Bioengineering:

1. Application of engineering principles to biological systems.

A bioengineer seeks
1. to understand basic physiological process,
2. To improve human health,

→ Applied problem solving.

Biomechanics:

Study of how physical forces interact with living systems.

Examples:

1. How do your bones "know" how big and strong to be

⬇

they can support your weight.

Evidence show:

Growth of bones is driven by **mechanical stimuli.**

Examples:

3. Locomotion of our body in everyday life

- Walking, running, jumping.

4. Locomotion happens on many scales:

from Unicellular to whole organisms.

5. Cells can **generate forces** and **sense** and **respond forces**.

→ **Hair cells in the ear.**

-outer hair cells do not send neural signals to the brain, but that they mechanically amplify low-level sound.

- The inner hair cells transform the sound vibrations in the fluids then into electrical signals that are then relayed via the nerve to the auditory brain stem.

Examples:

2. How do our arteries “know” how big to be:

They can deliver just the right amount of blood.

Evidence shows:

This is determined by **mechanical stress** exerted on the artery wall by flowing blood.

Hemodynamics and Hemorheology

References:

1- Applied Biofluid Mechanics (Waite-2007)

2- Biofluid Dynamics, Principles and selected applications (Clement Klieinstreuer-2006)

3- Biofluid Mechanics, The human circulation (Chandran & Yoganathan & Rittgers-2007)

4- Biofluid Mechanics (Mazumdar-1993)

5- Introductory Biomechanics, From Cells to Organisms(Ethier and Simmons, 2007)

Types of Flow in Cardiovascular Systems

- Unidirectional flows (Poiseuille)
- Pulsatile flow
- Non-Newtonian fluid flow
- Bend flows and time-varying curvature
- Bifurcations (symmetrical and asymmetrical)
- Wavy wall
- Entry flow
- Flow stability
- Flow separation
- Vessel deformability
- Wave propagation
- Particle flow
- Cardiac flow
- Flow in cardiac valves
- Microcirculation

Physical Properties of Blood

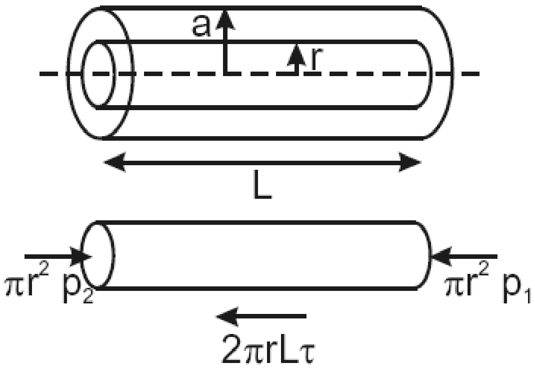
Table 4.2. Blood (Ht = 45%, $T = 37^\circ$C) and water ($T = 37^\circ$C) physical properties.

	μ ($\times 10^{-3}$ Pl)	ρ ($\times 10^{3}$ kg/m^3)	ν ($\times 10^{-6}$ m^2/s)
Blood	3–4	1.055	2.8–3.8
Plasma	1.2	$\sim$1.03	$\sim$1.2
Water	0.692	0.993	0.696

Poiseuille Flow

$$\Delta p . \pi r^2 = \tau . 2 \pi r \ell$$

$$\frac{\Delta p}{\ell} = \frac{2\tau}{r}$$

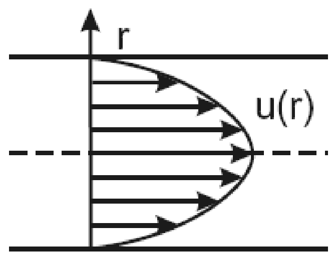


✓Assumptions

- the tube is a uniform circular cylinder;
- the tube is rigid;
- the fluid is Newtonian;
- the flow is steady, i.e., constant in time;
- the flow is laminar, i.e., not turbulent;
- the flow is not subject to entrance effects, i.e., non-uniformities associated with the entrance of fluid into the tube.

$$\tau = -\mu \frac{du}{dr} \Rightarrow \frac{du}{dr} = -\left(\frac{\Delta p}{2\mu\ell}\right) r \Rightarrow \int du = -\frac{\Delta p}{2\mu\ell} \int r\, dr$$

$$u = \frac{1}{4\mu} \frac{dP}{dx} [r^2 - R^2]$$



Wall Shear Stress Based on Poiseuille Flow

TABLE 1.1 Estimate of Wall Shear Stress in Various Vessels in the Human Circulatory System

Vessel	ID, cm	V_m, cm/s	Shear rate[3]	Shear stress,[4] N/m^2
Aorta	2.5	48	154	0.5
Large arteriole[5]	0.05	1.4	224	0.8
Arteriole (retinal microcirculation[6])	0.008	3	3000	10.5
Capillary	0.0008	0.7	7000	24.5

$$Q = \frac{\pi R^4}{8\mu}\frac{dp}{dx} = \frac{\pi R^4 \Delta p}{8\mu L} = \frac{\pi D^4 \Delta p}{128\mu L}$$

Shear rate: $\dot{\gamma} = \frac{8V_m}{D}$

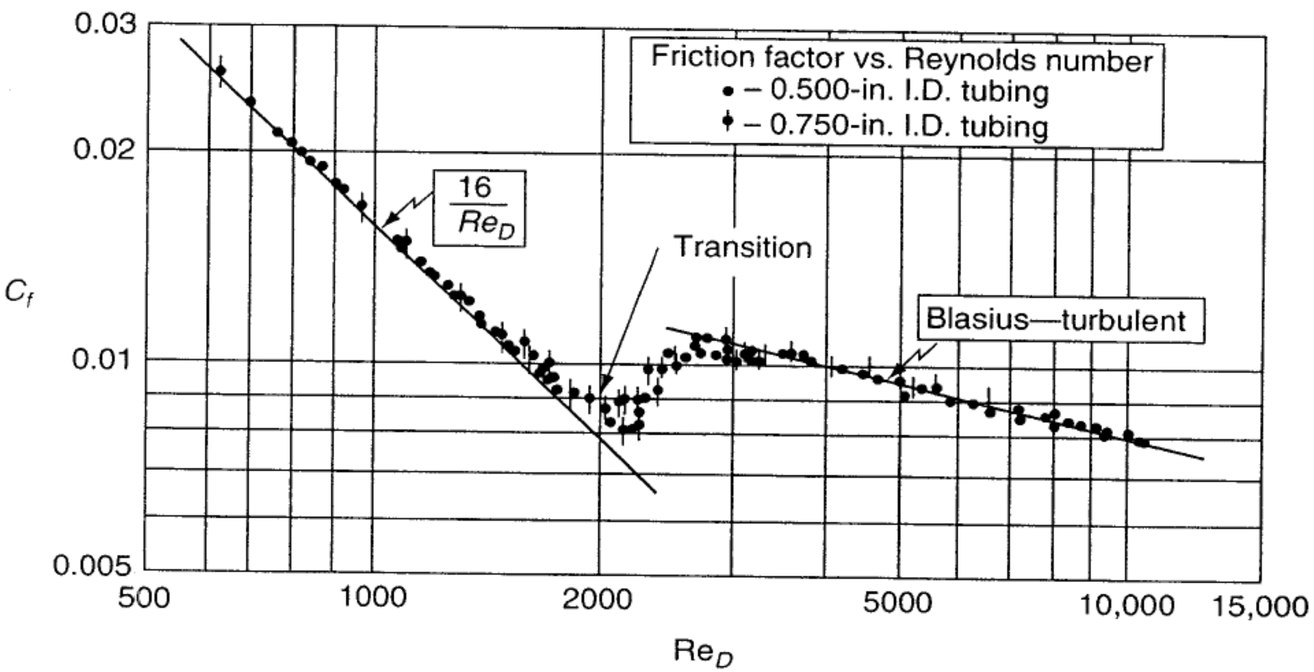
Friction Factor

Poiseuille number

$$Po = C_f Re_D = \frac{2\tau_w D}{\mu \bar{u}} = 16$$

$$\lambda = \frac{8\tau_w}{\rho \bar{u}^2} = \text{Darcy friction factor}$$

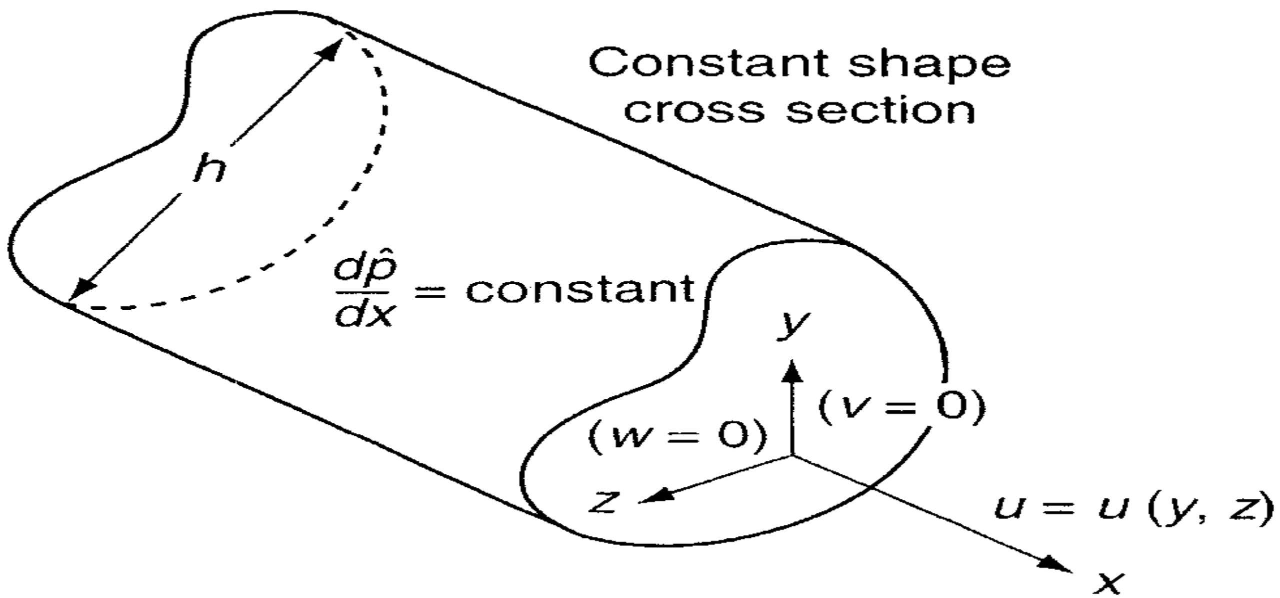
$$C_f = \frac{2\tau_w}{\rho \bar{u}^2} = \frac{1}{4}\lambda = \text{Fanning friction factor, or skin-friction coefficient}$$



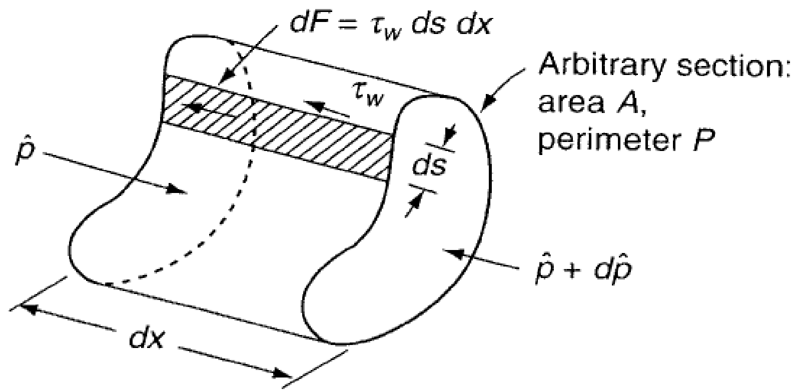
For laminar flow in a pipe

$$\lambda = \frac{64}{Re_D} \qquad C_f = \frac{16}{Re_D}$$

Poiseuille Flow Through Non-Circular Ducts



The Concept of Hydraulic Diameter



$$\bar{\tau}_w = \frac{1}{P}\int_0^P \tau_w \, ds$$

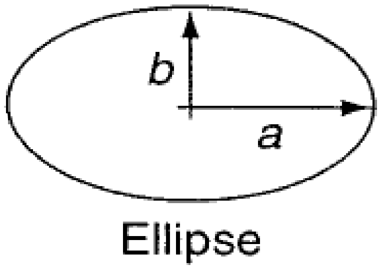
where ds = element of arc length
P = perimeter of section

$$\bar{\tau}_w = \frac{A}{P}\left(-\frac{d\hat{p}}{dx}\right)$$

$$D_h = \frac{4A}{P} = \frac{4 \times \text{area}}{\text{wetted perimeter}}$$

$$C_f = \frac{\lambda}{4} = \frac{\text{const}}{Re_{D_h}} \qquad Re_{D_h} = \frac{\rho \bar{u} D_h}{\mu}$$

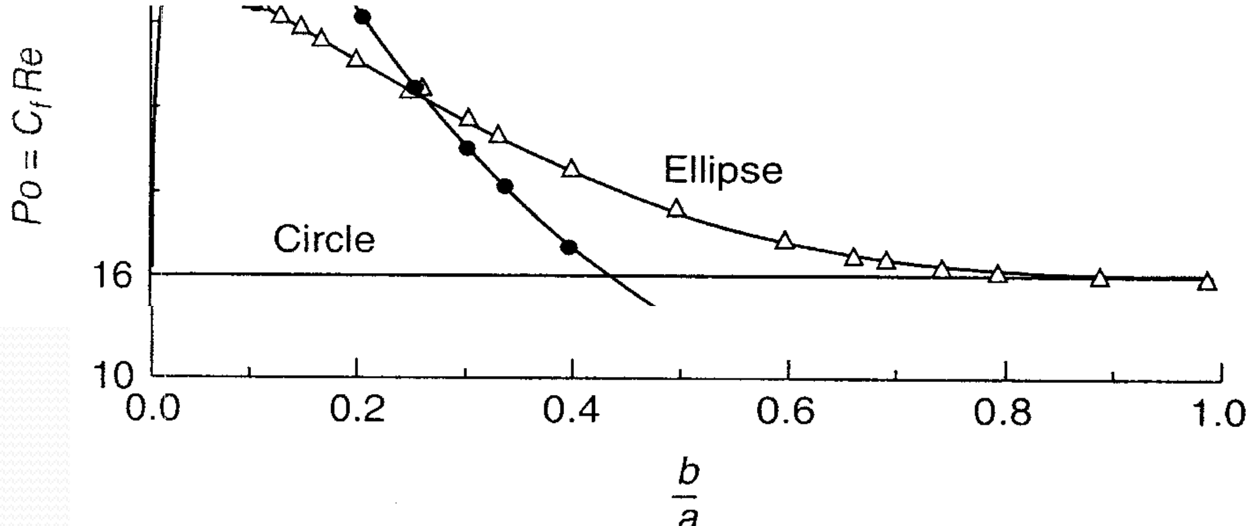
Flow Through an Ellipse



Elliptical section: $y^2/a^2 + z^2/b^2 \leq 1$:

$$u(y, z) = \frac{1}{2\mu}\left(-\frac{d\hat{p}}{dx}\right)\frac{a^2b^2}{a^2 + b^2}\left(1 - \frac{y^2}{a^2} - \frac{z^2}{b^2}\right)$$

$$Q = \frac{\pi}{4\mu}\left(-\frac{d\hat{p}}{dx}\right)\frac{a^3b^3}{a^2 + b^2}$$



Compliance

$$C = \frac{dV}{dp}$$

- Because blood vessels are elastic, there is a relationship between **pressure and volume (*V*)**.
- The blood vessel wall is stretched as a result of the **pressure difference between the interior and exterior** of the vessel.
- Based on the Laplace's law,

$$S_h = \frac{Pr}{h}$$

S_h is the hoop stress in the wall, P is the transmural pressure, r is the vessel radius and *h is the wall thickness.*

Transmural pressure is the difference in pressure between two sides of a wall

Laplace equation

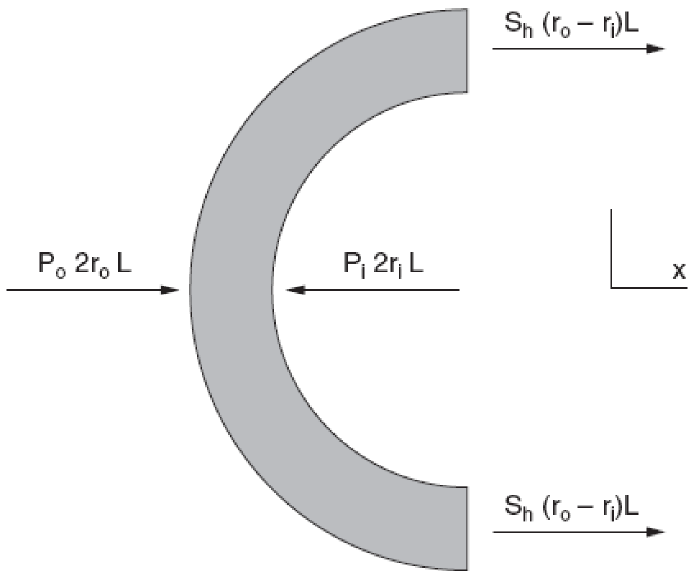
$$\Sigma F_x = 0$$

$$P_o 2 r_o L + 2 S_h (r_o - r_i) L = P_i 2 r_i L$$

$$S_h (r_o - r_i) = (P_i r_i) - (P_o r_o)$$

$$S_h h = (P_i r_i) - (P_o r_o)$$

S_h is the circumferential hoop stress.



$S_h h = pr$ Law of Laplace

$$p(x) - p_o = \frac{S_h h}{r}$$

- The simplest assumption one can make is that *V is linearly related to P, and thus*

$$V = V_0 + CP,$$

C, the compliance of the vessel, and V_o *is the volume of the* vessel at zero pressure.

C (**venous vessel**) = 24 times as great as **C(arterial system)**, because the **veins are both larger and weaker** than the arteries.
Then, large amounts of blood can be stored in the veins with only slight changes in venous pressure.

Also, it can be written, $A = A_0 + cP,$

c is the compliance per unit length,

in a cylindrical vessel of length *L and uniform internal pressure,* $V = AL$, *so that* $C = cL$.

However,* both *C and c are referred as compliance.

From Poiseuillie flow,

$$Q = -\frac{dp}{dx}\frac{A^2}{8\pi\mu}$$

For a compliance vessel,

$$8\pi\mu Q = -\frac{dp}{dx}A(p)$$

where *A(P) is the relationship between cross-sectional area and pressure.*

And by integrating,

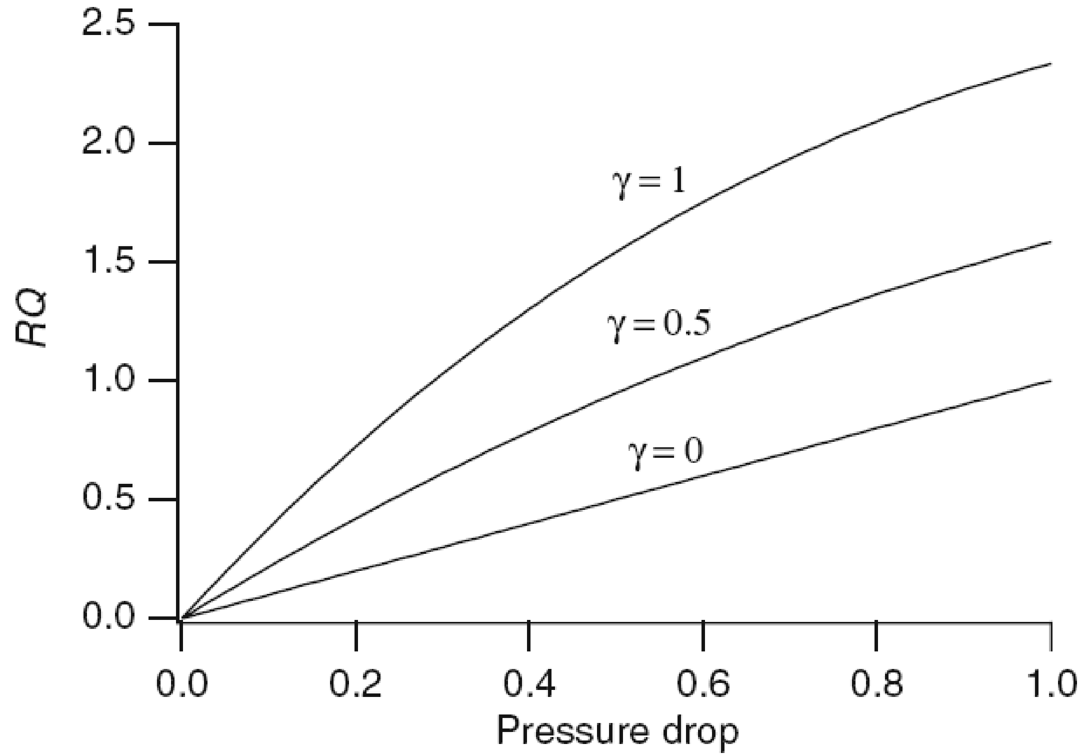
$$x = -\frac{1}{8\pi\mu Q}\int_{P_0}^{P(x)} A^2(P)\,dP$$

And we have

$$\begin{aligned} RQ &= \frac{1}{3\gamma}(1+\gamma P)^3\Big|_{P_1}^{P_0} \\ &= (P_0 - P_1)\left(1 + \gamma(P_0 + P_1) + \frac{\gamma^2}{3}(P_0^2 + P_0 P_1 + P_1^2)\right), \end{aligned}$$

where $R = 8\pi\mu L/A_o^2$ and $\gamma = c/A_o$. P_o input pressure *and* P_1 *output pressure.*

Mathematical Physiology, Keener, Sneyd, 2nd ed., Springer, 2009.



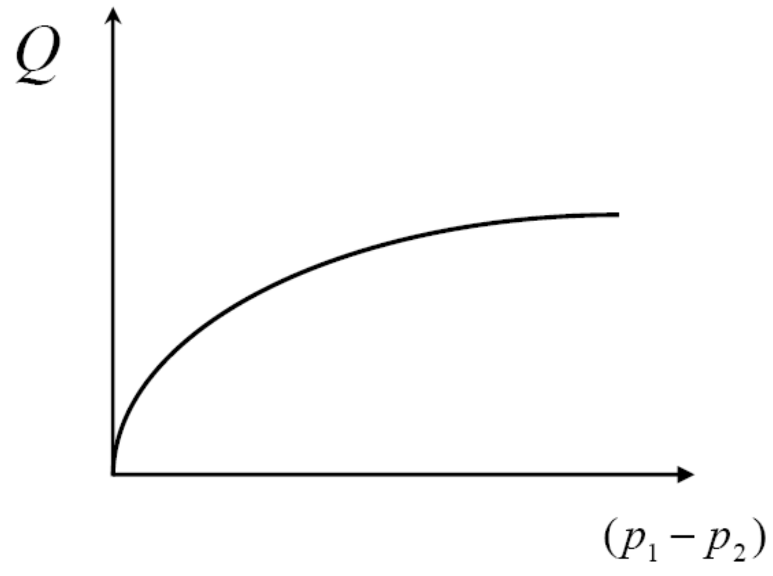
$\gamma = c/A_0$

Figure 11.3 Scaled flow RQ (with units of pressure) as a function of pressure drop $\Delta P = P_0 - P_1$ for different values of compliance γ. For all curves $P_0 = 1.0$.

The higher the compliance, the smaller the pressure drop.

The pressure drop in the veins can be much less than in the arteries, since the compliance of the veins is much greater than that of the arteries.

Observed flow of blood through the Veins



Relationship between Flow rate and pressure-difference in veins

When pressure difference becomes large, the flux gradually attains a maximum value, no longer increases.

Entrance Region of a pipe

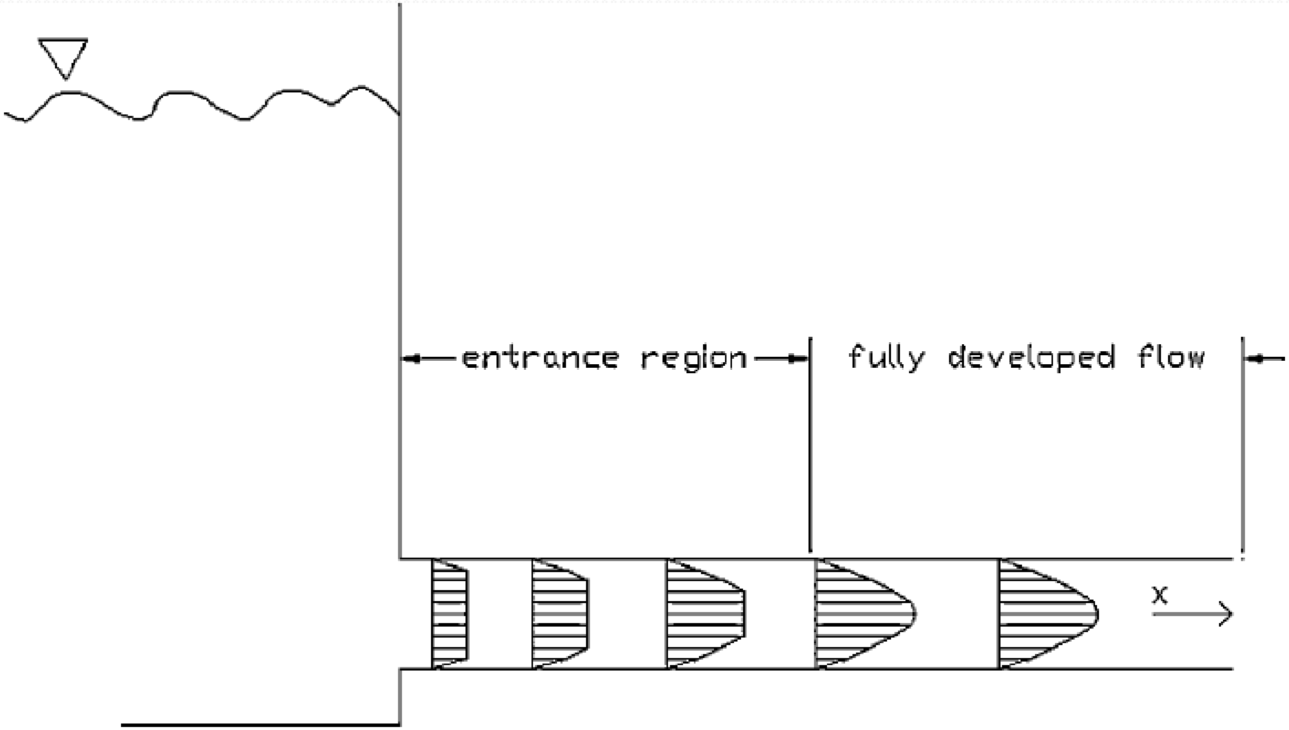
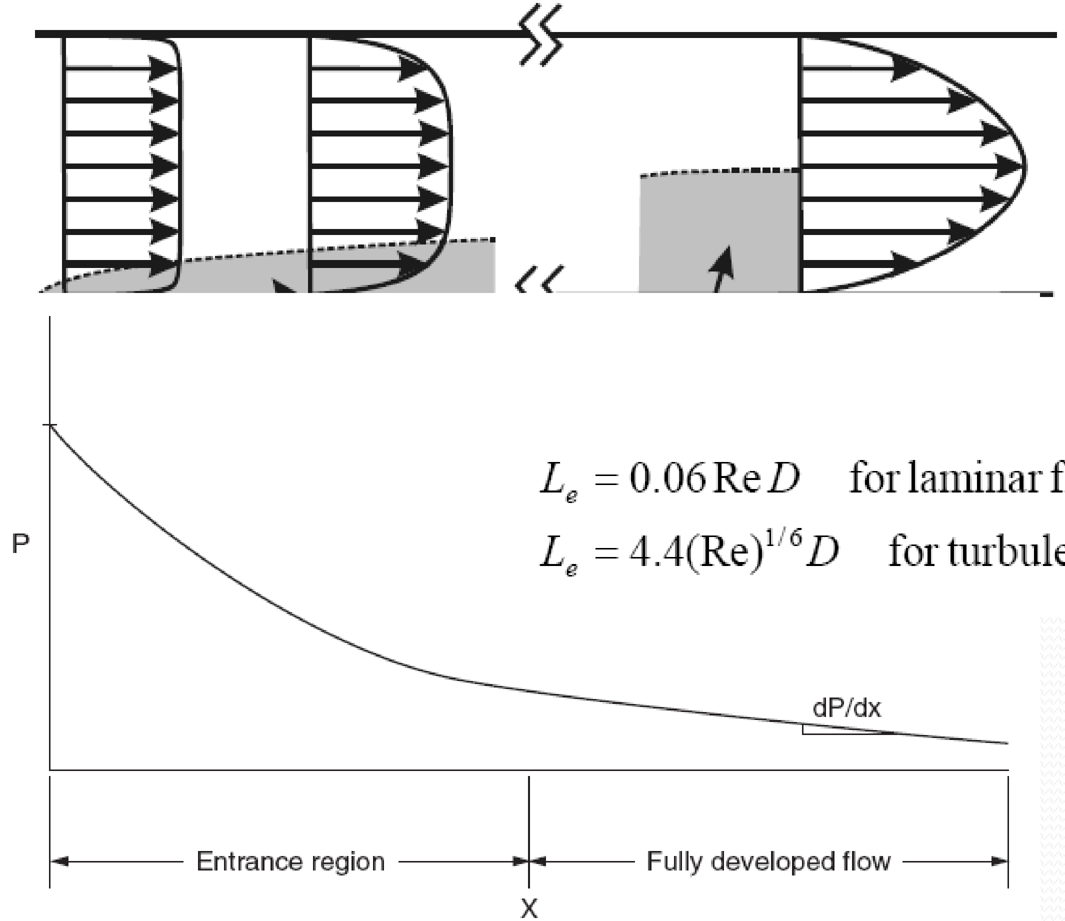


Figure 1.10 Entrance region and fully developed flow in a tube.

Entrance Region of a pipe



$$L_e = 0.06 \,\mathrm{Re}\, D \quad \text{for laminar flow}$$

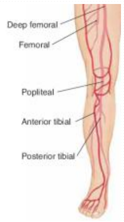
$$L_e = 4.4 (\mathrm{Re})^{1/6} D \quad \text{for turbulent flow}$$

Figure 1.11 Pressure as a function of the distance along a pipe. Note that the pressure gradient dP/dx is constant for fully developed flow.

Entrance lengths of arteries

Vessels	Diameter	Velocity	Re	Entrance Length (L_e)
Aorta	2.5 cm	50 cm/s	3200	4.8 m
Large artery	4 mm	15 cm/s	157	3.7 cm
Arteriols	30 micron	10 cm/s	0.78	0.18 mm
Capillaries	5 micron	0.5 mm/s	0.00065	$2*10^{-4}$ micron

Womersley Number



Womersly Number → $\alpha = \frac{d}{2}\sqrt{\frac{\omega\rho}{\mu}}$

Radius of the Vessel, Heart Rate, and the Womersley Parameter for the Various Species

Species	Weight (kg)	Vessel	Radius (cm)	Heart Rate	α^a (min^{-1})
Mouse	0.017	Aorta	0.035	500	1.4
Rat	0.6	Aorta	0.13	350	4.3
Cat	3.0	Aorta	0.21	140	4.4
Rabbit	4.0	Aorta	0.23	280	6.8
Dog	20.0	Aorta	0.78	90	13.1
Man	75.0	Aorta	1.5	70	22.2 ←
Ox	500.0	Aorta	2.0	52	25.6
Elephant	2000.0	Aorta	4.5	38	49.2
Rat	0.6	Femoral	0.04	350	1.5
Rabbit	4.0	Femoral	0.08	280	2.4
Dog	20.0	Femoral	0.23	90	3.9
Man	75.0	Femoral	0.27	70	4.0 ←

[a] At fundamental frequency.

1-variation of heart rate among the species.

2-Womersly no. is within one order of magnitude.

Pulsatile Flows-Unsteadiness

$$\underbrace{\rho\frac{\partial(u)}{\partial t}}_{\substack{\text{transient}\\ \text{inertia term}}} + \rho\underbrace{\left(\overbrace{u\frac{\partial(u)}{\partial x}}^{\text{nonlinear in dependent variable}} + v_r\frac{\partial(u)}{\partial r} + \frac{v_\theta}{r}\frac{\partial(u)}{\partial\theta}\right)}_{\text{convective inertia term}}$$

$$= \rho g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2(u)}{\partial x^2} + \frac{\partial^2(u)}{\partial r^2} + \frac{1}{r}\frac{\partial(u)}{\partial r} + \frac{1}{r^2}\frac{\partial^2(u)}{\partial\theta^2}\right)$$

$$\frac{\text{Unsteadiness effects}}{\text{Viscous effects}} = \frac{\rho\frac{\partial u}{\partial t}}{\mu\frac{\partial^2 u}{\partial r^2}} \approx \frac{\rho\frac{U}{T}}{\mu\frac{U}{r^2}} = \frac{\rho r^2}{\mu T} = \frac{\rho r^2\omega}{\mu}$$

Womersly No.: $\alpha = r\sqrt{\frac{\rho\omega}{\mu}}$

Pulsatile Flows(Sexl, 1930)

$$r^* = \frac{r}{r_0} \qquad \omega^* = \frac{\omega r_0^2}{\nu} \qquad u^* = \frac{u}{u_{\max}}$$

Small $\omega^* < 4$:

$$\frac{u}{u_{\max}} \approx (1 - r^{*2})\cos\omega t + \frac{\omega^*}{16}(r^{*4} + 4r^{*2} - 5)\sin\omega t + \mathcal{O}(\omega^{*2})$$

Large $\omega^* > 4$:

$$\frac{u}{u_{\max}} \approx \frac{4}{\omega^*}\left[\sin\omega t - \frac{e^{-B}}{\sqrt{r^*}}\sin(\omega t - B)\right] + \mathcal{O}(\omega^{*-2})$$

where

$$B = (1 - r^*)\sqrt{\frac{\omega^*}{2}}$$

ω^* is sometimes called the *kinetic Reynolds*

Pulsatile Flows (Sexl, 1930)

White, Viscous Fluid Flow, 2006

$$\frac{d\hat{p}}{dx} = -\rho K e^{i\omega t}$$

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

$$\rho\frac{\partial u}{\partial t} = -\frac{d\hat{p}}{dx} + \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

Its solution is a Bessel function.

$$u = \frac{K}{i\omega}e^{i\omega t}\left[1 - \frac{J_0\left(r\sqrt{-i\omega/\nu}\right)}{J_0\left(r_0\sqrt{-i\omega/\nu}\right)}\right]$$

Pulsatile Flows

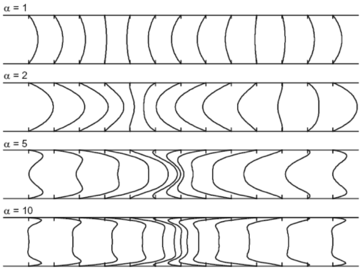
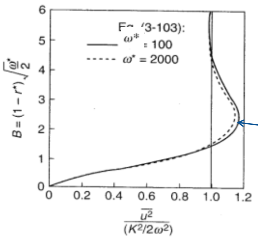


Figure 5. Sequences of velocity profiles in a tube with a sinusoidally varying pressure gradient, for indicated values of unsteadiness parameter α. Velocity profiles represent one complete cycle of the oscillation. Leftmost profile corresponds to moment of maximum pressure gradient.

Pulsatile Flows (Sexl, 1930)

Averaging over one cycle for large kinetic Reynolds

$$\frac{\overline{u^2}}{K^2/2\omega^2} = 1 - \frac{2}{\sqrt{r^*}} e^{-B} \cos B + \frac{e^{-2B}}{r^*}$$



Richardson annular effect

Fourier Series Representation

- Was published in 1822
- *A periodic function f(t) with a period T can be represented by the sum of a constant term, a fundamental of period T, and its harmonics.*
- Fourier series can be used to:

Expand any periodic function into an infinite sum of sines and cosines.

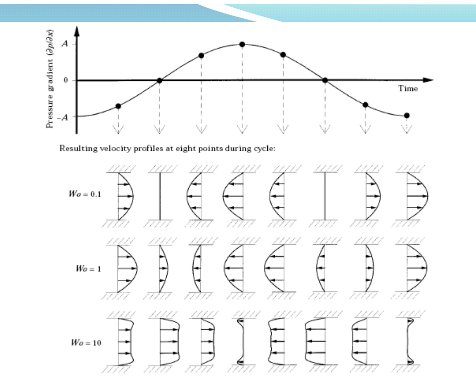


Fig. 2. Velocity profiles between two flat plates at eight points in time during a single cycle of a sinusoidally-varying pressure gradient for three values of the dimensionless Womersley number. The length of a horizontal arrow indicates the magnitude of the velocity for that location and is non-dimensionalized by dividing by the maximum velocity at any location during a complete cycle. A is assumed to be a positive constant.

Louden, Tordesillas, The Use of the Dimensionless Womersley Number to Characterize the Unsteady Nature of Internal Flow, *J. Theor. Biol., 191, 1998.*

Fourier Series

- It is also possible to use complex numbers:

$$\frac{\partial P}{\partial x} = \mathrm{Re}\left[\sum_{n=0}^{\infty} a_n e^{j\omega n t}\right]$$

$$a_n = A_n - B_n j,$$

n is the number of the harmonic,
ω *is the fundamental* frequency (rad/s),
A_o *is the mean pressure gradient.*

Fourier Series

Waite, Applied Biofluid Mechanics, 2007

Periodic functions can be written in the form

$$\frac{\partial P}{\partial x} = A_o + A_1\cos(\omega t) + A_2\cos(2\omega t) + A_3\cos(3\omega t) + \cdots$$
$$+ B_1\sin(\omega t) + B_2\sin(2\omega t) + B_3\sin(3\omega t) + \cdots$$

$$A_0 = \frac{1}{T_0}\int_0^{T_0} f(t)dt$$

$$A_n = \frac{2}{T_0}\int_0^{T_0} f(t)\cos(n\omega t)dt$$

$$B_n = \frac{2}{T_0}\int_0^{T_0} f(t)\sin(n\omega t)dt$$

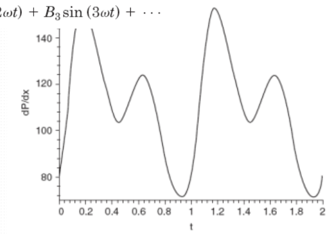


Figure 7.2 Plot of a pulsatile pressure waveform showing pressure drop versus time for a pulsatile flow condition.

Fourier Series

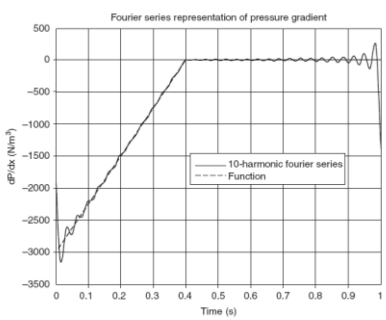


Figure 7.4 Comparison of a pressure gradient with its Fourier series.

Fourier Series

An example:

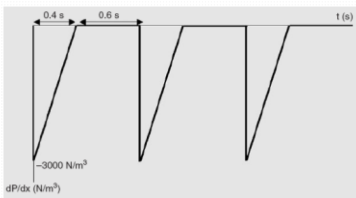


Figure 7.3 Schematic of pressure gradient in an artery.

In the first 0.4 s

$$\frac{dP}{dx}(t) = -3000 + \frac{3000}{0.4}t = -3000 + 7500t$$

The function is zero in the final 0.6 s.

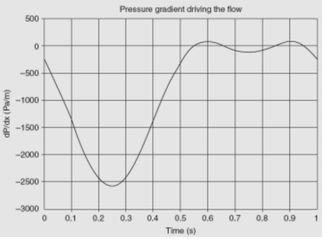
$$A_n = \frac{2}{T_0}\int_0^{T_0}\frac{dP}{dx}(t)\cos(n\omega t)dt = \int_0^{0.4}(-3000 + 7500t)\cos(n\omega t)dt$$

$$A_n = \frac{379.95\,(\cos(2.5133n) - 1)}{n^2}$$

Womersley Solution

A typical pressure gradient

$$\frac{dP}{dx}(t) = -796 - 1250\sin(2\pi t) + 531\cos(4\pi t)$$



Pulsatile Flow -Womersley Solution(1955)

$$\left.\frac{\partial P}{\partial x}\right|_n = a_n e^{j\omega n t}$$

For each harmonic →

$$\frac{\partial P}{\partial x} = \mathrm{Re}\left[\sum_{n=0}^{\infty} a_n e^{j\omega n t}\right]$$

Simplifies the N-S eq. →

$$\frac{a_n e^{j\omega n t}}{\rho} = \nu\frac{\partial^2(u_n)}{\partial r^2} + \frac{\nu}{r}\frac{\partial(u_n)}{\partial r} - \frac{\partial(u_n)}{\partial t}$$

one possible solution

$$u_n = \underbrace{f_n(r)}_{f\text{ is not time dependent}}\ e^{j\omega n t}$$

Ordinary Diff. eq. →

$$\frac{a_n}{\rho} = \nu\frac{d^2 f_n(r)}{dr^2} + \frac{\nu}{r}\frac{df_n(r)}{dr} - j\omega n f_n(r)$$

$$f_n(r) = \frac{a_n}{j\rho\omega n}\left[\frac{J_o(\lambda r)}{J_o(\lambda R)} - 1\right]$$

$$u_n = \mathrm{Re}\left[\frac{a_n}{j\rho\omega n}\left\{\frac{J_o(\lambda r)}{J_o(\lambda R)} - 1\right\}e^{j\omega n t}\right]$$

Velocity for each harmonic

Womersley Solution

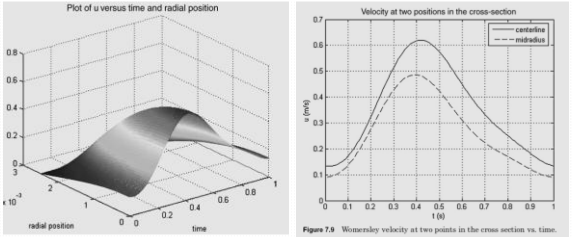


Figure 7.9 Womersley velocity at two points in the cross section vs. time.

1-Velocity lags the pressure gradient
2- Velocity peaks at the mid-radius position slightly before the centerline.

Womersley Solution

Fourier Series of pressure gradient (3 harmonic)

$$\frac{dP}{dx}(t) = \mathrm{Re}(-796 + 1250j e^{2\pi j t} + 531 e^{4\pi j t})$$

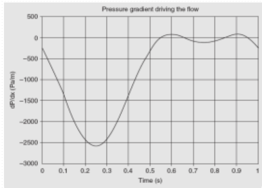
$$u(r,t) = u_0 + u_1 + u_2$$

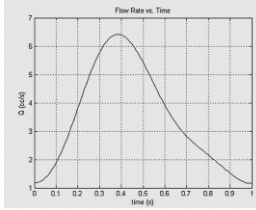
$$u_0 = \frac{1}{4\mu}a_0(r^2 - R^2) = -56840r^2 + 0.35525$$

$$u_1 = \mathrm{Re}[((-0.03225 - 0.06772j)J_0((975.4 - 975.4j)r) - 0.1877)e^{6.283jt}]$$

$$u_2 = \mathrm{Re}[((-0.0007274 + 0.006844j)J_0((1379 - 1379j)r) + 0.03983)e^{12.566jt}]$$

Womersley Solution





$$\frac{dP}{dx}(t) = \mathrm{Re}(-796 + 1250je^{2\pi jt} + 531e^{4\pi jt})$$

$$a = 3.5$$

$$Q_n = ((\pi R^2))\,\mathrm{Re}\left[\frac{a_n}{j\omega n\rho}\left\{\frac{2J_1(\lambda R)}{\lambda R J_0(\lambda R)} - 1\right\}e^{j\omega nt}\right]$$

The peak velocity = 0.62 m/s.

$$\mathrm{Re} = \frac{\rho U_{\max} D}{\mu} = \frac{(1060)(0.62)(0.005)}{0.0033} = 940$$

MBW:Womersley Arterial Flow

Author: Tracey Morland

This article summarizes the results and approach introduced by Womersley in his famous paper on pulsatile flow in arteries [1]. His approach uses concepts from fluid mechanics, including Poiseuille Flow, to model the pressure gradient and flow velocity in an arterial pulse. The model considers the flow of blood in a rigid tube. Womersley's number is also defined, and the usefulness of Womersley flow is discussed.

Contents
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Overview

This is a model of blood flow as a Newtonian fluid through a rigid tube propagated only by a pressure gradient. The Newtonian fluid is described by the Navier-Stokes second-order PDE's, and simplified by assuming a Poiseuille flow. A Fourier series accounts for the periodicity of the pressure gradient, and the coefficients are determined by the Fast Fourier Transform (FFT). Simplifications and substitutions allow for the system of PDE's to be approximated by Ordinary Differential Equations with solutions in the form of Bessel functions.

Biological Context

The human heart beats over 2.5 billion times in an average lifetime and about 100,000 times per day, and in one day your blood travels 12,000 miles, [2] or roughly the distance of traveling from Denver to Tokyo and back again. The heart is composed of four chambers. The sinoatrial nodes (SA) nodes in the right atrium (RA) initiate the electric pulse and cause the right ventricle (RV) to fill with blood. The action potential is propagated through the atria via the atrial cells [3]. The RV then contracts, sending blood to the pulmonary artery where it is then sent to the lungs. The blood, now fully oxygenated, now returns to the heart and fills the left atrium (LA). The LA contracts sending blood into the left ventricle (LV). Blood is then pumped from the LV into the aortic artery which sends the blood to the rest of the body. For an animation of the contraction of the heart valves, see NOVA: Map of the Human Heart.

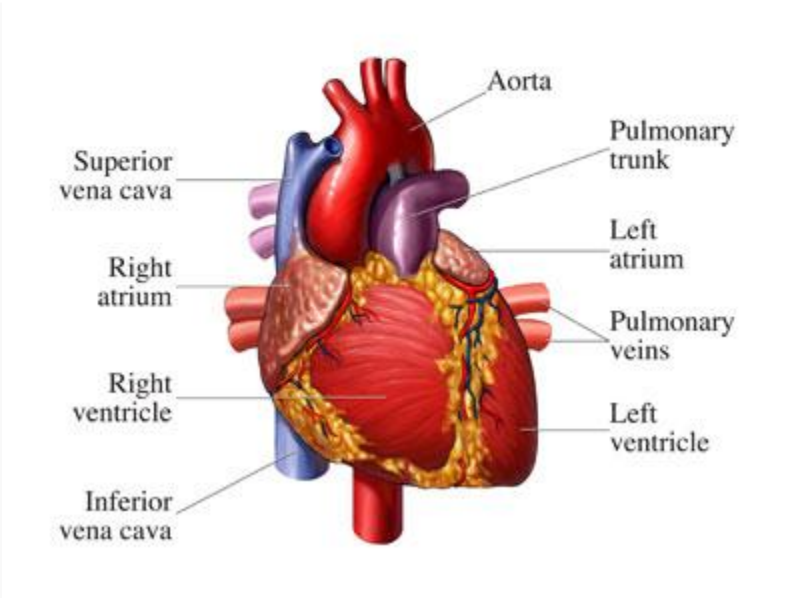


Figure 1: The Human Heart, Nucleus Medical Art, Inc./Getty Images [2]

How fast does the blood flow through the arteries with each beat? Is there any phase-lag between the shock wave (the pulse you feel in your wrist or neck) and the flow of blood? These are questions addressed in Womersley's paper reviewed in this article. The more we can understand the nature of how the arterial pulse works and perform accurate calculations of arterial blood flow, the more we can detect heart disease and defects, such as in patients with diabetes and atherosclerosis [3].

History

Most of the information on the nature of the arterial pulse has evolved from the study of fluid mechanics. Various aspects of the pulse can be included in a model, including the elasticity of the artery, fluid viscosity, pressure gradient, and the presence of arterial bifurcations (branches) to name a few. In 1808, Thomas Young, a British scientist and physician, connected the elastic nature of the arteries to pulse wave velocity [4]. Then, in 1878, Moens and Korteweg independently derived a mathematical model relating arterial elasticity, or stiffness, to pulse wave velocity. Today it is called the Moens-Korteweg equation and it is dervied from Newton's second law of motion $F = ma$.

The simplest model for pulsatile flow is the [Windkessel model], developed by Otto Frank in 1899 [5]. This approach is not accurate enough to be used for quantitative analysis, but it provides a simple foundation on which to build more complicated models. In the model, blood storage is simplified to a single chamber, called a Windkessel, and the pressure of this chamber varies periodically over time [6]. Inflow into the chamber is from the heart, and outflow is to the outer arteries, veins and capillaries and is represented as a simple resistance vessel [3]. There are two parts to this model: compliance (representing the elastic nature of arteries) and resistance. The resistance in the system causes blood to enter the arteries at a higher rate than it flows out. Thus, there is storage of blood in the arteries [7].

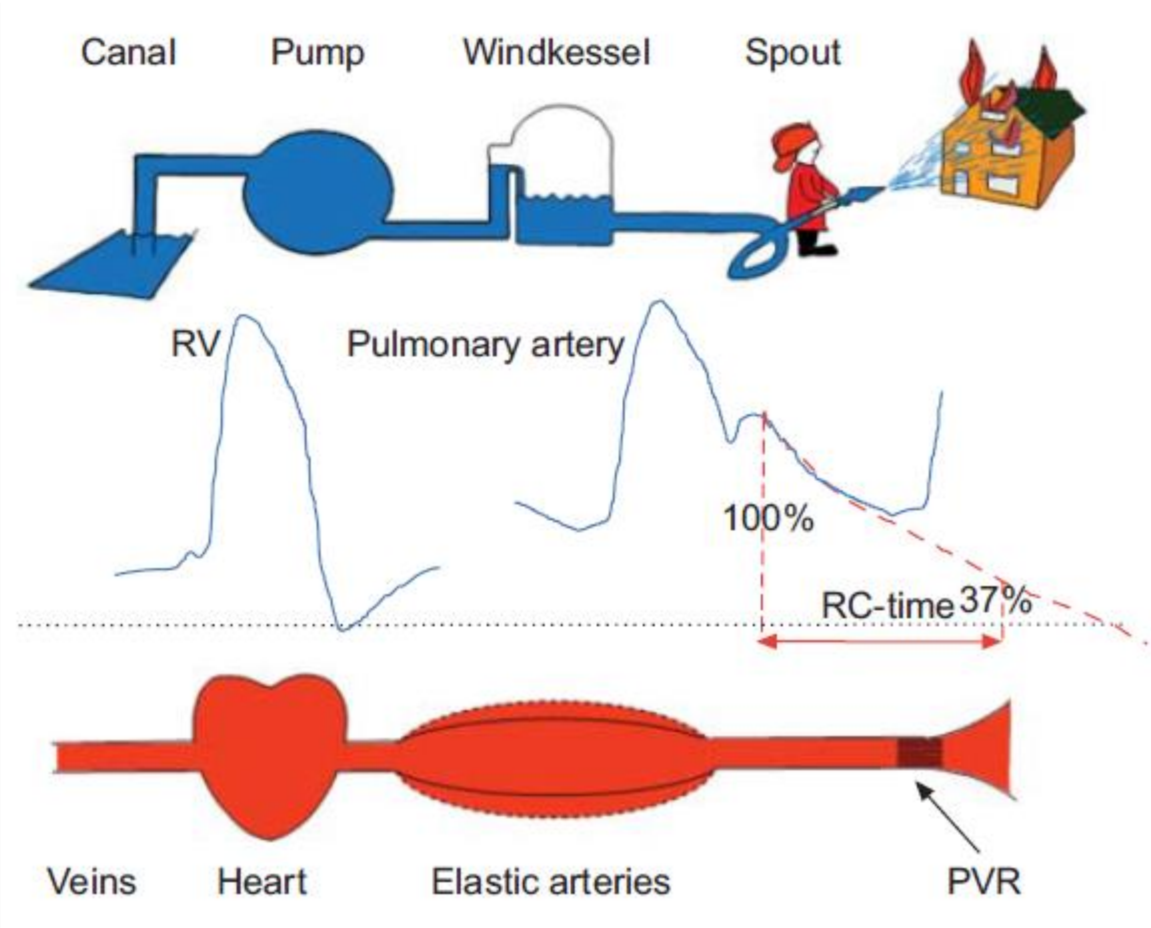


Figure 2: Windkessel Model [8]

In 1970, a three-part Windkessel model was created by Westerhof [9]. Westerhof's model incorporates impedence which, combined with Frank's model, includes aspects of wave propagation. Westerhof's work was inspired by the previous work on pulsatile flow established by Womersley in his 1955 paper, as well as McDonald in his classic book [10] [11].

Mathematical Background

As mentioned above, Womersely's model makes use of Poiseuille flow, and is a simplification of the Navier-Stokes equations. Therefore, it is important to have some background on these approaches. The Navier-Stokes equations can be used to completely model the motion of incompressible, Newtonian fluids. However, these equations are very difficult to analyze since they are non-linear, second order partial differential equations, and only in a few special cases can their exact solutions be found [12]. The equations, simplified using the continuity equation, for the $x,\ y,\ z$ directions are listed below [12]. In the right-hand side of Equation (1), and similarly for Equations (2) and (3), the term $\frac{\partial \omega}{\partial x}$ represents the pressure force, the term ρg_x represents the weight of the fluid, and the second-order partials in parentheses represents the viscous forces.

Navier-Stokes Equations:

$$\rho(\frac{\partial \omega}{\partial t} + \omega\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} + w\frac{\partial \omega}{\partial z}) = \rho g_x - \frac{\partial P}{\partial x} + \mu(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2})(1)$$

$$\rho(\frac{\partial \omega}{\partial t} + \omega\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} + w\frac{\partial \omega}{\partial z}) = \rho g_y - \frac{\partial P}{\partial y} + \mu(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2})(2)$$

$$\rho(\frac{\partial \omega}{\partial t} + \omega\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} + w\frac{\partial \omega}{\partial z}) = \rho g_z - \frac{\partial P}{\partial z} + \mu(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2})(3)$$

These equations are frequently written in cylindrical form. For example, Equation (1) can be written as:

$$\rho(\frac{\partial \omega}{\partial t} + \omega\frac{\partial \omega}{\partial x} + v_r\frac{\partial \omega}{\partial r} + \frac{v_\theta}{r}\frac{\partial \omega}{\partial \theta}) = -\frac{\partial p}{\partial x} + \rho g_x + \mu(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \omega}{\partial \theta^2})$$

(4)

A simplification of the Navier-Stokes equations can be made assuming a Poiseuille flow in which the velocity of the fluid is described by the following equation.

$$\omega = \frac{1}{4\mu}\frac{dP}{dx}[r^2 - R^2] \quad (5)$$

A Poiseuille flow assumes that the flow is steady, uniform (over a cross-section), laminar, and axially symmetric within a cylindrical tube. Under these assumptions, the term $\frac{\partial \omega}{\partial t} = 0$ (since there is no change in velocity over time). Furthermore, the terms $\frac{\partial \omega}{\partial x}$ and $\frac{\partial \omega}{\partial \theta}$ equal zero.

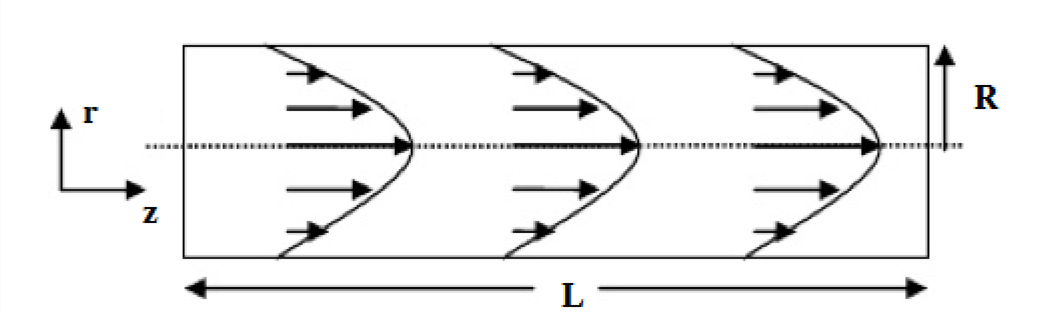


Figure 3: Poiseuille Flow in a Rigid Tube [13]

(For another project using Poiseuille flow, see MBW:Optimum Design of Blood Vessel Bifurcation)

Notation

Before examining Womersley's model, it is important to define some notation that will be used.

ρ	liquid density
μ	viscosity
$v = \frac{\mu}{\rho}$	kinematic viscosity
$p_1,\ p_2$	pressures at ends of pipe
R	radius of pipe
r	distance from center of pipe
l	length of pipe
ω	longitudinal velocity of liquid
$f = \frac{n}{2\pi}$	fundamental frequency (typically the heart rate in radians)
T_0	period of the pressure gradient wave

Womersley Flow Defined

We now take an in-depth look at Womersly's paper and model for pulsatile flow. To provide the basis for Womersley's model, we begin with a more complete derivation of Poiseuille's formula for steady flow and also include the pressure gradient. A constant pressure gradient throughout a pipe of length l is defined as:

$$\frac{p_1 - p_2}{l} \quad (6)$$

After accounting for the simplifications of the Navier-Stokes equations based on Poiseuille flow, the equation of motion is then:

$$\frac{d^2\omega}{dr^2} + \frac{1}{r}\frac{d\omega}{dr} + \frac{p_1 - p_2}{\mu l} = 0 \quad (7)$$

This has Equation (5) as its solution.

Now, Womersley expresses the pressure gradient as a periodic function of time with frequency $f = n/2\pi$ to represent the arterial pulse. The representation of this function is done using Fourier series.

Using Fourier Series to Represent Pressure Gradient

Since the change in pressure gradient is periodic, it can be expressed using the following function:

$$\frac{\partial P}{\partial x} = Re[\sum_{n=0}^{\infty} a_n e^{ifnt}] \quad (8)$$

where $a_n = A_n + iB_n$.

Using Euler's formula, this can be expressed as a Fourier series with Fourier coefficients $A_0,\ A_1, ...,\ A_n,\ B_0,\ B_1, ...,\ B_n$.

$$\frac{\partial P}{\partial x} = A_0 + \sum_{n=1}^{N} A_n cos(fnt) + \sum_{n=1}^{N} B_n sin(fnt) \quad (9)$$

The coefficients are calculated as follows:

$$A_n = \frac{2}{T_0}\int_0^{T_0} \frac{dP}{dx} cos(fnt)dt \quad (10)$$

$$B_n = \frac{2}{T_0}\int_0^{T_0} \frac{dP}{dx} sin(fnt)dt \quad (11)$$

These coefficients can be easily calculated using Matlab or Mathematica. A Fast Fourier Transform (FFT) algorithm can also be used instead to calculate these coefficients. See [14] page 196 for more information and sample Matlab code.

Solving for Flow Velocity (u)

Since we are assuming a Poiseuille flow that changes over time, we have that $\frac{\partial \omega}{\partial x} = 0$ and $\frac{\partial \omega}{\partial \theta} = 0$. After some algebra we get that for a single harmonic, n:

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{1}{v}\frac{\partial w}{\partial t} = -\frac{a_n e^{ifnt}}{\mu} \quad (12)$$

Let's consider the following simple solution to Equation (12) since the velocity also changes periodically over time:

$$\omega = u(r)e^{ifnt} \quad (13)$$

Next we substitute this equation for ω into Equation (12) and divide both sides by e^{ifnt} to get the following ordinary differential equation that is not dependent on time:

$$\frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{ifn}{v}u = \frac{-a_n}{\mu} \quad (14)$$

This equation can be rewritten to be a Bessel zero-order differential equation using the fact that $-i = i^3$:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \frac{i^3 fn}{v}u = \frac{-a_n}{\mu} \quad (15)$$

Solving the Bessel Differential Equation

The general form for a zero-order Bessel differential equation is:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \lambda^2 u = 0 \quad (16)$$

In our case note that $\lambda^2 = \frac{i^3 fn}{v}$. So the homogeneous version of Equation (15) fits the above form and its solution is of the general form:

$$u_n(r) = c_1 J_0(ri^{\frac{3}{2}}\sqrt{\frac{fn}{v}}) + c_2 Y_0(ri^{\frac{3}{2}}\sqrt{\frac{fn}{v}}) \quad (17)$$

The term The term Y_0 must be discarded since $u(r)$ has the requirement that it must be finite at the center (origin) of the pipe. Therefore, $c_2 = 0$. Since Equation (15) is non-homogeneous, we use the technique described in [14] and try the simple solution $u(r) = c_3$. Then we have that the terms involving the derivative of u are zero, and thus we have that $\frac{i^3 fn}{v}u = \frac{-a_n}{\mu}$. Therefore, using the fact that $v = \frac{\mu}{\rho}$, we have that:

$$c_3 = \frac{a_n}{i\rho fn} \quad (18)$$

Then the solution looks like:

$$u_n(r) = c_1 J_0(r i^{\frac{3}{2}} \sqrt{\frac{fn}{v}}) + \frac{a_n}{i\rho fn} \quad (19)$$

We can now solve for c_1 by using the no-slip condition that $u = 0$ at the boundary $r = R$. Therefore we have:

$$0 = c_1 J_0(r i^{\frac{3}{2}} \sqrt{\frac{fn}{v}}) + \frac{a_n}{i\rho fn} \quad (20)$$

$$c_1 = \frac{a_n}{i\rho fn J_0(R i^{\frac{3}{2}} \sqrt{\frac{fn}{v}})} \quad (21)$$

Then finally we get that:

$$u_n(r) = \frac{a_n}{i\rho fn}[\frac{J_0(\lambda r)}{J_0(\lambda R)} - 1] \quad (22)$$

The last step is to add the steady flow velocity term u_0 and thus we have:

$$u(r) = u_0 + \sum_{n=1}^{\infty} u_n(r) \quad (23)$$

Womersley Number

$$\alpha = r\sqrt{\frac{\omega}{v}}$$

The quantity $r\sqrt{\frac{f}{v}}$ in Equation (19) is called the Womersley Number, α. It is a dimensionless parameter that represents the ratio of transient forces, originating from the pulse wave, to the viscous force, or shear force. To get a feel for the magnitude of α, an example problem is presented.

Example 1: A 20 kg dog has an average heart rate of 90 beats per minute (bpm) and a 70 kg human has an average of 70 bpm [15].The density of blood is 1060 kg/m^3 . Using the following calculation ([15] Equation 3.7, p. 27) we calculate the diameter of the aorta to be 2.03 cm for humans, and 1.33 cm for dogs.

$$D = .48W^{.34}$$

where W is the weight of the animal. From [16] we get that the blood viscosity for a human is 0.006 Ns/m^2 and for a dog is 0.0056 Ns/m^2 .

Now we calculate Womersley's number: $r\sqrt{\frac{f}{v}} = r\sqrt{\frac{f\rho}{\mu}}$. Note that we need to convert bpm to rad/s so we multiply the bpm by $\frac{2\pi}{60} = \frac{\pi}{30}$.

For a human, we have: $\alpha = \frac{2.03}{100}m \cdot \sqrt{\frac{70(\frac{\pi}{30})rad/s \cdot 1060kg/m^3}{0.006Ns/m^2}} = 23.1$

For a dog, we have: $\alpha = \frac{1.33}{100}m \cdot \sqrt{\frac{90(\frac{\pi}{30})rad/s \cdot 1060kg/m^3}{0.0056Ns/m^2}} = 17.76$

This tells us that the oscillatory inertial forces become more important than the viscous force as the size of the animal increases, or particulary as the size of the blood vessels increases. (Note that this example is adapted from a similar example found in [14], p. 30).

Flow Rate (Q)

In the next part of the paper, Womersley derives the flow rate (Q) of the fluid passing through a cross-sectional area of the pipe, or artery in this case. This is accomplished by the integrating the velocity over a differential area:

$$Q = 2\pi \int_0^R u(r)rdr \qquad (21)$$

For steady flow, recall that $u = \frac{p_1 - p_2}{\mu l}(r^2 - R^2)$. After integrating, we get that:

$$Q = \frac{p_1 - p_2}{8\mu l}\pi R^4 \quad (22)$$

which is Poisueille's formula. Now, we can substitute in the formula we found for $u(r)$, and then use the fact that $\int xJ_0(x)dx = xJ_1(x)$, then after some calculus and algebraic simplification we get that:

$$Q_n = \frac{\pi R^2}{\mu} Re[\frac{a_n}{ifn}\{1 - \frac{2\alpha i^{3/2}}{i^3\alpha^2}\frac{J_1(\alpha i^{3/2})}{J_0(\alpha i^{3/2})}\}e^{ifnt}] \quad (23)$$

Recall that the Q_n need to be summed for each $n = 1, 2, 3...$, and added to the average flow rate given by the constant term in the Fourier series $Q_0 = a_0\frac{8\mu}{\pi R^4}$ [14] to get:

$$Q = Q_0 + \sum_{n=1}^{\infty} Q_n \quad (24)$$

Womersley now derives a way to calculate Q without Bessel functions, perhaps because back in 1955 these functions were numerically more challenging to calculate. He uses modulus and phase functions [1] to accomplish this. The details of these functions are out of the scope of this review, but the following relations are used:

$$J_0(xi^{3/2}) = M_0(x)e^{i\theta_0} \quad (25)$$

$$J_1(\alpha i^{3/2}) = M_1 e^{i\theta_1} \quad (26)$$

He also uses the fact that the real part of $a_n e^{ifnt}$ is $M_n\ cos(fnt + \phi)$ where $M_n = \sqrt{A^2 + B^2} = |a_n|$ since $a_n = A_n + iB_n$ [14]. Then, after some simplifications Womersley gets the following, simplified formula for Q (not including the steady flow term Q_0):

$$Q = \frac{\pi M R^4}{\mu} \frac{M'_{10}}{\alpha^2}\ sin(fmt + \phi + \epsilon_{10}) \quad (27)$$

where $m = 1, 2, ..., 6$ and the values for α^2, M, M'_{10}, ϕ, and ϵ_{10} are all given in a table in Womersley's paper. Let's look at an example problem presented in the paper to get more of an idea of the values for Q.

Example 2: The radius of an artery is 0.15 cm, the viscosity is $\mu = .04P$, and the pulse frequency is 180 bpm. The value for α^2, M, M'_{10}, ϕ, and ϵ_{10} are given in the following two tables:

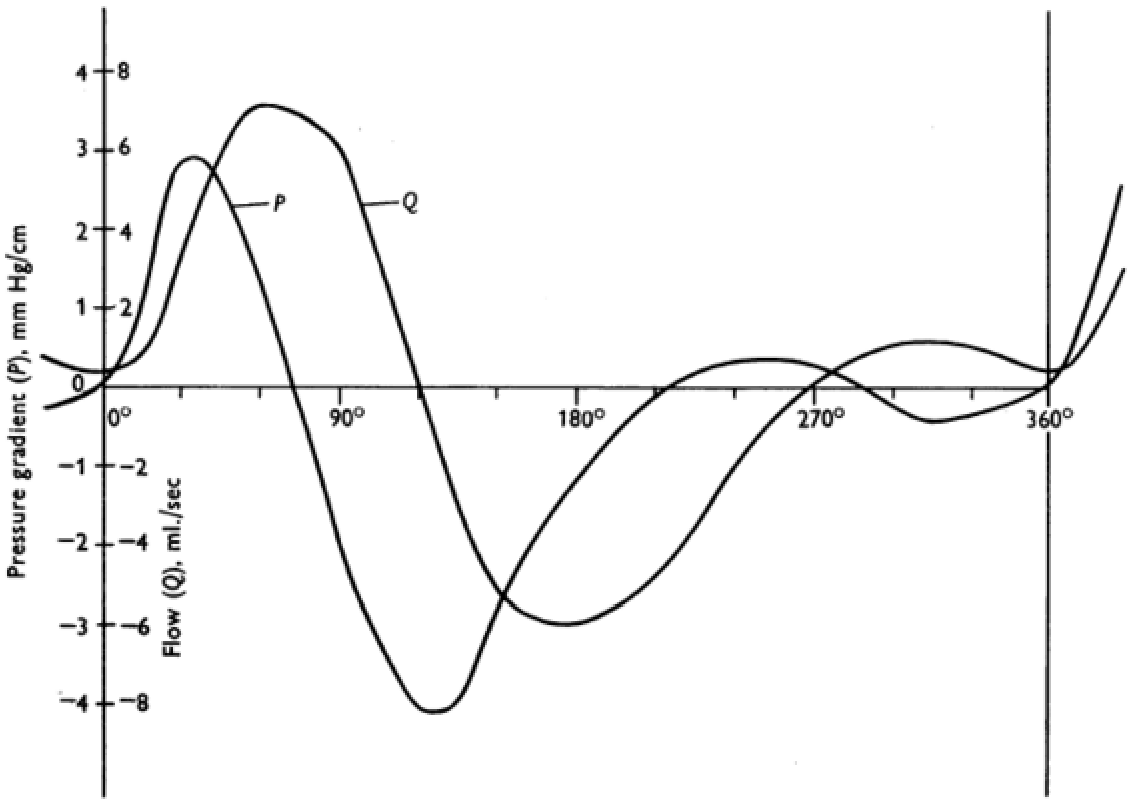
m	cosine term	sine term	M_m	ϕ_m
1	+0·8781	−0·7432	+1·1050	+40° 14′
2	+0·5415	+1·4327	+1·5316	−69° 17′
3	−0·7946	+0·5508	−0·9668	+34° 44′
4	−0·2375	−0·1588	−0·2857	−33° 47′
5	+0·0125	−0·2818	+0·2821	+87° 31′
6	−0·1917	−0·0167	−0·1924	− 4° 58′

m	α	α^2	M'_{10}	ϵ_{10}
1	3·34	11·13	0·6551	30° 59′
2	4·72	22·27	0·7436	19° 57′
3	5·78	33·40	0·7839	15° 49′
4	6·67	44·53	0·8096	13° 30′
5	7·46	55·67	0·8278	11° 57′
6	8·17	66·80	0·8416	10° 45′

The expression for Q is then then sum of the six terms:

$$Q = 3.56\ sin(t + 71°13') + 2.71\ sin(2t - 49°13')$$
$$-1.2\ sin(3t + 50°33') - .28\ sin(4t - 20°17')$$
$$+.22\ sin(5t + 99°28') - .13\ sin(6t + 5°47')$$

This result for Q is plotted against the pressure gradient below:



Analysis and Conclusions

Womersley's arterial flow model gives formulas to calculate the flow rate of a viscous fluid through a rigid tube under a periodic pressure gradient, described by a Fourier series. The solution is then extended to calculate the flow rate over a cross-sectional area of the tube.

The graph in Example 2 depicts the pressure gradient (P) juxtaposed with the flow rate (Q), and clearly shows the phase-lag between the two curves. This implies that the pulse wave is first sent through the body (seen as an increase in the pressure gradient) and then the blood flow follows. Typically the phase-lag is about 90 degrees, except at the boundary layer where it is about 45 degrees [17]. Also noteworthy is that this phase-lag is only present in large arteries; in smaller arteries less than 4mm in diameter, the flow behavior closely follows Poiseuille's formula [17]. Interestingly, the pulse wave travels about 5 times the maximum blood velocity [17].

Also evident from the graph is that the direction of flow velocity is actually reversed as seen when the velocity becomes negative. However, this is the topic of another paper written by Womersley et. al in 1955 [18].

Womersley's number has provided both fluid mechanics and biological sciences with a means to measure the inertial forces versus the viscous forces. It is as significant in analyzing unsteady flow as the Reynolds number is in measuring steady flow [19].

Womersley's model has been foundational to many models of arterial blood flow, but itself is limited to modeling laminar flow through a rigid, cylindrical tube. In reality, there are elastic effects in the blood vessel and the pressure gradient may depend on other factors besides time. In addition, Womersley's model considers only longitudinal velocity, whereas there may also be a radial part.

Recent Extension

In 2011, the Womersley article was cited in a study on Effects of vessel wall elasticity and non-Newtonian rheology on blood flow regime and hemodynamic parameters distribution by Foad Kabinejadian and Dhanjoo N. Ghista. This paper is a follow up of an earlier study, wherein they computationally simulated blood flow using Womersley's model. They were focused on vessel intersections and junctions. The current paper revisits the computational simulation with compliant walls and non-Newtonian fluid. They compared the results of their two studies and found that the compliant and non-Newtonian model was consistent with the results of earlier studies and observational data. They conclude that the inclusion of wall compliance and non-Newtonian rheology in flow simulation of blood vessels can be essential in quantitative and comparative analyses [20]. For more information on human blood flow rheology, read Viscoelastic Versus Newtonian Behavior from the Physical Review Letters.

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