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Solutions to Atiyah Macdonald

Chapter 1 : Rings and Ideals

1.1. **Show that the sum of a nilpotent element and a unit is a unit.**

If x is nilpotent, then $1 - x$ is a unit with inverse $\sum_{i=0}^{\infty} x^i$. So if u is a unit and x is nilpotent, then $v = 1 - (-u^{-1}x)$ is a unit since $-u^{-1}x$ is nilpotent. Hence, $u + x = uv$ is a unit as well.

1.2. **Let A be a ring with $f = a_0 + a_1x + \cdots + a_nx^n$ in $A[x]$.**

a. **Show that f is a unit iff a_0 is a unit and $a_1, \ldots, a_n$ are nilpotent.**

If $a_1, \ldots, a_n$ are nilpotent in A, then $a_1x, \ldots, a_nx^n$ are nilpotent in $A[x]$. Since the sum of nilpotent elements is nilpotent, $a_1x + \cdots + a_nx^n$ is nilpotent. So $f = a_0 + (a_1x + \cdots + a_nx^n)$ is a unit when a_0 is a unit by exercise 1.1.

Now suppose that f is a unit in $A[x]$ and let $g = b_0 + b_1x + \cdots + b_mx^m$ satisfy $fg = 1$. Then $a_0b_0 = 1$, and so a_0 is a unit in $A[x]$. Notice that $a_nb_m = 0$, and suppose that $0 \le r \le m - 1$ satisfies

$$a_n^{r+1}b_{m-r} = a_n^r b_{m-r-1} = \cdots = a_nb_m = 0$$

Notice that

$$0 = fg = \sum_{i=0}^{m+n} \left(\sum_{j=0}^{i} a_j b_{i-j} \right) x^i = \sum_{i=0}^{m+n} c_i x^i$$

where we define $a_j = 0$ for $j > n$ and $b_j = 0$ for $j > m$. This means that each $c_i = 0$, and so

$$0 = a_n^{r+1}c_{m+n-r-1} = \sum_{j=0}^{n} a_j a_n^{r+1} b_{m+n-r-1-j} = a_n^{r+2} b_{m-r-1}$$

since $m + n - r - 1 - j \ge m - r$ for $j \le n - 1$. So by induction $a_n^{m+1}b_0 = 0$. Since b_0 is a unit, we conclude that a_n is nilpotent. This means that $f - a_nx^n$ is a unit since a_nx^n is nilpotent and f is a unit. By induction, $a_1, \ldots, a_n$ are all nilpotent.

b. **Show that f is nilpotent iff $a_0, \ldots, a_n$ are nilpotent.**

Clearly $f = a_0 + a_1x + \ldots + a_nx^n$ is nilpotent if $a_0, \ldots, a_n$ are nilpotent. Assume f is nilpotent and that $f^m = 0$ for $m \in \mathbb{N}$. Then in particular $(a_nx^n)^m = 0$, and so a_nx^n is nilpotent. Thus, $f - a_nx^n$ is nilpotent. By induction, a_kx^k is nilpotent for all k. This means that $a_0, \ldots, a_n$ are nilpotent.

c. **Show that f is zero-divisor iff $bf = 0$ for some $b \neq 0$.**

If there is $b \neq 0$ for which $bf = 0$, then f is clearly a zero-divisor. So suppose f is a zero-divisor and choose a nonzero $g = b_0 + b_1x + \cdots + b_mx^m$ of minimal degree for which $fg = 0$. Then in particular, $a_nb_m = 0$. Since $a_ng \cdot f = 0$ and $a_ng = a_nb_0 + \cdots + a_nb_{m-1}x^{m-1}$, we conclude that $a_ng = 0$ by minimality. Hence, $a_nb_k = 0$ for all k. Suppose that

$$a_{n-r}b_k = a_{n-r+1}b_k = \cdots = a_nb_k = 0 \qquad \text{for all } k$$

Then as in part a we obtain the equation

$$0 = \sum_{j=0}^{m+n-r-1} a_{m+n-r-1-j} b_j = a_{n-r-1} b_m$$

Again we conclude that $a_{n-r-1}g = 0$. Hence, by induction $a_j b_k = 0$ for all j, k. Choose k so that $b = b_k \neq 0$. Then $bf = 0$ with $b \neq 0$.

d. **Prove that f, g are primitive iff fg is primitive.**

Let h be any polynomial in $A[x]$. If h is not primitive then there is a maximal $\mathfrak{m}$ in A containing the coefficients of h. Let k be the residue field of $\mathfrak{m}$ and consider the natural map $\pi : A[x] \to k[x]$. Then $\pi(h) = 0$. This condition is also sufficient for showing that h is not a primitive polynomial.

So if fg is not primitive, then $\pi(fg) = 0$ as above for some maximal $\mathfrak{m}$. But $\pi(fg) = \pi(f)\pi(g)$ and $k[x]$ is an integral domain so that $\pi(f) = 0$ or $\pi(g) = 0$. In other words, either f is not primitive or g is not primitive. The converse follows similarly.

1.3. **Generalize the results of exercise 2 to $A[x_1, \ldots, x_r]$ where $r \geq 2$.**

Let $f \in A[x_1, \ldots, x_r]$. Use multi-index notation to write

$$f = \sum_{I \in \mathbb{N}^r} \alpha_I x^I \qquad \text{where} \qquad x^I = x_1^{I_1} \cdots x_r^{I_r}$$

We can also write

$$f = \sum_{i=0}^{n} g x_r^i \quad \text{where} \quad g \in A[x_1, \ldots, x_{r-1}]$$

b. **Show that f is nilpotent iff each α_I is nilpotent.**

Suppose that f is nilpotent. Then $g_0, \ldots, g_n$ are nilpotent polynomials in $A[x_1, \ldots, x_{r-1}]$ by exercise 1.2. So by induction each a_α is nilpotent. If each α_I is nilpotent then each $\alpha_I x^I$ is nilpotent, so that f is nilpotent.

a. **Show that f is a unit iff the constant coefficient is a unit and each α_I is nilpotent for $|I| > 0$.**

Suppose that f is a unit. Then in $A[x_1, \ldots, x_{r-1}]$ we know that g_0 is a unit and $g_1, \ldots, g_n$ are nilpotent. So by part b we see that α_I is nilpotent whenever $I(r) > 0$. By symmetry α_I is nilpotent whenever $|I| > 0$. The constant coefficient is clearly a unit. On the other hand, if the constant coefficient is a unit and all other coefficients are nilpotent, then f is clearly a unit.

c. **Show that f is a zero-divisor iff $bf = 0$ for some $b \neq 0$.**

Let $\mathfrak{a}$ be any ideal in $A[x_1, \ldots, x_n]$ and suppose $g\mathfrak{a} = 0$ for some non-zero $g \in A[x_1, \ldots, x_n]$. Since $A[x_1, \ldots, x_n] = A[x_1, \ldots, x_{n-1}][x_n]$, exercise 1.2 allows us to assume that $g \in A[x_1, \ldots, x_{n-1}]$. Now given $f \in \mathfrak{a}$ we can write $f = \sum f_i x_n^i$ where each $f_i \in A[x_1, \ldots, x_{n-1}]$. Let $\mathfrak{b}$ be the subset of $A[x_1, \ldots, x_{n-1}]$ consisting of all such f_i, as f ranges across $\mathfrak{a}$. Then $\mathfrak{b}$ is an ideal since $\mathfrak{a}$ is an ideal, and $g\mathfrak{b} = 0$ since $g \in A[x_1, \ldots, x_{n-1}]$ by hypothesis. So by induction, there is $b \neq 0$ satisfying $b\mathfrak{b} = 0$, and hence $b\mathfrak{a} = 0$. Now we apply this result to $\mathfrak{a} = (f)$ to get the desired conclusion.

d. **Show that f and g are primitive iff fg is primitive.**

Let h be any polynomial in $A[x_1, \ldots, x_r]$. If h is not primitive then there is a maximal $\mathfrak{m}$ in A containing the coefficients of h. Let k be the residue field of $\mathfrak{m}$ and consider the natural map $\pi : A[x_1, \ldots, x_r] \to$

$k[x_1, \ldots, x_r]$. Then $\pi(h) = 0$ in $k[x_1, \ldots, x_r]$. This condition is also sufficient for showing that h is not a primitive polynomial.

So if fg is not primitive, then $\pi(fg) = 0$ as above for some maximal $\mathfrak{m}$. But $\pi(fg) = \pi(f)\pi(g)$ and $k[x_1, \ldots, x_r]$ is an integral domain so that $\pi(f) = 0$ or $\pi(g) = 0$. In other words, either f is not primitive or g is not primitive. The converse is obvious.

1.4. **Show that $\mathfrak{R}(A[x]) = \mathfrak{N}(A[x])$ for every ring A.**

As with any ring $\mathfrak{N}(A[x]) \subseteq \mathfrak{R}(A[x])$. So suppose that $f \in \mathfrak{R}(A[x])$. Then $1-fx$ is a unit. If $f = a_0 + \ldots + a_n x^n$ this means that $1 - a_0 x - \ldots - a_n x^{n+1}$ is a unit, so that $a_0, \ldots, a_n$ are nilpotent by exercise 1.2. By exercise 1.2 this means that f is nilpotent, and so $f \in \mathfrak{N}(A[x])$. Hence $\mathfrak{R}(A[x]) \subseteq \mathfrak{N}(A[x])$, giving the desired result.

1.5. **Let A be a ring with $f = \sum_0^\infty a_n x^n$ in $A[[x]]$.**

a. **Show that$f$ is a unit iff a_0 is a unit.**

Suppose f is a unit. Then there is $g(x) = \sum_0^\infty b_n x^n$ satisfying $fg = 1$. In particular, $a_0 b_0 = 1$, implying that a_0 is a unit. Conversely, suppose that a_0 is a unit. We wish to find b_n for which $fg = 1$. This is equivalent to finding b_n satisfying $a_0 b_0 = 1$ and

$$a_0 b_n + \sum_{i=0}^{n-1} a_{n-i} b_i = 0 \quad \text{for } n > 0$$

So we define $b_0 = a_0^{-1}$ and

$$b_n = -a_0^{-1} \sum_{i=0}^{n-1} a_{n-i} b_i \quad \text{for } n > 0$$

This constructively shows that f is a unit.

b. **Show that each a_i is nilpotent if f is nilpotent, and that the converse is false.**

Suppose that f is nilpotent and choose $n > 0$ for which $f^n = 0$. Then $a_0^n = 0$. Hence a_0 is nilpotent, as is $f - a_0$. Now by induction we see that every a_n is nilpotent. The converse need not be true though. We can define

$$A = \mathbb{Z}_4 \times \mathbb{Z}_8 \times \mathbb{Z}_{16} \times \cdots$$

and then let

$$a_0 = (2, 0, 0, \ldots) \quad a_1 = (0, 2, 0, \ldots) \quad \ldots$$

Observe that $a_j a_k = 0$ for $j \neq k$, and so

$$f^n = a_0^n + a_1^n x^n + a_2^n x^{2n} + \cdots \quad \text{for all } n > 0$$

Obviously each a_k is nilpotent, and yet f is not nilpotent. The problem here is that there is no N for which $a_k^N = 0$ for all k. This issue does not occur when $\mathfrak{N}(A)$ is a nilpotent ideal, as for instance when A is Noetherian.

c. **Show that $f \in \mathfrak{R}(A[[x]])$ iff $a_0 \in \mathfrak{R}(A)$.**

Assume $a_0 \in \mathfrak{R}(A)$ and suppose $g \in A[[x]]$ with constant coefficient b_0. Then there is $h \in A[[x]]$ satisfying $1 - fg = 1 - a_0 b_0 + hx$. Since $1 - a_0 b_0$ is a unit in A, we see by part a that $1 - fg$ is a unit in $A[[x]]$, so that $f \in \mathfrak{R}(A[[x]])$. On the other hand, if $f \in \mathfrak{R}(A[[x]])$ and $b \in A$, then $1 - fb$ is a unit in $A[[x]]$. Again by part a this means that $1 - a_0 b$ is a unit in A, so that $a_0 \in \mathfrak{R}(A)$.

d. **Show that the contraction of a maximal ideal $\mathfrak{m}$ of $A[[x]]$ is a maximal ideal of A, and that $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and x.**

By part c we have $(x) \subseteq \mathfrak{R}(A[x]) \subseteq \mathfrak{m}$ since $0 \in \mathfrak{R}(A)$. Now if $f = a + gx$ is in $\mathfrak{m}$ then $a = f - gx \in \mathfrak{m}$ since $x \in \mathfrak{m}$, so that $a \in \mathfrak{m} \cap A$. In other words, $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and x.

Notice that $\mathfrak{m}^c = \mathfrak{m} \cap A$, and that $A/\mathfrak{m}^c$ naturally embeds into $A[[x]]/\mathfrak{m}$ via the map $a + \mathfrak{m}^c \mapsto a + \mathfrak{m}$. I claim that $A/\mathfrak{m}^c$ is a subfield of the field $A[[x]]/\mathfrak{m}$. So suppose that $a + \mathfrak{m}^c \neq \mathfrak{m}^c$ and choose $f \in A[[x]]$ for which $(a+\mathfrak{m})(f+\mathfrak{m}) = 1+\mathfrak{m}$, so that $af - 1 \in \mathfrak{m}$. Write $f = a_0 + gx$ for some $g \in A[[x]]$ and observe that $af - 1 = aa_0 - 1 + agx \in \mathfrak{m}$, implying that $aa_0 - 1 \in \mathfrak{m}$ since $x \in \mathfrak{m}$. So we see that $aa_0 - 1 \in \mathfrak{m}^c$, and hence $a + \mathfrak{m}^c$ has the inverse $a_0 + \mathfrak{m}^c$. This means that $A/\mathfrak{m}^c$ is a subfield of $A[[x]]/\mathfrak{m}$, and hence $\mathfrak{m}^c$ is a maximal ideal in A.

e. **Show that every prime ideal $\mathfrak{p}$ of A is the contraction of a prime ideal $\mathfrak{q}$ of $A[[x]]$.**

Let $\mathfrak{q}$ be the ideal in $A[[x]]$ consisting of all $\sum a_k x^k$ for which $a_0 \in \mathfrak{p}$. If $fg \in \mathfrak{q}$ with $f = \sum a_k x^k$ and $g = \sum b_k x^k$, then $a_0 b_0 \in \mathfrak{p}zz$. Hence, $a_0 \in \mathfrak{p}$ or $b_0 \in \mathfrak{p}$, implying that $f \in \mathfrak{q}$ or $g \in \mathfrak{q}$. So $\mathfrak{q}$ is a prime ideal in $A[[x]]$ and $\mathfrak{p} = A \cap \mathfrak{q}$, so that $\mathfrak{p}$ is the contraction of $\mathfrak{q}$.

1.6. **Let A be a ring such that every ideal not contained in $\mathfrak{N}(A)$ contains a nonzero nilpotent. Show that $\mathfrak{N}(A) = \mathfrak{R}(A)$.**

As always $\mathfrak{N}(A) \subseteq \mathfrak{R}(A)$. Now suppose that $\mathfrak{N}(A) \subsetneq \mathfrak{R}(A)$. By hypothesis, there is an idempotent $e \neq 0$ in $\mathfrak{R}(A)$. Now $(1-e)e = e - e^2 = 0$. Since $e \in \mathfrak{R}(A)$ we know that $1 - e$ is a unit in A, so that $e = 0$. But this contradicts our choice of e, showing that $\mathfrak{N}(A) = \mathfrak{R}(A)$.

1.7. **Let A be a ring such that every $x \in A$ satisfies $x^n = x$ for some $n > 1$. Show that every prime ideal $\mathfrak{p}$ in A is maximal.**

For $x \in A$ choose $n > 1$ satisfying $x^n = x$. Then $\bar{x}(\bar{x}^{n-1} - \bar{1}) = \bar{0}$ in $A/\mathfrak{p}$. Since $A/\mathfrak{p}$ is an integral domain we have $\bar{x} = \bar{0}$ or $\bar{x}^{n-1} = \bar{1}$. In the second case $\bar{x}$ is a unit in $A/\mathfrak{p}$ since $n > 1$. This shows that $A/\mathfrak{p}$ is a field, so that $\mathfrak{p}$ is in fact a maximal ideal.

1.8. **Let $A \neq 0$ be a ring. Show that the set of prime ideals of A has minimal elements with respect to inclusion.**

Suppose that $\mathfrak{p}_\alpha$ are prime ideals for $\alpha \in I$. Suppose further that I has a linear ordering $\prec$ for which $\mathfrak{p}_\alpha \supset \mathfrak{p}_\beta$ whenever $\alpha \prec \beta$. Define $\mathfrak{p} = \bigcap_{\alpha \in I} \mathfrak{p}_\alpha$, and suppose that $\mathfrak{p}$ is not prime. Then there are x, y for which $xy \in \mathfrak{p}$, and yet $x, y \notin \mathfrak{p}$. Hence, there are α, β for which $x \notin \mathfrak{p}_\alpha$ and $y \notin \mathfrak{p}_\beta$. But either $\alpha \prec \beta$ or $\beta \prec \alpha$, implying that $x \notin \mathfrak{p}_\beta$ or $y \notin \mathfrak{p}_\alpha$. Either case leads to a contradiction as $\mathfrak{p}_\alpha$ and $\mathfrak{p}_\beta$ are prime ideals containing xy. So $\mathfrak{p}$ is a prime ideal, contained in every $\mathfrak{p}_\alpha$. This means, by Zorn's Lemma, that the set of prime ideals in A has minimal elements.

1.9. **Let $\mathfrak{a} \neq (1)$ be an ideal in A. Show that $\mathfrak{a} = r(\mathfrak{a})$ if and only if $\mathfrak{a}$ is the intersection of a collection of prime ideals.**

Suppose $\mathfrak{a} = \bigcap_I \mathfrak{p}_\alpha$ is the intersection of prime ideals. Notice that we always have $\mathfrak{a} \subseteq r(\mathfrak{a})$. Now if $x \in r(\mathfrak{a})$, then $x^n \in \mathfrak{a}$ for some n, and so $x^n \in \mathfrak{p}_\alpha$ for all α. Therefore, $x \in \mathfrak{p}_\alpha$ by the definition of prime ideals, implying that $x \in \mathfrak{a}$. Hence $\mathfrak{a} = r(\mathfrak{a})$. The converse is trivial.

1.10. **Show that the following are equivalent for any ring A.**

a. A **has exactly one prime ideal.**

b. **Every element of A is either a unit or nilpotent.**

c. $A/\mathfrak{N}(A)$ **is a field.**

(a $\Rightarrow$ b) Suppose that $x \in A$ is neither nilpotent nor invertible. Let $\mathfrak{m}$ be a maximal ideal in A containing x. Then $\mathfrak{N}(A) \subsetneq \mathfrak{m}$. But $\mathfrak{m}$ is a prime ideal, so that A has more than one prime ideal.

(b $\Rightarrow$ c) By hypothesis x is a unit in A whenever $x \notin \mathfrak{N}(A)$. This shows that $A/\mathfrak{N}(A)$ is a field.

(c $\Rightarrow$ a) If $A/\mathfrak{N}(A)$ is a field, then $\mathfrak{N}(A)$ is a maximal ideal. But $\mathfrak{N}(A)$ is contained in every prime ideal in A, and prime ideals are proper by definition. So $\mathfrak{N}(A)$ is the only prime ideal in A.

1.11. **Prove the following about a Boolean ring A.**

a. $2x = 0$ **for every $x \in A$.**

Notice that $2x = (2x)^2 = 4x^2 = 4x = 2x + 2x$, so that $2x = 0$ for every $x \in A$.

b. **For every prime ideal $\mathfrak{p}$, $A/\mathfrak{p}$ is a field with two elements.**

If $x \notin \mathfrak{p}$ then from the equation $(x + \mathfrak{p})^2 = x + \mathfrak{p}$ we conclude that $x + \mathfrak{p} = 1 + \mathfrak{p}$. Hence, $A/\mathfrak{p}$ is the field with two elements. This means in particular that every prime ideal in A is maximal.

c. **Every finitely generated ideal in A is principal.**

Suppose $x_1, x_2 \in A$ and define $y = x_1 + x_2 + x_1x_2$. Notice that

$$x_1 y = x_1 + x_1x_2 + x_1x_2 = x_1 + 2x_1x_2 = x_1$$

Similarly $x_2 y = x_2$. This shows that

$$(y) = (x_1, x_2) = (x_1) + (x_2)$$

The result now follows by induction.

1.12. **Show that a local ring contains no idempotents $\neq 0$ or 1.**

Suppose $e \in A$ is idempotent, so that $e(1-e) = 0$. If $e \neq 0$ or 1, then e and $1-e$ are nonunits. Since A is a local ring, the nonunits form an ideal. But this means that $e + (1-e) = 1$ is a nonunit, a contradiction.

1.13. **Given a field K construct an algebraic closure of K.**

Suppose that K is a field so that $K[x]$ is factorial. Let Σ consist of all irreducible polynomials in $K[x]$. Define A to be the polynomial ring generated by indeterminates x_f over K, one for each $f \in \Sigma$. Also define $\mathfrak{a}$ to be the ideal in A generated by $f(x_f)$ for $f \in \Sigma$. Suppose that $\mathfrak{a} = A$. Then there are $f_1, \ldots, f_n \in \Sigma$ and $g_1, \ldots, g_n \in A$ for which

$$g_1 f_1(x_{f_1}) + \cdots + g_n f_n(x_{f_n}) = 1$$

Let K' be a field containing K and roots α_i of f_i, noting that each f_i is a non-constant polynomial. Letting $x_{f_i} = \alpha_i$ yields $0 = 1$ in K', an impossibility. Therefore, $\mathfrak{a}$ is a proper ideal of A. Let $\mathfrak{m}$ be a maximal ideal in A containing $\mathfrak{a}$. Define $K_1 = A/\mathfrak{m}$. Then K_1 is an extension field of K. For $g \in K[x]$ let $f \in \Sigma$ be an irreducible factor of g. Then $f(x_f + \mathfrak{m}) = f(x_f) + \mathfrak{m} = \mathfrak{m}$, implying that f, and hence g, has a root in K_1. Hence, every polynomial over K has a root in K_1.

Now given the field K_n, choose an extension field K_{n+1} of K_n so that every polynomial over K_n has a root in K_{n+1}. Proceed in this way to obtain K_n for all $n \in \mathbb{N}^+$, and let $L = \bigcup_{n=1}^{\infty} K_n$. Then L is an extension field of K and every polynomial over Σ of degree m splits completely over K_m, and hence splits completely over L. Finally, let $\bar{L}$ be the set of all elements in L that are algebraic over K. Then $\bar{L}$ is algebraic over K and every monic polynomial over K can be written as $g = \prod_{k=1}^{\deg(g)} (x - \alpha_i)$, where α_i are the roots of g in L. But then each α_i is algebraic over K and hence lies in $\bar{L}$. So g has roots in $\bar{L}$. This means that $\bar{L}$ is an algebraic closure of K.

1.14. **In a ring A, let Σ be the set of all ideals in which every element is a zero-divisor. Show that Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence, the set D of zero-divisors in A is a union of prime ideals.**

It is clear by Σ is chain complete. Hence, Zorn's Lemma tells us that Σ has maximal elements. Suppose that $\mathfrak{a} \in \Sigma$ is not a prime ideal. Let $x, y \in A - \mathfrak{a}$ satisfy $xy \in \mathfrak{a}$ so that $\mathfrak{a} \subsetneq (\mathfrak{a} : x)$. If $(\mathfrak{a} : x) \notin \Sigma$ then there is $z \in (\mathfrak{a} : x)$ so that z is not a zero-divisor. I now claim that $(\mathfrak{a} : z) \in \Sigma$. If $w \in (\mathfrak{a} : z)$ then $wz \in \mathfrak{a}$, so that $vwz = 0$ for some $v \neq 0$. Since z is not a zero-divisor $vz \neq 0$, and hence w is a zero-divisor. Thus $\mathfrak{a} \subsetneq (\mathfrak{a} : z) \in \Sigma$ since $x \in (\mathfrak{a} : z) - \mathfrak{a}$. This means that $\mathfrak{a}$ is not a maximal element in Σ. So maximal elements in Σ are indeed prime ideals.

Now if D is the set of zero-divisors in A and $x \in D$ then $(x) \subseteq D$, and hence $(x) \in \Sigma$. It is clear from Zorn's Lemma that there is a maximal $\mathfrak{a} \in \Sigma$ containing (x), so that $x \in \mathfrak{a} \subseteq D$. This means that D is the union of some of the prime ideals of A.

1.15. **Suppose A is a ring and let $\operatorname{Spec}(A)$ be the set of all prime ideals of A. For each $E \subseteq A$, let $V(E) \subseteq \operatorname{Spec}(A)$ consist of all prime ideals containing E. Prove the following.**

a. **If $\mathfrak{a} = \langle E \rangle$ then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.**

Since $E \subseteq \mathfrak{a} \subseteq r(\mathfrak{a})$ we have

$$V(r(\mathfrak{a})) \subseteq V(\mathfrak{a}) \subseteq V(E)$$

Suppose $\mathfrak{p} \in V(E)$ so that $E \subseteq \mathfrak{p}$. Then $\mathfrak{a} = AE \subseteq A\mathfrak{p} = \mathfrak{p}$ and $r(\mathfrak{a}) \subseteq r(\mathfrak{p}) = \mathfrak{p}$. So we have $V(r(\mathfrak{a})) \subseteq V(E)$. We are finished.

b. **$V(0) = \operatorname{Spec}(A)$ and $V(1) = \emptyset$.**

Every prime ideal contains 0, and so $V(0) = \operatorname{Spec}(A)$. Also, no prime ideal equals all of A, by definition, and so $V(1) = \emptyset$.

c. **If $(E_i)_{i \in I}$ is a family of subsets of A then $V(\bigcup E_i) = \bigcup V(E_i)$.**

Any ideal contains $\bigcup E_i$ iff it contains each E_i.

d. **For ideals $\mathfrak{a}$, $\mathfrak{b}$ we have $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.**

By part a we have

$$V(\mathfrak{a} \cap \mathfrak{b}) = V(r(\mathfrak{a} \cap \mathfrak{b})) = V(r(\mathfrak{a}\mathfrak{b})) = V(\mathfrak{a}\mathfrak{b})$$

Clearly $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$ whenever $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$. The converse holds since $\mathfrak{p}$ is a prime ideal. So $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

1.16? **Describe the following**

a. Spec($\mathbb{Z}$)

It is not hard to see that $\text{Spec}(\mathbb{Z}) = \{(0)\} \cup \{(p) : p > 1 \ \text{ prime}\}$.

b. Spec($\mathbb{R}$)

Since $\mathbb{R}$ is a field, it has precisely one prime ideal, namely (0).

c. Spec($\mathbb{C}[x]$)

Since $\mathbb{C}$ is a field, $\mathbb{C}[x]$ is a PID, and so its nonzero prime ideals are of the form (p) for some monic irreducible polynomial p. The only monic polynomials that are irreducible over $\mathbb{C}$ are of the form $p = x - c$ for some $c \in \mathbb{C}$. Of course, the zero ideal is prime as well.

d. Spec($\mathbb{R}[x]$)

Since $\mathbb{R}$ is a field, $\mathbb{R}[x]$ is a PID, and so its nonzero prime ideals are of the form (p) for some monic irreducible polynomial p. Since every odd polynomial has a root, no polynomial of odd degree at least three is irreducible. Suppose p is a monic irreducible polynomial of even degree $2d > 2$. In $\mathbb{C}[x]$ write $p(z) = \prod_{i=1}^{2d}(z - \alpha_i)$. Letting α_i^* be the complex conjugate of α_i, we see that $p(\alpha_i^*) = p(\alpha_i)^* = 0$ since $p \in \mathbb{R}[x]$. This means that $p = \prod_{i=1}^{2d}(z - \alpha_i^*)$. So there is $\sigma \in \Sigma_{2d}$ so that $\alpha_i^* = \alpha_{\sigma(i)}$ for every i. Since p has no real roots, we cannot have $\sigma(i) = i$ for any i. Also, $\alpha_{\sigma(i)}^* = \alpha_i$ so that $\sigma^2 = \text{id}$, and hence σ is a product of 2-cycles. Thus

$$p(z) = \prod_{i=1}^{d}(z - \alpha_i)(z - \alpha_{\sigma(i)}) = \prod_{i=1}^{d}(z - \alpha_i)(z - \alpha_i^*) = \prod_{i=1}^{d}(z^2 - 2\mathfrak{Re}(\alpha_i)z + |\alpha_i|^2)$$

Since each of these quadratics is in $\mathbb{R}[x]$, we see that p is reducible in $\mathbb{R}[x]$, a contradiction. Consequently, the irreducible elements in $\mathbb{R}[x]$ are of the form $x - a$ and $x^2 + bx + c$ where $b^2 - 4c < 0$. These elements correspond bijectively with the non-zero prime ideals in $\mathbb{R}[x]$.

e. Spec($\mathbb{Z}[X]$)

Notice that $\mathbb{Z}[x]$ is factorial. If p is an irreducible polynomial over $\mathbb{Z}$ then (p) is a prime ideal in $\mathbb{Z}[x]$. Since $\mathbb{Z}[x]$ is an integral domain we see that (0) is a prime ideal in $\mathbb{Z}[x]$ as well. Suppose $\mathfrak{p}$ is a non-zero prime ideal in $\mathbb{Z}[x]$ that is not principal. Suppose $\mathfrak{p}$ has the property that, given $f, g \in \mathfrak{p}$, either $(f) \subseteq (g)$ or $(g) \subseteq (f)$. From this I will derive a contradiction. Let $f_1 \in \mathfrak{p}$ and choose $f_2 \in \mathfrak{p} - (f_1)$, making use of the fact that $\mathfrak{p}$ is not principal. Then $(f_1) \subsetneq (f_2)$. We can choose $f_3 \in \mathfrak{p} - (f_2)$. Then $(f_2) \subsetneq (f_3)$. We proceed in this way to get a properly ascending sequence of ideals in $\mathfrak{p}$. This is impossible since Hilbert's Theorem tells us that $\mathbb{Z}[x]$ is Noetherian. Therefore, there are nonzero $f, g \in \mathfrak{p}$ with $(f) \not\subseteq (g)$

and $(g) \not\subseteq (f)$.

We can consider f and g as elements of $\mathbb{Q}[x]$. Suppose, for the sake of contradiction, that $f = f'h$ and $g = g'h$ for some $f', g', h \in \mathbb{Q}[x]$ with $\deg(h) \geq 1$. We can write $f' = af''$ with $a \in \mathbb{Q}$ and $f'' \in \mathbb{Z}[x]$ so that the coefficients of f'' have no prime number in common. Similarly write $g' = bg''$ and $h = ch'$. We see that f'', g'', and h' are all primitive elements of $\mathbb{Z}[x]$. Exercise 1.2 tells us that $f''h'$ and $g''h'$ are primitive elements of $\mathbb{Z}[x]$. But $f = (ac)(f''h')$ so that $ac \in \mathbb{Z}$. Similarly, $g = (bc)(g''h')$ so that $bc \in \mathbb{Z}$. This means that h' is a common factor of f and g in $\mathbb{Z}[x]$; our sought after contradiction. Therefore, f and g have no common factor in $\mathbb{Q}[x]$.

Now $\mathbb{Q}[x]$ is a PID since $\mathbb{Q}$ is a field. So there are $j, k \in \mathbb{Q}[x]$ satisfying $jf + kg = 1$. Clearing the denominators in this equation we get a $0 \neq c \in \mathbb{Z}$ such that $(cj)f + (ck)g = c$, with $cj, ck \in \mathbb{Z}[x]$. This means that $(f, g) \cap \mathbb{Z} \neq (0)$, and hence $\mathfrak{p} \cap \mathbb{Z} = (p)$ is a non-zero prime ideal in $\mathbb{Z}$. But every nonzero prime ideal of $\mathbb{Z}$ is a maximal ideal. Choose $d \in \mathfrak{p} - p\mathbb{Z}[x]$.

1.17. **For $f \in A$ let $X_f = \operatorname{Spec}(A) - V(f)$. Show that $\{X_f : f \in A\}$ forms a basis of $X = \operatorname{Spec}(A)$.**

Each X_f is clearly open. Now if $X - V(E)$ is a general open set then

$$X - V(E) = X - V\Big(\bigcup_{f \in E} \{f\}\Big) = X - \bigcap_{f \in E} V(f) = \bigcup_{f \in E} X_f$$

We conclude that $\{X_f : f \in X\}$ is a basis for $\operatorname{Spec}(X)$.

a. **Show that $X_f \cap X_g = X_{fg}$ for all f, g.**

The equalities

$$X - V(fg) = X - V((f) \cap (g)) = X - V((f)) \cup V((g)) = (X - V(f)) \cap (X - V(g))$$

give us the result immediately.

b. **Show that $X_f = \emptyset$ iff f is nilpotent.**

$X_f = \emptyset$ precisely when f is contained in every prime ideal in A. This occurs precisely when f is in the nilradical of A, and hence precisely when f is nilpotent.

c. **Show that $X_f = X$ iff f is a unit in A.**

If f is a unit, then f is not contained in any prime ideal, and so $X_f = X$. If f is a nonunit, then f is contained in some maximal ideal, and hence $X_f \neq X$.

d. **Show that $X_f = X_g$ iff $r(f) = r(g)$.**

If $r(f) = r(g)$ then $V(f) = V(r(f)) = V(r(g)) = V(g)$ so that $X_f = X_g$. Suppose that $X_f = X_g$. Then every prime ideal containing f contains g, and vice versa. But $r(f)$ is the intersection of all prime ideals containing f, and similarly for g. So $r(f) = r(g)$.

e. **Show that $\operatorname{Spec}(A)$ is compact.**

Suppose $X = \bigcup U_\alpha$ with each U_α open, and write $U_\alpha = \bigcup_{\beta \in J_\alpha} X_{f_{\alpha,\beta}}$. Then $X = \bigcup X_{f_{\alpha,\beta}}$ so that $\emptyset = \bigcap V(f_{\alpha,\beta}) = V(\bigcup f_{\alpha,\beta})$. This means that $\{f_{\alpha,\beta}\}$ generates A. So we can write $1 = \sum a_{\alpha,\beta} f_{\alpha,\beta}$ with cofinitely many of the $a_{\alpha,\beta}$ non-zero. Working backwards, we see that X is the union of the $X_{f_{\alpha,\beta}}$ for which $a_{\alpha,\beta} \neq 0$. So in turn, X is the union of finitely many U_α. Thus, X is compact.

f. **Show that each X_f is compact.**

Suppose that $X_f \subseteq \bigcup U_\alpha$ and write $U_\alpha = \bigcup_{\beta \in J_\alpha} X_{g_{\alpha,\beta}}$. Then $X_f \subseteq \bigcup X_{g_{\alpha,\beta}}$. This gives us $V(\bigcup g_{\alpha,\beta}) \subseteq V(f)$. Suppose $\mathfrak{a}$ is the ideal generated by the $g_{\alpha,\beta}$. Then $f \in r(\mathfrak{a})$, so that there is an equation $f^n = \sum a_{\alpha,\beta} g_{\alpha,\beta}$ with cofinitely many of the $a_{\alpha,\beta}$ non-zero. Let $g_1, \ldots, g_n$ be the $g_{\alpha,\beta}$ with $a_{\alpha,\beta} \neq 0$. Then $V(\bigcup_1^n g_i) \subseteq V(f^n) = V(f)$ so that $X_f \subseteq \bigcup_1^n X_{g_i}$. It follows that X_f is the union of finitely many U_α. Thus, X_f is compact.

g. **Show that an open subspace of X is compact if and only if it is the union of finitely many of the basic open sets X_f.**

Clearly, the union of finitely many X_f is open and compact. So suppose $\mathcal{U}$ is compact and open. Then since $\mathcal{U}$ is the union of some X_f, it is the union of finitely many X_f.

1.18. **Show the following about $X = \mathrm{Spec}(A)$.**

a. **The set $\{\mathfrak{p}\}$ is closed iff $\mathfrak{p}$ is a maximal ideal.**

If $\mathfrak{p}$ is a maximal ideal, then $V(\mathfrak{p}) = \{\mathfrak{p}\}$, and so $\{\mathfrak{p}\}$ is closed. If $\{\mathfrak{p}\}$ is closed then $\{\mathfrak{p}\} = V(E)$ for some $E \subsetneq A$. Let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{p}$ so that $\mathfrak{m} \in V(E)$. Then $\mathfrak{m} = \mathfrak{p}$, so that $\mathfrak{p}$ is a maximal ideal.

b. **Cl**$(\{\mathfrak{p}\}) = V(\mathfrak{p})$

Notice that $\mathrm{Cl}(\mathfrak{p}) \subseteq V(\mathfrak{p})$ since $V(\mathfrak{p})$ is a closed set containing $\mathfrak{p}$ and $\mathrm{Cl}(\mathfrak{p})$ is the intersection of all closed sets containing $\mathfrak{p}$. Conversely, suppose that $\mathfrak{q}$ is a prime ideal not in $\mathrm{Cl}(\mathfrak{p})$, and choose a neighborhood U of $\mathfrak{q}$ that does not intersect $\{\mathfrak{p}\}$. Then there is $E \subset A$ for which $X - U = V(E)$. Consequently, $\mathfrak{p} \in V(E)$ and $\mathfrak{q} \notin V(E)$. Since $\mathfrak{p}$ contains E and $\mathfrak{q}$ does not, we conclude in particular that $\mathfrak{q}$ does not contain $\mathfrak{p}$. This means that $\mathfrak{q} \notin V(\mathfrak{p})$. So $\mathrm{Cl}(\mathfrak{p}) = V(\mathfrak{p})$.

c. $\mathfrak{q} \in$ **Cl**$(\{\mathfrak{p}\})$ **if and only if** $\mathfrak{p} \subseteq \mathfrak{q}$**.**

Obvious from part b.

d. **X is a T_0 space.**

Suppose that $\mathfrak{p} \neq \mathfrak{q}$. If $\mathfrak{p} \subsetneq \mathfrak{q}$ then $X - V(\mathfrak{q})$ is an open set containing $\mathfrak{p}$ but not containing $\mathfrak{q}$; otherwise $\mathfrak{p} \not\subseteq \mathfrak{q}$ and hence $X - V(\mathfrak{p})$ is an open set containing $\mathfrak{q}$ but not containing $\mathfrak{p}$.

1.19. **Show that $\mathrm{Spec}(A)$ is an irreducible topological space iff $\mathfrak{N}(A)$ is a prime ideal in A.**

Suppose that $\mathfrak{N}(A)$ is not a prime ideal. Then there are $f, g \in A$ for which $fg \in \mathfrak{N}(A)$ and yet $f, g \notin \mathfrak{N}(A)$. Since f and g are not nilpotent, we see that X_f and X_g are nonempty open sets. But $X_f \cap X_g = X_{fg} = \emptyset$ since fg is nilpotent. Hence, $\mathrm{Spec}(A)$ is not irreducible.

Suppose that $\mathrm{Spec}(A)$ is not irreducible. Choose nonempty open U, V for which $U \cap V = \emptyset$. Then there are f, g for which $\emptyset \neq X_f \subseteq U$ and $\emptyset \neq X_g \subseteq V$. So fg is nilpotent since $X_{fg} = X_f \cap X_g = \emptyset$. But neither f nor g is nilpotent. This means that $\mathfrak{N}(A)$ is not a prime ideal.

1.20. **Let X be a general topological space. Prove the following.**

a. **If Y is an irreducible subspace of X, then the closure $\bar{Y}$ of Y in X is irreducible.**

Suppose U and V are open in X, and that $U \cap \bar{Y}$ and $V \cap \bar{Y}$ are nonempty. Choose $x \in U \cap \bar{Y}$. Since U is a neighborhood of x, and since $x \in \bar{Y}$, we see that U intersects Y nontrivially. So $U \cap Y$, and similarly $V \cap Y$, are nonempty. Since Y is irreducible, $U \cap Y$ intersects $V \cap Y$ nontrivially, and therefore $U \cap \bar{Y}$ intersects $V \cap \bar{Y}$ nontrivially. Hence, $\bar{Y}$ is irreducible as well.

b. **Every irreducible subspace of X is contained in a maximal irreducible subspace.**

Suppose that Σ consists of all irreducible subspaces of X and that Σ is partially ordered by inclusion. Let $C = \{Y_\alpha : \alpha \in I\}$ be an ascending chain in Σ. Define $Y = \bigcup_{\alpha \in I} Y_\alpha$, and suppose that U, V open in X are such that $U \cap Y$ and $V \cap Y$ are nonempty. There are α, β for which $U \cap Y_\alpha$ and $V \cap Y_\beta$ are nonempty. We may assume that $\alpha \leq \beta$. Notice then that $U \cap Y_\beta \supseteq U \cap Y_\alpha$ is nonempty. Since Y_β is irreducible, we conclude that $U \cap Y_\beta$ and $V \cap Y_\beta$ intersect nontrivially. But then $U \cap Y$ and $V \cap Y$ intersect nontrivially. That is, Y is irreducible. So by Zorn's Lemma, Σ has maximal elements. Thus, every irreducible subspace of X is contained in a maximal irreducible subspace of X.

c. **The maximal irreducible subspaces of X are closed and cover X. What are the irreducible components of a Hausdorff space?**

If Y is a maximal irreducible subspace of X, then $Y = \bar{Y}$ since $\bar{Y}$ is irreducible. In other words, Y is closed. If $x \in X$, then $\{x\}$ is irreducible, and so x is contained in some maximal irreducible subspace of X. This means that X is covered by the irreducible components.

If X is a Hausdorff space and $Y \subseteq X$ contains two distinct points x and y, then we can choose disjoint open U and V for which $x \in U$ and $y \in V$. Then $U \cap Y$ and $V \cap Y$ are nonempty disjoint open sets in Y, implying that Y is not irreducible. So the irreducible components of a Hausdorff space are precisely the one point sets.

d. **The irreducible components of** $\mathrm{Spec}(A)$ **are of the form** $V(\mathfrak{p})$ **for some minimal prime ideal** $\mathfrak{p}$**.**

Let $\mathfrak{p}$ be a prime ideal and suppose $f \in A$. Then $X_f \cap V(\mathfrak{p}) \neq \emptyset$ if and only if $f \notin \mathfrak{q}$ for some prime ideal $\mathfrak{q} \supseteq \mathfrak{p}$, and this occurs if and only if $f \notin \mathfrak{p}$. Now assume that $X_f \cap V(\mathfrak{p})$ and $X_g \cap V(\mathfrak{p})$ are nonempty open subsets of $V(\mathfrak{p})$. Then $f, g \notin \mathfrak{p}$ so that $fg \notin \mathfrak{p}$, and hence

$$\mathfrak{p} \in X_{fg} \cap V(\mathfrak{p}) = (X_f \cap V(\mathfrak{p})) \cap (X_g \cap V(\mathfrak{p}))$$

This means that $V(\mathfrak{p})$ is an irreducible subspace of $\mathrm{Spec}(A)$. Now any irreducible subspace of $\mathrm{Spec}(A)$ is of the form $V(r(\mathfrak{a}))$ for some ideal $\mathfrak{a}$. Suppose $r(\mathfrak{a})$ is not prime. Then there are $f, g \notin r(\mathfrak{a})$ for which $fg \in r(\mathfrak{a})$. So there is $\mathfrak{p} \in V(\mathfrak{a})$ not containing f and there is $\mathfrak{q} \in V(\mathfrak{a})$ not containing g. This means that $X_f \cap V(r(\mathfrak{a}))$ and $X_g \cap V(r(\mathfrak{a}))$ are nonempty. But $X_{fg} \cap V(r(\mathfrak{a})) = \emptyset$ since every prime ideal containing $r(\mathfrak{a})$ contains fg. Hence, $V(r(\mathfrak{a}))$ is not irreducible. So the irreducible subspaces of X are precisely of the form $V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$. Further, $V(\mathfrak{p})$ is maximal among all sets of the form $V(\mathfrak{q})$, where $\mathfrak{q}$ is prime, if and only if $\mathfrak{p}$ is a minimal prime ideal. So we are done.

1.21. **Let $\phi : A \to B$ be a ring homomorphism, with $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. Define $\phi^* : Y \to X$ by $\phi^*(\mathfrak{q}) = \phi^{-1}(\mathfrak{q})$. Prove the following.**

a. **If $f \in A$ then ${\phi^*}^{-1}(X_f) = Y_{\phi(f)}$ and so ϕ^* is continuous.**

Notice that ${\phi^*}^{-1}(X_f)$ consists of all $\mathfrak{q} \in Y$ for which $f \notin \phi^{-1}(\mathfrak{q})$. Also, $Y_{\phi(f)}$ consists of all $\mathfrak{q} \in Y$ for which $\phi(f) \notin \mathfrak{q}$. But $\phi(f) \in \mathfrak{q}$ if and only if $f \in \phi^{-1}(\mathfrak{q})$, and so ${\phi^*}^{-1}(X_f) = Y_{\phi(f)}$. In turn, this implies that ϕ^* is continuous since $\{X_f | f \in A\}$ is a basis of X and ${\phi^*}^{-1}(X_f)$ is open for every $f \in A$.

b. **If $\mathfrak{a}$ is an ideal in A and then ${\phi^*}^{-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$.**

The following long chain of equalities

$$\begin{aligned}
{\phi^*}^{-1}(V(\mathfrak{a})) &= {\phi^*}^{-1}(V(\cup_{x\in\mathfrak{a}}\{x\})) \\
&= {\phi^*}^{-1}(\cap_{x\in\mathfrak{a}}V(x)) \\
&= \cap_{x\in\mathfrak{a}}{\phi^*}^{-1}(V(x)) \\
&= \cap_{x\in\mathfrak{a}}{\phi^*}^{-1}(X - X_x) \\
&= \cap_{x\in\mathfrak{a}}[Y - {\phi^*}^{-1}(X_x)] \\
&= \cap_{x\in\mathfrak{a}}[Y - Y_{\phi(x)}] \\
&= \cap_{x\in\mathfrak{a}}V(\phi(x)) \\
&= V(\phi(\mathfrak{a})) \\
&= V(\mathfrak{a}^e)
\end{aligned}$$

gives us the desired result.

c. **If $\mathfrak{b}$ is an ideal in B then $\mathrm{Cl}(\phi^*(V(\mathfrak{b}))) = V(\mathfrak{b}^c)$.**

Any $\mathfrak{p} \in \phi^*(V(\mathfrak{b}))$ is of the form $\mathfrak{q}^c$ for some $\mathfrak{q} \supseteq \mathfrak{b}$. Then $\mathfrak{p} \supseteq \mathfrak{b}^c$, so that $\phi^*(V(\mathfrak{b})) \subseteq V(\mathfrak{b}^c)$, and hence

$$\mathrm{Cl}(\phi^*(V(\mathfrak{b}))) \subseteq \mathrm{Cl}(V(\mathfrak{b}^c)) = V(\mathfrak{b}^c)$$

On the other hand, suppose $\mathfrak{p} \in V(\mathfrak{b}^c)$ and that X_f is a basic open set in X containing $\mathfrak{p}$. Then $\mathfrak{b}^c \subseteq \mathfrak{p}$ and $f \notin \mathfrak{p}$ so that $f \notin r(\mathfrak{b}^c) = r(\mathfrak{b})^c$. Hence, $\phi(f) \notin r(\mathfrak{b})$, implying the existence of a prime ideal $\mathfrak{q} \in V(\mathfrak{b})$ for which $\phi(f) \notin \mathfrak{q}$. Then $f \notin \phi^*(\mathfrak{q})$ and so $\phi^*(\mathfrak{q}) \in X_f$. This means that $\phi^*(V(\mathfrak{b})) \cap X_f \neq \emptyset$, so that $\mathfrak{p} \in \mathrm{Cl}(\phi^*(V(\mathfrak{b})))$. Thus $\mathrm{Cl}(\phi^*(V(\mathfrak{b}))) = V(\mathfrak{b}^c)$.

d. **If ϕ is surjective then ϕ^* is a homeomorphism of Y onto the closed subset $V(\mathrm{Ker}(\phi))$ of X. In particular, $\mathrm{Spec}(A)$ and $\mathrm{Spec}(A/\mathfrak{N}(A))$ are naturally isomorphic.**

If $\mathfrak{q} \in Y$, then $\phi^*(\mathfrak{q})$ contains $\mathrm{Ker}(\phi)$. If $\mathfrak{p} \in V(\mathrm{Ker}(\phi))$ then $\mathfrak{p}/\mathrm{Ker}(\phi)$ is isomorphic with a prime ideal $\mathfrak{q}$ of Y, under the isomorphism $\bar{\phi} : A/\mathrm{Ker}(\phi) \to B$. Thus, $\mathfrak{p} = \phi^*(\mathfrak{q})$ so that ϕ^* maps Y onto $V(\mathrm{Ker}(\phi))$. Now if $\phi^*(\mathfrak{p}) = \phi^*(\mathfrak{q})$, then $\phi^{-1}(\mathfrak{p}) = \phi^{-1}(\mathfrak{q})$, and so $\mathfrak{p} = \mathfrak{q}$ since ϕ is surjective. So ϕ^* is injective. We already know by part a that ϕ^* is continuous. To show that ϕ^* is a homeomorphism it suffices to show that ϕ^{-1} is continuous. To do this, it suffices to show that ϕ^* is a closed map. By part c we know that $\phi^*(V(\mathfrak{b})) \subseteq V(\mathfrak{b}^c)$ for any ideal $\mathfrak{b}$ in Y. If $\mathfrak{p} \in V(\mathfrak{b}^c)$ then $\phi(\mathfrak{p}) \supseteq \phi(\mathfrak{b}^c) = \mathfrak{b}$ by surjectivity of ϕ, and $\phi(\mathfrak{p}) \in Y$. But then $\mathfrak{p} = \phi^*(\phi(\mathfrak{p})) \in \phi^*(V(\mathfrak{b}))$. So $\phi^*(V(\mathfrak{b})) = V(\mathfrak{b}^c) = \mathrm{Cl}(\phi^*(V(\mathfrak{b})))$ by part c. Hence, ϕ^* is indeed a closed map. So ϕ^* is a homeomorphism between Y and $V(\mathrm{Ker}(\phi))$.

Finally, the natural homomorphism $A \to A/\mathfrak{N}(A)$ is surjective with kernel $\mathfrak{N}(A)$. Therefore, $\mathrm{Spec}(A/\mathfrak{N}(A)$ is homeomorphic with $V(\mathfrak{N}(A)) = \mathrm{Spec}(A)$.

e. **The image $\phi^*(Y)$ of Y is dense in X if and only if $\mathrm{Ker}(\phi) \subseteq \mathfrak{N}(A)$.**

Notice that $\mathrm{Cl}(\phi^*(Y)) = \mathrm{Cl}(\phi^*(V(0))) = V(0^c) = V(\mathrm{Ker}(\phi))$. Consequently, $\phi^*(Y)$ is dense in X if and only if $V(\mathrm{Ker}\,\phi) = X$, and this occurs precisely when $\mathrm{Ker}(\phi) \subseteq \mathfrak{N}(A)$, and in particular when ϕ is 1-1.

f. **Let $\psi : B \to C$ be another ring homomorphism. Show that $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.**

We have $(\psi \circ \phi)^*(\mathfrak{r}) = (\psi \circ \phi)^{-1}(\mathfrak{r}) = \phi^{-1}(\psi^{-1}(\mathfrak{r})) = \phi^*(\psi^*(\mathfrak{r}))$ for every $\mathfrak{r} \in \mathrm{Spec}(C)$.

g. **Let A be an integral domain with only one nonzero prime ideal $\mathfrak{p}$, and suppose that K is the field of fractions of A. Define $B = (A/\mathfrak{p}) \times K$ and let $\phi : A \to B$ by $\phi(x) = (\bar{x}, x)$. Show that ϕ^* is bijective but not a homeomorphism.**

First, $A/\mathfrak{p}$ is a field since $\mathfrak{p}$ is a maximal ideal in A. Now let $\mathfrak{q}_1$ consist of all $(\bar{x}, 0) \in B$ and let $\mathfrak{q}_2$ consist of all $(0, x) \in B$. Then $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are maximal ideals in B since $B/\mathfrak{q}_1 \cong K$ and $B/\mathfrak{q}_2 \cong A/\mathfrak{p}$. If $\mathfrak{q}$ is another prime ideal of B, then $\mathfrak{q}_1\mathfrak{q}_2 = 0$ is contained in $\mathfrak{q}$, and so $\mathfrak{q}_1 \subseteq \mathfrak{q}$ or $\mathfrak{q}_2 \subseteq \mathfrak{q}$. So $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are the only prime ideals of B. Hence, $\mathrm{Spec}(A) = \{0, \mathfrak{p}\}$ and $\mathrm{Spec}(B) = \{\mathfrak{q}_1, \mathfrak{q}_2\}$ are two-point spaces. It is easy to see that $\phi^*(\mathfrak{q}_1) = 0$ and $\phi^*(\mathfrak{q}_2) = \mathfrak{p}$, so that ϕ^* is a bijection. But ϕ^* is not a homeomorphism. After all, $\mathrm{Spec}(B)$ is Hausdorff since all prime ideals are maximal, but $\mathrm{Spec}(A)$ is not Hausdorff since 0 is a non-maximal prime ideal.

1.22. **Suppose that $A_1, \ldots, A_n$ are rings and $A = \prod_{j=1}^n A_j$. Show that $\mathrm{Spec}(A)$ is the disjoint union of open (and closed) subspaces X_j, where X_j is canonically homeomorphic with $\mathrm{Spec}(A_j)$.**

Let $\pi_j : A \to A_j$ and $i_j : A_j \to A$ be the canonical maps. If $\mathfrak{q}$ is a prime ideal in A_j, then $\pi_j^{-1}(\mathfrak{q})$ is a prime ideal in A. Conversely, suppose $\mathfrak{p}$ is a prime ideal in A. Define $e_j = i_j(1_{A_j})$ so that $\sum_1^n e_j = 1_A$ and $e_j e_k = 0$ if $j \neq k$. Some $e_{j^*} \notin \mathfrak{p}$ since $\mathfrak{p} \neq A$. For $j \neq j^*$ we have $e_j e_{j^*} = 0 \in \mathfrak{p}$ so that $e_j \in \mathfrak{p}$. From this we see that $\mathfrak{p} = \pi_{j^*}^{-1}(\mathfrak{q})$ for some ideal $\mathfrak{q}$ in A_{j^*}, and it is easy to see that $\mathfrak{q}$ is a prime ideal in A_{j^*}.

Therefore, $\mathrm{Spec}(A)$ is the disjoint union of the subsets X_j, where X_j is the set of all $\pi_j^{-1}(\mathfrak{q})$, where $\mathfrak{q}$ is a prime ideal in A_j. Notice that each X_j is closed since $X_j = V(\pi_j^{-1}(0))$. This also shows that each X_j is open since $X_j = \bigcap_{k \neq j} X_k^c$. Since π_j is surjective, exercise 1.22 tells us that $\pi_j^* : \mathrm{Spec}(A_j) \to \mathrm{Spec}(A)$ is a homeomorphism of $\mathrm{Spec}(A_i)$ onto $V(\mathrm{Ker}(\pi_j)) = V(\pi_j^{-1}(0)) = X_j$. In particular, X_j and $\mathrm{Spec}(A_j)$ are canonically homeomorphic.

Conversely, prove that the following are equivalent for any ring A. Deduce that the spectrum of a local ring is always connected.

a. $X = \mathrm{Spec}(A)$ **is disconnected.**
b. $A \cong A_1 \times A_2$ **where A_1 and A_2 are nonzero rings.**
c. A **has an idempotent $e \neq 0, 1$.**

$(a \Rightarrow c)$ We can write $X = V(\mathfrak{a}) \coprod V(\mathfrak{b})$ where $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in A. Then $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b}) = X$ implying that $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{N}(A)$. Also, $\emptyset = V(\mathfrak{a}) \cap V(\mathfrak{b}) = V(\mathfrak{a} \cup \mathfrak{b})$, implying that $A = \langle \mathfrak{a} \cup \mathfrak{b} \rangle$, and hence $A = \mathfrak{a} + \mathfrak{b}$. Now write $1 = a + b$ with $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Notice that $ab \in \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{N}(A)$ so that $(ab)^n = 0$ for some $n > 0$. Now $1 = (a+b)^n = a^n + b^n + abx$ for some $x \in A$. Since $abx \in \mathfrak{N}(A)$ we conclude that $a^n + b^n$ is a unit in A. Let u be the inverse of $a^n + b^n$ and notice that $ua^n b^n = 0$ so that $ua^n = ua^n(u(a^n + b^n)) = (ua^n)^2$ and similarly $ub^n = (ub^n)^2$. If $ua^n = 0$ then $a^n = 0$ and $1 = b(b^{-1} + ax) \in \mathfrak{b}$, which is not possible since $V(\mathfrak{b}) \neq \emptyset$. So ua^n and ub^n are nonzero. On the other hand, if $1 = ua^n = ub^n$ then $1 = u(a^n + b^n) = 2$ so that $1 = 0$. Hence, one of ua^n, ub^n is a nontrivial idempotent.

$(b \Rightarrow a)$ We already know that $X = X_1 \coprod X_2$ where $X_i = \mathrm{Spec}(A_i)$ is a non-empty open subset of X, since $A_i \neq 0$. So X is disconnected.

$(b \Rightarrow c)$ Take $e = (0, 1)$ or $e = (1, 0)$.

$(c \Rightarrow b)$ Define non-zero subrings of A by $A_1 = (e)$ and $A_2 = (1-e)$. Then $A = A_1 + A_2$ since $a = ae + a(1-e)$ for any $a \in A$. If $x \in A_1 \cap A_2$, then $x = ae$ and $x = b(1-e)$ for some a and b. But $ae = aee = b(1-e)e = 0$, and so $x = 0$. Therefore, $A \cong A_1 \times A_2$.

Exercise 1.12 shows that a local ring A has no idempotent $e \neq 0$ or 1, so that $\mathrm{Spec}(A)$ is always connected by the above.

1.23. **Let A be a Boolean ring. Prove the following.**

a. **For each $f \in A$, the set X_f is open and closed in $\mathrm{Spec}(A)$.**

By definition, $X_f = V(f)^c$ is open. If $\mathfrak{p}$ is a prime ideal, then $f \in \mathfrak{p}$ or $1 - f \in \mathfrak{p}$ since $f(1-f) = 0$. It follows from this that $X_f = V(1-f)$, so that X_f is closed in $\mathrm{Spec}(A)$.

b. **If $f_1, \ldots, f_n \in A$ then $X_{f_1} \cup \cdots \cup X_{f_n} = X_f$ for some $f \in A$.**

Choose f, as in exercise 1.11, so that $(f_1, \ldots, f_n) = (f)$. Then $V(f) = V(\bigcup_1^n (f_j)) = \bigcap_1^n V(f_j)$, implying that $X_f = \bigcup_1^n X_{f_j}$.

c. **If Y is both open and closed, then $Y = X_f$ for some $f \in A$.**

Since Y is closed in the compact space $\mathrm{Spec}(A)$, we see that Y itself is compact. Exercise 1.17 now says that Y is the union of finitely many sets of the form X_f. We now apply part b.

d. **$\mathrm{Spec}(A)$ is a compact Hausdorff space.**

Suppose that $\mathfrak{p}, \mathfrak{q}$ are distinct prime ideals in X. We may suppose that there is $f \in \mathfrak{p} - \mathfrak{q}$. Then $1 - f \in \mathfrak{q} - \mathfrak{p}$ since $f(1-f) = 0$. So X_{1-f} and X_f are open sets containing $\mathfrak{p}$ and $\mathfrak{q}$, respectively. These sets are disjoint since $X_{1-f} \cap X_f = X_{(1-f)f} = X_0 = \emptyset$. Therefore, X is compact Hausdorff.

1.24. **Show that every Boolean lattice becomes a Boolean ring, and that every Boolean ring becomes a Boolean lattice. Deduce that Boolean lattices and Boolean rings are equivalent.**

A lattice L is a partially ordered set such that, if a and b are in L, then there is an element $a \wedge b$ that is the largest element in the non-empty set $\{c \in L : c \leq a \text{ and } c \leq b\}$, and there is an element $a \vee b$ that is the smallest element in the non-empty set $\{c \in L : c \geq a \text{ and } c \geq b\}$. We say that L is Boolean provided that the following hold.

a. There is a smallest element 0 in L, and a largest element 1.
b. For $a, b, c \in L$ we have $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and also $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. In other words, we have distribution.
c. For each a there is a unique a' such that $a \wedge a' = 1$ and $a \vee a' = 0$.

Lets make a few observations about $\wedge$ and $\vee$. We first have

$$a \wedge 0 = 0 \qquad a \vee 0 = a \qquad a \wedge 1 = a \qquad a \vee 1 = 1$$

This implies that $0' = 1$ and $1' = 0$. Clearly $a'' = a$. We also have

$$a \wedge b = b \wedge a \qquad a \vee b = b \vee a \qquad a \wedge a = a \qquad a \vee a = a$$

We have the associativity relations

$$(a \wedge b) \wedge c = a \wedge (b \wedge c) \qquad (a \vee b) \vee c = a \vee (b \vee c)$$

We also have DeMorgan's Laws

$$(a \wedge b)' = a' \vee b' \qquad (a \vee b)' = a' \wedge b'$$

To prove the first of DeMorgan's Laws we note that

$$(a \wedge b) \wedge (a' \vee b') = (a \wedge b \wedge a') \vee (a \wedge b \wedge b') = 0 \vee 0 = 0$$

and also

$$(a \wedge b) \vee (a' \vee b') = (a \vee a' \vee b') \wedge (b \vee a' \vee b') = 1 \wedge 1 = 1$$

The first of Demorgan's Laws now follows from the uniqueness in b. The second of DeMorgan's Laws follows very similarly. Now for $a, b \in L$ we define operations of addition and multiplication by

$$a + b = (a \wedge b') \vee (a' \wedge b) \quad \text{and} \quad a \cdot b = a \wedge b$$

Notice that $a+0 = (a \wedge 1) \vee (a' \wedge 0) = a \vee 0 = a$ so that 0 is the additive identity in L. Addition is commutative since

$$\begin{aligned} b + a &= (b \wedge a') \vee (b' \wedge a) \\ &= (b' \wedge a) \vee (b \wedge a') \\ &= (a \wedge b') \vee (a' \wedge b) = a + b \end{aligned}$$

Every $a \in L$ has an additive inverse since $a + a' = (a \wedge a') \vee (a' \wedge a) = a \wedge a' = 0$ by definition of a'. Lastly, addition is associative. This is tedious to check, so I will not include that calculation. Notice that $a \cdot 1 = a \wedge 1 = a$ so that 1 is the multiplicative identity. Clearly, multiplication is commutative and associative. Lastly, multiplication distributes over addition since

$$\begin{aligned} a \cdot c + b \cdot c &= (a \wedge c) + (b \wedge c) \\ &= ((a \wedge c) \wedge (b \wedge c)') \vee ((a \wedge c)' \wedge (b \wedge c)) \end{aligned}$$

Summarizing, we see that L has a ring structure. L is a boolean ring since $a \cdot a = a \wedge a = a$. Now suppose that A is a Boolean ring. Define an ordering on A by $a \leq b$ if and only if $a = ab$. Then $\leq$ is reflexive since $a = a^2$. If $a \leq b$ and $b \leq a$ then $a = ab = ba = b$, so that $\leq$ is anti-symmetric. If $a \leq b$ and $b \leq c$ then $a = ab = abc = ac$ so that $a \leq c$, and hence $\leq$ is transitive. So A is partially ordered.

Now let a and b be arbitrary elements of A, and notice that $a, b \leq a+b+ab$ since $a(a+b+ab) = a+ab+ab = a$ and $b(a+b+ab) = ab+b+ab = b$. If $a \leq c$ and $b \leq c$, then $a = ac$ and $b = bc$, so that $(a+b+ab)c = a+b+ab$ and hence $a+b+ab \leq c$. This means that $\{c \in A : a, b \leq c\}$ is a non-empty set with $a+b+ab$ as its smallest element. So define $a \vee b = a + b + ab$.

Again let a and b be arbitrary elements of A, and notice that $ab \leq a$ and $ab \leq b$. If $c \leq a$ and $c \leq b$, then $c = ac$ and $c = bc$, so that $(ab)c = ac = c$ and hence $c \leq ab$. This means that $\{c \in A : c \leq a, b\}$ is a non-empty set with ab as its largest element. So define $a \vee b = a + b + ab$. Now that A is seen to be a lattice, I claim that A is a Boolean lattice. Notice that $0 \leq a \leq 1$ for every $a \in L$ since $0 = a0$ and $a = a1$. We see that $\vee$ and $\wedge$ distribute over one another since

$$
\begin{aligned}
a \vee (b \wedge c) &= a + (b \wedge c) + a(b \wedge c) \\
&= a + bc + abc \\
&= (a + 2ac) + (ab + bc + abc) + (ab + 2abc) \\
&= a(a + c + ac) + b(a + c + ac) + ab(a + c + ac) \\
&= (a + b + ab)(a + c + ac) \\
&= (a \vee b)(a \vee c) \\
&= (a \vee b) \wedge (a \vee c)
\end{aligned}
$$

and similarly $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Now define $a' = 1 - a$ so that $a \wedge a' = a(1-a) = 0$ and $a \vee a' = a + (1-a) + a(1-a) = 1$. If $b \in A$ satisfies $0 = a \wedge b$ and $1 = a \vee b = a + b + ab = a + b$, then $b = 1 - a = a'$. So a' is unique. Thus, A is indeed a Boolean lattice.

Now suppose that we started with a Boolean lattice $(L, \leq)$ and made it into a Boolean ring $(L, +, \cdot)$, then made this ring into a new Boolean lattice $(L, \preccurlyeq)$. If $a \leq b$ then $ab = a \wedge b = a$, so that $a \preccurlyeq b$. If $a \preccurlyeq b$ then $a = ab = a \wedge b$, so that $a \leq b$. Hence, $(L, \leq)$ and $(L, \preccurlyeq)$ are isomorphic Boolean lattices under the identity map $\mathrm{id} : L \to L$.

On the other hand, suppose we started with a ring $(A, +, \cdot)$ and made it into a Boolean lattice $(A, \leq)$, then made this Boolean lattice into a new Boolean ring $(A, \dot{+}, \times)$. Then $a \times b = a \wedge b = a \cdot b$ and

$$
\begin{aligned}
a \dot{+} b &= (a \wedge b') \vee (a' \wedge b) \\
&= (a \wedge (1 - b)) \vee ((1 - a) \wedge b) \\
&= a(1 - b) \vee (1 - a)b \\
&= a(1 - b) + (1 - a)b + a(1 - b)(1 - a)b \\
&= a + b
\end{aligned}
$$

Therefore, $(A, +, \cdot)$ and $(A, \dot{+}, \times)$ are isomorphic rings Boolean rings under the identity map $\mathrm{id} : A \to A$. Suppose $f : A \to B$ is a ring isomorphism of Boolean rings. Let $(A, \leq)$ and $(B, \preccurlyeq)$ be the resulting Boolean lattices. The bijection f is order-preserving since $a \leq b$ implies that $a = ab$, and hence $f(a) = f(a)f(b)$, implying that $f(a) \preccurlyeq f(b)$. This means that the two resulting lattices are isomorphic.

On the other hand, if $(L, \leq)$ and $(\bar{L}, \preccurlyeq)$ are two Boolean lattices, isomorphic under $f : L \to \bar{L}$, then let $(L, +, \cdot)$ and $(\bar{L}, +, \cdot)$ be the resulting Boolean rings. Notice that $f^{-1} : \bar{L} \to L$ is order-preserving as well. It follows easily that $f(a \wedge b) = f(a) \barwedge f(b)$ and $f(a \vee b) = f(a) \veebar f(b)$. So $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$. In other words, $(L, +, \cdot)$ and $(\bar{L}, +, \cdot)$ are isomorphic Boolean rings. Summarizing, there is a bijective correspondence between (isomorphism classes of) Boolean rings and (isomorphism classes of) Boolean lattices.

1.25. **Deduce Stone's Theorem, that every Boolean lattice is isomorphic to the lattice of open-and-closed subsets of some compact Hausdorff topological space.**

Suppose L is a Boolean lattice and make L into a Boolean ring A as in exercise 1.24. Then $X = \mathrm{Spec}(A)$ is a compact Hausdorff space. Let $\mathscr{L}$ consist of all subsets of X that are both open and closed. We order $\mathscr{L}$ by set-theoretic inclusion. $\mathscr{L}$ is clearly a partially ordered set. If $Y, Y' \in \mathscr{L}$ then $Y \cup Y', Y \cap Y' \in \mathscr{L}$ so that $\mathscr{L}$ is a lattice. The emptyset $\emptyset$ is the smallest element in $\mathscr{L}$ and full space X is the largest element of $\mathscr{L}$. Also, if $Y \in \mathscr{L}$ then Y^c is an open and closed subset of X, with $Y \cap Y^c = \emptyset$ and $Y \cup Y^c = X$, with Y^c uniquely determined by these equations. This means that $\mathscr{L}$ is in fact a Boolean lattice. Exercise 1.23 tells us that $Y \in \mathscr{L}$ if and only if $Y = X_f$ for some $f \in L$. So we have a surjective map $\psi : L \to \mathscr{L}$ given by $\psi(f) = X_f$. If $f \leq g$ then $f = fg$ so that $X_f = X_f \cap X_g$ and hence $X_f \subseteq X_g$. This means that ψ is an order-preserving map. On the other hand, if $X_f = X_g$ then

$$\emptyset = X_{1-f} \cap X_f = X_{1-f} \cap X_g = X_{(1-f)g}$$

so that $(1-f)g \in \mathfrak{N}(A)$. But then $0 = [(1-f)g]^n$ for some $n > 0$ so that $(1-f)g = 0$, and hence $g = fg$. Similarly, $f = fg$ and hence $f = g$. This shows that ψ is an isomorphism of lattices.

1.26. **Let X be a compact Hausdorff space, let $C(X)$ consists of all continuous real-valued functions defined on X, and define $\tilde{X}$ as the set of all maximal ideals in $C(X)$. We have a map $\mu : X \to \tilde{X}$ given by $x \mapsto \mathfrak{m}_x$, where $\mathfrak{m}_x$ consists of all $f \in C(X)$ that vanish at the point x. Prove the following.**

a. **The map μ is surjective.**

Suppose that $\mathfrak{m}$ is a maximal ideal in $C(X)$. Let V consist of all $x \in X$ such that $f(x) = 0$ whenever $f \in \mathfrak{m}$. If V is nonempty and $x \in V$, then $\mathfrak{m} \subseteq \mathfrak{m}_x$, and so $\mathfrak{m} = \mathfrak{m}_x = \mu(x)$ by maximality. So assume that V is empty. Then given $x \in X$ there is $f \in \mathfrak{m}$ for which $f(x) \neq 0$. By continuity, there is a neighborhood U_x of x on which f_x is nonzero. These neighborhoods cover X since $V = \emptyset$, and so by compactness there are $\{x_i\}_1^n$ so that $X = \bigcup_1^n U_{x_i}$. Let $f = \sum_1^n f_{x_i}^2$ and notice that f is a continuous function that is everywhere positive. But then f is a unit in $C(X)$, having multiplicative inverse $1/f$, and so $\mathfrak{m} = C(X)$; a contradiction. Therefore, V is nonempty and $\mathfrak{m} = \mu(x)$ for some $x \in V$.

b. **The map μ is injective.**

Recall that every compact Hausdorff space is normal. Let x, y be distinct points of X. Since $\{x\}$ and $\{y\}$ are disjoint closed sets, we can apply Urysohn's Lemma to deduce the existence of an $f \in C(X)$ for which $f(x) = 0$ and $f(y) = 1$. Then $f \in \mathfrak{m}_x$ and $f \notin \mathfrak{m}_y$. So $\mathfrak{m}_x \neq \mathfrak{m}_y$. This shows that μ is injective.

c. **The bijection μ is a homeomorphism when $\tilde{X}$ is given the subspace topology of $\mathrm{Spec}(C(X))$.**

Suppose $f \in C(X)$ and define $U_f = f^{-1}(\mathbf{R}^*)$ and $\tilde{U}_f = \{\mathfrak{m} \in \tilde{X} : f \notin \mathfrak{m}\}$. Every $\mathfrak{m} \in \tilde{X}$ is of the form $\mathfrak{m}_x$ for a unique $x \in X$. So $f \in \mathfrak{m}$ if and only if $f(x) = 0$. It follows that $\mu(U_f) = \tilde{U}_f$.

Now U_f is open in X since f is continuous. So suppose that $U \subseteq X$ is open and that $x \in U$. By normality there is a neighborhood V of x such that $\mathrm{Cl}(V) \subseteq U$. By Urysohn's Lemma there is $f \in C(X)$ such that $f(\mathrm{Cl}(V)) = \{1\}$ and $f(X \setminus U) = \{0\}$. But then $U_f \subseteq \mathrm{Cl}(V) \subseteq U$. This shows that $\{U_f\}_{f \in C(X)}$ is a basis for the topology on X.

Notice that $\tilde{U}_f = \tilde{X} \cap X_f$ is open in subspace topology. This also shows that $\{\tilde{U}_f\}_{f \in C(X)}$ is a basis for the topology of $\tilde{X}$ since $\{X_f\}_{f \in C(X)}$ is a basis for the topology of $\mathrm{Spec}(X)$ by exercise 1.17.

Now the fact that μ takes basis elements to basis elements shows that μ is a homeomorphism. Consequently, X and $\tilde{X}$ are homeomorphic topological spaces.

1.27. **Let k be an algebraically closed field and X an affine variety in k^n. Show that there is a natural bijection between the elements of X and the maximal ideals of $P(X)$, where $P(X) = k[t_1, \ldots, t_n]/I(X)$ is the coordinate ring of X.**

Let $x \in X$ and consider the map $k[t_1, \ldots, t_n] \to k$ given by $f \mapsto f(x)$. That is, consider the map given by evaluation at x. This map is surjective since $k[t_1, \ldots, t_n]$ contains all of the constant functions. If $f - g \in I(X)$ then $f(x) = g(x)$ since $x \in X$, and so the map $k[t_1, \ldots, t_n] \to k$ induces a surjective map $P(X) \to k$. The kernel of this map is a maximal ideal, denoted by $\mathfrak{m}_x$. We now have a map $\mu : X \to \mathrm{Max}(P(X))$ given by $\mu(x) = \mathfrak{m}_x$. If $\mathfrak{m}_x = \mathfrak{m}_y$ and $x = (x_1, \ldots, x_n)$ while $y = (y_1, \ldots, y_n)$, then $t_i - x_i \in \mathfrak{m}_y$ for every i as

$t_i - x_i \in \mathfrak{m}_x$ for every i. But this means that $y_i - x_i = 0$ and so $y_i = x_i$ for all i, so that $x = y$. In other words, μ is injective. The less trivial part of this exercise is showing that μ is surjective. So let $\mathfrak{m}$ be a maximal ideal in $P(X)$. Then $\mathfrak{m} = \mathfrak{n}/I(X)$ where $\mathfrak{n}$ is a maximal ideal in $k[t_1, \ldots, t_n]$ containing $I(X)$. Since k is algebraically closed, the Weak Nullstellensatz implies that $\mathfrak{n} = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_i \in k$. Suppose $(a_1, \ldots, a_n) \notin X$. Since X is an affine variety, we can easily verify that $x \in X$ if and only if $f(x) = 0$ for every $f \in I(X)$. So there is some $f \in I(X)$ for which $f(a_1, \ldots, a_n) \neq 0$. Since every $g \in \mathfrak{n}$ satisfies $g(a_1, \ldots, a_n) = 0$, we see that $f \notin \mathfrak{n}$; a contradiction. Therefore, $(a_1, \ldots, a_N) \in X$ and thus $\mathfrak{m} = \mu(a_1, \ldots, a_n)$, showing that μ is surjective. Hence, μ is a bijection between X and $\mathrm{Max}(P(X))$.

1.28? **Let X and Y be affine varieties in k^n and k^m. Show that there is a bijective correspondence Ψ between the regular mappings $X \to Y$ and the k-algebra homomorphisms $P(Y) \to P(X)$.**

By definition, $P(X)$ consists of all polynomial maps $X \to k$. There is a natural multiplication on $P(X)$ that makes $P(X)$ into a k-algebra. Suppose that $\phi : X \to Y$ is a regular mapping and that $\eta \in P(Y)$ so that $\eta \circ \phi \in P(X)$. Then $\eta \mapsto \eta \circ \phi$ is a k-linear map $P(Y) \to P(X)$. If $\eta, \theta \in P(Y)$ then

$$((\eta \circ \phi) \cdot (\theta \circ \phi))(x) = \eta(\phi(x))\theta(\phi(x)) = (\eta \cdot \theta)(\phi(x)) = ((\eta \cdot \theta) \circ \phi)(x)$$

This means that the map $P(Y) \to P(X)$ induced by ϕ is a k-algebra homomorphism. Now suppose that ϕ' induces the same k-algebra homomorphism $P(Y) \to P(X)$ as ϕ. Let $\eta_i : Y \to k$ be the ith coordinate function on Y, so that $\eta_i \circ (\phi - \phi') = 0$ for all i. Then $\phi(x) = \phi'(x)$ for all $x \in X$. So Ψ is an injective map. Now suppose that $f : P(Y) \to P(X)$ is a k-algebra homomorphism. Define $f_i : X \to k$ by $f_i = f(\eta_i)$ where η_i is ith coordinate function on Y, and let $\phi : X \to k^m$ by $\phi = (f_1, \ldots, f_m)$.

Chapter 2 : Modules

2.1. **Show that $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$ is the zero ring if gcd$(m,n) = 1$.**

Choose integers s and t for which $sm + tn = 1$. Then the identity element of $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$ satisfies

$$[1]_m \otimes [1]_n = [sm+tn]_m \otimes [1]_n = [tn]_m \otimes [1]_n = tn \cdot [1]_m \otimes [1]_n = [1]_m \otimes tn \cdot [1]_n = 0$$

Therefore our whole ring $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n = 0$.

2.2. **Let A be a ring with ideal $\mathfrak{a}$ and A-module M. Show that $A/\mathfrak{a} \otimes_A M \cong M/\mathfrak{a}M$.**

Tensoring the short exact sequence of A-modules

$$0 \longrightarrow \mathfrak{a} \xrightarrow{j} A \xrightarrow{\pi} A/\mathfrak{a} \longrightarrow 0$$

with M yields the exact sequence of A-modules

$$\mathfrak{a} \otimes_A M \xrightarrow{j \otimes 1} A \otimes_A M \xrightarrow{\pi \otimes 1} A/\mathfrak{a} \otimes_A M \longrightarrow 0$$

Since the map $f : A \otimes M \to M$ given by $f(a \otimes m) = am$ is an isomorphism of A-modules, we can define $g = (\pi \otimes 1) \circ f^{-1} : M \to A/\mathfrak{a} \otimes M$. Then $\mathrm{Im}(g) = \mathrm{Im}(\pi \otimes 1) = A/\mathfrak{a} \otimes M$ and $\mathrm{Ker}(g) = f(\mathrm{Ker}(\pi \otimes 1)) = f(\mathrm{Im}(j \otimes 1)) = \mathfrak{a}M$. So we have an isomorphism $\bar{g} : M/\mathfrak{a}M \to A/\mathfrak{a} \otimes M$ of A-modules.

2.3. **Let $(A, \mathfrak{m}, k)$ be a local ring. Show that, if M and N are finitely generated A-modules satisfying $M \otimes_A N = 0$, then $M = 0$ or $N = 0$.**

For every A-module P define a k-vector space $P_k = k \otimes_A P$. Then P_k and $P/\mathfrak{m}P$ are isomorphic by exercise 2.2. Now suppose that M and N are finitely generated A-modules for which $M \otimes N = 0$, so that $(M \otimes N)_k = 0$. Then

$$\begin{aligned} M_k \otimes_k N_k &= (M \otimes_A k) \otimes_k (N \otimes_A k) \\ &\cong M \otimes_A (k \otimes_k (N \otimes_A k)) \\ &\cong M \otimes_A (k \otimes_k (k \otimes_A N)) \\ &\cong M \otimes_A ((k \otimes_k k) \otimes_A N) \cong (M \otimes_A N)_k \end{aligned}$$

Therefore $M_k \otimes_k N_k = 0$. Since M_k and N_k are k-vector spaces, we see that $M_k = 0$ or $N_k = 0$. So either $\mathfrak{m}M = M$ or $\mathfrak{m}N = N$. By Nakayama's lemma, either $M = 0$ or $N = 0$.

2.4. **Suppose M_i are A-modules and let $M = \bigoplus_i M_i$. Prove that M is flat iff each M_i is flat.**

I claim that, for every A-module N, the A-modules $N \otimes \bigoplus M_i$ and $\bigoplus (N \otimes M_i)$ are isomorphic. Define $\phi : N \times M \to \bigoplus (N \otimes M_i)$ by $\phi(n, (x_i)) = (n \otimes x_i)$. Then ϕ is A-bilinear and so induces a homomorphism $\Phi : N \otimes M \to \bigoplus (N \otimes M_i)$ for which $\Phi(n \otimes (x_i)) = (n \otimes x_i)$. Suppose now that $j_i : M_i \to M$ corresponds to canonical injection. The map $n \otimes x_i \mapsto n \otimes j_i(x_i)$ is a homomorphism of $N \otimes M_i$ into $N \otimes M$. Consequently, $\Psi : \bigoplus (N \otimes M_i) \to N \otimes M$ by $\Psi((n_i \otimes x_i)) = \sum n_i \otimes j_i(x_i)$ is a homomorphism. It is easy to show that Φ and Ψ are inverse to one another, and so are isomorphisms.

Suppose now that $f : N' \to N$ is injective and consider the mapping $f \otimes 1 : N' \otimes M \to N \otimes M$. As above, $N' \otimes M$ is isomorphic with $\bigoplus (N' \otimes M_i)$ under Ψ', and $\bigoplus (N \otimes M_i)$ is isomorphic with $N \otimes M$ under Φ.

Therefore, $f \otimes 1$ is injective if and only if the induced map $g = \Phi \circ (f \otimes 1_M) \circ \Psi'$ from $\bigoplus(N' \otimes M_i)$ to $\bigoplus(N \otimes M_i)$ is injective.

$$\begin{array}{ccc} N' \otimes \bigoplus_\alpha M_\alpha & \xrightarrow{f\otimes 1_M} & N \otimes \bigoplus_\alpha M_\alpha \\ \uparrow \Psi' & & \downarrow \Phi \\ \bigoplus_\alpha (N' \otimes M_\alpha) & \xrightarrow{g} & \bigoplus_\alpha (N \otimes M_\alpha) \end{array}$$

Notice that $g((n_\alpha \otimes x_\alpha)) = (f(n_\alpha) \otimes x_\alpha)$. Put differently $g = (f \otimes 1_\alpha)$ where 1_α is identity on M_α. Therefore, g is injective if and only if each of its coordinate functions $f \otimes 1_\alpha$ is injective. Hence, M is flat if and only if each M_α is flat.

2.5. **Prove that $A[x]$ is a flat A-module for every ring A.**

Let M_i be the A-submodule of $A[x]$ generated by x^k. Then $M_i = Ax^i \cong A$ so that M_i is flat. Consequently, $A[x]$ is a flat A-module since $A[x] = \bigoplus_0^\infty M_i$.

2.6. **For any A-module M, let $M[x]$ denote the set of all polynomials in x with coefficients in M. Then $M[x]$ is an $A[x]$-module. Show that $M[x] \cong A[x] \otimes_A M$ as $A[x]$-modules.**

It is clear that as A-modules $A[x] \cong \bigoplus_{i=0}^\infty Ax^i$. Therefore, we have the isomorphism of A-modules

$$A[x] \otimes_A M \cong \bigoplus_{i=0}^{\infty} (Ax^i \otimes_A M) \cong \bigoplus_{i=0}^{\infty} Mx^i = M[x]$$

Here the isomorphism θ is given by $\theta(\sum a_i x^i \otimes m) = \sum (a_i m) x^i$. All we have to do now is verify that θ is $A[x]$-linear. Omitting indices we compute

$$\begin{aligned} \theta\Big(\sum a_i' x_i \cdot \Big(\sum a_i x^i \otimes m\Big)\Big) &= \theta\Big(\Big(\sum a_i' x_i \cdot \sum a_i x^i\Big) \otimes m\Big) \\ &= \theta\Big(\sum x^n \sum a_i a_{n-i}' \otimes m\Big) \\ &= \sum \Big(\sum a_i a_{n-i}' m\Big) x^n \\ &= \sum a_i' x^i \cdot \sum (a_i m) x^i \\ &= \sum a_i' x^i \cdot \theta\Big(\sum a_i x^i \otimes m\Big) \end{aligned}$$

Hence, θ is an isomorphism of $A[x]$-modules.

2.7. **Let $\mathfrak{p}$ be a prime ideal in A and show that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. If $\mathfrak{m}$ is a maximal ideal in A, must $\mathfrak{m}[x]$ be a maximal ideal in $A[x]$?**

Is $\pi : A \to A/\mathfrak{p}$ denotes the natural map, then π induces a map $A[x] \to (A/\mathfrak{p})[x]$ given by $\sum a_k x^k \mapsto \sum \pi(a_k) x^k$. This map is surjective and has kernel $\mathfrak{p}[x]$. So $A[x]/\mathfrak{p}[x] \cong (A/\mathfrak{p})[x]$. But $(A/\mathfrak{p})[x]$ is an integral domain since $A/\mathfrak{p}$ is an integral domain. So $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. If $\mathfrak{m}$ is a maximal ideal in A, then $A[x]/\mathfrak{m}[x] \cong (A/\mathfrak{m})[x]$ with $A/\mathfrak{m}$ a non-zero field. So $(A/\mathfrak{m})[x]$ is not a field, implying that $\mathfrak{m}[x]$ is not a maximal ideal in $A[x]$.

2.8. **Suppose that M and N are flat A-modules. Show that $M \otimes_A N$ is a flat A-module.**

Let $\mathcal{S}_0$ be an exact sequence. We may tensor $\mathcal{S}_0$ with M to get an exact sequence $\mathcal{S}_1$, and we may tensor $\mathcal{S}_1$ with N to get an exact sequence $\mathcal{S}_2$. But the tensor product is associative, and so the sequence $\mathcal{S}_2$ is the same one as would have been obtained had we tensored $\mathcal{S}_0$ with $M \otimes_A N$. This shows that $M \otimes_A N$ is flat.

Let B be a flat A-algebra and N a flat B-module. Show that N is a flat A-module.

Let $\mathcal{S}_0$ be an exact sequence of A-modules. We may tensor $\mathcal{S}_0$ with B to get an exact sequence $\mathcal{S}_1$ of A-modules. This is an exact sequence of B-modules, since B is an (A, B)-bimodule. Tensoring this sequence with N yields an exact sequence $\mathcal{S}_2$ of B-modules. Also, $\mathcal{S}_2$ is an exact sequence of A-modules. So N is a flat A-module.

2.9. **Suppose we have the short exact sequence of A-modules**

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

with M' and M'' finitely generated. Show that M is finitely generated as well.

Suppose that M' is generated by $\{x_i\}$ and M'' is generated by $\{z_i\}$. Clearly $\mathrm{Im}(f)$ is generated by $\{f(x_i)\}$. Since g is surjective, there are $y_i \in M$ for which $g(y_i) = z_i$. Let N be the submodule of M generated by $\{y_i\}$, so that $g(N) = M''$. So for $y \in M$ there is $y' \in N$ with $g(y) = g(y')$, and hence $y = y' + (y - y')$ where $y - y' \in \mathrm{Ker}(g) = \mathrm{Im}(f)$. We conclude that M is generated by $\{f(x_i)\} \cup \{y_i\}$.

2.10. **Let A be a ring with the ideal $\mathfrak{a} \subseteq \mathfrak{R}(A)$. Suppose M is an A-module and N is a finitely generated A-module, with $u : M \to N$ a homomorphism. Show that u is surjective provided the induced homomorphism $\bar{u} : M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective.**

We define $\bar{u}$ by $\bar{u}(\bar{m}) = \overline{u(m)}$. We have the commutative diagram

$$\begin{array}{ccc} A/\mathfrak{a} \otimes M & \xrightarrow{\bar{a} \otimes m \mapsto \bar{a} \otimes u(m)} & A/\mathfrak{a} \otimes N \\ {\scriptstyle \bar{a}\otimes m \mapsto \overline{am}} \downarrow & & \downarrow {\scriptstyle \bar{a}\otimes n \mapsto \overline{an}} \\ M/\mathfrak{a}M & \xrightarrow{\bar{m} \mapsto \overline{u(m)}} & N/\mathfrak{a}N \end{array}$$

Define $L = N/\mathrm{Im}(u)$. We have an exact sequence $M \to N \to L \to 0$. We can tensor this with $A/\mathfrak{a}$ to get an exact sequence. Using the canonical isomorphism above we get the exact sequence

$$M/\mathfrak{a}M \xrightarrow{\bar{u}} N/\mathfrak{a}N \xrightarrow{\bar{\pi}} L/\mathfrak{a}L \longrightarrow 0$$

But $\bar{u}$ is surjective so that $\bar{\pi}$ is the zero map, and hence $L/\mathfrak{a}L = 0$. Nakayama's lemma yields $L = 0$. In other words, u is surjective, as claimed.

2.11. **Suppose A is a nonzero ring. Show that $m = n$ if A^m and A^n are isomorphic A-modules. Show that $m \geq n$ if A^n is a homomorphic image of A^m. Must $m \leq n$ if there is an injective homomorphism $A^m \to A^n$ of A-modules?**

Let $\mathfrak{m}$ be a maximal ideal in A with residue field $k = A/\mathfrak{m}$. If $\phi : A^m \to A^n$ is an isomorphism of A-modules, then $1 \otimes \phi : k \otimes_A A^m \to k \otimes_A A^n$ is an isomorphism of A-modules, and so is an isomorphism of k-vector spaces. These vector spaces have dimension m and n, respectively. We conclude that $m = n$. We prove similarly that $m \geq n$ if there is a surjection $A^m \to A^n$, and that $m \leq n$ if there is an injection $A^m \to A^n$.

2.12. **Let M be a finitely generated A-module and $\phi : M \to A^n$ a surjective A-module homomorphism. Show that $\mathrm{Ker}(\phi)$ is finitely generated.**

Let A^n be free on $\{e_1, \ldots, e_n\}$, and choose $u_i \in M$ so that $\phi(u_i) = e_i$. Then for $x \in M$ there are $a_i \in A$ satisfying $\phi(x) = \phi(\sum_1^n a_i u_i)$, and hence $x - \sum_1^n a_i u_i \in \text{Ker}(\phi)$. So if we let N be the submodule of M generated by $\{u_i\}$, then $M = N + \text{Ker}(\phi)$. Obviously $N \cap \text{Ker}(\phi) = 0$ since $0 = \phi(\sum_1^n a_i u_i) = \sum_1^n a_i e_i$ implies that each $a_i = 0$, and hence $\sum_1^n a_i u_i = 0$. Therefore, $M = N \oplus \text{Ker}(\phi)$. Now $\text{Ker}(\phi)$ is isomorphic with M/N, so that $\text{Ker}(\phi)$ is finitely generated.

2.13. **Let $f : A \to B$ be a ring homomorphism, and let N be a B-module. Regarding N as an A-module by restriction of scalars, form the B-module $N_B = B \otimes_A N$. Define $g : N \to N_B$ by $g(n) = 1 \otimes n$. Show that g is an injective homomorphism and that $g(N)$ is a direct summand of N_B.**

In general, the map $M \to M_B$ need not be injective. So we are proving that it is injective in the special case where A acts on M by restriction of scalars. Now (presumably) the action of B on N_B is given by

$$b' \cdot (b \otimes n) = b'b \otimes n$$

Of course the action of A on N is given by $a.n = f(a) \cdot n$. Define $p' : B \times N \to N$ by $p'(b, n) = b \cdot n$. Obviously p' is additive in both variables. Also, p' is A-bilinear since

$$p'(a.b, n) = p'(f(a)b, n) = f(a)b \cdot n = f(a) \cdot (b \cdot n) = a.(b \cdot n) = a.p'(b, n)$$
$$p'(b, a.n) = b \cdot (a.n) = b \cdot (f(a) \cdot n) = f(a) \cdot (b \cdot n) = a.(b \cdot n) = a.p'(b, n)$$

So there is a unique A-linear map $p : B \otimes_A N \to N$ satisfying $p(b \otimes n) = b \cdot n$. Since p is A-linear we see that p is a A-submodule of N_B, and since g is A-linear, we see that $\text{Im}(g)$ is an A-submodule of N_B. Now g is injective since $p \circ g = 1_N$. If $y \in \text{Im}(g) \cap \text{Ker}(p)$ with $y = g(x)$ then $x = p(g(x)) = p(y) = 0$, so that $y = 0$. In other words, $\text{Im}(g) \cap \text{Ker}(p) = 0$. On the other hand, for $x \in N_B$

$$x = g(p(x)) + (x - g(p(x)))$$

where $g(p(x)) \in \text{Im}(g)$ and $x - g(p(x)) \in \text{Ker}(p)$ since

$$p(x - g(p(x))) = p(x) - p(g(p(x))) = 0$$

Therefore, $N_B = \text{Im}(g) \oplus \text{Ker}(p)$ as an A-module. On the other hand, if we define the action of B on N_B by $b' \cdot (b \otimes n) = b \otimes b' \cdot n$ then p and g are both B-linear so that $N_B = \text{Im}(g) \oplus \text{Ker}(p)$ as a B-module.

2.15. **Use the notation of exercise 14 to show the following.**

a. **Every element in M is of the form $\mu_j(x_j)$ for some $j \in I$.**

The general element of M is of the form $\sum_{i \in F} x_i + C$, where $x_i \in M_i$ and F is a finite subset of I. Choose $j \in I$ so that $i \leq j$ whenever $i \in F$. By definition of C we have $\sum_{i \in F} x_i + C = \sum_{i \in F} \mu_{ij}(x_i) + C$. But $\sum_{i \in F} \mu_{ij}(x_i) \in M_j$ since each $\mu_{ij} : M_i \to M_j$. So elements in M are of the form $x_j + C = \mu_j(x_j)$ for some $j \in I$ and $x_j \in M_j$.

b. **If $\mu_i(x_i) = 0$ then $\mu_{il}(x_i) = 0$ for some $l \geq i$.**

Notice that $x_i \in C$ since $\mu_i(x_i) = 0$. So write

$$x_i = \sum_{j \in I} \sum_{k \geq j} (x_{jk} - \mu_{jk}(x_{jk}))$$

Where $x_{jk} \in M_j$ equals 0 for all but finitely many j, k. We can choose $l \geq i$ so that $x_{jk} = 0$ if $j > l$ or $k > l$. I claim that $\mu_{il}(x_i) = 0$. Now we play with indices to get

$$\begin{aligned}
\mu_{il}(x_i) &= ((-x_i) - \mu_{il}(-x_i)) + x_i \\
&= ((-x_i) - \mu_{il}(-x_i)) + \sum_{j \le l} \sum_{j \le k \le l} (x_{jk} - \mu_{jk}(x_{jk})) \\
&= \sum_{j \le l} \sum_{j \le k \le l} (x'_{jk} - \mu_{jk}(x'_{jk})) \\
&= \sum_{j \le l} \sum_{j \le k \le l} \Big[(x'_{jk} - \mu_{jl}(x'_{jk})) + (\mu_{jl}(x'_{jk}) - \mu_{jk}(x'_{jk})) \Big] \\
&= \sum_{j \le l} \sum_{j \le k \le l} \Big[(x'_{jk} - \mu_{jl}(x'_{jk})) + (\mu_{kl}(\mu_{jk}(x'_{jk})) - \mu_{jk}(x'_{jk})) \Big] \\
&= \sum_{j \le l} \sum_{j \le k \le l} (x''_{jk} - \mu_{jl}(x''_{jk})) \\
&= \sum_{j \le l} \left[\Big(\sum_{j \le k \le l} x''_{jk} \Big) - \mu_{jl} \Big(\sum_{j \le k \le l} x''_{jk} \Big) \right] \\
&= \sum_{j \le l} (x'''_j - \mu_{jl}(x'''_j)) \\
&= \sum_{j < l} (x'''_j - \mu_{jl}(x'''_j)) + (x'''_j - \mu_{ll}(x'''_j)) \\
&= \sum_{j < l} (x'''_j - \mu_{jl}(x'''_j))
\end{aligned}$$

since μ_{ll} is the identity. Since this identity holds in $\bigoplus_j M_j$, we see that $x'''_j = 0$ for all $j < l$. This implies that $\mu_{il}(x_i) = 0$, as desired.

2.16. Suppose that N is an A-module paired with A-module homomorphisms $\alpha_i : M_i \to N$, indexed by I, with the property that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \le j$. Define a A-module homomorphism $\phi : \bigoplus_{i \in I} M_i \to N$ by $\phi(\sum x_i) = \sum \alpha_i(x_i)$. Notice that $\phi(x_i - \mu_{ij}(x_i)) = \alpha_i(x_i) - \alpha_j(\mu_{ij}(x_i)) = 0$ for every $j > i$, and of course $\phi(x_i - \mu_{ii}(x_i)) = \phi(0) = 0$. So ϕ is identically zero on the submodule generated by $\{x_i - \mu_{ij}(x_i) : j \ge i\}$. This means that ϕ induces an A-module homomorphism Φ on $\varinjlim M_i$ for which $\Phi(\sum x_i + C) = \sum \alpha_i(x_i)$. Obviously $\Phi \circ \mu_i = \alpha_i$ for all $i \in I$. If Φ' were a homomorphism on M for which $\Phi' \circ \mu_i = \alpha_i$, then we would have $\Phi'(\sum x_i + C) = \sum \Phi'(\mu_i(x_i)) = \sum \alpha_i(x_i) = \Phi(\sum x_i = C)$, so that $\Phi' = \Phi$. Therefore, M has the desired universal mapping property.

Suppose that M is an A-module and $\nu_i : M_i \to M$ are A-module homomorphisms for which $\nu_i = \nu_j \circ \mu_{ij}$ whenever $j \ge i$. Suppose also that whenever N is an A-module and $\alpha_i : M_i \to N$ are A-module homomorphisms for which $\alpha_i = \alpha_j \circ \mu_{ij}$ for every $j \ge i$, then there is a unique A-module homomorphism $\Psi : M \to N$ such that $\Psi \circ \nu_i = \alpha_i$ holds for every $i \in I$. It is easy to show that M and $\varinjlim M_i$ are isomorphic as A-modules. After all, choose $\Psi : M \to \varinjlim M_i$ so that $\Psi \circ \nu_i = \mu_i$ for every i. Also, choose $\Phi : \varinjlim M_i \to M$ so that $\Phi \circ \mu_i = \nu_i$ for every i. Then $\Phi \circ \Psi : M \to M$ is an A-module homomorphism for which $(\Phi \circ \Psi) \circ \nu_i = \nu_i$. But i_M is another map from M to M with this property. So by uniqueness $\Phi \circ \Psi = i_M$. Similarly $\Psi \circ \Phi$ is identity on $\varinjlim M_i$. Therefore, Φ and Ψ are inverse isomorphisms.

2.17. Let $(M_i)_{i \in I}$ be a family of submodules of an A-module, such that for every i, j there is k for which $M_i + M_j \subseteq M_k$. Define $i \le j$ if $M_i \subseteq M_j$, and in this case let μ_{ij} correspond to inclusion. Notice that I is a directed set under this ordering. So we may speak of $\varinjlim M_i$.

Consider the submodule $\bigcup M_i$. Let N be an A-module and $\alpha_i : M_i \to N$ an A-module homomorphism for which $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \le j$. Define $\alpha : \bigcup M_i \to N$ by $\alpha(x) = \alpha_i(x)$, where $x \in M_i$. If $x \in M_i$ and

$x \in M_j$ then choose k for which $i \leq k$ and $j \leq k$. Then $\alpha_k(x) = \alpha_i(x)$ and $\alpha_k(x) = \alpha_j(x)$ since μ_{ik} and μ_{ij} correspond to inclusion. Therefore, α is a well-defined map. It is an A-module homomorphism for which $\alpha \circ \mu_i = \alpha_i$. It is also the unique A-module homomorphism with this property. Therefore, $\bigcup M_i$ is isomorphic with $\varinjlim M_i$. It is easy to see that $\bigcup M_i = \sum M_i$.

Suppose M is an arbitrary A-module. Let $\mathcal{F}$ consist of all finitely generated submodules of M. If M_1 and M_2 are finitely generated then so is $M_1 + M_2$. So we can consider the direct limit of the elements of $\mathcal{F}$. Also, if $x \in M$ then $Ax \in \mathcal{F}$. Consequently M equals the union of all the finitely generated submodules of M. The previous paragraph shows that M is isomorphic with the direct limit of its finitely generated submodules.

2.18. Let $\mathbf{M} = (M_i, \mu_{ij})$ and $\mathbf{N} = (N_i, \nu_{ij})$ be direct systems of A-modules over the same directed set I. Suppose that $\phi_i : M_i \to N_i$ are A-module homomorphisms such that $\phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$ whenever $i \leq j$. Let M and N be the direct limits of $\mathbf{M}$ and $\mathbf{N}$, with associated homomorphisms μ_i and ν_i. Define $\alpha_i : M_i \to N$ by $\alpha_i = \nu_i \circ \phi_i$. Notice that $\alpha_j \circ \mu_{ij} = \nu_j \circ \nu_{ij} \circ \phi_i = \nu_i \circ \phi_i = \alpha_i$ whenever $i \leq j$. By exercise 17 there is an A-module homomorphism $\phi : M \to N$ for which $\phi \circ \mu_i = \alpha_i = \nu_i \circ \phi_i$ for every i. So ϕ is the desired homomorphism. By exercise 16 we see that Φ is the unique A-module homomorphism with this property.

2.19. The sequence $\mathbf{M} \to \mathbf{N} \to \mathbf{P}$ of direct systems over the same directed set I is said to be exact provided that the corresponding sequence of modules and module homomorphisms is exact for every $i \in I$. Let M, N, P be the direct limits of these directed systems and let $\phi : M \to N$ and $\psi : N \to P$ be the homomorphisms induced by the homomorphisms of the directed systems. For all $i \leq j$ we have the commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{\phi} & N & \xrightarrow{\psi} & P \\ \uparrow{\scriptstyle \mu_j} & & \uparrow{\scriptstyle \nu_j} & & \uparrow{\scriptstyle \xi_j} \\ M_j & \xrightarrow{\phi_j} & N_j & \xrightarrow{\psi_j} & P_j \\ \uparrow{\scriptstyle \mu_{ij}} & & \uparrow{\scriptstyle \nu_{ij}} & & \uparrow{\scriptstyle \xi_{ij}} \\ M_i & \xrightarrow{\phi_i} & N_i & \xrightarrow{\psi_i} & P_i \end{array}$$

Suppose that $x \in M$. Choose j and $x_j \in M_j$ for which $x = \mu_j(x_j)$. Then $\psi(\phi(x)) = \psi(\phi(\mu_j(x_j))) = \xi_j(\psi_j(\phi_j(x_j))) = \xi_j(0) = 0$ since $\mathrm{Im}(\phi_j) = \mathrm{Ker}(\psi_j)$. Thus $\mathrm{Im}(\phi) \subseteq \mathrm{Ker}(\psi)$.

Suppose that $\psi(y) = 0$ where $y \in N$. Choose i and $y_i \in N_i$ for which $y = \nu_i(y_i)$. Then $0 = \psi(\nu_i(x_i)) = \xi_i(\psi_i(y_i))$. But then there is $j \geq i$ for which $\xi_{ij}(\psi_i(y_i)) = 0$. Then $\psi_j(\nu_{ij}(y_i)) = 0$, implying the existence of $x_j \in M_j$ for which $\nu_{ij}(y_i) = \phi_j(x_j)$. Now notice that $y = \nu_i(y_i) = \nu_j(\nu_{ij}(y_i)) = \nu_j(\phi_j(x_j)) = \phi(\mu_j(x_j))$. Thus $\mathrm{Ker}(\psi) \subseteq \mathrm{Im}(\psi)$ and hence $\mathrm{Ker}(\psi) = \mathrm{Im}(\psi)$. We conclude that $M \to N \to P$ is an exact sequence.

2.20. **Let M be a directed system of A-modules and N an A-module. $\{(M_i \otimes N, \mu_{ij} \otimes 1) : i \in I\}$ is a directed system of A-modules; let P be its direct limit with associated homomorphisms ν_i. For each $i \in I$ we have a homomorphism $\mu_i \otimes 1 : M_i \otimes N \to M \otimes N$. Clearly $\mu_i \otimes 1 = (\mu_j \otimes 1) \circ (\mu_{ij} \otimes 1)$. So there is a unique homomorphism $\psi : P \to M \otimes N$ satisfying $\psi \circ \nu_i = \mu_i \otimes 1$. Show ψ is an isomorphism.**

Assume $(m, n) \in M \times N$ and write $m = \mu_i(m_i)$. Define $g(m, n) = \nu_i(m_i \otimes n)$. I claim that g is well-defined. So suppose that $\mu_i(m_i) = \mu_j(m_j)$ with $j \geq i$. Then $\mu_i(m_i) = \mu_j(\mu_{ij}(m_i))$ so that

2.21. **Let (A_i, α_{ij}) be a directed system of $\mathbb{Z}$-modules so that each A_i is a ring and each α_{ij} is a ring homomorphism. Show that $A = \varinjlim A_i$ inherits a ring structure so that each associated homomorphism α_i is a ring homomorphism. In case $A = 0$, show that some $A_i = 0$.**

Let ξ and η be elements of A. We can write $\xi = \mu_i(x)$ and $\eta = \mu_j(y)$. Choose $k \geq i, j$ and notice that $\xi = \mu_k(\mu_{ik}(x))$ and $\eta = \mu_k(\mu_{jk}(y))$. Define $\xi * \eta = \mu_k(\mu_{ik}(x)\mu_{jk}(y))$. I claim that this defines a multiplication of A that makes A into a ring and each μ_i into a ring homomorphism. The hardest part of this is to show that $\xi * \eta$ is actually well-defined. Suppose first that $l \geq i, j$ and $m \geq l, k$. Then

$$\begin{aligned}
\mu_k(\mu_{jk}(x)\mu_{ik}(y)) &= \mu_m(\mu_{km}(\mu_{jk}(x)\mu_{jk}(y))) \\
&= \mu_m(\mu_{km}(\mu_{ik}(x))\mu_{km}(\mu_{jk}(y))) \\
&= \mu_m(\mu_{im}(x)\mu_{jm}(y)) \\
&= \mu_m(\mu_{lm}(\mu_{il}(x))\mu_{lm}(\mu_{jl}(y))) \\
&= \mu_m(\mu_{lm}(\mu_{jl}(x)\mu_{jl}(y))) \\
&= \mu_l(\mu_{jl}(x)\mu_{il}(y))
\end{aligned}$$

This shows that $\xi * \eta$ is independent of k. Now suppose that $\xi = \mu_{i'}(x')$ and $\eta = \mu_{j'}(y')$. Choose $k \geq i, i', j, j'$ and observe that

$$\mu_k(\mu_{ik}(x) - \mu_{i'k}(x')) = 0 \quad \text{and} \quad \mu_k(\mu_{jk}(y) - \mu_{j'k}(y')) = 0$$

By exercise 15 part b we can choose $l \geq k$ for which

$$\mu_{kl}(\mu_{ik}(x) - \mu_{i'k}(x')) = 0 \quad \text{and} \quad \mu_{kl}(\mu_{jk}(y) - \mu_{j'k}(y')) = 0$$

But this means that $\mu_{il}(x) = \mu_{i'l}(x')$ and $\mu_{jl}(y) = \mu_{j'l}(y')$. Hence

$$\mu_l(\mu_{il}(x)\mu_{jl}(y)) = \mu_l(\mu_{i'l}(x')\mu_{j'l}(y'))$$

This shows that $\xi * \eta$ is well-defined. It is clear that the multiplication is associative, commutative, and unital. Lastly, multiplication distributes over addition : suppose $i, j, k \leq m$ and notice that

$$\begin{aligned}
(\mu_i(x) + \mu_j(y)) * \mu_k(z) &= (\mu_m(\mu_{im}(x)) + \mu_m(\mu_{jm}(y))) * \mu_k(z) \\
&= \mu_m(\mu_{im}(x) + \mu_{jk}(y)) * \mu_k(z) \\
&= \mu_m((\mu_{im}(x) + \mu_{jk}(y))\mu_{km}(z)) \\
&= \mu_m(\mu_{im}(x)\mu_{km}(z)) + \mu_m(\mu_{jm}(y)\mu_{km}(z)) \\
&= \mu_i(x) * \mu_k(z) + \mu_j(y) * \mu_k(z)
\end{aligned}$$

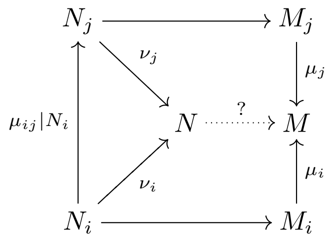
Further, each μ_i is a ring homomorphism since

$$\mu_i(x) * \mu_i(y) = \mu_i(\mu_{ii}(x)\mu_{ii}(y)) = \mu_i(xy)$$

So A is indeed a ring and each μ_i is a map of rings. Now suppose that $A = 0$. Let the zero and identity elements in A_i be represented by 0_i and 1_i respectively. Since $\alpha_i(1_i) = 0_A$, exercise 15 part b tells us that there is $j \geq i$ for which $0_j = \alpha_{ij}(1_i) = 1_j$. This forces $A_j = 0$.

2.22. **Suppose (A_i, α_{ij}) is a directed system of rings and let $\mathfrak{N}_i$ be the nilradical of A_i. Show that $\varinjlim \mathfrak{N}_i$ is the nilradical of $\varinjlim A_i$.**

Lets work in the general setting for the moment. Assume that (M_i, μ_{ij}) is a direct system of A-modules, with direct limit M and maps $\mu_i : M_i \to M$. Suppose that for each $i \in I$ there is a submodule N_i of M_i, and that $\mu_{ij}(N_i) \subseteq N_j$. Then $(N_i, \mu_{ij}|N_i)$ is a direct system as well. Let N be the direct limit with maps $\nu_i : N_i \to N$. Now we have a commutative diagram



By exercise 16 there is a unique $\alpha : N \to M$ that makes the diagram commute with $? = \alpha$. Notice that $\alpha(x + C') = x + C$ if we construct $N = \bigoplus N_i/C'$ and $M = \bigoplus N_i/C$ as in exercise 14. This means that there is a natural way of considering N as a submodule of M. Now lets return to the specific case given in the problem statement. It is clear that $\mathfrak{N}(M_i)$ is a $\mathbb{Z}$-submodule of A_i and that $\mu_{ij}(\mathfrak{N}(A_i)) \subseteq \mathfrak{N}(A_j)$ for $i \leq j$ since μ_{ij} is a ring homomorphism. Write

$$N = \varinjlim \mathfrak{N}(A_i) = \bigoplus \mathfrak{N}(A_i)/C' \quad \text{and} \quad A = \varinjlim A_i = \bigoplus A_i/C$$

as in exercise 14. Let $\nu_i : \mathfrak{N}(A_i) \to N$ and $\mu_i : A_i \to A$ be the natural maps and let $\alpha : N \to A$ as above. Giving N the obvious ring structure I claim that α is a ring homomorphism and that $\mathfrak{N}(A) = \alpha(N)$. So suppose that $\nu_i(x), \nu_j(y) \in N$ and that $k \geq i, j$. Then

$$\begin{aligned}
\alpha(\nu_i(x)) * \alpha(\nu_j(y)) &= \mu_i(x) * \mu_j(y) \\
&= \mu_k(\mu_{ik}(x)\mu_{jk}(y)) \\
&= \alpha(\nu_k(\mu_{ik}(x)\mu_{jk}(y))) \\
&= \alpha(\nu_k(\nu_{ik}(x)\nu_{jk}(y))) \\
&= \alpha(\nu_i(x) * \nu_j(y))
\end{aligned}$$

Consequently, α is a ring homomorphism. Now every element of N is of the form $\nu_i(x)$ for some $x \in N_i$. So every element of N is nilpotent (since every element of N_i is nilpotent by definition). Since α is a ring homomorphism we conclude that $\alpha(N) \subseteq A$. On the other hand suppose that $\mu_i(x) \in \mathfrak{N}(A)$. Then $\mu_i(x)^n = 0$ for some $n > 0$, so that $\mu_i(x^n) = 0$. There is some $j \geq i$ satisfying $\mu_{ij}(x^n) = 0$; implying that $\mu_{ij}(x) \in N_j$. This means that $\mu_i(x) = \mu_j(\mu_{ij}(x)) = \alpha(\nu_j(\mu_{ij}(x))) \in \alpha(N)$. Thus, $\alpha(N) = \mathfrak{N}(A)$ as claimed. This has can be written more suggestively as

$$\varinjlim \mathfrak{N}_i = \mathfrak{N}(\varinjlim A_i)$$

2.23. **Let B_λ be a collection of A-algebras for $\lambda \in \Lambda$. When J is a finite subset of Λ, let B_J denote the tensor product of the B_λ for $\lambda \in J$. Then B_J is an A-algebra and if $J \subset J'$ are finite sets, then there is a canonical map $B_J \to B_{J'}$. Let B denote the direct limit of the B_J as J ranges over the finite subsets of Λ. Show that B has an A-algebra structure for which the maps $B_J \to B$ are A-algebra homomorphisms.**

Suppose J is a finite subset with n elements $\lambda_1, \ldots, \lambda_n$. Then the A-algebra structure of A on $B_J = \bigotimes_A B_{\lambda_i}$ is given by

$$a \cdot (b_1 \otimes \cdots \otimes b_n) = a.b_1 \otimes b_2 \otimes \cdots \otimes b_n$$

If $J \subset J'$ are finite, then let $\mu_{JJ'} : B_J \to B_{J'}$ be the obvious inclusion map. Notice that $\{J \subset \Lambda : J \text{ is finite}\}$ is a directed set under inclusion, and that $\mu_{JJ''} = \mu_{J'J''} \circ \mu_{JJ'}$ whenever $J \subset J' \subset J''$. Clearly $\mu_{JJ} = \text{id}$. This means that we can define the direct limit B and the maps $\mu_J : B_J \to B$. Moreover, B has a natural ring structure so that each μ_J is a ring homomorphism. Now suppose that $f_\lambda : A \to B_\lambda$ gives the A-algebra structure of B_λ. Define $f : A \to B$ by $f = \mu_\lambda \circ f_\lambda$ for any $\lambda \in \Lambda$. This is well-defined: let $J_1 = \{\lambda_1\}$, $J_2 = \{\lambda_2\}$, and $J = \{\lambda_1, \lambda_2\}$. Then

$$\begin{aligned} \mu_{J_1}(f_{\lambda_1}(a)) &= \mu_J(\mu_{J_1 J})(f_{\lambda_1}(a)) \\ &= \mu_J(f_{\lambda_1}(a) \otimes 1) \\ &= \mu_J(1 \otimes f_{\lambda_2}(a)) \\ &= \mu_J(\mu_{J_2 J}(f_{\lambda_2}(a))) \\ &= \mu_{J_2}(f_{\lambda_2}(a)) \end{aligned}$$

So B has a natural A-algebra structure. Lastly, each μ_i is a map of A-algebras since we have (for each λ) the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f_\lambda}\downarrow & \nearrow {\scriptstyle \mu_\lambda} & \\ B_\lambda & & \end{array}$$

2.24. **Let M an A-module and show that TFAE**

a. M **is flat.**

b. $\mathrm{Tor}_n^A(M, N) = 0$ **for every A-module N and every $n > 0$.**

c. $\mathrm{Tor}_1^A(M, N) = 0$ **for every A-module N.**

$(a \Rightarrow b)$ Take a projective resolution $P \xrightarrow{\varepsilon} N$ of N. Since M is flat, $P \otimes_A M$ is exact in degree n, for $n > 0$. But $\mathrm{Tor}_n^A(M, N)$ is defined as the nth homology group of $P \otimes_A M$, so that $\mathrm{Tor}_n^A(M, N) = 0$ for $n > 0$.

$(b \Rightarrow c)$ O.K.

$(c \Rightarrow a)$ Assume that we have an exact sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

Then we have the exact sequence

$$\mathrm{Tor}_1^A(M, N'') \longrightarrow M \otimes N' \longrightarrow M \otimes N \longrightarrow M \otimes N'' \longrightarrow 0$$

But $\mathrm{Tor}_1^A(M, N'') = 0$ so that we have the exact sequence

$$0 \longrightarrow M \otimes N' \longrightarrow M \otimes N \longrightarrow M \otimes N'' \longrightarrow 0$$

This shows that M is a flat A-module.

2.25. **Suppose we have an exact sequence of A-modules**

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

with N'' flat. Show that N' is flat iff N is flat.

Let M be an A-module. Since N'' is flat, we can take a projective resolution of M, and argue as above to get

$$\mathrm{Tor}_2^A(M, N'') = \mathrm{Tor}_1^A(M, N'') = 0$$

So we have the short exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^A(M, N') \longrightarrow \mathrm{Tor}_1^A(M, N) \longrightarrow 0$$

Now $\mathrm{Tor}_1^A(M, N') = 0$ if and only if $\mathrm{Tor}_1^A(M, N) = 0$. Since this holds for every A-module M, we are done. I have used here the fact that in computing Tor we can take a projective resolution in either variable. This seemed like a reasonably elementary fact to assume.

2.26. **Let N be an A-module. Show that N is flat if and only if $\mathrm{Tor}_1^A(A/\mathfrak{a}, N) = 0$ whenever $\mathfrak{a}$ is a finitely generated ideal in A.**

We already know that $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$ when N is flat. We prove the converse through a series of reductions. So suppose that $\mathrm{Tor}_1(M, N) = 0$ whenever M is a finitely generated A-module. Let $f : M' \to M$ be injective with M and M' finitely generated A-modules. Then we have the short exact sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{\pi} M/f(M') \longrightarrow 0$$

So we have the exact sequence

$$\mathrm{Tor}_1(M/f(M'), N) \longrightarrow M' \otimes_A N \xrightarrow{f\otimes \mathrm{id}} M \otimes_A N$$

But $M/f(M')$ is finitely generated so that $\mathrm{Tor}_1(M/f(M'), N) = 0$. This means that $f \otimes \mathrm{id}$ is injective. Proposition 2.19 now tells us that N is flat. Now suppose that $\mathrm{Tor}_1(M, N) = 0$ whenever M is generated by a single element, and let M be an arbitrary finitely generated A-module. Assume $x_1, \ldots, x_n$ generate M and let M' be the submodule of M generated by $x_1, \ldots, x_{n-1}$. We have the short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M/M' \longrightarrow 0$$

This yields the exact sequence

$$\mathrm{Tor}_1(M', N) \longrightarrow \mathrm{Tor}_1(M, N) \longrightarrow \mathrm{Tor}_1(M/M', N)$$

But M/M' is generated by a single element so that $\mathrm{Tor}_1(M/M', N) = 0$. By induction on n we see that $\mathrm{Tor}_1(M', N) = 0$. Hence $\mathrm{Tor}_1(M, N) = 0$. Now assume that $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$ whenever $\mathfrak{a}$ is any ideal in A. If M is an A-module generated by the element x, then M and $A/\operatorname{Ann}(x)$ are isomorphic, so that $\mathrm{Tor}_1(M, N) = \mathrm{Tor}_1(A/\operatorname{Ann}(x), N) = 0$. Now suppose that $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$ whenever $\mathfrak{a}$ is a finitely generated ideal in A. Let $\mathfrak{b}$ be an arbitrary ideal in A. If $\mathfrak{a}$ is a finitely generated ideal of A contained in $\mathfrak{b}$, then we have the short exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow A \longrightarrow A/\mathfrak{a} \longrightarrow 0$$

From this we get the long exact sequence

$$\mathrm{Tor}_1(A/\mathfrak{a}, N) \longrightarrow \mathfrak{a} \otimes_A N \longrightarrow A \otimes_A N \longrightarrow A/\mathfrak{a} \otimes_A N \longrightarrow 0$$

Since $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$, we conclude that the map $\mathfrak{a} \otimes_A N \to A \otimes_A N$ is injective. Analysing the proof to Proposition 2.19, we see that more is proved than is stated. In particular, it is demonstrated that $\mathfrak{b} \otimes_A N \to A \otimes_A N$ is injective since $\mathfrak{a} \otimes_A N \to A \otimes_A N$ is injective for every finitely generated ideal $\mathfrak{a}$ contained in $\mathfrak{b}$. So from the short exact sequence

$$0 \longrightarrow \mathfrak{b} \longrightarrow A \longrightarrow A/\mathfrak{b} \longrightarrow 0$$

we get the long exact sequence

$$\mathrm{Tor}_1(A,N) \longrightarrow \mathrm{Tor}_1(A/\mathfrak{b},N) \longrightarrow \mathfrak{b}\otimes_A N \longrightarrow A\otimes_A N \longrightarrow A/\mathfrak{b}\otimes_A N \longrightarrow 0$$

with $\mathrm{Tor}_1(A,N) = 0$ since $A = A/0$ with 0 a finitely generated ideal, and the map $\mathfrak{b}\otimes_A N \to A\otimes_A N$ injective. These two observations imply that $\mathrm{Tor}_1(A/\mathfrak{b},N) = 0$. Summarizing, we have shown that N is flat provided $\mathrm{Tor}_1(A/\mathfrak{a},N) = 0$ whenever $\mathfrak{a}$ is a finitely generated ideal in A.

2.27. **Show that the following conditions are equivalent for a ring A**

a. A is absolutely flat (i.e. every A-module is flat).

b. Every principal ideal in A is idempotent.

c. Every finitely generated ideal in A is a direct summand of A.

$(a \Rightarrow b)$ Let (x) be a principal ideal in A so that $A/(x)$ is a flat A-module. Then from the inclusion $(x) \to A$ we get an inclusion $(x)\otimes_A A/(x) \to A\otimes_A A/(x)$. But this map is the zero map since $x\otimes\bar{1} \mapsto x\otimes\bar{1} = 1\otimes x\cdot\bar{1} = 0$. Hence $(x)\otimes_A A/(x) = 0$, so that $(x)/(x^2) \cong A/(x)\otimes_A (x) = 0$ by exercise 2.2. This shows that $(x) = (x^2) = (x)^2$, as desired.

$(b \Rightarrow c)$ Let $\mathfrak{a}$ be a finitely generated ideal in A and write $\mathfrak{a} = (x_1,\dots,x_n)$. For each i there is $a_i \in A$ for which $x_i = a_i x_i^2$. But then $e_i = a_i x_i$ satisfies $e_i^2 = a_i(a_i x_i^2) = a_i x_i = e_i$. That is, each e_i is idempotent and $(e_i) = (x_i)$. Now $(x_1,\dots,x_n) = (x_1)+\cdots+(x_n) = (e_1)+\cdots+(e_n) = (e_1,\dots,e_n)$. In general, if e and f are idempotent elements then $(e+f-ef) \subseteq (e,f)$, and also $(e,f) \subseteq (e+f-ef)$ since $e = e(e+f-ef)$ and $f = f(e+f-ef)$. Hence, $(e,f) = (e+f-ef)$. By induction on n there is an idempotent element e^* for which $(e_1,\dots,e_n) = (e^*)$. Finally, $A = (e^*)+(1-e^*)$ for every idempotent element e^*, as was shown in exercise 1.22, or as can be seen directly.

$(c \Rightarrow a)$ Let M be an A-module and suppose $\mathfrak{a}$ is a finitely generated ideal of A. Choose an ideal $\mathfrak{b}$ of A so that $A = \mathfrak{a}\oplus\mathfrak{b}$. Then in particular $\mathfrak{b}$ is a projective A-module. Thus $\mathrm{Tor}_1^A(A/\mathfrak{a},M) = \mathrm{Tor}_1^A(\mathfrak{b},M) = 0$. So M is flat by exercise 1.26. Hence, A is an absolutely flat ring.

2.28. **Establish the following.**

Every Boolean ring A is absolutely flat.

If (x) is a principal ideal in A, then $(x)^2 = (x^2) = (x)$ since $x^2 = x$. So A is absolutely flat by exercise 1.27.

The ring A is absolutely flat if, for every $x \in A$, there is $n > 1$ for which $x^n = x$.

Let (x) be an arbitrary principal ideal in A. Write $x^n = x$ for some $n > 1$. Then $(x^n) = (x)$. But $(x^n) \subseteq (x^2) \subseteq (x)$ since $n \geq 2$. We conclude that $(x) = (x^2) = (x)^2$ so that A is absolutely flat.

If A is absolutely flat and $f : A \to B$ is surjective, then B is absolutely flat.

A principal ideal of B has the form $(f(a))$ for some $a \in A$. Clearly $(f(a))^2 \subseteq (f(a))$. On the other hand, if $bf(a)$ is an arbitrary element of $(f(a))$ and choose $\tilde{a} \in A$ satisfying $a = \tilde{a}a^2$. Such an $\tilde{a}$ exists since $(a^2) = (a)$. Then $bf(a) = bf(\tilde{a})f(a)^2 \in (f(a))^2$. Hence, $(f(a)) = (f(a))^2$ so that B is absolutely flat.

If a local ring A is absolutely flat, then A is a field.

Since A is absolutely flat, every principal ideal is generated by an idempotent element, as demonstrated in the course of establishing exercise 2.27. But in a nonzero local ring, there are precisely two idempotents, namely 0 and 1. So the only principal ideals in A are 0 and A, implying that A is a field.

If A is an absolutely flat ring and $x \in A$, then x is a zero-divisor or x is a unit.

Choose $a \in A$ for which $x = ax^2$. Then $x(ax - 1) = 0$. If $ax - 1 = 0$, then x is a unit. Otherwise, $ax - 1 \neq 0$, and hence x is a zero-divisor.

Chapter 3 : Rings and Modules of Fractions

3.1. **Let M be a finitely generated A-module and S a multiplicatively closed subset of A. Show that $S^{-1}M = 0$ iff $sM = 0$ for some s.**

Suppose $x_1, \ldots, x_n$ generate M. If $S^{-1}M = 0$ then $s_i x_i = 0$ for some $s_i \in S$. Defining $s = s_1 \cdots s_n$ yields an element $s \in S$ such that $sx_i = 0$ for each i, and hence $sM = 0$. The converse is obvious.

3.2. **Let $\mathfrak{a}$ be an ideal in A and let $S = 1 + \mathfrak{a}$. Show that $S^{-1}\mathfrak{a} \subseteq \mathfrak{R}(S^{-1}A)$.**

Clearly S is a multiplicatively closed subset of A since

$$(1+a)(1+a') = 1 + (a + a' + aa') \in 1 + \mathfrak{a}$$

We also have $S^{-1}\mathfrak{a} \subseteq \mathfrak{R}(S^{-1}A)$ since

$$1 - \frac{a_1}{1+a_2} \cdot \frac{x}{1+a_3} = \frac{1 + a_2 + a_3 + a_2a_3 - a_1x}{(1+a_2)(1+a_3)} = \frac{1+a_4}{(1+a_2)(1+a_3)}$$

is a unit in $S^{-1}A$ for all $a_1, a_2, a_3 \in \mathfrak{a}$ and $x \in A$.

Use this result and Nakayama's Lemma to give a different proof of Proposition 2.5

Now suppose that M is a finitely generated A-module for which $\mathfrak{a}M = M$ with $\mathfrak{a} \subseteq \mathfrak{R}(A)$. Then $(S^{-1}\mathfrak{a})(S^{-1}M) = S^{-1}M$ where again $S = 1 + \mathfrak{a}$. After all, given $m/s \in S^{-1}M$ there is $a \in \mathfrak{a}$ and $m' \in M$ for which $am' = m$, implying that $(a/1)(m'/s) = m/s$, and hence showing that $S^{-1}M \subseteq (S^{-1}\mathfrak{a})(S^{-1}M)$. Since $S^{-1}\mathfrak{a} \subseteq \mathfrak{R}(S^{-1}A)$ and since $S^{-1}M$ is a finitely generated $S^{-1}A$-module, Nakayama's Lemma yields $S^{-1}M = 0$. By exercise 3.1 there is $a \in \mathfrak{a}$ satisfying $(1+a)M = 0$.

3.3. **Let A be a ring with multiplicatively closed subsets S and T. Define U to be the image of T in $S^{-1}A$. Show that $(ST)^{-1}A$ and $U^{-1}(S^{-1}A)$ are isomorphic rings.**

Notice that ST is a multiplicatively closed subset of A. Now we apply the universal mapping property for the ring of fractions three times.

Define a map from A to $(ST)^{-1}A$ by $a \mapsto a/1$. Since this is a homomorphism and since the image $s/1$ of s in S has the inverse $1/s$, we conclude that there is a homomorphism from $S^{-1}A$ to $(ST)^{-1}A$ sending a/s to a/s. But this map sends t/s to t/s, which has inverse s/t in $(ST)^{-1}A$. So there is a homomorphism $F : U^{-1}(S^{-1}A) \to (ST)^{-1}A$ satisfying $F((a/s)/(t/s')) = as'/st$.

Similarly, the map from A into $U^{-1}(S^{-1}A)$ given by $a \mapsto (a/1)/(1/1)$ is such that the image $(st/1)/(1/1)$ of st has inverse $(1/s)/(t/1)$. So there is a homomorphism $G : (ST)^{-1}A \to U^{-1}(S^{-1}A)$ satisfying $G(a/st) = (a/s)/(t/1)$.

It is straightforward to check that $F \circ G$ is the identity map for $(ST)^{-1}A$ and that $G \circ F$ is the identity map for $U^{-1}(S^{-1}A)$. So F and G are isomorphisms, and hence $U^{-1}(S^{-1}A)$ and $(ST)^{-1}A$ are isomorphic rings.

3.4. **Let $f : A \to B$ be a ring homomorphism, suppose that S is a multiplicatively closed subset of A, and define $T = f(S)$. Show that $S^{-1}B$ and $T^{-1}B$ are isomorphic as $S^{-1}A$-modules.**

First, it is clear that T is a multiplicatively closed subset of B since $1 = f(1)$ and $f(s)f(s') = f(ss')$. We make $T^{-1}B$ into an $S^{-1}A$-module by defining $a/s \cdot b/f(s') = f(a)b/f(s)f(s')$. Now define $\Phi : S^{-1}B \to T^{-1}B$

by $\Phi(b/s) = b/f(s)$. I claim that Φ is an isomorphism. First, suppose that $b/s = b'/s'$ in $S^{-1}B$. Then for some $s'' \in S$ we have

$$0 = s'' \cdot (s' \cdot b - s \cdot b') = f(s'')(f(s')b - f(s)b')$$

so that $b/f(s) = b'/f(s')$ in $T^{-1}B$. Hence, Φ is well-defined. Notice that

$$\begin{aligned}\Phi(b/s + b'/s') &= \Phi((s' \cdot b + s \cdot b')/ss') \\ &= \Phi((f(s')b + f(s)b')/ss') \\ &= (f(s')b + f(s)b')/f(ss') \\ &= (f(s')b + f(s)b')/f(s)f(s') \\ &= b/f(s) + b'/f(s') \\ &= \Phi(b/s) + \Phi(b'/s')\end{aligned}$$

We also have the relation

$$\Phi(a/s \cdot b/s') = \Phi(f(a)b/ss') = f(a)b/f(ss') = f(a)b/f(s)f(s') = a/s \cdot b/f(s') = a/s \cdot \Phi(b/s')$$

So Φ is a homomorphism of $S^{-1}A$-modules. Clearly Φ is surjective. Now if $\Phi(b/s) = \Phi(b'/s')$ then for some $t \in T$ we have

$$t(f(s')b - f(s)b') = 0$$

Choose $s'' \in S$ satisfying $t = f(s'')$. Then

$$s'' \cdot (s' \cdot b - s \cdot b') = 0$$

This means that $b/s = b'/s'$ in $S^{-1}A$. So Φ is injective as well. Thus, Φ is an isomorphism of $S^{-1}A$-modules, as claimed.

3.5. **Suppose that for each prime ideal $\mathfrak{p}$, the ring $A_\mathfrak{p}$ has no nilpotent element $\neq 0$. Show that A has no nilpotent element $\neq 0$.**

For every prime ideal $\mathfrak{p}$ we have $\mathfrak{N}(A)_\mathfrak{p} = \mathfrak{N}(A_\mathfrak{p}) = 0$, so that $\mathfrak{N}(A) = 0$.

Must A be an integral domain if $A_\mathfrak{p}$ is an integral domain for every prime ideal $\mathfrak{p}$?

Let $A = k \times k$ where k is an field. Obviously A is not an integral domain. From exercise 1.23 we know that $\mathfrak{p} = 0 \times k$ and $\mathfrak{q} = k \times 0$ are the prime ideals of A. Since $(1, 0) \in A - \mathfrak{p}$ and $(1, 0)\mathfrak{p} = 0$ we see that $\mathfrak{p}_\mathfrak{p} = 0$. But $\mathfrak{p}_\mathfrak{p}$ is a prime ideal in $A_\mathfrak{p}$, so that $A_\mathfrak{p}$ is an integral domain. Similarly, $A_\mathfrak{q}$ is an integral domain as well. Thus, the property of being an integral domain is not a local property.

3.6. **Let A be a nonzero ring and let Σ be the set of all multiplicatively closed subsets S of A for which $0 \notin S$. Show that Σ has maximal elements and that $S \in \Sigma$ is maximal if and only if $A - S$ is a minimal prime ideal of A.**

That Σ has maximal elements follows from a straightforward application of Zorn's Lemma since Σ is chain complete. Now suppose that $S \in \Sigma$ is maximal. Since $0 \notin S$ we know that $1/1 \neq 0/1$ in $S^{-1}A$. So $S^{-1}A$ is a nonzero ring, and hence has a maximal ideal, which is of course a prime ideal. But this prime ideal corresponds to a prime ideal $\mathfrak{p}$ in A that does not meet S. In other words, there is $\mathfrak{p}$ for which $S \subseteq A - \mathfrak{p}$. But $A - \mathfrak{p}$ is in Σ, so that $S = A - \mathfrak{p}$ by maximality. Further, if $\mathfrak{q} \subseteq \mathfrak{p}$ is a prime ideal, then $A - \mathfrak{p} \subseteq A - \mathfrak{q}$ and $A - \mathfrak{q}$

is in Σ, so that $S = A - \mathfrak{q}$ again by maximality. This means that $\mathfrak{p} = \mathfrak{q}$, so that $\mathfrak{p}$ is a minimal prime ideal in A.

On the other hand, if $\mathfrak{p}$ is a minimal prime ideal in A, then $S = A - \mathfrak{p}$ is an element of Σ. Choose a maximal $S' \in \Sigma$ for which $S \subseteq S'$. By the above $A - S'$ is a minimal prime ideal in A. But $A - S' \subseteq \mathfrak{p}$, implying that $A - S' = \mathfrak{p}$, since $\mathfrak{p}$ is minimal. So $S = S'$, showing that $A - \mathfrak{p}$ is a maximal element of Σ whenever $\mathfrak{p}$ is a minimal prime ideal in A.

3.7. **A multiplicatively closed subset S in A is called saturated if x and y are in S whenever xy is in S. Prove the following.**

a. **S is saturated iff $A - S$ is a union of prime ideals of A.**

Suppose that $A - S = \bigcup \mathfrak{p}$ is a union of prime ideals of A. If $xy \notin S$ then xy is in some $\mathfrak{p}$, implying that $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$, so that $x \notin S$ or $y \notin S$. If $x \notin S$ or $y \notin S$, then $xy \notin S$ since $A - S$ is a union of ideals. So S is a saturated multiplicatively closed subset of A.

Now suppose S is a saturated multiplicatively closed subset of A. It suffices to show that every $x \in A - S$ is contained in a prime ideal that does not intersect S. If $x \in A - S$, then $(x) \cap S = \emptyset$ since S is saturated. But then $(x)^e \neq (1)$ in $S^{-1}A$, so that $x/1$ is not a unit in $S^{-1}A$ and $S^{-1}A \neq 0$. So there is a maximal ideal $\mathfrak{m}$ in $S^{-1}A$ containing $x/1$. We can choose a prime ideal $\mathfrak{p}$ that does not meet S and is such that $\mathfrak{p}^e = \mathfrak{m}$. Then $x \in \mathfrak{p}$ since $\mathfrak{p} = \mathfrak{m}^c$. So $A - S$ is indeed a union of prime ideals.

b. **If S is any multiplicatively closed subset of A then there is a unique smallest saturated multiplicatively closed subset S^* of A containing S. S^* is the complement in A of the union of the prime ideals in A that do not intersect S.**

Let Σ consist of all saturated multiplicatively closed subsets of A containing S. Then $\Sigma \neq \emptyset$ since $A \in \Sigma$. Let $S^* = \bigcap_{S' \in \Sigma} S'$, and notice that S^* is the desired set. We can choose prime ideals $\mathfrak{p}_{\alpha,S'}$ so that $A - S' = \bigcup \mathfrak{p}_{\alpha,S'}$ for each $S' \in \Sigma$. Then $S^* = A - \bigcup_{S' \in \Sigma} \bigcup \mathfrak{p}_{\alpha,S'}$. So clearly each $\mathfrak{p}_{\alpha,S'}$ has empty intersection with S. Further, if $\mathfrak{p}$ is a prime ideal that does not meet S, then $A - \mathfrak{p} \in \Sigma$, so that $\mathfrak{p} \subseteq A - S^*$. Hence, S^* is the complement in A of the prime ideals that do not intersect S.

c. **Find S^* if $S = 1 + \mathfrak{a}$ for some ideal $\mathfrak{a}$.**

If $\mathfrak{p}$ meets S then $1 + a \in \mathfrak{p}$ for some $a \in \mathfrak{a}$, and hence $1 \in \mathfrak{p} + \mathfrak{a}$. Conversely, if $1 \in \mathfrak{p} + \mathfrak{a}$ then $\mathfrak{p}$ meets S. Therefore $S^* = A - \bigcup_{\mathfrak{p}:1 \notin \mathfrak{p}+\mathfrak{a}} \mathfrak{p}$. If $\mathfrak{m}$ is a maximal ideal containing $\mathfrak{a}$, then $\mathfrak{m}$ is a prime ideal satisfying $1 \notin \mathfrak{m} + \mathfrak{a}$. Conversely, if $\mathfrak{p}$ is a prime ideal satisfying $1 \notin \mathfrak{p} + \mathfrak{a}$, then there is a maximal (and hence prime) ideal $\mathfrak{m}$ containing $\mathfrak{p} + \mathfrak{a}$, so that $1 \notin \mathfrak{m} + \mathfrak{a}$. These two observations give us $S^* = A - \bigcup_{\mathfrak{m} \supseteq \mathfrak{a}} \mathfrak{m}$.

3.8. **Let S and T be multiplicatively closed subsets of A such that $S \subseteq T$. Let $\phi : S^{-1}A \to T^{-1}A$ be the obvious inclusion. Show that the following conditions are equivalent.**

a. **ϕ is bijective**
b. **For each $t \in T$ the element $t/1$ is a unit in $S^{-1}A$.**
c. **For each $t \in T$ there is $x \in A$ for which $xt \in S$.**
d. **T is contained in the saturation of S.**
e. **Every prime ideal which meets T also meets S.**

Notice that the map $a \mapsto a/1$ from A to $T^{-1}A$ is a homomorphism such that the image $s/1$ of $s \in S$ has inverse $1/s$ (since $s \in T$). Thus, there is a unique homomorphism $\phi : S^{-1}A \to T^{-1}A$ for which $\phi(a/s) = a/s$ whenever $a \in A$ and $s \in S$.

(a $\Rightarrow$ b) As always, $t/1$ is a unit in $T^{-1}A$. So if ϕ is bijective, then ϕ is a ring isomorphism, so that $t/1 = \phi^{-1}(t/1)$ is a unit in $S^{-1}A$.

(b $\Rightarrow$ c) Choose $a \in A$ and $s \in S$ so that $t/1 \cdot a/s = 1/1$. Then $s'(at - s) = 0$ for some $s' \in S$. But then $(as')t = ss' \in S$.

(c $\Rightarrow$ d) For $t \in T$ choose $x \in A$ so that $xt \in S \subseteq S^*$. Then $x \in S^*$ and $t \in S^*$, and hence $T \subseteq S^*$.

(d $\Rightarrow$ e) If $\mathfrak{p}$ is a prime ideal in A that does not meet S, then $\mathfrak{p}$ does not meet S^* by exercise 3.7. Therefore, $\mathfrak{p}$ does not meet T. So every prime ideal in A that meets T also meets S.

(e $\Rightarrow$ c) If b does not hold then $(t) \cap S = \emptyset$ for some $t \in T$. But then there is a prime ideal $\mathfrak{p}$ containing (t) such that $\mathfrak{p} \cap S = \emptyset$. Since $t \in \mathfrak{p} \cap T$ we see that e does not hold.

(c $\Rightarrow$ b) Let $t \in T$ and choose $x \in A$ satisfying $xt \in S$. Then $t/1$ has inverse x/xt in $S^{-1}A$.

(b $\Rightarrow$ a) Suppose that $\phi(a/s) = \phi(a'/s')$ in $T^{-1}A$ so that $t(as' - a's) = 0$ for some $t \in T$. Choose $x \in A$ for which $xt \in S$. Then $(xt)(as' - a's) = 0$, so that $a/s = a'/s'$ in $S^{-1}A$. In other words, ϕ is injective. Now let $t \in T$ and choose $a \in A$ and $s \in S$ for which $t/1 \cdot a/s = 1/1$ in $S^{-1}A$. Then $s'(at - s) = 0$ for some $s' \in S$. But $S \subseteq T$ so that $1/t = a/s$ in $T^{-1}A$. In other words, $1/t = \phi(a/s) \in \mathrm{Im}(\phi)$, so that ϕ is surjective. Thus, ϕ is a bijection.

3.9. **For $A \neq 0$ let S_0 consist of all regular elements of A. Show that S_0 is a saturated mutliplicatively closed subset of A and that every minimal prime ideal of A is contained in $D = A - S_0$. The ring $S_0^{-1}A$ is called the total ring of fractions of A. Prove assertions a,b, and c below.**

Suppose $x \notin S_0$ or $y \notin S_0$. Then there is $z \neq 0$ such that $xz = 0$ or $yz = 0$. But then $xyz = 0$ so that $xy \notin S_0$. On the other hand, if $xy \notin S_0$ then there is $z \neq 0$ satisfying $xyz = 0$. If $yz = 0$ then $y \notin S_0$, and if $yz \neq 0$ then $x \notin S_0$. Thus, S_0 is a saturated multiplicatively closed subset of A.

Now let $\mathfrak{p}$ be a prime ideal in A and suppose that $x \in \mathfrak{p}$ is regular. We see that $\{x^i y : y \in A - \mathfrak{p} \text{ and } i \in \mathbb{N}\}$ is a multiplicatively closed subset of A properly containing $A - \mathfrak{p}$. This subset of A does not contain 0 since x is not a zero-divisor. Therefore, $A - \mathfrak{p}$ is not maximal in Σ, and hence $\mathfrak{p}$ is not a minimal prime ideal. In other words, every minimal prime ideal in A consists entirely of zero-divisors and so is contained in D. From this it follows easily that D is the union of the minimal prime ideals in A.

a. **S_0 is the largest multiplicatively closed subset S of A so that the map $A \to S^{-1}A$ is 1-1.**

Suppose that $a/1 = 0/1$ in $S_0^{-1}A$. Then $ax = 0$ for some $x \in S_0$. But x is not a zero-divisor, and so $a = 0$. So the natural map is 1-1. Now assume that S is a multiplicatively closed subset of A with this property. Suppose that $x \in S$ and $a \in A$ satisfy $ax = 0$. Then $a/1 = 0/1$ in $S^{-1}A$ so that $a = 0$. In other words x is a regular element, and so $S \subseteq S_0$.

b. **Every element in $S_0^{-1}A$ is a unit or a zero-divisor.**

Suppose that $x/y \in S_0^{-1}A$. If $x \in S_0$ then x/y is a unit in $S_0^{-1}A$ with inverse y/x. If $x \notin S_0$, then there is $z \neq 0$ satisfying $xz = 0$, implying that $(x/y)(z/1) = 0/1$. Since $z/1 \neq 0/1$ we see that x/y is a zero-divisor in $S_0^{-1}A$. So we are done.

c. **If every element in A is a unit or a zero-divisor then the natural map $f : A \to S_0^{-1}A$ is an isomorphism.**

We already know that f is injective. Now if $x \in S_0$ then x is a unit. So f is surjective since $a/x = ax^{-1}/(xx^{-1}) = ax^{-1}/1 = f(ax^{-1})$ for $a \in A$ and $x \in S_0$. Thus, f is bijective, and hence an isomorphism.

3.10. **Show that $S^{-1}A$ is an absolutely flat ring if A is an absolutely flat ring.**

Suppose that M is an $S^{-1}A$-module. Let $N = M$, where we consider N as an A-module with $a.m = a/1 \cdot m$. Then $S^{-1}N$ is an $S^{-1}A$-module. I claim that $S^{-1}N$ and M are isomorphic as $S^{-1}A$-modules. Assuming this, we see that N is a flat A-module since A is absolutely flat, and so $S^{-1}N$ is a flat $S^{-1}A$-module. This means that M is a flat $S^{-1}A$-module, and so $S^{-1}A$ is absolutely flat. Now we finish the stickier part of this exercise by defining $f : S^{-1}N \to M$ by $f(m/s) = 1/s \cdot m$. Notice first that f is additive since

$$f(m/s + m'/s') = f((s'.m + s.m')/ss') = 1/ss' \cdot (s'/1 \cdot m + s/1 \cdot m') = f(m/s) + f(m'/s')$$

Further, f preserves the action of $S^{-1}A$ since

$$f(a/s \cdot m/t) = f(a.m/st) = 1/st \cdot a.m = 1/st \cdot a/1 \cdot m = a/s \cdot 1/t \cdot m = a/s \cdot f(m/t)$$

So f will be a homomorphism provided that f is well-defined. Suppose $m/s = 0/1$ in $S^{-1}N$. Then $t.m = 0$ for some $t \in S$, so that $t/1 \cdot m = 0$. But now $m = 0$ since $t/1$ is a unit in $S^{-1}A$. Hence, f is well-defined and thus is a homomorphism. Clearly f is surjective with $f(m/1) = m$. Lastly, suppose that $f(m/s) = f(m'/s')$. Then $1/s \cdot m = 1/s' \cdot m'$ so that $s'/1 \cdot m = s/1 \cdot m'$, implying that $1.(s'.m - s.m') = 0$. In other words, $m/s = m'/s'$ in $S^{-1}N$. Consequently, f is an isomorphism of $S^{-1}A$-modules.

Show that A is an absolutely flat ring if and only if $A_\mathfrak{m}$ is a field for every maximal $\mathfrak{m}$.

If A is absolutely flat and $\mathfrak{m}$ is a maximal ideal in A, then $A_\mathfrak{m}$ is absolutely flat by the above. But $A_\mathfrak{m}$ is a local ring so that $A_\mathfrak{m}$ is a field by exercise 2.28. So suppose that $A_\mathfrak{m}$ is a field whenever $\mathfrak{m}$ is a maximal ideal in A. Let M be an A-module so that $M_\mathfrak{m}$ is an $A_\mathfrak{m}$-module. This means that $M_\mathfrak{m}$ is an $A_\mathfrak{m}$-vector space. But now $M_\mathfrak{m}$ is flat as an $A_\mathfrak{m}$-module. Hence, M is flat as an A-module, implying that A is absolutely flat.

3.11. **Let A be a ring. Show that the following are equivalent.**

a. $A/\mathfrak{N}(A)$ **is absolutely flat.**
b. **Every prime ideal in A is a maximal ideal.**
c. **In** Spec(A) **every one point set is closed.**
d. Spec(A) **is Hausdorff.**

(a $\Rightarrow$ b) Let $\mathfrak{p}$ be a prime ideal in A. Since $\mathfrak{N}(A) \subseteq \mathfrak{p}$ we have a surjective homomorphism $A/\mathfrak{N}(A) \to A/\mathfrak{p}$. In other words, $A/\mathfrak{p}$ is the homomorphic image of an absolutely flat ring, and so is an absolutely flat ring. But then every non-unit in $A/\mathfrak{p}$ is a zero-divisor by exercise 2.28. Since $A/\mathfrak{p}$ is an integral domain, this means that $A/\mathfrak{p}$ is a field, and so $\mathfrak{p}$ is a maximal ideal in A.

(b $\Rightarrow$ a) A maximal ideal $\mathfrak{q}$ in $A/\mathfrak{N}(A)$ is of the form $\mathfrak{q} = \mathfrak{p}/\mathfrak{N}(A)$ for some prime ideal $\mathfrak{p}$ in A. Now $A/\mathfrak{N}(A)$ is a reduced ring. Since localization commutes with taking the nilradical, we see that $(A/\mathfrak{N}(A))_\mathfrak{q}$ is a reduced ring as well. But Spec$((A/\mathfrak{N}(A))_\mathfrak{q}) \cong V(\mathfrak{q})$ and $V(\mathfrak{q}) = \{\mathfrak{q}\}$ since prime ideals in $A/\mathfrak{N}(A)$ are maximal. So $\mathfrak{q}_\mathfrak{q} = 0$, and hence $(A/\mathfrak{N}(A))_\mathfrak{q}$ is a field. Exercise 3.10 now implies that $A/\mathfrak{N}(A)$ is absolutely flat.

(b $\Leftrightarrow$ c) If $\mathfrak{p}$ is maximal then $\{\mathfrak{p}\} = V(\mathfrak{p})$ so that $\{\mathfrak{p}\}$ is a closed set. If $\{\mathfrak{p}\}$ is closed then $\{\mathfrak{p}\} = V(E)$ for some $E \subseteq A$. Clearly $\mathfrak{p} \supseteq E$ and no other prime ideal in A contains E. In particular, no prime ideal in A strictly contains $\mathfrak{p}$. So $\mathfrak{p}$ is a maximal ideal in A.

(d $\Rightarrow$ c) This is elementary point-set topology.

(b $\Rightarrow$ d) Suppose that $\mathfrak{p}$ and $\mathfrak{q}$ are distinct elements of Spec(A).

If these conditions hold, show that Spec(A) **is compact Hausdorff and totally disconnected.**

It is always true that Spec(A) is compact, and by hypothesis Spec(A) is Hausdorff.

3.12. **Let M be an A-module and A an integral domain. Show that the set of all $x \in M$ for which** $\text{Ann}(x) \neq 0$ **forms an A-submodule of M, denoted $T(M)$. An element $x \in T(M)$ is called a torsion element. Prove assertions a-d.**

Suppose that $x, y \in T(M)$ and $a, a' \neq 0$ satisfy $ax = a'y = 0$. Then $aa'(x - y) = 0$ and $aa' \neq 0$ since A has no zero-divisors. Therefore, $x - y \in T(M)$. Also, if $a'' \neq 0$, then $a''x \in T(M)$ since $a(a''x) = 0$ and $aa'' \neq 0$. Therefore $T(M)$ is a submodule of M.

a. $M/T(M)$ **is torsion free.**

Suppose that $\bar{x}$ is a torsion element in $M/T(M)$. Choose $a \neq 0$ for which $0 = a\bar{x} = \overline{ax}$, so that $ax \in T(M)$. Then there is $a' \neq 0$ for which $a'ax = 0$. But $a'a \neq 0$, and hence $x \in T(M)$, so that $\bar{x} = 0$.

b. $f(T(M)) \subseteq T(N)$ **if $f : M \to N$ is an A-module homomorphism.**

If $x \in T(M)$ and $a \neq 0$ satisfies $ax = 0$, then $af(x) = f(ax) = 0$, so that $f(x) \in T(N)$.

c. **Suppose we have an exact sequence**

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

of A-modules. Then we get a new exact sequence obtained by restricting f and g

$$0 \longrightarrow T(M') \xrightarrow{f} T(M) \xrightarrow{g} T(M'')$$

This sequence is clearly exact at $T(M')$. Suppose that $m \in T(M)$ and $g(m) = 0$. Choose $m' \in M'$ for which $f(m') = m$, and suppose $a \neq 0$ satisfies $am = 0$. Then $0 = am = af(m') = f(am')$. By injectivity of f we conclude that $am' = 0$, and hence $m' \in T(M')$. This means that $\text{Ker}(g|_{T(M)}) \subseteq \text{Im}(f|_{T(M')})$. The oppositive inclusion follows from $g \circ f = 0$. Therefore, the resulting sequence is exact at $T(M)$, and hence is exact.

d. $T(M)$ **is the kernel of the A-module homomorphism $x \mapsto 1 \otimes x$ of M into $K \otimes_A M$, where K is the field of fractions of A.**

Let $S = A - \{0\}$ so that $K = S^{-1}A$. Recall that the mapping $a/s \otimes m \mapsto am/s$ of $S^{-1}A \otimes_A M$ into $S^{-1}M$ is an isomorphism. So the kernel of the map $M \to K \otimes_A M$ is precisely the kernel of the canonical map $M \to S^{-1}M$ given by $x \mapsto x/1$. Now $x/1 = 0/1$ in $S^{-1}M$ precisely when there is $s \in S$ for which $sx = 0$. Since $S = A - \{0\}$, this occurs precisely when $x \in T(M)$.

3.13. **Let A be an integral domain with a multiplicatively closed subset S, and let M be an A-module. Show that $T(S^{-1}M) = S^{-1}(TM)$.**

We may assume that $0 \notin S$ since otherwise $S^{-1}M = S^{-1}(TM) = 0$. If $m/s \in T(S^{-1}M)$, then there is $a/s' \neq 0/1$ in $S^{-1}A$ so that $0/1 = (a/s')(m/s) = am/(ss')$. But then there is $s'' \in S$ for which $s''am = 0$. Now $s''a \neq 0$ since $s'', a \neq 0$. So $m \in T(M)$, and hence $m/s \in S^{-1}(TM)$. In other words, $T(S^{-1}M) \subseteq S^{-1}(TM)$.

On the other hand, if $m \in TM$ then there is $a \neq 0$ for which $am = 0$. Then $a/1 \neq 0/1$ since $0 \notin S$. Since $(a/1)(m/s) = 0/1$ for any $s \in S$, we see that $m/s \in T(S^{-1}M)$. In other words, $S^{-1}(TM) \subseteq T(S^{-1}M)$.

Deduce that the following conditions are equivalent.

a. M **is torsion free.**
b. $M_{\mathfrak{p}}$ **is torsion free for all prime ideals $\mathfrak{p}$.**
c. $M_{\mathfrak{m}}$ **is torsion free for all maximal ideals $\mathfrak{m}$.**

(a ⇒ b) $T(M_{\mathfrak{p}}) = (TM)_{\mathfrak{p}}$ by the above, and $(TM)_{\mathfrak{p}} = 0$ when $TM = 0$.
(b ⇒ c) O.K.
(c ⇒ a) $(TM)_{\mathfrak{m}} = T(M_{\mathfrak{m}})$ by the above, and $T(M_{\mathfrak{m}}) = 0$ by hypothesis. Therefore $TM = 0$.

3.14. **Let M be an A-module and $\mathfrak{a}$ an ideal of A. Suppose that $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \supseteq \mathfrak{a}$. Prove that $M = \mathfrak{a}M$.**

If $M \neq \mathfrak{a}M$, then there is $x \in M - \mathfrak{a}M$. Define an ideal $\mathfrak{b} = (\mathfrak{a}M : x)$. Then $\mathfrak{a} \subseteq \mathfrak{b} \subsetneq A$ since $1 \notin \mathfrak{b}$. So we can choose a maximal $\mathfrak{m}$ that contains $\mathfrak{b}$. By hypothesis $M_{\mathfrak{m}} = 0$, and so $x/1 = 0/1$ in $M_{\mathfrak{m}}$. So there is $a \in A - \mathfrak{m}$ for which $ax = 0$. But $0 \in \mathfrak{a}M$ so that $a \in \mathfrak{b} \subseteq \mathfrak{m}$. This contradiction shows that $M = \mathfrak{a}M$, as claimed.

3.15. **Let A be a ring and let $F = A^n$. Show that every set of n generators of F is a basis of F. Deduce that every set of generators of F has at least n elements.**

Suppose $\{x_i\}_1^n$ generates F and let $\{e_i\}_1^n$ be the standard basis. Choose b_{ij} and c_{ij} in A for which

$$x_i = \sum_{j=1}^{n} b_{ij} e_j \qquad e_i = \sum_{j=1}^{n} c_{ij} x_j$$

Define matrices $B = (b_{ij})$ and $C = (c_{ij})$. Notice that

$$e_i = \sum_{j=1}^{n} \sum_{k=1}^{n} c_{ij} b_{jk} e_k = \sum_{k=1}^{n} e_k \cdot \sum_{j=1}^{n} c_{ij} b_{jk}$$

Since $\{e_1, \ldots, e_n\}$ is linearly independent we conclude that

$$\sum_{j=1}^{n} c_{ij} b_{jk} = \delta_{ik}$$

This means that $CB = I$, so that $\det(C)\det(B) = 1$. But now $\det(B)$ is a unit in A, so that B (and hence B^T) is an invertible matrix. So suppose that $\sum_{i=1}^{n} \lambda_i x_i = 0$ for some λ_i. Then

$$0 = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i b_{ij} e_j = \sum_{j=1}^{n} e_j \cdot \sum_{i=1}^{n} b_{ij} \lambda_i$$

We see that each $\sum_{i=1}^{n} b_{ij}\lambda_i = 0$, so that $B^T\lambda = 0$. But now $\lambda = 0$ since B^T is invertible. This means that $\{x_i\}_1^n$ is linearly independent set, and hence is a basis. Further, if F is generated by m elements $x_1, \dots, x_m$ with $m < n$, then F is generated by the n elements $\{x_1, \dots, x_m, 0, \dots, 0\}$ and this is a basis by the above; a contradiction. So F is generated by no fewer than n elements.

3.16. **Let $f : A \to B$ be a ring homomorphism and assume that B is flat as an A-algebra. Show that the following are equivalent.**

a. $\mathfrak{a}^{ec} = \mathfrak{a}$ **for all ideals** $\mathfrak{a}$ **in** A.

b. $f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ **is surjective.**

c. **For every maximal ideal** $\mathfrak{m}$ **in** A **we have** $\mathfrak{m}^e \neq (1)$.

d. **If** M **is a nonzero** A**-module then** M_B **is nonzero as well.**

e. **For every** A**-module** M **the natural map** $M \to M_B$ **is injective.**

(a $\Rightarrow$ b) Assume that $\mathfrak{p} \in \operatorname{Spec}(A)$. Then $\mathfrak{p}$ is the contraction of a prime ideal in B by Proposition 3.16. This means that $\mathfrak{p}$ is in the image of f^*. In particular $\mathfrak{p} = f^*(\mathfrak{p}^e)$.

(b $\Rightarrow$ c) Since $\mathfrak{m}$ is maximal and since f^* is surjective we know that $\mathfrak{m} = \mathfrak{q}^c$ for some $\mathfrak{q} \in \operatorname{Spec}(B)$. But then $\mathfrak{m}^{ec} = \mathfrak{q}^{cec} = \mathfrak{q}^c = \mathfrak{m}$. So $\mathfrak{m}^e = (1)$ implies that $\mathfrak{m} = \mathfrak{m}^{ec} = B^c = A$, a contradiction.

(c $\Rightarrow$ d) Let $0 \neq x \in M$ so that $M' = Ax$ is a nonzero submodule of M. Then the sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M/M' \longrightarrow 0$$

is exact. Since B is flat as an A-module we have the exact sequence

$$0 \longrightarrow M'_B \longrightarrow M_B \longrightarrow (M/M')_B \longrightarrow 0$$

Since the map $M'_B \to M_B$ is injective, $M_B \neq 0$ provided that $M'_B \neq 0$. Now $M' \cong A/\operatorname{Ann}(x)$ where $\operatorname{Ann}(x) \neq A$ since $1 \notin \operatorname{Ann}(x)$. Choose a maximal ideal $\mathfrak{m}$ containing $\operatorname{Ann}(x)$. Then $\operatorname{Ann}(x)^e \subseteq \mathfrak{m}^e \subsetneq B$. Now $M'_B \cong A/\operatorname{Ann}(x) \otimes_A B \cong B/\operatorname{Ann}(x)^e \neq 0$, as claimed.

(d $\Rightarrow$ e) Let M' be the kernel of the natural map $M \to M_B$ given by $x \mapsto 1 \otimes x$. The sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M_B \longrightarrow 0$$

is exact. Since B is flat as an A-module we have an exact sequence

$$0 \longrightarrow M'_B \longrightarrow M_B \longrightarrow (M_B)_B \longrightarrow 0$$

Now the map $M_B \to (M_B)_B$ is injective by 2.13. So the image of the map $M'_B \to M_B$ is trivial. Since this map is injective, we see that $M'_B = 0$, so that $M' = 0$ by hypothesis. In other words, the natural map $M \to M_B$ is injective.

(e $\Rightarrow$ a) Let $\mathfrak{a}$ be an ideal in A. The natural map $A/\mathfrak{a} \to A/\mathfrak{a} \otimes_A B$ is injective by hypothesis. Suppose $x \in \mathfrak{a}^{ec} \subseteq A$ so that $f(x) = \sum f(a_i)b_i$ for some $a_i \in \mathfrak{a}$. Then in $A/\mathfrak{a} \otimes_A B$ we have

$$\bar{x} \otimes 1 = x \cdot \bar{1} \otimes 1 = \bar{1} \otimes x \cdot 1 = \bar{1} \otimes f(x)$$

and from this we get

$$\bar{x} \otimes 1 = \bar{1} \otimes \sum f(a_i)b_i = \sum \bar{a}_i \otimes b_i = 0$$

since each $a_i \in \mathfrak{a}$. By injectivity $\bar{x} = \bar{0}$, so that $x \in \mathfrak{a}$. Therefore $\mathfrak{a}^{ec} \subseteq \mathfrak{a}$, and hence $\mathfrak{a} = \mathfrak{a}^{ec}$.

3.17. **Let $f : A \to B$ and $g : B \to C$ be ring homomorphisms. Suppose that $g \circ f$ is flat and g is faithfully flat. Show that f is flat.**

Let $M \to N$ be an injection of A-modules. Then we have the commutative diagram

$$\begin{array}{ccc}
M \otimes_A B & \longrightarrow & N \otimes_A B \\
\downarrow {\scriptstyle x \mapsto x \otimes 1} & & \downarrow {\scriptstyle x \mapsto x \otimes 1} \\
(M \otimes_A B) \otimes_A C & \longrightarrow & (N \otimes_A B) \otimes_A C \\
\downarrow & & \downarrow \\
M \otimes_A (B \otimes_B C) & \longrightarrow & N \otimes_A (B \otimes_B C) \\
\downarrow & & \downarrow \\
M \otimes_A C & \longrightarrow & N \otimes_A C
\end{array}$$

where the last four vertical maps are natural isomorphisms, and the top two vertical maps are injections since g is faithfully flat. Finally, horizontal map on the bottom row is injective since $g \circ f$ is flat. This shows that the horizontal map on the top row is injective as well. This means that f is flat.

3.18. **Suppose $f : A \to B$ is a flat ring homomorphism. If $\mathfrak{q}$ is a prime ideal in B let $\mathfrak{p} = \mathfrak{q}^c$. Show that $f^* : \mathrm{Spec}(B_\mathfrak{q}) \to \mathrm{Spec}(A_\mathfrak{p})$ is onto.**

Since B is a flat A-module, we know that $B_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module. In fact, $B_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-algebra since $B_\mathfrak{p}$ has the obvious multiplicative structure. Since $f(A - \mathfrak{p})$ is a multiplicatively closed subset of B that does not meet $\mathfrak{q}$, we see that $B_\mathfrak{q}$ is a localization of $B_\mathfrak{p}$, so that $B_\mathfrak{q}$ is a flat $B_\mathfrak{p}$-algebra. Now exercise 2.8 tells us that $B_\mathfrak{q}$ is a flat $A_\mathfrak{p}$-algebra. The only maximal ideal of $A_\mathfrak{p}$ is $\mathfrak{p}_\mathfrak{p}$ whose contraction to $B_\mathfrak{q}$ is $\mathfrak{q}_\mathfrak{q} \neq B_\mathfrak{q}$. It follows that the map $f : A_\mathfrak{p} \to B_\mathfrak{q}$ is faithfully flat, and so the induced map $f^* : \mathrm{Spec}(B_\mathfrak{q}) \to \mathrm{Spec}(A_\mathfrak{p})$ is onto.

3.19. **Suppose M is an A-module and define $\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(A) : M_\mathfrak{p} \neq 0\}$. Show the following.**

a. $\mathrm{Supp}(M) \neq \emptyset$ **if** $M \neq 0$

If $M_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$ then $M = 0$.

b. $V(\mathfrak{a}) = \mathrm{Supp}(A/\mathfrak{a})$

Notice that $(A/\mathfrak{a})_\mathfrak{p} = 0$ iff $\bar{1}/1 = \bar{0}/1$ in $(A/\mathfrak{a})_\mathfrak{p}$. This occurs precisely when there is $x \in A - \mathfrak{p}$ satisfying $\bar{0} = x\bar{1} = \bar{x}$. But this occurs precisely when $(A - \mathfrak{p}) \cap \mathfrak{a} \neq \emptyset$. This is equivalent to $\mathfrak{a} \not\subseteq \mathfrak{p}$. Hence, $(A/\mathfrak{a})_\mathfrak{p} \neq 0$ if and only if $\mathfrak{a} \subseteq \mathfrak{p}$.

c. **Suppose we have an exact sequence**

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

and show that $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$**.**

We have the exact sequence

$$0 \longrightarrow M'_\mathfrak{p} \xrightarrow{f_\mathfrak{p}} M_\mathfrak{p} \xrightarrow{g_\mathfrak{p}} M''_\mathfrak{p} \longrightarrow 0$$

If $M_{\mathfrak{p}} = 0$ then $M'_{\mathfrak{p}} = 0$ since $f_{\mathfrak{p}}$ is injective, and $M''_{\mathfrak{p}} = 0$ since $g_{\mathfrak{p}}$ is surjective. If $M'_{\mathfrak{p}} = 0$ and $M''_{\mathfrak{p}} = 0$ then $0 = \mathrm{Im}(f_{\mathfrak{p}})$ and $\mathrm{Ker}(g_{\mathfrak{p}}) = M_{\mathfrak{p}} = 0$, implying that $M_{\mathfrak{p}} = 0$. Therefore $M_{\mathfrak{p}} \neq 0$ iff $M'_{\mathfrak{p}} \neq 0$ or $M''_{\mathfrak{p}} \neq 0$. This gives $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$.

d. **If $M = \sum M_i$ then $\mathrm{Supp}(M) = \bigcup \mathrm{Supp}(M_i)$.**

Suppose that $M_{\mathfrak{p}} = 0$ and that $m_i/s \in (M_i)_{\mathfrak{p}}$. Since m_i/s is zero in $M_{\mathfrak{p}}$, there is $x \notin \mathfrak{p}$ for which $xm_i = 0$. But then m_i/s is zero in $(M_i)_{\mathfrak{p}}$. In other words each $(M_i)_{\mathfrak{p}} = 0$. Now suppose that each $(M_i)_{\mathfrak{p}} = 0$. If $(\sum m_i)/s \in M_{\mathfrak{p}}$, then there are $x_i \notin \mathfrak{p}$ for which $x_i m_i = 0$, so that $(\prod x_i) \sum m_i = 0$. In other words $M_{\mathfrak{p}} = 0$. So $M_{\mathfrak{p}} = 0$ iff each $(M_i)_{\mathfrak{p}} = 0$. This yields $\mathrm{Supp}(M) = \bigcup \mathrm{Supp}(M_i)$.

e. **If M is finitely generated then $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$.**

Since M is finitely generated $(A - \mathfrak{p})^{-1}M = 0$ iff $xM = 0$ for some $x \in A - \mathfrak{p}$. This occurs iff $(A - \mathfrak{p}) \cap \mathrm{Ann}(M) \neq \emptyset$, or equivalently iff $\mathrm{Ann}(M) \not\subseteq \mathfrak{p}$. So $M_{\mathfrak{p}} \neq 0$ iff $\mathrm{Ann}(M) \subseteq \mathfrak{p}$.

f. **If M and N are finitely generated then $\mathrm{Supp}(M \otimes_A N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$.**

Recall that $(M \otimes_A N)_{\mathfrak{p}}$ and $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}}$ are isomorphic as $A_{\mathfrak{p}}$-modules. Since M, N are finitely generated A-modules we see that $M_{\mathfrak{p}}, N_{\mathfrak{p}}$ are finitely generated $A_{\mathfrak{p}}$-modules. So exercise 2.3 tells us that $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} = 0$ iff $M_{\mathfrak{p}} = 0$ or $N_{\mathfrak{p}} = 0$.

g. **If M is finitely generated and $\mathfrak{a}$ is an ideal in A, then $\mathrm{Supp}(M/\mathfrak{a}M) = V(\mathfrak{a} + \mathrm{Ann}(M))$.**

Since M is finitely generated, $M/\mathfrak{a}M$ and $A/\mathfrak{a} \otimes_A M$ are isomorphic as A-modules by exercise 2.2. Further, $A/\mathfrak{a}$ is generated by the single element $1 + \mathfrak{a}$ as an A-module. So

$$\begin{aligned} \mathrm{Supp}(M/\mathfrak{a}M) &= \mathrm{Supp}(A/\mathfrak{a} \otimes_A M) \\ &= \mathrm{Supp}(A/\mathfrak{a}) \cap \mathrm{Supp}(M) \\ &= V(\mathfrak{a}) \cap V(\mathrm{Ann}(M)) \\ &= V(\mathfrak{a} + \mathrm{Ann}(M)) \end{aligned}$$

h. **If $f : A \to B$ is a ring homomorphism and if M is a finitely generated A-module, then $\mathrm{Supp}(B \otimes_A M) = f^{*-1}(\mathrm{Supp}(M))$.**

Since M is a finitely generated A-module we have $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$, and since M_B is a finitely generated B-module we have $\mathrm{Supp}(M_B) = V(\mathrm{Ann}(M_B))$. So we need to prove that a prime ideal $\mathfrak{q}$ in B contains $\mathrm{Ann}(M_B)$ if and only if $f^{-1}(\mathfrak{q})$ contains $\mathrm{Ann}(M)$. Suppose $\mathfrak{q} \supseteq \mathrm{Ann}(M_B)$ and $a \in \mathrm{Ann}(M)$ so that $a \cdot m = 0$ for every $m \in M$. Then $f(a)$ annihilates M_B since $f(a)(b \otimes m) = f(a)b \otimes m = a \cdot b \otimes m = b \otimes a \cdot m = 0$ for all $b \in B$ and $m \in M$. By hypothesis, $f(a) \in \mathfrak{q}$. This means that $\mathrm{Ann}(M) \subseteq f^{-1}(\mathfrak{q})$. Now suppose that $\mathrm{Ann}(M) \subseteq f^{-1}(\mathfrak{q})$ and let $b \in \mathrm{Ann}(M_B)$.

3.20. **Let $f : A \to B$ be a ring homomorphism. Show the following.**

a. **Every prime ideal in A is a contracted ideal $\Leftrightarrow$ f^* is onto.**

Suppose $\mathfrak{p}$ is a prime ideal in A. Proposition 1.17 and 3.16 yield: $\mathfrak{p}$ is a contracted ideal in A iff $\mathfrak{p}$ satisfies $\mathfrak{p}^{ec} = \mathfrak{p}$ iff $\mathfrak{p}$ is the contraction of a prime ideal in B iff $\mathfrak{p}$ lies in the image of f^*.

b. **Every prime ideal in B is an extended ideal $\Rightarrow f^*$ is 1-1.**

Assume that every prime ideal in B is an extended ideal. Suppose that $f^*(\mathfrak{p}) = f^*(\mathfrak{q})$, so that $\mathfrak{p}^c = \mathfrak{q}^c$. Then $\mathfrak{p} = \mathfrak{p}^{ce} = \mathfrak{q}^{ce} = \mathfrak{q}$ by Proposition 1.17. But this means that f^* is 1-1.

c. **Is the converse to part b true?**

The converse to part b is false. Let $j : \mathbb{Z} \to \mathbb{Z}[i]$ be the natural inclusion map. If p is a prime congruent to 3 modulo 4, then (p) is a prime ideal in $\mathbb{Z}[i]$. If p is a prime congruent to 1 modulo 4, then there are unique $a, b > 0$ such that $a^2 + b^2 = p$, and $(a + bi)$ is a prime ideal in $\mathbb{Z}[i]$. Also, $(1 + i)$ is a prime ideal in $\mathbb{Z}[i]$. These are all of the prime ideals in $\mathbb{Z}[i]$. Now the contraction of (p) equals (p), the contraction of $(a + bi)$ equals $(a^2 + b^2)$, and the contraction of $(1 + i)$ equals (2). This means that j^* is an injective map. However, the extension of (2) and (p) are not prime ideals, for p a prime congruent to 1 modulo 4. Also, the extension of (p) equals (p), for p a prime congruent to 3 modulo 4. This means that $(1 + i)$ and prime ideals of the form $(a + bi)$ are not extended ideals in $\mathbb{Z}[i]$.

3.21. **Throughout, $f : A \to B$ is a ring homomorphism, $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$, S is a multiplicatively closed subset of A, and $\phi_A : A \to S^{-1}A$ is the canonical homomorphism. Establish the following facts.**

a. **$\phi^* : \text{Spec}(S^{-1}A) \to X$ is a homeomorphism onto its image, which we denote by $S^{-1}X$.**

Notice that $S^{-1}X$ consists of all prime ideals in A that have empty intersection with S. Now every ideal in $S^{-1}A$ is an extended ideal so that ϕ^* is 1-1 by exercise 2.20. As always, ϕ^* is continuous. I claim that ϕ^* is a closed map. Let $\mathfrak{a}$ be an ideal in A and notice that

$$\phi^*(V(S^{-1}\mathfrak{a})) = S^{-1}X \cap V(\mathfrak{a}^{ec})$$

After all, if $\mathfrak{p} \in \phi^*(V(S^{-1}\mathfrak{a}))$ then $\mathfrak{p} \cap S = \emptyset$ and $S^{-1}\mathfrak{a} \subseteq S^{-1}\mathfrak{p}$ so that $\mathfrak{a}^{ec} \subset \mathfrak{p}^{ec} = \mathfrak{p}$. Conversely, if $\mathfrak{p} \in S^{-1}X \cap V(\mathfrak{a}^{ec})$ then $\mathfrak{p} \cap S = \emptyset$ and $\mathfrak{a} = S^{-1}\mathfrak{a}^{ec} \subseteq S^{-1}\mathfrak{p}$. So ϕ is a homeomorphism onto its image.

b. **Identify $\text{Spec}(S^{-1}A)$ with its image $S^{-1}X$, and identify $\text{Spec}(S^{-1}B)$ with its image $S^{-1}Y$. Then $(S^{-1}f)^*$ is the restriction of f^* to $S^{-1}Y$, and $S^{-1}Y = f^{*-1}(S^{-1}X)$.**

Notice that $S^{-1}B = f(S)^{-1}B$ as in exercise 3.4 and that $S^{-1}f(a/s) = f(a)/f(s)$. So we have the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \downarrow{\scriptstyle \phi_A} & & \downarrow{\scriptstyle \phi_B} \\ S^{-1}A & \xrightarrow{\quad S^{-1}f \quad} & S^{-1}B \end{array}$$

This yields the commutative diagram

$$\begin{array}{ccc} \text{Spec}(S^{-1}B) & \xrightarrow{\quad (S^{-1}f)^* \quad} & \text{Spec}(S^{-1}A) \\ \downarrow{\scriptstyle \phi_B^*} & & \downarrow{\scriptstyle \phi_A^*} \\ \text{Spec}(B) & \xrightarrow{\quad f^* \quad} & \text{Spec}(A) \end{array}$$

as desired. Now obviously $S^{-1}Y \subseteq f^{*-1}(S^{-1}X)$. So suppose that $\mathfrak{q} \in Y$ and $f^*(\mathfrak{q}) \in S^{-1}X$. Then $f^{-1}\mathfrak{q}$ is a prime ideal in A that has empty intersection with $f(S)$. If $x \in \mathfrak{q} \cap f(S)$ with $x = f(s)$ then $s \in f^{-1}(\mathfrak{q}) \cap S$, which is not possible. So $\mathfrak{q} \cap f(S) = \emptyset$, implying that $\mathfrak{q} \in S^{-1}Y$. Hence

$S^{-1}Y = f^{*-1}(S^{-1}X)$.

c. **Let $\mathfrak{a}$ be an ideal in A and write $\mathfrak{b} = B\mathfrak{a}$. Then f induces a map $\bar{f} : A/\mathfrak{a} \to B/\mathfrak{b}$. If $\mathrm{Spec}(A/\mathfrak{a})$ is identified with its image $V(\mathfrak{a})$ in X and $\mathrm{Spec}(B/\mathfrak{b})$ is identified with its image $V(\mathfrak{b})$ in Y, then $\bar{f}^*$ is the restriction of f^* to $V(\mathfrak{a})$.**

We have the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_B} \\ A/\mathfrak{a} & \xrightarrow{\quad \bar{f} \quad} & B/\mathfrak{b} \end{array}$$

This yields the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(B/\mathfrak{b}) & \xrightarrow{\quad \bar{f}^* \quad} & \mathrm{Spec}(A/\mathfrak{a}) \\ \downarrow{\scriptstyle \pi_B^*} & & \downarrow{\scriptstyle \pi_A^*} \\ \mathrm{Spec}(B) & \xrightarrow{\quad f^* \quad} & \mathrm{Spec}(A) \end{array}$$

Now exercise 1.21 tells us that π_B^* maps $\mathrm{Spec}(B/\mathfrak{b})$ homeomorphically onto $V(\mathrm{Ker}(\pi_B)) = V(\mathfrak{b})$, and π_A^* maps $\mathrm{Spec}(A/\mathfrak{a})$ homeomorphically onto $V(\mathrm{Ker}(\pi_A)) = V(\mathfrak{a})$. We are done.

d. **Let $\mathfrak{p}$ be a prime ideal in A and define $S = A - \mathfrak{p}$. Then the subspace $f^{*-1}(\mathfrak{p})$ of Y is homeomorphic with $\mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}B_\mathfrak{p}) = \mathrm{Spec}(k(\mathfrak{p}) \otimes_A B)$, where $k(\mathfrak{p})$ is the residue field of $A_\mathfrak{p}$.**

We use part c with $\mathfrak{a} = \mathfrak{p}_\mathfrak{p}$ and $\mathfrak{b} = \mathfrak{p}_\mathfrak{p}^e = \mathfrak{p}_\mathfrak{p}B_\mathfrak{p} = (\mathfrak{p}B)_\mathfrak{p}$ to get the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}B_\mathfrak{p}) & \xrightarrow{\quad \overline{f_\mathfrak{p}}^* \quad} & \mathrm{Spec}(A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}) \\ \downarrow{\scriptstyle \pi_B^*} & & \downarrow{\scriptstyle \pi_A^*} \\ \mathrm{Spec}(B_\mathfrak{p}) & \xrightarrow{\quad (f_\mathfrak{p})^* \quad} & \mathrm{Spec}(A_\mathfrak{p}) \\ \downarrow{\scriptstyle \phi_B^*} & & \downarrow{\scriptstyle \phi_A^*} \\ \mathrm{Spec}(B) & \xrightarrow{\quad f^* \quad} & \mathrm{Spec}(A) \end{array}$$

Now $\mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}B_\mathfrak{p})$ is homeomorphic with $V((\mathfrak{p}B)_\mathfrak{p})$, which is homeomorphic with $\phi_B^*(V((\mathfrak{p}B)_\mathfrak{p}))$. I claim that $\phi_B^*(V((\mathfrak{p}B)_\mathfrak{p})) = f^{*-1}(\mathfrak{p})$, establishing the first result. So suppose that $\mathfrak{q} \in f^{*-1}(\mathfrak{p})$. Since $\mathfrak{p} \in \mathrm{Im}(\phi_A^*)$ we see that $\mathfrak{q} \in \mathrm{Im}(\phi_B^*)$. Also, $\mathfrak{p} = f^{-1}(\mathfrak{q})$, so that $f(\mathfrak{p}) \subseteq \mathfrak{q}$, and hence $\mathfrak{p}B \subseteq \mathfrak{q}$. But now $\mathfrak{q}_\mathfrak{p}$ is a prime ideal in $B_\mathfrak{p}$ containing the ideal $(\mathfrak{p}B)_\mathfrak{p}$. Conversely, assume that $\mathfrak{q} \in \phi_B^*(V((\mathfrak{p}B)_\mathfrak{p}))$. Then $(\mathfrak{p}B)_\mathfrak{p} \subseteq \mathfrak{q}_\mathfrak{p}$ so that $\mathfrak{p}B \subseteq \mathfrak{q}_\mathfrak{p}^c = \mathfrak{q}$, and hence $f(\mathfrak{p}) \subseteq \mathfrak{q}$. So we see that $\mathfrak{p} \subseteq f^{-1}(\mathfrak{q})$. On the other hand, it is trivial to check that $f^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}$ since $\mathfrak{q} \cap f(A - \mathfrak{p}) = \emptyset$. So the claim is established. Now we have a chain of isomorphisms between $A_\mathfrak{p}$-modules

$$\begin{aligned} B_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}B_{\mathfrak{p}} &= B_{\mathfrak{p}}/(\mathfrak{p}B)_{\mathfrak{p}} \\ &\cong (B/\mathfrak{p}B)_{\mathfrak{p}} \\ &\cong A_{\mathfrak{p}} \otimes_A B/\mathfrak{p}B \\ &\cong A_{\mathfrak{p}} \otimes_A (A/\mathfrak{p} \otimes_A B) \\ &\cong (A/\mathfrak{p} \otimes_A A_{\mathfrak{p}}) \otimes_A B \\ &\cong A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_A B \\ &= A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}} \otimes_A B \\ &= k(\mathfrak{p}) \otimes_A B \end{aligned}$$

Specifically, the map is given by

$$b/f(x) + \mathfrak{p}_{\mathfrak{p}}B_{\mathfrak{p}} \mapsto (1/x + \mathfrak{p}_{\mathfrak{p}}) \otimes b$$

It is easy to see that this preserves the product structure of our rings. Consequently, $\operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}B_{\mathfrak{p}}) = \operatorname{Spec}(k(\mathfrak{p}) \otimes_A B)$.

3.22. **Let A be a ring and $\mathfrak{p}$ a prime ideal in A. Show that the canonical image $X_{\mathfrak{p}}$ of $\operatorname{Spec}(A_{\mathfrak{p}})$ in $X = \operatorname{Spec}(A)$ is equal to the intersection of all open neighborhoods of $\mathfrak{p}$ in X.**

As in 3.21, $X_{\mathfrak{p}}$ consists of all prime ideals in A that have empty intersection with $S = A - \mathfrak{p}$, that is, the prime ideals contained in $\mathfrak{p}$. Suppose $\mathfrak{q} \not\subseteq \mathfrak{p}$, so that $\mathfrak{p} \notin V(\mathfrak{q})$. Then $\mathfrak{q} \notin X - V(\mathfrak{q})$, even though $X - V(\mathfrak{q})$ is an open neighborhood of $\mathfrak{p}$ in X. Conversely, if $\mathfrak{q} \subseteq \mathfrak{p}$, then $\mathfrak{p} \in X - V(E)$ implies that $E \not\subseteq \mathfrak{p}$, and consequently $E \not\subseteq \mathfrak{q}$, so that $\mathfrak{q} \in X - V(E)$. So we are done.

3.23? **Let A be a ring with $X = \operatorname{Spec}(A)$ and assume that $U = X_f = A - V(f)$ for some $f \in A$. Show the following.**

a. **The ring $A(U) := A_f$ is independent of f.**

Suppose that $X_f = X_g$, so that $f \in r((g))$ and $g \in r((f))$, as according to exercise 1.17. Then $f^m = ag$ and $g^n = bf$ for some $a, b \in A$ and $m, n > 0$. Define

$$F : A_f \to A_g \quad \text{by} \quad F(x/f^p) = xb^p/g^{np}$$

and define

$$G : A_g \to A_f \quad \text{by} \quad G(x/g^p) = xa^p/f^{mp}$$

Notice that

$$\begin{aligned} G(F(x/f^p)) &= G(xb^p/g^{np}) \\ &= G(xb^p a^{np}/f^{mnp}) \\ &= xb^p a^{np}/a^{np}g^{np} \\ &= xb^p/b^p f^p \\ &= x/f^p \end{aligned}$$

Similarly, $F(G(x/g^p)) = x/g^p$. Thus, F and G are bijections and inverse to one another. Another tedious calculation reveals that F is additive since

$$\begin{aligned}F(x/f^p + x'/f^q) &= F((f^q x + f^p x')/f^{p+q})\\ &= (f^q x + f^p x')b^{p+q}/g^{n(p+q)}\\ &= (f^q x + f^p x')b^{p+q}/b^{p+q}f^{p+q}\\ &= x/f^p + x'/f^q\end{aligned}$$

Clearly F and G respect the multiplication. Lastly, F and G are well-defined: suppose $x/f^p = 0/1$ in A_f so that $f^q x = 0$ for some q. Then clearly $b^p b^q f^q x = 0$, so that $g^{nq} x b^p = 0$, implying that $xb^p/g^{np} = 0/1$ in A_g. Hence, A_f and A_g are isomorphic, as desired.

b. **Suppose $U' = X_g$ satisfies $U' \subseteq U$. There is a natural homomorphism $\rho : A(U) \to A(U')$ that is independent of f, g.**

If $U' \subseteq U$ then $V(f) \subseteq V(g)$, so that any prime ideal containing f contains g. This means that $g \in r(f)$, so that $g^m = af$ for some $m > 0$ and some $a \in A$. As in part a, we define a map

$$\rho : A_f \to A_g \quad \text{by} \quad \rho(x/f^r) = xa^r/g^{mr}$$

This is a well-defined ring homomorphism. Now suppose $X_f = X_{f'}$ and $X_g = X_{g'}$. Then we have equations

$$(f')^n = bf \qquad (g')^p = cg \qquad (g')^q = df'$$

Define maps

$$F : X_f \to X_{f'} \quad \text{by} \quad F(x/f^r) = xb^r/f'^{nr}$$

and

$$G : X_g \to X_{g'} \quad \text{by} \quad G(x/g^r) = xc^r/g'^{pr}$$

we also need

$$\rho' : X_{f'} \to X_{g'} \quad \text{by} \quad \rho'(x/f'^r) = xd^r/g'^{qr}$$

To say that ρ is independent of f and g is to say that $\rho' \circ F = G \circ \rho$. But $\rho'(F(x/f^r)) = xb^r d^{nr}/g'^{qnr}$ and $G(\rho_{fg}(x/f^r)) = xa^r c^{mr}/g'^{mpr}$. Using the equations above we see that

$$(b^r d^{nr})g'^{mpr} - (a^r c^{mr})g'^{qnr} = 0$$

So equality follows, showing that ρ is independent of f, g.

c. **If $U' = U$ then $\rho = \text{id}$.**

This follows from part b.

d. **If $U'' \subseteq U' \subseteq U$ then ρ acts 'functorially'.**

Write $U'' = X_h, U' = X_g, U = X_f$. We can write $g^m = af$ and $h^n = bg$.

e. **If $\mathfrak{p} \in X$ then $\lim\limits_{\mathfrak{p} \in U} A(U) \cong A_{\mathfrak{p}}$.**

3.24?

3.25. **Let $f : A \to B$ and $g : A \to C$ be ring homomorphisms. Suppose $h : A \to B \otimes_A C$ is defined by $h(a) = f(a) \otimes 1 = 1 \otimes g(a)$. Define X, Y, Z, T to be the spectra of $A, B, C, B \otimes_A C$ respectively. Show that $h^*(T) = f^*(Y) \cap g^*(Z)$.**

Let $\mathfrak{p} \in X$ and define $k = k(\mathfrak{p})$. We have a natural homeomorphism between $h*^{-1}(\mathfrak{p})$ and $\operatorname{Spec}((B \otimes_A C) \otimes_A k)$, and also

$$\begin{aligned}(B \otimes_A C) \otimes_A k &\cong B \otimes_A k \otimes_A C \\ &\cong B \otimes_A (k \otimes_k k) \otimes_A C \\ &\cong B \otimes_A (k \otimes_k (k \otimes_A C)) \\ &\cong B \otimes_A (k \otimes_k (C \otimes_A k)) \\ &\cong (B \otimes_A k) \otimes_k (C \otimes_A k)\end{aligned}$$

Now $\mathfrak{p} \in h^*(T)$ precisely when $h^{*-1}(\mathfrak{p}) \neq \emptyset$. By the natural homeomorphism this occurs precisely when $\operatorname{Spec}((B \otimes_A C) \otimes_A k) \neq \emptyset$. Now the spectrum of any ring is nonempty if and only if that ring is nonzero. Since $B \otimes_A k$ and $C \otimes_A k$ are vector spaces over k, we see that $(B \otimes_A k) \otimes_k (C \otimes_A k) \neq 0$ if and only if $B \otimes_A k \neq 0$ and $C \otimes_A k \neq 0$. Again, this occurs precisely when $\mathfrak{p} \in f^*(Y)$ and $\mathfrak{p} \in g^*(Z)$. So we are done.

3.26. **Let $(B_\alpha, g_{\alpha\beta})$ be a direct system of rings and B the direct limit. For each α let $f_\alpha : A \to B_\alpha$ be a ring homomorphism satisfying $g_{\alpha\beta} \circ f_\alpha = f_\beta$ whenever $\alpha \leq \beta$. Then there is an induced map $f : A \to B$. Show that**

$$f^*(\operatorname{Spec}(B)) = \bigcap_\alpha f_\alpha^*(\operatorname{Spec}(B_\alpha))$$

Let $\mathfrak{p} \in \operatorname{Spec}(A)$. Then $\mathfrak{p} \notin f^*(\operatorname{Spec}(B))$ precisely when $f^*(\mathfrak{p}) = \emptyset$. This occurs precisely when $\operatorname{Spec}(B \otimes_A k(\mathfrak{p})) = \emptyset$. As in exercise 25, this happens if and only if $B \otimes_A k(\mathfrak{p}) = 0$. But we have the isomorphism

$$B \otimes_A k(\mathfrak{p}) \cong \varinjlim (B_\alpha \otimes_A k(\mathfrak{p}))$$

since the direct limit commutes with tensor products. So $B \otimes_A k(\mathfrak{p}) = 0$ if and only if some $B_\alpha \otimes_A k(\mathfrak{p}) = 0$. Again, this occurs precisely when $\mathfrak{p} \notin f_\alpha^*(\operatorname{Spec}(B_\alpha))$ for some α. So we are done.

3.27? **Prove the following.**

a. **Let $f_\alpha : A \to B_\alpha$ be any family of A-algebras and let $f : A \to B$ be their tensor product over A. Then**

$$f^*(\operatorname{Spec}(B)) = \bigcap_\alpha f_\alpha^*(\operatorname{Spec}(B_\alpha))$$

b. **Let $f_\alpha : A \to$**

c.

d. **The space X endowed with the constructible topology (denoted hereafter as X_C) is compact.**

3.28? **Prove the following results.**

a. **X_g is open and closed in the constructible topology.**

b. **Let C' denote the smallest topology on X for which the sets X_g are both open and closed, and let $X_{C'}$ denote the set X with this topology. Show that $X_{C'}$ is Hausdorff.**

c. **Deduce that the identity map $X_C \to X_{C'}$ is a homeomorphism. Hence, a subset E of X is of the form $f^*(\mathrm{Spec}(B))$ for some $f : A \to B$ if and only if it is closed in C'.**

d. X_C **is compact Hausdorff and totally disconnected.**

3.29? **Show that, for $f : A \to B$, the map $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a continuous and closed mapping, when $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ are given the constructible topology.**

3.30? **Show that the Zariski topology and the constructible topology on $\mathrm{Spec}(A)$ coincide iff $A/\mathfrak{N}(A)$ is absolutely flat.**

If the two topologies coincide, then $\mathrm{Spec}(A)$ is Hausdorff in the Zariski topology, and so $A/\mathfrak{N}(A)$ is absolutely flat. Suppose then that $A/\mathfrak{N}(A)$ is absolutely flat. Let $f : A \to B$ be a ring homomorphism so that $f^*(\mathrm{Spec}(A))$ is closed in the constructible topology.

Chapter 4 : Primary Decomposition

4.1. **If the ideal $\mathfrak{a}$ has a primary decomposition in A, then $\operatorname{Spec}(A/\mathfrak{a})$ has finitely many irreducible components.**

The minimal elements in the set of all prime ideals containing $\mathfrak{a}$ is precisely the set of isolated primes belonging to $\mathfrak{a}$ in any primary decomposition of $\mathfrak{a}$. But the isolated primes belonging to $\mathfrak{a}$ are uniquely determined, so that there are finitely many minimal elements in the set of all prime ideals containing $\mathfrak{a}$. This means that there are finitely many minimal prime ideals in $A/\mathfrak{a}$. Also, the irreducible components of $\operatorname{Spec}(A/\mathfrak{a})$ are of the form $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal in $A/\mathfrak{a}$. So $\operatorname{Spec}(A/\mathfrak{a})$ has finitely many irreducible components.

4.2. **If $\mathfrak{a} = r(\mathfrak{a})$ then $\mathfrak{a}$ has no embedded prime ideals.**

Let Σ consist of all the prime ideals containing $\mathfrak{a}$, and let $\Sigma' \subseteq \Sigma$ consist of the minimal elements in Σ. Then

$$\mathfrak{a} = r(\mathfrak{a}) = \bigcap_{\mathfrak{p}\in\Sigma} \mathfrak{p} = \bigcap_{\mathfrak{p}\in\Sigma'} \mathfrak{p}$$

Since $\mathfrak{a}$ is decomposable, Σ' is finite. By using proposition 1.11 we see that $\mathfrak{a}$ has the minimal primary decomposition

$$\mathfrak{a} = \bigcap_{\mathfrak{p}\in\Sigma'} \mathfrak{p}$$

But the first uniqueness theorem tells us that $\{\mathfrak{p} : \mathfrak{p} \in \Sigma'\}$ is uniquely determined by $\mathfrak{a}$. We conclude that $\mathfrak{a}$ has no embedded prime ideals.

4.3. **Every primary ideal in A is maximal if A is absolutely flat.**

Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal in A. If A is absolutely flat then so is $A/\mathfrak{N}(A)$, since it is a homomorphic image of A. This tells us that every prime ideal in A is maximal. In particular $A_\mathfrak{p}$ is a field. This means that (0) is the only primary ideal in $A_\mathfrak{p}$. Now the correspondence in Prop 4.8 tells us that $\mathfrak{q} = \mathfrak{p}$.

After all, if $\mathfrak{p}' \cap (A - \mathfrak{p}) = \emptyset$ with $\mathfrak{p}'$ a prime ideal, then $\mathfrak{p}' \subseteq \mathfrak{p}$, so that $\mathfrak{p}' = \mathfrak{p}$. So the $\mathfrak{p}$-primary ideals are in a bijective correspondence with the primary ideals in $A_\mathfrak{p}$. But there is only one primary ideal in $A_\mathfrak{p}$, and we already know that $\mathfrak{p}$ is a $\mathfrak{p}$-primary ideal since $\mathfrak{p}$ is a maximal ideal. This forces us to conclude that $\mathfrak{q} = \mathfrak{p}$.

4.4. **In the polynomial ring $\mathbb{Z}[t]$, the ideal $\mathfrak{m} = (2, t)$ is maximal and the ideal $\mathfrak{q} = (4, t)$ is $\mathfrak{m}$-primary, but $\mathfrak{q}$ is not a power of $\mathfrak{m}$.**

$\mathfrak{m}$ is a maximal ideal since $\mathbb{Z}[t]/\mathfrak{m} \cong \mathbb{Z}_2$ is a field. Clearly $\mathfrak{q} \subseteq \mathfrak{m} \subseteq r(\mathfrak{q})$. Since $\mathfrak{m}$ is a prime ideal we have $\mathfrak{m} = r(\mathfrak{q})$. Since $\mathfrak{m}$ is maximal we conclude that $\mathfrak{q}$ is $\mathfrak{m}$-primary. Now $(4, 4t, t^2) = \mathfrak{m}^2 \subseteq \mathfrak{q} \subseteq \mathfrak{m}$. The first inclusion is strict since $t \in \mathfrak{q} - \mathfrak{m}^2$, and the second inclusion is strict since $2 \in \mathfrak{m} - \mathfrak{q}$. So $\mathfrak{q}$ is not a power of $\mathfrak{m}$.

4.5. **Let K be a field and $A = K[x, y, z]$. Write $\mathfrak{p}_1 = (x, y), \mathfrak{p}_2 = (x, z)$, and $\mathfrak{m} = (x, y, z)$, so that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime ideals, while $\mathfrak{m}$ is maximal. Let $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2$. Show that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a minimal primary decomposition of $\mathfrak{a}$. Which components are isolated and which are embedded?**

Notice that $\mathfrak{a} = (x^2, xy, xz, yz)$ so that $\mathfrak{a} \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ by inspection. Suppose that $p \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Since $p \in \mathfrak{m}^2$ we can write

$$p = ax^2 + by^2 + cz^2 + dxy + exz + fyz$$

where $a, b, \ldots \in A$. But $c = 0$ since $p \in \mathfrak{p}_1$ and $b = 0$ since $p \in \mathfrak{p}_2$. Hence

$$p = ax^2 + dxy + exz + fyz \in \mathfrak{a}$$

so that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Now we know by proposition 4.2 that $\mathfrak{m}^2$ is a primary ideal, as are all prime ideals. So $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a primary decomposition of $\mathfrak{a}$. It satisfies the first condition for minimality since $r(\mathfrak{p}_i) = \mathfrak{p}_i$ and $r(\mathfrak{m}^2) = \mathfrak{m}$ are all distinct. The second condition is satisfied since

$$z^2 \in (\mathfrak{p}_2 \cap \mathfrak{m}^2) - \mathfrak{p}_1 \qquad y^2 \in (\mathfrak{p}_1 \cap \mathfrak{m}^2) - \mathfrak{p}_2 \qquad x \in (\mathfrak{p}_1 \cap \mathfrak{p}_2) - \mathfrak{m}^2$$

Thus, the primary decomposition is indeed minimal. Lastly, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are the isolated components and $\mathfrak{m}^2$ is the embedded component.

4.6. **Let X be an infinite compact Hausdorff space and $C(X)$ the ring of all real-valued continuous functions on X. Is the zero ideal decomposable in this ring?**

Let $\mathfrak{m}_x$ consist of all $f \in C(X)$ for which $f(x) = 0$. Then $\mathfrak{m}_x$ is a maximal ideal in X since $C(X)/\mathfrak{m}_x$ is isomorphic with $\mathbb{R}$ under the map $f + \mathfrak{m}_x \mapsto f(x)$. If Σ_x is the set of all prime ideals in $C(X)$ contained in $\mathfrak{m}_x$, then $\mathfrak{m}_x \in \Sigma_x$, and so Σ_x is nonempty. Let $\mathfrak{p}_x$ be a minimal element in Σ_x. This exists by a straightforward application of Zorn's Lemma. If 0 is decomposable, then there are finitely many minimal prime ideals in $C(X)$, by proposition 4.6. So to show that 0 is not decomposable it suffices to show that $\mathfrak{p}_x \neq \mathfrak{p}_{x'}$ whenever $x \neq x'$. Here we use the fact that X is infinite.

So assume that $x \neq x'$. Choose a neighborhood U of x not containing x'. Notice that X is normal since it is compact Hausdorff. Hence, there is a neighborhood V of x so that $\mathrm{Cl}(V) \subset U$. By Urysohn's Lemma there is $f \in C(X)$ so that $f = 0$ on $\mathrm{Cl}(V)$ and $f(x') = 1$. Similarly, there is $g \in C(X)$ so that $g = 0$ on $X - V$ and $g(x) = 1$. Then $f \in \mathfrak{p}_x$ since $fg = 0 \in \mathfrak{p}_x$ and $g \notin \mathfrak{p}_x$. Since $f \notin \mathfrak{p}_{x'}$ we see that $\mathfrak{p}_x \neq \mathfrak{p}_{x'}$, as claimed.

4.7. **If $\mathfrak{a}$ is an ideal of the ring A, let $\mathfrak{a}[x]$ consist of all polynomials in $A[x]$ with coefficients in $\mathfrak{a}$. Show the following.**

a. **The extension of $\mathfrak{a}$ to $A[x]$ equals $\mathfrak{a}[x]$.**

By definition $\mathfrak{a}^e = \mathfrak{a}A[x]$. A moment's worth of thought though shows that $\mathfrak{a}A[x] = \mathfrak{a}[x]$.

b. **If $\mathfrak{p}$ is a prime ideal in A then $\mathfrak{p}[x]$ is a prime ideal in $A[x]$.**

Define a ring homomorphism

$$A[x] \to (A/\mathfrak{p})[x] \quad \text{by} \quad \sum a_k x^k = \sum (a_k + \mathfrak{p}) x^k$$

This is a surjective map with kernel $\mathfrak{p}[x]$. So $A[x]/\mathfrak{p}[x]$ is isomorphic with $(A/\mathfrak{p})[x]$. But $(A/\mathfrak{p})[x]$ is an integral domain since $A/\mathfrak{p}$ is an integral domain. Therefore, $\mathfrak{p}[x]$ is a prime ideal in $A[x]$.

c. **If $\mathfrak{q}$ is $\mathfrak{p}$-primary in A then $\mathfrak{q}[x]$ is $\mathfrak{p}[x]$-primary in $A[x]$.**

First $A[x]/\mathfrak{q}[x] \neq 0$ since $1 \notin \mathfrak{q}[x]$. As above, $A[x]/\mathfrak{q}[x]$ is isomorphic with $(A/\mathfrak{q})[x]$. So if $\sum a_k x^k + \mathfrak{q}[x]$ is a zero-divisor in $A[x]/\mathfrak{q}[x]$, then $\sum (a_k + \mathfrak{q}) x^k$ is a zero-divisor in $(A/\mathfrak{q})[x]$. Hence, there is $b \in A - \mathfrak{q}$ satisfying $\bar{b} \sum (a_k + \mathfrak{q}) x^k = 0$. This means that $ba_k \in \mathfrak{q}$ for all k. So for every k there is $n > 0$ satisfying $a_k^n \in \mathfrak{q}$. This means that $a_k + \mathfrak{q}$ is nilpotent in $A/\mathfrak{q}$, and hence $\sum (a_k + \mathfrak{q}) x^k$ is nilpotent in $(A/\mathfrak{q})[x]$ as well. Consequently, $\sum a_k x^k + \mathfrak{q}[x]$ is nilpotent in $A[x]/\mathfrak{q}[x]$. So every zero-divisor in $A[x]/\mathfrak{q}[x]$ is nilpotent, implying that $\mathfrak{q}[x]$ is primary.

Notice that $\sum (a_k + \mathfrak{q}) x^k \in (A/\mathfrak{q})[x]$ is nilpotent iff each $a_k + \mathfrak{q}$ is nilpotent in $A/\mathfrak{q}$. This occurs precisely when $a_k \in \mathfrak{p}$. So $\mathfrak{N}((A/\mathfrak{q})[x]) = (\mathfrak{p}/\mathfrak{q})[x]$, and hence $\mathfrak{N}(A[x]/\mathfrak{q}[x]) = \mathfrak{p}[x]/\mathfrak{q}[x]$. This means that

$$r(\mathfrak{q}[x]) = \pi^{-1}(\mathfrak{N}(A[x]/\mathfrak{q}[x])) = \pi^{-1}(\mathfrak{p}[x]/\mathfrak{q}[x]) = \mathfrak{p}[x]$$

d. **If $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ is a minimal primary decomposition in A then $\mathfrak{a}[x] = \bigcap_{i=1}^n \mathfrak{q}_i[x]$ is a minimal primary decomposition in $A[x]$.**

Notice that $\mathfrak{a}[x] = \mathfrak{a}^e \subseteq \bigcap_1^n \mathfrak{q}_k^e = \bigcap_1^n \mathfrak{q}_k[x]$. On the other hand, if $\sum a_k x^k \notin \mathfrak{a}[x]$, then some $a_k \notin \mathfrak{a}$, and so $a_k \notin \mathfrak{q}_j$ for some j. But then $\sum a_k x^k \notin \mathfrak{q}_j[x]$. Therefore, $\mathfrak{a}[x] = \bigcap_1^n \mathfrak{q}_k[x]$ is a primary decomposition of $\mathfrak{a}[x]$. Notice that $\mathfrak{p}_k[x] \neq \mathfrak{p}_j[x]$ whenever $\mathfrak{p}_k \neq \mathfrak{p}_j$. Also, $\mathfrak{q}_k[x] \supseteq \bigcap_{j\neq k} \mathfrak{q}_j[x]$ would imply that

$$\mathfrak{q}_k = \mathfrak{q}_k[x]^c \supseteq \left(\bigcap_{j\neq k} \mathfrak{q}_j[x]\right)^c = \bigcap_{j\neq k} \mathfrak{q}_j[x]^c = \bigcap_{j\neq k} \mathfrak{q}_j$$

Thus, the primary decomposition for $\mathfrak{a}[x]$ is minimal.

e. **If $\mathfrak{p}$ is a minimal prime ideal of $\mathfrak{a}$, then $\mathfrak{p}[x]$ is a minimal prime ideal of $\mathfrak{a}[x]$.**

Obviously $\mathfrak{p}[x]$ is a prime ideal contained in $\mathfrak{a}[x]$. So suppose that $\mathfrak{q}$ is a prime ideal for which $\mathfrak{q} \subseteq \mathfrak{p}[x]$. Then $\mathfrak{q}^c \subseteq \mathfrak{p}$ and $\mathfrak{q}^c$ is a prime ideal, so that $\mathfrak{q}^c = \mathfrak{p}$. But now $\mathfrak{p}[x] = \mathfrak{p}^e = \mathfrak{q}^{ce} \subseteq \mathfrak{q} \subseteq \mathfrak{p}[x]$, and hence $\mathfrak{q} = \mathfrak{p}[x]$. Thus, $\mathfrak{p}[x]$ is a minimal prime ideal of $\mathfrak{a}[x]$.

4.8? **Let k be a field. Show that in $k[x_1, \ldots, x_n]$ the ideals $\mathfrak{p}_i = (x_1, \ldots, x_i)$ are prime and that all their powers are primary.**

Write $A_n = k[x_1, \ldots, x_n]$. Each $\mathfrak{p}_i$ is a prime ideal since $A_n/\mathfrak{p}_i \cong A_{n-i}$ is an integral domain. Now since (x) is maximal in $k[x]$, every power of (x) is primary in $k[x]$. So the result holds for A_1. We proceed by induction by assuming the result holds for A_n. Every power of $\mathfrak{p}_{n+1}$ is primary in A_{n+1} since $\mathfrak{p}_{n+1}$ is maximal in A_{n+1}. If $i < n+1$ then every power of $\mathfrak{p}_i$ is primary in A_n by induction.

4.9. **In a ring A, let $D(A)$ consist of all prime ideals $\mathfrak{p}$ that satisfy the following condition: there is $a \in A$ so that $\mathfrak{p}$ is minimal in the set of prime ideals containing $\operatorname{Ann}(a)$. Show the following.**

Notice that $\operatorname{Ann}(a)$ is a proper ideal in A for $a \neq 0$ (and $A \neq 0$) since $1 \notin \operatorname{Ann}(a)$. So there is a maximal ideal containing $\operatorname{Ann}(a)$, implying that the set of all prime ideals containing $\operatorname{Ann}(a)$ is non-empty. If we order this set by reverse inclusion, then it is clearly chain complete. So Zorn's Lemma yields minimal elements.

a. **x is a zero-divisor iff $x \in \mathfrak{p}$ for some $\mathfrak{p} \in D(A)$.**

Suppose $xy = 0$ with $y \neq 0$. Then $x \in (0 : y) \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in D(A)$. Conversely, suppose $\mathfrak{p} \in D(A)$. We have to show that $\mathfrak{p}$ consists of zero-divisors.

b. **After identifications, $D(S^{-1}A) = D(A) \cap \operatorname{Spec}(S^{-1}A)$.**

Let $\mathfrak{p} \in D(A) \cap \operatorname{Spec}(S^{-1}A)$ so that $\mathfrak{p}$ is a minimal element in the set of all prime ideals containing $(0 : a)$ for some $a \in A$, and $\mathfrak{p} \cap S = \emptyset$. Define a prime ideal $\mathfrak{q} = S^{-1}\mathfrak{p}$ in $S^{-1}A$ and notice that $(0 : a/1) \subseteq \mathfrak{q}$. Suppose $(0 : a/1) \subseteq S^{-1}\mathfrak{r} \subseteq \mathfrak{q}$, with $\mathfrak{r}$ a prime ideal in A that does not meet S. Then $(0 : a) \subseteq (0 : a/1)^c \subseteq \mathfrak{r} \subseteq \mathfrak{p}$ so that $\mathfrak{r} = \mathfrak{p}$, and hence $S^{-1}\mathfrak{r} = \mathfrak{q}$. It follows that $\mathfrak{q}$ is minimal in the set of prime ideals in $S^{-1}A$ containing $(0 : a/1)$, and hence $\mathfrak{q} \in D(S^{-1}A)$. Thus $D(A) \cap \operatorname{Spec}(S^{-1}A) \subseteq D(S^{-1}A)$. Conversely, suppose that $\mathfrak{q} \in D(S^{-1}A)$ so that $\mathfrak{q}$ is a minimal element in the set of prime ideals in $S^{-1}A$ containing $(0 : a/s)$. Write $\mathfrak{q} = S^{-1}\mathfrak{p}$ with $\mathfrak{p}$ a prime ideal in A that does not meet S. Since $(0 : a/1) = (0 : a/s)$ we have $(0 : a) \subseteq (0 : a/1)^c \subseteq \mathfrak{p}$. Suppose $(0 : a) \subseteq \mathfrak{r} \subseteq \mathfrak{p}$

with $\mathfrak{r}$ a prime ideal in A. Then $\mathfrak{r}$ does not meet S, and hence $(0 : a/1) \subseteq S^{-1}\mathfrak{r} \subseteq \mathfrak{q}$. After all, if $a/1 \cdot b/t = 0/1$ so that $abu = 0$ for some $u \in S$, then $bu \in (0 : a) \subseteq \mathfrak{r}$, and hence $b/t = bu/tu \in S^{-1}\mathfrak{r}$. Thus, $S^{-1}\mathfrak{r} = \mathfrak{q}$, implying that $\mathfrak{r} = \mathfrak{p}$; showing that $\mathfrak{p}$ is minimal in the set of all prime ideals containing $(0 : a)$. Therefore, $\mathfrak{q} \in D(A) \cap \operatorname{Spec}(S^{-1}A)$. Hence, $D(S^{-1}A) = D(A) \cap \operatorname{Spec}(S^{-1}A)$ after our identifications.

c. **If the zero ideal has a primary decomposition, then $D(A)$ is the set of all prime ideals belonging to 0.**

Suppose $\mathfrak{p}$ is a prime ideal belonging to 0 so that $\mathfrak{p}$ is a minimal element in the set of all prime ideals containing $0 = (0 : 1)$. Then $\mathfrak{p}$ is an element of $D(A)$. Conversely, suppose $\mathfrak{p} \in D(A)$ and $\mathfrak{p}$ is minimal in the set of all prime ideals containing $(0 : a)$.

4.10. **For any prime $\mathfrak{p}$, let $S_{\mathfrak{p}}(0) = \operatorname{Ker}(A \to A_{\mathfrak{p}})$. Prove the following.**

a. **We have the containment $S_{\mathfrak{p}}(0) \subseteq \mathfrak{p}$.**

If a is in $S_{\mathfrak{p}}(0)$, then $a/1 = 0$ in $A_{\mathfrak{p}}$. So there is $s \in A - \mathfrak{p}$ for which $as = 0 \in \mathfrak{p}$. But then $a \in \mathfrak{p}$ since $s \notin \mathfrak{p}$. Thus, $S_{\mathfrak{p}}(0) \subseteq \mathfrak{p}$.

b. **$r(S_{\mathfrak{p}}(0)) = \mathfrak{p}$ if and only if $\mathfrak{p}$ is a minimal prime ideal in A.**

The prime ideals of $A_{\mathfrak{p}}$ are in a bijective correspondence with the prime ideals that don't meet $S = A - \mathfrak{p}$. That is, they correspond bijectively with prime ideals contained in $\mathfrak{p}$. When $\mathfrak{p}$ is minimal, we see that $A_{\mathfrak{p}}$ has precisely one prime ideal, namely $\mathfrak{p}_{\mathfrak{p}}$. Hence, $\mathfrak{p}_{\mathfrak{p}}$ is the nilradical of $A_{\mathfrak{p}}$. So if $a \in \mathfrak{p}$ then $(a/1)^n = 0$ in $A_{\mathfrak{p}}$ for some $n > 0$, and therefore $a^n \in S_{\mathfrak{p}}(0)$. Hence $\mathfrak{p} \subseteq r(S_{\mathfrak{p}}(0))$. On the other hand, $r(S_{\mathfrak{p}}(0)) \subseteq r(\mathfrak{p}) = \mathfrak{p}$. Hence $\mathfrak{p} = r(S_{\mathfrak{p}}(0))$.

Suppose that $\mathfrak{p}$ is not minimal. Then there is prime $\mathfrak{q} \subsetneq \mathfrak{p}$. So by the correspondence in the above paragraph, $\mathfrak{N}(A_{\mathfrak{p}}) \subsetneq \mathfrak{p}_{\mathfrak{p}}$. There is thus $a \in \mathfrak{p}$ for which $(a/1)^n \neq 0$ in $A_{\mathfrak{p}}$ for any $n > 0$. This means that $a \notin r(S_{\mathfrak{p}}(0))$, and so $\mathfrak{p} \neq r(S_{\mathfrak{p}}(0))$.

c. **If $\mathfrak{p}' \subseteq \mathfrak{p}$ are prime ideals, then $S_{\mathfrak{p}}(0) \subseteq S_{\mathfrak{p}'}(0)$.**

If $a \in S_{\mathfrak{p}}(0)$ then $as = 0$ for some $s \in A - \mathfrak{p} \subseteq A - \mathfrak{p}'$, and hence $a \in S_{\mathfrak{p}'}(0)$. Therefore $S_{\mathfrak{p}}(0) \subseteq S_{\mathfrak{p}'}(0)$.

d. **The intersection $\bigcap_{\mathfrak{p} \in D(\mathfrak{a})} S_{\mathfrak{p}}(0)$ equals 0.**

Suppose that $x \neq 0$ and notice that $(0 : x) \neq (1)$. So there is a minimal $\mathfrak{p}$ in the set of prime ideals containing $(0 : x)$. If $x \in S_{\mathfrak{p}}(0)$, then for some $s \in A - \mathfrak{p}$ we have $sx = 0$. This contradicts the equation $(0 : x) \subseteq \mathfrak{p}$. Therefore, $x \notin S_{\mathfrak{p}}(0)$; and hence $\bigcap_{\mathfrak{p} \in D(A)} S_{\mathfrak{p}}(0) = 0$.

4.11. **If $\mathfrak{p}$ is a minimal prime ideal in A, show that $S_{\mathfrak{p}}(0)$ is the smallest $\mathfrak{p}$-primary ideal. Let $\mathfrak{a}$ be the intersection of the ideals $S_{\mathfrak{p}}(0)$ as $\mathfrak{p}$ runs through the minimal prime ideals in A. Show that $\mathfrak{a} \subseteq \mathfrak{N}(A)$. Suppose that the zero ideal is decomposable. Prove that $\mathfrak{a} = 0$ iff every prime ideal of 0 is isolated.**

As above $r(S_{\mathfrak{p}}(0)) = \mathfrak{p}$ whenever $\mathfrak{p}$ is a minimal prime ideal in A. Now suppose that $xy \in S_{\mathfrak{p}}(0)$ with $x \notin S_{\mathfrak{p}}(0)$. Choose $s \in A - \mathfrak{p}$ with $sxy = 0$. Then $sy \in \mathfrak{p}$ (for otherwise $x \in S_{\mathfrak{p}}(0)$), and so $y \in \mathfrak{p} = r(S_{\mathfrak{p}}(0))$. This means that $y^n \in S_{\mathfrak{p}}(0)$ for some $n > 0$. Hence, $S_{\mathfrak{p}}(0)$ is $\mathfrak{p}$-primary.

Now let $\mathfrak{q}$ be any $\mathfrak{p}$-primary ideal, with $\mathfrak{p}$ a minimal prime ideal. If $x \in S_{\mathfrak{p}}(0)$ then $0 = sx \in \mathfrak{q}$ for some $s \in A - \mathfrak{p}$. If $x \notin \mathfrak{q}$ then $s^n \in \mathfrak{q}$ for some $n > 0$. But this is impossible since $A - \mathfrak{p}$ is multiplicatively closed. Therefore $S_{\mathfrak{p}}(0) \subseteq \mathfrak{q}$.

It is clear that $\mathfrak{a} \subseteq \mathfrak{N}(A)$ since we always have $S_{\mathfrak{p}}(0) \subseteq \mathfrak{p}$ and since $\mathfrak{N}(A)$ is the intersection of all the minimal prime ideals in A.

Suppose that the zero ideal is decomposable and that $\mathfrak{a} = 0$. Then there are finitely many minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ in A. Notice that $0 = \mathfrak{a} = \bigcap_{i=1}^n S_{\mathfrak{p}_i}(0)$ is a primary decomposition since each $S_{\mathfrak{p}_i}(0)$ is a $\mathfrak{p}_i$-primary ideal. From this we see that the prime ideals belonging to 0 are all isolated.

Suppose that the zero ideal is decomposable and that every prime ideal belonging to 0 is isolated. Write $0 = \bigcap_{i=1}^n \mathfrak{q}_i$ and let $\mathfrak{p}_i = r(\mathfrak{q}_i)$. Then each $\mathfrak{p}_i$ is a minimal prime ideal in A. Therefore $S_{\mathfrak{p}_i}(0) \subseteq \mathfrak{q}_i$ so that $\mathfrak{a} = 0$.

4.12? **Let S be a multiplicatively closed subset of A. For any ideal $\mathfrak{a}$, let $S(\mathfrak{a})$ denote the contraction of $S^{-1}\mathfrak{a}$ in A. The ideal $S(\mathfrak{a})$ is called the saturation of $\mathfrak{a}$ with respect to S. Prove the following.**

a. $S(\mathfrak{a}) \cap S(\mathfrak{b}) = S(\mathfrak{a} \cap \mathfrak{b})$

This follows directly from proposition 1.18.

b. $S(r(\mathfrak{a})) = r(S(\mathfrak{a}))$

This follows directly from proposition 1.18.

c. $S(\mathfrak{a}) = (1)$ **iff $\mathfrak{a}$ meets S.**

This follows directly from proposition 3.11.

d. $S_1(S_2(\mathfrak{a})) = (S_1 S_2)(\mathfrak{a})$

Notice that $S_1 S_2$ is a multiplicatively closed subset of A. Suppose $x \in S_1(S_2(\mathfrak{a}))$ so that $x/1 = y/s_1$ for some $y \in S_2(\mathfrak{a})$ and $y/1 = a/s_2$ for some $a \in A$. Choose s_1', s_2' with $s_1'(xs_1 - y) = 0$ and $s_2'(ys_2 - a) = 0$. Then $s_1' s_2'(s_1 s_2 x - a) = s_1' s_2 s_2' y - s_1' s_2' a = 0$ so that $x/1 = a/s_1 s_2$ and hence $x \in (S_1 S_2)(\mathfrak{a})$. Conversely, if $x/1 = a/s_1 s_2$ then ????

e. **If $\mathfrak{a}$ is decomposable then the set of $S(\mathfrak{a})$ is finite.**

4.13. **Let A be a ring and $\mathfrak{p}$ a prime ideal in A. Define the nth symbolic power $\mathfrak{p}^{(n)}$ of $\mathfrak{p}$ by $\mathfrak{p}^{(n)} = S_{\mathfrak{p}}(\mathfrak{p}^n)$. Prove the following.**

a. **$\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary ideal.**

Notice first that $r(S_{\mathfrak{p}}(\mathfrak{p}^n)) = S_{\mathfrak{p}}(r(\mathfrak{p}^n)) = S_{\mathfrak{p}}(\mathfrak{p}) = \mathfrak{p}$. Now $r((\mathfrak{p}^n)_{\mathfrak{p}}) = (r(\mathfrak{p}^n))_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}$ is the maximal ideal in $A_{\mathfrak{p}}$ so that $(\mathfrak{p}^n)_{\mathfrak{p}}$ is primary in $A_{\mathfrak{p}}$. This means that its contraction (i.e. $\mathfrak{p}^{(n)}$) is primary in A, and hence is $\mathfrak{p}$-primary.

b. **If $\mathfrak{p}^n$ has a primary decomposition, then $\mathfrak{p}^{(n)}$ is its $\mathfrak{p}$-component.**

Suppose $\mathfrak{p}^n = \bigcap_{i=1}^m \mathfrak{q}_i$ is a minimal primary decomposition of $\mathfrak{p}^n$, and write $\mathfrak{p}_i = r(\mathfrak{q}_i)$. Assume that $\mathfrak{p}_i$ does not meet $A - \mathfrak{p}$ for $1 \leq i \leq n$ and that $\mathfrak{p}_i$ meets $S - \mathfrak{p}$ for $n < i \leq m$. Then $\mathfrak{p}^{(n)} = \bigcap_{i=1}^n \mathfrak{q}_i$ is a primary decomposition of $\mathfrak{p}^{(n)}$. Now $\mathfrak{p} = r(\mathfrak{p}^{(n)}) = \bigcap_{i=1}^n \mathfrak{p}_i$. But $\mathfrak{p}_i \subseteq \mathfrak{p}$ for $1 \leq i \leq n$, and $\mathfrak{p}_i \neq \mathfrak{p}_j$ for $i \neq j$. Therefore, $n = 1$ and $\mathfrak{p}_1 = \mathfrak{p}$. This means that $\mathfrak{q}_1 = \mathfrak{p}^{(n)}$. In other words, $\mathfrak{p}^{(n)}$ is the $\mathfrak{p}$-component of $\mathfrak{a}$, as claimed.

c. **If $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$ has a primary decomposition, then $\mathfrak{p}^{(m+n)}$ is its $\mathfrak{p}$-primary component.**

Let $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)} = \bigcap_{i=1}^m \mathfrak{q}_i$ be a minimal primary decomposition, and write $\mathfrak{p}_i = r(\mathfrak{q}_i)$. Assume that $\mathfrak{p}_i$ does not meet $A - \mathfrak{p}$ for $1 \leq i \leq n$ and that $\mathfrak{p}_i$ meets $S - \mathfrak{p}$ for $n < i \leq m$. Then $S_\mathfrak{p}(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}) = \bigcap_{i=1}^n \mathfrak{q}_i$ so that $\bigcap_{i=1}^n \mathfrak{p}_i = r(S_\mathfrak{p}(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)})) = S_\mathfrak{p}(r(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)})) = S_\mathfrak{p}(\mathfrak{p}) = \mathfrak{p}$. So again, $n = 1$ and $\mathfrak{p}_1 = \mathfrak{p}$. Using Proposition 1.18 we see that $S_\mathfrak{p}(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}) = \mathfrak{p}^{(m+n)}$. So $\mathfrak{q}_1 = \mathfrak{p}^{(m+n)}$, showing that $\mathfrak{p}^{(m+n)}$ is the $\mathfrak{p}$-primary component of $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$.

d. **$\mathfrak{p}^{(n)} = \mathfrak{p}^n$ if and only if $\mathfrak{p}^n$ is $\mathfrak{p}$-primary.**

If $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ then $\mathfrak{p}^n$ is $\mathfrak{p}$-primary by part a. Assume $\mathfrak{p}^n$ is $\mathfrak{p}$-primary so that $\mathfrak{p}^n = \mathfrak{p}^n$ is a minimal primary decomposition of $\mathfrak{p}^n$, implying that $\mathfrak{p}^n = \mathfrak{p}^{(n)}$ by part c.

4.14. **Let $\mathfrak{a}$ be a decomposable ideal in the ring A and let $\mathfrak{p}$ be a maximal element in $\Sigma = \{(\mathfrak{a} : x) : x \notin \mathfrak{a}\}$. Show that $\mathfrak{p}$ is a prime ideal belonging to $\mathfrak{a}$.**

Let $\mathfrak{p} = (\mathfrak{a} : x)$ be a maximal element in Σ. Suppose $ab \in \mathfrak{p}$ and $b \notin \mathfrak{p}$, so that $abx \in \mathfrak{a}$ and $bx \notin \mathfrak{a}$. Then $(\mathfrak{a} : x) \subseteq (\mathfrak{a} : bx) \in \Sigma$ so that $(\mathfrak{a} : x) = (\mathfrak{a} : bx)$ by maximality. Then $a \in (\mathfrak{a} : bx) = (\mathfrak{a} : x) = \mathfrak{p}$. Therefore, $\mathfrak{p}$ is a prime ideal in A. Also, $\mathfrak{p} = r(\mathfrak{p}) = r(\mathfrak{a} : x)$ is a prime ideal in the set $\{r(\mathfrak{a} : x) | x \in A\}$. Since $\mathfrak{a}$ is a decomposable ideal, the first uniqueness theorem tells us that $\mathfrak{p}$ belongs to $\mathfrak{a}$.

4.15? **Let $\mathfrak{a}$ be a decomposable ideal, Σ an isolated set of prime ideals belonging to $\mathfrak{a}$, and q_Σ the intersection of the corresponding primary components. Suppose f is an element of A such that, if $\mathfrak{p}$ belongs to $\mathfrak{a}$, then $f \in \mathfrak{p}$ if and only if $\mathfrak{p} \notin \Sigma$. Show that $q_\Sigma = S_f(\mathfrak{a}) = (\mathfrak{a} : f^n)$ for all large n.**

If $\mathfrak{p}$ belongs to A, then $\mathfrak{p}$ meets $S_f = \{1, f, f^2, \ldots\}$ if and only if $\mathfrak{p} \notin \Sigma$. Therefore, $S_f(\mathfrak{a}) = \bigcap_{\mathfrak{p} \cap S_f = \emptyset} \mathfrak{q} = \mathfrak{q}_\Sigma$. Now $S_f(\mathfrak{a}) = \mathfrak{a}^{ec} = \bigcup_{0 \leq n} (\mathfrak{a} : f^n)$ so that $(\mathfrak{a} : f^n) \subseteq S_f(\mathfrak{a})$ for all n.

4.16. **Suppose A is a ring in which every proper ideal has a primary decomposition. Show that the same holds for $S^{-1}A$.**

This follows from proposition 4.9 and the fact that every proper ideal in $S^{-1}A$ is of the form $S^{-1}\mathfrak{a}$ for some proper ideal $\mathfrak{a}$ in A.

4.17? **Let A be a ring satisfying (L1) For every proper ideal $\mathfrak{a}$ and every prime ideal $\mathfrak{p}$, there exists $x \notin \mathfrak{p}$ such that $S_\mathfrak{p}(\mathfrak{a}) = (\mathfrak{a} : x)$. Show that every proper ideal $\mathfrak{a}$ in A is an intersection of (perhaps infinitely many) primary ideals.**

Let $\mathfrak{p}_1$ be a minimal element in the set of all prime ideals containing $\mathfrak{a}$. Then $\mathfrak{q}_1 = S_{\mathfrak{p}_1}(\mathfrak{a})$ is $\mathfrak{p}_1$-primary. By hypothesis, $\mathfrak{q}_1 = (\mathfrak{a} : x)$ for some $x \notin \mathfrak{p}_1$.

4.18? **Show that every proper ideal in A has a primary decomposition if and only if A satisfies the following two conditions.**

L1. **If $\mathfrak{a}$ is a proper ideal and $\mathfrak{p}$ is a prime ideal, then there exists $x \notin \mathfrak{p}$ such that $S_\mathfrak{p}(\mathfrak{a}) = (\mathfrak{a} : x)$.**

L2. **If $\mathfrak{a}$ is a proper ideal and $S_1 \supseteq S_2 \supseteq \ldots$ is a descending sequence of multiplicatively closed subsets of A, then there exists an N such that $S_n(\mathfrak{a}) = S_N(\mathfrak{a})$ for all $n \geq N$.**

Suppose that every proper ideal in A has a primary decomposition. Let $\mathfrak{a}$ be a proper ideal in A, so that $\mathfrak{a}$ has a primary decomposition, and hence the saturations of $\mathfrak{a}$ in A form a finite set by exercise 4.12. This shows that L2 holds for $\mathfrak{a}$. Let $\mathfrak{p}$ a prime ideal.

4.19? **Show that every $\mathfrak{p}$-primary ideal contains $S_{\mathfrak{p}}(0)$. Suppose that A satisfies the following condition: for every prime ideal $\mathfrak{p}$, the intersection of all $\mathfrak{p}$-primary ideals equals $S_{\mathfrak{p}}(0)$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be distinct non-minimal prime ideals in A. Show that there is an ideal $\mathfrak{a}$ whose associated prime ideals are $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$.**

Suppose that $\mathfrak{p}$ is a prime ideal in A. Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal and suppose $a \in S_{\mathfrak{p}}(0)$. Then $a/1 = 0/1$ so that $ab = 0$ for some $b \notin \mathfrak{p}$. Since $b^n \notin \mathfrak{q}$ for any $n > 0$, we see that $a \in \mathfrak{q}$. In other words, $S_{\mathfrak{p}}(0) \subseteq \mathfrak{q}$, as claimed.

Now let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be distinct prime ideals in A, where A satisfies the hypothesis as in the problem statement. If $n = 1$ then we can take $\mathfrak{a} = \mathfrak{p}_1$. Suppose then that $n > 1$, and assume $\mathfrak{p}_n$ is a maximal element in $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. By induction, there is an ideal $\mathfrak{b}$ and a minimal primary decomposition $\mathfrak{b} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_{n-1}$ with each $\mathfrak{q}_i$ a $\mathfrak{p}_i$-primary ideal. Suppose for the sake of contradiction that $\mathfrak{b} \subseteq S_{\mathfrak{p}_n}(0)$. Let $\mathfrak{p}$ be a minimal prime ideal in A contained in $\mathfrak{p}_n$ so that $S_{\mathfrak{p}_n}(0) \subseteq S_{\mathfrak{p}}(0)$ by exercise 4.10. Then $\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_{n-1} = r(\mathfrak{b}) \subseteq r(S_{\mathfrak{p}}(0)) = \mathfrak{p}$ so that $\mathfrak{p}_i \subseteq \mathfrak{p}$ for some i. By minimality, $\mathfrak{p}_i = \mathfrak{p}$ is a minimal prime ideal; a contradiction. Therefore, $\mathfrak{b} \not\subseteq S_{\mathfrak{p}_n}(0)$. Since $S_{\mathfrak{p}_n}(0)$ is the intersection of all $\mathfrak{p}_n$-primary ideals in A, there is a $\mathfrak{p}_n$-primary ideal $\mathfrak{q}_n$ such that $\mathfrak{b} \not\subseteq \mathfrak{q}_n$. Now define $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{q}_n$. Obviously $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$ is a primary decomposition of $\mathfrak{a}$. We know that $r(\mathfrak{q}_i) = \mathfrak{p}_i \neq \mathfrak{p}_j = r(\mathfrak{q}_j)$ for $i \neq j$, and that $\mathfrak{q}_n \not\supseteq \bigcap_{i \neq n} \mathfrak{q}_i = \mathfrak{b}$. Suppose then that $\mathfrak{q}_i \supseteq \bigcap_{j \neq i} \mathfrak{q}_j$ for $1 \leq i < n$.

Taking radicals we see that $\bigcap_{j \neq i,n} \mathfrak{p}_j \cap \mathfrak{p}_n \subseteq \mathfrak{p}_i$. Either $\bigcap_{j \neq i,n} \mathfrak{p}_j \subseteq \mathfrak{p}_i$ or $\mathfrak{p}_n \subseteq \mathfrak{p}_i$. In the latter case, $\mathfrak{p}_n = \mathfrak{p}_i$ since $\mathfrak{p}_n$ is a maximal element in $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. But $\mathfrak{p}_n \neq \mathfrak{p}_i$, so that $\bigcap_{j \neq i,n} \mathfrak{p}_j \subseteq \mathfrak{p}_i$.

4.20. **Let M be a fixed A-module with submodules N and N'. The radical $r_M(N)$ of N in M is defined to be the set of all $x \in A$ so that $x^q M \subseteq N$ for some $q > 0$. Establish the following.**

a. $r_M(N) = r(N : M) = r(\mathrm{Ann}(M/N))$

It is clear that $r_M(N) = r(N : M)$ so that $r_M(N)$ is an ideal in A. We also know that $(N : M) = \mathrm{Ann}((N + M)/N) = \mathrm{Ann}(M/N)$ so that the last equality holds as well.

b. $r(r_M(N)) = r_M(N)$

We have $r(r_M(N)) = r(r(N : M)) = r(N : M) = r_M(N)$.

c. $r_M(N \cap N') = r_M(N) \cap r_M(N')$

This follows from

$$\begin{aligned} r_M(N \cap N') &= r(N \cap N' : M) \\ &= r((N : M) \cap (N' : M)) \\ &= r(N : M) \cap r(N' : M) \\ &= r_M(N) \cap r_M(N') \end{aligned}$$

d. $r_M(N) = A$ **if and only if** $N = M$**.**

Since $r_M(N)$ is an ideal, $r_M(N) = A$ iff $1 \in r_M(N)$ iff $M = N$.

e. $r_M(N+N') \supseteq r(r_M(N)+r_M(N'))$

Suppose that $x^n \in r_M(N)+r_M(N')$. Write $x^n = y+y'$ with $y^qM \subseteq N$ and $y'^rM \subseteq N'$. Then $x^{n(q+r)}M \subseteq y^qM + y'^rM \subseteq N+N'$ so that $x \in r_M(N+N')$.

4.21. **Each $a \in A$ defines an endomorphism $\phi_a : M \to M$. a is called a zero-divisor if ϕ_a is not injective, and a is called nilpotent if ϕ_a is nilpotent. A submodule $Q \neq M$ is called primary if every zero-divisor in M/Q is nilpotent. Prove the following.**

a. **If Q is primary in M then $(Q : M)$ is a primary ideal.**

Suppose that $ab \in (Q : M)$ with $a \notin (Q : M)$. Choose $x \in M$ with $ax \notin Q$ so that the image of ax in M/Q is nonzero. Then $b(ax) \in Q$ since $abM \subseteq Q$. Since Q is primary, we see that $b^qM \subseteq Q$ for some $q > 0$. This means that $b^q \in (Q : M)$. Therefore, $(Q : M)$ is a primary ideal in A.

b. **If $Q_1, \ldots, Q_n$ are $\mathfrak{p}$-primary in M then so is $Q = \bigcap_1^n Q_i$.**

We know that $r(Q) = \bigcap_1^n r(Q_i) = \mathfrak{p}$. Suppose $a \in A$ satisfies $ax \in Q$ for some $x \in M$. If $a^qQ \neq Q$ for any q, then $a \notin r_M(Q) = \mathfrak{p}$. Since Q_i is $\mathfrak{p}$-primary and $ax \in Q_i$, we conclude that $x \in Q_i$. Thus, $x \in \bigcap_1^n Q_i = Q$. This means that Q is a primary ideal in A.

c. **If Q is $\mathfrak{p}$-primary and $x \notin Q$ then $(Q : x)$ is $\mathfrak{p}$-primary.**

Suppose $a \in (Q : x)$ so $ax \in Q$. Hence, $a^qM \subseteq Q$ for some $q > 0$. This means that $a \in r_M(Q) = \mathfrak{p}$. So $(Q : M) \subseteq (Q : x) \subseteq \mathfrak{p}$, and hence $r(Q : x) = \mathfrak{p}$, after taking radicals. Now let $ab \in (Q : x)$. If $a \notin \mathfrak{p}$ then $bx \in Q$. After all, $a(bx) \in Q$ and if $bx \notin Q$ then $a \in r(Q : M) = \mathfrak{p}$ since Q is a primary submodule. Thus, either $a \in \mathfrak{p} = r(Q : x)$ or $b \in (Q : x)$. This means that $(Q : x)$ is a $\mathfrak{p}$-primary ideal in A.

4.22. **Let N be a submodule of M. We say that N is decomposable if $N = \bigcap_{i=1}^n Q_i$ where each Q_i is a primary submodule of Q. This decomposition is said to be minimal if $r_M(Q_i) \neq r_M(Q_j)$ for $i \neq j$ and if every i we have $Q_i \not\supseteq \bigcap_{j\neq i} Q_j$. Supposing N is a decomposable submodule, show that the primes belonging to N are uniquely determined, and that they are the primes belonging to 0 in M/N.**

Let $N = \bigcap_{i=1}^n Q_i$ be a minimal primary decomposition. For $x \in M$

$$(N : x) = (\bigcap Q_i : x) = \bigcap (Q_i : x)$$

Taking radicals yields

$$r(N : x) = \bigcap r(Q_i : x) = \bigcap_{x \notin Q_i} r(Q_i : x) = \bigcap_{x \notin Q_i} \mathfrak{p}_i$$

where $\mathfrak{p}_i = r_M(Q_i)$. So if $r(N : x)$ is a prime ideal, then $r(N : x) = \mathfrak{p}_i$ for some i. Conversely, choose $x_i \in \bigcap_{j\neq i} Q_j - Q_i$ and notice that $r(N : x_i) = \mathfrak{p}_i$. Therefore, the $\mathfrak{p}_i$ are precisely the prime ideals in the set of all $r(N : x)$ as x ranges over M. This means that the primes belonging to N are unique, defined independently of the particular primary decomposition of $\mathfrak{a}$. Notice that $N \subseteq Q_i$ for each i, and so $0 = \bigcap_1^n Q_i/N$ is a primary decomposition of 0 in M/N. This is clearly a minimal primary decomposition with $r_{M/N}(Q_i/N) = r_M(Q_i)$. So the primes belonging to N are precisely the primes belonging to 0 in M/N, by the uniqueness theorem proved above.

4.23. **Prove analogues of Propositions 4.6 to 4.11.**

Let N be a decomposable submodule of M, with minimal primary decomposition $N = \bigcap Q_i$. Write $\mathfrak{p}_i = r(Q_i : M)$ and notice that $(N : M) = \bigcap (Q_i : M) \subseteq (Q_i : M) \subseteq \mathfrak{p}_i$ for every i. Suppose $\mathfrak{p}$ be a prime ideal in A containing $(N : M)$. Then $\mathfrak{p} \supseteq r(N : M) = \bigcap r(Q_i : M) = \bigcap \mathfrak{p}_i$ so that $\mathfrak{p}_i \subseteq \mathfrak{p}$ for some i. This means that the minimal elements in the set of all prime ideals containing $(N : M)$ are precisely the minimal elements in the set of prime ideals belonging to N.

Suppose that 0 is a decomposable submodule with minimal primary decomposition $0 = \bigcap Q_i$ and $\mathfrak{p}_i = r_M(Q_i)$. Notice that $a \in A$ is a zero-divisor in M iff $a \in \bigcup_{0 \neq x \in M} \mathrm{Ann}(x)$. The set $D(M)$ of $a \in A$ that are zero-divisors clearly satisfies $r(D(M)) = D(M)$ so that $D(M) = \bigcup_{0 \neq x \in M} r(0 : x)$. From the work done in exercise 4.22, we know that $r(0 : x) = \bigcap_{x \notin Q_i} \mathfrak{p}_i$, and hence $r(0 : x) \subseteq \mathfrak{p}_j$ for some j, since x is assumed to be nonzero. Therefore, $D(M) \subseteq \bigcup_1^n \mathfrak{p}_i$. We have $\bigcup_1^n \mathfrak{p}_i \subseteq D(M)$ since $\mathfrak{p}_i = r(0 : x)$ for some $x \neq 0$. Thus, we have the equality $\bigcup_1^n \mathfrak{p}_i = D(M)$.

Let S be a multiplicatively closed subset of A. Suppose Q is a $\mathfrak{p}$-primary submodule of M. Assume $\mathfrak{p}$ meets S at s, so that $s^n M \subseteq Q$ for some n. Then $S^{-1}Q$ contains $m/t = (s^n m)/(s^n t)$ for every $m \in M$ and $t \in S$. This means that $S^{-1}Q = S^{-1}M$. On the other hand, assume that $\mathfrak{p} \cap S = \emptyset$. Then $S^{-1}Q$ is an $S^{-1}\mathfrak{p}$-primary submodule of $S^{-1}M$. We have the canonical map $f : M \to S^{-1}M$ that is a homomorphism of A-modules. Then $f^{-1}(S^{-1}Q) = Q$.

Let N be a decomposable submodule of M, with minimal primary decomposition $N = \bigcap_1^n Q_i$. Suppose S is a multiplicatively closed subset of A. Write $\mathfrak{p}_i = r_M(Q_i)$ and assume that $\mathfrak{p}_i \cap S = \emptyset$ for $1 \leq i \leq m$, and that $\mathfrak{p}_i$ meets S for $m < i \leq n$. By the above paragraph, $S^{-1}N = \bigcap_1^n S^{-1}Q_i = \bigcap_1^m S^{-1}Q_i$ is a primary decomposition of $S^{-1}N$ in $S^{-1}M$. Since the $\mathfrak{p}_i$ are distinct, so are the $S^{-1}\mathfrak{p}_i$ for $1 \leq i \leq m$. If $S^{-1}Q_m \supseteq \bigcap_{1 \leq i < m} S^{-1}Q_i = S^{-1}(\bigcap_{1 \leq i < m} Q_i)$ then $Q_m = (S^{-1}Q_m)^c \supseteq (S^{-1} \bigcap_{1 \leq i < m} Q_i)^c \supseteq \bigcap_{1 \leq i < m} Q_i$. So $S^{-1}N = \bigcap_1^m S^{-1}Q_i$ is a minimal primary decomposition. Contracting this, we get $S(N) = (S^{-1}N)^c = \bigcap_1^m (S^{-1}Q_i)^c = \bigcap_1^m Q_i$. This is a minimal primary decomposition of $S(N)$ in M.

Let N be a decomposable submodule of M, with minimal primary decomposition $N = \bigcap_1^n Q_i$. Suppose Σ is an isolated set of prime ideals belonging to N, where we write $\mathfrak{p}_i = r_M(Q_i)$ as usual. Define $Q_\Sigma = \bigcap_{\mathfrak{p}_i \in \Sigma} Q_i$. Clearly, $S = A - \bigcup_{\mathfrak{p}_i \in \Sigma}$ is a multiplicatively closed subset of A. Then $Q_\Sigma = S(N)$ depends only on Σ, and is independent of the minimal primary decomposition of N. In particular, the isolated components of N are uniquely determined.

Chapter 5 : Integral Dependence and Valuations

5.1. **Let $f : A \to B$ be an integral homomorphism of rings. Show that f^* is a closed map.**

Let $\mathfrak{q}$ be a prime ideal in B. I claim that $f^*(V(\mathfrak{q})) = V(f^*(\mathfrak{q}))$. Clearly $f^*(V(\mathfrak{q})) \subseteq V(f^*(\mathfrak{q}))$. Now if $\mathfrak{p} \in V(f^*(\mathfrak{q}))$ then $\operatorname{Ker}(f) \subseteq f^*(\mathfrak{q}) \subseteq \mathfrak{p}$ so that $f(f^*(\mathfrak{q})) \subseteq f(\mathfrak{p})$ is a chain of prime ideals in $f(A)$. Observe that $\mathfrak{q} \cap f(A) = f(f^{-1}(\mathfrak{q})) = f(f^*(\mathfrak{q}))$. Since B is integral over $f(A)$, there is a prime ideal $\mathfrak{r}$ in B containing $\mathfrak{q}$ with $\mathfrak{r} \cap f(A) = f(\mathfrak{p})$. So $\mathfrak{p} = f^{-1}(f(\mathfrak{p})) = f^{-1}(\mathfrak{r} \cap f(A)) = f^{-1}(\mathfrak{r}) = f^*(\mathfrak{r})$ with $\mathfrak{r} \in V(\mathfrak{q})$. This means that f^* is a surjective map, and hence $f^*(V(\mathfrak{q})) = V(f^*(\mathfrak{q}))$, showing that f^* is a closed map.

5.2. **Let A be a subring of B so that B is integral over A, and let $f : A \to \Omega$ be a homomorphism of A into an algebraically closed field Ω. Show that f can be extended to a map $B \to \Omega$.**

By a straightforward application of Zorn's Lemma there is a subring C of B containing A so that f can be extended to a map $C \to \Omega$ but such that f cannot be extended to a map defined on a subring of B properly containing C. So assume that $C \neq B$ so that we can derive a contradiction. If $b \notin C$ then $p(b) = 0$ for some monic $p \in C[x]$, where x is an indeterminate. Assume that p is chosen to have minimal degree, so that p is an irreducible polynomial. Since Ω is algebraically closed, p has a root ξ in Ω. Now define $\bar{f} : C[x] \to \Omega$ by $\bar{f}(\sum c_i x^i) = \sum f(c_i)\xi^i$. Then $\bar{f}$ is a ring homomorphism whose kernel contains (p). Hence, $\bar{f}$ induces a ring homomorphism $C[x]/(p) \to \Omega$ given by $\sum c_i x^i + (p) \to \sum f(c_i)\xi^i$. But $C[b]$ and $C[x]/(p)$ are isomorphic rings, so that there is a ring homomorphism $C[b] \to \Omega$ given by $\sum c_i b^i \mapsto \sum f(c_i)\xi^i$. This map extends f to the subring $C[b]$ of B that properly contains C; a contradiction. Hence, f can indeed be extended to a map $B \to \Omega$.

5.3. **Let $f : B \to B'$ be a homomorphism of A-algebras, and let C be an A-algebra. If f is integral, prove that $f \otimes 1 : B \otimes A \to B' \otimes C$ is integral.**

Let $b' \otimes c$ be a generator of $B' \otimes C$. It suffices to show that $b' \otimes c$ is integral over $(f \otimes 1)(B \otimes C)$. Suppose b' is a root of the polynomial $p(x) = \sum_{i=0}^n f(b_i)x^i$. Define a polynomial $q(x) = \sum_{i=0}^n (f \otimes 1)(b_i \otimes c^{n-i})x^i$. Then $q(b' \otimes c) = p(b') \otimes c^n = 0$. So we are done.

5.4. **Suppose $A \subseteq B$ are rings with B integral over A. Let $\mathfrak{n}$ be a maximal ideal of B and let $\mathfrak{m} = A \cap \mathfrak{n}$ be the corresponding maximal ideal of A. Must $B_\mathfrak{n}$ be integral over $A_\mathfrak{m}$?**

Let k be a field and consider the subring $k[x^2 - 1]$ of $k[x]$. Since the polynomial $x - 1$ is irreducible over k, and since $k[x]$ is a principal ideal domain, the ideal $\mathfrak{n} = (x - 1)$ is maximal in $k[x]$. Thus, $\mathfrak{m} = k[x^2 - 1] \cap \mathfrak{n} = (\text{l.c.m.}\{x^2 - 1, x - 1\}) = (x^2 - 1)$ is a maximal ideal in $k[x^2 - 1]$.

Notice that $x \in k[x]$ is integral over $k[x^2 - 1]$ since x is a root of the polynomial $p(\xi) = \xi^2 - [(x^2 - 1) + 1]$. Since the set of all elements integral over $k[x^2 - 1]$ form a subring of $k[x]$, and since x is integral over $k[x^2 - 1]$, we see that $k[x]$ is indeed integral over $k[x^2 - 1]$.

For the sake of deriving a contradiction, suppose $k[x]_\mathfrak{n}$ is integral over $k[x^2 - 1]_\mathfrak{m}$. Then in particular, $1/(x+1)$ is integral over $k[x^2 - 1]_\mathfrak{m}$ since $x + 1 \in k[x] - \mathfrak{n}$. This means that there are polynomials $p_1, \ldots, p_n \in k[x^2 - 1]$ and polynomials $q_1, \ldots, q_n \in k[x^2 - 1] - \mathfrak{m}$ for which

$$(x + 1)^{-n} + (x + 1)^{-(n-1)} \frac{p_{n-1}}{q_{n-1}} + \cdots + (x + 1)^{-1} \frac{p_1}{q_1} + \frac{p_0}{q_0} = 0$$

Define $\hat{q}_i = \prod_{j \neq i} q_j$ and $q = \prod_1^n q_i$. Clearing the denominators in the above equation yields

$$(x + 1)^n p_0 \hat{q}_0 + (x + 1)^{n-1} p_1 \hat{q}_1 + \cdots + (x + 1) p_{n-1} \hat{q}_{n-1} + q = 0$$

This shows that $x + 1$ divides q. Since $q \in k[x^2 - 1]$, we can choose scalars $r_0, \ldots, r_m \in k$ satisfying

$$q = r_0 + r_1(x^2 - 1) + \cdots + r_m(x^2 - 1)^{2m}$$

Since $x+1$ divides q and x^2-1, we see that $r_0 = 0$, and so $q \in \mathfrak{m}$. But q is the product of elements $q_1, \ldots, q_n \notin \mathfrak{m}$, and so we have a contradiction. This contradiction shows us that $1/(x+1)$ is not integral over $k[x^2-1]_{\mathfrak{m}}$. Hence, $k[x]_{\mathfrak{n}}$ is not integral over $k[x^2-1]_{\mathfrak{m}}$.

5.5. **Let $A \subseteq B$ be rings with B integral over A. Prove the following.**

a. **If $x \in A$ is a unit in B then x is a unit in A.**

Since $x^{-1} \in B$ we have an equation of the form

$$x^{-n} + a_{n-1}x^{-n+1} + \cdots + a_1 x^{-1} + a_0 =$$

with $n > 0$ and each $a_i \in A$. Then

$$x^{-1} = -(a_0 x^{n-1} + a_1 x^{n-2} + \cdots + a_{n-1}) \in A$$

since $x \in A$. That is, x is invertible in A.

b. **$\mathfrak{R}(A) = A \cap \mathfrak{R}(B)$.**

If $\mathfrak{m}$ is a maximal ideal in B then $\mathfrak{m} \cap A$ is a maximal ideal in A. If $\mathfrak{n}$ is a maximal ideal in A, then $\mathfrak{n}$ is a prime ideal in A, so that $\mathfrak{n} = A \cap \mathfrak{m}$ for some prime ideal $\mathfrak{m}$ in B. But now $\mathfrak{m}$ is a maximal ideal in B. So

$$\mathfrak{R}(A) = \bigcap \mathfrak{m} = \bigcap (\mathfrak{m} \cap A) = \bigcap \mathfrak{m} \cap A = \mathfrak{R}(B) \cap A$$

where the first intersection is taken over all maximal ideals in A and the last intersection is taken over all maximal ideals in B.

5.6. **Let $B_1, \ldots, B_n$ be integral A-algebras. Show that $B = \prod_{i=1}^n B_i$ is an integral A-algebra as well.**

It suffices to assume $n = 2$. If B_i is given the A-algebra structure induced by $f_i : A \to B_i$, then B is given the A-algebra structure induced by $f : A \to B$ with $f(a) = (f_1(a), f_2(a))$. Suppose $(b_1, b_2) \in B$ so that b_1 is integral over $f_1(A)$. Choose a monic polynomial

$$p(x) = x^m + \sum_{i=0}^{m-1} f_1(a_i)x^i \qquad \text{such that } p(b_1) = 0$$

Then define a new monic polynomial with coefficients in $f(A)$ by

$$p'(x) = x^m + \sum_{i=0}^{m-1} f(a_i)x^i$$

so that $p'(b_1, b_2) = (0, b_2')$ for some $b_2' \in B$. Choose a monic polynomial

$$q(x) = x^n + \sum_{i=0}^{n-1} f_2(a_i')x^i \qquad \text{such that } q(b_2') = 0$$

Then define a new monic polynomial with coefficients in $f(A)$ by

$$q'(x) = x^n + \sum_{i=0}^{n-1} f(a_i')x^i$$

so that $q'(0, b_2') = (f(a_0'), 0)$. Now define a monic polynomial r with coefficients in $f(A)$ by the equation

$$r(x) = x^2 + (-f(a_0'), -f(a_0'))x$$

so that $r(f(a_0'), 0) = (0, 0)$. To summarize, (b_1, b_2) is integral over $f(A)[(0, b_2'), (f(a_0'), 0)]$, the element $(0, b_2')$ is integral over $f(A)[(f(a_0'), 0)]$, and lastly $(f(a_0'), 0)$ is integral over $f(A)$. Working backwards reveals that (b_1, b_2) is integral over $f(A)$. Hence, $B_1 \times B_2$ is integral over A.

5.7. **Let $A \subset B$ be rings so that $B - A$ is closed under multiplication. Show that A is integrally closed in B.**

Let C be the integral closure of A in B and suppose that $A \subsetneq C$. Define

$$n = \min\{d : \text{the irreducible polynomial of some } x \in C - A \text{ has degree } d\}$$

Clearly $n > 1$. Suppose $x \in C - A$ has the irreducible polynomial

$$x^n + a_1x^{n-1} + \cdots + a_n = 0$$

Then by minimality $x^{n-1} + a_1x^{n-2} + \cdots + a_{n-1} \notin A$. But

$$x(x^{n-1} + a_1x^{n-2} + \cdots + a_{n-1}) = -a_n \in A$$

showing that $B - A$ is not closed under multiplication.

5.8. **Suppose $A \subseteq B$ are rings and let C be the integral closure of A in B. Let f, g be monic polynomials in $B[x]$ so that $fg \in C[x]$. Show that $f, g \in C[x]$.**

Suppose for the moment that there is a ring D containing B over which f and g split completely into linear factors. Then we can write $f = \prod(x - a_j)$ and $g = \prod(x - b_j)$ for appropriate a_j, b_j in D. Notice that a_j, b_j are roots of fg in D. Since fg is a monic polynomial in $C[x]$, this means that the a_j, b_j are integral over C. Now the coefficients of f and g are polynomials in terms of the a_j, b_j. So these coefficients are themselves integral over C, and are hence integral over A. Since the coefficients of f and g lie in B, they are in C by definition of C. In other words, f and g are in $C[x]$.

So now it suffices to prove that for every ring B and every $f \in B[x]$, there is a ring D containing B over which f splits completely into linear factors. Of course we proceed by induction on $\deg(f) > 0$. Let $D' = B[x]/(f)$, and consider the natural map $B \to B[x] \to B[x]/(f) = D'$. This map is injective since f is monic and has degree greater than 0. Hence, we can consider B as being a subring of D', and we can consider f as being an element of $D'[x]$. As such, f has the root $x + (f)$. Denote this root by a. Notice that we can choose a monic $q \in D'[x]$ satisfying $f(x) = q(x)(x - a)$ and $\deg(q) = \deg(f) - 1$. By induction there is a ring D containing D' over which q splits completely into linear factors. Now B is a subring of D and f splits completely over D into linear factors. So we are done.

5.9. **Suppose $A \subseteq B$ are rings with C the integral closure of A in B. Show that $C[x]$ is the integral closure of $A[x]$ in $B[x]$.**

Let $cx^m \in C[x]$ and suppose that c is a root of the polynomial $\sum_{i=0}^n a_i\xi^i \in A[\xi]$. Then cx^m is a root of the polynomial $\sum_{i=0}^n (a_ix^{mn-im})\xi^i \in A[x][\xi]$ so that cx^m is integral over $A[x]$. Consequently, $C[x]$ is contained in the integral closure of $A[x]$ in $B[x]$. Now suppose that $f \in B[x]$ is integral over $A[x]$ and choose $g_0, \ldots, g_m \in A[x]$ satisfying

$$f^m + g_{m-1}f^{m-1} + \cdots + g_1 f + g_0 = 0$$

Let r be an integer that is greater than m and every $\deg(g_i)$. Define

$$\tilde{f} = f - x^r$$

Of course $-\tilde{f}$ is a monic polynomial in $B[x]$ of degree r and

$$(\tilde{f} + x^r)^m + g_{m-1}(\tilde{f} + x^r)^{m-1} + \cdots + g_1(\tilde{f} + x^r) + g_0 = 0$$

We can rewrite this as

$$\tilde{f}^m + h_{m-1}\tilde{f}^{m-1} + \cdots + h_1\tilde{f} + h_0 = 0$$

for appropriate $h_i \in B[x]$. Observe that

$$(-\tilde{f})(\tilde{f}^{m-1} + h_{m-1}\tilde{f}^{m-2} + \cdots + h_1) = h_0$$

But $h_0 = x^{rm} + g_{m-1}x^{r(m-1)} + \cdots + g_1 x^r + g_0 \in A[x] \subseteq C[x]$ and $\deg(h_0) = rm$ with leading coefficient equal to 1. After all

$$\deg(g_i x^{ri}) = \deg(g_i) + ri < r(i+1) \leq rm \quad \text{for } 0 \leq i \leq m-1$$

So h is a monic polynomial. This implies that

$$\tilde{f}^{m-1} + h_{m-1}\tilde{f}^{m-2} + \cdots + h_1$$

is monic as well. Now exercise 5.8 tells us that $-\tilde{f} \in C[x]$. Since $x^r \in C[x]$ we see that $f \in C[x]$. So we are done.

5.10. **Consider the following conditions and show that $a \Rightarrow b \Leftrightarrow c$.**

a. **The map f^* is closed.**
b. **The map f has the going-up property.**
c. **The map $f^* : \mathrm{Spec}(B/\mathfrak{q}) \to \mathrm{Spec}(A/\mathfrak{p})$ is onto whenever $\mathfrak{q}$ is a prime ideal in B and $\mathfrak{p} = f^*(\mathfrak{q})$.**

$(a \Rightarrow b)$ Suppose that $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ is a chain of prime ideals in $f(A)$ with $\mathfrak{p}_1 = f(A) \cap \mathfrak{q}_1$, where $\mathfrak{q}_1$ is a prime ideal in B. Then $f^{-1}(\mathfrak{p}_2) \in V(f^*(\mathfrak{q}_1))$ since $f^*(\mathfrak{q}_1) = f^{-1}(\mathfrak{p}_1) \subseteq f^{-1}(\mathfrak{p}_2)$. Since $f^*(V(\mathfrak{q}_1)) = V(f^*(\mathfrak{q}_1))$ there is a prime ideal $\mathfrak{q}_2$ in B containing $\mathfrak{q}_1$ such that $f^{-1}(\mathfrak{p}_2) = f^*(\mathfrak{q}_2) = f^{-1}(f(A) \cap \mathfrak{q}_2)$. This means that $\mathfrak{p}_2 = f(A) \cap \mathfrak{q}_2$. Therefore, B and $f(A)$ satisfy the conclusions of the going-up theorem, showing that f has the going-up property.

$(b \Rightarrow c)$ Let $\mathfrak{q}$ be a prime ideal in B and write $\mathfrak{p} = \mathfrak{q}^c$. We have to show that the map $f^* : V(\mathfrak{q}) \to V(\mathfrak{p})$ is surjective. If $\mathfrak{p}' \in V(\mathfrak{p})$ then $\mathrm{Ker}(f) \subseteq \mathfrak{p} \subseteq \mathfrak{p}'$ so that $f(\mathfrak{p}) \subseteq f(\mathfrak{p}')$ is a chain of prime ideals in $f(A)$ with $\mathfrak{q} \cap f(A) = f(\mathfrak{p})$. Since f has the going-up property, there is a prime ideal $\mathfrak{q}'$ in B containing $\mathfrak{q}$ so that $\mathfrak{q}' \cap f(A) = f(\mathfrak{p}')$. Now $f^*(\mathfrak{q}') = f^{-1}(f(\mathfrak{p}')) = \mathfrak{p}'$. This means that f^* is surjective.

$(c \Rightarrow b)$ Let $\mathfrak{p}$ be a prime ideal in $f(A)$ so that $f^{-1}(\mathfrak{p})$ is a prime ideal in A.

5.10′. **Consider the following conditions and show that $a \Rightarrow b \Leftrightarrow c$.**

a. **The map f^* is open.**

b. **The map f has the going-down property.**

c. **The map $f^* : \mathrm{Spec}(B_\mathfrak{q}) \to \mathrm{Spec}(A\mathfrak{p})$ is onto whenever $\mathfrak{q}$ is a prime ideal in B and $\mathfrak{p} = f^*(\mathfrak{q})$.**

$(a \Rightarrow b)$

$(b \Rightarrow c)$

$(c \Rightarrow b)$

5.11. **Let $f : A \to B$ be a flat homomorphism of rings. Then f has the going-down property.**

By exercise 3.18 we know that $f^* : \mathrm{Spec}(B_\mathfrak{q}) \to \mathrm{Spec}(A_\mathfrak{p})$ is a closed map whenever $\mathfrak{q}$ is a prime ideal of B and $\mathfrak{p} = \mathfrak{q}^c$. But now exercise 3.10 tells us that f has the going-down property.

5.12. **Let G be a finite group of automorphisms of the ring A. Prove that A is integral over A^G. Let S be a multiplicatively closed subset of A such that $\sigma(S) = S$ for every $\sigma \in G$. Define $S^G = S \cap A^G$. Show that the action of G on A extends to an action on $S^{-1}A$, and that $(S^G)^{-1}A^G \cong (S^{-1}A)^G$.**

It is clear that A^G is a subring of A. Let $a \in A$ and consider

$$p(x) = \prod_{\sigma \in G} (x - \sigma(a))$$

Notice that $p(a) = 0$ since 1_G induces the identity autmorphism on A. Label the elements of G as $\sigma_1, \dots, \sigma_n$ assuming that σ_1 is the identity map of A, and observe that $p(x) = x^n - a_1x^{n-1} + \cdots + (-1)^{n-1}a_{n-1}x + (-1)^n a_n$ where

$$a_k = \sum_{i_1 < \cdots < i_k} \sigma_{i_1}(a) \cdots \sigma_{i_k}(a)$$

It follows that $\tau(a_k) = a_k$ for any $\tau \in G$. In other words, the coefficients of p are elements of A^G. Consequently, A is integral over A^G.

Clearly $S^G = \{s \in S : \sigma(s) = s \text{ for every } \sigma \in G\}$ is a multiplicatively closed subset of A. Now given $\sigma \in G$ and $a/s \in S^{-1}A$, define $\sigma(a/s) = \sigma(a)/\sigma(s)$. Suppose that $a/s = a'/s'$ in $S^{-1}A$ so that $s''(as' - a's) = 0$ for some $s'' \in S$. Then $\sigma(s'')(\sigma(a)\sigma(s') - \sigma(a')\sigma(s))$ and $\sigma(s'') \in S$ so that $\sigma(a)/\sigma(s) = \sigma(a')/\sigma(s')$ in $S^{-1}A$. This means that σ extends to a well-defined map $S^{-1}A \to S^{-1}A$. Clearly this extension is a surjective homomorphism of rings. Now suppose that $0/1 = \sigma(a/s) = \sigma(a)/\sigma(s)$ so that $s'\sigma(a) = 0$ for some $s' \in S$. Now $\sigma(S) = S$ so that $s' = \sigma(s'')$ for some $s'' \in S$, implying that $\sigma(s''a) = 0$ and hence $s''a = 0$. This means that $a/s = 0/1$ in $S^{-1}A$. In other words, σ extends to an automorphism of $S^{-1}A$. It is also clear that the extension of the composition equals the composition of the extensions, so that G is a group of automorphisms of $S^{-1}A$.

Since the natural map $A^G \to S^{-1}A$ sends elements of S^G to units of $S^{-1}A$, there is a map $(S^G)^{-1}A^G \to S^{-1}A$ given by $a/s \mapsto a/s$. I claim that this map is injective. If $a \in A^G$ and $s \in S^G$ are such that $a/s = 0/1$ in $S^{-1}A$ then $ta = 0$ for some $t \in S$. In particular, $t\prod_{\sigma \in G^*} \sigma(t)a = 0$ where $t\prod_{\sigma \in G^*} \sigma(t) \in S^G$. So $a/s = 0/1$ in $(S^G)^{-1}A^G$. This means that the map $(S^G)^{-1}A^G \to S^{-1}A$ is injective. Clearly $\sigma(a/s) = a/s$ whenever $a \in A^G$ and $s \in S^G$, and hence the image of $(S^G)^{-1}A^G$ in $S^{-1}A$ is contained in $(S^{-1}A)^G$.

Now suppose that $x = a/s \in (S^{-1}A)^G$. Notice that $a/s = as'/ss'$ with $s' = \prod_{\sigma \neq \sigma_1} s$, and that $\sigma(ss') = ss'$ for every $\sigma \in G$. We still have $x = as'/ss'$. Since

$$as'/ss' = x = \sigma(x) = \sigma(as')/\sigma(ss') = \sigma(as')/ss'$$

there is, for every $\sigma \in G$, an element $u_\sigma \in S$ satisfying

$$u_\sigma(as'ss' - \sigma(as')ss') = 0$$

Defining $u = \prod_{\sigma \in G} u_\sigma$ we see that

$$uss'(as' - \sigma(as')) = 0 \quad \text{for every } \sigma \in G$$

Define $v = \prod_{\sigma \neq \sigma_1} \sigma(u)$ so that

$$uvss'(as' - \sigma(as')) = 0 \quad \text{and } uvss' \in S^G$$

Then $\sigma(as'uvss') = as'uvss'$ for all $\sigma \in G$. This means that $as'uvss' \in A^G$. Since

$$x = as'uvss'/uvss'ss'$$

with $as'uvss' \in A^G$ and $uvss'ss' \in S^G$ we conclude that x is in the image of the map $(S^G)^{-1}A^G \to S^{-1}A$. So we have the desired isomorphism $(S^G)^{-1}A^G \cong S^{-1}A$.

5.13. **In the situation above, let $\mathfrak{p}$ be a prime ideal in A^G and define P as the set of prime ideals in A whose contraction is $\mathfrak{p}$. Show that G acts transitively on P. In particular, P is finite.**

Suppose $\mathfrak{q} \in P$ and $\sigma \in G$ so that $\sigma(\mathfrak{q})$ is a prime ideal in A. It is easy to check that $\sigma(\mathfrak{q}) \cap A^G = \mathfrak{p}$. After all, if $a \in \sigma(\mathfrak{q}) \cap A^G$ with $a = \sigma(a')$ and $a' \in \mathfrak{q}$, then $a' = \sigma^{-1}(a) = a$ so that $a \in \mathfrak{q} \cap A^G = \mathfrak{p}$. Similarly, if $a \in \mathfrak{p} = \mathfrak{q} \cap A^G$ then $a = \sigma(a)$ so that $a \in \sigma(\mathfrak{q}) \cap A^G$. This means that G acts on P.

Now let $\mathfrak{q}_1$ and $\mathfrak{q}_2$ be elements in P. Suppose $x \in \mathfrak{q}_1$ and consider $y = \prod_{\sigma \in G} \sigma(x)$. Clearly $y \in A^G$ and $y \in \mathfrak{q}_1$ since 1_G induces the identity automorphism of A. Therefore, $y \in \mathfrak{q}_1 \cap A^G = \mathfrak{p} \subseteq \mathfrak{q}_2$. Since $\mathfrak{q}_2$ is a prime ideal, we see that $\sigma(x) \in \mathfrak{q}_2$ for some $\sigma \in G$. This means that $\mathfrak{q}_2 \subseteq \bigcup_{\sigma \in G} \sigma(\mathfrak{q}_1)$. Now $\sigma(\mathfrak{q}_1)$ is a prime ideal for each $\sigma \in G$, allowing us to conclude that $\mathfrak{q}_2 \subseteq \sigma(\mathfrak{q}_1)$ for some $\sigma \in G$. Since A is integral over A^G and $\sigma(\mathfrak{q}_1) \cap A^G = \mathfrak{p} = \mathfrak{q}_2 \cap A^G$, we see by Corollary 5.9 that $\sigma(\mathfrak{q}_1) = \mathfrak{q}_2$. In other words, G acts transitively on P. Finally, P is a finite set since G is finite and acts transitively on P.

5.14. **Let A be an integrally closed domain, K its field of fractions, and L a finite normal separable extension of K. Let G be the Galois group of L over K, and let B be the integral closure of A in L. Show that $\sigma(B) = B$ for every $\sigma \in G$, and that $A = B^G$.**

Suppose that $b \in B$, let b satisfy the integral dependence relation $b^n + \sum_{i=0}^{n-1} a_i b^i = 0$ where each $a_i \in A$, and let $\sigma \in G$. Then $\sigma(b)$ satisfies the integral dependence relation $\sigma(b)^n + \sum_{i=0}^{n-1} a_i \sigma(b)^i = 0$ since σ fixes K and $A \subseteq K$. This means that $\sigma(B) \subseteq B$. Similarly, $\sigma^{-1}(B) \subseteq B$ so that $B \subseteq \sigma(B)$, and hence $\sigma(B) = B$ for every $\sigma \in G$. Now A is clearly contained in B^G, and $B^G \subseteq L^G = K$. But elements in B^G are integral over A, and A is algebraically closed in K, implying that $B^G = A$.

5.15. **Let A be an integrally closed domain, K its field of fractions, L a finite extension field of K, and B the integral closure of A in L. Show that, if $\mathfrak{p}$ is any prime ideal in A, then the set of prime ideals $\mathfrak{q}$ in B that contract to $\mathfrak{p}$ is finite.**

Suppose for the moment that we can establish this result in the case that L/K is a separable extension or in the case that L/K is a purely inseparable extension. We know from field theory that there is an intermediate field $K \subset J \subset L$ so that J/K is a finite separable extension and L/J is a finite purely inseparable extension. Let C be the integral closure of A in J and notice that B is the integral closure of C in L. So by hypothesis, if $\mathfrak{p}$ is any prime ideal in A then there are finitely many prime ideals in C that contract to $\mathfrak{p}$, label these $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$. Again by hypothesis, for each i there are finitely many prime ideals in B that contract to $\mathfrak{q}_i$.

These are precisely the prime ideals of B that contract to $\mathfrak{p}$, and so finitely many prime ideals in B contract to $\mathfrak{p}$, establishing the claim. So it suffices to tackle the problem in the two special cases.

So suppose first that L is a finite separable extension of K. If $x_1, \dots, x_n$ generate L over K, then let $p_1, \dots, p_n$ be the minimal polynomials of x_i over K. Assuming L is embedded in its algebraic closure $\bar{L}$, let L' be the subfield of $\bar{L}$ generated by K and all of the roots of $p_1, \dots, p_n$. Then L' is an extension of L, L' is a finite extension over K since each root of $p_1, \dots, p_n$ is algebraic over K, and L' is a normal extension of K since it is generated over K by roots of irreducible polynomials. Further, since L is a separable extension of K, we know that each p_i is a separable polynomial, and so L' is separable over K as well. Now define G to be the Galois group of L' over K so that $L'^G = K$. Define B' to be the integral closure of A in L'. Exercise 5.14 tells us that the set of prime ideals P of B' lying over $\mathfrak{p}$ is finite. By the Going Up Theorem, if there is a prime ideal $\mathfrak{q}$ in B that lies over $\mathfrak{p}$, then there is a prime ideal $\mathfrak{r}$ in P that contracts to $\mathfrak{q}$. This means that there are finitely many prime ideals in B that contract to $\mathfrak{p}$.

Now assume that L is a finite purely inseparable extension of A. Let $\mathfrak{q}$ be a prime ideal of B that contracts to A, where B is the integral closure of A in L. As we may assume that $L \neq K$ we conclude that char(K) is a prime p. If $x^{p^m} \in \mathfrak{p}$ for some $m \geq 0$, then $x^{p^m} \in \mathfrak{q}$ so that $x \in \mathfrak{q}$. On the other hand, if $x \in \mathfrak{q}$ then $x^{p^m} \in K$ for some $m \geq 0$ since L/K is purely inseparable. But now $x^{p^m} \in K \cap \mathfrak{q} = \mathfrak{p}$. This means that $\mathfrak{q}$ consists of all $x \in L$ satisfying $x^{p^m} \in \mathfrak{p}$ for some $m \geq 0$. Hence, there is precisely one prime ideal of B lying over $\mathfrak{p}$. So we are done.

5.16. **Suppose k is an infinite field and A a finitely generated k-algebra. Show that there exist $y_1, \dots, y_s \in A$ algebraically independent over k such that A is integral over $k[y_1, \dots, y_r]$.**

Suppose A is generated by $x_1, \dots, x_n$ as a k-algebra. Renumber the $\{x_i\}$ and choose $r \geq 0$ so that $x_1, \dots, x_r$ are algebraically independent and each x_i is algebraic over $k[x_1, \dots, x_r]$ for $r < i \leq n$. Proceed by induction on $n - r$. If $n - r = 0$ then there is nothing to show. So suppose $n - r > 0$ and choose a non-trivial algebraic dependence relation $f(x_1, \dots, x_n) = 0$. Let F be the homogeneous part of highest degree in f. Since k is infinite, there exist $\lambda_1, \dots, \lambda_{n-1} \in k$ such that $\mu := F(\lambda_1, \dots, \lambda_{n-1}, 1) \neq 0$. After all, $F(\cdot, \dots, \cdot, 1)$ is a non-zero polynomial in $n-1$ variables, and so it cannot induce the zero function on k^{n-1} when k is infinite. Now define $x_i' = x_i - \lambda_i x_n$ for $1 \leq i < n$, and let $A' = k[x_1', \dots, x_{n-1}']$. I claim that x_n is integral over A'. Let $d = \deg(F)$ and choose polynomials G_j in $n-1$ variables so that

$$F(\xi_1, \dots, \xi_n) = \sum_{j=0}^{d} \xi_n^j G_j(\xi_1, \dots, \xi_{n-1})$$

Notice that each G_j is a homogeneous polynomial of degree $d - j$. Now let $\xi_i' = \xi_i - \lambda_i \xi_n$ and compute

$$\begin{aligned} F(\xi_1, \dots, \xi_n) &= \sum_{j=0}^{d} \xi_n^j G_j(\xi_1' + \lambda_1 \xi_n, \dots, \xi_{n-1}' + \lambda_{n-1} \xi_n) \\ &= \sum_{j=0}^{d} \xi_n^j \left[\xi_n^{d-j} G_j(\lambda_1, \dots, \lambda_{n-1}, 1) + H_j(\xi_1', \dots, \xi_{n-1}', \xi_n) \right] \\ &= \xi_n^d F(\lambda_1, \dots, \lambda_{n-1}, 1) + \sum_{j=0}^{d} \xi_n^j H_j(\xi_1', \dots, \xi_{n-1}', \xi_n) \end{aligned}$$

where each H_j is a polynomial in the variables $\xi_1', \dots, \xi_{n-1}', \xi_n$ with degree strictly less than $d - j$ in ξ_n, and with coefficients in k. Define a new polynomial $\tilde{F}$ by

$$\tilde{F}(\xi) = \xi^d + \frac{1}{\mu} \sum_{j=0}^{d} \xi^j H_j(x_1', \dots, x_{n-1}', \xi_n)$$

Then $\tilde{F}$ is a monic polynomial in ξ with coefficients in A' and such that $\tilde{F}(x_n) = F(x_1, \ldots, x_{n-1}, x_n) = 0$. Therefore, x_n is indeed integral over A'. This means that $A = k[x_1, \ldots, x_n] = k[x'_1, \ldots, x'_{n-1}, x_n]$ is integral over A'. By the induction hypothesis, there are $y_1, \ldots, y_s \in A'$ algebraically independent over k such that A' is integral over $A'[y_1, \ldots, y_s]$. Now $y_1, \ldots, y_s \in A$ are algebraically independent over k and A is integral over $A[y_1, \ldots, y_s]$. We are finished.

5.16.′ **Suppose that k is an algebraically closed field and that X is an affine algebraic variety in k^n with coordinate ring $A \neq 0$. Show that there is a linear subspace L of dimension r in k^n and a linear mapping of k^n onto L that maps X onto L.**

5.17. **Let k be algebraically closed. Show that, if $\mathfrak{a} \neq (1)$ is an ideal in $A = k[t_1, \ldots, t_n]$, then $V(\mathfrak{a}) \neq \emptyset$. Deduce that every maximal ideal in A is of the form $(t_1 - a_1, \ldots, t_n - a_n)$ for some $a_i \in k$.**

Let $\mathfrak{m}$ be a maximal ideal in A containing $\mathfrak{a}$. Then $A/\mathfrak{m} \neq 0$ is a finitely generated k-algebra, since it is generated by $t_1 + \mathfrak{m}, \ldots, t_n + \mathfrak{m}$ as a k-algebra. By Noether's Normalization Lemma, there are $y_1, \ldots, y_s \in A/\mathfrak{m}$ algebraically independent over k such that $A/\mathfrak{m}$ is integral over $k[y_1, \ldots, y_s]$. But $A/\mathfrak{m}$ is a field, so that $k[y_1, \ldots, y_s]$ is a field by Proposition 5.7. Since $k[y_1, \ldots, y_s]$ is a polynomial ring over k, we must have $s = 0$ and $k[y_1, \ldots, y_s] \cong k$. So $A/\mathfrak{m}$ is a finite algebraic extension of k. Since k is algebraically closed, we conclude that $A/\mathfrak{m} = k$. More precisely, $A/\mathfrak{m}$ is generated by $1 + \mathfrak{m}$ as a k-vector space. Now let a_i be the unique element in k satisfying $a_i + \mathfrak{m} = t_i + \mathfrak{m}$, so that $t_i - a_i \in \mathfrak{m}$. Then $\mathfrak{n} = (t_1 - a_1, \ldots, t_n - a_n) \subseteq \mathfrak{m}$. But $A/\mathfrak{n} \cong k$ so that $\mathfrak{n}$ is a maximal ideal, and hence $\mathfrak{n} = \mathfrak{m}$. Now $(a_1, \ldots, a_n) \in V(\mathfrak{m}) \subseteq V(\mathfrak{a})$. In particular, this means that $V(\mathfrak{a}) \neq \emptyset$.

5.18. **Let k be a field and B a finitely generated k-algebra. Suppose B is a field. Show that B is a finite algebra extension of k.**

Assume B is generated by $x_1, \ldots, x_n$ as a k-algebra. If $n = 1$ and $x_1 \neq 0$, then $x_1^{-1} = p(x_1)$ where p is some polynomial with coefficients in k, so that $x_1 p(x_1) = 1$. If $d = \deg(p)$ then we can write x_1^{d+1} as a k-linear combination of $\{1, x_1, \ldots, x_1^d\}$ so that B is finitely generated as a k-vetor space, and hence B is a finite algebraic extension of k.

Therefore, assume that $n > 1$. Define an integral subdomain $A = k[x_1]$ of B, and $K = k(x_1)$ as the field of fractions of A, contained in B since B is a field. Now B is a K-algebra generated by $\{x_2, \ldots, x_n\}$. By induction, B is a finite algebraic extension of K. In particular, $x_2, \ldots, x_n$ satisfy monic polynomial equations with coefficients in K. Coefficients in K are of the form a/b for $a, b \in A$. Let f be the product of the denominators of all these coefficients. Then the coefficients a/b are elements of A_f when we consider $A \subset A_f \subset K \subset B$. So $x_2, \ldots, x_n$ are integral over A_f. Since B is an A_f-algebra generated by $\{x_2, \ldots, x_n\}$, we see that B is integral over A_f, and hence that K is integral over A_f.

For the sake of deriving a contradiction, suppose that x_1 is trascendental over k. Then A is a Euclidean domain since k is a field, and so A is a unique factorization domain. As such, A is integrally closed in K. By 5.12 this means that A_f is integrally closed in $K_f = K$. By the above, integral closure of A_f in K equals K, implying that $A_f = K$. In other words, $k[x]_f = k(x)$ for some $f \in k[x]$. This is impossible: let $p \in k[x]$ be irreducible, then $1/p = g/f^n$ for some $n \in \mathbb{N}$ and some $g \in k[x]$ having no factor in common with f, implying that p is a factor of f, and in particular implying that $k[x]$ has finitely many irreducible elements. But an adaptation of Euclid's proof shows that $k[x]$ has infinitely many irreducible elements.

Therefore, x_1 is algebraic over k. As a result, $K = k(x_1)$ is a finite algebraic extension of k. As B is a finite algebraic extension of K, we conclude that B is a finite algebraic extension of k, as claimed.

5.19. **Deduce the result of exercise 17 from exercise 18.**

Choose a maximal ideal $\mathfrak{m}$ in A containing $\mathfrak{a}$. Notice that $A/\mathfrak{m}$ is a finitely generated k-algebra, which is itself a field. So $A/\mathfrak{m}$ is a finite algebraic extension of k by Corollary 5.24. But k is algebraically closed, so that $A/\mathfrak{m}$ is generated by $1 + \mathfrak{m}$ as a k-vector space. Let a_i be the unique element in k satisfying $a_i + \mathfrak{m} = t_i + \mathfrak{m}$, so that $t_i - a_i \in \mathfrak{m}$. Then $\mathfrak{n} = (t_1 - a_1, \ldots, t_n - a_n) \subseteq \mathfrak{m}$. But $A/\mathfrak{n} \cong k$ so that $\mathfrak{n}$ is a maximal ideal, and hence $\mathfrak{n} = \mathfrak{m}$. Now $(a_1, \ldots, a_n) \in V(\mathfrak{m}) \subseteq V(\mathfrak{a})$. In particular, this means that $V(\mathfrak{a}) \neq \emptyset$.

5.20. **Let A be a subring of an integral domain B so that B is finitely generated over A. Show that there exists $0 \neq s \in A$ and elements $y_1, \ldots, y_n \in B$ algebraically independent over A such that B_s is integral over $(B')_s$, where $B' = A[y_1, \ldots, y_n]$.**

Let F be the field of fractions of B, let $S = A - \{0\}$, and define $K \subset F$ by $K = S^{-1}A$ so that K is the field of fractions of A. Supposing that B is generated by $\{z_1, \ldots, z_m\}$ as an A-algebra, we easily see that $S^{-1}B$ is generated by $\{z_1, \ldots, z_m\}$ as a K-algebra. Hence, we can apply Noether's Normalization Lemma to deduce the existence of $y_1/s_1, \ldots, y_n/s_n \in S^{-1}B$ algebraically independent over K and such that $S^{-1}B$ is integral over $K[y_1/s_1, \ldots, y_n/s_n] = K[y_1, \ldots, y_n]$. If s is any element in S, then we have a commutative diagram as below.

Now suppose p is some polyomial in n indeterminates with coefficients in A such that $p(y_1, \ldots, y_n) = 0$. We can write

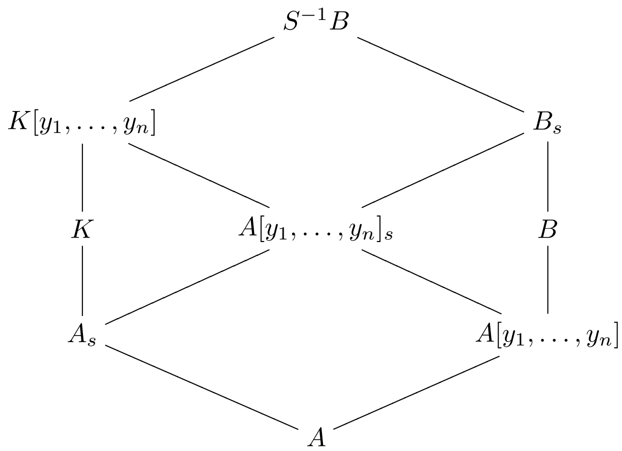
$$p(\xi_1, \ldots, \xi_n) = \sum_{\alpha:\{1,\cdots,n\}\to A} a_\alpha \xi_1^{\alpha(1)} \cdots \xi_n^{\alpha(n)} \qquad \text{with } a_\alpha \in A$$

Define a polynomial $\tilde{p}$ in n indeterminates with coefficients in K by

$$\tilde{p}(\xi_1, \ldots, \xi_n) = \sum_{\alpha:\{1,\cdots,n\}\to A} (a_\alpha s_1^{\alpha(1)} \cdots s_n^{\alpha(n)}) \xi_1^{\alpha(1)} \cdots \xi_n^{\alpha(n)}$$

Then $\tilde{p}(y_1/s_1, \ldots, y_n/s_n) = p(y_1, \ldots, y_n) = 0$. Since $y_1/s_1, \ldots, y_n/s_n$ are algebraically independent over K, we see that $\tilde{p} = 0$ and so $a_\alpha s_1^{\alpha(1)} \cdots s_n^{\alpha(n)} = 0$ for all α. But every $s_i \in S = A - \{0\}$ so that each $a_\alpha = 0$. This means that $p = 0$, and hence $y_1, \ldots, y_n$ are algebraically independent over A.

Now $z_1, \ldots, z_m$ satisfy integral dependence relations $q_i(z_i) = 0$ with coefficients from $K[y_1, \ldots, y_n]$. Define $d_i = \deg(q_i)$. Clearing denominators in all of the q_i simultaneously gives us an $s \in S$ and polynomials r_i with coefficients from $A[y_1, \ldots, y_n]$ so that $z_i^{d_i} + r_i(z_i)/s = 0$ and $\deg(r_i) < d_i$ for every i. In particular, each z_i is integral over $(B')_s$. Consequently, B_s is integral over $(B')_s$ since $B_s = (B')_s[z_1, \ldots, z_m]$.



5.21. **Let A and B be as in exercise 5.20. Show that there is $0 \neq s \in A$ such that, if Ω is an algebraically closed field and $f : A \to \Omega$ is a homomorphism satisfying $f(s) \neq 0$, then f can be extended to a homomorphisms $B \to \Omega$.**

We use the same notation as in exercise 2. Since $y_1, \ldots, y_n$ are algebraically independent over A, we have an extension $f : A[y_1, \ldots, y_n] \to \Omega$ induced by defining $f(y_i) = 0$ for every i. Now $f(s)$ is a unit in Ω since $f(s) \neq 0$. By the Universal Mapping Property for $A[y_1, \ldots, y_n]_s$, we have an extension $f : A[y_1, \ldots, y_n] \to \Omega$. Since B_s is integral over $A[y_1, \ldots, y_n]_s$ and since Ω is algebraically closed, exercise 5.2 tells us that we have an extension $f : B_s \to \Omega$. Now restriction yields a map $f : B \to \Omega$ that is an extension of the original map $A \to \Omega$.

5.22. **Let A and B be as in exercise 5.20. Show that the Jacobson radical $\mathfrak{R}(B)$ of B equals zero if $\mathfrak{R}(A) = 0$.**

Let $0 \neq v \in B$ and notice that A is a subring of the integral domain B_v. By exercise 5.21 there is $0 \neq s \in A$ such that, if Ω is an algebraically closed field and $f : A \to \Omega$ is a homomorphism satisfying $f(s) \neq 0$, then f can be extended to a homomorphism $B \to \Omega$. Let $\mathfrak{m}$ be a maximal ideal of A not containing s. This exists since $s \notin \mathfrak{R}(A) = 0$. Write $k = A/\mathfrak{m}$ and embed k in its algebraic closure Ω. Then the composition of the maps $A \to k \to \Omega$ is a homomorphism not sending s to 0. So we can extend this to a map $g : B_v \to \Omega$. Clearly $g(v) \neq 0$ since $v = v/1$ is a unit in B_v with inverse $1/v$. Hence, $v \notin \mathrm{Ker}(g) \cap B$.

5.23. **Show that the following are equivalent for a ring A.**

a. **Each prime ideal in A is an intersection of maximal ideals.**

b **In each homomorphic image of A, the nilradical equals the Jacobson radical.**

c. **Each non-maximal prime ideal in A equals the intersection of the prime ideals that strictly contain it.**

$(a \Rightarrow b)$ Let $\mathfrak{a}$ be a proper ideal in A. Every prime ideal in $A/\mathfrak{a}$ is of the form $\mathfrak{p}/\mathfrak{a}$ where $\mathfrak{p}$ is a prime ideal in A. By hypothesis, $\mathfrak{p}$ is an intersection of maximal ideals (containing $\mathfrak{p}$). These maximal ideals correspond to maximal ideals in $A/\mathfrak{a}$. So every prime ideal in $A/\mathfrak{a}$ is an intersection of maximal ideals. Hence, it suffices to show that $\mathfrak{N}(A) = \mathfrak{R}(A)$. As always $\mathfrak{N}(A) \subseteq \mathfrak{R}(A)$. Now every prime ideal in A contains $\mathfrak{R}(A)$ so that $\mathfrak{N}(A) \supseteq \mathfrak{R}(A)$, and therefore $\mathfrak{N}(A) = \mathfrak{R}(A)$.

$(a \Rightarrow c)$ Let $\mathfrak{p}$ be a non-maximal prime ideal. By hypothesis, $\mathfrak{p}$ is the intersection of all maximal ideals containing $\mathfrak{p}$. But these ideals strictly contain $\mathfrak{p}$ since $\mathfrak{p}$ is not a maximal ideal. Therefore, $\mathfrak{p}$ equals the intersection of all prime ideals strictly containing $\mathfrak{p}$.

$(b \Rightarrow c)$ Let $\mathfrak{p}$ be a non-maximal prime ideal in A so that $A/\mathfrak{p}$ is an integral domain that is not a field. Then 0 is not a maximal ideal in $A/\mathfrak{p}$. Since $0 = \mathfrak{N}(A/\mathfrak{p}) = \mathfrak{R}(A/\mathfrak{p})$ we see that 0 is the intersection of all maximal ideals in $A/\mathfrak{p}$. This means that $\mathfrak{p}$ is the intersection of all maximal ideals in A containing $\mathfrak{p}$, and hence is the intersection of all the prime ideals in A strictly containing $\mathfrak{p}$.

$(c \Rightarrow b)$ If b does not hold, then a does not hold, so that there is a prime ideal $\mathfrak{p}$ that is properly contained in the intersection I of all maximal ideals in A containing $\mathfrak{p}$. Choose $f \in I - \mathfrak{p}$ and notice that $A_f \neq 0$, since $1/1 = 0/1$ in A_f implies that $f^n = 0 \in \mathfrak{p}$ for some $n \geq 0$. Also, $\mathfrak{p}$ does not meet $\{1, f, f^2, \ldots\}$ so that $\mathfrak{p}_f \neq A_f$. Let $\mathfrak{m}$ be a maximal ideal in A_f containing $\mathfrak{p}_f$, so that $\mathfrak{m}^c$ is a prime ideal $\mathfrak{q}$ in A containing $\mathfrak{p}$. Observe that $f \in \mathfrak{q}$ implies that $f/1 \in \mathfrak{m}$ and hence $\mathfrak{m}$ contains a unit in A_f. Thus, $f \notin \mathfrak{q}$. If $\mathfrak{q}$ were a maximal ideal, then $f \in \mathfrak{q}$ since $f \in I$, but this is not the case. Suppose that $\mathfrak{r} \supseteq \mathfrak{q}$ is another prime ideal in A not containing f, so that $\mathfrak{r}$ does not meet $\{1, f, f^2, \ldots\}$, and hence $A_f \neq \mathfrak{r}_f \supseteq \mathfrak{q}_f = \mathfrak{m}$. Then $\mathfrak{r}_f = \mathfrak{m}$, and hence $\mathfrak{r} = \mathfrak{q}$. So if $\mathfrak{r}$ is a prime ideal strictly containing $\mathfrak{q}$, then $f \in \mathfrak{r}$. Hence, $\mathfrak{q}$ is not the intersection of the prime ideals in A strictly containing $\mathfrak{q}$, since this intersection contains $f \notin \mathfrak{q}$. Therefore, c does not hold when b does not hold.

5.24. **Let A be a Jacobson ring (as in exercise 5.23) and B an A-algebra. Show that if B is either integral over A or finitely generated as an A-algebra, then B is a Jacobson ring as well.**

Suppose that B is integral over A. Let $\mathfrak{p}$ be a prime ideal in B so that $A \cap \mathfrak{p}$ is a prime ideal in A. For every maximal ideal $\mathfrak{q}$ in A containing $A \cap \mathfrak{p}$, choose a maximal ideal $\mathfrak{r}$ in B with $A \cap \mathfrak{r} = \mathfrak{q}$. Then $A \cap \mathfrak{p} = \bigcap_{A\cap\mathfrak{p}\subseteq\mathfrak{q}} \mathfrak{q} = A \cap \bigcap_{A\cap\mathfrak{p}\subseteq\mathfrak{q}} \mathfrak{r}$ so that

Suppose that B is finitely generated as an A-algebra. Let $\mathfrak{p}$ be a prime ideal in B so that $\mathfrak{q} = A \cap \mathfrak{p}$ is a prime ideal in A, and $A/\mathfrak{q}$ is a subring of the integral domain $B/\mathfrak{p}$. Then $B/\mathfrak{p}$ is finitely generated over $A/\mathfrak{q}$. Since A is a Jacobson ring, $\mathfrak{R}(A/\mathfrak{q}) = \mathfrak{N}(A/\mathfrak{q}) = 0$. By exercise 5.22, $\mathfrak{R}(B/\mathfrak{p}) = 0$ as well, implying that $\mathfrak{q}$ is the intersection of all the maximal ideals in B containing $\mathfrak{q}$. Therefore, B is Jacobson.

5.25? **Show that A is a Jacobson ring if and only if every finitely generated A-algebra B which is a field is finite over A.**

5.26? **Show that the following are equivalent for a ring A.**

a. **A is a Jacobson ring.**

b **The maximal ideals are very dense in** Spec(A)**.**

c. **A singleton set in** Spec(A) **is closed if it's locally closed.**

$(a \Rightarrow b)$

$(b \Rightarrow c)$

$(c \Rightarrow a)$

5.27. **We say that the local ring $(B, \mathfrak{n})$ dominates the local ring $(A, \mathfrak{m})$ if $A \subseteq B$ and $\mathfrak{m} = A \cap \mathfrak{n}$. Let K be a field and let Σ consist of all local rings $(A, \mathfrak{m})$ of K, partially ordered by the above relation. Show that Σ has maximal elements and that $(A, \mathfrak{m})$ is a maximal element of Σ iff A is a valuation ring of K.**

Let $C = \{A_\alpha : \alpha \in I\}$ be a chain in Σ. Define $A = \bigcup_{\alpha \in I} A_\alpha$ and $\mathfrak{m} = \bigcup_{\alpha \in I} \mathfrak{m}_\alpha$. As usual, A is a ring with ideal $\mathfrak{m}$. If $x \in A \setminus \mathfrak{m}$, then $x \in A_\alpha \setminus \mathfrak{m}_\alpha$ for some α, and so x is a unit in A_α. But then x is a unit in A. Thus, $(A, \mathfrak{m})$ is a local ring dominating each $(A_\alpha, \mathfrak{m}_\alpha)$. Therefore, Σ is chain complete, and so Σ has maximal elements.

Suppose that $(A, \mathfrak{m}) \in \Sigma$ is a maximal element. Let Ω be the algebraic closure of $A/\mathfrak{m}$ and $\eta : A \to \Omega$ the canonical map. Denote Σ' as the set of all (B, f) with B a subring of K and f a map $B \to \Omega$. We order Σ' in the obvious way. Choose $(B, f) \in \Sigma'$ as a maximal element dominating (A, η). Then B is a local ring with maximal ideal $\mathfrak{n} = \mathrm{Ker}(f)$. Now $\mathfrak{m} = \mathrm{Ker}(\eta) = A \cap \mathrm{Ker}(f) = A \cap \mathfrak{n}$ so that $(B, \mathfrak{n}) \in \Sigma$ dominates $(A, \mathfrak{m})$. Therefore, $A = B$ by maximality. Consequently, Theorem 5.21 tells us that A is a valuation ring of K.

Suppose $(A, \mathfrak{m})$ is a valuation ring of K strictly dominated by $(B, \mathfrak{n})$. Choose $x \in B \setminus A$ so that $x^{-1} \in A$. Then $x^{-1} \in \mathfrak{m}$ since x^{-1} is a non-unit in A. But $x^{-1} \notin \mathfrak{n}$ since x^{-1} is invertible in B. This contradicts $\mathfrak{m} \subseteq \mathfrak{n}$. Thus, every valuation ring of K is maximal in Σ.

5.28. **Let K be the field of fractions of the integral domain A. Show that A is a valuation ring of K if and only if the ideals of A are totally ordered by inclusion. Deduce that, if A is a valuation ring and if $\mathfrak{p}$ is a prime ideal in A, then $A_\mathfrak{p}$ and $A/\mathfrak{p}$ are valuation rings in their field of fractions.**

Assume A is a valuation ring of K. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals in A. Suppose there is $x \in \mathfrak{a} - \mathfrak{b}$ and let $0 \neq y \in \mathfrak{b}$. Then $x/y \notin A$ since $\mathfrak{b}$ is an ideal. So we have $y/x \in A$, and hence $y \in \mathfrak{a}$. In other words $\mathfrak{b} \subseteq \mathfrak{a}$.

Now assume that $\mathfrak{a} \subseteq \mathfrak{b}$ or $\mathfrak{b} \subseteq \mathfrak{a}$ whenever $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in A. Suppose that $a, b \in A$ with $b \neq 0$ are such that $a/b \in K - A$. Then $a \neq 0$. Define ideals in A by $\mathfrak{a} = (a)$ and $\mathfrak{b} = (b)$. If $\mathfrak{a} \subseteq \mathfrak{b}$ then there is $c \in A$ with $bc = a$ so that $a/b = c \in A$; a contradiction. Thus, $\mathfrak{b} \subseteq \mathfrak{a}$, implying the existence of $c \in A$ with $ac = b$, so that $b/a = c \in A$. Hence, A is a valuation ring of K.

Now let $\mathfrak{p}$ be a prime ideal in A. Any two ideals in $A_\mathfrak{p}$ are of the form $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{b}_\mathfrak{p}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in A. Either $\mathfrak{a} \subseteq \mathfrak{b}$ or $\mathfrak{b} \subseteq \mathfrak{a}$ so that $\mathfrak{a}_\mathfrak{p} \subseteq \mathfrak{b}_\mathfrak{p}$ or $\mathfrak{b}_\mathfrak{p} \subseteq \mathfrak{a}_\mathfrak{p}$. This means that $A_\mathfrak{p}$ is a valuation ring in its field of fractions. Any two ideals in $A/\mathfrak{p}$ are of the form $\mathfrak{a}/\mathfrak{p}$ and $\mathfrak{b}/\mathfrak{p}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals in A containing $\mathfrak{p}$. Either $\mathfrak{a} \subseteq \mathfrak{b}$ or $\mathfrak{b} \subseteq \mathfrak{a}$ so that $\mathfrak{a}/\mathfrak{p} \subseteq \mathfrak{b}/\mathfrak{p}$ or $\mathfrak{b}/\mathfrak{p} \subseteq \mathfrak{a}/\mathfrak{p}$. This means that $A/\mathfrak{p}$ is a valuation ring in its field of fractions.

5.29? **Let A be a valuation ring of the field K. Show that every subring B of K containing A is local.**

What is the problem asking?

5.30. **Let A be a valuation ring of the field K. Assign to (A, K) a valuation $v : K \to \Gamma$ of K with values in Γ.**

Notice that $K^* = K - \{0\}$ is an abelian group under multiplication, and that the set U of units in A is a subgroup of K^*. Define an abelian group $\Gamma = K^*/U$. For $xU, yU \in \Gamma$, we say that $xU \geq yU$ provided $xy^{-1} \in A$. If $xU = x'U$ and $yU = y'U$ so that $xx'^{-1} \in U$ and $y^{-1}y' \in U$, then $xy^{-1} = x'y'^{-1} \cdot xx'^{-1}y^{-1}y' \in A$, and hence $xy^{-1} \in A$ if and only if $x'y'^{-1} \in A$. This means that our relation $\geq$ is well-defined. Clearly $xU \geq xU$ since $xx^{-1} \in A$. So $\geq$ is a reflexive relation. If $xU \geq yU \geq zU$ then $xy^{-1} \in A$ and $yz^{-1} \in A$, so that $xz^{-1} \in A$, and hence $xU \geq zU$. So $\geq$ is a transitive relation. Suppose $xU \geq yU$ and $yU \geq xU$, so that $xy^{-1} \in A$ and $yx^{-1} \in A$, implying that $xy^{-1} \in U$, and hence $xU = yU$. So $\geq$ is an antisymmetric relation. If $xU, yU \in \Gamma$ then $xy^{-1} \in A$ or $yx^{-1} \in A$, so that $xU \geq yU$ or $yU \geq xU$. So any two elements of Γ are comparable. All of these observations imply that $\geq$ is a total order on Γ. If $xU \geq yU$ and $zU \in \Gamma$, then $(xz)(yz)^{-1} = xy^{-1} \in A$ so that $xU + zU \geq yU + zU$. This means that Γ is a totally ordered abelian group. Define $v : K^* \to \Gamma$ and notice finally that $v(x+y) \geq \min\{v(x), v(y)\}$ since. This means that v is a valuation of K. Lastly, suppose x and y are non-zero elements such that $x \neq -y$. Either $xy^{-1} \in A$ or $yx^{-1} \in A$, so that either $(x+y)y^{-1} = 1 + xy^{-1} \in A$ or $(x+y)x^{-1} = 1 + yx^{-1} \in A$, and hence either $v(x+y) \geq v(x)$ or $v(x+y) \geq v(x)$. This means that $v(x+y) \geq \min\{v(x), v(y)\}$ for $x \neq y \in K^*$.

5.31. **Let $v : K^* \to \Gamma$ be a valuation. Show that K has the valuation ring $A = \{x \in K^* : v(x) \geq 0\} \cup \{0\}$. Thus, the concepts of valuation ring and valuations are equivalent.**

Lets make a few observations. Notice that $v(1)+v(1) = v(1)$ so that $v(1) = 0$. Suppose that $v(-1) < 0 = v(1)$ so that $v(-1) = v(1) + v(-1) > v(-1) + v(-1) = v(1)$, a contradiction. Thus, $v(-1) \geq v(1) = 0$. Finally, if $x \in K^*$ then $0 = v(1) = v(xx^{-1}) = v(x) + v(x^{-1})$.

From the above $1, -1 \in A$. If $x, y \in A - \{0\}$ then $v(xy) = v(x) + v(y) \geq v(x) + v(1) \geq v(1) + v(1) = 0$ so that $xy \in A$. Hence, A is closed under multiplication. If $x \neq y \in A$ then $x + y \in A$ since $v(x+y) \geq \min\{v(x), v(y)\} \geq 0$. So A is closed under addition. Finally, A is closed under additive inversion since $-1 \in A$ and A is closed under multiplication. These remarks show that A is a subring of K.

Now suppose that $x, x^{-1} \in K - A$ for some $x \neq 0$. Then $v(x), v(x^{-1}) < 0$ so that $v(x), v(x^{-1}) < v(1)$. Thus $0 = v(x)+v(x^{-1}) < v(1)+v(x^{-1}) < v(1)+v(1) = 0$. So all of these inequalities are equalities, implying that $v(x^{-1}) = 0 = v(x)$, a contradiction. We conclude that A is a valuation ring in K.

Now to show how these two concepts are equivalent in a precise manner. If we start with a field K and a valuation ring A, lets assign the valuation $v : K^* \to \Gamma = K^*/U$ as in exercise 5.20. Then $0 \neq x \in A$ if and only if $v(x) \geq v(1)$. But $v(1) = 0$ since $1 \in A$. Therefore, A equals the valuation ring of K assigned to v.

Conversely, suppose we start with a valuation $v : K^* \to \Gamma$ of the field K. Let A be the valuation ring of K consisting of 0 and all $x \in K^*$ such that $v(x) \geq 0$. Define $\Gamma' = K^*/U$ where U is the group of units in A, and let $v' : K^* \to \Gamma'$ by $v'(x) = xU$. Suppose that $v(x) = 0$ so that $0 = v(x) + v(x^{-1}) = v(x^{-1})$. Conversely, suppose that $x \in U$ so that $x^{-1} \in U$, and hence $v(x), v(x^{-1}) \geq 0$. Then $0 = v(x) + v(x^{-1}) = \min\{v(x), v(x^{-1})\}$, implying that $v(x) = 0$ or $v(x^{-1}) = 0$, and hence $v(x) = v(x^{-1}) = 0$. Combining these two remarks reveals that $U = \{x \in K^* : v(x) = 0\}$. Obviously $U = \{x \in K^* : v'(x) = 0\}$. Now define a map $f : \Gamma' \to \Gamma$ by $f(v'(x)) = v(x)$. If $v'(x) = v'(y)$ so that $xy^{-1} \in U$, then $v(xy^{-1}) = 0$, and hence $0 = v(x) + v(y^{-1}) = v(x) - v(y)$, implying that $v(x) = v(y)$. Therefore, ψ is well-defined. Similarly, ψ is injective. Obviously $\psi \circ v' = v$. Lastly, $\mathrm{Im}(v)$ is a totally order subgroup of Γ, and $\psi : \Gamma \to \mathrm{Im}(v)$ is an isomorphism of totally ordered groups.

5.32? **Suppose A is a valuation ring of K with value group Γ. Show that, if $\mathfrak{p}$ is a prime ideal in A, then there is an isolated subgroup Δ of Γ such that $v(A - \mathfrak{p})$ consists of all $\xi \in \Gamma$ with $v(\xi) \geq 0$. Show that this defines a bijective correspondence between $\mathrm{Spec}(A)$ and the set of all isolated subgroups of Γ. If $\mathfrak{p}$ is prime, then describe the values groups of $A/\mathfrak{p}$ and $A_\mathfrak{p}$.**

5.33. **Let Γ be a totally ordered abelian group. Construct a field K and a valuation v of K with Γ as**

the value group.

First let k be any field and $A = k[\Gamma]$ the group algebra of Γ over k. I claim that A is an integral domain. So suppose that $x = \sum_{\alpha \in S} a_\alpha \alpha$ and $y = \sum_{\beta \in T} b_\beta \beta$ are nonzero elements in $k[\Gamma]$, where S and T are finite subsets of Γ. Let $\alpha_1 < \cdots < \alpha_m$ be the elements of S, and $\beta_1 < \cdots < \beta_n$ be the elements of T, where we can assume that each a_{α_i} and b_{β_i} is nonzero. The smallest coefficient xy is $a_{\alpha_1} b_{\beta_1}(\alpha_1 + \beta_1)$, which is non-zero since k is a field. Therefore, $xy \neq 0$, and hence A is an integral domain.

Now letting x and y be as before, define $v_0 : A - \{0\} \to \Gamma$ by $v_0(x) = \alpha_1$. Notice that $v_0(xy) = \alpha_1 + \beta_1 = v_0(x) + v_0(y)$ and $v_0(x + y) =$.

5.34. **Let A be a valuation ring in its field of fractions K. Suppose $f : A \to B$ is such that f^* is a closed map. Show that, if $g : B \to K$ is a map of A-algebras, then $g(B) = A$.**

Since g is a map of A-algebras, $g \circ f = i$ where $i : A \to K$ is the inclusion map. Define $C = g(B)$ so that $A = g(f(A)) \subset g(B) = C$. Let $\mathfrak{n}$ be a maximal ideal in C, and define $\mathfrak{q} = g^{-1}(\mathfrak{n})$, so that $\mathfrak{q}$ is maximal in B. Since f^* is a closed map, $f^* : \operatorname{Spec}(B/\mathfrak{q}) \to \operatorname{Spec}(A/\mathfrak{p})$ is surjective, where $\mathfrak{p} = f^{-1}(\mathfrak{q})$. But 0 is the only prime ideal in $B/\mathfrak{q}$, so that $A/\mathfrak{p}$ is an integral domain with precisely one prime ideal. This means that $A/\mathfrak{p}$ is a field, and hence $\mathfrak{p}$ is a maximal ideal in A. Now we have $A \subset C \subset C_\mathfrak{n} \subset K$ with $(C_\mathfrak{n}, \mathfrak{n})$ a local ring. We also have $\mathfrak{p} = f^{-1}(\mathfrak{q}) = f^{-1}(g^{-1}(\mathfrak{n})) = i^{-1}(\mathfrak{n}) = A \cap \mathfrak{n}$ showing that $(C, \mathfrak{n})$ dominates $(A, \mathfrak{p})$. But A is a valuation ring in K, so that $A = C$ by exercise 5.27. In other words, $g(B) = A$, as claimed.

5.35? **Let B be an integral domain and f a map $A \to B$ such that $(f \otimes 1)^* : \operatorname{Spec}(B \otimes_A C) \to \operatorname{Spec}(A \otimes_A C)$ is a closed map for every A-algebra C. Show that f is an integral mapping.**

Chapter 6 : Chain Conditions

6.1. **Let M be an A-module and $u \in \mathbf{End}_A(M)$. Show the following.**

a. **If M is Noetherian and u is surjective then u is injective.**

Clearly $\operatorname{Ker}(u) \subseteq \operatorname{Ker}(u^2) \subseteq \ldots$ is a chain of submodules in M. So there is $n > 0$ with $\operatorname{Ker}(u^{n+1}) = \operatorname{Ker}(u^n)$. Suppose that $x \in \operatorname{Ker}(u)$. Since u is surjective, we can choose x' for which $u^n(x') = x$. Then $u^{n+1}(x') = u(x) = 0$ so that $u^n(x') = 0$. But now $x = 0$, and hence u is injective.

b. **If M is Artinian and u is injective then u is surjective.**

Clearly $\operatorname{Im}(u) \supseteq \operatorname{Im}(u^2) \supseteq \ldots$ is a chain of submodules in M. So there is $n > 0$ with $\operatorname{Im}(u^{n+1}) = \operatorname{Im}(u^n)$. Suppose that $x \in M$ and choose y for which $u^n(x) = u^{n+1}(y) = u^n(u(y))$. Since u is injective, we see that $u(y) = x$. This means that u is surjective.

6.2. **Let M be an A-module. If every non-empty set of finitely generated submodules of M has a maximal element, then M is Noetherian.**

Suppose that N is a submodule of M that is not finitely generated. Then given $x_1, \ldots, x_n \in N$ there is $x_{n+1} \in N$ not lying in the submodule N_n of N generated by $x_1, \ldots, x_n$. But then $N_1 \subset N_2 \subset \ldots$ is a strictly increasing sequence of finitely generated submodules of M, which has no maximal element. This contradiction shows that every submodule of M is finitely generated, and so M is Noetherian.

6.3. **Let M be an A-module, and let N_1, N_2 be submodules of M. If M/N_1 and M/N_2 are Noetherian, then so is $M/(N_1 \cap N_2)$. Similarly with Artinian in place of Noetherian.**

Define $\varphi : M/(N_1 \cap N_2) \to M/N_1 \oplus M/N_2$ by $\varphi(x + N_1 \cap N_2) = (x + N_1, x + N_2)$. This yields a well-defined A-module monomorphism. Now if $M/N_1, M/N_2$ are Noetherian (Artinian) then is $M/N_1 \oplus M/N_2$, and hence so is every submodule of $M/N_1 \oplus M/N_2$. Since φ is injective, this means that $M/(N_1 \cap N_2)$ is Noetherian (Artinian) as well.

6.4. **Let M be a Noetherian A-module and let $\mathfrak{a}$ be the annihilator of M in A. Prove that $A/\mathfrak{a}$ is Noetherian. Does a similar result hold with Artinian in place of Noetherian?**

Let M be Noetherian and suppose M is generated as an A-module by $\{x_1, \ldots, x_n\}$. Notice that $M^n = \bigoplus_1^n M$ is a Noetherian A-module and that the map $A \to M^n$ given by $a \mapsto (ax_1, \ldots, ax_n)$ is a homomorphism of A-modules. Clearly $\mathfrak{a} = \operatorname{Ann}(M)$ is precisely the kernel of this map. So $A/\mathfrak{a}$ is isomorphic with a submodule of M^n. From this we conclude that $A/\mathfrak{a}$ is a Noetherian A-module, and so is a Noetherian $A/\mathfrak{a}$-module, and is therefore a Noetherian ring.

This result does not hold with Artinian in place of Noetherian. As a counterexample, let p be a fixed prime number, take $A = \mathbb{Z}$, and define G as the subgroup of $\mathbb{Q}/\mathbb{Z}$ consisting of all $[a/b]$ with b a power of p. Then the subgroups of G are generated by $[1/p^n]$ for some $n \in \mathbb{N}$. Hence, G is an Artinian $\mathbb{Z}$-module. Now suppose that $n \in \mathbb{Z}$ annihilates G. Then $n/p^m \in \mathbb{Z}$ for every $m \geq 0$. This means that $n = 0$, and thus $\operatorname{Ann}(G) = 0$. But $\mathbb{Z}/\operatorname{Ann}(G) = \mathbb{Z}$ is not Artinian. So we have a counterexample.

6.5. **Show that every subspace Y of a Noetherian topological space X is Noetherian, and that X is compact.**

Let $U_1 \subseteq U_2 \subseteq \ldots$ be open sets in Y. Choose V_k open in X such that $U_k = V_k \cap Y$. Define $W_k = \bigcup_{1 \leq i \leq k} V_i$, and note that $W_k \cap Y = \bigcup_{1 \leq i \leq k} U_i = U_k$. Since $W_1 \subseteq W_2 \subseteq \ldots$ we deduce the existence of an N for which $W_n = W_N$ whenever $n \geq N$. But then $U_n = U_N$ whenever $n \geq N$. Therefore, Y is itself Noetherian.

Let $\mathcal{C}$ be a collection of closed subsets of X such that any intersection of finitely many members of $\mathcal{C}$ is non-empty. Let $\mathcal{I}$ denote the set of all intersections of finitely many members of $\mathcal{C}$ so that $\mathcal{I}$ is a collection of closed subsets of X. Then $\mathcal{I}$ has minimal elements. Since $\mathcal{I}$ is closed under finite intersections, it must be that $\mathcal{I}$ has a minimum element. Since this element is non-empty, we see that $\bigcap \mathcal{C}$ is non-empty. This implies that X is compact.

6.6. **Let X be a topological space. Show that X is Noetherian if and only if every open subspace is compact, and that this occurs if and only if every subspace of X is compact.**

Suppose that X is Noetherian. Then every subspace of X is Noetherian in the subspace topology, and so every subspace of X is compact.

If every subspace of X is compact then so is every open subspace.

Suppose that every open subspace of X is compact. Let $U_1 \subseteq U_2 \subseteq \ldots$ be a sequence of open subsets of X. Then $\{U_i\}_1^\infty$ is an open cover of $U = \bigcup_1^\infty U_i$. Since U is compact, $\{U_i\}_1^\infty$ has a finite subcover. This means that our sequence of open sets becomes stationary. Therefore, X is a Noetherian topological space.

6.7. **Show that a Noetherian topological space X is a union of finitely many irreducible closed subspaces. Conclude that X has finitely many irreducible components.**

Suppose that X is not the union of finitely many closed irreducible subspaces. Let Σ be the collection of all closed subsets of X that cannot be written as the union of finitely many closed irreducible subspaces of X. By hypothesis, $X \in \Sigma$ and so Σ is non-empty. Since X is Noetherian, Σ has a minimal element Y. Now Y is not irreducible, so Y is the union of two proper closed subsets, each of these being closed in X since Y is closed in X. By minimality of Y, each of these closed subsets can be written as the union of finitely many closed irreducible subspaces of X. This means that $Y \notin \Sigma$, a contradiction. Therefore, X is the union of finitely many irreducible closed subspaces.

This means that X is the union of finitely many irreducible components, say $Y_1, \ldots, Y_n$. If Y is an irreducible component of X, then $Y \subseteq \bigcup_1^n Y_i$. I claim that $Y \subseteq Y_i$ for some i. Otherwise, there is a set $S \subseteq \{1, \ldots, n\}$ minimal with respect to the property that $Y \subseteq \bigcup_{i \in S} Y_i$, with $|S| \geq 2$. But then $Y = \bigcup_{i \in S} Y \cap Y_i$ with each $Y \cap Y_i$ a proper closed subset of Y, contradicting the assumption that Y is irreducible. Therefore, $Y \subseteq Y_i$ for some i, and hence $Y = Y_i$ for some i. This means that X has finitely many irreducible components.

6.8. **Show that $\operatorname{Spec}(A)$ is a Noetherian topological space whenever A is a Noetherian ring. Is the converse true?**

Let A be a Noetherian ring. Suppose we have a descending sequence of closed subsets of $\operatorname{Spec}(A)$. This sequence has the form $V(\mathfrak{a}_1) \supseteq V(\mathfrak{a}_2) \supseteq \ldots$ for some ideals $\mathfrak{a}_i$ in A. The relation $V(\mathfrak{a}_i) \supseteq V(\mathfrak{a}_{i+1})$ implies that $r(\mathfrak{a}_i) \subseteq r(\mathfrak{a}_{i+1})$. This means that $r(\mathfrak{a}_1) \subseteq r(\mathfrak{a}_2) \subseteq \ldots$ is an increasing sequence of ideals in A. So we can choose N satisfying $r(\mathfrak{a}_n) = r(\mathfrak{a}_N)$ for all $n \geq N$. Then $V(\mathfrak{a}_n) = V(r(\mathfrak{a}_n)) = V(r(\mathfrak{a}_N)) = V(\mathfrak{a}_n)$ for all $n \geq N$. Therefore, $\operatorname{Spec}(A)$ is Noetherian.

It is not true that A needs to be a Noetherian ring when $\operatorname{Spec}(A)$ is a Noetherian topological space. As a counterexample, let $B = k[x_1, x_2, \ldots]$ be the polynomial ring in countably many variables, suppose we have the ideal $\mathfrak{a} = (x_1, x_2^2, x_3^3, \ldots)$ in B, and define $A = B/\mathfrak{a}$. Also define an ideal $\mathfrak{b} = (x_1, x_2, x_3, \ldots)$ in B. Then $\mathfrak{b}$ is a maximal ideal in B containing $\mathfrak{a}$, so that $\mathfrak{b}/\mathfrak{a}$ is a maximal ideal in A. But $\mathfrak{b}/\mathfrak{a} \subseteq \mathfrak{N}(A) \subset A$ so that $\mathfrak{N}(A) = \mathfrak{b}/\mathfrak{a}$. Therefore, A has exactly one prime ideal. This means that $\operatorname{Spec}(A)$ is a one-point space, and hence is trivially Noetherian. But A is not Noetherian since there is no $k \in \mathbb{N}$ satisfying $\mathfrak{N}(A)^k = 0$. After all, such a k would yield $\mathfrak{b}^k \subseteq \mathfrak{a}$, which cannot hold since $x_{k+1}^k \in \mathfrak{b}^k - \mathfrak{a}$ by inspection.

6.9. **Deduce from exercise 6.8 that a Noetherian ring A has finitely many minimal prime ideals.**

Since A is Noetherian, $\text{Spec}(A)$ is Noetherian, and so $\text{Spec}(A)$ has finitely many irreducible components. But the minimal prime ideals of A and the irreducible components of $\text{Spec}(A)$ are in a bijective correspondence under the map $\mathfrak{p} \mapsto V(\mathfrak{p})$. So A has finitely many minimal prime ideals.

6.10. **Let M be a Noetherian A-module. Show that $\text{Supp}(M)$ is a closed Noetherian subspace of $\text{Spec}(A)$.**

Since M is finitely generated, $\text{Supp}(M) = V(\text{Ann}(M))$. Therefore $\text{Supp}(M)$ is closed in $\text{Spec}(A)$. Also, $V(\text{Ann}(M))$ is homeomorphic with $\text{Spec}(A/\text{Ann}(M))$ as topological spaces. Exercise 6.4 shows that $A/\text{Ann}(M)$ is a Noetherian ring, so that $\text{Supp}(M)$ is a Noetherian space.

6.11. **Let $f : A \to B$ be a ring homomorphism and suppose that $\text{Spec}(B)$ is Noetherian. Prove that $f^* : \text{Spec}(B) \to \text{Spec}(A)$ is a closed mapping if and only if f has the going-up property.**

Suppose that f^* is a closed mapping. Let $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ be a chain of prime ideals in $f(A)$ with $\mathfrak{p}_1 = f(A) \cap \mathfrak{q}_1$, where $\mathfrak{q}_1$ is a prime ideal in B. Then $f^{-1}(\mathfrak{p}_2) \in V(f^*(\mathfrak{q}_1))$ since $f^*(\mathfrak{q}_1) = f^{-1}(\mathfrak{p}_1) \subseteq f^{-1}(\mathfrak{p}_2)$. Since $f^*(V(\mathfrak{q}_1)) = V(f^*(\mathfrak{q}_1))$ there is a prime ideal $\mathfrak{q}_2$ in B containing $\mathfrak{q}_1$ such that $f^{-1}(\mathfrak{p}_2) = f^*(\mathfrak{q}_2) = f^{-1}(f(A) \cap \mathfrak{q}_2)$. This means that $\mathfrak{p}_2 = f(A) \cap \mathfrak{q}_2$. Therefore, B and $f(A)$ satisfy the conclusions of the going-up theorem, showing that f has the going-up property.

Now suppose that f has the going up-property. Notice that $\text{Spec}(B/\mathfrak{b})$ is homeomorphic with $V(\mathfrak{b})$. So $V(\mathfrak{b})$ a Noetherian space, since it is a subspace of the Noetherian space $\text{Spec}(B)$. Exercise 6.9 tells us that there are finitely many prime ideals in B containing $\mathfrak{b}$ minimal with respect to inclusion. Label these primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ and write $\mathfrak{q}_i = \mathfrak{p}_i^c$. If $\mathfrak{r} \in f^*(V(\mathfrak{b}))$ then $\mathfrak{r} = \mathfrak{p}^c$ for some $\mathfrak{p}$ containing $\mathfrak{b}$, so that $\mathfrak{r} = \mathfrak{p}_i$ for some i. In other words, $f^*(V(\mathfrak{b})) \subseteq \bigcup_{i=1}^n V(\mathfrak{q}_i)$. Now suppose that $\mathfrak{r} \in V(\mathfrak{q}_i)$ for some i. Then $f(\mathfrak{q}_i) \subseteq f(\mathfrak{r})$ is a chain of prime ideals in $f(A)$ with $f(A) \cap \mathfrak{p}_i = f(\mathfrak{q}_i)$. So we can choose a prime ideal $\mathfrak{p}$ containing $\mathfrak{p}_i$ so that $f(A) \cap \mathfrak{p} = f(\mathfrak{r})$. But now $\mathfrak{r} = f^{-1}(\mathfrak{p})$ with $\mathfrak{p} \in V(\mathfrak{b})$, so that $\mathfrak{r} \in f^*(V(\mathfrak{b}))$. Thus, $f^*(V(\mathfrak{b})) = \bigcup_{i=1}^n V(\mathfrak{q}_i)$ is a closed set, so that f^* is a closed mapping.

6.12. **Let A be a ring such that $\text{Spec}(A)$ is a Noetherian space. Show that the set of prime ideals of A satisfies the ascending chain condition. Is the converse true?**

Let $\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \ldots$ be an ascending sequence of prime ideals in A. Then $V(\mathfrak{p}_1) \supseteq V(\mathfrak{p}_2) \supseteq \ldots$ is a descending sequence of closed subset in $\text{Spec}(A)$. Choose N with $V(\mathfrak{p}_n) = V(\mathfrak{p}_N)$ for all $n \geq N$. It follows immediately that $\mathfrak{p}_n = \mathfrak{p}_N$ for all $n \geq N$.

The converse does not hold. As a counterexample, take $A = \prod_{i=0}^{\infty} \mathbb{Z}_2(e_i)$. Suppose $\mathfrak{p} \subsetneq \mathfrak{q}$ are prime ideals in A, and let $x \in \mathfrak{q} - \mathfrak{p}$. Then $x^2 = x$ so that $x(1-x) = 0 \in \mathfrak{p}$, and hence $1 - x \in \mathfrak{p}$. But then $1 - x \in \mathfrak{q}$ so that $1 \in \mathfrak{q}$, a contradiction. This means that every prime ideal in A is maximal, so that the prime ideals in A satisfy the ascending chain condition. Now define an ideal $\mathfrak{a}_n$ in A by $\mathfrak{a}_n = \prod_{i=1}^n \mathbb{Z}_2(e_i)$ so that $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \ldots$ and hence $V(\mathfrak{a}_1) \supseteq V(\mathfrak{a}_2) \supseteq \ldots$ is a descending sequence of closed subsets of $\text{Spec}(A)$. Now $\prod_{j \neq n+1} \mathbb{Z}_2(e_i)$ is a prime ideal in A containing $\mathfrak{a}_n$ but not containing $\mathfrak{a}_{n+1}$ so that $V(\mathfrak{a}_n) \supsetneq V(\mathfrak{a}_{n+1})$ for all n. This shows that $\text{Spec}(A)$ is not a Noetherian space.

Chapter 7 : Noetherian Rings

7.1. **Suppose A is a non-Noetherian ring and let Σ consist of all ideals in A that are not finitely generated, so that $\Sigma \neq \emptyset$. Show that Σ has maximal elements and that every maximal element is a prime ideal. So if every prime ideal is finitely generated, then A is Noetherian.**

A straightforward application of Zorn's Lemma tells us that Σ has maximal elements since Σ is chain complete. Let $\mathfrak{a}$ be a maximal element in Σ and suppose that there are $x, y \notin \mathfrak{a}$ for which $xy \in \mathfrak{a}$. Then $\mathfrak{a} \subsetneq \mathfrak{a} + (x)$. By maximality, $\mathfrak{a} + (x)$ is finitely generated, by elements of the form $a_i + b_i x$, where a_i are elements of $\mathfrak{a}$ and b_i are elements of A. Let $\mathfrak{a}_0$ be the ideal of $\mathfrak{a}$ generated by the a_i. Clearly $\mathfrak{a}_0 + (x) = \mathfrak{a} + (x)$. Also clear is that $\mathfrak{a}_0 + x(\mathfrak{a} : x) \subseteq \mathfrak{a}$. So suppose that $a \in \mathfrak{a}$. Then $a + x = \sum c_i(a_i + b_i x)$ for appropriate $c_i \in A$. Hence, $a = \sum c_i a_i + x(\sum b_i c_i - 1)$ where $\sum b_i c_i - 1$ is in $(\mathfrak{a} : x)$. Consequently $\mathfrak{a} = \mathfrak{a}_0 + x(\mathfrak{a} : x)$. Observe that $(\mathfrak{a} : x)$ strictly contains $\mathfrak{a}$ since $y \in (\mathfrak{a} : x) - \mathfrak{a}$. By maximality of $\mathfrak{a}$ we see that $(\mathfrak{a} : x)$ is finitely generated. But then $\mathfrak{a} = \mathfrak{a}_0 + x(\mathfrak{a} : x)$ is itself finitely generated; a contradiction. So every maximal element in Σ is prime. Therefore, a ring in which every prime ideal is finitely generated must be Noetherian.

7.2. **Suppose A is a Noetherian ring and let $f = \sum_{i=0}^{\infty} a_i x^i \in A[[x]]$. Show that f is nilpotent if and only if each a_i is nilpotent.**

From exercise 1.5 each a_i is nilpotent if f is nilpotent. So suppose that each a_i is nilpotent. Then each $a_i \in \mathfrak{N}(A)$. Since A is Noetherian there is $n > 0$ for which $\mathfrak{N}(A)^n = 0$. By induction each coefficient of f^n is an element of $\mathfrak{N}(A)^n$, so that $f^n = 0$. Hence, f is nilpotent.

7.3. **Let $\mathfrak{a}$ be a proper irreducible ideal in a ring A. Prove that the following are equivalent.**

a. **The ideal $\mathfrak{a}$ is $\mathfrak{p}$-primary for some prime ideal $\mathfrak{p}$.**
b. **For every S the saturation $S(\mathfrak{a}) = (\mathfrak{a} : s)$ for some $s \in S$.**
c. **The sequence $(\mathfrak{a} : x^n)$ is stationary for every $x \in A$.**

(a $\Rightarrow$ b) If it occurs that $r(\mathfrak{a}) \cap S = \emptyset$, then since $r(\mathfrak{a})$ is a prime ideal, we can deduce that $S(\mathfrak{a}) = \mathfrak{a}$ with of course $\mathfrak{a} = (\mathfrak{a} : 1)$. So suppose then that $s \in r(\mathfrak{a}) \cap S$. Choose $n > 0$ for which $s^n \in \mathfrak{a}$. Then $S(\mathfrak{a}) = (1)$ and $\mathfrak{a} = (\mathfrak{a} : s^n)$ with $s^n \in S$. So we are done.

(b $\Rightarrow$ c) Let $x \in A$ and define $S = \{1, x, x^2, \ldots\}$. Then $\bigcup_{n=0}^{\infty} (\mathfrak{a} : x^n) = S(\mathfrak{a}) = (\mathfrak{a} : x^N)$ for some N. Thus $(\mathfrak{a} : x^N) = (\mathfrak{a} : x^n)$ for $n \geq N$.

(c $\Rightarrow$ a) We can imitate the proof of Lemma 7.12, noting that the ascending chain of ideals becomes stationary by hypothesis (instead of assuming that the ring A is Noetherian).

7.4. **Which of the following rings A are Noetherian?**

a. **The ring A of rational functions having no pole on S^1.**

Let S be the set of all $f \in \mathbb{C}[z]$ so that f has no zero on S^1. It is clear that S is a multiplicatively closed subset of $\mathbb{C}[z]$, and that $A = S^{-1}\,\mathbb{C}[z]$. Since $\mathbb{C}[z]$ is a Noetherian ring, we see that A is a Noetherian ring.

b. **The ring A of powers series in z with a positive radius of convergence.**

Notice that A is the ring of germs of functions defined at 0. Let $\mathfrak{a}$ be an ideal in A. If $0 \neq f \in \mathfrak{a}$ then write $f(z) = \sum_{i=n}^{\infty} a_i z^i$ with $n \geq 0$ and $a_n \neq 0$. Define $g(z) = \sum_{i=0}^{\infty} a_{i+n} z^i$ so that $f(z) = z^n g(z)$ and $g(0) = a_n \neq 0$. Complex analysis tells us that $g \in A$ and $1/g \in A$, so that g is invertible in A. In particular, $z^n = f \cdot 1/g \in \mathfrak{a}$. Assume n is the smallest number satisfying $z^n \in \mathfrak{a}$. From what we have shown, $\mathfrak{a} = (z^n)$. So the ideals in A are $A \supset (z) \supset (z^2) \supset \ldots \supset (0)$. We see that A is Noetherian.

c. **The ring A of power series in z with an infinite radius of convergence.**

Notice that A is the same as the ring of entire functions on $\mathbb{C}$. More precisely, an element of A yields an entire function on $\mathbb{C}$ via evaluation, and every entire function on $\mathbb{C}$ yields an element of A by taking the Taylor expansion of the function at the origin. Now by Weierstrass' Theorem for complex analysis, there is, for every $n \in \mathbb{N}$, an entire function f_n defined on $\mathbb{C}$ having simple zeros precisely at $n, n+1, n+2, \dots$ and no zeros elsewhere. Suppose that g is an entire function with zeros at $n, n+1, n+2, \dots$. Then g/f_n is an entire function, so that $g \in (f_n)$ and hence (f_n) is the set of all entire functions that vanish at $n, n+1, n+2, \dots$. Defining $\mathfrak{a}_n = (f_n)$, we have $\mathfrak{a}_0 \subsetneq \mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \dots$ is a properly ascending sequence of ideals in A, showing that A is non-Noetherian.

d. **The ring A of polynomials in z whose first k derivatives vanish at the origin, where k is a fixed natural number.**

It is easy to see that A is the set of all polynomials $c + z^{k+1}p(z)$ where $c \in \mathbb{C}$ and $p \in \mathbb{C}[z]$. Therefore, A is generated over $\mathbb{C}$ by $\{1, z^{k+1}, z^{k+2}, \dots, z^{2k+1}\}$. In other words, A is finitely generated over the Noetherian ring $\mathbb{C}$, and therefore A is itself Noetherian.

e. **The ring A of polynomials in z and w all of whose partial derivatives with respect to w vanish at $z = 0$.**

Define $B = \mathbb{C}[z, zw, zw^2, zw^3, \dots]$ so that B is a subring of $\mathbb{C}[z, w]$. It is clear that $zw^i \in A$ for every $i \geq 0$. Since A is a ring containing $\mathbb{C}$, we see that $B \subseteq A$. On the other hand, let p be a general element of A. We can choose $n \in \mathbb{N}$ and $p_0, \dots, p_n \in \mathbb{C}[z]$ satisfying

$$p(z, w) = p_0(z) + p_1(z)w + p_2(z)w^2 + \cdots + p_n(z)w^n$$

Notice that

$$\frac{\partial p}{\partial w}(z, w) = p_1(z) + 2p_2(z)w + \cdots + np_n(z)w^{n-1}$$

Our condition on p is that

$$p_1(0) + 2p_2(0)w + \cdots + np_n(0)w^{n-1} = 0$$

Since this holds for all $w \in \mathbb{C}$ we conclude that $p_1(0) = p_2(0) = \dots = p_n(0) = 0$. In other words, $z \mid p_i(z)$ for $1 \leq i \leq n$. From this we see that $p \in B$, and hence $B = A$. Now let I_n be the ideal generated by $z, zw, zw^2, \dots, zw^n$. Then $I_1 \subseteq I_2 \subseteq \dots$ is a sequence of ideals in A. Suppose, for the sake of contradiction, that $zw^{n+1} \in I_n$. Then we can write

$$zw^{n+1} = \sum_{j=0}^{n} \lambda_j(z, w) zw^j \quad \text{for some } \lambda_j(z, w) \in B$$

Now we can write

$$\lambda_j(z, w) = q_0(z) + zr_0(z)t_0(w)$$

Combining these relations yields

$$zw^{n+1} = \sum_{j=0}^{n} q_0(z) zw^j + z^2 \sum_{j=0}^{n} r_0(z)t_0(w)w^j$$

This equality is impossible by inspection. So $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \dots$ is a properly ascending sequence of ideals in A. This means that A is non-Noetherian.

7.5. **Let A be a Noetherian ring, B a finitely generated A-algebra, and G a finite group of A-automorphisms of B. Show that B^G is a finitely generated A-algebra as well.**

Suppose $f : A \to B$ induces the A-algebra structure of B. Notice that B^G is an A-subalgebra of B containing $f(A)$. By exercise 5.12 we know that B is integral over B^G. So we have the sequence $f(A) \subseteq B^G \subseteq B$ with $f(A)$ a Noetherian ring, B a finitely generated $f(A)$-algebra, and B integral over B^G. So proposition 7.8 tells us that B^G is finitely generated as an $f(A)$-algebra, and hence as an A-algebra, as desired.

7.6. **Show that a finitely generated field K is finite.**

Suppose that $\mathrm{char}(K) = 0$ so that $\mathbb{Z} \subset \mathbb{Q} \subseteq K$. Then K is finitely generated over $\mathbb{Q}$ since K is finitely generated over $\mathbb{Z}$ by hypothesis. So K is finitely generated as a $\mathbb{Q}$-module by proposition 7.9. Since $\mathbb{Z}$ is Noetherian, proposition 7.8 tells us that $\mathbb{Q}$ is finitely generated over $\mathbb{Z}$, say by $\{a_1/b_1, \ldots, a_n/b_n\}$. But if p is a prime number not dividing any b_i, then $1/p$ is not in $\mathbb{Z}[a_1/b_1, \ldots, a_n/b_n] \subseteq \mathbb{Z}[1/b_1 \cdots b_n]$. Hence, the characteristic of K is a prime number p. Again, proposition 7.9 tells us that K is finitely generated as an $\mathbb{F}_p$-module, so that K is a finite field.

7.7. **Suppose k is an algebraically closed field and I an ideal of $k[x_1, \ldots, x_n]$. Let $X \subset k^n$ consist of all x so that $f(x) = 0$ for every $f \in I$. Show that there is a finite subset $I_0 \subset I$ so that $x \in X$ if and only if $f(x) = 0$ for every $x \in I_0$.**

Obviously k is Noetherian, so that $k[x_1, \ldots, x_n]$ is Noetherian. Hence, I is a finitely generated ideal. Suppose I is generated by $f_1, \ldots, f_n$. If $x \in X$ then $f_i(x) = 0$ for every i. Conversely, let $f \in I$ and write $f = \sum_1^n g_i f_i$ with $g_i \in k[x_1, \ldots, x_n]$. Then $f(x) = 0$ provided that $f_i(x) = 0$ for every i. Hence, $I_0 = \{f_i\}_1^n$ is the desired subset of I.

7.8. **If $A[x]$ is Noetherian, must A be Noetherian as well?**

Define a ring homomorphism $A[x] \to A$ by $\sum_0^n a_k x^k \mapsto a_0$. Since this map is surjective, A is Noetherian.

7.9. **Show that the ring A is Noetherian if the following hold**

a. **For each maximal ideal $\mathfrak{m}$, the ring $A_\mathfrak{m}$ is Noetherian.**
b. **For each $x \neq 0$ in A, there are finitely many maximal ideals in A containing x.**

Let $\mathfrak{a} \neq 0$ be any ideal in A and suppose $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are the maximal ideals in A containing $\mathfrak{a}$. Suppose $x_0 \in \mathfrak{a}$ is nonzero and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r, \ldots, \mathfrak{m}_{r+s}$ be the maximal ideals in A containing x. Since $\mathfrak{a} \not\subseteq \mathfrak{m}_{r+j}$ for $j > 0$ there is $x_j \in \mathfrak{a} - \mathfrak{m}_{r+j}$. Now $\mathfrak{a} = \mathfrak{a}_{\mathfrak{m}_i}^c$ for $1 \le i \le r$ since $\mathfrak{a} \cap (A - \mathfrak{m}_i) = \emptyset$. But each $\mathfrak{a}_{\mathfrak{m}_i}$ is an ideal in $A_{\mathfrak{m}_i}$ and so is finitely generated, since $A_{\mathfrak{m}_i}$ is Noetherian. If $\mathfrak{a}_{\mathfrak{m}_i}$ is generated by $\xi_1^{(i)}, \ldots, \xi_q^{(i)}$ then we can choose $a_1^{(i)}, \ldots, a_q^{(i)} \in \mathfrak{a}$ with $a_j^{(i)}/1 = \xi_i^{(j)}$ so that $\mathfrak{a}_{\mathfrak{m}_i}$ is generated by the images of $a_1^{(i)}, \ldots, a_q^{(i)}$ in $A_{\mathfrak{m}_i}$. Now choose some $t > 0$ and some $x_{s+1}, \ldots, x_t \in \mathfrak{a}$ so that

$$\{x_{s+1}, \ldots, x_t\} = \{\xi_j^{(i)} | 1 \le j \le q \quad \text{and} \quad 1 \le i \le r\}$$

So the images of $x_{s+1}, \ldots, x_t$ in $A_{\mathfrak{m}_i}$ generate $\mathfrak{a}_{\mathfrak{m}_i}$ for every $1 \le i \le r$. Now define $\mathfrak{b} = (x_0, x_1, \ldots, x_t)$. We have the inclusion map $\phi : \mathfrak{b} \to \mathfrak{a}$. To show that $\mathfrak{b} = \mathfrak{a}$ it is enough to show that ϕ is surjective. So it suffices to show that $\phi_\mathfrak{m} : \mathfrak{b}_\mathfrak{m} \to \mathfrak{a}_\mathfrak{m}$ is surjective whenever $\mathfrak{m}$ is a maximal ideal in A. That is, it suffices to show that $\mathfrak{b}_\mathfrak{m} = \mathfrak{a}_\mathfrak{m}$. We already know this to be true when $\mathfrak{m}$ contains $\mathfrak{a}$. So suppose that $\mathfrak{a} \not\subseteq \mathfrak{m}$. If $x_0 \in \mathfrak{m}$ then $\mathfrak{m} = \mathfrak{m}_{r+i}$ for some $i > 0$ so that $\mathfrak{b}_\mathfrak{m} = A_\mathfrak{m}$ (since $x_i/1 \in \mathfrak{b}_\mathfrak{m}$ is a unit in $A_\mathfrak{m}$) and hence $\mathfrak{b}_\mathfrak{m} = \mathfrak{a}_\mathfrak{m}$. If $x_0 \notin \mathfrak{m}$ then $\mathfrak{b}_\mathfrak{m} = A_\mathfrak{m}$ (since $x_0/1 \in \mathfrak{b}_\mathfrak{m}$ is a unit in $A_\mathfrak{m}$) so that $\mathfrak{b}_\mathfrak{m} = \mathfrak{a}_\mathfrak{m}$. Therefore, $\mathfrak{a} = \mathfrak{b}$ is finitely generated, proving that A is a Noetherian ring.

7.10. **Let M be a Noetherian A-module. Show that $M[x]$ is a Noetherian $A[x]$-module.**

Suppose N is an $A[x]$-submodule of $M[x]$. For $n \geq 0$, let M_n be the set of all $m \in M$ so that $mx^n + p \in N$ where $p \in M[x]$ is some polynomial of degree at most $n-1$. Then M_n is an A-submodule of M, so that $M_0 \subseteq M_1 \subseteq \ldots$ is an ascending sequence of submodules. Since M is a Noetherian A-module, there is N^* such that $M_n = M_{N^*}$ for all $n \geq N^*$. Again since M is Noetherian, there are $m_{i,j} \in M_i$ such that $\{m_{i,1}, \ldots, m_{i,r}\}$ generates M_i for $1 \leq i \leq N$. Clearly, $\{m_{N^*,1}, \ldots, m_{N^*,r}\}$ generates M_n for $n \geq N^*$. For each i, j choose $p_{i,j}$ of degree at most $i-1$ so that $m_{i,j}x^i + p_{i,j} \in N$ and define $q_{i,j} = m_{i,j}x^i + p_{i,j}$.

Assume $0 \neq p \in N$ has degree d and let m be the leading coefficient of p. Suppose $d > N^*$, and let $m = \sum_{i=1}^r a_i m_{N^*,i}$ with $a_i \in A$. Then defining $p' = p - \sum_{i=1}^r a_i x^{d-N^*} q_{N^*,i}$ yields $p' \in N$ with p' having degree less than d. By induction, there is $p' \in N$ with $\deg(p-p') \leq N^*$. Now we proceed analogously to write $p - p'$ as an A-linear sum of the $q_{i,j}$. So p is an $A[x]$-linear sum of the $q_{i,j}$. This means that $\{q_{i,j}\}$ generates N as an $A[x]$-module, and hence N is finitely generated. Consequently, $M[x]$ is a Noetherian $A[x]$-module.

7.11. **Let A be a ring such that each local ring $A_\mathfrak{p}$ is Noetherian. Must A itself be Noetherian?**

Define A to be the internal direct product $A = \prod_{k=1}^{\infty} \mathbb{Z}_2(e_k)$. Let $\mathfrak{a}_n$ be the ideal generated by $e_1, \ldots, e_n \in A$. Then A is not Noetherian since we have a countable properly increasing sequence of ideals in A

$$\mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \mathfrak{a}_3 \subsetneq$$

Let $\mathfrak{p}$ be any prime ideal in A. Suppose $x \in \mathfrak{p}$ so that $1 - x \notin \mathfrak{p}$, for otherwise $1 \in \mathfrak{p}$. Then $x/1 = 0/1$ in $A_\mathfrak{p}$ since $(1-x)x = x - x^2 = 0$. Therefore, $A_\mathfrak{p}$ is a local ring whose maximal ideal $\mathfrak{p}_\mathfrak{p} = 0$. This means that $A_\mathfrak{p}$ is a field, and is hence Noetherian. This shows that A need not be Noetherian even if each of its localizations is Noetherian, so that being Noetherian is not a local property.

7.12. **Let A be a ring and B a faithfully flat A-algebra. If B is Noetherian, show that A is Noetherian.**

Suppose that $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \ldots$ is an ascending chain of ideals in A. Since extension is order preserving, $\mathfrak{a}_1^e \subseteq \mathfrak{a}_2^e \subseteq \ldots$ is an ascending chain of ideals in B. But then there is N for which $\mathfrak{a}_n^e = \mathfrak{a}_N^e$ whenever $n \geq N$. Because B is faithfully flat we see that $\mathfrak{a}_n = \mathfrak{a}_n^{ec} = \mathfrak{a}_N^{ec} = \mathfrak{a}_N$ whenever $n \geq N$. Hence, A is Noetherian as well.

7.13. **Let $f : A \to B$ be a ring homomorphism of finite type. Show that the fibers of f^* are Noetherian subspaces of B.**

Let $\mathfrak{p}$ be a prime ideal in B. By hypothesis, B is a finitely generated A-algebra. So $B \otimes_A k(\mathfrak{p})$ is a finitely generated $k(\mathfrak{p})$-algebra. But this means that $B \otimes_A k(\mathfrak{p})$ is a Noetherian ring since $k(\mathfrak{p})$ is a field. Hence, $\mathrm{Spec}(B \otimes_A k(\mathfrak{p}))$ is a Noetherian topological space by exercise 6.8. So we are done.

7.14. **Suppose k is an algebraically closed field and $\mathfrak{a}$ is an ideal in the ring $A = k[t_1, \ldots, t_n]$. Show that $I(V(\mathfrak{a})) = r(\mathfrak{a})$.**

Suppose that $f \in r(\mathfrak{a})$ so that $f^n \in \mathfrak{a}$ for some $n > 0$. If $x \in V(\mathfrak{a})$ then $0 = f^n(x) = f(x)^n$, so that $f(x) = 0$. We see that $f \in I(V(\mathfrak{a}))$, and hence $r(\mathfrak{a}) \subseteq I(V(\mathfrak{a}))$.

Now suppose that $f \notin r(\mathfrak{a})$ and choose a prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$ so that $f \notin \mathfrak{p}$. Let $\bar{f} \neq 0$ be the image of f in $B = A/\mathfrak{p}$, and define $C = B_{\bar{f}}$. Notice that $C \neq 0$ since B is an integral domain and $\bar{f} \neq 0$. Let $\mathfrak{m}$ be a maximal ideal in C. Now A is generated as a k-algebra by $\{t_1, \ldots, t_n\}$ so that B is generated as a k-algebra by $\{\bar{t_1}, \ldots, \bar{t_n}\}$. We see that C is generated as a k-algebra by $\{\bar{1}/\bar{f}, \bar{t_1}/\bar{1}, \ldots, \bar{t_n}/\bar{1}\}$. In particular, C is a finitely generated k-algebra. Since k is algebraically closed, we have $C/\mathfrak{m} \cong k$. More precisely, $1 + \mathfrak{m}$ generates $C/\mathfrak{m}$ as a k-vector space. Now we have a series of maps

$$A \xrightarrow{\pi_A} B \xrightarrow{\varphi} C \xrightarrow{\pi_C} C/\mathfrak{m} \cong k$$

Let ψ denote the composition of these maps, and let $x_i = \psi(t_i)$. Then we can consider $x = (x_1, \ldots, x_n)$ as being a point in k^n. More precisely, we choose x_i to be the unique point in k satisfying $x_i + \mathfrak{m} = \psi(t_i)$. Let g be any element in A, so that $\psi(g)$ can be considered as a point in k^n as well. I claim that $\psi(g) = g(x)$. This holds for each of $t_1, \ldots, t_n \in A$ and so it holds for any $g \in A$ since all maps involved are maps of k-algebras, including valuation at the point x.

Now let g be any element of $\mathfrak{a}$. Then $g \in \mathfrak{p}$ so that $\pi_A(g) = 0$, and hence $g(x) = \psi(g) = 0$. This means that $x \in V(\mathfrak{a})$. On the other hand, $\varphi(\pi_A(f)) = \bar{f}/\bar{1}$ is a unit in C so that $\varphi(\pi_A(f)) \notin \mathfrak{m}$, and hence $\psi(f) \neq 0$. This means that $f(x) \neq 0$, and hence $f \notin I(V(\mathfrak{a}))$. Consequently, $I(V(\mathfrak{a})) \subseteq r(\mathfrak{a})$, and therefore $I(V(\mathfrak{a})) = r(\mathfrak{a})$.

7.15. **Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring and M a finitely generated A-module. Show that the following four conditions on M are equivalent**

a. **M is free.**

b. **M is flat.**

c. **The map $\mathfrak{m} \otimes_A M \to A \otimes_A M$ is injective.**

d. **$\mathbf{Tor}_1^A(k, M) = 0$.**

(a $\Rightarrow$ b) O.K.

(b $\Rightarrow$ c) O.K.

(c $\Rightarrow$ d) From the short exact sequence

$$0 \longrightarrow \mathfrak{m} \xrightarrow{i} A \longrightarrow k \longrightarrow 0$$

we get the long exact sequence

$$\mathrm{Tor}_1^A(A, M) \longrightarrow \mathrm{Tor}_1^A(k, M) \longrightarrow \mathfrak{m} \otimes_A M \xrightarrow{i \otimes \mathrm{Id}} A \otimes_A M$$

But $\mathrm{Tor}_1^A(A, M) = 0$ and so $\mathrm{Tor}_1^A(k, M)$ is isomorphic with $\mathrm{Ker}(i \otimes \mathrm{Id}) = 0$. Hence, d holds.

(d $\Rightarrow$ a) Since M is finitely generated, $M/\mathfrak{m}M$ is finitely generated as an A-module, and thus finite dimensional as a k-vector space. Let $\{x_1, \ldots, x_n\}$ be a basis of $M/\mathfrak{m}M$. Then M is generated by $\{x_1, \ldots, x_n\}$ and $k \otimes_A M \cong M/\mathfrak{m}M$ is an n-dimensional vector space over k. Now let F be the free A-module of rank n with basis $\{e_1, \ldots, e_n\}$ and define a map $\phi : F \to M$ by $\phi(e_i) = x_i$. If E is the kernel of this map, then we have a short exact sequence

$$0 \longrightarrow E \longrightarrow F \xrightarrow{\phi} M \longrightarrow 0$$

Since $\mathrm{Tor}_1^A(k, M) = 0$, we have the short exact sequence

$$0 \longrightarrow k \otimes_A E \longrightarrow k \otimes_A F \xrightarrow{\mathrm{id} \otimes \phi} k \otimes_A M \longrightarrow 0$$

But $k \otimes_A F \cong \bigoplus_1^n k$ is an n-dimensional k-vector space. Since $\mathrm{id} \otimes \phi$ is surjective, we see that $\mathrm{id} \otimes \phi$ is an isomorphism. Therefore, $k \otimes_A E = 0$. Since A is a Noetherian ring, E is a finitely generated A-module. Exercise 2.3 now tells us that $E = 0$. This means that $F \cong M$ and so M is a free A-module.

7.16. **Let A be a Noetherian ring and M a finitely generated A-module. Show that the following are equivalent**

a. **M is flat.**

b. **$M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module whenever $\mathfrak{p}$ is a prime ideal.**

c. **$M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module whenever $\mathfrak{m}$ is a maximal ideal.**

Notice that $S^{-1}M$ is a finitely generated $S^{-1}A$-module for every multiplicatively closed subset S of A, since M is a finitely generated A-module. Also, $A_{\mathfrak{p}}$ is a local Noethering ring for every prime ideal $\mathfrak{p}$ in A. Finally, Proposition 3.10 tells us that flatness is a local condition.

(a ⇒ b) Each $M_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$-module and so is a free $A_{\mathfrak{p}}$-module by exercise 7.15.

(b ⇒ c) O.K.

(c ⇒ a) Each $M_{\mathfrak{m}}$ is a flat $A_{\mathfrak{m}}$-module by exercise 5.15, and so M is a flat A-module.

7.17. **Let A be a ring and M a Noetherian A-module. Show that every submodule $N \neq M$ of M has a primary decomposition.**

A submodule P of M is said to be irreducible if it cannot be expressed as the intersection of two submodules of M properly containing P. Since M is Noetherian, every submodule of M is the intersection of finitely many irreducible submodules (the proof of 7.11 easily carries over to modules). So it suffices to show that every proper irreducible submodule of M is primary.

Let $Q \neq M$ be an irreducible submodule. Then 0 is an irreducible submodule of M/Q. If 0 is primary in M/Q, then Q is primary in M. So we may take $Q = 0$. Suppose $ax = 0$ with $0 \neq x \in M$. Let M_n consist of all $y \in M$ so that $a^n y = 0$. Then $M_1 \subseteq M_2 \subseteq \dots$ is a chain of submodules in M. Since M is Noetherian, we can choose N such that $M_n = M_N$ for $n \geq N$. Now suppose that $y \in a^N M \cap Ax$. Then $ay = 0$ since $y \in Ax$, and $y = a^N x'$ for some $x' \in M$, so that $0 = ay = a^{N+1}x'$. Since $x' \in M_{N+1} = M_N$, we must have $0 = a^N x' = y$. In other words, $a^N M \cap Ax = 0$. Since $Ax \neq 0$ and 0 is an irreducible submodule of M, we conclude that $a^N M = 0$, so that a is nilpotent. This shows that 0 is primary in M.

7.18. **Let A be a Noetherian ring, $\mathfrak{p}$ a prime ideal of A, and M a finitely generated A-module. Show that the following are equivalent**

a. **The ideal $\mathfrak{p}$ belongs to 0 in M.**

b. **There exists $x \in M$ so that $\operatorname{Ann}(x) = \mathfrak{p}$.**

c. **There exists a submodule N of M isomorphic with $A/\mathfrak{p}$.**

($a \Rightarrow b$) Let $\bigcap_{i=1}^n Q_i = 0$ be a minimal primary decomposition of 0. We may assume that Q_1 is $\mathfrak{p}$-primary, and we can choose a nonzero $x \in \bigcap_{i=2}^n Q_i$. Then clearly $\operatorname{Ann}(x) = (Q_1 : x)$. But $(Q_1 : M)$ is a $\mathfrak{p}$-primary ideal in A, and so $\mathfrak{p}^n M \subseteq Q_1$ for some $n > 0$. This implies that $\mathfrak{p}^n x = 0$. Take $n \geq 0$ to be such that $\mathfrak{p}^{n+1} x = 0$ and $\mathfrak{p}^n x \neq 0$, and choose $y \in \mathfrak{p}^n x$. Then $\mathfrak{p} \subseteq \operatorname{Ann}(y)$ and $y \notin Q_1$ since $y \in \bigcap_{i=2}^n Q_i$. Now if $a \in \operatorname{Ann}(y)$ then a annihilates $0 \neq y + Q_1 \in M/Q_1$ so that $a \in \mathfrak{p}$. This means that $\mathfrak{p} = \operatorname{Ann}(y)$.

($b \Rightarrow a$)

($b \Rightarrow c$) The submodule Ax of M is isomorphic with $A/\operatorname{Ann}(x) \cong A/\mathfrak{p}$.

($c \Rightarrow b$) Let $x \in N$ correspond with $1_{A/\mathfrak{p}} = 1 + \mathfrak{p} \in A/\mathfrak{p}$. Then $\operatorname{Ann}(x) = \operatorname{Ann}(1_{A/\mathfrak{p}}) = \mathfrak{p}$.

Deduce that there exists a chain of submodules $0 = M_0 \subset M_1 \subset \dots \subset M_n = M$ of M with each M_{i+1}/M_i isomorphic with $A/\mathfrak{p}_i$, for some prime ideal $\mathfrak{p}_i$ in A.

7.19? **Let $\mathfrak{a}$ be an ideal in the Noetherian ring A. Let**

$$\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{b}_i = \bigcap_{i=1}^{s} \mathfrak{c}_i$$

be two minimal decompositions of $\mathfrak{a}$ as intersections of irreducible ideals. Prove that $r = s$ and that $r(\mathfrak{b}_i) = r(\mathfrak{c}_i)$ after reindexing. State and prove analogous results for modules.

7.20. **Let X be a topological space and let $\mathcal{F}$ be the smallest collection of subsets of X which contains all open subsets of X and is closed with respect to the formation of finite intersections and complements. Show the following.**

a. **A subset E of X belongs to $\mathcal{F}$ iff E is a finite union of sets of the form $U \cap C$, where U is open and C is closed.**

Let $\mathbb{F}$ consist of all sets expressible as the finite union of sets of the form $U \cap C$, where U is open and C is closed. By DeMorgan's Law $\mathcal{F}$ is closed under finite unions. As the complement of an open set is closed, and as $\mathcal{F}$ contains all open sets, we see that $\mathcal{F}$ contains all closed sets. So $\mathcal{F}$ contains all sets that are finite unions of sets of the form $U \cap C$, where U is open and C is closed. Hence, $\mathbb{F} \subseteq \mathcal{F}$. Now $\mathbb{F}$ contains all open sets since $U \cap X = U$ and X. $\mathbb{F}$ is closed under complements since

$$\left[\bigcup_{k=1}^{n} (U_k \cap C_k)\right]^c = \bigcap_{k=1}^{n} (U_k^c \cup C_k^c) = \bigcup_{s+t=n} \left[\bigcap_{i_1,\ldots,i_s} C_{i_k}^c \cap \bigcap_{j_1,\ldots,j_t} U_{j_k}^c\right]$$

It is obvious that $\mathbb{F}$ is closed under finite unions, and so $\mathbb{F}$ is also closed under finite intersections. Therefore $\mathbb{F} = \mathcal{F}$.

b. **If X is irreducible and $E \in \mathcal{F}$, then E is dense in X if and only if E contains a non-empty open subset of X.**

If E contains a non-empty open subset of X, then E is dense in X since X is irreducible. So suppose that $E = \bigcup_1^n (U_i \cap C_i)$ satisfies $\mathrm{Cl}(E) = X$. Then $\mathrm{Cl}(E) = \bigcup_1^n \mathrm{Cl}(U_i \cap C_i) = X$ so that $\mathrm{Cl}(U_i \cap C_i) = X$ for some i, since X is irreducible. But then $X = \mathrm{Cl}(U_i \cap C_i) \subseteq \mathrm{Cl}(U_i) \cap \mathrm{Cl}(C_i) = C_i$ so that $U_i \cap C_i = U_i$ is open in X. Thus, E contains a non-empty open subset of X.

7.21. **Let X be a Noetherian space and $E \subseteq X$. Show that $E \in \mathcal{F}$ iff, for each irreducible closed $X_0 \subseteq X$, either $\mathrm{Cl}(E \cap X_0) \neq X_0$ or $E \cap X_0$ contains a non-empty open subset of X_0.**

Suppose that $E \in \mathcal{F}$ and let X_0 be a closed irreducible subspace of X such that $\mathrm{Cl}(E \cap X_0) = X_0$. Notice that $E \cap X_0$ is a union of locally closed subspaces of X_0. So by exercise 7.21, we conclude that $E \cap X_0$ contains a non-empty open subset of X_0.

Now suppose that $E \notin \mathcal{F}$. Define Σ as the set of all closed subsets X' of X such that $E \cap X' \notin \mathcal{F}$. Then Σ is non-empty since $X \in \Sigma$. Since X is a Noetherian space, there is a minimal element X_0 of Σ. Suppose, for the sake of contradiction, that X_0 is reducible, with $X_0 = C_1 \cup C_2$ and each C_i a proper closed subset of X_0. Then $E \cap C_i \in \mathcal{F}$ so that $E \cap X_0 = (E \cap C_1) \cup (E \cap C_2)$ is an element of $\mathcal{F}$; a contradiction. This means that X_0 is a closed irreducible subspace of X. Now suppose that $\mathrm{Cl}(E \cap X_0) = X_0$.

7.22. **Let X be a Noetherian space and E a subset of X. Show that E is open in X iff, for each irreducible closed X_0 in X, either $E \cap X_0 = \emptyset$ or $E \cap X_0$ contains a non-empty open subset of X_0.**

Suppose E is open in X and let X_0 be an irreducible closed subset of X. Either $E \cap X_0 = \emptyset$ or $E \cap X_0$ is a non-empty open subset of X_0. Now suppose that E is not an open subspace of X. Then the collection Σ of all closed $X' \subseteq X$ such that $E \cap X'$ is not open in X' is non-empty, since $X \in \Sigma$. Since X is a Noetherian space, we can choose a minimal $X_0 \in \Sigma$. Suppose $X_0 = C_1 \cup C_2$ where each C_i is a proper closed subset of X_0. Then $E \cap X_0 = (E \cap C_1) \cup (E \cap C_2)$ is open in X_0 by minimality; a contradiction.

7.23? **Let A be a Noetherian ring and $f : A \to B$ a homomorphism of finite type. Show that $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ maps constructible sets into constructible sets.**

We can write $E = \bigcup_1^n (U_i \cap C_i)$ so that $f^*(E) = \bigcup_1^n f^*(U_i \cap C_i)$. If each $f^*(U_i \cap C_i)$ is a constructible subset of $\mathrm{Spec}(A)$, then $f^*(E)$ is a constructible subset of $\mathrm{Spec}(A)$. So assume that $E = U \cap C$.

7.24? **Let A be a Noetherian ring and $f : A \to B$ be a homomorphism of finite type. Show that f^* is an open mapping if and only if f^* has the going-down property.**

7.25? **Let A be a Noetherian ring and $f : A \to B$ a flat homomorphism of finite type. Show that $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an open mapping.**

7.26. **Suppose A is Noetherian and let $F(A)$ denote the set of all isomorphism classes of finitely generated A-modules. Let C be the free abelian group generated by $F(A)$. With each short exact sequence of finitely generated A-modules**

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

we associate the element $[M'] - [M] + [M'']$ of C. Let D be the subgroup of C generated by these elements. The quotient group C/D is called the Grothendieck group of A, and is denoted by $K(A)$. If M is a finitely generated A-module, let $\gamma_A(M)$ or $\gamma(M)$ denote the image of $[M]$ in $K(A)$. Prove the following concerning $K(A)$.

a. **For each additive function λ defined on $F(A)$ with values in the abelian group G, there is a unique homomorphism $\lambda_0 : K(A) \to G$ satisfying $\lambda_0 \circ \gamma = \lambda$.**

We can obviously extend $\lambda : F(A) \to G$ to a map $\lambda : C \to G$ of abelian groups in the obvious way. Since λ is additive, we know that $D \subseteq \mathrm{Ker}(\lambda)$. So λ induces a map $\lambda_0 : C/D \to G$ satisfying $\lambda_0 \circ \gamma = \lambda$. Clearly this λ_0 is unique since $K(A)$ is generated by $\gamma(F(A))$ as an abelian group.

b. **The elements $\gamma(A/\mathfrak{p})$ with $\mathfrak{p}$ a prime ideal generate $K(A)$.**

Let M be a finitely generated A-module and choose a chain of submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

so that M_{i+1}/M_i is isomorphic with $A/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. Then we have the short exact sequence

$$0 \longrightarrow M_{r-1} \longrightarrow M \longrightarrow M/M_{r-1} \longrightarrow 0$$

of finitely generated A-modules, so that $[M] = [M_{r-1}] + [A/\mathfrak{p}_r]$. By induction $[M] = \sum_{i=1}^r [A/\mathfrak{p}_i]$. Applying γ yields $\gamma(M) = \sum_{i=1}^r \gamma(A/\mathfrak{p}_i)$. So we are done.

c. **If $A \neq 0$ is a principal ideal domain, then $K(A) \cong \mathbb{Z}$.**

Let $\mathfrak{p} = (a)$ be a non-zero prime ideal in A. Define $f : A \to \mathfrak{p}$ by $f(b) = ab$. Then f is a surjective homomorphism of A-modules. If $f(b) = 0$ then $a = 0$ or $b = 0$, so that $b = 0$ since $\mathfrak{p} \neq 0$. This means that f is an isomorphism of A-modules. From the short exact sequence

$$0 \longrightarrow \mathfrak{p} \longrightarrow A \longrightarrow A/\mathfrak{p} \longrightarrow 0$$

we see that $[A/\mathfrak{p}] = [A] - [\mathfrak{p}] = 0$. The only other prime ideal of A is 0, with $[A/0] = [A]$. So C is the abelian group generated by $[A]$, and hence $C \cong \mathbb{Z}$. Since $[A]$ has infinite order, we get $K(A) \cong \mathbb{Z}$.

d. **Let $f : A \to B$ be a finite ring homomorphism. The restriction of scalars yields a homomorphism $f_! : K(B) \to K(A)$ such that $f_!(\gamma_B(N)) = \gamma_A(N)$ for every finitely generated B-module N. If $g : B \to C$ is another finite ring homomorphism, then $(g \circ f)_! = f_! \circ g_!$.**

Let N be a finitely generated B-module so that N is a finitely generated A-module. If N and N' are isomorphic B-modules, then $f_!(N)$ and $f_!(N')$ are isomorphic as well. Also, a short exact sequence of B-modules turns into a short exact sequence of A-modules under restriction. Therefore, there is a map $f_! : K(B) \to K(A)$ satisfying $f_!(\gamma_B(N)) = \gamma_A(N)$. Suppose $g : B \to C$ is another finite ring homomorphism and let P be a finitely generated C-module. The pullback of P along $g \circ f$ is the same as the pullback of N along f, where N is the pullback of P along g. From this it follows that $(g \circ f)_! = f_! \circ g_!$.

7.27? **Let A be a Noetherian ring and let $F_1(A)$ denote the set of all isomorphism classes of finitely generated flat A-modules. Repeating the construction of exercise 7.26, we obtain a group $K_1(A)$. Let $\gamma_1(M)$ denote the image (M) in $K_1(A)$, when M is a finitely generated flat A-module. Prove the following concerning $K_1(A)$.**

a. **The tensor product induces a commutative ring structure on $K_1(A)$ such that $\gamma_1(M) \cdot \gamma_1(N) = \gamma_1(M \otimes_A N)$. The identity element is $\gamma_1(A)$.**

The tensor product of two finitely generated flat A-modules is clearly a finitely generated flat A-module. The tensor product is commutative, associative, respects direct sums, and has identity A. We get a multiplicative structure on $F_1(A)$ since $M \cong M'$ and $N \cong N'$ implies that $M \otimes_A N \cong M' \otimes_A N'$. By linearity we get a multiplicative structure on $C_1(A)$, where $C_1(A)$ is the free abelian group generated by $F_1(A)$. Let $D_1(A)$ be the subgroup of $C_1(A)$ generated by all elements of the form $(M) - (M') - (M'')$ where M', M, and M'' fit into the obvious short exact sequence. To get a multiplicative structure on $K_1(A)$, we need to verify that $x \cdot y = x' \cdot y'$ whenever $x - x', y - y' \in D_1(A)$. By linearity, we simply need to check that $(N) \cdot ((M) - (M') - (M'')) \in D_1(A)$ whenever $(N) \in C_1(A)$ and $(M) - (M') - (M'') \in D_1(A)$. But this is immediate since N is a flat A-module. So $K_1(A)$ is a commutative ring, with identity $\gamma_1(A)$, and γ_1 satisfies the desired relation.

b. **Show that the tensor product induces a $K_1(A)$-module structure on $K(A)$ such that $\gamma_1(M) \cdot \gamma(N) = \gamma(M \otimes N)$.**

We see that $C(A)$ has a $K_1(A)$-module structure induced from the tensor product. Also, $K_1(A)$ annihilates $D(A)$ since all modules in $F_1(A)$ are flat over A. So $K_1(A)$ induces the desired module structure on $K(A)$.

c. **If $(A, \mathfrak{m})$ is a Noetherian local ring, then $K_1(A) \cong \mathbb{Z}$.**

d. **Let $f : A \to B$ be a ring homomorphism with B Noetherian. Prove that extension of scalars gives rise to a ring homomorphism $f^! : K_1(A) \to K_1(B)$ such that $f^!(\gamma_1(M)) = \gamma_1(M \otimes_A B)$. If $g : B \to C$ with C Noetherian, then $(g \circ f)^! = g^! \circ f^!$.**

If M is a finitely generated flat A-module, then $M_B = M \otimes_A B$ is a finitely generated flat B-module. Also, if $M \cong N$ then $M_B \cong N_B$. So there is a map $F_1(A) \to F_1(B)$ that extends to a group homomorphism $C_1(A) \to C_1(B)$. In fact, this is a ring homomorphism since $M_B \cdot N_B = (M \otimes_A B) \otimes_B (N \otimes_A B) \cong (M \otimes_A N) \otimes_B B = (M \cdot N)_B$.

e. **If $f : A \to B$ is a finite ring homomorphism then $f_!(f^!(x)y) = x f_!(y)$ for $x \in K_1(A)$ and $y \in K(B)$.**

Chapter 8 : Artin Rings

8.1. **Assume A is Noetherian and that 0 has the minimal primary decomposition $0 = \bigcap_{i=1}^n \mathfrak{q}_i$, with $\mathfrak{p}_i = r(\mathfrak{q}_i)$. Show that for every i there is $r_i > 0$ with $\mathfrak{p}_i^{(r_i)} \subseteq \mathfrak{q}_i$. Suppose $\mathfrak{q}_i$ is an isolated primary component. Show that $A_{\mathfrak{p}_i}$ is a local Artin ring, and that if $\mathfrak{m}_i$ is the maximal ideal of $A_{\mathfrak{p}_i}$, then $\mathfrak{m}_i^r = 0$ for some r. Also prove that $\mathfrak{q}_i = \mathfrak{p}_i^{(r)}$ for all large r.**

Let $\mathfrak{q}$ be any $\mathfrak{p}$-primary ideal. Since A is Noetherian, there is $r > 0$ with $\mathfrak{p}^r \subseteq \mathfrak{q}$. Then $(\mathfrak{p}^r)_\mathfrak{p} \subseteq \mathfrak{q}_\mathfrak{p}$ so that $\mathfrak{p}^{(r)} = (\mathfrak{p}^r)^c_\mathfrak{p} \subseteq \mathfrak{q}^c_\mathfrak{p} = \mathfrak{q}$ (after all, $\mathfrak{p} \cap S_\mathfrak{p} = \emptyset$). This holds in particular with $\mathfrak{q} = \mathfrak{q}_i$ for some i. Now suppose that $\mathfrak{q}_i$ is one of the isolated primary components of 0. Clearly $A_{\mathfrak{p}_i}$ is a Noetherian ring. Any prime ideal in $A_{\mathfrak{p}_i}$ is of the form $\mathfrak{p}_{\mathfrak{p}_i}$ where $\mathfrak{p}$ is a prime ideal in A contained in $\mathfrak{p}_i$. But $\mathfrak{p}_i$ is a minimal element in the set of all prime ideals in A. This means that $A_{\mathfrak{p}_i}$ has precisely one prime ideal, namely $\mathfrak{m}_i = (\mathfrak{p}_i)_{\mathfrak{p}_i}$. Therefore, $A_{\mathfrak{p}_i}$ is a local Artin ring. Since $\mathfrak{N}(A_{\mathfrak{p}_i}) = \mathfrak{m}_i$ we see that $\mathfrak{m}_i^r = 0$ for all sufficiently large r. Finally, $\mathfrak{p}_i^{(r)} \subseteq \mathfrak{q}_i$ for all large r, so that $0 = \mathfrak{p}_i^{(r)} \cap \bigcap_{j \neq i} \mathfrak{q}_j$. Since isolated components are uniquely determined, we see that $\mathfrak{p}_i^{(r)} = \mathfrak{q}_i$ for all large r.

8.2. **Let A be Noetherian. Prove that the following are equivalent.**

a. **A is Artinian.**
b. **$\mathrm{Spec}(A)$ is discrete and finite.**
c. **$\mathrm{Spec}(A)$ is discrete.**

(a ⇒ b) Notice that $\mathrm{Spec}(A)$ is Hausdorff since each prime ideal in A is maximal. Also, $\mathrm{Spec}(A)$ is finite since there are finitely many maximal ideals in A. Hence, $\mathrm{Spec}(A)$ has the discrete topology.

(b ⇒ c) O.K.

(c ⇒ a) Each prime ideal in A is maximal since $\mathrm{Spec}(A)$ is discrete. Therefore, A has Krull dimension 0. Hence, A is Artinian.

8.3. **Let k be a field and A a finitely generated k-algebra. Prove that the following two conditions are equivalent.**

a. **A is Artinian.**
b. **A is a finite k-algebra.**

(a ⇒ b) Write $A = \prod_{j=1}^n A_j$, where each A_j is an Artin local ring, and let $\pi_j : A \to A_j$ be the canonical projection. Notice that there is a unique way to make each A_j into a k-algebra in such that a way that π_j is a homomorphism of k-algebras. Also observe that if A is finitely generated as a k-algebra by $\{x_i\}_{i=1}^m$ then A_j is finitely generated as a k-algebra by $\{\pi_j(x_i)\}_{i=1}^m$. So if we prove that the result holds for the local Artin rings A_j, then the result holds for A since $\dim_k(A) = \sum_{j=1}^n \dim_k(A_j)$.

So assume that $(A, \mathfrak{m})$ is an Artin local ring. Then $A/\mathfrak{m}$ is a finite algebraic extension of k since $A/\mathfrak{m}$ is a finitely generated field extension of k. Since A is Noetherian, we see that $\mathfrak{m}$ is a finitely generated A-module, and since $\mathfrak{m}$ is the only prime ideal in A, we know by exercise 7.18 that there is a chain of ideals

$$0 = \mathfrak{m}_0 \subset \mathfrak{m}_1 \subset \ldots \subset \mathfrak{m}_r = \mathfrak{m}$$

in A with each $\mathfrak{m}_{i+1}/\mathfrak{m}_i \cong A/\mathfrak{m}$. Since each $\mathfrak{m}_{i+1}/\mathfrak{m}_i$ is a finite-dimensional k-vector space, the same is true for $\mathfrak{m}$, and therefore the same can be said about A.

(b ⇒ a) If $\mathfrak{a}$ is an ideal in A, then $k\mathfrak{a} \subseteq \mathfrak{a}$, where we identify k with its isomorphic image in A. So $\mathfrak{a}$ is a k-vector subspace of A. Since A is finite dimensional as a k-vector space, the vector subspaces of A satisfy the d.c.c. This means that ideals in A satisfy the d.c.c. In other words, A is an Artin ring.

8.4. **Let $f : A \to B$ be a ring homomorphism of finite type. Consider the following conditions and show that $a \Rightarrow b \Leftrightarrow c \Rightarrow d$. Also, if $f : A \to B$ is integral and the fibers of f^* are finite, is f finite?**

a. **The map f is finite.**

b. **The fibers of f^* are discrete subspaces of** $\mathrm{Spec}(B)$**.**

c. **For prime $\mathfrak{p}$ in A, the ring $B \otimes_A k(\mathfrak{p})$ is a finite $k(\mathfrak{p})$-algebra.**

d. **The fibers of f^* are finite.**

By hypothesis, B is a finitely generated A-algebra, so that $B \otimes_A k(\mathfrak{p})$ is a finitely generated $k(\mathfrak{p})$-algebra.

(a $\Rightarrow$ b) If B is generated as an A-module by $\{b_i\}_1^n$, then $B \otimes_A k(\mathfrak{p})$ is generated as a $k(\mathfrak{p})$-vector space by $\{b_i \otimes 1\}_1^n$, and hence $B \otimes_A k(\mathfrak{p})$ is Artinian by exercise 8.3. So by exercise 8.2, $\mathrm{Spec}(B \otimes_A k(\mathfrak{p}))$ is discrete. This shows that every fiber of f^* is a discrete subspace of $\mathrm{Spec}(B)$.

(b $\Rightarrow$ c) We know that $B \otimes_A k(\mathfrak{p})$ is a finitely generated $k(\mathfrak{p})$-algebra, so that $B \otimes_A k(\mathfrak{p})$ is a Noetherian ring. Now by hypothesis $\mathrm{Spec}(B \otimes_A k(\mathfrak{p}))$ is discrete, and so exercise 8.2 tells us that $B \otimes_A k(\mathfrak{p})$ is an Artinian ring. But exercise 8.3 nows tells us that $B \otimes_A k(\mathfrak{p})$ is a finite $k(\mathfrak{p})$-algebra.

(c $\Rightarrow$ b) Whenever $\mathfrak{p}$ is a prime ideal in A, the ring $B \otimes_A k(\mathfrak{p})$ is Artinian by exercise 8.3. So by exercise 8.2, the fiber $\mathrm{Spec}(B \otimes_A k(\mathfrak{p}))$ of f^* over $\mathfrak{p}$ is discrete.

(c $\Rightarrow$ d) Whenever $\mathfrak{p}$ is a prime ideal in A, the ring $B \otimes_A k(\mathfrak{p})$ is Artinian, again by exercise 8.3. So again by exercise 2, the fiber $\mathrm{Spec}(B \otimes_A k(\mathfrak{p}))$ of f^* over $\mathfrak{p}$ is finite.

8.5? **In exercise 5.16 show that X is a finite covering of L.**

8.6? **Let A be a Noetherian ring and $\mathfrak{q}$ a $\mathfrak{p}$-primary ideal. Consider chains of primary ideals from $\mathfrak{q}$ to $\mathfrak{p}$. Show that all such chains are of finite bounded length, and that all maximal chains have the same length.**

If $\mathfrak{q} \subseteq \mathfrak{r} \subseteq \mathfrak{p}$ then $r(\mathfrak{r}) = \mathfrak{p}$. So we can restrict attention to chains of $\mathfrak{p}$-primary ideals from $\mathfrak{q}$ to $\mathfrak{p}$. Clearly all such chains are of finite length since A is Noetherian.

Chapter 9 : Discrete Valuation Rings and Dedekind Domains

9.1. **Let A be a Dedekind domain, S a multiplicatively closed subset of A not containing 0. Show that $S^{-1}A$ is either a Dedekind domain or the field of fractions K of A.**

If $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ is a chain of prime ideals in $S^{-1}A$, then $\mathfrak{p}_0^c \subset \mathfrak{p}_1^c \subset \ldots \subset \mathfrak{p}_n^c$ is a chain of prime ideals in A. So in general, the Krull dimension of $S^{-1}A$ is less than or equal to the Krull dimension of A. Now A has dimension 1 since A is a Dedekind domain. Hence, $S^{-1}A$ has dimension equal to 1 or 0. Since A is an integral domain and $0 \notin S$, we can consider $A \subseteq S^{-1}A \subseteq K$. If $S^{-1}A$ has dimension 0, then $S^{-1}A$ is a field, and so $S^{-1}A = K$.

Now assume that $S^{-1}A$ has dimension 1. Clearly $S^{-1}A$ is Noetherian, and K is the field of fractions of $S^{-1}A$. Since the integral closure of A in K equals A, the integral closure of $S^{-1}A$ in $S^{-1}(K) = K$ is $S^{-1}A$. This means that $S^{-1}A$ is integrally closed as well. Therefore, $S^{-1}A$ is a Dedekind domain.

Suppose again that $0 \notin S$, and let H, H' be the ideal class groups of A and $S^{-1}A$ respectively. Show that extension of ideals induces a surjective homomorphism $H \to H'$.

Suppose that $\mathfrak{a}$ is a non-zero fractional ideal of A. It is clear that $S^{-1}\mathfrak{a}$ is a non-zero ideal of $S^{-1}A$ since S has no zero-divisors. If $x \in A$ is such that $x\mathfrak{a} \subseteq A$, then $xS^{-1}\mathfrak{a} \subseteq S^{-1}A$. Hence $S^{-1}\mathfrak{a}$ is a fractional ideal of $S^{-1}A$. Therefore, if we let I be the group of non-zero fractional ideals of A, and I' the group of non-zero fractional ideals of $S^{-1}A$, then we have a map $I \to I'$ given by $\mathfrak{a} \mapsto S^{-1}\mathfrak{a}$. In other words, this map is given by extension. This map is a group homomorphism since localization commutes with taking finite products. Let P be the image of the canonical map $K^* \to I$, and P' the image of the canonical map $K^* \to I'$. If $x \in K^*$ then $S^{-1}(x) = (x)$, and hence the map $I \to I'$ carries P into P'. Consequently, the map $I \to I'$ induces a map $H \to H'$. If $\mathfrak{b}I' \in H'$ then there is $0 \neq x \in A$ satisfying $x\mathfrak{b} \subseteq S^{-1}A$. We can write $(x)\mathfrak{b} = S^{-1}\mathfrak{a}$ for some non-zero ideal $\mathfrak{a}$ in A. Since $\mathfrak{a}$ is an integral ideal, it is clearly a fractional ideal of A, and so is an element of I. This means that the map $H \to H'$ is surjective.

9.2. **Let A be a Dedekind domain. If $f = a_0 + a_1x + \cdots + a_nx^n$ then the content $c(f)$ of f is defined by $c(f) = (a_0, \ldots, a_n)$. Prove Gauss's Lemma that $c(fg) = c(f)c(g)$ for all f, g.**

Suppose that A is in fact a discrete valuation ring, with maximal ideal $\mathfrak{m}$, where $\mathfrak{m} = (y)$. Each a_i is of the form $u_iy^{v(a_i)}$ where u_i is a unit in A and v is the appropriate discrete valuation. Let $a \geq 0$ be the biggest a' so that $y^{a'}$ divides each a_i. Similarly, let $b \geq 0$ be the biggest b' so that $y^{b'}$ divides each coefficient of g. Then f/y^a and g/y^b are primitive polynomials since some coefficient of f and g is a unit. Exercise 1.2 tells us that fg/y^{a+b} is primitive as well. Now $c(fg) = (y^{a+b}) = (y^a)(y^b) = c(f)c(g)$ so that Gauss's Lemma holds for discrete valuation rings.

Now suppose that A is a general Dedekind domain. Let $\mathfrak{m}$ be a maximal ideal in A so that $A_\mathfrak{m}$ is a discrete valuation ring. The canonical map $A \to A_\mathfrak{m}$ extends naturally to a map $A[x] \to A_\mathfrak{m}[x]$. Denote this map by $f \mapsto f_\mathfrak{m}$. It is clear that $c(f_\mathfrak{m}) = c(f)_\mathfrak{m}$. Now there is an inclusion map $j : c(fg) \to c(f)c(g)$. We see that the map $j_\mathfrak{m} : c(fg)_\mathfrak{m} \to (c(f)c(g))_\mathfrak{m} = c(f)_\mathfrak{m}c(g)_\mathfrak{m} = c(f_\mathfrak{m})c(g_\mathfrak{m})$ is the natural inclusion map. By the work done above, we see that $j_\mathfrak{m}$ is the identity, and in particular is surjective. This means that j is surjective, and hence $c(fg) = c(f)c(g)$. This means that Gauss's Lemma holds for Dedekind domains.

9.3. **Suppose that $(A, \mathfrak{m}, K)$ is a valuation ring, with $A \neq K$. Show that A is Noetherian if and only if A is a discrete valuation ring.**

If A is a DVR then A is clearly Noetherian. So suppose that A is Noetherian. If $\mathfrak{a}$ is an ideal in A then we can write $\mathfrak{a} = (a_1, \ldots, a_n)$ for some a_i. Since A is a valuation ring, the ideals in A are totally ordered. So there is some i for which $(a_j) \subseteq (a_i)$ for all $1 \leq j \leq n$. This means that $\mathfrak{a} = (a_j)$, and so $\mathfrak{a}$ is a principal ideal. This means that A is a PID. Now write $\mathfrak{m} = (x)$, where $x \neq 0$ since A is not a field. Let y be an arbitrary non-zero element of $\mathfrak{m}$.

I claim that $y = ux^k$ for some unit u and some $k > 0$. If not, then for every i there is $a_i \in \mathfrak{m}$ satisfying $y = a_i x^i$. Notice that $a_i = a_{i+1}x$ since $x \neq 0$, and so $(a_i) \subseteq (a_{i+1})$. But if $(a_{i+1}) = (a_i)$ then there is b for which $a_{i+1} = ba_i$, and hence $y = a_{i+1}x^{i+1} = (xb)(a_i x^i)$ so that $xb = 1$, implying that x is a unit. Consequently, we have a properly ascending sequence of ideals $(a_1) \subset (a_2) \subset \ldots$ in the Noetherian ring A, a contradiction.

Now let $\mathfrak{a}$ be any proper ideal in A. Choose y for which $\mathfrak{a} = (y)$ and notice that $y \in \mathfrak{m}$ since y is not a unit. Write $y = ux^k$ as above, so that $\mathfrak{a} = (x^k)$. Now we argue as in $(f \Rightarrow a)$ from Proposition 9.2 to conclude that A is a discrete valuation ring (noting that this portion of Proposition 9.2 does not require the assumption that A have dimension 1).

9.4. **Let A be a local domain which is not a field. Suppose the non-zero maximal ideal $\mathfrak{m} = (x)$ of A is principal and satisfies $\bigcap_{i=1}^{\infty} \mathfrak{m}^i = 0$. Prove that A is a DVR.**

If $0 \neq y \in \mathfrak{m}$ then I claim that $y = ux^k$ for some unit u and some $k > 0$. If not, then there are $a_i \in \mathfrak{m}$ satisfying $y = a_i x^i$ for all i. But then $y \in \bigcap_{i=1}^{\infty} \mathfrak{m}^i = 0$ so that $y = 0$, contrary to our assumption on y. Now let $\mathfrak{a}$ be a proper non-zero ideal in A, so that $\mathfrak{a} \subseteq \mathfrak{m}$. For every nonzero $y \in \mathfrak{a}$ write $y = ux^k$ as above. Let k^* be the minimal k that arises in this fashion. Then clearly $\mathfrak{a} \subseteq (x^{k^*})$ since every nonzero $y \in \mathfrak{a}$ can be written as $y = ux^k$ for some unit u and some $k \geq k^*$. On the other hand, there is some unit u such that $ux^{k^*} \in \mathfrak{a}$, and hence $(x^{k^*}) = \mathfrak{a}$. Now we argue as in $(f \Rightarrow a)$ from Proposition 9.2 to conclude that A is a discrete valuation ring (noting that this portion of Proposition 9.2 only requires that $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ for all n, and that this holds true since x is a non-unit in A).

9.5. **Let M be a finitely generated module over a Dedekind domain A. Prove that M is flat if and only if M is torsion free.**

Exercise 7.16 tells us that M is a flat A-module if and only if $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module whenever $\mathfrak{m}$ is a maximal ideal in A. But $A_{\mathfrak{m}}$ is a principal ideal domain whenever $\mathfrak{m}$ is a maximal ideal in A. So the structure theorem of finitely generated modules over a PID tells us that $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module if and only if $M_{\mathfrak{m}}$ is torsion free. Exercise 3.13 now tells us that each $M_{\mathfrak{m}}$ is torsion free if and only if M is torsion free. Summarizing, M is a flat A-module if and only if M is torsion free.

9.6? **Let M be a finitely generated torsion module over the Dedekind domain A. Prove that M is uniquely representable as a finite direct sum of modules $A/\mathfrak{p}_i^{n_i}$ where $\mathfrak{p}_i$ are non-zero prime ideals in A.**

9.7? **Let A be a Dedekind domain and $\mathfrak{a} \neq 0$ an ideal in A. Show that every ideal in $A/\mathfrak{a}$ is principal. Deduce that every ideal in A can be generated by at most 2 elements.**

Since A is a Dedekind domain we can write $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_n^{e_n}$ where $\mathfrak{p}_i$ are distinct prime ideals in A and each $e_i \geq 0$. Since each $\mathfrak{p}_i$ is maximal, we know that $\mathfrak{p}_i$ and $\mathfrak{p}_j$ are coprime for $i \neq j$. Hence, $\mathfrak{p}_i^{e_i}$ and $\mathfrak{p}_j^{e_j}$ are coprime for $i \neq j$. This means that $A/\mathfrak{a} \cong \prod_{i=1}^{n} A/\mathfrak{p}_i^{e_i}$. I claim that every ideal in $A/\mathfrak{p}_i^{e_i}$ is principal. Suppose that $\mathfrak{b}$ is an ideal in $A/\mathfrak{a}$

9.8. **Let $\mathfrak{a}, \mathfrak{b}$, and $\mathfrak{c}$ be ideals in the Dedekind domain A. Prove that**

$$\mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}) = \mathfrak{a} \cap \mathfrak{b} + \mathfrak{a} \cap \mathfrak{c} \qquad \text{and} \qquad \mathfrak{a} + \mathfrak{b} \cap \mathfrak{c} = (\mathfrak{a} + \mathfrak{b}) \cap (\mathfrak{a} + \mathfrak{c})$$

Suppose first that A is in fact a discrete valuation ring. Let $\mathfrak{m}$ be the maximal ideal in A and write $\mathfrak{m} = (x)$. If any of the three ideals are zero, then we clearly have equality. So we may suppose that all three ideals are nonzero. Then we can choose $a, b, c \geq 0$ for which $\mathfrak{a} = (x^a)$, $\mathfrak{b} = (x^b)$, and $\mathfrak{c} = (x^c)$. Now $(x^j) \cap (x^k) = (x^{\max\{j,k\}})$ and $(x^j) + (x^k) = (x^{\min\{j,k\}})$ for all $j, k \geq 0$. So the equalities that we need to verify are as follows

$$\max\{a, \min\{b, c\}\} = \min\{\max\{a, b\}, \max\{a, c\}\}$$
$$\min\{a, \max\{b, c\}\} = \max\{\min\{a, b\}, \min\{a, c\}\}$$

To do this requires a straightforward case-by-case analysis, and so is omitted. Now assume that A is a general Dedekind domain. We have an inclusion map $j : \mathfrak{a} \cap \mathfrak{b} + \mathfrak{a} \cap \mathfrak{c} \to \mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c})$. In the field of fractions of A we have the equality $(\mathfrak{a} \cap \mathfrak{b} + \mathfrak{a} \cap \mathfrak{c})_\mathfrak{p} = \mathfrak{a}_\mathfrak{p} \cap \mathfrak{b}_\mathfrak{p} + \mathfrak{a}_\mathfrak{p} \cap \mathfrak{c}_\mathfrak{p}$ of sets, and similarly $(\mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}))_\mathfrak{p} = \mathfrak{a}_\mathfrak{p} \cap (\mathfrak{b}_\mathfrak{p} + \mathfrak{c}_\mathfrak{p})$. Further, the induced map $j_\mathfrak{p}$ corresponds to inclusion. Since $A_\mathfrak{p}$ is a PID, the work above shows that $j_\mathfrak{p}$ is surjective. Therefore, j is surjective, and hence $\mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}) = \mathfrak{a} \cap \mathfrak{b} + \mathfrak{a} \cap \mathfrak{c}$. The second equality follows analogously.

9.9. **Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals and let $x_1, \ldots, x_n$ be elements in the Dedekind domain A. Show that the system of congruences $x \equiv_{\mathfrak{a}_i} x_i$ has a solution x iff $x_i \equiv_{\mathfrak{a}_i + \mathfrak{a}_j} x_j$ whenever $i \neq j$.**

Consider the following sequence

$$A \xrightarrow{\quad\phi\quad} \bigoplus_{i=1}^n A/\mathfrak{a}_i \xrightarrow{\quad\psi\quad} \bigoplus_{i<j} A/(\mathfrak{a}_i + \mathfrak{a}_j)$$

where $\phi(x) = (x + \mathfrak{a}_1, \ldots, x + \mathfrak{a}_n)$ and $\psi(x_1 + \mathfrak{a}_1, \ldots, x_n + \mathfrak{a}_n)$ has (i, j) component $x_i - x_j + \mathfrak{a}_i + \mathfrak{a}_j$. Notice first that ψ is well-defined. Suppose that this sequence is exact, and let $x_1, \ldots, x_n \in A$. If the system of congruences $x \equiv_{\mathfrak{a}_i} x_i$ has a solution x then $(x_1 + \mathfrak{a}_1, \ldots, x_n + \mathfrak{a}_n) = \phi(x)$ so that $\psi(x_1 + \mathfrak{a}_1, \ldots, x_n + \mathfrak{a}_n) = 0$. This means that $x_i \equiv_{\mathfrak{a}_i + \mathfrak{a}_j} x_j$ whenever $i \neq j$. Conversely, if this holds then $\psi(x_1 + \mathfrak{a}_1, \ldots, x_n + \mathfrak{a}_n) = 0$ so that $(x_1 + \mathfrak{a}_1, \ldots, x_n + \mathfrak{a}_n) = \phi(x)$ for some $x \in A$, and hence our system of congruences has a solution. So it suffices to demonstrate that the sequence is exact. To do this it suffices to show that the sequence is exact whenever it is localized at a maximal ideal $\mathfrak{m}$ of A. Hence, we simply need to show that the sequence is exact in the special case that A is a discrete valuation ring. We may assume that the ideals are ordered by $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \ldots$. Clearly $\psi \circ \phi = 0$, so suppose that $\psi(x_1 + \mathfrak{a}_1, \ldots, x_n + \mathfrak{a}_n) = 0$. Then $x_1 - x_i \in \mathfrak{a}_1 + \mathfrak{a}_i = \mathfrak{a}_i$ for $1 < i$, and hence $x_i + \mathfrak{a}_i = x_1 + \mathfrak{a}_i$ for all i. But this means that $(x_1 + \mathfrak{a}_1, \ldots, x_n + \mathfrak{a}_n) = \phi(x_1)$. Therefore, the sequence is indeed exact when A is a discrete valuation ring. Thus, we are done.

Chapter 10 : Completions

10.1. **Let $\alpha_n : \mathbb{Z}_p \to \mathbb{Z}_{p^n}$ be the obvious injection, and let $\alpha : A \to B$ be the direct sum of all the α_n, where $A = \bigoplus_{n=1}^{\infty} \mathbb{Z}_p$ and $B = \bigoplus_{n=1}^{\infty} \mathbb{Z}_{p^n}$. Show that the p-adic completion of A is just A, but that the completion of A for the topology induced from the p-adic topology on B is $\prod_{n=1}^{\infty} \mathbb{Z}_p$. Deduce that the p-adic completion is not a right-exact functor on the category of all $\mathbb{Z}$-modules.**

Let M be an arbitrary module with the filtration $M = M_0 \supseteq M_1 \supseteq \ldots$. Suppose that N satisfies $M_n = M_N$ for $n \geq N$. Then the maps $M/M_{n+1} \to M/M_n$ are the identity maps for $n \geq N$. So an element $\xi \in \hat{M} \subseteq \prod_{n=1}^{\infty} M/M_n$ is completely determined by ξ_N. This means that the canonical map $M \to \hat{M}$ given by $x \mapsto (x + M_0, x + M_1, \ldots)$ is surjective. Clearly, the kernel of this map is M_N. Therefore, $\hat{M}$ and M/M_N are isomorphic.

Now if $A = \bigoplus_{n=1}^{\infty} \mathbb{Z}_p$ then $pA = 0$, and so the standard p-adic filtration of A is given by $A \supset 0 = 0 = \ldots$. By the general considerations from above, we see that the p-adic completion $\hat{A}$ of A is isomorphic with $A/0 = A$.

On the other hand, we have an injection $\alpha : A \to B$ and we have the p-adic filtration $B \supset pB \supset p^2B \supset \ldots$ of B. This gives a p-adic filtration $A \supset \alpha^{-1}(pB) \supset \alpha^{-1}(p^2B) \supset \ldots$ of A. Now $\alpha(x_1, x_2, x_3, \ldots) = (x_1, px_2, p^2x_3, \ldots)$ so that $(x_1, x_2, x_3, \ldots) \in \alpha^{-1}(p^nB)$ if and only if $x_i = 0$ for $1 \leq i \leq n$. We see that $A/\alpha^{-1}(p^nB) \cong \bigoplus_{i=1}^{n} \mathbb{Z}_p$ and that under these identifications the map $A/\alpha^{-1}(p^{n+1}B) \to A/\alpha^{-1}(p^nB)$ is given by $(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n)$. Now the general element of $\prod_{n=1}^{\infty} A/\alpha^{-1}(p^nB)$ under these identifications is of the form

$$((x_{11}), (x_{12}, x_{22}), (x_{13}, x_{23}, x_{33}), (x_{14}, x_{24}, x_{34}, x_{44}), \ldots)$$

where x_{ij} are arbitrary elements of $\mathbb{Z}_p$. For this element to be in $\hat{A}$, it is necessary and sufficient that $x_{ij} = x_{ik}$ for any $k \geq j$. So $\hat{A}$ can be identified with $\prod_{n=1}^{\infty} \mathbb{Z}_p$. Now p-adic completion is an exact functor on the category of all finitely generated $\mathbb{Z}$-modules, but A is not finitely generated. Now we have the short exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow B/A \longrightarrow 0$$

10.2. **In the notation of exercise 10.1 let $A_n = \alpha^{-1}(p^nB)$. Consider the short exact sequences**

$$0 \longrightarrow A_n \longrightarrow A \longrightarrow A/A_n \longrightarrow 0$$

to show that $\varprojlim$ is not right exact, and compute $\varprojlim^1 A_n$.

We see that $\{A_n\}_1^{\infty}$ is an inverse system with inclusion as the map $A_m \to A_n$ for $m \geq n$. Clearly $\{A\}_1^{\infty}$ is an inverse system with identity $A \to A$. Finally, $\{A/A_n\}_1^{\infty}$ is an inverse system with the induced maps $A/A_m \to A/A_n$ for $m \geq n$. Now we have the commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_{n+1} & \longrightarrow & A & \longrightarrow & A/A_{n+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_n & \longrightarrow & A & \longrightarrow & A/A_n & \longrightarrow & 0 \end{array}$$

with exact rows. So Proposition 10.2 gives us the exact sequence

$$0 \longrightarrow \varprojlim A_n \longrightarrow \varprojlim A \xrightarrow{f} \varprojlim A/A_n$$

I claim that f is not surjective. Using the identification from exercise 10.1 and the isomorphism $\varprojlim A/A_n \cong \prod_{n=1}^{\infty} \mathbb{Z}_p$ we see that f can be identified with the inclusion map $\bigoplus_{n=1}^{\infty} \mathbb{Z}_p \to \prod_{n=1}^{\infty} \mathbb{Z}_p$. So f is not surjective.

10.3. **Let A be a Noetherian ring, $\mathfrak{a}$ an ideal, and M a finitely generated A-module. Prove that**

$$\bigcap_{n=1}^{\infty} \mathfrak{a}^n M = \bigcap_{\mathfrak{m}\supseteq\mathfrak{a}} \mathrm{Ker}(M \to M_{\mathfrak{m}})$$

By Krull's Theorem, the elements of $\bigcap_{n=1}^{\infty} \mathfrak{a}^n M$ are precisely the elements in M annihilated by some element of $1+\mathfrak{a}$. So suppose first that $x \in M$ satisfies $(1+a)x = 0$ for some $a \in \mathfrak{a}$. If $\mathfrak{m}$ is a maximal ideal in A containing $\mathfrak{a}$, then $a \in \mathfrak{m}$ so that $1+a \notin \mathfrak{m}$. Since $(1+a)x = 0$ and $1+a \in A-\mathfrak{m}$, we see that $x/1 = 0/1$ in $M_{\mathfrak{m}}$. This means that $\bigcap_{n=1}^{\infty} \mathfrak{a}^n M \subseteq \bigcap_{\mathfrak{m}\supseteq\mathfrak{a}} \mathrm{Ker}(M \to M_{\mathfrak{m}})$. Now let $x \in \bigcap_{\mathfrak{m}\supseteq\mathfrak{a}} \mathrm{Ker}(M \to M_{\mathfrak{m}})$ so that $(x)_{\mathfrak{m}} = 0$ whenever $\mathfrak{m}$ is a maximal ideal containing $\mathfrak{a}$. Then exercise 3.14 tells us that $(x) = \mathfrak{a}(x)$. So in particular we can write $x = -ax$ for some $a \in \mathfrak{a}$. This means that $(1+a)x = 0$, and hence $x \in \bigcap_{n=1}^{\infty} \mathfrak{a}^n M$. So we are done.

Deduce that $\hat{M} = 0$ if and only if $\mathrm{Supp}(M) \cap V(\mathfrak{a}) = \emptyset$.

10.4. **Let A be a Noetherian ring, $\mathfrak{a}$ an ideal, and $\hat{A}$ the $\mathfrak{a}$-adic completion. For any $x \in A$ let $\hat{x}$ be the image of x in $\hat{A}$. Show that $\hat{x}$ is not a zero-divisor in $\hat{A}$ if x is not a zero-divisor in A. Does this imply that $\hat{A}$ is an integral domain provided A is an integral domain?**

If x is not a zero-divisor in A then we have a short exact sequence

$$0 \longrightarrow A \xrightarrow{\ x\ } A \longrightarrow A/xA \longrightarrow 0$$

Proposition 10.12 tells us that we have a new short exact sequence

$$0 \longrightarrow \hat{A} \xrightarrow{\ \hat{x}\ } \hat{A} \longrightarrow \hat{A}/\hat{x}\hat{A} \longrightarrow 0$$

This means that $\hat{x}$ is not a zero-divisor in $\hat{A}$. Now $\mathbb{Z}_{(6)}$ is not an integral domain even though $\mathbb{Z}$ is an integral domain.

10.5. **Let A be Noetherian with ideals $\mathfrak{a}$ and $\mathfrak{b}$. If M is an A-module, let $M^{\mathfrak{a}}, M^{\mathfrak{b}}$ denote the $\mathfrak{a}$-adic and $\mathfrak{b}$-adic completions of M. If M is finitely generated, prove that $(M^{\mathfrak{a}})^{\mathfrak{b}} \cong M^{\mathfrak{a}+\mathfrak{b}}$.**

For every n we have a short exact sequence

$$0 \longrightarrow \mathfrak{b}^n M \longrightarrow M \longrightarrow M/\mathfrak{b}^n M \longrightarrow 0$$

Since M is finitely generated and A is Noetherian, all modules in this sequence are finitely generated. So we have a new short exact sequence

$$0 \longrightarrow (\mathfrak{b}^n M)^{\mathfrak{a}} \longrightarrow M^{\mathfrak{a}} \longrightarrow (M/\mathfrak{b}^n M)^{\mathfrak{a}} \longrightarrow 0$$

10.6. **Let A be a Noetherian ring and $\mathfrak{a}$ an ideal in A. Prove that $\mathfrak{a} \subseteq \mathfrak{R}(A)$ if and only if every maximal ideal $\mathfrak{m}$ in A is closed when A is given the $\mathfrak{a}$-adic topology.**

Suppose that $\mathfrak{a} \subseteq \mathfrak{R}(A)$ and let $\mathfrak{m}$ be a maximal ideal in A. Then the quotient topology of $A/\mathfrak{m}$ is the same as the $\mathfrak{a}$-adic topology of $A/\mathfrak{m}$. Since $A/\mathfrak{m}$ is a finite A-module, Corollary 10.19 tells us that the $\mathfrak{a}$-adic topology of $A/\mathfrak{m}$ is Hausdorff. By the definition of the quotient topology, this means that $\mathfrak{m}$ is closed in the $\mathfrak{a}$-adic topology on A.

Suppose now that $\mathfrak{m}$ is closed in the $\mathfrak{a}$-adic topology on A whenever $\mathfrak{m}$ is a maximal ideal in A. Then $\mathfrak{m} = \mathrm{Cl}(\mathfrak{m}) = \bigcap_{n=1}^{\infty}(\mathfrak{m} + \mathfrak{a}^n)$.

10.7?

10.8?

10.9?

10.10? a.

b.

c.

10.11. **Find a non-Noetherian local ring A with an ideal $\mathfrak{a}$ such that the $\mathfrak{a}$-adic completion $\hat{A}$ of A is a Noetherian ring that is finitely generated over A.**

Let A be the ring of germs of C^∞ functions of x at $x = 0$, and let $\mathfrak{a}$ be the ideal of all germs that vanish at $x = 0$. Then A is a local ring with maximal ideal $\mathfrak{a}$. Now A is not Noetherian since we have the properly ascending sequence of ideals

$$(e^{-1/x^2}) \subset (e^{-1/x^2}/x) \subset (e^{-1/x^2}/x^2) \subset \dots$$

10.12? **Assuming that A is Noetherian, show that $A[[x_1, \dots, x_n]]$ is a faithfully flat A-algebra.**

1. **Let $f \in k[x_1, \dots, x_n]$ be an irreducible polynomial over the algebraically closed field k. A point P on the variety defined by (f) is said to be non-singular if not all derivatives $\partial f/\partial x_i$ vanish at P. Let $A = k[x_1, \dots, x_n]/(f)$ and let $\mathfrak{m}$ be the maximal ideal of A corresponding to the point P. Prove that P is non-singular if and only if $A_\mathfrak{m}$ is a regular ring.**

   Write $P = (a_1, \dots, a_n)$ and define $\mathfrak{n} = (x_1 - a_1, \dots, x_n - a_n)$ so that $\mathfrak{m} = \mathfrak{n}/(f)$. Then $A_\mathfrak{m} \cong k[x_1, \dots, x_n]_\mathfrak{n}/(f)_\mathfrak{n} = k[x_1, \dots, x_n]_\mathfrak{n}/(f/1)$ as rings. Now f vanishes at P so that $f \in \mathfrak{n}$, and hence $f/1$ is in the (unique) maximal ideal $\mathfrak{n}_\mathfrak{n}$ of $k[x_1, \dots, x_n]_\mathfrak{n}$. Also, $f/1$ is not a zero-divisor in $k[x_1, \dots, x_n]_\mathfrak{n}$ since

2.

3.

4. **Give an example of a Noetherian ring A that has infinite Krull dimension.**

5. **Reformulate the Hilbert-Serre Theorem in terms of the Grothendieck group $K(A_0)$.**

   Let γ be the map that sends a finitely generated A_0-module M to its image in $K(A_0)$. The Hilbert-Serre Theorem states that if $\lambda : K(A_0) \to \mathbb{Z}$ is a homomorphism of groups then $P(M, t) := \sum_{n=0}^\infty \lambda(M_n) t^n$ is of the form $P(M, t) = f(t)\{\prod_{i=1}^s (1 - t^{k_i})\}^{-1}$ for some $f(t) \in \mathbb{Z}[t]$.

6. **Let A be a ring and prove that $1 + \dim(A) \leq \dim A[x] \leq 1 + 2\dim(A)$.**

   Let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ be a chain of prime ideals in A. Then $\mathfrak{p}_i[x]$ is a prime ideal in $A[x]$ since $A[x]/\mathfrak{p}_i[x] \cong (A/\mathfrak{p}_i)[x]$ is an integral domain. So we have a chain of prime ideals $\mathfrak{p}_0[x] \subseteq \cdots \subseteq \mathfrak{p}_n[x]$ in A. But $\mathfrak{p}_i[x] \neq \mathfrak{p}_{i+1}[x]$ since $\mathfrak{p}_i[x] \cap A = \mathfrak{p}_i$ for all i. Now $1 \notin \mathfrak{p}_n$ since $\mathfrak{p}_n \neq A$, and so $\mathfrak{p}_n[x] \subsetneq (\mathfrak{p}_n[x], x)$. Also, $(\mathfrak{p}_n[x], x)$ is a prime ideal in $A[x]$ since $A[x]/(\mathfrak{p}_n[x], x) \cong A/\mathfrak{p}_n$. From this we see that $\dim A[x] \geq \dim A + 1$.

7. **Show that $\dim A[x] = \dim(A) + 1$ if A is Noetherian.**

   It suffices to show that $\dim A[x] \leq \dim(A) + 1$.