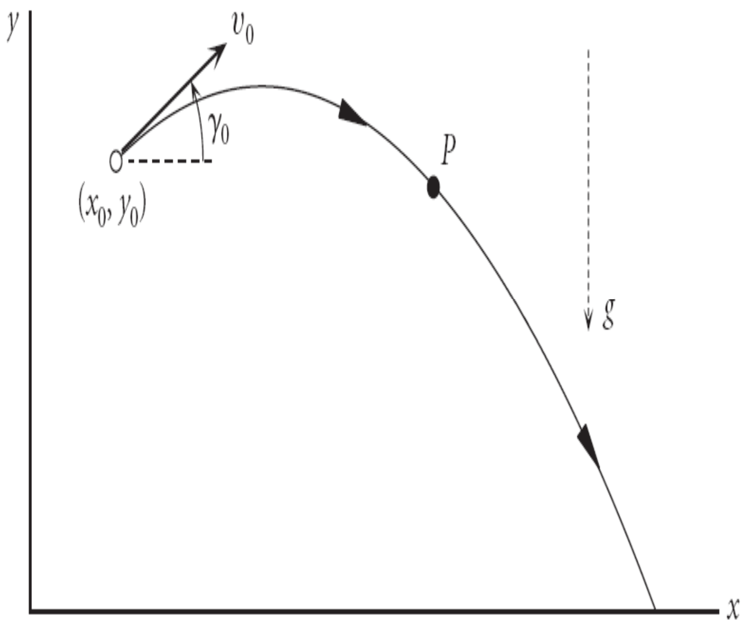


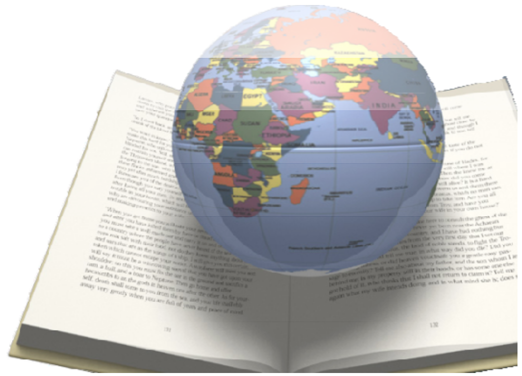
2-NEWTON'S LAW OF GRAVITATION

EXAMPLE 2.2

An airplane flies a parabolic trajectory so that the passengers will experience free fall (weightlessness)

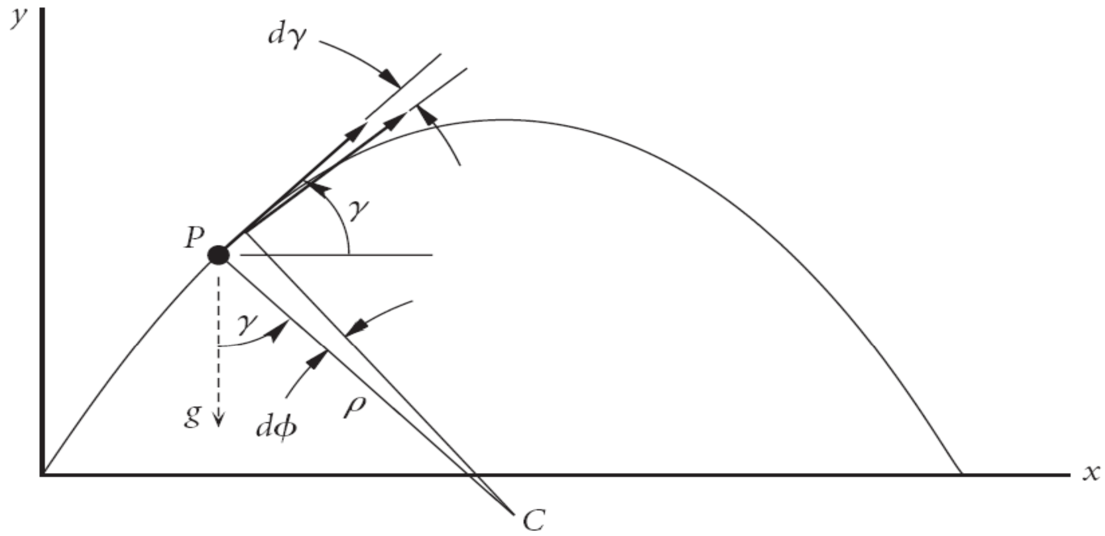
What is the required variation of the flight path angle γ with speed v?





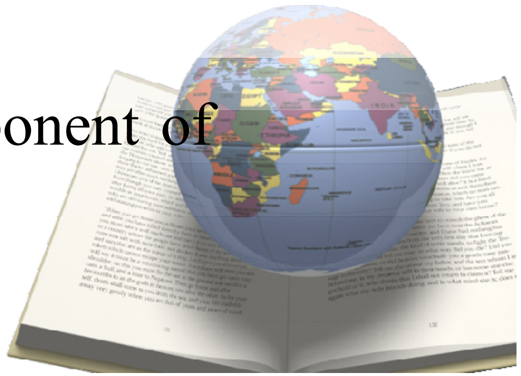
2-NEWTON'S LAW OF GRAVITATION

Solution: EXAMPLE 2.2



- For "flat" earth $d\gamma = -d\varphi$ → $\dot{\gamma} = -\dot{\varphi}$
- We have had: $\rho\dot{\gamma} = v$ (g)
- The normal acceleration a_n is just the component of the gravitational acceleration g then:

$$a_n = g\cos\gamma \qquad \text{(a)}$$



2-NEWTON'S LAW OF GRAVITATION

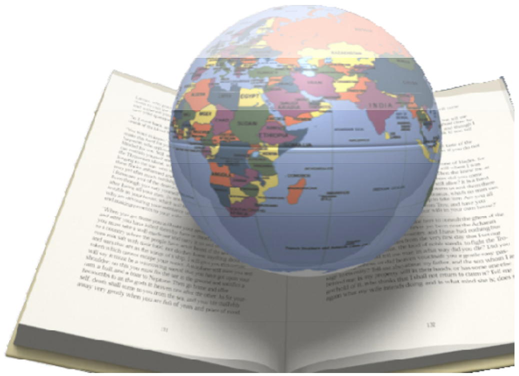
EXAMPLE 2.2

✶ Substituting $a_n = \frac{v^2}{\rho}$ into (a) and solving for the radius of curvature yields:

$$\varrho = \frac{v^2}{g \cos\gamma} \qquad \text{(b)}$$

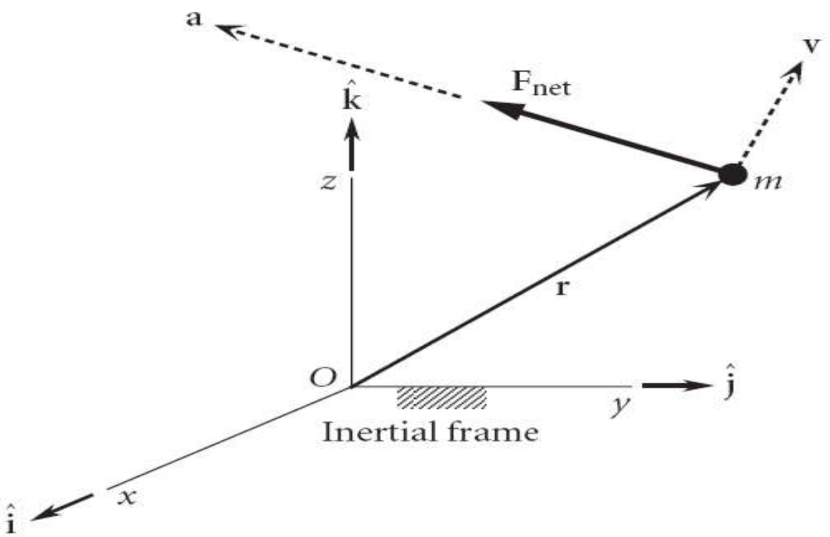
✶ Combining equations (g) and (b) we find:

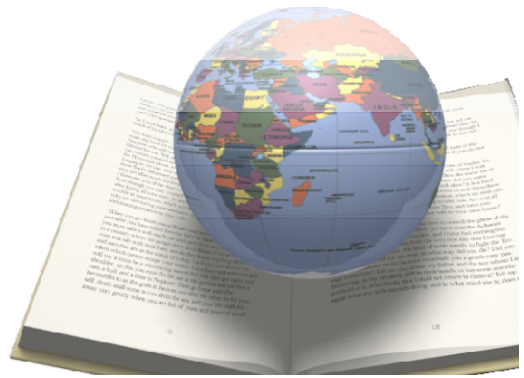
$$\dot{\gamma} = -\frac{g \cos\gamma}{v}$$



2-NEWTON'S LAW OF GRAVITATION

- ✶ Force is not primitive concept like mass because it is connected with the concepts of motion and inertia
- ✶ The only way to alter the motion of a body is to * exert a force on it.
- ✶ If the resultant or net force on a body of mass m is, then $F_{net} = ma$ (10)





2-NEWTON'S LAW OF GRAVITATION

- The integral of a force F over a time interval is called the impulse I of the force.

$$\mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}\, dt \quad (11)$$

(10) , (11) →

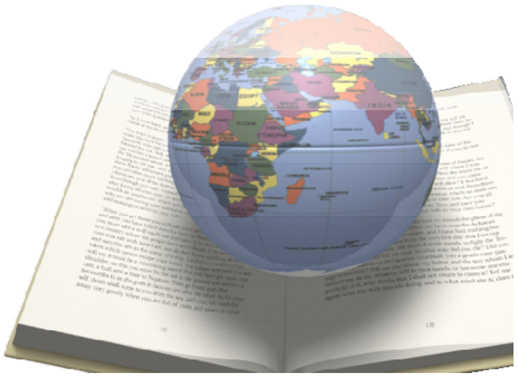
$$\mathbf{I}_{\text{net}} = \int_{t_1}^{t_2} m \frac{d\mathbf{v}}{dt} dt = m\mathbf{v}_2 - m\mathbf{v}_1 \quad (12)$$

- That is the net impulse on a body yields a change $m\Delta v$ in its liner momentum so that:

$$\Delta\mathbf{v} = \frac{\mathbf{I}_{\text{net}}}{m} \quad (13)$$

- If F_{net} is <u>constant</u>, then $I_{net} = F_{net}\Delta t$
- So, equation (13) becomes:

$$\Delta\mathbf{v} = \frac{\mathbf{F}_{\text{net}}}{m}\Delta t \quad (14)$$

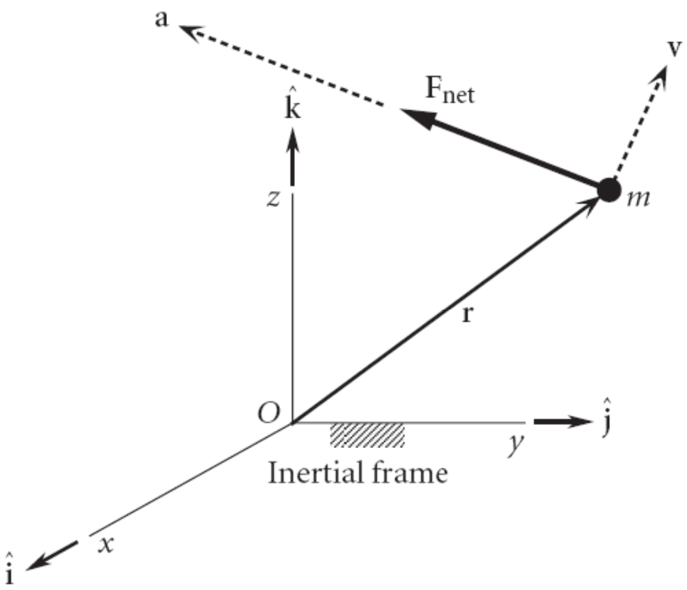


2-NEWTON'S LAW OF GRAVITATION

- The moment of the net force about O is

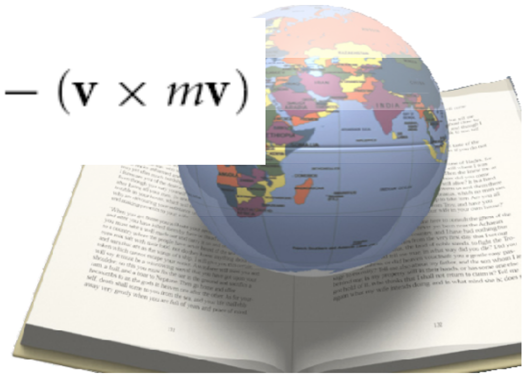
$$\mathbf{M}_{O_{\text{net}}} = \mathbf{r} \times \mathbf{F}_{\text{net}}$$

$$\mathbf{M}_{O_{\text{net}}} = \mathbf{r} \times m\mathbf{a} = \mathbf{r} \times m\frac{d\mathbf{v}}{dt}$$



- If the mass m is constant:

$$\mathbf{r} \times m\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) - \left(\frac{d\mathbf{r}}{dt} \times m\mathbf{v}\right) = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) - (\mathbf{v} \times m\mathbf{v})$$



2-NEWTON'S LAW OF GRAVITATION

★ Since $\mathbf{v} \times m\mathbf{v} = m(\mathbf{v} \times \mathbf{v}) = \mathbf{0}$, so

(15) → $$\mathbf{M}_{O_{\text{net}}} = \frac{d\mathbf{H}_O}{dt} \qquad (16)$$

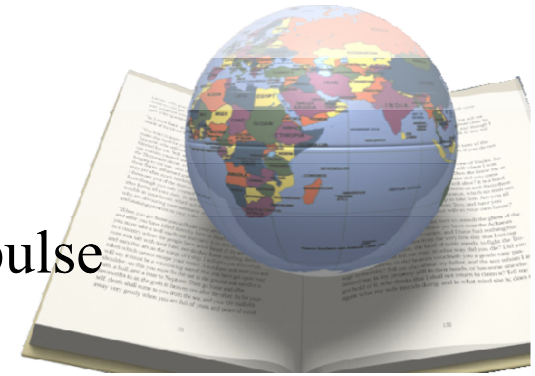
★ Where H_o is the angular momentum about O

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} \qquad (17)$$

★ Thus, just as the net force on a particle changes its linear momentum mv, the moment of that force about a fixed point changes the moment of its linear momentum about that point.

$$\int_{t_1}^{t_2} \mathbf{M}_{O_{\text{net}}} dt = \mathbf{H}_{O_2} - \mathbf{H}_{O_1} \quad (18)$$

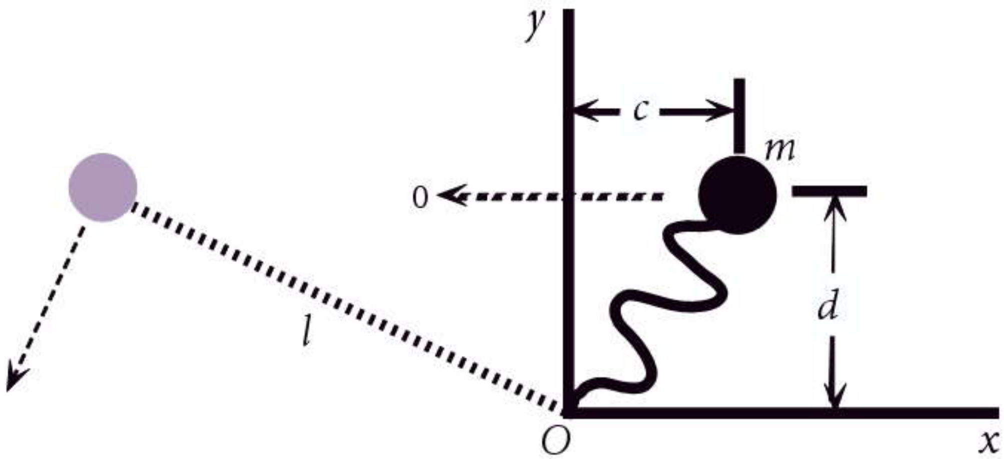
★ The integral on the left is the net angular impulse

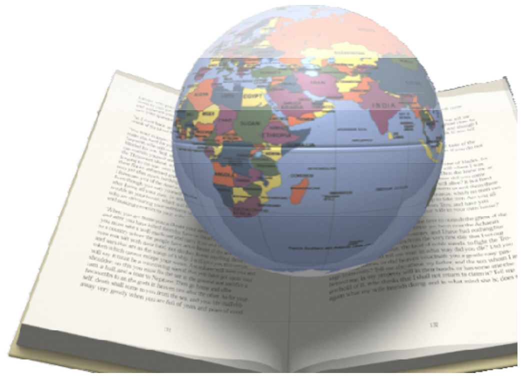


2-NEWTON'S LAW OF GRAVITATION

EXAMPLE 2.3

- A particle of mass m is attached to point O by an inextensible string of length l. initially the string is slack when m is moving to the left with a speed v_0 in the position shown. Calculate the speed of m just after the string becomes taut. Also, compute the average force in the string over the small time interval Δt required to change the direction of the particle's motion.





2-NEWTON'S LAW OF GRAVITATION

EXAMPLE 2.3

✶ Initially, the position and velocity of the particle are

$$\mathbf{r}_1 = c\hat{\mathbf{i}} + d\hat{\mathbf{j}} \qquad \mathbf{v}_1 = -v_0\hat{\mathbf{i}}$$

✶ The angular momentum is

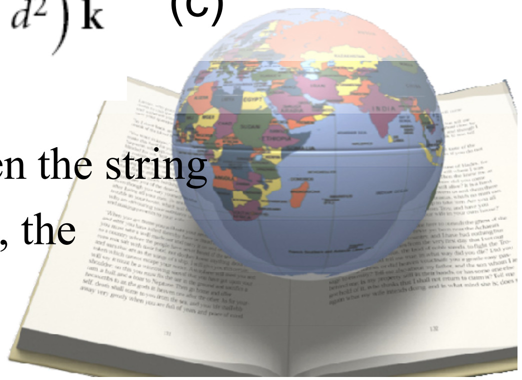
$$\mathbf{H}_1 = \mathbf{r}_1 \times m\mathbf{v}_1 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ c & d & 0 \\ -mv_0 & 0 & 0 \end{vmatrix} = mv_0\hat{\mathbf{k}} \qquad \text{(a)}$$

✶ Just after the string becomes taut

$$\mathbf{r}_2 = -\sqrt{l^2 - d^2}\hat{\mathbf{i}} + d\hat{\mathbf{j}} \qquad \mathbf{v}_2 = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} \qquad \text{(b)}$$

✶ And the angular momentum is

$$\mathbf{H}_2 = \mathbf{r}_2 \times m\mathbf{v}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sqrt{l^2 - d^2} & d & 0 \\ v_x & v_y & 0 \end{vmatrix} = \left(-mv_x d - mv_y\sqrt{l^2 - d^2}\right)\hat{\mathbf{k}} \qquad \text{(c)}$$



✶Initially the force exerted on m by the slack string is zero. When the string becomes taut, the force exerted on m passes through O, therefore , the moment of the net force on m about O remains zero. $\mathbf{H}_2 = \mathbf{H}_1$

2-NEWTON'S LAW OF GRAVITATION

EXAMPLE 2.3

✶ Substituting (a) and (c) yields

$$v_x d + \sqrt{l^2 - d^2}\, v_y = -v_0 d \qquad \text{(d)}$$

✶ The string is inextensible, so the component of the velocity of m along the string must be zero

$$\mathbf{v}_2 \cdot \mathbf{r}_2 = 0$$

✶ Substituting v_2 and r_2 from (b) and solving for v_y we get

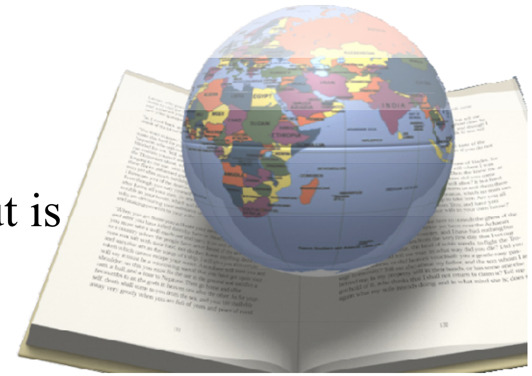
$$v_y = v_x \sqrt{\frac{l^2}{d^2} - 1} \qquad \text{(e)}$$

✶ Solving (d) and (e) for v_x and v_y leads to

$$v_x = -\frac{d^2}{l^2} v_0 \qquad v_y = -\sqrt{1 - \frac{d^2}{l^2}}\, \frac{d}{l} v_0 \qquad \text{(f)}$$

✶thus, the speed, $v = \sqrt{v_x^2 + v_y^2}$,after the string becomes taut is

$$v = \frac{d}{l} v_0$$



2-NEWTON'S LAW OF GRAVITATION

EXAMPLE 2.3

✶ From equation 12, the impulse on m during the time it takes the string become taut is

$$\mathbf{I} = m(\mathbf{v}_2 - \mathbf{v}_1) = m\left[\left(-\frac{d^2}{l^2}v_0\hat{\mathbf{i}} - \sqrt{1-\frac{d^2}{l^2}}\frac{d}{l}v_0\hat{\mathbf{j}}\right) - (-v_0\hat{\mathbf{i}})\right]$$

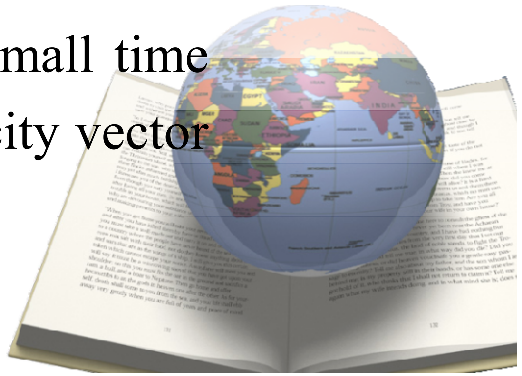
$$= \left(1-\frac{d^2}{l^2}\right)mv_0\hat{\mathbf{i}} - \sqrt{1-\frac{d^2}{l^2}}\frac{d}{l}mv_0\hat{\mathbf{j}}$$

✶ The magnitude of this impulse, which is directed along the string, is

$$I = \sqrt{1-\frac{d^2}{l^2}}mv_0$$

✶ Hence, the average force in the string during the small time interval Δt required to change the direction of the velocity vector turns out to be

$$F_{\text{avg}} = \frac{I}{\Delta t} = \sqrt{1-\frac{d^2}{l^2}}\frac{mv_0}{\Delta t}$$



IN THE NAME OF GOD

ORBITAL MECHANICS

CHAPTER 4

The two-body problem

Student version

By: M.Mirshams

2010

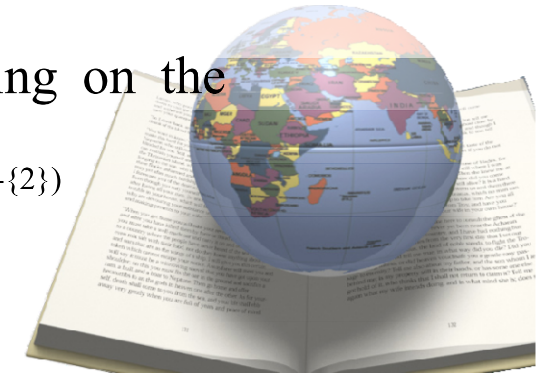
4-THE TWO-BODY PROBLEM

- Now that we have a general expression for the relative motion of two bodies perturbed by other bodies it would be a simple matter to reduce it to an equation for only two bodies.

- There are two assumptions we will make with regard to our model:

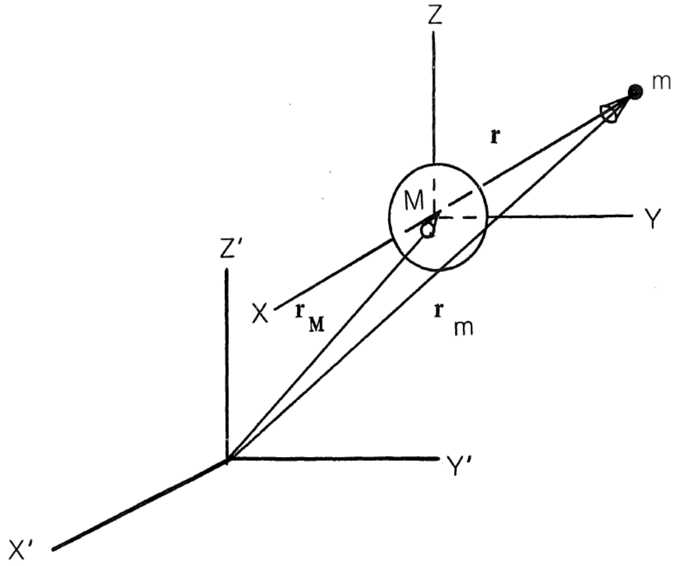
  1- The bodies are spherically symmetric (Note 3-page11-{2})

  2- There are no external non internal forces acting on the system other than the gravitational forces (Note 4-page12-{2})

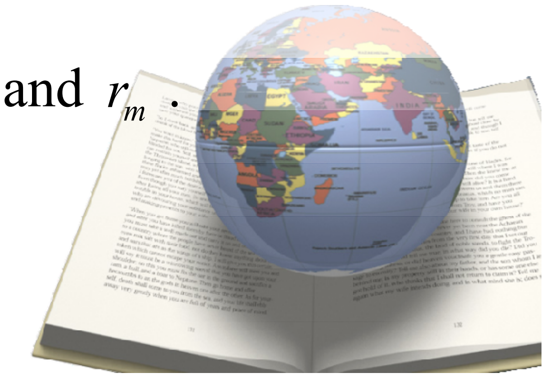


4-THE TWO-BODY PROBLEM

- (Note 5 Page12 {2})
- Consider the system of two bodies of mass M and m
- Let (x', y', z') be an internal set of rectangular cartesian coordinates.



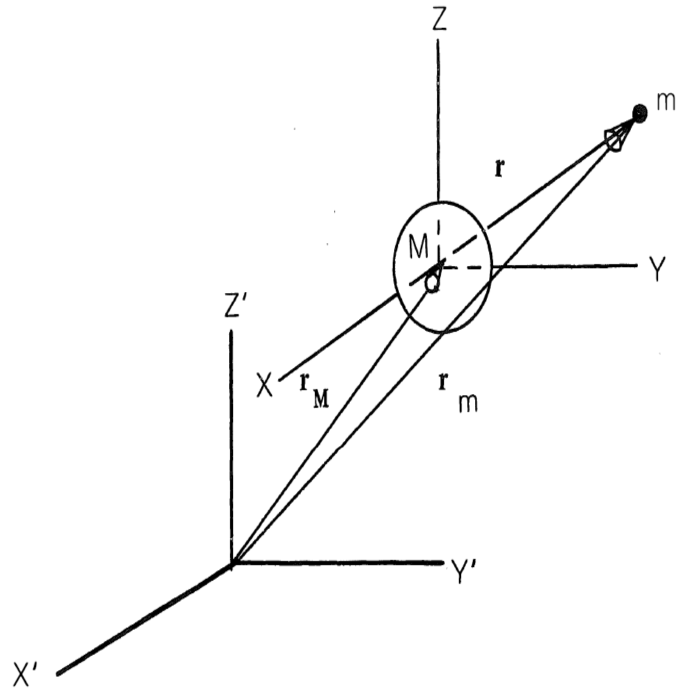
- Let (x', y', z') be a set of nonrotating coordinates parallel to (x, y, z) and having an origin coincident with the body of mass M.
- The position vectors of the bodies M and m are r_M and r_m .



4-THE TWO-BODY PROBLEM

- We have defined

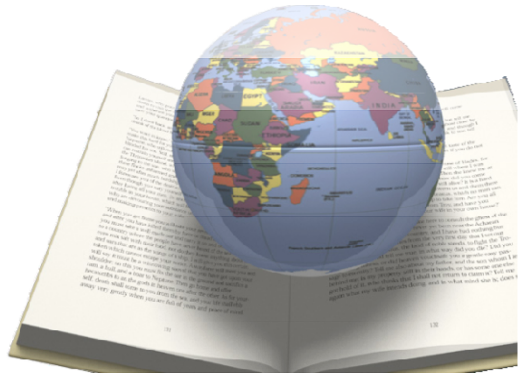
$$\mathbf{r} = \mathbf{r}_m - \mathbf{r}_M .$$



- By applying Newton's laws in the inertial frame (x', y', z') we will obtain:

$$m\ddot{\mathbf{r}}_m = -\frac{GMm}{r^2}\frac{\mathbf{r}}{r}$$

$$M\ddot{\mathbf{r}}_M = \frac{GMm}{r^2}\frac{\mathbf{r}}{r}$$



4-THE TWO-BODY PROBLEM

- The above equations may be written:

$$\ddot{\mathbf{r}}_m = -\frac{GM}{r^3}\,\mathbf{r} \qquad (1)$$

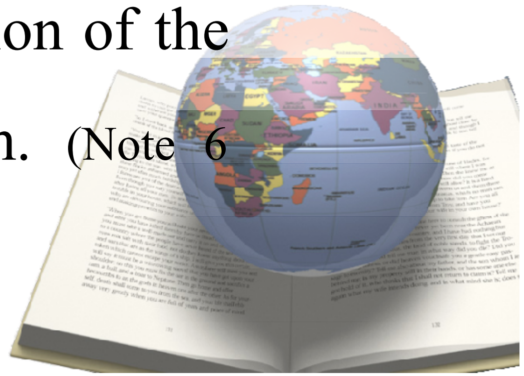
$$\ddot{\mathbf{r}}_M = \frac{Gm}{r^3}\,\mathbf{r}\,. \qquad (2)$$

- Subtracting equation (2) from (1) we have

$$\ddot{r} = \ddot{r}_m - \ddot{r}_M$$

$$\ddot{\mathbf{r}} = -\frac{G(M+m)}{r^3}\,\mathbf{r}\,. \qquad (3)$$

- Equation (3) is the vector differential equation of the relative motion for the two-body problem. (Note 6 Page13 {2})



4-THE TWO-BODY PROBLEM

✶ Since our efforts will be devoted to studying the motion of satellites . Ballistic missiles or space probes orbiting about some planet or the sun, Hence we see that:

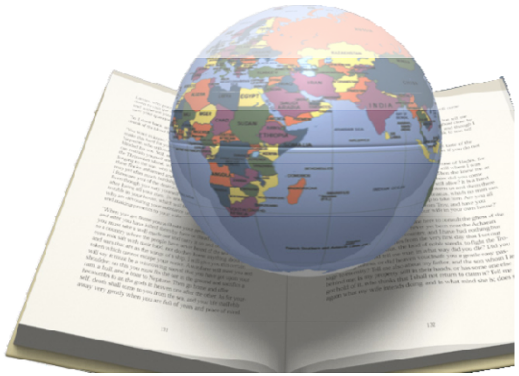
$$G(M+m) \approx GM.$$

✶ It is convenient to define a parameter μ ,called the gravitational parameter as:

$$\mu \equiv GM.$$

✶ Then the equation 3 becomes:

$$\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} = 0. \qquad (4)$$

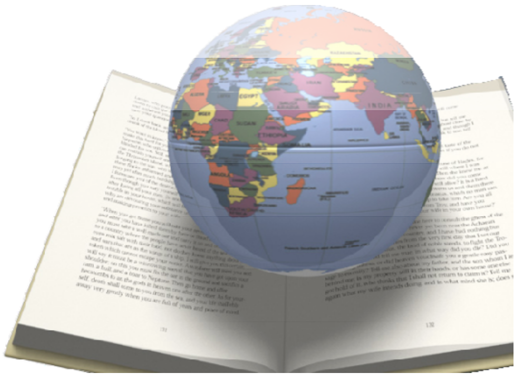


4-THE TWO-BODY PROBLEM

$$\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} = 0. \qquad (4)$$

- Equation(4) is the two-body equation of motion
- Remember the results obtained from equation(4) will be only as accurate as the assumptions (1),(2) and the assumption that $M \gg m$
- If m is not much less than M.

  then $G(M + m)$ must be used in place of μ



4-THE TWO-BODY PROBLEM

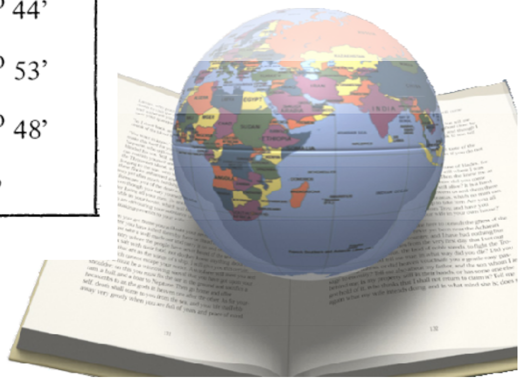
★ μ will have a different value for each major attracting body

Table 8.2-3

PHYSICAL CHARACTERISTICS OF THE SUN AND PLANETS*

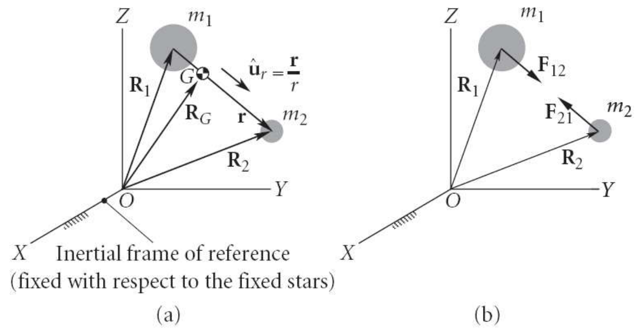
Planet	Orbital Period years	Mean distance 10^6 km	Orbital speed km/sec	Mass Earth = 1	μ km^3/sec^2	Equatorial radius km	Inclination of equator to orbit
Sun	–	–	–	333432	1.327×10^{11}	696000	7° 15'
Mercury	.241	57.9	47.87	.056	2.232×10^4	2487	?
Venus	.615	108.1	35.04	.817	3.257×10^5	6187	32°
Earth	1.000	149.5	29.79	1.000	3.986×10^5	6378	23° 27'
Mars	1.881	227.8	24.14	.108	4.305×10^4	3380	23° 59'
Jupiter	11.86	778	13.06	318.0	1.268×10^8	71370	3° 04'
Saturn	29.46	1426	9.65	95.2	3.795×10^7	60400	26° 44'
Uranus	84.01	2868	6.80	14.6	5.820×10^6	23530	97° 53'
Neptune	164.8	4494	5.49	17.3	6.896×10^6	22320	28° 48'
Pluto	247.7	5896	4.74	.9?	3.587×10^5?	7016?	?

*From reference 3



4-THE TWO-BODY PROBLEM

Equations of motion in an inertial frame:



- ✶ Above figure shows two point masses acted upon only by the force of gravity between them. (Note 7. P 34.{1})

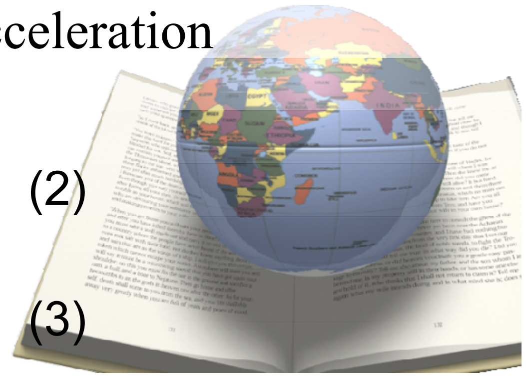
- ✶ The position vector R_G of the center of mass G of the system is defined by the formula:

$$\mathbf{R}_G = \frac{m_1\mathbf{R}_1 + m_2\mathbf{R}_2}{m_1 + m_2} \qquad (1)$$

- ✶ Therefore the absolute velocity and the absolute acceleration of G are: (Note 8. P 35.{1})

$$\mathbf{v}_G = \dot{\mathbf{R}}_G = \frac{m_1\dot{\mathbf{R}}_1 + m_2\dot{\mathbf{R}}_2}{m_1 + m_2} \qquad (2)$$

$$\mathbf{a}_G = \ddot{\mathbf{R}}_G = \frac{m_1\ddot{\mathbf{R}}_1 + m_2\ddot{\mathbf{R}}_2}{m_1 + m_2} \qquad (3)$$



4-THE TWO-BODY PROBLEM

- Let **r** be the position vector m_2 relative to m_1 , then:

$$\mathbf{r} = \mathbf{R}_2 - \mathbf{R}_1 \qquad (4)$$

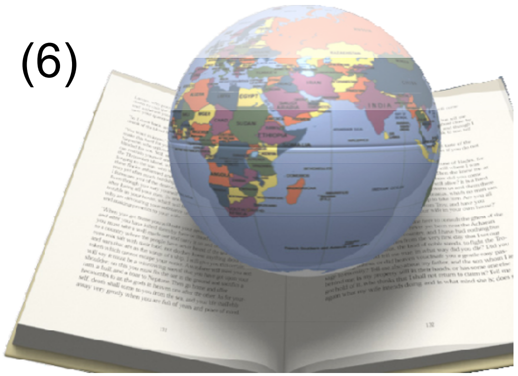
- Furthermore, let $\hat{u}_r$ be the unit vector pointing from m_1

  towards m_2 , so that $\hat{\mathbf{u}}_r = \frac{\mathbf{r}}{r}$ (5)

- Where $r = \|r\|$ the magnitude of **r**

- The gravitational attraction force exerted on m_2 by m_1 is

$$\mathbf{F}_{21} = \frac{Gm_1m_2}{r^2}(-\hat{\mathbf{u}}_r) = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{u}}_r \qquad (6)$$



Note 9. P 36.{1}

4-THE TWO-BODY PROBLEM

- Newton's second law of motion as applied to body m_2 is

  $\mathbf{F}_{21} = m_2\ddot{\mathbf{R}}_2$, where $\ddot{\mathbf{R}}_2$ is the absolute acceleration of m_2

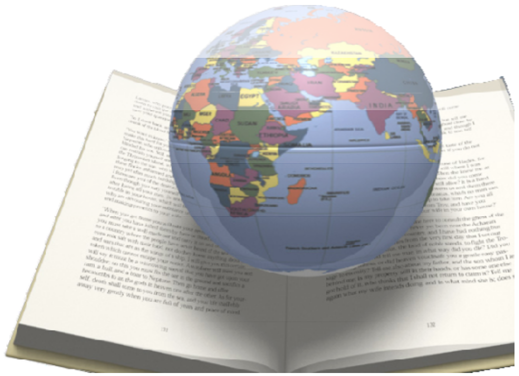
  thus:

$$-\frac{Gm_1m_2}{r^2}\hat{\mathbf{u}}_r = m_2\ddot{\mathbf{R}}_2 \qquad (7)$$

- By Newton's third law $\mathbf{F}_{12} = -\mathbf{F}_{21}$, so that for m_1 we have

$$\frac{Gm_1m_2}{r^2}\hat{\mathbf{u}}_r = m_1\ddot{\mathbf{R}}_1 \qquad (8)$$

- Equations (7) and (8) are the equations of motion of the two bodies in inertial space



4-THE TWO-BODY PROBLEM

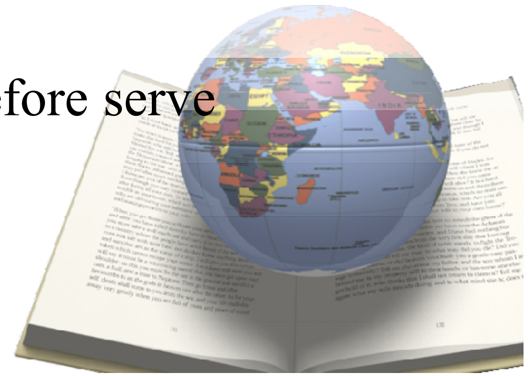
- By adding each side of these equations together we find:

$$m_1 \ddot{\mathbf{R}}_1 + m_2 \ddot{\mathbf{R}}_2 = 0$$

- According to Equ.(3), that means the acceleration of the center of mass G of the system of two bodies m_1 and m_2 is zero.
- G moves with a constant velocity V_G in a straight lines, so that its position vector relative to XYZ given by

$$\mathbf{R}_G = \mathbf{R}_{G_0} + \mathbf{v}_G t \qquad (9)$$

- Where R_{G_0} is the position of G at time $t = 0$
- The center of mass of a two-body system may therefore serve as the origin of an inertial frame.



4-THE TWO-BODY PROBLEM

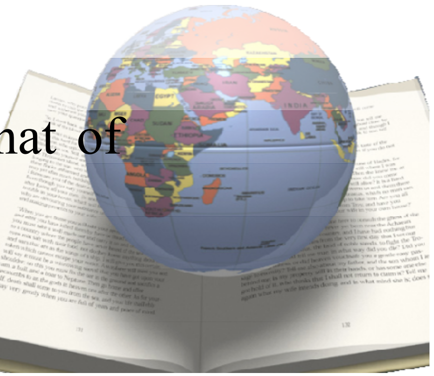
Example

Use the equations of motion to show why orbiting astronauts experience weightlessness.

Solution:

- ✶ We sense weight by feeling the contact forces that developed wherever our body is supported.
- ✶ Consider an astronaut of mass m_A strapped into the space shuttle of mass m_S in orbit about the earth.
- ✶ The distance between the center of the earth and spacecraft is **r**, and the mass of the earth is m_E
- ✶ Sense the only external force on the space shuttle is that of gravity $\mathbf{F}_S)_g$ the equation of motion of the shuttle is:

$$\mathbf{F}_S)_g = m_S \mathbf{a}_S \quad \text{(a)}$$



4-THE TWO-BODY PROBLEM

EXAMPLE

- According to equation (6)

$$\mathbf{F}_S)_g = -\frac{GM_E m_S}{r^2}\hat{\mathbf{u}}_r \qquad \text{(b)}$$

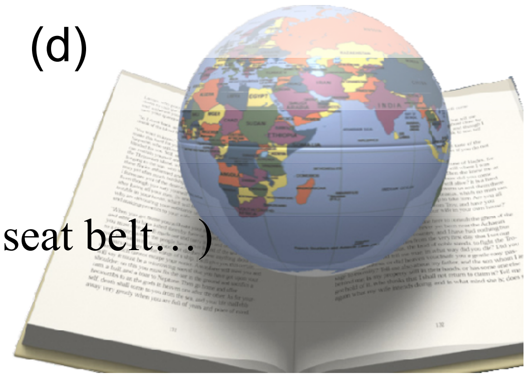
$\hat{u}_r$: is the unit vector pointing outward from the earth to space shuttle.

- Thus (a) and (b) imply:

$$\mathbf{a}_S = -\frac{GM_E}{r^2}\hat{\mathbf{u}}_r \qquad \text{(c)}$$

- The equation of motion of the astronaut is:

$$\mathbf{F}_A)_g + \mathbf{C}_A = m_A \mathbf{a}_A \qquad \text{(d)}$$



$F_{A)g}$:the weight of the astronaut

C_A : the net contact force on the astronaut from restraints (seat, seat belt…)

a_A : the astronaut's acceleration.

4-THE TWO-BODY PROBLEM

EXAMPLE

- According to Equ.(6)

$$\mathbf{F}_A)_g = -\frac{GM_E m_A}{r^2}\hat{\mathbf{u}}_r \qquad \text{(e)}$$

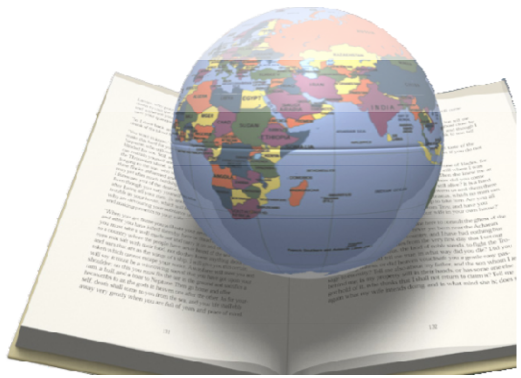
- Since the astronaut is moving with the shuttle we have:

$$\mathbf{a}_A = \mathbf{a}_S = -\frac{GM_E}{r^2}\hat{\mathbf{u}}_r \qquad \text{(f)}$$

- Substituting (e) and (f) into (d) yields:

$$-\frac{GM_E m_A}{r^2}\hat{\mathbf{u}}_r + \mathbf{C}_A = m_A\left(-\frac{GM_E}{r^2}\hat{\mathbf{u}}_r\right)$$

- From which it is clear that $C_A = 0$



IN THE NAME OF GOD

ORBITAL MECHANICS

CHAPTER 4

THE N-BODY PROBLEM

Student version

By: M.Mirshams

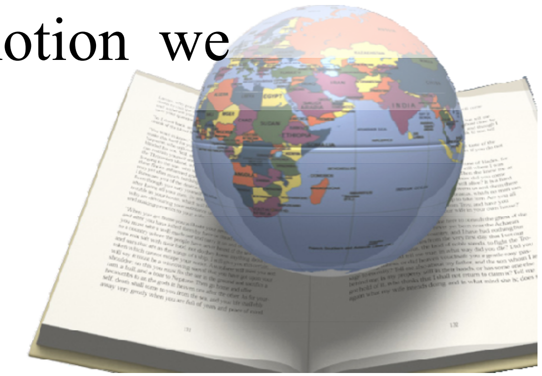
2010

3-THE N-BODY PROBLEM

✶ In this section we shall explain the motion of a body which is being acted upon by several gravitational masses and may even be experiencing other forces such as drag, thrust and solar radiation pressure.

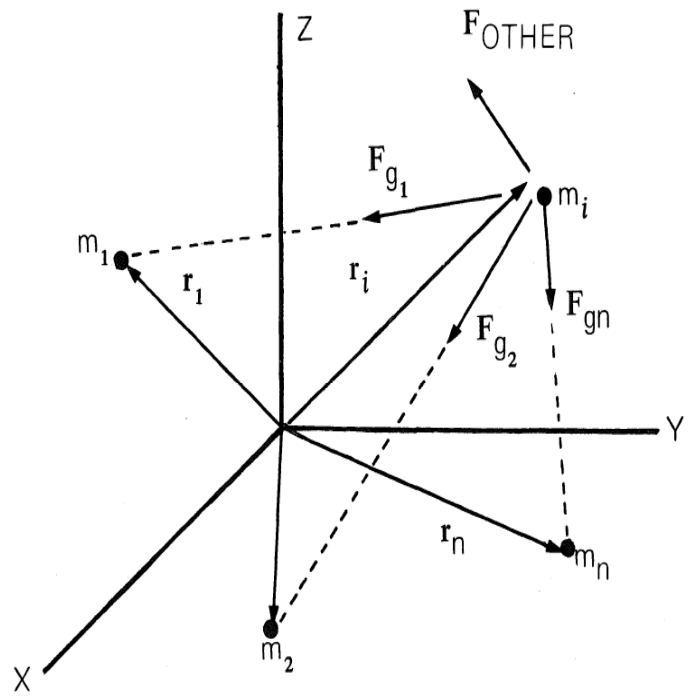
✶ For this we shall assume a "system" of n-bodies $(m_1, m_2, m_3, \ldots, m_n)$

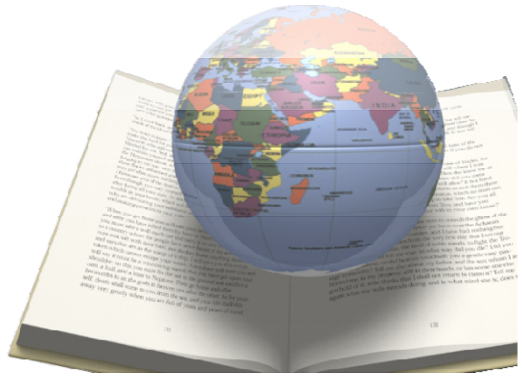
✶ One of these bodies is the body whose motion we wish to study-call it the i^{th} body, m_i



3-THE N-BODY PROBLEM

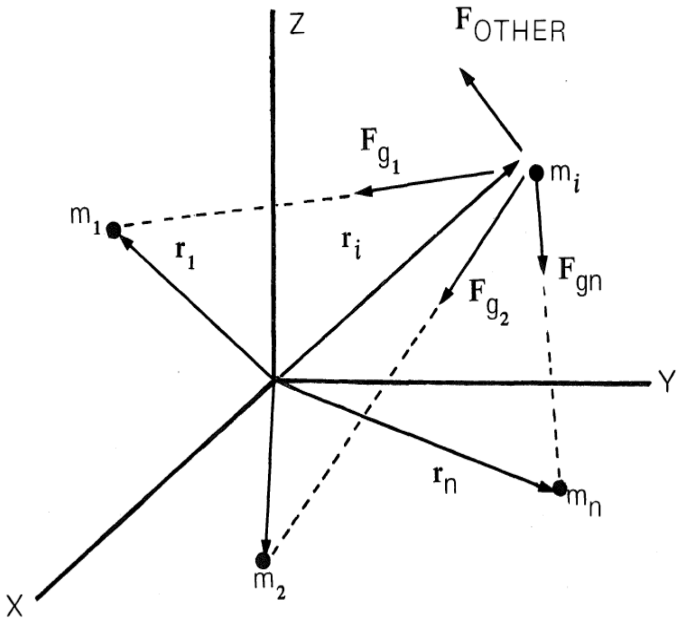
- The vector sum of all gravitational forces and other external forces acting on m_i will be used to determine the equation of motion
- To determine the gravitational forces we shall apply Newton’s law of universal gravitation. (Note1,page5,{2})





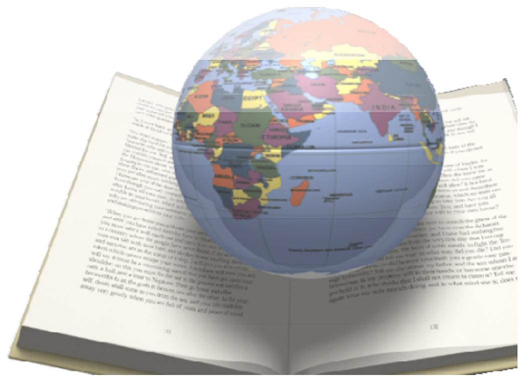
3-THE N-BODY PROBLEM

✶ The first step in our analysis will be to choose a "suitable" coordinate system. This system is illustrated below:



✶ In (X,Y,Z) coordinate system, the position of the n masses are known

$(r_1, r_2, \ldots, r_n)$



3-THE N-BODY PROBLEM

✶The force F_{gn} exerted on m_i by m_n is: (Newton's law of universal gravitation)

$$\mathbf{F}_{gn} = -\frac{Gm_i m_n}{r_{ni}^3}(\mathbf{r}_{ni}) \qquad (1)$$

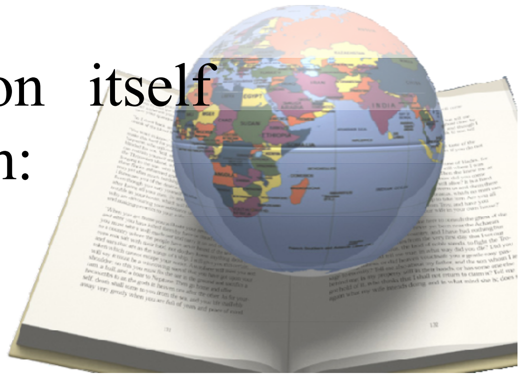
$$\mathbf{r}_{ni} = \mathbf{r}_i - \mathbf{r}_n \qquad (2)$$

✶ The vector sum, F_g , of all gravitational forces acting on the i^{th} body may be written:

$$\mathbf{F}g = -\frac{Gm_i m_1}{r_{1i}^3}(\mathbf{r}_{1i}) - \frac{Gm_i m_2}{r_{2i}^3}(\mathbf{r}_{2i}) \cdots\cdots - \frac{Gm_i m_n}{r_{ni}^3}(\mathbf{r}_{ni}) \qquad (3)$$

✶ Since the body cannot exert a force on itself obviously, equation (3) does not contain the term:

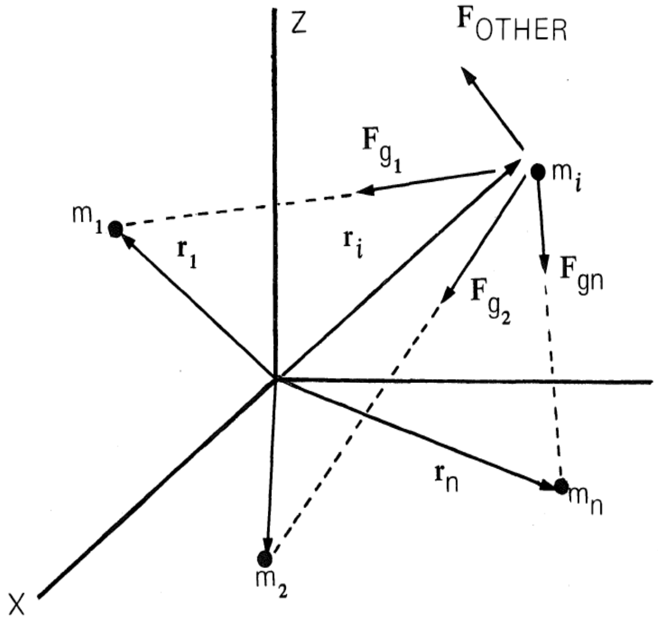
$$-\frac{Gm_i m_i}{r_{ii}^3}(\mathbf{r}_{ii}) \qquad (4)$$



3-THE N-BODY PROBLEM

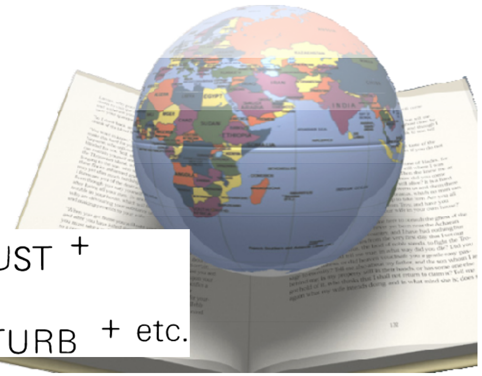
- ✶ We may simplify the equation No3 by using the summation notation so that

$$\mathbf{F}_g = -Gm_i \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_j}{r_{ji}^3} (\mathbf{r}_{ji}) \quad (5)$$



- ✶ The other external force, F_{OTHER} , is composed of drag, thrust, solar radiation pressure, perturbations due to nonspherical shapes, etc.

$$\mathbf{F}_{OTHER} = \mathbf{F}_{DRAG} + \mathbf{F}_{THRUST} + \mathbf{F}_{SOLAR\ PRESSURE} + \mathbf{F}_{PERTURB} + \text{etc.}$$



3-THE N-BODY PROBLEM

- The combined force acting on the i^{th} body we will call F_{TOTAL}

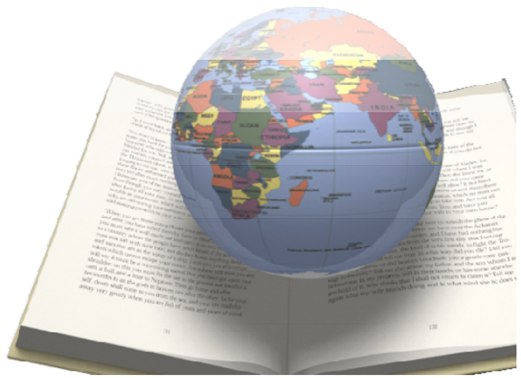
$$\mathbf{F}_{TOTAL} = \mathbf{F}_g + \mathbf{F}_{OTHER} \,. \qquad (6)$$

- By applying the Newton's second law of motion, we will have

$$\frac{d}{dt}(m_i \mathbf{v}_i) = \mathbf{F}_{TOTAL}. \qquad (7)$$

- The time derivative may be expanded to:

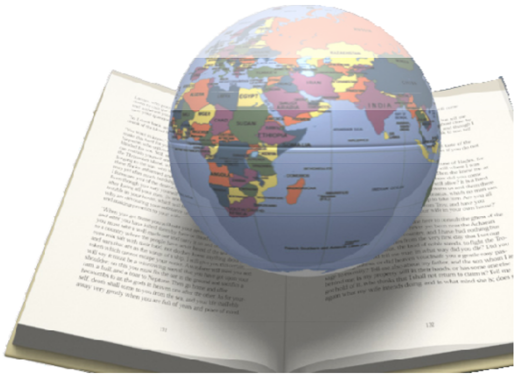
$$m_i \frac{d\mathbf{v}_i}{dt} + \mathbf{v}_i \frac{dm_i}{dt} = \mathbf{F}_{TOTAL} \,. \qquad (8)$$



3-THE N-BODY PROBLEM

- If the body is expelling some mass, (for example to produce thrust) the second term of equation(8) would not be zero.
- Certain relativistic effects would also give rise to changes in the mass m_i as a function of time.
- In other words, in space dynamics, it is not true that F=ma.
- Dividing through by the mass m_i gives the most general equation of motion for the i^{th} body

$$\ddot{\mathbf{r}}_i = \frac{\mathbf{F}_{TOTAL}}{m_i} - \dot{\mathbf{r}}_i \frac{\dot{m}_i}{m_i} \qquad (9)$$



3-THE N-BODY PROBLEM

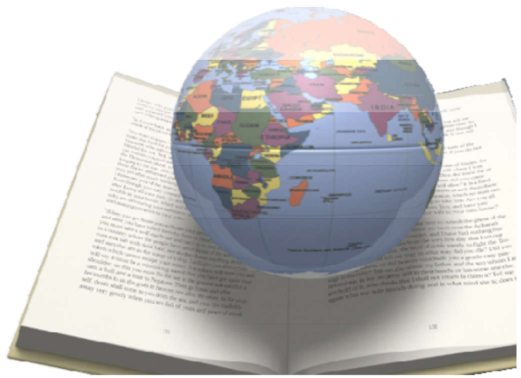
$\ddot{r}_i$: The vector acceleration of the i^{th} body relative to the x,y,z coordinate system.

m_i :The mass of i^{th} body

F_{TOTAL} :The vector sum of all gravitational forces and all other external forces.

$\dot{r}_i$: The velocity vector of the i^{th} body relative to the x,y,z coordinate system.

$\dot{m}_i$:The time rate of change of mass of the i^{th} body (due to expelling mass or relativistic effects)



3-THE N-BODY PROBLEM

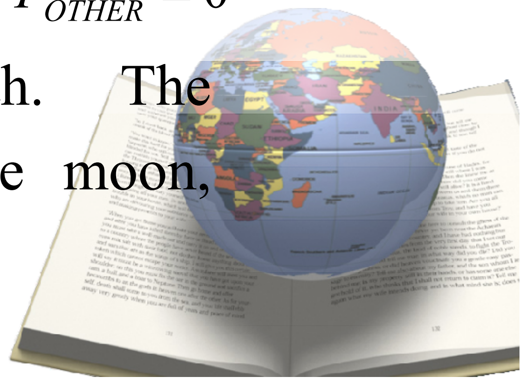
$$\ddot{\mathbf{r}}_i = \frac{\mathbf{F}_{TOTAL}}{m_i} - \dot{\mathbf{r}}_i \frac{\dot{m}_i}{m_i}$$

- ✶ Equation (9) is a second order, nonlinear vector, differential equation of motion which has defied solution in its present form.
- ✶ So we make some simplifying assumptions:

  1- The mass of the i^{th} body remains constant (i.e., unpowered flight $\dot{m}_i = 0$)

  2- The all other external forces are not present $F_{OTHER} = 0$

  3- m_2 is an earth satellite and m_1 is the earth. The remaining masses $m_3, m_4, \ldots, m_n$ may be the moon, sun and planets.



3-THE N-BODY PROBLEM

- From the first 2 assumptions we will write equation 9 in the following form:

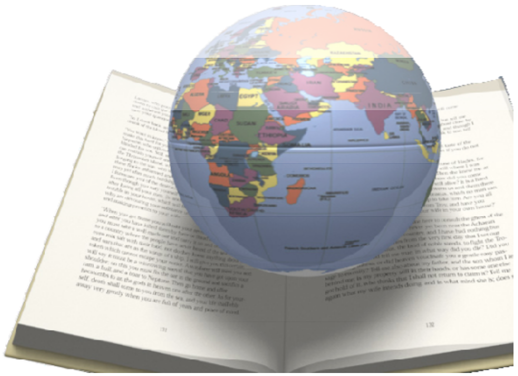
$$\ddot{\mathbf{r}}_i = -G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_j}{r_{ji}^3} (\bar{\mathbf{r}}_{ji}) . \qquad (10)$$

- By using the 3 assumption for i=1 we will have

$$\ddot{\mathbf{r}}_1 = -G \sum_{j=2}^{n} \frac{m_j}{r_{j1}^3} (\mathbf{r}_{j1}) . \qquad (11)$$

- And for i=2 equation 10 becomes

$$\ddot{\mathbf{r}}_2 = -G \sum_{\substack{j=1 \\ j \neq 2}}^{n} \frac{m_j}{r_{j2}^3} (\mathbf{r}_{j2}) . \qquad (12)$$



3-THE N-BODY PROBLEM

- From equation 2 we see that:

$$\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1 \qquad (13)$$

- So that:

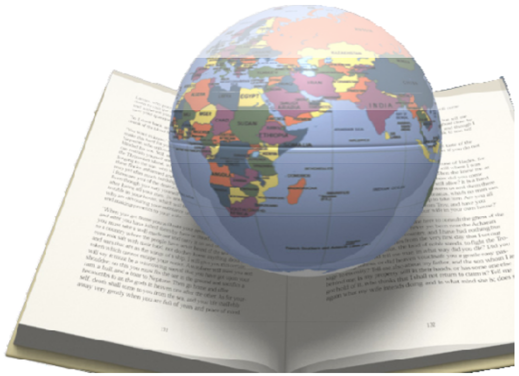
$$\ddot{\mathbf{r}}_{12} = \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 \qquad (14)$$

- Substituting equations (11) and (12) into equation (14) gives:

$$\ddot{\mathbf{r}}_{12} = -G\sum_{\substack{j=1\\ j\neq 2}}^{n} \frac{m_j}{r_{j2}^{3}}(\mathbf{r}_{j2}) + G\sum_{j=2}^{n} \frac{m_j}{r_{j1}^{3}}(\mathbf{r}_{j1}) \qquad (15)$$

- Or expanding

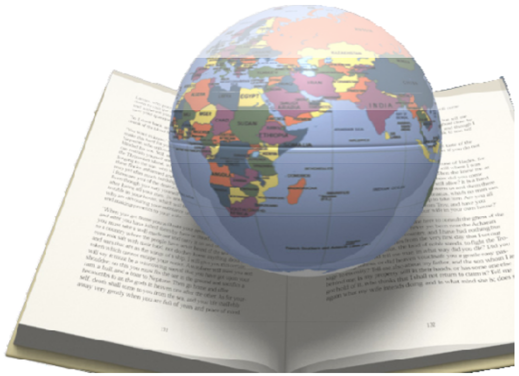
$$\ddot{\mathbf{r}}_{12} = -\left[\frac{Gm_1}{r_{12}^{3}}(\mathbf{r}_{12}) + G\sum_{j=3}^{n} \frac{m_j}{r_{j2}^{3}}(\mathbf{r}_{j2})\right] - \left[-\frac{Gm_2}{r_{21}^{3}}(\mathbf{r}_{21}) - G\sum_{j=3}^{n} \frac{m_j}{r_{j1}^{3}}(\mathbf{r}_{j1})\right] \qquad (16)$$



3-THE N-BODY PROBLEM

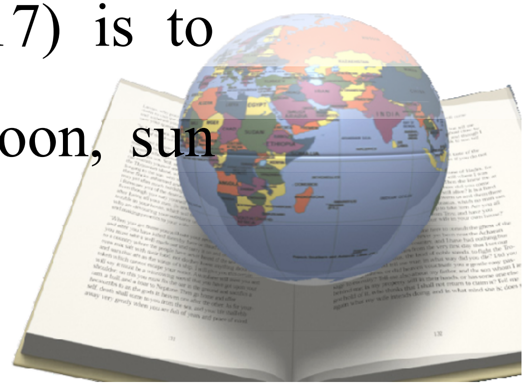
- Since $\mathbf{r}_{12} = -\mathbf{r}_{21}$ we may combine the first terms in each bracket. Hence:

$$\ddot{\mathbf{r}}_{12} = -\frac{G(m_1 + m_2)}{r_{12}^3}(\mathbf{r}_{12}) - \sum_{j=3}^{n} Gm_j\left(\frac{\mathbf{r}_{j2}}{r_{j2}^3} - \frac{\mathbf{r}_{j1}}{r_{j1}^3}\right) \qquad (17)$$



3-THE N-BODY PROBLEM

- If we are going to study the motion of a near earth satellite, so we could assume that, m_2 is the mass of the satellite and m_1 is the mass of the earth. In equation (17).
- Then from equation (17) $\ddot{r}_{12}$ is the acceleration of the satellite relative to earth.
- The effect of the last term of equation (17) is to account for the perturbing effects of the moon, sun and planets on a near earth satellite.

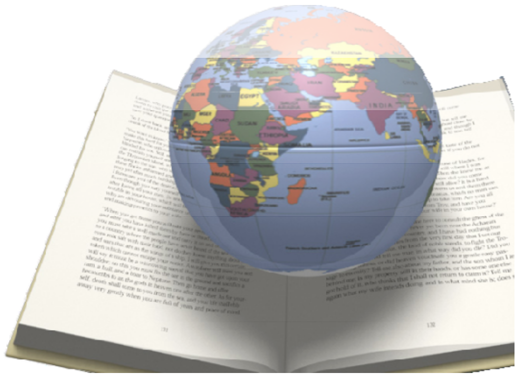


3-THE N-BODY PROBLEM

- To further simplify this equation it is necessary to determine the magnitude of the perturbing effects compared to the force between earth and satellite. (note no2 , page11 {2})

COMPARISON OF RELATIVE ACCELERATION (IN G's) FOR A 200 NM EARTH SATELLITE

	Acceleration in G's on 200 nm Earth Satellite
Earth	.89
Sun	$6x10^{-4}$
Mercury	$2.6x10^{-10}$
Venus	$1.9x10^{-8}$
Mars	$7.1x10^{-10}$
Jupiter	$3.2x10^{-8}$
Saturn	$2.3x10^{-9}$
Uranus	$8x10^{-11}$
Neptune	$3.6x10^{-11}$
Pluto	10^{-12}
Moon	$3.3x10^{-6}$
Earth Oblateness	10^{-3}



IN THE NAME OF GOD

ORBITAL MECHANICS

CHAPTER 7

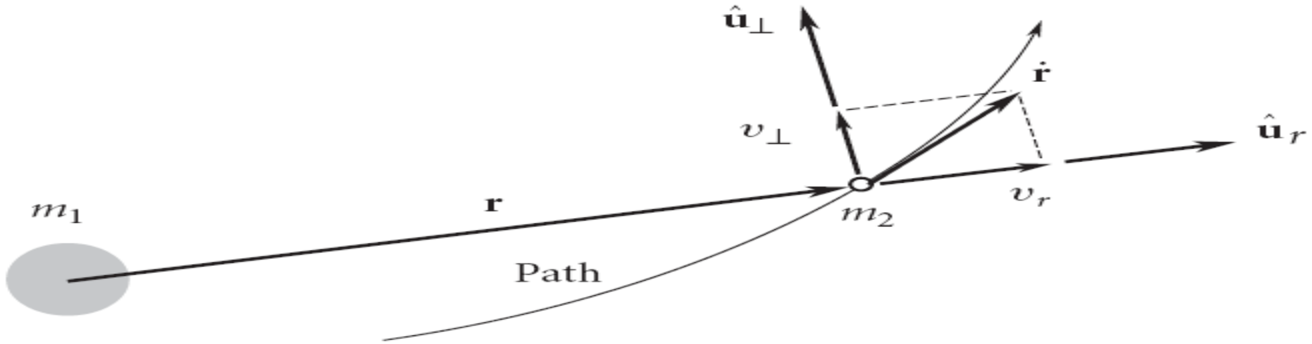
Angular momentum and the orbit formulas

Student version

By: M.Mirshams

2010

7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS



- The angular momentum of m_2 relative to m_1 is:

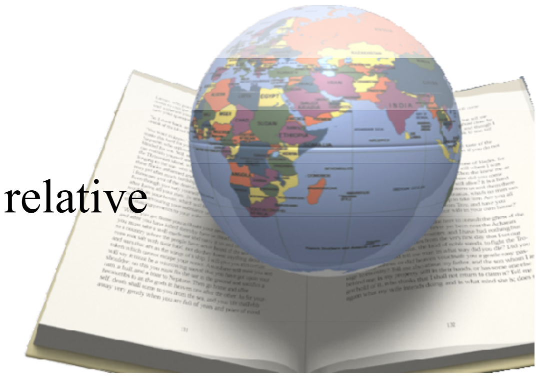
$$\mathbf{H}_{2/1} = \mathbf{r} \times m_2\dot{\mathbf{r}} \qquad (1)$$

$\dot{\mathbf{r}}$ The velocity of m_2 relative to m_1

- Let us divide this equation through by m_2 and let $\mathbf{h} = \mathbf{H}_{2/1}/m_2$, so that

$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} \qquad (2)$$

h: the relative momentum of m_2 per unit mass (the specific relative angular momentum), $\left[\frac{km^2}{s}\right]$



7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS

✶ Taking the time derivative of h yields:

$$\frac{d\mathbf{h}}{dt} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} \qquad (3)$$

$\dot{\mathbf{r}} \times \dot{\mathbf{r}} = 0$

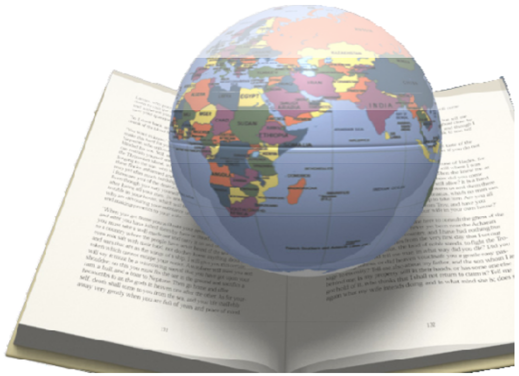
According to previous lecture $\ddot{\mathbf{r}} = -(\mu/r^3)\mathbf{r}$

✶ So that:

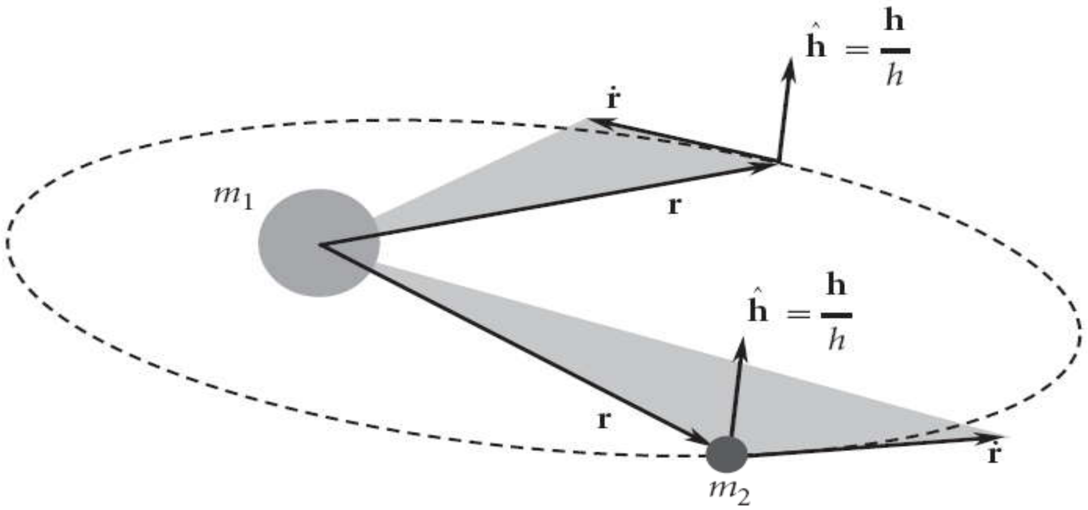
$$\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times \left(-\frac{\mu}{r^3}\mathbf{r}\right) = -\frac{\mu}{r^3}(\mathbf{r} \times \mathbf{r}) = 0$$

✶ Therefore:

$$\frac{d\mathbf{h}}{dt} = 0 \longrightarrow \mathbf{r} \times \dot{\mathbf{r}} = \text{constant} \qquad (4)$$



7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS

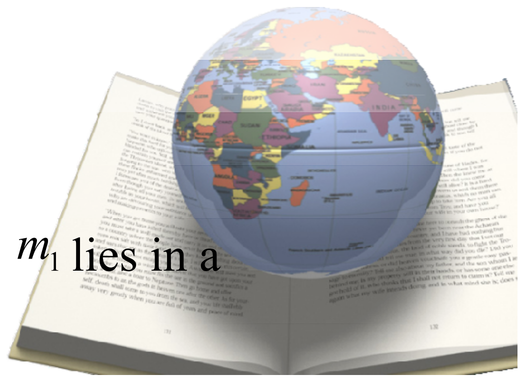


- ✶ At any given time, the position vector r and the velocity vector $\dot{\mathbf{r}}$ lie in the same plane
- ✶ Their cross product $\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}$ is perpendicular to that plane

$$\hat{\mathbf{h}} = \frac{\mathbf{h}}{h} \qquad (5)$$

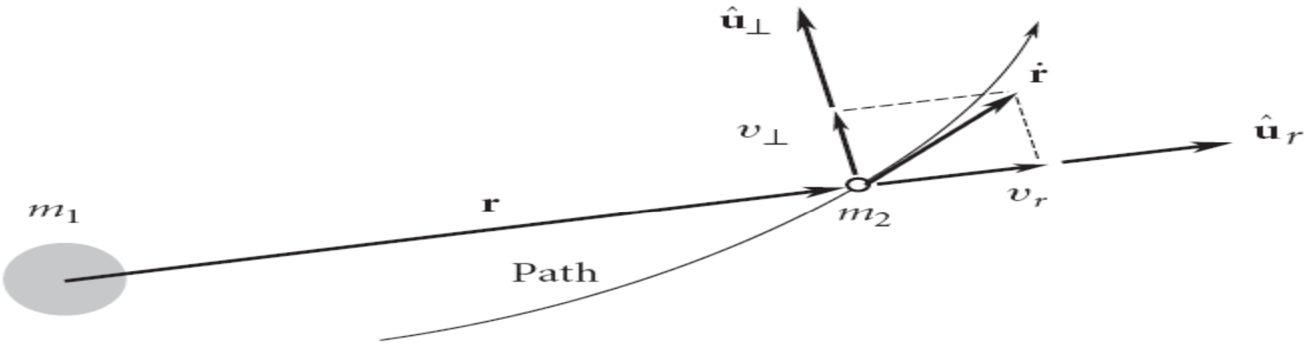
$\hat{h}$: The unit vector normal to the plane

$\frac{d\mathbf{h}}{dt} = 0$ ⟶ $\hat{\mathbf{h}} = \text{constant}$ ⟶ The path of m_2 around m_1 lies in a single plane



7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS

✶ Let us resolve the relative velocity vector $\dot{\mathbf{r}}$ into components $\mathbf{v}_r$ and $\mathbf{v}_\perp$ along the outward radial from m_1 and perpendicular to it:



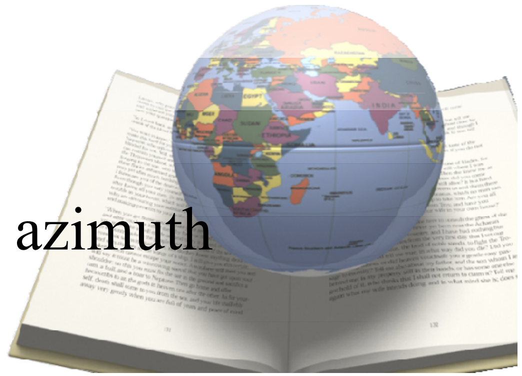
✶ We can write equation (2) as:

$$\mathbf{h} = r\hat{\mathbf{u}}_r \times (v_r\hat{\mathbf{u}}_r + v_\perp\hat{\mathbf{u}}_\perp) = rv_\perp\hat{\mathbf{h}}$$

✶ That is:

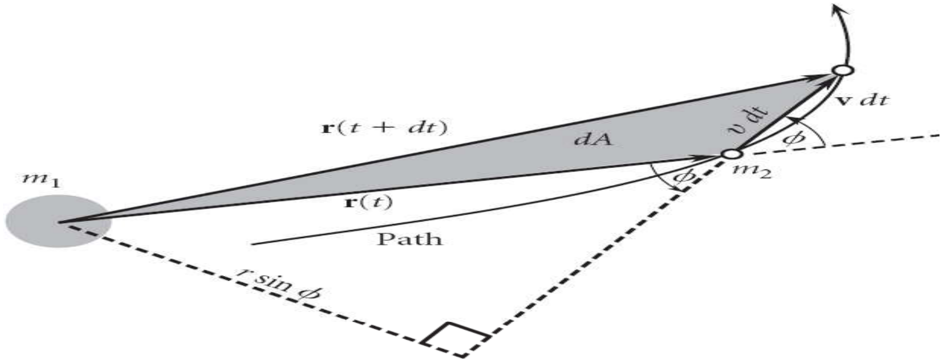
$$h = rv_\perp \qquad (6)$$

✶ The angular momentum depends only on the azimuth component of the relative velocity.



7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS

✶ During the differential time interval dt the position vector **r** sweeps out area dA



✶ From the figure it is clear that triangular area dA is given by:

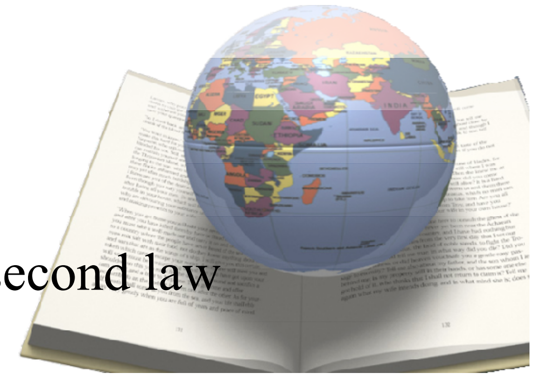
$$dA = \frac{1}{2} \times \text{base} \times \text{altitude} = \frac{1}{2} \times v\,dt \times r\sin\phi = \frac{1}{2} r(v\sin\phi)dt = \frac{1}{2} r v_{\perp} dt$$

✶ Therefore, using equation (6) we have:

$$\frac{dA}{dt} = \frac{h}{2} \qquad (7)$$

$\frac{dA}{dt}$:areal velocity

✶ According to (7) areal velocity is constant kepler's second law (equal area) are swept out in equal times (1571-1630)



7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS

✶ Now, we are going to integrate the equation of motion of m_2 relative to m_1

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \qquad (8)$$

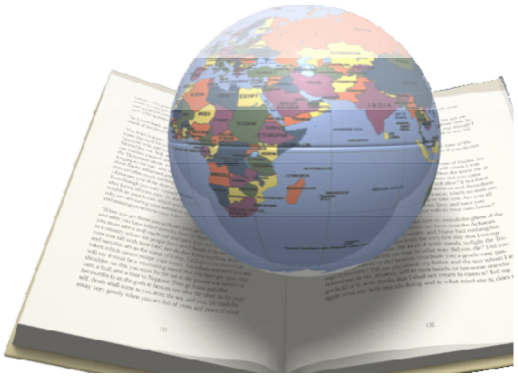
✶ Before that, recall several useful vector identities :

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \qquad (9)$$

$$\mathbf{r} \cdot \mathbf{r} = r^2 \qquad (10)$$

$$\mathbf{r} \cdot \dot{\mathbf{r}} = r\dot{r} \qquad (11)$$

$$\mathbf{r} \cdot \mathbf{v} = \|\mathbf{r}\| \frac{d\|\mathbf{r}\|}{dt} \qquad (12)$$



7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS

✶ Now let us take the cross product of both sides of equation (8) with the specific angular momentum h:

$$\ddot{\mathbf{r}} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times \mathbf{h} \qquad (13)$$

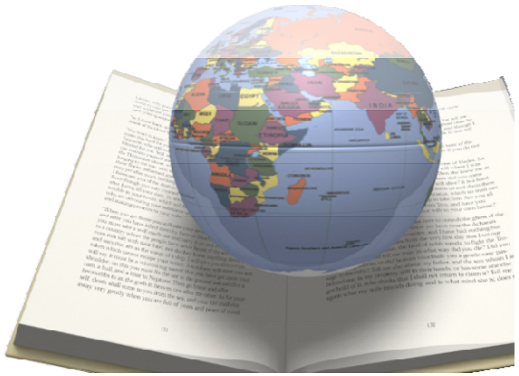
Since: $\frac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) = \ddot{\mathbf{r}} \times \mathbf{h} + \dot{\mathbf{r}} \times \dot{\mathbf{h}}$

so the left hand side of equation (13) can be written:

$$\ddot{\mathbf{r}} \times \mathbf{h} = \frac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) - \dot{\mathbf{r}} \times \dot{\mathbf{h}}$$

✶ But we have had $\dot{\mathbf{h}} = 0$ (Equ.4), so finally the left hand side of equation (13) can be written as:

$$\ddot{\mathbf{r}} \times \mathbf{h} = \frac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) \qquad (14)$$



7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS

✶ The right- hand side of equation (13) can be transformed by the following sequence of substitutions:

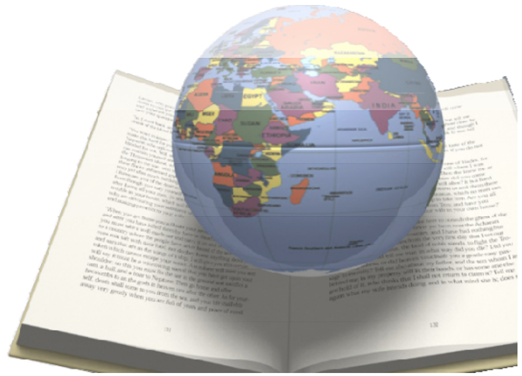
$$\begin{aligned}\frac{1}{r^3}\mathbf{r}\times\mathbf{h} &= \frac{1}{r^3}[\mathbf{r}\times(\mathbf{r}\times\dot{\mathbf{r}})]\\ &= \frac{1}{r^3}[\mathbf{r}(\mathbf{r}\cdot\dot{\mathbf{r}})-\dot{\mathbf{r}}(\mathbf{r}\cdot\mathbf{r})]\\ &= \frac{1}{r^3}[\mathbf{r}(r\dot{r})-\dot{\mathbf{r}}r^2]\\ &= \frac{\mathbf{r}\dot{r}-\dot{\mathbf{r}}r}{r^2}\end{aligned}$$

✶ But

$$\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right)=\frac{r\dot{\mathbf{r}}-\mathbf{r}\dot{r}}{r^2}=-\frac{\mathbf{r}\dot{r}-r\dot{\mathbf{r}}}{r^2}$$

✶ Therefore

$$\frac{1}{r^3}\mathbf{r}\times\mathbf{h}=-\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) \qquad (15)$$



7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS

✶ Substituting equation (15) , (14) into Equation (13) we get:

$$\frac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) = \frac{d}{dt}\left(\mu\frac{\mathbf{r}}{r}\right)$$

or

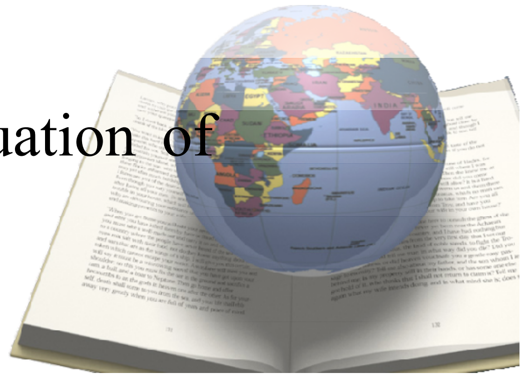
$$\frac{d}{dt}\left(\dot{\mathbf{r}} \times \mathbf{h} - \mu\frac{\mathbf{r}}{r}\right) = 0$$

that is:

$$\dot{\mathbf{r}} \times \mathbf{h} - \mu\frac{\mathbf{r}}{r} = \mathbf{C} \qquad (16)$$

✶Where the vector C is an arbitrary constant of integration having the dimensions of μ.

✶Equation (16) is the first integral of the equation of motion $\ddot{\mathbf{r}} = -(\mu/r^3)\mathbf{r}$.



7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS

✶ Taking the dot product of both sides of equation (16) with the vector h yields:

$$(\dot{\mathbf{r}} \times \mathbf{h}) \cdot \mathbf{h} - \mu \frac{\mathbf{r} \cdot \mathbf{h}}{r} = \mathbf{C} \cdot \mathbf{h}$$

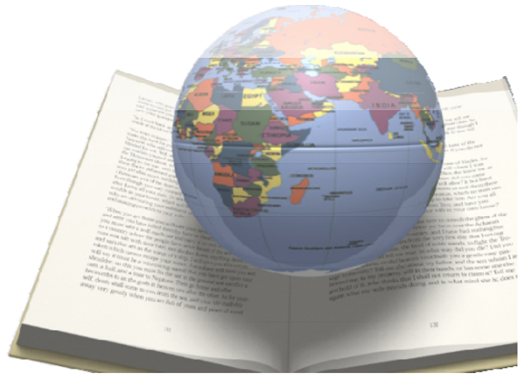
$\dot{\mathbf{r}} \times \mathbf{h} \perp \dot{\mathbf{r}}$ & $\mathbf{h} \rightarrow (\dot{\mathbf{r}} \times \mathbf{h}) \cdot \mathbf{h} = 0.$

$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} \perp \mathbf{r}$ & $\dot{\mathbf{r}} \rightarrow \mathbf{r} \cdot \mathbf{h} = 0.$

$\rightarrow \mathbf{C} \cdot \mathbf{h} = 0$

$\rightarrow \mathbf{C} \perp \mathbf{h} \rightarrow$

Since h is normal to the orbital plane so C must lie in the orbital plane.



7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS

✶ Let us rearrange equation (16) and write it as:

$$\frac{\mathbf{r}}{r} + \mathbf{e} = \frac{\dot{\mathbf{r}} \times \mathbf{h}}{\mu} \qquad (17)$$

$\mathbf{e} = \mathbf{C}/\mu$: The dimensionless vector "eccentricity"

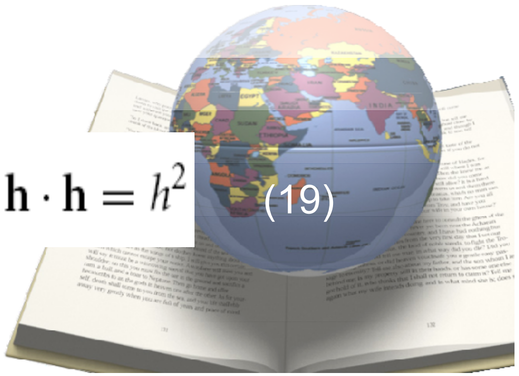
✶The line defined by the vector **e** commonly called the apse line.

✶ In order to obtain a scalar equation, let us take the dot product of both sides of equation (17) with **r**

$$\frac{\mathbf{r} \cdot \mathbf{r}}{r} + \mathbf{r} \cdot \mathbf{e} = \frac{\mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{h})}{\mu} \qquad (18)$$

$$\mathrm{A} \cdot (\mathrm{B} \times \mathrm{C}) = (\mathrm{A} \times \mathrm{B}) \cdot \mathrm{C} \rightarrow \mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{h}) = (\mathbf{r} \times \dot{\mathbf{r}}) \cdot \mathbf{h} = \mathbf{h} \cdot \mathbf{h} = h^2 \qquad (19)$$

$$\mathbf{r} \cdot \mathbf{r} = r^2 \qquad (20)$$



7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS

- ✶ Substituting expressions (19), (20) in (18) yields:

$$r + \mathbf{r} \cdot \mathbf{e} = \frac{h^2}{\mu} \qquad (21)$$

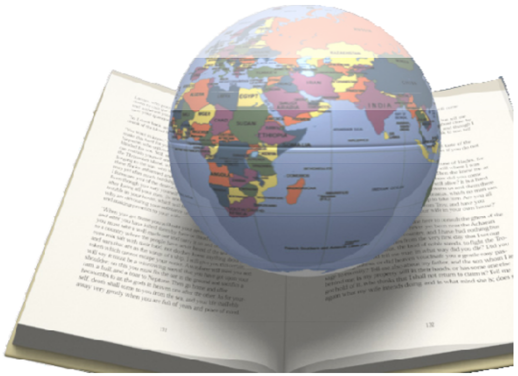
(Note7, P46, {1})

- ✶ $\mathbf{r} \cdot \mathbf{e} = re\cos\theta$
- ✶ substituting this expression into equation (21), we get

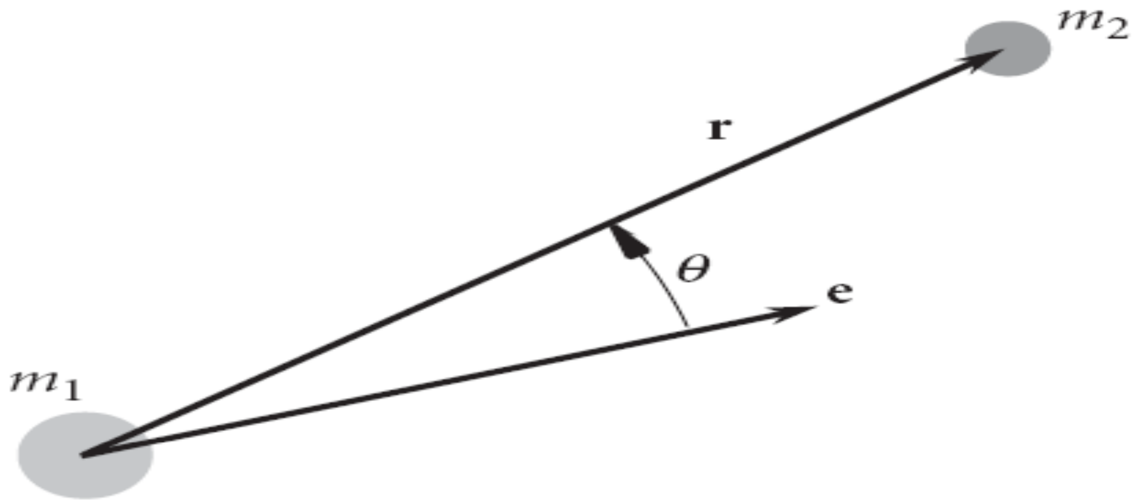
$$r + re\cos\theta = \frac{h^2}{\mu} \qquad (22)$$

or

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta} \qquad (23)$$

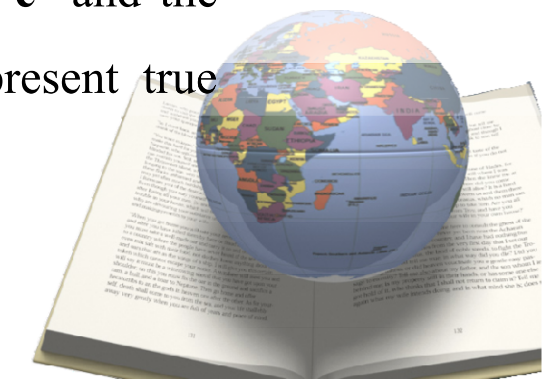


7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS



e: The magnitude of the eccentricity vector **e**

θ: is the true anomaly (the angle between the fixed vector **e** and the variable position vector **r**. (other symbols used to represent true anomaly include ν, f, ϕ....)



7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS

$$r = \frac{h^2}{\mu} \frac{1}{1 + e\cos\theta} \qquad (23)$$

✶This is the orbit equation, and it defines the path of the body m_2 around m_1, relative to m_1.

$* Remember \quad \mu, h, e = constants \quad \& \quad e \geq 0$

✶Since the orbit equation describes conic sections including ellipses, it is a mathematical statement of Kepler's first law, namely, that the planets follow elliptical around the sun.

✶ Two- body orbits are often referred to as Keplerian orbits.

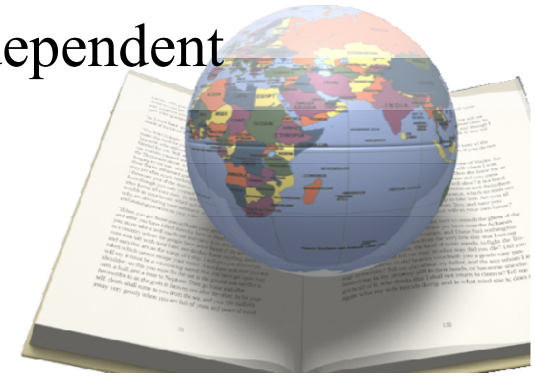
7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS

✶ Integration of the equation of relative motion, leads to six constants of integration.

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$$

✶In this section it would seem that we have arrived at those constants, namely the three components of the angular momentum **h** and the three components of the eccentricity vector **e**.

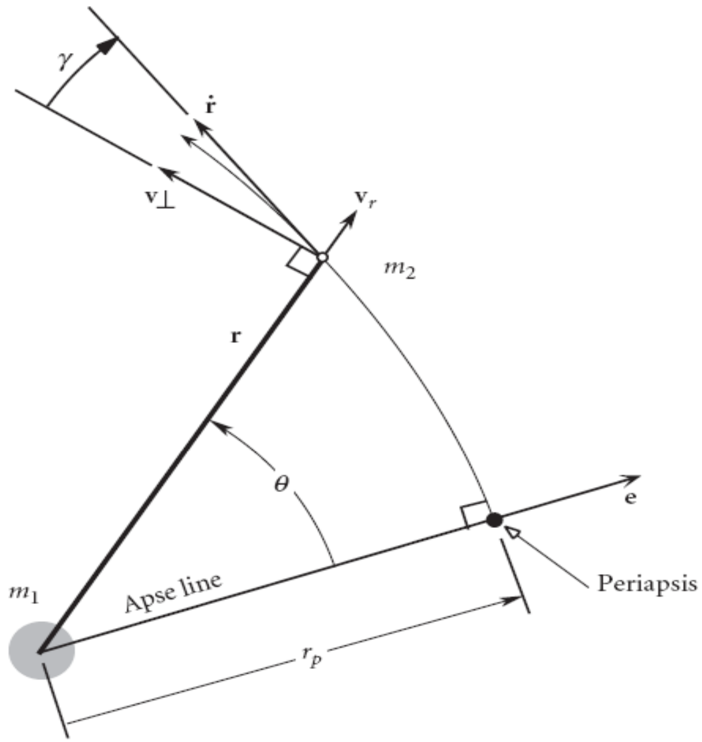
✶ However we showed that **h** is perpendicular to **e**. this places a condition, namely **h.e=o**, on the components of **h** and **e**, so that we really have just five independent constants of integration. (Note8,P47,{1})



7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS

✶ The angular velocity of the position vector **r** is $\dot{\theta}$, the rate of change the true anomaly.

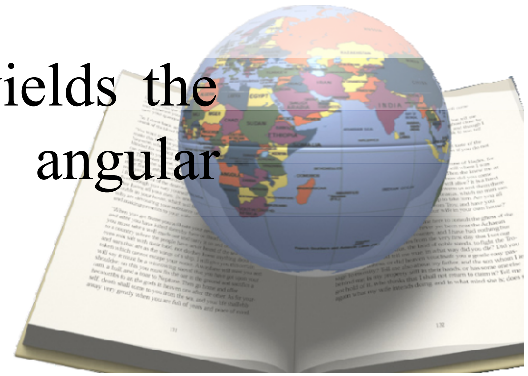
✶The component of velocity normal to the position vector is found in terms of the angular velocity by the formula



$$v_{\perp} = r\dot{\theta} \qquad (24)$$

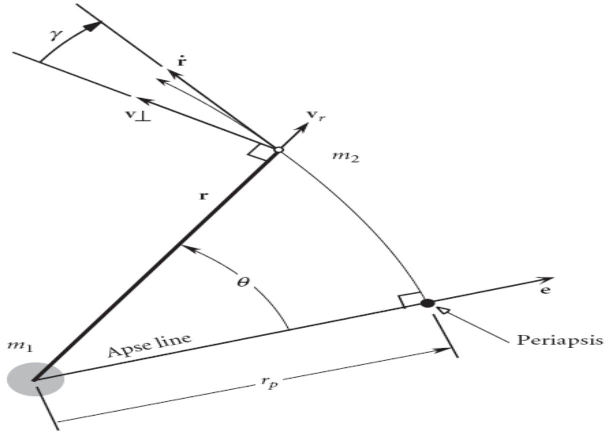
✶ Substituting this into equation $h = rv_{\perp}$ yields the specific angular momentum in terms of the angular velocity.

$$h = r^2\dot{\theta} \qquad (25)$$



7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS

✶ It is convenient to have formulas for computing the radial and azimuth components of velocity.



✶For azimuth components we have:

$$h = r v_\perp \;\rightarrow\; v_\perp = \frac{h}{r}$$

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta}$$

$$\rightarrow\; v_\perp = \frac{\mu}{h}(1 + e\cos\theta) \qquad (26)$$

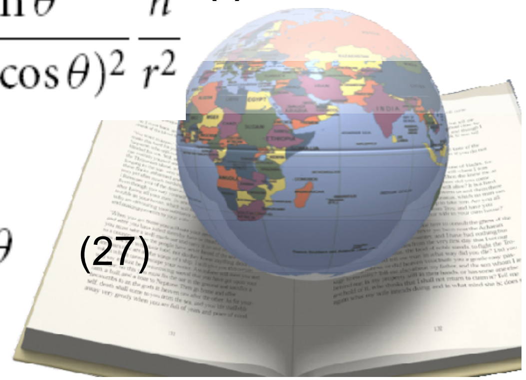
✶ For radial components we will have

$$v_r = \dot{r} \;\rightarrow\; \dot{r} = \frac{dr}{dt} = \frac{h^2}{\mu}\left[-\frac{e(-\dot{\theta}\sin\theta)}{(1 + e\cos\theta)^2}\right] = \frac{h^2}{\mu}\frac{e\sin\theta}{(1 + e\cos\theta)^2}\frac{h}{r^2} \qquad (*)$$

$$\dot{\theta} = h/r^2$$

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta}$$

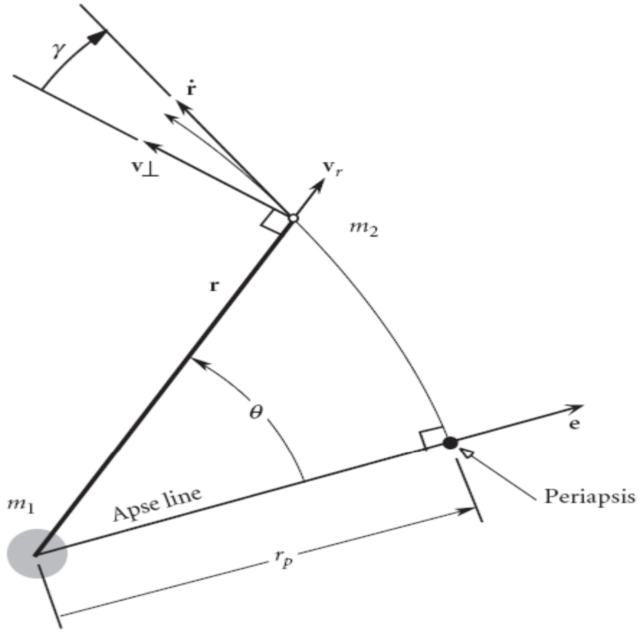
$$\rightarrow\; (*) \;\rightarrow\; v_r = \frac{\mu}{h}e\sin\theta \qquad (27)$$



7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS

✶ From equation:

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta}$$



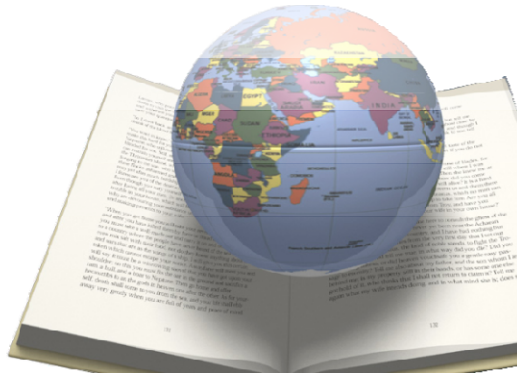
✶ we see that m_2 comes closest to m_1 (r is smallest) when $\theta = 0$ (unless $e = 0$, in which case the distance between m_1 and m_2 is constant)

✶The point of closest approach lies on the apse line and is called periapsis.

✶ The distance r_p to periapsis is: $\theta = 0$

$$r_p = \frac{h^2}{\mu}\frac{1}{1 + e} \qquad (28)$$

✶ $v_r = 0$ at periapsis

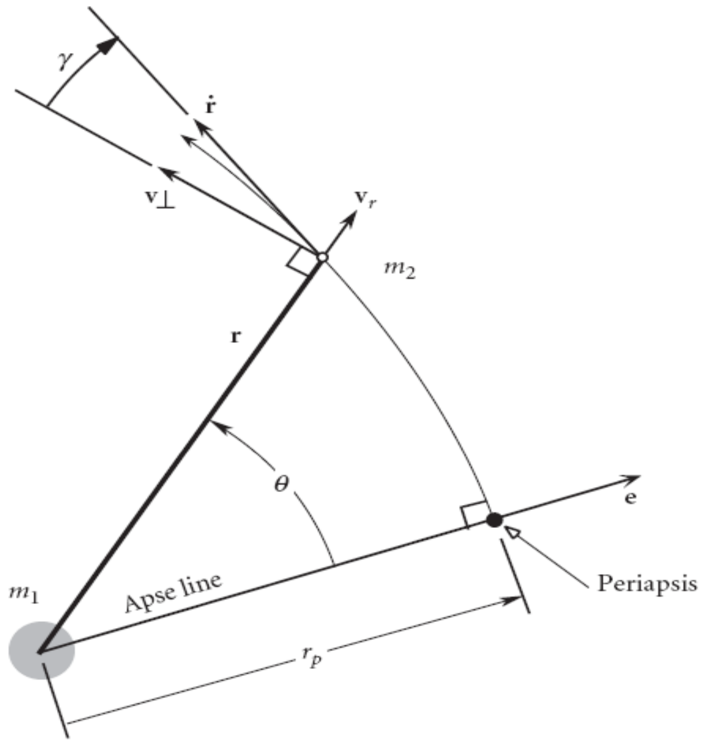


7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS

γ : Flight path angle

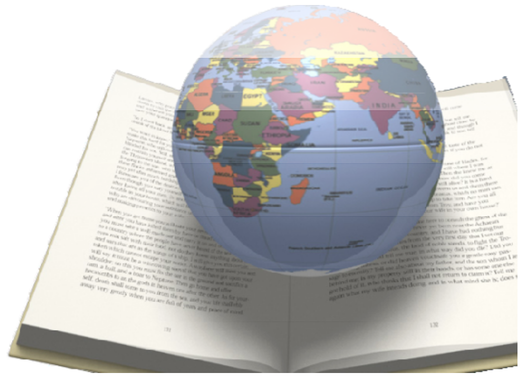
✶ γ is the angle that the velocity vector **v** makes with the normal to the position vector.

✶ the normal to the position vector points in the direction of $\mathbf{v}_\perp$, and it is called the local horizon.



✶ It is clear that:

$$\tan\gamma = \frac{v_r}{v_\perp} \qquad (29)$$



7-ANGULAR MOMENTUM AND THE ORBIT FORMULAS

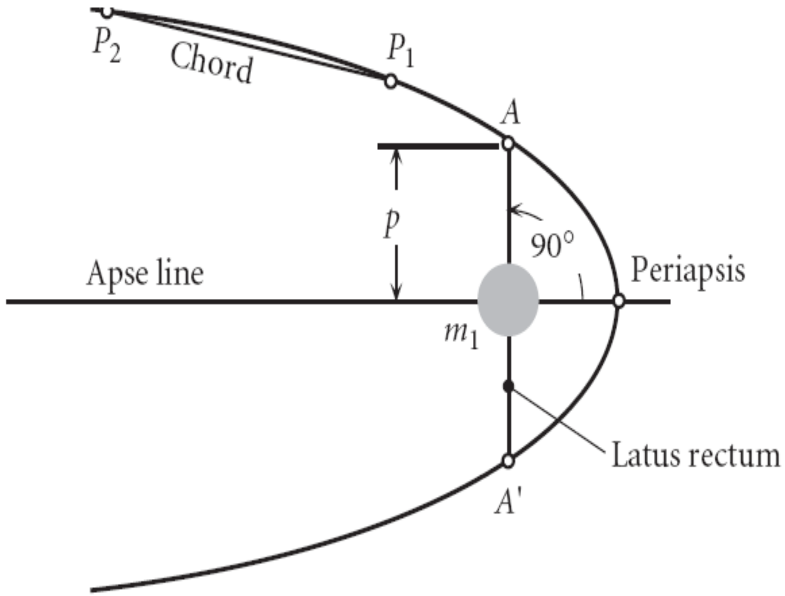
✶ Substituting v_r and $v_\perp$ we will have: $\tan\gamma = \dfrac{e\sin\theta}{1+e\cos\theta}$ (30)

✶ the trajectory described by the orbit equation is symmetric about the apse line. Why?

✶ because $\cos(-\theta) = \cos\theta$.

✶ Chord: the straight line connecting any two points on the orbit

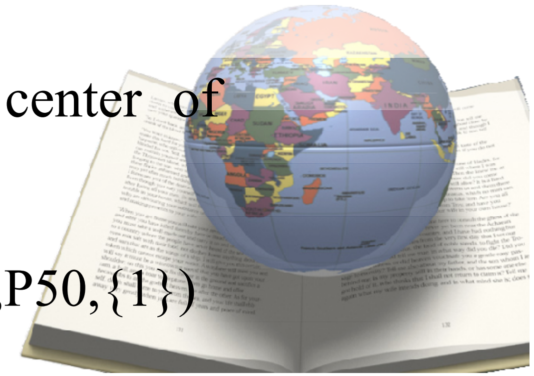
✶ The latus rectum: the chord through the center of attraction perpendicular to the apse line.



✶ parameter P: two equal parts divided by the center of attraction on the latus rectum.

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta} \rightarrow p = \frac{h^2}{\mu} \quad (31)$$

Note:9,P50,{1})



8- THE ENERGY LAW

✶(Note:10,P50,{1})

✶ Let us see what result from taking the dot product of equation:

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \qquad (*)$$

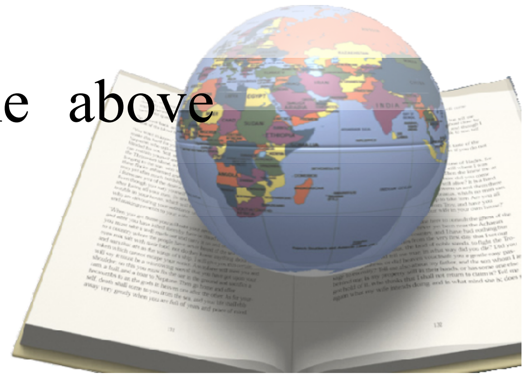
✶ We will do it with the relative linear momentum per unit mass.

✶ The relative linear momentum per unit mass is just the relative velocity:

$$\frac{m_2\dot{\mathbf{r}}}{m_2} = \dot{\mathbf{r}}$$

✶ Thus, carrying out the dot product in the above mention equation (*) yields:

$$\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = -\mu\frac{\mathbf{r} \cdot \dot{\mathbf{r}}}{r^3} \qquad (1)$$



8- THE ENERGY LAW

✶For the left hand side we observe that:

$$\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \frac{1}{2}\frac{d}{dt}(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) = \frac{1}{2}\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \frac{1}{2}\frac{d}{dt}(v^2) = \frac{d}{dt}\left(\frac{v^2}{2}\right) \qquad (2)$$

✶ For the right- hand side of equation (1) we have:

$$\left.\begin{aligned} &d(1/r)/dt = (-1/r^2)(dr/dt), \\ &\mathbf{r} \cdot \mathbf{r} = r^2 \end{aligned}\right] \longrightarrow \mu\frac{\mathbf{r} \cdot \dot{\mathbf{r}}}{r^3} = \mu\frac{r\dot{r}}{r^3} = \mu\frac{\dot{r}}{r^2} = -\frac{d}{dt}\left(\frac{\mu}{r}\right)$$

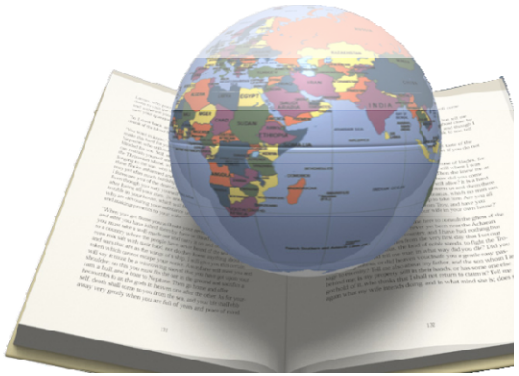
(2)

✶ Substituting Equations (3) , (2) into equation (1) yields:

$$\frac{d}{dt}\left(\frac{v^2}{2} - \frac{\mu}{r}\right) = 0$$

or

$$\frac{v^2}{2} - \frac{\mu}{r} = \varepsilon \quad \text{(constant)} \qquad (4)$$



8- THE ENERGY LAW

$$\frac{v^2}{2} - \frac{\mu}{r} = \varepsilon \qquad (5)$$

$v^2/2$: the relative kinetic energy per unit mass.

$-\mu/r$: the potential energy per unit mass of the body m_2 in the gravitational field of m_1

ε : constant (the total mechanical energy per unit mass)

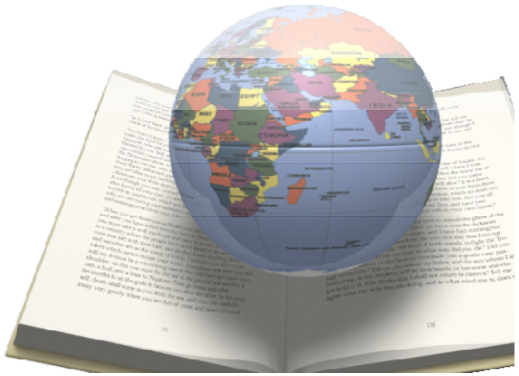
✶ Equation (5) is a statement of conservation of energy, namely, that the specific mechanical energy is the same at all points of the trajectory.

✶ Let us evaluate equ.(5) at periapsis $(\theta = 0)$

$$\varepsilon = \varepsilon_p = \frac{v_p^2}{2} - \frac{\mu}{r_p} \qquad (6)$$

r_p and v_p : the position and speed at periapsis

at periapsis → $v_r = 0$ → $v_p = v_\perp = h/r_p$ (7)



8- THE ENERGY LAW

✶ By substituting (7) in equation (6) we have:

$$\varepsilon = \frac{1}{2}\frac{h^2}{r_p^2} - \frac{\mu}{r_p} \qquad (8)$$

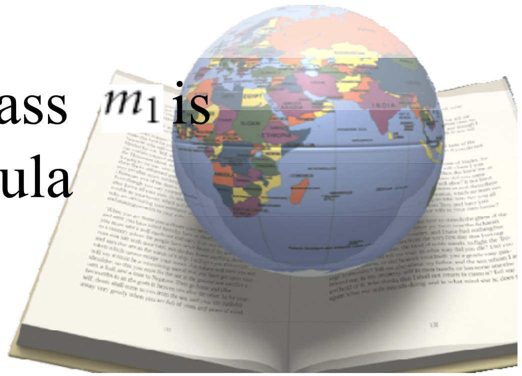
✶ Substituting Equ. for r_p into (8) yields a formula for the orbital specific energy in terms of the orbital constants **h** and **e**,

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}(1 - e^2) \qquad (9)$$

✶ Clearly, the orbital energy is not an independent orbital parameter.

✶ The mechanical energy $\mathcal{E}$ of a satellite of mass m_1 is obtained from the specific energy ε by the formula

$$\mathcal{E} = m_1 \varepsilon \qquad (10)$$



Chapter 1

Circular orbits (e=0)

CHAPTER CONTENT

9- CIRCULAR ORBITS (E=0)

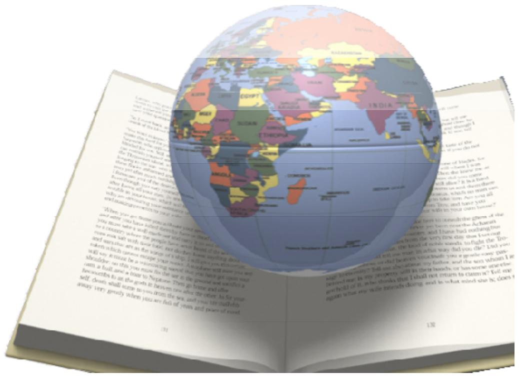
⋆ Setting $e=0$ in the orbital equation yields:

$$e=0 \longrightarrow r=(h^2/\mu)/(1+e\cos\theta) \longrightarrow r=\frac{h^2}{\mu} \qquad (1)$$

⋆ That is $r=\text{constant}$, which means the orbit of m_2 around m_1 is a circle.

$$(1) \longrightarrow \dot{r}=0 \longrightarrow v=v_\perp \longrightarrow h=rv_\perp \longrightarrow h=rv \longrightarrow (1)$$

$$\longrightarrow v_{\text{circular}}=\sqrt{\frac{\mu}{r}} \qquad (2)$$



9- CIRCULAR ORBITS (E=0)

- ✶ The time T required for one orbit is known as the period.
- ✶ Because the speed is constant, the period of a circular orbit is easy to compute:

$$T = \frac{\text{circumference}}{\text{speed}} = \frac{2\pi r}{\sqrt{\dfrac{\mu}{r}}}$$

- ✶ So that

$$T_{\text{circular}} = \frac{2\pi}{\sqrt{\mu}} r^{\frac{3}{2}} \qquad (3)$$

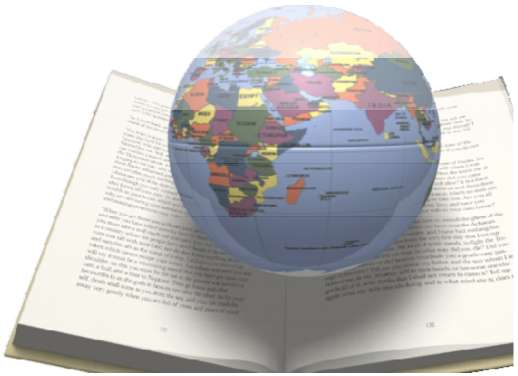
- ✶ The specific energy of a circular orbit is found by setting $e = 0$ in the equation of orbital specific energy:

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}$$

- ✶ Employing $r = \dfrac{h^2}{\mu}$ yields:

$$\varepsilon_{\text{circular}} = -\frac{\mu}{2r} \qquad (4)$$

(NOTE11.P52.{1})



9- CIRCULAR ORBITS (e=0)

EXAMPLE 9.1

- Plot the speed v and period T of a satellite in circular LEO as a function of altitude z.

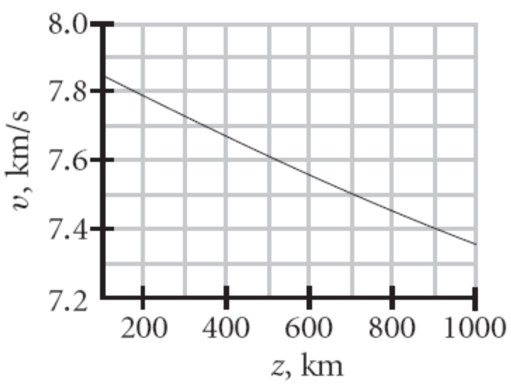
Solution:

- Equation (2) and (3) give the speed and period,

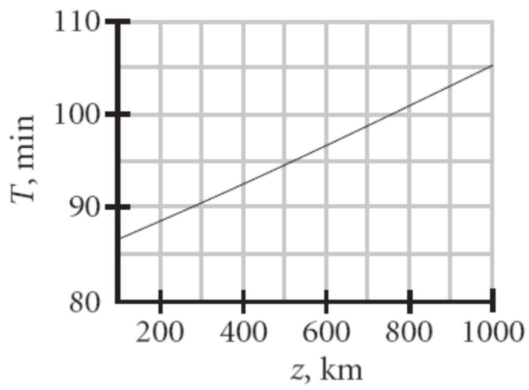
$$v = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{\mu}{R_E + z}} = \sqrt{\frac{398\,600}{6378 + z}}$$

$$T = \frac{2\pi}{\sqrt{\mu}} r^{\frac{3}{2}} = \frac{2\pi}{\sqrt{398\,600}} (6378 + z)^{\frac{3}{2}}$$

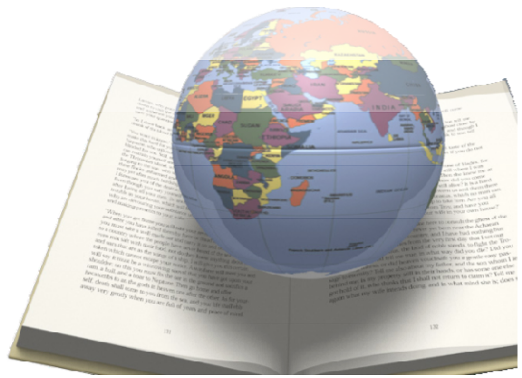
- These relation are graphed in the below figures:



(a)



(b)



9- CIRCULAR ORBITS (E=0)

EXAMPLE 9.2

✶ Calculate the altitude z_{GEO} and speed v_{GEO} of a geostationary earth satellite: (NOTE12.P53.{1})

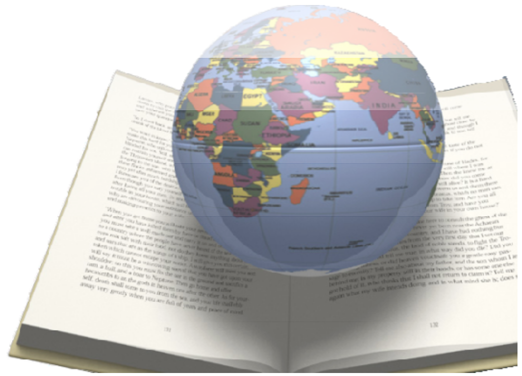
- GEO (Geostationary Equatorial Orbit)

- Sidereal day: the time it takes the earth to complete or rotation relative to inertial space. (the fixed stars)

- Synodic day: (the ordinary 24-hour day), the time it takes the sun to apparently rotate once around the earth, from high noon one day to high noon the next.

- Earth inertial angular velocity ω_E is:

$$\omega_E = 72.9217 \times 10^{-6} \text{ rad/s} \tag{7}$$



9- CIRCULAR ORBITS (E=0)

EXAMPLE 9.2

- **Solution:**
- The speed of the satellite in its circular GEO of radius r_{GEO} s:

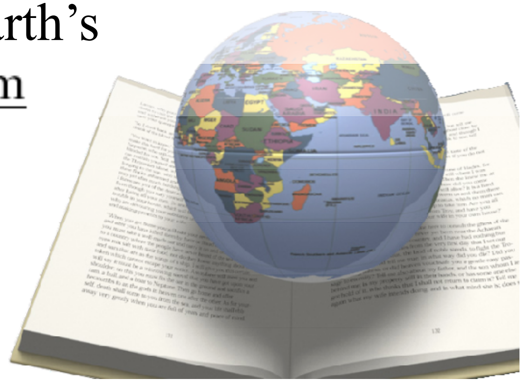
$$v_{GEO} = \sqrt{\frac{\mu}{r_{GEO}}} \qquad \text{(a)}$$

- On other hand: $v_{GEO} = \omega_E r_{GEO}$
- Solving for r_{GEO} yields: $r_{GEO} = \sqrt[3]{\frac{\mu}{\omega_E^2}}$
- Substituting Equ.(7) we get:

$$r_{GEO} = \sqrt[3]{\frac{398\,600}{(72.9217 \times 10^{-6})^2}} = 42\,164\ \text{km} \qquad (8)$$

- Therefore, the distance of the satellite above the earth's surface is: $z_{GEO} = r_{GEO} - R_E = 42\,164 - 6378 = \underline{35\,786\ \text{km}}$
- Substituting Equ.(8) into (a) yields the speed:

$$v_{GEO} = \sqrt{\frac{398\,600}{42\,164}} = \underline{3.075\ \text{km/s}} \qquad (9)$$

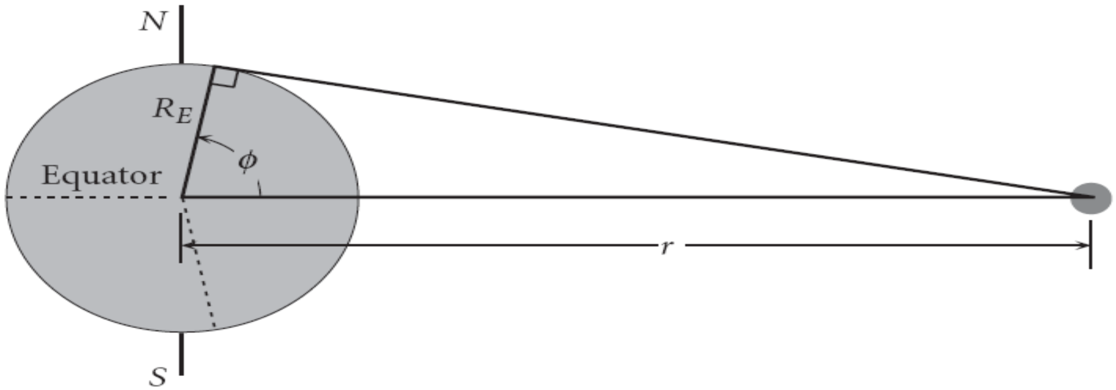


9- CIRCULAR ORBITS (E=0)

EXAMPLE 9.3

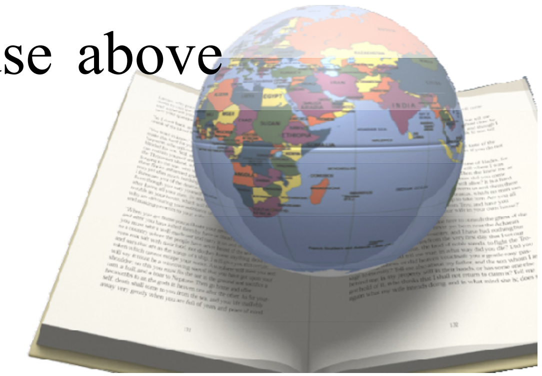
✶ Calculate the maximum latitude and the percentage of the earth's surface visible from GEO.

Solution:



To find the maximum viewable latitude ϕ use above figure, from which it is apparent that:

$$\phi = \cos^{-1} \frac{R_E}{r} \qquad \text{(a)}$$

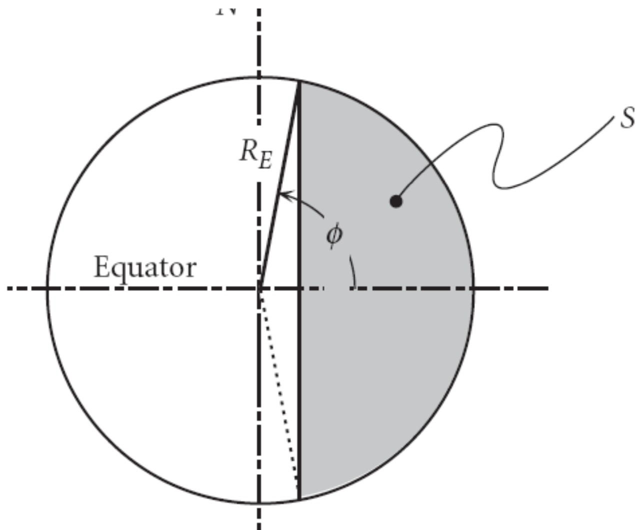


9- CIRCULAR ORBITS (E=0)

EXAMPLE 9.3

Where $R_E = 6378$ km, $r = 42\,164$ km, therefore:

$$\phi = \cos^{-1}\frac{6378}{42\,164} = \underline{81.30^\circ} \quad \text{Maximum visible north or south latitude.}$$ (b)



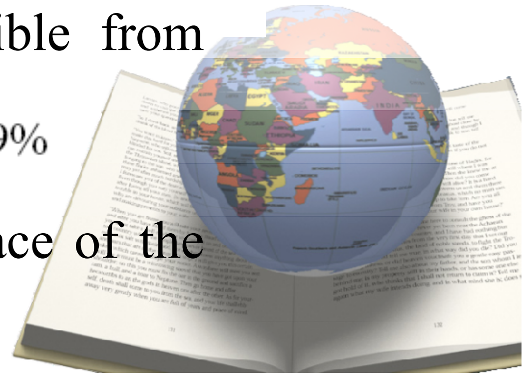
- The surface area S visible from GEO is the shaded region illustrated in figure
- It can be shown that the area S is given by:

$$S = 2\pi R_E^2(1 - \cos\phi)$$

- Therefore, the percentage of the hemisphere visible from GEO is:

$$\frac{S}{2\pi R_E^2} \times 100 = (1 - \cos 81.30^\circ) \times 100 = 84.9\%$$

- Which of course means that 42.4 of the total surface of the earth can be seen from GEO.



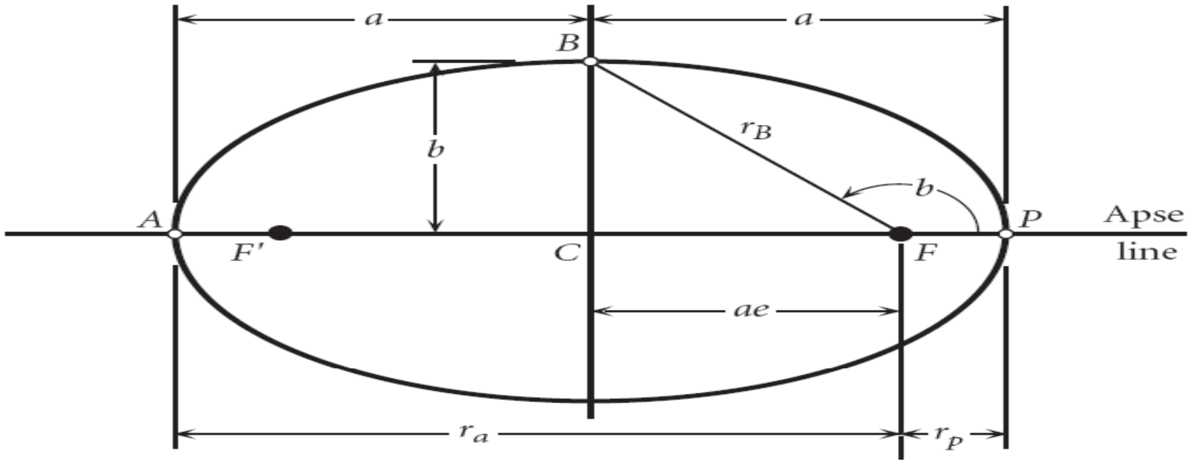
Chapter 8

Elliptical orbits $(0<e<1)$

CHAPTER CONTENT:

8- ELLIPTICAL ORBITS (0<e<1)

★ (NOTE13,P55,{1})



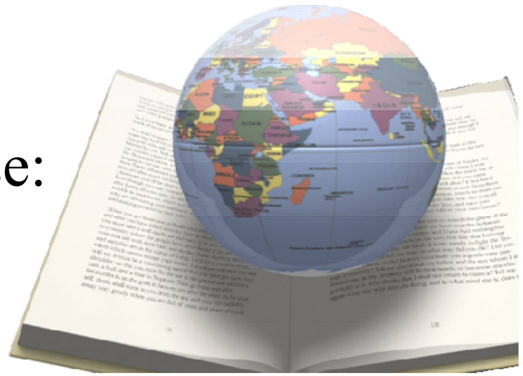
★ If $0 < e < 1 \longrightarrow r = (h^2/\mu)/(1 + e\cos\theta) = f(\theta) \longrightarrow r \rangle 0$

★ $$r_p = \frac{h^2}{\mu}\frac{1}{1+e} \langle r \langle r_a = \frac{h^2}{\mu}\frac{1}{1-e} \qquad (1)$$

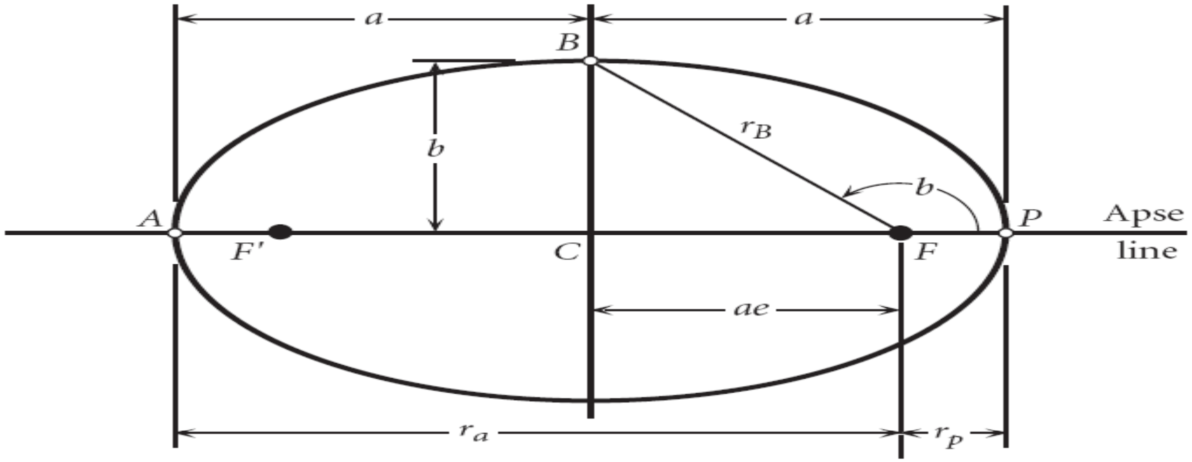
r_p: $\theta = 0$ $\qquad$ r_a: $\theta = 180$

★ The curve defined by orbit equation is an ellipse:

$$r = (h^2/\mu)/(1 + e\cos\theta)$$



8- ELLIPTICAL ORBITS $(0<e<1)$



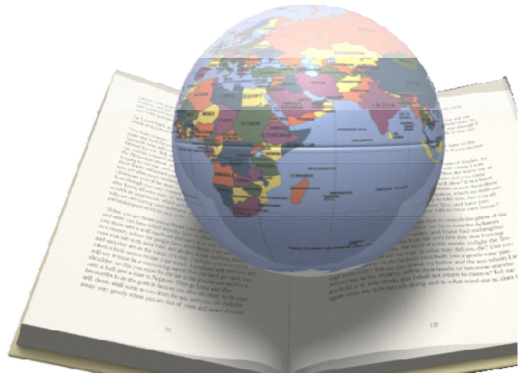
- Let 2a be the distance measured along the apse line from periapsis *P* to apoapsis *A*, as illustrated in figure, then

$$2a = r_p + r_a \qquad (2)$$

- Substituting r_p and r_a values into (2), we get:

$$a = \frac{h^2}{\mu}\frac{1}{1-e^2} \qquad (3)$$

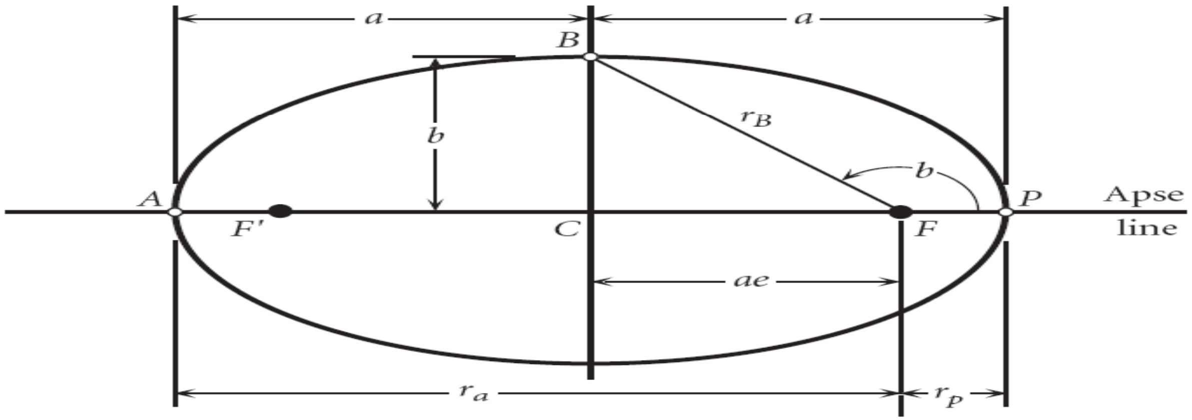
- a is the semimajor axis of the ellipse.



8- ELLIPTICAL ORBITS (0<e<1)

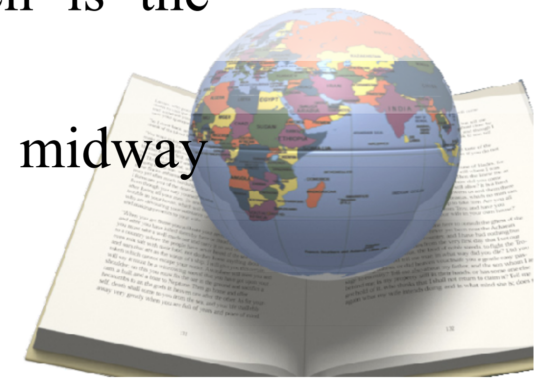
- Solving equation (3) for h^2/μ and putting the result into orbit equation yields an alternative form of the orbit equation:

$$r = a\frac{1-e^2}{1+e\cos\theta} \qquad (4)$$

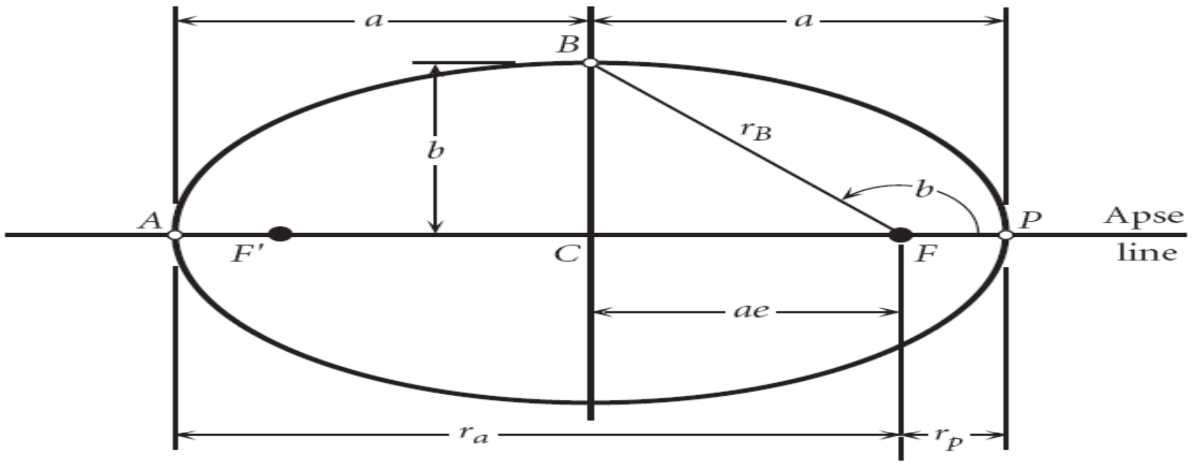


- Let F denote the location of the body m_1, which is the origin of the r, θ polar coordinate system.
- The center C of the ellipse is the point lying midway between the apoapsis and periapsis.

$$CF = a - FP = a - r_p$$



8- ELLIPTICAL ORBITS ($0<e<1$)



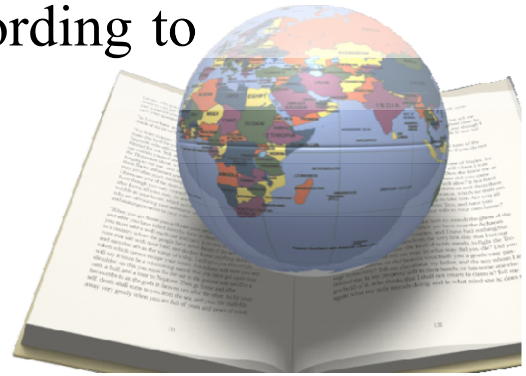
✶ From equation (4) we have:

$$r_p = a(1 - e) \qquad (5)$$

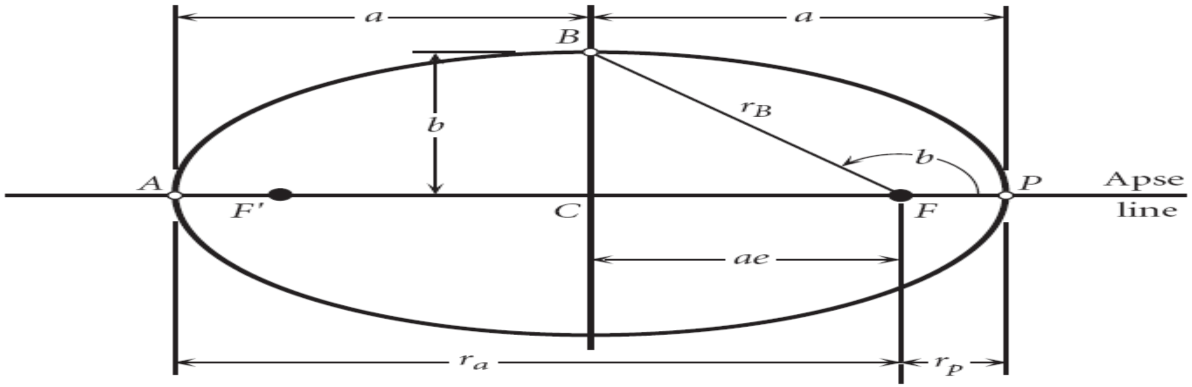
✶ So $CF = ae$, as indicated in the previous figure.

✶ If the true anomaly of point B is β , then according to equation (4), the radial coordinate of B is:

$$r_B = a\frac{1 - e^2}{1 + e\cos\beta} \qquad (6)$$



8- ELLIPTICAL ORBITS ($0<e<1$)



- The projection of r_B onto the apse line is ae:

$$ae = r_B \cos(180 - \beta) = -r_B \cos\beta = -\left(a\frac{1-e^2}{1+e\cos\beta}\right)\cos\beta$$

- Solving this expression for e, we obtain

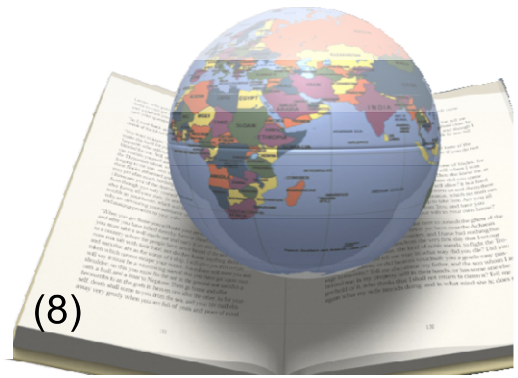
$$e = -\cos\beta \qquad (7)$$

- Substituting this result into equation (6) we get

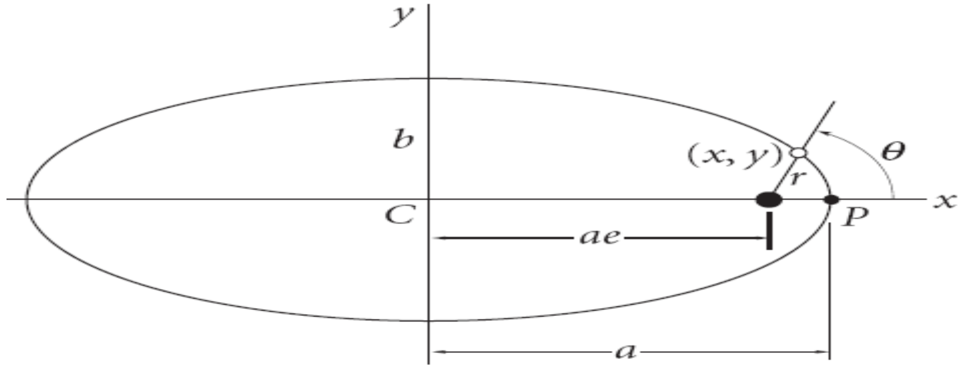
$$r_B = a$$

- According to the Pythagorean theorem,

$$b^2 = r_B^2 - (ae)^2 = a^2 - a^2e^2 \longrightarrow b = a\sqrt{1-e^2} \qquad (8)$$



8- ELLIPTICAL ORBITS (0<e<1)



- Let an xy cartesian coordinate system be centered at C,
- In terms of r and θ., we see that:

$$x = ae + r\cos\theta = ae + \left(a\frac{1-e^2}{1+e\cos\theta}\right)\cos\theta = a\frac{e+\cos\theta}{1+e\cos\theta}$$

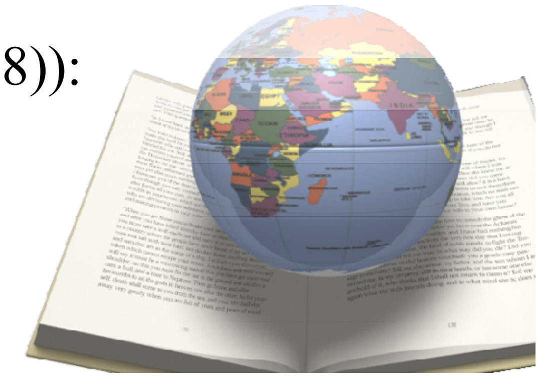
- From, this we have:

$$\frac{x}{a} = \frac{e+\cos\theta}{1+e\cos\theta} \qquad (9)$$

- For the y coordinate we have (by using equation (8)):

$$y = r\sin\theta = \left(a\frac{1-e^2}{1+e\cos\theta}\right)\sin\theta = b\frac{\sqrt{1-e^2}}{1+e\cos\theta}\sin\theta$$

- Therefore: $$\frac{y}{b} = \frac{\sqrt{1-e^2}}{1+e\cos\theta}\sin\theta \qquad (10)$$



8- ELLIPTICAL ORBITS $(0<e<1)$

✶ Using equations (10) and (9), we find:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{(1+e\cos\theta)^2}\left[(e+\cos\theta)^2+(1-e^2)\sin^2\theta\right]$$

$$=\frac{1}{(1+e\cos\theta)^2}\left[e^2+2e\cos\theta+\cos^2\theta+\sin^2\theta-e^2\sin^2\theta\right]$$

$$=\frac{1}{(1+e\cos\theta)^2}\left[e^2+2e\cos\theta+1-e^2\sin^2\theta\right]$$

$$=\frac{1}{(1+e\cos\theta)^2}\left[e^2(1-\sin^2\theta)+2e\cos\theta+1\right]$$

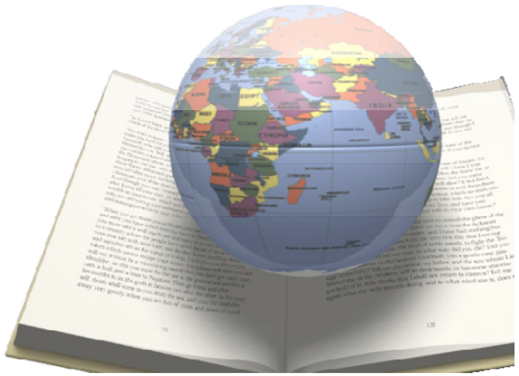
$$=\frac{1}{(1+e\cos\theta)^2}\left[e^2\cos^2\theta+2e\cos\theta+1\right]$$

$$=\frac{1}{(1+e\cos\theta)^2}(1+e\cos\theta)^2$$

✶ That is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (11)$$

(NOTE14,P59,{1})



8- ELLIPTICAL ORBITS (0<e<1)

✶ The specific energy of an elliptical orbit is negative, and it is found by substituting the specific angular momentum and eccentricity into equation:

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}(1 - e^2)$$

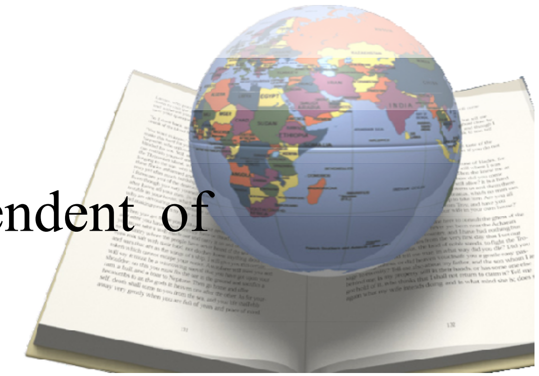
✶ We have had:

$$a = \frac{h^2}{\mu}\frac{1}{1 - e^2} \longrightarrow h^2 = \mu a(1 - e^2)$$

✶ So that:

$$\varepsilon = -\frac{\mu}{2a} \qquad (12)$$

✶ This shows that the specific energy is independent of the eccentricity and depends only on the a:



8- ELLIPTICAL ORBITS ($0<e<1$)

- For an elliptical orbit, the conservation of energy may therefore be written:

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

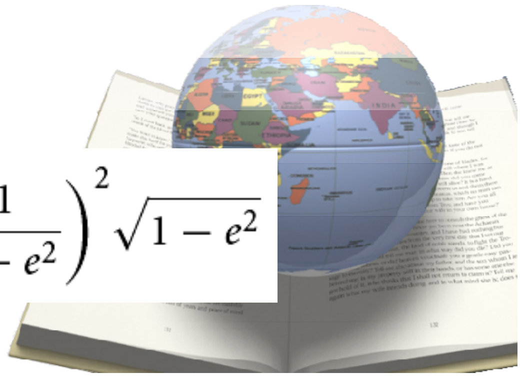
- To find the period T of the elliptical orbit, we employ Kepler's second law,

$$dA/dt = h/2, \longrightarrow \Delta A = \frac{h}{2}\Delta t \qquad (*)$$

- For a complete revolution

$$\left.\begin{array}{l} \Delta A = \pi ab \\ \Delta t = T. \end{array}\right\} \longrightarrow (*) \longrightarrow T = \frac{2\pi ab}{h} \qquad (**)$$

$$\left.\begin{array}{l} b = a\sqrt{1-e^2} \\ a = \dfrac{h^2}{\mu}\dfrac{1}{1-e^2} \end{array}\right\} \longrightarrow (**) \longrightarrow T = \frac{2\pi}{h}a^2\sqrt{1-e^2} = \frac{2\pi}{h}\left(\frac{h^2}{\mu}\frac{1}{1-e^2}\right)^2\sqrt{1-e^2}$$



8- ELLIPTICAL ORBITS ($0<e<1$)

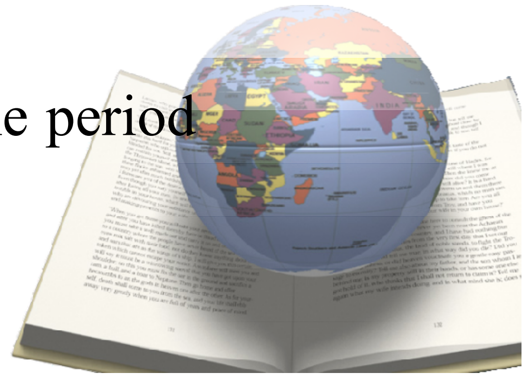
✶ So that the formula for the period of an elliptical orbit, in terms of the orbital parameters h and e, becomes:

$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1-e^2}}\right)^3 \qquad (13)$$

✶ We can substitute $h=\sqrt{\mu a(1-e^2)}$ into this equation, thereby obtaining an alternative expression for the period:

$$T = \frac{2\pi}{\sqrt{\mu}}a^{\frac{3}{2}} \qquad (14)$$

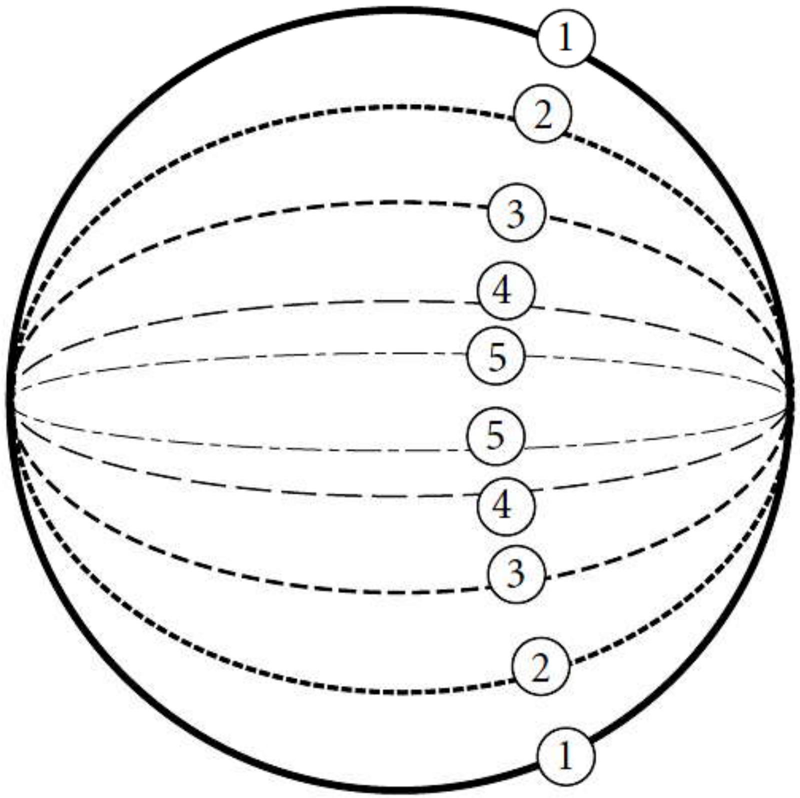
✶ This expression, reveals that, like the energy, the period of an elliptical orbit is independent of the e.

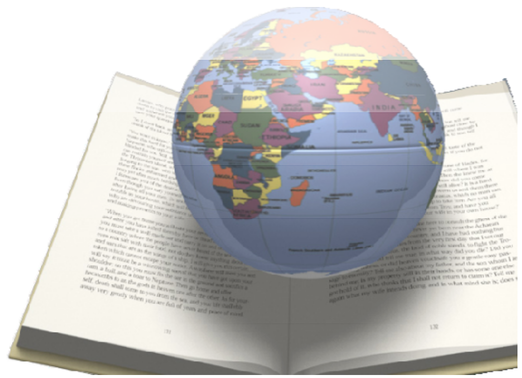


8- ELLIPTICAL ORBITS ($0<e<1$)

$$T = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}} \qquad (14)$$

✶ Equation (14) embodies Kepler's third law, the period of a planet is proportional to the three-halves power of its semimajor axis.

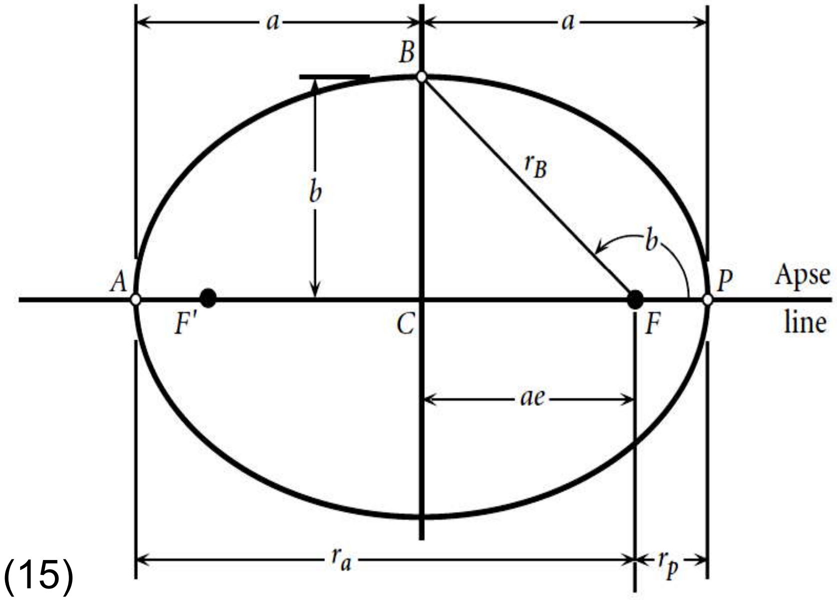




8- ELLIPTICAL ORBITS (0<e<1)

- Dividing equations r_p by r_a yields:

$$\frac{r_p}{r_a} = \frac{\dfrac{h^2}{\mu}\dfrac{1}{1+e}}{\dfrac{h^2}{\mu}\dfrac{1}{1-e}} = \frac{1-e}{1+e}$$

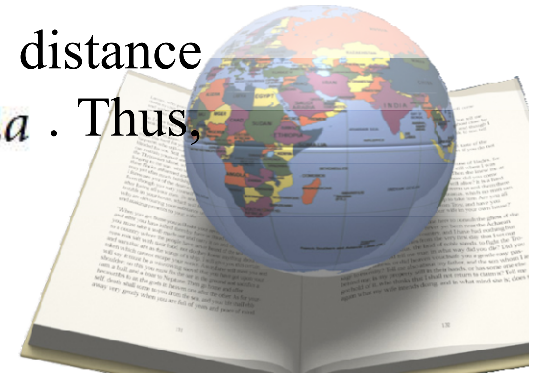


- Solving this for e result in a useful formula for calculating the eccentricity of an elliptical orbit, namely:

$$e = \frac{r_a - r_p}{r_a + r_p} \qquad (15)$$

- From figure it is apparent that $r_a - r_p = \overline{F'F}$, the distance between the foci. As previously noted $r_a + r_p = 2a$. Thus, equation (15) has the geometrical interpretation:

$$eccentricity = \frac{\text{distance between the foci}}{\text{length of the major axis}}$$



8- ELLIPTICAL ORBITS (0<e<1)

- What is the average distance of m_2 from m_1 in the course of one complete orbit?
- To answer this question, we divide the range of the true anomaly (2π) into n equal segments $\Delta\theta$, so that:

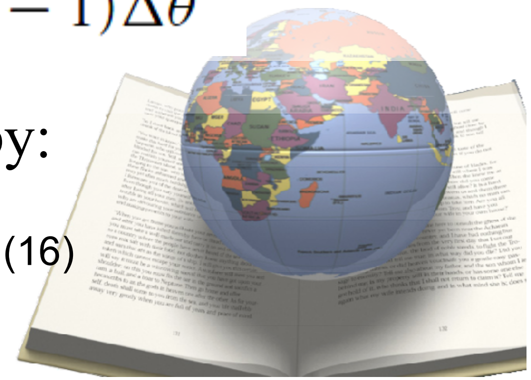
$$n = \frac{2\pi}{\Delta\theta}$$

- We then use $r = (h^2/\mu)/(1 + e\cos\theta)$ to evaluate $r(\theta)$ at n equally spaced values of true anomaly starting at periapsis:

$$\theta_1 = 0, \quad \theta_2 = \Delta\theta, \quad \theta_3 = 2\Delta\theta, \ldots, \theta_n = (n-1)\Delta\theta$$

- The average of this set of n values r is given by:

$$\bar{r}_\theta = \frac{1}{n}\sum_{i=1}^{n} r(\theta_i) = \frac{\Delta\theta}{2\pi}\sum_{i=1}^{n} r(\theta_i) = \frac{1}{2\pi}\sum_{i=1}^{n} r(\theta_i)\Delta\theta \quad (16)$$



8- ELLIPTICAL ORBITS (0<e<1)

- Now let $n \to \infty$ equation (16) becomes:

$$\bar{r}_\theta = \frac{1}{2\pi}\int_0^{2\pi} r(\theta)d\theta \qquad \text{(17)}$$

- We know that: $$r = a\frac{1-e^2}{1+e\cos\theta}$$

- So, substituting into the integrand yields:

$$\bar{r}_\theta = \frac{1}{2\pi}a(1-e^2)\int_0^{2\pi}\frac{d\theta}{1+e\cos\theta}$$

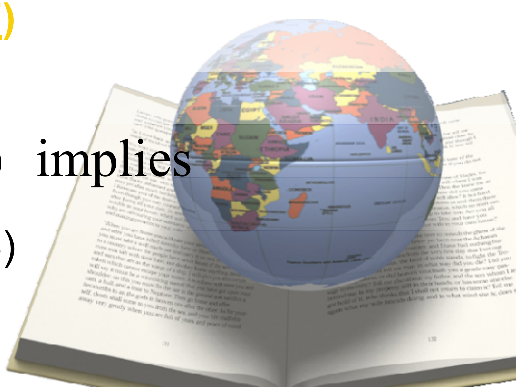
- The integral can be found in integral tabels, which yields.

$$\bar{r}_\theta = \frac{1}{2\pi}a(1-e^2)\left(\frac{2\pi}{\sqrt{1-e^2}}\right) = a\sqrt{1-e^2} \qquad \text{(17)}$$

- Since $r_p = a(1-e)$ and $r_a = a(1+e)$, equation (17) implies that

$$\bar{r}_\theta = \sqrt{r_p r_a} \qquad (18)$$

- (NOTE15,P61,{1})



8- ELLIPTICAL ORBITS ($0<e<1$)

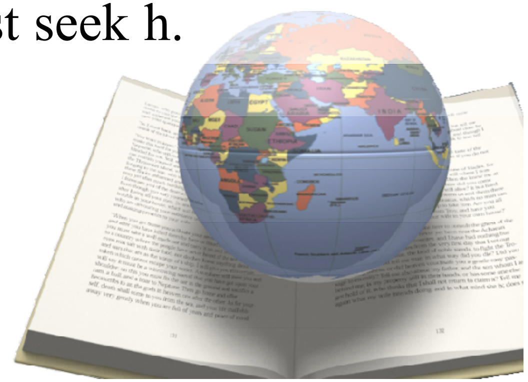
EXAMPLE 8.1

An earth satellite is in an orbit with perigee altitude $z_p = 400\,\text{km}$ and an eccentricity $e = 0.6$. Find (a) the perigee velocity, v_p; (b) the apogee, r_a; (c) the semimajor axis, a; (d) the true-anomaly –averaged radius $\bar{r}_\theta$; (e) the apogee velocity; (f) the period of the orbit; (g) the true anomaly when $r = \bar{r}_\theta$; (h) the satellite speed when $r = \bar{r}_\theta$; (i) the flight path angle γ when $r = \bar{r}_\theta$; (j) the maximum flight path angle $\gamma_{\max}$ and the true anomaly at which it occurs.

the strategy is always to go after the primary orbital parameters, eccentricity and angular momentum, first. In this problem we are given the eccentricity, so we will first seek h. recall that also that $R_E = 6378\,\text{km}$.

(a) the perigee radius is

$$r_p = R_E + z_p = 6378 + 400 = 6778\,\text{km}$$



8- ELLIPTICAL ORBITS $(0<e<1)$

EXAMPLE 8.1

Evaluating the orbit formula, equation $r = \dfrac{h^2}{\mu}\dfrac{1}{1 + e\cos\theta}$ at (perigee) $\theta = 0$ we get

$$r_p = \frac{h^2}{\mu}\frac{1}{1 + e}$$

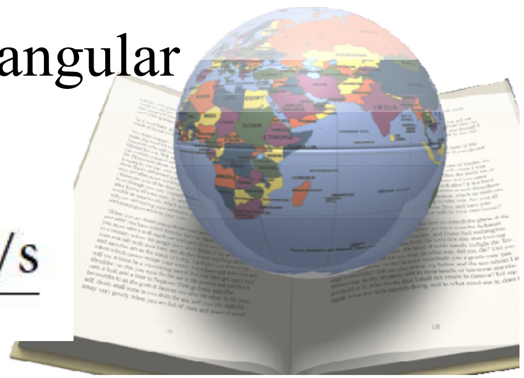
We use this to evaluate the angular momentum

$$6778 = \frac{h^2}{398\,600}\frac{1}{1 + 0.6}$$

$$h = 65\,750\,\mathrm{km^2/s}$$

Now we can find the perigee velocity using the angular momentum formula, equation $h = rv_\perp$

$$v_p = v_\perp)_{\mathrm{perigee}} = \frac{h}{r_p} = \frac{65\,750}{6778} = \underline{9.700\,\mathrm{km/s}}$$



8- ELLIPTICAL ORBITS ($0<e<1$)

EXAMPLE 8.1

(b) the apogee radius is found by evaluating the orbit equation $\theta = 180°$ (apogee):

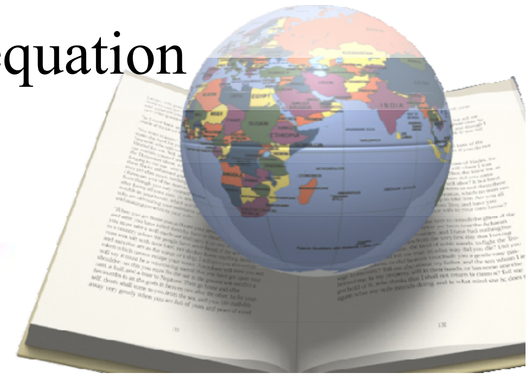
$$r_a = \frac{h^2}{\mu}\frac{1}{1-e} = \frac{65\,750^2}{398\,600}\frac{1}{1-0.6} = \underline{27\,110\,\text{km}}$$

(c) the semimajor axis is the average of the perigee and apogee radii:

$$a = \frac{r_p + r_a}{2} = \frac{6778 + 27\,110}{2} = \underline{16\,940\,\text{km}}$$

(d) the azimuth- averaged radius is given by equation (18):

$$\bar{r}_\theta = \sqrt{r_p r_a} = \sqrt{6778 \cdot 27\,110} = \underline{13\,560\,\text{km}}$$



8- ELLIPTICAL ORBITS (0<e<1)

EXAMPLE 8.1

(e) the apogee velocity, like that at perigee, is obtained from the angular momentum formula,

$$v_a = v_\perp)_{\text{apogee}} = \frac{h}{r_a} = \frac{65\,750}{27\,110} = \underline{2.425\ \text{km/s}}$$

(f) to find the orbit period, use equation (14)

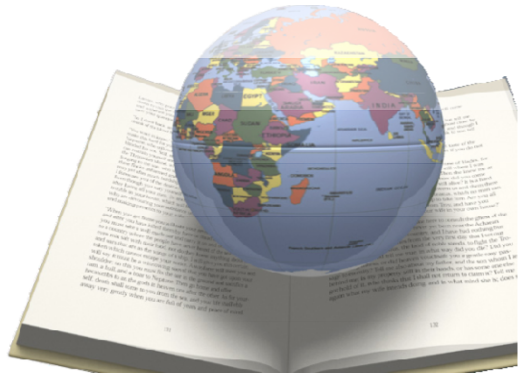
$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1-e^2}}\right)^3 = \frac{2\pi}{398\,600^2}\left(\frac{65\,750}{\sqrt{1-0.6^2}}\right)^3 = 21\,950\ \text{s} = \underline{6.098\ \text{hr}}$$

(g) to find the true anomaly when $r = \bar{r}_\theta$, we again use the orbit formula

$$\bar{r}_\theta = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$$

$$13\,560 = \frac{65\,750^2}{398\,600}\frac{1}{1+0.6\cos\theta}$$

$$\cos\theta = -0.3333$$



8- ELLIPTICAL ORBITS (0<e<1)

EXAMPLE 8.1

This means:

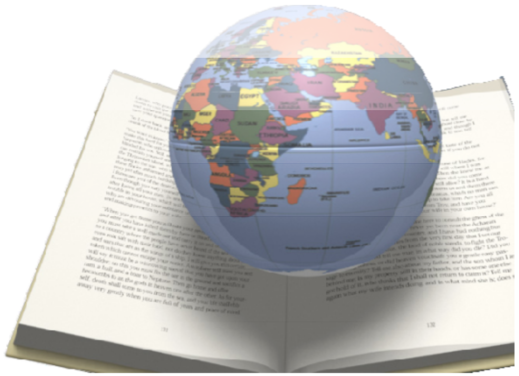
$\underline{\theta = 109.5°}$, where the satellite passes through $\bar{r}_\theta$ on its way from perigee

and :

$\underline{\theta = 250.5°}$, where the satellite passes through on $\bar{r}_\theta$ its way towards perigee

(h) To find the speed of the satellite $r = \bar{r}_\theta$, we first calculate the radial and transverse components of velocity:

$$v_\perp = \frac{h}{\bar{r}_\theta} = \frac{65\,750}{13\,560} = 4.850\,\text{km/s}$$



8- ELLIPTICAL ORBITS (0<e<1)

EXAMPLE 8.1

For the radial velocity component, use equation $v_\perp = \frac{\mu}{h}(1 + e\cos\theta)$

$$v_r = \frac{\mu}{h} e \sin\theta = \frac{398\,600}{65\,750} \cdot 0.6 \cdot \sin(109.5°) = 3.430\,\text{km/s}$$

or

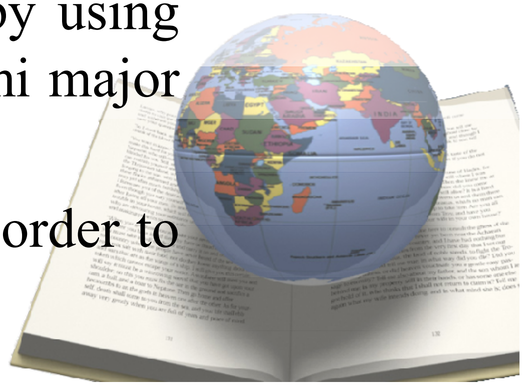
$$v_r = \frac{\mu}{h} e \sin\theta = \frac{398\,600}{65\,750} \cdot 0.6 \cdot \sin(250.5°) = -3.430\,\text{km/s}$$

The magnitude of the velocity can now be found as

$$v = \sqrt{v_r^2 + v_\perp^2} = \sqrt{3.430^2 + 4.850^2} = \underline{5.940\,\text{km/s}}$$

We could have obtained the speed v more directly by using conservation of energy ($\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$), since the semi major axis is available from part (c) above.

however we would still need to compute, v_r and $v_\perp$ in order to solve next part of this problem.



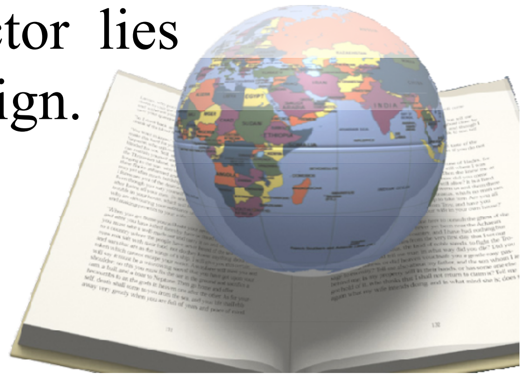
8- ELLIPTICAL ORBITS (0<e<1)

EXAMPLE 8.1

(i) use equation $v_r = \frac{\mu}{h} e \sin\theta$ to calculate the flight path an$r = \bar{r}_\theta$,t

$$\gamma = \tan^{-1}\frac{v_r}{v_\perp} = \tan^{-1}\frac{3.430}{4.850} = \underline{35.26^\circ} \text{ at } \theta = 109.5^\circ$$

γ is positive, meaning the velocity vector is above the local horizon, indicating the spacecraft is flying away from the attracting force. At $\theta = 250.5^\circ$, Where the spacecraft is flying towards perigee, $\gamma = -35.26^\circ$. since the satellite is approaching the attracting body, the velocity vector lies below the local horizon, as indicated by the minus sign.



8- ELLIPTICAL ORBITS ($0<e<1$)

EXAMPLE 8.1

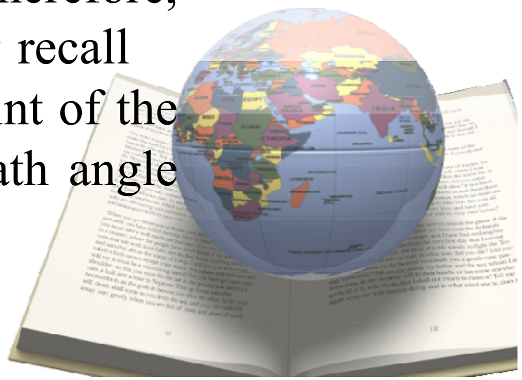
(j) equation $\tan\gamma = \dfrac{e\sin\theta}{1+e\cos\theta}$ gives the flight path angle in terms of the true anomaly,

$$\gamma = \tan^{-1}\frac{e\sin\theta}{1+e\cos\theta} \qquad \text{(a)}$$

To find where γ is a maximum, we must take the derivative of this expression with respect to θ and set the result equal to zero. Using the rules of calculus,

$$\frac{d\gamma}{d\theta} = \frac{1}{1+\left(\dfrac{e\sin\theta}{1+e\cos\theta}\right)^2}\frac{d}{d\theta}\left(\frac{e\sin\theta}{1+e\cos\theta}\right) = \frac{e(e+\cos\theta)}{(1+e\cos\theta)^2+e^2\sin^2\theta}$$

For $e<1$, the denominator is positive for all values of θ. therefore, $d\gamma/d\theta=0$ only if the numerator vanishes, that is, if $\cos\theta=-e$ recall from equation (7) that this true anomaly locates the end-point of the minor axis of the ellipse. The maximum positive flight path angle therefore occurs at the true anomaly



$$\theta = \cos^{-1}(-0.6) = \underline{126.9^\circ}$$

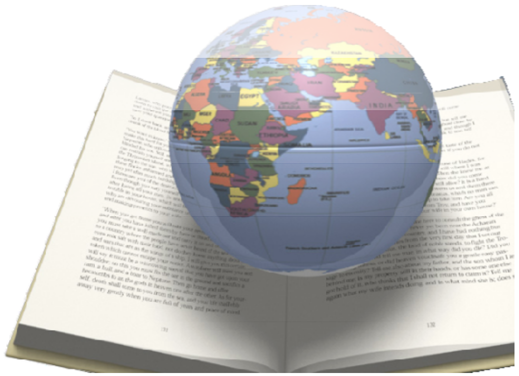
8- ELLIPTICAL ORBITS ($0<e<1$)

EXAMPLE 8.1

Substituting this into (a), we find the value of the flight path angle to be

$$\gamma_{max} = \tan^{-1}\frac{0.6\sin 126.9^\circ}{1 + 0.6\cos 126.9^\circ} = \underline{36.87^\circ}$$

After attaining this greatest magnitude, the flight path angle starts to decrease steadily towards its value at apogee (zero).



8- ELLIPTICAL ORBITS $(0<e<1)$

EXAMPLE 8.2

At two point on a geocentric orbit the altitude and true anomaly $z_1 = 1545\ \text{km}, \theta_1 = 126°$ and $z_2 = 852\ \text{km}, \theta_2 = 58°$, respectively.

Find (a) the eccentricity; (b) the altitude of perigee; (c) the semi major; and (d) the period.

(a) The radii of the two points are

$$r_1 = R_E + z_1 = 6378 + 1545 = 7923\ \text{km}$$

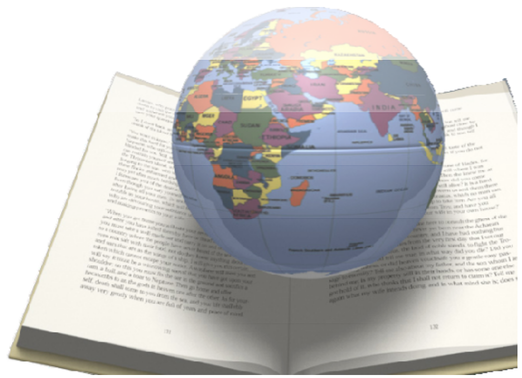
$$r_2 = R_E + z_2 = 6378 + 852 = 7230\ \text{km}$$

Applying the orbit formula, equation $r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$, to both of these points yields two equations for the primary orbital parameters, angular momentum *h* and eccentricity *e*

$$r_1 = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta_1}$$

$$7923 = \frac{h^2}{398\,600}\frac{1}{1+e\cos 126°}$$

$$h^2 = 3.158\times10^9 - 1.856\times10^9 e \qquad \text{(a)}$$



8- ELLIPTICAL ORBITS (0<e<1)

EXAMPLE 8.2

$$r_2 = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta_2}$$

$$7230 = \frac{h^2}{398\,600}\frac{1}{1 + e\cos 58°}$$

$$h^2 = 2.882 \times 10^9 + 1.527 \times 10^9 e \qquad \text{(b)}$$

Equation (a) and (b), the two expressions for h^2, yields single equation for the eccentricity e,

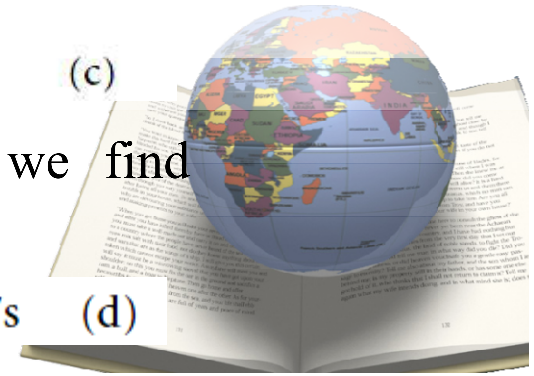
$$3.158 \times 10^9 - 1.856 \times 10^9 e = 2.882 \times 10^9 + 1.527 \times 10^9 e \Rightarrow 3.384 \times 10^9 e = 276.2 \times 10^6$$

Therefore,

$$\underline{e = 0.08164} \text{ (an ellipse)} \qquad \text{(c)}$$

(b) By substituting the eccentricity back into (a) [or (b)] we find the angular momentum,

$$h^2 = 3.158 \times 10^9 - 1.856 \times 10^9 \cdot 0.08164 \Rightarrow h = 54\,830\,\text{km}^2/\text{s} \qquad \text{(d)}$$



8- ELLIPTICAL ORBITS (0<e<1)

EXAMPLE 8.2

Now we can use the orbit equation to obtain the perigee radius

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e\cos(0)} = \frac{54\,830^2}{398\,600}\frac{1}{1+0.08164} = 6974\,\text{km}$$

and perigee altitude

$$z_p = r_p - R_E = 6974 - 6378 = \underline{595.5\,\text{km}}$$

(c) the semimajor axis can be found after we calcute the apogee radius by means of the orbit equation, just we did for perigee radius:

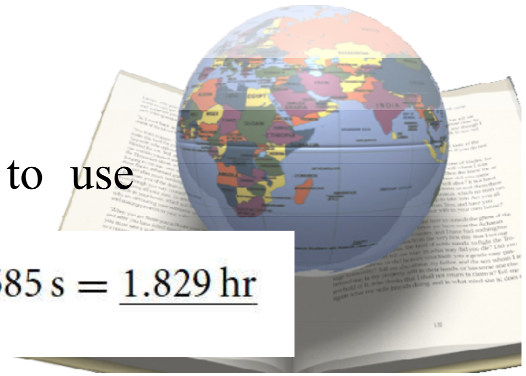
$$r_a = \frac{h^2}{\mu}\frac{1}{1+e\cos(180°)} = \frac{54\,830^2}{398\,600}\frac{1}{1-0.08164} = 8213\,\text{km}$$

Hence

$$a = \frac{r_p + r_a}{2} = \frac{8213+6974}{2} = \underline{7593\,\text{km}}$$

(d) Since the semi major axis is available, it is convenient to use equation (15) to find the period

$$T = \frac{2\pi}{\sqrt{\mu}}a^{\frac{3}{2}} = \frac{2\pi}{\sqrt{398\,600}}7593^{\frac{3}{2}} = 6585\,\text{s} = \underline{1.829\,\text{hr}}$$



Chapter 9

Parabolic trajectories

(e =1)

CHAPTER CONTENT

9- PARABOLICTRAJECTORIES (e =1)

- If the eccentricity equals 1. then the orbit equation becomes:

$$r = \frac{h^2}{\mu}\frac{1}{1+\cos\theta} \qquad (1)$$

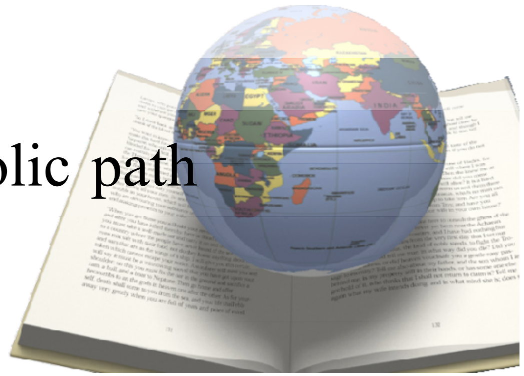
- If $\theta \longrightarrow 180°$ → $r \longrightarrow \infty$

- For a parabolic trajectory the conservation of energy is

$$\left.\begin{aligned} &\frac{v^2}{2} - \frac{\mu}{r} = \varepsilon \\ &\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}(1-e^2) \end{aligned}\right\} \rightarrow \frac{v^2}{2} - \frac{\mu}{r} = 0$$

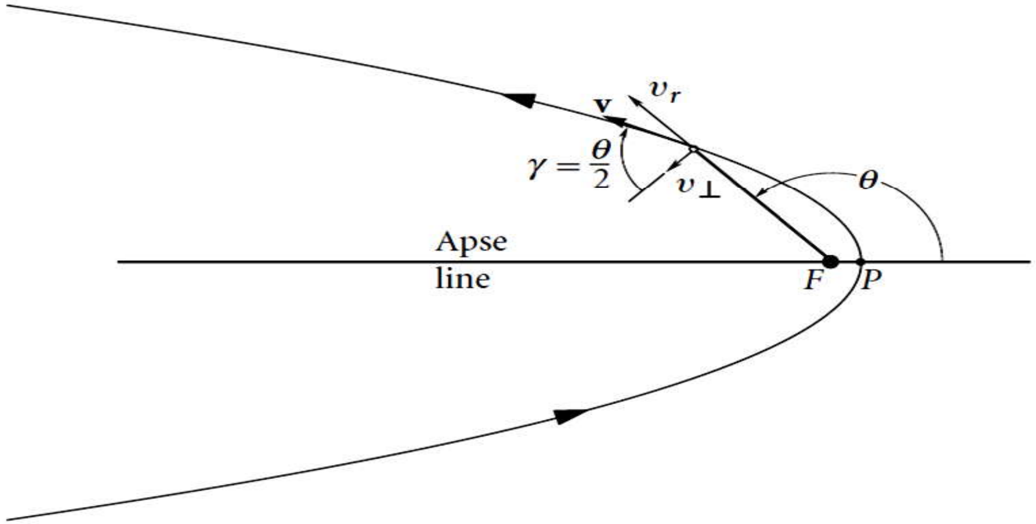
- It means that the speed anywhere on a parabolic path is:

$$v = \sqrt{\frac{2\mu}{r}} \qquad (2)$$



9- PARABOLICTRAJECTORIES (e =1)

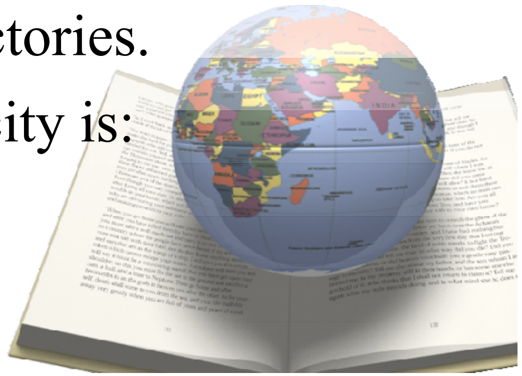
✶ If the body m_2 is launched on a parabolic trajectory; it will coast to infinity, arriving there with zero velocity relative to m_1. It will not return



✶ Parabolic paths are therefore called escape trajectories.

✶ At a given distance r from m_1 the escape velocity is:

$$v_{esc} = \sqrt{\frac{2\mu}{r}} \quad (4)$$



9- PARABOLICTRAJECTORIES (e =1)

✶ Let v_o be the speed of a satellite in a circular orbit of radius r. then :

$$v_{esc} = \sqrt{2} v_o \qquad (4)$$

✶(NOTE 16, P66, {1})

✶ For the parabola, the flight path angle takes the form:

$$\tan\gamma = \frac{\sin\theta}{1+\cos\theta}$$

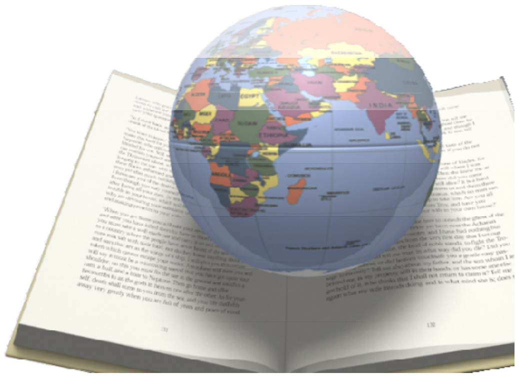
✶ Using the trigonometric identities

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

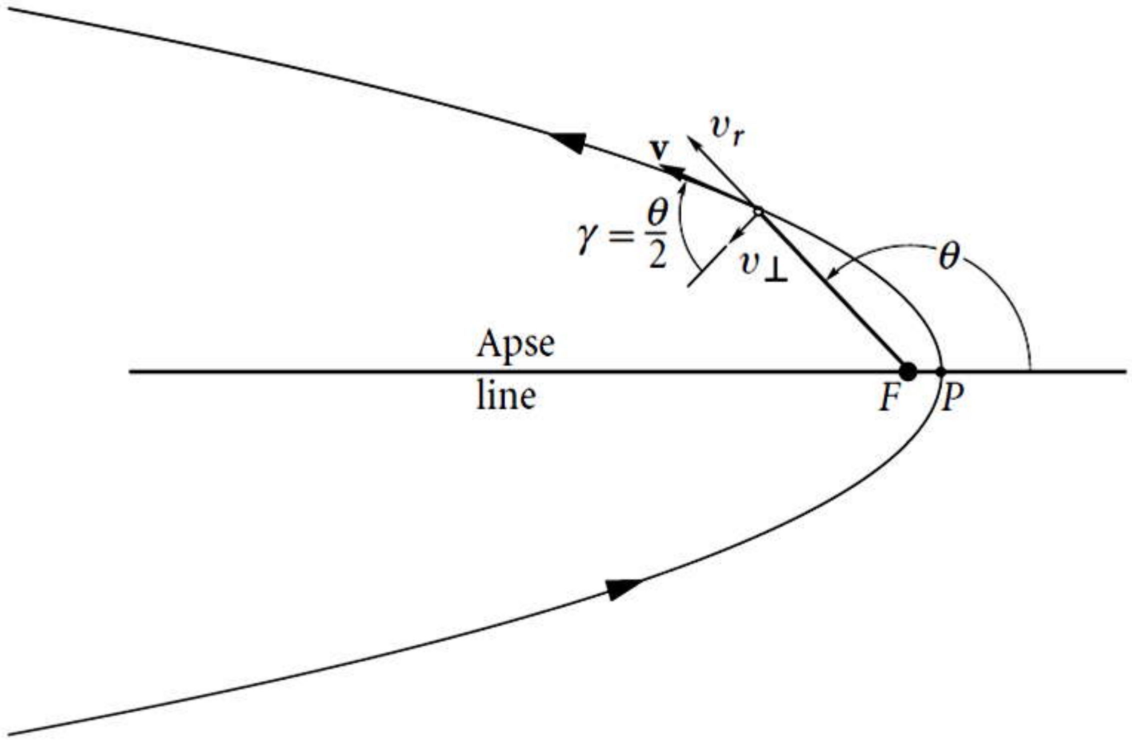
$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1$$

✶ We can write

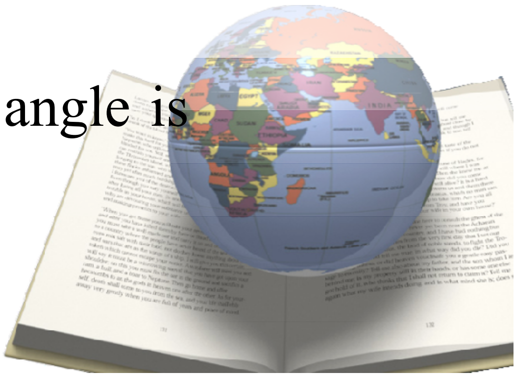
$$\tan\gamma = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2} \longrightarrow \gamma = \frac{\theta}{2} \qquad (5)$$



9- PARABOLICTRAJECTORIES (e =1)



✶ That is, on parabolic trajectories the flight path angle is one-half the true anomaly

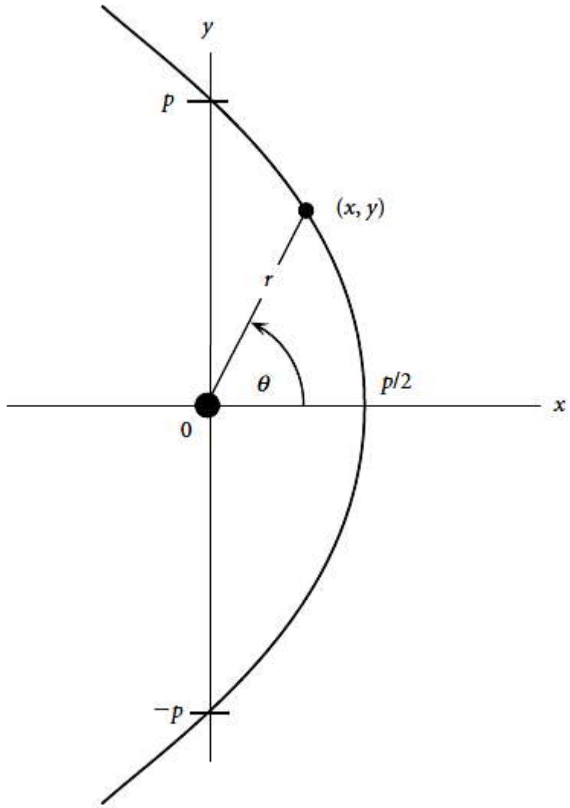


9- PARABOLICTRAJECTORIES (e =1)

✶ Recall that the parameter p of an orbit:

$$p = \frac{h^2}{\mu}$$

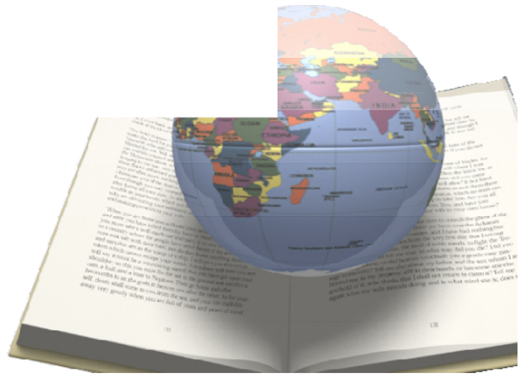
✶ Substitute this expression into equation(1) and then plot $r = \dfrac{2a}{1+\cos\theta}$ in a cartesian coordinate system centered at the focus, we will get:



✶ From the figure it is clear that:

$$x = r\cos\theta = p\frac{\cos\theta}{1+\cos\theta} \qquad (6)$$

$$y = r\sin\theta = p\frac{\sin\theta}{1+\cos\theta} \qquad (7)$$



9- PARABOLICTRAJECTORIES (e =1)

- Therefore

$$\frac{x}{p/2} + \left(\frac{y}{p}\right)^2 = 2\frac{\cos\theta}{1+\cos\theta} + \frac{\sin^2\theta}{(1+\cos\theta)^2}$$

- Working to simplify the right-hand side, we get:

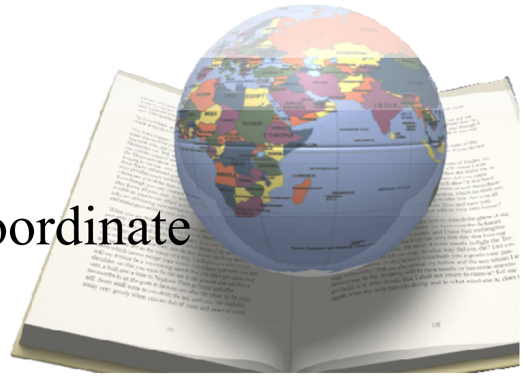
$$\frac{x}{p/2} + \left(\frac{y}{p}\right)^2 = \frac{2\cos\theta(1+\cos\theta)+\sin^2\theta}{(1+\cos\theta)^2} = \frac{2\cos\theta + 2\cos^2\theta + (1-\cos^2\theta)}{(1+\cos\theta)^2}$$

$$= \frac{1+2\cos\theta+\cos^2\theta}{(1+\cos\theta)^2} = \frac{(1+\cos\theta)^2}{(1+\cos\theta)^2} = 1$$

- It follows that:

$$x = \frac{p}{2} - \frac{y^2}{2p} \qquad (10)$$

- This is the equation of a parabola in a cartesian coordinate system whose origin serves as the focus.

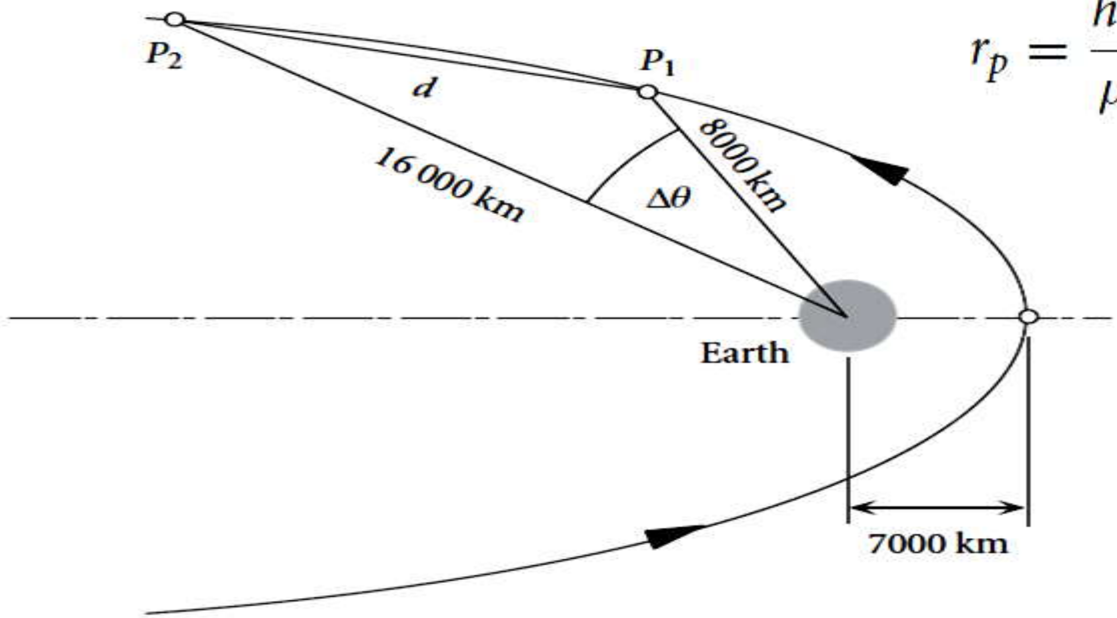


9- PARABOLICTRAJECTORIES (e =1)

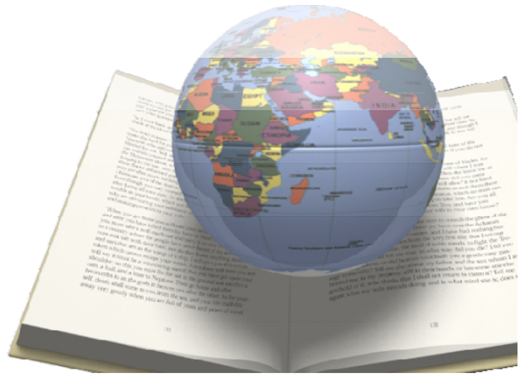
EXAMPLE ?.1

- The perigee of a satellite in parabolic geocentric trajectory is 7000km. Find the distance *d* between point P_1 and P_2 n the orbit which are 8000km and 16000km, respectively, from the center of the earth.

  first, let us calculate the angular momentum of the satellite by evaluating the orbit equation at perigee,



$$r_p = \frac{h^2}{\mu}\frac{1}{1+\cos(0)} = \frac{h^2}{2\mu}$$



9- PARABOLICTRAJECTORIES (e =1)

EXAMPLE ?.1

- From which

$$h = \sqrt{2\mu r_p} = \sqrt{2 \cdot 398\,600 \cdot 7000} = 74\,700\,\text{km}^2/\text{s} \qquad \text{(a)}$$

- To find the length of the chord $\overline{P_1P_2}$, we must use the law of cosines from trigonometry,

$$d^2 = 8000^2 + 16\,000^2 - 2 \cdot 8000 \cdot 16\,000 \cos \Delta\theta \qquad \text{(b)}$$

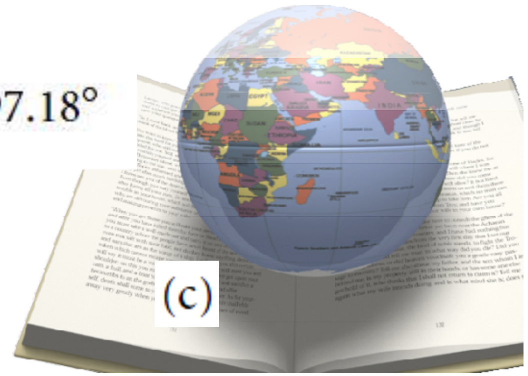
- The true anomalies of points P_1 and P_2 are found using the orbit equation:

$$8000 = \frac{74\,700^2}{398\,600}\frac{1}{1+\cos\theta_1} \Rightarrow \cos\theta_1 = 0.75 \Rightarrow \theta_1 = 41.41^\circ$$

$$16\,000 = \frac{74\,700^2}{398\,600}\frac{1}{1+\cos\theta_2} \Rightarrow \cos\theta_2 = -0.125 \Rightarrow \theta_2 = 97.18^\circ$$

- Therefore,; $\Delta\theta = 97.18^\circ - 41.41^\circ = 55.78^\circ$, so that (b) yields

$$\underline{d = 13\,270\,\text{km}} \qquad \text{(c)}$$



Chapter 10

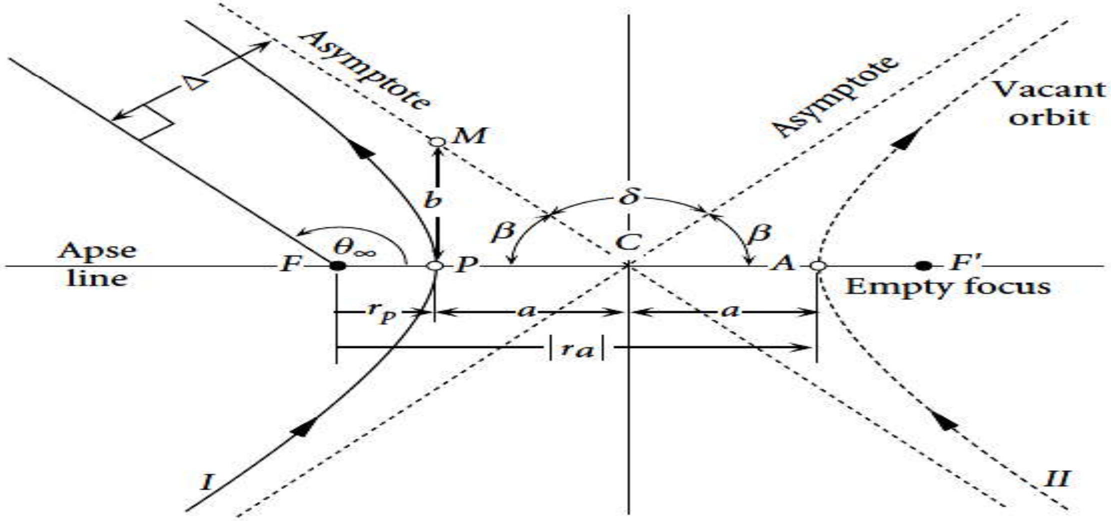
Hyperbolic trajectories ($e > 1$)

CHAPTER CONTENT

10- HYPERBOLICTRAJECTORIES (e >1)

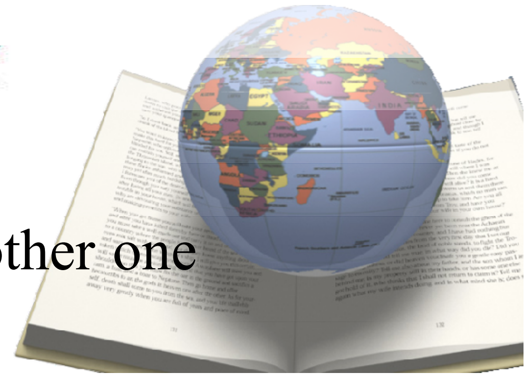
★ If $e > 1$, the orbit formula describes the geometry of the hyperbola

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta} \quad (1)$$

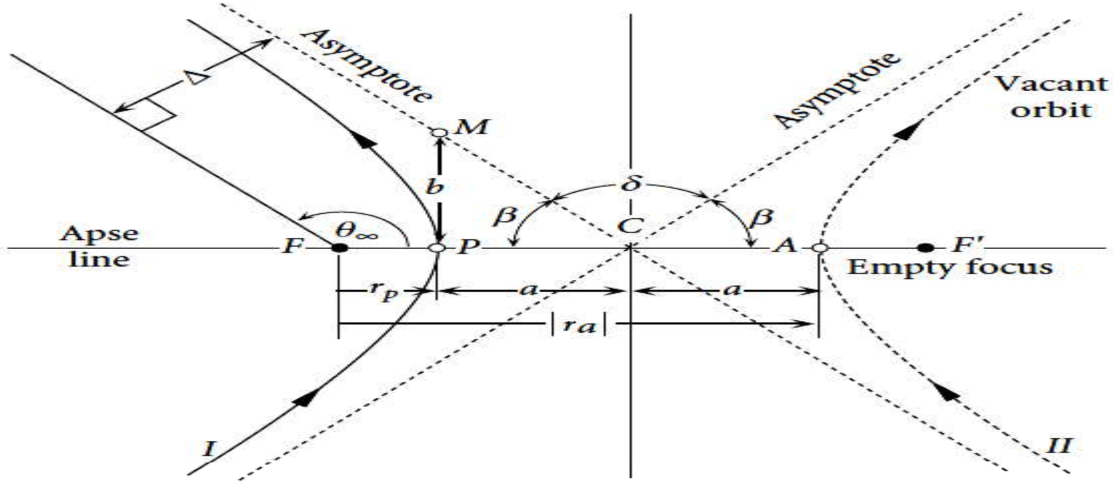


★ The system consist of two symmetric curves

★ One of the occupied by the orbiting body, the other one is its empty, mathematical image



10- HYPERBOLICTRAJECTORIES (e >1)



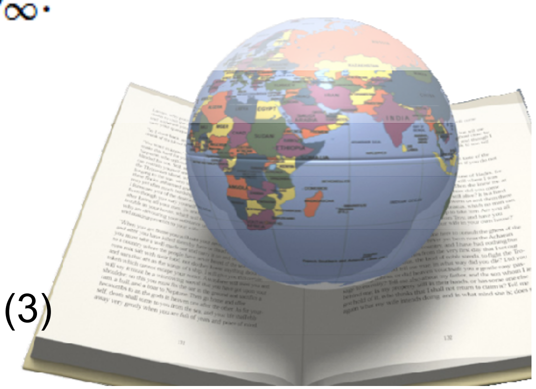
- Clearly: $\lim r \to \&$

  $\cos\theta \to -1/e$

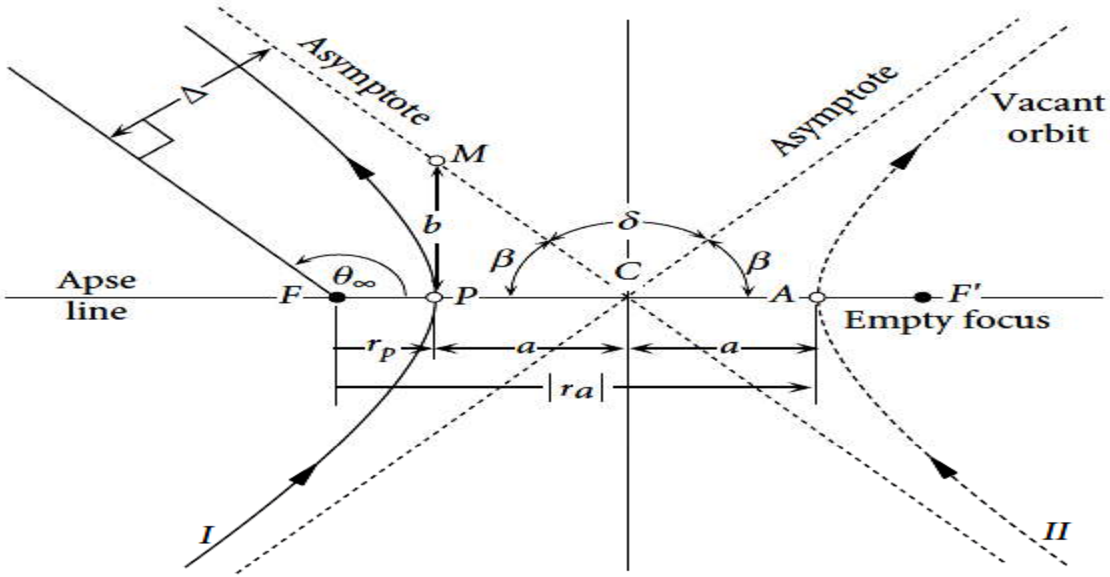
- We denote this value of true anomaly since the radial distance approaches infinity as the true anomaly approaches θ_∞.

$$\theta_\infty = \cos^{-1}(-1/e) \qquad (2)$$

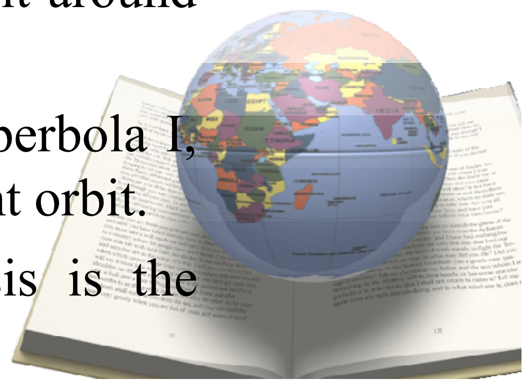
- θ_∞ is known as the true of the asymptote.
- Observe that θ_∞ lies between $90°$ and $180°$
- From trigonometry it follow that $\sin\theta_\infty = \dfrac{\sqrt{e^2-1}}{e}$ (3)



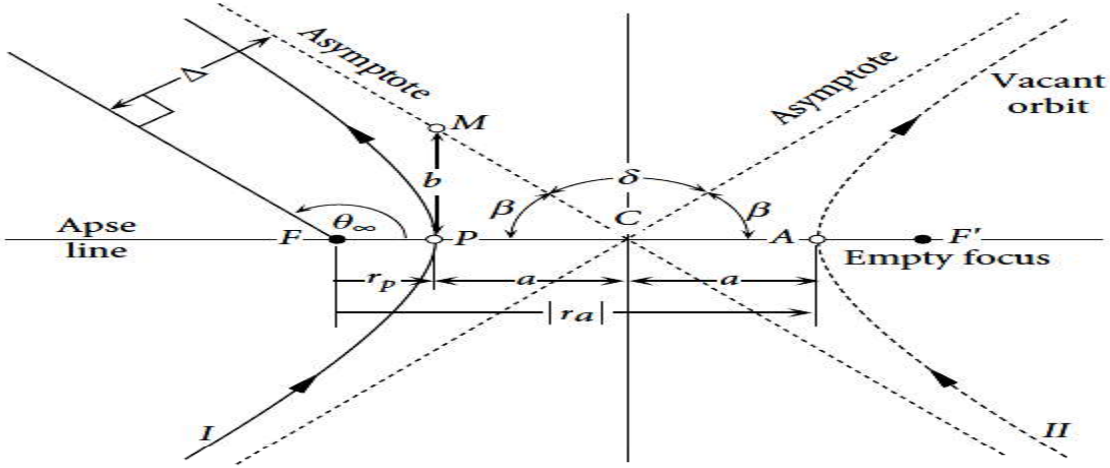
10- HYPERBOLICTRAJECTORIES (e >1)



- ✶ For $-\theta_\infty < \theta < \theta_\infty$, the physical trajectory is the occupied hyperbola I (on the left)
- ✶ For $\theta_\infty < \theta < (360° - \theta_\infty)$, hyperbola II- the vacant orbit around the empty focus F' - is traced out. (NOTE17,P69,{1})
- ✶ Periapsis P lies on the apse line on the physical hyperbola I, whereas apoapsis A lies on the apse line on the vacant orbit.
- ✶ The point halfway between periapsis and apoapsis is the center C of the hyperbola.



10- HYPERBOLICTRAJECTORIES (e >1)

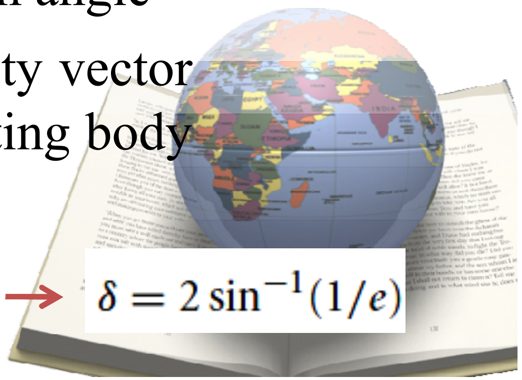


- ✶ The asymptotes intersect at C, making angle β with the apse line.

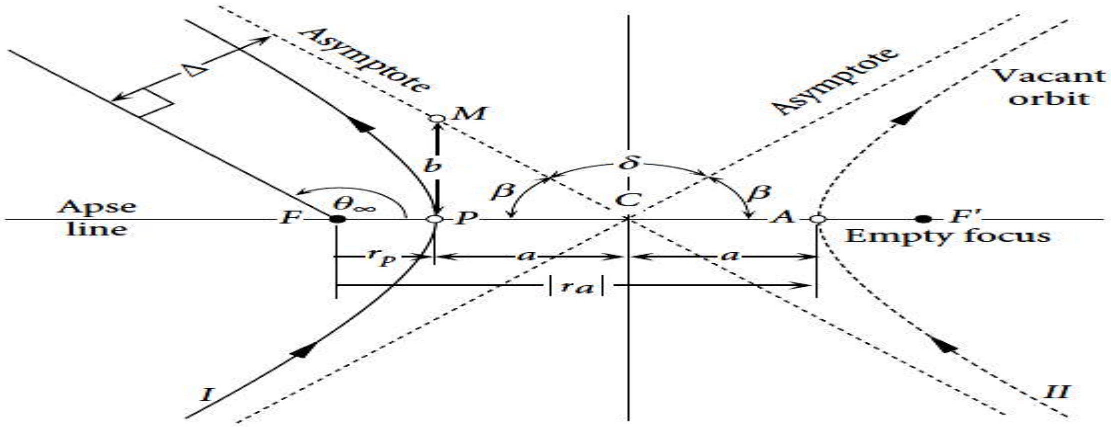
$$\beta = 180^\circ - \theta_\infty \;\longrightarrow\; \cos\beta = -\cos\theta_\infty \;\longrightarrow\; \beta = \cos^{-1}(1/e) \qquad (2)$$

- ✶ The angle δ between the asymptotes is called the turn angle
- ✶ The turn angle is the angle through which the velocity vector of the orbiting body is rotated as it rounds the attracting body at F and heads back towards infinity.

$$\delta = 180^\circ - 2\beta, \;\rightarrow\; \sin\frac{\delta}{2} = \sin\left(\frac{180^\circ - 2\beta}{2}\right) = \sin(90^\circ - \beta) = \cos\beta \overset{\text{Eq. 2.89}}{=} \frac{1}{e} \;\rightarrow\; \delta = 2\sin^{-1}(1/e)$$



10- HYPERBOLICTRAJECTORIES (e >1)



- The distance r_p from the focus F to the periapsis is given by equation:

$$r_a = \frac{h^2}{\mu}\frac{1}{1+e} \qquad (6)$$

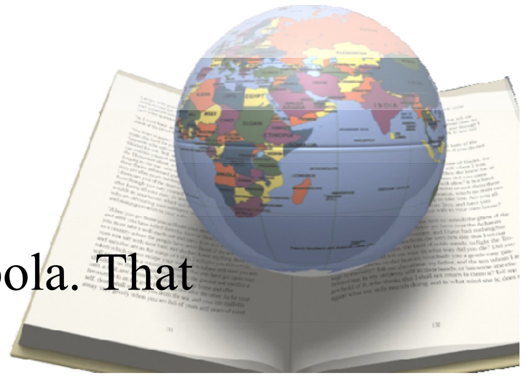
- The radial coordinate r_a of apoapsis is found by setting $\theta = 180°$ in equation:

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta} \qquad (7)$$

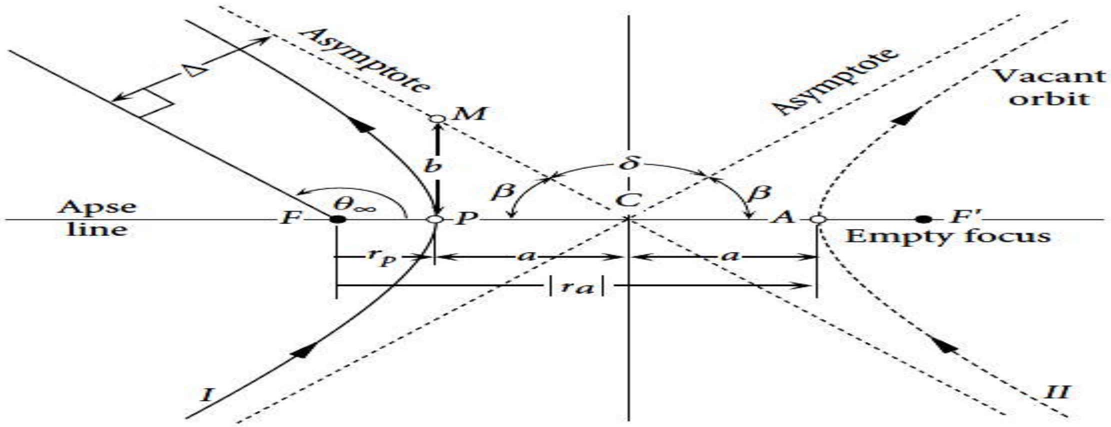
- so

$$r_a = \frac{h^2}{\mu}\frac{1}{1-e}$$

- Observe that r_a is negative, since $e > 1$ for the hyperbola. That means the apoapse lies to the right of the focus F



10- HYPERBOLICTRAJECTORIES (e >1)



- ✶ We see that the distance 2a from periapse P to apoapse A is:

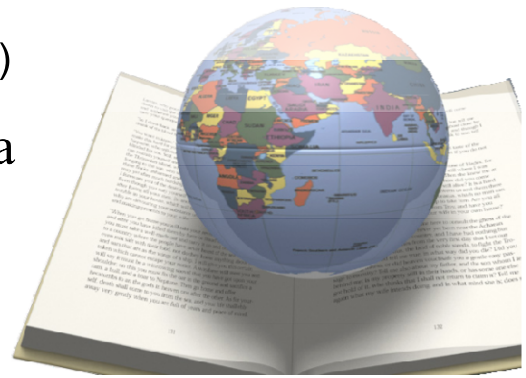
$$2a = |r_a| - r_p = -r_a - r_p$$

- ✶ Substituting equation (6) , (7) yields

$$2a = -\frac{h^2}{\mu}\left(\frac{1}{1-e} + \frac{1}{1+e}\right) \longrightarrow a = \frac{h^2}{\mu}\frac{1}{e^2-1} \qquad (8)$$

- ✶ So the orbit formula may be written for the hyperbola

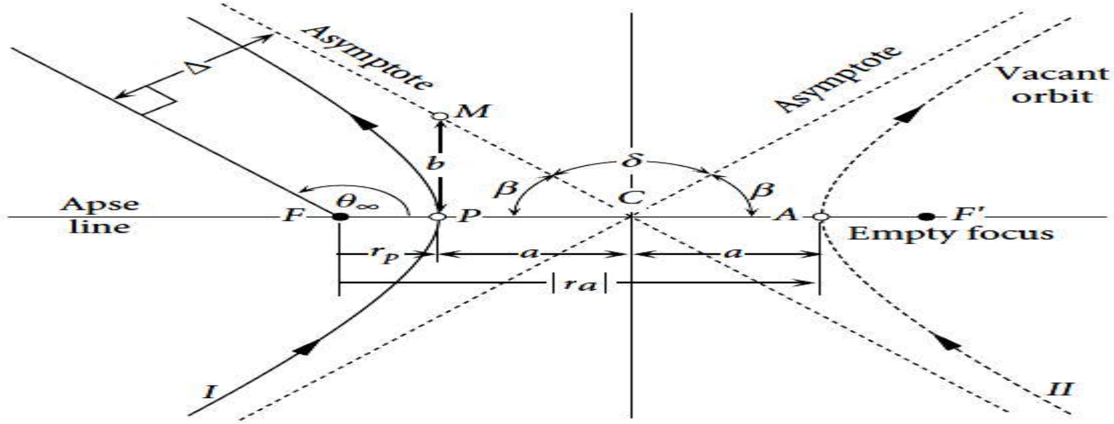
$$r = a\frac{e^2-1}{1+e\cos\theta} \qquad (9)$$



10- HYPERBOLICTRAJECTORIES ($e > 1$)

- From equation (g) it follows that: $r_p = a(e-1)$ (10)

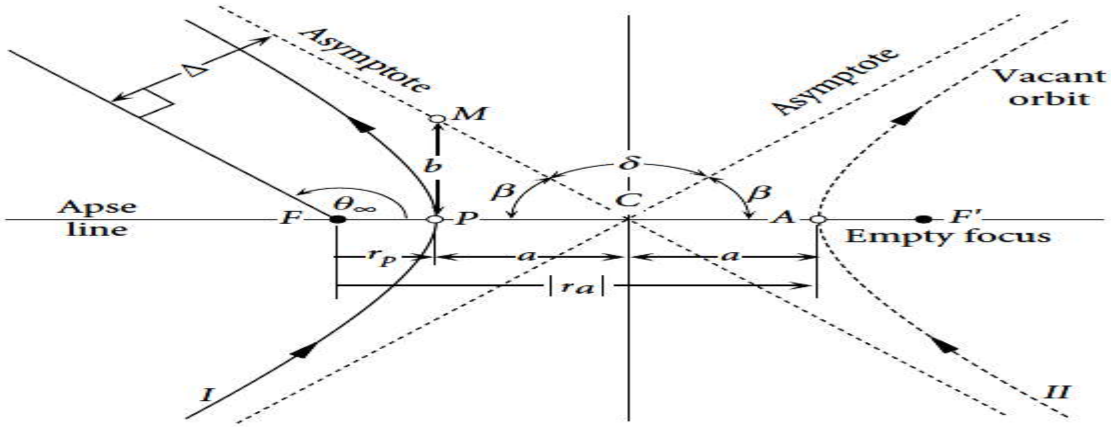
$$r_a = -a(e+1) \quad (11)$$



- The distance b, from periapsis to an asymptote measured perpendicular to the apse line; is the semiminor axis of the hyperbola
- The length b is

$$b = a\tan\beta = a\frac{\sin\beta}{\cos\beta} = a\frac{\sin(180-\theta_\infty)}{\cos(180-\theta_\infty)} = a\frac{\sin\theta_\infty}{-\cos\theta_\infty} = a\frac{\dfrac{\sqrt{e^2-1}}{e}}{-\left(-\dfrac{1}{e}\right)} \quad (12)$$

10- HYPERBOLICTRAJECTORIES (e >1)



- The distance Δ between the asymptote and a parallel line through the focus is called the aiming radius
- We see that

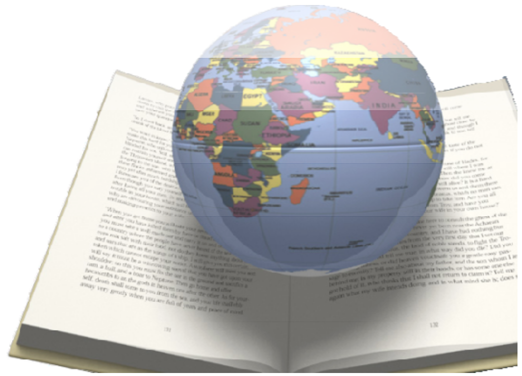
$$\Delta = (r_p + a)\sin\beta$$

(10) → $\Delta = ae\sin\beta$

(4) → $\Delta = ae\dfrac{\sqrt{e^2-1}}{e}$

(3) → $\Delta = ae\sin\theta_\infty = ae\sqrt{1-\cos^2\theta_\infty}$

(2) → $\Delta = ae\sqrt{1-\dfrac{1}{e^2}}$

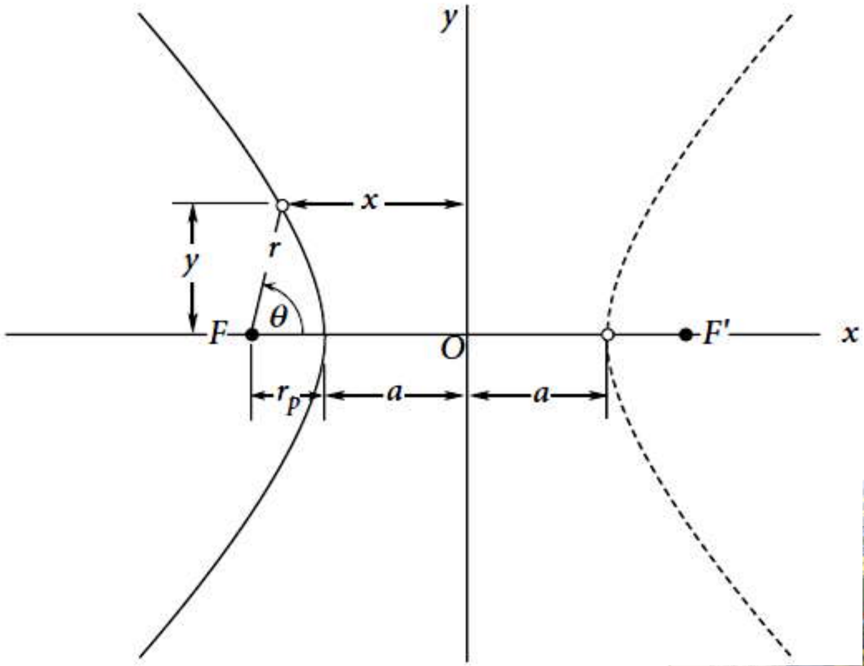


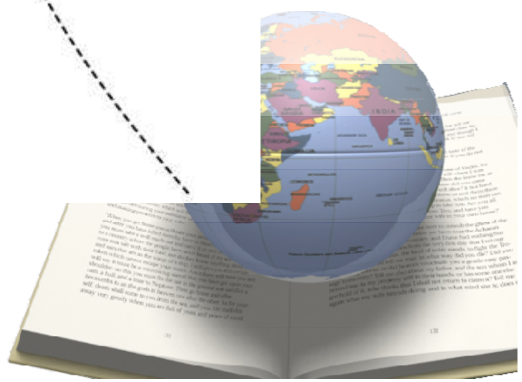
10- HYPERBOLICTRAJECTORIES (e >1)

✶ Finally: $$\Delta = a\sqrt{e^2 - 1} \qquad (13)$$

✶ Comparing this result with equation 12, it is clear that the aiming radius equals the length of the seminor axis of the hyperbola.

✶As with the ellipse and the parabola, we can express the polar form of the equation of the hyperbola in a cartesian coordinate system whose origin is in this case midway between the two foci.



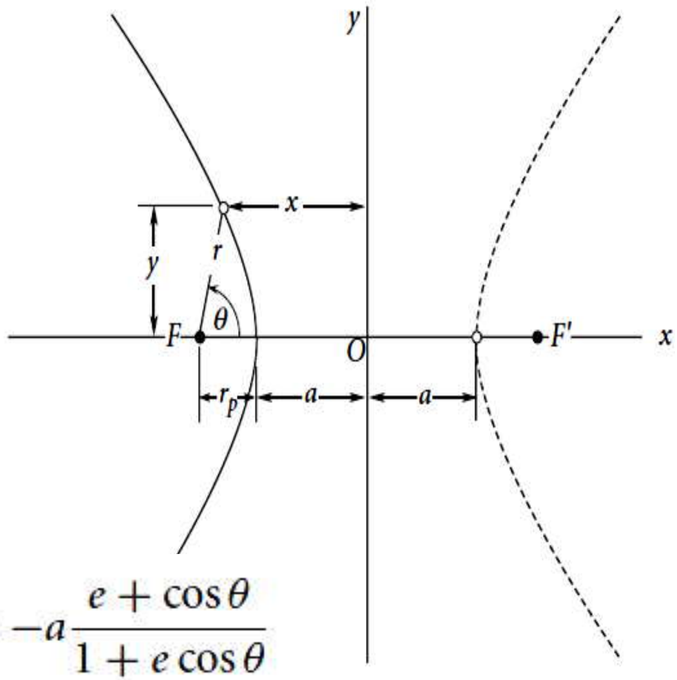


10- HYPERBOLICTRAJECTORIES (e >1)

✶ From the figure it is apparent that:

$$x = -a - r_p + r\cos\theta \qquad (14)$$

$$y = r\sin\theta \qquad (15)$$

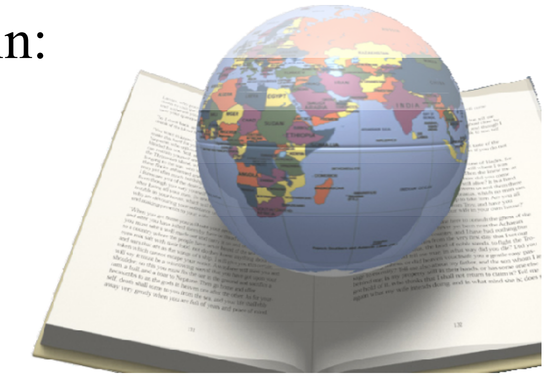


✶ Using equation (9),(10), (14) we obtain:

$$x = -a - a(e-1) + a\frac{e^2-1}{1+e\cos\theta}\cos\theta = -a\frac{e+\cos\theta}{1+e\cos\theta}$$

✶ substituting equation (9) and (12) in (15) we obtain:

$$y = \frac{b}{\sqrt{e^2-1}}\frac{e^2-1}{1+e\cos\theta}\sin\theta = b\frac{\sqrt{e^2-1}\sin\theta}{1+e\cos\theta}$$



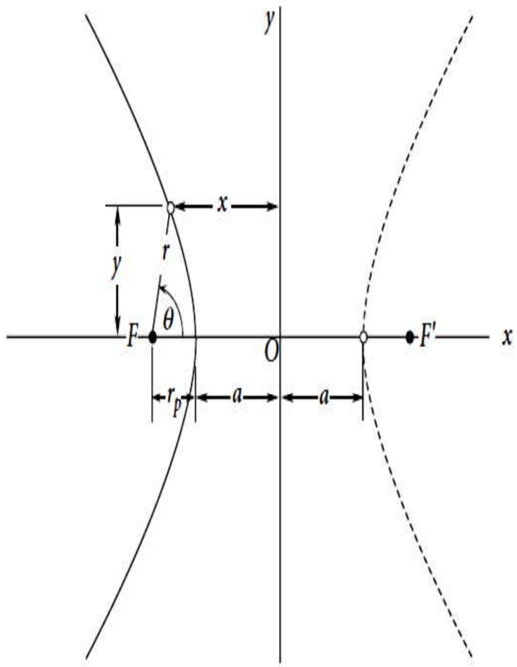
10- HYPERBOLICTRAJECTORIES (e >1)

✶ It follows that:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \left(\frac{e + \cos\theta}{1 + e\cos\theta}\right)^2 - \left(\frac{\sqrt{e^2 - 1}\sin\theta}{1 + e\cos\theta}\right)^2$$

$$= \frac{e^2 + 2e\cos\theta + \cos^2\theta - (e^2 - 1)(1 - \cos^2\theta)}{(1 + e\cos\theta)^2}$$

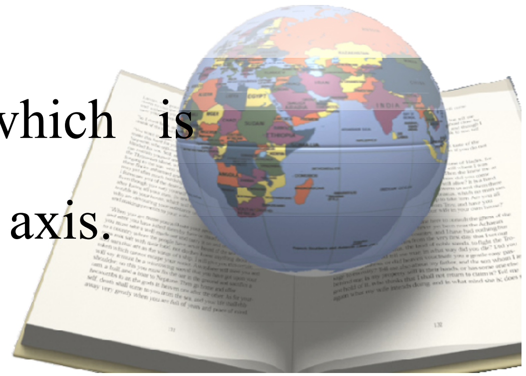
$$= \frac{1 + 2e\cos\theta + e^2\cos^2\theta}{(1 + e\cos\theta)^2} = \frac{(1 + e\cos\theta)^2}{(1 + e\cos\theta)^2}$$



✶ That is,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (16)$$

✶ this is the familiar equation of hyperbola which is symmetric about x and y exes, with intercept on the x axis.



10- HYPERBOLICTRAJECTORIES (e >1)

- The specific energy of the hyperbolic trajectory is:

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}(1 - e^2)$$

$$a = \frac{h^2}{\mu}\frac{1}{e^2 - 1}$$

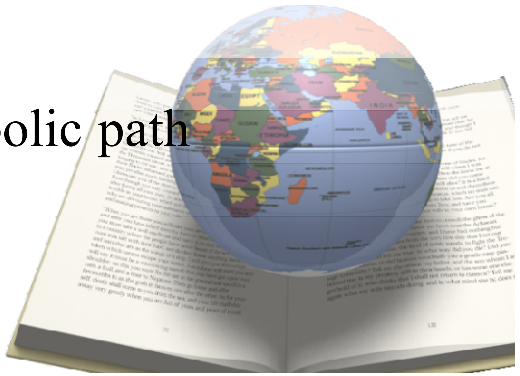
$$\longrightarrow \quad \varepsilon = \frac{\mu}{2a} \qquad (17)$$

- The specific energy of a hyperbolic orbit is clearly positive and independent of the eccentricity.
- The conservation of energy for a hyperbolic trajectory is:

$$\frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a} \qquad (18)$$

- Let v_∞ denote the speed at which a body on a hyperbolic path arrives at infinity so:

$$(18) \longrightarrow v_\infty = \sqrt{\frac{\mu}{a}} \qquad (19)$$



10- HYPERBOLICTRAJECTORIES (e >1)

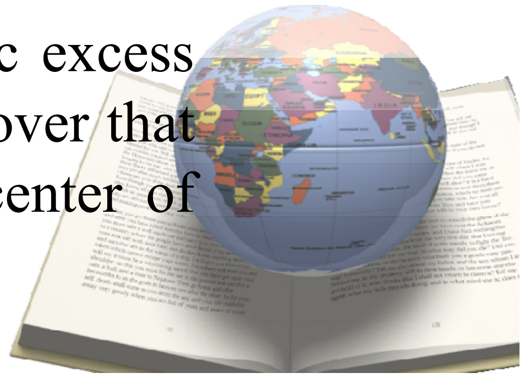
- In terms of v_∞ we may write equation (18) as:

$$\frac{v^2}{2} - \frac{\mu}{r} = \frac{v_\infty^2}{2}$$

- v_∞ is called the hyperbolic excess speed.
- Substituting the expression for escape speed, we obtain for a hyperbolic trajectory

$$v^2 = v_{esc}^2 + v_\infty^2 \qquad (19)$$

- This equation clearly shows that the hyperbolic excess speed v_∞ represent the excess kinetic energy over that which is required to simply escape from the center of attraction.



10- HYPERBOLICTRAJECTORIES (e >1)

- The square of v_∞ is denoted C_3 , and is known as the characteristic energy

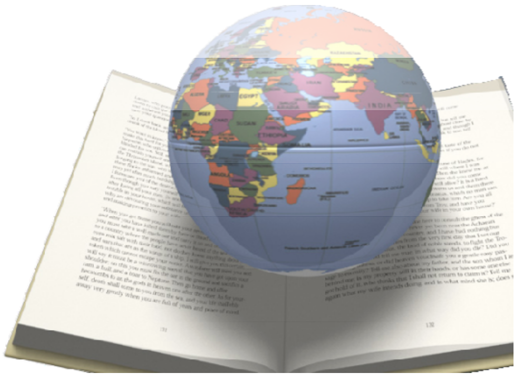
$$C_3 = v_\infty^2 \qquad (20)$$

- C_3 is a measure of the energy required for an interplanetary mission and C_3 is also a measure of maximum energy a launch vehicle can import to a spacecraft of a given mass

$$C_3)_{\text{launchvehicle}} > C_3)_{\text{mission}}$$

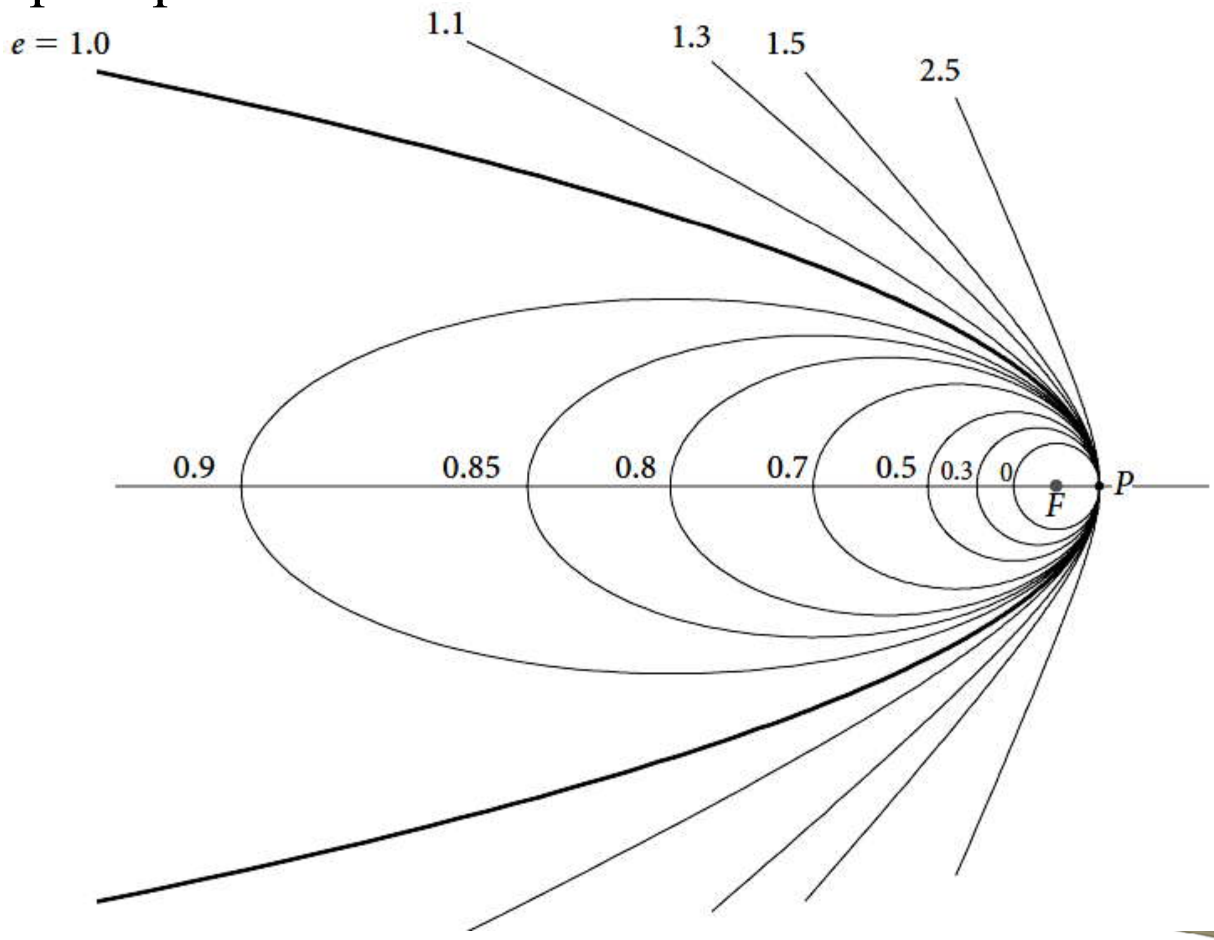
- v_∞ can be find also:

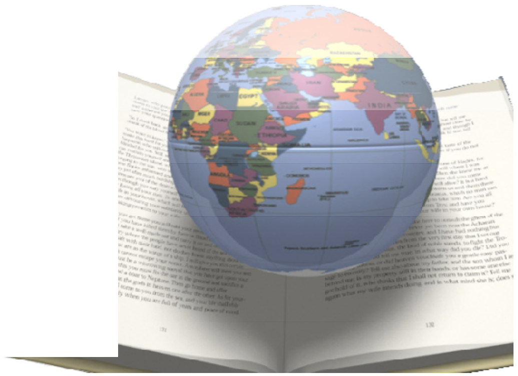
$$v_\infty = \frac{\mu}{h} e \sin\theta_\infty = \frac{\mu}{h}\sqrt{e^2 - 1} \qquad (21)$$



10- HYPERBOLICTRAJECTORIES (e >1)

- The figure shows a range of trajectories, from a circle through hyperbolas, all having common focus and periapsis





10- HYPERBOLICTRAJECTORIES (e >1)

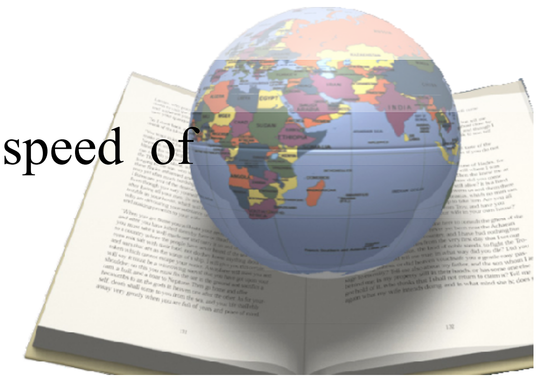
EXAMPLE ?.1

- At given point of a spacecraft's geocentric trajectory, the radius is 14600km, the speed is 8.6km/s, and the flight path angle is $50°$. Show that the path is a hyperbola and calculate the following: (a) C_3, (b) angular momentum, © true anomaly, (d) eccentricity, (e) radius of perigee, (f) turn angle, (g) semimajor axis, and (h) aiming radius.

  to determine the type of the trajectory, calculate the escape speed at the given radius.

$$v_{esc} = \sqrt{\frac{2\mu}{r}} = \sqrt{\frac{2 \cdot 398\,600}{14\,600}} = 7.389\,\text{km/s}$$

Since the escape speed is less than the spacecraft's speed of 8.6km/s, the path is a hyperbola.



10- HYPERBOLICTRAJECTORIES (e >1)

EXAMPLE ?.1

(a) the hyperbolic excess velocity v_∞ is found from equation (19),

$$v_\infty^2 = v^2 - v_{esc}^2 = 8.6^2 - 7.389^2 = 19.36\,\text{km}^2/\text{s}^2$$

From equation (20) it follows that

$$\underline{C_3 = 19.36\,\text{km}^2/\text{s}^2}$$

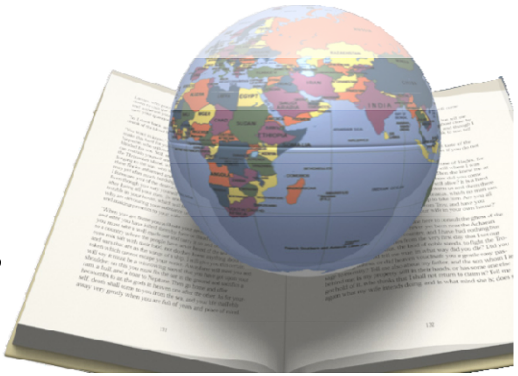
(b) Knowing the speed and the flight path angle, we can obtain both v_r and $v_\perp$:

$$v_r = v\sin\gamma = 8.6\sin 50° = 6.588\,\text{km/s} \qquad \text{(a)}$$

$$v_\perp = v\cos\gamma = 8.6\cdot\cos 50° = 5.528\,\text{km/s} \qquad \text{(b)}$$

Then equation * provides us with the angular momentum,

$$h = rv_\perp = 14\,600\cdot 5.528 = \underline{80\,710\,\text{km}^2/\text{s}} \qquad \text{(c)}$$



10- HYPERBOLICTRAJECTORIES (e >1)

EXAMPLE ?.1

(c) Evaluating the orbit equation at the given location on the trajectory, we get

$$14\,600 = \frac{80\,710^2}{398\,600}\frac{1}{1 + e\cos\theta}$$

From which

$$e\cos\theta = 0.1193 \qquad \text{(d)}$$

The radial component of velocity is given by equation $v_r = \frac{\mu}{h} e\sin\theta$, $v_r = \mu e \sin\theta / h,$ so that with (a) and (c), we obtain

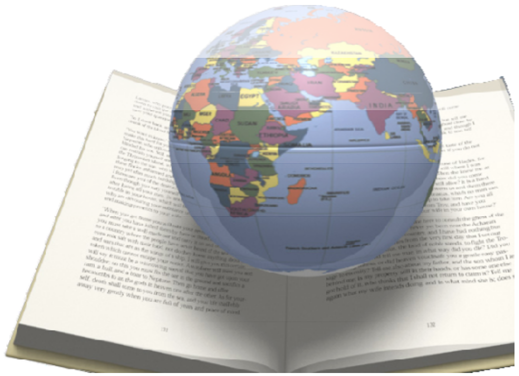
$$6.588 = \frac{398\,600}{80\,170} e\sin\theta$$

or

$$e\sin\theta = 1.334 \qquad \text{(e)}$$

Computing the ratio of (e) to (d) yields

$$\tan\theta = \frac{1.334}{0.1193} = 11.18 \quad \Rightarrow \quad \underline{\theta = 84.89^\circ}$$



10- HYPERBOLICTRAJECTORIES ($e > 1$)

EXAMPLE ?.1

(d) We substitute the true anomaly back into either (d) or (e) to find the eccentricity,

$$\underline{e = 1.339}$$

(e) The radius of perigee can now be found from the orbit equation,

$$r_p = \frac{h^2}{\mu}\frac{1}{1 + e\cos(0)} = \frac{80\,710^2}{398\,600}\frac{1}{1 + 1.339} = \underline{6986\,\text{km}}$$

(f) The formula for turn angle is equation $\delta = 2\sin^{-1}(1/e)$, from which

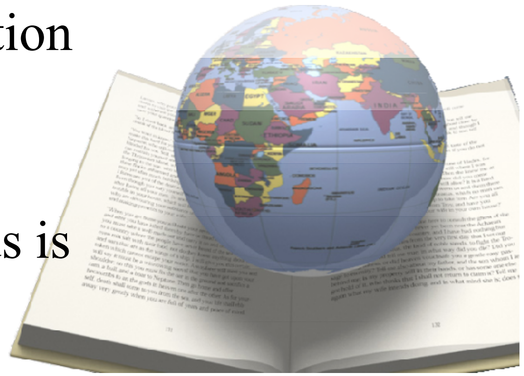
$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right) = 2\sin^{-1}\left(\frac{1}{1.339}\right) = \underline{96.60^\circ}$$

(g) The semimajor axis of the hyperbola is found in equation

$$a = \frac{h^2}{\mu}\frac{1}{e^2 - 1}$$

(h) According to equation $b = a\sqrt{e^2 - 1}$, the aiming radius is

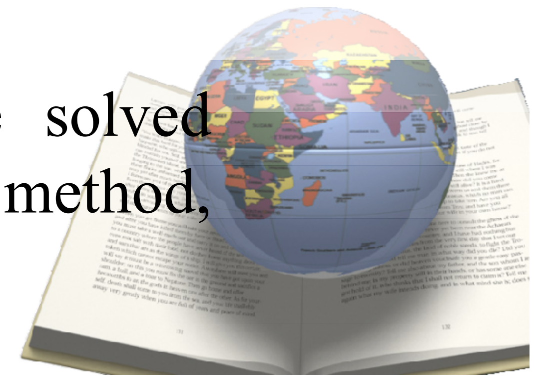
$$\Delta = a\sqrt{e^2 - 1} = 20\,590\sqrt{1.339^2 - 1} = \underline{18\,340\,\text{km}}$$



11- ORBITAL POSITION AS A FUNCTION OF TIME

11- ORBITAL POSITION AS A FUNCTION OF TIME

- In preview chapter we found the relationship between position and true anomaly for the two-body problem.
- The only place time appeared explicitly was in the expression for the period of an ellipse.
- Obtaining position as a function of time is a simple matter for circular orbits.
- For elliptical, parabolic and hyperbolic paths we are led to the various forms of Kepler’s equation relating position to time.
- These transcendental equations must be solved iteratively using a procedure like Newton’s method, which is presented in this chapter.



11- ORBITAL POSITION AS A FUNCTION OF TIME

11.1 Time Since Periapsis

- The orbit formula, gives the position of body m_2 in its orbit around m_1 as a function of the true anomaly

$$r = (h^2/\mu)/(1 + e\cos\theta)$$

- For many practical reasons we need to be able to determine the position of m_2 as a function of time.
- For elliptical orbits we have formula for the period T:

$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1-e^2}}\right)^3$$

- But we cannot yet calculate the time required to fly between any two anomalies. The purpose of this section is to come up with formulas that allow us to do that calculation

11- ORBITAL POSITION AS A FUNCTION OF TIME

- The one equation which relates true anomaly directly to time is:

$$h = r^2\dot{\theta} \longrightarrow \frac{d\theta}{dt} = \frac{h}{r^2}$$

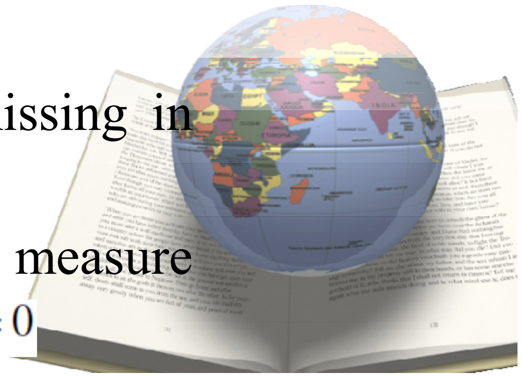
- Substituting r from orbit formula, after separating variables we find:

$$\frac{\mu^2}{h^3}dt = \frac{d\theta}{(1 + e\cos\theta)^2}$$

- Integrating both sides of this equation yields:

$$\frac{\mu^2}{h^3}(t - t_p) = \int_0^{\theta} \frac{d\vartheta}{(1 + e\cos\vartheta)^2} \qquad (1)$$

- t_p : time at periapse passage ($\theta = 0$)
- t_p is the sixth constant of the motion that was missing in previous chapter.
- The origin of time is arbitrary. It is convenient to measure time from periapse passage so we will usually set $t_p = 0$



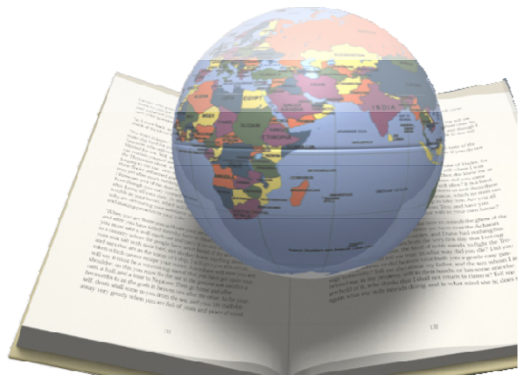
11- ORBITAL POSITION AS A FUNCTION OF TIME

✶ If $t_p = 0$, in that case we have

$$\frac{\mu^2}{h^3} t = \int_0^{\theta} \frac{d\vartheta}{(1 + e\cos\vartheta)^2} \qquad (2)$$

✶ The integral on the right maybe found in any standard mathematical handbook.

✶ the specific form of the integral depends on whether the value of the eccentricity e corresponds to a circle, ellipse, parabola or hyperbola



11- ORBITAL POSITION AS A FUNCTION OF TIME

11.2 Circular Orbits

- For a circle, $e = 0$ so the integral in Equation (3) is simply:

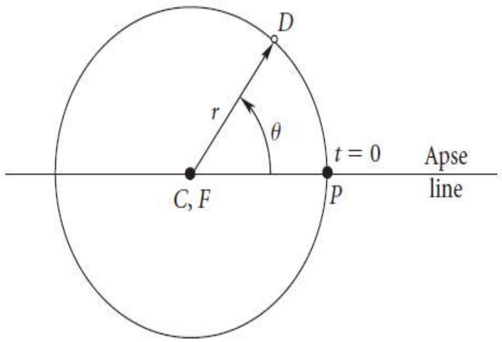
$$\int_0^\theta d\vartheta. \longrightarrow t = \frac{h^3}{\mu^2}\theta$$

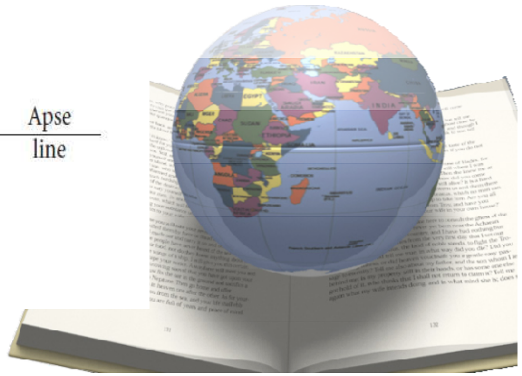
- Recall that for a circle:

$$r = h^2/\mu \longrightarrow h^3 = r^{\frac{3}{2}}\mu^{\frac{3}{2}} \longrightarrow t = \frac{r^{\frac{3}{2}}}{\sqrt{\mu}}\theta$$

- Substituting the formula for the period T of a circular orbit yields: (NOTE 18 P(109),{1})

$$\theta = \frac{2\pi}{T}t$$





11- ORBITAL POSITION AS A FUNCTION OF TIME

11.3 Elliptical Orbits

- For 0<e<1, we find in integral tables that

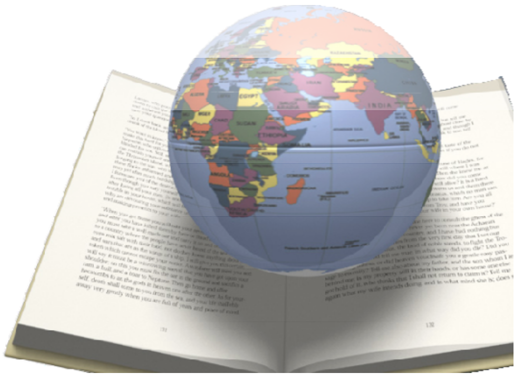
$$\int_0^{\theta} \frac{d\vartheta}{(1+e\cos\vartheta)^2} = \frac{1}{(1-e^2)^{\frac{3}{2}}}\left[2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}\right]$$

- Therefore, Equation (2) in this case becomes:

$$\frac{\mu^2}{h^3}t = \frac{1}{(1-e^2)^{\frac{3}{2}}}\left[2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}\right]$$

- Or

$$M_e = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta} \qquad (3)$$

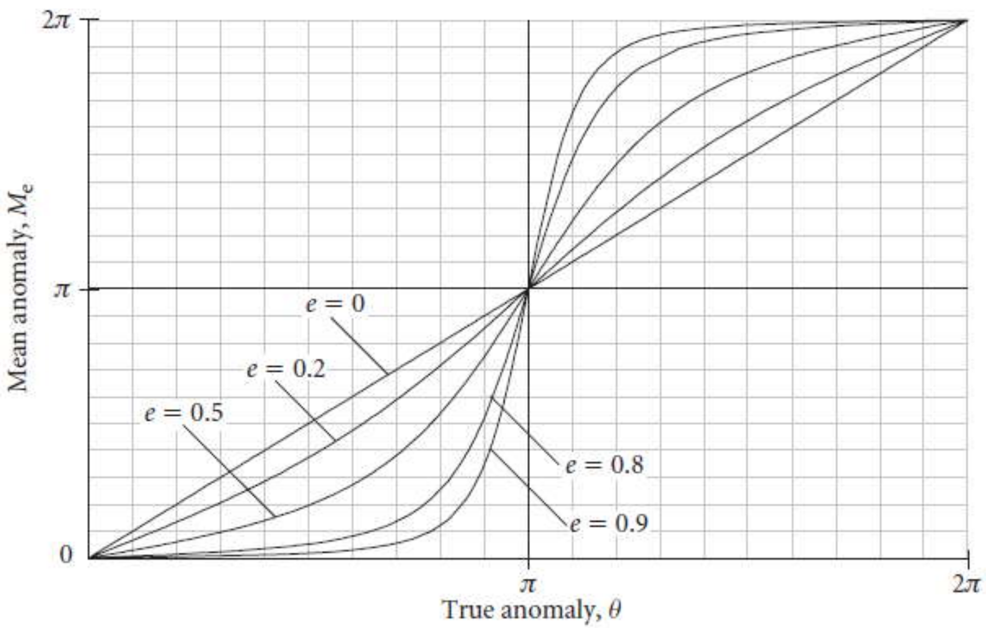


11- ORBITAL POSITION AS A FUNCTION OF TIME

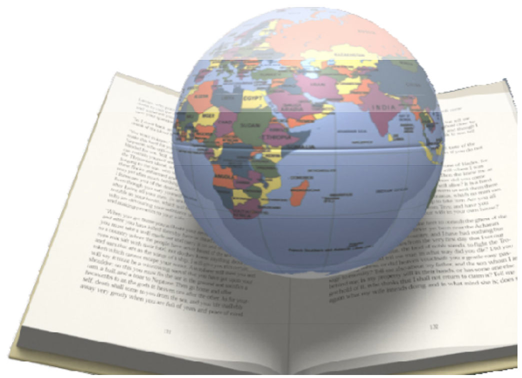
- In equation (3), M_e is: called the mean anomaly:

$$M_e = \frac{\mu^2}{h^3}(1 - e^2)^{\frac{3}{2}} t \qquad (4)$$

- Equation (3) in plotted in the below figure:



- (NOTE 19 PAGE 110, {1})



11- ORBITAL POSITION AS A FUNCTION OF TIME

- From the formula for the period T of an elliptical orbit we have:

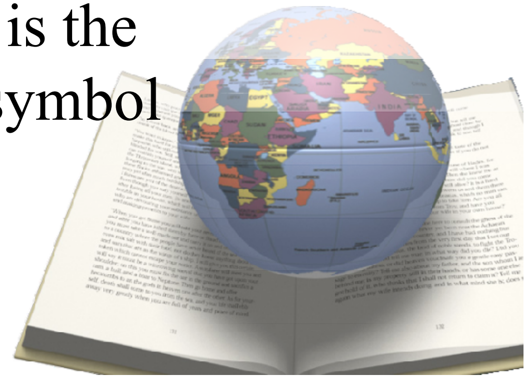
$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1-e^2}}\right)^3 \longrightarrow \mu^2(1-e^2)^{\frac{3}{2}}/h^3 = 2\pi/T$$

- So that the mean anomaly can be written much more simply as:

$$M_e = \frac{2\pi}{T}t \qquad (5)$$

- The angular velocity of the position vector of an elliptical orbit is not constant, but since 2π radians are swept out per period T, the ratio $2\pi/T$ is the averagee angular velocity which is given the symbol n and called the mean motion.

$$n = \frac{2\pi}{T} \qquad (6)$$

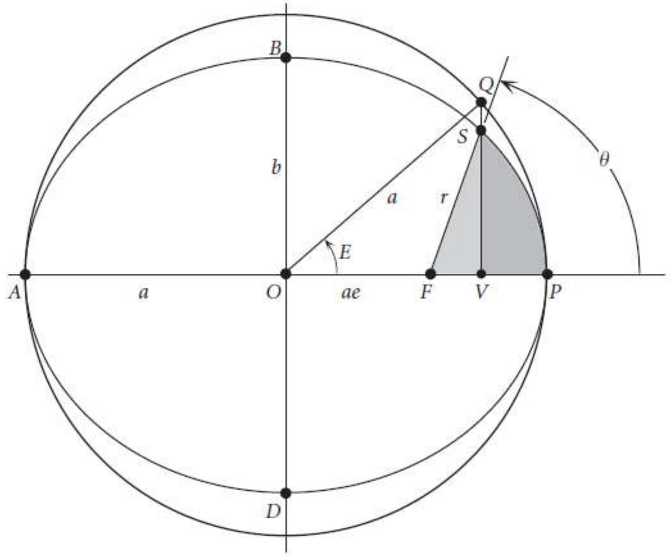


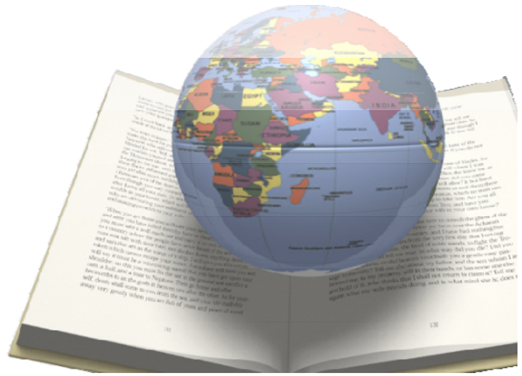
11- ORBITAL POSITION AS A FUNCTION OF TIME

- In terms of the mean motion, Equation (5) can be written simpler still:

$$M_e = nt$$

- (NOTE 20 P 111, {1})
- It is convenient to simplify Equation (3), by introducing an auxiliary angle E called the eccentric anomaly. (NOTE 21, P 111, {1})





11- ORBITAL POSITION AS A FUNCTION OF TIME

✶ To find E as a function of θ , we first observe from previous figure that:

$$\overline{OV} = a\cos E$$

$$\overline{OV} = ae + r\cos\theta$$

✶ Thus:

$$a\cos E = ae + r\cos\theta$$

✶ Using Equation:

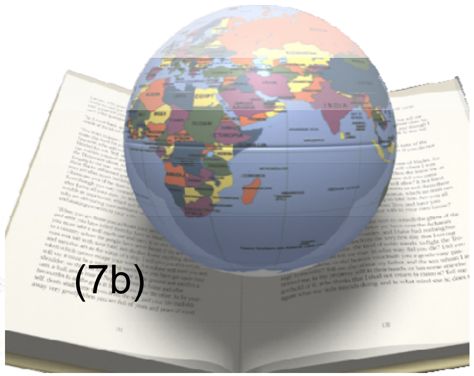
$$r = a(1-e^2)/(1+e\cos\theta)$$

✶ We can write this as:

$$a\cos E = ae + \frac{a(1-e^2)\cos\theta}{1+e\cos\theta}$$

✶ Simplifying the right-hand side, we get

$$\cos E = \frac{e+\cos\theta}{1+e\cos\theta} \quad (7a) \longrightarrow \quad \cos\theta = \frac{e-\cos E}{e\cos E - 1} \quad (7b)$$

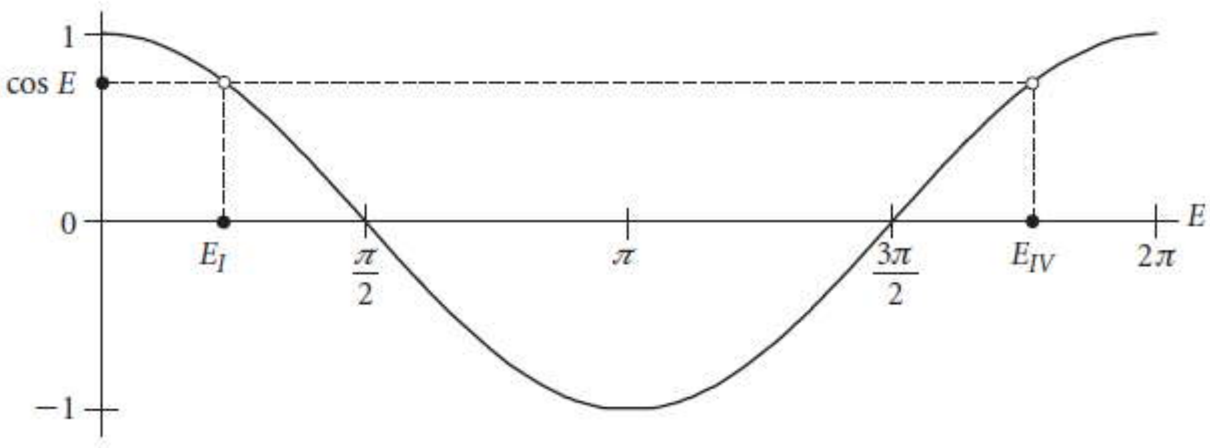


11- ORBITAL POSITION AS A FUNCTION OF TIME

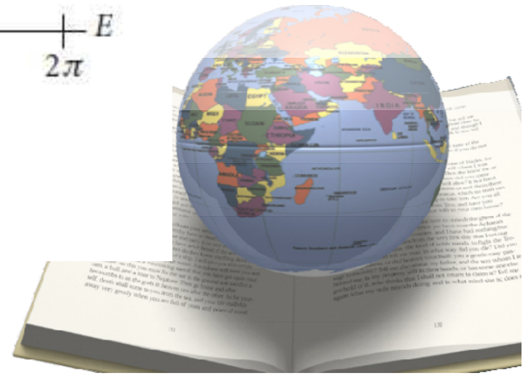
- ✶ Substituting Equation(7a) into the trigonometric identity($\sin^2 E + \cos^2 E = 1$) and solving for $\sin E$
- ✶ yields:

$$\sin E = \frac{\sqrt{1-e^2}\sin\theta}{1+e\cos\theta} \qquad (8)$$

- ✶ Equation (7a) would be find for obtaining E from θ except that, given a value of $\cos E$ between -1 and 1, there are two values of E between $0°$ and $360°$



- ✶ The same comments hold for Equation(8)



11- ORBITAL POSITION AS A FUNCTION OF TIME

- To resolve this quadrant ambiguity, we use the following trigonometric identitiy:

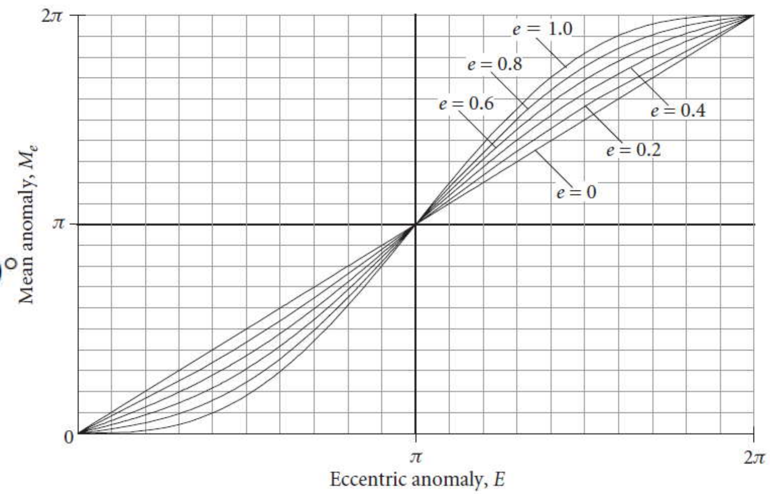
$$\tan^2\frac{E}{2} = \frac{1-\cos E}{1+\cos E} \qquad (9)$$

- By the use of Equation's (9) and (7a), we obtain:

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2} \qquad (10a)$$

- Or $$E = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) \qquad (10b)$$

- Observe from the above figure that for any value of $\tan(E/2)$, there is only one value of E between $0°$ and $360°$ there is no quadrant ambiguity.

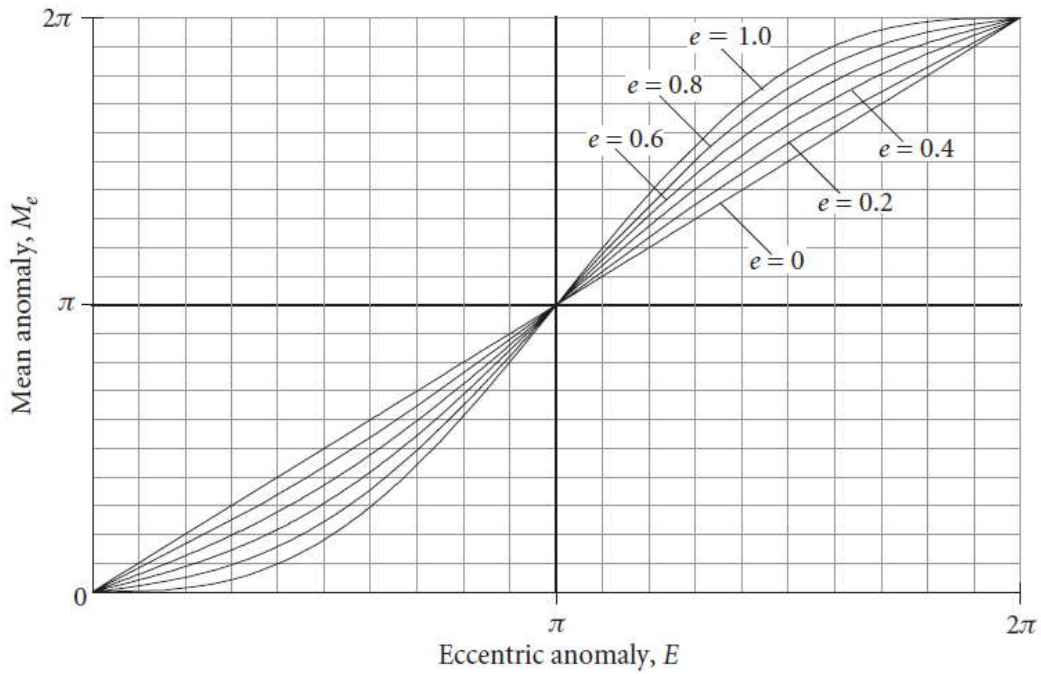


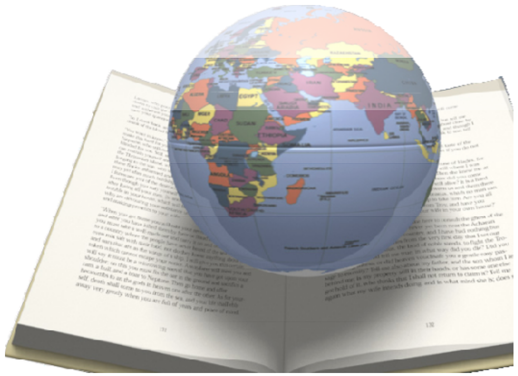
11- ORBITAL POSITION AS A FUNCTION OF TIME

- ✶ Substituting Equation(8) and (10b) into Equation (3) yield's Kepler's equation:

$$M_e = E - e \sin E \quad (11)$$

- ✶ This monotonically increasing relationship between mean anomaly and eccentric anomaly is plotted for several values of eccentricity.





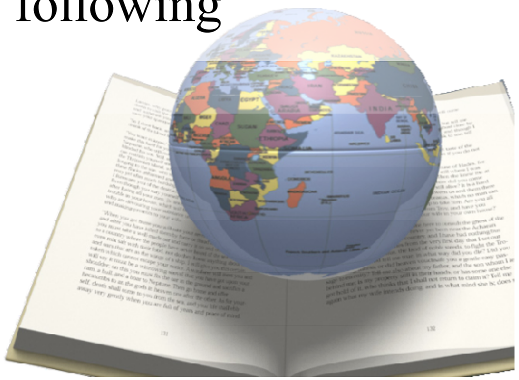
11- ORBITAL POSITION AS A FUNCTION OF TIME

- Given the true anomaly θ, we calculate the eccentric anomaly E using Equations(10)
- Substituting E into Kepler's formula "Equ.(11)" yields the mean anomaly directly.
- From the mean anomaly and the period T we find the time (since periapsis) from Equ:

$$t = \frac{M_e}{2\pi} T \qquad (12)$$

- On the other hand, if we are given the time, then Equation 12 yields the mean anomaly M_e
- Substituting M_e into Kepler's equation we get the following expression for the eccentric anomaly.

$$E - e \sin E = M_e$$



11- ORBITAL POSITION AS A FUNCTION OF TIME

- We cannot solve this transcendental equation directly for E. (A rough value of E might be read of previous figure)
- However, an accurate solution requires an iterative, “trial and error” procedure.
- Newton’s method, or one of its variants, is one of the more common and efficient ways of finding the root of a well-behaved function.

  (NOTE21,P114,{1})
- To apply Newton’s method to the solution of Kepler’s equation, we form the function,

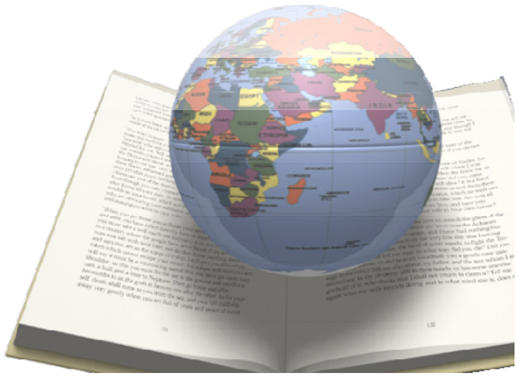
$$f(E) = E - e\sin E - M_e$$

And seek the value of eccentric anomaly that makes

$f(E) = 0$. Since $f'(E) = 1 - e\cos E$

- For this problem we have:

$$E_{i+1} = E_i - \frac{E_i - e\sin E_i - M_e}{1 - e\cos E_i} \tag{13}$$



Chapter 12

Parabolic trajectories

12- PARABOLIC TRAJECTORIES

⋆ For the parabola (e=1) Equation:

$$\frac{\mu^2}{h^3}t = \int_0^\theta \frac{d\vartheta}{(1+e\cos\vartheta)^2}$$

⋆ becomes:

$$\frac{\mu^2}{h^3}t = \int_0^\theta \frac{d\vartheta}{(1+\cos\vartheta)^2} \qquad (1)$$

⋆ In integral tables we find that:

$$\int_0^\theta \frac{d\vartheta}{(1+\cos\vartheta)^2} = \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2}$$

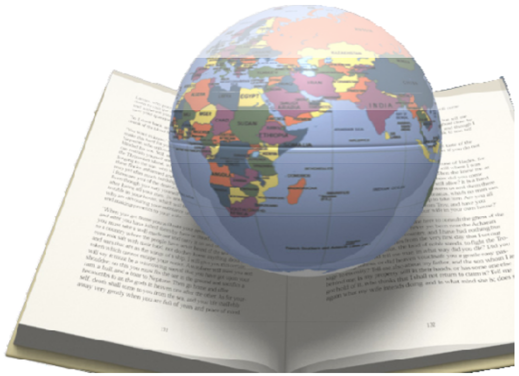
⋆ Therefore equation (1) may be written as:

(Barker's Equation)

$$M_p = \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2} \qquad (2)$$

⋆ Where

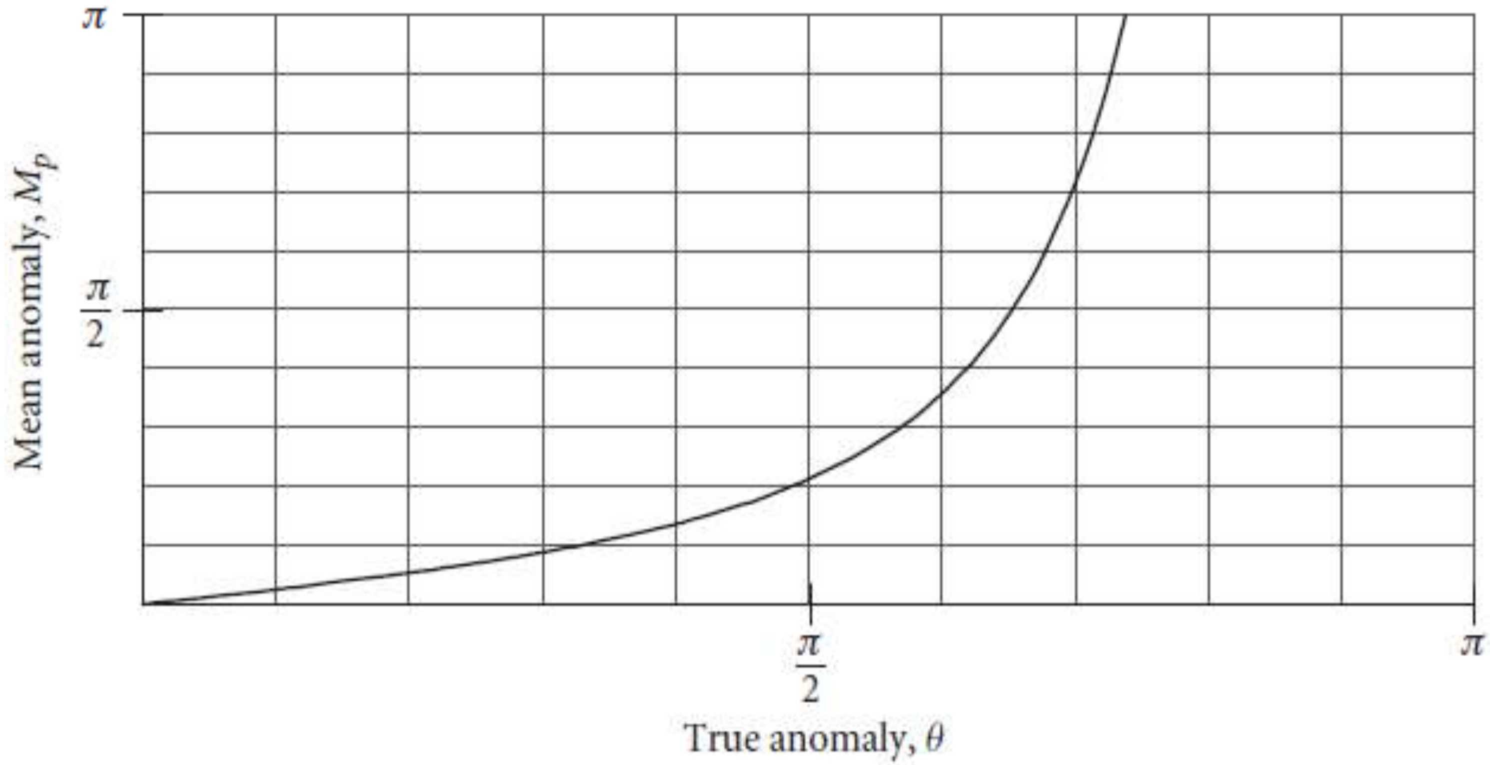
$$M_p = \frac{\mu^2 t}{h^3} \qquad (3)$$



12- PARABOLIC TRAJECTORIES

$$M_p = \frac{\mu^2 t}{h^3} \qquad (3)$$

- ✶ M_p Is dimensionless, and it may be thought of as the “mean anomaly” for the parabola.



12- PARABOLIC TRAJECTORIES

- Given the anomaly θ, we find the time directly from Equations (3) , (2).

$$M_p = \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2}$$

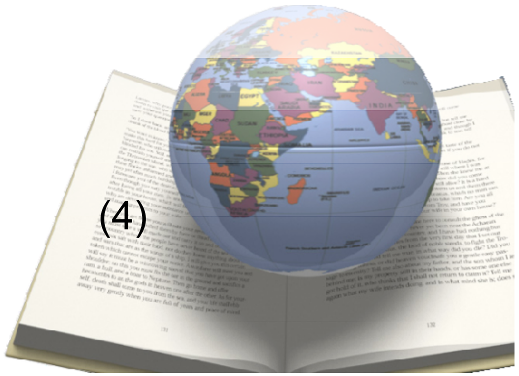
$$M_p = \frac{\mu^2 t}{h^3}$$

- If time is the given variable, then we must solve the cubic equation:

$$\frac{1}{6}\left(\tan\frac{\theta}{2}\right)^3 + \frac{1}{2}\tan\frac{\theta}{2} - M_p = 0$$

- Which has but one real root, namely:

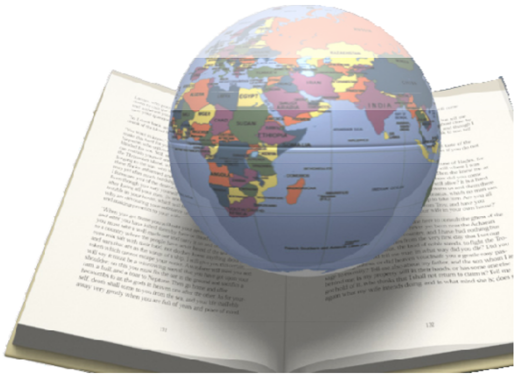
$$\tan\frac{\theta}{2} = \left[3M_p + \sqrt{(3M_p)^2 + 1}\right]^{\frac{1}{3}} - \left[(3M_p + \sqrt{(3M_p)^2 + 1})\right]^{-\frac{1}{3}} \quad (4)$$



12- PARABOLIC TRAJECTORIES

EXAMPLE 12.1

✶ A geocentric parabola has a perigee velocity of $10\,\text{km/s}$. How far is the satellite from the center of the earth six hours after perigee passage?



12- PARABOLIC TRAJECTORIES

EXAMPLE 12.1

- **Solution:**
- We will find the perigee radius from equation:

$$r_p = \frac{2\mu}{v_p^2} = \frac{2 \cdot 398\,600}{10^2} = 7972\,\text{km}$$

- So that the angular momentum is

$$h = r_p v_p = 7972 \cdot 10 = 79\,720\,\text{km}^2/\text{s}$$

- Now we can calculate the parabolic mean anomaly:

$$M_p = \frac{\mu^2 t}{h^3} = \frac{398\,600^2 \cdot (6 \cdot 3600)}{79\,720^3} = 6.7737\,\text{rad}$$

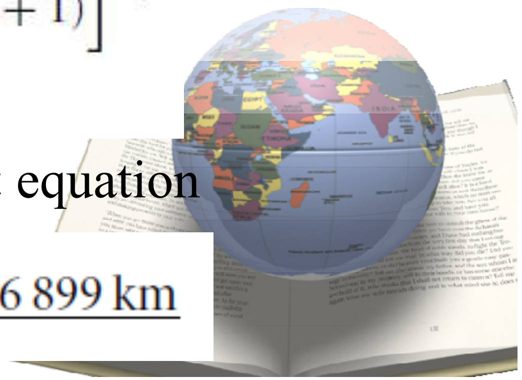
- So that $3M_p = 20.321$ rad. Equation(4) yields the true anomaly:

$$\tan\frac{\theta}{2} = \left[20.321 + \sqrt{20.321^2 + 1}\right]^{\frac{1}{3}} - \left[(20.321 + \sqrt{20.321^2 + 1})\right]^{-\frac{1}{3}}$$

$$= 3.1481 \quad \Rightarrow \quad \theta = 144.75°$$

- Finally, we substitute the true anomaly into the orbit equation to find the radius:

$$r = \frac{79\,720^2}{398\,600}\frac{1}{1 + \cos(144.75°)} = \underline{86\,899\,\text{km}}$$



12- HYPERBOLIC TRAJECTORIES

✶For the hyperbola (e>1) the Equation:

$$\frac{\mu^2}{h^3}(t - t_p) = \int_0^\theta \frac{d\vartheta}{(1 + e\cos\vartheta)^2}$$

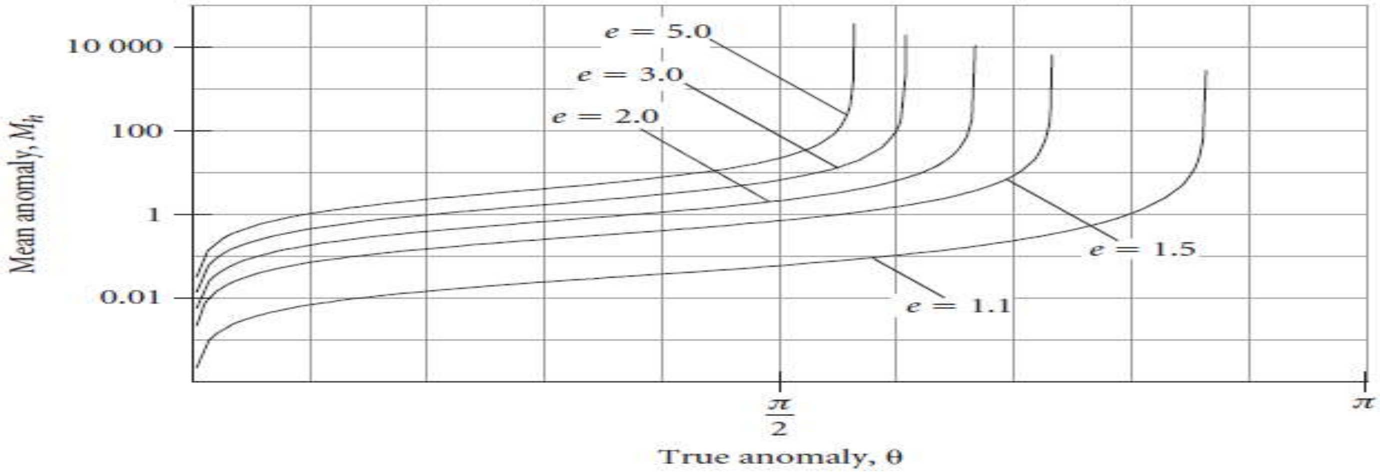
✶After some substitutions becomes:

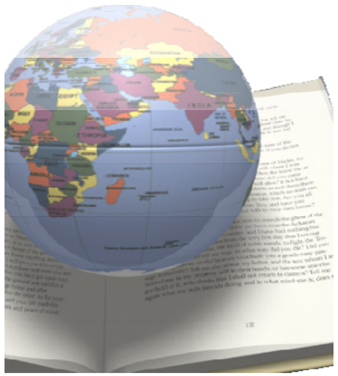
$$M_h = \frac{e\sqrt{e^2-1}\sin\theta}{1+e\cos\theta} - \ln\left(\frac{\sqrt{e+1}+\sqrt{e-1}\tan(\theta/2)}{\sqrt{e+1}-\sqrt{e-1}\tan(\theta/2)}\right) \qquad (1)$$

✶Where, M_h is the hyperbolic mean anomaly:

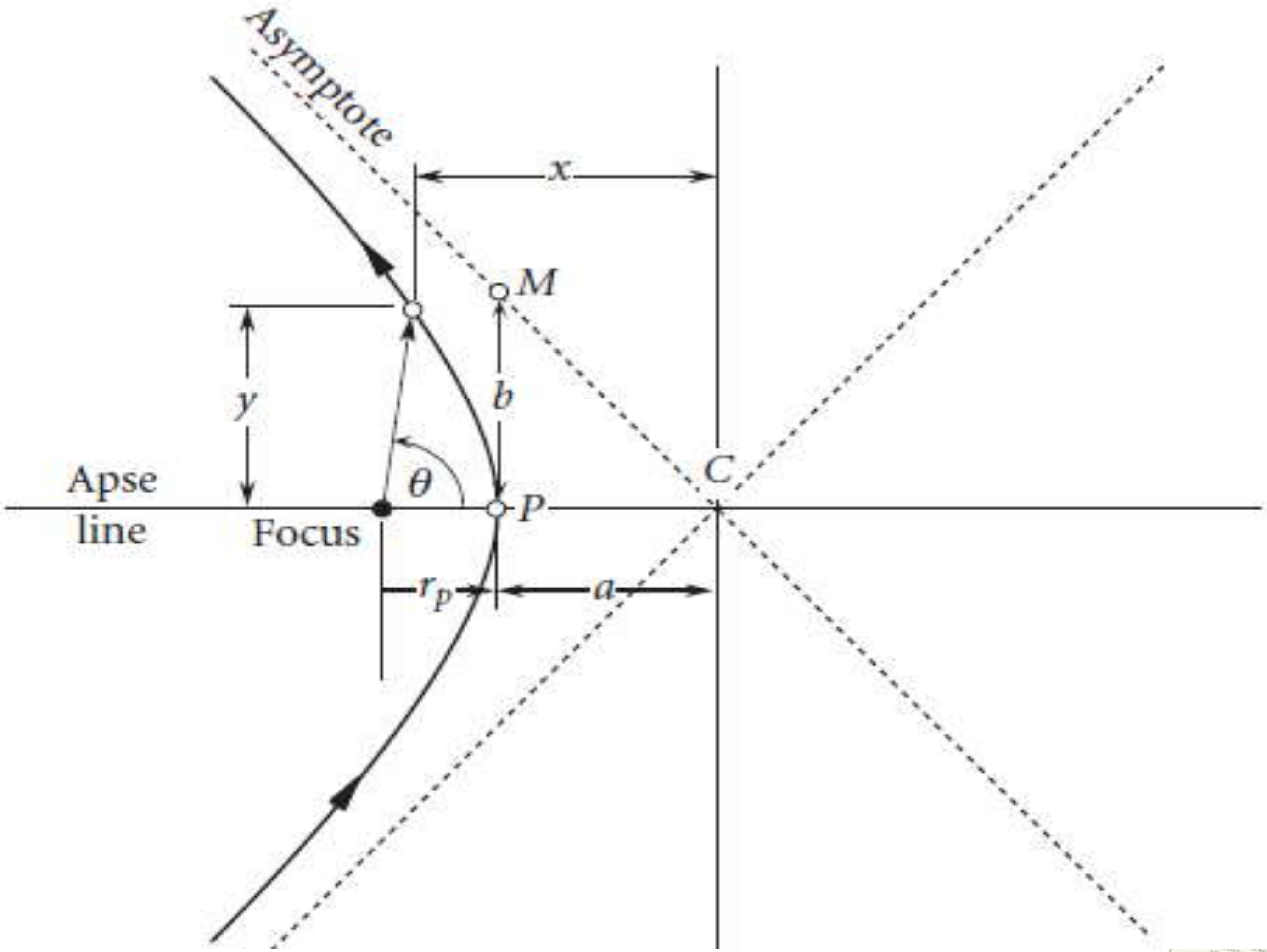
$$M_h = \frac{\mu^2}{h^3}(e^2-1)^{\frac{3}{2}}t \qquad (2)$$

✶Equation (1) is plotted in the below figure

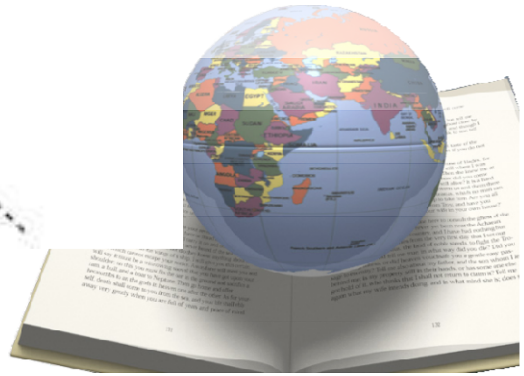




12- HYPERBOLIC TRAJECTORIES



✶(NOTE 22,P126,{1})



12- HYPERBOLIC TRAJECTORIES

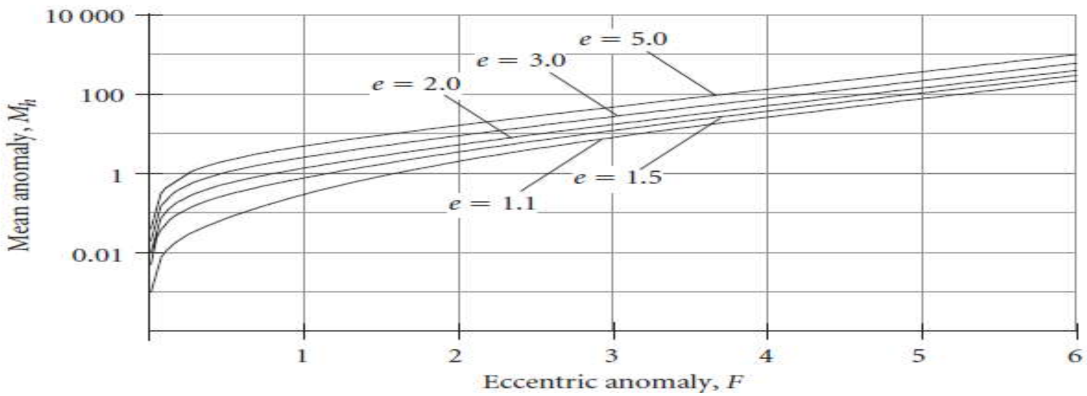
- ✶We define F to be such that: $\sinh F = \frac{y}{b}$ (3)
- ✶It is consistent with the definition of $\sinh F$ to define the hyperbolic cosine as: $\cosh F = \frac{x}{a}$ (4)
- ✶We can prove that:

$$\sinh F = \frac{\sqrt{e^2-1}\sin\theta}{1+e\cos\theta} \qquad (5)$$

$$F = \sinh^{-1}\left(\frac{\sqrt{e^2-1}\sin\theta}{1+e\cos\theta}\right) \qquad (6)$$

$$F = \ln\left[\frac{\sqrt{e+1}+\sqrt{e-1}\tan(\theta/2)}{\sqrt{e+1}-\sqrt{e-1}\tan(\theta/2)}\right] \qquad (7)$$

- ✶Substituting equation(7),(5) into equation(1), yields Kepler's equation for the hyperbola, $M_h = e\sinh F - F$ (8)
- ✶this equation is plotted for several different eccentricities in below figure:



12- HYPERBOLIC TRAJECTORIES

- If time is the given quantity, then equation(8), must be solved for F by an iterative procedure, as was the case for the ellipse
- To apply Newton's procedure to the solution of Kepler's equation for the hyperbola, we from the function:

$$f(F) = e \sinh F - F - M_h$$

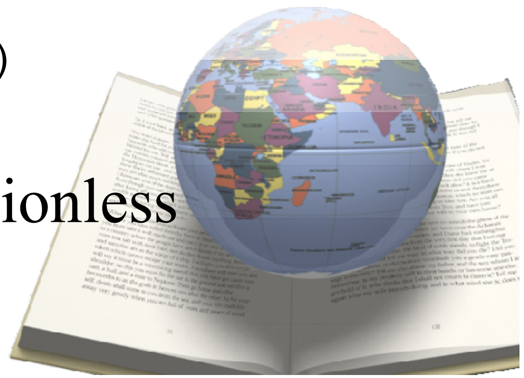
- And seek the value of F that makes *f*(F)=0 since

$$f'(F) = e \cosh F - 1$$

- Equation becomes

$$F_{i+1} = F_i - \frac{e \sinh F_i - F_i - M_h}{e \cosh F_i - 1} \qquad (9)$$

- All quantities in this formula are dimensionless (radians, not degrees).



12- HYPERBOLIC TRAJECTORIES

- When determining orbital position as a function of time with the aid of Kepler's equation, it is convenient to have position r as a function of eccentric anomaly.
- This is obtained by substituting equation:

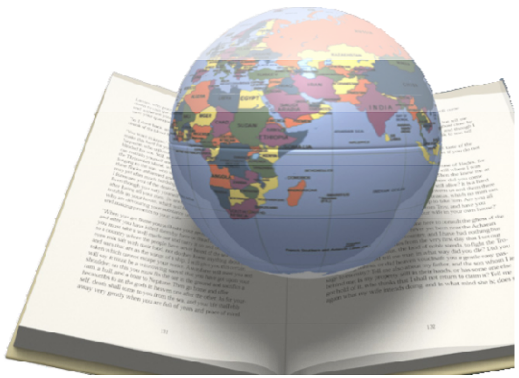
$$\cos\theta = \frac{\cosh F - e}{1 - e\cosh F}$$

- Into equation

$$r = a\frac{e^2 - 1}{1 - e\cosh F}$$

- This reduces to:

$$r = a(e\cosh F - 1) \qquad (10)$$



CHAPTER 13

ORBITAL MANEUVERS

CHAPTER CONTENT

13- ORBITAL MANEUVERS

✶ Orbital maneuvers transfer a spacecraft from one orbit to another.

✶ Orbital changes can be:

- The transfer from law-earth parking orbit to an interplanetary trajectory. (Big maneuver)
- The rendezvous of one spacecraft with another. (Small maneuver)

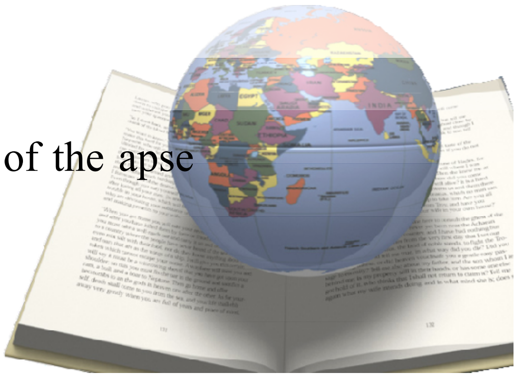
✶ Changing orbits required the firing of onboard spacecrafts engines.

✶ We will use impulsive maneuvers, in which the rockets fire in relatively short bursts to produce the required velocity change Δv

✶ In this chapter we will consider:

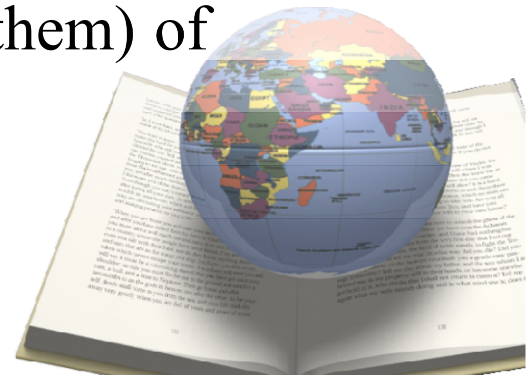
✶ (NOTE23,P255,{1})

- Classical, energy-efficient Hohmann transfer maneuvers.
- The bi-elliptic Hohmann transfer
- The phasing maneuver. (a from of Hohmann transfer)
- The non-Hohmann transfer maneuvers with and without rotation of the apse line
- Chase maneuvers
- Plane charge maneuvers (introduction)



13- ORBITAL MANEUVERS

- impulsive maneuvers are those in which brief firings of onboard rocket motors change the magnitude and direction of the velocity vector instantaneously.
- During an impulsive maneuver, the position of the spacecraft is considered to be fixed; only the velocity changes.
- (NOTE24,P256,{1})
- Each impulsive maneuver result in a change in the velocity (magnitude Δv; pumping maneuver; direction “cranking maneuver”, or both of them) of spacecraft.



13- ORBITAL MANEUVERS

- The magnitude Δv of the velocity increment is related to Δm, the mass of propellant consumed by the formula

$$\frac{\Delta m}{m} = 1 - e^{-\frac{\Delta v}{I_{sp} g_o}} \qquad (1)$$

m : is the mass of the spacecraft before the burn

g_o : is the sea-level acceleration of gravity

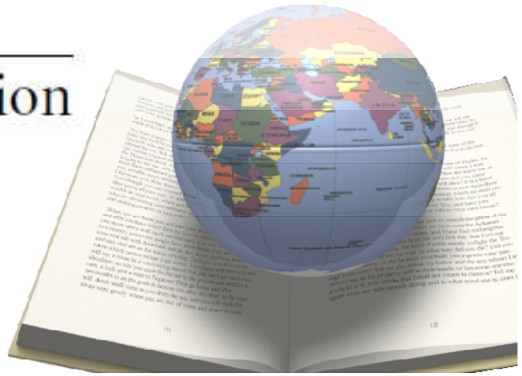
I_{sp} : is the specific impulse of the propellants.

- Specific impulse is defined as follows:

$$I_{sp} = \frac{\text{thrust}}{\text{sea-level weight rate of fuel consumption}}$$

$I_{SP} : [s]$

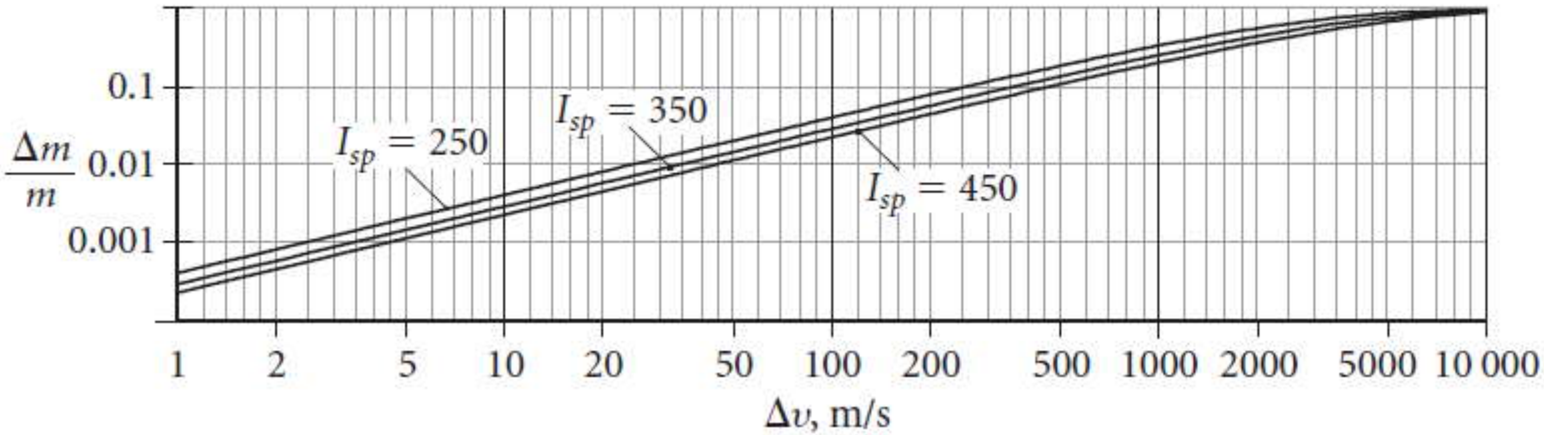
- (NOTE25,P256,{1})



13- ORBITAL MANEUVERS

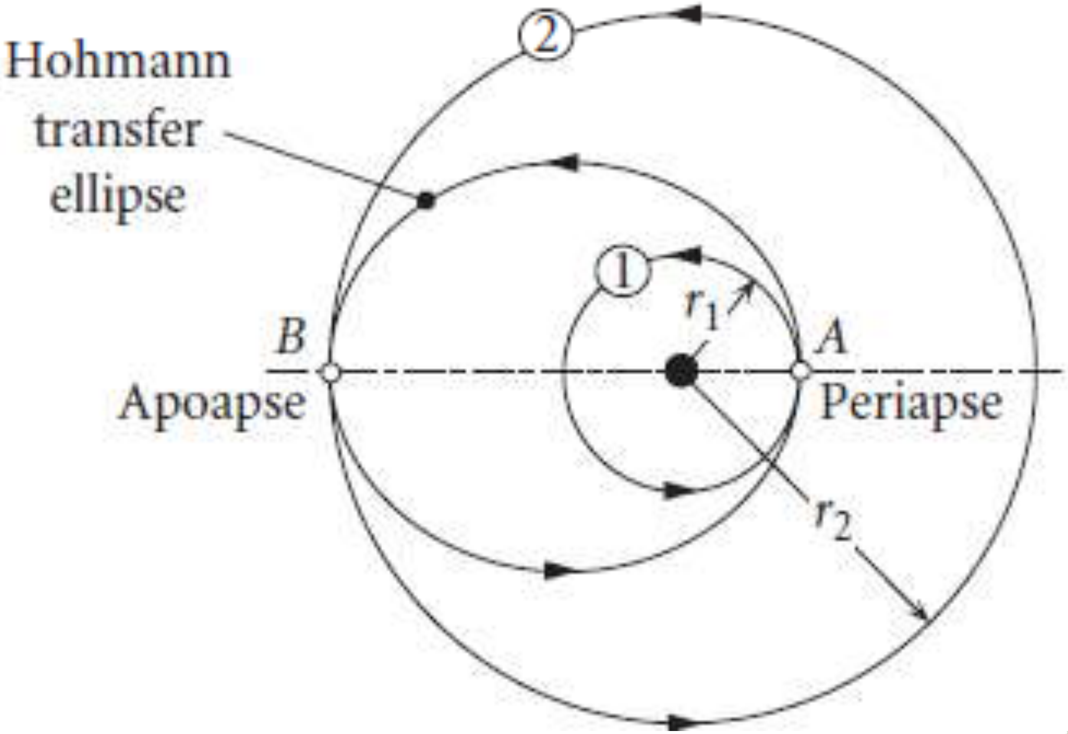
- I_{sp} for some common propellant combinations are shown in below table:
- (NOTE25,P256,{1})

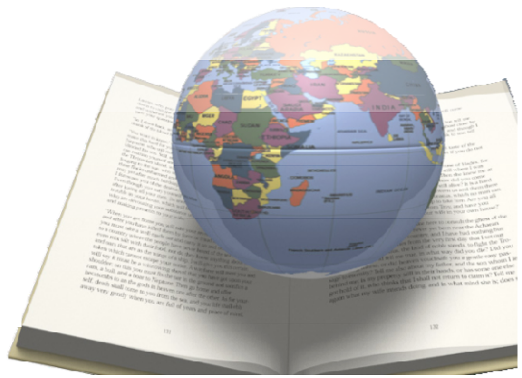
Propellant	**I_{sp} (seconds)**
Cold gas	50
Monopropellant hydrazine	230
Solid propellant	290
Nitric acid/monomethylhydrazine	310
Liquid oxygen/liquid hydrogen	455



13- HOHMANN TRANSFER

- ✶ The Hohmann transfer is the most efficient two-impulse maneuver for transferring between two coplanar circular orbits sharing a common focus.
- ✶ (NOTE26,P257,{1})



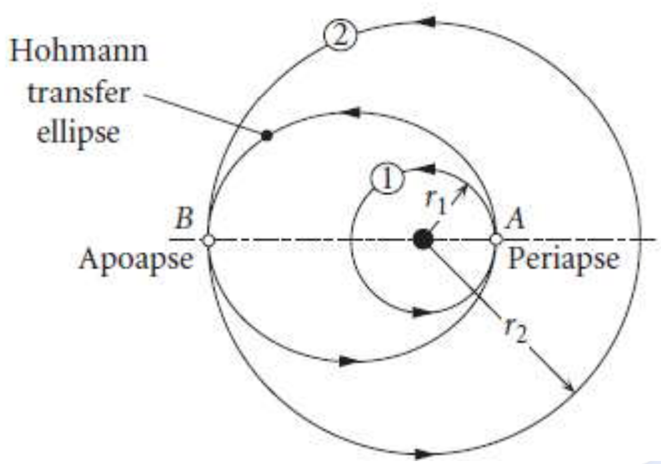


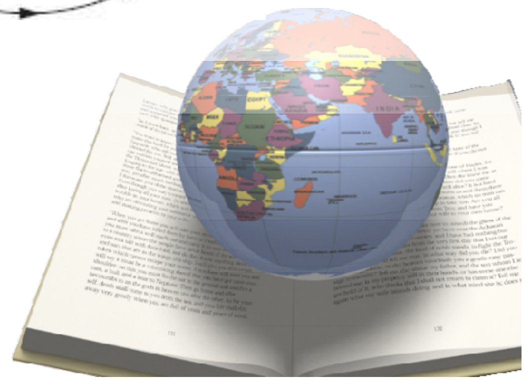
13- HOHMANN TRANSFER

- Recall that for an ellipse the specific energy is negative:

$$\varepsilon = -\frac{\mu}{2a}$$

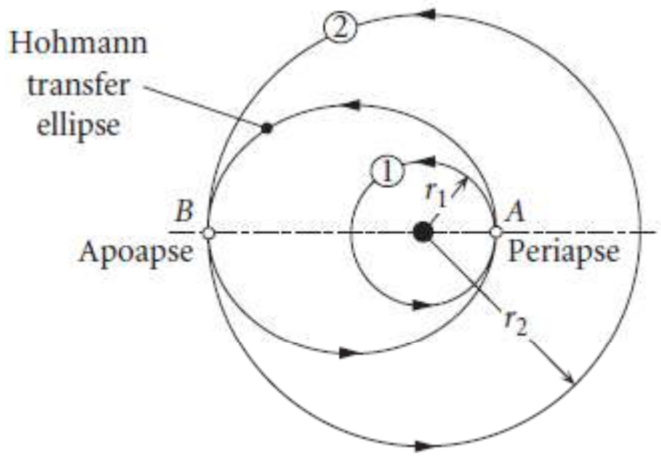
- Increasing the energy requires reducing its magnitude, in order to make ε less negative.
- Therefore, the larger the semimajor axis is, the more the energy the orbit has, the energies increase as we more from the inner to the outer circle.





13- HOHMANN TRANSFER

- Starting at A on the inner circle, a velocity increment Δv_A in the direction of flight is required to boost the vehicle onto the higher-energy elliptical trajectory.

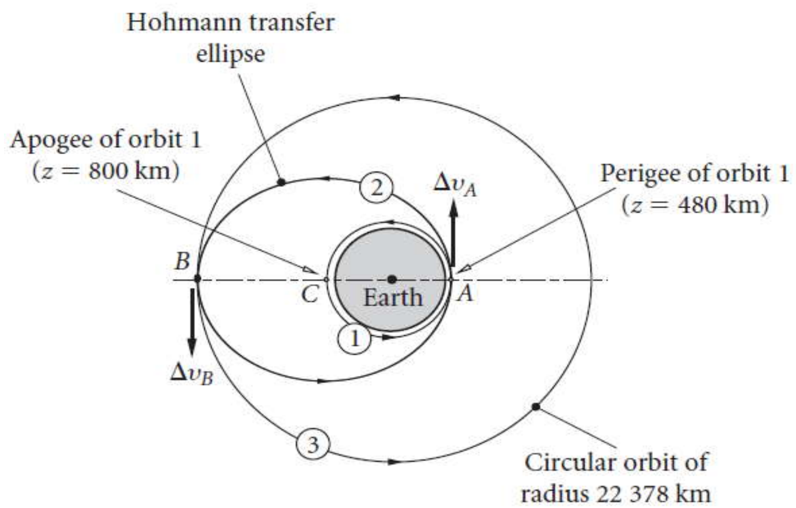


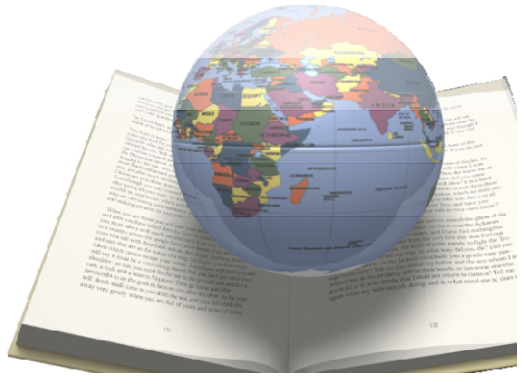
- After coasting from A to B, another forward velocity increment Δv_B places the vehicle on the outer circular orbit.
- The total energy expenditure is: $\Delta v_{\text{total}} = \Delta v_A + \Delta v_B.$
- The same Δv_{total} is required if the transfer begins at B. in this case Δv_s must be accomplished by retrofires.

13- HOHMANN TRANSFER

EXAMPLE 13.1

- A spacecraft in a 480km by 800km earth orbit. Find (a) the Δv required at perigee A to place the spacecraft in 480km by 16000km transfer orbit (orbit2); and (b) the Δv (apogee kick) required at B of the transfer orbit to establish a circular orbit of 16000km altitude (orbit3)





13- HOHMANN TRANSFER

EXAMPLE 13.1

- (a) first, let us establish the primary orbital parameters of the original orbit 1. the perigee and apogee radii are

$$r_A = R_E + z_A = 6378 + 480 = 6858\,\text{km}$$

$$r_C = R_E + z_C = 6378 + 800 = 7178\,\text{km}$$

- Therefore, the eccentricity of orbit 1 is

$$e_1 = \frac{r_C - r_A}{r_C + r_A} = 0.022799$$

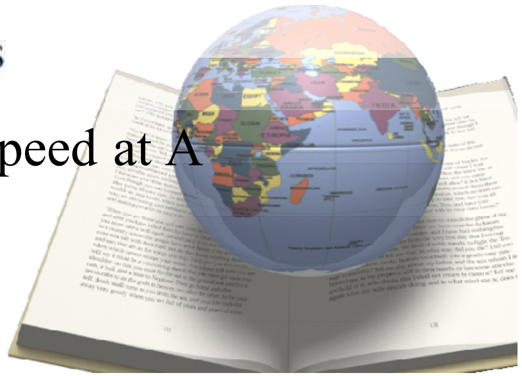
- Applying the orbit equation at perigee of orbit 1, we calculate the angular momentum,

$$r_A = \frac{h_1^2}{\mu}\frac{1}{1 + e_1\cos(0)} \Rightarrow h_1 = 52\,876\,\text{km}^2/\text{s}$$

- With the angular momentum, we can calculate the speed at A on orbit 1.

$$v_A)_1 = \frac{h_1}{r_A} = 7.7102\,\text{km/s}$$

(a)



13- HOHMANN TRANSFER

EXAMPLE 13.1

- Moving to the transfer orbit 2, we proceed in a similar fashion to get

$$r_B = R_E + z_B = 6378 + 16\,000 = 22\,378\ \text{km}$$

$$e_2 = \frac{r_B - r_A}{r_B + r_A} = 0.53085$$

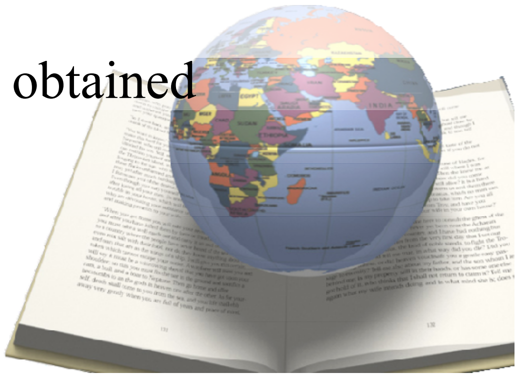
$$r_A = \frac{h_2^2}{\mu}\frac{1}{1 + e_2\cos(0)} \quad \Rightarrow \quad h_2 = 64\,690\ \text{km}$$

- Thus, the speed at A on orbit 2 is

$$v_A)_2 = \frac{h_2}{r_A} = \frac{64\,690}{6858} = 9.4327\ \text{km/s} \quad \text{(b)}$$

- The required forward velocity increment at A is now obtained from (a) and (b) as

$$\Delta v_A = v_A)_2 - v_A)_1 = \underline{1.7225\ \text{km/s}}$$



13- HOHMANN TRANSFER

EXAMPLE 13.1

- (b) we use angular momentum formula to find the speed at B on orbit 2,

$$v_B)_2 = \frac{h_2}{r_B} = \frac{64\,690}{22\,378} = 2.8908\,\text{km/s} \qquad \text{(c)}$$

- Orbit3 is circular, so its constant orbital speed is obtained from equation ? ,

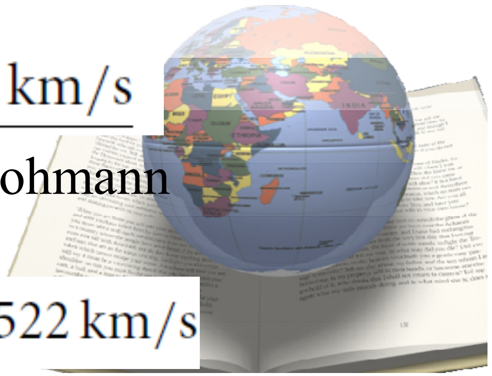
$$v_B)_3 = \sqrt{\frac{398\,600}{22\,378}} = 4.2204\,\text{km/s} \qquad \text{(d)}$$

- Thus, the delta-v requirement at B to climb from orbit 2 to orbit 3 is

$$\Delta v_B = v_B)_3 - v_B)_2 = 4.2204 - 2.8908 = \underline{1.3297\,\text{km/s}}$$

- Observe that the total delta-v requirement for this Hohmann transfer is

$$\Delta v_{\text{total}} = |\Delta v_A| + |\Delta v_B| = 1.7225 + 1.3297 = 3.0522\,\text{km/s}$$



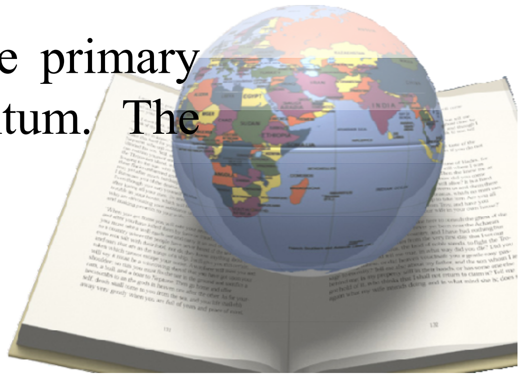
13- HOHMANN TRANSFER

EXAMPLE 13.2

- A spacecraft returning from a lunar mission approaches earth on a hyperbolic trajectory. At its closest approach A it is an altitude of 5000km, traveling at 10km/s. at A retrorockets are fired to lower the spacecraft into a 500km altitude circular orbit, where it is to rendezvous with a space station. Find the location of the space station at retrofire so that rendezvous will occur at B.
- The time of flight from A to B is one-half the period T_2 of the elliptical transfer orbit 2. while the spacecraft coasts from A to B, the space station coasts through the angle ϕ_{CB} from C to B. Hence, this mission has to be carefully planned and executed, going all the way back to lunar departure, so that the two vehicles meet at B.
- To calculate the period T_2, , we must first obtain the primary orbital parameters, eccentricity and angular momentum. The apogee and perigee of orbit2, the transfer ellipse, are

$$r_A = 5000 + 6378 = 11\,378 \text{ km}$$

$$r_B = 500 + 6378 = 6878 \text{ km}$$



13- HOHMANN TRANSFER

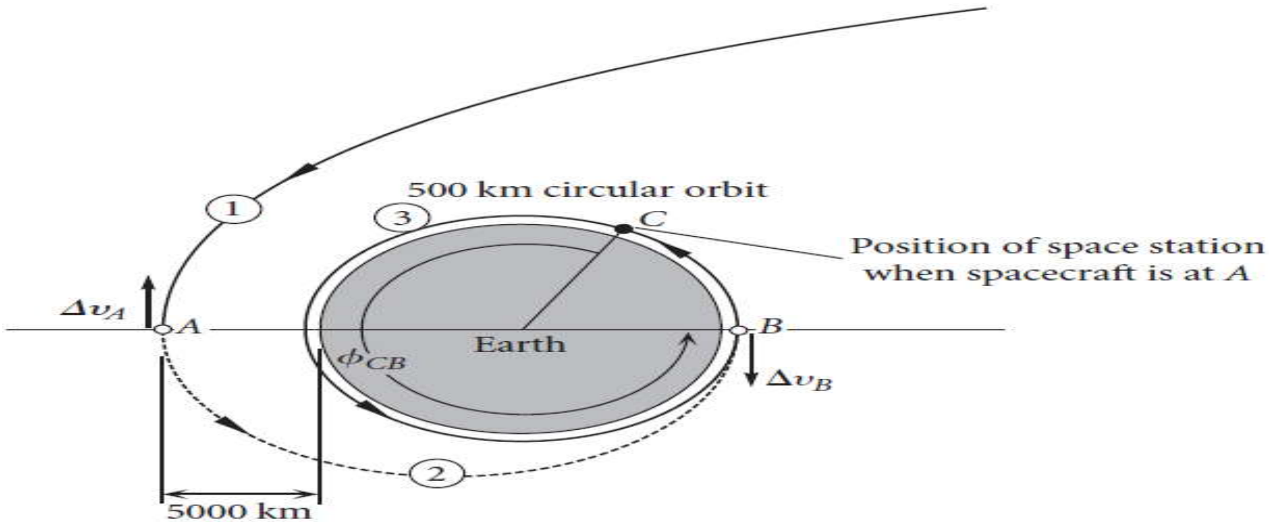
EXAMPLE 13.2

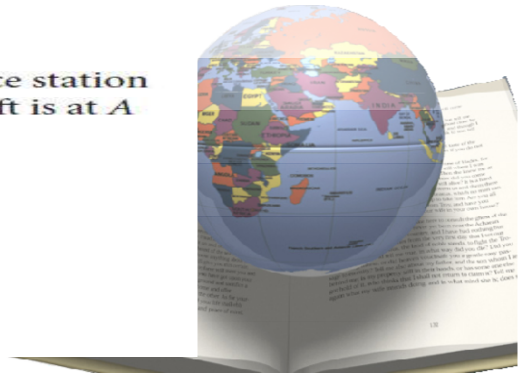
- Therefore, the eccentricity is

$$e_2 = \frac{11\,378 - 6878}{11\,378 + 6878} = 0.24649$$

- Evaluating the orbit equation at perigee yields the angular momentum,

$$r_B = \frac{h_2^2}{\mu}\frac{1}{1+e_2} \Rightarrow 6878 = \frac{h_2^2}{398\,600}\frac{1}{1+0.24649} \Rightarrow h_2 = 58\,458\,\text{km}^2/\text{s}$$





13- HOHMANN TRANSFER

EXAMPLE 13.2

- Now we can use equation ? To find the period of the transfer ellipse,

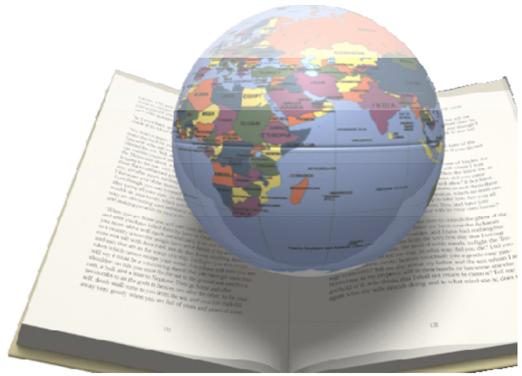
$$T_2 = \frac{2\pi}{\mu^2}\left(\frac{h_2}{\sqrt{1-e_2^2}}\right)^3 = \frac{2\pi}{398\,600^2}\left(\frac{58\,458}{\sqrt{1-0.246492^2}}\right)^3 = 8679.1\ \text{s} \quad \text{(a)}$$

- The period of circular orbit3 is, according to equation ?

$$T_3 = \frac{2\pi}{\sqrt{\mu}} r_B^{\frac{3}{2}} = \frac{2\pi}{\sqrt{398\,600}} 6878^{\frac{3}{2}} = 5676.8\ \text{s} \quad \text{(b)}$$

- The time of flight from C to B on orbit3 must equal the time of flight from A to B on orbit2.

$$\Delta t_{CB} = \frac{1}{2}T_2 = \frac{1}{2}\cdot 8679.1 = 4339.5\ \text{s}$$



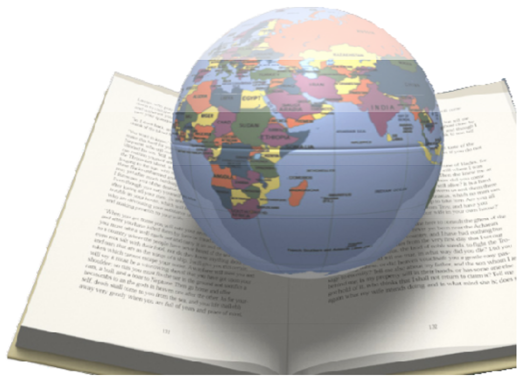
13- HOHMANN TRANSFER

EXAMPLE 13.2

- Since orbit3 is a circle, its angular velocity, unlike an ellipse, is constant. Therefore, we can write

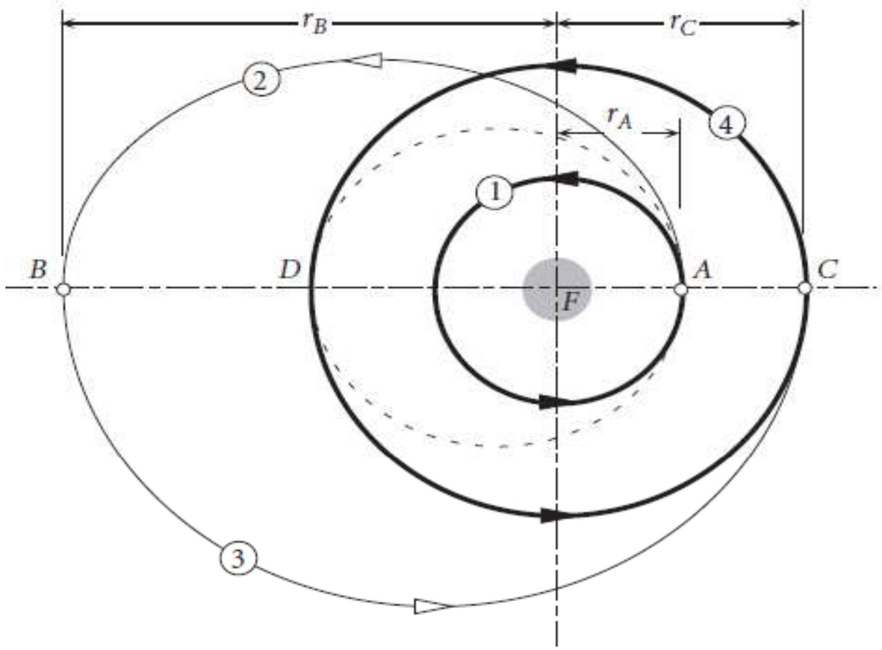
$$\frac{\phi_{CB}}{\Delta t_{CB}} = \frac{360°}{T_3} \Rightarrow \phi_{CB} = \frac{4339.5}{5676.8} \cdot 360 = \underline{275.2°}$$

- (the student should verify that the total delta-v required to lower the spacecraft from the hyperbola into the parking orbit is 6.415km/s. A glance at figure ? Reveals the tremendous amount of propellant this would require.)



13- BI-ELLIPTIC HOHMANN TRANSFER

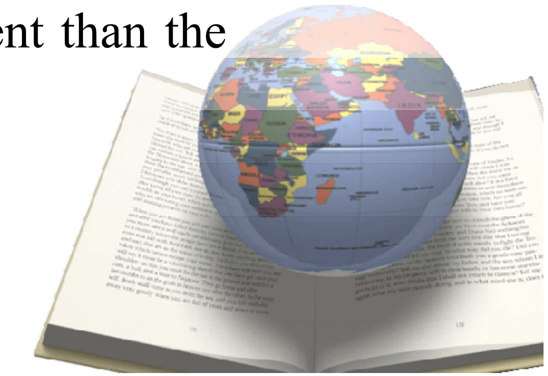
- (NOTE26,P264,{1})
- A Hohmann transfer is the dotted ellips.
- The bi-elliptical Hohmann transfer uses two coaxial semi-ellipses, 2 and 3 (A,B,C)
- The idea is to places B sufficiently far from the focus that the Δv_B will be very small.



$$r_B \longrightarrow \infty \Longrightarrow \Delta v_B \longrightarrow 0$$

- For the bi-elliptical scheme to be more energy efficient than the Hohmann transfer, it must be true that

$$\Delta v_{\text{total}}\big)_{\text{bi-elliptical}} < \Delta v_{\text{total}}\big)_{\text{Hohmann}}$$



13- BI-ELLIPTIC HOHMANN TRANSFER

- Δv analyses of the Hohmann and bi-elliptical transfers lead to the following results:

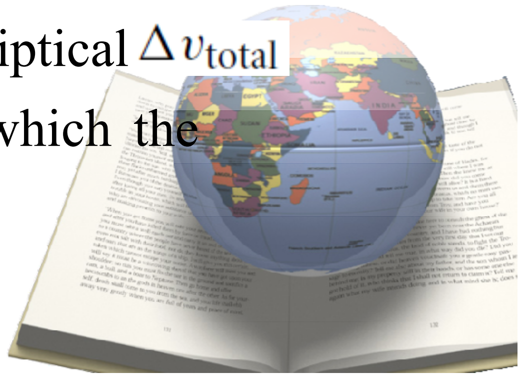
$$\Delta v)_{\text{Hohmann}} = \left[\frac{1}{\sqrt{\alpha}} - \frac{\sqrt{2}(1-\alpha)}{\sqrt{\alpha(1+\alpha)}} - 1\right]\sqrt{\frac{\mu}{r_A}}$$

$$\Delta v)_{\text{bi-elliptical}} = \left[\sqrt{\frac{2(\alpha+\beta)}{\alpha\beta}} - \frac{1+\sqrt{\alpha}}{\sqrt{\alpha}} - \sqrt{\frac{2}{\beta(1+\beta)}}(1-\beta)\right]\sqrt{\frac{\mu}{r_A}}$$

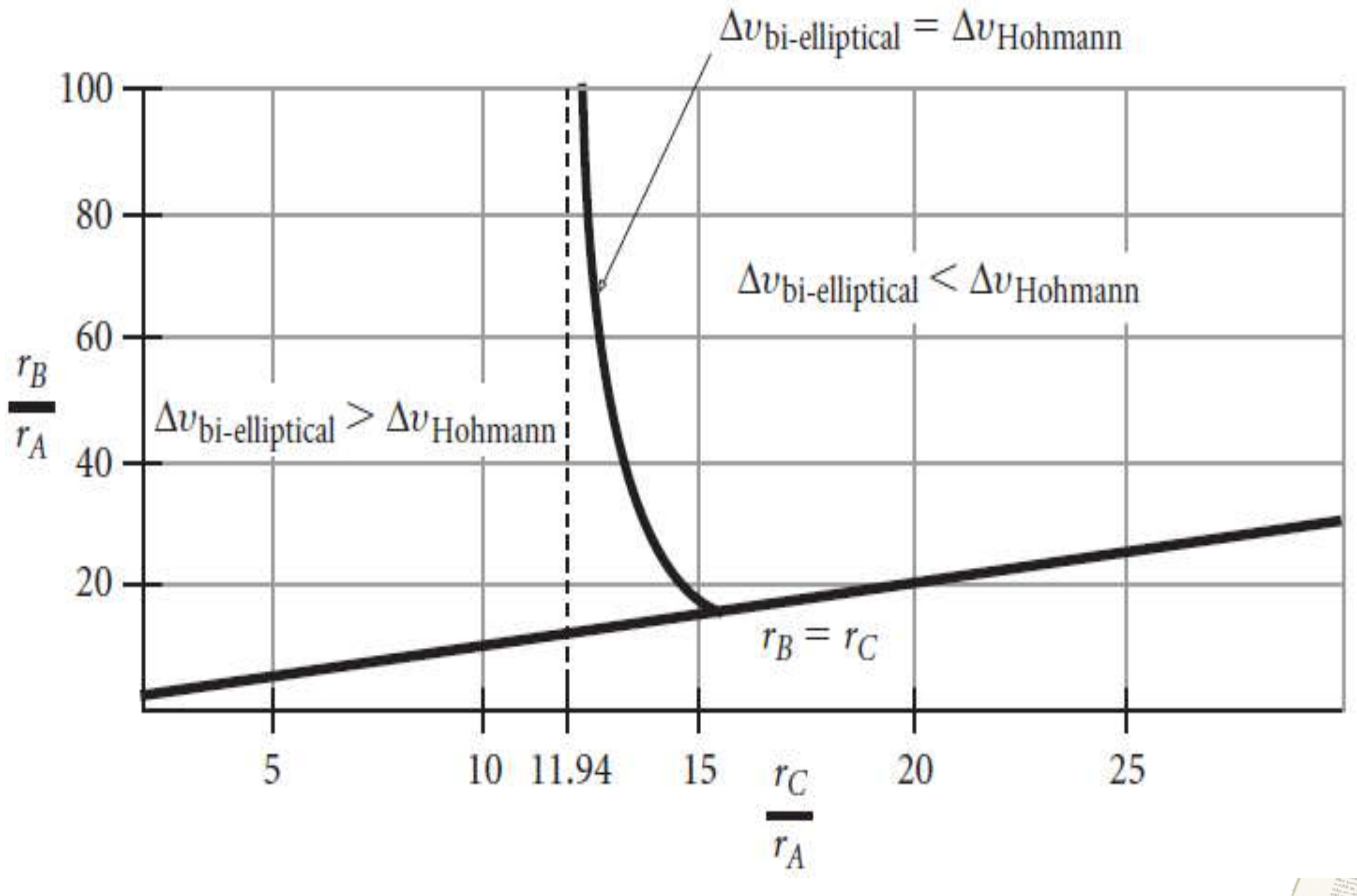
- Where

$$\alpha = \frac{r_C}{r_A} \qquad \beta = \frac{r_B}{r_A}$$

- Plotting the difference between Hohmann and bi-elliptical Δv_{total} as a function of a and β reveals the regions in which the difference is positive, negative and zero



13- BI-ELLIPTIC HOHMANN TRANSFER



⋆ (NOTE27,P265,{1})

13- BI-ELLIPTIC HOHMANN TRANSFER

EXAMPLE 13.3

- Find the total delta-v requirement for a bi-ellptical Hohmann transfer from a geocentric circular orbit of 7000km radius to one of 105000km radius. Let the apogee of the first ellipse be 210000km. Compare the delta-v schedule and total flight time with that for an ordinary single Hohmann transfer ellipse.

  Since $r_A = 7000\,\text{km}$ $\quad r_B = 210\,000\,\text{km}$ $\quad r_C = r_D = 105\,000\,\text{km}$

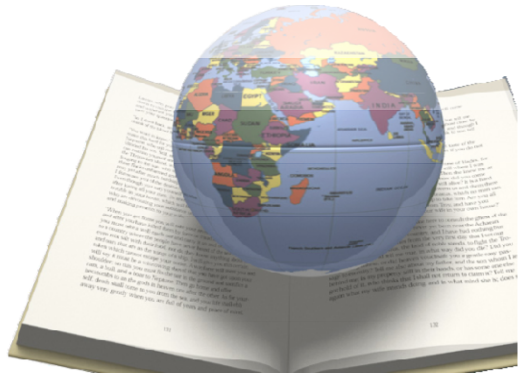
We have $r_B/r_A = 30$ and $r_C/r_A = 15$, so that from figure ? It is apparent right away that the bi-elliptic transfer will be the more energy efficient.

To do the delta-v analysis requires analyzing each of the five orbits.

Orbit 1:

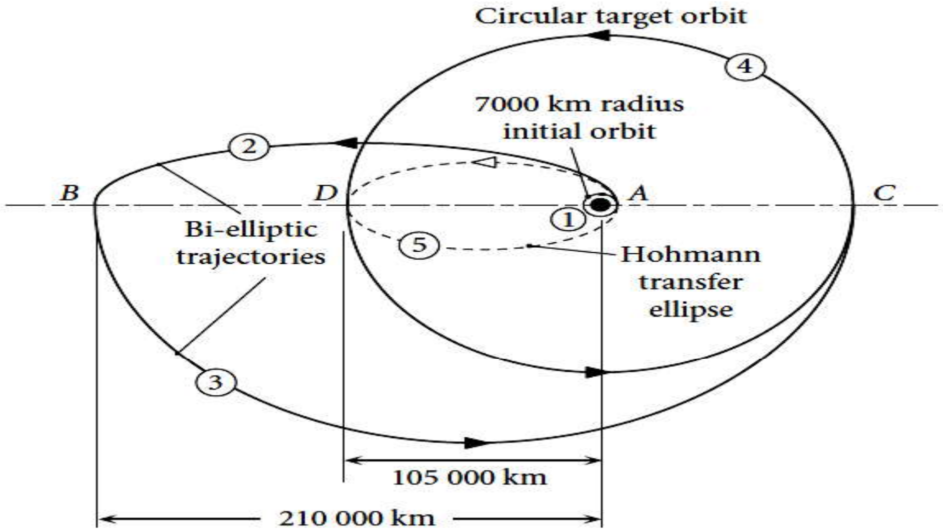
Since this is a circular orbit, we have, simply,

$$v_A)_1 = \sqrt{\frac{\mu}{r_A}} = \sqrt{\frac{398\,600}{7000}} = 7.546\,\text{km/s} \qquad \text{(a)}$$



13- HOHMANN TRANSFER

EXAMPLE 13.3



⋆ Orbit 2:

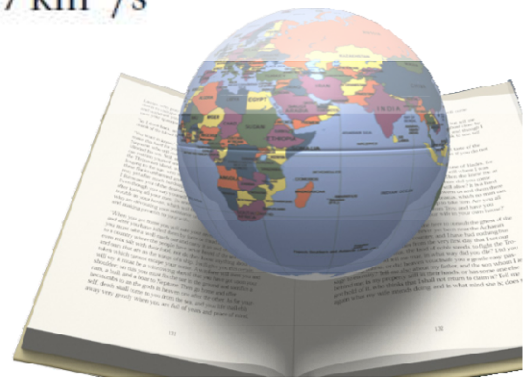
For this transfer ellipse, equation ? yields

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_A r_B}{r_A + r_B}} = \sqrt{2 \cdot 398\,600}\sqrt{\frac{7000 \cdot 210\,000}{7000 + 210\,000}} = 73\,487 \text{ km}^2/\text{s}$$

⋆ Therefore,

$$v_A)_2 = \frac{h_2}{r_A} = \frac{73\,487}{7000} = 10.498 \text{ km/s} \qquad \text{(b)}$$

$$v_B)_2 = \frac{h_2}{r_B} = \frac{73\,487}{210\,000} = 0.34994 \text{ km/s} \qquad \text{(c)}$$



13- HOHMANN TRANSFER

EXAMPLE 13.3

- Orbit 3:

For the second transfer ellipse, we have

$$h_3 = \sqrt{2 \cdot 398\,600}\sqrt{\frac{105\,000 \cdot 210\,000}{105\,000 + 210\,000}} = 236\,230\ \mathrm{km^2/s}$$

- From this we obtain

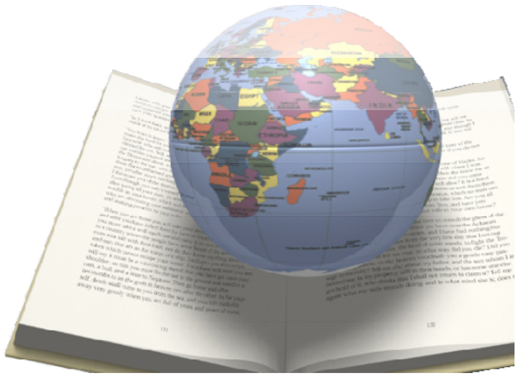
$$v_B)_3 = \frac{h_3}{r_B} = \frac{236\,230}{210\,000} = 1.1249\ \mathrm{km/s} \quad \text{(d)}$$

$$v_C)_3 = \frac{h_3}{r_C} = \frac{236\,230}{105\,000} = 2.2498\ \mathrm{km/s} \quad \text{(e)}$$

- Orbit 4:

The target orbit, like orbit1, is a circle, which means

$$v_C)_4 = v_D)_4 = \sqrt{\frac{398\,600}{105\,000}} = 1.9484\ \mathrm{km/s} \quad \text{(f)}$$



13- HOHMANN TRANSFER

EXAMPLE 13.3

- For the bi-elliptical maneuver, the total delta-v is, therefore,

$$\begin{aligned}\Delta v_{\text{total}})_{\text{bi-elliptical}} &= \Delta v_A + \Delta v_B + \Delta v_C \\ &= |v_A)_2 - v_A)_1| + |v_B)_3 - v_B)_2| + |v_C)_4 - v_C)_3| \\ &= |10.498 - 7.546| + |1.1249 - 0.34994| + |1.9484 - 2.2498| \\ &= 2.9521 + 0.77496 + 0.30142\end{aligned}$$

- Or, $\underline{\Delta v_{\text{total}})_{\text{bi-elliptical}} = 4.0285\ \text{km/s}}$ (g)

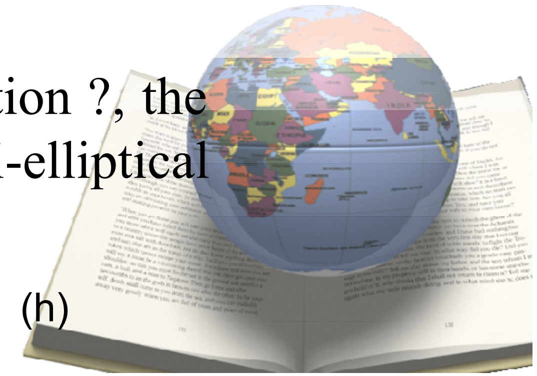
- The semimajor axes of transfer orbits 2 and 3 are

$$a_2 = \frac{1}{2}(7000 + 210\,000) = 108\,500\ \text{km}$$

$$a_3 = \frac{1}{2}(105\,000 + 210\,000) = 157\,500\ \text{km}$$

- With this information and the period formula, equation ?, the time of flight for the two semi-ellipses of the bi-elliptical transfer is found to be

$$t_{\text{bi-elliptical}} = \frac{1}{2}\left(\frac{2\pi}{\sqrt{\mu}}a_2^{\frac{3}{2}} + \frac{2\pi}{\sqrt{\mu}}a_3^{\frac{3}{2}}\right) = 488\,870\ \text{s} = \underline{5.66\ \text{days}} \qquad \text{(h)}$$



13- HOHMANN TRANSFER

EXAMPLE 13.3

- For the Hohmann transfer ellipse 5,

$$h_5 = \sqrt{2 \cdot 398\,600}\sqrt{\frac{7000 \cdot 105\,000}{7000 + 105\,000}} = 72\,330\ \text{km}^2/\text{s}$$

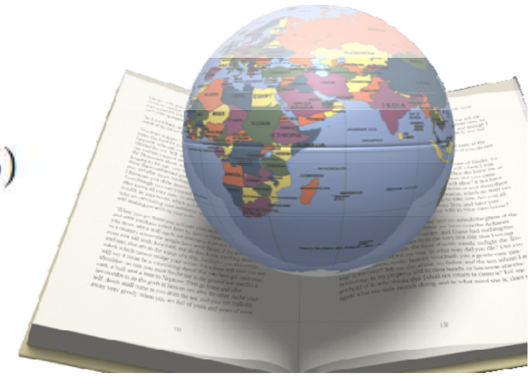
- Hence,

$$v_A)_5 = \frac{h_5}{r_A} = \frac{72\,330}{7000} = 10.333\ \text{km/s} \qquad \text{(i)}$$

$$v_D)_5 = \frac{h_5}{r_D} = \frac{72\,330}{105\,000} = 0.68886\ \text{km/s} \qquad \text{(j)}$$

- It follows that

$$\begin{aligned} \Delta v_{\text{total}})_{\text{Hohmann}} &= |v_A)_5 - v_A)_1| + |v_D)_5 - v_D)_1| \\ &= (10.333 - 7.546) + (1.9484 - 0.68886) \\ &= 2.7868 + 1.2595 \end{aligned}$$



13- HOHMANN TRANSFER

EXAMPLE 13.3

- or

$$\underline{\Delta v_{\text{total}})_{\text{Hohmann}} = 4.0463\ \text{km/s}} \qquad \text{(k)}$$

- This is only slightly (0.44 percent) larger than that of the bi-elliptical transfer.

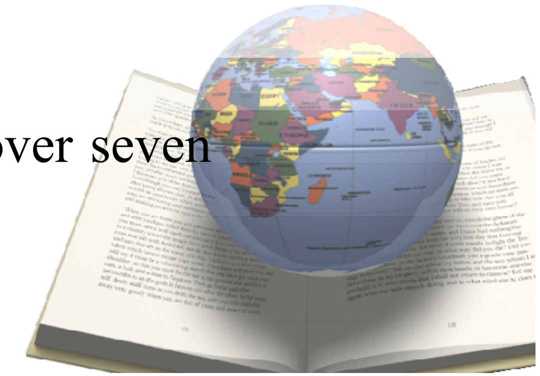
  since the semimajor axis of the Hohmann semi-ellipse is

$$a_5 = \frac{1}{2}(7000 + 105\,000) = 56\,000\ \text{km}$$

- The time of flight from A to D is

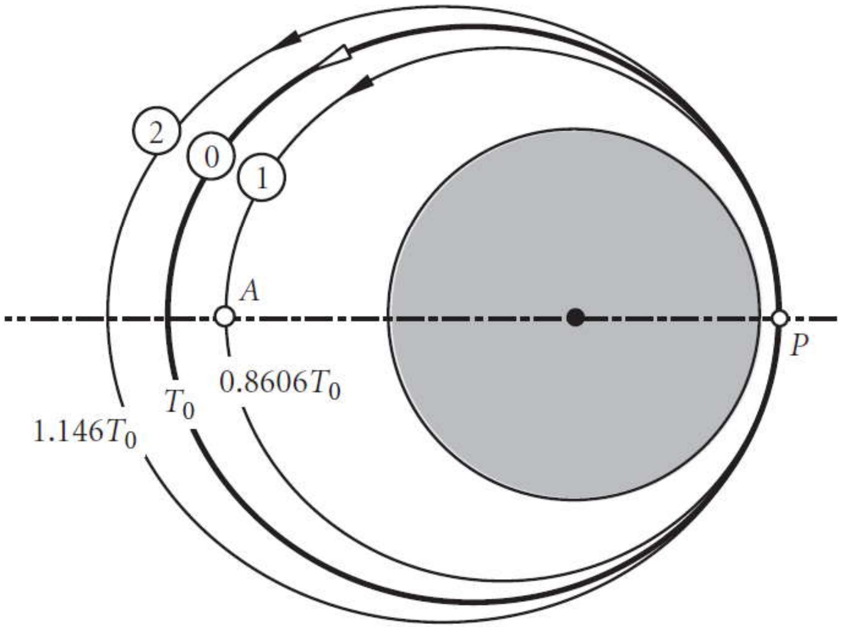
$$t_{\text{Hohmann}} = \frac{1}{2}\left(\frac{2\pi}{\sqrt{\mu}}a_5^{\frac{3}{2}}\right) = 65\,942\ \text{s} = \underline{0.763\ \text{days}} \qquad \text{(l)}$$

- The time of flight of the bi-elliptical maneuver is over seven times longer than that of the Hohmann transfer.



13- PHASING MANEUVERS

- A phasing maneuver is a two-impulse Hohmann transfer from and back to the same orbit, as illustrated in figure:



- Phasing maneuvers are used to change the position of a spacecraft in its orbit.
- (NOTE28,P268,{1}

13- PHASING MANEUVERS

- Once the period T of the phasing orbit is established, then the following equation should be used to determine the semimajor axis of the phasing ellipse:

$$a = \left(\frac{T\sqrt{\mu}}{2\pi}\right)^{\frac{2}{3}} \qquad (5)$$

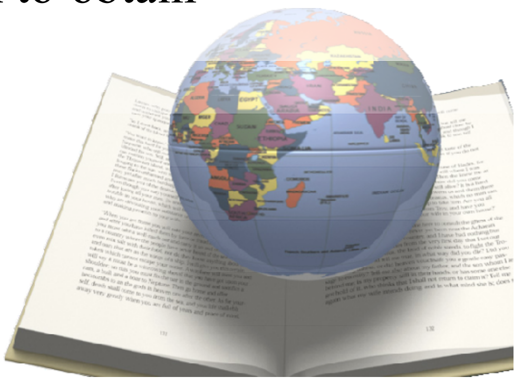
- With the semimajor axis established, r_A opposite to P is obtained from: $2a = r_P + r_A$
- Then we can calculate the eccentricity of phasing orbit from equation:

$$e_3 = \frac{r_B - r_A}{r_B + r_A}$$

- Then the orbit equation may be applied at either P or A to obtain the angular momentum

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta}$$

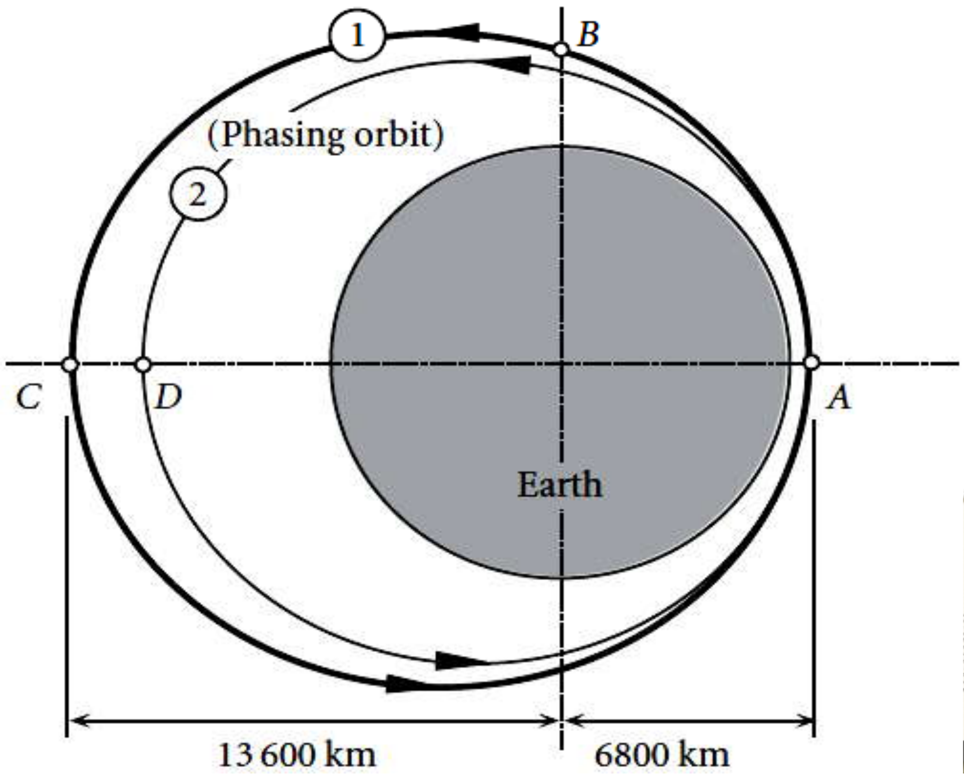
- The phasing orbit is characterized completely

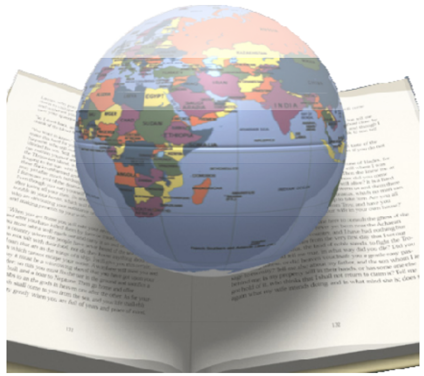


13- PHASING MANEUVERS

<u>EXAMPLE 13.4</u>

- Spacecraft at A and B are in the same orbit (1). At the instant shown, the chaser vehicle at A executes a phasing maneuver so as to catch the target spacecraft back at A after just one revolution of the chaser's phasing orbit (2). What is the required total delta-v?





13- PHASING MANEUVERS

EXAMPLE 13.4

- From the figure, $r_A = 6800\,\text{km} \qquad r_C = 13\,600\,\text{km}$
- Orbit 1:

The eccentricity of orbit 1 is $$e_1 = \frac{r_C - r_A}{r_C + r_A} = 0.33333$$

Evaluating the orbit equation at A, we find

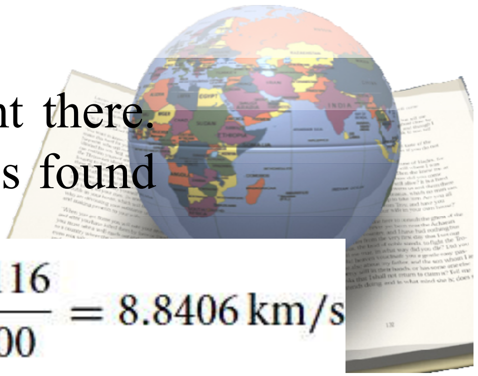
$$r_A = \frac{h_1^2}{\mu}\frac{1}{1 + e_1\cos(0)} \Rightarrow 6800 = \frac{h_1^2}{398\,600}\frac{1}{1 + 0.3333} \Rightarrow h_1 = 60\,116\,\text{km}^2/\text{s}$$

The period is found using equation ?

$$T_1 = \frac{2\pi}{\mu^2}\left(\frac{h_1}{\sqrt{1 - e_1^2}}\right)^3 = \frac{2\pi}{398\,600^2}\left(\frac{60\,116}{\sqrt{1 - 0.33333^2}}\right)^3 = 10\,252\,\text{s}$$

- Since A is perigee, there is no radial velocity component there. The speed, directed entirely in the transverse direction, is found from the angular momentum formula,

$$v_{A1} = \frac{h_1}{r_A} = \frac{60\,116}{6800} = 8.8406\,\text{km/s}$$



13- PHASING MANEUVERS

EXAMPLE 13.4

The phasing orbit must have a period T_2 equal to the time it takes the target vehicle at B to coast around to point A on orbit 1. we can determine the flight time by calculating Δt_{AB} the time from A to B and subtracting that result from the period T_1 of orbit 1. At B the true anomaly is $\theta_A = 90°$.therefore, according to equation ?

$$\tan\frac{E_B}{2} = \sqrt{\frac{1-e_1}{1+e_1}}\tan\frac{\theta_B}{2} = \sqrt{\frac{1-0.33333}{1+0.33333}}\tan\frac{90°}{2}$$

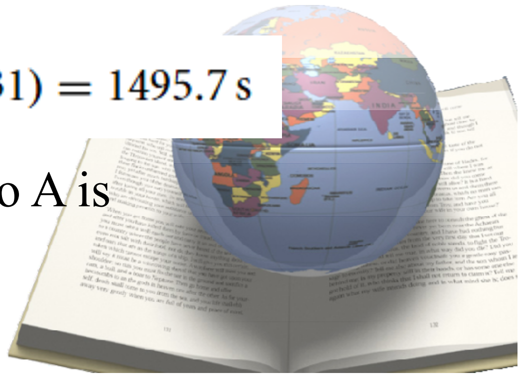
$$= 0.70711 \Rightarrow E_B = 1.2310\,\text{rad}$$

- Then, from Kepler's equation (?), we get

$$\Delta t_{AB} = \frac{T_1}{2\pi}(E_B - e_1\sin E_B) = \frac{10\,252}{2\pi}(1.231 - 0.33333\cdot\sin 1.231) = 1495.7\,\text{s}$$

- Thus, the time of flight of the target spacecraft from B to A is

$$\Delta t_{BA} = T_1 - \Delta t_{AB} = 10\,252 - 1495.7 = 8756.3\,\text{s}$$



13- PHASING MANEUVERS

EXAMPLE 13.4

- Orbit 2:

The period of orbit 2 must equal Δt_{BA} so that the chaser will arrive at A when the target does. That is,

$$T_2 = 8756.3\,\mathrm{s}$$

This, together with the period formula, equation ?, yields the semimajor axis of orbit 2,

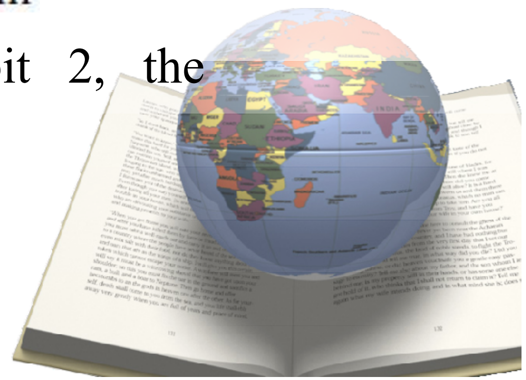
$$T_2 = \frac{2\pi}{\sqrt{\mu}} a_2^{\frac{3}{2}} \;\Rightarrow\; 8756.2 = \frac{2\pi}{\sqrt{398\,600}} a_2^{\frac{3}{2}} \;\Rightarrow\; a_2 = 9182.1\,\mathrm{km} \qquad \text{(a)}$$

Since $2a_2 = r_A + r_D$, we find

$$r_D = 2a_2 - r_A = 2 \cdot 9182.1 - 6800 = 11\,564\,\mathrm{km}$$

Therefore, point A is indeed the perigee of orbit 2, the eccentricity of which can now be determined:

$$e_2 = \frac{r_D - r_A}{r_D + r_A} = 0.25943$$



13- PHASING MANEUVERS

EXAMPLE 13.4

Evaluating the orbit equation at point A orbit 2 yields its angular momentum,

$$r_A = \frac{h_2^2}{\mu}\frac{1}{1+e_2\cos(0)} \Rightarrow 6800 = \frac{h_2^2}{398\,600}\frac{1}{1+0.25943} \Rightarrow h_2 = 58\,426\ \text{km}^2/\text{s}$$

Finally, we can calculate the speed at perigee of orbit 2,

$$v_{A_2} = \frac{h_2}{r_A} = \frac{58\,426}{6800} = 8.5921\ \text{km/s}$$

At the beginning of the phasing maneuver,

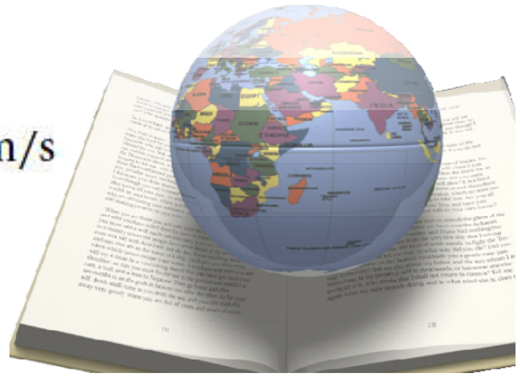
$$\Delta v_A = v_{A_2} - v_{A_1} = 8.5921 - 8.8406 = -0.24851\ \text{km/s}$$

At the end of the phasing maneuver,

$$\Delta v_A = v_{A_1} - v_{A_2} = 8.8406 - 8.5921 = 0.24851\ \text{km/s}$$

The total delta-v, therefore, is

$$\Delta v_{\text{total}} = |-0.24851| + |0.24851| = \underline{0.4970\ \text{km/s}}$$



13- PHASING MANEUVERS

EXAMPLE 13.5

- It is desired to shift the longitude of a GEO satellite $12°$ westward in three revolutions of its phasing orbit. Calculate the delta-v requirement.

  this problem is illustrated in ? . It may be recalled from equation ?, ? and ? That the angular velocity of the earth, the radius to GEO and the speed in GEO are, respectively,

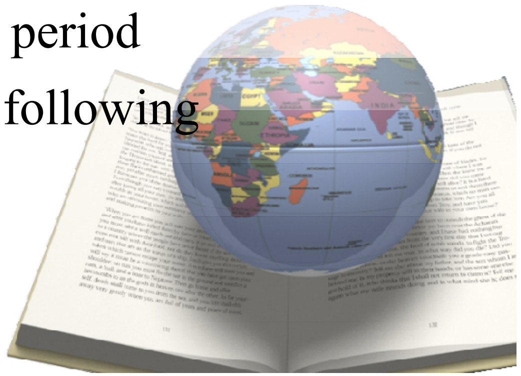
$$\omega_E = \omega_{GEO} = 72.922 \times 10^{-6} \text{ rad/s}$$

$$r_{GEO} = 42\,164 \text{ km} \quad \text{(a)}$$

$$v_{GEO} = 3.0747 \text{ km/s}$$

- Let $\Delta\Lambda$ be the change in longitude in radians. Then the period T_2 of the phasing orbit can be obtained from the following formula,

$$\omega_E(3T_2) = 3 \cdot 2\pi + \Delta\Lambda \quad \text{(b)}$$

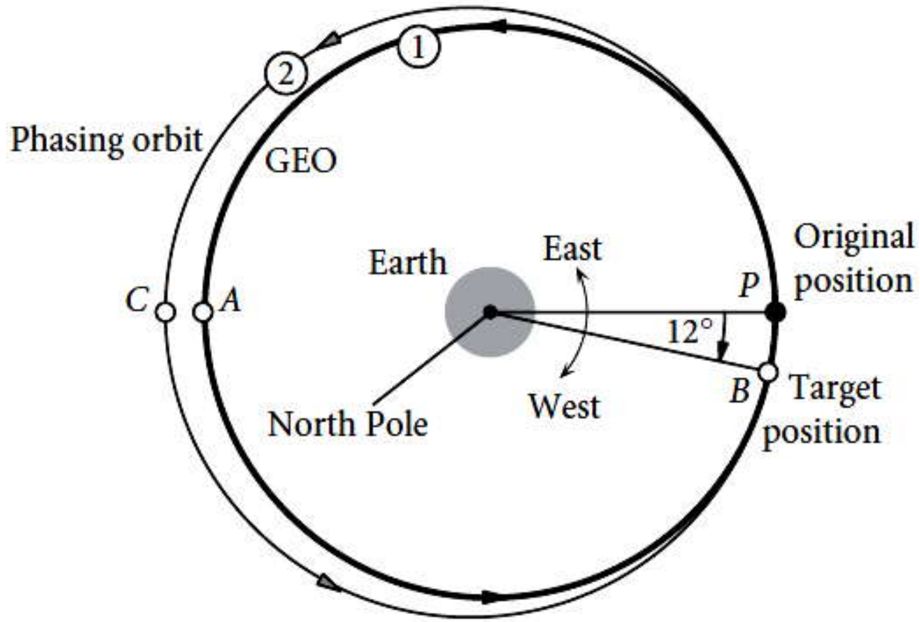


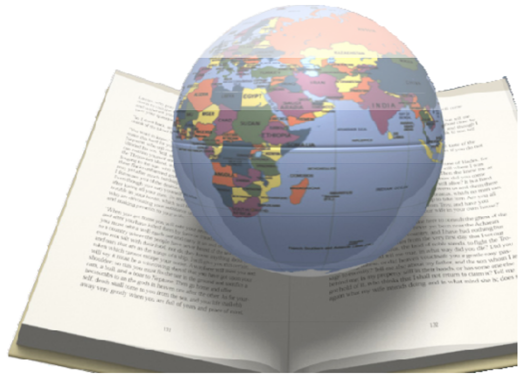
13- PHASING MANEUVERS

EXAMPLE 13.5

which states that after three circuits of the phasing orbit, the original position of the satellite will be $\Delta\Lambda$ radians east of P. in other words, the satellite will end up $\Delta\Lambda$ radians west of its original position in GEO, as desired. From (b) we obtain,

$$T_2 = \frac{1}{3}\frac{\Delta\Lambda + 6\pi}{\omega_E} = \frac{1}{3}\cdot\frac{12° \cdot \frac{\pi}{180°} + 6\pi}{72.922 \times 10^{-6}} = 87\,121\ \text{s}$$





13- PHASING MANEUVERS

EXAMPLE 13.5

Note that the period of GEO is

$$T_{GEO} = \frac{2\pi}{\omega_{GEO}} = 86\,163 \text{ s}$$

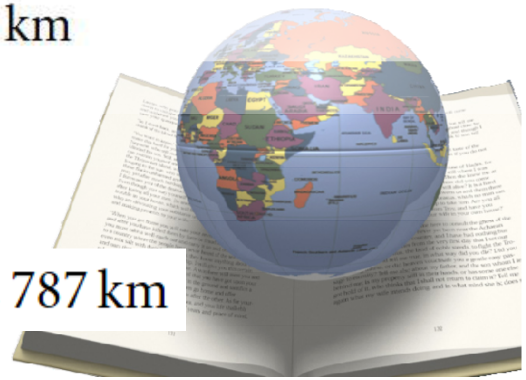
The satellite in its slower phasing orbit appears to drift westward at the rate

$$\dot{\Lambda} = \frac{\Delta\Lambda}{3T_2} = 8.0133 \times 10^{-7} \text{ rad/s} = 3.9669^\circ/\text{day}$$

Having the period, we can use equation ? To obtain the semimajor axis of orbit 2,

$$a = \left(\frac{T\sqrt{\mu}}{2\pi}\right)^{\frac{2}{3}} = \left(\frac{87\,121\sqrt{398\,600}}{2\pi}\right)^{\frac{2}{3}} = 42\,476 \text{ km}$$

From this we find the radial coordinate of C,



$$2a_2 = r_P + r_C \quad \Rightarrow \quad r_C = 2 \cdot 42\,476 - 42\,164 = 42\,787 \text{ km}$$

13- PHASING MANEUVERS

EXAMPLE 13.5

Now we can find the eccentricity of orbit 2,

$$e_2 = \frac{r_C - r_A}{r_C + r_A} = \frac{42\,787 - 42\,164}{42\,787 + 42\,164} = 0.0073395$$

And the angular momentum follows from applying the orbit equation at P (or C) of orbit 2:

$$r_P = \frac{h_2^2}{\mu}\frac{1}{1 + e_2\cos(0)} \Rightarrow 42\,164 = \frac{h_2^2}{398\,600}\frac{1}{1 + 0.0073395} \Rightarrow h_2 = 130\,120\,\text{km}^2/\text{s}$$

at P the speed in orbit 2 is

$$v_{P_2} = \frac{130\,120}{42\,164} = 3.0859\,\text{km/s}$$

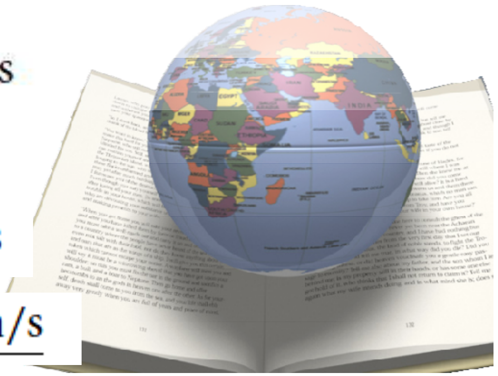
therefore, at the beginning of the phasing orbit,

$$\Delta v = v_{P_2} - v_{\text{GEO}} = 3.0859 - 3.0747 = 0.01126\,\text{km/s}$$

at the end of the phasing maneuver,

$$\Delta v = v_{\text{GEO}} - v_{P_2} = 3.0747 - 3.08597 = -0.01126\,\text{km/s}$$

Therefore, $\Delta v_{\text{total}} = |0.01126| + |-0.01126| = \underline{0.022525\,\text{km/s}}$

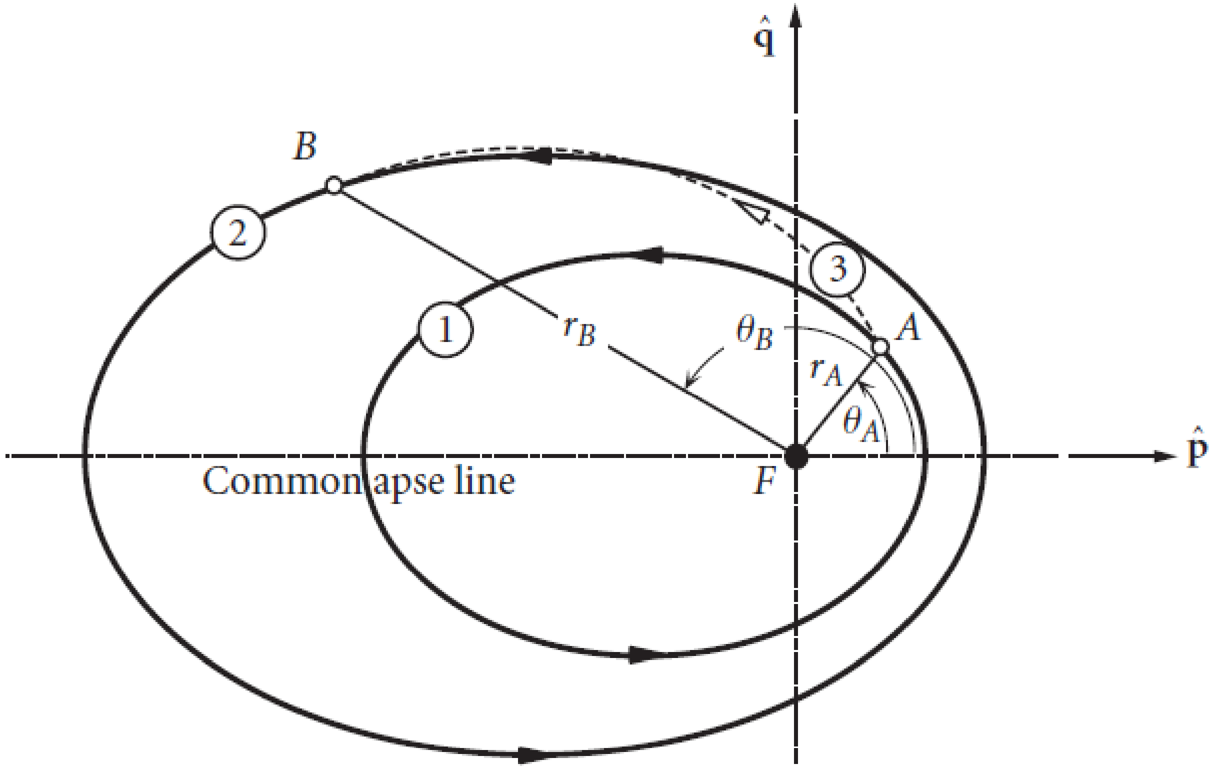


Chapter 14

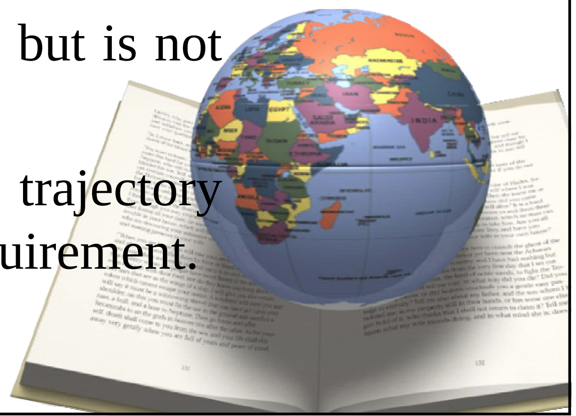
Non-Hohmann transfers with a common apse line

CHAPTER CONTENT

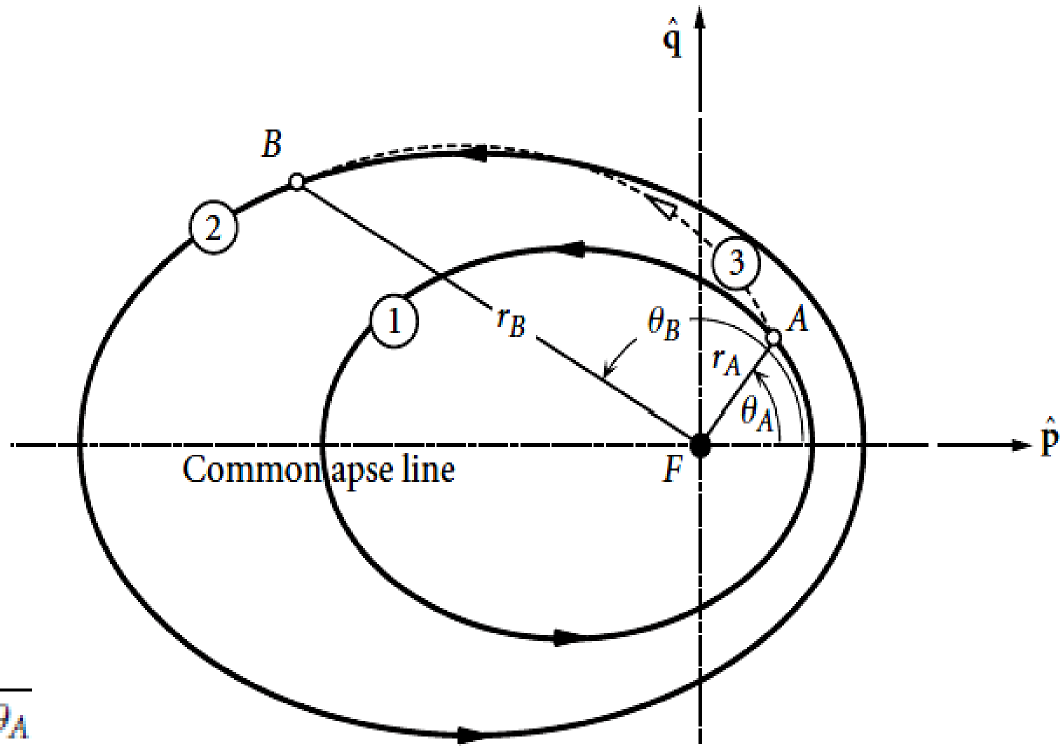
14- NON-HOHMANN TRANSFERS WITH A COMMON APSE LINE



- **A**bove figure illustrates a transfer between two coaxial elliptical orbits in which the transfer trajectory shares the apse line but is not necessarily tangent to either the initial or target orbit.
- The problem is to determine whether there exists such a trajectory joining points A and B, and if so to find the total Δv requirement.



14- NON-HOHMANN TRANSFERS WITH A COMMON APSE LINE



- ✶ r_A And r_B are given, as are the true anomalies θ_A and θ_B.
- ✶ Applying the orbit equation to A and B on orbit 3 yields:

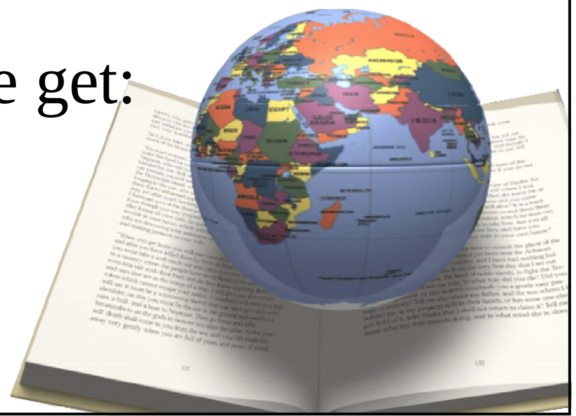
$$r_A = \frac{h_3^2}{\mu} \frac{1}{1 + e_3 \cos\theta_A}$$

$$r_B = \frac{h_3^2}{\mu} \frac{1}{1 + e_3 \cos\theta_B}$$

- ✶ Solving these two equations for e_3 and h_3, we get:

$$e_3 = \frac{r_B - r_A}{r_A \cos\theta_A - r_B \cos\theta_B}$$

$$h_3 = \sqrt{\mu r_A r_B}\sqrt{\frac{\cos\theta_A - \cos\theta_B}{r_A \cos\theta_A - r_B \cos\theta_B}} \qquad (1)$$



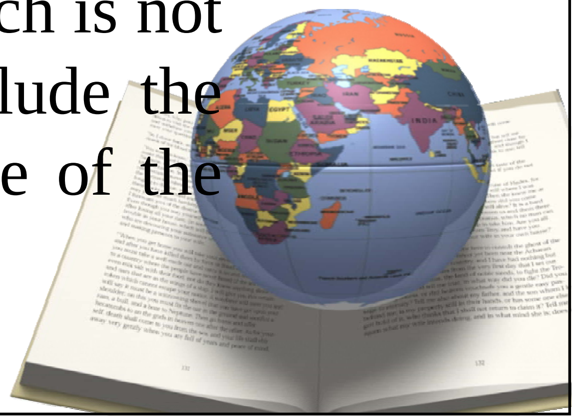
14- NON-HOHMANN TRANSFERS WITH A COMMON APSE LINE

- With these, the transfer orbit is determined and velocity may be found at any true anomaly.
- For a Hohmann transfer, in which $\theta_A = 0$ and $\theta_B = \pi$, equation (1) become:

$$e_3 = \frac{r_B - r_A}{r_B + r_A}$$

$$h_3 = \sqrt{2\mu}\sqrt{\frac{r_A r_B}{r_A + r_B}} \quad \text{(Hohmann transfer)} \qquad (2)$$

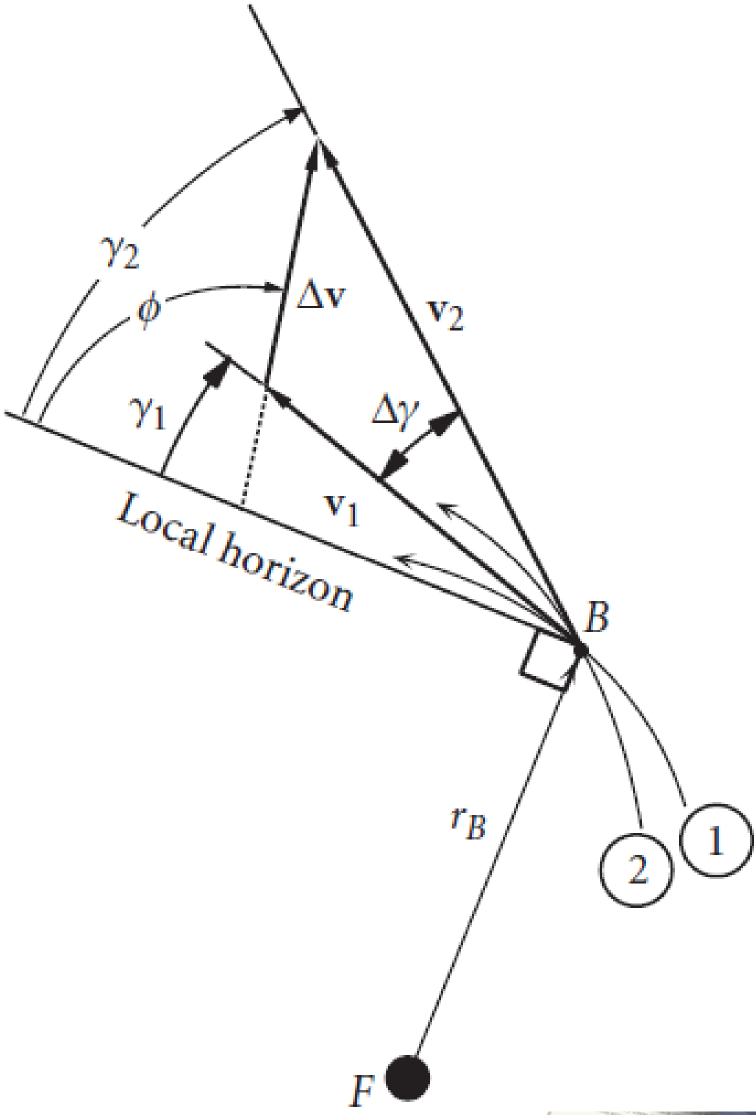
- When a Δv calculation is done at a point which is not on the apse line, care must be taken to include the change in direction as well as the magnitude of the velocity vector.



14- NON-HOHMANN TRANSFERS WITH A COMMON APSE LINE

- Figure shows a point where an impulsive $\mathbf{v}_1$ maneuver changes the velocity vector from

  on orbit 1 to $\mathbf{v}_2$ on orbit 2.

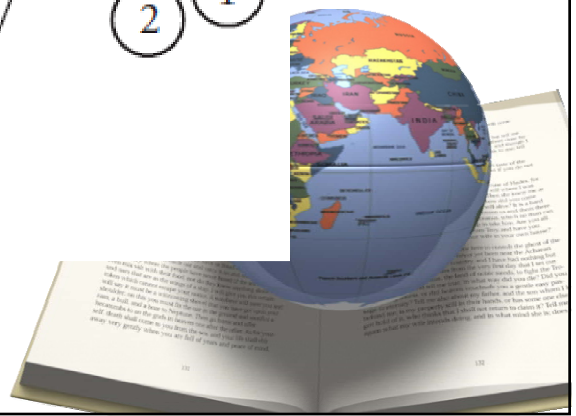


- It is important to observe that the Δv we seek is the magnitude of the change in the velocity vector. Not the change in its magnitude (speed). That is:

$$\Delta v = ||\mathbf{v}_2 - \mathbf{v}_1|| \qquad (3)$$

- Only if $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel, as in Hohmann transfers, is it true that

$$\Delta v = \|v_2\| - \|v_1\|$$



14- NON-HOHMANN TRANSFERS WITH A COMMON APSE LINE

- From the figure and the law of cosines, we find that,

$$\Delta v = \sqrt{v_1^2 + v_2^2 - 2 v_1 v_2 \cos \Delta\gamma} \qquad (4)$$

- $v_1 = \|\mathbf{v}_1\|$;

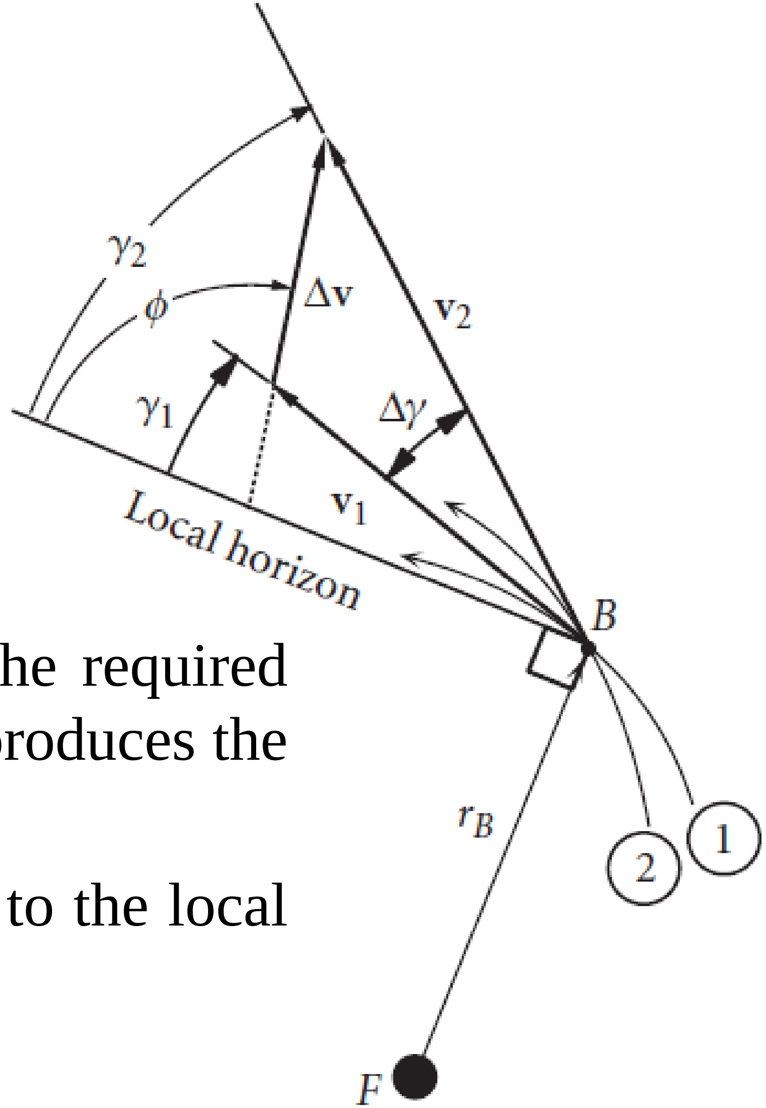
  $v_2 = \|\mathbf{v}_2\|$

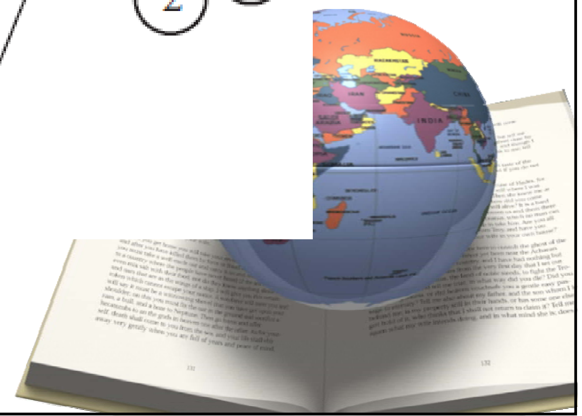
  $\Delta\gamma = \gamma_2 - \gamma_1$

- the direction of Δv shows the required alignment of the thruster that produces the impulse.
- The orientation of Δv relative to the local horizon is found by equation:

$$\tan\phi = \frac{\Delta v_r}{\Delta v_\perp} \qquad (5)$$

- ϕ : the angle from the local horizon to the Δv vector





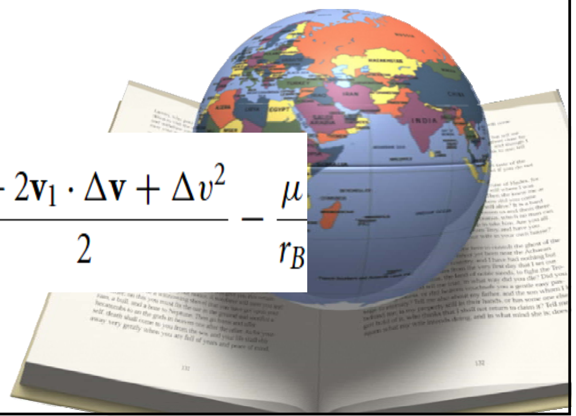
14- NON-HOHMANN TRANSFERS WITH A COMMON APSE LINE

- An impulsive maneuver results in a change of orbit and, therefore, a change in the specific energy ε.
- If the expenditure of propellant Δm is negligible compared to the initial mass m_1 of the vehicle then:

$$\Delta\varepsilon = \varepsilon_2 - \varepsilon_1$$

- Recall the formula for specific mechanical energy of an orbit, for the situation illustrated in previous figure:

$$\varepsilon = \frac{\mathbf{v}\cdot\mathbf{v}}{2} - \frac{\mu}{r} \quad (v^2 = \mathbf{v}\cdot\mathbf{v}) \Rightarrow \begin{cases} \varepsilon_1 = \dfrac{v_1^2}{2} - \dfrac{\mu}{r_B} \\ \varepsilon_2 = \dfrac{(\mathbf{v}_1 + \Delta\mathbf{v})\cdot(\mathbf{v}_1 + \Delta\mathbf{v})}{2} - \dfrac{\mu}{r_B} = \dfrac{v_1^2 + 2\mathbf{v}_1\cdot\Delta\mathbf{v} + \Delta v^2}{2} - \dfrac{\mu}{r_B} \end{cases}$$

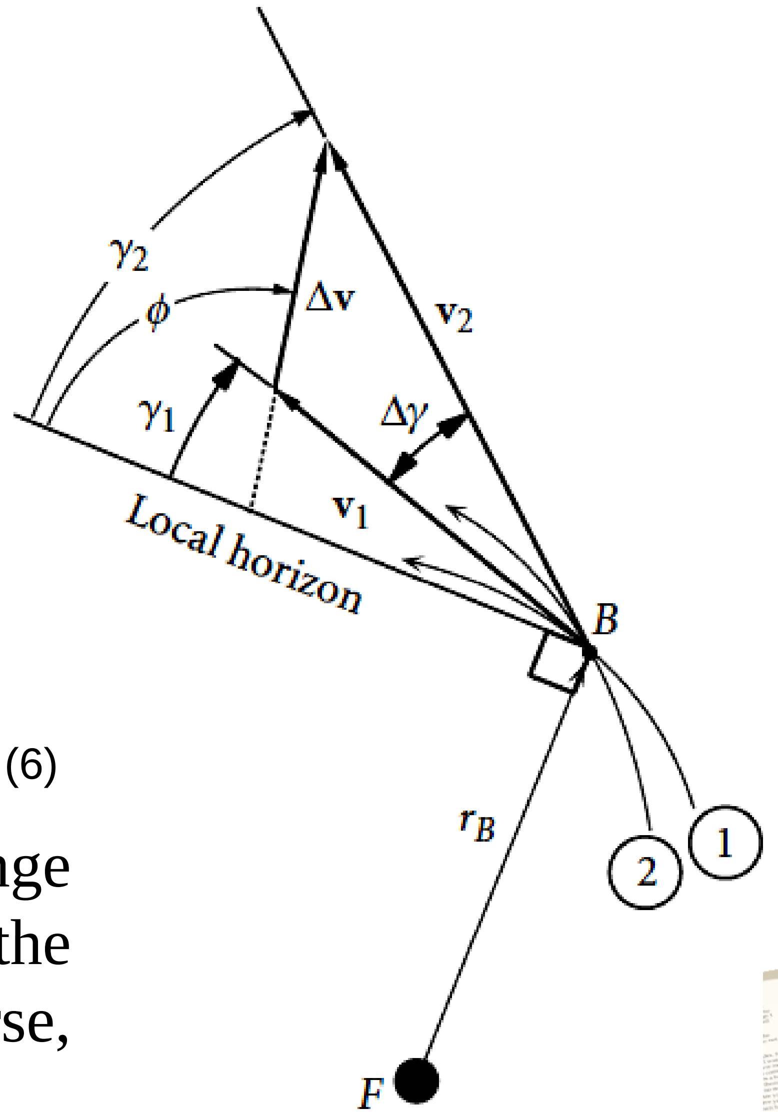


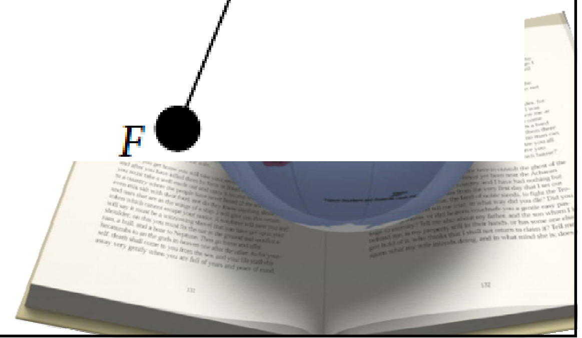
14- NON-HOHMANN TRANSFERS WITH A COMMON APSE LINE

- Hence $\Delta\varepsilon = \mathbf{v}_1 \cdot \Delta\mathbf{v} + \dfrac{\Delta v^2}{2}$
- From figure, it is apparent that: $\mathbf{v}_1 \cdot \Delta\mathbf{v} = v_1 \Delta v \cos\Delta\gamma$;
- So that:

$$\Delta\varepsilon = v_1 \Delta v \cos\Delta\gamma + \frac{\Delta v^2}{2} = v_1 \Delta v \left(\cos\Delta\gamma + \frac{1}{2}\frac{\Delta v}{v_1} \right)$$

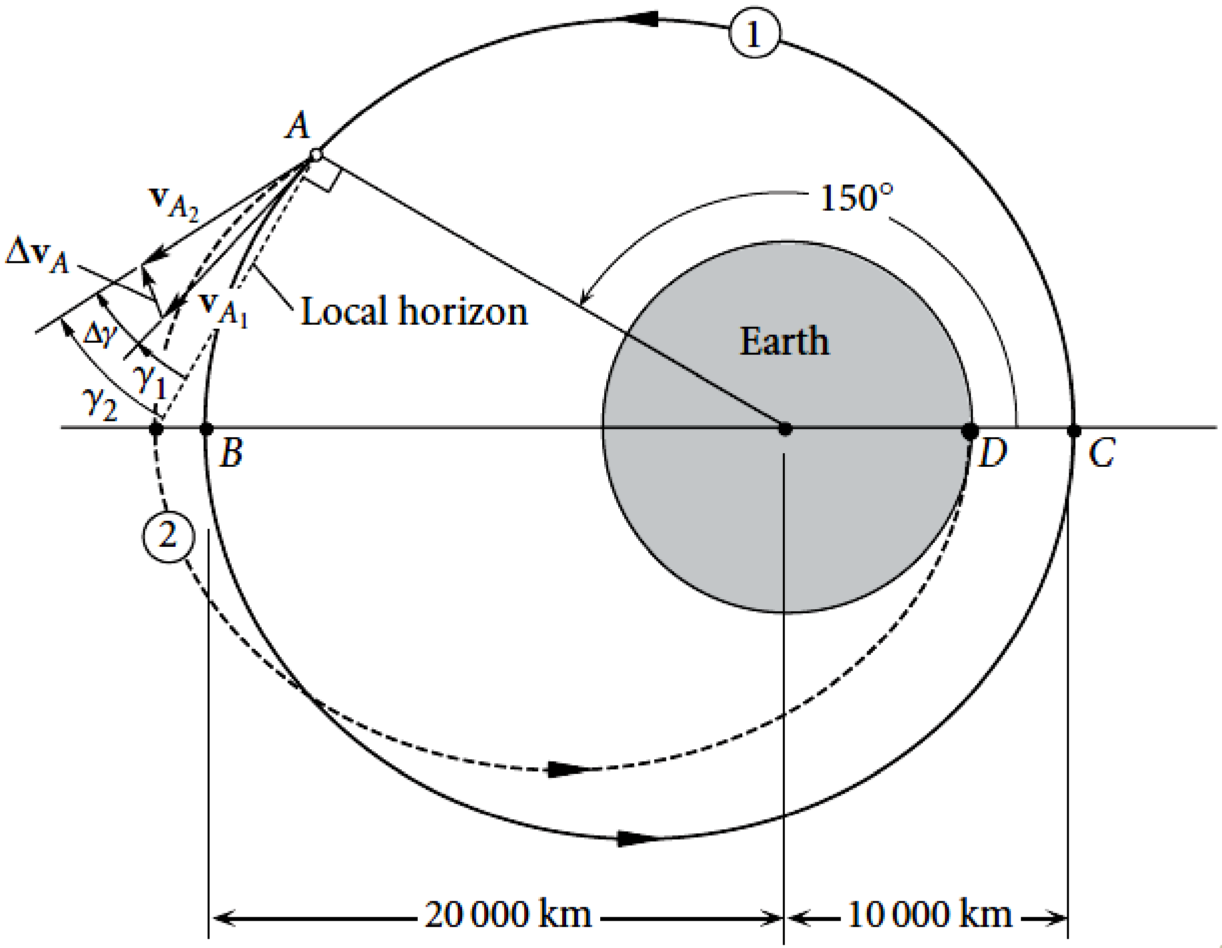
- (our assumption) $\Delta m << m_1$ ⟹ $\Delta v << v_1$ ⟹ $\Delta\varepsilon \approx v_1 \Delta v \cos\Delta\gamma$ (6)
- It shows that, for a given Δv, the change is specific energy is larger the faster the spacecraft is moving (unless, of course, the change in flight path angle is $90°$)
- (NOTE 29,P276,{1})





EXAMPLE 14.1

- A geocentric satellite in orbit 1 of below executes a delta-v maneuver at A which places it on orbit 2, for re-entry at D. calculate Δv at A and its direction relative to the local horizon.



14- NON-HOHMANN TRANSFERS WITH A COMMON APSE LINE

EXAMPLE 14.1

- From the figure we see that

$$r_B = 20\,000\,\text{km} \qquad r_C = 10\,000\,\text{km} \qquad r_D = 6378\,\text{km}$$

- Orbit 1:

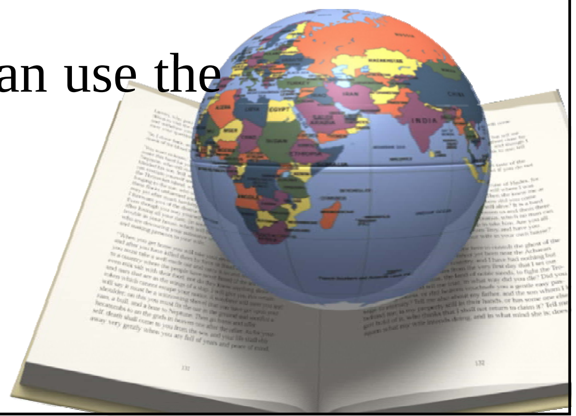
  The eccentricity is $$e_1 = \frac{r_B - r_C}{r_B + r_C} = 0.33333$$

  The angular momentum is obtained from the orbit equation, noting that C is perigee:

$$r_C = \frac{h_1^2}{\mu}\frac{1}{1 + e_1\cos(0)} \Rightarrow 10\,000 = \frac{h_1^2}{398\,600}\frac{1}{1 + 0.33333} \Rightarrow h_1 = 72\,902\,\text{km}^2/\text{s}$$

- With the angular momentum and the eccentricity, we can use the orbit equation to find the radial coordinate of point A,

$$r_A = \frac{72\,902^2}{398\,600}\frac{1}{1 + 0.33333\cdot\cos 150°} = 18\,744\,\text{km}$$



14- NON-HOHMANN TRANSFERS WITH A COMMON APSE LINE

EXAMPLE 14.1

- $h = rv_\perp$ And $v_\perp = \frac{\mu}{h}(1 + e\cos\theta)$ Yields the transverse and radial components of velocity at A on orbit 1,

$$v_{\perp_A})_1 = \frac{h_1}{r_A} = 3.8893\,\text{km/s} \qquad \text{(a)}$$

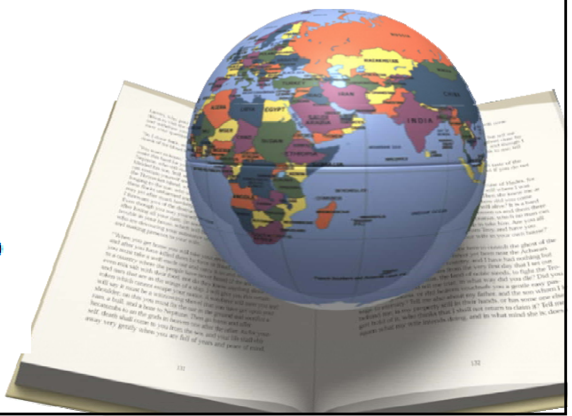
$$v_{r_A})_1 = \frac{\mu}{h_1} e_1 \sin 150^\circ = 0.91127\,\text{km/s}$$

- From these we find the speed at A

$$v_A)_1 = \sqrt{v_{\perp_A})_1^2 + v_{r_A})_1^2} = 3.9946\,\text{km/s}$$

- And the flight path angle,

$$\gamma_1 = \tan^{-1}\frac{v_{r_A})_1}{v_{\perp_A})_1} = \tan^{-1}\frac{0.91127}{3.8893} = 13.187^\circ$$



14- NON-HOHMANN TRANSFERS WITH A COMMON APSE LINE

EXAMPLE 14.1

- Orbit 2:

  the radius and true anomaly of points A and D on orbit 2 are known. Applying the orbit equation at A, we get

$$18\,744 = \frac{h_2^2}{398\,600}\frac{1}{1 + e_2 \cos 150°} \Rightarrow h_2^2 = 7.4715 \times 10^9 - 6.4705 \times 10^9 e_2 \qquad \text{(b)}$$

- Likewise, at point D, which is perigee of orbit 2,

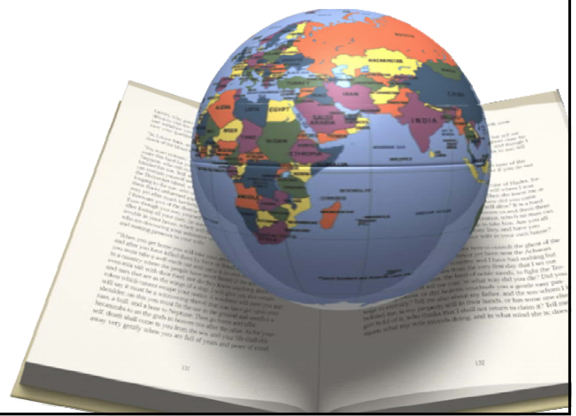
$$6378 = \frac{h_2^2}{398\,600}\frac{1}{1 + e_2} \Rightarrow h_2^2 = 2.5423 \times 10^9 + 2.5423 \times 10^9 e_2 \qquad \text{(c)}$$

- Equating the expressions for h_2^2 in (b) and (c), and solving for e_2, yields

$$e_2 = 0.54692$$

- Whereupon either (b) or (c) may be used to find

$$h_2 = 62\,711 \text{ km}^2/\text{s}$$



14- NON-HOHMANN TRANSFERS WITH A COMMON APSE LINE

EXAMPLE 14.1

- Now we can calculate the radial and perpendicular components of velocity on orbit 2 at point A:

$$v_{\perp_A})_2 = \frac{h_2}{r_A} = 3.3456\,\text{km/s}$$

$$v_{r_A})_2 = \frac{\mu}{h_2} e_2 \sin 150^\circ = 1.7381\,\text{km/s} \qquad \text{(d)}$$

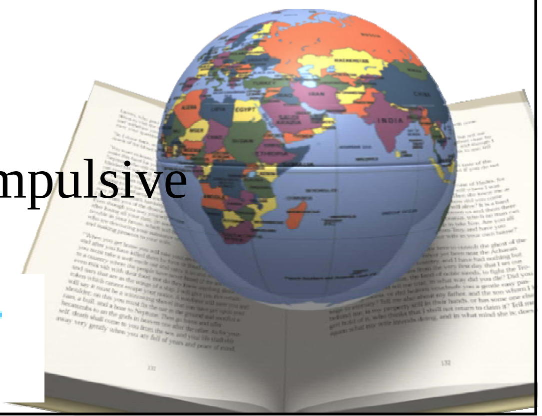
- Hence, the speed and flight path angle at A on orbit 2 are

$$v_A)_2 = \sqrt{v_{\perp_A})_2^2 + v_{r_A})_2^2} = 3.7702\,\text{km/s}$$

$$\gamma_2 = \tan^{-1}\frac{v_{r_A})_2}{v_{\perp_A})_2} = \tan^{-1}\frac{1.7381}{3.3456} = 27.453^\circ$$

- The change in the flight path angle as a result of the impulsive maneuver is

$$\Delta\gamma = \gamma_2 - \gamma_1 = 27.453^\circ - 13.187^\circ = 14.266^\circ$$



14- NON-HOHMANN TRANSFERS WITH A COMMON APSE LINE

EXAMPLE 14.1

- With this we can use equation ? To finally obtain Δv_A,

$$\Delta v_A = \sqrt{v_A)_1^2 + v_A)_2^2 - 2v_A)_1 v_A)_2 \cos \Delta\gamma}$$

$$= \sqrt{3.9946^2 + 3.7702^2 - 2 \cdot 3.9946 \cdot 3.7702 \cdot \cos 14.266}$$

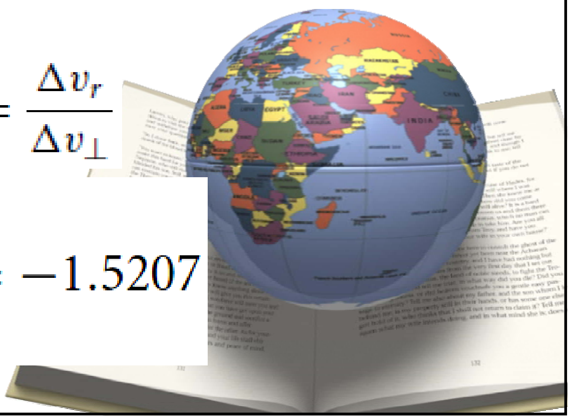
$$\underline{\Delta v_A = 0.9896\,\text{km/s}} \qquad \text{(e)}$$

- Note that Δv_A, is the magnitude of the change in velocity vector Δv_A, at A. that is not the same as the change in the magnitude of the velocity (i.e., the change in speed), which is

$$v_A)_2 - v_A)_1 = 3.9946 - 3.7702 = 0.2244\,\text{km/s}.$$

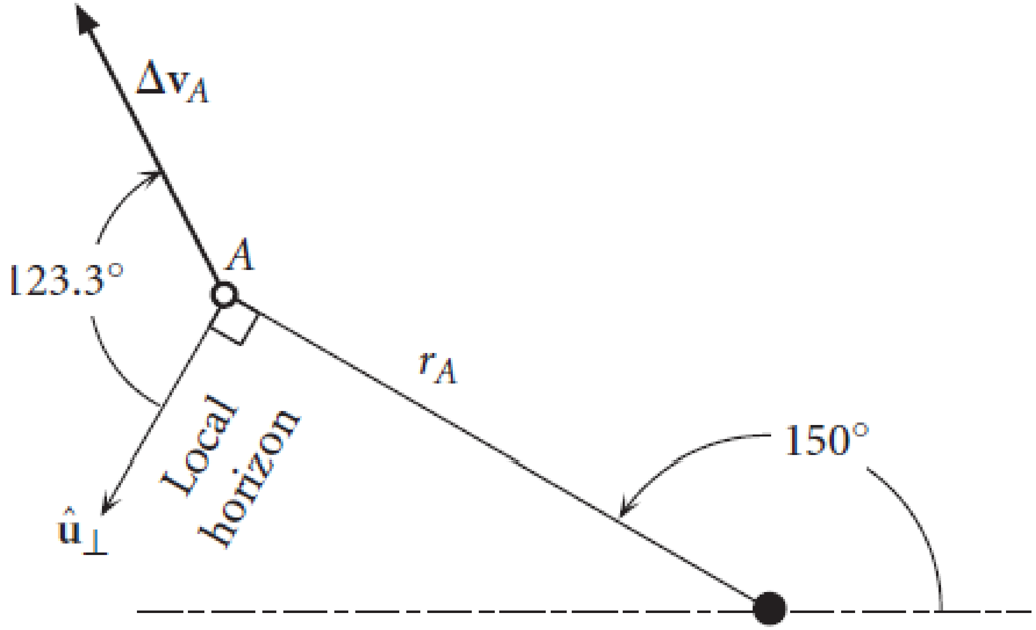
- To find the orientation of Δv_A, we use equation $\tan\phi = \dfrac{\Delta v_r}{\Delta v_\perp}$

$$\tan\phi = \frac{\Delta v_r)_A}{\Delta v_\perp)_A} = \frac{v_{r_A})_2 - v_{r_A})_1}{v_{\perp_A})_2 - v_{\perp_A})_1} = \frac{1.7381 - 0.9113}{3.3456 - 3.8893} = -1.5207$$



14- NON-HOHMANN TRANSFERS WITH A COMMON APSE LINE

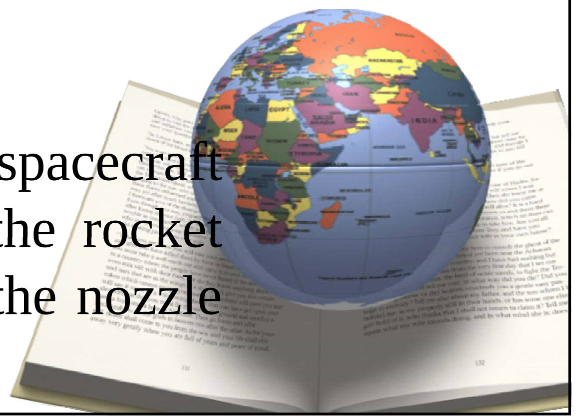
EXAMPLE 14.1



- So that

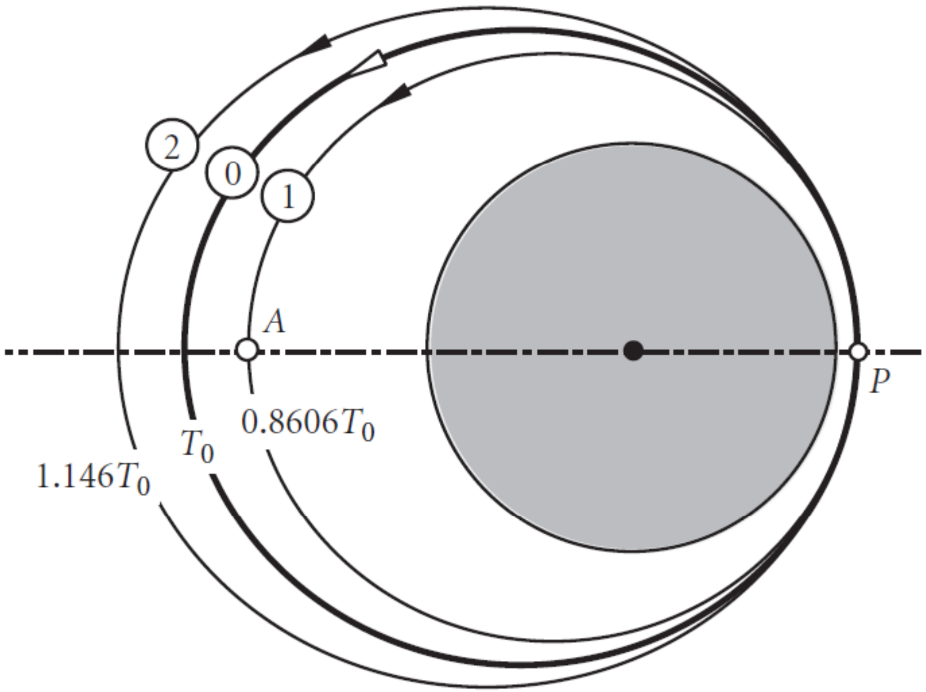
$$\underline{\phi = 123.3^\circ}$$

- This angle is illustrated in above. Prior to firing, the spacecraft would have to be rotated so that the centerline of the rocket motor coincides with the line of action of $\Delta \mathbf{v}_A$, with the nozzle aimed in the opposite direction.

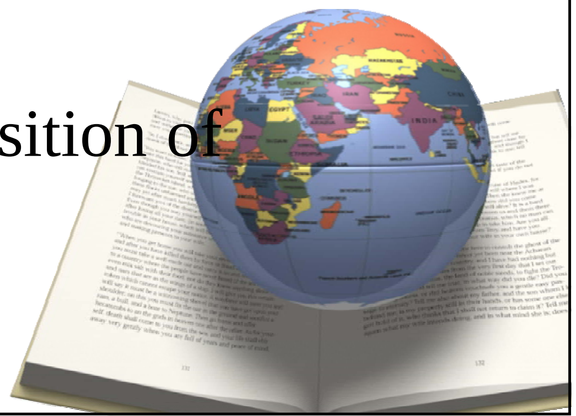


14- PHASING MANEUVERS

- A phasing maneuver is a two-impulse Hohmann transfer from and back to the same orbit, as illustrated in figure:



- Phasing maneuvers are used to change the position of a spacecraft in its orbit.
- (NOTE28,P268,{1}



14- PHASING MANEUVERS

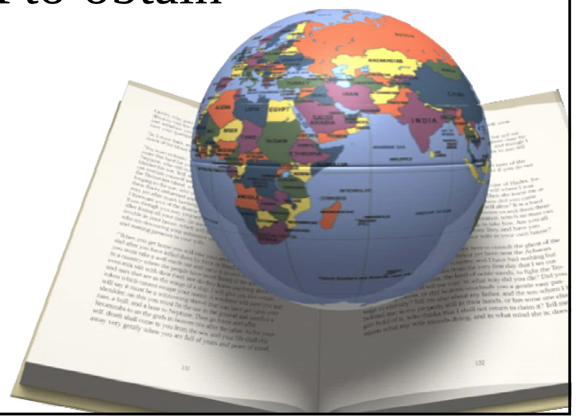
- Once the period T of the phasing orbit is established, then the following equation should be used to determine the semimajor axis of the phasing ellipse:

$$a = \left(\frac{T\sqrt{\mu}}{2\pi}\right)^{\frac{2}{3}} \qquad (5)$$

- With the semimajor axis established, r_A opposite to P is obtained from: $$2a = r_P + r_A$$
- Then we can calculate the eccentricity of phasing orbit from equation: $$e_3 = \frac{r_B - r_A}{r_B + r_A}$$
- Then the orbit equation may be applied at either P or A to obtain the angular momentum

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta}$$

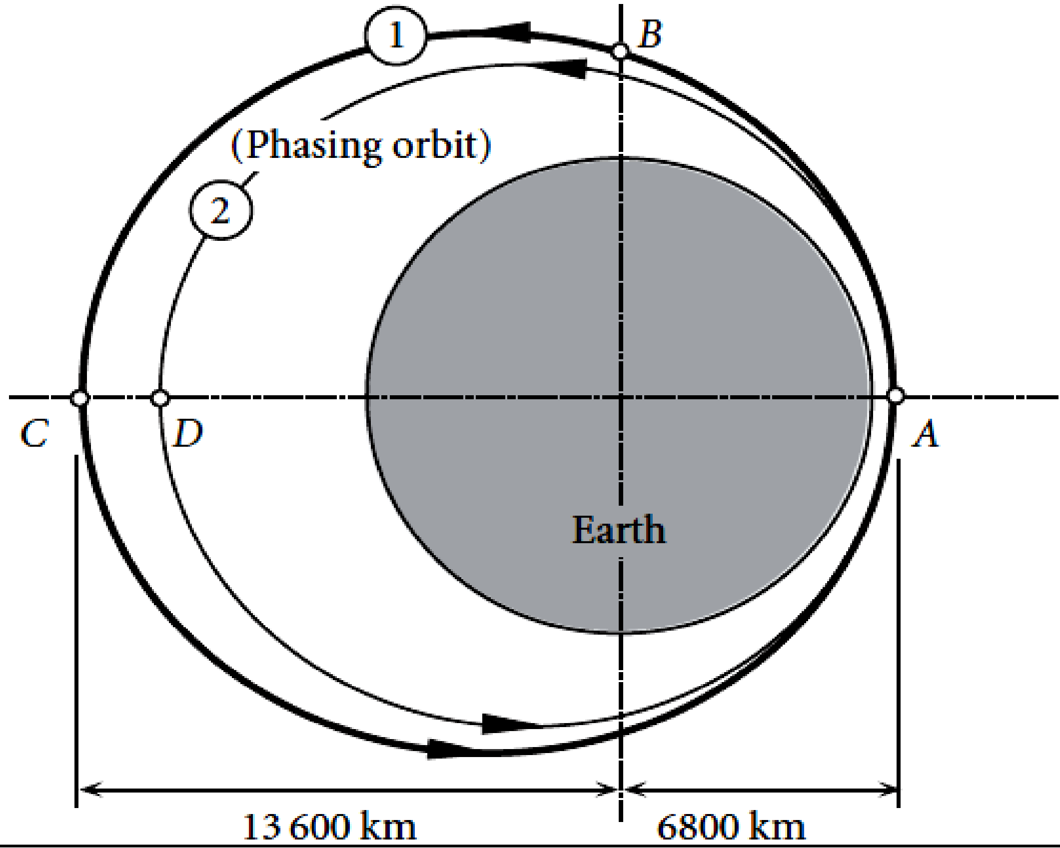
- The phasing orbit is characterized completely

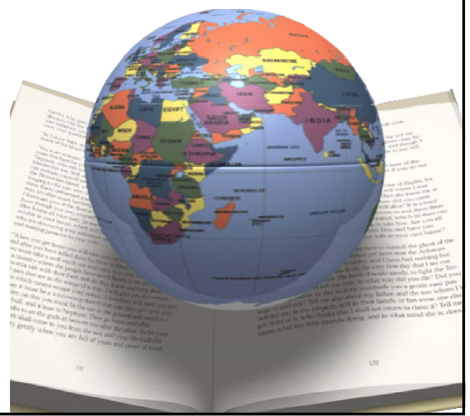


14- PHASING MANEUVERS

EXAMPLE 14.2

- Spacecraft at A and B are in the same orbit (1). At the instant shown, the chaser vehicle at A executes a phasing maneuver so as to catch the target spacecraft back at A after just one revolution of the chaser's phasing orbit (2). What is the required total delta-v?





14- PHASING MANEUVERS

EXAMPLE 14.2

- From the figure, $r_A = 6800\,\text{km} \qquad r_C = 13\,600\,\text{km}$
- Orbit 1:

The eccentricity of orbit 1 is $e_1 = \dfrac{r_C - r_A}{r_C + r_A} = 0.33333$

Evaluating the orbit equation at A, we find

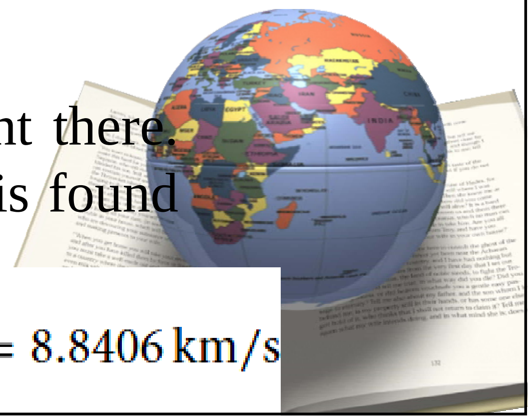
$$r_A = \frac{h_1^2}{\mu}\frac{1}{1 + e_1\cos(0)} \Rightarrow 6800 = \frac{h_1^2}{398\,600}\frac{1}{1 + 0.3333} \Rightarrow h_1 = 60\,116\,\text{km}^2/\text{s}$$

The period is found using equation ?

$$T_1 = \frac{2\pi}{\mu^2}\left(\frac{h_1}{\sqrt{1 - e_1^2}}\right)^3 = \frac{2\pi}{398\,600^2}\left(\frac{60\,116}{\sqrt{1 - 0.33333^2}}\right)^3 = 10\,252\,\text{s}$$

- Since A is perigee, there is no radial velocity component there. The speed, directed entirely in the transverse direction, is found from the angular momentum formula,

$$v_{A1} = \frac{h_1}{r_A} = \frac{60\,116}{6800} = 8.8406\,\text{km/s}$$



14- PHASING MANEUVERS

EXAMPLE 14.2

The phasing orbit must have a period T_2 equal to the time it takes the target vehicle at B to coast around to point A on orbit 1. we can determine the flight time by calculating Δt_{AB} time from A to B and subtracting that result from the period T_1 of orbit 1. At B the true anomaly $\theta_A = 90°$. therefore, according to equation ?

$$\tan\frac{E_B}{2} = \sqrt{\frac{1-e_1}{1+e_1}}\tan\frac{\theta_B}{2} = \sqrt{\frac{1-0.33333}{1+0.33333}}\tan\frac{90°}{2}$$

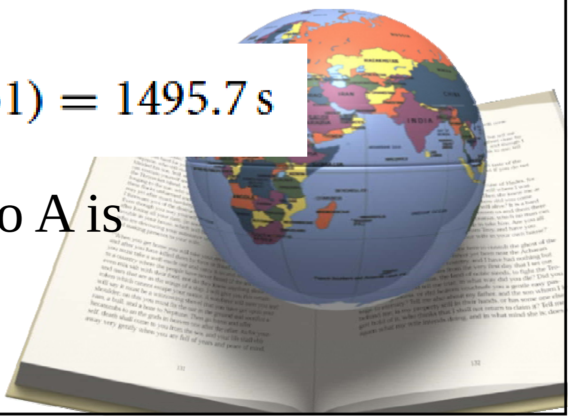
$$= 0.70711 \Rightarrow E_B = 1.2310\,\text{rad}$$

- Then, from Kepler's equation (?), we get

$$\Delta t_{AB} = \frac{T_1}{2\pi}(E_B - e_1 \sin E_B) = \frac{10\,252}{2\pi}(1.231 - 0.33333 \cdot \sin 1.231) = 1495.7\,\text{s}$$

- Thus, the time of flight of the target spacecraft from B to A is

$$\Delta t_{BA} = T_1 - \Delta t_{AB} = 10\,252 - 1495.7 = 8756.3\,\text{s}$$



14- PHASING MANEUVERS

EXAMPLE 14.2

- Orbit 2:

The period of orbit 2 must equal Δt_{BA} so that the chaser will arrive at A when the target does. That is,

$$T_2 = 8756.3\ \text{s}$$

This, together with the period formula, equation ?, yields the semimajor axis of orbit 2,

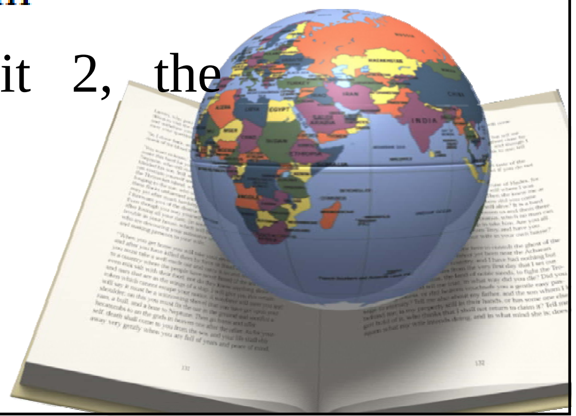
$$T_2 = \frac{2\pi}{\sqrt{\mu}} a_2^{\frac{3}{2}} \Rightarrow 8756.2 = \frac{2\pi}{\sqrt{398\,600}} a_2^{\frac{3}{2}} \Rightarrow a_2 = 9182.1\ \text{km} \qquad \text{(a)}$$

Since $2a_2 = r_A + r_D$, we find

$$r_D = 2a_2 - r_A = 2 \cdot 9182.1 - 6800 = 11\,564\ \text{km}$$

Therefore, point A is indeed the perigee of orbit 2, the eccentricity of which can now be determined:

$$e_2 = \frac{r_D - r_A}{r_D + r_A} = 0.25943$$



14- PHASING MANEUVERS

EXAMPLE 14.2

Evaluating the orbit equation at point A orbit 2 yields its angular momentum,

$$r_A = \frac{h_2^2}{\mu}\frac{1}{1 + e_2\cos(0)} \Rightarrow 6800 = \frac{h_2^2}{398\,600}\frac{1}{1 + 0.25943} \Rightarrow h_2 = 58\,426\ \text{km}^2/\text{s}$$

Finally, we can calculate the speed at perigee of orbit 2,

$$v_{A_2} = \frac{h_2}{r_A} = \frac{58\,426}{6800} = 8.5921\ \text{km/s}$$

At the beginning of the phasing maneuver,

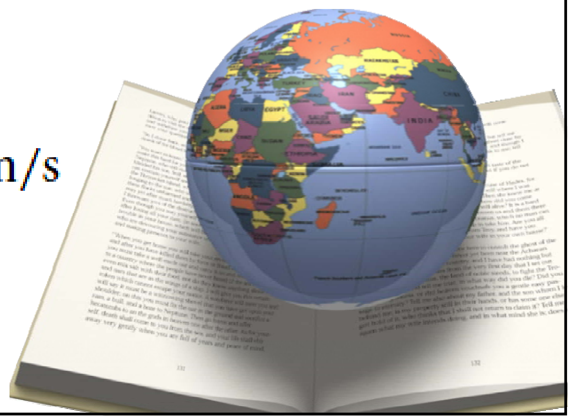
$$\Delta v_A = v_{A_2} - v_{A_1} = 8.5921 - 8.8406 = -0.24851\ \text{km/s}$$

At the end of the phasing maneuver,

$$\Delta v_A = v_{A_1} - v_{A_2} = 8.8406 - 8.5921 = 0.24851\ \text{km/s}$$

The total delta-v, therefore, is

$$\Delta v_{\text{total}} = |-0.24851| + |0.24851| = \underline{0.4970\ \text{km/s}}$$



14- PHASING MANEUVERS

EXAMPLE 14.3

- It is desired to shift the longitude of a GEO satellite $12°$ westward in three revolutions of its phasing orbit. Calculate the delta-v requirement.

  this problem is illustrated in ? . It may be recalled from equation ?, ? and ? That the angular velocity of the earth, the radius to GEO and the speed in GEO are, respectively,

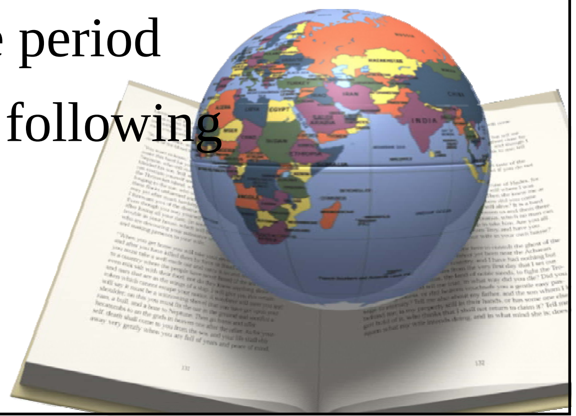
$$\omega_E = \omega_{GEO} = 72.922 \times 10^{-6} \text{ rad/s}$$

$$r_{GEO} = 42\,164 \text{ km} \qquad \text{(a)}$$

$$v_{GEO} = 3.0747 \text{ km/s}$$

- Let $\Delta\Lambda$ be the change in longitude in radians. Then the period T_2 of the phasing orbit can be obtained from the following formula,

$$\omega_E(3T_2) = 3 \cdot 2\pi + \Delta\Lambda \qquad \text{(b)}$$

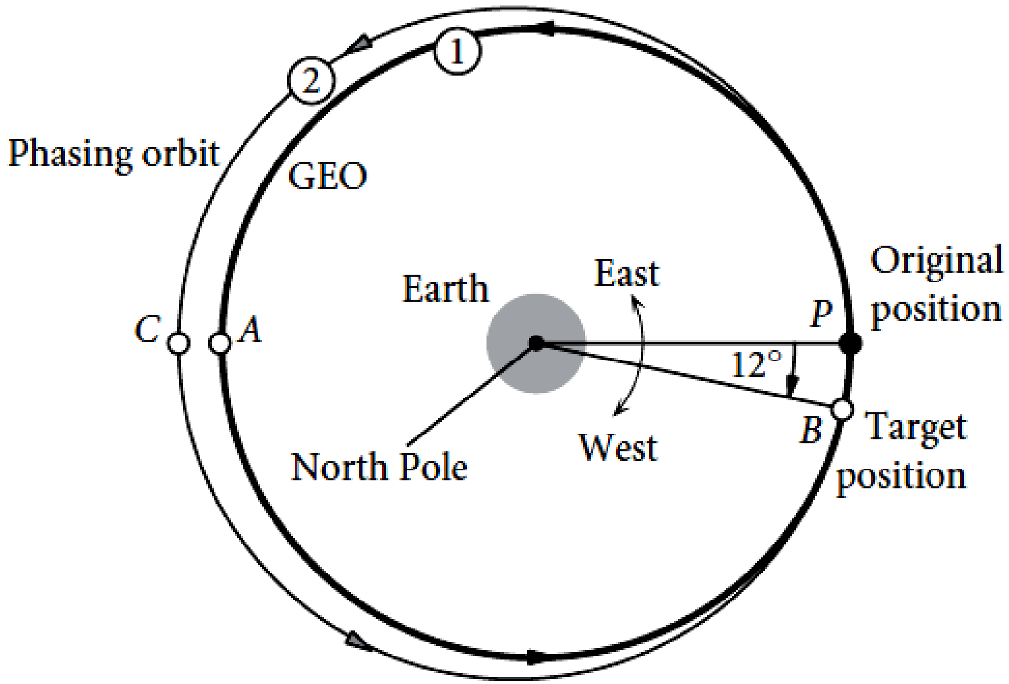


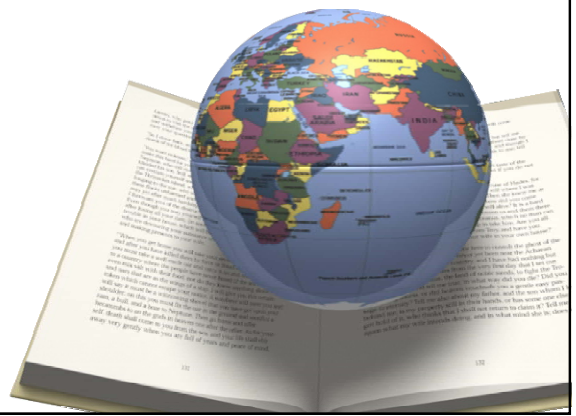
14- PHASING MANEUVERS

EXAMPLE 14.3

which states that after three circuits of the phasing orbit, the original position of the satellite will be $\Delta\Lambda$ radians east of P. in other words, the satellite will end up $\Delta\Lambda$ radians west of its original position in GEO, as desired. From (b) we obtain,

$$T_2 = \frac{1}{3}\frac{\Delta\Lambda + 6\pi}{\omega_E} = \frac{1}{3}\cdot\frac{12^\circ\cdot\frac{\pi}{180^\circ} + 6\pi}{72.922\times10^{-6}} = 87\,121\ \text{s}$$





14- PHASING MANEUVERS

EXAMPLE 14.3

Note that the period of GEO is

$$T_{\mathrm{GEO}} = \frac{2\pi}{\omega_{\mathrm{GEO}}} = 86\,163\ \mathrm{s}$$

The satellite in its slower phasing orbit appears to drift westward at the rate

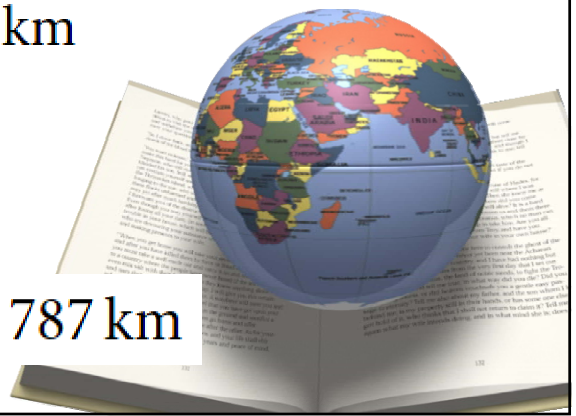
$$\dot{\Lambda} = \frac{\Delta\Lambda}{3T_2} = 8.0133 \times 10^{-7}\ \mathrm{rad/s} = 3.9669^\circ/\mathrm{day}$$

Having the period, we can use equation ? To obtain the semimajor axis of orbit 2,

$$a = \left(\frac{T\sqrt{\mu}}{2\pi}\right)^{\frac{2}{3}} = \left(\frac{87\,121\sqrt{398\,600}}{2\pi}\right)^{\frac{2}{3}} = 42\,476\ \mathrm{km}$$

From this we find the radial coordinate of C,

$$2a_2 = r_P + r_C \quad \Rightarrow \quad r_C = 2 \cdot 42\,476 - 42\,164 = 42\,787\ \mathrm{km}$$



14- PHASING MANEUVERS

EXAMPLE 14.3

Now we can find the eccentricity of orbit 2,

$$e_2 = \frac{r_C - r_A}{r_C + r_A} = \frac{42\,787 - 42\,164}{42\,787 + 42\,164} = 0.0073395$$

And the angular momentum follows from applying the orbit equation at P (or C) of orbit 2:

$$r_P = \frac{h_2^2}{\mu}\frac{1}{1 + e_2\cos(0)} \Rightarrow 42\,164 = \frac{h_2^2}{398\,600}\frac{1}{1 + 0.0073395} \Rightarrow h_2 = 130\,120\,\text{km}^2/\text{s}$$

at P the speed in orbit 2 is

$$v_{P_2} = \frac{130\,120}{42\,164} = 3.0859\,\text{km/s}$$

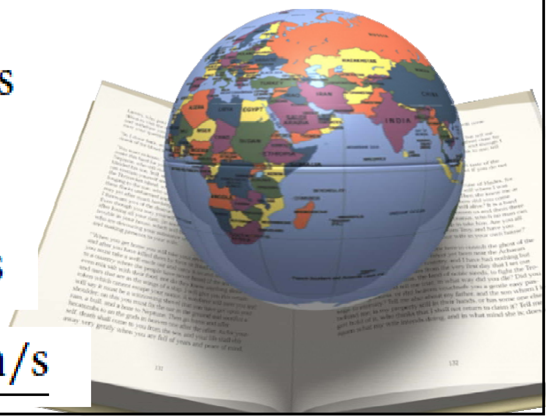
therefore, at the beginning of the phasing orbit,

$$\Delta v = v_{P_2} - v_{\text{GEO}} = 3.0859 - 3.0747 = 0.01126\,\text{km/s}$$

at the end of the phasing maneuver,

$$\Delta v = v_{\text{GEO}} - v_{P_2} = 3.0747 - 3.08597 = -0.01126\,\text{km/s}$$

Therefore, $\Delta v_{\text{total}} = |0.01126| + |-0.01126| = \underline{0.022525\,\text{km/s}}$



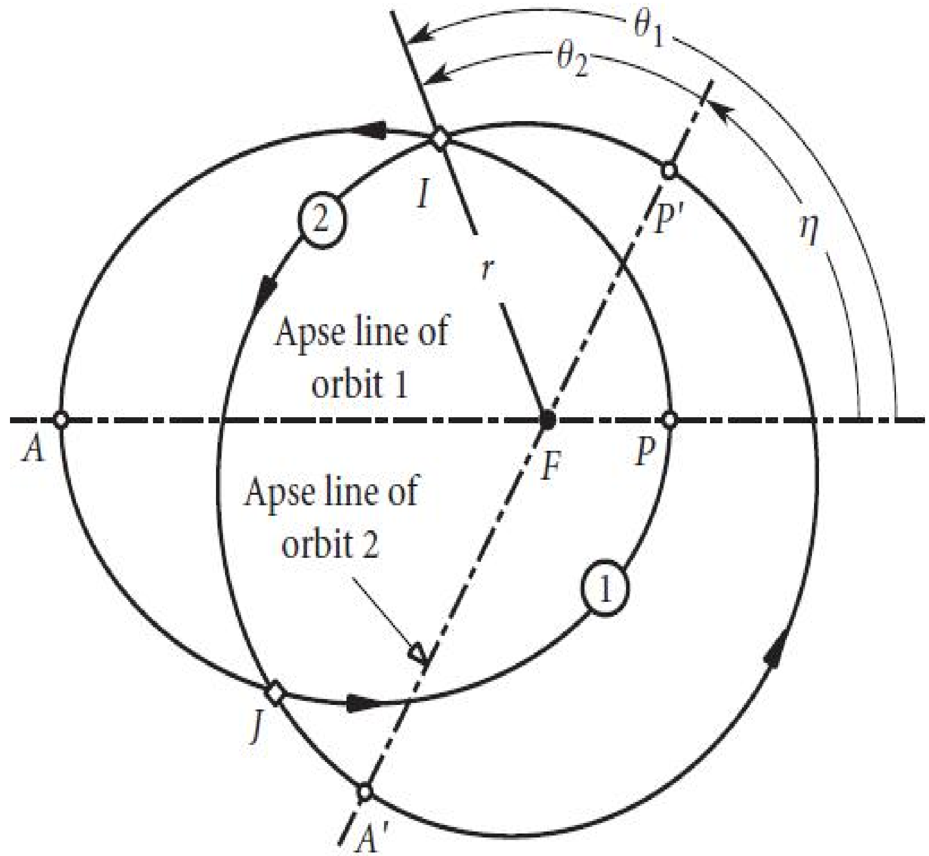
CHAPTER 15

Apse line rotation

CHAPTER CONTENT

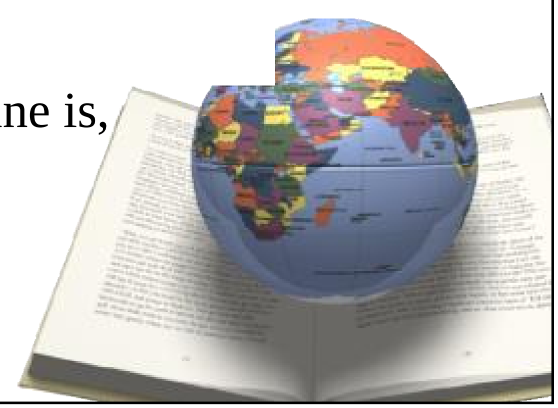
15- APSE LINE ROTATION

- **T**he above figure shows two intersecting orbits which have a common focus, but their apse lines are not collinear.
- A Hohmann transfer between them is clearly impossible
- The opportunity for transfer from one orbit to the other by a single impulsive maneuver occurs at points I and j



- As can be seen from the figure, the rotation o η the apse line is,

$$\eta = \theta_1 - \theta_2 \qquad (1)$$

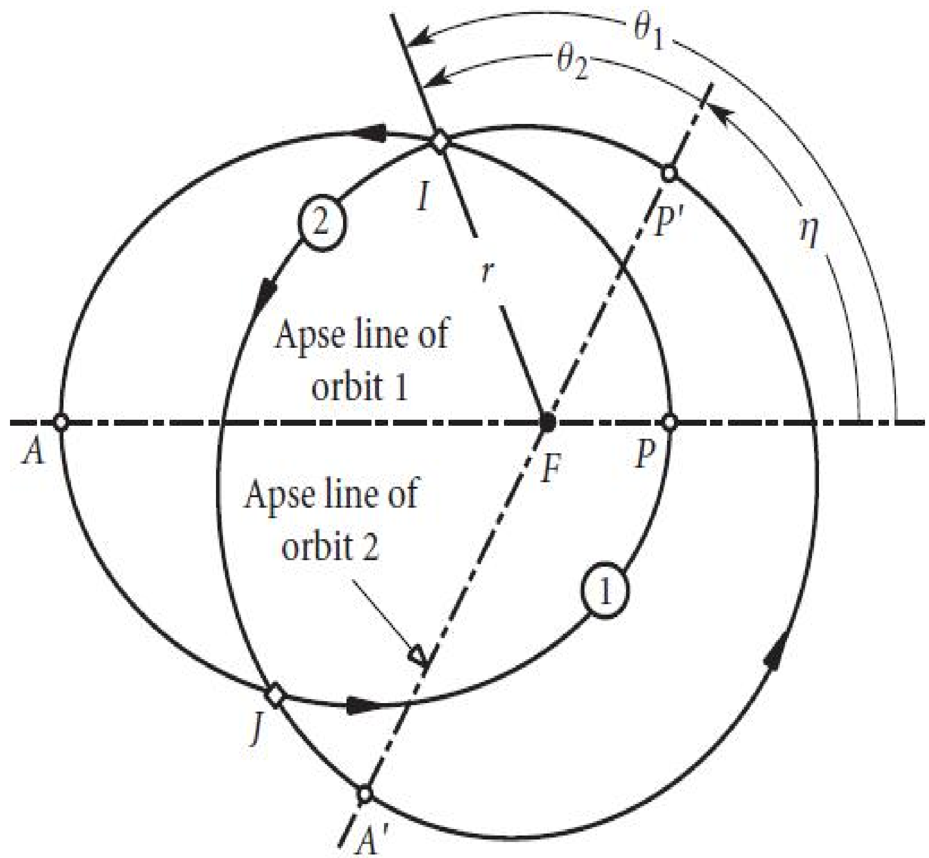


15- APSE LINE ROTATION

- we will consider two cases of apse line rotation.

1- the first case is that in which the apse line rotation η is given as well as e and h for both orbits.

2- the second case is that in which the impulsive maneuver takes place at a given true anomaly θ_1 on orbit 1.



- In the first case the problem is to find the true anomaly of I and J to both orbits, and in the second case the problem is to determine the angle of rotation η and the eccentricity of the new orbit.

15- APSE LINE ROTATION

- Now we will consider the first case.
- The radius of the point I is given by either of the following:

$$r_I)_1 = \frac{h_1^2}{\mu}\frac{1}{1+e_1\cos\theta_1}$$

$$r_I)_2 = \frac{h_2^2}{\mu}\frac{1}{1+e_2\cos\theta_2}$$

$$r_I)_1 = r_I)_2$$

$\Rightarrow$

$$e_1h_2^2\cos\theta_1 - e_2h_1^2\cos\theta_2 = h_1^2 - h_2^2$$

$$\theta_2 = \theta_1 - \eta$$

$$\cos(\theta_1 - \eta) = \cos\theta_1\cos\eta + \sin\theta_1\sin\eta$$

$$\Rightarrow \quad a\cos\theta_1 + b\sin\theta_1 = c \qquad \text{(2a)}$$

- where:

$$a = e_1h_2^2 - e_2h_1^2\cos\eta \qquad b = -e_2h_1^2\sin\eta \qquad c = h_1^2 - h_2^2 \qquad \text{(2b)}$$

15- APSE LINE ROTATION

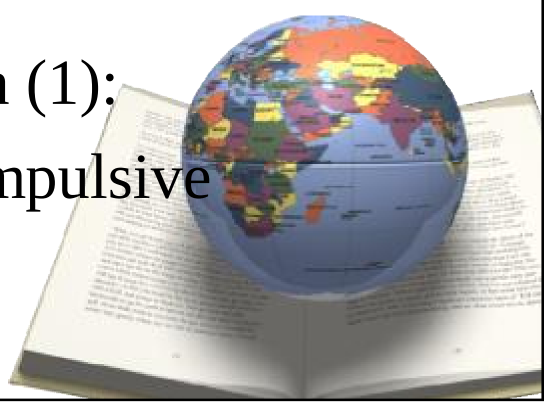
- Equation (2a) has two roots, corresponding to two points I and J:

$$\theta_1 = \phi \pm \cos^{-1}\left(\frac{c}{a}\cos\phi\right) \qquad \text{(3a)}$$

- where:

$$\phi = \tan^{-1}\frac{b}{a} \qquad \text{(3b)}$$

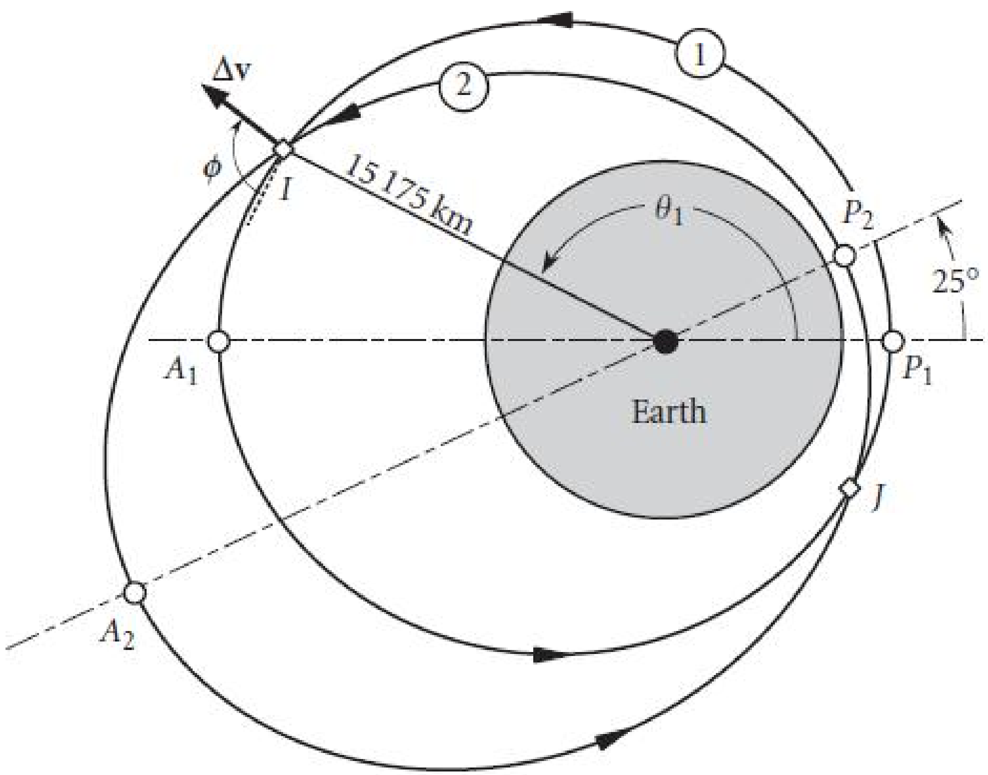
- Having found θ_1 we obtain θ_2 from equation (1):
- Now we can also compute Δv for impulsive maneuver.



15- APSE LINE ROTATION

EXAMPLE 15.1

- An earth satellite is in an 8000km radius orbit (orbit 1 of below figure) calculate the delta-v and the true anomaly θ_1 required to obtain a 7000km by 21000km radius orbit (orbit 2) whose apse line is rotated $25°$ counterclockwise. Indicate the orientation ϕ of $\Delta \mathbf{v}$ to the local horizon.



15- APSE LINE ROTATION

EXAMPLE 15.1

- The eccentricities of the two orbits are

$$e_1 = \frac{r_{A_1} - r_{P_1}}{r_{A_1} + r_{P_1}} = \frac{16\,000 - 8000}{16\,000 + 8000} = 0.33333 \qquad \text{(a)}$$

$$e_2 = \frac{r_{A_2} - r_{P_2}}{r_{A_2} + r_{P_2}} = \frac{21\,000 - 7000}{21\,000 + 7000} = 0.5$$

- The orbit equation yields the angular momenta

$$r_{P_1} = \frac{h_1^2}{\mu}\frac{1}{1 + e_1\cos(0)} \Rightarrow 8000 = \frac{h_1^2}{398\,600}\frac{1}{1 + 0.33333} \Rightarrow h_1 = 65\,205\ \text{km}^2/\text{s}$$

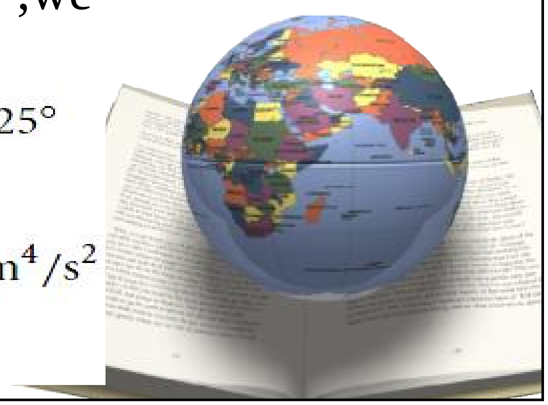
$$r_{P_2} = \frac{h_2^2}{\mu}\frac{1}{1 + e_2\cos(0)} \Rightarrow 7000 = \frac{h_2^2}{398\,600}\frac{1}{1 + 0.5} \Rightarrow h_2 = 64\,694 \qquad \text{(b)}$$

- Using these orbital parameters and the fact that $\eta = 25°$,we calculate the terms in equation (2b)

$$a = e_1 h_2^2 - e_2 h_1^2 \cos\eta = 0.3333 \cdot 64\,694^2 - 0.5 \cdot 65\,205^2 \cdot \cos 25°$$
$$= -5.3159 \times 10^8\ \text{km}^4/\text{s}^2$$

$$b = -e_2 h_1^2 \sin\eta = -0.5 \cdot 65\,205^2 \sin 25° = -8.9843 \times 10^8\ \text{km}^4/\text{s}^2$$

$$c = h_1^2 - h_2^2 = 65\,205^2 - 64\,694^2 = 6.6433 \times 10^7\ \text{km}^4/\text{s}^2$$



15- APSE LINE ROTATION

EXAMPLE 15.1

- Then equation (3) yields

$$\phi = \tan^{-1}\frac{-8.9843 \times 10^8}{-5.3159 \times 10^8} = 59.39°$$

$$\theta_1 = 59.39° \pm \cos^{-1}\left(\frac{6.6433 \times 10^7}{-5.3159 \times 10^8}\cos 59.39°\right) = 59.39° \pm 93.65°$$

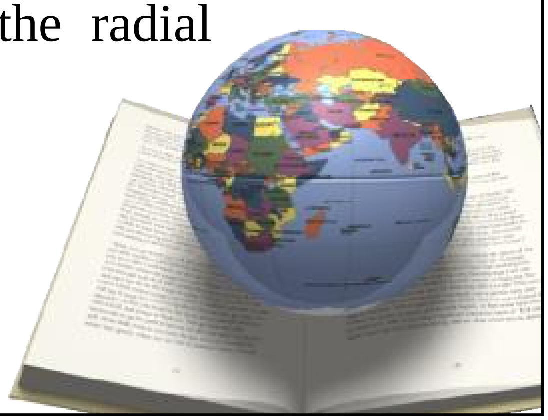
- Thus, the true anomaly of point I, the point of interest, is

$$\theta_1 = 153.04° \qquad \text{(c)}$$

(For point J, $\theta_1 = 325.74°$.)

- With the true anomaly available, we can evaluate the radial coordinate of the maneuver point,

$$r = \frac{h_1^2}{\mu}\,\frac{1}{1 + e_1\cos 153.04°} = 15\,175\,\text{km}$$



15- APSE LINE ROTATION

EXAMPLE 15.1

- The velocity components and flight path angle for orbit 1 at point I are

$$v_{\perp_1} = \frac{h_1}{r} = \frac{65\,205}{15\,175} = 4.2968\,\text{km/s}$$

$$v_{r_1} = \frac{\mu}{h_1} e_1 \sin 153.04^\circ = \frac{398\,600}{65\,205} \cdot 0.33333 \cdot \sin 153.04^\circ = 0.92393\,\text{km/s}$$

$$\gamma_1 = \tan^{-1} \frac{v_{r_1}}{v_{\perp_1}} = 12.135^\circ$$

- The speed of the satellite in orbit 1 is, therefore

$$v_1 = \sqrt{v_{r_1}^2 + v_{\perp_1}^2} = 4.3950\,\text{km/s}$$

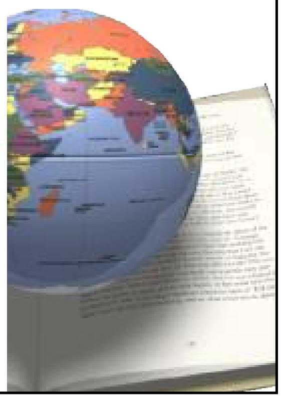
- Likewise, for orbit 2,

$$v_{\perp_2} = \frac{h_2}{r} = \frac{64\,694}{15\,175} = 4.2631\,\text{km/s}$$

$$v_{r_2} = \frac{\mu}{h_2} e_2 \sin(153.04^\circ - 25^\circ) = \frac{398\,600}{64\,694} \cdot 0.5 \cdot \sin 128.04^\circ = 2.4264\,\text{km/s}$$

$$\gamma_2 = \tan^{-1} \frac{v_{r_2}}{v_{\perp_2}} = 29.647^\circ$$

$$v_2 = \sqrt{v_{r_2}^2 + v_{\perp_2}^2} = 4.9053\,\text{km/s}$$



15- APSE LINE ROTATION

EXAMPLE 15.1

- Equation ? Is used to find Δv,

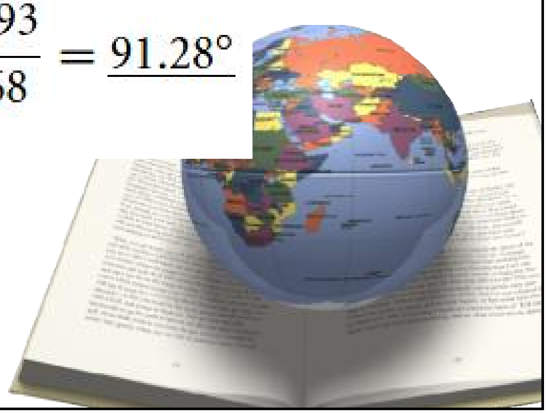
$$\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos(\gamma_2 - \gamma_1)}$$

$$= \sqrt{4.3950^2 + 4.9053^2 - 2 \cdot 4.3950 \cdot 4.9053 \cos(29.647° - 12.135°)}$$

$$\underline{\Delta v = 1.503\,\text{km/s}}$$

- The angle ϕ which the vector $\Delta \mathrm{v}$ males with the local horizon is given by following equation

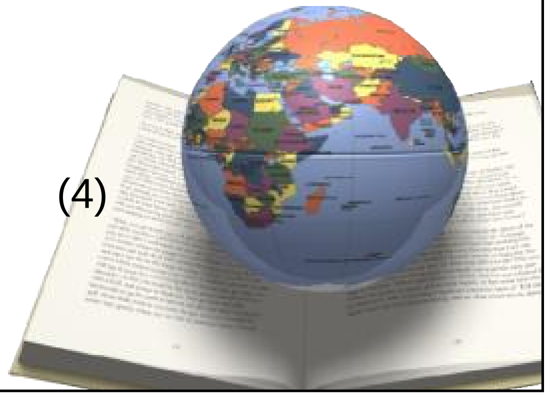
$$\phi = \tan^{-1}\frac{\Delta v_r}{\Delta v_\perp} = \tan^{-1}\frac{v_{r_2} - v_{r_1}}{v_{\perp_2} - v_{\perp_1}} = \tan^{-1}\frac{2.4264 - 0.92393}{4.2631 - 4.2968} = \underline{91.28°}$$



15- APSE LINE ROTATION

- The second case of apse line rotation is that in which the impulsive maneuver takes place at a given true anomaly θ_1 on orbit 1.
- The problem is to determine the angle of rotation η and the eccentricity e_2 of the new orbit.
- The impulsive maneuver creates a change in the radial and transverse velocity components at point I of orbit 1.
- From the angular momentum formula, we obtain the angular momentum of orbit 2.

$$h = rv_{\perp} \Rightarrow h_2 = r(v_{\perp} + \Delta v_{\perp}) = h_1 + r\Delta v_{\perp} \quad (4)$$



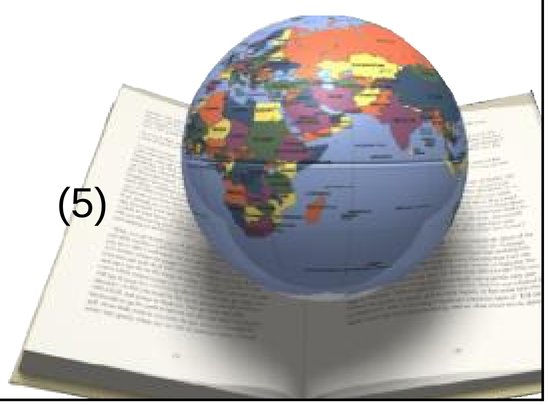
15- APSE LINE ROTATION

- The formula for radial velocity $v_r = (\mu/h) e \sin\theta$ applied to orbit 2 at point I, where:

$$\left.\begin{aligned} v_{r_2} &= v_{r_1} + \Delta v_r \\ \theta_2 &= \theta_1 - \eta \end{aligned}\right\} \Rightarrow \quad v_{r_1} + \Delta v_r = \frac{\mu}{h_2} e_2 \sin\theta_2$$

- Substituting equation (4) into this expression and solving for $\sin\theta_2$ leads to:

$$\sin\theta_2 = \frac{1}{e_2} \frac{(h_1 + r\Delta v_\perp)(\mu e_1 \sin\theta_1 + h_1 \Delta v_r)}{\mu h_1} \qquad (5)$$



15- APSE LINE ROTATION

- From the orbit equation, we have at point I

$$r = \frac{h_1^2}{\mu} \frac{1}{1 + e_1 \cos\theta_1} \quad \text{(orbit 1)}$$

$$r = \frac{h_2^2}{\mu} \frac{1}{1 + e_2 \cos\theta_2} \quad \text{(orbit 2)}$$

- Equating these two expressions for r, substituting equation (4) and solving for $\cos\theta_2$ yields:

$$\cos\theta_2 = \frac{1}{e_2} \frac{(h_1 + r\Delta v_\perp)^2 e_1 \cos\theta_1 + (2h_1 + r\Delta v_\perp) r\Delta v_\perp}{h_1^2} \quad (6)$$

15- APSE LINE ROTATION

- Finally we obtain:

(6)

$$\Rightarrow \quad \tan\theta_2 = \frac{h_1}{\mu}\frac{(h_1 + r\Delta v_\perp)(\mu e_1 \sin\theta_1 + h_1 \Delta v_r)}{(h_1 + r\Delta v_\perp)^2 e_1 \cos\theta_1 + (2h_1 + r\Delta v_\perp) r\Delta v_\perp} \qquad \text{(7a)}$$

(5)

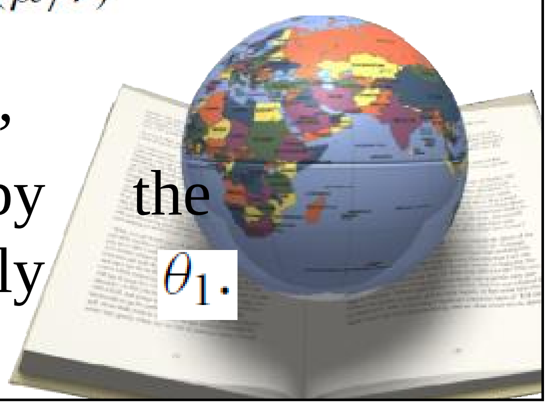
- Equation (7a) can be simplified a bit by the next replacements:

$$\mu e_1 \sin\theta_1 \Rightarrow h_1 v_{r_1} \qquad h_1 \Rightarrow r v_{\perp_1}$$

- So that:

$$\tan\theta_2 = \frac{(v_{\perp_1} + \Delta v_\perp)(v_{r_1} + \Delta v_r)}{(v_{\perp_1} + \Delta v_\perp)^2 e_1 \cos\theta_1 + (2v_{\perp_1} + \Delta v_\perp)\Delta v_\perp}\frac{v_{\perp_1}^2}{(\mu/r)} \qquad \text{(7b)}$$

- Equations (7) show how the apse line rotation, $\eta = \theta_1 - \theta_2$,is completely determined by the components of $\Delta\mathbf{v}$mparted at the true anomaly θ_1.



15- APSE LINE ROTATION

- After solving equation 7 (a or b), we will:

(7) ⇒ θ_2 ⇒ (6) , (5) ⇒ e_2

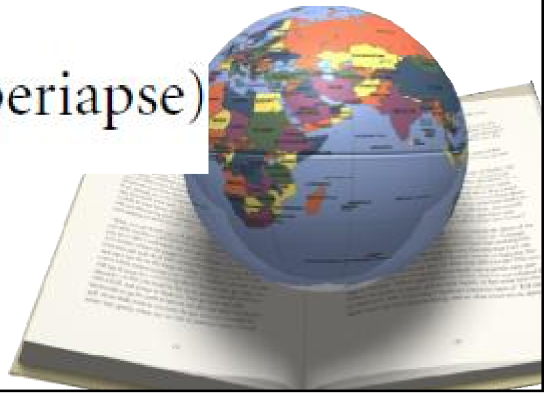
(4) ⇒ h_2

➡ Rotate orbit 2 is completely specified

- If the impulsive maneuver takes place at the periapse of orbit 1, so that $\theta_1 = v_r = 0$ and if it is also true that $\Delta v_\perp = 0$, then equation (7c) yields.

$$\tan\eta = -\frac{r v_{\perp_1}}{\mu e_1}\Delta v_r \quad \text{(with radial impulse at periapse)}$$

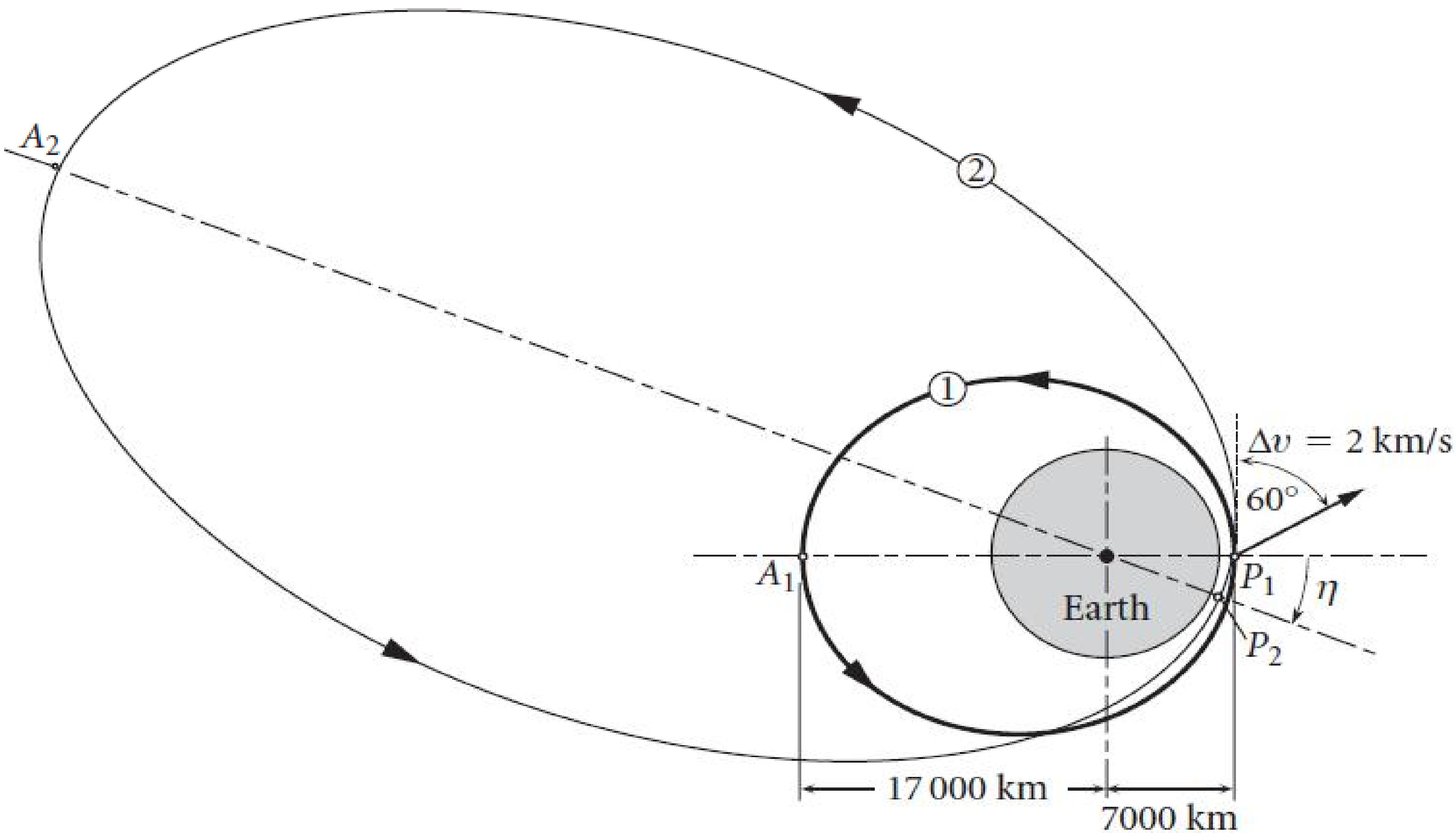
- (NOTE30,P283,{1})

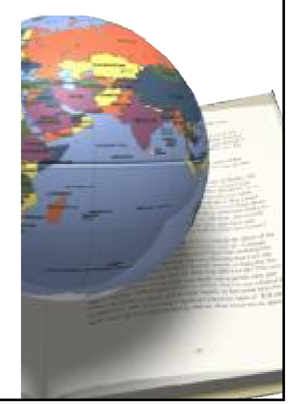


15- APSE LINE ROTATION

EXAMPLE 15.2

- An earth satellite in orbit 1 of bellow figure undergoes the indicated delta-v maneuver at its perigee. Determine thη rotation of its apse line.





15- APSE LINE ROTATION

EXAMPLE 15.2

- From the figure

$$r_{A_1} = 17\,000\,\text{km} \qquad r_{P_1} = 7000\,\text{km}$$

- The eccentricity of orbit 1 is

$$e_1 = \frac{r_{A_1} - r_{P_1}}{r_{A_1} + r_{P_1}} = 0.41667 \qquad \text{(a)}$$

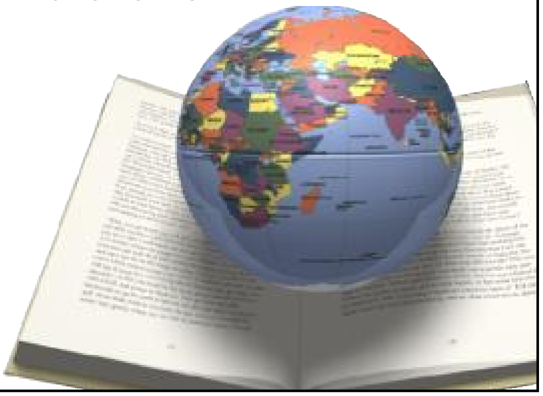
- As usual, we use the orbit equation to find the angular momentum,

$$r_{P_1} = \frac{h_1^2}{\mu}\frac{1}{1 + e_1\cos(0)} \Rightarrow 7000 = \frac{h_1^2}{398\,600}\frac{1}{1 + 0.41667} \Rightarrow h_1 = 62\,871\,\text{km}^2/\text{s}$$

- At the maneuver point P_1, the angular momentum formula and the fact that P_1, is perigee of orbit 1 $(\theta_1 = 0)$ imply that

$$v_{\perp_1} = \frac{h_1}{r_{P_1}} = \frac{62\,871}{7000} = 8.9816\,\text{km/s} \qquad \text{(b)}$$

$$v_{r_1} = 0$$



15- APSE LINE ROTATION

EXAMPLE 15.2

- From the above figure It is clear that

$$\Delta v_\perp = \Delta v \cos 60^\circ = 1\,\text{km/s}$$
$$\Delta v_r = \Delta v \sin 60^\circ = 1.7321\,\text{km/s} \qquad \text{(c)}$$

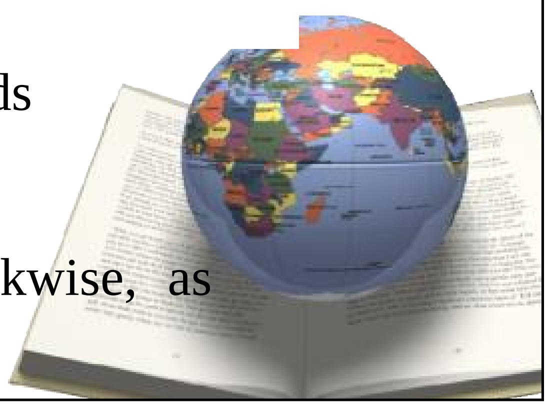
- The compute θ_2, we use equation (7b) Together with (a), (b) and (c):

$$\tan\theta_2 = \frac{(v_{\perp_1} + \Delta v_\perp)(v_{r_1} + \Delta v_r)}{(v_{\perp_1} + \Delta v_\perp)^2 e_1 \cos\theta_1 + (2v_{\perp_1} + \Delta v_\perp)\Delta v_\perp} \frac{v_{\perp_1}^2}{(\mu/r_{p_1})}$$
$$= \frac{(8.9816 + 1)(0 + 1.7321)}{(8.9816 + 1)^2 \cdot 0.41667 \cdot \cos(0) + (2 \cdot 8.9816 + 1) \cdot 1} \cdot \frac{8.9816^2}{(398\,600/7000)}$$
$$= 0.4050$$

- The follows that $\theta_2 = 22.047^\circ$, so that equation (1) yields

$$\underline{\eta = -22.05^\circ}$$

- Which means the rotation of the apse line is clockwise, as indicated in the presented figure.



Chapter 17

Geocentric right ascension-declination frame

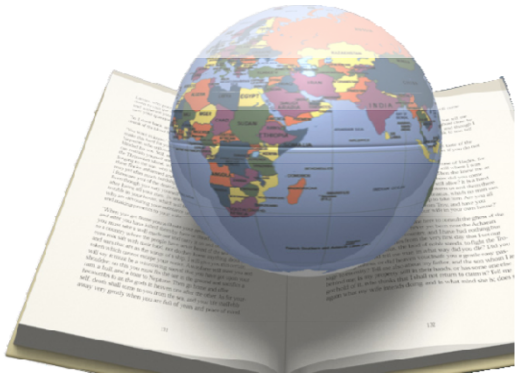
CHAPTER CONTENT

17-GEOSENTRIC RIGHT ASCENSION- DECLINATION FRAME

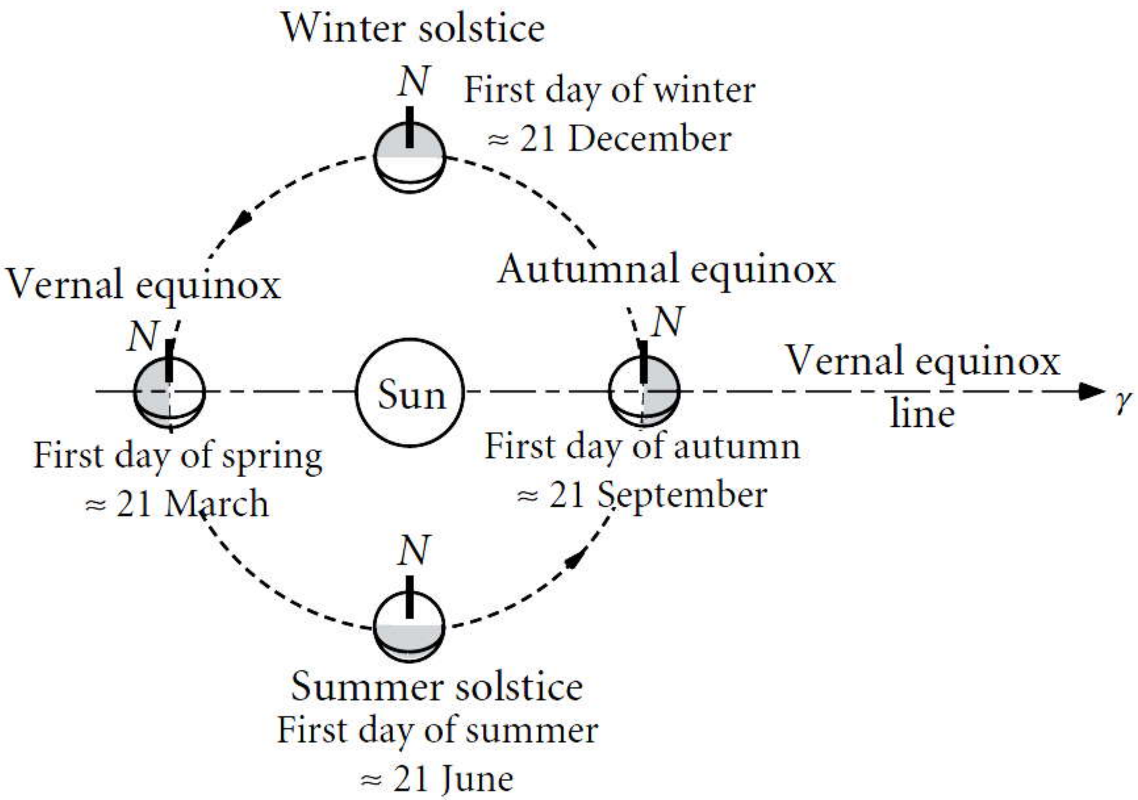
✶The discussion of orbital mechanics up to now has been confined to two dimensions, i.e., to the plane of the orbits themselves.
In this chapter we will see orbits in three-dimensional (real missions and orbital maneuvers)
Our focus will be on the orbits of earth satellites, but the applications are to any two-body trajectories

The coordinate system used to describe earth orbits in three dimensions is defined in terms of:

- ✓ Earth's equatorial plane,
- ✓ The ecliptic plane,
- ✓ The earth's axis of rotation.



17- GEOSENTRIC RIGHT ASCENSION- DECLINATION FRAME



ε is approximately 23.4°.

ε = obliquity of the ecliptic

The earth's equatorial plane and the ecliptic intersect along a line, which is known as the vernal equinox line.

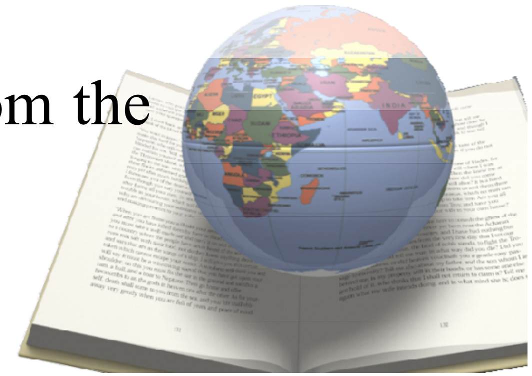
17- GEOSENTRIC RIGHT ASCENSION- DECLINATION FRAME

‘vernal equinox’ is the first day of spring in the northern hemisphere, when the noontime sun crosses the equator from south to north

symbol γ ‘vernal equinox’ : The position of the sun at that instant defines the location of a point in the sky called the vernal equinox

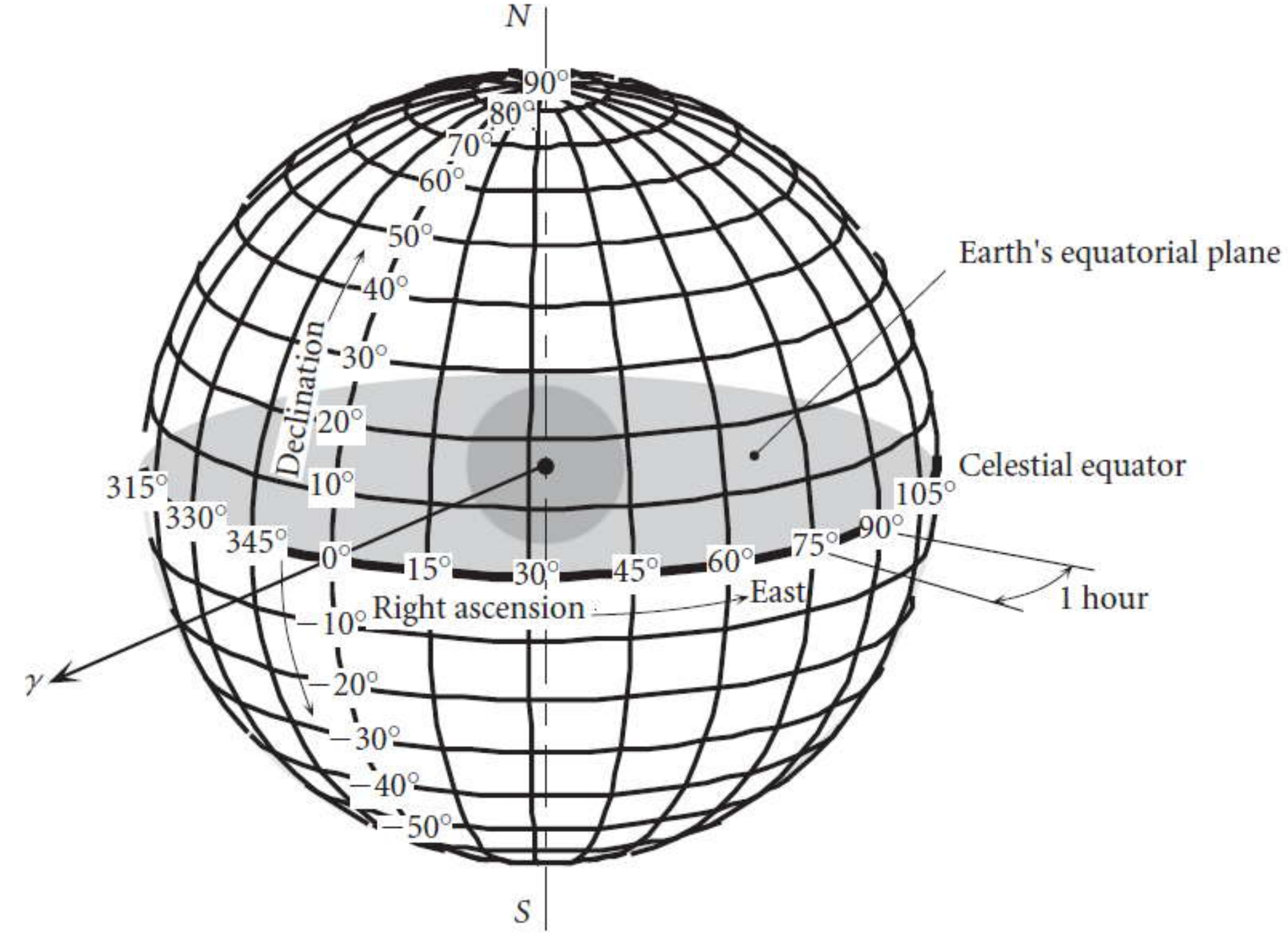
The vernal equinox lies today in the constellation Pisces

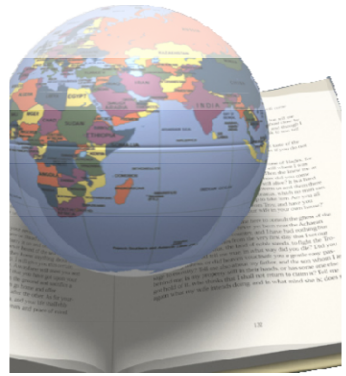
The direction of the vernal equinox line is from the earth towards γ,



17- GEOSENTRIC RIGHT ASCENSION- DECLINATION FRAME

To the human eye, objects in the night sky appear as points on a celestial sphere surrounding the earth





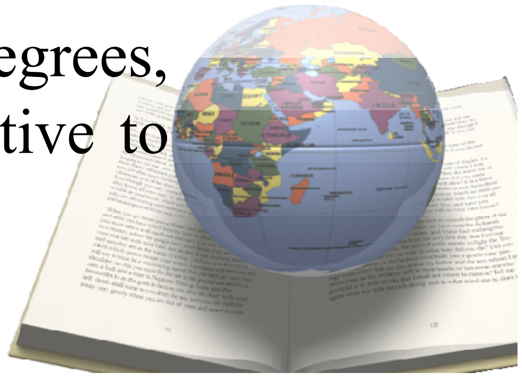
17-GEOSENTRIC RIGHT ASCENSION- DECLINATION FRAME

The vernal equinox γ, which lies on the celestial equator, is the origin for measurement of longitude, which in astronomical parlance is called right ascension. Right ascension (RA or α)

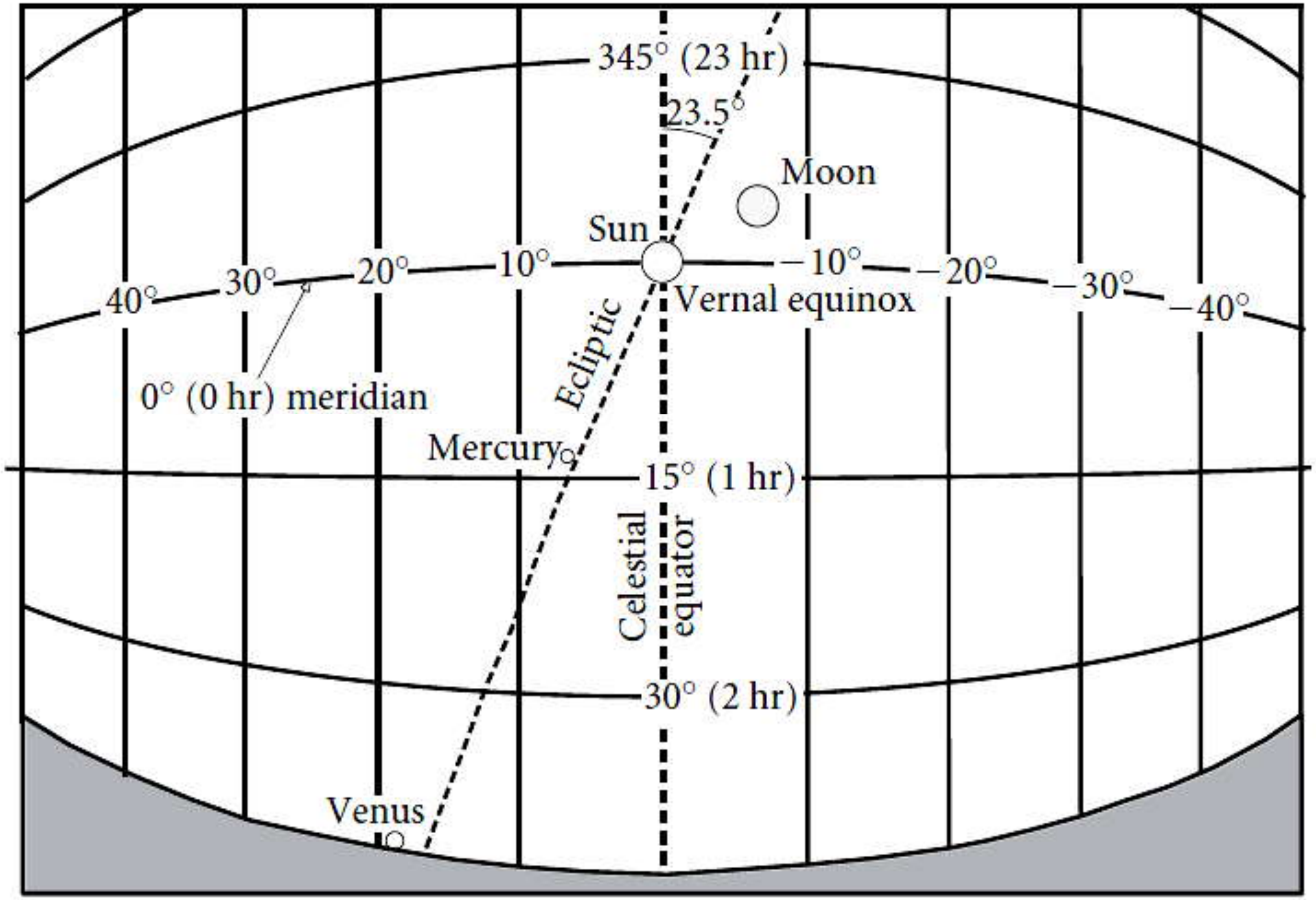
(RA or α) is measured along the celestial equator in degrees east from the vernal equinox.

Latitude on the celestial sphere is called declination. Declination (Dec or δ)

(Dec or δ) is measured along a meridian in degrees, positive to the north of the equator and negative to the south



17-GEOSENTRIC RIGHT ASCENSION- DECLINATION FRAME



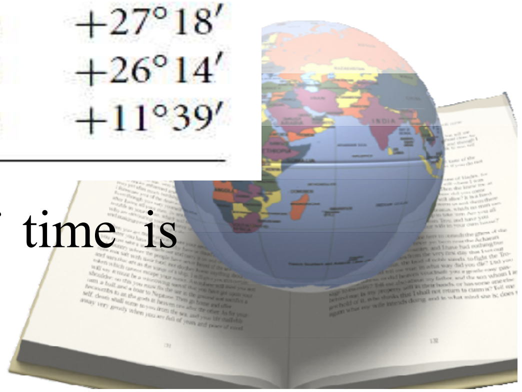
A view of the sky above the eastern horizon from 0° longitude on the equator at 9 am local time, 20 March, 2004. (Precession epoch AD 2000.)

17-GEOSENTRIC RIGHT ASCENSION- DECLINATION FRAME

Venus and moon ephemeris for 0 hours universal time (Precession epoch: AD 2000)

	Venus		Moon	
Date	RA	Dec	RA	Dec
1 Jan 2004	21 hr 05.0 min	−18°36′	1 hr 44.9 min	+8°47′
1 Feb 2004	23 hr 28.0 min	−04°30′	4 hr 37.0 min	+24°11′
1 Mar 2004	01 hr 30.0 min	+10°26′	6 hr 04.0 min	+08°32′
1 Apr 2004	03 hr 37.6 min	+22°51′	9 hr 18.7 min	+21°08′
1 May 2004	05 hr 20.3 min	+27°44′	11 hr 28.8 min	+07°53′
1 Jun 2004	05 hr 25.9 min	+24°43′	14 hr 31.3 min	−14°48′
1 Jul 2004	04 hr 34.5 min	+17°48′	17 hr 09.0 min	−26°08′
1 Aug 2004	05 hr 37.4 min	+19°04′	21 hr 05.9 min	−21°49′
1 Sep 2004	07 hr 40.9 min	+19°16′	00 hr 17.0 min	−00°56′
1 Oct 2004	09 hr 56.5 min	+12°42′	02 hr 20.9 min	+14°35′
1 Nov 2004	12 hr 15.8 min	+00°01′	05 hr 26.7 min	+27°18′
1 Dec 2004	14 hr 34.3 min	−13°21′	07 hr 50.3 min	+26°14′
1 Jan 2005	17 hr 12.9 min	−22°15′	10 hr 49.4 min	+11°39′

The coordinates of celestial bodies as a function of time is called an ephemeris



18- STATE VECTOR AND GEOCENTRIC EQUATORIAL FRAME

At any given time, the state vector of a satellite comprises its velocity **v and acceleration a.**

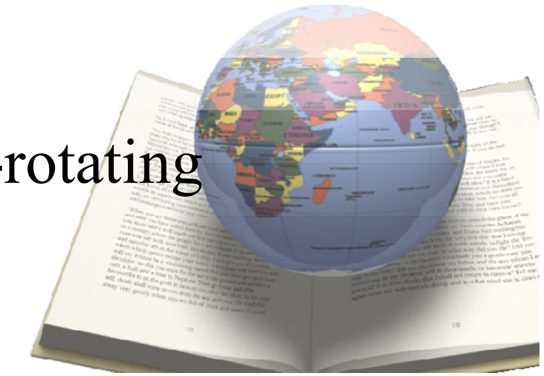
Orbital mechanics is concerned with specifying or predicting state vectors over intervals of time

the equation governing the state vector of a satellite traveling around the earth is,

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$$

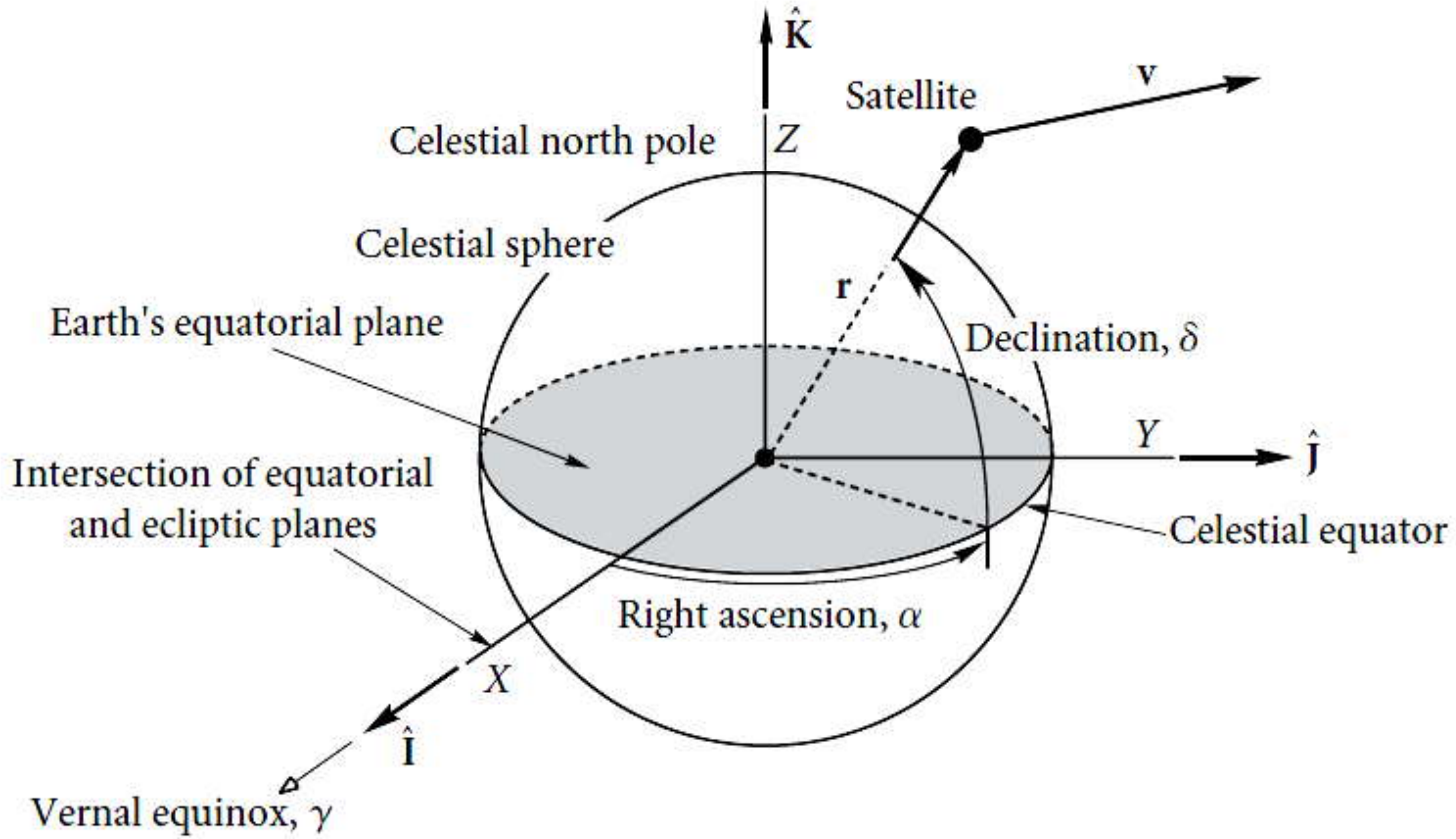
r is the position vector of the satellite relative to the center of the earth.

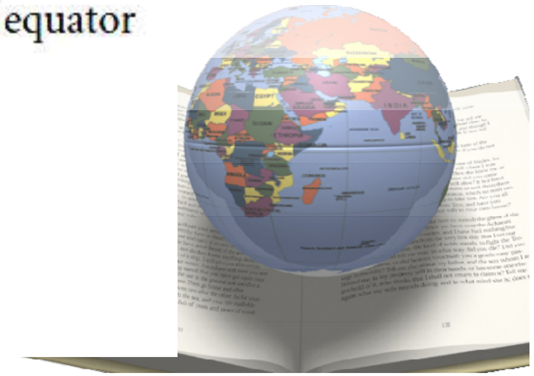
$\dot{r}$ =v and $\ddot{r}$ =a, must be measured in a non-rotating frame attached to the earth.



18- STATE VECTOR AND GEOCENTRIC EQUATORIAL FRAME

A commonly used nonrotating right-handed cartesian coordinate system is the geocentric equatorial frame





18- STATE VECTOR AND GEOCENTRIC EQUATORIAL FRAME

In the geocentric equatorial frame the state vector is given in component form by

$$\mathbf{r} = X\hat{\mathbf{I}} + Y\hat{\mathbf{J}} + Z\hat{\mathbf{K}}$$

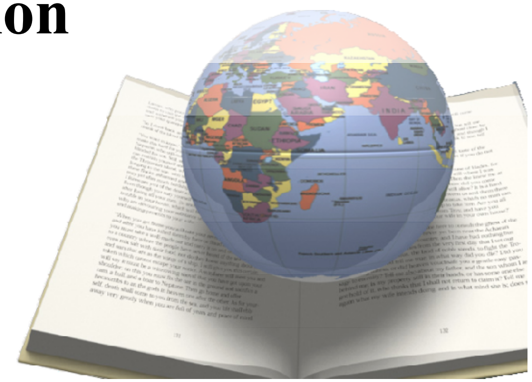
$$\mathbf{v} = v_X\hat{\mathbf{I}} + v_Y\hat{\mathbf{J}} + v_Z\hat{\mathbf{K}}$$

For the magnitude of the position vector we have:

$$\mathbf{r} = r\hat{\mathbf{u}}_r$$

we see that the components of ˆ**ur (the direction cosines of r) are** found in terms of the right ascension α and declination δ as follows :

$$\hat{\mathbf{u}}_r = \cos\delta\cos\alpha\hat{\mathbf{I}} + \cos\delta\sin\alpha\hat{\mathbf{J}} + \sin\delta\hat{\mathbf{K}}$$



18- STATE VECTOR AND GEOCENTRIC EQUATORIAL FRAME

EXAMPLE 18.1

If the position vector of the International Space Station is

$$\mathbf{r} = -5368\hat{\mathbf{I}} - 1784\hat{\mathbf{J}} + 3691\hat{\mathbf{K}} \text{ (km)}$$

what are its right ascension and declination?

The magnitude of $\mathbf{r}$ is

$$r = \sqrt{(-5368)^2 + (-1784)^2 + 3691^2} = 6754 \text{ km}$$

Hence,

$$\hat{\mathbf{u}}_r = \frac{\mathbf{r}}{r} = -0.7947\hat{\mathbf{I}} - 0.2642\hat{\mathbf{J}} + 0.5464\hat{\mathbf{K}} \qquad \text{(a)}$$

From this and Equation 4.5 we see that $\sin\delta = 0.5464$ which means

$$\delta = \sin^{-1} 0.5464 = \underline{33.12^\circ}$$

There is no quadrant ambiguity since, by definition, the declination lies between -90° and $+90^\circ$, which is precisely the range of the principal values of the arcsin function. It also follows that $\cos\delta$ cannot be negative.

18- STATE VECTOR AND GEOCENTRIC EQUATORIAL FRAME

EXAMPLE 18.1

From Equation 4.5 and Equation (a) just above we have

$$\cos\delta\cos\alpha = -0.7947 \quad \text{(b)}$$

$$\cos\delta\sin\alpha = -0.2642 \quad \text{(c)}$$

Therefore

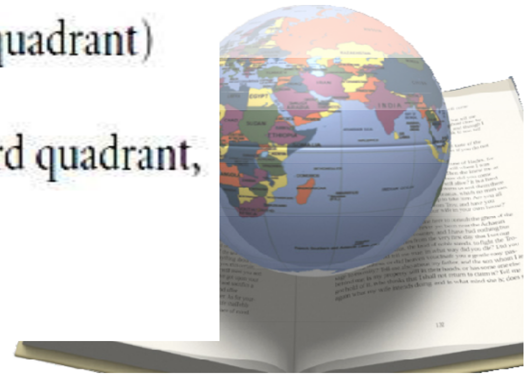
$$\cos\alpha = \frac{-0.7947}{\cos 33.12^\circ} = -0.9489$$

which implies

$$\alpha = \cos^{-1}(-0.9489) = 161.6^\circ \text{ (second quadrant) or } 198.4^\circ \text{ (third quadrant)}$$

From (c) we observe that $\sin\alpha$ is negative, which means α lies in the third quadrant,

$$\alpha = \underline{198.4^\circ}$$



Chapter 19

Orbital elements and the state vector

CHAPTER CONTENT

19- ORBITAL ELEMENTS AND THE STATE VECTOR

⋆ TO define an orbit in the plane requires two parameters:

- The semimajor axis (a)
- The specific energy (ε)

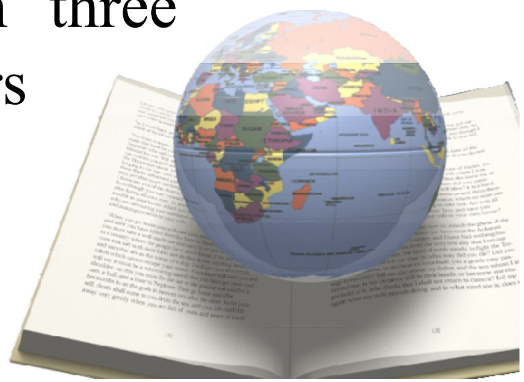
⋆Note G P158 {1}

⋆To locate a point on the orbit requires a third parameter:

- The true anomaly (o)

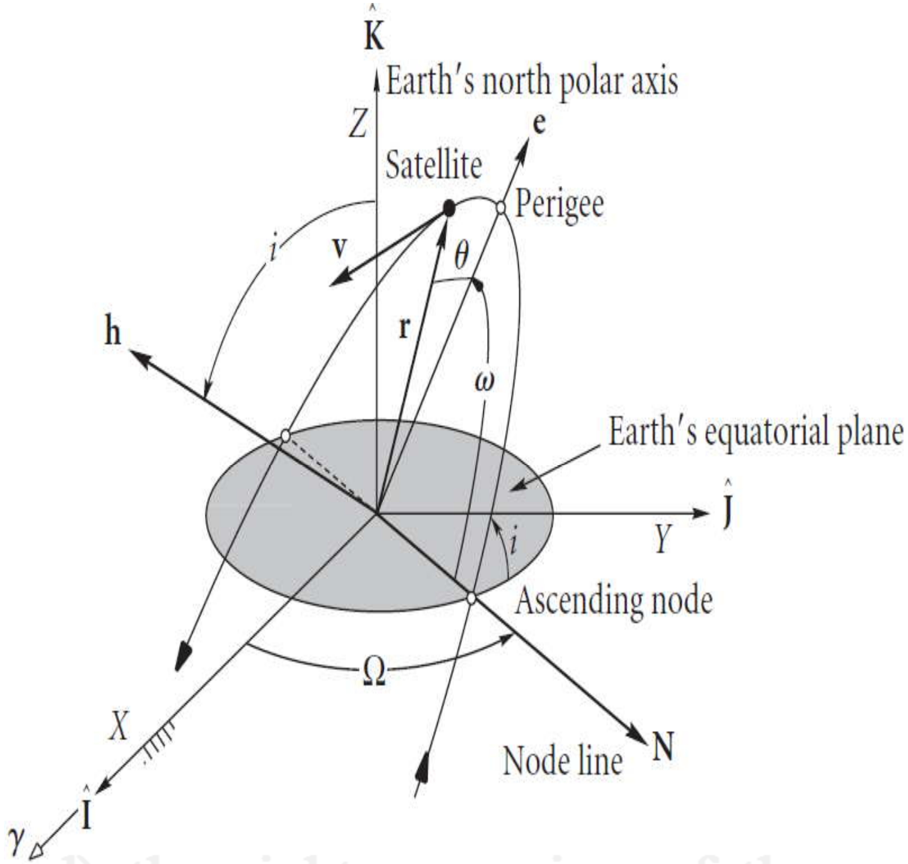
⋆Describing the orientation of an orbit in three dimensions requires three additional parameters

- The Eulers angels, (i), (O), (w)

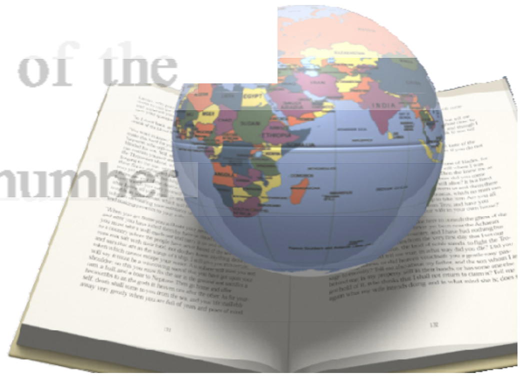


19- ORBITAL ELEMENTS AND THE STATE VECTOR

- Note E P158 {1}
- Node line
- Ascending node
- Node line vector N
- Descending node

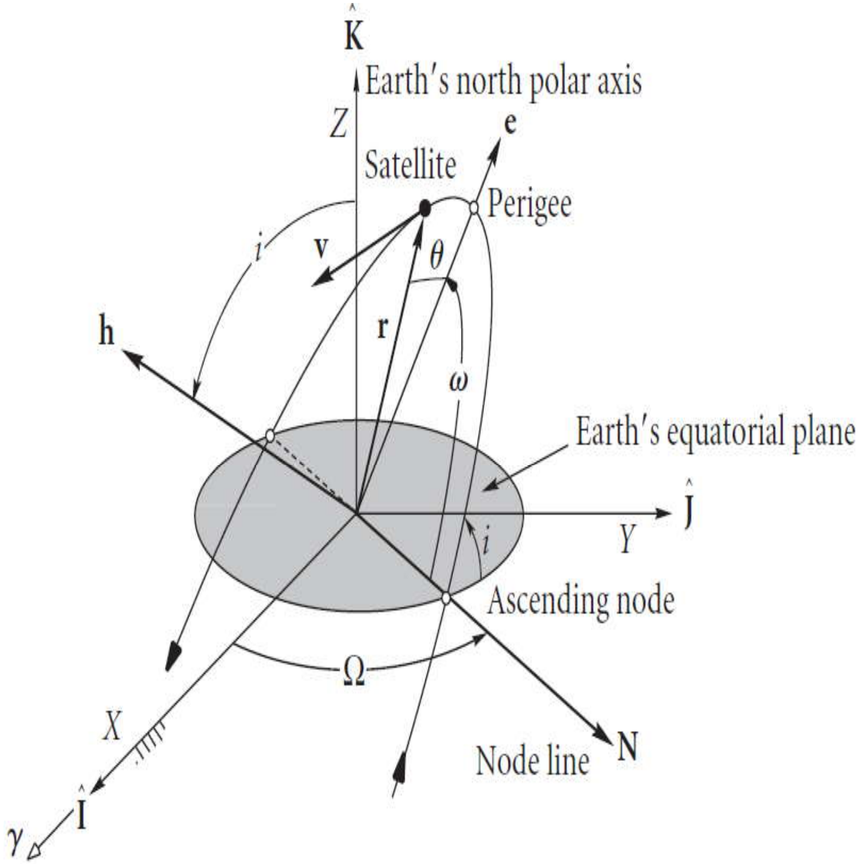


- (the first Euler angel) the right ascension of the ascending node (Ω). RAAN is a positive number (0°÷360°)



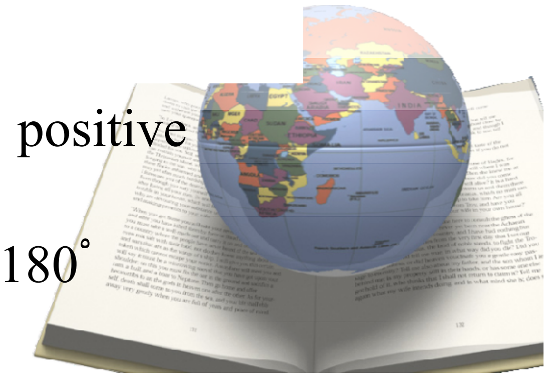
19- ORBITAL ELEMENTS AND THE STATE VECTOR

- (the second Euler angle) inclination (i), measured according to the right-hand rule, i is also the angle between the positive z axis and the normal to the plane of the orbit.
- Recall from previous chapters, that the angular momentum vector h is normal to the plane of the orbit.



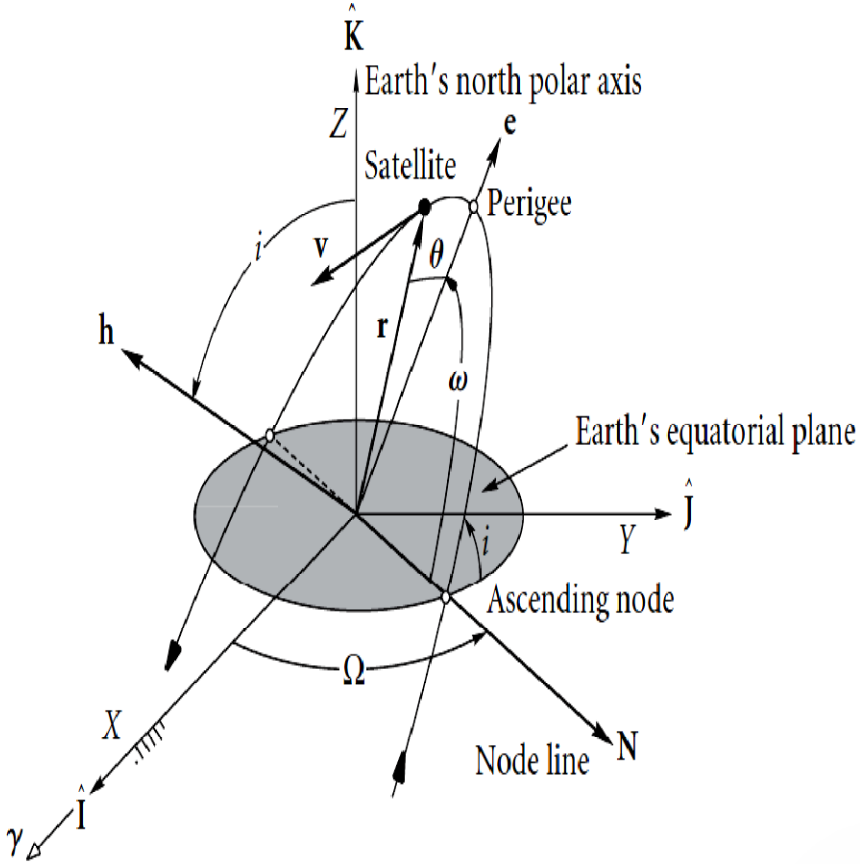
Therefore the inclination i is the angle between the positive z axis and h.

- The inclination is a positive number between 0° and 180°

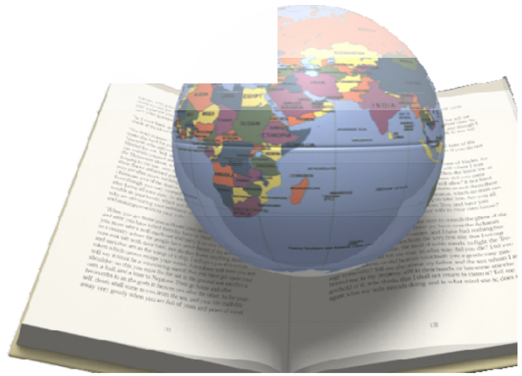


19- ORBITAL ELEMENTS AND THE STATE VECTOR

- (the third Euler angle) the argument of perigee ω, ($\bar{N}$ ^ $\bar{e}$), is a positive number between 0° and 360°, and measured in the plane of the orbit.
- In summary, the six orbital elements are:



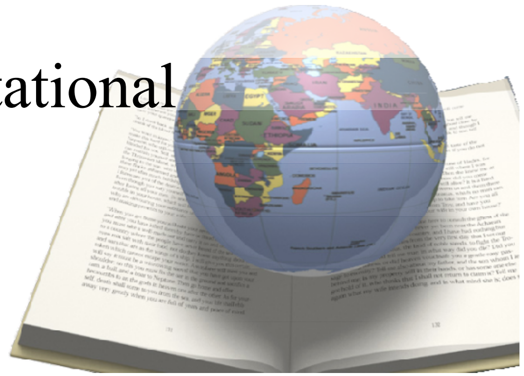
- h specific angular momentum (or semimajor axis a)
- i inclination
- Ο right ascension (RA) of the ascending node
- e eccentricity
- W argument of perigee
- ο true anomaly (or mean anomaly M)



19- ORBITAL ELEMENTS AND THE STATE VECTOR

- Given the position r and velocity v of a satellite in the geocentric equatorial frame, how do we obtain the orbital elements? In other words how do we obtain orbital elements from the state vector?

- The step-by-step procedure is outlined bellow: (we can also use this procedure for other planets and sun, buy defining the frame of reference and substituting the appropriate gravitational parameterμ)



19- ORBITAL ELEMENTS AND THE STATE VECTOR

ALGORITHM 4.1

- ✶ Obtain orbital elements from the state vector.

1- Calculate the distance,

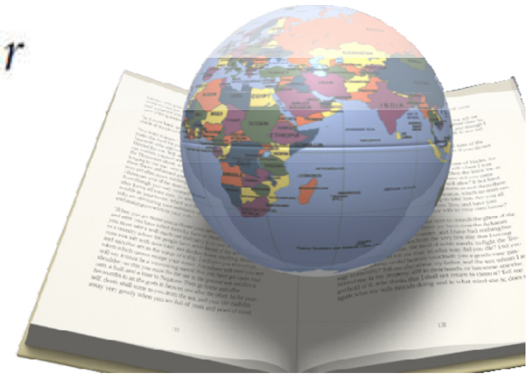
$$r = \sqrt{\mathbf{r} \cdot \mathbf{r}} = \sqrt{X^2 + Y^2 + Z^2}$$

2- Calculate the speed,

$$v = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_X^2 + v_Y^2 + v_Z^2}$$

3- Calculate the radial velocity,

$$v_r = \mathbf{r} \cdot \mathbf{v}/r = (X v_X + Y v_Y + Z v_Z)/r$$



- ✶ Note that if $v_r > 0$ the satellite is flying away from peigee.
- ✶ If $v_r < 0$, it is flying towards perigee.

19- ORBITAL ELEMENTS AND THE STATE VECTOR

4- Calculate the specific angular momentum,

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ X & Y & Z \\ v_X & v_Y & v_Z \end{vmatrix}$$

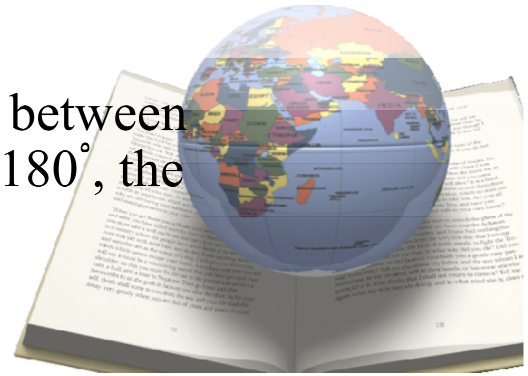
5- Calculate the magnitude of the specific angular momentum, the first orbital element.

$$h = \sqrt{\mathbf{h} \cdot \mathbf{h}}$$

6- Calculate the inclination,

$$i = \cos^{-1}\left(\frac{h_Z}{h}\right)$$

This is the second orbital element. Recall that I must lie between 0°and 180°, so there is no quadrant ambiguity. If 90°<i≤180°, the orbit is retrograde.



19- ORBITAL ELEMENTS AND THE STATE VECTOR

7- Calculate

$$\mathbf{N} = \hat{\mathbf{K}} \times \mathbf{h} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0 & 0 & 1 \\ h_X & h_Y & h_Z \end{vmatrix}$$

This vector defines the node line.

8- Calculate the magnitude of N,

$$N = \sqrt{\mathbf{N} \cdot \mathbf{N}}$$

9- Calculate the RA of the ascending node,

$$\Omega = \cos^{-1}(N_X/N)$$

the third orbital element. If $(N_X/N) > 0$,, then Ω lies in either the first of fourth quadrant. If $(N_X/N) < 0$, then Ω lies in either the second or third quadrant. To place Ω in the proper quadrant, observe that the ascending node lies on the positive side of the vertical XZ plane ($0 \leq \Omega < 180°$) if $N_Y > 0$. On the other hand, the ascending node lies on the negative side of the XZ plane ($180° \leq \Omega < 360°$) if $N_Y < 0$. Therefore, $N_Y > 0$ implies that $0 < \Omega < 180°$, whereas $N_Y < 0$ implies that $180° < \Omega < 360°$. In summary,

19- ORBITAL ELEMENTS AND THE STATE VECTOR

$$\Omega = \begin{cases} \cos^{-1}\left(\dfrac{N_X}{N}\right) & (N_Y \geq 0) \\ 360^\circ - \cos^{-1}\left(\dfrac{N_X}{N}\right) & (N_Y < 0) \end{cases}$$

10. Calculate the eccentricity vector, starting with equation 2.30,

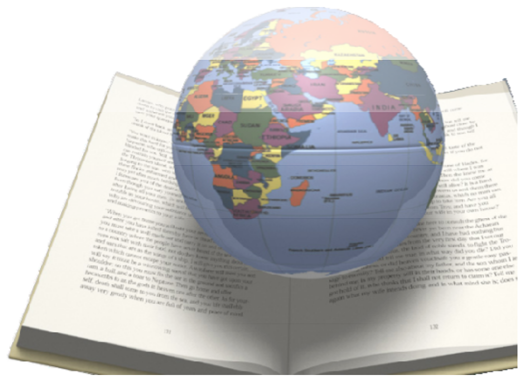
$$\mathbf{e} = \frac{1}{\mu}\left[\mathbf{v} \times \mathbf{h} - \mu\frac{\mathbf{r}}{r}\right] = \frac{1}{\mu}\left[\mathbf{v} \times (\mathbf{r} \times \mathbf{v}) - \mu\frac{\mathbf{r}}{r}\right] = \frac{1}{\mu}\left[\overbrace{\mathbf{r}v^2 - \mathbf{v}(\mathbf{r} \cdot \mathbf{v})}^{bac-cab \text{ rule}} - \mu\frac{\mathbf{r}}{r}\right]$$

So that

$$\mathbf{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\mathbf{r} - r v_r \mathbf{v}\right]$$

11- Calculate the eccentricity

$$e = \sqrt{\mathbf{e} \cdot \mathbf{e}}$$



19- ORBITAL ELEMENTS AND THE STATE VECTOR

The fourth orbital element. Substituting equation 4.10 leads to a form depending only on the scalars obtained thus far.

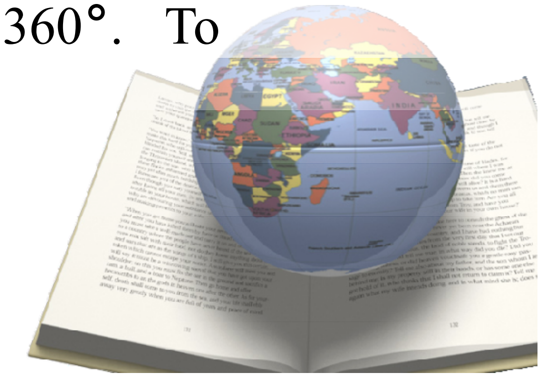
$$e = \frac{1}{\mu}\sqrt{(2\mu - rv^2)rv_r^2 + (\mu - rv^2)^2}$$

12- Calculate the argument of perigee,

$$\omega = \cos^{-1}(\mathbf{N}\cdot\mathbf{e}/Ne)$$

the fifth orbital element. If N. e>0, then w lies in either the first of fourth quadrant. If N. e<0, then w lies in either the second or third quadrant. To place w in the proper quadrant, observe that perigee lies above the equatorial plane (0≤w<180°)if e points up (in the positive Z direction), and perigee lies below the plane (180° ≤w<360°) if e points down. Therefore $e_Z \geq 0$ implies that 0 <w<180°, whereas $e_Z < 0$ implies that 180° <w<360°. To summarize,

$$\omega = \begin{cases} \cos^{-1}\left(\dfrac{\mathbf{N}\cdot\mathbf{e}}{Ne}\right) & (e_Z \geq 0) \\ 360^\circ - \cos^{-1}\left(\dfrac{\mathbf{N}\cdot\mathbf{e}}{Ne}\right) & (e_Z < 0) \end{cases}$$



19- ORBITAL ELEMENTS AND THE STATE VECTOR

13- Calculate the true anomaly,

$$\theta = \cos^{-1}\left(\frac{\mathbf{e}\cdot\mathbf{r}}{er}\right)$$

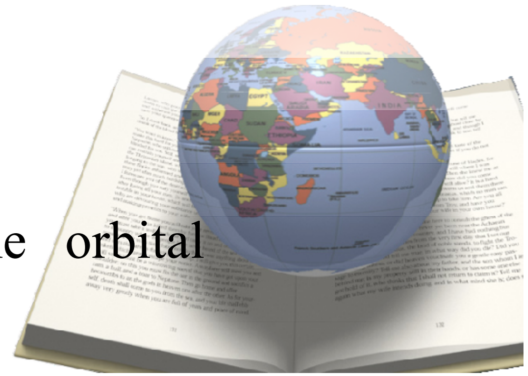
the sixth orbital element. If e.r>0, then θ lies in the first of fourth quadrant. If e.r<0, then θ lies in the second or third quadrant. To place θ in the proper quadrant, note that if flying away from perigee (r.v>0), then $0 \le \theta < 180°$, whereas if the satellite is flying towards perigee (r.v<0), then $180° \le \theta < 360°$. Therefore, using the result of step 3 above

$$\theta = \begin{cases} \cos^{-1}\left(\dfrac{\mathbf{e}\cdot\mathbf{r}}{er}\right) & (v_r \ge 0) \\ 360° - \cos^{-1}\left(\dfrac{\mathbf{e}\cdot\mathbf{r}}{er}\right) & (v_r < 0) \end{cases}$$

Substituting equation 4.10 yields an alternative form of this expression,

$$\theta = \begin{cases} \cos^{-1}\left[\dfrac{1}{e}\left(\dfrac{h^2}{\mu r} - 1\right)\right] & (v_r \ge 0) \\ 360° - \cos^{-1}\left[\dfrac{1}{e}\left(\dfrac{h^2}{\mu r} - 1\right)\right] & (v_r < 0) \end{cases}$$

The procedure described above for calculating the orbital elements is not unique.



19- ORBITAL ELEMENTS AND THE STATE VECTOR

EXAMPLE 19.1

- ✶ Given the state vector,

$$\mathbf{r} = -6045\hat{\mathbf{I}} - 3490\hat{\mathbf{J}} + 2500\hat{\mathbf{K}} \text{ (km)}$$

$$\mathbf{v} = -3.457\hat{\mathbf{I}} + 6.618\hat{\mathbf{J}} + 2.533\hat{\mathbf{K}} \text{ (km/s)}$$

- ✶ Find the orbital elements h,i,O,e,w and o using algorithm 4.1.

STEP 1:

$$r = \sqrt{\mathbf{r}\cdot\mathbf{r}} = \sqrt{(-6045)^2 + (-3490)^2 + 2500^2} = 7414 \text{ km} \qquad \text{(a)}$$

STEP 2:

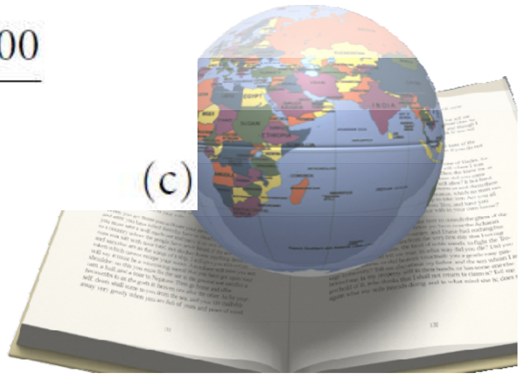
$$v = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{(-3.457)^2 + 6.618^2 + 2.533^2} = 7.884 \text{ km/s} \qquad \text{(b)}$$

STEP 3:

$$v_r = \frac{\mathbf{v}\cdot\mathbf{r}}{r} = \frac{(-3.457)\cdot(-6045) + 6.618\cdot(-3490) + 2.533\cdot 2500}{7414}$$

$$= 0.5575 \text{ km/s} \qquad \text{(c)}$$

Since $v_r > 0$, the satellite is flying away from perigee



19- ORBITAL ELEMENTS AND THE STATE VECTOR

EXAMPLE 19.1

STEP 4:

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ -6045 & -3490 & 2500 \\ -3.457 & 6.618 & 2.533 \end{vmatrix} = -25\,380\hat{\mathbf{I}} + 6670\hat{\mathbf{J}} - 52\,070\hat{\mathbf{K}}\ (\text{km}^2/\text{s}) \tag{d}$$

STEP 5:

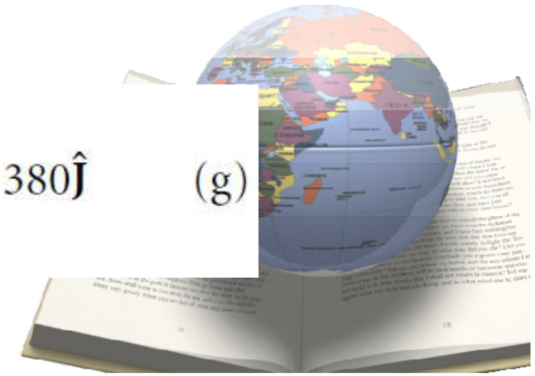
$$h = \sqrt{\mathbf{h} \cdot \mathbf{h}} = \sqrt{(-25\,380)^2 + 6670^2 + (-52\,070)^2} = \underline{58\,310\ \text{km}^2/\text{s}} \tag{e}$$

STEP 6:

$$i = \cos^{-1}\frac{h_Z}{h} = \cos^{-1}\left(\frac{-52\,070}{58\,310}\right) = \underline{153.2^\circ} \tag{f}$$

STEP 7:

$$\mathbf{N} = \hat{\mathbf{K}} \times \mathbf{h} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0 & 0 & 1 \\ -25\,380 & 6670 & -52\,070 \end{vmatrix} = -6670\hat{\mathbf{I}} - 25\,380\hat{\mathbf{J}} \tag{g}$$



EXAMPLE 19.1

STEP 8:

$$N = \sqrt{\mathbf{N} \cdot \mathbf{N}} = \sqrt{(-6670)^2 + (-25\,380)^2} = 26\,250 \qquad \text{(h)}$$

Using (g) and (h), we compute the right ascension of the node.

STEP 9:

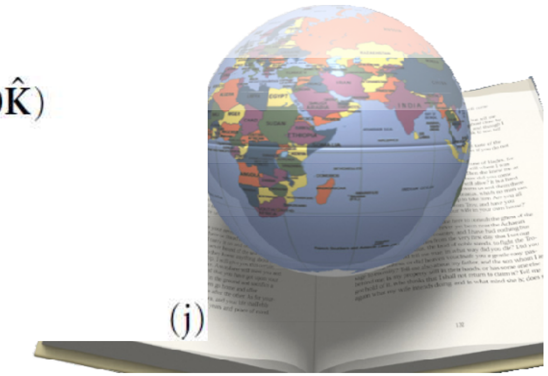
$$\Omega = \cos^{-1} \frac{N_X}{N} = \cos^{-1} \left(\frac{-6670}{26\,250} \right) = 104.7^\circ \text{ or } 255.3^\circ$$

From (g) we know that $N_Y < 0$; therefore, O must lie in the third quadrant

$$\underline{\Omega = 255.3^\circ} \qquad \text{(i}$$

STEP 10:

$$\begin{aligned}
\mathbf{e} &= \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\mathbf{r} - (\mathbf{r} \cdot \mathbf{v})\mathbf{v}\right] \\
&= \frac{1}{398\,600}\left[\left(7.884^2 - \frac{398\,600}{7414}\right)(-6045\hat{\mathbf{I}} - 3490\hat{\mathbf{J}} + 2500\hat{\mathbf{K}}) \right. \\
&\qquad \left. -4133(-3.457\hat{\mathbf{I}} + 6.618\hat{\mathbf{J}} + 2.533\hat{\mathbf{K}})\right] \\
&= -0.09160\hat{\mathbf{I}} - 0.1422\hat{\mathbf{J}} + 0.02644\hat{\mathbf{K}} \qquad \text{(j)}
\end{aligned}$$



19- ORBITAL ELEMENTS AND THE STATE VECTOR

EXAMPLE 19.1

STEP 11:

$$e = \sqrt{\mathbf{e} \cdot \mathbf{e}} = \sqrt{(-0.09160)^2 + (-0.1422)^2 + (0.02644)^2} = \underline{0.1712} \qquad \text{(k)}$$

Clearly, the orbit is an ellipse.

STEP 12:

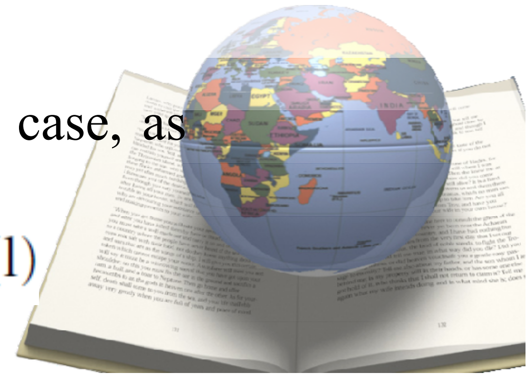
$$\omega = \cos^{-1} \frac{\mathbf{N} \cdot \mathbf{e}}{Ne}$$

$$= \cos^{-1} \left[\frac{(-6670)(-0.09160) + (-25\,380)(-0.1422) + (0)(0.02644)}{(26\,250)(0.1712)} \right]$$

$$= 20.07^\circ \text{ or } 339.9^\circ$$

w lies in the first quadrant if $e_Z > 0$, which is true in this case, as we see from (j). Therefore,

$$\underline{\omega = 20.07^\circ} \qquad \text{(l)}$$



19- ORBITAL ELEMENTS AND THE STATE VECTOR

EXAMPLE 19.1

STEP 13:

$$\theta = \cos^{-1}\left(\frac{\mathbf{e}\cdot\mathbf{r}}{er}\right)$$

$$= \cos^{-1}\left[\frac{(-0.09160)(-6045) + (-0.1422)\cdot(-3490) + (0.02644)(2500)}{(0.1712)(7414)}\right]$$

$$= 28.45^\circ \text{ or } 331.6^\circ$$

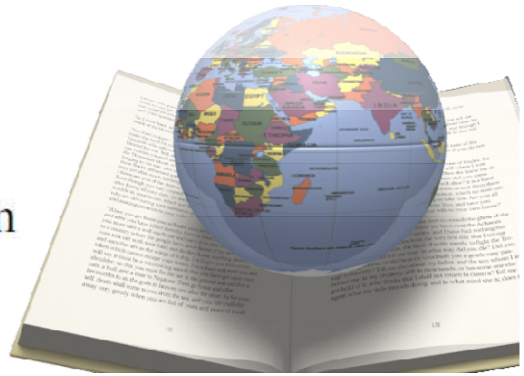
From (c) we know that $v_r > 0$, which means $0 \leq o < 180°$. Therefore,

$$\underline{\theta = 28.45^\circ}$$

Having found orbital elements, we can go on to compute other parameters. The perigee and apogee radii are

$$r_p = \frac{h^2}{\mu}\frac{1}{1 + e\cos(0)} = \frac{58\,310^2}{398\,600}\frac{1}{1 + 0.1712} = 7284\text{ km}$$

$$r_a = \frac{h^2}{\mu}\frac{1}{1 + e\cos(180^\circ)} = \frac{58\,310^2}{398\,600}\frac{1}{1 - 0.1712} = 10\,290\text{ km}$$



19- ORBITAL ELEMENTS AND THE STATE VECTOR

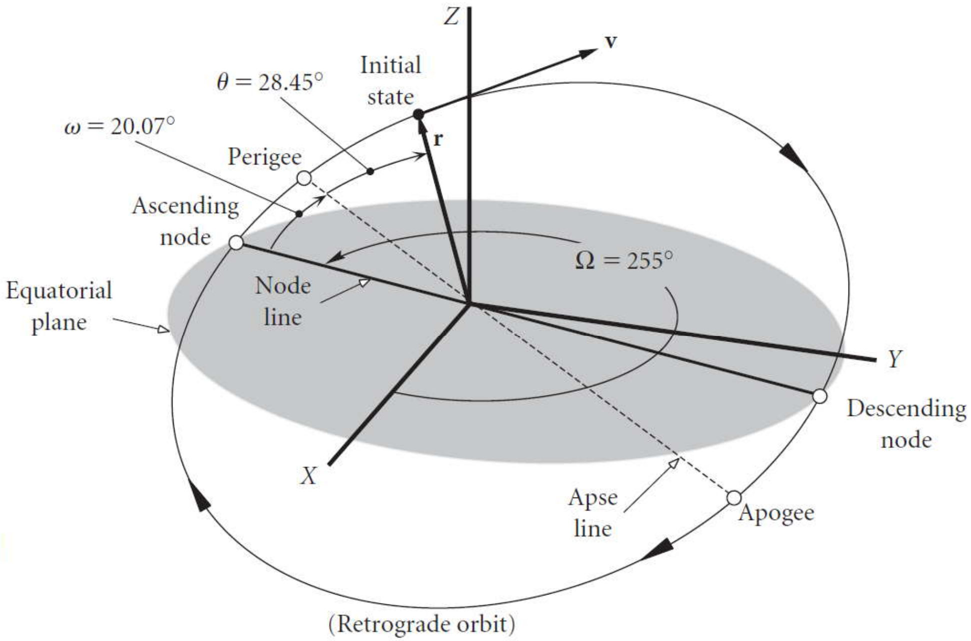
EXAMPLE 19.1

From these it follows that the semimajor axis of the ellipse is

$$a = \frac{1}{2}(r_p + r_a) = 8788\ \text{km}$$

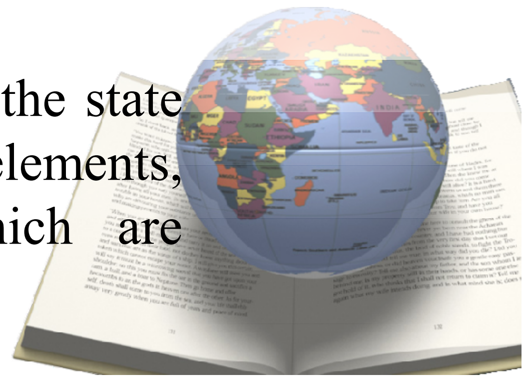
This leads to the period,

$$T = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}} = 2.278\ \text{hr}$$



The orbit is illustrated in figure 4.8

We have seen how to obtain the orbital elements from the state vector. To arrive at the state vector, given the orbital elements, requires performing coordinate transformations, which are discussed in the next section



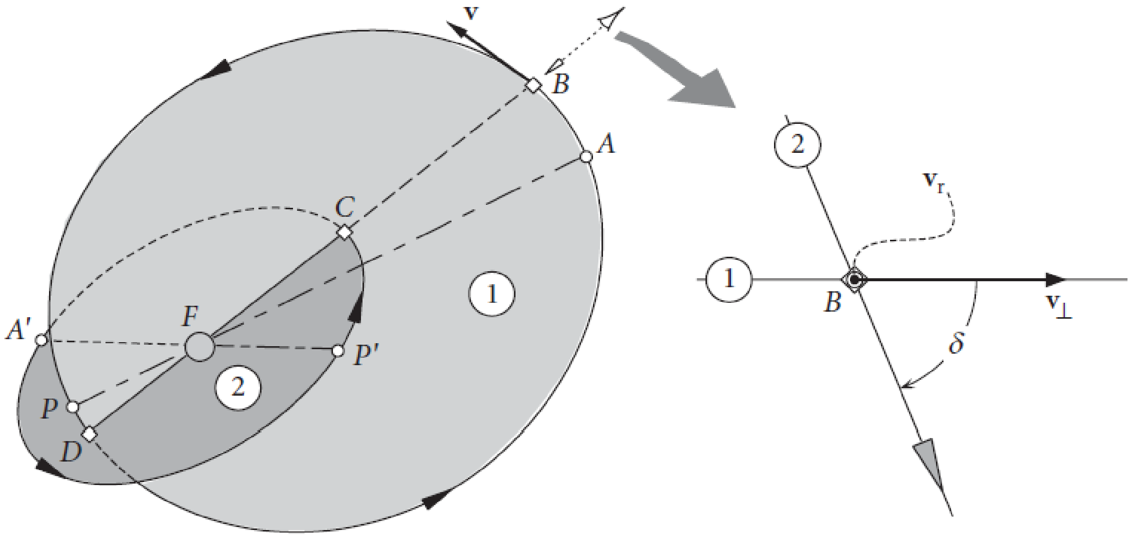
Chapter 20

Plane change maneuvers

CHAPTER CONTENT

20- PLANE CHANGE MANEUVERS

⋆**O**rbits having a common focus F generally do not lie in a common plane.



(a) (b)

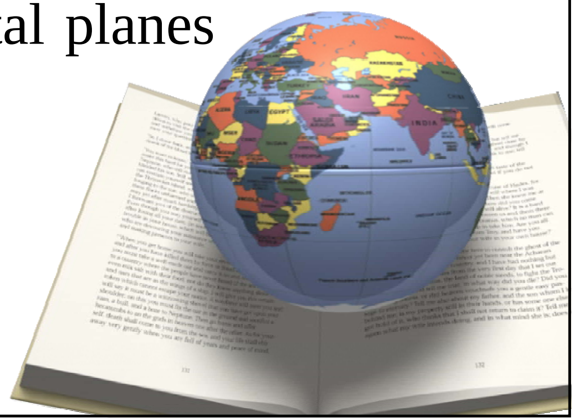
Since the common focus lies in every orbital plane it must lie on the line of intersection of any two orbits.

⋆For a spacecraft in orbit 1 to change its plane to that of orbit 2 by means of a single ΔV maneuver (cranking maneuver), it must do so when it is on the line of intersection of the orbital planes (points B and D)

d – dihedral angle

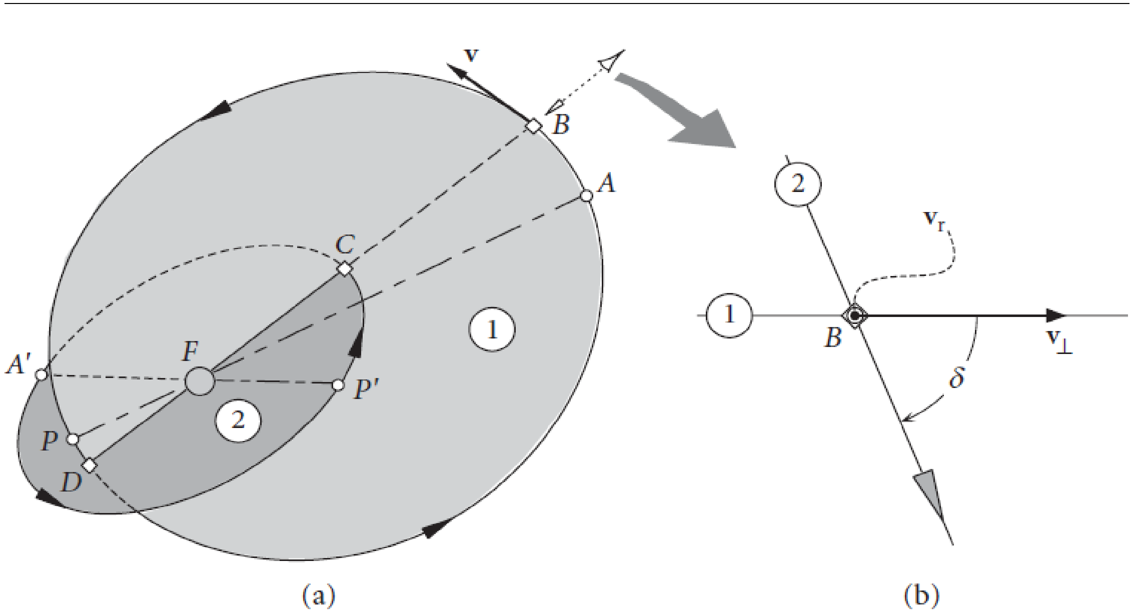
$V_{\perp}$ - transverse component of velocity

V_r - radial component of velocity



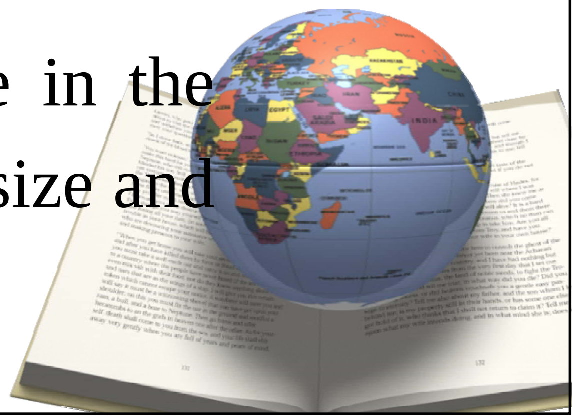
20- PLANE CHANGE MANEUVERS

⋆ Changing the plane of orbit 1 requires simply rotating $V_{\perp}$ around the intersection line through the d angle .

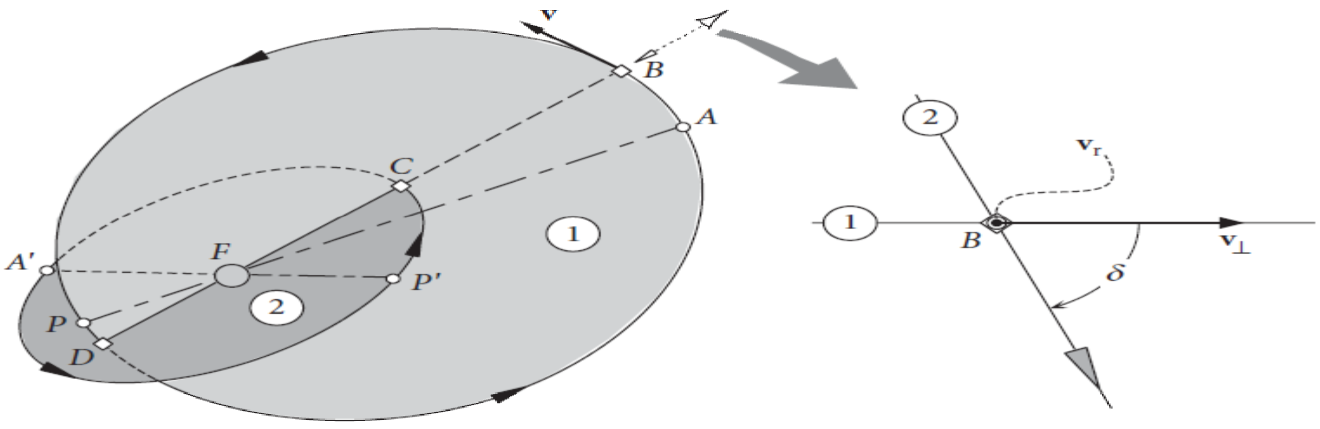


⋆ If $V_{\perp}$ & V_r = constant, then the orbit remains unchanged except for its new orientation in space.

⋆ If the magnitudes of $V_{\perp}$ & V_r change in the process, then the rotated orbit acquires a new size and shape.



20- PLANE CHANGE MANEUVERS



(a) (b)

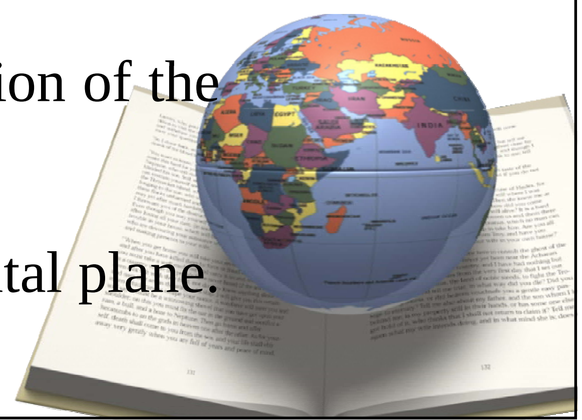
⋆ To find ΔV associated with a plane change, let $\mathbf{v}_1$ be the velocity before and $\mathbf{v}_2$ the velocity after the impulsive maneuver.

$$\mathbf{v}_1 = v_{r_1}\hat{\mathbf{u}}_r + v_{\perp_1}\hat{\mathbf{u}}_{\perp_1}$$

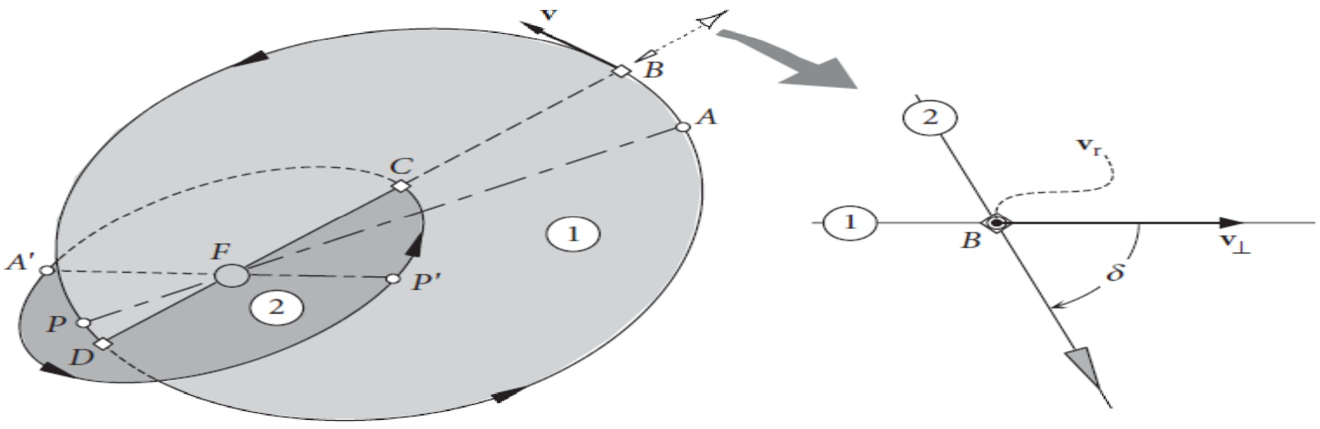
$$\mathbf{v}_2 = v_{r_2}\hat{\mathbf{u}}_r + v_{\perp_2}\hat{\mathbf{u}}_{\perp_2}$$

$\hat{\mathbf{u}}_r$: radial unit vector directed along the line of intersection of the two orbital planes.

$\hat{\mathbf{u}}_\perp$: unit vector is perpendicular to $\hat{\mathbf{u}}_r$ and lies in the orbital plane. ($\hat{\mathbf{u}}_{\perp_1}$, $\hat{\mathbf{u}}_{\perp_2}$)



20- PLANE CHANGE MANEUVERS



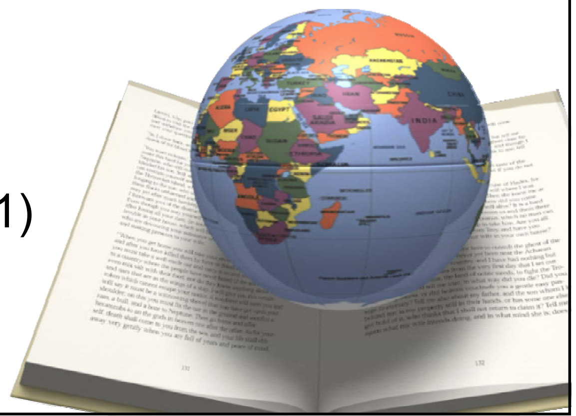
(a) (b)

- ✶ The change ΔV in the velocity vector is

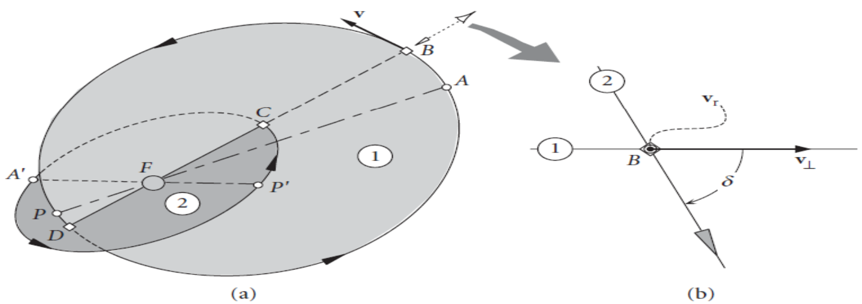
$$\Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 = (v_{r_2} - v_{r_1})\hat{\mathbf{u}}_r + v_{\perp_2}\hat{\mathbf{u}}_{\perp_2} - v_{\perp_1}\hat{\mathbf{u}}_{\perp_1}$$

- ✶ ΔV is found by taking the dot product of Δ**V** with itself
- ✶ The general formula for ΔV with plane change is:

$$\Delta v = \sqrt{(v_{r_2} - v_{r_1})^2 + v_{\perp_1}^2 + v_{\perp_2}^2 - 2v_{\perp_1}v_{\perp_2}\cos\delta} \quad (1)$$



20- PLANE CHANGE MANEUVERS



⋆ From the definition of the flight path angle:

$$v_{r_1} = v_1 \sin\gamma_1 \qquad v_{\perp_1} = v_1 \cos\gamma_1$$

$$v_{r_2} = v_2 \sin\gamma_2 \qquad v_{\perp_2} = v_2 \cos\gamma_2$$

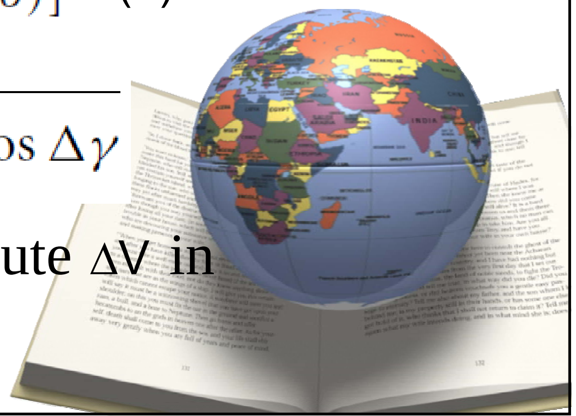
✶ Substituting these relations into equation1 expanding and collecting terms, and using the trig identities, leads to another version of the same equation.

$$\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1v_2[\cos\Delta\gamma - \cos\gamma_2\cos\gamma_1(1-\cos\delta)]} \quad (2)$$

$$\Delta\gamma = \gamma_2 - \gamma_1$$

⋆ If d=0 → cos d=1 → (2) ⇨ $\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\Delta\gamma}$

⋆ Which is the cosine law we have been using to compute ΔV in coplanar maneuvers



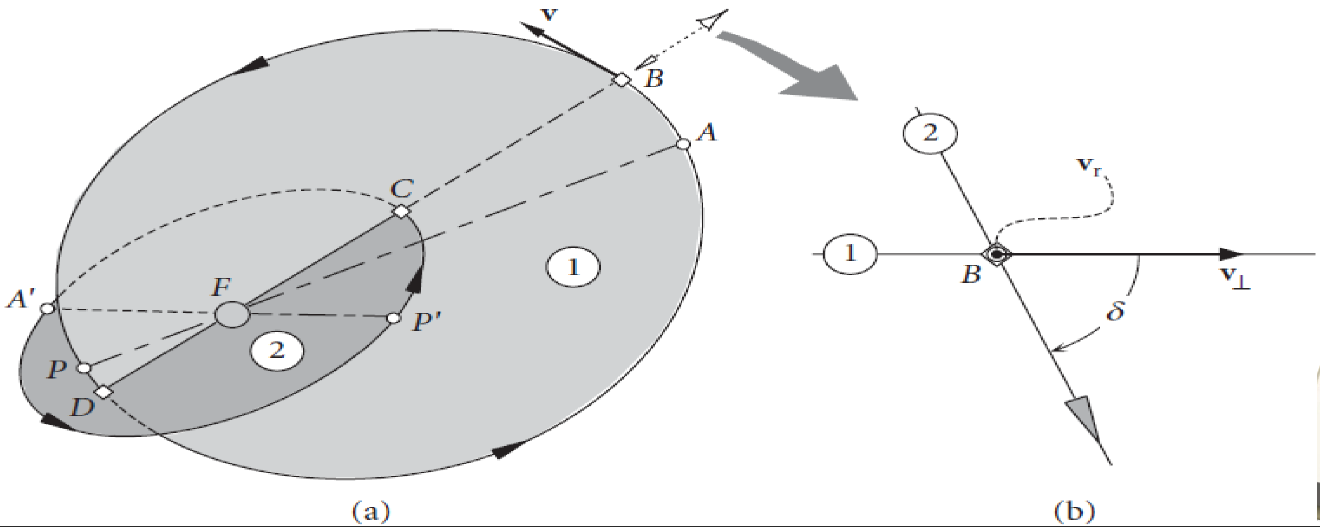
20- PLANE CHANGE MANEUVERS

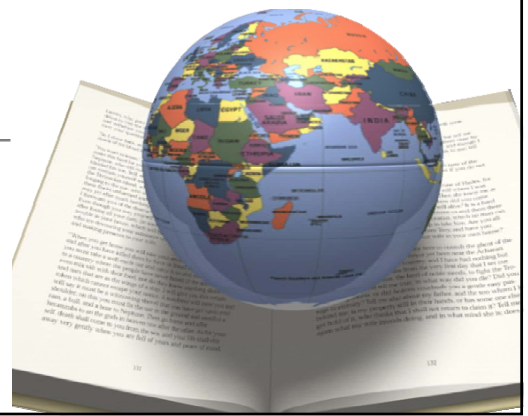
⋆ To keep ΔV at a minimum, the radial velocity should remain unchanged during a plane change maneuver.

⋆ It is clear from equation(1)

$$\Delta v = \sqrt{(v_{r_2} - v_{r_1})^2 + v_{\perp_1}^2 + v_{\perp_2}^2 - 2v_{\perp_1}v_{\perp_2}\cos\delta}$$

⋆ For the same reason it is apparent that the maneuver should occur where $v_\perp$ is smallest, which is at apoapse. (figure)





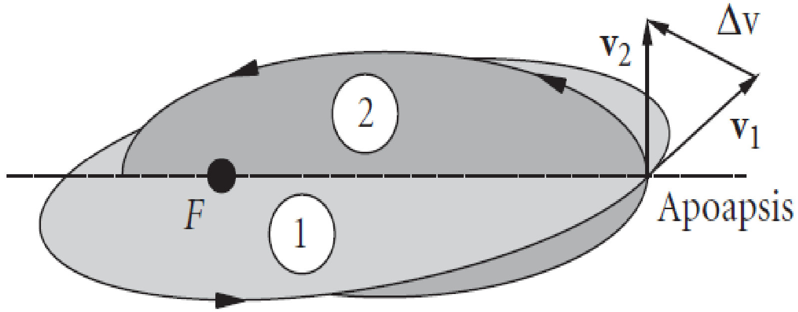
20- PLANE CHANGE MANEUVERS

✶ In this case:

$v_{r_1} = v_{r_2} = 0$ ⟶ $v_{\perp_1} = v_1$ and $v_{\perp_2} = v_2$

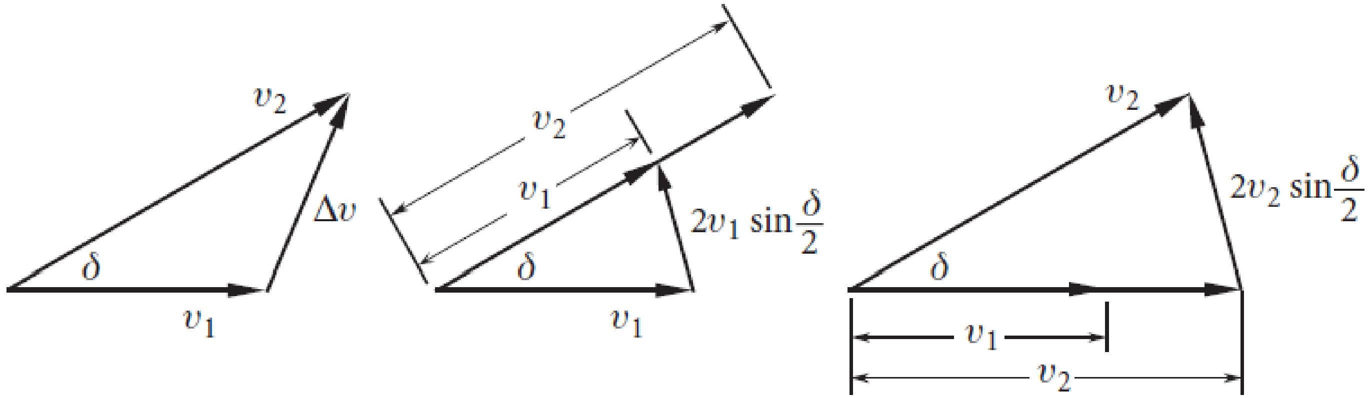
⟶ equ(1) ⟶

$$\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos\delta} \quad (3)$$

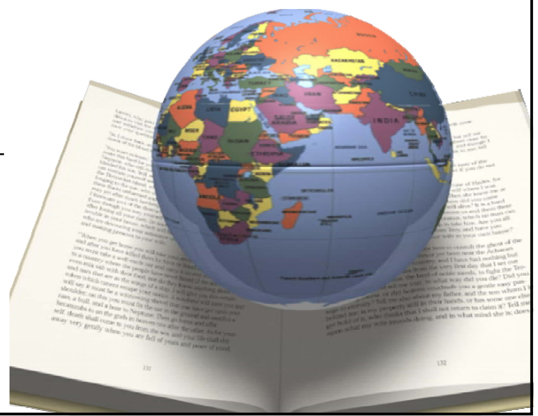


Plane change at apoapse (or periapse)

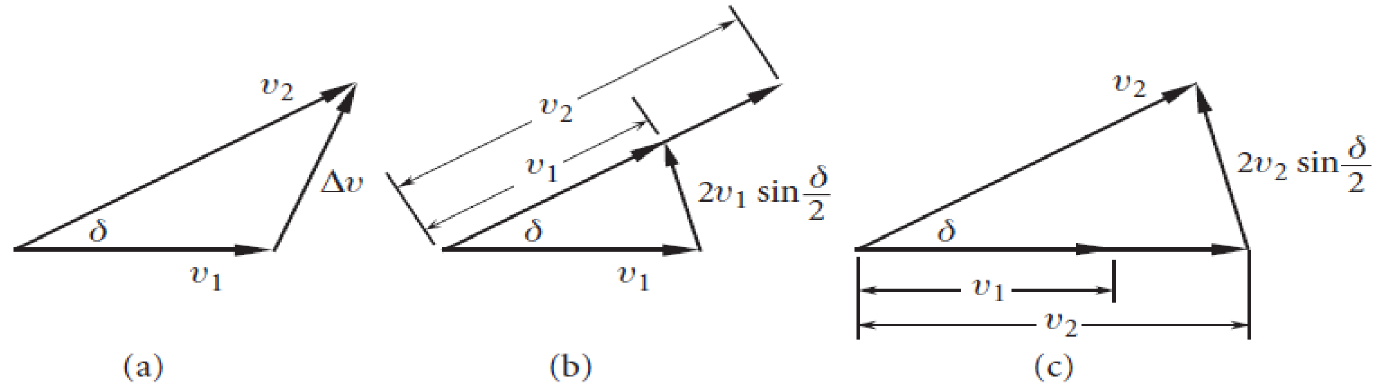
✶ Equation (3) is for a speed change accompanied by a plane change



(a) (b) (c)



20- PLANE CHANGE MANEUVERS



✶ As we mentioned equation(3) is for a speed change accompanied by a plane change.

So using the trig identity: $\cos\delta = 1 - 2\sin^2\dfrac{\delta}{2}$

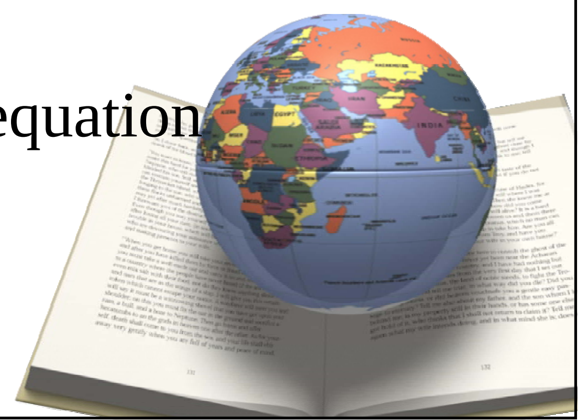
✶We can rewrite equation(3) as follows for a plane change together with a speed change at apoapse or periapse:

$$\Delta v_{\mathrm{I}} = \sqrt{(v_2 - v_1)^2 + 4v_1 v_2 \sin^2\frac{\delta}{2}} \qquad (4)$$

✶ If there is no change in the speed ($v_2 = v_1$) equation (4) yields

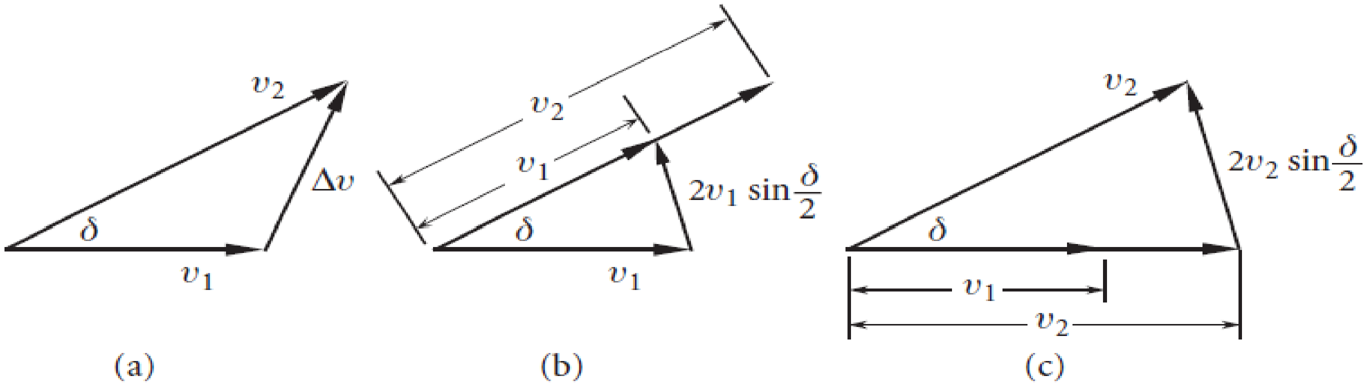
$$\Delta v_\delta = 2v\sin\frac{\delta}{2} \qquad (5)$$

d: pure rotation of the velocity vector



20- PLANE CHANGE MANEUVERS

⋆ Another plane-change strategy is to rotate the velocity vector and then change its magnitude. (figure (b))

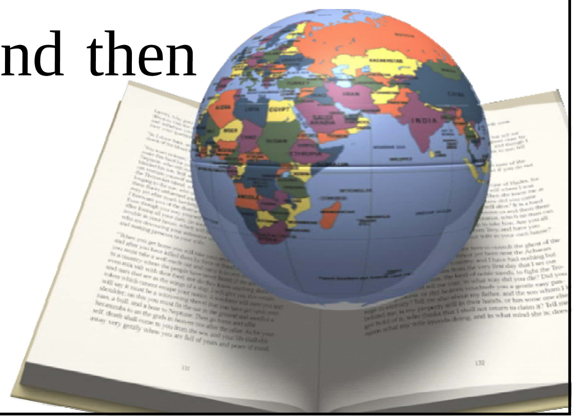


⋆ In that case, the ΔV is:

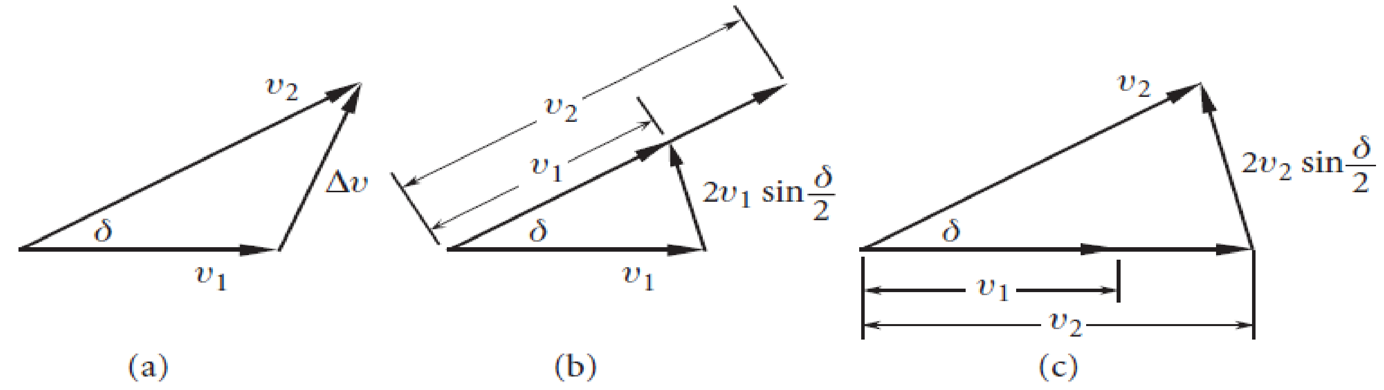
$$\Delta v_{II} = 2v_1 \sin\frac{\delta}{2} + |v_2 - v_1|$$

⋆Another possibility is to change the speed first and then rotate the velocity vector (figure(c)), then:

$$\Delta v_{III} = |v_2 - v_1| + 2v_2 \sin\frac{\delta}{2}$$



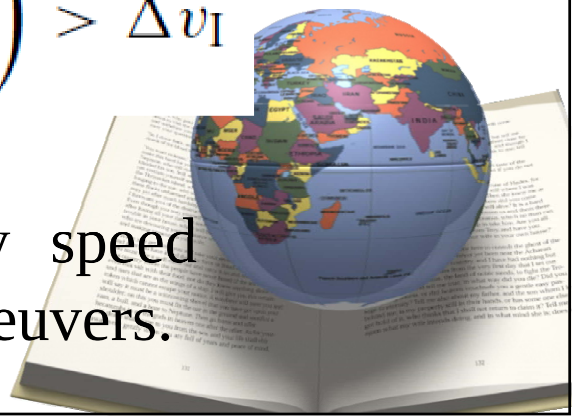
20- PLANE CHANGE MANEUVERS



⋆ It is easy to show that

$$\Delta v_{\mathrm{II}} = \sqrt{\Delta v_1^2 + 4v_1|v_2 - v_1|\sin\frac{\delta}{2}\left(1 - \sin\frac{\delta}{2}\right)} > \Delta v_{\mathrm{I}}$$

$$\Delta v_{\mathrm{III}} = \sqrt{\Delta v_1^2 + 4v_2|v_2 - v_1|\sin\frac{\delta}{2}\left(1 - \sin\frac{\delta}{2}\right)} > \Delta v_{\mathrm{I}}$$



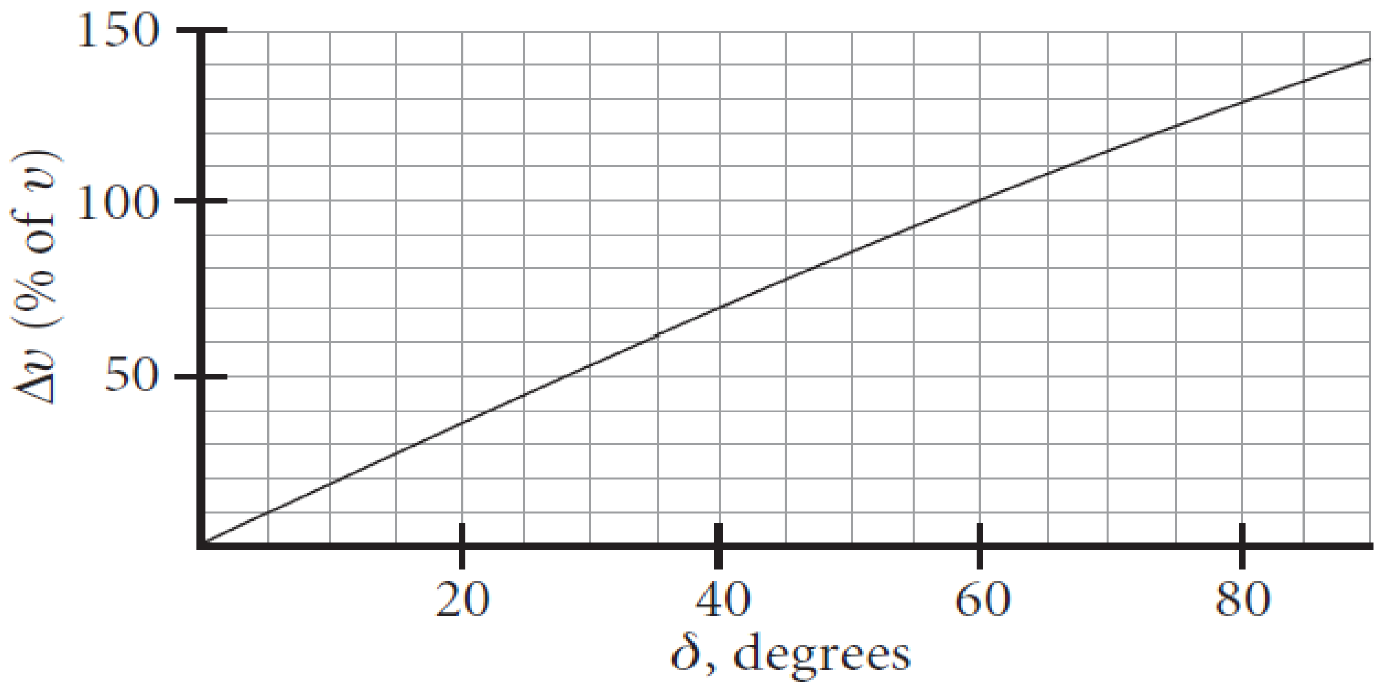
⋆It follows that plane change accompanied by speed change is the most efficient of the above three maneuvers.

20- PLANE CHANGE MANEUVERS

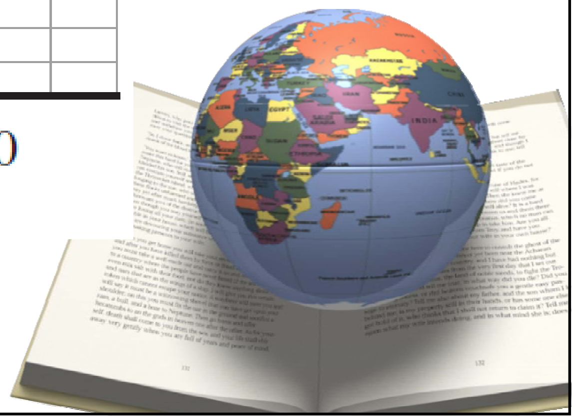
★ The ΔV for pure rotation of the velocity vector, according to equ(5)

$$\Delta v_\delta = 2v \sin \frac{\delta}{2} \qquad (5)$$

★Is ploted in the bellow figure:

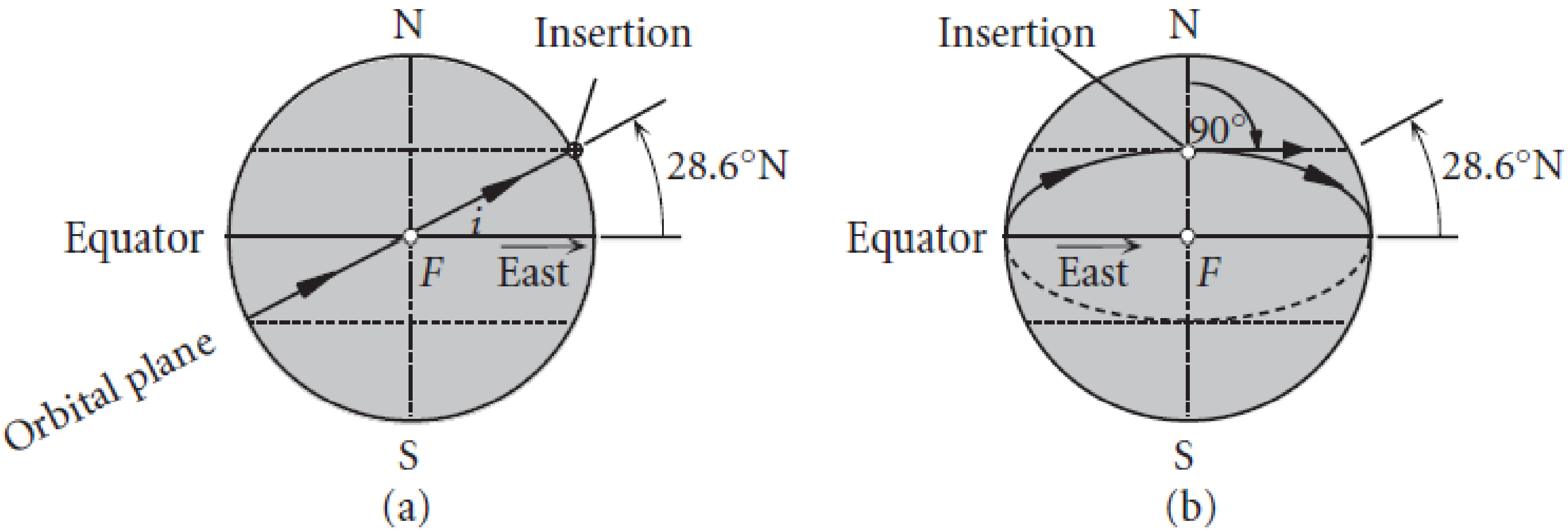


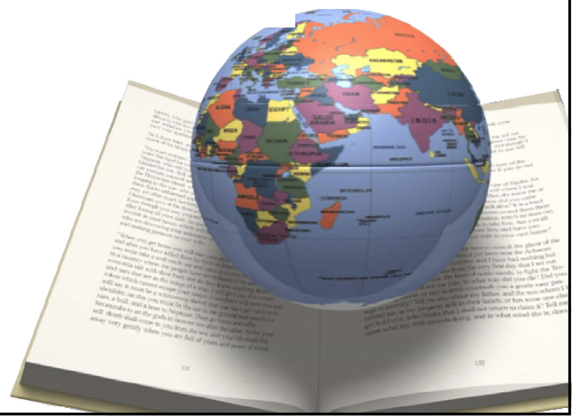
★Note 32 P293 (1)



20- PLANE CHANGE MANEUVERS

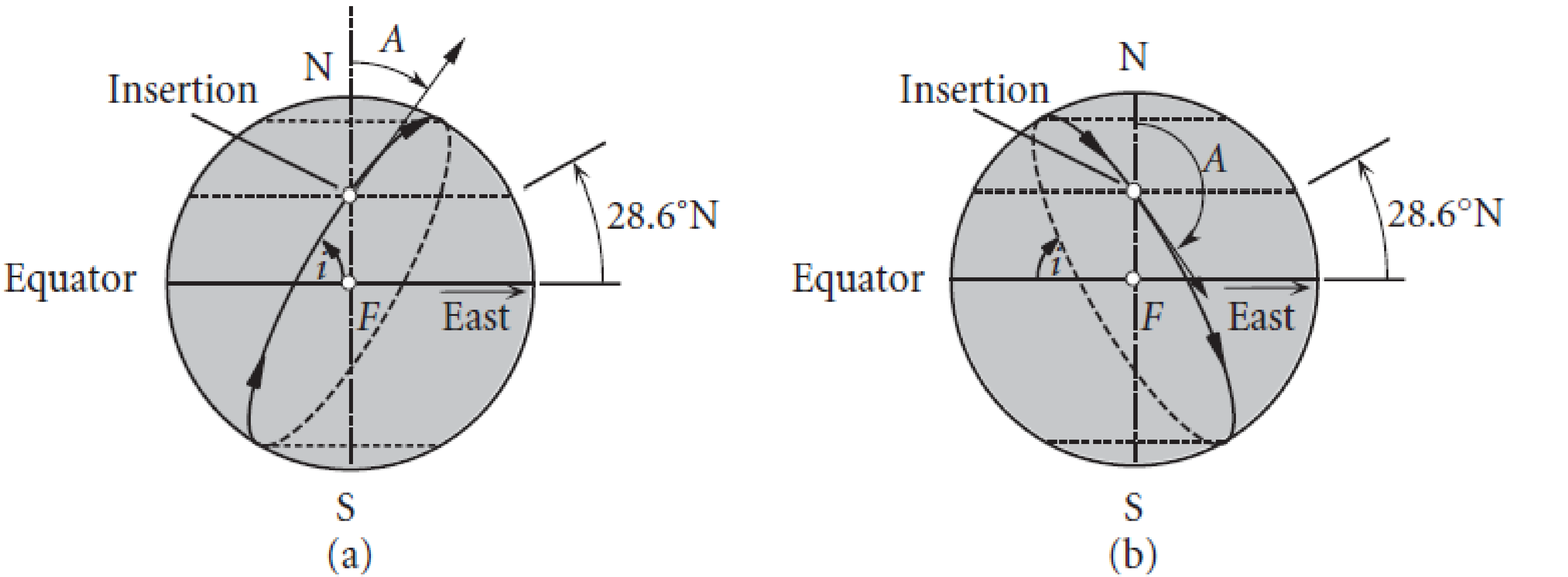
- Note 33 P293 (1)

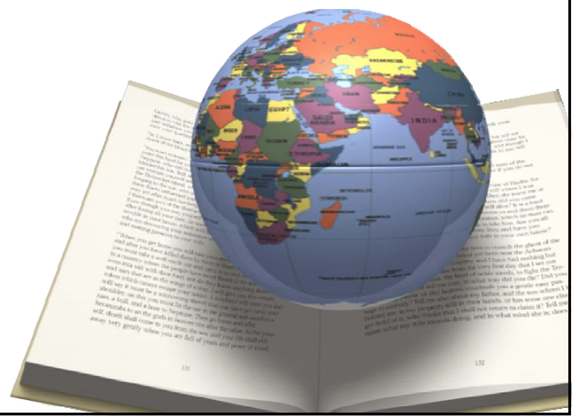




20- PLANE CHANGE MANEUVERS

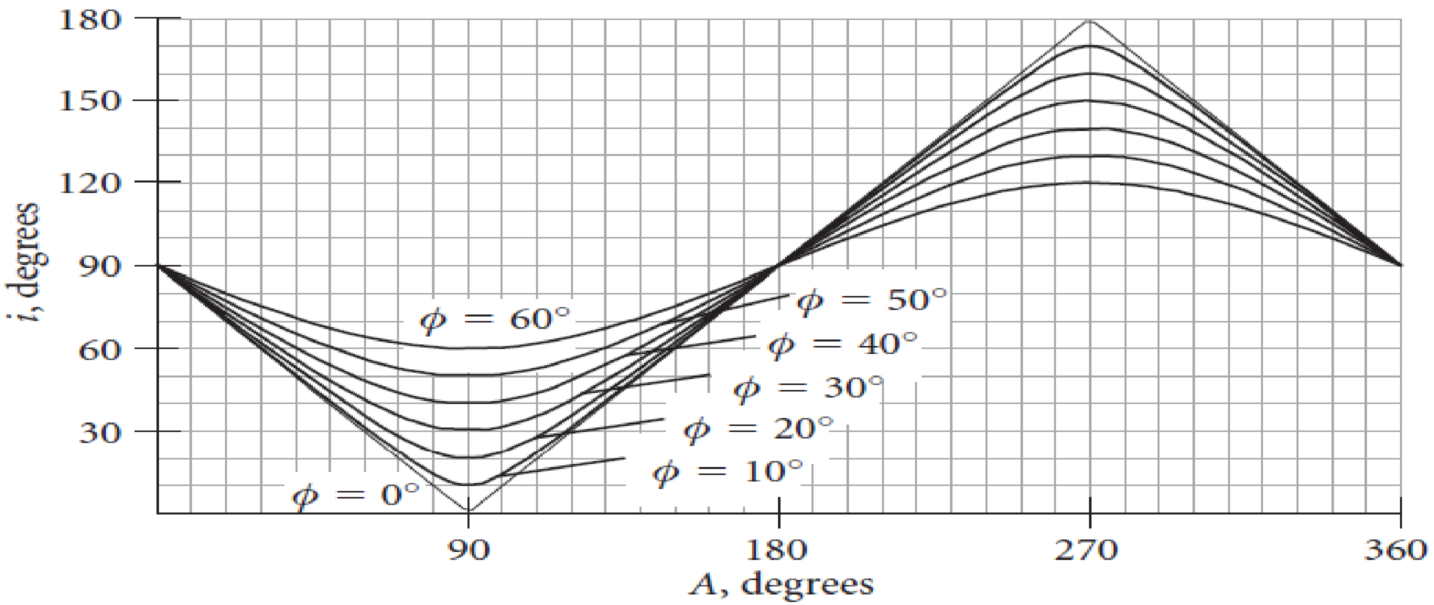
⋆Note 34 P294 (1)

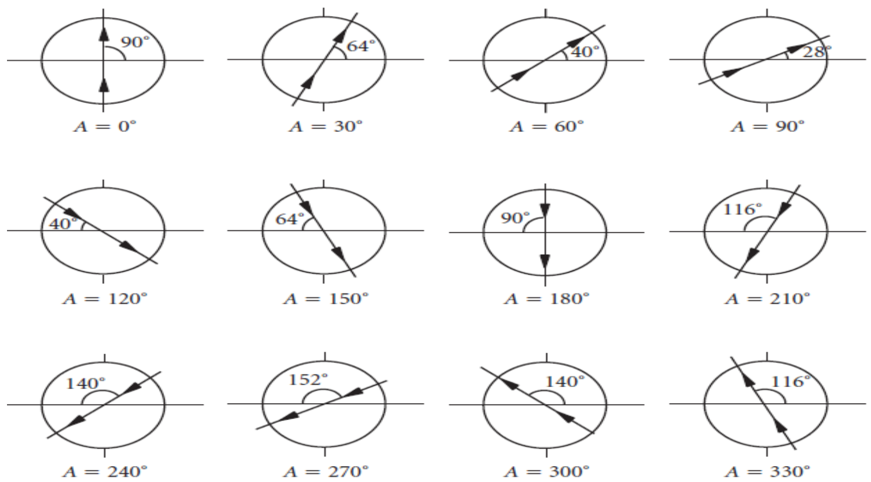


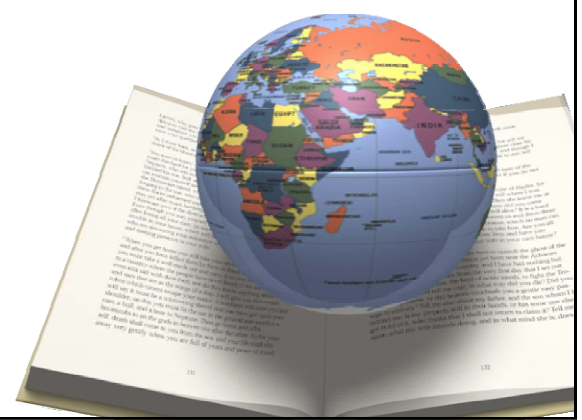


20- PLANE CHANGE MANEUVERS

★Note 35 P296 (1)



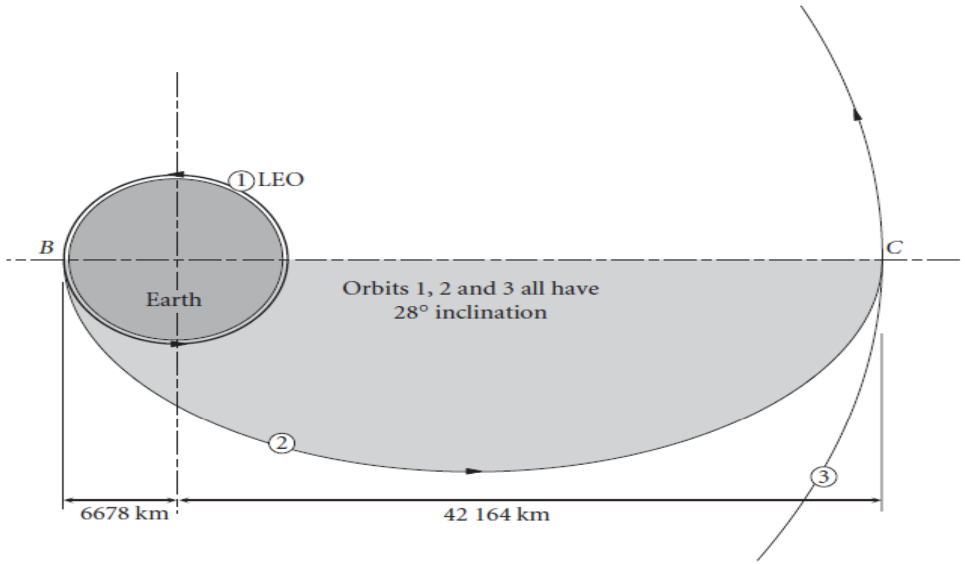


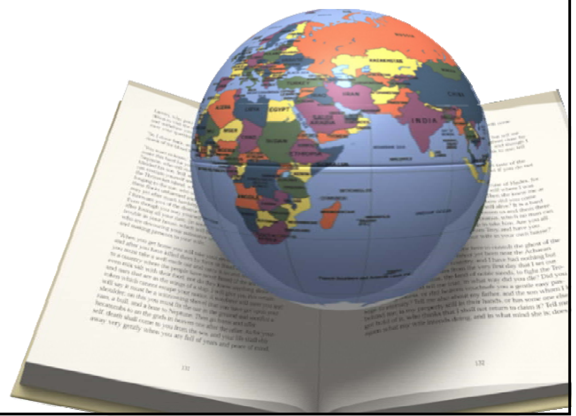


19- PALNE CHANGE MANEUVERS

EXAMPLE 19.1

- Find the ΔV required to transfer a satellite from a circular, 300km altitude law earth orbit of 28° inclination to a geostationary equatorial orbit. Circularize and change the inclination at altitude. Compare that ΔV requirement with the one in which the plane change is done in the low-earth orbit.
- figure bellow shows the 28° inclined low-earth parking orbit (1), the coplanar Hohmann transfer ellipse (2), and the coplanar GEO orbit(3)



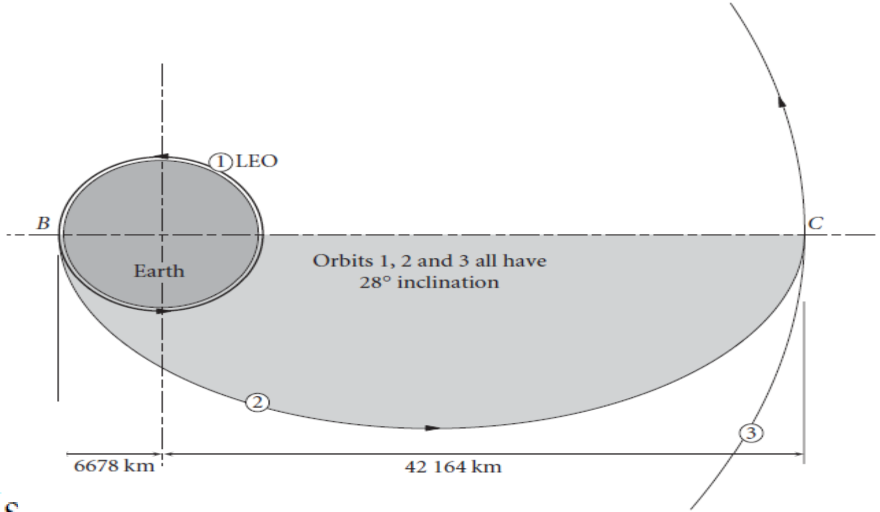


19-PALNE CHANGE MANEUVERS

EXAMPLE 19.1

* From the figure we see that:

$$r_B = 6678\,\text{km} \qquad r_c = 42\,164\,\text{km}$$



* Orbit 1:

For this circular orbit the speed at

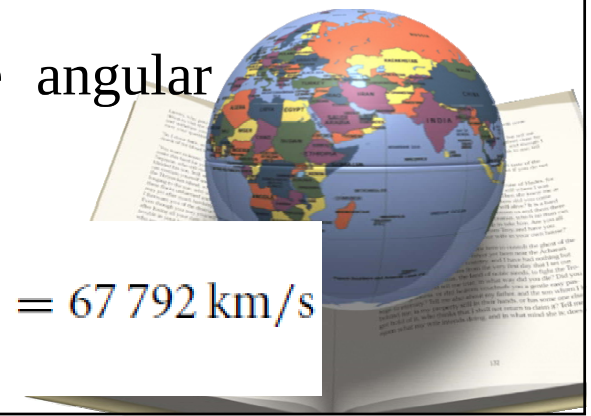
$$v_{B_1} = \sqrt{\frac{\mu}{r_B}} = \sqrt{\frac{398\,600}{6678}} = 7.7258\,\text{km/s}$$

* Orbit 2:

The eccentricity of the transfer orbit is $e_2 = \dfrac{r_C - r_B}{r_C + r_B} = 0.72655$

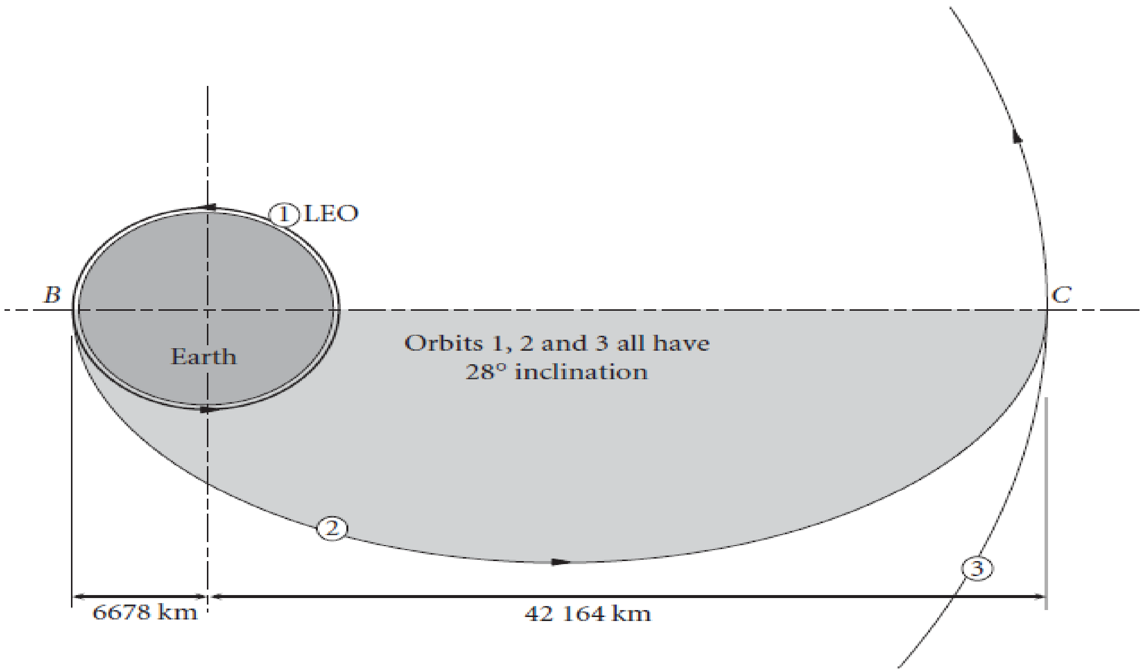
Let us evaluate the orbit equation at B to find the angular momentum of the Hohmann transfer orbit 2,

$$r_B = \frac{h_2^2}{\mu}\frac{1}{1 + e_2\cos(0)} \Rightarrow 6678 = \frac{h_2^2}{398\,600}\frac{1}{1 + 0.72655} \Rightarrow h_2 = 67\,792\,\text{km/s}$$



19-PALNE CHANGE MANEUVERS

EXAMPLE 19.1

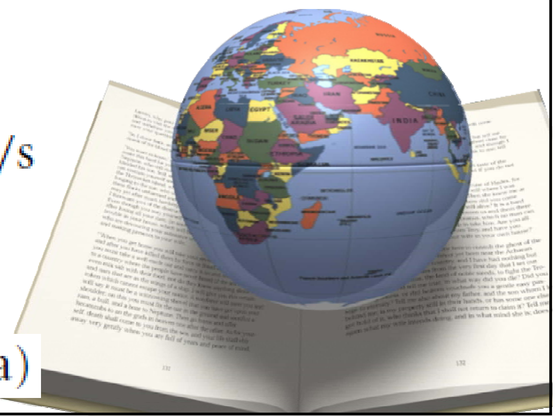


The velocities at perigee and apogee of orbit 2 are, from the angular momentum formula,

$$v_{B_2} = \frac{h_2}{r_B} = 10.152\,\text{km/s} \qquad v_{C_2} = \frac{h_2}{r_C} = 1.6078\,\text{km/s}$$

At this point we can calculate Δv_B,

$$\Delta v_B = v_{B_2} - v_{B_1} = 10.152 - 7.7258 = 2.4258\,\text{km/s} \qquad \text{(a)}$$

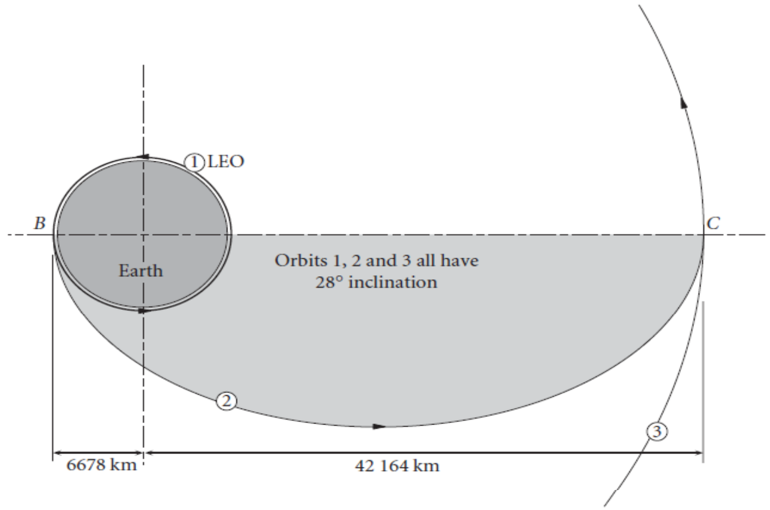


19- PALNE CHANGE MANEUVERS

EXAMPLE 19.1

* Orbit 3:

For this orbit, which is circular, the speed at C is



$$v_{C_3} = \sqrt{\frac{\mu}{r_C}} = 3.0747\,\text{km/s}$$

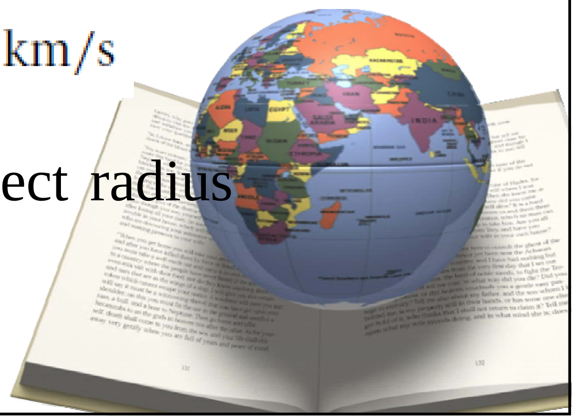
So that

$$\Delta v_C = v_{C_3} - v_{C_2} = 3.0747 - 1.6078 = 1.4668\,\text{km/s} \qquad \text{(b)}$$

We can now calculate the total ΔV for the Hohmann transfer:

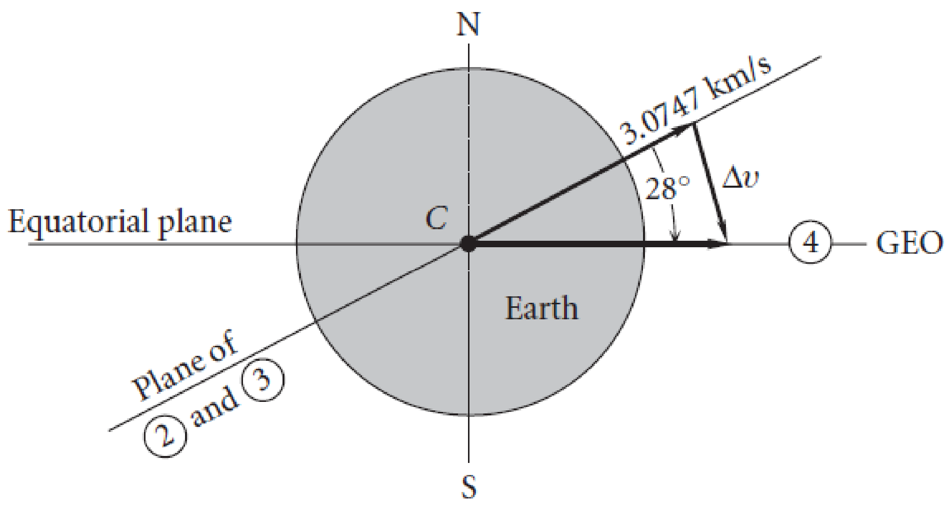
$$\Delta v_{\text{Hohmann}} = \Delta v_B + \Delta v_C = 2.4258 + 1.4668 = 3.8926\,\text{km/s}$$

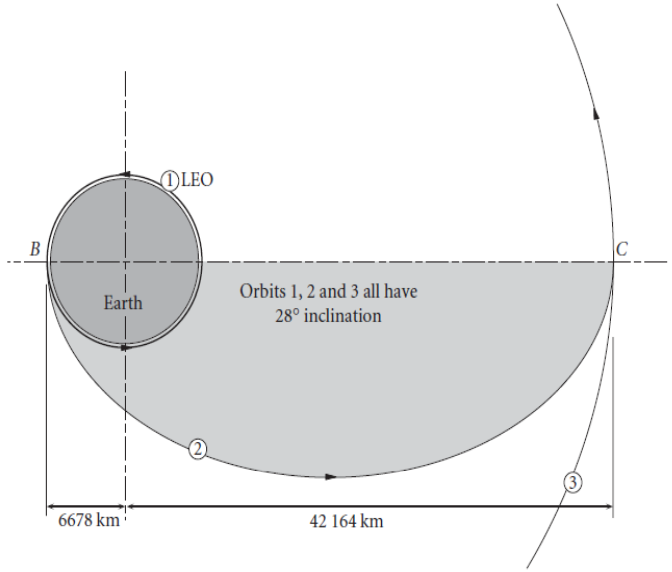
* This places the satellite in a circular orbit of the correct radius but the wrong inclination



19-PALNE CHANGE MANEUVERS

EXAMPLE 19.1





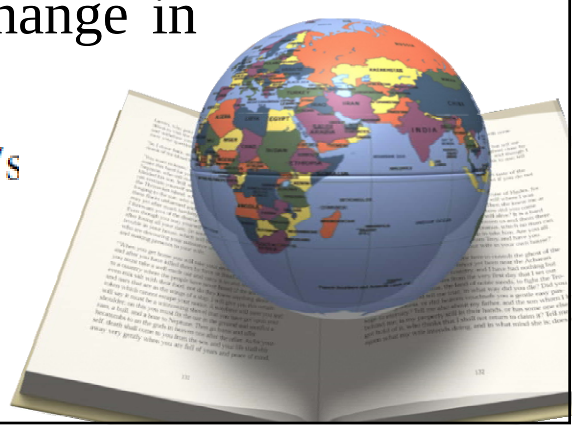
* The velocity vector at C must be rotated into the plane of the equator, as illustrated in the above figure.

* The ΔV requires to rotate that velocity through the change in inclination of 28° is:

$$\Delta v_{iC} = 2v_{C_3}\sin\frac{\Delta i}{2} = 2\cdot 3.0747\cdot\sin\frac{28^\circ}{2} = 1.4877\ \text{km/s}$$

* Therefore, the total ΔV requirement is.

$$\Delta v_{\text{total}} = \Delta v_{\text{Hohmann}} + \Delta v_{iC} = \underline{5.3803\ \text{km/s}}$$



19-PALNE CHANGE MANEUVERS

EXAMPLE 19.1

* Suppose we make the change at LEO instead of at GEO, to rotate the velocity vector $\mathbf{v}_{B_1}$ through 28° requires

$$\Delta v_{B_i} = 2v_{B_1} \sin\frac{\Delta i}{2} = 2 \cdot 7.7258 \cdot \sin\frac{28^\circ}{2} = 3.7381\,\text{km/s}$$

* This, together with (a) and (b), yields the total ΔV schedule for insertion into GEO:

$$\Delta v_{\text{total}} = \Delta v_{B_i} + \Delta v_B + \Delta v_C = 3.7381 + 2.4258 + 1.4668 = \underline{7.6307\,\text{km/s}}$$

* This is a 42 percent increase over the total ΔV with plane change at GEO. Clearly it is best to do plane change maneuvers at the largest possible distance (apoapse) from the primary attractor, where the velocities are smallest.