Flow in Porous Media

Module 2.a
A quick review on method of separation of variables and Laplace transformation for solution of PDEs

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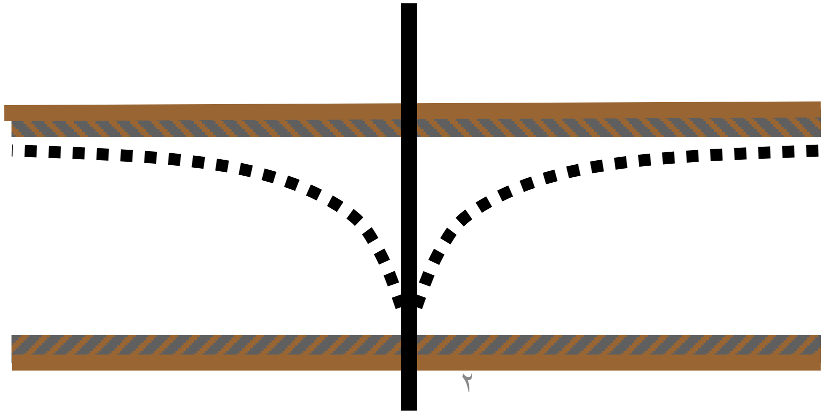
Non-Steady State Filtration in Infinite Acting Systems

Radial Systems with Constant Production Rate

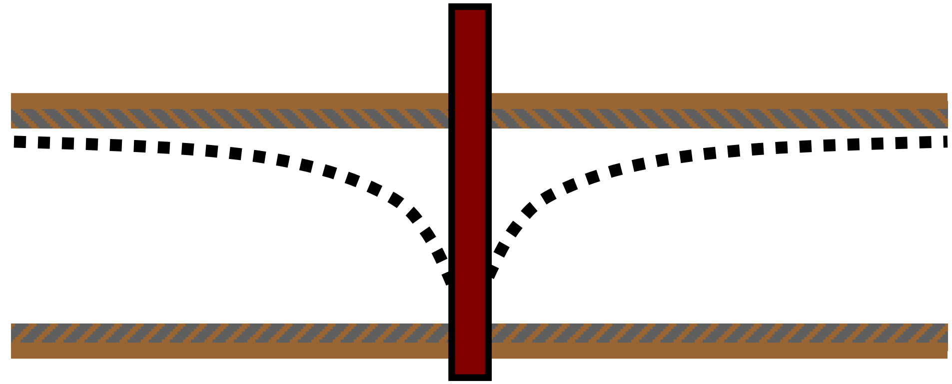
Line-source: the well has zero radius

$$\left(\frac{\partial p}{\partial r}\right)_{r\to 0} = -\frac{\mu q B_o}{2\pi r_w h k}$$



Finite-wellbore

$$\left(\frac{\partial p}{\partial r}\right)_{r_w} = -\frac{\mu q B_o}{2\pi r_w h k}$$



Non-Steady State Filtration in Infinite Acting Systems

Radial Systems with Constant Production Rate

Solution method: Combination of variables

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} = \frac{1}{K}\frac{\partial p}{\partial t} \qquad K = \frac{k}{\mu\phi c} \tag{3.30}$$

$$p = p_i \quad r > r_w \quad t = 0 \tag{3.31}$$

$$\frac{2\pi r_w hk}{\mu}\left(\frac{\partial p}{\partial r}\right)_{r = r_w} = -qB \tag{3.32}$$

$$p = p_i \quad r = \infty \quad t > 0 \tag{3.33}$$

$$p = p(r,t) \text{ for } r_w < r < \infty \quad t > 0 \tag{3.34}$$

$$z = \frac{r^2}{K \cdot t} \tag{3.38}$$

Based on this assumption we get:

$$\frac{1}{K}\frac{\partial p}{\partial t} = \frac{1}{K}\frac{\partial p}{\partial z}\frac{\partial z}{\partial t} = \frac{r^2}{K^2 t^2}\frac{dp}{dz} = -\frac{z}{Kt}\frac{dp}{dz} \tag{3.39}$$

$$\frac{\partial p}{\partial r} = \frac{dp}{dz}\frac{\partial z}{\partial r} = \frac{2r}{Kt}\frac{dp}{dz} = \frac{2r^2}{Kt}\frac{1}{r}\frac{dp}{dz} = \frac{2z}{r}\frac{dp}{dz} \tag{3.40}$$

$$\frac{\partial^2 p}{\partial r^2} = \frac{2}{Kt}\frac{dp}{dz} + \frac{4r^2}{K^2 t^2}\frac{d^2 p}{dz^2} = \frac{2}{Kt}\frac{dp}{dz} + \frac{4z}{Kt}\frac{d^2 p}{dz^2} \tag{3.41}$$

Substituting Eq. 3.39 - Eq. 3.41 into Eq. 3.30 yields:

$$4z\frac{d^2 p}{dz^2} + (4 + z)\frac{dp}{dz} = 0 \tag{3.42}$$

To solve Eq. 3.42 the boundary conditions must also be transformed. Using Eq. 3.38 the B.C. in Eq. 3.33 can be transferred into:

$$p = p_i \quad z = \infty \quad t > 0 \tag{3.43}$$

Using Eq. 3.38 and Eq. 3.40, the first B.C. in Eq. 3.32 can be transformed as well to:

$$z\frac{dp}{dz} = -\frac{\mu q B}{4\pi h k} \tag{3.44}$$

Eq. 3.42 can be written as:

$$z\frac{d^2p}{dz^2} + \frac{dp}{dz} = -\frac{z}{4}\frac{dp}{dz} \tag{3.45}$$

or:

$$\frac{d}{dz}\left(z\frac{dp}{dz}\right) = -\frac{1}{4}z\frac{dp}{dz} \tag{3.46}$$

Let $y = z\frac{dp}{dz}$

then Eq. 3.46 becomes:

$$\frac{d}{dz}y = -y/4 \tag{3.47}$$

By separation of variables:

$$\frac{d}{dz}y = -\frac{y}{4} \tag{3.48}$$

By integrating Eq. 3.48 yields:

$$\ln y - \ln A_1 = -\frac{z}{4} \tag{3.49}$$

where A_1 is a constant of integration, taking the exponential of Eq. 3.49

$$y = A_1 e^{-z/4} \tag{3.50}$$

or

$$z\frac{dp}{dz} = A_1 e^{-z/4} \tag{3.51}$$

and the second substitution with the notation $\xi = z/4$ and by separation of variables, Eq. 3.51 becomes:

$$dp = A_1 \frac{e^{-\xi}}{\xi} d\xi \tag{3.52}$$

Integrating Eq. 3.52 yields:

$$p_i - p = A_1 \int_{z/4}^{\infty} \frac{e^{-\xi}}{\xi} d\xi \tag{3.53}$$

Using Eq. 3.51 and the boundary condition in Eq. 3.44 and assuming that r_w is very small so that the condition $\frac{r_w^2}{4Kt} \approx 0$ is valid, A_1 can be determined as:

$$A_1 = -\frac{\mu q B}{4\pi h k} \tag{3.54}$$

The integral in Eq. 3.53 cannot be solved in a closed form. This integral is defined as the so called *exponential integral* and its numerical solution can be found in any mathematical handbook:

$$Ei(-x) = -\int_{x}^{\infty} \frac{e^{-\xi}}{\xi} d\xi \tag{3.55}$$

Table 6-1
Values of the $-E_i(-x)$ as a function of x
(After Craft, Hawkins, and Terry, 1991)

x	$-E_i(-x)$	x	$-E_i(-x)$	x	$-E_i(-x)$
0.1	1.82292	4.3	0.00263	8.5	0.00002
0.2	1.22265	4.4	0.00234	8.6	0.00002
0.3	0.90568	4.5	0.00207	8.7	0.00002
0.4	0.70238	4.6	0.00184	8.8	0.00002
0.5	0.55977	4.7	0.00164	8.9	0.00001
0.6	0.45438	4.8	0.00145	9.0	0.00001
0.7	0.37377	4.9	0.00129	9.1	0.00001
0.8	0.31060	5.0	0.00115	9.2	0.00001
0.9	0.26018	5.1	0.00102	9.3	0.00001
1.0	0.21938	5.2	0.00091	9.4	0.00001
1.1	0.18599	5.3	0.00081	9.5	0.00001
1.2	0.15841	5.4	0.00072	9.6	0.00001
1.3	0.13545	5.5	0.00064	9.7	0.00001
1.4	0.11622	5.6	0.00057	9.8	0.00001
1.5	0.10002	5.7	0.00051	9.9	0.00000
1.6	0.08631	5.8	0.00045	10.0	0.00000
1.7	0.07465	5.9	0.00040		
1.8	0.06471	6.0	0.00036		
1.9	0.05620	6.1	0.00032		
2.0	0.04890	6.2	0.00029		
2.1	0.04261	6.3	0.00026		
2.2	0.03719	6.4	0.00023		
2.3	0.03250	6.5	0.00020		
2.4	0.02844	6.6	0.00018		
2.5	0.02491	6.7	0.00016		
2.6	0.02185	6.8	0.00014		
2.7	0.01918	6.9	0.00013		
2.8	0.01686	7.0	0.00012		
2.9	0.01482	7.1	0.00010		
3.0	0.01305	7.2	0.00009		
3.1	0.01149	7.3	0.00008		
3.2	0.01013	7.4	0.00007		
3.3	0.00894	7.5	0.00007		
3.4	0.00789	7.6	0.00006		
3.5	0.00697	7.7	0.00005		
3.6	0.00616	7.8	0.00005		
3.7	0.00545	7.9	0.00004		
3.8	0.00482	8.0	0.00004		
3.9	0.00427	8.1	0.00003		
4.0	0.00378	8.2	0.00003		
4.1	0.00335	8.3	0.00003		
4.2	0.00297	8.4	0.00002		

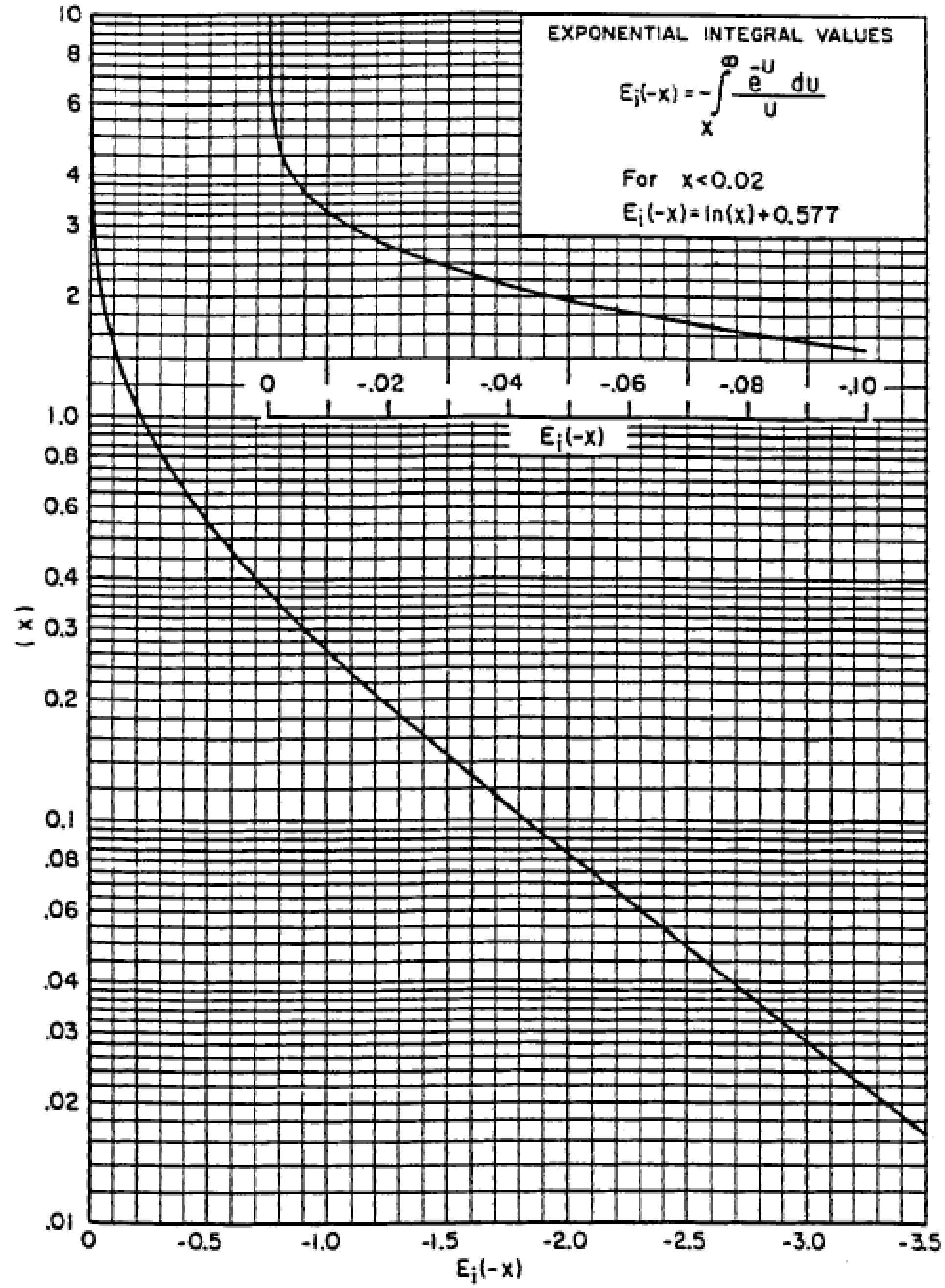


Figure 6-19. The E_i-function. *(After Craft, Hawkins, and Terry, 1991.)*

Finally Eq. 3.53 becomes:

$$p_i - p(r,t) = \frac{\mu q B}{4\pi h k} Ei\left(-\frac{r^2}{4Kt}\right) \tag{3.56}$$

The calculation of the pressure drop at the well bottom is made by substituting $r = r_w$ and $p(r,t) = p_{wf}(t)$ into Eq. 3.56:

$$p_i - p_{wf} = \frac{\mu q B}{4\pi h k} Ei\left(-\frac{r_w^2}{4Kt}\right) \tag{3.57}$$

3.2.2 Properties of the *Ei*-Function

The function $-Ei(-z)$ is illustrated in Figure 3.4. In the vicinity of $z = 0$ the TAYLOR-Series of $-Ei(-z)$ is defined as:

$$-Ei(-z) = -\gamma - \ln z + z - \frac{z^2}{4} + \ldots \tag{3.58}$$

where $\gamma = 0,57722$ is the EULER-Constant.

If $z \ll 1$ the series in Eq. 3.58 will have very small values in terms higher than the third term, which makes the following approximation valid:

$$-Ei(-z) = -0,57722 - \ln z \tag{3.59}$$

then:

$$-Ei\left(-\frac{r^2}{4Kt}\right) = -0,57722 - \ln\frac{4Kt}{r^2} = 0,80907 + \ln\frac{Kt}{r^2} \tag{3.60}$$

Analytical Solution of PDEs

$$A(x,y)\frac{\partial^2}{\partial x^2}u + B(x,y)\frac{\partial^2}{\partial x \partial y}u + C(x,y)\frac{\partial^2}{\partial y^2}u = \phi\left(u, \frac{\partial}{\partial x}u, \frac{\partial}{\partial y}u, x, y\right) \qquad (10.1)$$

$$Au_{xx} + Bu_{xy} + Cu_{yy} = \phi(u, u_x, u_y, x, y) \qquad (10.2)$$

$$u_{xx} = \frac{\partial^2}{\partial x^2}u \qquad u_{xy} = \frac{\partial^2}{\partial x \partial y}u \qquad \text{and} \qquad u_{yy} = \frac{\partial^2}{\partial y^2}u$$

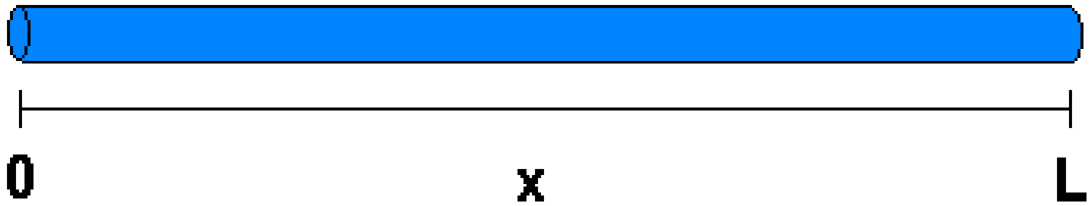
The most common PDE classification scheme identifies the PDE as either a hyperbolic, parabolic, or elliptic equation depending on the sign of the term $B^2 - 4AC$ (which can vary with x and y). In particular, we have the following classification scheme:

$$B^2 - 4AC \begin{cases} > 0 & \text{hyperbolic} \\ = 0 & \text{parabolic} \\ < 0 & \text{elliptic} \end{cases}$$

These types of systems give rise to significantly different characteristic behavior and, as mentioned above, the solution scheme for each method can also differ. An example of each type of PDE is summarized below:

Application	Differential Equation	Value of Coefficients	Sign of $B^2 - 4AC$	PDE Class
Wave Equation	$u_{tt} = \alpha^2 u_{xx}$	$A = 1, B = 0, C = -\alpha^2$	positive	hyperbolic
Diffusion Equation	$u_t = \alpha^2 u_{xx}$	$A = 0, B = 0, C = -\alpha^2$	zero	parabolic
Poisson's Equation	$u_{xx} + u_{yy} = f(x,y)$	$A = 1, B = 0, C = 1$	negative	elliptic

Introduction to the 1-D Heat Equation



Fourier's law of heat transfer: rate of heat transfer proportional to negative temperature gradient,

$$\frac{\text{Rate of heat transfer}}{\text{area}} = -K_0 \frac{\partial u}{\partial x} \qquad (1)$$

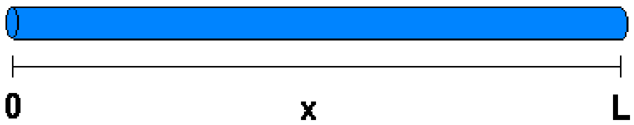
Governing Equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \qquad (2)$$

$$\kappa = \frac{K_0}{c\rho} \qquad (3)$$

Initial Condition and Boundary Conditions

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$$



To make use of the Heat Equation, we need more information:

1. **Initial Condition(IC):** in this case ,the initial temperature distribution in the rod u(x,0).

2. **Boundary Conditions(BC):** in this case, the temperature of the rod is affected by what happens at the ends, x =0,l. What happens to the temperature at the end of the rod must be specified. In reality, the BCs can be complicated. Here we consider three simple cases for the boundary at x =0.

(I) Temperature prescribed at a boundary. For $t > 0$,

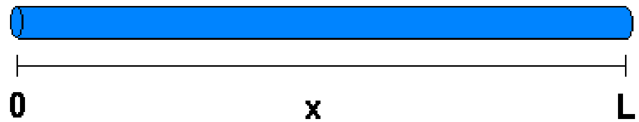
Drichlit BC $$u(0,t) = u_1(t).$$

(II) Insulated boundary. The heat flow can be prescribed at the boundaries,

Neumann BC $$-K_0 \frac{\partial u}{\partial x}(0,t) = \phi_1(t)$$

(III) Mixed condition: an equation involving $u(0,t)$, $\partial u/\partial x\,(0,t)$, etc.

Example 1: Consider a rod of length l with insulated sides is given an initial temperature distribution of f (x) degree C, for 0 <x<l. Find u(x,t) at subsequent times t> 0 if end of rod are kept at 0^{o} C. The **Heat Eqn** and corresponding **IC** and **BCs** are thus



PDE:	$u_t = \kappa u_{xx},$	$0 < x < l,$	(4)
IC:	$u(x,0) = f(x),$	$0 < x < l,$	(5)
BC:	$u(0,t) = u(L,t) = 0,$	$t > 0.$	(6)

Physical intuition: we expect $u \to 0 \text{ as } t \to \infty.$

Non-dimensionalization

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \qquad (2)$$

Dimensional (or physical) terms in the PDE (2): k, l, x, t, u. Others could be introduced in IC and BCs. To make the solution more meaningful and simpler, we group as many physical constants together as possible. Let the characteristic length, time and temperature be L_*, T_* and U_*, respectively, with dimensions $[L_*] = L$, $[T_*] = T$, $[U_*] = U$. Introduce dimensionless variables via

$$\hat{x} = \frac{x}{L_*}, \qquad \hat{t} = \frac{t}{T_*}, \qquad \hat{u}\left(\hat{x}, \hat{t}\right) = \frac{u\left(x,t\right)}{U_*}, \qquad \hat{f}\left(\hat{x}\right) = \frac{f\left(x\right)}{U_*}. \qquad (7)$$

The variables $\hat{x}$, $\hat{t}$, $\hat{u}$ are dimensionless (i.e. no units, $[\hat{x}] = 1$). The sensible choice for the characteristic length is $L_* = l$, the length of the rod. While x is in the range $0 < x < l$, $\hat{x}$ is in the range $0 < \hat{x} < 1$.

The choice of dimensionless variables is an ART. Sometimes the statement of the problem gives hints: e.g. the length l of the rod (1 is nicer to deal with than l, an unspecified quantity). Often you have to solve the problem first, look at the solution, and try to simplify the notation.

From the chain rule,

$$
\begin{aligned}
u_t &= \frac{\partial u}{\partial t} = U_* \frac{\partial \hat{u}}{\partial \hat{t}} \frac{\partial \hat{t}}{\partial t} = \frac{U_*}{T_*} \frac{\partial \hat{u}}{\partial \hat{t}}, \\
u_x &= \frac{\partial u}{\partial x} = U_* \frac{\partial \hat{u}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} = \frac{U_*}{L_*} \frac{\partial \hat{u}}{\partial \hat{x}} \\
u_{xx} &= \frac{U_*}{L_*^2} \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}
\end{aligned}
$$

Substituting these into the Heat Eqn (4) gives

$$
u_t = \kappa u_{xx} \qquad \Rightarrow \qquad \frac{\partial \hat{u}}{\partial \hat{t}} = \frac{T_* \kappa}{L_*^2} \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}
$$

To make the PDE simpler, we choose $T_* = L_*^2/\kappa = l^2/\kappa$, so that

$$
\frac{\partial \hat{u}}{\partial \hat{t}} = \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}, \qquad 0 < \hat{x} < 1, \qquad \hat{t} > 0.
$$

The characteristic (diffusive) time scale in the problem is $T_* = l^2/\kappa$. For different substances, this gives time scale over which diffusion takes place in the problem. The IC (5) and BC (6) must also be non-dimensionalized:

$$
\begin{aligned}
&\text{IC:} \qquad \hat{u}(\hat{x}, 0) = \hat{f}(\hat{x}), \qquad && 0 < \hat{x} < 1, \\
&\text{BC:} \qquad \hat{u}\left(0, \hat{t}\right) = \hat{u}\left(1, \hat{t}\right) = 0, \qquad && \hat{t} > 0.
\end{aligned}
$$

Dimensionless Problem

Dropping hats, we have the dimensionless problem

$$\begin{aligned} &\text{PDE:} \quad && u_t = u_{xx}, && 0 < x < 1, && (8) \\ &\text{IC:} \quad && u(x,0) = f(x), && 0 < x < 1, && (9) \\ &\text{BC:} \quad && u(0,t) = u(1,t) = 0, && t > 0, && (10) \end{aligned}$$

where x, t are dimensionless scalings of physical position and time.

Separation of Variables

We look for a solution to the dimensionless Heat Equation (8) – (10) of the form

$$u(x,t) = X(x)\,T(t) \tag{11}$$

Take the relevant partial derivatives:

$$u_{xx} = X''(x)\,T(t)\,, \qquad u_t = X(x)\,T'(t)$$

where primes denote differentiation of a single-variable function. The PDE (8), $u_t = u_{xx}$, becomes

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

The left hand side (l.h.s.) depends only on t and the right hand side (r.h.s.) only depends on x. Hence if t varies and x is held fixed, the r.h.s. is constant, and hence T'/T must also be constant, which we set to $-\lambda$ by convention:

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda, \qquad \lambda = \text{constant}. \tag{12}$$

The BCs become, for $t > 0$,

$$\begin{aligned} u(0,t) &= X(0)\,T(t) = 0 \\ u(1,t) &= X(1)\,T(t) = 0 \end{aligned}$$

Taking $T(t) = 0$ would give $u = 0$ for all time and space (called the trivial solution), from (11), which does not satisfy the IC unless $f(x) = 0$. If you are lucky and $f(x) = 0$, then $u = 0$ is the solution (this has to do with uniqueness of the solution, which we'll come back to). If $f(x)$ is not zero for all $0 < x < 1$, then $T(t)$ cannot be zero and hence the above equations are only satisfied if

$$X(0) = X(1) = 0. \tag{13}$$

Solving for X(x) – Case(1)

We obtain a boundary value problem for $X(x)$, from (12) and (13),

$$X''(x) + \lambda X(x) = 0, \qquad 0 < x < 1, \tag{14}$$

$$X(0) = X(1) = 0. \tag{15}$$

This is an example of a Sturm-Liouville problem (from your ODEs class).

There are 3 cases: $\lambda > 0$, $\lambda < 0$ and $\lambda = 0$.

(i) $\lambda < 0$. Let $\lambda = -k^2 < 0$. Then the solution to (14) is

$$X = Ae^{kx} + Be^{-kx}$$

for integration constants A, B found from imposing the BCs (15),

$$X(0) = A + B = 0, \qquad X(1) = Ae^{k} + Be^{-k} = 0.$$

The first gives $A = -B$, the second then gives $A\left(e^{2k} - 1\right) = 0$, and since $|k| > 0$ we have $A = B = u = 0$, which is the trivial solution. Thus we discard the case $\lambda < 0$.

Solving for X(x) – Case(2, 3)

(ii) $\lambda = 0$. Then $X(x) = Ax + B$ and the BCs imply $0 = X(0) = B$, $0 = X(1) = A$, so that $A = B = u = 0$. We discard this case also.

(iii) $\lambda > 0$. In this case, (14) is the simple harmonic equation whose solution is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right). \qquad (16)$$

The BCs imply $0 = X(0) = A$, and $B\sin\sqrt{\lambda} = 0$. We don't want $B = 0$, since that would give the trivial solution $u = 0$, so we must have

$$\sin\sqrt{\lambda} = 0. \qquad (17)$$

Thus $\sqrt{\lambda} = n\pi$, for any nonzero integer n ($n = 1, 2, 3, ...$). We use subscripts to label the particular n-value. The values of λ are called the eigenvalues of the Sturm-Liouville problem (14),

$$\lambda_n = n^2\pi^2, \qquad n = 1, 2, 3, ...$$

and the corresponding solutions of (14) are called the eigenfunctions of the Sturm-Liouville problem (14),

$$X_n(x) = b_n \sin(n\pi x), \qquad n = 1, 2, 3, ... \qquad (18)$$

We have assumed that $n > 0$, since $n < 0$ gives the same solution as $n > 0$.

Solving for T (t)

When solving for $X(x)$, we found that non-trivial solutions arose for $\lambda = n^2\pi^2$ for all nonzero integers n. The equation for $T(t)$ is thus, from (12),

$$T'(t) = -n^2\pi^2 T(t)$$

and, for n, the solution is

$$T_n = c_n e^{-n^2\pi^2 t}, \qquad n = 1, 2, 3, ... \tag{19}$$

where the c_n's are constants of integration.

Full Solution u(x,t) - Principle of Superposition

Putting things together, we have, from (11), (18) and (19),

$$u_n(x,t) = B_n \sin(n\pi x)\, e^{-n^2\pi^2 t}, \qquad n = 1, 2, 3, \ldots \tag{20}$$

where $B_n = c_n b_n$. Each function $u_n(x,t)$ is a solution to the PDE (8) and the BCs (10). But, in general, they will not individually satisfy the IC (9),

$$u_n(x,0) = B_n \sin(n\pi x) = f(x).$$

We now apply the principle of superposition: if u_1 and u_2 are two solutions to the PDE (8) and BC (10), then $c_1 u_1 + c_2 u_2$ is also a solution, for any constants c_1, c_2. This relies on the linearity of the PDE and BCs. We will, of course, soon make this more precise....

Full Solution u(x,t) - Fourier Sine Series

Since each $u_n(x,0)$ is a solution of the PDE, then the principle of superposition says any finite sum is also a solution. To solve the IC, we will probably need all the solutions u_n, and form the infinite sum (convergence properties to be checked),

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t). \tag{21}$$

$u(x,t)$ satisfies the BCs (10) since each $u_n(x,t)$ does. Assuming term-by-term differentiation holds (to be checked) for the infinite sum, then $u(x,t)$ also satisfies the PDE (8). To satisfy the IC, we need to find B_n's such that

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} u_n(x,0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x). \tag{22}$$

This is the Fourier Sine Series of $f(x)$.

Full Solution u(x,t) - Orthogonality Relation

To solve for the B_n's, we use the orthogonality property for the eigenfunctions $\sin(n\pi x)$,

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \end{cases} = \frac{1}{2}\delta_{mn} \tag{23}$$

where δ_{mn} is the kronecker delta,

$$\delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

The orthogonality relation (23) is derived by substituting

$$2\sin(m\pi x)\sin(n\pi x) = \cos((m-n)\pi x) - \cos((m+n)\pi x)$$

into the integral on the left hand side of (23) and noting

$$\int_0^1 \cos(m\pi x)\,dx = \delta_{m0}.$$

The orthogonality of the functions $\sin(n\pi x)$ is analogous to that of the unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ in 2-space; the integral from 0 to 1 in (23) above is analogous to the dot product in 2-space.

Full Solution u(x,t) - Solve for B_n's

To solve for the B_n's, we multiply both sides of (22) by $\sin(m\pi x)$ and integrate from 0 to 1:

$$\int_0^1 \sin(m\pi x) f(x)\, dx = \sum_{n=1}^{\infty} B_n \int_0^1 \sin(n\pi x)\sin(m\pi x)\, dx$$

Substituting (23) into the right hand side yields

$$\int_0^1 \sin(m\pi x) f(x)\, dx = \sum_{n=1}^{\infty} B_n \frac{1}{2}\delta_{nm}$$

By definition of δ_{nm}, the only term that is non-zero in the infinite sum is the one where $n = m$, thus

$$\int_0^1 \sin(m\pi x) f(x)\, dx = \frac{1}{2} B_m$$

Rearranging yields

$$B_m = 2\int_0^1 \sin(m\pi x) f(x)\, dx. \tag{24}$$

The full solution is, from (20) and (21),

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, e^{-n^2\pi^2 t}, \tag{25}$$

where B_n are given by (24).

To derive the solution (25) of the Heat Equation (8) and corresponding BCs (10) and IC (9), we used properties of linear operators and infinite series that need justification.

The Classical Separation of Variables Method

- It can only be applied directly to linear homogeneous problems with homogeneous boundary conditions.
- The basic idea is to assume that the original function of two variables can be written as a product of two functions, each of which is only dependent upon a single independent variable.
- The separated form of the solution is inserted into the original linear PDE and, after some manipulation, one obtains two homogeneous ODEs that can be solved by traditional means.
- If the original boundary conditions (BCs) for the problem are homogeneous, one of the ODEs will give a Sturm-Liouville type problem, which leads to a set of orthogonal eigen-functions as solutions.
- Because the original PDE is linear, its final solution is formed as a linear combination of the individual solutions - which gives rise to a solution written in the form of an infinite series.
- A final condition imposed on the problem (either an initial condition or a remaining BC that has not yet been used) is used to determine the unknown expansion coefficients in the infinite series solution. Since the basis functions are orthogonal, these coefficients are readily determined.
- The analytical solution is complete once the coefficients have been determined. However, since the solution is still written in the form of an infinite series expansion, it is often evaluated and plotted using computer techniques - thus completing the overall problem.

Example : Cooling of a Rod from a Constant Initial Temperature

Suppose the initial temperature distribution $f(x)$ in the rod is constant, i.e. $f(x) = u_0$. The solution for the temperature in the rod is (25),

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, e^{-n^2\pi^2 t},$$

where, from (24), the Fourier coefficients are given by

$$B_n = 2\int_0^1 \sin(n\pi x)\, f(x)\, dx = 2u_0 \int_0^1 \sin(n\pi x)\, dx.$$

Calculating the integrals gives

$$B_n = 2u_0 \int_0^1 \sin(n\pi x)\, dx = -2u_0 \frac{\cos(n\pi) - 1}{n\pi} = -\frac{2u_0}{n\pi}\left((-1)^n - 1\right) = \begin{cases} 0 & n \text{ even} \\ \frac{4u_0}{n\pi} & n \text{ odd} \end{cases}$$

In other words,

$$B_{2n} = 0, \qquad B_{2n-1} = \frac{4u_0}{(2n-1)\pi}$$

and the solution becomes

$$u(x,t) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x)}{(2n-1)} \exp\left(-(2n-1)^2 \pi^2 t\right). \qquad (26)$$

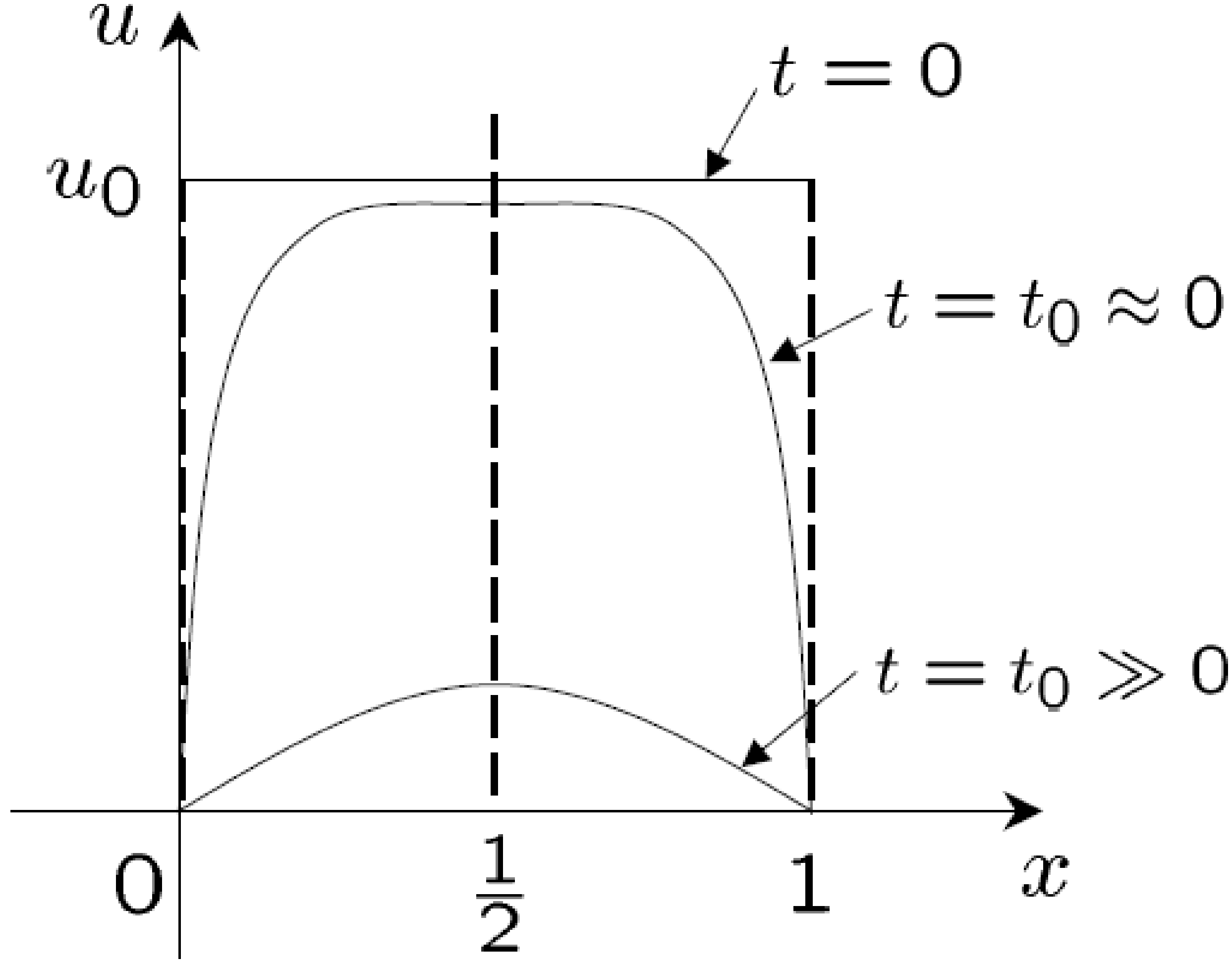
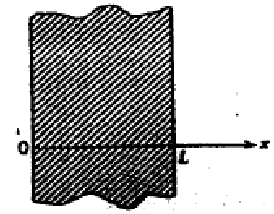


Figure 1: Spatial temperature profiles $u(x, t_0)$.

TABLE 2-2 The Solution $X(\beta_m, x)$, the Norm $N(\beta_m)$ and the Eigenvalues β_m of the Differential Equation

$$\frac{d^2X(x)}{dx^2}+\beta^2X(x)=0 \quad \text{in} \quad 0<x<L$$



Subject to the Boundary Conditions Shown in the Table Below

No.	Boundary Condition at $x=0$	Boundary Condition at $x=L$	$X(\beta_m, x)$	$1/N(\beta_m)$	Eigenvalues β_m's are Positive Roots of
1	$-\frac{dX}{dx}+H_1X=0$	$\frac{dX}{dx}+H_2X=0$	$\beta_m\cos\beta_m x+H_1\sin\beta_m x$	$2\left[(\beta_m^2+H_1^2)\left(L+\frac{H_2}{\beta_m^2+H_2^2}\right)+H_1\right]^{-1}$	$\tan\beta_m L=\frac{\beta_m(H_1+H_2)}{\beta_m^2-H_1H_2}$
2	$-\frac{dX}{dx}+H_1X=0$	$\frac{dX}{dx}=0$	$\cos\beta_m(L-x)$	$2\frac{\beta_m^2+H_1^2}{L(\beta_m^2+H_1^2)+H_1}$	$\beta_m\tan\beta_m L=H_1$
3	$-\frac{dX}{dx}+H_1X=0$	$X=0$	$\sin\beta_m(L-x)$	$2\frac{\beta_m^2+H_1^2}{L(\beta_m^2+H_1^2)+H_1}$	$\beta_m\cot\beta_m L=-H_1$
4	$\frac{dX}{dx}=0$	$\frac{dX}{dx}+H_2X=0$	$\cos\beta_m x$	$2\frac{\beta_m^2+H_2^2}{L(\beta_m^2+H_2^2)+H_2}$	$\beta_m\tan\beta_m L=H_2$
5	$\frac{dX}{dx}=0$	$\frac{dX}{dx}=0$	*$\cos\beta_m x$	$\frac{2}{L}$ for $\beta_m\neq0$; $\frac{1}{L}$ for $\beta_0=0$*	$\sin\beta_m L=0$*
6	$\frac{dX}{dx}=0$	$X=0$	$\cos\beta_m x$	$\frac{2}{L}$	$\cos\beta_m L=0$
7	$X=0$	$\frac{dX}{dx}+H_2X=0$	$\sin\beta_m x$	$2\frac{\beta_m^2+H_2^2}{L(\beta_m^2+H_2^2)+H_2}$	$\beta_m\cot\beta_m L=-H_2$
8	$X=0$	$\frac{dX}{dx}=0$	$\sin\beta_m x$	$\frac{2}{L}$	$\cos\beta_m L=0$
9	$X=0$	$X=0$	$\sin\beta_m x$	$\frac{2}{L}$	$\sin\beta_m L=0$

*For this particular case $\beta_0=0$ is also an eigenvalue corresponding to $X=1$.

Introduction to Laplace Transforms

•The method of Laplace transform has been widely used in time-dependent heat conduction problems, because the partial derivative with respect to the time-variable can be removed from the differential equation by Laplace transformation.

Definition

$$\mathcal{L}[f(t)] = \tilde{f}(s) = \int_0^\infty e^{-st} f(t)\, dt, \quad \text{for } s > 0. \tag{1.1}$$

Linear operator

$$\mathcal{L}[af + bg] = a\mathcal{L}[f] + b\mathcal{L}[g]$$

•Conditions for the existence of Laplace transform:

- Function F(t) is continuous or piecewise continuous in any interval between t_1 and t_2 for $t_1>0$
- The term $t^n|F(t)|$ is bounded as *t* approaches to 0^- for some number n when n<1.
- Function F(t) is of exponential order, namely $e^{-\gamma t}|F(t)|$ is bounded for some positive number *γ*, as t approaches to infinity.

Some Examples

$$\text{For } f(t) = 1,\ \ \mathcal{L}[1] = \int_0^\infty e^{-st}\,dt \ = \left[-\frac{1}{s}e^{-st}\right]_0^\infty = \frac{1}{s}.$$

$$\text{For } f(t) = t,\ \ \mathcal{L}[t] = \int_0^\infty e^{-st} t\,dt \ = \left[-\frac{1}{s}e^{-st} t\right]_0^\infty + \int_0^\infty \frac{1}{s}e^{-st} = \frac{1}{s^2}.$$

$$\text{For } f(t) = \frac{dy}{dt},\ \mathcal{L}\left[\frac{dy}{dt}\right] = \int_0^\infty e^{-st}\frac{dy}{dt}\,dt \ = \left[e^{-st}y\right]_0^\infty + \int_0^\infty se^{-st}y\,dt \ = -y(0) + s\tilde{y}(s).$$

For $f(t) = e^{at}$, a constant,

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-st}e^{at}\,dt \ = \int_0^\infty e^{-(s-a)t}\,dt \ = \left[-\frac{1}{s-a}e^{-(s-a)t}\right]_0^\infty = \frac{1}{s-a},\ \ s > a.$$

Laplace Transform of the Derivative

Suppose that the Laplace transform of y(t) is Y(s). Then the Laplace Transform of y'(t) is :

$$L[y'(t)](s) = sY(s) - y(0)$$

For the second derivative we have

$$L[y''(t)](s) = s^2Y(s) - sy(0) - y'(0)$$

For the n'th derivative we have

$$L[y^{(n)}(t)](s) = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - y^{(n-1)}(0)$$

Derivatives of the Laplace Transform

Let Y(s) be the Laplace Transform of y(t). Then

$$L[t^n y(t)](s) = (-1)^n \frac{d^nY}{ds^n}(s)$$

Table of Laplace Transforms

$$\bar{v}(p) = \int_0^\infty e^{-pt} v(t)\, dt.$$

We write $q = \sqrt{(p/\kappa)}$. κ and x are always real and positive. α and h are unrestricted.

	$\bar{v}(p)$	$v(t)$
1.	$\frac{1}{p}$	1
2.	$\frac{1}{p^{\nu+1}}, \quad \nu > -1$	$\frac{t^\nu}{\Gamma(\nu+1)}$
3.	$\frac{1}{p+\alpha}$	$e^{-\alpha t}$
4.	$\frac{\omega}{p^2+\omega^2}$	$\sin \omega t$
5.	$\frac{p}{p^2+\omega^2}$	$\cos \omega t$
6.	e^{-qx}	$\frac{x}{2\sqrt{(\pi\kappa t^3)}} e^{-x^2/4\kappa t}$
7.	$\frac{e^{-qx}}{q}$	$\left(\frac{\kappa}{\pi t}\right)^{\frac{1}{2}} e^{-x^2/4\kappa t}$
8.	$\frac{e^{-qx}}{p}$	$\operatorname{erfc} \frac{x}{2\sqrt{(\kappa t)}}$
9.	$\frac{e^{-qx}}{pq}$	$2\left(\frac{\kappa t}{\pi}\right)^{\frac{1}{2}} e^{-x^2/4\kappa t} - x \operatorname{erfc} \frac{x}{2\sqrt{(\kappa t)}}$
10.	$\frac{e^{-qx}}{p^2}$	$\left(t+\frac{x^2}{2\kappa}\right) \operatorname{erfc} \frac{x}{2\sqrt{(\kappa t)}} - x\left(\frac{t}{\pi\kappa}\right)^{\frac{1}{2}} e^{-x^2/4\kappa t}$
11.	$\frac{e^{-qx}}{p^{1+\frac{1}{2}n}}, \quad n = 0, 1, 2,...$	$(4t)^{\frac{1}{2}n}\, \mathrm{i}^n \operatorname{erfc} \frac{x}{2\sqrt{(\kappa t)}}$
12.	$\frac{e^{-qx}}{q+h}$	$\left(\frac{\kappa}{\pi t}\right)^{\frac{1}{2}} e^{-x^2/4\kappa t} - h\kappa e^{hx+\kappa t h^2} \times$ $\times \operatorname{erfc}\left\{\frac{x}{2\sqrt{(\kappa t)}} + h\sqrt{(\kappa t)}\right\}$
13.	$\frac{e^{-qx}}{q(q+h)}$	$\kappa e^{hx+\kappa t h^2} \operatorname{erfc}\left\{\frac{x}{2\sqrt{(\kappa t)}} + h\sqrt{(\kappa t)}\right\}$
14.	$\frac{e^{-qx}}{p(q+h)}$	$\frac{1}{h} \operatorname{erfc} \frac{x}{2\sqrt{(\kappa t)}} - \frac{1}{h} e^{hx+\kappa t h^2} \times$ $\times \operatorname{erfc}\left\{\frac{x}{2\sqrt{(\kappa t)}} + h\sqrt{(\kappa t)}\right\}$

	$\bar{v}(p)$	$v(t)$
15.	$\dfrac{e^{-qx}}{pq(q+h)}$	$\frac{2}{h}\left(\frac{\kappa t}{\pi}\right)^{\frac{1}{2}} e^{-x^2/4\kappa t}-\frac{(1+hx)}{h^2}\operatorname{erfc}\frac{x}{2\sqrt{(\kappa t)}}+$ $+\frac{1}{h^2}e^{hx+\kappa t h^2}\operatorname{erfc}\left\{\frac{x}{2\sqrt{(\kappa t)}}+h\sqrt{(\kappa t)}\right.$
16.	$\dfrac{e^{-qx}}{q^{n+1}(q+h)}$	$\frac{\kappa}{(-h)^n}e^{hx+\kappa t h^2}\operatorname{erfc}\left\{\frac{x}{2\sqrt{(\kappa t)}}+h\sqrt{(\kappa t)}\right\}-$ $-\frac{\kappa}{(-h)^n}\sum_{r=0}^{n-1}[-2h\sqrt{(\kappa t)}]^r\, i^r\operatorname{erfc}\frac{x}{2\sqrt{(\kappa t}}$
17.	$\dfrac{e^{-qx}}{(q+h)^2}$	$-2h\left(\frac{\kappa^3 t}{\pi}\right)^{\frac{1}{2}}e^{-x^2/4\kappa t}+\kappa(1+hx+2h^2\kappa t)e^{hx+\kappa t h}$ $\times\operatorname{erfc}\left\{\frac{x}{2\sqrt{(\kappa t)}}+h\sqrt{(\kappa t)}\right.$
18.	$\dfrac{e^{-qx}}{p(q+h)^2}$	$\frac{1}{h^2}\operatorname{erfc}\frac{x}{2\sqrt{(\kappa t)}}-\frac{2}{h}\left(\frac{\kappa t}{\pi}\right)^{\frac{1}{2}}e^{-x^2/4\kappa t}-$ $-\frac{1}{h^2}\{1-hx-2h^2\kappa t\}e^{hx+\kappa t h^2}\times$ $\times\operatorname{erfc}\left\{\frac{x}{2\sqrt{(\kappa t)}}+h\sqrt{(\kappa t)}\right\}$
19.	$\dfrac{e^{-qx}}{p-\alpha}$	$\frac{1}{2}e^{\alpha t}\left\{e^{-x\sqrt{(\alpha/\kappa)}}\operatorname{erfc}\left[\frac{x}{2\sqrt{(\kappa t)}}-\sqrt{(\alpha t)}\right]+\right.$ $\left.+e^{x\sqrt{(\alpha/\kappa)}}\operatorname{erfc}\left[\frac{x}{2\sqrt{(\kappa t)}}+\sqrt{(\alpha t)}\right]\right\}$
20.	$\dfrac{1}{p^{\frac{3}{4}}}e^{-qx}$	$\frac{1}{\pi}\left(\frac{x}{2t\kappa^{\frac{1}{2}}}\right)^{\frac{1}{2}}e^{-x^2/8\kappa t}K_{\frac{1}{4}}\left(\frac{x^2}{8\kappa t}\right)$
21.	$\dfrac{1}{p^{\frac{1}{2}}}K_{2\nu}(qx)$	$\frac{1}{2\sqrt{(\pi t)}}e^{-x^2/8\kappa t}K_\nu\left(\frac{x^2}{8\kappa t}\right)$
22.	$\left.\begin{matrix} I_\nu(qx')K_\nu(qx), & x>x' \\ I_\nu(qx)K_\nu(qx'), & x<x' \end{matrix}\right\}$	$\frac{1}{2t}e^{-(x^2+x'^2)/4\kappa t}I_\nu\left(\frac{xx'}{2\kappa t}\right), \quad \nu\geqslant 0$
23.	$K_0(qx)$	$\frac{1}{2t}e^{-x^2/4\kappa t}$
24.	$\dfrac{1}{p}e^{x/p}$	$I_0[2\sqrt{(xt)}]$
25.	$\dfrac{\exp\{xp-x[(p+a)(p+b)]^{\frac{1}{2}}\}}{[(p+a)(p+b)]^{\frac{1}{2}}}$	$e^{-\frac{1}{2}(a+b)(t+x)}I_0\{\frac{1}{2}(a-b)[t(t+2x)]^{\frac{1}{2}}\}$
26.	$p^{\frac{1}{2}\nu-1}K_\nu(x\sqrt{p})$	$x^{-\nu}2^{\nu-1}\int_{x^2/4t}^{\infty}e^{-u}u^{\nu-1}\,du$
27.	$[p-\sqrt{(p^2-x^2)}]^\nu, \quad \nu>0.$	$\nu x^\nu I_\nu(xt)/t$
28.	$\dfrac{\exp\{x[(p+a)^{\frac{1}{2}}-(p+b)^{\frac{1}{2}}]^2\}}{(p+a)^{\frac{1}{2}}(p+b)^{\frac{1}{2}}[(p+a)^{\frac{1}{2}}+(p+b)^{\frac{1}{2}}]^{2\nu}}, \quad \nu\geqslant 0$	$\dfrac{t^{\frac{1}{2}\nu}e^{-\frac{1}{2}(a+b)t}I_\nu[\frac{1}{2}(a-b)t^{\frac{1}{2}}(t+4x)^{\frac{1}{2}}]}{(a-b)^\nu(t+4x)^{\frac{1}{2}\nu}}$

	$\bar{v}(p)$	$v(t)$
29.	$\dfrac{e^{-qx}}{(p-\alpha)^2}$	$\tfrac{1}{2}e^{\alpha t}\Big\{\Big(t-\dfrac{x}{2\sqrt{(\kappa\alpha)}}\Big)\times$ $\times e^{-x\sqrt{(\alpha/\kappa)}}\operatorname{erfc}\Big[\dfrac{x}{2\sqrt{(\kappa t)}}-\sqrt{(\alpha t)}\Big]+$ $+\Big(t+\dfrac{x}{2\sqrt{(\kappa\alpha)}}\Big)e^{x\sqrt{(\alpha/\kappa)}}\operatorname{erfc}\Big[\dfrac{x}{2\sqrt{(\kappa t)}}+\sqrt{(\alpha t)}\Big]\Big\}$
30.	$\dfrac{e^{-qx}}{q(p-\alpha)}$	$\tfrac{1}{2}e^{\alpha t}\Big(\dfrac{\kappa}{\alpha}\Big)^{\frac{1}{2}}\Big\{e^{-x\sqrt{(\alpha/\kappa)}}\operatorname{erfc}\Big[\dfrac{x}{2\sqrt{(\kappa t)}}-\sqrt{(\alpha t)}\Big]-$ $-e^{x\sqrt{(\alpha/\kappa)}}\operatorname{erfc}\Big[\dfrac{x}{2\sqrt{(\kappa t)}}+\sqrt{(\alpha t)}\Big]\Big\}$
31.	$\dfrac{e^{-qx}}{(p-\alpha)(q+h)},\quad \alpha\neq\kappa h^2$	$\tfrac{1}{2}e^{\alpha t}\Big\{\dfrac{\kappa^{\frac{1}{2}}}{h\kappa^{\frac{1}{2}}+\alpha^{\frac{1}{2}}}e^{-x\sqrt{(\alpha/\kappa)}}\operatorname{erfc}\Big[\dfrac{x}{2\sqrt{(\kappa t)}}-\sqrt{(\alpha t)}\Big]+$ $+\dfrac{\kappa^{\frac{1}{2}}}{h\kappa^{\frac{1}{2}}-\alpha^{\frac{1}{2}}}e^{x\sqrt{(\alpha/\kappa)}}\operatorname{erfc}\Big[\dfrac{x}{2\sqrt{(\kappa t)}}+\sqrt{(\alpha t)}\Big]\Big\}-$ $-\dfrac{h\kappa}{h^2\kappa-\alpha}e^{hx+h^2\kappa t}\operatorname{erfc}\Big[\dfrac{x}{2\sqrt{(\kappa t)}}+h\sqrt{(\kappa t)}\Big]$
32.	$\dfrac{1}{p}\ln p$	$-\ln(Ct),\quad \ln C=\gamma=0{\cdot}5772...$
33.	$p^{\frac{1}{2}\nu}K_\nu(x\sqrt{p})$	$\dfrac{x^\nu}{(2t)^{\nu+1}}e^{-x^2/4t}$

Numerical Laplace Inversion Methods

H. Hassanzadeh, M. Pooladi-Darvish / Applied Mathematics and Computation 189 (2007) 1966–1981

Using the Laplace transform for solving differential equations, however, sometimes leads to solutions in the Laplace domain that are not readily invertible to the real domain by analytical means. Numerical inversion methods are then used to convert the obtained solution from the Laplace domain into the real domain.

- **Some Numerical inversion methods**:
  - **Stehfest's method**: It is used in many engineering applications is easy to implement and leads to accurate results for many problems including diffusion-dominated ones and solutions that behave like e^{-t} type functions. However, this method fails to predict e^{+t} type functions or those with an oscillatory response, such as sine and wave functions.

  - **Zakian's method**: Zakian's algorithm is accurate for e^{+t} functions, diffusion problems, and fractional functions in the Laplace domain.

  - **Fourier series method** : the most powerful but also the most computationally expensive.

  - **Schapery's method**: It is an analytical inversion method, may be used to estimate the global behavior of the solution before applying a numerical inversion method.

Stehfest's Method

This numerical Laplace inversion technique was first introduced by Graver [4] and its algorithm then offered by Stehfest [5]. This method has been used extensively in petroleum engineering literature [6].

Stehfest's algorithm approximates the time domain solution using the following equation [5,6]:

$$f(t) = \frac{\ln 2}{t} \sum_{i=1}^{n} V_i F\left(\frac{\ln 2}{t} i\right), \tag{3}$$

where V_i is given by the following equation:

$$V_i = (-1)^{\left(\frac{n}{2}+1\right)} \sum_{k=\left(\frac{i+1}{2}\right)}^{\min\left(i,\frac{n}{2}\right)} \frac{k^{\left(\frac{n}{2}+1\right)}(2k)!}{\left(\frac{n}{2}-k\right)!k!(i-k)!(2k-1)!}. \tag{4}$$

The parameter n is the number of terms used in the summation in Eq. (3) and should be optimized by trial and error. Increasing n increases the accuracy of the result up to a point, and then the accuracy declines because of increasing round-off errors. An optimal choice of $10 \leqslant n \leqslant 14$ has been reported by Lee et al. for some problem of their interest [6]. This method results in accurate solutions when the time function is in the form of e^{-t}. It is very simple to implement, but it leads to inaccurate solutions for some functions.

Zakian's method

Zakian's method [7,8] approximates the time domain function using the following infinite series of weighted evaluations of domain function [9]:

$$f(t) = \frac{2}{t}\sum_{i=1}^{n} \mathrm{Re}\left\{K_i F\left(\frac{\alpha_i}{t}\right)\right\}. \tag{5}$$

The constants K_i and α_i for $n = 5$ are given in Table 1.

This method is fast and easy to implement, and there is one free parameter, n, to be determined. The parameter n should be optimized to obtain accurate solutions. Zakian's method is suitable for time domain solutions that have a positive exponential term, e^t. The method requires using complex arithmetic. Lee et al. [6] found that an accurate solution is obtained in a well-testing application of single well pulse testing when $n = 10$.

Re	real part of a complex number
s	Laplace variable

Table 1
Five constants for α and K for the Zakian method [10]

i	α	K
1	12.83767675 + j1.666063445	−36902.08210 + j196990.4257
2	12.22613209 + j5.012718792	+61277.02524 − j95408.62551
3	10.93430308 + j8.409673116	−28916.56288 + j18169.18531
4	8.776434715 + j11.92185389	+4655.361138 − j1.901528642
5	5.225453361 + j15.72952905	−118.7414011 − j141.3036911

Fourier Series Method

Dubner and Abate [10] were the first to use the Fourier series technique for the Laplace inversion. The technique is based on choosing the contour of integration in the inversion integral, converting the inversion integral into the Fourier transform, and, then, approximating the transform by a Fourier series. This method approximates the inversion integral using the following equation:

$$f(t) = \frac{e^{at}}{t}\left\{\frac{1}{2}F(a) + \mathrm{Re}\sum_{k=1}^{n} F\left(a + \mathrm{j}\frac{k\pi}{t}\right)(-1)^k\right\},$$

where

$$\mathrm{j} = \sqrt{-1}. \tag{6}$$

The parameters a and n must be optimized for increased accuracy. Lee et al. [6] suggested values of at between 4 and 5.

Schapery's method

Schapery's method [11] is a simple analytical inversion technique that is very simple to implement. It leads to approximate solutions, and therefore is suitable for the initial evaluation of the time domain solution: for example, when the global behaviour of the time domain solution needs to be predicted. Jelmert [12] reported an accuracy of 5% in a petroleum engineering application of linear flow. This method is applicable when the Laplace domain solution is in the form of $sF(s) = As^m$ and $m < 1$. In this case, the following relationship exists between the real time solution and its form in the Laplace domain [12]:

$$f(t) \approx [sF(s)]_{s=\frac{1}{\gamma t}}, \tag{7}$$

where $\gamma = 1.781$.

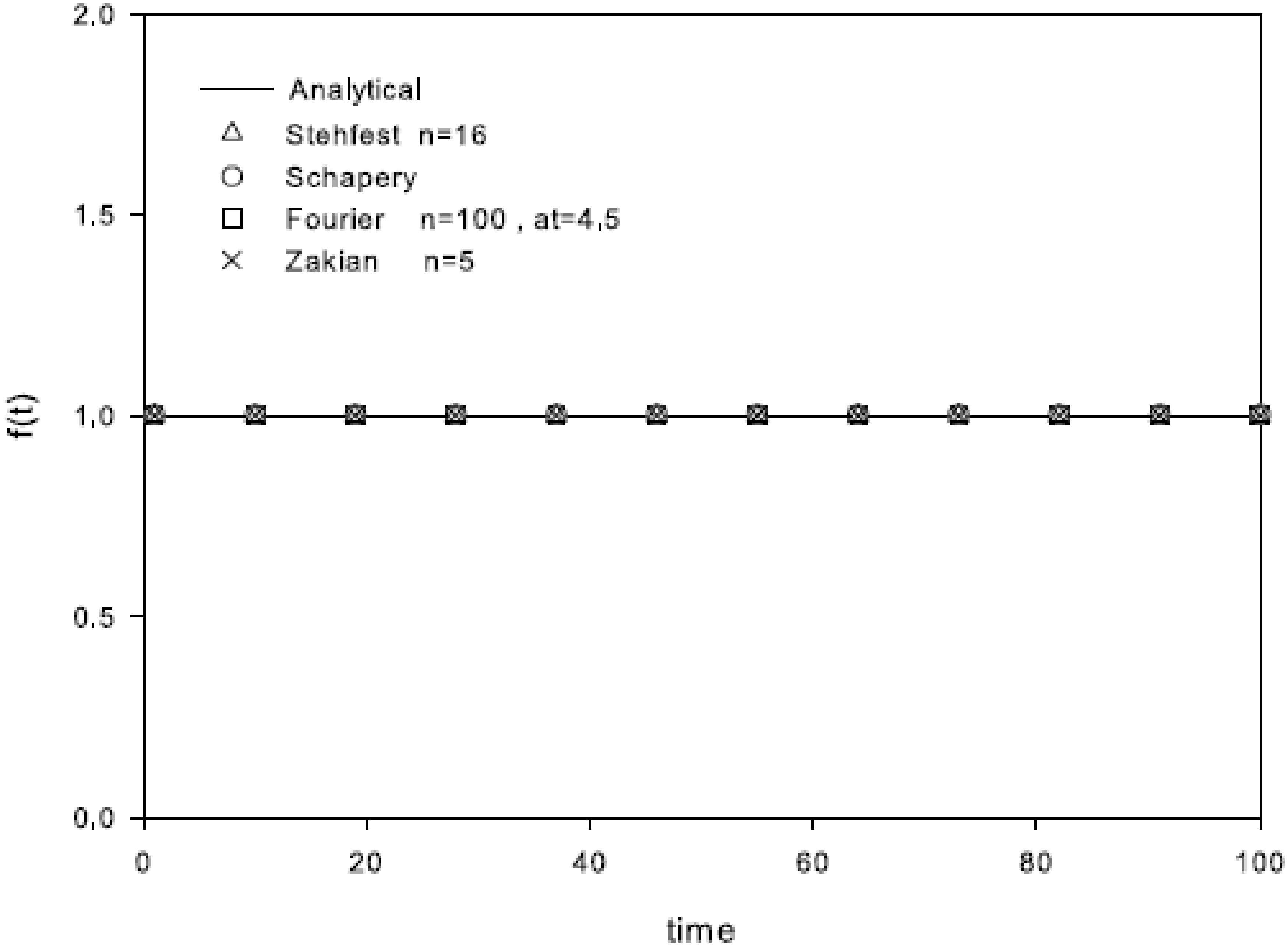


Fig. 1. Comparison of different numerical inversion methods for $f(t) = 1$.

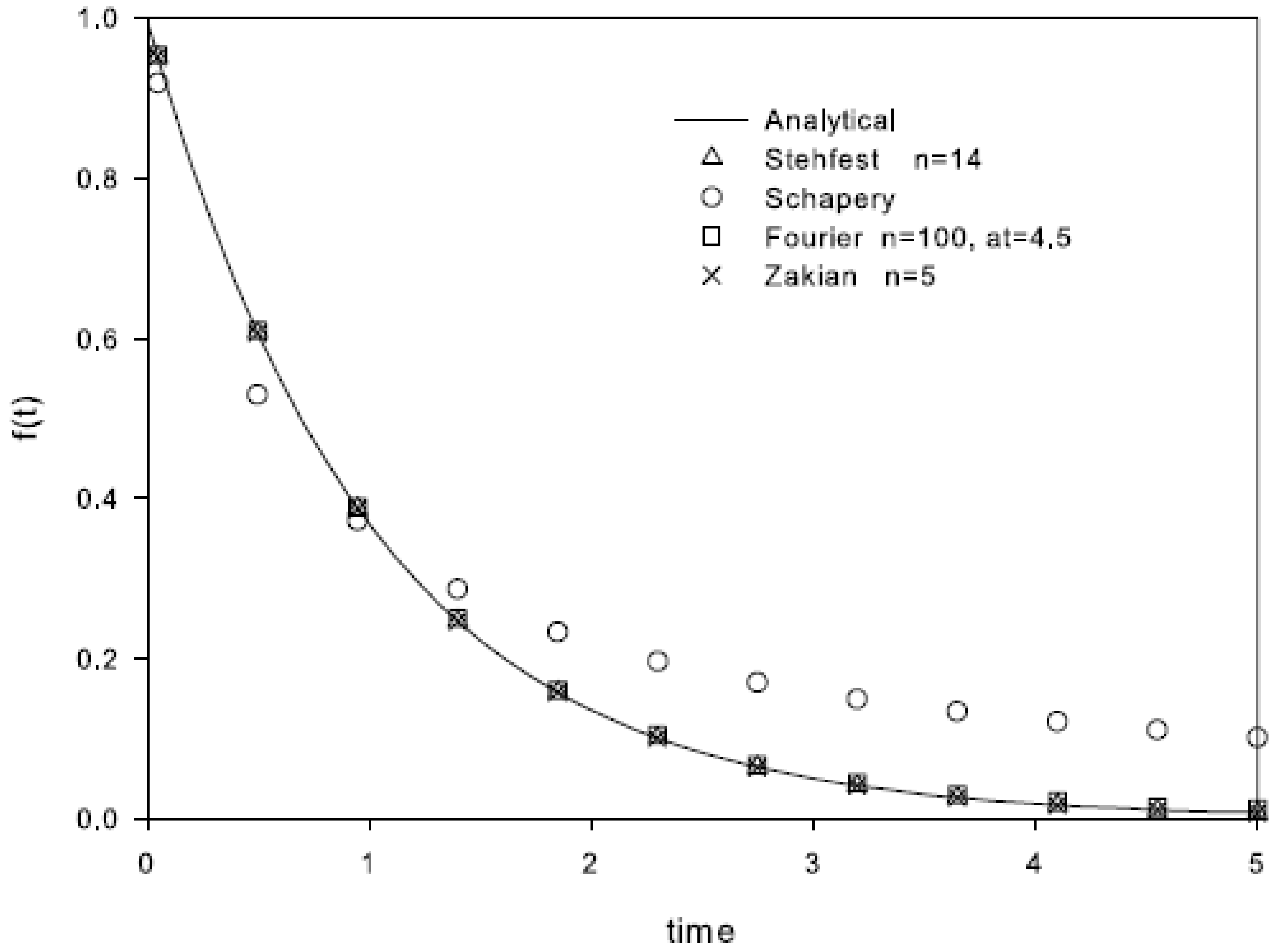


Fig. 2. Comparison of different numerical inversion methods for $f(t) = e^{-t}$.

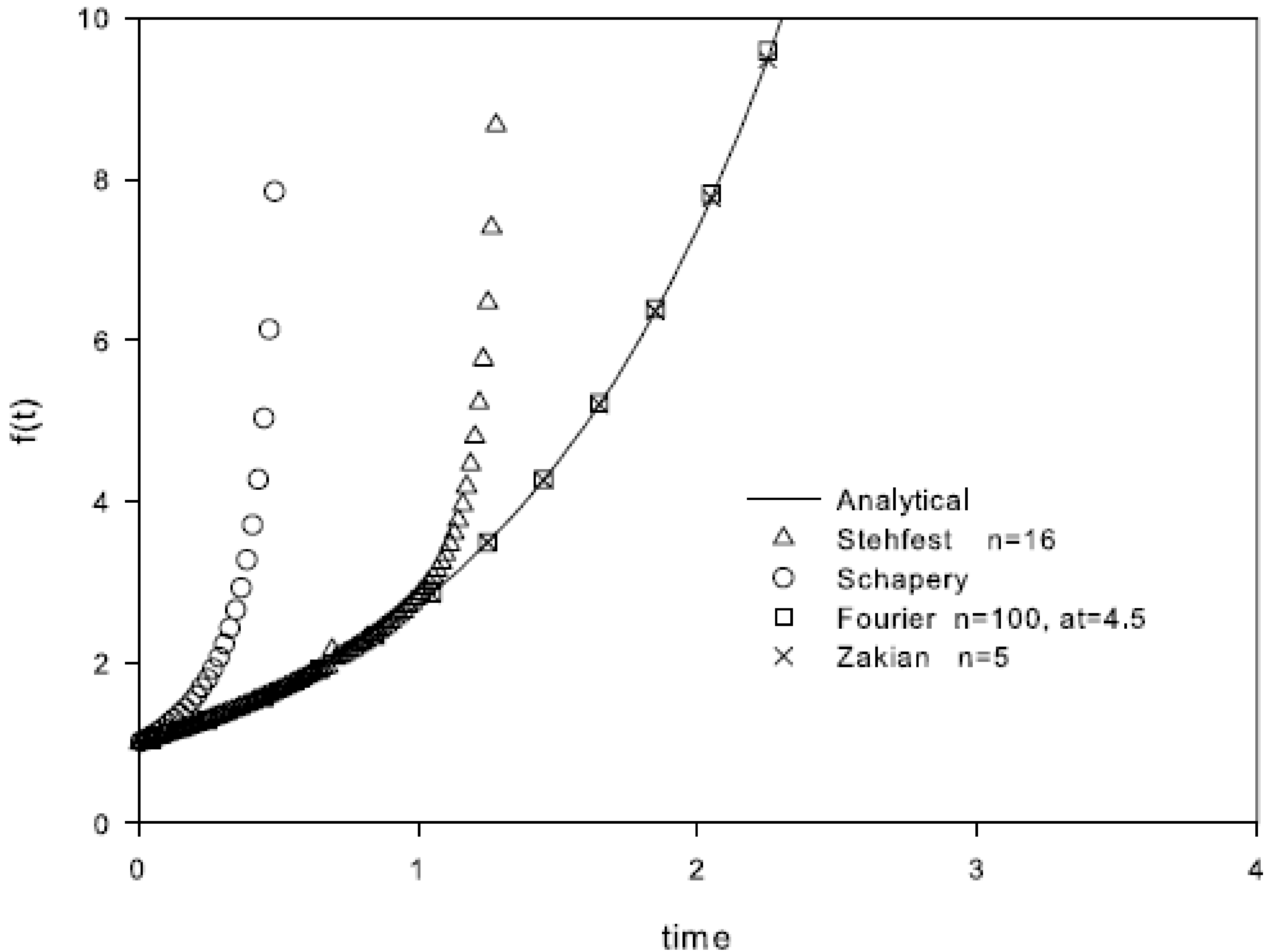


Fig. 3. Comparison of different numerical inversion methods for $f(t) = e^t$.

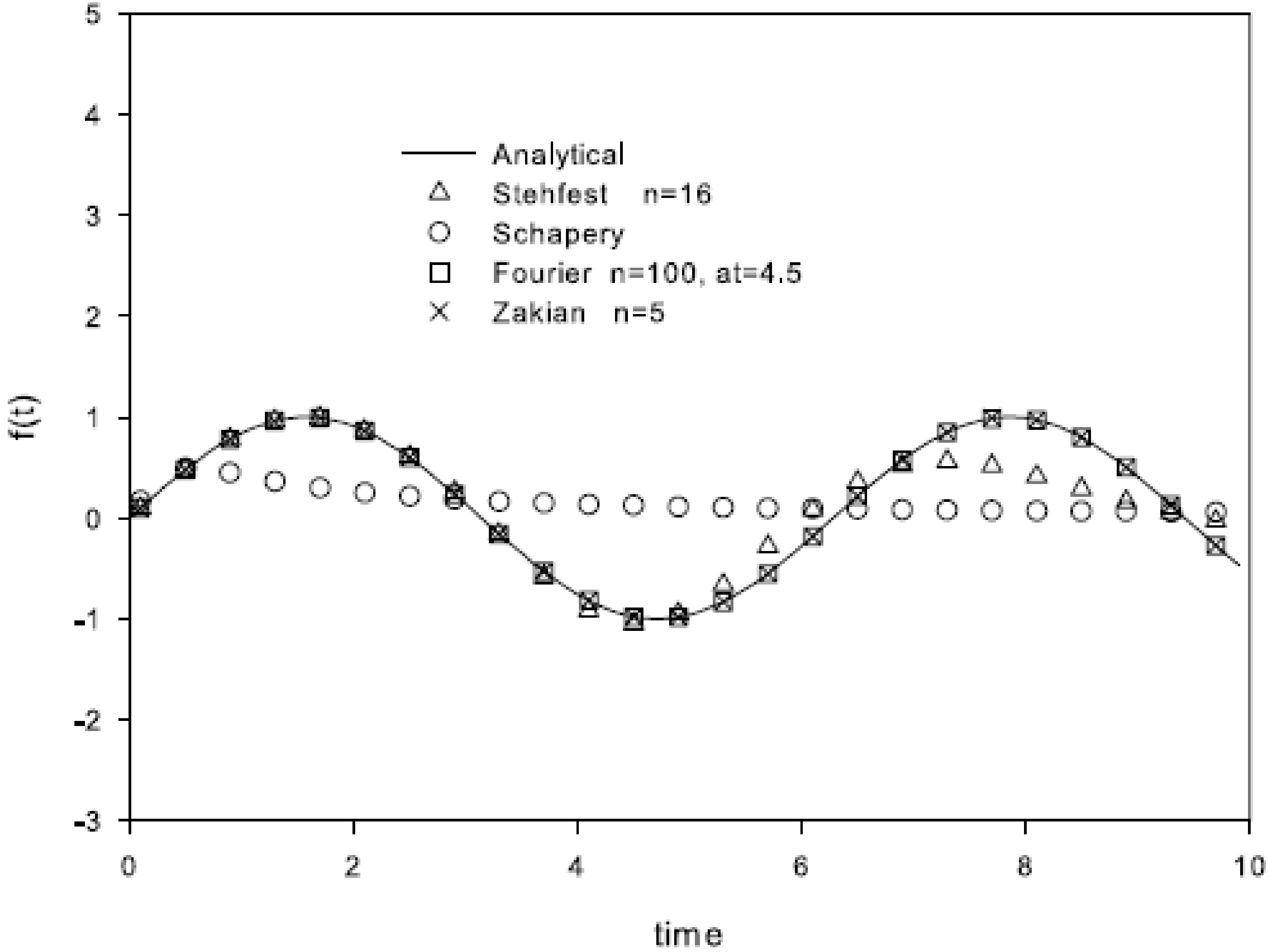


Fig. 4. Comparison of different numerical inversion methods for $f(t) = \sin(t)$.

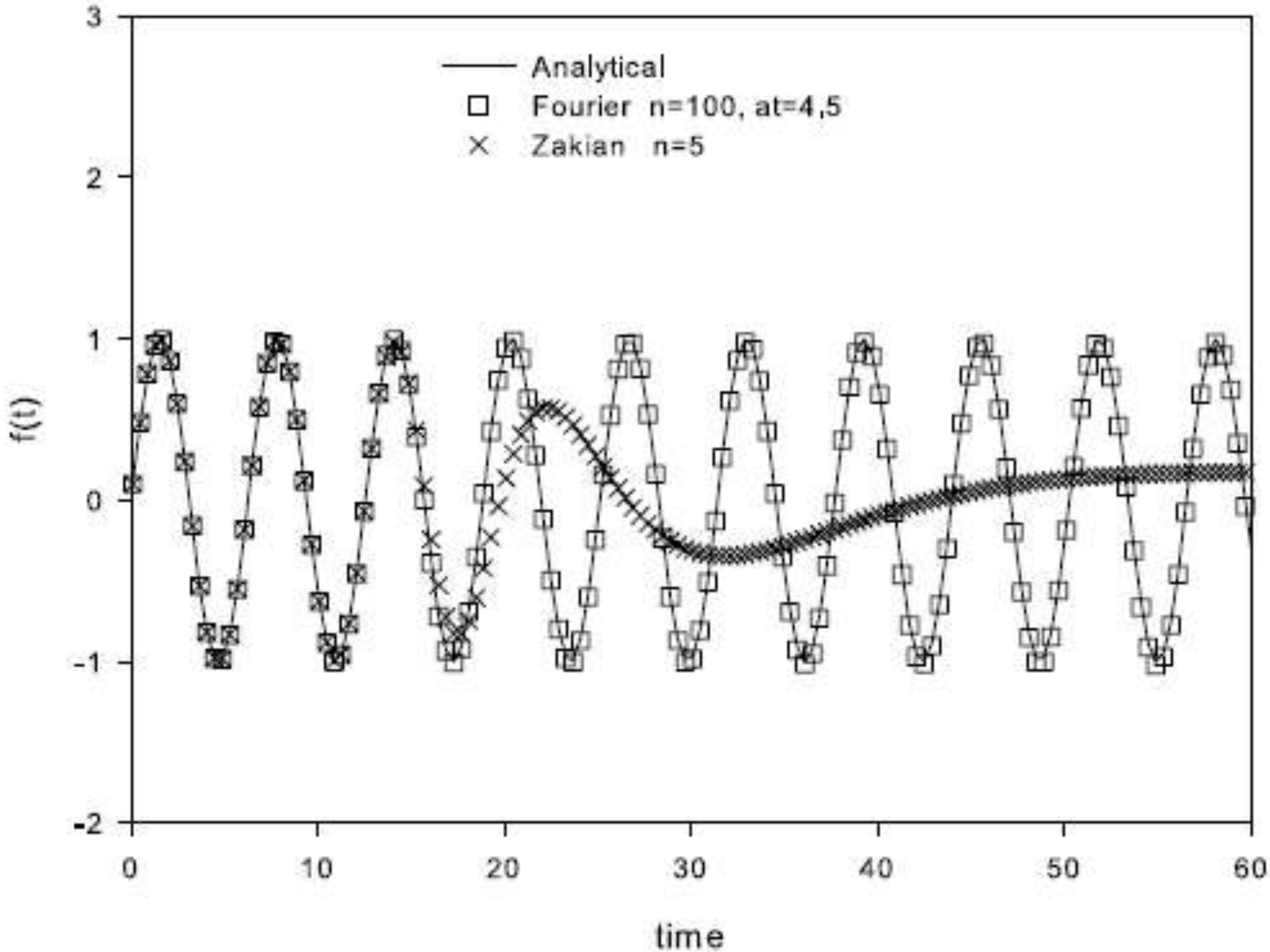


Fig. 5. Comparison of different numerical inversion methods for $f(t) = \sin(t)$.

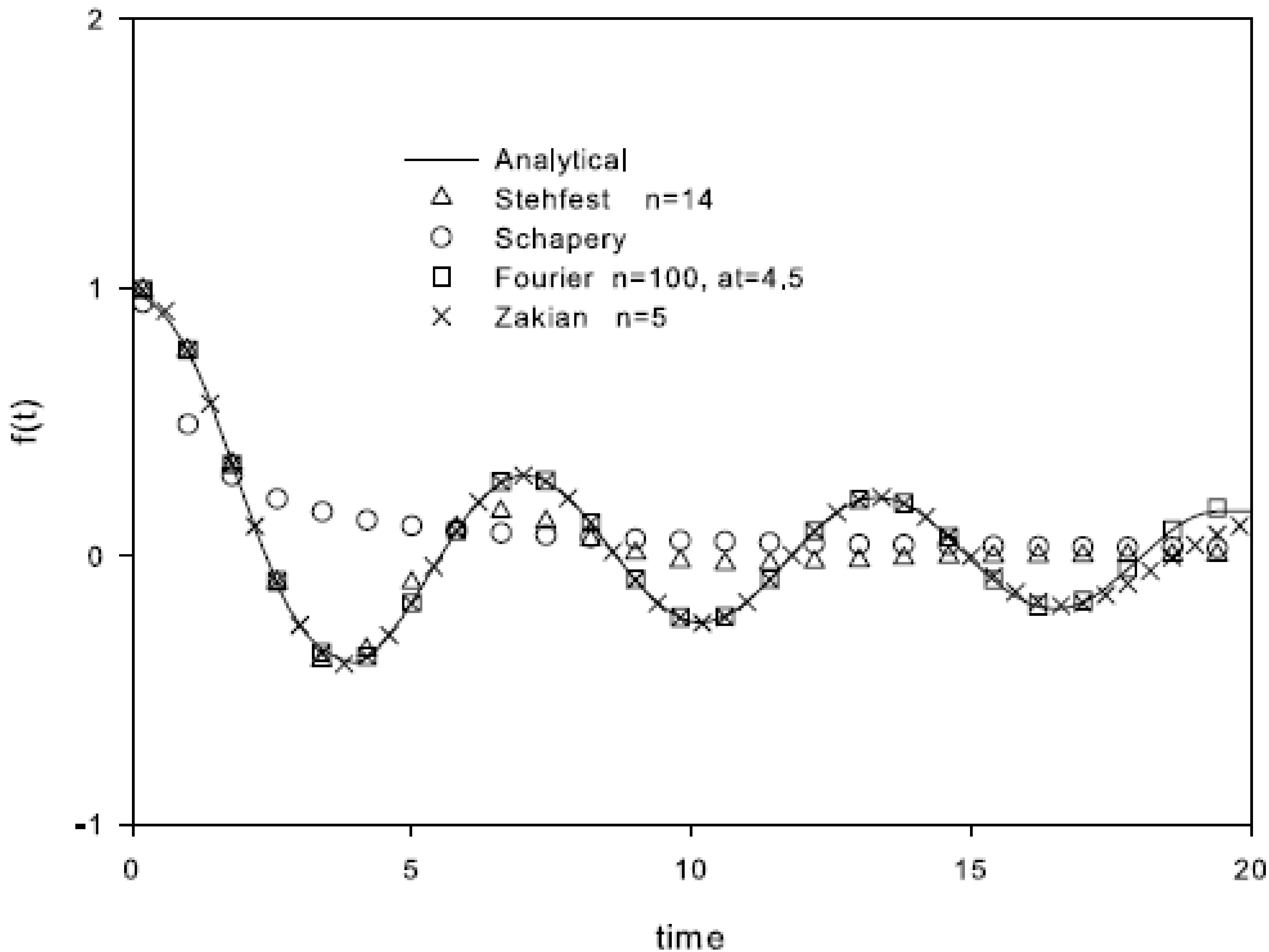


Fig. 6. Comparison of different numerical inversion methods for $f(t) = J_0(t)$.