A COURSE IN HOMOLOGICAL ALGEBRA

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Contents

Chapter 1

HOMOLOGY

1.1 Homology Functors

Definition 1.1.1. Let R be a ring. By a **(chain) complex** $(\mathbf{X}, d^{\mathbf{X}})$ of R-modules we mean a sequence

$$(\mathbf{X}, d^{\mathbf{X}}) =: \ldots \longrightarrow X_{n+1} \xrightarrow{d^{X}_{n+1}} X_n \xrightarrow{d^{\mathbf{X}}_n} X_{n-1} \longrightarrow \ldots$$

of R-modules $\{X_n\}$ and R-module homomorphisms $\{d^X_n : X_n \longrightarrow X_{n-1}\}$ such that $d^{\mathbf{X}}_n d^{\mathbf{X}}_{n+1} = 0$ for all $n \in \mathbb{Z}$. X_n and $d^{\mathbf{X}}_n$ are called the **module in degree** n and the n-th **differential** of $(\mathbf{X}, d^{\mathbf{X}})$, respectively.

We usually simplify the notation and write $\mathbf{X}$ instead of $(\mathbf{X}, d^{\mathbf{X}})$.

Remark 1.1.2. An R-module M is considered as the complex

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots,$$

where the module M is sitting in degree 0.

Definition 1.1.3. Suppose $\mathbf{X}$ and $\mathbf{Y}$ are two complexes. Then we can define a **morphism** between them, $f : \mathbf{X} \longrightarrow \mathbf{Y}$, as a family of homomorphisms

$f_n : X_n \longrightarrow Y_n$ such that for all $n \in \mathbb{Z}$, the following diagram commutes:

$$\begin{tikzcd} X_n \arrow[r, "d_n^{\mathbf{X}}"] \arrow[d, "f_n"'] & X_{n-1} \arrow[d, "f_{n-1}"] \\ Y_n \arrow[r, "d_n^{\mathbf{Y}}"] & Y_{n-1} \end{tikzcd}$$

It is easy to see that the collection of complexes and their morphisms (with the obvious composition) forms a category. We denote this category by ${}_R\text{Comp}$ (or Comp).

Definition 1.1.4. If $(\mathbf{X}, d^{\mathbf{X}})$ is a complex, define

$$\begin{aligned} n\textbf{-cycle} = Z_n(\mathbf{X}) &= \ker d_n^{\mathbf{X}}, \\ n\textbf{-boundaries} = B_n(\mathbf{X}) &= \operatorname{im} d_{n+1}^{\mathbf{X}}, \\ n\textbf{-homology} = H_n(\mathbf{X}) &= Z_n(\mathbf{X})/B_n(\mathbf{X}). \end{aligned}$$

Since the equation $d_n^{\mathbf{X}} d_{n+1}^{\mathbf{X}} = 0$ in the complex $\mathbf{X}$ is equivalent to the condition $\operatorname{im} d_{n+1}^{\mathbf{X}} \subseteq \ker d_n^{\mathbf{X}}$, we have $B_n(\mathbf{X}) \subseteq Z_n(\mathbf{X})$, and so the quotient module $Z_n(\mathbf{X})/B_n(\mathbf{X})$ does make sense. An element of $H_n(\mathbf{X})$ is a coset $z_n + B_n(\mathbf{X})$; we call this element a *homology class*, and often denote it by $[z_n]$.

Lemma 1.1.5. *Let $f : \mathbf{X} \longrightarrow \mathbf{Y}$ be a morphism of complexes. Then*
(1) $f_n(Z_n(\mathbf{X})) \subseteq Z_n(\mathbf{Y})$,
(2) $f_n(B_n(\mathbf{X})) \subseteq B_n(\mathbf{Y})$.

Proof. Consider the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & X_{n+1} \arrow[r, "d_{n+1}^{\mathbf{X}}"] \arrow[d, "f_{n+1}"] & X_n \arrow[r, "d_n^{\mathbf{X}}"] \arrow[d, "f_n"] & X_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\ \cdots \arrow[r] & Y_{n+1} \arrow[r, "d_{n+1}^{\mathbf{Y}}"] & Y_n \arrow[r, "d_n^{\mathbf{Y}}"] & Y_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

(1): Let $x \in Z_n(\mathbf{X})$. Then $d_n^{\mathbf{X}}(x) = 0$. By commutativity of the above diagram, $d_n^{\mathbf{Y}} f_n(x) = f_{n-1} d_n^{\mathbf{X}}(x) = f_{n-1}(0) = 0$, so that $f_n(x) \in Z_n(\mathbf{Y})$. Thus $f_n(Z_n(\mathbf{X})) \subseteq Z_n(\mathbf{Y})$.

(2): Let $y \in B_n(\mathbf{X})$. Then there exists $x \in X_{n+1}$ such that $d_{n+1}^{\mathbf{X}}(x) = y$. By commutativity of the above diagram, $f_n(y) = f_n d_{n+1}^{\mathbf{X}}(x) = d_{n+1}^{\mathbf{Y}} f_{n+1}(x)$, so that $f_n(y) \in B_n(\mathbf{Y})$. Thus $f_n(B_n(\mathbf{X})) \subseteq B_n(\mathbf{Y})$. □

Theorem 1.1.6. *Let $f : \mathbf{X} \longrightarrow \mathbf{Y}$ be a morphism of complexes and let $n \in \mathbb{Z}$. Define*

$$\begin{aligned} H_n(f) : H_n(X) &\longrightarrow H_n(Y) \\ [z_n] &\longmapsto [f_n z_n]. \end{aligned}$$

Then $H_n : {}_R\mathit{Comp} \longrightarrow {}_R\mathit{Mod}$ is an additive functor.

Proof. First of all, we show that $H_n(f)$ is well defined. Let $[z] = [y]$. Then $z - y \in B_n(X)$ and so there exists $x \in X_{n+1}$ such that $z - y = d_{n+1}^X(x)$. By the part (2) of the above lemma we have

$$f_n z - f_n y = f_n(z - y) \in B_n(Y).$$

Therefore $[f_n z] = [f_n y]$, and hence $H_n(f)$ is well defined.

Now, we show that H_n is a functor. It is clear that $H_n(1_X)$ is the identity. If f and g are morphisms whose composite gf is defined, then

$$H_n(gf)[z] = [(gf)_n z] = [(g_n f_n) z]) = [g_n(f_n z)] = H_n(g)[f_n z] = H_n(g)H_n(f)[z].$$

Finally, we show that H_n is additive. If $f, g : \mathbf{X} \longrightarrow \mathbf{Y}$ are two morphisms of complexes, then

$$H_n(f+g)[z] = [(f+g)_n z] = [(f_n + g_n)z] = [f_n z] + [g_n z] = H_n(f)[z] + H_n(g)[z].$$

□

Definition 1.1.7. We say that the sequence

$$0 \longrightarrow \mathbf{X} \xrightarrow{f} \mathbf{Y} \xrightarrow{g} \mathbf{W} \longrightarrow 0$$

is an exact sequence of complexes if the sequences

$$0 \longrightarrow X_n \xrightarrow{f_n} Y_n \xrightarrow{g_n} W_n \longrightarrow 0$$

are exact for every $n \in \mathbb{Z}$.

Theorem 1.1.8. (Connecting Homomorphism). *If* $0 \longrightarrow \mathbf{X} \xrightarrow{f} \mathbf{Y} \xrightarrow{g} \mathbf{W} \longrightarrow 0$ *is an exact sequence of complexes, then, for each* $n \in \mathbb{Z}$*, there is a homomorphism*

$$\begin{aligned} \partial_n : H_n(\mathbf{W}) &\longrightarrow H_{n-1}(\mathbf{X}) \\ [w_n] &\longmapsto [x_{n-1}] \quad (x_{n-1} \in f_{n-1}^{-1} d_n^Y g_n^{-1}(w_n)). \end{aligned}$$

Proof. Consider the commutative diagram with exact rows:

$$\begin{tikzcd}
 & \mathbf{X} & \mathbf{Y} & \mathbf{W} & \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & X_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d_{n+1}^{\mathbf{X}}"] & Y_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d_{n+1}^{\mathbf{Y}}"] & W_{n+1} \arrow[r] \arrow[d, "d_{n+1}^{\mathbf{W}}"] & 0 \\
0 \arrow[r] & X_n \arrow[r, "f_n"] \arrow[d, "d_n^{\mathbf{X}}"] & Y_n \arrow[r, "g_n"] \arrow[d, "d_n^{\mathbf{Y}}"] & W_n \arrow[r] \arrow[d, "d_n^{\mathbf{W}}"] & 0 \\
0 \arrow[r] & X_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & Y_{n-1} \arrow[r, "g_{n-1}"] \arrow[d] & W_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

We only show that ∂_n is well defined; the other verifications are also routine and are left to the reader. For this, we first show that $f_{n-1}^{-1} d_n^Y g_n^{-1}(w_n) \neq \emptyset$, where $w_n \in \ker d_n^{\mathbf{W}}$. Let $y_n \in g_n^{-1}(w_n)$. Then $g_n(y_n) = w_n$. By commutativity of the above diagram,

$$g_{n-1} d_n^{\mathbf{Y}}(y_n) = d_n^{\mathbf{W}} g_n(y_n) = d_n^{\mathbf{W}}(w_n) = 0.$$

It follows that $d_n^{\mathbf{Y}}(y_n) \in \ker g_{n-1} = \operatorname{im} f_{n-1}$. Thus $d_n^{\mathbf{Y}} g_n^{-1}(w_n) \subseteq \operatorname{im} f_{n-1}$ and hence $f_{n-1}^{-1} d_n^{Y} g_n^{-1}(w_n) \neq \emptyset$. Let $w_n \in \ker d_n^{\mathbf{W}}$ and $x_{n-1} \in f_{n-1}^{-1} d_n^{Y} g_n^{-1}(w_n)$. We must show that $[x_{n-1}] \in H_{n-1}(\mathbf{X})$. Suppose that $f_{n-1}(x_{n-1}) = d_n^{\mathbf{Y}}(y_n)$ for some $y_n \in g_n^{-1}(w_n)$. By commutativity of the above diagram,

$$f_{n-2} d_{n-1}^{\mathbf{X}}(x_{n-1}) = d_{n-1}^{\mathbf{Y}} f_{n-1}(x_{n-1}) = d_{n-1}^{\mathbf{Y}} d_n^{\mathbf{Y}} y_n = 0.$$

Since f_{n-2} is injective, we have $d_{n-1}^{\mathbf{X}}(x_{n-1}) = 0$ and hence $[x_{n-1}] \in H_{n-1}(\mathbf{X})$.

Now let $x_{n-1}, \overline{x}_{n-1} \in f_{n-1}^{-1} d_n^{\mathbf{Y}} g_n^{-1}(w_n)$. Then there exist $y_n, \overline{y}_n \in g_n^{-1}(w_n)$ such that $x_{n-1} = f_{n-1}^{-1} d_n^{\mathbf{Y}}(y_n)$ and $\overline{x}_{n-1} = f_{n-1}^{-1} d_n^{\mathbf{Y}}(\overline{y}_n)$. Since $g_n(y_n) = g_n(\overline{y}_n)$, we have $y_n - \overline{y}_n \in \ker g_n = \operatorname{im} f_n$ and hence there exists $x_n \in X_n$ such that $y_n - \overline{y}_n = f_n(x_n)$. Therefore

$$\begin{aligned}
[x_{n-1}] &= [f_{n-1}^{-1} d_n^{\mathbf{Y}}(y_n)] = [f_{n-1}^{-1} d_n^{\mathbf{Y}}(\overline{y}_n + f_n(x_n))] \\
&= [f_{n-1}^{-1} d_n^{\mathbf{Y}}(\overline{y}_n) + f_{n-1}^{-1} d_n^{\mathbf{Y}}(f_n(x_n))] \\
&= [\overline{x}_{n-1}] + [f_{n-1}^{-1} f_{n-1} d_n^{\mathbf{X}}(x_n)] = [\overline{x}_{n-1}] + [d_n^{\mathbf{X}}(x_n)] \\
&= [\overline{x}_{n-1}].
\end{aligned}$$

This proves that ∂_n is well defined. □

Definition 1.1.9. The homomorphisms $\partial_n : H_n(W) \longrightarrow H_{n-1}(X)$ are called **connecting homomorphisms**.

Theorem 1.1.10. (Long Exact Sequence). *If* $0 \longrightarrow \mathbf{X} \xrightarrow{f} \mathbf{Y} \xrightarrow{g} \mathbf{W} \longrightarrow 0$ *is a sequence of complexes, then there is an exact sequence of modules*

$$\cdots \longrightarrow H_n(\mathbf{X}) \xrightarrow{H_n(f)} H_n(\mathbf{Y}) \xrightarrow{H_n(g)} H_n(\mathbf{W}) \xrightarrow{\partial_n} H_{n-1}(\mathbf{X}) \xrightarrow{H_{n-1}(f)} H_{n-1}(\mathbf{Y}) \longrightarrow \cdots .$$

Proof. Consider the commutative diagram with exact rows:

$$\begin{tikzcd}
& \mathbf{X} & \mathbf{Y} & \mathbf{W} & \\
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & X_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d^{\mathbf{X}}_{n+1}"] & Y_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d^{\mathbf{Y}}_{n+1}"] & W_{n+1} \arrow[r] \arrow[d, "d^{\mathbf{W}}_{n+1}"] & 0 \\
0 \arrow[r] & X_n \arrow[r, "f_n"] \arrow[d, "d^{\mathbf{X}}_{n}"] & Y_n \arrow[r, "g_n"] \arrow[d, "d^{\mathbf{Y}}_{n}"] & W_n \arrow[r] \arrow[d, "d^{\mathbf{W}}_{n}"] & 0 \\
0 \arrow[r] & X_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & Y_{n-1} \arrow[r, "g_{n-1}"] \arrow[d] & W_{n-1} \arrow[r] \arrow[d] & 0 \\
& \vdots & \vdots & \vdots &
\end{tikzcd}$$

There are six inclusions to verify.

(1) $\mathrm{im} H_n(f) \subseteq \ker H_n(g)$: Because $H_n(g)H_n(f) = H_n(gf) = 0$, we have $\mathrm{im} H_n(f) \subseteq \ker H_n(g)$.

(2) $\ker H_n(g) \subseteq \mathrm{im} H_n(f)$: Let $[y_n] \in \ker H_n(g)$. Then $g_n y_n \in B_n(\mathbf{W})$ and hence there is $w_{n+1} \in W_{n+1}$ such that $g_n y_n = d^{\mathbf{W}}_{n+1}(w_{n+1})$. Since g_{n+1} is surjective, there exists $y_{n+1} \in Y_{n+1}$ such that $g_{n+1} y_{n+1} = w_{n+1}$. Therefore, by commutativity of the above diagram,

$$\begin{aligned}
g_n(y_n - d^{\mathbf{Y}}_{n+1}(y_{n+1})) &= g_n y_n - g_n d^{\mathbf{Y}}_{n+1}(y_{n+1}) \\
&= g_n y_n - d^{\mathbf{W}}_{n+1} g_{n+1} y_{n+1} \\
&= g_n y_n - d^{\mathbf{W}}_{n+1}(w_{n+1}) = 0.
\end{aligned}$$

It follows that there exists $x_n \in X_n$ such that $y_n - d^{\mathbf{Y}}_{n+1}(y_{n+1}) = f_n(x_n)$. Hence

$$H_n(f)[x_n] = [f_n(x_n)] = [y_n - d^{\mathbf{Y}}_{n+1}(y_{n+1})] = [y_n].$$

(3) $\mathrm{im}H_n(g) \subseteq \ker\partial_n$: Let $H_n(g)[y_n] = [g_n y_n] \in \mathrm{im}H_n(g)$. Then $\partial_n[g_n y_n] = [x_{n-1}]$, where $x_{n-1} = f_{n-1}^{-1} d_n^{\mathbf{Y}} y_n \in f_{n-1}^{-1} d_n^{\mathbf{Y}} g_n^{-1} g_n y_n$. Therefore $f_{n-1}x_{n-1} = d_n^{\mathbf{Y}} y_n = 0$, and hence $x_{n-1} = 0$, because f_{n-1} is injective. It follows that $H_n(g)[y_n] \in \ker\partial_n$.

(4) $\ker\partial_n \subseteq \mathrm{im}H_n(g)$: Let $\partial_n[w_n] = 0$. Since g_n is surjective, there exists $y_n \in Y_n$ such that $w_n = g_n(y_n)$. Let $x_{n-1} = f_{n-1}^{-1} d_n^{\mathbf{Y}} y_n \in f_{n-1}^{-1} d_n^{\mathbf{Y}} g_n^{-1}(w_n)$. By definition of ∂_n, we have $\partial_n[w_n] = [x_{n-1}] = 0$. Hence there exists $x_n \in X_n$ such that $x_{n-1} = d_n^{\mathbf{X}} x_n$. We have

$$d_n^{\mathbf{Y}}(y_n - f_n(x_n)) = d_n^{\mathbf{Y}}(y_n) - f_{n-1} d_n^{\mathbf{X}} x_n = 0.$$

Therefore $y_n - f_n(x_n) \in \ker d_n^{\mathbf{Y}}$ and

$$H_n(g)[y_n - f_n(x_n)] = [g_n y_n - g_n f_n(x_n)] = [g_n y_n] = [w_n].$$

(5) $\mathrm{im}\partial_n \subseteq \ker H_{n-1}(f)$: Let $\partial_n[w_n] \in \mathrm{im}\partial_n$. Then there exists $y_n \in g_n^{-1}(w_n)$ such that $\partial_n[w_n] = [x_{n-1}]$, where $x_{n-1} = f_{n-1}^{-1} d_n^{\mathbf{Y}} y_n \in f_{n-1}^{-1} d_n^{\mathbf{Y}} g_n^{-1}(w_n)$. Therefore

$$H_{n-1}(f)[x_{n-1}] = [f_{n-1}x_{n-1}] = [f_{n-1} f_{n-1}^{-1} d_n^{\mathbf{Y}} y_n] = [d_n^{\mathbf{Y}} y_n] = 0.$$

(6) $\ker H_{n-1}(f) \subseteq \mathrm{im}\partial_n$: Let $H_{n-1}(f)[x_{n-1}] = [f_{n-1}x_{n-1}] = 0$. Then there exists $y_n \in Y_n$ such that $f_{n-1}x_{n-1} = d_n^{\mathbf{Y}} y_n$. Therefore $x_{n-1} = f_{n-1}^{-1} d_n^{\mathbf{Y}} y_n \in f_{n-1}^{-1} d_n^{\mathbf{Y}} g_n^{-1}(g_n y_n)$ and hence $\partial_n[g_n y_n] = [x_{n-1}]$. □

Theorem 1.1.11. (Naturality of ∂_n). *Consider the commutative diagram with exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{X} \arrow[r, "f"] \arrow[d, "\alpha"] & \mathbf{Y} \arrow[r, "g"] \arrow[d, "\beta"] & \mathbf{W} \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \mathbf{X}' \arrow[r, "f'"] & \mathbf{Y}' \arrow[r, "g'"] & \mathbf{W}' \arrow[r] & 0
\end{tikzcd}$$

Then there is a commutative diagram of modules with exact rows:

$$
\begin{tikzcd}
\cdots \arrow[r] & H_n(\mathbf{X}) \arrow[r, "H_n(f)"] \arrow[d, "H_n(\alpha)"] & H_n(\mathbf{Y}) \arrow[r, "H_n(g)"] \arrow[d, "H_n(\beta)"] & H_n(\mathbf{W}) \arrow[r, "\partial_n"] \arrow[d, "H_n(\gamma)"] & H_{n-1}(\mathbf{X}) \arrow[r, "H_{n-1}(f)"] \arrow[d, "H_{n-1}(\alpha)"] & \cdots \\
\cdots \arrow[r] & H_n(\mathbf{X}') \arrow[r, "H_n(f')"] & H_n(\mathbf{Y}') \arrow[r, "H_n(g')"] & H_n(\mathbf{W}') \arrow[r, "\partial'_n"] & H_{n-1}(\mathbf{X}') \arrow[r, "H_{n-1}(f')"] & \cdots
\end{tikzcd}
$$

Proof. Exactness of the rows is Theorem 1.0.12 (Long Exact Sequence). The first two squares commute because H_n is a functor. Now we show that the commutativity of the square involving the connecting homomorphism. Consider the commutative three-dimensional diagram:

$$
\begin{tikzcd}
X_n \arrow[rr, "f_n"] \arrow[dr, "d_n^X"] \arrow[dd, "\alpha_n"'] & & Y_n \arrow[rr, "g_n"] \arrow[dr, "d_n^Y"] \arrow[dd, "\beta_n"' near start] & & W_n \arrow[dr, "d_n^W"] \arrow[dd, "\gamma_n"' near start] & \\
& X_{n-1} \arrow[rr, "f_{n-1}" near start] \arrow[dd, "\alpha_{n-1}"' near start] & & Y_{n-1} \arrow[rr, "g_{n-1}" near start] \arrow[dd, "\beta_{n-1}" near start] & & W_{n-1} \arrow[dd, "\gamma_{n-1}"] \\
X'_n \arrow[rr, "f'_n" near end] \arrow[dr, "d_n^{X'}"] & & Y'_n \arrow[rr, "g'_n" near end] \arrow[dr, "d_n^{Y'}"] & & W'_n \arrow[dr, "d_n^{W'}"] & \\
& X'_{n-1} \arrow[rr, "f'_{n-1}"] & & Y'_{n-1} \arrow[rr, "g'_{n-1}"] & & W'_{n-1}
\end{tikzcd}
$$

Let $[w_n] \in H_n(\mathbf{W})$. We show that $H_{n-1}(\alpha)\partial_n[w_n] = \partial'_n H_n(\gamma)[w_n]$. Let $y_n \in g_n^{-1}(w_n)$ and $x_{n-1} = f_{n-1}^{-1} d_n^{\mathbf{Y}} y_n$. Then

$$H_{n-1}(\alpha)\partial_n[w_n] = H_{n-1}(\alpha)[x_{n-1}] = [\alpha_{n-1}x_{n-1}].$$

Let $x'_{n-1} = f_{n-1}^{\prime -1} d_n^{\mathbf{Y}'} \beta_n y_n$. Since $\gamma_n(w_n) = \gamma_n(g_n y_n) = g'_n \beta_n y_n$, we have

$$\partial'_n H_n(\gamma)[w_n] = \partial'_n[\gamma_n w_n] = [x'_{n-1}].$$

On the other hand,

$$f'_{n-1}(\alpha_{n-1}x_{n-1}) = \beta_{n-1} f_{n-1} x_{n-1} = \beta_{n-1} d_n^{\mathbf{Y}} y_n = d_n^{\mathbf{Y}'} \beta_n y_n = f'_{n-1} x'_{n-1}$$

Since f'_{n-1} is injective, it follows that $\alpha_{n-1}x_{n-1} = x'_{n-1}$, which completes the proof. $\square$

Theorem 1.1.12. (Snake Lemma). *Consider the commutative diagram of modules with exact rows:*

$$\begin{tikzcd}
 & M' \arrow[r, "f"] \arrow[d, "\alpha"] & M \arrow[r, "g"] \arrow[d, "\beta"] & M'' \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & N' \arrow[r, "f'"] & N \arrow[r, "g'"] & N'' &
\end{tikzcd}$$

Then there is the following exact sequence

$$\ker\alpha \xrightarrow{\overline{f}} \ker\beta \xrightarrow{\overline{g}} \ker\gamma \xrightarrow{\partial} \mathrm{coker}\alpha \xrightarrow{\overline{f'}} \mathrm{coker}\beta \xrightarrow{\overline{g'}} \mathrm{coker}\gamma$$

Proof. It is easy to see that $\overline{f} = f|_{\ker\alpha} : \ker\alpha \longrightarrow \ker\beta$, $\overline{g} = g|_{\ker\beta} : \ker\beta \longrightarrow \ker\gamma$,

$$\begin{aligned} \overline{f'} : \mathrm{coker}\alpha &\longrightarrow \mathrm{coker}\beta \\ n' + \mathrm{im}\alpha &\longmapsto f'(n') + \mathrm{im}\beta \end{aligned}$$

and

$$\begin{aligned} \overline{g'} : \mathrm{coker}\beta &\longrightarrow \mathrm{coker}\gamma \\ n + \mathrm{im}\beta &\longmapsto g'(n) + \mathrm{im}\gamma \end{aligned}$$

are well defined. There are eight inclusions to verify.

(1) $\mathrm{im}\overline{f} \subseteq \ker\overline{g}$: Let $m' \in M'$, then

$$\overline{g}\overline{f}(m') = \overline{g}f(m') = gf(m') = 0.$$

Hence $\mathrm{im}\overline{f} \subseteq \ker\overline{g}$.

(2) $\ker\overline{g} \subseteq \mathrm{im}\overline{f}$: Let $m \in \ker\overline{g}$. Then $m \in \ker\beta$ and $g(m) = 0$. Therefore there exists $m' \in M'$ such that $m = f(m')$. Since $f'\alpha(m') = \beta f(m') = \beta(m) = 0$, we have $m' \in \ker\alpha$ and hence $m = f(m') = \overline{f}(m') \in \mathrm{im}\overline{f}$.

Define

$$\begin{aligned} \partial : \mathrm{ker}\gamma &\longrightarrow \mathrm{coker}\alpha \\ m'' &\longmapsto n' + \mathrm{im}\alpha \quad (n' \in {f'}^{-1}\beta g^{-1}(m'')). \end{aligned}$$

We show that ∂ is well defined. Let ${n'}_1, {n'}_2 \in {f'}^{-1}\beta g^{-1}(m'')$. Then there are $m_1, m_2 \in g^{-1}(m'')$ such that $f'({n'}_1) = \beta(m_1)$ and $f'({n'}_2) = \beta(m_2)$. Therefore $g(m_1) = g(m_2) = m''$ and hence $m_1 - m_2 \in \mathrm{ker}g = \mathrm{im}f$. Hence there exists $m' \in M'$ such that $f(m') = m_1 - m_2$. By commutativity of the above diagram,

$$f'({n'}_1 - {n'}_2) = \beta(m_1 - m_2) = \beta f(m') = f'\alpha(m').$$

Hence ${n'}_1 - {n'}_2 = \alpha(m')$ and so ${n'}_1 + \mathrm{im}\alpha = {n'}_1 + \mathrm{im}\alpha$.

(3) $\mathrm{ker}\partial \subseteq \mathrm{im}\overline{g}$: Let $\lambda \in \mathrm{ker}\partial$. Then $\lambda \in \mathrm{ker}\gamma \subseteq M''$. Therefore there exists $m \in M$ such that $m \in g^{-1}(\lambda)$. Since $0 = \partial(\lambda) = {f'}^{-1}\beta(m) + \mathrm{im}\alpha$, there exists $m' \in M'$ such that ${f'}^{-1}\beta(m) = \alpha(m')$. By commutativity of the above diagram,

$$\beta f(m') = f'\alpha(m') = \beta(m).$$

Hence $m - f(m') \in \mathrm{ker}\beta$. Now we have $\overline{g}(m - f(m')) = g(m - f(m')) = g(m) = \lambda$ and hence $\mathrm{ker}\partial \subseteq \mathrm{im}\overline{g}$.

(4) $\mathrm{im}\overline{g} \subseteq \mathrm{ker}\partial$: Let $\overline{g}(m) \in \mathrm{im}\overline{g}$, where $m \in \mathrm{ker}\beta$. Then $\partial(\overline{g}(m)) = n' + \mathrm{im}\alpha$, where $n' = {f'}^{-1}\beta(m) \in {f'}^{-1}\beta g^{-1}(\overline{g}(m))$. Therefore $\partial(\overline{g}(m)) = {f'}^{-1}\beta(m) + \mathrm{im}\alpha = \mathrm{im}\alpha$. and hence $\mathrm{im}\overline{g} \subseteq \mathrm{ker}\partial$.

(5) $\mathrm{im}\partial \subseteq \mathrm{ker}\overline{f'}$: Let $\partial(\lambda) = {f'}^{-1}\beta(m) + \mathrm{im}\alpha \in \mathrm{im}\partial$, where $m \in g^{-1}(\lambda)$. Then

$$\overline{f'}\partial(\lambda) = \overline{f'}({f'}^{-1}\beta(m) + \mathrm{im}\alpha) = f'{f'}^{-1}\beta(m) + \mathrm{im}\beta = \mathrm{im}\beta.$$

It follows that $\mathrm{im}\partial \subseteq \mathrm{ker}\overline{f'}$.

(6) $\mathrm{ker}\overline{f'} \subseteq \mathrm{im}\partial$: Let $\overline{f'}(n' + \mathrm{im}\alpha) = 0$. Then $f'(n') + \mathrm{im}\beta = \mathrm{im}\beta$. Therefore there exists $m \in M$ such that $f'(n') = \beta(m)$. By commutativity of the above diagram,

$$\gamma g(m) = g'\beta(m) = g'f'(n') = 0.$$

Therefore $g(m) \in \ker\gamma$. Since $n' = {f'}^{-1}\beta(m) \in {f'}^{-1}\beta g^{-1}(g(m))$, we have $\partial(g(m)) = n' + \mathrm{im}\alpha$. Thus $\ker\overline{f'} \subseteq \mathrm{im}\partial$.

(7) $\mathrm{im}\overline{f'} \subseteq \ker\overline{g'}$: Let $n' \in N'$. Then

$$\overline{g'}\,\overline{f'}(n' + \mathrm{im}\alpha) = \overline{g'}(f'(n') + \mathrm{im}\beta) = g'f'(n') + \mathrm{im}\gamma = 0.$$

Hence $\mathrm{im}\overline{f'} \subseteq \ker\overline{g'}$.

(8) $\ker\overline{g'} \subseteq \mathrm{im}\overline{f'}$: Let $n + \mathrm{im}\beta \in \ker\overline{g'}$. Then $g'(n) \in \mathrm{im}\gamma$. Therefore there exists $m'' \in M''$ such that $g'(n) = \gamma(m'')$. Since g in surjective, there exists $m \in M$ such that $g(m) = m''$. By commutativity of the above diagram,

$$g'(n) = \gamma(m'') = \gamma g(m) = g'\beta(m).$$

Hence $n - \beta(m) \in \ker g' = \mathrm{im} f'$ and so there exists $n' \in N'$ such that $n - \beta(m) = f'(n')$. It follows that $n + \mathrm{im}\beta = f'(n') + \mathrm{im}\beta \in \mathrm{im}\overline{f'}$ and hence $\ker\overline{g'} \subseteq \mathrm{im}\overline{f'}$. □

Remark 1.1.13. The snake is

$$\begin{tikzcd}
 & \ker\alpha \arrow[r, "\overline{f}"] \arrow[d] & \ker\beta \arrow[r, "\overline{g}"] \arrow[d] & \ker\gamma \arrow[d] \arrow[dddll, bend left] & \\
 & M' \arrow[r, "f"] \arrow[d, "\alpha"] & M \arrow[r, "g"] \arrow[d, "\beta"] & M'' \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & N' \arrow[r, "f'"] \arrow[d] & N \arrow[r, "g'"] \arrow[d] & N'' \arrow[d] & \\
 & \mathrm{coker}\alpha \arrow[r, "\overline{f'}"] & \mathrm{coker}\beta \arrow[r, "\overline{g'}"] & \mathrm{coker}\gamma &
\end{tikzcd}$$

Definition 1.1.14. Two morphisms $f, g : \mathbf{X} \longrightarrow \mathbf{Y}$ are **homotopic**, denoted by $f \simeq g$, if for all $n \in \mathbb{Z}$, there are homomorphism $s_n : X_n \longrightarrow Y_{n+1}$ so that

$$f_n - g_n = d^{\mathbf{Y}}_{n+1} s_n + s_{n-1} d^{\mathbf{X}}_n,$$

as illustrated in the diagram below:

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r, "d^{\mathbf{X}}_{n+1}"] \arrow[d, "\theta_{n+1}"'] & X_n \arrow[r, "d^{\mathbf{X}}_n"] \arrow[d, "\theta_n"] \arrow[dl, dotted, "s_n"] & X_{n-1} \arrow[r] \arrow[d, "\theta_{n-1}"] \arrow[dl, dotted, "s_{n-1}"] & \cdots \\
\cdots \arrow[r] & Y_{n+1} \arrow[r, "d^{\mathbf{Y}}_{n+1}"] & Y_n \arrow[r, "d^{\mathbf{Y}}_n"] & Y_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

where $\theta_n = f_n - g_n$.

Theorem 1.1.15. (Homotopic Morphisms Theorem). *If $f, g : \mathbf{X} \longrightarrow \mathbf{Y}$ are homotopic morphisms, then*

$$H_n(f) = H_n(g) \ \textit{for all} \ n \in \mathbb{Z}.$$

Proof. Let $z_n \in \ker d^{\mathbf{X}}_n$. Then

$$\begin{aligned}
H_n(f)[z_n] &= [f_n z_n] = [(g_n + d^{\mathbf{Y}}_{n+1} s_n + s_{n-1} d^{\mathbf{X}}_n) z_n] \\
&= [g_n z_n] + [d^{\mathbf{Y}}_{n+1} s_n z_n] + [s_{n-1} d^{\mathbf{X}}_n z_n] \\
&= H_n(g)[z_n].
\end{aligned}$$

This completes the proof. □

Definition 1.1.16. A **Free resolution** of a module M is an exact sequence

$$\mathbf{F} : \cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

in which each F_i is free. Also then the sequence (no longer exact at F_0)

$$\mathbf{F_M} : \cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} 0$$

is called the **deleted free resolution** of the resolution $\mathbf{F}$.

Projective resolution and flat resolution are defined similarly.

Definition 1.1.17. An injective resolution of a module M is an exact sequence

$$\mathbf{E} : 0 \longrightarrow M \xrightarrow{\varepsilon} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \longrightarrow \cdots$$

in which each E^i is injective. Also then the sequence (no longer exact at E^0)

$$\mathbf{E_M} : 0 \longrightarrow E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \longrightarrow \cdots$$

is called the **deleted injective resolution** of the resolution $\mathbf{E}$.

We may, in fact, define the deleted complex of any complex:

Definition 1.1.18. Let $\mathbf{X}$ be a complex of the form

$$\mathbf{X} : \cdots \longrightarrow X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Then the complex

$$\mathbf{X_M} : \cdots \longrightarrow X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} 0$$

is called the **deleted complex** of the complex $\mathbf{X}$. Similarly, if $\mathbf{Y}$ is a complex of the form

$$\mathbf{Y} : 0 \longrightarrow N \xrightarrow{\varepsilon} Y^0 \xrightarrow{d^0} Y^1 \xrightarrow{d^1} Y^2 \longrightarrow \cdots,$$

then the complex

$$\mathbf{Y_N} : 0 \longrightarrow Y^0 \xrightarrow{d^0} Y^1 \xrightarrow{d^1} Y^2 \longrightarrow \cdots$$

is called the **deleted complex** of the complex $\mathbf{Y}$.

Theorem 1.1.19. *Every module M has a free resolution (which is necessarily a projective resolution and a flat resolution).*

Proof. There is a free module F_0 and an exact sequence

$$0 \longrightarrow K_1 \xrightarrow{i_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Similarly, there is a free module F_1, and an exact sequence

$$0 \longrightarrow K_2 \xrightarrow{i_2} F_1 \xrightarrow{\varepsilon_1} K_1 \longrightarrow 0,$$

and, by induction, a free module F_n, and an exact sequence

$$0 \longrightarrow K_{n+1} \xrightarrow{i_{n+1}} F_n \xrightarrow{\varepsilon_n} K_n \longrightarrow 0$$

Assemble all these sequences into the diagram

$$\begin{tikzcd}
\cdots F_3 \arrow[rr, "d_3"] \arrow[dr, "\varepsilon_3"'] & & F_2 \arrow[rr, "d_2"] \arrow[dr, "\varepsilon_2"'] & & F_1 \arrow[rr, "d_1"] \arrow[dr, "\varepsilon_1"'] & & F_0 \arrow[r, "\varepsilon"] & M \arrow[r] & 0 \\
0 \arrow[r] & K_3 \arrow[ur, "i_3"] & & K_2 \arrow[ur, "i_2"] & & K_1 \arrow[ur, "i_1"'] & & &
\end{tikzcd}$$

where $d_n : F_n \longrightarrow F_{n-1}$ is the composite $i_n\varepsilon_n$. Because $\ker\varepsilon = K_1 = \mathrm{im}d_1$, and for every n, $\ker d_n = K_{n+1}$ and $\mathrm{im}d_n = K_n$, we have that the top row is exact. □

Theorem 1.1.20. *Every module M has an injective resolution.*

Proof. Every module can be imbedded as a submodule of an injective module. Thus, there is an injective module E^0, an injection $\varepsilon : M \longrightarrow E^0$ and an exact sequence

$$0 \longrightarrow M \xrightarrow{\varepsilon} E^0 \xrightarrow{\pi^0} C^0 \longrightarrow 0.$$

Similarly, there is an injective module E^1, and an exact sequence

$$0 \longrightarrow C^0 \xrightarrow{\varepsilon^1} E^1 \xrightarrow{\pi^1} C^1 \longrightarrow 0,$$

and, by induction, an injective module E^n, and an exact sequence

$$0 \longrightarrow C^{n-1} \xrightarrow{\varepsilon^n} E^n \xrightarrow{\pi^n} C^n \longrightarrow 0,$$

Assemble all these sequences into the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\varepsilon"] & E^0 \arrow[rr, "d^0"] \arrow[dr, "\pi^0"'] & & E^1 \arrow[rr, "d^1"] \arrow[dr, "\pi^1"'] & & E^2 \cdots \\
& & & C^0 \arrow[ur, "\varepsilon^1"'] & & C^1 \arrow[ur, "\varepsilon^2"'] \arrow[r] & 0
\end{tikzcd}$$

where $d^n : E^n \longrightarrow E^{n+1}$ is the composite $\varepsilon^{n+1}\pi^n$. Because $\mathrm{im}\varepsilon = M = \ker d^0$, and for every n, $\ker d^n = C^{n-1}$ and $\mathrm{im}d^n = C^n$, we have that the top row is exact. □

Theorem 1.1.21. (Comparison Theorem). *Consider the diagram*

$$\begin{tikzcd}
\mathbf{P} : \ldots \arrow[r] & P_{n+1} \arrow[r, "d^{\mathbf{P}}_{n+1}"] \arrow[d, dashed, "\alpha_{n+1}"] & P_n \arrow[r] \arrow[d, dashed, "\alpha_n"] & \ldots \arrow[r] & P_0 \arrow[r, "d^{\mathbf{P}}_0"] \arrow[d, dashed, "\alpha_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\mathbf{Q} : \ldots \arrow[r] & Q_{n+1} \arrow[r, "d^{\mathbf{Q}}_{n+1}"] & Q_n \arrow[r] & \ldots \arrow[r] & Q_0 \arrow[r, "d^{\mathbf{Q}}_0"] & N \arrow[r] & 0
\end{tikzcd}$$

where the rows are complexes. If each P_n in the top row is projective, and if the bottom row is exact, then there exists a morphism $\alpha : \mathbf{P_M} \longrightarrow \mathbf{Q_N}$ (the *dashed arrows) making the completed diagram commute. Moreover, any two such morphisms are homotopic.*

Proof. (1) The existence of α. We prove the existence of $\alpha = \{\alpha_n\}$ by induction on $n \geq 0$. For the base step $n = 0$, consider the diagram

$$\begin{tikzcd} & P_0 \arrow[dl, dotted, "\alpha_0"'] \arrow[d, "fd_0^{\mathbf{P}}"] & \\ Q_0 \arrow[r, "d_0^{\mathbf{Q}}"] & N \arrow[r] & 0 \end{tikzcd}$$

Since P_0 is projective and $d_0^{\mathbf{Q}}$ is surjective, there exists an R-module homomorphism $\alpha_0 : P_0 \longrightarrow Q_0$ such that $d_0^{\mathbf{Q}}\alpha_0 = fd_0^{\mathbf{P}}$. Suppose that $n \geq 0$ and that we have already constructed R-homomorphisms $\alpha_i : P_i \longrightarrow Q_i$, $0 \leq i \leq n$ such that

$$d_{i+1}^{\mathbf{Q}}\alpha_{i+1} = \alpha_i d_{i+1}^{\mathbf{P}} \text{ for } 0 \leq i \leq n-1.$$

We have $d_n^{\mathbf{Q}}\alpha_n d_{n+1}^{\mathbf{P}} = \alpha_{n-1}d_n^{\mathbf{P}}d_{n+1}^{\mathbf{P}} = 0$. Therefore $\mathrm{im}\alpha_n d_{n+1}^{\mathbf{P}} \subseteq \ker d_n^{\mathbf{Q}} = \mathrm{im}d_{n+1}^{\mathbf{Q}}$ and hence we have the following diagram.

$$\begin{tikzcd} & P_{n+1} \arrow[dl, dotted, "\alpha_{n+1}"'] \arrow[d, "\alpha_n d_{n+1}^{\mathbf{P}}"] & \\ Q_{n+1} \arrow[r, "d_{n+1}^{\mathbf{Q}}"] & \mathrm{im}d_{n+1}^{\mathbf{Q}} \arrow[r] & 0 \end{tikzcd}$$

Since P_{n+1} is projective, there exists an R-module homomorphism $\alpha_{n+1} : P_{n+1} \longrightarrow Q_{n+1}$ such that $d_{n+1}^{\mathbf{Q}}\alpha_{n+1} = \alpha_n d_{n+1}^{\mathbf{P}}$. This completes induction and therefore, the existence of a morphism $\alpha = \{\alpha_n\}$ is achieved.

(2) Uniqueness of α to homotopy. Assume $\beta = \{\beta_n\} : \mathbf{P_M} \longrightarrow \mathbf{Q_N}$ is another morphism satisfying $d_0^{\mathbf{Q}}\beta_0 = fd_0^{\mathbf{P}}$ and

$$d_{n+1}^{\mathbf{Q}}\beta_{n+1} = \beta_n d_{n+1}^{\mathbf{P}} \text{ for } n \geq 0.$$

We construct a homotopy s by induction. Let $P_{-1} = Q_{-1} = 0$. Take s_{-1} :

$P_{-1} \longrightarrow Q_0$ to be the zero map. Now consider the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r, "d^{\mathbf{P}}_{n+1}"] & P_n \arrow[r] \arrow[dl, dotted, "s_n"'] & \ldots P_1 \arrow[r] & P_0 \arrow[r, "d^{\mathbf{P}}_0"] \arrow[dl, dotted, "s_0"'] & P_{-1}=0 \arrow[dl, dotted, "s_{-1}"'] \\
\cdots \arrow[r] & Q_{n+1} \arrow[r, "d^{\mathbf{Q}}_{n+1}"] & Q_n \arrow[r] & \ldots Q_1 \arrow[r] & Q_0 \arrow[r, "d^{\mathbf{Q}}_0"] & Q_{-1}=0
\end{tikzcd}$$

Since P_0 is projective, there exists an R-module homomorphism $s_0 : P_0 \longrightarrow Q_1$ such that $\alpha_0 - \beta_0 = d^{\mathbf{Q}}_1 s_0$ and hence $\alpha_0 - \beta_0 = d^{\mathbf{Q}}_1 s_0 + s_{-1} d^{\mathbf{P}}_0$. We have

$$d^{\mathbf{Q}}_{n+1}(\alpha_{n+1} - \beta_{n+1} - s_n d^{\mathbf{P}}_{n+1}) = \alpha_n d^{\mathbf{P}}_{n+1} - \beta_n d^{\mathbf{P}}_{n+1} - d^{\mathbf{Q}}_{n+1} s_n d^{\mathbf{P}}_{n+1} =$$
$$(\alpha_n - \beta_n) d^{\mathbf{P}}_{n+1} - d^{\mathbf{Q}}_{n+1} s_n d^{\mathbf{P}}_{n+1} = (d^{\mathbf{Q}}_{n+1} s_n - s_{n-1} d^{\mathbf{P}}_n) d^{\mathbf{P}}_{n+1} - d^{\mathbf{Q}}_{n+1} s_n d^{\mathbf{P}}_{n+1} = 0$$

Therefore $\mathrm{im}(\alpha_{n+1} - \beta_{n+1} - s_n d^{\mathbf{P}}_{n+1}) \subseteq \mathrm{im} d^{\mathbf{Q}}_{n+2}$ and hence we have the following diagram.

$$\begin{tikzcd}
 & P_{n+1} \arrow[dl, dotted, "s_{n+1}"'] \arrow[d, "\alpha_{n+1} - \beta_{n+1} - s_n d^{\mathbf{P}}_{n+1}"] & \\
Q_{n+2} \arrow[r, "d^{\mathbf{Q}}_{n+2}"] & \mathrm{im} d^{\mathbf{Q}}_{n+2} \arrow[r] & 0
\end{tikzcd}$$

Since P_{n+1} is projective, there exists an R-module homomorphism $s_{n+1} : P_{n+1} \longrightarrow Q_{n+2}$ such that $\alpha_n - \beta_n - s_n d^{\mathbf{P}}_{n+1} = d^{\mathbf{Q}}_{n+2} s_{n+1}$ or $\alpha_{n+1} - \beta_{n+1} = d^{\mathbf{Q}}_{n+2} s_{n+1} + s_n d^{\mathbf{P}}_{n+1}$. This completes induction and hence $\alpha \simeq \beta$. $\square$

Theorem 1.1.22. (Horseshoe Lemma). *Let* $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ *be a short exact sequence and let* $\mathbf{P}'$, $\mathbf{P}''$ *be projective resolutions for* M' *and*

M'' respectively, as shown in the diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & P_1' \arrow[d] & & P_1'' \arrow[d] & \\
 & P_0' \arrow[d, "\varepsilon'"] & & P_0'' \arrow[d, "\varepsilon''"] & \\
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

Then there exists a projective resolution $\mathbf{P}$ of M and morphisms $\alpha : \mathbf{P}_{M'} \longrightarrow \mathbf{P}_M$ and $\beta : \mathbf{P}_M \longrightarrow \mathbf{P}_{M''}$ such that $0 \longrightarrow \mathbf{P}_{M'} \xrightarrow{\alpha} \mathbf{P}_M \xrightarrow{\beta} \mathbf{P}_{M''} \longrightarrow 0$ is an exact sequence of complexes.

Proof. We show first that there is a projective P_0 and a commutative 3×3

diagram with exact columns and rows:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_0' \arrow[r] \arrow[d] & K_0 \arrow[r] \arrow[d] & K_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0' \arrow[r, "\alpha_0"] \arrow[d, "\varepsilon'"] & P_0 \arrow[r, "\beta_0"] \arrow[d, "\varepsilon"] & P_0'' \arrow[r] \arrow[d, "\varepsilon''"] \arrow[dl, dotted, "h"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Take $P_0 = P_0' \oplus P_0''$ and define $\alpha_0 : P_0' \longrightarrow P_0$ by $x' \longmapsto (x', 0)$, and $\beta_0 : P_0 \longrightarrow P_0''$ by $(x', x'') \longmapsto x''$. It is clear that P_0 is projective and that

$$0 \longrightarrow P_0' \xrightarrow{\alpha_0} P_0 \xrightarrow{\beta_0} P_0'' \longrightarrow 0$$

is exact. Since P_0'' is projective and g is surjective, there exists an R-module homomorphism $h : P_0'' \longrightarrow M$ such that $gh = \varepsilon''$. Now define

$$\begin{aligned} \varepsilon : P_0 &\longrightarrow M \\ (x', x'') &\longmapsto f\varepsilon' x' + hx''. \end{aligned}$$

Surjectivity of ε follows from the Five Lemma. It is an easy verification that, if $K_0 = \ker\varepsilon$, $K_0' = \ker\varepsilon'$, and $K_0'' = \ker\varepsilon''$, the resulting 3×3 diagram commutes. Exactness of the top row is the 3×3 Lemma.

We now prove, by induction on $n \geq 0$, that the bottom n rows of the desired diagram can be constructed. Consider the following commutative diagrams with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K'_n \arrow[r] \arrow[d] & K_n \arrow[r] \arrow[d] & K''_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P'_n \arrow[r, "\alpha_n"] \arrow[d] & P_n \arrow[r, "\beta_n"] \arrow[d] & P''_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K'_{n-1} \arrow[r] \arrow[d] & K_{n-1} \arrow[r] \arrow[d] & K''_{n-1} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K'_{n+1} \arrow[r] \arrow[d] & K_{n+1} \arrow[r] \arrow[d] & K''_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P'_{n+1} \arrow[r, "\alpha_{n+1}"] \arrow[d] & P_{n+1} \arrow[r, "\beta_{n+1}"] \arrow[d] & P''_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K'_n \arrow[r] \arrow[d] & K_n \arrow[r] \arrow[d] & K''_n \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Combining the above diagrams, we get the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
0 \arrow[r] & P'_{n+1} \arrow[r] \arrow[d] & P_{n+1} \arrow[r] \arrow[d] & P''_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K'_n \arrow[r] \arrow[d] & K_n \arrow[r] \arrow[d] & K''_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P'_n \arrow[r] & P_n \arrow[r] & P''_n \arrow[r] & 0
\end{tikzcd}$$

By defining $d^{\mathbf{P}}_{n+1} : P_{n+1} \longrightarrow P_n$ as the as the composite $P_{n+1} \longrightarrow K_n \longrightarrow P_n$, we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & P'_{n+1} \arrow[r] \arrow[d] & P_{n+1} \arrow[r] \arrow[d, "d^{\mathbf{P}}_{n+1}"] & P''_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P'_n \arrow[r] & P_n \arrow[r] & P''_n \arrow[r] & 0
\end{tikzcd}$$

It is easy to see that $\mathrm{im} d^{\mathbf{P}}_{n+1} = \ker d^{\mathbf{P}}_n$ and hence the proof is completed. □

We finally make some remarks about the dual notion.

Definition 1.1.23. Let R be a ring. By a *cochain complex* $(\mathbf{X}, d_{\mathbf{X}})$ of R-modules we mean a sequence

$$(\mathbf{X}, d_{\mathbf{X}}) =: \ldots \longrightarrow X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_{\mathbf{X}}^n} X^{n+1} \longrightarrow \ldots$$

of R-modules $\{X^n\}$ and R-module homomorphisms $\{d_X^n : X^n \longrightarrow X^{n+1}\}$ such that $d_{\mathbf{X}}^n d_{\mathbf{X}}^{n-1} = 0$ for all $n \in \mathbb{Z}$. X^n and $d_{\mathbf{X}}^n$ are called the *module in degree* n and the n-th *differential* of $(\mathbf{X}, d_{\mathbf{X}})$, respectively. We usually simplify the notation and write $\mathbf{X}$ instead of $(\mathbf{X}, d_{\mathbf{X}})$. Morphisms of cochain complexes are defined analogously to chain complexes. Given a cochain complex $(\mathbf{X}, d_{\mathbf{X}})$ we define its cohomology $H^n(\mathbf{X})$ by

$$H^n(\mathbf{X}) = \mathrm{ker} d^n / \mathrm{im} d^{n-1} \quad \text{for all } \; n \in \mathbb{Z}$$

With the obvious definition of induced maps, $H^n(-)$ then becomes a functor, the **cohomology functor**. In case of a cochain complex we will speak of cocycles, coboundaries, cohomology classes. All the theorems we have established for homology therefore work for cohomology without requiring separate proofs. Indeed, given a chain complex $(\mathbf{X}, d^{\mathbf{X}})$ we obtain a cochain complex $(\mathbf{Y}, \delta_{\mathbf{Y}})$ by setting $Y^n = X_{-n}$, $\delta^n = d_{-n}$. Conversely given a cochain complex we obtain a chain complex by this procedure.

Exercises

1. (i) Let $T : {}_R\mathrm{Mod} \longrightarrow {}_R\mathrm{Mod}$ be an exact covariant functor. For each $n \in \mathbb{Z}$ and every complex $\mathbf{X}$ of R-modules, prove that $H_n(T\mathbf{X}) \cong TH_n(\mathbf{X})$.

   (ii) Let $T : {}_R\mathrm{Mod} \longrightarrow {}_R\mathrm{Mod}$ be an exact contravariant functor. For each $n \in \mathbb{Z}$ and every complex $\mathbf{X}$ of R-modules, prove that $H_n(T\mathbf{X}) \cong TH_{-n}(\mathbf{X})$.

2. State and prove the dual of Comparison Theorem.

3. State and prove the dual of Horseshoe Lemma.

Chapter 2

DERIVED FUNCTORS

2.1 Covariant Left Derived Functors

Suppose for the time being that for every R-module M we have chosen exactly one deleted projective resolution $\mathbf{P_M}$.

Definition 2.1.1. Let S be another ring and $T : {}_R\text{Mod} \longrightarrow {}_S\text{Mod}$ be a covariant functor. For $n \in \mathbb{Z}$, define

$$(L_nT)M = H_n(T\mathbf{P_M}) = \ker Td_n/\text{im}Td_{n+1}.$$

To complete the definition of L_nT, we must describe its action on homomorphism $f : M \longrightarrow N$. By the Comparison Theorem, there is a morhism $\alpha : \mathbf{P_M} \longrightarrow \mathbf{P_N}$ over f. Then $T\alpha : T\mathbf{P_M} \longrightarrow T\mathbf{P_N}$ is also a morphism, and we define $(L_nT)f : (L_nT)M \longrightarrow (L_nT)N$ by

$$(L_nT)f = H_n(T\alpha).$$

In more detail,

$$\begin{aligned} (L_nT)f : (L_nT)M &\longrightarrow (L_nT)N \\ [z] &\longmapsto [(T\alpha_n)z] \end{aligned}$$

In pictures, look at the chosen projective resolutions:

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r, "d^{\mathbf{P}}_{n+1}"] \arrow[d, dashed, "\alpha_{n+1}"] & P_n \arrow[r] \arrow[d, dashed, "\alpha_n"] & \cdots \arrow[r] & P_0 \arrow[r, "d^{\mathbf{P}}_0"] \arrow[d, dashed, "\alpha_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & {P'}_{n+1} \arrow[r, "d^{\mathbf{P}'}_{n+1}"] & {P'}_n \arrow[r] & \cdots \arrow[r] & {P'}_0 \arrow[r, "d^{\mathbf{P}'}_0"] & N \arrow[r] & 0
\end{tikzcd}$$

Fill in the dashed arrows, delete M and N, apply T to this diagram, and then take the map induced by $T\alpha$ in homology.

Theorem 2.1.2. *Let S be another ring and $T : {}_R Mod \longrightarrow {}_S Mod$ be an additive covariant functor. Then*

$$L_n T : {}_R Mod \longrightarrow {}_S Mod$$

is an additive covariant functor for every $n \in \mathbb{Z}$.

Proof. We will prove that $(L_nT)f$ is well defined on homorphism f. If $\beta : \mathbf{P_M} \longrightarrow \mathbf{P_N}$ is another morphism over f, then the Comparison Theorem says that $\alpha \simeq \beta$, so that $T\alpha \simeq T\beta$ (Exercise 1). It follows from Homotopic Morphism Theorem that $H_n(T\alpha) = H_n(T\beta)$. Thus $(L_nT)f$ is independent of the choice of the morphim α.

By taking $1_{P_n} : P_n \longrightarrow P_n$, the identity map for every $n \in \mathbb{Z}$, we get a morphism $1_{P_M} = \{1_{P_n}\} : \mathbf{P_M} \longrightarrow \mathbf{P_M}$ and we have

$$(L_nT)(1_M) = H_n(T(1_{\mathbf{P_M}})) = H_n(1_{\mathbf{TP_M}}) = 1_{H_n(T\mathbf{P_M})} = 1_{(L_nT)M}.$$

Let $g : N \longrightarrow L$ be an R-homomorphism and $\{\beta_n\} : \mathbf{P_N} \longrightarrow \mathbf{P_L}$ be a morphism over g. Then $\{\beta_n\alpha_n\} : \mathbf{P_M} \longrightarrow \mathbf{P_L}$ is a morphism over $gf : M \longrightarrow L$. By definition, we have

$$\begin{aligned}
(L_nT)(gf)[x] &= [T(\beta_n\alpha_n)(x)] = [(T(\beta_n)T(\alpha_n))(x)] \\
&= [T(\beta_n)(T(\alpha_n))(x)] = (L_nT)g[T(\alpha_n)(x)] \\
&= (L_nT)g((L_nT)f[x]) = ((L_nT)g(L_nT)f)[x]
\end{aligned}$$

This implies that $(L_nT)(gf) = (L_nT)g(L_nT)f$. Therefore L_nT is a covariant functor. Finally, we show that L_nT is an additive covariant functor. Let h :

$M \longrightarrow N$ be another R-homomorphism and $\{\gamma_n\} : \mathbf{P_M} \longrightarrow \mathbf{P_N}$ be a morphism over h. Then $\{\alpha_n + \gamma_n\} : \mathbf{P_M} \longrightarrow \mathbf{P_N}$ is a morphism over $f + h$. By definition, we have

$$\begin{aligned} L_nT(f+h)[x] &= [T(\alpha_n + \gamma_n)(x)] = [(T(\alpha_n) + T(\gamma_n))(x)] \\ &= [T(\alpha_n)(x) + T(\gamma_n)(x)] = [T(\alpha_n)(x)] + [T(\gamma_n)(x)] \\ &= (L_nT)f[x] + (L_nT)h[x] = ((L_nT)f + (L_nT)h)[x] \end{aligned}$$

which implies that $L_nT(f+h) = L_nT(f) + L_nT(h)$. □

Definition 2.1.3. L_nT is called the nth **left derived functor** of T.

Definition 2.1.4. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $T, U : \mathcal{C} \longrightarrow \mathcal{D}$ be two covariant functors. We say that $\tau : T \longrightarrow U$ is a **natural transformation** (of functors) if for every object $M \in \mathcal{C}$ there is a morphism $\tau_M : T(M) \longrightarrow U(M)$ in $\mathcal{D}$ such that for every morphism $f : M \longrightarrow N$ in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} T(M) & \xrightarrow{\tau_M} & U(M) \\ {\scriptstyle T(f)}\downarrow & & \downarrow{\scriptstyle U(f)} \\ T(N) & \xrightarrow{\tau_N} & U(N) \end{array}$$

is commutative. There is a similar definition if both T and U are contravariant. If for each $M \in \mathcal{C}$, $\tau_M : T(M) \longrightarrow U(M)$ is an equivalence, then τ is called **naturallly equivalence**. Also then T and U are called **naturally equivalent** functors and we write $T \approx U$.

Assume that new choices

$$\ldots \longrightarrow \overline{P}_2 \longrightarrow \overline{P}_1 \longrightarrow \overline{P}_0 \longrightarrow M \longrightarrow 0$$

of projective resolutions (one for each module M) have been made, and denote the left derived functors arising from these new choices by $\overline{L}_nT$. our next project is to show that L_nT and $\overline{L}_nT$ are essentially the same.

Theorem 2.1.5. *Given an additive covariant functor $T : {}_R\mathit{Mod} \longrightarrow {}_S\mathit{Mod}$, where R and S are rings, then the functors L_nT and $\overline{L}_nT$ are naturally equivalent. In particular, for each M,*

$$(L_nT)M \cong (\overline{L}_nT)M.$$

i.e., there modules are independent of the choice of projective resolution of M.

Proof. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r] \arrow[d, dashed, "i_{n+1}"] & P_n \arrow[r] \arrow[d, dashed, "i_n"] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, dashed, "i_0"] & M \arrow[r] \arrow[d, "1_M"] & 0 \\
\cdots \arrow[r] & \overline{P}_{n+1} \arrow[r] & \overline{P}_n \arrow[r] & \cdots \arrow[r] & \overline{P}_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

where the top row is the chosen projective resolution of M used to define L_nT and the bottom is that used to define $\overline{L}_nT$. By the Comparison Theorem, there is a morphism $i : \mathbf{P_M} \longrightarrow \overline{\mathbf{P}}_\mathbf{M}$ over 1_M. Similarly, there is a morphism $j : \overline{\mathbf{P}}_\mathbf{M} \longrightarrow \mathbf{P_M}$ over 1_M. Therefore $ji : \mathbf{P_M} \longrightarrow \mathbf{P_M}$ and $ij : \overline{\mathbf{P}}_\mathbf{M} \longrightarrow \overline{\mathbf{P}}_\mathbf{M}$ are morphisms over 1_M. Since $1_{\mathbf{P_M}} : \mathbf{P_M} \longrightarrow \mathbf{P_M}$ and $1_{\overline{\mathbf{P}}_\mathbf{M}} : \overline{\mathbf{P}}_\mathbf{M} \longrightarrow \overline{\mathbf{P}}_\mathbf{M}$ are also morphisms over 1_M, we have $ji \simeq 1_{\mathbf{P_M}}$ and $ij \simeq 1_{\overline{\mathbf{P}}_\mathbf{M}}$. It follows that $T(j)T(i) = T(ji) \simeq 1_{\mathbf{TP_M}}$ and $T(i)T(j) = T(ij) \simeq 1_{\mathbf{T\overline{P}_M}}$. Since $H_n : {}_R\text{Comp} \longrightarrow {}_R\text{Mod}$ is an additive functor for every $n \geq 0$;

$$\begin{aligned}
1_{L_nT(M)} &= 1_{H_n(TP_M)} = H_n(1_{TP_M}) = H_n(T(ji)) = H_n(T(j))H_n(T(i)), \\
1_{\overline{L}_nT(M)} &= 1_{H_n(T\overline{P}_M)} = H_n(1_{T\overline{P}_M}) = H_n(T(ij)) = H_n(T(i))H_n(T(j)).
\end{aligned}$$

If we define

$$\tau_M = H_n(T(i)) : (L_nT)M \longrightarrow (\overline{L}_nT)M,$$

then τ_M is an isomorphism with $H_n(T(j))$ as its inverse.

We now prove that the isomorphisms τ_M constitute a natural isomorphism: if $f : M \longrightarrow N$, we must show commutativity of

$$\begin{tikzcd}
(L_nT)M \arrow[r, "\tau_M"] \arrow[d, "(L_nT)f"'] & (\overline{L}_nT)M \arrow[d, "(\overline{L}_nT)f"] \\
(L_nT)N \arrow[r, "\tau_N"] & (\overline{L}_nT)N
\end{tikzcd}$$

Consider the diagrams

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r] \arrow[d, dashed, "\alpha_{n+1}"] & P_n \arrow[r] \arrow[d, dashed, "\alpha_n"] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, dashed, "\alpha_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & Q_{n+1} \arrow[r] \arrow[d, dashed, "k_{n+1}"] & Q_n \arrow[r] \arrow[d, dashed, "k_{n+1}"] & \cdots \arrow[r] & Q_0 \arrow[r] \arrow[d, dashed, "k_0"] & N \arrow[r] \arrow[d, "1_N"] & 0 \\
\cdots \arrow[r] & \overline{Q}_{n+1} \arrow[r] & \overline{Q}_n \arrow[r] & \cdots \arrow[r] & \overline{Q}_0 \arrow[r] & N \arrow[r] & 0,
\end{tikzcd}$$

and

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r] \arrow[d, dashed, "i_{n+1}"] & P_n \arrow[r] \arrow[d, dashed, "i_n"] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, dashed, "i_0"] & M \arrow[r] \arrow[d, "1_M"] & 0 \\
\cdots \arrow[r] & \overline{P}_{n+1} \arrow[r] \arrow[d, dashed, "\overline{\alpha}_{n+1}"] & \overline{P}_n \arrow[r] \arrow[d, dashed, "\overline{\alpha}_n"] & \cdots \arrow[r] & \overline{P}_0 \arrow[r] \arrow[d, dashed, "\overline{\alpha}_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & \overline{Q}_{n+1} \arrow[r] & \overline{Q}_n \arrow[r] & \cdots \arrow[r] & \overline{Q}_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Applying the Comparison Theorem yields morphisms $\alpha : \mathbf{P_M} \longrightarrow \mathbf{P_N}$ and $\overline{\alpha} : \overline{\mathbf{P}}_\mathbf{M} \longrightarrow \overline{\mathbf{P}}_\mathbf{N}$ over the homomorphism f. Let $k : \mathbf{P_N} \longrightarrow \overline{\mathbf{P}}_\mathbf{N}$ be a morphism over 1_N. Then we have morphism $k\alpha : \mathbf{P_M} \longrightarrow \overline{\mathbf{P}}_\mathbf{N}$ over $1_N f = f$ and $\overline{\alpha}i$: $\mathbf{P_M} \longrightarrow \overline{\mathbf{P}}_\mathbf{N}$ over $f1_M = f$. Therefore $k\alpha \simeq \overline{\alpha}i$ and so $T(k\alpha) \simeq T(\overline{\alpha}i)$. Hence

$$H_nT(k)H_nT(\alpha) = H_nT(k\alpha) = H_nT(\overline{\alpha}i) = H_nT(\overline{\alpha})H_nT(i).$$

It follows that $\tau_N(L_nT)f = (\overline{L}_nT)f\tau_M$. This completes the proof. □

Theorem 2.1.6. *Let $0 \longrightarrow K \longrightarrow P \xrightarrow{\varepsilon} M \longrightarrow 0$ be an exact sequence of R-modules, where P is projective. Then if T is covariant*

$$(L_{n+1}T)M \cong (L_nT)K \quad (n \geq 0)$$

Proof. Let

$$\mathbf{P} : \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 = P \xrightarrow{\varepsilon} M \longrightarrow 0$$

be a projective resolution for M. By exactness of $\mathbf{P}$, we have $K = \ker\varepsilon = \mathrm{im} d_1$, and so

$$\cdots \longrightarrow P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} K \longrightarrow 0$$

is a projective resolution of K. Since the indices are no longer correct, relabel the indices, and define $Q_n = P_{n+1}$ $(n \geq 0)$, $\Delta_n = d_{n+1}$ $(n \geq 1)$. Therefore we have the following projective resolution for K.

$$\cdots \longrightarrow Q_2 \xrightarrow{\Delta_2} Q_1 \xrightarrow{\Delta_1} Q_0 \xrightarrow{d_1} K \longrightarrow 0.$$

By definition, we have

$$(L_{n+1}T)M \cong \ker T d_{n+1}/\mathrm{im} T d_{n+2} = \ker T\Delta_n/\mathrm{im} T\Delta_{n+1} \cong (L_n T)K.$$

This completes the proof. □

Corollary 2.1.7. *Let*

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

be a projective resolution of M, and define $K_0 = \ker\varepsilon$ and $K_n = \ker d_n$ for all $n \geq 1$. Then if T is covariant,

$$(L_{n+1}T)M \cong (L_n T)K_0 \cong (L_{n-1}T)K_1 \cong \ldots \cong (L_1 T)K_{n-1}.$$

Proof. Let $K_{-1} = M$. Consider the following short exact sequences

$$0 \longrightarrow K_i \longrightarrow P_i \longrightarrow K_{i-1} \longrightarrow 0 \qquad (i \geq 0)$$

In view of the above theorem, we have

$$(L_{n+1}T)K_{i-1} \cong (L_n T)K_i \quad (n \geq 0, i \geq 0)$$

This completes the proof. □

Theorem 2.1.8. *Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be an exact sequence of modules. If $T : {}_R Mod \longrightarrow {}_S Mod$ is an additive covariant functor, then there is a long exact sequence:*

$$\begin{aligned} \cdots \longrightarrow (L_n T)M' \xrightarrow{(L_n T)f} (L_n T)M \xrightarrow{(L_n T)g} (L_n T)M'' \xrightarrow{\partial_n} \cdots \\ \longrightarrow (L_0 T)M' \xrightarrow{(L_0 T)f} (L_0 T)M \xrightarrow{(L_0 T)g} (L_0 T)M'' \longrightarrow 0 \end{aligned}$$

Proof. Let $\mathbf{P_{M'}}$ and $\mathbf{P_{M''}}$ be the chosen deleted projective resolutions of M' and of M'', respectively. By the Horseshoe Lemma, there is a projective resolution $\overline{\mathbf{P}}_{\mathbf{M}}$ of M with

$$0 \longrightarrow \mathbf{P_{M'}} \xrightarrow{\alpha} \overline{\mathbf{P}}_{\mathbf{M}} \xrightarrow{\beta} \mathbf{P_{M''}} \longrightarrow 0.$$

Applying T gives another exact sequence of complexes (because, additive functors preserve split short exact sequences)

$$0 \longrightarrow T\mathbf{P_{M'}} \xrightarrow{T\alpha} T\overline{\mathbf{P}}_{\mathbf{M}} \xrightarrow{T\beta} T\mathbf{P_{M''}} \longrightarrow 0.$$

Thus there is a long exact sequence

$$\cdots \longrightarrow H_n(T\mathbf{P_{M'}}) \xrightarrow{H_n(T\alpha)} H_n(T\overline{\mathbf{P}}_{\mathbf{M}}) \xrightarrow{H_n(T\beta)} (H_n(T\mathbf{P_{M''}}) \xrightarrow{\partial_n} \cdots;$$

that is, there is an exact sequence

$$\cdots \longrightarrow (L_nT)M' \xrightarrow{(L_nT)f} (\overline{L}_nT)M \xrightarrow{(L_nT)g} (L_nT)M'' \xrightarrow{\partial_n} \cdots;$$

Notice that we have $(\overline{L}_nT)M$ instead of $(L_nT)M$ for the projective resolution of M constructed with the Horseshoe Lemma need not be the projective resolution originally chosen.

There are morphisms $i : \mathbf{P_M} \longrightarrow \overline{\mathbf{P}}_{\mathbf{M}}$ and $j : \overline{\mathbf{P}}_{\mathbf{M}} \longrightarrow \mathbf{P_M}$, where both i, j are morphisms over 1_M in opposite directions. In fact, $H_n(Ti) : H_n(T\mathbf{P_M}) \longrightarrow H_n(T\overline{\mathbf{P}}_{\mathbf{M}})$ is the inverse of $H_n(Tj) : H_n(T\overline{\mathbf{P}}_{\mathbf{M}}) \longrightarrow H_n(T\mathbf{P_M})$. Therefore, by Exercise 3, we have the following exact sequence

$$\cdots \longrightarrow H_n(T\mathbf{P_{M'}}) \xrightarrow{H_n(Tj)H_n(T\alpha)} H_n(T\mathbf{P_M}) \xrightarrow{H_n(T\beta)H_n(Ti)} (H_n(T\mathbf{P_{M''}}) \xrightarrow{\partial_n} \cdots;$$

Let $\delta : \mathbf{P_{M'}} \longrightarrow \mathbf{P_M}$ and $\epsilon : \mathbf{P_M} \longrightarrow \mathbf{P_{M''}}$ be morphisms over $f : M' \longrightarrow M$ and $g : M \longrightarrow M''$, respectively. Now $TjT\alpha \simeq T\delta$, because both are morphisms over Tf. Similarly, $T\beta Ti \simeq T\epsilon$. Then we have exact sequence

$$\cdots \longrightarrow H_n(T\mathbf{P_{M'}}) \xrightarrow{H_n(T\delta)} H_n(T\mathbf{P_M}) \xrightarrow{H_n(T\epsilon)} (H_n(T\mathbf{P_{M''}}) \xrightarrow{\partial_n} \cdots.$$

We conclude there is an exact sequence

$$\cdots \longrightarrow (L_nT)M' \xrightarrow{(L_nT)f} (L_nT)M \xrightarrow{(L_nT)g} (L_nT)M'' \xrightarrow{\partial_n} \cdots.$$

Finally, the sequence does terminate at 0, for $L_nT = 0$ for negative n. Indeed, every P_n, hence every TP_n, is 0 for negative n. □

Corollary 2.1.9. *Let $T : {}_R\mathit{Mod} \longrightarrow {}_S\mathit{Mod}$ be an additive covariant functor. Then L_0T is right exact.*

Proof. We have just seen that exactness of $M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ yields the exactness of $(L_0T)M' \longrightarrow (L_0T)M \longrightarrow (L_0T)M'' \longrightarrow 0$. □

Theorem 2.1.10. *Let $T : {}_R\mathit{Mod} \longrightarrow {}_S\mathit{Mod}$ be an additive covariant functor. Then $L_0T \approx T$ if and only if T is right exact.*

Proof. The "only if" parts comes from the right exactness of L_0T. For the converse, let

$$\mathbf{P} : \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

be the chosen projective resolution of M. But right exactness of T gives an exact sequence

$$TP_1 \xrightarrow{Td_1} TP_0 \xrightarrow{T\varepsilon} TM \longrightarrow 0.$$

This exact sequences induce isomorphism

$$\tau_M : TP_0/\mathrm{ker}T\varepsilon \longrightarrow TM.$$

By definition

$$(L_0T)M = \mathrm{ker}Td_0/\mathrm{im}Td_1 = TP_0/\mathrm{im}Td_1 = TP_0/\mathrm{ker}T\varepsilon \cong TM.$$

Let N be another R-module and $f : M \longrightarrow N$ be a homomorphism. Let

$$\mathbf{Q} : \cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \xrightarrow{\varepsilon'} N \longrightarrow 0$$

be a projective resolution of N. By the Comparison Theorem, there is a morphism $\alpha : \mathbf{P_M} \longrightarrow \mathbf{P_N}$ over f. We then get commutative diagram with exact rows.

$$\begin{tikzcd}
TP_1 \arrow[r] \arrow[d, "T\alpha_1"] & TP_0 \arrow[r, "T\varepsilon"] \arrow[d, "T\alpha_0"] & TM \arrow[r] \arrow[d, "Tf"] & 0 \\
TQ_1 \arrow[r] & TQ_0 \arrow[r, "T\varepsilon'"] & TN \arrow[r] & 0
\end{tikzcd}$$

This commutative diagram induces commutative diagram

$$\begin{tikzcd}
(L_0T)M \cong TP_0/\ker T\varepsilon \arrow[r, "\tau_M"] \arrow[d, "(L_0T)f"'] & TM \arrow[r] \arrow[d, "Tf"] & 0 \\
(L_0T)N \cong TQ_0/\ker T\varepsilon' \arrow[r, "\tau_N"] & TN \arrow[r] & 0
\end{tikzcd}$$

This completes the proof that L_0T is naturally equivalent to T. $\square$

Definition 2.1.11. Let M be an R-module and $a \in C(R)$. Then $a_\bullet : M \longrightarrow M$ defined by $x \longmapsto ax$ is an R-module homomorphism, called **multiplication by** a (or homothety). We say that a functor $T : {}_R\text{Mod} \longrightarrow {}_R\text{Mod}$ **preserves multiplications** if $T(a_\bullet) = a_\bullet$ for all $a \in C(R)$.

Theorem 2.1.12. *If $T : {}_R Mod \longrightarrow {}_R Mod$ is an additive covariant functor which preserves multiplications, then $L_nT : {}_R Mod \longrightarrow {}_R Mod$ also preserves multiplications.*

Proof. Let

$$\mathbf{P} : \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

be a projective resolution of M. Let $a \in C(R)$ and consider the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "a_\bullet"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "a_\bullet"] & M \arrow[r] \arrow[d, "a_\bullet"] & 0 \\
\cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r, "\varepsilon"] & M \arrow[r] & 0
\end{tikzcd}$$

Applying T gives

$$\begin{tikzcd}
\cdots \arrow[r] & TP_1 \arrow[r] \arrow[d, "a_\bullet"] & TP_0 \arrow[r, "T\varepsilon"] \arrow[d, "a_\bullet"] & TM \arrow[r] \arrow[d, "a_\bullet"] & 0 \\
\cdots \arrow[r] & TP_1 \arrow[r] & TP_0 \arrow[r, "T\varepsilon"] & TM \arrow[r] & 0
\end{tikzcd}$$

Now we have

$$\begin{aligned}(L_nT)(a_\bullet) : H_n(T\mathbf{P_M}) &\longrightarrow H_n(T\mathbf{P_M}) \\ [z_n] &\longmapsto [az_n] = a[z_n].\end{aligned}$$

That is $(L_nT)(a_\bullet) = a_\bullet$. □

As you would expect, the case for contravariant functors is done similarly and the process produces contravariant left derived functors.

2.2 Right Derived Functors

We are now going to define right derived functors R^nT, where $T : {}_R\text{Mod} \longrightarrow {}_S\text{Mod}$ is an additive covariant (contravariant) functor.

Definition 2.2.1. Let S be another ring and $T : {}_R\text{Mod} \longrightarrow {}_S\text{Mod}$ be a covariant functor. For $n \in \mathbb{Z}$, define

$$(R^nT)M = H^n(T\mathbf{E_M}) = \ker Td^n / \operatorname{im} Td^{n-1}.$$

To complete the definition of R^nT, we must describe its action on homomorphism $f : M \longrightarrow N$. By the dual of the Comparison Theorem, there is a morphism $\alpha : \mathbf{E_M} \longrightarrow \mathbf{E_N}$ over f. Then $T\alpha : T\mathbf{E_M} \longrightarrow T\mathbf{E_N}$ is also a morphism, and we define

$$\begin{aligned}(R^nT)f : (R^nT)M &\longrightarrow (R^nT)N \\ [z] &\longmapsto [(T\alpha_n)z].\end{aligned}$$

Definition 2.2.2. Let S be another ring and $T : {}_R\text{Mod} \longrightarrow {}_S\text{Mod}$ be a contravariant functor. For $n \in \mathbb{Z}$, define

$$(R^nT)M = H^n(T\mathbf{P_M}) = \ker Td_{n+1} / \operatorname{im} Td_n.$$

To complete the definition of R^nT, we must describe its action on homomorphism $f : M \longrightarrow N$. By the Comparison Theorem, there is a morphism

$\alpha : \mathbf{P_M} \longrightarrow \mathbf{P_N}$ over f. Then $T\alpha : T\mathbf{P_N} \longrightarrow T\mathbf{P_M}$ is also a morphism, and we define

$$\begin{aligned}(R^nT)f : (R^nT)N &\longrightarrow (R^nT)M\\ [z] &\longmapsto [(T\alpha_n)z].\end{aligned}$$

Let $T : {}_R\text{Mod} \longrightarrow {}_S\text{Mod}$ be an additive covariant (contravariant) functor. Then the proof of the following results are dual (similar) to the proof of results in previous section.

Theorem 2.2.3. *Let $T : {}_RMod \longrightarrow {}_SMod$ be an additive covariant (contravariant) functor, where R and S are rings, then*

$$R^nT : {}_RMod \longrightarrow {}_SMod$$

is an additive covariant (contravariant) functor for every $n \in \mathbb{Z}$.

Definition 2.2.4. Let $T : {}_R\text{Mod} \longrightarrow {}_S\text{Mod}$ be an additive covariant (contravariant) functor, where R and S are rings. Then R^nT is called the nth **right derived functor** of T.

Theorem 2.2.5. *If $T : {}_RMod \longrightarrow {}_SMod$ is an additive covariant (contravariant) functor which preserves multiplications, then $R^nT : {}_RMod \longrightarrow {}_SMod$ also preserves multiplications.*

Theorem 2.2.6. *Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be an exact sequence of modules.*

(1) If $T : {}_RMod \longrightarrow {}_SMod$ is an additive covariant functor, then there is a long exact sequence:

$$\begin{aligned}0 \longrightarrow (R^0T)M' \xrightarrow{(R^0T)f} (R^0T)M \xrightarrow{(R^0T)g} (R^0T)M'' \longrightarrow \cdots\\ \cdots \longrightarrow (R^nT)M' \xrightarrow{(R^nT)f} (R^nT)M \xrightarrow{(R^nT)g} (R^nT)M'' \longrightarrow \cdots\end{aligned}$$

(2) If $T : {}_RMod \longrightarrow {}_SMod$ is an additive contravariant functor, then there

is a long exact sequence:

$$0 \longrightarrow (R^0T)M'' \overset{(R^0T)g}{\longrightarrow} (R^0T)M \overset{(R^0T)f}{\longrightarrow} (R^0T)M' \longrightarrow \cdots$$
$$\cdots \longrightarrow (R^nT)M'' \overset{(R^nT)g}{\longrightarrow} (R^nT)M \overset{(R^nT)f}{\longrightarrow} (R^nT)M' \longrightarrow \cdots$$

Corollary 2.2.7. *If $T : {}_RMod \longrightarrow {}_SMod$ is an additive covariant (contravariant) functor, then the functor R^0T is left exact.*

Theorem 2.2.8. *Let $T : {}_RMod \longrightarrow {}_SMod$ be an additive covariant (contravariant) functor. Then $R^0T \approx T$ if and only if T is left exact.*

Theorem 2.2.9. *(1) Let*

$$0 \longrightarrow M \overset{\varepsilon}{\longrightarrow} E \longrightarrow V \longrightarrow 0$$

be an exact sequence of R-modules, where E is injective. Then if T is covariant,

$$(R^{n+1}T)M \cong (R^nT)V \quad (n \geq 0)$$

(2) Let

$$0 \longrightarrow K \longrightarrow P \overset{\varepsilon}{\longrightarrow} M \longrightarrow 0$$

be an exact sequence of R-modules, where P is projective. Then if T is contravariant,

$$(R^{n+1}T)M \cong (R^nT)K.$$

Corollary 2.2.10. *(1) Let*

$$0 \longrightarrow M \overset{\varepsilon}{\longrightarrow} E^0 \overset{d^0}{\longrightarrow} E^1 \overset{d^1}{\longrightarrow} E^2 \longrightarrow \dots$$

be an injective resolution of M, and define $V_0 = im\varepsilon$ and $V_n = imd^{n-1}$ for all $n \geq 1$. Then if T is covariant,

$$(R^{n+1}T)M \cong (R^nT)V_0 \cong (R^{n-1}T)V_1 \cong \dots \cong (R^1T)V_{n-1}.$$

(2) Let

$$\cdots \longrightarrow P_2 \overset{d_2}{\longrightarrow} P_1 \overset{d_1}{\longrightarrow} P_0 \overset{\varepsilon}{\longrightarrow} M \longrightarrow 0$$

be a projective resolution of M, and define $K_0 = \ker\varepsilon$ and $K_n = \ker d_n$ for all $n \geq 1$. Then if T is contravariant,

$$(R^{n+1}T)M \cong (R^nT)K_0 \cong (R^{n-1}T)K_1 \cong \ldots \cong (R^1T)K_{n-1}.$$

Exercises

1. Let $f, g : \mathbf{X} \longrightarrow \mathbf{Y}$ be morphisms, and let $T : {}_R\text{Comp} \longrightarrow {}_R\text{Comp}$ be an additive functor. If $f \simeq g$, prove that $Tf \simeq Tg$.

2. Let $T : {}_R\text{Mod} \longrightarrow {}_S\text{Mod}$ be a covariant functor. Show that the following are equivalent:

   (1) T is additive,

   (2) $T(M \oplus N) \cong T(M) \oplus T(N)$ for all $M, N \in {}_R\text{Mod}$,

   (3) $T(M \oplus M) \cong T(M) \oplus T(M)$ for all $M \in {}_R\text{Mod}$.

3. Consider the exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C.$$

   If $i : B \longrightarrow B'$ is an isomorphism with inverse $j : B' \longrightarrow B$, prove exactness of

$$A \xrightarrow{if} B' \xrightarrow{gj} C.$$

4. Let $T : {}_R\text{Mod} \longrightarrow {}_R\text{Mod}$ be an additive functor and $n \geq 1$.

   (1) If T is covariant, prove that $(L_nT)P = 0$ for all projective $P \in {}_R\text{Mod}$,

   (2) If T is covariant, prove that $(R^nT)E = 0$ for all injective $E \in {}_R\text{Mod}$,

   (3) If T is contravariant, prove that $(R^nT)P = 0$ for all projective $P \in {}_R\text{Mod}$.

5. Let $T : {}_R\text{Mod} \longrightarrow {}_S\text{Mod}$ be a covariant functor.

   (1) Show that $L_n(L_mT) = 0$ if $m > 0$,

   (2) Show that $L_n(L_0T)M \cong (L_nT)M$ for all $M \in {}_R\text{Mod}$.

6. Set $R = \mathbb{Z}_4$, $S = \mathbb{Z}$ and

$$\begin{aligned} T : {}_R\mathrm{Mod} &\longrightarrow {}_S\mathrm{Mod} \\ M &\longmapsto \mathrm{Hom}(\mathbb{Z}_2, M). \end{aligned}$$

   Write down a projective resolution of $\mathbb{Z}_2$ and compute $(L_nT)\mathbb{Z}_2$.

Chapter 3

Tor AND Ext

3.1 Elementary Properties

Definition 3.1.1. Let M be a right R-module and N be a left R-module. Then

(1) If $T(-) = M \otimes_R -$, then $\mathrm{Tor}_n^R(M,-) := L_n T(-)$.

(2) If $T(-) = - \otimes_R N$, then $\mathrm{tor}_n^R(-,N) := L_n T(-)$.

(3) If $T(-) = \mathrm{Hom}(N,-)$, then $\mathrm{Ext}_R^n(N,-) := R^n T(-)$.

(4) If $T(-) = \mathrm{Hom}(-,N)$, then $\mathrm{ext}_R^n(-,N) := R^n T(-)$.

Proposition 3.1.2. *Let M be a right R-module and N be a left R-module. Then the following hold.*

(1) $Tor_0^R(M,-) \approx M \otimes_R -$.

(2) $tor_0^R(-,N) \approx - \otimes_R N$.

(3) $Ext_R^0(N,-) \approx Hom(N,-)$.

(4) $ext_R^0(-,N) \approx Hom(-,N)$.

Proof. Follows from Theorem 2.1.10 and Theorem 2.2.8. □

Proposition 3.1.3. *(1) Let M and P are right R-modules with P projective, and let N and Q are left R-modules with Q projective. Then*

$$Tor_n^R(M,Q) = tor_n^R(P,N) = 0.$$

(2) Let N, P and E are left R-modules with P projective and E injective. Then

$$Ext_R^n(N,E) = ext_R^n(P,N) = 0.$$

Proof. Follows from Exercise 4 of Chapter 2. □

Proposition 3.1.4. *The following hold.*

(1) Let $\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} N \longrightarrow 0$ be a projective resolution of a left R-module N, and define $K_0 = \ker\varepsilon$ and $K_n = \ker d_n$ for all $n \geq 1$. If M is a right R-module, then

$$Tor_{n+1}^R(M,N) \cong Tor_n^R(M,K_0) \cong \ldots \cong Tor_1^R(M,K_{n-1}).$$

(2) Let $\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$ be a projective resolution of a right R-module M, and define $K_0 = \ker\varepsilon$ and $K_n = \ker d_n$ for all $n \geq 1$. If N is a left R-module, then

$$tor_{n+1}^R(M,N) \cong tor_n^R(K_0,N) \cong \ldots \cong tor_1^R(K_{n-1},N).$$

(3) Let $0 \longrightarrow M \xrightarrow{\varepsilon} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \longrightarrow \ldots$ be an injective resolution of a left R-module N, and define $V_0 = im\varepsilon$ and $V_n = imd^{n-1}$ for all $n \geq 1$. If M is a left R-module, then

$$Ext_R^{n+1}(M,N) \cong Ext_R^n(M,V_0) \cong \ldots \cong Ext_R^1(M,V_{n-1}).$$

(4) Let $\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$ be a projective resolution of a left R-module M, and define $K_0 = \ker\varepsilon$ and $K_n = \ker d_n$ for all $n \geq 1$. If N is a left R-module, then

$$ext_R^{n+1}(M,N) \cong ext_R^n(K_0,N) \cong \ldots \cong ext_R^1(K_{n-1},N).$$

Proof. Follows from Corollary 2.1.7 and Corollary 2.2.10. □

Proposition 3.1.5. *Let $0 \longrightarrow K' \longrightarrow K \longrightarrow K'' \longrightarrow 0$ be an exact sequence of modules. Then there are the long exact sequences*

$$\begin{aligned} (1) \quad \cdots \longrightarrow Tor_n^R(M,K') \longrightarrow Tor_n^R(M,K) \longrightarrow Tor_n^R(M,K'') \longrightarrow \cdots \\ \longrightarrow M \otimes_R K' \longrightarrow M \otimes_R K \longrightarrow M \otimes_R K'' \longrightarrow 0 \end{aligned}$$

$$(2)\quad \cdots \longrightarrow tor_n^R(K',N) \longrightarrow tor_n^R(K,N) \longrightarrow tor_n^R(K'',N) \longrightarrow \cdots$$
$$\longrightarrow K' \otimes_R N \longrightarrow K \otimes_R N \longrightarrow K'' \otimes_R N \longrightarrow 0$$

$$(3)\quad 0 \longrightarrow Hom(N,K') \longrightarrow Hom(N,K) \longrightarrow Hom(N,K'') \longrightarrow \cdots$$
$$\cdots \longrightarrow Ext_R^n(N,K') \longrightarrow Ext_R^n(N,K) \longrightarrow Ext_R^n(N,K'') \longrightarrow \cdots$$

$$(4)\quad 0 \longrightarrow Hom(K'',N) \longrightarrow Hom(K,N) \longrightarrow Hom(K',N) \longrightarrow \cdots$$
$$\cdots \longrightarrow ext_R^n(K'',N) \longrightarrow ext_R^n(K,N) \longrightarrow ext_R^n(K',N) \longrightarrow \cdots$$

Proof. Follows from Theorem 2.1.8 and Theorem 2.2.6. □

Theorem 3.1.6. *(1) Let M be a right R-module, let N be a left R-module. Then $Tor_n^R(M,N) \cong tor_n^R(M,N)$ for all $n \geq 0$.*

(2) Let M and N be left R-modules. Then $Ext_R^n(M,N) \cong ext_R^n(M,N)$ for all $n \geq 0$.

Proof. We only prove (1); the proof of the dual (2) is similar.

(1): The proof is by induction on $n \geq 0$. By Theorem 2.1.10, $\mathrm{Tor}_0^R(M,-) \approx M \otimes_R -$ and $\mathrm{tor}_0^R(-,N) \approx - \otimes_R N$. Therefore

$$\mathrm{Tor}_0^R(M,N) \cong M \otimes_R N \cong \mathrm{tor}_0^R(M,N).$$

We now suppose that $n \geq 1$. Let

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

be a projective resolution of M and

$$\cdots \longrightarrow Q_2 \xrightarrow{d_2'} Q_1 \xrightarrow{d_1'} Q_0 \xrightarrow{\varepsilon'} N \longrightarrow 0$$

be a projective resolution of N. Set

$$K_{-1} = M,\ K_0 = \mathrm{ker}\varepsilon,\ K_i = \mathrm{ker}d_i\ \ (i \geq 1),$$
$$H_{-1} = N,\ H_0 = \mathrm{ker}\varepsilon',\ H_i = \mathrm{ker}d_i'\ \ (i \geq 1).$$

Since tensor is a bifunctor the exact sequences

$$\begin{aligned} 0 \longrightarrow K_i \longrightarrow P_i \xrightarrow{d_i} K_{i-1} \longrightarrow 0, \\ 0 \longrightarrow H_j \longrightarrow Q_j \xrightarrow{d'_j} H_{j-1} \longrightarrow 0. \end{aligned}$$

give a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & 0 \arrow[d] & \\
 & & X \arrow[d] & 0 \arrow[d] & W \arrow[d] & \\
0 \arrow[r] & Y \arrow[r] & K_i \otimes H_j \arrow[r] \arrow[d] & K_i \otimes Q_j \arrow[r] \arrow[d] & K_i \otimes H_{j-1} \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & P_i \otimes H_j \arrow[r] \arrow[d] & P_i \otimes Q_j \arrow[r] \arrow[d] & P_i \otimes H_{j-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z \arrow[r] & K_{i-1} \otimes H_j \arrow[r] \arrow[d] & K_{i-1} \otimes Q_j \arrow[r] \arrow[d] & K_{i-1} \otimes H_{j-1} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

where $X = \mathrm{tor}_1^R(K_{i-1}, H_j)$, $Y = \mathrm{Tor}_1^R(K_i, H_{j-1})$, $W = \mathrm{tor}_1^R(K_{i-1}, H_{j-1})$, and $Z = \mathrm{Tor}_1^R(K_{i-1}, H_{j-1})$. By Exercise 1, we conclude, for all $i, j \geq 0$,

$$\begin{aligned} \mathrm{tor}_1^R(K_{i-1}, H_j) &\cong \mathrm{Tor}_1^R(K_i, H_{j-1}), \\ \mathrm{tor}_1^R(K_{i-1}, H_{j-1}) &\cong \mathrm{Tor}_1^R(K_{i-1}, H_{j-1}). \end{aligned}$$

Therefore the theorem has been proved for $n = 1$. By Theorem 2.1.6, we have

$$\begin{aligned} \mathrm{Tor}_{n+1}^R(M, N) \cong \mathrm{Tor}_n^R(M, H_0) \cong \ldots \cong \mathrm{Tor}_1^R(M, H_{n-1}) = \mathrm{Tor}_1^R(K_{-1}, H_{n-1}), \\ \mathrm{tor}_{n+1}^R(M, N) \cong \mathrm{tor}_n^R(K_0, N) \cong \ldots \cong \mathrm{tor}_1^R(K_{n-1}, N) = \mathrm{tor}_1^R(K_{n-1}, H_{-1}). \end{aligned}$$

Then

$$
\begin{aligned}
\mathrm{Tor}_{n+1}^R(M,N) &\cong \mathrm{Tor}_1^R(K_{-1},H_{n-1}),\\
\mathrm{tor}_1^R(K_{-1},H_{n-1}) &\cong \mathrm{Tor}_1^R(K_0,H_{n-2}),\\
&\vdots\\
\mathrm{tor}_1^R(K_{n-2},H_0) &\cong \mathrm{Tor}_1^R(K_{n-1},H_{-1}),\\
\mathrm{tor}_1^R(K_{n-1},H_{-1}) &\cong \mathrm{tor}_{n+1}^R(M,N).
\end{aligned}
$$

This completes the proof. □

Remark 3.1.7. In view of the above theorem, we have

(1)

$$\boxed{\mathrm{Tor}_n^R(M,N) \cong H_n(\mathbf{P_M} \otimes_R N) \cong H_n(M \otimes_R \mathbf{P_N}),}$$

where $\mathbf{P_M}$ is a deleted projective resolution of a right R-module M and $\mathbf{P_N}$ is a deleted projective resolution of a left R-module N.

(2)

$$\boxed{\mathrm{Ext}_R^n(M,N) \cong H^n(\mathrm{Hom}(\mathbf{P_M},N)) \cong H^n(\mathrm{Hom}(M,\mathbf{E_N})),}$$

where $\mathbf{P_M}$ is a deleted projective resolution of a left R-module M and $\mathbf{E_N}$ is a deleted injective resolution of a left R-module N.

Theorem 3.1.8. *Let R be a commutative ring and M, N be R-modules. Then*

(1) $Tor_n^R(M,N)$ is an R-module,

(2) $Ext_R^n(M,N)$ is an R-module.

Proof. We only prove (1); the proof of the dual (2) is similar.

(1): Since $\mathrm{Tor}_0^R(M,N) \cong M \otimes_R N$ is an R-module, we may assume that $n \geq 1$. Let $\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$ be a projective resolution of M. For any $n \geq 1$, $P_n \otimes_R N$ is an R-module. Also for $n \geq 1$, $x \in P_n$, $y \in N$ and $a \in R$,

$$
\begin{aligned}
(d_n \otimes 1)(a(x \otimes y)) &= (d_n \otimes 1)((ax) \otimes y) = d_n(ax) \otimes y = ad_n(x) \otimes y\\
&= a(d_n(x) \otimes y) = a(d_n \otimes 1)(x \otimes y)
\end{aligned}
$$

which proves that $d_n \otimes 1$ is an R-homomorphism. Therefore $\ker(d_n \otimes 1)$ and $\operatorname{im}(d_{n+1} \otimes 1)$ are R-submodule of $P_n \otimes_R N$ and hence

$$\operatorname{Tor}_n^R(M, N) \cong \ker(d_n \otimes 1)/\operatorname{im}(d_{n+1} \otimes 1)$$

is an R-module. $\square$

Theorem 3.1.9. *(1) If R is a ring, M is a right R-module, and N is a left R-module, then*

$$Tor_n^R(M, N) \cong Tor_n^{R^{op}}(N, M)$$

for all $n \geq 0$, where R^{op} is the opposite ring of R.

(2) If R is a commutative ring and M and N are R-modules, then for all $n \geq 0$,

$$Tor_n^R(M, N) \cong Tor_n^R(N, M).$$

Proof. (1): Choose a deleted projective resolution $\mathbf{P_M}$ of the right R-module M. Then $\mathbf{P_M}$ is also a deleted projective resolution of the left R^{op}-module M. Now the morphism $\alpha : \mathbf{P_M} \otimes_R N \longrightarrow N \otimes_{R^{op}} \mathbf{P_M}$ given by

$$\begin{aligned} \alpha_n : P_n \otimes_R N &\longrightarrow N \otimes_{R^{op}} P_n \\ x_n \otimes b &\longmapsto b \otimes x_n \end{aligned}$$

is an isomorphism of compelexes, because each α_n is an isomorphism of abelian groups (its inverse is $b \otimes x_n \longmapsto x_n \otimes b$). Since isomorphic complexes have the same homology,

$$H_n(\mathbf{P_M} \otimes_R N) \cong H_n(N \otimes_{R^{op}} \mathbf{P_M}).$$

Hence $\operatorname{Tor}_n^R(M, N) \cong \operatorname{Tor}_n^{R^{op}}(N, M)$ for all $n \geq 0$.

(2): This is obvious from part (1). $\square$

Theorem 3.1.10. *Let R be a commutative ring and N an R-module, and $\{M_i\}$ a family of R-modules. Then*

(1) $Tor_n^R(\coprod_i M_i, N) \cong \coprod_i Tor_n^R(M_i, N)$,

(2) $Ext_R^n(\coprod_i M_i, N) \cong \prod_i Ext_R^n(M_i, N)$,

(3) $Ext_R^n(N, \prod_i M_i) \cong \prod_i Ext_R^n(N, M_i)$,

Proof. We shall prove (1); the proofs of (2) and (3) are similar.

(1): We use induction on n. The case $n = 0$ is Corollary 3.1.2. For each i, construct an exact sequence

$$0 \longrightarrow K_i \longrightarrow P_i \longrightarrow M_i \longrightarrow 0,$$

where P_i is projective. There is an exact sequence

$$0 \longrightarrow \coprod_i K_i \longrightarrow \coprod_i P_i \longrightarrow \coprod_i M_i \longrightarrow 0,$$

in which $\coprod_i P_i$ being direct sum of projective modules is projective. There is a commutative diagram with exact rows:

$$\begin{tikzcd}
0 = \mathrm{Tor}_1^R(\coprod_i P_i, N) \arrow[r] & \mathrm{Tor}_1^R(\coprod_i M_i, N) \arrow[r] \arrow[d, dashed, "h"] & (\coprod_i K_i) \otimes N \arrow[r] \arrow[d, "g"] & (\coprod_i P_i) \otimes N \arrow[d, "f"] \\
0 = \coprod_i \mathrm{Tor}_1^R(P_i, N) \arrow[r] & \coprod_i \mathrm{Tor}_1^R(M_i, N) \arrow[r] & \coprod_i (K_i \otimes N) \arrow[r] & \coprod_i (P_i \otimes N)
\end{tikzcd}$$

Where the vertical arrows are the isomorphisms and the maps in the bottom row are the maps of Corollary 3.1.5 at each coordinate. Now $\mathrm{Tor}_1^R(\coprod_i P_i, N) = 0 = \coprod_i \mathrm{Tor}_1^R(P_i, N)$, because $\coprod_i P_i$ and each P_i are projective; and so by Exercise 4, there exists an isomorphism $\mathrm{Tor}_1^R(\coprod_i M_i, N) \longrightarrow \coprod_i \mathrm{Tor}_1^R(M_i, N)$ making the augmented diagram commute. Thus the theorem is true for $n = 1$. Suppose that $n > 1$ and that $\mathrm{Tor}_n^R(\coprod_i L_i, N) \cong \coprod_i \mathrm{Tor}_n^R(L_i, N)$ for every family of R-modules $\{L_i\}$. Then by Corollary 3.1.4, we have

$$\mathrm{Tor}_{n+1}^R(\coprod_i M_i, N) \cong \mathrm{Tor}_n^R(\coprod_i K_i, N) \cong \coprod_i \mathrm{Tor}_n^R(K_i, N) \cong \coprod_i \mathrm{Tor}_{n+1}^R(M_i, N).$$

This completes induction. $\square$

Theorem 3.1.11. *Let N be a left R-module, and (M_i, f_{ji}) be a direct system of right R-modules. Then*

$$Tor_n^R(\varinjlim M_i, N) \cong \varinjlim Tor_n^R(M_i, N).$$

Proof. We use induction on n. The case $n = 0$ is Corollary 3.1.2. For each i, construct an exact sequence

$$0 \longrightarrow K_i \longrightarrow P_i \longrightarrow M_i \longrightarrow 0,$$

where P_i is projective. There is an exact sequence

$$0 \longrightarrow \varinjlim K_i \longrightarrow \varinjlim P_i \longrightarrow \varinjlim M_i \longrightarrow 0,$$

Now $\varinjlim P_i$ is flat, for every projective module is flat, and a direct limit of flat modules is flat. Therefore Exercise 2 implies that

$$\mathrm{Tor}_1^R(\varinjlim P_i, N) = 0 = \varinjlim \mathrm{Tor}_1^R(P_i, N).$$

So, there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 = \mathrm{Tor}_1^R(\varinjlim P_i, N) \arrow[r] & \mathrm{Tor}_1^R(\varinjlim M_i, N) \arrow[r] \arrow[d, dashed, "h"] & (\varinjlim K_i) \otimes N \arrow[r] \arrow[d, "g"] & (\varinjlim P_i) \otimes N \arrow[d, "f"] \\
0 = \varinjlim \mathrm{Tor}_1^R(P_i, N) \arrow[r] & \varinjlim \mathrm{Tor}_1^R(M_i, N) \arrow[r] & \varinjlim (K_i \otimes N) \arrow[r] & \varinjlim (P_i \otimes N)
\end{tikzcd}$$

where the vertical arrows are the isomorphisms and the maps in the bottom row are the maps of Corollary 3.1.5 at each coordinate. By Exercise 4, there exists an isomorphism $\mathrm{Tor}_1^R(\varinjlim M_i, N) \xrightarrow{h} \varinjlim \mathrm{Tor}_1^R(M_i, N)$ making the augmented diagram commute. Thus the theorem is true for $n = 1$. Suppose that $n > 1$ and that $\mathrm{Tor}_n^R(\varinjlim L_i, N) \cong \varinjlim \mathrm{Tor}_n^R(L_i, N)$ for every family of R-modules $\{L_i\}$. Then by Corollary 3.1.4, we have

$$\mathrm{Tor}_{n+1}^R(\varinjlim M_i, N) \cong \mathrm{Tor}_n^R(\varinjlim K_i, N) \cong \varinjlim \mathrm{Tor}_n^R(K_i, N) \cong \varinjlim \mathrm{Tor}_{n+1}^R(M_i, N).$$

This completes induction. $\square$

3.2 Natural Isomorphisms

Various natural isomorphisms involving tensor and Hom can be extended to isomorphisms involving Tor and Ext.

Theorem 3.2.1. *Let R and S be commutative rings, and let $\varphi : R \longrightarrow S$ be a homomorphism. Let M be a finitely generated free R-module. If N is an R-module, then*

$$Hom_R(M, N) \otimes_R S \cong Hom_S(M \otimes_R S, N \otimes_R S).$$

Proof. Exercise. □

Theorem 3.2.2. *Let R and S be commutative rings.*

(1) **(Adjoint Isomorphism)** *Consider the situation $(L_R, {}_RM_S, N_S)$. Then there is a natural isomorphism*

$$\varphi : Hom_S(M \otimes_R L, N) \longrightarrow Hom_R(L, Hom_S(M, N)),$$

defined for each $f : M \otimes_R L \longrightarrow N$ by $(\varphi(f)l)(m) = f(m \otimes l)$.

(2) Consider the situation $({}_RL, {}_RM_S, N_S)$. If L is a finitely generated free R-module, then there is a natural isomorphism

$$\varphi : Hom_S(M, N) \otimes_R L \longrightarrow Hom_S(Hom_R(L, M), N),$$

defined by $\varphi(f \otimes l)(g) = f(g(l))$.

(3) **(Associativity)** *Consider the situation $(L_R, {}_RM_S, {}_SN)$. Then there is a natural isomorphism*

$$L \otimes_R (M \otimes_S N) \longrightarrow (L \otimes_R M) \otimes_S N,$$

defined by $l \otimes (m \otimes n) \longrightarrow (l \otimes m) \otimes n$.

(4) Consider the situation $({}_RL, {}_RM_S, N_S)$. If L is a finitely generated free R-module, then there is a natural isomorphism

$$\varphi : Hom_R(L, M) \otimes_S N \longrightarrow Hom_R(L, M \otimes_S N),$$

defined by $\varphi(f \otimes n)(l) = f(l) \otimes n$.

Proof. Exercise. □

Theorem 3.2.3. *Let R and S be commutative rings, and let $\varphi : R \longrightarrow S$ be a flat homomorphism. If M is an R-module and N is an S-module, then*

$$Tor_n^R(M,N) \cong Tor_n^S(M \otimes_R S, N).$$

Proof. Since $\mathbf{P_M} \otimes_R S$ is a deleted projective resolution for $M \otimes_R S$, Theorem 3.2.2(3), implies that

$$\begin{aligned} \mathrm{Tor}_n^R(M,N) &\cong H_n(\mathbf{P_M} \otimes_R S) \cong H_n(\mathbf{P_M} \otimes_R (S \otimes_S N)) \\ &\cong H_n((\mathbf{P_M} \otimes_R S) \otimes_S N) \cong \mathrm{Tor}_n^S(M \otimes_R S, N). \end{aligned}$$

This completes the proof. □

Theorem 3.2.4. *Let R and S be commutative rings, and let $\varphi : R \longrightarrow S$ be a flat homomorphism. If M and N are R-modules, then*

$$S \otimes_R Tor_n^R(M,N) \cong Tor_n^S(M \otimes_R S, N \otimes_R S).$$

Proof. By the above theorem, we have

$$\begin{aligned} S \otimes_R \mathrm{Tor}_n^R(M,N) &\cong S \otimes_R H_n(\mathbf{P_M} \otimes_R N) \cong H_n((\mathbf{P_M} \otimes_R N) \otimes_R S) \\ &\cong H_n(\mathbf{P_M} \otimes_R (N \otimes_R S)) \cong \mathrm{Tor}_n^R(M, N \otimes_R S) \\ &\cong \mathrm{Tor}_n^S(M \otimes_R S, N \otimes_R S) \end{aligned}$$

This completes the proof. □

Theorem 3.2.5. *Let R be a Noetherian ring and S be commutative rings and let let $\varphi : R \longrightarrow S$ be a flat homomorphism. If M is a finitely generated R-module and N is a R-module, then*

$$S \otimes_R Ext_R^n(M,N) \cong Ext_S^n(S \otimes_R M, S \otimes_R N).$$

Proof. By Theorem 3.2.2, we have

$$\begin{aligned} S \otimes_R \mathrm{Ext}_R^n(M,N) &\cong S \otimes_R H^n(\mathrm{Hom}(\mathbf{P_M}, N)) \cong H^n(S \otimes_R (\mathrm{Hom}_R(\mathbf{P_M}, N))) \\ &\cong H^n(\mathrm{Hom}_S(S \otimes_R \mathbf{P_M}, S \otimes_R N)) \\ &\cong \mathrm{Ext}_S^n(S \otimes_R M, S \otimes_R N). \end{aligned}$$

This completes the proof. □

Theorem 3.2.6. *Let R and S be commutative rings.*

(1): Consider the situation $(L_R, {}_RM_S, E_S)$. If E is injective, then

$$Ext^n_R(L, Hom_S(M, E)) \cong Hom_S(Tor^R_n(L, M), E).$$

(2): Let R be a Noetherian ring. Consider the situation $(L_R, {}_RM_S, E_S)$. If L is finitely generated and E is injective, then

$$Hom_S(Ext^n_R(L, M), E) \cong Tor^R_n(Hom_S(M, E), L).$$

(3): Consider the situation $(L_R, {}_RM_S, F_S)$. If F is flat, then

$$Tor^R_n(L, M \otimes_S F) \cong Tor^R_n(L, M) \otimes_S F.$$

(4): Let R be a Noetherian ring. Consider the situation $(L_R, {}_RM_S, F_S)$. If L is finitely generated and F is flat, then

$$Ext^n_R(L, M) \otimes_S F \cong Ext^n_R(L, M \otimes_S F).$$

Proof. (1): It follows from Theorem 3.2.2(1) that

$$\begin{aligned}
\operatorname{Ext}^n_R(L, \operatorname{Hom}_S(M, E)) &\cong H^n\operatorname{Hom}_R(\mathbf{P_L}, \operatorname{Hom}_S(M, E)) \\
&\cong H^n\operatorname{Hom}_S(\mathbf{P_L} \otimes_R M, E) \\
&\cong \operatorname{Hom}_S(H_n(\mathbf{P_L} \otimes_R M), E) \\
&\cong \operatorname{Hom}_S(\operatorname{Tor}^R_n(L, M), E).
\end{aligned}$$

(2): Since L is finitely generated and R is Noetherian, there exists a free resolution

$$\mathbf{F}: \quad \ldots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \ldots \longrightarrow F_0 \longrightarrow L \longrightarrow 0$$

in which every F_n is a finitely generated free R-module. It follows from Theorem 3.2.2(2) that

$$\begin{aligned}
\operatorname{Hom}_S(\operatorname{Ext}^n_R(L, M), E)) &\cong \operatorname{Hom}_S(H^n\operatorname{Hom}_R(\mathbf{F_L}, M), E) \\
&\cong H_n(\operatorname{Hom}_S(\operatorname{Hom}_R(\mathbf{F_L}, M), E)) \\
&\cong H_n(\operatorname{Hom}_S(M, E) \otimes_R \mathbf{F_L}) \\
&\cong \operatorname{Tor}^R_n(\operatorname{Hom}_S(M, E), L).
\end{aligned}$$

(3): It follows from Theorem 3.2.2(3) that

$$\begin{aligned} \mathrm{Tor}_n^R(L, M \otimes_S F) &\cong H_n(\mathbf{P_L} \otimes_R (M \otimes_S F)) \\ &\cong H_n((\mathbf{P_L} \otimes_R M) \otimes_S F) \\ &\cong H_n(\mathbf{P_L} \otimes_R M) \otimes_S F \\ &\cong \mathrm{Tor}_n^R(L, M) \otimes_S F. \end{aligned}$$

(4): Since L is finitely generated and R is Noetherian, there exists a free resolution

$$\mathbf{F} : \quad \ldots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \ldots \longrightarrow F_0 \longrightarrow L \longrightarrow 0$$

in which every F_n is a finitely generated free R-module. It follows from Theorem 3.2.2(4) that

$$\begin{aligned} \mathrm{Ext}_R^n(L, M) \otimes_S F &\cong H^n(\mathrm{Hom}(\mathbf{F_L}, M)) \otimes_S F \\ &\cong H^n(\mathrm{Hom}(\mathbf{F_L}, M) \otimes_S F) \\ &\cong H^n(\mathrm{Hom}(\mathbf{F_L}, M \otimes_S F)) \\ &\cong \mathrm{Ext}_R^n(L, M \otimes_S F). \end{aligned}$$

□

3.3 Tor and Torsion

In this section, R denotes an integral domain, Q denotes its quotient field, denotes the module $K = Q/R$.

Definition 3.3.1. The **torsion submodule** $T(M)$ of an R-module M is defined by

$$T(M) = \{x \in M | rx = 0 \ \text{ for some nonzero } \ r \in R\}.$$

M is called **torsion** if $T(M) = M$ and M is called **torsion-free** if $T(M) = 0$.

It is easy to check that $M/T(M)$ is torsion-free and $T(M)$ is torsion.

Remark 3.3.2. Were R not an integral domain, then $T(M)$ might not be a submodule.

The torsion submodule actually defines a functor: if $f : M \longrightarrow N$, define $T(f) = f|_{T(M)}$.

Proposition 3.3.3. *If R is an integral domain with quotient field Q, then every torsion-free R-module M can be embedded in a vector space over Q. If M is a finitely generated torsion-free R-module, then M can be embedded in a finitely generated free R-module.*

Proof. Left to the reader as an exercise, or can be found in Rotman's book. □

Lemma 3.3.4. *(1) For every R-module M, we have $Tor_1^R(K, T(M)) \cong T(M)$.*

(2) For every R-module M, we have $Tor_n^R(K, M) = 0$ for all $n \geq 2$.

(3) If M is a torsion-free R-module, then $Tor_1^R(K, M) = 0$.

Proof. (1) Exactness of $0 \longrightarrow R \longrightarrow Q \longrightarrow K \longrightarrow 0$ gives exactness of

$$\mathrm{Tor}_1^R(Q, T(M)) \longrightarrow \mathrm{Tor}_1^R(K, T(M)) \longrightarrow R \otimes_R T(M) \longrightarrow Q \otimes_R T(M).$$

$\mathrm{Tor}_1^R(Q, T(M)) = 0$ since Q is a flat R-module, and $Q \otimes_R T(M) = 0$ because $T(M)$ is torsion. It follows that $\mathrm{Tor}_1^R(K, T(M)) \cong R \otimes_R T(M) \cong T(M)$.

(2) The sequence

$$\mathrm{Tor}_n^R(Q, M) \longrightarrow \mathrm{Tor}_n^R(K, M) \longrightarrow \mathrm{Tor}_{n-1}^R(R, M)$$

is exact. Since $n \geq 2$, we have $n - 1 \geq 1$ and so the outside terms are, because Q and R are flat. Thus, exactness gives $\mathrm{Tor}_n^R(K, M) = 0$.

(3) By Proposition 3.3.3 there is a vector space V over Q containing M as a submodule. Since every vector space has a basis, V is a direct sum of copies of Q. We conclude that V is a flat R-module. Exactness of $0 \longrightarrow M \longrightarrow V \longrightarrow V/M \longrightarrow 0$ gives exactness of

$$\mathrm{Tor}_2^R(K, V/M) \longrightarrow \mathrm{Tor}_1^R(K, M) \longrightarrow \mathrm{Tor}_1^R(K, V).$$

Now $\mathrm{Tor}_2^R(K, V/M) = 0$, by part (2), and $\mathrm{Tor}_1^R(K, V) = 0$, because V is flat. We conclude from exactness that $\mathrm{Tor}_1^R(K, M) = 0$. □

The reason for the name Tor is:

Theorem 3.3.5. *$Tor_1^R(K, M) \cong T(M)$ for all R-modules M.*

Proof. Exactness of $0 \longrightarrow T(M) \longrightarrow M \longrightarrow M/T(M) \longrightarrow 0$ gives exactness of

$$\mathrm{Tor}_2^R(K, M/T(M)) \longrightarrow \mathrm{Tor}_1^R(K, T(M)) \longrightarrow \mathrm{Tor}_1^R(K, M) \longrightarrow \mathrm{Tor}_1^R(K, M/T(M)).$$

The first term is 0 by Lemma 3.3.4 (2); the last term is 0 by Lemma 3.3.4 (3). It follows that $\mathrm{Tor}_1^R(K, M) \cong \mathrm{Tor}_1^R(K, T(M)) \cong T(M)$. □

As an immediate consequence of Theorem 3.3.5, we have the following

Corollary 3.3.6. *(1) For every module A, there is an exact sequence*

$$0 \longrightarrow T(M) \longrightarrow M \longrightarrow Q \otimes_R M \longrightarrow K \otimes_R M \longrightarrow 0.$$

(2) A module M is torsion if and only if $Q \otimes_R M = 0$.

Exercises

1. Consider the commutative diagram with exact rows and columns

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & 0 \arrow[d] & \\
 & & X \arrow[d] & 0 \arrow[d] & W \arrow[d] & \\
0 \arrow[r] & Y \arrow[r] & L' \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z \arrow[r] & L'' \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & N'' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}
$$

   Prove that $X \cong Y$ and $W \cong Z$.

2. If a right R-module F is flat, prove that $\mathrm{Tor}_1^R(F, N) = 0$ for all $n \geq 1$ and every left R-module N. Conversely, if $\mathrm{Tor}_1^R(F, N) = 0$ for every left R-module N, prove that F is flat.

   The following exercise shows that we may use flat resolutions, not merely projective resolutions, to compute Tor.

3. Let $\mathbf{F_M}$ be a deleted flat resolution of a right R-module M and $\mathbf{F_N}$ a deleted flat resolution of a left R-module N. If $n \geq 0$, prove that

$$H_n(\mathbf{F_M} \otimes_R N) \cong \mathrm{Tor}_n^R(M, N) \cong H_n(M \otimes_R \mathbf{F_N}).$$

4. Given a commutative diagram with exact rows,

$$
\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, dashed, "h"] & M \arrow[r] \arrow[d, "g"] & N \arrow[d, "f"] \\
0 \arrow[r] & L' \arrow[r] & M' \arrow[r] & N'
\end{tikzcd}
$$

there exists a unique map $h : L \longrightarrow L'$ making the augmented diagram commute. Moreover, h is an isomorphism if f and g are isomorphisms.

5. Compute $\mathrm{Tor}_n^{\mathbb{Z}_8}(\mathbb{Z}_4, \mathbb{Z}_4)$.

6. If I is a right ideal in a ring R and J a left ideal, then

   (1) $\mathrm{Tor}_1^R(R/I, R/J) \cong (I \cap J)/IJ$,

   (2) $\mathrm{Tor}_n^R(R/I, R/J) \cong \mathrm{Tor}_{n-2}^R(I, J)$ for all $n > 2$,

   (3) $\mathrm{Tor}_2^R(R/I, R/J) \cong \ker(I \otimes J \longmapsto IJ)$.

7. Let M be an R-module and $a \in R$. Show that

$$\mathrm{Tor}_1^R(R/(a), M) \cong_{R/(a)} \{x \in M | ax = 0\}.$$

8. Let R be an integral domain with quotient field Q, and let $K = Q/R$. Show that

$$\mathrm{Tor}_1^R(K, -) \approx T(-).$$

9. **(Axioms for Tor)**. Let $\{T_n : {}_R\mathrm{Mod} \longrightarrow {}_{\mathbb{Z}}\mathrm{Mod}\}_{n \geq 0}$ be a sequence of additive covariant functors. If,

   (1) for every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of left R-modules, there is a long exact sequence with natural connecting homomorphisms

$$\longrightarrow T_{n+1}(C) \xrightarrow{\partial_{n+1}} T_n(A) \longrightarrow T_n(B) \longrightarrow T_n(C) \xrightarrow{\partial_n} T_{n-1}(A) \longrightarrow,$$

   (2) $T_0(-)$ is naturally isomorphic to $M \otimes_R (-)$ for some right R-module M, (3) $T_n(P) = 0$ for all projective left R-modules P and all $n \geq 1$, show that $T_n(-)$ is naturally isomorphic to $Tor_n^R(M, -)$ for all $n \geq 0$.

10. **(Axioms for Covariant Ext)**. Let $\{F^n : {}_R\mathrm{Mod} \longrightarrow {}_{\mathbb{Z}}\mathrm{Mod}\}_{n \geq 0}$ be a sequence of additive covariant functors. If,

   (1) for every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of left

R-modules, there is a long exact sequence with natural connecting homomorphisms

$$\longrightarrow F^{n-1}(C) \xrightarrow{\partial_{n-1}} F^n(A) \longrightarrow F^n(B) \longrightarrow F^n(C) \xrightarrow{\partial_n} F^{n-1}(A) \longrightarrow,$$

(2) there is a left R-module M such that $F^0(-)$ is naturally isomorphic to $\mathrm{Hom}_R(M,-)$,

(3) $F^n(E) = 0$ for all injective left R-modules E and all $n \geq 1$,

show that $F^n(-)$ is naturally isomorphic to $\mathrm{Ext}^n_R(M,-)$ for all $n \geq 0$.

11. **(Axioms for Contravariant Ext)**. Let $\{G^n : {}_R\mathrm{Mod} \longrightarrow {}_{\mathbb{Z}}\mathrm{Mod}\}_{n\geq 0}$ be a sequence of additive covariant functors. If,

(1) for every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of left R-modules, there is a long exact sequence with natural connecting homomorphisms

$$\longrightarrow G^{n-1}(C) \xrightarrow{\partial_{n-1}} G^n(A) \longrightarrow F^n(B) \longrightarrow F^n(C) \xrightarrow{\partial_n} G^{n-1}(A) \longrightarrow,$$

(2) there is a left R-module M such that $G^0(-)$ is naturally isomorphic to $\mathrm{Hom}_R(-,M)$,

(3) $G^n(P) = 0$ for all projective left R-modules P and all $n \geq 1$,

show that $G^n(-)$ is naturally isomorphic to $\mathrm{Ext}^n_R(-,M)$ for all $n \geq 0$.

Chapter 4

DIMENSIONS

4.1 Homological Dimensions

Definition 4.1.1. A projective resolution

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of the R-module M is said to be of **length** n. The **projective dimension** of R-module M is denoted by $\mathrm{pd}_R M$ and is defined by

$$\mathrm{pd}_R M = \min\{n | M \text{ has a projective resolusion of length } n\}.$$

If M has no finite projective resolution, we set $\mathrm{pd}_R M = \infty$.

Definition 4.1.2. An injective resolution

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^{n-1} \longrightarrow E^n \longrightarrow 0$$

of the R-module M is said to be of **length** n. The **injective dimension** of R-module M is denoted by $\mathrm{id}_R M$ and is defined by

$$\mathrm{id}_R M = \min\{n | M \text{ has an injective resolusion of length } n\}.$$

If M has no finite injective resolution, we set $\mathrm{id}_R M = \infty$.

Definition 4.1.3. A flat resolution

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

of the R-module M is said to be of **length** n. The **flat dimension** of R-module M is denoted by $\mathrm{fd}_R M$ and is defined by

$$\mathrm{fd}_R M = \min\{n | M \text{ has a flat resolusion of length } n\}.$$

If M has no finite flat resolution, we set $\mathrm{fd}_R M = \infty$.

Example 4.1.4. (1) $\mathrm{pd}(M) = 0$ if and only if M is projective,

(2) $\mathrm{id}(M) = 0$ if and only if M is injective,

(3) $\mathrm{fd}(M) = 0$ if and only if M is flat.

Theorem 4.1.5. *The following are equivalent for a left R-module P:*

(1) P is projective,

(2) $Ext_R^n(P, N) = 0$ for all modules N and all $n \geq 1$,

(3) $Ext_R^1(P, N) = 0$ for all modules N.

Proof. (1) $\Longrightarrow$ (2) : Follows from Corollary 3.1.3(2).

(2) $\Longrightarrow$ (3) : Trivial.

(3) $\Longrightarrow$ (1) : Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence of R-modules. Then by Corollary 3.1.5(3), we have the following long exact sequence

$$0 \longrightarrow \mathrm{Hom}(P, L) \longrightarrow \mathrm{Hom}(P, M) \longrightarrow \mathrm{Hom}(P, N) \longrightarrow \underbrace{\mathrm{Ext}_R^1(P, L)}_{0} \longrightarrow \cdots .$$

Therefore P is projective. □

Lemma 4.1.6. *A left R-module E is injective if and only if $Ext_R^1(R/I, E) = 0$ for all left ideals I.*

Proof. Use Baer criterion. □

As an immediate consequence of the above lemma, we have the following

Theorem 4.1.7. *The following are equivalent for a left R-module E:*

(1) E is injective,

(2) $Ext_R^n(M, E) = 0$ for all modules M and all $n \geq 1$,

(3) $Ext_R^1(M, E) = 0$ for all modules M,

(4) $Ext_R^1(R/I, E) = 0$ for all left ideals I.

Lemma 4.1.8. *A left R-module F is flat if and only if $Tor_1^R(R/I, F) = 0$ for every finitely generated right ideal I.*

Proof. Exercise. □

As an immediate consequence of the above lemma, we have the following

Theorem 4.1.9. *The following are equivalent for a left R-module P:*

(1) F is flat,

(2) $Tor_n^R(M, F) = 0$ for all modules M and all $n \geq 1$,

(3) $Tor_1^R(M, F) = 0$ for all modules M,

(4) $Tor_1^R(R/I, F) = 0$ for all finitely generated right ideals I.

The next theorems generalize the above theorems.

Theorem 4.1.10. (Projective Dimension Theorem) *For a left R-module M, the following conditions are equivalent:*

(1) $\mathrm{pd}_R M \leq n$,

(2) $Ext_R^k(M, N) = 0$ for all modules N and all $k \geq n+1$,

(3) $Ext_R^{n+1}(M, N) = 0$ for all modules N,

(4) If $0 \longrightarrow K_{n-1} \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ is an exact sequence of R-modules, where P_i is projective, then K_{n-1} is projective.

Proof. (1) $\Longrightarrow$ (2) : There is a projective resolution of M with $P_k = 0$ for all $k \geq n+1$. Therefore $\mathrm{Hom}(P_k, N) = 0$ for all $k \geq n+1$, and so $\mathrm{Ext}_R^k(M, N) = 0$ for all $k \geq n+1$.

(2) $\Longrightarrow$ (3) : Trivial.

(3) $\Longrightarrow$ (4) : We have $0 = \mathrm{Ext}_R^{n+1}(M, N) \cong \mathrm{Ext}_R^1(K_{n-1}, N)$ for all modules N. Then K_{n-1} is projective by Theorem 4.1.5.

(4) $\Longrightarrow$ (1) : Let

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} P_{n-2} \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be a projective resolution for M. If $K_{n-1} = \ker d_{n-1}$, then by hypothesis the sequence

$$0 \longrightarrow K_{n-1} \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is a projective resolution of M and hence $\mathrm{pd}_R M \leq n$. $\square$

We next state without proof results for injective and flat dimensions of modules corresponding to the results obtained for projective dimensions.

Theorem 4.1.11. (Injective Dimension Theorem) *For a left R-module N, the following conditions are equivalent:*

(1) $\mathrm{id}_R N \leq n$,

(2) $Ext_R^k(M, N) = 0$ for all modules M and all $k \geq n+1$,

(3) $Ext_R^{n+1}(M, N) = 0$ for all modules M,

(4) $Ext_R^{n+1}(R/I, N) = 0$ for all left ideals I,

(5) If $0 \longrightarrow N \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^{n-1} \longrightarrow V^{n-1} \longrightarrow 0$ is an exact sequence of R-modules, where E^i is injective, then V^{n-1} is injective.

Theorem 4.1.12. (Flat Dimension Theorem) *For a left R-module N, the following conditions are equivalent:*

(1) $\mathrm{fd}_R N \leq n$,

(2) $Tor_k^R(M, N) = 0$ for all modules M and all $k \geq n+1$,

(3) $Tor_{n+1}^R(M, N) = 0$ for all modules M,

(4) $Tor_{n+1}^R(R/I, N) = 0$ for all finitely generated right ideals I,

(5) If $0 \longrightarrow Y_{n-1} \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0$ is an exact sequence of R-modules, where F_i is flat, then Y_{n-1} is flat.

Theorem 4.1.13. *(1) Let M be an (R, S)-bimodule and E be an injective S-module. Then*

$$\mathrm{id}_R Hom_S(M, E) \leq \mathrm{fd}_R M.$$

In particular, if M is a flat R-module, then $Hom_S(M,E)$ is an injective R-module.

(2) Let R be a Noetherian ring. Let M be an (R,S)-bimodule and E be an injective S-module. Then

$$\mathrm{fd}_R Hom_S(M,E) \leq \mathrm{id}_R M.$$

In particular, if M is an injective R-module, then $Hom_S(M,E)$ is a flat R-module.

(3) Let M be an (R,S)-bimodule and F be a flat S-module. Then

$$\mathrm{fd}_R(M \otimes_S F) \leq \mathrm{fd}_R M.$$

In particular, if M is a flat R-module, then $M \otimes_S F$ is a flat R-module.

(4) Let R be a Noetherian ring. Let M be an (R,S)-bimodule and F be a flat S-module. Then

$$\mathrm{id}_R(M \otimes_S F) \leq \mathrm{id}_R M.$$

In particular, if M is an injective R-module, then $M \otimes_S F$ is an injective R-module.

Proof. (1): Let $n \in \mathbb{N}$. If $n \leq \mathrm{id}_R \mathrm{Hom}_S(M,E)$, then there exists an R-module L such that $\mathrm{Ext}^n_R(L, \mathrm{Hom}_S(M,E)) \neq 0$. Therefore $\mathrm{Hom}_S(\mathrm{Tor}^R_n(L,M),E) \neq 0$, by Theorem 3.2.6 (1) and hence $\mathrm{Tor}^R_n(L,M) \neq 0$. It follows that $n \leq \mathrm{fd}_R M$.

(2): Let $n \in \mathbb{N}$. If $n \leq \mathrm{fd}_R \mathrm{Hom}_S(M,E)$, then there exists a finitely generated R-module L such that $\mathrm{Tor}^R_n(L, \mathrm{Hom}_S(M,E)) \neq 0$. Therefore $\mathrm{Hom}_S(\mathrm{Ext}^n_R(L,M),E) \neq 0$, by Theorem 3.2.6(2) and hence $\mathrm{Ext}^n_R(L,M) \neq 0$. It follows that $n \leq \mathrm{id}_R M$.

(3): Let $n \in \mathbb{N}$. If $n \leq \mathrm{fd}_R(M \otimes_S F)$, then there exists an R-module L such that $\mathrm{Tor}^R_n(L, M \otimes_S F) \neq 0$. Therefore $\mathrm{Tor}^R_n(L,M) \neq 0$, by Theorem 3.2.6(3). It follows that $n \leq \mathrm{fd}_R M$.

(4): Let $n \in \mathbb{N}$. If $n \leq \mathrm{id}_R(M \otimes_S F)$, then there exists a finitely generated R-module L such that $\mathrm{Ext}^n_R(L,M) \otimes_S F \neq 0$. Therefore $\mathrm{Ext}^n_R(L,M) \neq 0$, by Theorem 3.2.6(4). It follows that $n \leq \mathrm{id}_R M$. □

Corollary 4.1.14. *(1) Let M be an (R, S)-bimodule and E be a faithfully injective S-module. Then*

$$\mathrm{id}_R Hom_S(M, E) = \mathrm{fd}_R M.$$

(2) Let R be a Noetherian ring. Let M be an (R, S)-bimodule and E be a faithfully injective S-module. Then

$$\mathrm{fd}_R Hom_S(M, E) = \mathrm{id}_R M.$$

(3) Let M be an (R, S)-bimodule and F be a faithfully flat S-module. Then

$$\mathrm{fd}_R(M \otimes_S F) = \mathrm{fd}_R M.$$

(4) Let R be a Noetherian ring. Let M be an (R, S)-bimodule and F be a faithfully flat S-module. Then

$$\mathrm{id}_R(M \otimes_S F) = \mathrm{id}_R M.$$

Proof. (1): Let $n \in \mathbb{N}$. If $n \leq \mathrm{fd}_R M$, then there exists an R-module L such that $\mathrm{Tor}_n^R(L, M) \neq 0$. Since E is a faithfully injective R-module, $\mathrm{Hom}_S(\mathrm{Tor}_n^R(L, M), E) \neq 0$ and hence $\mathrm{Ext}_R^n(L, \mathrm{Hom}_S(M, E)) \neq 0$, by Theorem 3.2.6(1). It follows that $n \leq \mathrm{id}_R \mathrm{Hom}_S(M, E)$.

(2): Let $n \in \mathbb{N}$. If $n \leq \mathrm{id}_R M$, then, by Theorem 4.1.11, there exists a finitely generated (cyclic) R-module L such that $\mathrm{Ext}_R^n(L, M) \neq 0$. Since E is a faithfully injective S-module, $\mathrm{Hom}_S(\mathrm{Ext}_R^n(L, M), E) \neq 0$ and hence $\mathrm{Tor}_n^R(L, \mathrm{Hom}_S(M, E)) \neq 0$, by Theorem 3.2.6(2). It follows that $n \leq \mathrm{fd}_R \mathrm{Hom}_S(M, E)$.

(3): Let $n \in \mathbb{N}$. If $n \leq \mathrm{fd}_R M$, then there exists an R-module L such that $\mathrm{Tor}_n^R(L, M) \neq 0$. Since F is a faithfully flat S-module, $\mathrm{Tor}_n^R(L, M) \otimes_S F \neq 0$ and hence $\mathrm{Tor}_n^R(L, M \otimes_S F) \neq 0$, by Theorem 3.2.6(3). It follows that $n \leq \mathrm{fd}_R(M \otimes_S F)$.

(4): Let $n \in \mathbb{N}$. If $n \leq \mathrm{id}_R M$, then, by Theorem 4.1.11, there exists a finitely generated (cyclic) R-module L such that $\mathrm{Ext}_R^n(L, M) \neq 0$. Since F is a faithfully flat S-module, $\mathrm{Ext}_R^n(L, M) \otimes_S F \neq 0$ and hence $\mathrm{Ext}_R^n(L, M \otimes_S F) \neq 0$, by Theorem 3.2.6(4). It follows that $n \leq \mathrm{id}_R(M \otimes_S F)$. $\square$

Proposition 4.1.15. *Let $\varphi : R \longrightarrow S$ be a homomorphism of rings. Then*

(1) If E is an injective S-module, then $\mathrm{id}_R E \leq \mathrm{fd}_R S$. Moreover if E is a faithfully injective S-module, then the inequality is equality.

(2) If R is a Noetherian ring and E is an injective S-module, then $\mathrm{fd}_R E \leq \mathrm{id}_R S$. Moreover if E is a faithfully injective S-module, then the inequality is equality.

(3) If F is a flat S-module, then $\mathrm{fd}_R F \leq \mathrm{fd}_R S$. Moreover if F is a faithfully flat S-module, then the inequality is equality.

(4) If R is a Noetherian ring and F is a flat S-module, then $\mathrm{id}_R F \leq \mathrm{id}_R S$. Moreover if F is a faithfully flat S-module, then the inequality is equality

Proof. Take $M = S$ in Theorem 4.1.13. $\square$

Proposition 4.1.16. *Let $\varphi : R \longrightarrow S$ be a homomorphism of rings. Then*

(1) If E is an injective R-module, then $Hom_R(S, E)$ is an injective S-module.

(2) If S is a Noetherian ring and E is an injective R-module, then $\mathrm{fd}_R Hom_R(S, E) \leq \mathrm{id}_R S$. Moreover if E is a faithfully injective S-module, then the inequality is equality.

(3) If F is a flat R-module, then $S \otimes_R F$ is a flat S-module.

(4) If S is a Noetherian ring and F is a flat R-module, then $S \otimes_R F$ is an injective S-module.

Proof. Take $R = S$, $S = R$ and $M = S$ in Theorem 4.1.13. $\square$

Theorem 4.1.17. *Let R be a Noetherian ring and $\varphi : R \longrightarrow S$ be a homomorphism, and let M be an S-module. Then*

(1) If $\mathrm{id}_S M < \infty$, *then* $\mathrm{fd}_R M \leq \mathrm{id}_R S$.

(2) If $\mathrm{fd}_S M < \infty$, *then* $\mathrm{id}_R M \leq \mathrm{id}_R S$.

Proof. We shall prove (1); the proof of (2) is similar.

(1): We use induction on $n = \mathrm{id}_S M$. If $n = 0$, then M is an injective S-module and the assertion follows from Proposition 4.1.13(2). Now let $n \geq 1$. Consider the exact sequence of S-modules

$$0 \longrightarrow M \longrightarrow E \longrightarrow L \longrightarrow 0,$$

where E is an injective S-module. Since M is not an injective S-module, we have $\mathrm{id}_S L = \mathrm{id}_S M - 1$. Therefore $\mathrm{fd}_R L \leq \mathrm{id}_R S$, by induction hypothesis. It suffices to show that if $\mathrm{id}_R S < m$ for some $m \in \mathbb{N}$, then $\mathrm{fd}_R < m$. Consider the following long exact sequence

$$\ldots \longrightarrow \mathrm{Tor}^R_{m+1}(L, N) \longrightarrow \mathrm{Tor}^R_m(M, N) \longrightarrow \mathrm{Tor}^R_m(E, N) \longrightarrow \mathrm{Tor}^R_m(L, N) \longrightarrow \ldots.$$

Since $\mathrm{fd}_R L \leq \mathrm{id}_R S < m$ and $\mathrm{fd}_R E \leq \mathrm{id}_R S < m$ (by Proposition 4.1.5(2)), we have

$$\mathrm{Tor}^R_{m+1}(L, N) = 0 = \mathrm{Tor}^R_m(E, N).$$

Therefore $\mathrm{Tor}^R_m(M, N) = 0$ and hence $\mathrm{fd}_R M < m$. Thus $\mathrm{fd}_R M \leq \mathrm{id}_R S$ and the proof is complete. $\square$

Corollary 4.1.18. *Let R be a Noetherian ring and $\varphi : R \longrightarrow S$ be a homomorphism of rings. Then the following are equivalent.*

(1) $\mathrm{id}_R S < \infty$,

(2) if M is an S-module, then $\mathrm{id}_S M < \infty$ *implies that* $\mathrm{fd}_R M < \infty$,

(3) if M is an S-module, then $\mathrm{fd}_S M < \infty$ *implies that* $\mathrm{id}_R M < \infty$,

(4) there is a faithfully injective S-module E such that $\mathrm{fd}_R E < \infty$,

(5) there is a faithfully flat S-module F such that $\mathrm{id}_R F < \infty$.

Proof. (1) $\Longrightarrow$ (2) : Follows from Theorem 4.1.17(1).

(1) $\Longrightarrow$ (3) : Follows from Theorem 4.1.17(2).

(4) $\Longrightarrow$ (1) : Follows from Proposition 4.1.15(2).

(5) $\Longrightarrow$ (1) : Follows from Proposition 4.1.15(4).

(2) $\Longrightarrow$ (4) and (3) $\Longrightarrow$ (5) are trivial. □

Definition 4.1.19. Let R be a ring. R is **Gorenstein** if $\mathrm{id}_R R < \infty$.

Corollary 4.1.20. *Let M be module over a Noetherian Gorenstein ring R. Then $\mathrm{id}_R M < \infty$ if and only if $\mathrm{fd}_R M < \infty$.*

Proof. Follows easily from the above theorem. □

4.2 Change of Rings Theorems

Theorem 4.2.1. (General Change of Rings Theorem). *Let $\varphi : R \longrightarrow S$ be a ring homomorphism, and let M be an S-module. Then*

$$\mathrm{pd}_R M \leq \mathrm{pd}_S M + \mathrm{pd}_R S.$$

Proof. If $\mathrm{pd}_S M = \infty$, there is nothing to prove, so we assume $\mathrm{pd}_S M = n < \infty$ and proceed by induction on n. If $n = 0$, then M is a projective S-module; thus there exists an S-module N such that $M \oplus N = \coprod S$. Exercise 1(i) applies to give

$$\mathrm{pd}_R M \leq \sup\{\mathrm{pd}_R M, \mathrm{pd}_R N\} = \mathrm{pd}_R(M \oplus N) = \mathrm{pd}_R(\coprod S) = \mathrm{pd}_R S.$$

Suppose $n > 0$. There is an exact sequence of S-modules

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0,$$

where F is a free S-module. By Exercise 5(iii)

$$\mathrm{pd}_R M \leq \max\{1 + \mathrm{pd}_R K, \mathrm{pd}_R F\} = \max\{1 + \mathrm{pd}_R K, \mathrm{pd}_R S\}.$$

Since M is not a projective S-module, Exercise 2(i) gives $\mathrm{pd}_S K = n - 1$, so that induction gives

$$\mathrm{pd}_R K \leq \mathrm{pd}_S K + \mathrm{pd}_R S.$$

Combining these inequalities:

$$\mathrm{pd}_R M \leq \max\{1+\mathrm{pd}_S K+\mathrm{pd}_R S, \mathrm{pd}_R S\} \leq \max\{n+\mathrm{pd}_R S, \mathrm{pd}_R S\} = n+\mathrm{pd}_R S.$$

□

Proposition 4.2.2. *Let R be a commutative ring and $a \in R$ a non-zero divisor element which is not a unit. Then*

$$\mathrm{pd}_R R/(a) = 1.$$

Proof. From the exact sequence $0 \longrightarrow R \xrightarrow{a.} R \longrightarrow R/(a) \longrightarrow 0$ an Exercise 3(iii), we deduce that $\mathrm{pd}_R R/(a) \leq 1$. If $\mathrm{pd}_R R/(a) = 0$, then there exists an R-module N such that $R/(a) \oplus N = \coprod R$. Then since a is not a unit,

$$a \in Z(R/(a) \oplus N) = Z(\coprod R) = Z(R),$$

which is a contradiction. Thus $\mathrm{pd}_R R/(a) = 1$ and the proof is complete.

□

Lemma 4.2.3. *Let I be an ideal of a commutative ring R.*

(1) If F is a free R-module, then F/IF is a free R/I-module,

(2) If P is a projective R-module, then P/IP is a projective R/I-module.

Proof. The proof is left to the reader. □

The converse of Lemma 4.2.3(1) is:

Lemma 4.2.4. *Let R be a commutative Noetherian ring with maximal ideal $\mathfrak{m}$, and let M be a finitely generated R-module. Let $a \in \mathfrak{m}$ be a non-zero divisor element on both R and M. If M/aM is a free $R/(a)$-module, then M is a free R-module.*

Proof. If $M/aM = 0$ then $M = 0$ by Nakayama's lemma. So, suppose $M/aM \neq 0$. Let $\{x_1 + aM, x_2 + aM, \ldots, x_n + aM\}$ be a basis for free $R/(a)$-module M/aM. We claim that $\{x_1, x_2, \ldots, x_n\}$ is a basis for R-module M. Since $Rx_1 + Rx_2 + \ldots + Rx_n + aM = M$, it follows from Nakayama's lemma that

$Rx_1+Rx_2+\ldots+Rx_n=M$. To show that x_i are linearly independent, suppose that $\sum_{i=1}^n r_ix_i=0$. Then $\sum_{i=1}^n(r_i+(a))(x_i+aM)=0$. Since x_i+aM are linearly independent over $R/(a)$, we have $r_i\in aR$ for all i. As a is non-zero divisor on R and M we can divide to get a well-defined quotient $r_i/a\in R$ such that $\sum_{i=1}^n(r_i/a)x_i=0$ in M. Continuing this process, we get a sequence of elements $r_i, r_i/a, r_i/a^2,\ldots$. Now consider the following ascending chain of ideals of R.

$$(r_i/a)\subseteq(r_i/a^2)\subseteq\ldots.$$

Since R is Noetherian there exists $k\in\mathbb{N}$ such that $(r_i/a^k)=(r_i/a^{k+1})$. Therefore, there exists $r\in R$ such that $r_i=r_ira$. Therefore $r_i=0$, which completes the proof of this lemma. □

Theorem 4.2.5. (First Change of Rings Theorem). *Let a be a central non-zero divisor in a ring R. If $M\neq 0$ is a $R/(a)$-module with $\mathrm{pd}_{R/(a)}M$ finite, then*

$$\mathrm{pd}_RM=1+\mathrm{pd}_{R/(a)}M.$$

Proof. We proceed by induction on $n=\mathrm{pd}_{R/(a)}M$. As a is a regular element and $aM=0$, it follows that M cannot be a projective R-module, so $\mathrm{pd}_RM\geq 1$. Let $n=0$. Then by Theorem 4.2.1 and Proposition 4.2.2 we see that $\mathrm{pd}_RM=\mathrm{pd}_RR/(a)=1$. If $n=1$, then $\mathrm{pd}_RM\leq 2$ and strict inequality means $\mathrm{pd}_RM\leq 1$. We have already shown $\mathrm{pd}_RM\neq 0$; we claim that $\mathrm{pd}_RM\neq 1$. Otherwise there is an exact sequence of R-modules

$$0\longrightarrow K\longrightarrow F\longrightarrow M\longrightarrow 0,$$

where F is a free R-module and K is a projective R-module. We have an exact sequence of $R/(a)$-modules

$$0\longrightarrow \mathrm{Tor}_1^R(R/(a),M)\longrightarrow K/aK\longrightarrow F/aF\longrightarrow M/aM\longrightarrow 0.$$

Since $\mathrm{pd}_{R/(a)}M=1\leq 2$, $\mathrm{Tor}_1^R(R/(a),M)$ is a projective $R/(a)$-module. But

$$\mathrm{Tor}_1^R(R/(a),M)\cong_{R/(a)}\{x\in M|ax=0\}=M,$$

so $\mathrm{pd}_{R/(a)}M = 0$, which is a contradiction. Now, let $n \geq 2$. Consider an exact sequence of $R/(a)$-modules

$$0 \longrightarrow K_1 \longrightarrow F_1 \longrightarrow M \longrightarrow 0,$$

where F_1 is a free $R/(a)$-module. Since $\mathrm{pd}_{R/(a)}M \neq 0$, it follows from Lemma 4.2.3(2) and Exercise 2(i) that $\mathrm{pd}_R M = 1 + \mathrm{pd}_R K_1$ and hence

$$\mathrm{pd}_R M = 1 + \mathrm{pd}_R K_1 = 1 + 1 + \mathrm{pd}_{R/(a)} K_1 = 1 + \mathrm{pd}_{R/(a)} M.$$

□

Theorem 4.2.6. (Second Change of Rings Theorem). *Let a be a central non-zero divisor in a ring R. If M is an R-module and a is a non-zero divisor on M, then*

$$\mathrm{pd}_R M \geq \mathrm{pd}_{R/(a)}(M/aM).$$

Proof. If $\mathrm{pd}_R M = \infty$, there is nothing to prove, so we assume $n = \mathrm{pd}_R M < \infty$ and proceed by induction on n. If $\mathrm{pd}_R M = 0$, then M/aM is a projective $R/(a)$-module, so the result is true in the case $n = 0$. Now suppose $n \geq 1$ and consider the the exact sequence of R-modules $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$, where F is free. Since $\mathrm{Tor}_1^R(R/(a), M) \cong \{x \in M | ax = 0\} = 0$, we have the following exact sequence of $R/(a)$-modules.

$$0 \longrightarrow K/aK \longrightarrow F/aF \longrightarrow M/aM \longrightarrow 0.$$

$\mathrm{pd}_R K = \mathrm{pd}_R M - 1$, by Exercise 2(i). We have $a \notin Z(K)$, since $Z(K) \subseteq Z(F) = Z(R)$. So by the inductive hypothesis $\mathrm{pd}_R K \geq \mathrm{pd}_{R/(a)}(K/aK)$. If $\mathrm{pd}_{R/(a)}(M/aM) = 0$, we are done. Otherwise, $\mathrm{pd}_{R/(a)}(M/aM) = 1 + \mathrm{pd}_{R/(a)}(K/aK)$ and therefore,

$$\mathrm{pd}_{R/(a)}(M/aM) = 1 + \mathrm{pd}_{R/(a)}(K/aK) \leq 1 + \mathrm{pd}_R K = \mathrm{pd}_R M.$$

□

If R is a ring, not necessarily commutative, then $R[x]$ denotes the polynomial ring in which the indeterminate x commutes with every element in R (thus, x lies in the center of $R[x]$). If M is a left R-module, write

$$M[x] = R[x] \otimes_R M.$$

Corollary 4.2.7. *For every left R-module M,*

$$\mathrm{pd}_{R[x]} M[x] = \mathrm{pd}_R M.$$

Proof. Let $\mathrm{pd}_R M \leq n$, then there is a projective resolution of R-modules

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

Since $R[x]$ is a flat (free) R-module, there is an exact sequence of $R[x]$-modules

$$0 \longrightarrow P_n[x] \longrightarrow P_{n-1}[x] \longrightarrow \ldots \longrightarrow P_0[x] \longrightarrow M[x] \longrightarrow 0,$$

where the module $P_i[x]$ is projective. Therefore $\mathrm{pd}_{R[x]} M[x] \leq n$ and hence $\mathrm{pd}_{R[x]} M[x] \leq \mathrm{pd}_R M$. On the other hand, the Second Change of Ring Theorem implies that

$$\mathrm{pd}_R M = \mathrm{pd}_{\frac{R[x]}{xR[x]}} \left(\frac{M[x]}{xM[x]}\right) \leq \mathrm{pd}_{R[x]} M[x].$$

$\square$

Lemma 4.2.8. *Let R be a commutative Noetherian local ring, and let M be a finitely generated R-module. Then the following are equivalent.*

(1) M is free,

(2) M is projective,

(3) M is flat.

Proof. The proof is left to the reader. $\square$

Theorem 4.2.9. (Third Change of Rings Theorem). *Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, and let M be a finitely generated R-module. If $a \in \mathfrak{m}$ is a non-zero divisor on both R and M, then*

$$\mathrm{pd}_R M = \mathrm{pd}_{R/(a)}(M/aM).$$

Proof. We know $\mathrm{pd}_R M \geq \mathrm{pd}_{R/(a)}(M/aM)$ by the second Change of Rings Theorem, and we shall prove $\mathrm{pd}_R M \leq \mathrm{pd}_{R/(a)}(M/aM)$. If $\mathrm{pd}_{R/(a)}(M/aM) = \infty$, there is nothing to prove, so we assume $n = \mathrm{pd}_{R/(a)}(M/aM) < \infty$ and proceed by induction on n. If $n = 0$ then M/aM is projective, hence a free $R/(a)$-module since $R/(a)$ is local. It follows from the previous Lemma that M is a free R-module, so $\mathrm{pd}_R M = 0$. Now suppose $n \geq 1$ and consider the the exact sequence of R-modules $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$, where F is free. Since $\mathrm{Tor}_1^R(R/(a), M) \cong \{x \in M | ax = 0\} = 0$, we have the following exact sequence of $R/(a)$-modules.

$$0 \longrightarrow K/aK \longrightarrow F/aF \longrightarrow M/aM \longrightarrow 0.$$

By the Second Change of Ring $\mathrm{pd}_R M \geq \mathrm{pd}_{R/(a)}(M/aM) \geq 1$. Therefore, by Exercise 2(i) and induction hypothesis we have

$$\mathrm{pd}_R M = 1 + \mathrm{pd}_R K = 1 + \mathrm{pd}_{R/(a)} K/aK = \mathrm{pd}_{R/(a)} M/aM,$$

Which completes the proof. □

Corollary 4.2.10. *Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, and let M be a finitely generated R-module with $\mathrm{pd}_R M < \infty$. If $a \in \mathfrak{m}$ is a non-zero divisor on both R and M, then*

$$1 + \mathrm{pd}_{R/(a)} M = \mathrm{pd}_{R/(a)}(M/aM).$$

Proof. Combine the first and third Change of Rings Theorems. □

4.3 Global and Weak Dimension

Theorem 4.3.1. (Global Dimension Theorem) *The following numbers are the same for any ring R.*

(1) $a = \sup\{\mathrm{id}_R M | M \in {}_R\mathit{Mod}\}$,

(2) $b = \sup\{\mathrm{pd}_R M | M \in {}_R\mathit{Mod}\}$,

(3) $c = \sup\{\mathrm{pd}_R R/I | I \text{ is a left ideal of } R\}$,

(4) $d = \sup\{d | \mathit{Ext}_R^d(M, N) \neq 0 \text{ for some left modules } M, N\}$.

Proof. First of all, we show that $b = d$. Suppose

$$\begin{aligned} B &= \{\mathrm{pd}_R M | M \in {}_R\mathrm{Mod}\}, \\ D &= \{d | \mathrm{Ext}^d_R(M, N) \neq 0 \text{ for some left modules } M, N\}. \end{aligned}$$

If $t \in B$, then there exists $M \in {}_R\mathrm{Mod}$ such that $\mathrm{pd}_R M = t$. By Theorem 4.1.10, $\mathrm{Ext}^t_R(M, N) \neq 0$ for some $N \in {}_R\mathrm{Mod}$. Therefore $t \in D$,a nd hence $B \subseteq D$. Thus $b \leq d$. Now, let $t \in D$. Then there exist $M, N \in {}_R\mathrm{Mod}$ such that Hence $\mathrm{Ext}^t_R(M, N) \neq 0$. Therefore $\mathrm{pd}_R M \geq t$. It follows that $b \geq t$. Since t was an arbitrary element of D, we have $b \geq d$. Thus $b = d$. A similar argument shows that $a = d$. It is enough to show that $a \leq c$. Suppose $N \in {}_R\mathrm{Mod}$ and consider the following exact sequence

$$0 \longrightarrow N \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \ldots \longrightarrow E^{c-1} \longrightarrow M \longrightarrow 0,$$

where E^i is injective. Then Theorem 4.1.11 implies that

$$0 = \mathrm{Ext}^{c+1}_R(R/I, N) \cong \mathrm{Ext}^1_R(R/I, M),$$

for any left ideal I of R. This implies M is injective, and hence $\mathrm{id}_R N \leq c$. Therefore $a \leq c$ as required. □

Definition 4.3.2. The common numbers in the above theorem is called the **left global dimension** of R and is denoted $\ell.g.\mathrm{dim}R$.

We can also similarly define right global dimension $r.g.\mathrm{dim}R$ of R. The two global dimensions of R are not always equal.

Theorem 4.3.3. (Weak Dimension Theorem) *The following numbers are the same for any ring R.*

(1) $a = \sup\{\mathrm{fd}_R M | M \in {}_R Mod\}$,

(2) $b = \sup\{\mathrm{fd}_R R/I | I \text{ is a left ideal of } R\}$,

(3) $c = \sup\{\mathrm{fd}_R N | N \in Mod_R\}$,

(4) $d = \sup\{\mathrm{fd}_R R/I | I \text{ is a right ideal of } R\}$,

(5) $e = \sup\{d | \, Tor^R_d(M, N) \neq 0 \text{ for some right modules } M, N\}$.

Definition 4.3.4. The common numbers in the above theorem is called the **weak dimension** of R and is denoted $w.\mathrm{dim}R$.

Lemma 4.3.5. *If M is an $R[x]$-module, there is an exact sequence or $R[x]$-modules*

$$0 \longrightarrow M[x] \longrightarrow M[x] \longrightarrow M \longrightarrow 0.$$

Theorem 4.3.6. *If R is any ring, then*

$$\ell.g.\mathrm{dim}R[x] = \ell.g.\mathrm{dim}R + 1.$$

Proof. If $\ell.g.\mathrm{dim}R = \infty$, then Corollary 4.2.7 implies that $\ell.g.\mathrm{dim}R[x] = \infty$. Now suppose $n = \ell.g.\mathrm{dim}R < \infty$. Let M be an R-module such that $\mathrm{pd}_R M = n$. We can view M as an R-module by setting $(a_0+a_1x+\ldots+a_nx^n)m = a_0m$. It is a consequence of the first Change of Rings Theorem that $\mathrm{pd}_{R[x]}M = \mathrm{pd}_R M+1$. Hence $\ell.g.\mathrm{dim}R[x] \geq n+1$. Now let M be an $R[x]$-module and consider the following exact sequence of $R[x]$-modules.

$$0 \longrightarrow R[x] \otimes_R M \longrightarrow R[x] \otimes_R M \longrightarrow M \longrightarrow 0.$$

Then by Exercise 5 and Corollary 4.2.7

$$\mathrm{pd}_R M \leq \sup\{1 + \mathrm{pd}_{R[x]} M[x], \mathrm{pd}_{R[x]} M[x]\} = 1 + \mathrm{pd}_R M \leq n+1.$$

Hence $\ell.g.\mathrm{dim}R[x] \leq n+1$ as required. □

Corollary 4.3.7. (Hilbert's Theorem on Syzygies). *If k is a field, then*

$$\ell.g.\mathrm{dim}k[x_1, x_2 \ldots, x_n] = n.$$

Proof. Follows immediately from the above theorem. □

Exercises

1. Let $\{M_i\}_{i\in I}$ be a family of R-modules. Show that

   (i) $\mathrm{pd}_R(\coprod_{i\in I} M_i) = \sup\{\mathrm{pd}_R M_i | i \in I\}$,

   (ii) $\mathrm{id}_R(\prod_{i\in I} M_i) = \sup\{\mathrm{id}_R M_i | i \in I\}$.

2. (i) If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is an exact sequence with M is projective, prove that either all three modules are projective or $\mathrm{pd}_R N = 1 + \mathrm{pd}_R L$.

   (ii) If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is an exact sequence with M is injective, prove that either all three modules are injective or $\mathrm{id}_R L = 1 + \mathrm{id}_R N$.

3. Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of R-modules. Show that

   (i) if $\mathrm{pd}_R L < \mathrm{pd}_R M$, then $\mathrm{pd}_R N = \mathrm{pd}_R M$,

   (ii) if $\mathrm{pd}_R L > \mathrm{pd}_R M$, then $\mathrm{pd}_R N = 1 + \mathrm{pd}_R L$,

   (iii) if $\mathrm{pd}_R L = \mathrm{pd}_R M$, then $\mathrm{pd}_R N \leq 1 + \mathrm{pd}_R L$.

4. Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of R-modules. Show that

   (i) $\mathrm{pd}_R M \leq \max\{\mathrm{pd}_R L, \mathrm{pd}_R N\}$ with equality unless $\mathrm{pd}_R N = 1 + \mathrm{pd}_R L$,

   (ii) $\mathrm{id}_R M \leq \max\{\mathrm{id}_R L, \mathrm{id}_R N\}$ with equality unless $\mathrm{id}_R N = 1 + \mathrm{id}_R L$,

   (iii) $\mathrm{fd}_R M \leq \max\{\mathrm{fd}_R L, \mathrm{fd}_R N\}$ with equality unless $\mathrm{fd}_R N = 1 + \mathrm{fd}_R L$.

5. Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of R-modules. If any two of these modules have finite projective dimension, show that the third does also and

   (i) $\mathrm{pd}_R L \leq \max\{\mathrm{pd}_R M, \mathrm{pd}_R N\}$,

   (ii) $\mathrm{pd}_R M \leq \max\{1 + \mathrm{pd}_R L, \mathrm{pd}_R N\}$,

   (iii) $\mathrm{pd}_R N \leq \max\{1 + \mathrm{pd}_R L, \mathrm{pd}_R M\}$.

   Furthermore, if $\mathrm{pd}_R M = 1$ and $\mathrm{pd}_R N \geq 2$, prove that $\mathrm{pd}_R N = 1 + \mathrm{pd}_R L$.

6. Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of R-modules. If any two of these modules have finite injective dimension, show that the third does also. Furthermore, prove that

(i) $\mathrm{id}_R L \leq \max\{1 + \mathrm{id}_R M, 1 + \mathrm{id}_R N\}$,

(ii) $\mathrm{id}_R M \leq \max\{\mathrm{id}_R L, 1 + \mathrm{id}_R N\}$,

(iii) $\mathrm{id}_R N \leq \max\{\mathrm{id}_R L, \mathrm{id}_R M\}$.

7. Let R be a commutative Noetherian ring and let n be a non-negative integer. Show that the following are equivalent.

(i) $\mathrm{pd}_R M \leq n$ for all R-modules M,

(ii) $\mathrm{id}_R M \leq n$ for all R-modules M,

(iii) $\mathrm{pd}_R M \leq n$ for all finitely generated R-modules M,

(iv) $\mathrm{pd}_R M \leq n$ for all cyclic R-modules M,

(v) $\mathrm{id}_R M \leq n$ for all finitely generated R-modules M,

(vi) $\mathrm{id}_R M \leq n$ for all cyclic R-modules M.

8. **(Change of Rings Theorems for Injective Dimension).**

(i) **(First Change of Rings Theorem).** Let $M \neq 0$ be an $R/(a)$-module with $\mathrm{id}_{R/(a)} M$ finite. Then

$$\mathrm{id}_R M = 1 + \mathrm{id}_{R/(a)} M.$$

(ii) **(Second Change of Rings Theorem).** Let M be an R-module. If a is a non-zero divisor on M, then M is injective (in the case $M/aM = 0$) or

$$\mathrm{id}_R M \geq 1 + \mathrm{id}_{R/(a)}(M/aM).$$

(iii) **(Third Change of Rings Theorem).** Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, and let M be a finitely generated R-module. If $a \in \mathfrak{m}$ is a non-zero divisor on M, then

$$\mathrm{id}_R M = \mathrm{id}_R(M/aM) = 1 + \mathrm{id}_{R/(a)}(M/aM).$$

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