Naturally Fractured Reservoirs

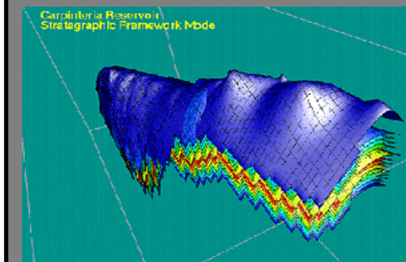
Well Testing of Naturally Fractured Reservoirs (Part A)

Shahab Gerami, Fall 2010

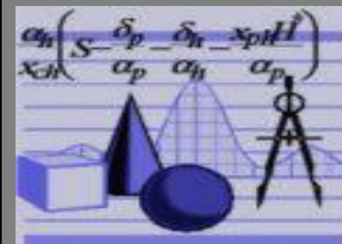
Outline

- Analysis of Production Data
- Roles of Reservoir Analysis Models
- Well-test Interpretation
- Components of Well Test Models
- Direct versus Inverse Solutions
- Flow of a Slightly Compressible Oil-Single Porosity Models
- Double Porosity Formulation
- Radial Well in a Naturally Fractured Reservoir
- Mathematical Model (Warren and Root)
- Pseudo-steady State & Transient Formulation

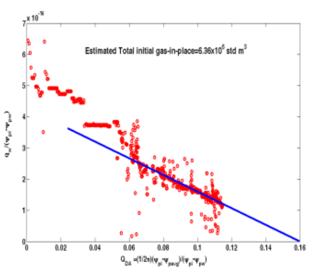
Analysis of Production Data



Reservoir Information



Predictive Models (forward solution)

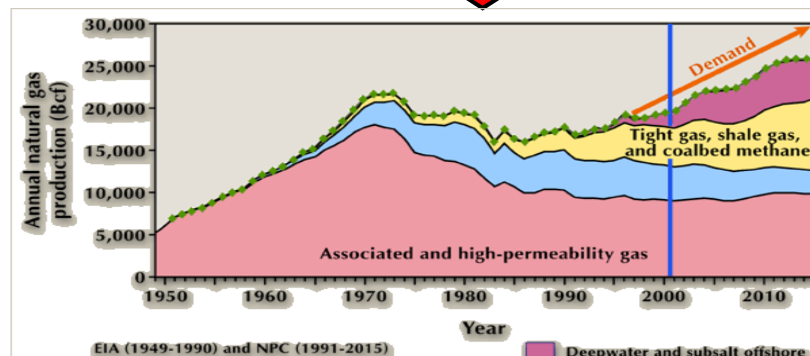


Production Analysis Models (backward solution)

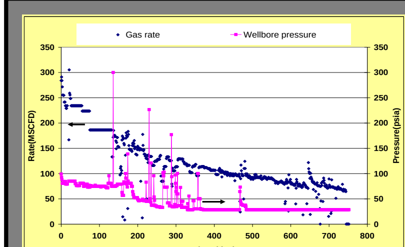
(i) Well test models

(ii) Material balance models

(iii) Decline curve analysis



Production Forecast



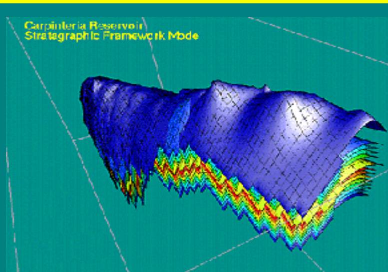
Field Data

(i) Well test data

(ii) Production data

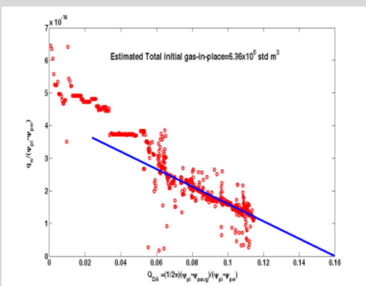
Economic Study and Decision Making for the Field Development

Roles of Reservoir Analysis Models



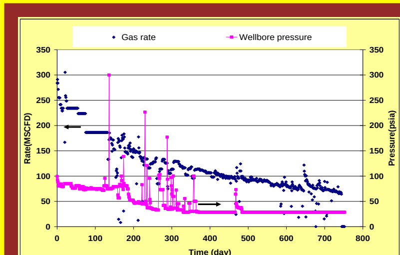
Reservoir
Information

K S IGIP

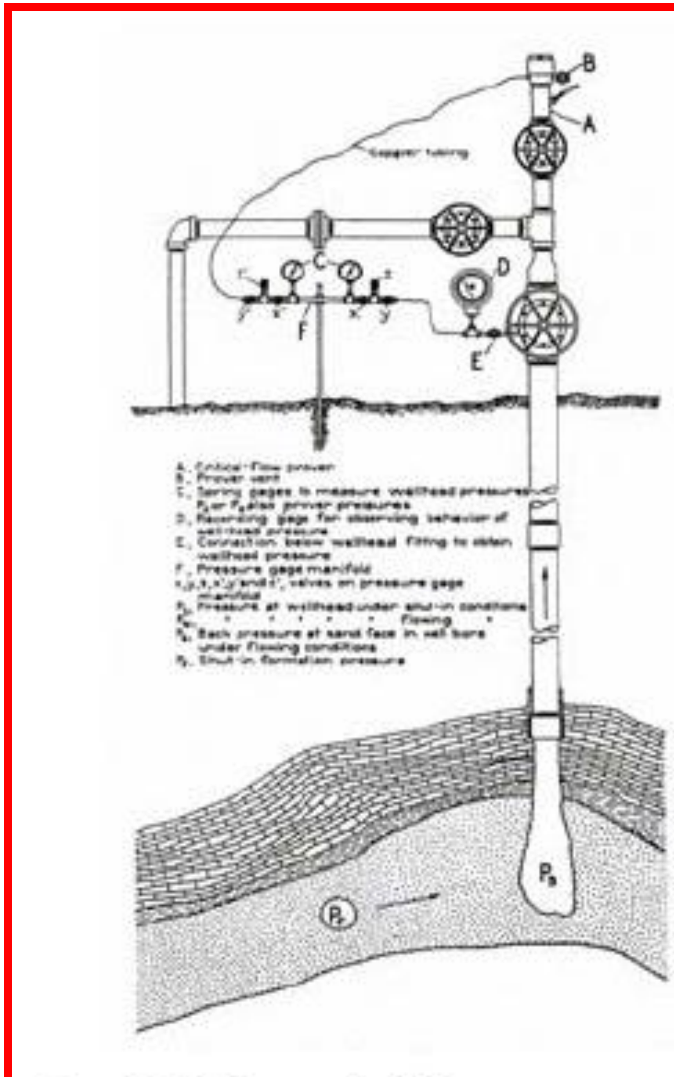


Production
Analysis Models

PTA RTA



Field Data
(i) Well test data
(ii) Production data



From: USBM Monograph, 1935

Well-test Interpretation

The aim of well-test interpretation is to obtain from the analysis of pressure versus time data or of simultaneously measured pressure and sandface flow-rate data the following parameters and functions:

-Average permeability for the drainage area of the well.

-Reservoir initial or average pressure.

-Sandface condition (damaged or stimulated).

-Volume of the drainage area.

-Degree of communication between wells.

-Validation of the geological model

- System identification (reservoir type and the mathematical model for its pressure drop as a function of time)

Transient Tests →	Reservoir Characterization
RFT®, WFT®, MDT® …	p_i, k, fluid samples
DST	p_i, k, fluid samples
Drawdown* / Injection	k, s (often un-interpretable)
Buildup* / Falloff	k, s, $\overline{p}_R$
Interference/Pulse	k, ϕc_t, lateral/vertical continuity

Stabilized Tests →	Deliverability Forecasting
IPR	q_{stab}
AOF	q_{stab}

A Typical Layout Used in Conducting a Well Test

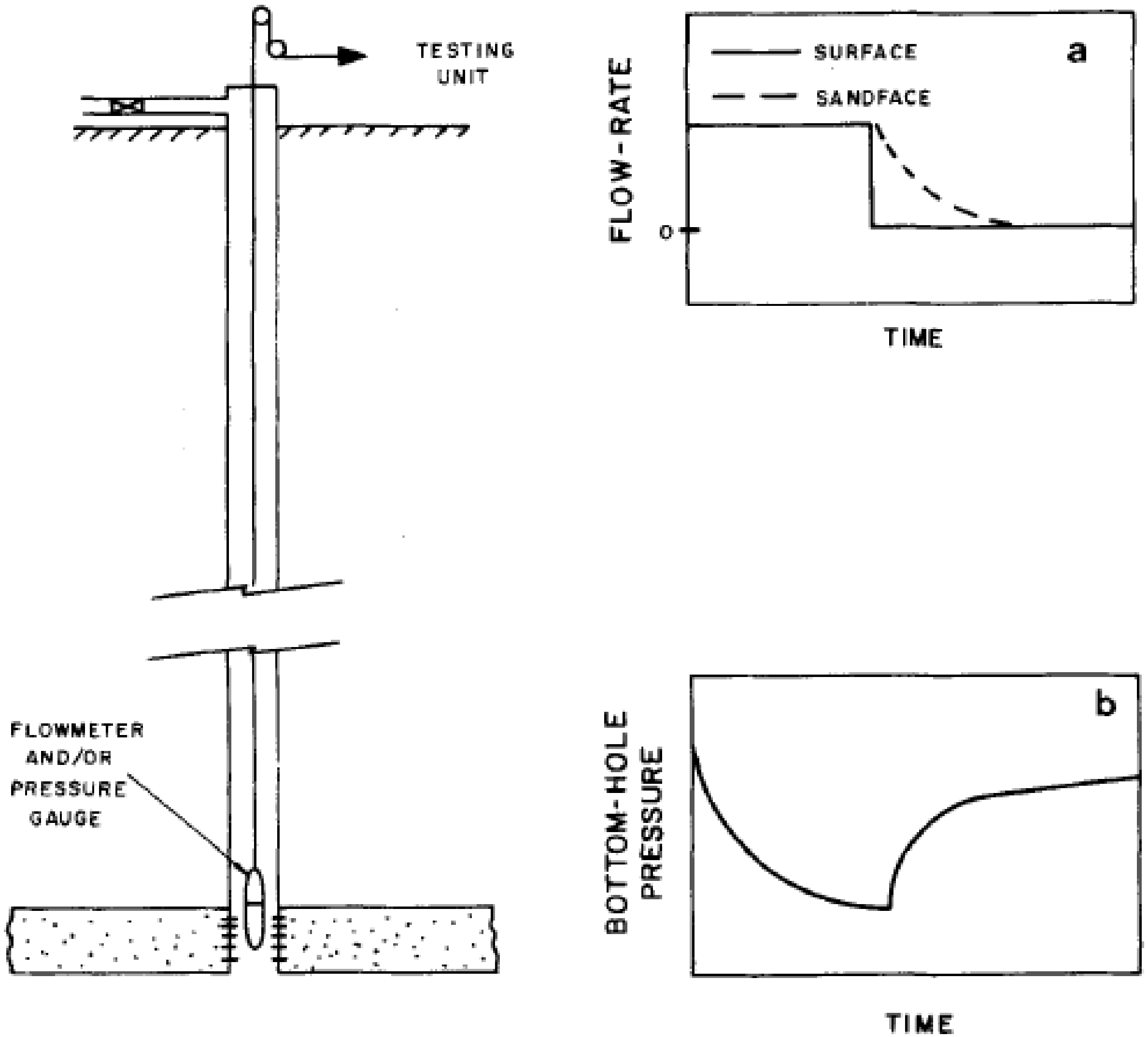
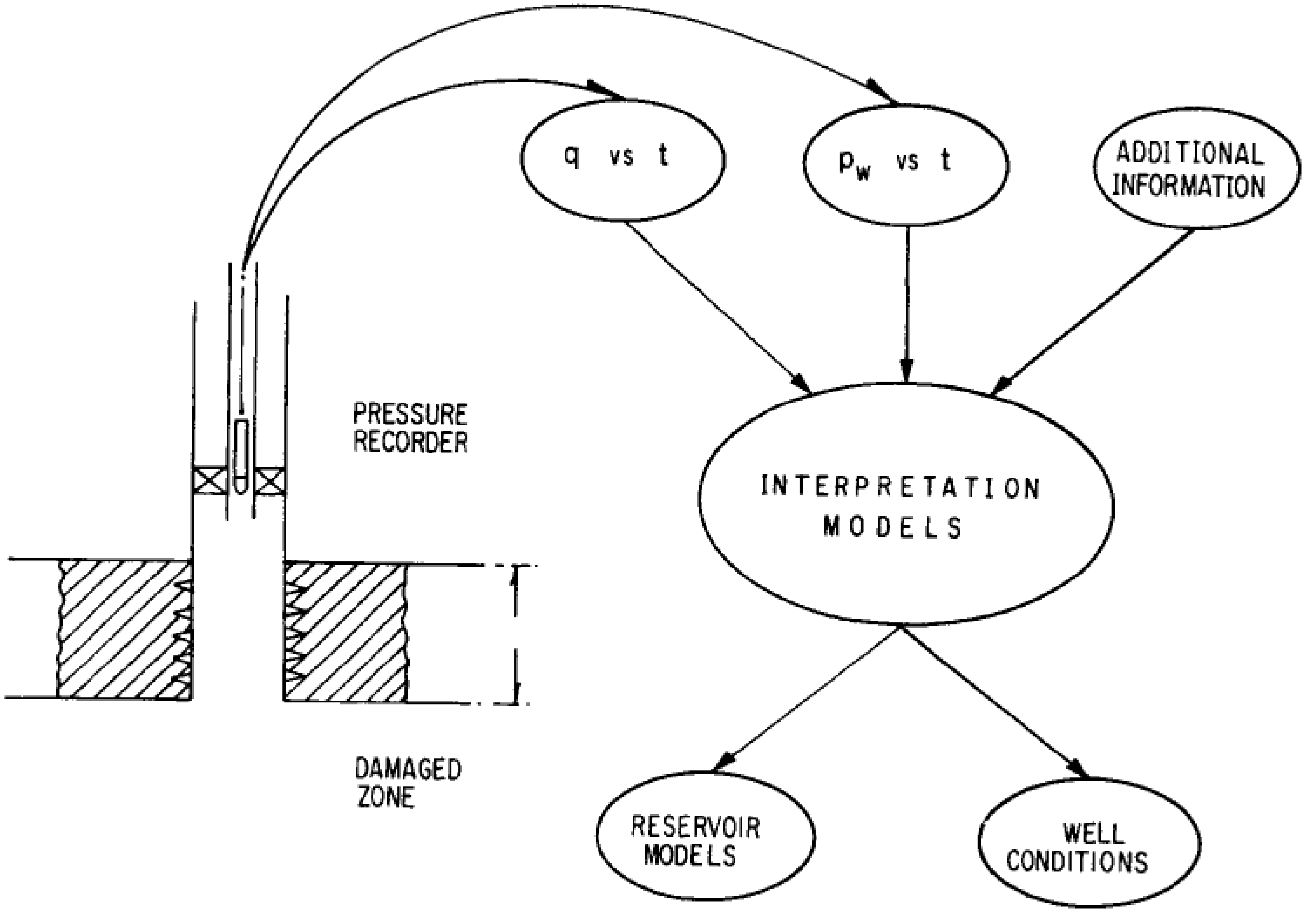


Fig. 1-1. Typical layout when conducting a production test. (a) Surface shut-in is not transmitted instantaneously to sandface. (b) Bottonhole pressure behavior for the flow and build-up period.

Interpretation of a Transient Pressure Test

Uniqueness Dilemma: different reservoir situations yield the same pressure behavior



Integrated approach: a combination of pressure transient data and geological and geophysical information, well logging, production data, core analysis, etc.

Components of Well Test Models

- Well
  - Direction (Vertical, Horizontal)
  - Storage (Constant, Changing)
  - Completion (Damaged, Fractured and Acidized)
- Reservoir
  - Homogeneous
  - Heterogeneous
  - Composite
  - Multilayer
  - Dual porosity
- Boundaries
  - Flow boundaries (No flow, Constant pressure, infinite)
  - Geometrical boundaries (Circular, Rectangular)

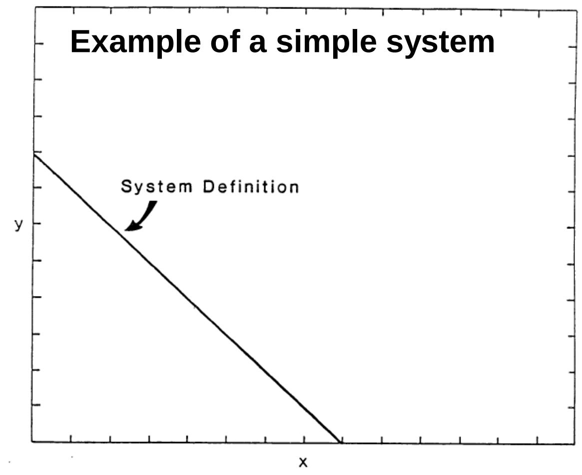
Direct versus Inverse Solutions

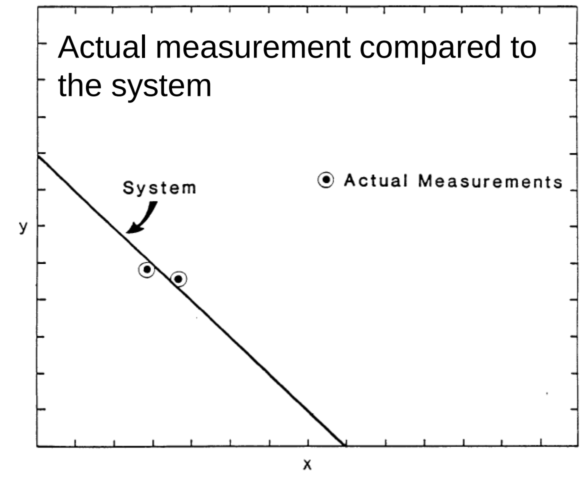
Direct solution (Convolution)

Input + **system** → **Output (?)**

Inverse solution (Deconvolution)

Input + **System (?)** → **Output**



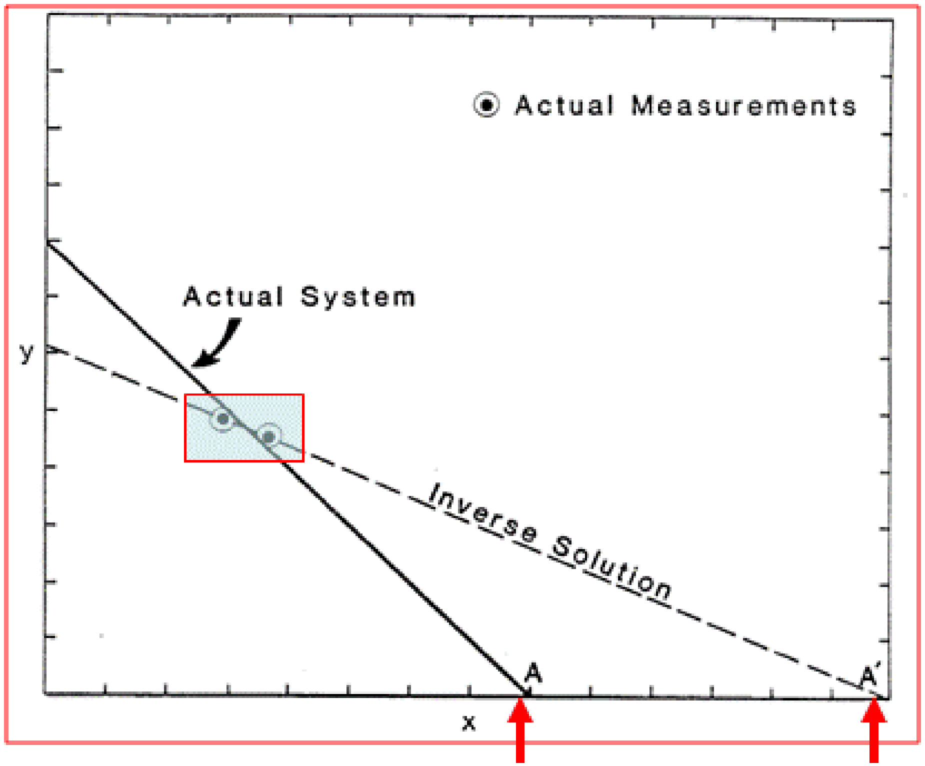


Inverse Solution Compared to Actual System

•Inverse solution can be used for the identification of system characteristics

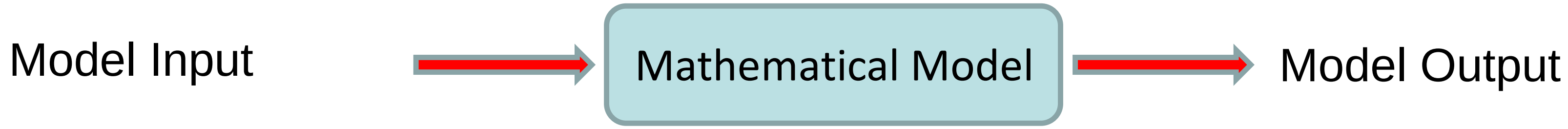
•Inverse solution can result in grossly erroneous answers

•Whereas the mathematics is correct, the utility of the results derived from this mathematically process is questionable.



Input-System-Response

Well test interpretation is essentially an inverse problem and in general is better suited to analytical solution.

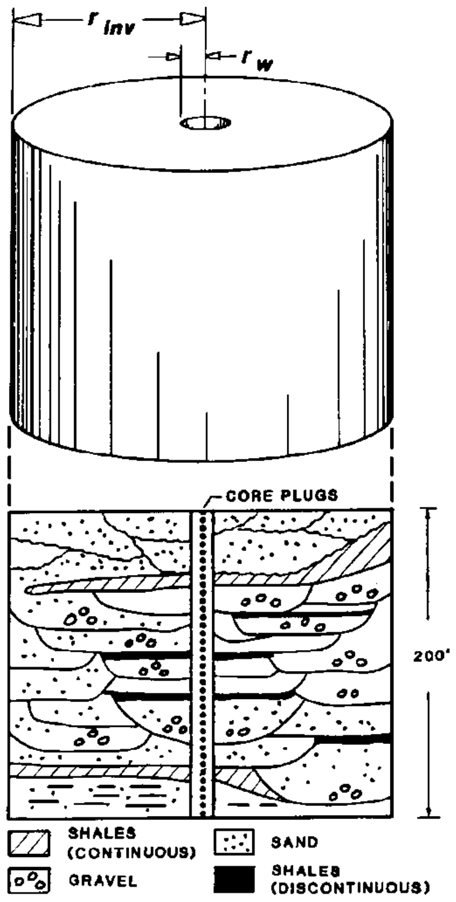


The objective of well test analysis is to describe an unknown system S (well + reservoir) by indirect measurement (O a pressure response to I a change of rate).

The unknown system (S) may be a type curve and its derivative to describe the reservoir characteristics.

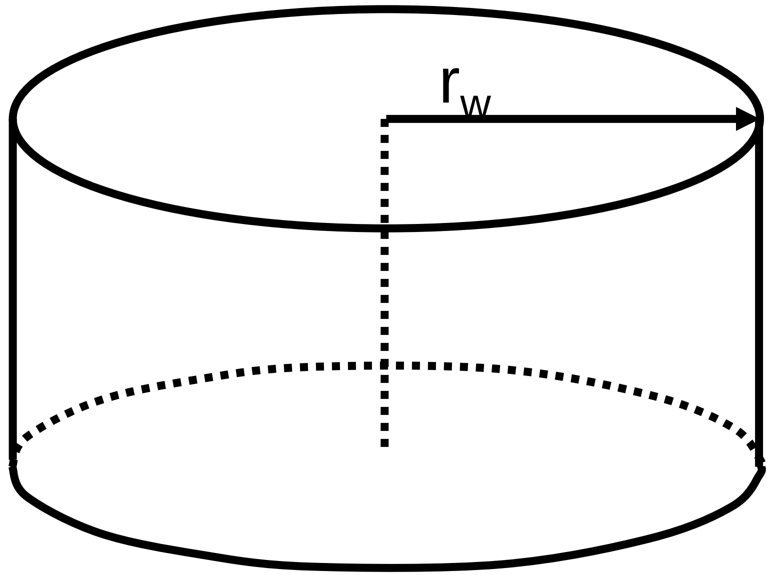
Flow of a Slightly Compressible Oil-Single Porosity Models

- Physical model
- Simplifying assumptions
- Mathematical model
  - Choosing an appropriate element
  - Governing equation
    - Mass balance
    - Momentum balance (Darcy's law)
    - Equation of state
  - Initial and Boundary conditions
    - Infinite acting
      - Constant rate production
      - Constant pressure production
    - Finite acting
      - Constant rate production
      - Constant pressure production
  - Solutions
    - Laplace space solutions
    - Time domain solutions
    - Simplified solutions
- Applications (Drawdown (single rate & multi rate), Reservoir limit test, Build up, Superposition (time & space), …),



Simplifying Assumptions

- Homogeneous
- Isotropic
- Ignore Gravity
- Constant Temperature
- Darcy's law applies
- Single phase fluid
- Radial flow
- Totally penetrating vertical well
- Constant net pay, saturation
- $(\partial p/\partial r)$ - gradient in reservoir - is small
- Constant wellbore storage
- Constant pressure throughout reservoir at time t = 0
- Constant production rate
- Closed circular reservoir
- Model complexities will be introduced as necessary



OIL/WATER

- Compressibility is small and constant
- Viscosity is constant
- Laminar flow

Mathematical Model-Governing Equation

- Mass balance

$$\left(-\rho_o A v\right)_{r+\Delta r} - \left(-\rho_o A v\right)_r = \frac{\left(\rho_o \Delta V\right)_{t+\Delta t} - \left(\rho_o \Delta V\right)_t}{\Delta t}$$

- Momentum balance (Darcy's law)

$$v_{gr} = -\frac{k}{\mu}\frac{\partial p}{\partial r}$$

- Equation of state

$$\rho_o = \rho_{ob} \exp\left(c_o \left(p - p_b\right)\right)$$

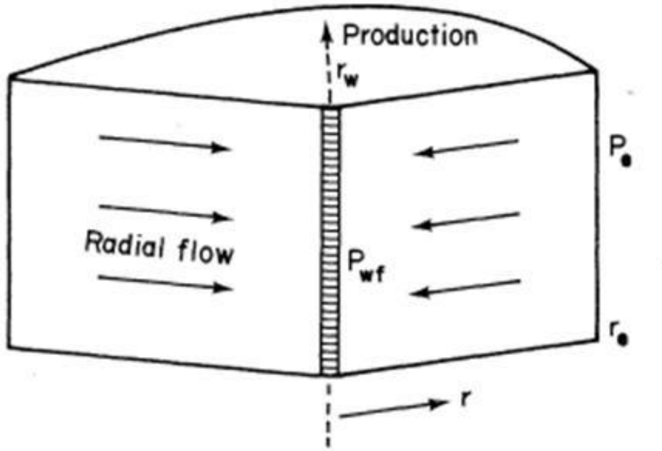
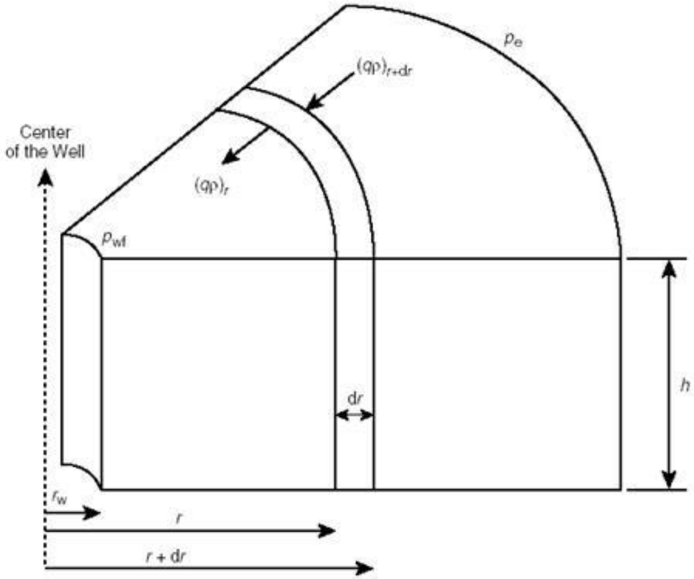
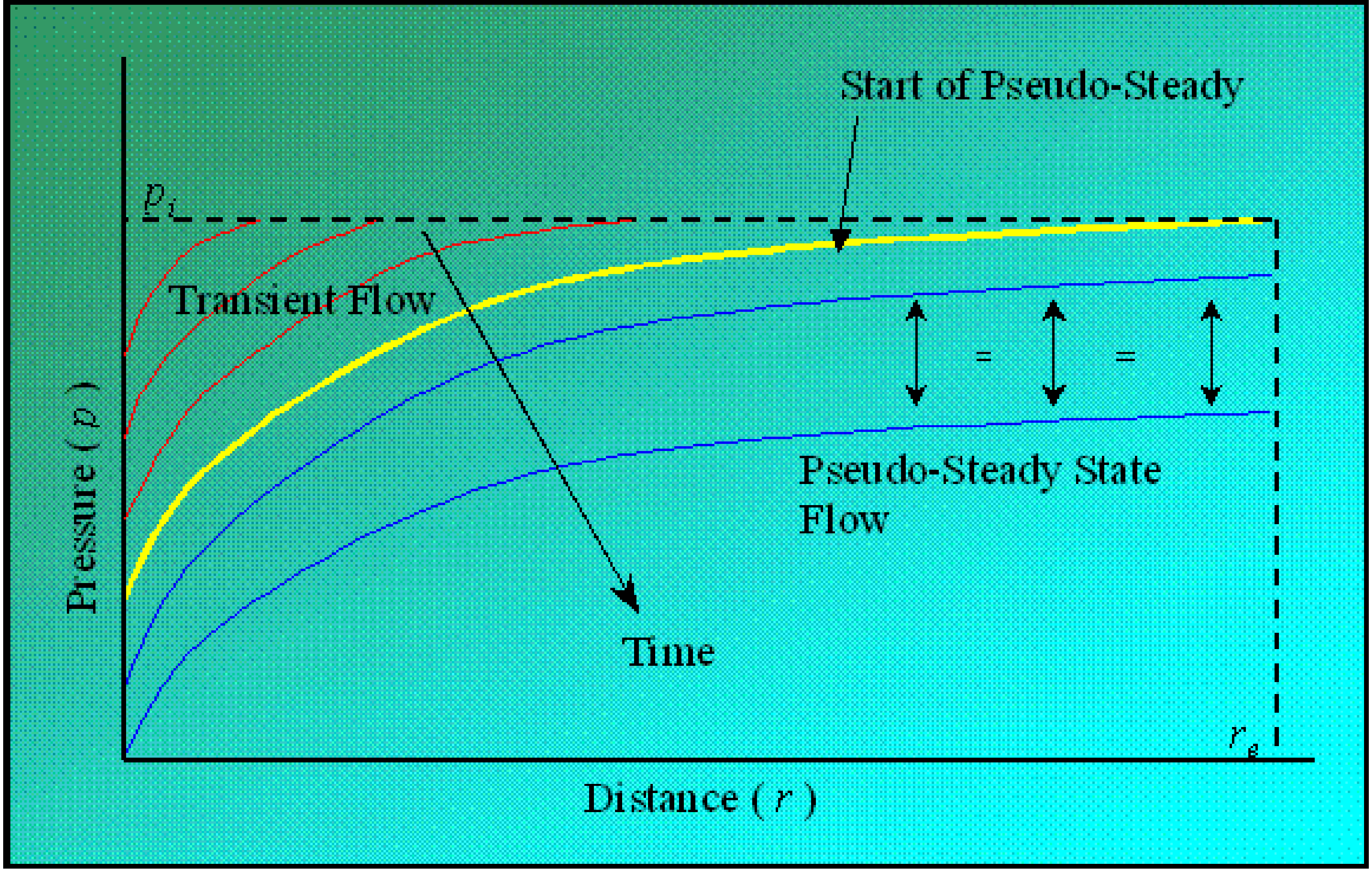


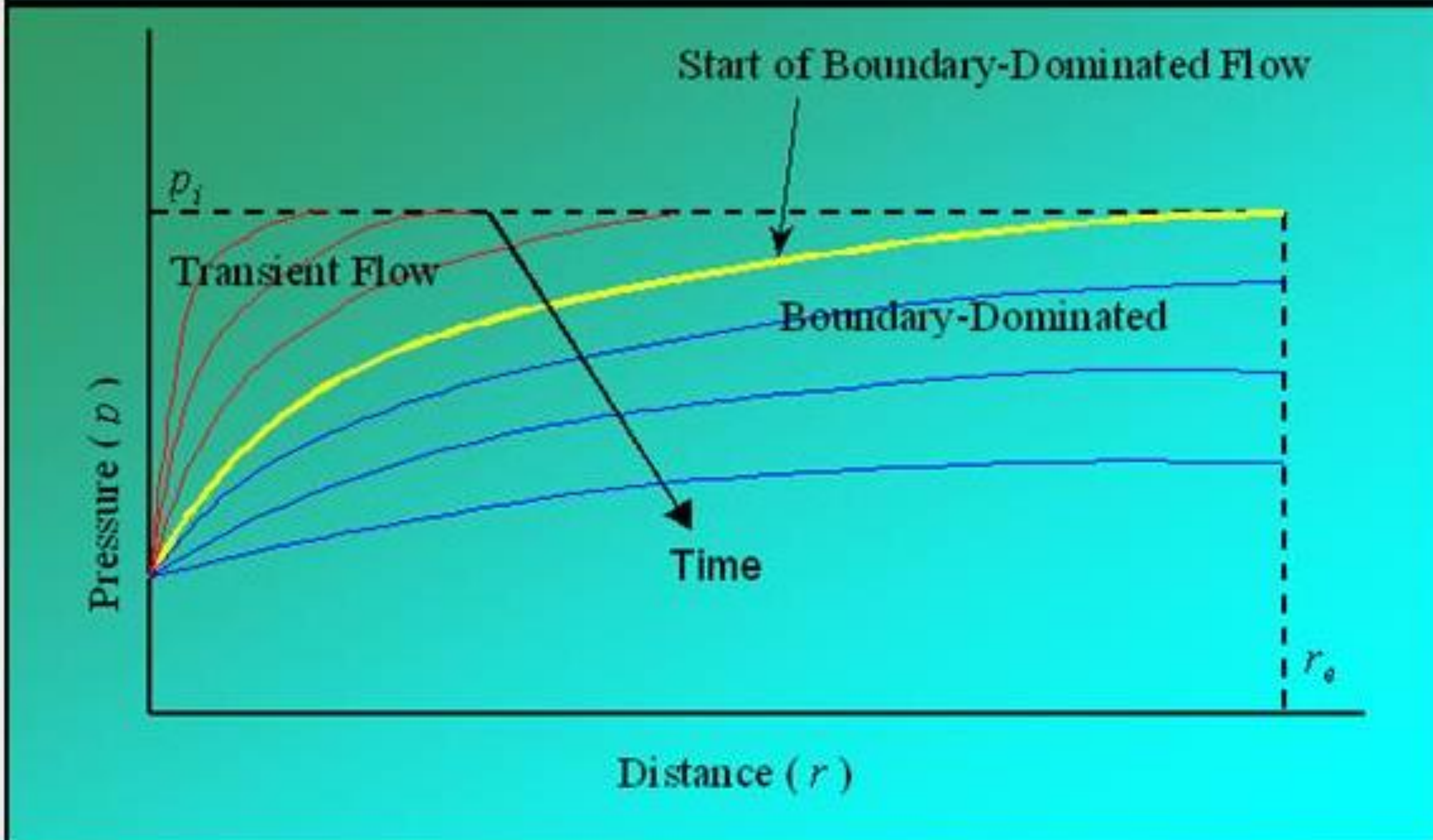
Fig. 9.1 Radial flow towards a well.



Pseudo-Steady State Flow



Boundary-Dominated Flow



Mathematical Model-Governing Equation

A reservoir model is the superposition of reservoir, inner, and outer boundary conditions

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)\right]=\frac{\varphi\mu c_t}{k}\frac{\partial p}{\partial t}$$

$$Initial\quad Condition:\quad p=p_i\,,\qquad t=0,\quad r\geq r_w$$

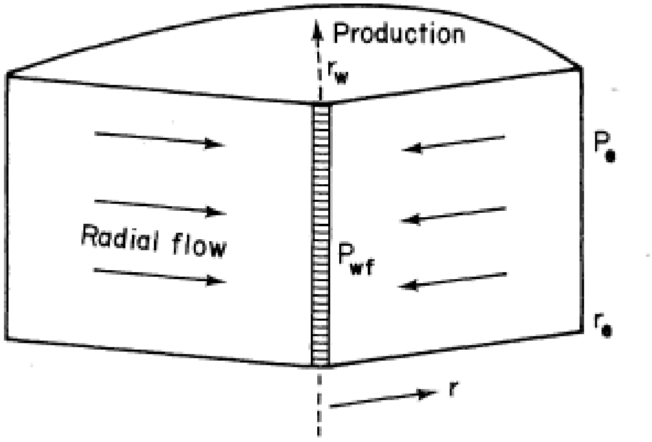


Fig. 9.1 Radial flow towards a well.

Well production	Flow regime	Inner Boundary Condition	Outer Boundary conditions
Constant rate	Finite acting (Bounded)	$\left(\frac{\partial p}{\partial r}\right)_{r_w}=-\frac{\mu qB_o}{2\pi r_w hk}$	$\left(\frac{\partial p}{\partial r}\right)_{r\to re}=0$
Constant pressure	Finite acting (Bounded)	$(p)_{r_w}=p_{wf}$	$\left(\frac{\partial p}{\partial r}\right)_{r\to re}=0$

Radial Hydraulic Diffusivity Equation

$$\frac{\partial^2 P_{r,t}}{\partial r^2} + \frac{1}{r}\frac{\partial P_{r,t}}{\partial r} = \frac{3{,}792\ S}{T}\frac{\partial P_{r,t}}{\partial t} \qquad (1\text{-}26)$$

Equation 1-26 is valid when oilfield units (ft, hr, STB/D, cp, md, psi) are used. Setting $S = \phi C_t h$, and $T = kh/\mu$ and changing to Darcy units (cm, sec, cc/sec, cp, darcy, atm) we obtain:

$$\frac{\partial^2 P_{r,t}}{\partial r^2} + \frac{1}{r}\frac{\partial P_{r,t}}{\partial r} = \frac{\phi\ \mu\ C_t}{k}\frac{\partial P_{r,t}}{\partial t} \qquad (11\text{-}1)$$

Let us define the following dimensionless parameters in Darcy units:

$$r_D = r/r_w \qquad (11\text{-}2)$$

$$t_D = kt/(\phi\mu C_t r_w^2) \qquad (11\text{-}3)$$

$$P_D = 2\pi kh\ (P_i - P_{r,t})/(q\mu) \qquad (11\text{-}4)$$

Then,

$$\frac{\partial r_D}{\partial r} = 1/r_w$$

$$\frac{\partial t_D}{\partial t} = k/(\phi \mu C_t r_w^2)$$

$$\frac{\partial P_D}{\partial t} = -\frac{2\ \pi\ k\ h}{q\ \mu}\frac{\partial P_{r,t}}{\partial t}$$

$$\frac{\partial P_D}{\partial t_D} = \frac{\partial P_D}{\partial t} \times \frac{\partial t}{\partial t_D}$$

$$= -\frac{2\ \pi\ k\ h}{q\ \mu}\frac{\phi\ \mu\ C_t\ r_w^2}{k}\frac{\partial P_{r,t}}{\partial t}$$

Also,

$$\frac{\partial P_D}{\partial r} = -\frac{2\ \pi\ k\ h}{q\ \mu}\frac{\partial P_{r,t}}{\partial r}$$

$$\frac{\partial P_D}{\partial r_D} = \frac{\partial P_D}{\partial r}\frac{\partial r}{\partial r_D}$$

$$= -\frac{2\ \pi\ k\ h}{q\ \mu} r_w \frac{\partial P_{r,t}}{\partial r}$$

$$\frac{\partial^2 P_D}{\partial r_D^2} = \frac{-2\ \pi\ k\ h\ r_w^2}{q\ \mu}\frac{\partial^2 P_{r,t}}{\partial r^2}$$

Now, by substituting in Equation 11-1 we get:

$$\frac{\partial^2 P_D}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial P_D}{\partial r_D} = \frac{\partial P_D}{\partial t_D} \qquad (11\text{-}5)$$

$$\frac{\partial^2 P_D}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial P_D}{\partial r_D} = \frac{\partial P_D}{\partial t_D} \qquad (11\text{-}5)$$

subject to the following conditions:

1. $P_D = 0$, for $t_D = 0$ at all r_D

2. $\left(\frac{\partial P_D}{\partial r_D}\right)_{@r_D = 1} = -1$, for all $t_D > 0$

3. $\left(\frac{\partial P_D}{\partial r_D}\right)_{@r_{De} = r_e/r_w} = 0$ for all t_D

Let,

$P_D(z) = P_D(r_D,z) = L[P_D(r_D,t_D)]$

and,

$P_D(0) = P_D(r_D,t_D = 0)$

The Laplace transform of a continuous time function, f(t), is given by:

$$L[f(t)] = \int_0^\infty e^{-zt} f(t)\, dt \qquad (11\text{-}8)$$

where z = Laplace parameter

then by Equation 11-11 the boundary value problem in the Laplace domain is stated as:

$$\frac{d^2P_D(z)}{dr_D^2} + \frac{1}{r_D}\frac{dP_D(z)}{dr_D} = z\,P_D(z) - P_D(0) \tag{11-20}$$

subject to the following conditions:

1. $P_D(z) = 0$, for $t_D = 0$ at all r_D (11-21)

2. $\left(\frac{dP_D(z)}{dr_D}\right)_{@r_D=1} = -1/z$ (11-22)

3. $\left(\frac{dP_D(z)}{dr_D}\right)_{@r_{De}=r_e/r_w} = 0$ (11-23)

Applying condition 1 to Equation 11-20 we obtain:

$$\frac{d^2P_D(z)}{dr_D^2} + \frac{1}{r_D}\frac{dP_D(z)}{dr_D} = z\,P_D(z) \tag{11-24}$$

Solution

Equation 11-24 is a form of Bessel's equation. Its general solution is given by:

$$P_D(z) = A\ I_0\ (r_D\ \sqrt{z}) + B\ K_0\ (r_D\ \sqrt{z}) \tag{11-25}$$

where $I_0(r_D\ \sqrt{z})$ and $K_0(r_D\ \sqrt{z})$, respectively, are zero order modified Bessel functions of the first and second kind; and A and B are constants to be determined by applying boundary conditions 2 and 3.

$$P_D(z) = \frac{K_1\ (r_{De}\sqrt{z})\ I_0\ (r_D\ \sqrt{z}) + I_1\ (r_{De}\ \sqrt{z})\ K_0\ (r_D\sqrt{z})}{z^{3/2}\ [K_1\ (\sqrt{z})\ I_1\ (r_{De}\ \sqrt{z}) - K_1\ (r_{De}\ \sqrt{z})\ I_1\ (\sqrt{z})]} \tag{11-28}$$

Naturally Farctured Reservoirs
Porosity and Permeability

- Whereas the matrix permeability is much smaller than the fracture permeability, the fracture porosity of a particular class of naturally fractured reservoirs seldom exceeds 1.5% or 2%, and usually falls below 1%.

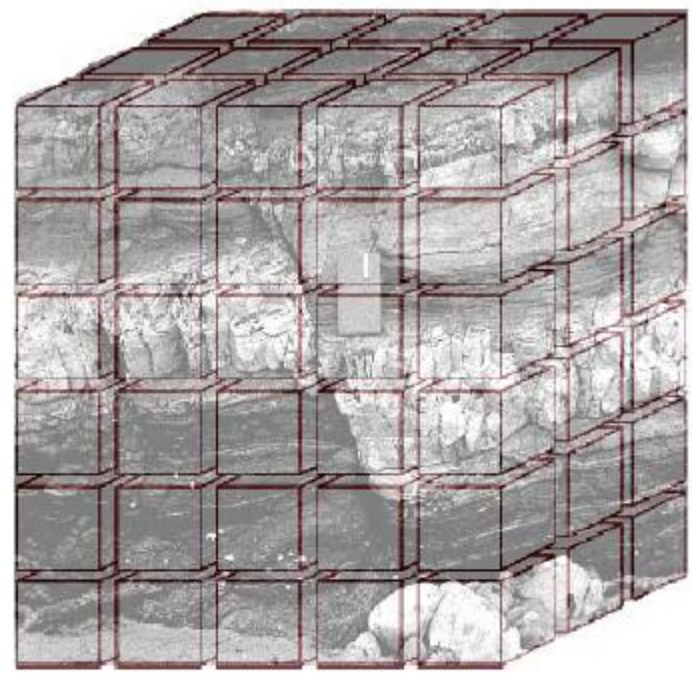
- The high permeability of a fracture results in a high diffusivity of the pressure propagation pulse along the fracture.

A fracture of 0.1 mm will have a permeability of 833 darcys, whereas the permeability of the limestone proper will usually be of the order of 0.01 darcy. (Muskat (1937),pp.425)

Double Porosity Formulation

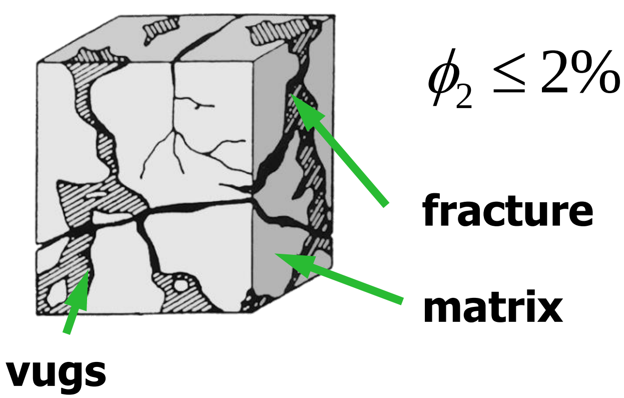
•The fractures which cut the reservoir rock in various directions, delineate a bulk unit referred to as the **matrix block** unit or simply the matrix block.

•The shape of the matrix block is irregular, but for practical work the block units are reduced to simplified geometrical volumes, such as cubes or as elongated or flat parallelepipeds.



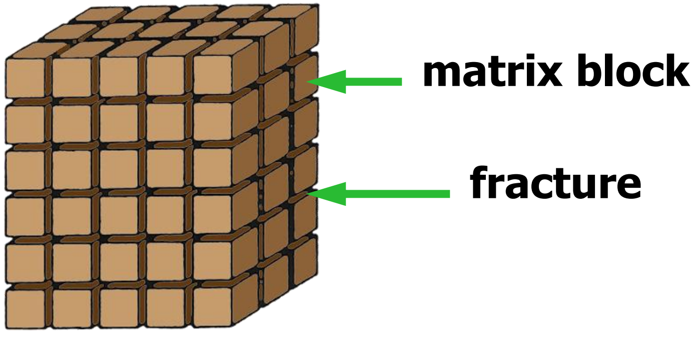
- Based on the theory of fluid flow in fractured porous media developed in the 1960's by Barrenblatt *et al.*, **Warren and Root** introduced the concept of dual-porosity models into petroleum reservoir engineering. Their idealized model of a highly interconnected set of fractures which is supplied by fluids from numerous small matrix blocks, is shown below:

ACTUAL RESERVOIR



$\phi_2 \leq 2\%$

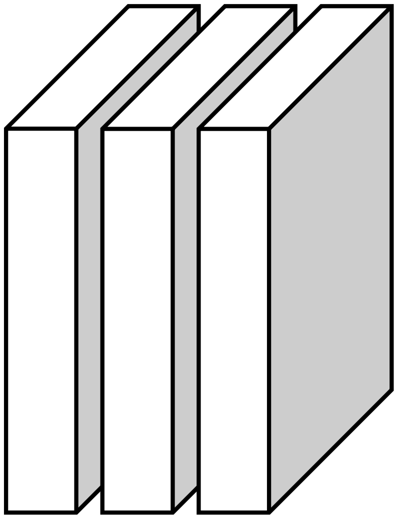
MODEL RESERVOIR



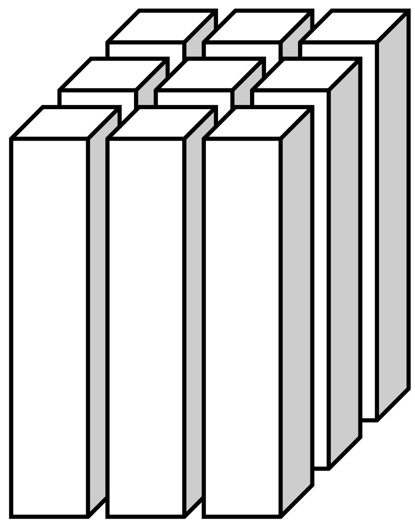
•A naturally fractured formation is generally represented by a tight matrix rock broken up by fractures of secondary origin.

• The fractures are assumed continuous throughout the formation and to represent the paths of principal permeability.

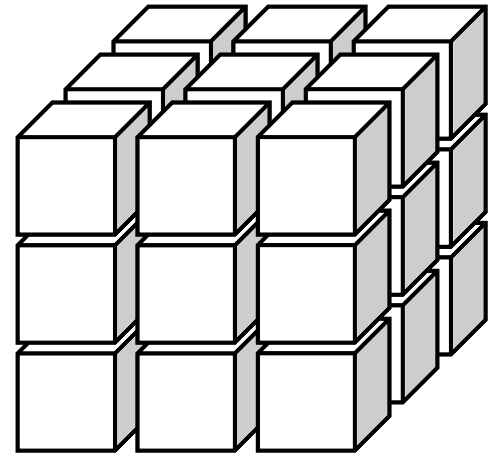
Idealized Fracture Geometries



Slab Geometry



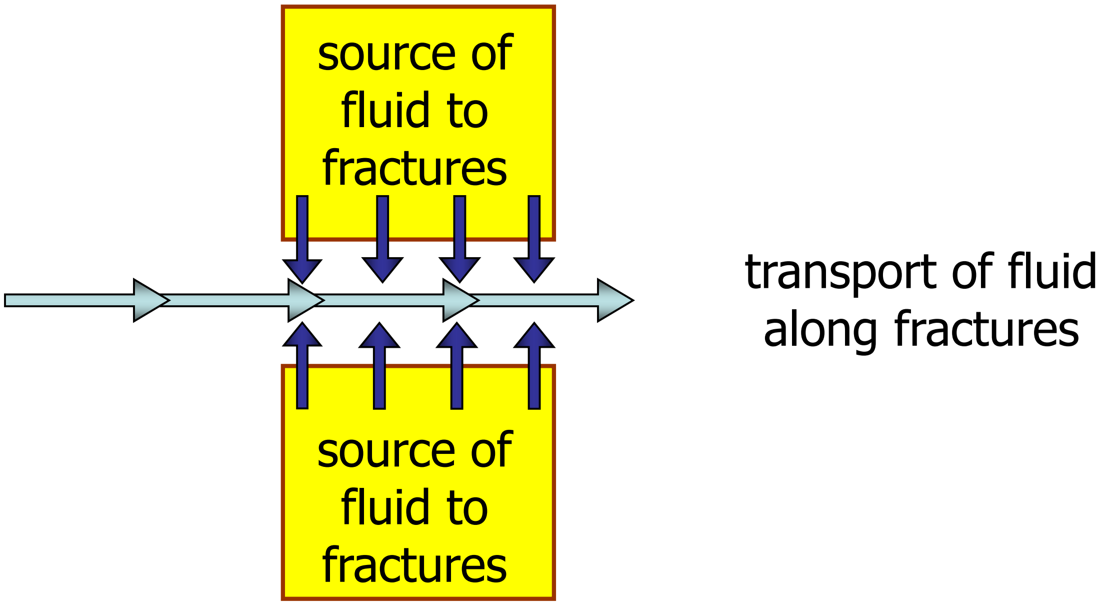
Column Geometry



Cube Geometry

Fluid Exchange

•Avery important characteristic of the double porosity system is the nature of the fluid exchange between the two distinct porous systems.



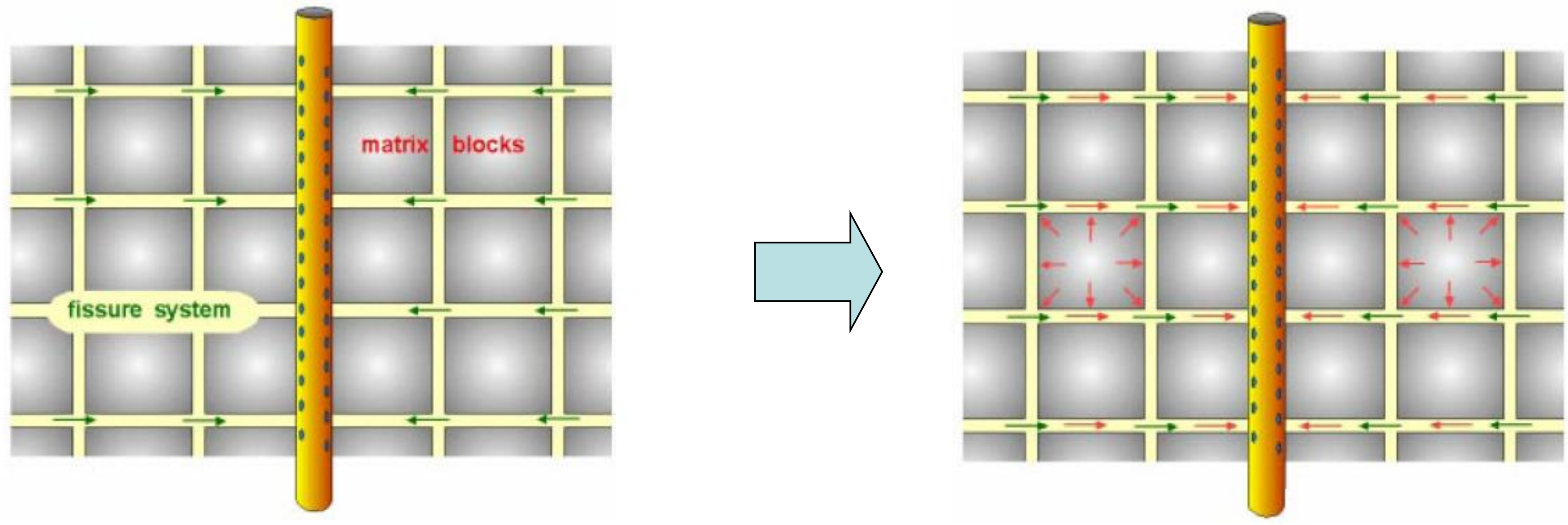
•The matrix system does not produce directly to the well but acts as a source of fluid to the fissure system.

•The high diffusivity of a fracture results in a rapid response along the fracture to any pressure change such as that caused by well production.

Radial Well in a Naturally Fractured Reservoir

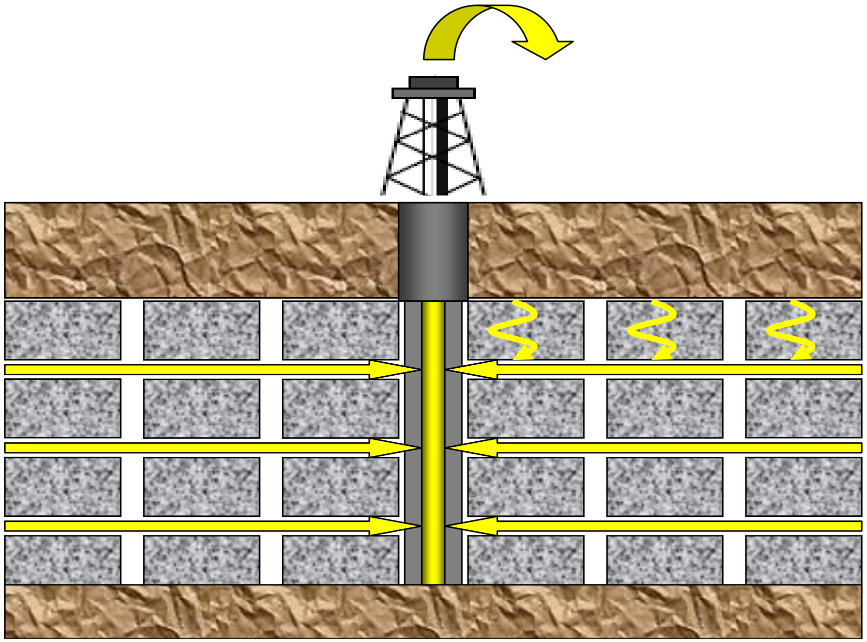
In general, the matrix releases the fluid into the fractures upon pressure decline (inter-porosity flow). Subsequently the fractures transport the fluid to the wellbore.

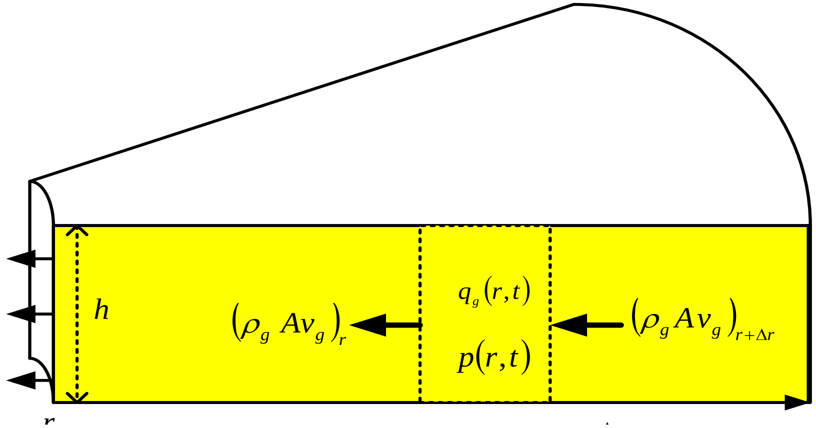
Early-time: Fissure System Flow **Late-time: Matrix contribution**



Due to significant contrast between matrix and fracture permeabilities, the matrix has a "delayed" response to pressure changes that occur in the surrounding fractures. Such a non-concurrent response induces matrix-to-fracture cross-flow.

Mathematical Model (Warren and Root)



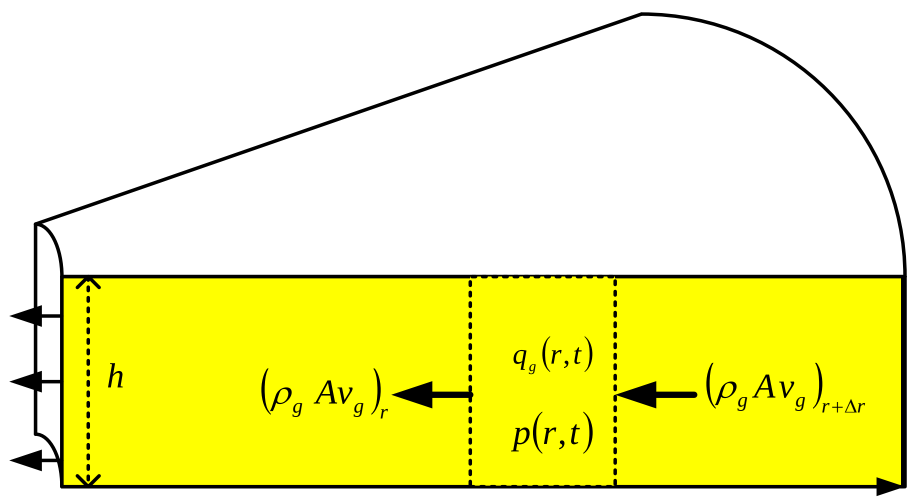


Elemental volume in naturally fractured reservoir (Warren and Root model)

Continuity Equation

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial \rho_f}{\partial r}\left(-\frac{k_f}{\mu_f}\frac{\partial \rho_f}{\partial r}\right)\right)\right]+q_g^{*}=\frac{\partial\left(\rho_f \phi_2\right)}{\partial t}$$

$$q_g^{*}=-\frac{\partial\left[\rho_m \phi_1\left(1-S_{wi}\right)\right]}{\partial t}$$



Elemental volume in naturally fractured gas reservoir

Warren & Root Equations
Pseudo-steady state Model

Warren and Root (1963) derived the following radial flow equation in Darcy units:

$$\frac{\partial^2 P_{Df}}{\partial {r_D}^2} + \frac{1}{r_D}\frac{\partial P_{Df}}{\partial r_D} = (1-\omega)\frac{\partial P_{Dm}}{\partial t_D} + \omega\frac{\partial P_{Df}}{\partial t_D} \qquad (11\text{-}60)$$

where

$$\omega = \frac{\phi_f C_f}{\phi_m C_m + \phi_f C_f} \qquad (11\text{-}61)$$

$$t_D = \frac{k_f\, t}{(\phi_m C_m + \phi_f C_f)\,\mu\, r_w^2} \qquad (11\text{-}62)$$

$$P_{Dm} = \frac{2\,\pi\, k_f h\,(P_i - P_{r,t|m})}{q\,\mu} \qquad (11\text{-}63)$$

$$P_{Df} = \frac{2\,\pi\, k_f\, h\,(P_i - P_{r,t|f})}{q\,\mu} \qquad (11\text{-}64)$$

Equation 11-60 is based on the assumption that semi-steady state conditions develop instantaneously in the matrix. This assumption is stated as follows:

$$\phi_m C_m \frac{\partial P_m}{\partial t} = \frac{\alpha\, k_m}{\mu} (P_f - P_m) \tag{11-65}$$

where α is a geometric factor which depends on number and orientation of fractures.

Warren and Root defined α as follows:

$$\alpha = \frac{4\, n\, (n + 2)}{L} \tag{11-66}$$

where n = number of orthogonal sets of fractures
L = geometrical factor characteristic of matrix

For slabs: n = 1; and $L = h_m^2$ (h_m = thickness of matrix). For spheres: n = 3; and $L = 4\, r_m^2$ (r_m = radius of sphere which approximates a matrix block).

$$\phi_m C_m \frac{\partial P_m}{\partial t} = \frac{\alpha\ k_m}{\mu} (P_f - P_m) \qquad (11\text{-}65)$$

Equation 11-65 can be written as follows:

$$\phi_m C_m \frac{\partial \Delta P_m}{\partial t} = \frac{\alpha\ k_m}{\mu} (\Delta P_f - \Delta P_m)$$

where $\Delta P_m = P_i - P_m$
$\Delta P_f = P_i - P_f$

$$t_D = \frac{k_f\ t}{(\phi_m C_m + \phi_f C_f)\ \mu\ r_w^2}$$

$$P_{Dm} = \frac{2\ \pi\ k_f h\ (P_i - P_{r,t|m})}{q\ \mu}$$

$$P_{Df} = \frac{2\ \pi\ k_f\ h\ (P_i - P_{r,t|f})}{q\ \mu}$$

Thus, Equation 11-65 can be written as:

$$\phi_m C_m \frac{\partial P_{Dm}}{\partial t_D} \times \frac{\partial t_D}{\partial t} = \frac{\alpha\ k_m}{\mu} (P_{Df} - P_{Dm})$$

Since,

$$\frac{\partial t_D}{\partial t} = \frac{k_f}{(\phi_m C_m + \phi_f C_f)\ \mu\ r_w^2} \qquad\qquad \omega = \frac{\phi_f\ C_f}{\phi_m C_m + \phi_f C_f}$$

then,

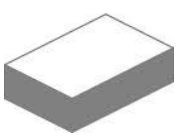
$$(1-\omega)\ \frac{\partial P_{Dm}}{\partial t_D} = \frac{\alpha\ k_m}{k_f}\ r_w^2\ (P_{Df} - P_{Dm})$$

or,

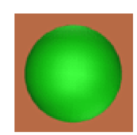
$$(1-\omega)\ \frac{\partial P_{Dm}}{\partial t_D} = \lambda\ (P_{Df} - P_{Dm}) \qquad (11\text{-}67)$$

where λ = dimensionless matrix-fracture flow coefficient

$$= (\alpha\ k_m\ r_w^2)/k_f \qquad (11\text{-}68)$$



For slabs: $\lambda = \frac{12}{h_m^2}\ r_w^2\ \frac{k_m}{k_f}$



For spheres: $\lambda = \frac{15}{r_m^2}\ r_w^2\ \frac{k_m}{k_f}$

Assuming that at $t_D = 0$, $P_{Dm} = 0$, the Laplace transform of Equation 11-67 is given by:

$$(1 - \omega)\ z\ P_{Dm}(z) = \lambda\ (P_{Df}(z) - P_{Dm}(z))$$

Thus,

$$P_{Dm}(z) = \frac{\lambda\ P_{Df}(z)}{(1 - \omega)\ z + \lambda} \qquad (11\text{-}69)$$

Again, assuming that at $t_D = 0$, $P_{Df} = 0$, the Laplace transform of Equation 11-60 is given by:

$$\frac{d^2P_{Df}(z)}{\partial r_D^2} + \frac{1}{r_D}\frac{dP_{Df}(z)}{\partial r_D} = (1 - \omega)\ z\ P_{Dm}(z) + \omega\ z\ P_{Df}(z)$$

Substituting Equation 11-69 in the above equation, we get:

$$\frac{d^2P_{Df}(z)}{dr_D^2} + \frac{1}{r_D}\frac{dP_{Df}(z)}{dr_D} = [z\ f(z)]\ P_{Df}(z) \qquad (11\text{-}70)$$

where $f(z) = \dfrac{\omega(1 - \omega)\ z + \lambda}{(1 - \omega)\ z + \lambda}$ (11-71)

It should be evident that Equation 11-70 is the same as Equation 11-24 except that z on the right-hand side of Equation 11-24 is replaced by z f(z) on the right-hand side of Equation 11-70. Therefore, all the solutions presented for Equation 11-24 are also solutions of Equation 11-70 providing that z is replaced by z f(z). Thus, the general solution of Equation 11-70 is obtained from Equation 11-25 by substituting z f(z) for z, and the result is as follows:

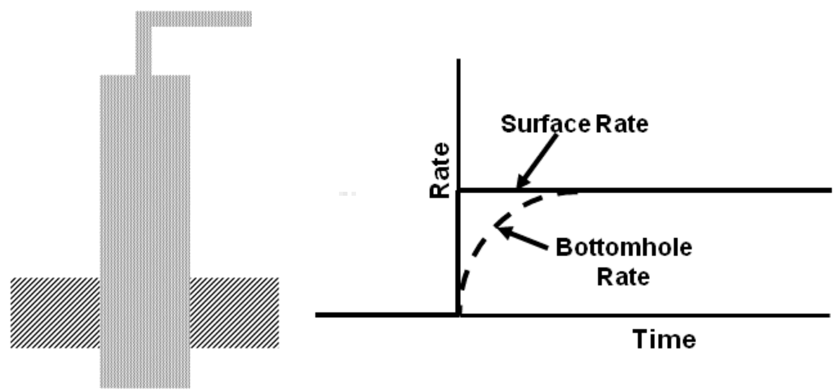
$$\frac{d^2P_{Df}(z)}{dr_D^2} + \frac{1}{r_D}\frac{dP_{Df}(z)}{dr_D} = [z\ f(z)]\ P_{Df}(z) \tag{11-70}$$

$$\frac{d^2P_D(z)}{dr_D^2} + \frac{1}{r_D}\frac{dP_D(z)}{dr_D} = z\ P_D(z) \tag{11-24}$$

$$P_D(z) = \frac{K_1(r_{De}\sqrt{zf(z)})\ I_0(r_D\sqrt{zf(z)}) + I_1(r_{De}\sqrt{zf(z)})\ K_0(r_D\sqrt{zf(z)})}{z\sqrt{zf(z)}\ [K_1(\sqrt{zf(z)})\ I_1(r_{De}\sqrt{zf(z)}) - K_1(r_{De}\sqrt{zf(z)})\ I_1\sqrt{zf(z)}]} \tag{11-73}$$

For the case of an infinite, naturally fractured reservoir with the well producing at a constant rate, and damage and skin included in the solution, we can obtain the solution from Equation 11-53. The result is as follows:

$$P_{wD}(z) = \frac{K_0(\sqrt{zf(z)}) + s\sqrt{zf(z)}\;K_1(\sqrt{zf(z)})}{zf(z)\;\{\sqrt{zf(z)}K_1(\sqrt{zf(z)}) + (zf(z))C_D[K_0\sqrt{zf(z)} + s\sqrt{zf(z)}K_1\sqrt{zf(z)}]\}} \qquad (11\text{-}74)$$



$$C = -\Delta V_w/\Delta P = c_f\,V_w,\ \text{bbl/psi} \qquad (4\text{-}2)$$

C_D = dimensionless wellbore storage coefficient defined by Equation 4-5

$$C = 144\,V_s/\rho,\ \text{bbl/psi} \qquad (4\text{-}3)$$

In Darcy units, C is expressed in res cc/atm, C_t is the total system compressibility at prevailing reservoir conditions in 1/atm, and h_z and r_W are expressed in cm. Note, however, that the expression, $C/(2\pi\phi hC_tr_w^2)$, is both dimensionless and unitless. If C is given in reservoir cu.ft/psi, and both h and r_W are given in ft, and C_t is in 1/psi, the expression $C/(2\pi\phi hC_tr_w^2)$ will remain unchanged. Therefore, we can define the dimensionless wellbore storage, C_D, as follows:

$$C_D = \frac{5.615\ C}{2\pi\phi hC_tr_w^2} \qquad (4\text{-}5)$$

where C = res bbl/psi

h, r_w = ft

Double Porosity
Pseudo-steady State Formulation

Deruyck et al. (1982) presented derivations and solutions for both the semi-steady state and transient flow models. It would be instructive to review their derivations here even though the semi-steady state case has already been presented.

We begin with the semi-steady state interporosity flow model. In this case the diffusivity equation for the fractures is given by:

$$\frac{\partial^2 P_f}{\partial r^2} + \frac{1}{r}\frac{\partial P_f}{\partial r} = \frac{1}{\eta_f}\frac{\partial P_f}{\partial t} - \frac{q^* \mu}{k_f} \qquad (11\text{-}75)$$

Fracture Flow Equation

where $\eta_f = \dfrac{k_f}{\phi_f \, C_f \, \mu}$

q^* = interporosity flow rate per unit bulk volume

In the matrix, the pressure is assumed to vary only in the vertical direction, v. Thus, the diffusivity equation is given by:

$$\frac{\partial^2 P_m}{\partial v^2} = \frac{1}{\eta_m} \frac{\partial P_m}{\partial t} + \frac{q^* \mu}{k_m}$$

The above equation can be written as follows:

$$\frac{k_m}{\mu} \frac{\partial^2 P_m}{\partial v^2} = \phi_m C_m \frac{\partial P_m}{\partial t} + q^* \qquad (11\text{-}76)$$

Matrix Flow Equation

When k_m is very small and $(\partial^2 P_m / \partial v^2)$ is negligible, Equation 11-76 takes the following form:

$$q^* = -\phi_m C_m \frac{\partial^2 P_m}{\partial t} \qquad (11\text{-}77)$$

The semi-steady state interporosity flow assumption is stated as follows:

$$q^* = \alpha \frac{k_m}{\mu} (P_m - P_f) \tag{11-78}$$

Thus, by equating Equations 11-77 and 11-78 we obtain Equation 11-65.

$$\begin{cases} q^* = -\phi_m C_m \dfrac{\partial^2 P_m}{\partial t} \\ q^* = \alpha \dfrac{k_m}{\mu} (P_m - P_f) \end{cases}$$

$$\phi_m C_m \frac{\partial P_m}{\partial t} = \frac{\alpha\, k_m}{\mu} (P_f - P_m) \tag{11-65}$$

In dimensionless form, Equation 11-75 becomes:

$$\frac{\partial^2 P_{Df}}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial P_{Df}}{\partial r_D} = \omega \frac{\partial P_{Df}}{\partial t_D} - \frac{\mu\, r_w^2}{k_f}\frac{2\,\pi\, k_f\, h}{q\,\mu} q^*$$

Let,

$$q_\cdot^* = \frac{2\,\pi\, k_f\, h}{q\,\mu} q^*$$

Thus, the above equation is written as follows:

$$\frac{\partial^2 P_{Df}}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial P_{Df}}{\partial r_D} = \omega \frac{\partial P_{Df}}{\partial t_D} - \frac{\mu\, r_w^2}{k_f} q_\cdot^* \qquad (11\text{-}79)$$

By assuming that at t = 0, $P_{Df} = 0$, the Laplace transform of Equation 11-79 is given by:

$$\frac{d^2 P_{Df}(z)}{dr_D^2} + \frac{1}{r_D}\frac{dP_{Df}(z)}{dr_D} = \omega\, z\, P_{Df}(z) - \frac{\mu\, r_w^2}{k_f} q_\cdot^*(z) \qquad (11\text{-}80)$$

By writing Equation 11-78 in dimensionless form and then taking the Laplace transform, we obtain:

$$q_{*}^{*}(z) = \alpha \frac{k_m}{\mu} (P_{Dm}(z) - P_{Df}(z)) \tag{11-81}$$

By Equation 11-69, Equation 11-81 becomes:

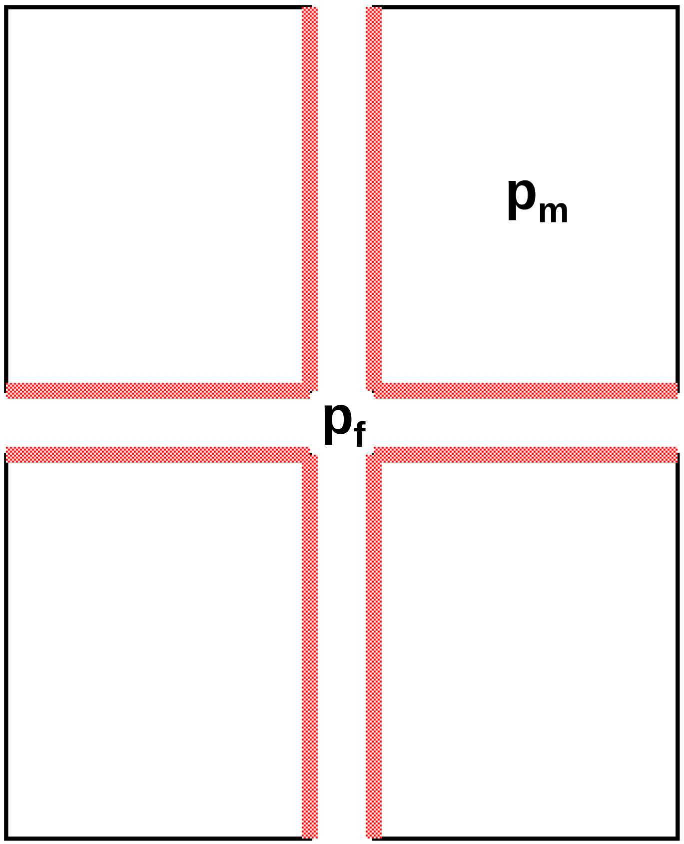
$$q_{*}^{*}(z) = -\alpha \frac{k_m}{\mu} \frac{(1-\omega) z}{(1-\omega)z+\lambda} P_{Df}(z) \tag{11-82}$$

Noting that $\lambda = (\alpha r_w^2 k_m)/k_f$, then substituting Equation 11-82 in Equation 11-80, we obtain:

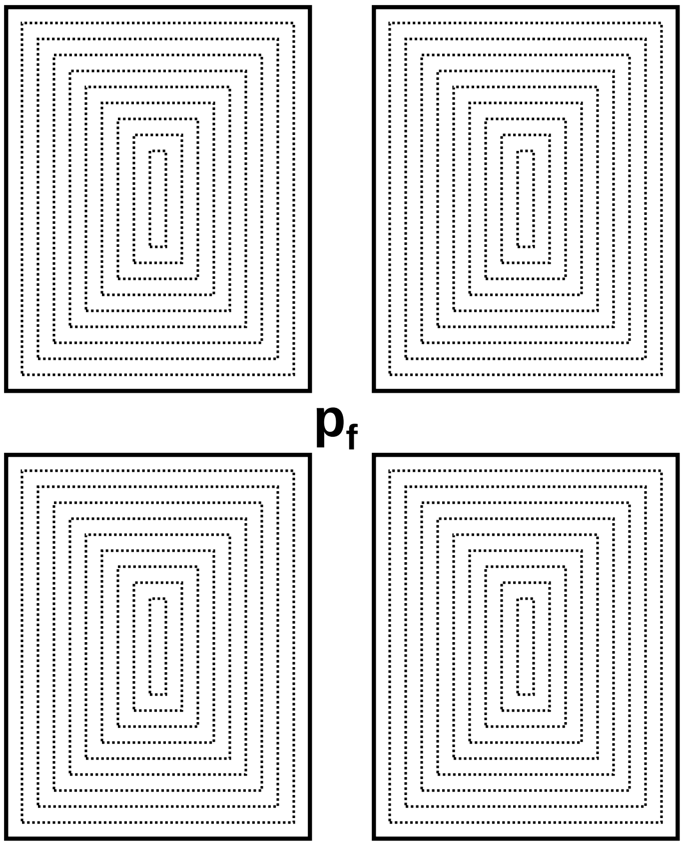
$$\frac{d^2 P_{Df}(z)}{dr_D^2} + \frac{1}{r_D} \frac{dP_{Df}(z)}{dr_D} = z\, f(z)\, P_{Df}(z) \tag{11-83}$$

Equation 11-83 is identical to Equation 11-70, and f(z) is the same as that defined by Equation 11-71.

Dual Porosity Models



Pseudosteady State



Transient

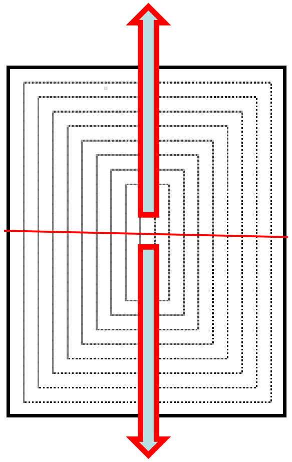
Double Porosity
Transient Formulation

In the case of the transient interporosity flow model, Equation 11-80 is still valid except that $q^*(z)$ is now defined as follows:

$$q^*(z) = \frac{2}{h_m} \frac{k_m}{\mu} \frac{dP_m(z)}{dv}\bigg|_{v=0} \qquad (11\text{-}84)$$

For the matrix, it is necessary to know whether the matrix can be approximated by slabs or by spheres. In the case of slabs, the following equation applies:

$$\frac{\partial^2 P_m}{\partial v^2} = \frac{1}{\eta_m} \frac{\partial P_m}{\partial t} \qquad (11\text{-}85)$$



Subject to the following conditions:

1. $P_m = P_i$ at $t = 0$
2. $\partial P_m/\partial v = 0$ at $v = h_m/2$, at all t
3. $P_m = P_f$ at $v = 0$, at all t

Before we take the Laplace transform of Equation 11-85, we need to distinguish between the Laplace parameters when the transform is taken with respect to t and when it is taken with respect to t_D.
Let,

$$f(z') = \int_0^\infty f(t) \exp(-z' t)\, dt$$

$$t_D = \frac{k_f\, t}{(\phi_m C_m + \phi_f C_f)\, \mu\, r_w^2}$$

and,

$$f(z) = \int_0^\infty f(t_D) \exp(-z t_D)\, dt_D$$

Since t_D is defined by Equation 11-62, z' and z are related as follows:

$$z' = \frac{k_f}{[\phi_m C_m + \phi_f C_t]\, \mu\, r_w^2}\, z \qquad (11\text{-}86)$$

Laplace Space Solution

In terms of z′, the Laplace transform of Equation 11-85 is given by:

$$\frac{d^2P_m(z')}{dv^2} = \frac{1}{\eta_m} z' \, P_m(z') \qquad (11\text{-}87)$$

The solution of Equation 11-87 is given by:

$$P_m(z') = P_f(z') \frac{\cosh\left[\left(\frac{h_m}{2} - v\right)\sqrt{z'/\eta_m}\right]}{\cosh\left[\frac{h_m}{2}\sqrt{z'/\eta_m}\right]} \qquad (11\text{-}88)$$

By Equation 11-84, Equation 11-88 is written as:

$$q^*(z) = \frac{2}{h_m}\frac{k_m}{\mu}\frac{dP_m(z)}{dv}\bigg|_{v=0}$$

$$q^*_\cdot(z') = -\frac{2}{h_m}\frac{k_m}{\mu}\sqrt{z'/\eta_m}\, P_{Df} \tanh\left[\frac{h_m}{2}\sqrt{z'/\eta_m}\right] \qquad (11\text{-}89)$$

In terms of the Laplace parameter, z, Equation 11-89 becomes:

$$q_{\cdot}^{\cdot}(z) = -\frac{2}{h_m}\frac{k_m}{\mu}\left[\frac{k_f}{k_m}\frac{(1-\omega)\ z}{r_w^2}\right]^{\frac{1}{2}} \tanh\left\{\frac{k_m}{2}\left[\frac{k_f}{k_m}\frac{(1-\omega)\ z}{r_w^2}\right]^{\frac{1}{2}}\right\} \qquad (11\text{-}90)$$

Substituting the above value of $q_{\cdot}^{\cdot}(z)$ in Equation 11-80 we obtain Equation 11-70 except that now f(z) is given by:

$$f(z) = \omega + \frac{2}{h_m}\left[r_w^2\frac{k_m}{k_f}\frac{(1-\omega}{z}\right]^{\frac{1}{2}} \tanh\left\{\frac{h_m}{2}\left[\frac{(1-\omega)}{r_w^2}\frac{k_f}{k_m}\ z\right]^{\frac{1}{2}}\right\} \qquad (11\text{-}91)$$

$$\frac{d^2P_{Df}(z)}{dr_D^2} + \frac{1}{r_D}\frac{dP_{Df}(z)}{dr_D} = \omega\ z\ P_{Df}(z) - \frac{\mu\ r_w^2}{k_f}\ q_{\cdot}^{\cdot}(z) \qquad (11\text{-}80)$$

$$\frac{d^2P_{Df}(z)}{dr_D^2} + \frac{1}{r_D}\frac{dP_{Df}(z)}{dr_D} = [z\ f(z)]\ P_{Df}(z) \qquad (11\text{-}70)$$

The inverse Laplace transform can be found by different ways. For example, we could prepare a table of transforms in which we list the transforms of many functions and refer to this table to find the inverse transform. We can use the table of transform in conjunction with Equation 11-17 and other known properties of the transform. Another technique relies on integration in the complex plane. However, in most problems related to well testing, this latter technique could lead to expressions that are very difficult to evaluate. For this reason, the present trend is to find the inverse transform numerically and present the results in the form of a type-curve.

The algorithm presented by Stehfest (1970) has gained wide acceptance by researchers in the field of well testing. We will discuss Stehfest's algorithm and with the exception of referring to a table of transforms, we will not discuss any of the other methods of finding the inverse Laplace transform.

Stehfest's algorithm is based on the following formulae:

$$V_i = (-1)^{n/2+i} \sum_{k=(i+1)/2}^{\min(i,n/2)} \frac{k^{n/2}\,(2k)!}{(n/2-k)!\,k!\,(k-1)!\,(i-k)!\,(2k-i)!} \tag{11-18}$$

$$f(t) = \frac{\ln 2}{t} \sum_{i=1}^{n} V_i\, P\left(\frac{\ln 2}{t}\, i\right) \tag{11-19}$$

The number, n, in these expressions should be optimized by trial and error. Increasing n increases the accuracy of the results up to a point, and then the accuracy declines because of roundoff errors, since the word length on the computer is finite. Note that $f(t) = L^{-1}P(z)$, and z is replaced by $i \ln 2/t$, where t is the time at which the inverse transform is required. Also note that for a given n the Stehfest algorithm requires calculation of V_i only once.

Program 11-1 is written in FORTRAN. It is written to find the inverse transform of $P(z) = 1/\sqrt{z}$, at t = 1, 2, 3,. . . . . .10. The program is suitable for finding the inverse transform of any given continuous function by making the necessary changes where indicated in the program. With n = 18, the program gave exact results up to 5 decimals. This was possible to check because we know that:

$$\frac{1}{\sqrt{z}} = L\left[\frac{1}{\sqrt{\pi t}}\right]$$

For a given f(z) for which we do not know the inverse transform, n can be optimized by referring to a table of transforms and choosing a function that is close to the function on hand. Also, if n is not properly selected, a plot of the inverse transform will tend to oscillate, whereas an appropriately chosen value of n will yield a smooth inverse transform.

Program 11-1
Inverse Laplace Transform by the Stehfest Algorithm

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION V(30),G(30),H(30)
C     N SHOULD BE OPTIMIZED
      N = 18
C
      DLN2 = 0.6931471805599453
      G(1) = 1.0
      NH = N/2
      DO 10 I = 2,N
 10   G(I) = G(I-1)*I
      H(1) = 2.0/G(NH-1)
      DO 100 I = 2,NH
      FI = I
      IF(I.EQ.NH) GO TO 50
      H(I) = FI**NH*G(2*I)/(G(NH-I)*G(I)*G(I-1))
      GO TO 100
 50   H(I) = FI**NH*G(2*I)/(G(I)*G(I-1))
100   CONTINUE
      SN = 2*(NH-NH/2*2)-1
      DO 200 I = 1,N
      V(I) = 0.0
      K = (I + 1)/2
      KK = I
      IF(KK.GT.NH) KK = NH
      DO 150 J = K,KK
      IF(2*J-I.EQ.0) GO TO 120
      IF(I.EQ.J) GO TO 130
```

(program continued on next page)

```
      V(I) = V(I) + H(J)/(G(I-J)*G(2*J-I))
      GO TO 150
120   V(I) = V(I) + H(J)/G(I-J)
      GO TO 150
130   V(I) = V(I) + H(J)/G(2*J-I)
150   CONTINUE
      V(I) = SN*V(I)
      SN = -SN
200   CONTINUE
      FT = 0.0
C     t SHOULD BE CHANGED AS DESIRED
      T =  1.0
C     HERE, WE EVALUATE THE INVERSE LAPLACE TRANSFORM AT
C     T = 1, 2, 3.......10.
C     THIS SHOULD BE CHANGED IF OTHER T VALUES ARE NEEDED.
      DO 300 I = 1,10
      A = DLN2/T
      DO 270 K = 1,N
      Z = A*K
C     THE FUNCTION BEING EVALUATED HERE IS: F(z) = 1/SQRT(z). THIS
C     SHOULD BE CHANGED.
C     FT = FT +  (1.0/DSQRT(Z))*V(K)
      FT = FT +  (1.0/DSQRT(Z))*V(k)
270   CONTINUE
      FT = A*FT
      WRITE (*,280)T,FT
280   FORMAT(5x,'T = ',F6.0,5X,'FT = ',F12.5)
      T = T + 1
      FT = 0.0
300   CONTINUE
      END
```

Spherical Coordinate- PDE

When the matrix blocks are assumed to be spheres, Equation 11-80 is still valid except that $q^*(z)$ is now defined by:

$$q^*(z) = \frac{2}{r_m}\frac{k_m}{\mu}\frac{dP_m(z)}{dr}\bigg|_{r=r_m} \quad (11\text{-}92)$$

and Equation 11-85, the matrix equation, is replaced by:

$$\frac{\partial^2 P_m}{\partial r^2} + \frac{2}{r}\frac{\partial P_m}{\partial r} = \frac{1}{\eta_m}\frac{\partial P_m}{\partial t} \quad (11\text{-}93)$$

with the following initial and boundary conditions:

1. $P_m = P_i$ at $t = 0$
2. $P_m = P_f$ at $r = r_m$, for all t and for each sphere

In the Laplace domain, Equation 11-93 is given by:

$$\frac{d^2 P_m(z')}{dr^2} + \frac{2}{r}\frac{dP_m(z')}{dr} = \frac{1}{\eta_m} z' P_m(z') \quad (11\text{-}94)$$

Spherical Coordinate-Solution

The solution to Equation 11-94 is given by:

$$P_m(z') = \frac{r_m}{r} P_f(z') \sinh\left[r\sqrt{z'/\eta_m}\right] \Big/ \sinh\left[r_m\sqrt{z'/\eta_m}\right] \tag{11-95}$$

From Equations 11-92 and 11-95, $q^{\bullet}(z')$ is given by:

$$q^{\bullet}(z') = -\frac{3}{r_m^2}\frac{k_m}{\mu} P_{Df}(z') \left\{\sqrt{z'/\eta_m}\coth\left[r_m\sqrt{z'/\eta_m}\right] - 1\right\} \tag{11-96}$$

By converting Equation 11-96 to the Laplace parameter, z, by Equation 11-86, and then by substituting in Equation 11-80 and noting that $\lambda = 15r_w^2/r_m^2(k_m/k_f)$, we obtain Equation 11-70 with f(z) now given by:

$$f(z) = \omega + \frac{1}{5}\frac{\lambda}{z}\left\{\left[\frac{15(1-\omega)\,z}{\lambda}\right]^{\frac{1}{2}}\coth\left[\frac{15(1-\omega)\,z}{\lambda}\right]^{\frac{1}{2}} - 1\right\} \tag{11-97}$$

Well Test Analysis on the Basis of Warren & Root Model