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Discrete Structure (Discrete Mathmatics)

Includes:
1. counting
2. Logic
3. Set Theory
4. Mathmatical Induction
5. Number Theory
6. Relations and Functions
7. Recurrencess
8. Graph Theory

Introduction

Methodology of mathmatics

Euclid (300.B.G)

Methed = Axiomatic

Axiomatic
- → primitives → Intutive (common sense) سطحی‌ترین اجزای علم
- → Definitions = Defining concepts based on primitives and previous concepts.
- → Axioms: → Intuitive
  - → complete [illegible]
  - → consistent
- → Theorems [illegible]

Counting:

The Rule of Sum (Axiom of Sum)

If a first task can be performed in m ways while a second task can be performed in n ways and two task cannot be performed simultaneously, then performing either task can be

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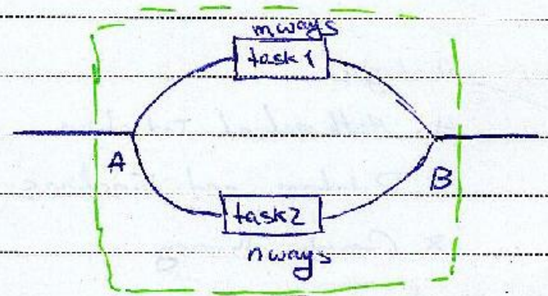
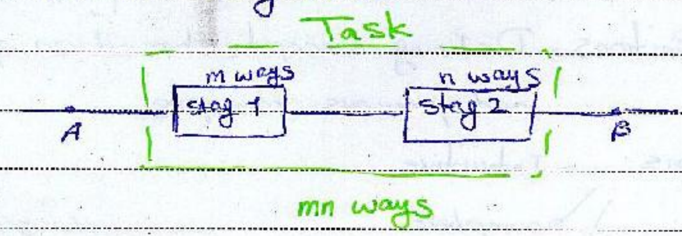
accomplished in m+n ways.



The Rule of Product (Axiom of Product)

If a procedure can broken down into first and Second stges and if there are m possible out comes for the first stage and n for the second stage, then total procedure can be carried out in designated.



Example) How many subsets of $A=\{a_1, \ldots, a_n\}$ does have?

procedure : n stages

stag1 , stage 2 , ... , stage n

$2 \times 2 \times \ldots \times 2 = 2^n$

after doing the procedure we have one outcome.

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Example) How many n-digit even numbers are there in $\mathbb{Z}^+$?
(Digits are distinc)

stage 2, stage 3, ..., stage n, stage 1

Task 1: $9 \times 8 \times \dots \times (9-n+2) \times 1$

0

Task 2: $8 \times 8 \times \dots \times (9-n+2) \times 4$

$2, 4, 6, 8$

To count the number of ways to select, or order, with or without repetitions, r of n distinct objects.

Order is relevent	repetitions are allowed	Typ of problem	FormulA
Yes	Yes	ترتیب arrangement	$A(n,r) = n^r$
Yes	NO	جایگشت permutation	$p(n,r) = \frac{n!}{(n-r)!}, r \le n$
NO	NO	ترکیب combination	$c(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$ $r \le n$
NO	Yes	combination with repetition	$H(n,r) = c(n+r-1, r)$

Arrangment

procedure: r stages

stage 1, stage 2, ..., stage n

$A(n,r) = n \times n \times \dots \times n = n^r$

permutation

procedure: r stages

stage 1, stage 2, ..., stage n

$p(n,r) = n \times (n-1) \times \dots \times (n-r+1)$

$$P(n,r) = \frac{n(n+1) \cdots (n-r+1)(n-r) \cdots \times 2 \times 1}{(n-r) \times \cdots \times 2 \times 1} = \frac{n!}{(n-r)!} ; r \le n$$

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Example) The number of repetitions of the letters in the word COMPUTER is $P(8,8) = 8!$

$P(n,n) = n!$ $n=r$ [illegible]

Example) The number of permutations of the letters in the word BALL is $\frac{4!}{2!}$

$B \; A \; L_1 \; L_2 \Rightarrow 4!$

Point The number of permutations of n objects wh... [illegible] indistingshable object is $\frac{n!}{r!}$

The number of permutations of n distinct object $= n!$

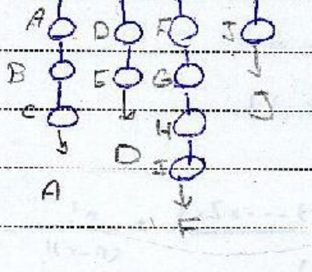
The number of permutations : N $\qquad N \cdot r! = n! \Rightarrow N = \frac{n!}{r!}$

The number of permutations of

$$\underbrace{a_1, a_1, \dots, a_1}_{n_1 \text{ times}}, \underbrace{a_2, a_2, \dots, a_2}_{n_2 \text{ times}}, \dots, \underbrace{a_r, a_r, \dots, a_r}_{n_r \text{ times}}$$

is $\frac{n!}{n_1! \times n_2! \times \cdots \times n_r!}$ where $n = n_1 + n_2 + \cdots + n_r$

Example)



[illegible] A,B [illegible]

[illegible] A,D [illegible]

ADFFFDJFAA $\Rightarrow \frac{10!}{3!\,2!\,4!\,1!}$

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Combination

combination $\begin{cases} \text{select} & \cancel{\text{with}} \\ & & \text{repetition} \\ \cancel{\text{order}} & \text{without} \end{cases}$

$C(n,r) = \binom{n}{r}$ [illegible] [illegible]

$$P(n,r) = \frac{n!}{(n-r)!} = C(n,r) \times r! \Longrightarrow C(n,r) = \frac{n!}{r!\,(n-r)!} \; ; \; r \leq n$$

Example) Show that

1) $\binom{n}{r} = \binom{n}{n-r}$

2) $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$

3) $r \times \binom{n}{r} = n\binom{n-1}{r-1}$

4) $\sum_{i=0}^{n} \binom{n}{i} = 2^n$

combination proof

problem

Solution 1 → Answer 1

Solution 2 → Answer 2

Answer 1 = Answer 2

2) $A = \{a_1, a_2, \dots, a_{n+1}\}$

$\binom{n+1}{r}$ with a_1: $\binom{n}{r-1}$; without a_1: $\binom{n}{r}$

$$\Longrightarrow \binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

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4) $A = \{a_1, a_2, \ldots, a_n\}$ The number of subsets $\longrightarrow 2^n$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

3) $r\binom{n}{r} = r \times \frac{n!}{(n-r)!\,r!} = n \times \frac{(n-1)!}{(n-r)!\,(r-1)!} = n \times \binom{n-1}{r-1}$

Example) Binomial expansion (Newton's expansion)

For each $n \geq 1$ and $x, y \in \mathbb{R}$,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x\, x^{n-i} y^i$$

$$(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ times}} = x^n + x^{n-1}y + \cdots + x^{3}y^{n-3} + \cdots + y^n$$

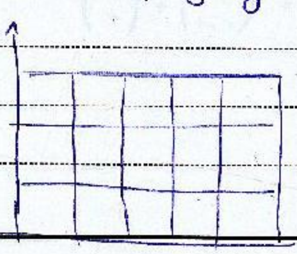
$$= \sum_{i=0}^{n} A_i\, x^{n-i} y^i = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

Multinomial Theorem

Generalization. For positive integers n and t, the coefficient of $(x_1^{n_1}, x_2^{n_2}, \ldots, x_t^{n_t})$ is $(x_1 + x_2 + \cdots + x_t)^n$ is

$$\frac{n!}{n_1!\, n_2! \cdots n_t!} \quad \text{where } n_1 + n_2 + \cdots + n_t = n$$

Example) Determine the number of (staircase) paths in xy plane from $(0,0)$ to $(5,3)$ where each such path is made of up individual step going 1 unit to the right (R) or 1 unit upward (U).



طول ۵، عرض ۳، U ۳ ، R ۵ تا

The number of paths = the number of word with 5 R's and 3 U's

$$\frac{8!}{5!\,3!}$$

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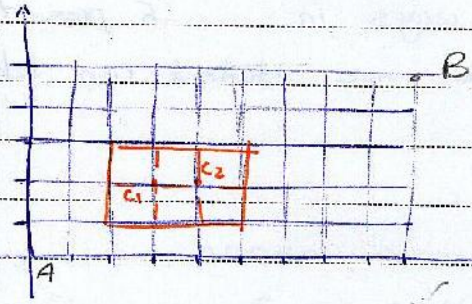
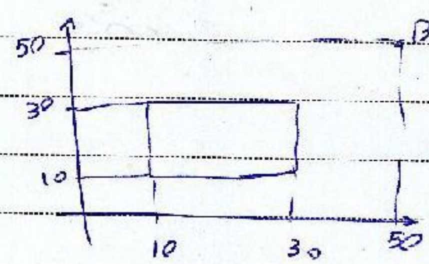
Example)



تعداد مسیرهایی که از C1 و C2 بگذرد دوبار کم کردیم

$$N_{AB} = N_{AC_1} N_{C_1B} - N_{AC_2} N_{C_2B} + N_{AC_1} N_{C_1C_2} N_{C_2B}$$

$$= \binom{8+5}{5} - \binom{3+2}{2}\binom{5+5}{5} - \binom{4+2}{4}\binom{4+3}{3} - \binom{3+2}{2}\binom{1+0}{1}\binom{4+3}{3}$$



combination with repetition

select	with	
~~order~~	~~without~~	repetion

Example) 3 distinct objects a,b,c to select 2 objects with repetition

aa bb cc

ab bc $\longrightarrow H(3,2) = 6$

ac

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Example) The number of ways ، 5 present between 3 students where one student can take 0,1,2,3,4,5 presents.

[illegible] a b c

aaabc, bbaac, aaaaa

$H(3,5) = ?$

$x_1+x_2+x_3=5$

xxxxx

$H(n,r)$ — $\{a_1, a_2, \dots, a_n\}$

The number of a_i's used in the selection

00 | 000 | 0 | ...... | | 0000 $(n-1)$ 1's

The number of a_i's used in the selection $\quad a_{n-1} \quad a_n$ $\quad r$ 0's

$$H(n,r) = \frac{(n-1+r)!}{r!\,(n-1)!} = \binom{n+r-1}{r}$$

Example) The number of nonnegative integer solutions to the equation

$$x_1+x_2+x_3+x_4 = 25$$

$$H(4,25) = \binom{4+25-1}{25} = \binom{28}{25} \qquad \frac{28!}{25!\,3!}$$

Example) Determine the number of positive integer solutions to the inequality

$$x_1+x_2+x_3+x_4+x_5+x_6+x_7 < 109$$

$x_1+x_2+\dots+x_7 \leq 108 \quad x_i \geq 1$

$y_1+y_2+\dots+y_7 \leq 101 \Rightarrow y_i = x_i - 1 \; ; y_i \geq 0 \Rightarrow x_i = y_i + 1$

PAPCO $\quad y_1+y_2+\dots+y_7+z = 101 \quad y_i, z \geq 0 \quad \frac{108!}{101!\,7!}$

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Example) In How many ways can you put n objects in r distinct containers, where

1) The objects are distinct?

$r \times r \times r \times \dots \times r = r^n$

2) The objects are indistinguishable? $\binom{n+r-1}{r} = \binom{n+r-1}{r-1}$

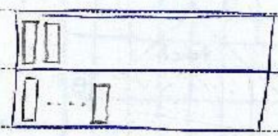
$x_1 + \dots + x_r = n$

3) The objects are distinct and their order in a container is relevant?

$1, 2 \mid 8, n, 5 \mid \dots \mid \dots \mid 3, 6, 10$ $\quad (r-1)$ 1's $\quad 1, 2, 3, \dots, n$

$$\frac{(n+r-1)!}{(r-1)!} = n! \frac{(n+r-1)!}{n!(r-1)!} = n! \binom{n+r-1}{n} = n! \cdot H(n, r)$$

Example) Determine the number of ways one can put 15 book in the shelves above so that each shelf contain at least one book.



حل $\binom{15}{1} 14! \, 1! + \binom{15}{2} 13! \, 2! + \dots + \binom{15}{14} 1! \, 14! = 14 \times 15!$

فرض $b_1 \, b_2 \dots b_{15} \, / \Rightarrow 16! - 2 \times 15! = 4 \times 15!$

$x_1 + x_2 = 15 \quad x_i \geq 1 \quad \frac{14!}{1! \, 13!} = 14 \Rightarrow 14 \times 15!$

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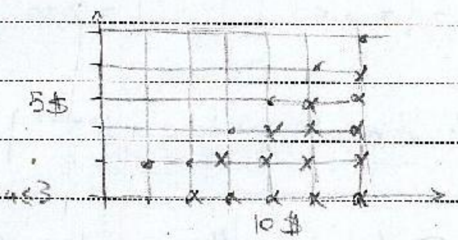
Example) The cost of admision at a neighberhood movie theater is 5\$. When Donna the cashier opens the ticket booth, she has one 10\$ bill and two 5\$ bill. The line for early admision contains 11 patrons, five of whom have a 5\$ bill each and Six others, each of whom have a 10\$. Each time Donna is unable to give the correct change she summon the manager a 5\$ bill. In How many ways can these 11 people be lined up so that Donna always has at least one 5\$ bill and never hasto Summon the manager

\$10, 2 5\$ bill

11 patrons

5 \$5 6 \$10



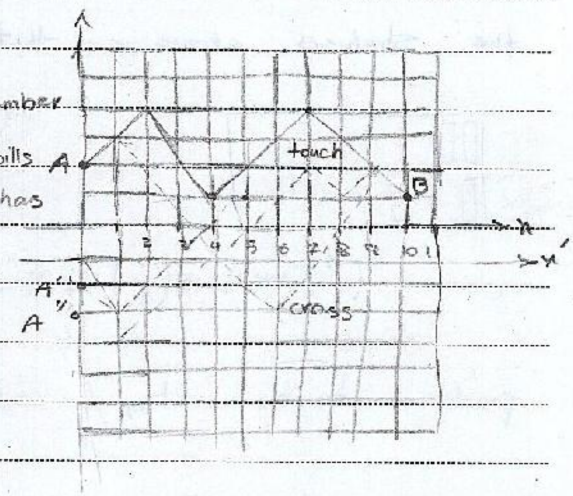
$x_1 + x_2 + \dots + x_7 = 5$

$0 \le x_1 \le 5$ $0 \le x_2 \le 5$ $0 \le x_3 \le 4$ $0 \le x_4 \le 3$

$0 \le x_5 \le 2$ $0 \le x_6 \le 1$ $x_7 = 0$

D 4, U 7 B(11, 5)

$N_{AB} = \dfrac{11!}{5!\,6!}$



The number of paths from A to B that touch or cross x axis:

$N_{AB} = N_{A'B} = \dfrac{11!}{7!\,4!}$
(touch or cross)

$U: (x,y) \to (x+1, y+1)$ $u+d=1$

$D: (x,y) \to (x+1, y-1)$ $u-d=3$

$\underset{\text{cross}}{N_{AB}} = N_{AB} = N_{A'B} = \dfrac{11!}{8!\,3!}$
toch or cross x'

$u-d=5$ $u=8$

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$x' = y = -1$

$u+d=11$ $d=3$

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$N_{AB} = N_{AB} - N_{A'B} = \binom{11}{5} - \binom{11}{7}$ حالت ب ـ n ترتیبی لتیم

non crossing cross

$N_{AB} = N_{AB} - N_{AB} = \binom{11}{5} - \binom{11}{7}$

not-touch or cross Touch or cross

Inclusion and Exclusion

S, A finite set

$c_1, c_2, c_3, \ldots, c_t$: t conditions (propertion defined on S.

$N(c_i)$ = The number of elements of S satisfying the condition c_i

$N(c_i c_j)$ = " " " " " " " both c_i & c_j

$N(\bar{c}_i)$ = " " " " " " " not satisfying c_i

$N(\bar{c}_i \bar{c}_j)$ = " " " " " " " none of c_i & c_j

$N(\bar{c}_i \bar{c}_j) \neq N(\overline{c_i c_j})$

Theorem - (The principle of Inclusion & exclusion)
consider a set S, with $|S| = N$, and conditions c_i, $1 \le i \le t$, satisfied by some of elements of S. The number of elements of S that satisfied none of the conditions c_i, $1 \le i \le t$, denoted by $\bar{N}$ or $N(\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_t)$ is

$$\bar{N} = N(\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_t) = N - \sum_{1 \le i \le t} N(c_i) + \sum_{1 \le i < j \le t} N(c_i c_j) - \cdots + (-1)^t N(c_1, c_2, \ldots, c_t)$$

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combinatorial: یعنی هر عضو که را از ست که برداریم بیاییم ببینیم چند بار در چپ چند بار در راست حساب شده

دو حالت داریم

proof: consider $x \in S$

(case 1) x satisfies none of $c_1, \dots, c_t$:

contribution of x in left side $= 1$

" " " " right " $= 1 - 0 + 0 - \dots + (-1)^t 0 = 1$

(case 2) x satisfies exactly r conditions where $1 \leq r \leq t$.

contribution of x in left side $= 0$

" " " " right " $= 1 - r + \binom{r}{2} - \binom{r}{3} \dots + (-1)^r \binom{r}{r} + 0 + 0$

$= (1-1)^r = 0$

Example) $S = \{1, 2, \dots, 1000\}$

The number of elements of S which are divided by none of $2, 3, \text{or } 5$.

c_1 = the number is divisible by 2

c_2 = " " " " by 3

c_3 = " " " " " 5

$$N(\bar{c}_1 \bar{c}_2 \bar{c}_3) = N - (N(c_1) + N(c_2) + N(c_3)) + (N(c_1c_2) + N(c_1c_3) + N(c_2c_3)) - N(c_1c_2c_3)$$

$$= 1000 - \left(\left[\frac{1000}{2}\right] + \left[\frac{1000}{3}\right] + \left[\frac{1000}{5}\right]\right) - \left(\left[\frac{1000}{6}\right] + \left[\frac{1000}{10}\right] + \left[\frac{1000}{15}\right]\right) - \left[\frac{1000}{30}\right] = \cdot$$

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Example) Determine the number of nonnegative integer solutions to the equation $x_1+x_2+x_3+x_4=18$ $x_i \le 7$ $1 \le i \le 4$

S = The set of all nonnegative integer solutions to the equation

$c_i = x_i \ge 8$ $1 \le i \le 4$

$N = |S| = \binom{18+3}{3} = \binom{21}{3}$

$N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4) = \binom{21}{3} - \binom{4}{1}\binom{10+4-1}{4-1} + \binom{4}{2}\binom{2+4-1}{4-1} - 0 + 0$

Example) (Euler's phi function)

$\Phi : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$

$\phi(n) = \left| \{ m \mid m \in \mathbb{Z}^+, 1 \le m \le n, (m,n)=1 \} \right|$

$|S| = n$

$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_3^{\alpha_3}$

$S = \{ m \mid m \in \mathbb{Z}, 1 \le m \le n \}$

c_1 = m is divisible by p_1

c_2 = " " " by p_2

$\vdots$

$\phi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)$

$N(\bar{c}_1 \bar{c}_2 \bar{c}_3) = n - \left(\frac{n}{p_1} + \frac{n}{p_2} + \frac{n}{p_3}\right) + \left(\frac{n}{p_1 p_2} + \frac{n}{p_1 p_3} + \frac{n}{p_2 p_3}\right) - \frac{n}{p_1 p_2 p_3} =$

$n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right)$

$\Phi(23100) = \Phi(2^2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11) = 2300\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{7}\right)\left(1 - \frac{1}{11}\right)$

$= 4800$

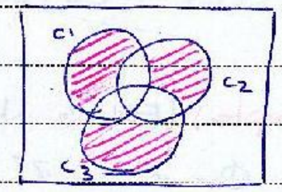
Generalization of the Inclusion & Exclusion

E_m = The number of elements in S that satisfy exactly m of the t conditions

$t=3$

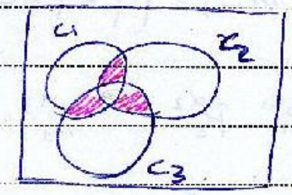
$E_0 = \bar{N}$

$$E_1 = N(C_1) + N(C_2) + N(C_3) - \left(2N(C_1C_2) + 2N(C_1C_3) + 2(C_2C_3)\right) + 3N(C_1C_2C_3)$$



$E_1 = S_1 - 2S_2 + 3S_3$

$$E_2 = \left(N(C_1C_2) + N(C_1C_3) + N(C_2C_3)\right) - 3N(C_1C_2C_3) = S_2 - 3S_3$$



Theorem - The number of elements in S that satisfy exactly m of the conditions $c_1, c_2, \ldots, c_t$ is given by

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^{t-m} \binom{t}{t-m} S_t$$

$$S_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le t} N(c_{i_1} c_{i_2} \ldots c_{i_k}) \quad \text{and } S_0 = N$$

$$N(\bar{c}_1 \ldots \bar{c}_t) = E_0 = S_0 - S_1 + S_2 - \cdots + (-1)^t S_t$$

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Proof. consider $x \in S$ and the following case.

case1) x satisfies fewer than m condition:

left side $= 0$

right side $= 0 - 0 + 0 - 0 \dots = 0$

case2) x satisfies exactly m conditions

left side $= 1$

right side $= 1 - 0 + 0 \dots = 1$

case3) x satisfies r conditions $m < r \le t$

left side $= 0$

right side $= \binom{r}{m} - \binom{m+1}{1}\binom{r}{m+1} + \binom{m+2}{2}\binom{r}{m+2} \dots + (-1)^{r-m}\binom{r}{r-m}\binom{r}{r}$

$- 0 + 0$

$$\binom{m+k}{k}\binom{r}{m+k} = \frac{(m+k)!}{k!\,m!} \cdot \frac{r!}{(m+k)!\,(r-m-k)!} = \frac{(r-m)!}{k!\,(r-m-k)!} \cdot \frac{r!}{m!\,(r-m)!} =$$

$$\binom{r-m}{k}\binom{r}{m}$$

$$\text{right side} = \sum_{k=0}^{r-m} (-1)^k \binom{m+k}{k}\binom{k}{m+k} = \sum_{k=0}^{r-m} (-1)^k \binom{r}{m}\binom{r-m}{r} =$$

$$\binom{r}{m}\sum_{k=0}^{r-m} (-1)^k \binom{r-m}{k} = \binom{r}{m}(1-1)^{r-m} = 0$$

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Example) Let $A = \{1, 2, \dots, 10\}$ & $B = \{1, 2, \dots, 7\}$ How many functions $f: A \to B$ satisfy $|f(A)| = 4$?

Domain $f: A \to B$, codomain

for any $x \in A$, there is $(x, y) \in f$ for some $y \in B$

Domain $= \mathbb{R} - \{0\}$

[illegible] exc inc [illegible] $c_i : i \in f(A)$ [illegible] ?

S = The set of all functions

$|S| = 7^{10}$

$c_i : i \in f(A)$, $1 \leq i \leq 7$

$$E_3 = S_3 - \binom{4}{1} S_4 + \binom{5}{2} S_5 - \binom{6}{3} S_6 + \binom{7}{4} S_7$$

$$= \binom{7}{3} 4^{10} - \binom{4}{1}\binom{7}{4} 3^{10} - \binom{5}{2}\binom{7}{5} 2^{10} - \binom{6}{3}\binom{7}{6} 1^{10} + 0$$

How many have $|f(A)| \leq 4$?

$$E_3 + E_4 + E_5 + E_6 =$$

$$|S| - E_0 - E_1 - E_2$$

[illegible] [illegible] [illegible]

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```
For i:=1 to 20 do
 for j:=1 to i do
 for k:=1 to j do
 writeln(i*j+k);
```

$i=1 \quad j=1 \quad k=1$

$i=2 \quad j=1 \quad k=1$

$\quad j=2 \quad k=1$

$\quad k=2$

(i,j,k) , $i \geq j \geq k$ تعداد سه‌تایی‌ها مثل ترکیب تکراری است

$$H(20,3) = \binom{20+3-1}{3} = \binom{22}{3}$$

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \dots + (-1)^{k-m} \binom{t}{t-m} S_t$$

m ترکیب دارند که یکی یا بیش از شرط S_m با ترتیب‌های متفاوت شمرده می‌شود

$E_m \leq S_m$ ← دقیقاً m ترکیب دارد

Derangement: Nothing is in right place

① ② ③ ④ ⑤

② ③ ④ ⑤ ①

1 2 3 4 5

③ ① ② ⑤ ④

S = The set of all permutations of the object ①, ②, ③, ④, ⑤

$|S| = 5!$

e_i = i is it's right place $\quad 1 \leq i \leq 5$

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$$n(\overline{C_1}\,\overline{C_2}\,\overline{C_3}\,\overline{C_4}\,\overline{C_5}) = 5! - \binom{5}{1}4! + \binom{5}{2}3! - \binom{5}{3}2! + \binom{5}{4}1! - \binom{5}{5}0!$$
$$= 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$$

The number of derangment of n objects is:

$$d_n = n! \sum_{k=1}^{n} (-1)^k \frac{1}{k!}$$

Maclaurin series:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots$$

$$f(x) = e^x \rightarrow f'(x) = f''(x) = \dots = e^x$$

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

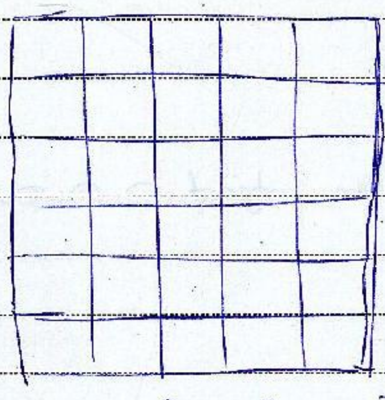
$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots$$

for large n: $d_n \approx n!\,e^{-1}$

Rook Polynomials

rook, castle

به جای رخ ها تیری نیز در شطرنج هم اند



$$\frac{30 \times 20}{2}$$

مثلا 5 تا رخ را می توان گذاشت که یکدیگر را نزنند

$$\frac{30 \times 20 \times 12 \times 6 \times 2}{5!}$$

$$\frac{n^2 \cdot (n-1)^2 \cdots 1^2}{n!} = n!$$

اگر صفحه‌ی شطرنج $n \times n$ باشد n

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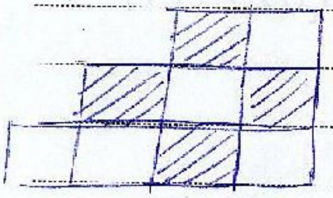


$r_0(C)=1$ $\quad$ $r_1(C)=5$

$r_2(C)=6$ $\quad$ $r_3(C)=2$

$r_4(C)=r_5(C)=\dots=0$

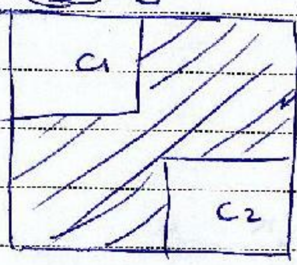
$r_k(C)=$ The number of ways of placing k non-taking rooks on C $(k\geq 0)$

$$r(C,x)=r_0(C)+r_1(C)x+r_2(C)x^2+\dots=\sum_{k=0}^{\infty} r_k(C)\,x^k$$

The rook polynomial of the board C

For the board above $r(C,x)=1+5x+6x^2+2x^3$

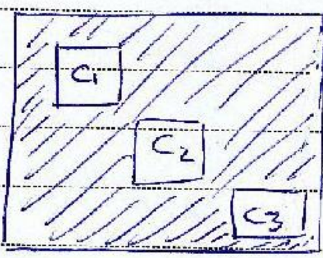
(I) C



forbidden

C_1 and C_2 have no column or rows in common

$$r(C,x)=r(C_1,x)\cdot r(C_2,x)$$



$$r(C,x)=r(C_1,x)\cdot r(C_2,x)\cdot r(C_3,x)$$

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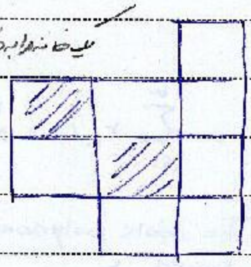
(3) $r_k(C) = r_0(C_1) r_k(C_2) + r_1(C_1) r_{(k-1)}(C_2) + \dots + r_k(C_1) r_0(C_2)$

$$r(C_1, x) r(C_2, x) = \left(\sum_{m=0}^{\infty} r_m(C_1) x^m \right) \left(\sum_{n=0}^{\infty} r_n(C_2) x^n \right)$$

$$= \sum_{k=0}^{\infty} \left(r_0(C_1) r_k(C_2) + \dots + r_k(C_1) r_0(C_2) x^k \right) = r(C, x)$$

یک خانه را بدلخواه در نظر بگیریم



$r_k(C)$

C_e = The subboard obtained from eliminating the designated square

C_s = The sub-board obtained from eliminating the row & colum of the designated square

$r_k(C) = r_k(C_e) + r_{k-1}(C_s)$

$r_k(C) x^k = r_k(C_e) x^k + r_{k-1}(C_s) x^k$

$$\sum_{k=1}^{\infty} r_k(C) x^k = \sum_{k=1}^{\infty} r_k(C_e) x^k + \sum_{k=1}^{\infty} r_{k-1}(C_s) x^k$$

$r(C, x) - 1 = r(C_e, x) - 1 + x r(C_s, x)$

$r(C, x) = r(C_e, x) + x r(C_s, x)$

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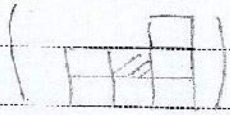
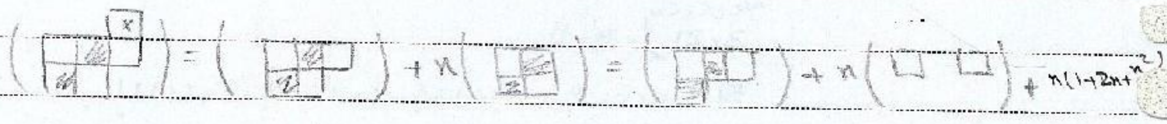
C

 = The rook polynomial of C.

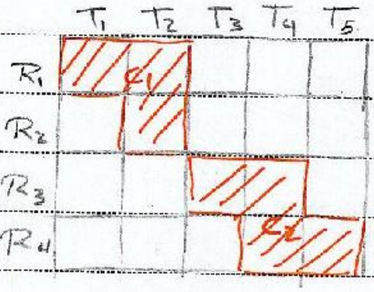
$(\boxed{}\boxed{}) = 1 + 2x$

 $+ x(1+2x+x^2)$

$= (1+2x) + x(1+2x) + x(1+2x+x^2) = 1 + 4x + 4x^2 + x^3$

Example) In how many ways can you make seat arrangement for 4 people denoted by R_i, $1 \le i \le 4$, in 5 tables (T_i's) with one open seat at each table provided that

1) R_1 will not sit at T_1 or T_2
2) R_2 will " " " T_2
3) R_3 " " " " T_3 or T_4
4) R_4 " " " " T_4 or T_5



سطر بیش از یکی ← نفر بیش از یک میز
ستون " " ← در یک میز بیش از یک نفر

Inclusion & exclusion. C_i: R_i sits at undersirable table(s) $1 \le i \le 4$

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S: The set of way in which 4 people sit at 5 tables where each table has only one open seat

$$|S| = 5\times4\times3\times2 = 5!$$

$$N(\bar{C}_1\bar{C}_2\bar{C}_3\bar{C}_4) = 5! - (\overset{N_{C_1}}{2\times4!} + \overset{N_{C_2}}{4!} + \overset{N_{C_3}}{2\times4!} + \overset{N_{C_4}}{2\times4!}) + (\overset{N_{C_1C_2}}{1\times3!} + \overset{N_{C_1C_3}}{4\times3!} + \overset{N_{C_1C_4}}{4\times3!} + \overset{N_{C_2C_3}}{2\times3!} + \overset{N_{C_2C_4}}{2\times3!} + \overset{N_{C_3C_4}}{3\times3!}) - (\overset{N_{C_1C_2C_3}}{2\times2!} + \overset{N_{C_1C_2C_4}}{2\times2!} + \overset{N_{C_1C_3C_4}}{6\times2!} + \overset{N_{C_2C_3C_4}}{3\times2!}) + 3\times1!$$

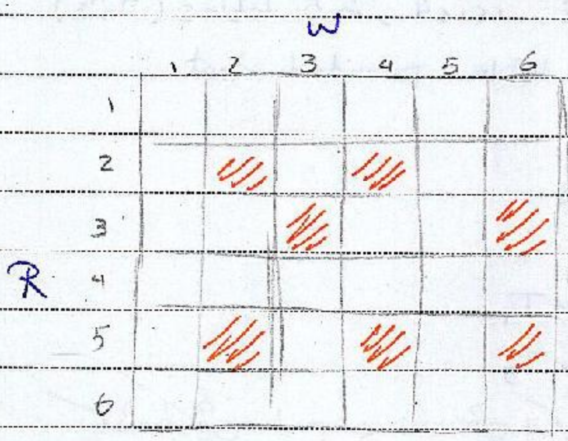
$$= 5! - r_1(C)\,4! + r_2(C)\,3! - r_3(C)\,2! + r_4(C)\,1!$$

$$= \sum_{n=0}^{4} (-1)^n r_n(C)\,(5-n)!$$

(4,3), (3,4), (5,5), (1,2)

[illegible]

[illegible]



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Logic

natural language: گاهی با یک کلمه بیان های کتبی متغیر می‌شوند مثلاً کلمات

Reasoning برای بعضی اتفاقات دلایلی می‌آوریم

This is a mathematical course

If he is generous, I can count on him → کلی نیست

None of the books is suitable for my intention

Any bird has wing
Joe is a bird ⟹ Therefor, Joe has wings

$\forall x \in B \; [P(x)]$

$a \in B$

$P(a)$

There is symbiotic relation between the natural language and formal logic

Proposition or statement:

is a declarative (جمله خبری) sentence that can be assigned one of true or false values. جمله‌ای که بتوانیم بگوییم درست است یا غلط است یا نسبت بدهیم یکی از ارزش درست یا غلط را

"This is a course of combinition"

"One man lives in Mars" is a proposition

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are propositions, while

"Is this proposition?"
" x is greater than 2." — [illegible]
" may I go out? "
they aren't propositions

A primitive or atomic proposition. [illegible] proposition [illegible]

There is no way to break them down into anything simple.

denoted by small Roman letters p.q.r. ...

compound statement are constructed using the atomic proposition. and some connectives

Syntax making sentences out of words or phrases.

conjuction: The conjunction of p.q is denoted by $p \land q$, and read (p and q)

Example) $\overbrace{\text{One man lives in Mars}}^{p}$ and $\overbrace{\text{this is a course of math}}^{q}$

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Disjunction: Denoted by $(P \vee q)$ and read $(P \text{ or } q)$

Implication: We say that "P implies q" (گزاره) and write "$p \rightarrow q$"
to designated the implication of q by P

در علوم این درست است : "2<3 implies 3<4" [illegible]

$P \rightarrow q$
- If P then q
- P is sufficient for q
- q is necessary for P — هیچ الزامی نیست که رابطه علی بین p و q برقرار باشد و در واقع نیازی نیست تا استلزام داشته باشند
- q if P
- P only if q

Negation: Denoted by "$\neg p$" or "$\bar{p}$" and read "not P"

Syntax: $(p \wedge q)$, $\neg(p \wedge q)$, $(p \vee q) \rightarrow \neg r$, $P \rightarrow q$, ...

Semantics: propositional logic — منطق براساس حکم نسبت می‌دهیم به گزاره‌ها
model - Theoretic — نسبت می‌دهیم true , false
{true , false} — حکم ← ساختار نمی‌شناسیم که درست باشد
{1 , 0} — اگر غلط باشد

V^* , set of primitive propositions $\{0, 1\}$

Valuation function (basic assignment (واگذاری))

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Extended valuation function (assignment):

v: The set of all propositions $\longrightarrow \{0,1\}$

defined as following

- $v(p) = v^*(p)$ if p is atomic
- $v(\neg\psi) = 1 - v(\psi)$ [illegible]
- $v(\psi \wedge \varphi) = v(\psi) \cdot v(\varphi)$
- $v(\psi \vee \varphi) = v(\psi) + v(\varphi) - v(\psi)\, v(\varphi)$
- $v(\psi \to \varphi) = 1 - v(\psi) + v(\varphi) - (1 - v(\psi))\, v(\varphi)$

$$= 1 - v(\psi) + v(\psi)\, v(\varphi)$$

$v(\psi) = 0$, $v(\varphi) = 0$

$$v(\psi \to \varphi) = 1 - 0 + 0 = 1$$

$v(\psi) = 1$, $v(\varphi) = 0$

$$v(\psi \to \varphi) = 1 - 1 + 0 = 0$$

$v(\psi) = 1$ $v(\varphi) = 1$

$$v(\psi \to \varphi) = 1 - 1 + 1 = 1$$

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Definition – two statements φ and ψ are said to be logically equivalent, written $\varphi \Longleftrightarrow \psi$, when the truth tables for φ and ψ are exactly the same.
(In other word for any possible basis assignment $v^*: \Omega^* \to \{0,1\}$, $v(\varphi) = v(\psi)$.)

$\neg(p \wedge q) \Longleftrightarrow \neg p \vee \neg q$

p	q	$\neg(p\wedge q)$	$\neg p \vee \neg q$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

The laws of logic

1. $\neg(\neg p) \Longleftrightarrow p$ the laws of double negation

2. $\neg(p \vee q) \Longleftrightarrow \neg p \wedge \neg q$
$\neg(p \wedge q) \Longleftrightarrow \neg p \vee \neg q$ } Demorgan's law

3. $p \vee q \Longleftrightarrow q \vee p$
$p \wedge q \Longleftrightarrow q \wedge p$ commutative laws

4. $p \vee (q \vee r) \Longleftrightarrow (p \vee q) \vee r$
$p \wedge (q \wedge r) \Longleftrightarrow (p \wedge q) \wedge r$ Associative laws

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5. $P \vee (q \wedge r) \iff (P \vee q) \wedge (P \vee r)$
$P \wedge (q \vee r) \iff (P \wedge q) \vee (P \wedge r)$
Distributive law

6. $P \vee P \iff P$
$P \wedge P \iff P$
Idempotent laws

7. $P \vee F_0 \iff P$
$P \wedge T_0 \iff P$
Identify laws

8. $P \vee \neg P \iff T_0$
$P \wedge \neg P \iff F_0$
Invense laws

9. $P \vee T_0 \iff T_0$
$P \wedge F_0 \iff F_0$
Domination laws

10. $P \vee (P \wedge q) \iff P$
$P \wedge (P \vee q) \iff P$
Absorption laws

11. $P \rightarrow q \iff \neg p \vee q$

12. $P \rightarrow q \iff \neg q \rightarrow \neg p$

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Example) Simplify

$$(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \Longleftrightarrow (\neg p \vee q) \wedge ((p \wedge p) \wedge q)$$
$$\Longleftrightarrow (\neg p \vee q) \wedge (p \wedge q)$$
$$\Longleftrightarrow ((\neg p \vee q) \wedge p) \wedge q$$
$$\Longleftrightarrow (p \wedge (\neg p \vee q)) \wedge q$$
$$\Longleftrightarrow ((p \wedge \neg p) \vee (p \wedge q)) \wedge q$$
$$\Longleftrightarrow (F_0 \vee (p \wedge q)) \wedge q$$
$$\Longleftrightarrow (p \wedge q) \wedge q$$
$$\Longleftrightarrow p \wedge q$$

Logical Implication Rules of Inference

$\overset{\varphi}{p} \rightarrow \overset{\psi}{q}$ استنتاج

Definition — If p and q are statements such that $p \rightarrow q$ is a tautology, then we say that p logically implies q written $p \Rightarrow q$

Theorem — $(p_1 \wedge p_2 \wedge \dots \wedge p_n) \Rightarrow q$

It is a cold day $\Rightarrow 2 = 5 - 3$

" " " " " $\rightarrow 2 = 5 - 3$ → چون q همیشه درست است

$P \rightarrow T_0$ این یک tautology است

proof —
با فرض درست بودن طرف چپ به راست می‌رسیم
می‌خواهیم tautology بودن "q" است پس باید ثابت کنیم اگر $p=1$ آنگاه $q=1$ چون در غیر اینصورت

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we simply logically

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Example) $P \Rightarrow (p \vee q)$

p	q	$p \vee q$	$p \to (p \vee q)$
0	0	0	1
0	1	1	1
1	0	1	1
1	1	1	1

$$p \to (p \vee q) \Leftrightarrow \neg p \vee (p \vee q)$$
$$\Leftrightarrow (\neg p \vee p) \vee q$$
$$\Leftrightarrow T_0 \vee q$$
$$\Leftrightarrow T_0$$

Modus Ponens (Rule of Detachment)

inference Rule $[p \wedge (p \to q)] \Rightarrow q$ $\Leftrightarrow [p \wedge (\neg p \vee q)] \to q$

$$\Leftrightarrow [(p \wedge \neg p) \vee (p \wedge q)] \to q$$
$$\Leftrightarrow [F_0 \vee (p \wedge q)] \to q$$
$$\Leftrightarrow (p \wedge q) \to q$$
$$\Leftrightarrow \neg (p \wedge q) \vee q$$
$$\Leftrightarrow (\neg p \vee \neg q) \vee q$$
$$\Leftrightarrow \neg p \vee (\neg q \vee q)$$
$$\Leftrightarrow \neg p \vee T_0$$
$$\Leftrightarrow T_0$$

$(p_1 \wedge p_2 \wedge \dots \wedge p_n) \Rightarrow q$

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

$\to$ provide a valid argument or proof for q

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Law of the syllogism

$$[(p \to q) \wedge (q \to r)] \to (p \to r)$$

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$$

Example) Is the following argument valid?

$$\begin{array}{l} p \\ p \to q \\ \underline{q \to r} \\ \therefore r \end{array}$$

1) p Hypothesis
2) $p \to q$ "
3) q (1), (2) & Modus ponens
4) $q \to r$ Hypothesis
5) r (3), (4) & Modus ponens

Modus Tollens

$$\begin{array}{l} p \to q \\ \underline{\neg q} \\ \therefore \neg p \end{array}$$

Example) Develop a proof for the following argument

$$\begin{array}{l} p \to r \\ r \to s \\ t \vee \neg s \\ \neg t \vee u \\ \underline{\neg u} \\ \neg p \end{array}$$

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1) $p \to r$ Hypothesis
2) $r \to s$ "
3) $p \to s$ (1), (2), law of syllogism
4) $t \vee \neg s$ Hypothesis
5) $s \to t$ (4), $\neg s \vee t \Leftrightarrow s \to t$
6) $p \to t$ (4), (5), law of syllogism
7) $\neg t \vee u$ Hypothesis
8) $t \to u$ (7), $\neg t \vee u \Leftrightarrow t \to u$
9) $p \to u$ (6), (8), law of syllogism
10) $\neg u$ Hypothesis
11) $\neg p$ (9), (10), and modus Tollens

Proof by contradiction (Reductio Absurdum)

$$\frac{\neg s \Rightarrow F_0}{\therefore s}$$

$(\neg s \Rightarrow F_0) \Rightarrow s$

Example) prove that $\sqrt{2}$ is irrational تمرین

$\sqrt{2}$ is rational $\Rightarrow \sqrt{2} = \frac{a}{b} \wedge (a, b) = 1$

$\Rightarrow 2 = \frac{a^2}{b^2}$

$\Rightarrow a^2 = 2b^2$

$\Rightarrow a$ is even

$\Rightarrow a = 2k$ for some $k \in \mathbb{Z}$

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$\Rightarrow 4k^2 = 2b^2$

$\Rightarrow b^2 = 2k^2$

$\Rightarrow b$ is even

$\Rightarrow \underbrace{(a,b) \neq 1}_{\neg p} \Rightarrow r \wedge \neg r \Rightarrow T_0$

$$\frac{\neg s \Rightarrow F_0}{\therefore s}$$

Example provide a proof the following argument if it is a valid argument, and give a counterexample if it's invalid. لا تناقض

$$\begin{array}{l} p \\ p \vee q \\ p \to (r \to s) \\ t \to r \\ \hline \therefore \neg s \to \neg t \end{array}$$

$\neg s \to \neg t : 0 \begin{cases} \neg s : 1 & s : 0 \\ \neg t : 0 & t : 1 \end{cases}$

$t \to r : 1 \quad r : 1$

$q \to (r \to s) : 1 \quad q : 0$

$p \vee q : 1 \quad p : 1$

argument لا تناقض يبقى

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Use of Quantifiers :

" x is greater than 2 "

Definition: A declarative sentense is an open statement if

1. it contains one or more variables
2. it is not a statement.
3. it becomes a statement when the variables are replaced by {certain allowable choices.} → universe of discourse

if $x \in \mathbb{R} \Rightarrow x^2 \geq 0$ is statement

$x^2 \geq 0 \Rightarrow$ not statement

$$\underbrace{\text{"For any } \overbrace{\text{integer}}^{\text{universe of discourse}} \text{ n,}}_{\text{quantifier}} \underbrace{\text{ n is greater than 2"}}_{\text{open statement}}$$

statement

open statement : $p(x)$, $q(x,y)$

[illegible] $\begin{cases} p(x) = x \text{ is prime} \Rightarrow \text{[illegible]} \\ q(x,y) : x^2 + y > 5 \end{cases}$

Quantifiers

universal ($\forall$)	Existantial ($\exists$)
For any	For some
For all	There is
For every	There exist
For each	

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$\forall x\, [x \in \mathbb{Z} \rightarrow x > 2]$ or $\forall x \in \mathbb{Z}\, [x > 2]$

$\exists x \in \mathbb{R}\, [x^2 \leq 0]$ or $\exists x\, [x \in \mathbb{R} \wedge x^2 \leq 0]$

For any real x, there is some real y, so that $x + y = 0$

$\forall x \in \mathbb{R}\, [\exists y \in \mathbb{R}\, [x + y = 0]]$

Semantics:

Logic (so far) + $\forall, \exists$

مثال) $\forall x\, [P(x) \rightarrow \exists x\, [q(x)]]$ با استفاده از Logic جدید می‌توانیم

مثال $\forall x\, [P(x) \rightarrow \neg q(x)] \rightarrow \exists y\, [P(y) \wedge q(y)]$ می‌توانیم جملات پیچیده‌تری بسازیم

$V : \Omega \longrightarrow \{0, 1\}$

universal discourse

Notation: $A_P = \{ x \in U \mid V(P(x)) = 1 \}$

$\bar{V} : \mathcal{F} \longrightarrow \{0, 1\}$

- $\bar{V}(\varphi) = V(\varphi)$ for all $\varphi \in \Omega$ (statement)
- $\bar{V}(\forall x \in U\, [P(x)]) = \begin{cases} 1 & \text{if } A_P = U \\ 0 & \text{otherwise} \end{cases}$
- $\bar{V}(\exists x \in U\, [P(x)]) = \begin{cases} 1 & \text{if } A_P \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$

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Example) $U = \{1, 2, 3\}$ $\quad p(x): x > 2$

$$v(p(1)) = v(1 > 2) = 1$$

$$v(p(2)) = v(2 > 2) = 0$$

$$v(p(3)) = v(3 > 2) = 1$$

$$A_p = \{x \in U \mid v(p(x)) = 1\} = \{1, 3\}$$

$v(\forall x \in U [p(x)]) = 0$ becomes $A_p \neq U$

$v(\exists x \in U [p(x)]) = 1$ becomes $A_p \neq \emptyset$

Example) Assume A is an array of 20 positive integers $A[1], A[2], \dots$ and $A[20]$ state the following statements in first order logic.

1. The integer $A[20]$ is the largest entry in the arraye.

$$\forall k [1 \leq k \leq 19 \longrightarrow A[k] < A[20]]$$

2. There exist two consecutive entires so that the larger entry is twice the smaller

$[A_i] = 2[A_{i+1}]$ $\vee$ $[A_{i+1}] = 2A[i]$

$$\exists k [1 \leq k \leq 19 \wedge (A[k] = 2A[k+1] \vee A[k+1] = 2A[k])]$$

3. The entries is the array are stored ascending order

$$\forall k [1 \leq k \leq 19 \longrightarrow A[k] < A[k+1]]$$

4. The entries in the array are distinct.

$$\forall m \forall n [1 \leq m, n \leq 20, n \neq m \longrightarrow A[m] \neq A[n]]$$

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Negation:

$$\neg \forall x\, [p(x)] \Longleftrightarrow \exists x\, [\neg p(x)]$$

$$\neg \exists x\, [p(x)] \Longleftrightarrow \forall x\, [\neg p(x)]$$

Example)

$$\neg \forall x\, [p(x) \rightarrow \neg q(x)]$$
$$\Longleftrightarrow \exists x\, [\neg (p(x) \rightarrow \neg q(x))]$$
$$\Longleftrightarrow \exists x\, [p(x) \wedge q(x)]$$

$\forall$, $\exists$, $\nexists$, $\exists!$ (uniqueness)

$\nexists x\, [p(x)] \Longleftrightarrow \forall x\, [\neg p(x)]$ هیچ x ای p(x) ندارد

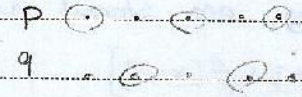
$\exists! x\, [p(x)] \Longleftrightarrow \exists x\{[p(x)] \wedge \forall y[y \neq x \rightarrow \neg p(y)]\}$ دقیقاً یک x وجود دارد که

این دو x هیچ [illegible]

Example) True or False

$\exists x\, [p(x) \wedge q(x)] \Longrightarrow \exists x\, [p(x)] \wedge \exists x\, [q(x)]$ True

$(\exists x\, [p(x)] \wedge \exists x\, [q(x)]) \Longrightarrow \exists x\, [p(x) \wedge q(x)]$ False



$\exists x\, [p(x) \vee q(x)] \Longleftrightarrow \exists x\, [p(x)] \vee \exists x\, [q(x)]$ True

اگر این درست باشد   این درست است

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OK

$\forall x\,[p(x) \wedge q(x)] \Longleftrightarrow \forall x\,[p(x)] \wedge \forall x\,[q(x)]$ True

$\forall x\,[p(x) \vee q(x)] \Longleftrightarrow \forall x\,[p(x)] \vee \forall x\,[q(x)]$ False

$\Longleftarrow$ True

p ⊙ . ⊙
q . ⊙ . ⊙
p∨q ⊙⊙ ⊙⊙

$\forall x\,\exists y\,[p(x,y)] \Longleftrightarrow \exists y\,\forall x\,[p(x,y)]$ False

برای هر x یک y وجود دارد   برای همه x ها یک y وجود دارد

$\forall n\,\exists m\,[m > n]$ برای هر عدد طبیعی عددی هست که از n بزرگتر است

$\exists m\,\forall n\,[m > n]$ یک عدد هست که از همه بزرگتر است

$\forall x\,\exists y\,[x+y=0]$ True

$\exists y\,\forall x\,[x+y=0]$ False

$\exists y\,\forall x\,[x+y=x]$ True
$\quad \hookrightarrow y=0$

$\forall x\,\forall y\,[p(x,y)] \Longleftrightarrow \forall y\,\forall x\,[p(x,y)]$

$\exists x\,\exists y\,[p(x,y)] \Longleftrightarrow \exists y\,\exists x\,[p(x,y)]$ $\Longrightarrow$ درست هستند چون می‌توان جای آن‌ها را عوض کنیم

Example: state $\lim_{x \to a} f(x) = l$ in first order logic.

$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x\,[0 < |x-a| < \delta \rightarrow |f(x) - l| < \varepsilon]$

$\exists \varepsilon > 0\ \forall \delta > 0\ \exists x\,[0 < |x-a| < \delta \wedge |f(x) - l| \geq \varepsilon]$

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Proportion of Integers (Elementry number theory)

well_ordering principle

partial order relation

$A = \{a, b, d, d\}$

$B = \{(a,b)\ (c,d)\ (a,a)\ (b,b)\ (c,c)\ (d,d)\}$

$aRa,\ aRb$ $\preceq$ precede

$a \prec b$ $\quad c \prec b$

از کوچکترین $a<b<d<c$ total order

"$\leq$" less than or equal to

اگر این رابطه برقرار باشد به این total order گویند.

$\mathbb{R},\ \mathbb{Z},\ \mathbb{Z}^+,\ \mathbb{Q}$ اگر روی هر کدام از این مجموعه‌ها بنویسیم عددی
حتماً در هم ایجاد می‌شود

$(0,1] \subseteq \mathbb{R}$

$(A, \leq)$ $\Longrightarrow$ اگر این مجموعه باشد در نظر بگیریم هیچ
عنصر کوچکترین عضو مجموعه نیست

$[0,1]$ ولی برای این مجموعه صفر کوچکترین است

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Smallest (least) element

For a set A with partial order relation R, a is the least element of A if ← [illegible]

$\forall x \in A\ [aRx]$

special case: R to be "$\leq$"

Axiom (well ordering).

Any nonempty subset of $\mathbb{Z}^+$ has least element

هر زیرمجموعه ناتهی از $\mathbb{Z}^+$ عضو مینیمم دارد

Theorem (Mathematical Induction)

Assume $A \neq \emptyset$ and $A \subseteq \mathbb{Z}^+$ and

1) $1 \in A$, and
2) $k \in A$ logically implies $(k+1) \in A$ for any $k \in \mathbb{Z}^+$

Then $A = \mathbb{Z}^+$

proof: Assume $A \neq \mathbb{Z}^+$. برهان خلف

$\Rightarrow \mathbb{Z}^+ - A \neq \emptyset \subseteq \mathbb{Z}^+$

well-ordering $\Rightarrow \exists a \in \mathbb{Z}-A\ \forall x \in \mathbb{Z}-A\ [a \leq x]$

consider $a-1$:

$(a-1) \notin \mathbb{Z}^+ - A \Rightarrow (a-1) \in A \overset{②}{\Rightarrow} (a-1)+1 = a \in A \Rightarrow$

$a \in A \wedge a \in \mathbb{Z}^+ - A \Leftrightarrow F \Rightarrow$

$\neg(A \neq \mathbb{Z}^+) \Leftrightarrow A = \mathbb{Z}^+$

$a-1 \in \mathbb{Z}^+$
$a \neq 1$
$1 \in A \Rightarrow 1 \notin \mathbb{Z}^+ - A \Rightarrow a \neq 1$

In brief, for a set $A \subseteq \mathbb{Z}^+$ استقرای ریاضی

$$\left(1 \in A \wedge \forall k \in \mathbb{Z}^+ [k \in A \Rightarrow (k+1) \in A]\right) \Rightarrow A = \mathbb{Z}^+$$

Theorem: Assume $p(n)$ is an open statement over $\mathbb{Z}^+$

$p : \mathbb{Z}^+ \to [0,1]$ تابع از $\mathbb{Z}^+$ به صفر و یک

Then

$$p(1) \wedge \forall k \in \mathbb{Z}^+ [p(k) \to p(k+1)] \Rightarrow \forall n \in \mathbb{Z}^+ [p(n)]$$

این open statement برای همه مقادیر $\mathbb{Z}^+$ درست است اگر این شرط ها برقرار باشد

proof _ As an excercise!

Theorem: Assume $A \subseteq \mathbb{Z}^+$, and $a_0 \in A$ is it's least element. Further, $p(n)$ is a predicate on $\mathbb{Z}^+$. Then $\left(p(a_0) \wedge \forall k \in \mathbb{Z}^+ [k \geq a_0 \to p(k) \to p(k+1)]\right) \Rightarrow \forall n \geq a_0 [p(n)]$

Theorem: (strong form) (constructive induction)

Let A be a nonempty subset of $\mathbb{Z}$ with the least element a_0 and $p(n)$ is a predicate over $\mathbb{Z}$. Then,

$$\left(\underbrace{p(a_0)}_{\text{basis}} \wedge \forall k \geq a_0 \underbrace{[\overbrace{p(a_0) \wedge p(a_0+1) \wedge \dots \wedge p(k)}^{\text{Induction hypothens}} \to p(k+1)]}_{\text{induction step}}\right) \Rightarrow \underbrace{\forall n \geq a_0 [p(n)]}_{\text{Induction conclusion}}$$

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Example) Assume a is a sequence on $\mathbb{Z}^{\geq 0}$ so that $a(0)=3$, $a(1)=7$, and $a(n)=3a(n-1)-2a(n-2)$ for $n\geq 2$. prove that $a(n)=2^{n+2}-1$ for $n\geq 0$

$p(n):$ $a(0)=3 \wedge a(1)=7 \wedge a(n)=3a(n-1)-2a(n-2)$ for $n\geq 2 \Longrightarrow$

$p(n): a(n)=2^{n+2}-1$ $\quad a(n)=2^{n+2}-1$

if

فرض تغییر کرد تا بتوانیم از استقرا استفاده کنیم.

$a(0)=3 \wedge a(1)=7 \wedge \forall n\geq 2\,[a(n)=3a(n-1)-2a(n-2)] \overset{\text{direct}}{\Longrightarrow}$ ①*

$\Big(p(0)\wedge \forall k\geq 0\,[p(0)\wedge p(1)\wedge \dots \wedge p(k) \longrightarrow p(k+1)]\Big) \overset{\text{mathematical induction}}{\Longrightarrow}$ ②*

$\forall n\geq 0\,[p(n)]$

پس از دو فرض تغییر (علامت‌دار) دو طرف را ثابت کنیم و سپس از استقرا استفاده کنیم

① $p(0): a(0)=2^{0+2}-1=3$ is true

حال می‌خواهیم نشان دهیم که اگر $p(0)$ تا $p(k)$ درست باشد $p(k+1)$ هم درست است

② $a(k+1)=3a(k)-2a(k-1)$ ①* $\quad k+1\geq 2$

$=3(2^{k+2}-1)-2(2^{k+1}-1)$ ②*

$=2^{k+3}-1$

③ $a(1)=2^{3}-1=7$ is true

پس ② و ③ برای اثبات ②* کافی است

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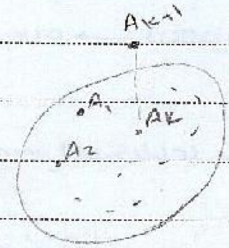
انسان‌های خبررسان

یک مجموعه n نفری (n≥4) از انسان‌ها را در نظر بگیرید که هر کدام از آنها در ابتدا یک خبر جدید را دارند. خبرهای جدید متمایز هستند. در هر مرحله ۲ نفر از این افراد به هم تلفن می‌زنند و اخبار را مبادله کنند. ثابت کنید این افراد می‌توانند با ۲n-۴ بار تلفن زدن همگی از همه‌ی اخبار مطلع شوند.

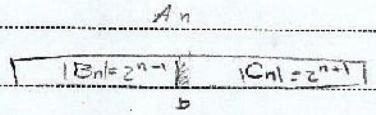
P(n): All of the n people in group will have every news by $2n-4$ calls.



ثابت کنید برای هر عدد طبیعی n می‌توان 2^n دایره به شعاع واحد را درون یک دایره به شعاع 3^n جا داد به طوری که هیچ دو دایره‌ای متقاطع نباشند.

a(n) =

Example.) For $n \geq 0$. Let $A_n \subseteq \mathbb{R}$, where $|A_n| = 2^n$ and the elements of A_n are listed in asscending order. IF $r \in \mathbb{R}$ prove that in order to determine whether $r \in A_n$ (by procedure developed below) we must compare r with no more than $n+1$ element in A_n.



از وسط نصف می‌کنیم

$A_n = B_n \cup C_n$

r با این عنصر مقایسه می‌کنیم اگر بزرگتر بود ← C_n

اگر کوچکتر مساوی بود ← B_n

اگر کوچکتر بود C_n نیازی نداریم دوباره کار را ادامه می‌دهیم

$n=0 \quad A_0 = \{a\} \Rightarrow p(0)$ is true

$n = k+1$ we should prove that the number of required comparisom is at most $(k+1)+1 = k+2$

$|A_n| = 2^{k+1}$

$B_{k+1} = A_k$