Flow in Porous Media

Module 2.b
Fundamental of Single Phase Flow in Porous Media

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Outline

- **Mathematical treatment of engineering problems**
- **Relations of Diffusive and Convective Flow**
- **Steady State Flow Behavior and Solution**
- **1-D Radial Steady State Flow**
- **Time-dependent Processes without Spatial Variation**
- **Development of Hydraulic Diffusivity Equation**
- **Flow Regimes**
- van Everdingen- Hurst Constant Terminal Rate Solution
- Constant pressure solution
- Superposition principle

Mathematical Treatment of Engineering Problems

A quantitative description of a physical process always requires a mathematical formulation. These mathematics aim at approximating these processes in a more or less sufficient way, but they will always refer only to the most important aspects of the process. These mathematics are summarized by the term mathematical model.

Step-1: The expression of the problem in mathematical language (formulation)

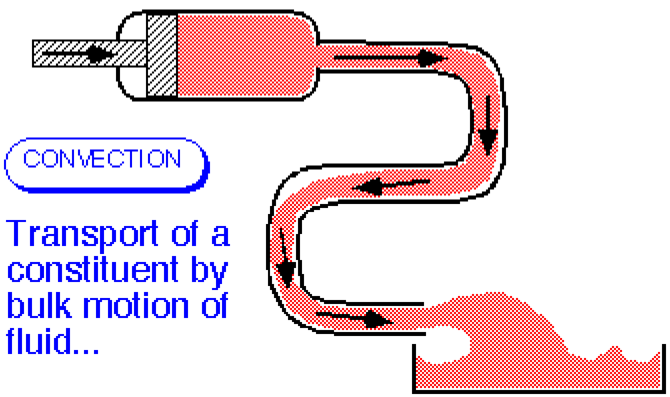
- The key step in formulation of the problem is expression of conservation laws
- It frequently involves setting of a differential equation

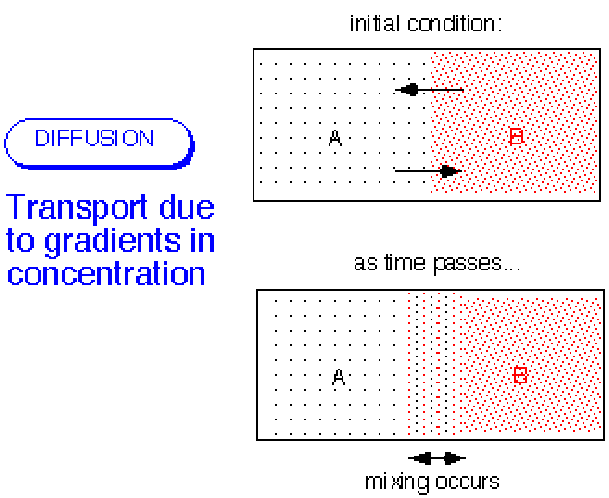
Step-2: The appropriate mathematical operations (end up with solution of the PDE)

Step-3: The interpretation of the results (from behavior of the solution)

Relations of Diffusive and Convective Flow

- **Flow of heat and fluid**
  - **Diffusive** (conductive): A potential (T, P,C,…) is transferred by diffusive flow (molecular movement)
    - Example: Molecular diffusion is the resulting net transport of molecules from a region of higher concentration to one of lower concentration.
  - **Convective**: Convective transport occurs when a constituent of the fluid (mass, energy, a component in a mixture) is carried along with the fluid (bulk movement of a fluid)





In the general case, it is necessary to account for both mechanisms of transport. The flux of each component of a mixture is determined by its concentration and both the "bulk" fluid velocity (the average of all the components) and the "diffusion" velocity of the component.

Steady State Flow Behavior and Solution

- **Steady State Flow**: A system is called steady state if <u>none</u> of the system variables change with time.

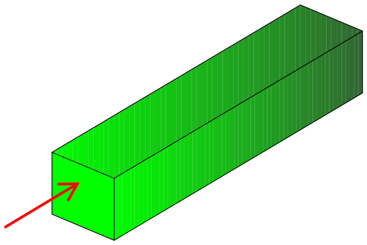
- Darcy's law for 1-D steady state flow

$$q = -\frac{kA}{\mu}\frac{\partial p}{\partial x}$$

- Separating variable and integrating lead to:

$$q = -\frac{kA}{\mu}\frac{p_1 - p(x)}{x}$$

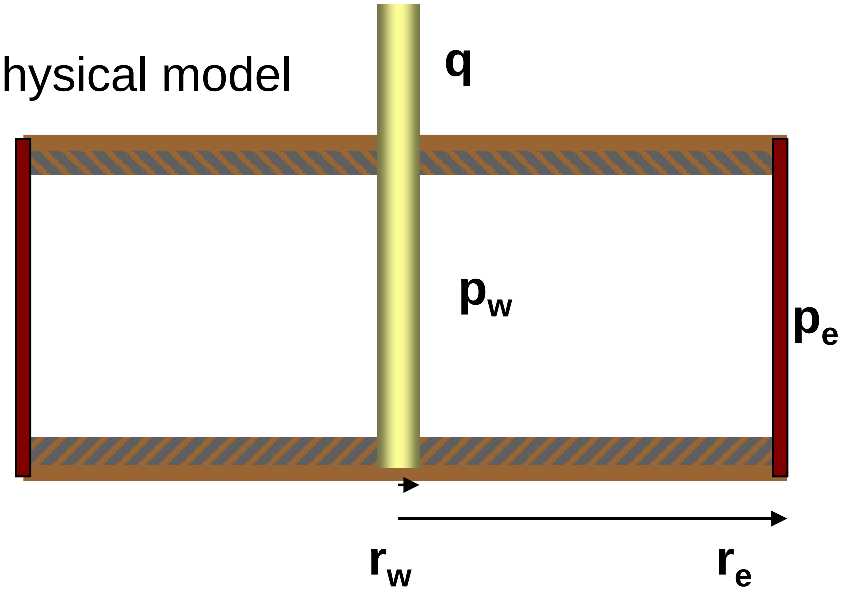
- Behavior: Linear relation between pressure and distance



$$p(x) = p_1 - \frac{q\mu}{kA}x$$

1-D Radial Steady State Flow

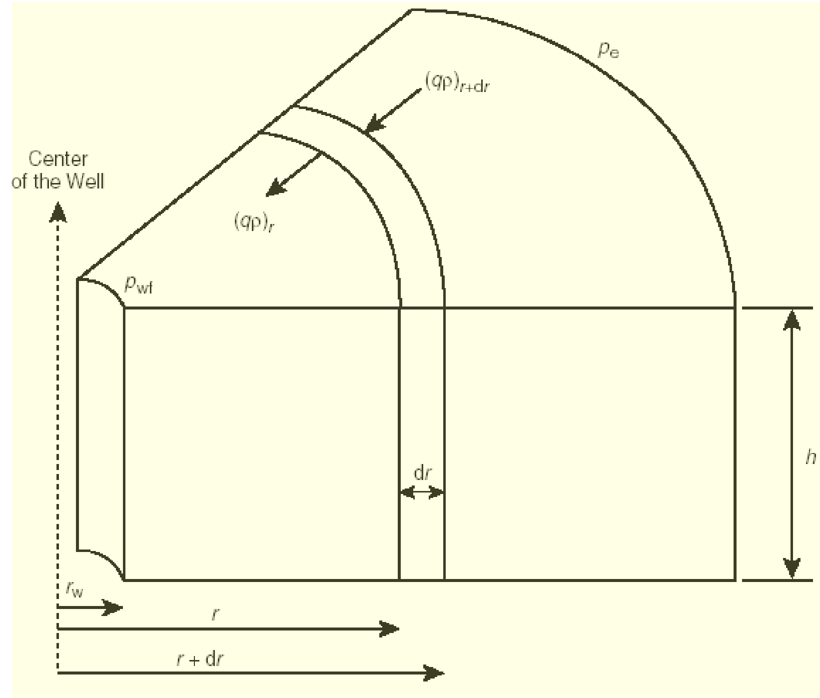
- Physical model



- Simplifying assumptions
  - Single phase fluid flow
  - Fluid has a small compressibility
  - Darcy's law applies
  - Flow is radial towards the wellbore
  - Rock and fluid properties are constant

Mathematical model Steady State, Radial Flow

- Choosing an appropriate element



- Governing equation

Mass balance

$$Input - Output = 0$$

$$\left(-\rho\ A\ v\ \right)_r - \left(-\rho\ A\ v\ \right)_{r+\Delta r} = 0$$

Darcy's law

$$v = -\frac{k}{\mu}\frac{\partial p}{\partial r}$$

Equation of state

$$\rho = \rho_b \exp\left(c\ \left(p - p_b\right)\right)$$

$$\begin{cases} \left(-\rho\ A\ \dfrac{k}{\mu}\dfrac{\partial p}{\partial r}\right)_r - \left(-\rho\ A\ \dfrac{k}{\mu}\dfrac{\partial p}{\partial r}\right)_{r+\Delta r} = 0 \\ \left(-\rho\ A\ \dfrac{k}{\mu}\dfrac{dp}{dr}\right)_{r+\Delta r} = \left(-\rho\ A\ \dfrac{k}{\mu}\dfrac{dp}{dr}\right)_r + \dfrac{d}{dr}\left(-\rho\ A\ \dfrac{k}{\mu}\dfrac{dp}{dr}\right)_r \Delta r + \ldots. \end{cases}$$

$$\Rightarrow \quad \frac{k}{\mu}\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho\frac{\partial p}{\partial r}\right) = 0$$

$$\frac{k}{\mu}\frac{1}{r}\left(\frac{\partial\rho}{\partial r}\left(r\frac{\partial p}{\partial r}\right)+\rho\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)\right)=0 \Rightarrow \frac{1}{r}\left(\frac{\partial\rho}{\partial p}\frac{\partial p}{\partial r}\left(r\frac{\partial p}{\partial r}\right)+\rho\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)\right)=0 \Rightarrow$$

$$\frac{\rho}{r}\left(\frac{1}{\rho}\frac{\partial\rho}{\partial p}\frac{\partial p}{\partial r}\left(r\frac{\partial p}{\partial r}\right)+\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)\right)=0 \xrightarrow{c=\frac{1}{\rho}\frac{\partial\rho}{\partial p}=-\frac{1}{V}\frac{\partial V}{\partial p}} \underbrace{\cancel{cr\left(\frac{\partial p}{\partial r}\right)^2}}_{\text{Negligible}}+\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)=0$$

Governing equation

$$\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)=0$$

or

$$\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}=0$$

Initial condition

$$p = p_i \,, \qquad t = 0, \quad r \geq r_w$$

Boundary conditions

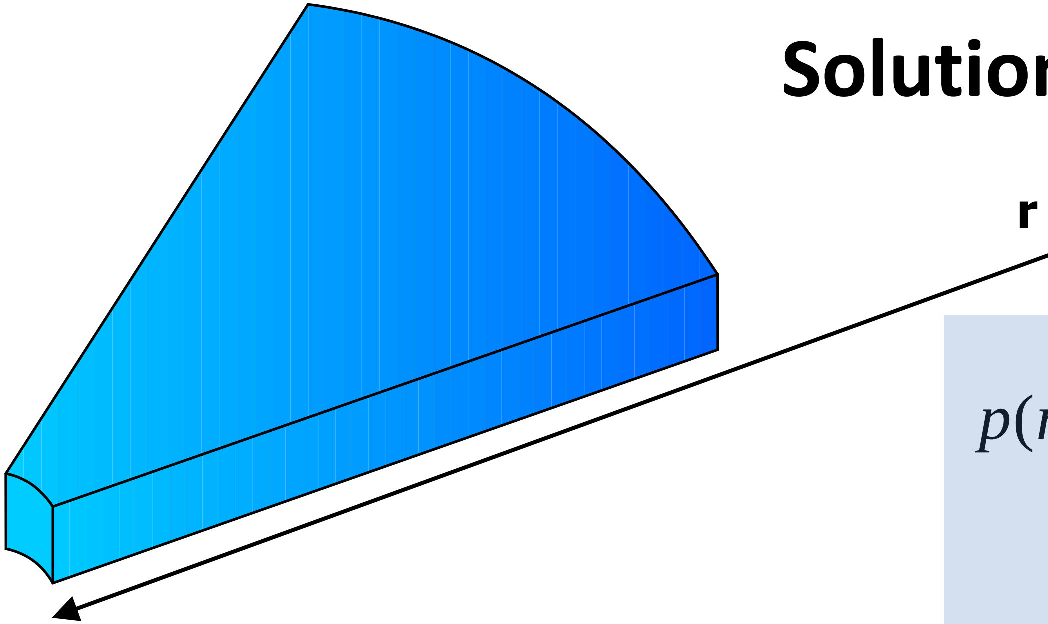
$$\begin{cases} (p)_{r_w} = p_{wf} \\ (p)_{r_e} = p_e \end{cases}$$

Solution

$$\left(r \frac{dp}{dr} \right) = C_1 \Rightarrow p = C_1 \ln(r) + C_2$$

$$C_1 = \frac{(p_e - p_w)}{\ln(\frac{r_e}{r_w})} \qquad C_2 = p_w - \frac{(p_e - p_w)}{\ln(\frac{r_e}{r_w})} \ln(r_w)$$

Solution



$$\begin{cases} q = 2\pi\, r\, hk \left.\dfrac{dp}{dr}\right|_{r_w} \\[2ex] \dfrac{dp}{dr} = \dfrac{\left(p_e - p_w\right)}{\ln(\dfrac{r_e}{r_w})}\,\dfrac{1}{r} \end{cases}$$

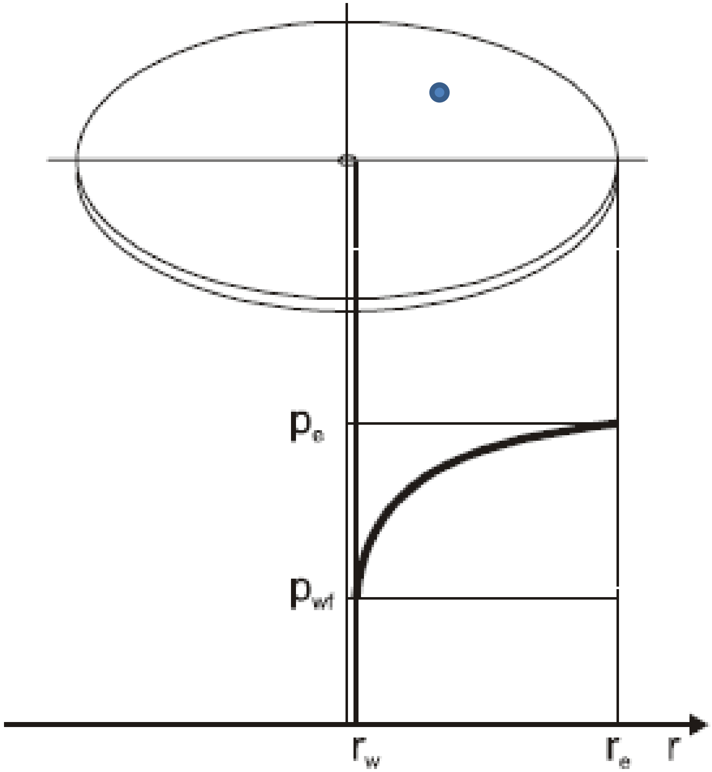
$$p(r) = p_w + \frac{\left(p_e - p_w\right)}{\ln(\dfrac{r_e}{r_w})} \ln(\frac{r}{r_w})$$

$$q = \frac{2\pi\, hk}{\mu}\,\frac{\left(p_e - p_w\right)}{\ln(\dfrac{r_e}{r_w})}$$

Steady State Flow- Radial System

$$p_{wf} - p = \frac{\mu q B}{2\pi h k} \ln \frac{r}{r_w}$$

$$q = -\frac{2\pi h k}{\mu B} \frac{p_e - p_{wf}}{\ln(r_e / r_w)}.$$



Off-centered well

$$q = -\frac{-2\pi k h}{\mu B} \frac{p_e - p_w}{ln\left(\frac{r_e^2 - \sigma^2}{r_e \quad r_w}\right)}$$

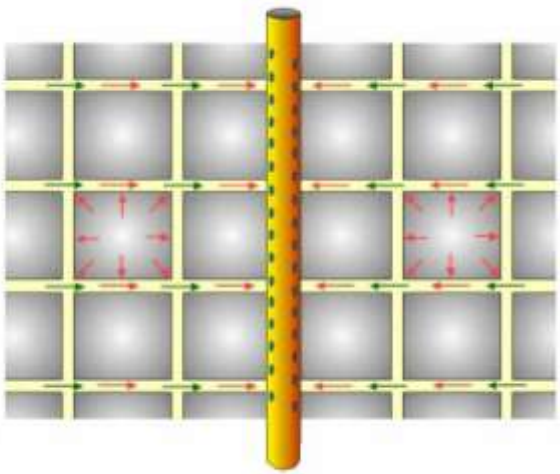
$$\sigma = \sqrt{x^2 + y^2}$$ Shape Factor: Depends only on system geometry

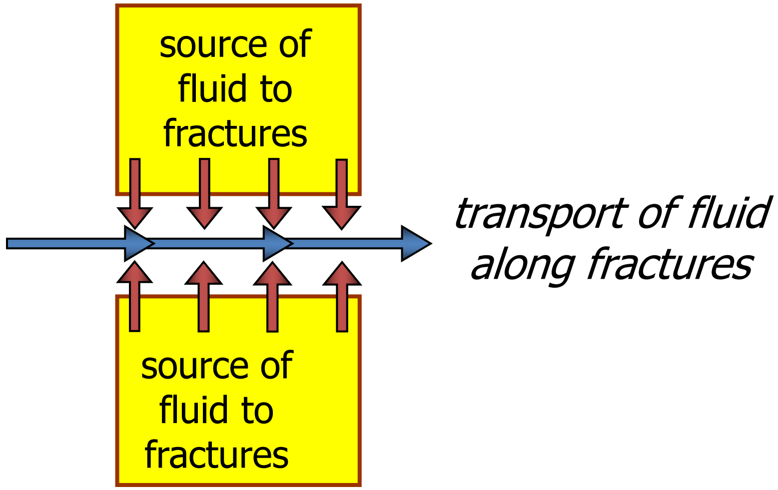
Time-dependent Processes without Spatial Variation

Lumped Analysis (zero-dimensional) of Transfer Function in NFRs

Here, we will consider problems where changes of a parameter such as pressure on a “ spatially averaged” is required.

In general, the matrix releases the fluid into the fractures upon pressure decline (inter-porosity flow). Subsequently the fractures transport the fluid to the wellbore.





$$\varphi_m c_m \frac{\partial p_m}{\partial t} = \alpha \frac{k_m}{\mu} \left(p_m - p_f \right)$$

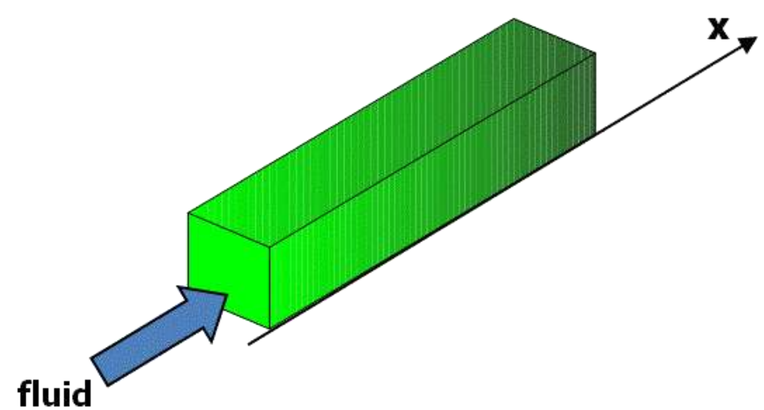
Time-dependent Processes with Spatial Variation

- The linear, one dimensional, horizontal, one phase, partial differential flow equation for a liquid, assuming constant permeability, viscosity and compressibility for transient or time dependent flow:

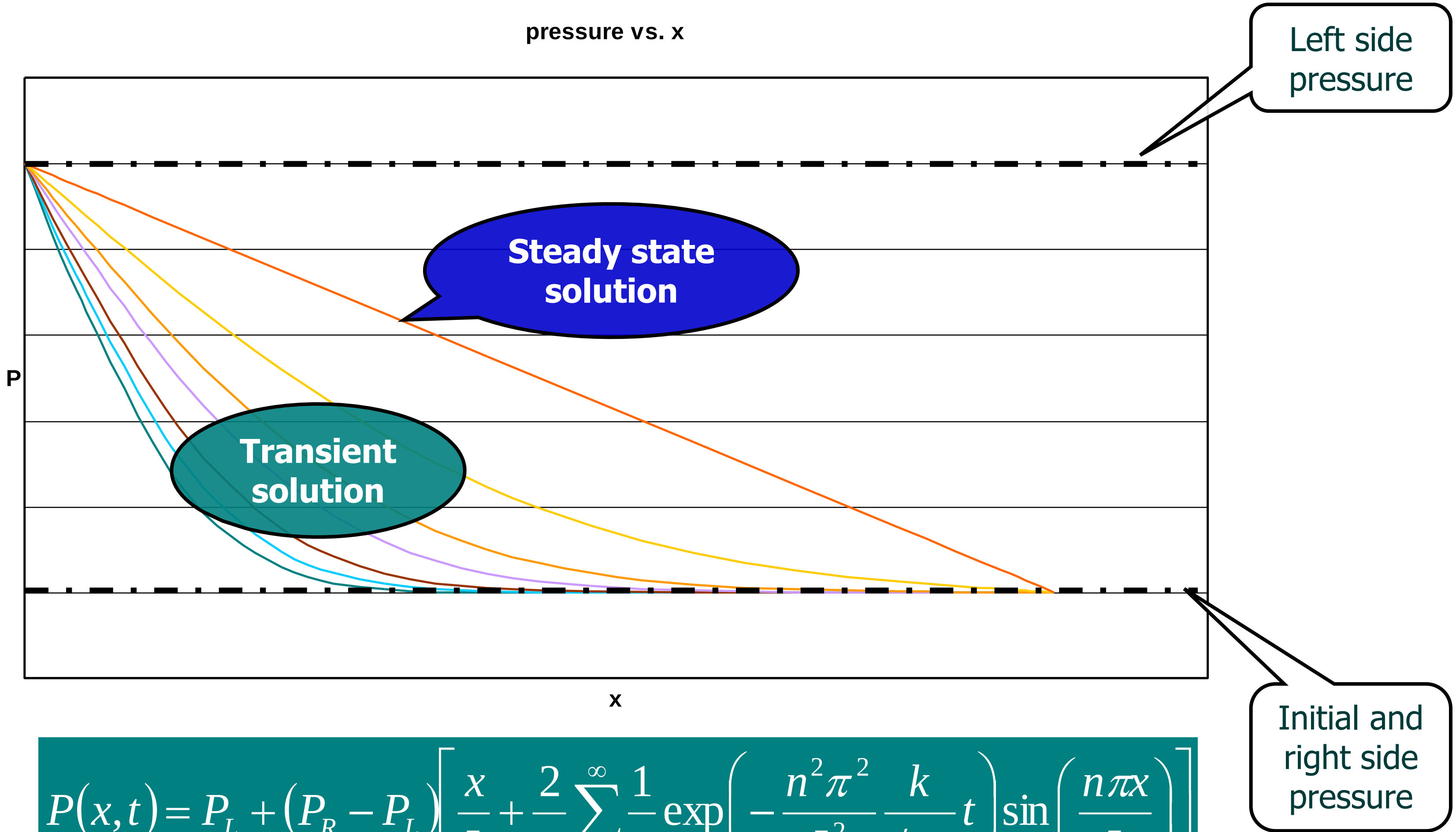
$$\boxed{\frac{\partial^2 P}{\partial x^2} = \left(\frac{\varphi \mu c}{k}\right)\frac{\partial P}{\partial t}}$$

- If the flow reaches a state where it is no longer time dependent, we denote the flow as steady state. The equation then simplifies to:

$$\frac{\partial^2 P}{\partial x^2} = 0$$



- Transient and steady state pressure distributions are illustrated graphically in the figure below for a system where initial and right hand pressures are equal:



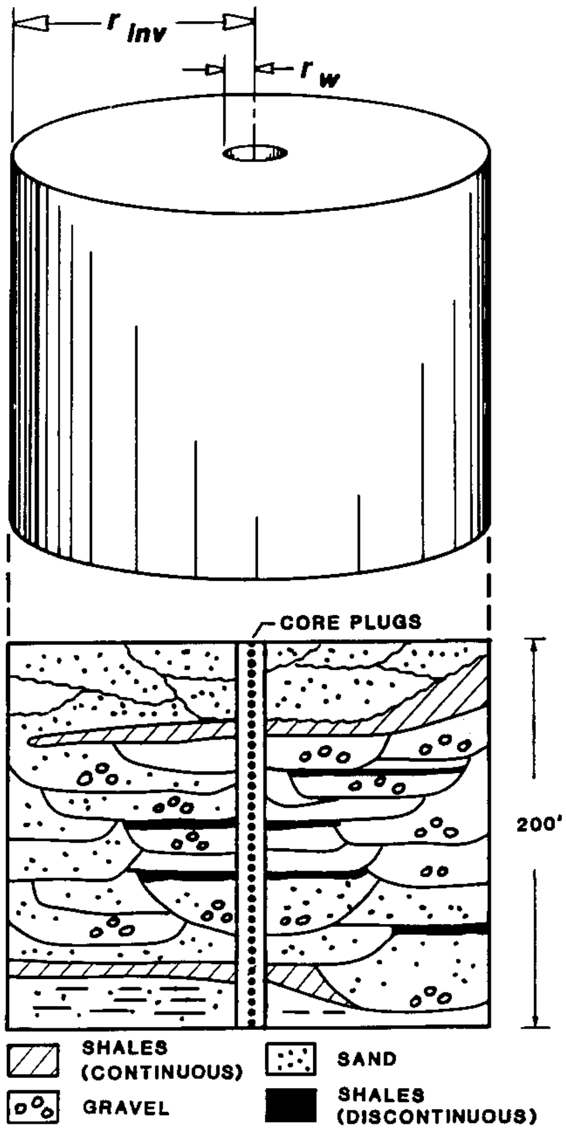
$$P(x,t)=P_L+(P_R-P_L)\left[\frac{x}{L}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-\frac{n^2\pi^2}{L^2}\frac{k}{\phi\mu c}t\right)\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$P(x,t)=P_L+(P_R-P_L)\frac{x}{L}$$

Development of Hydraulic Diffusivity Equation for Flow of a Slightly Compressible Oil and Its Solution Subjected to Different Boundary Conditions

- **Physical model**
- **Simplifying assumptions**
- **Mathematical model**
  - **Choosing an appropriate element**
  - **Governing equation**
    - **Mass balance**
    - **Momentum balance (Darcy's law)**
    - **Equation of state**
  - **Initial and Boundary conditions**
    - **Infinite acting**
      - **Constant rate production**
      - **Constant pressure production**
    - **Finite acting**
      - **Constant rate production**
      - **Constant pressure production**
  - **Solutions**
    - **Laplace space solutions**
    - **Time domain solutions**
    - **Simplified solutions**
- **Applications (Drawdown (single rate & multi rate), Reservoir limit test, Build up, Superposition (time & space), …),**

Physical Model



Reservoir Engineering Model

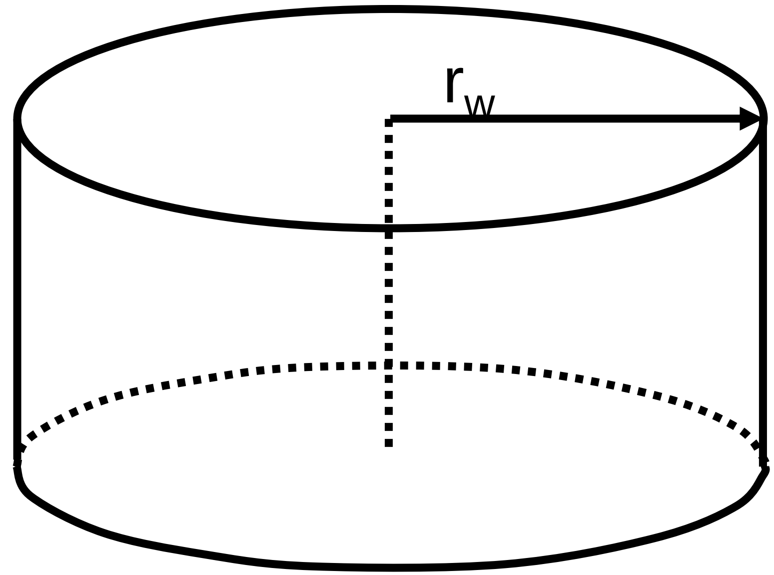
- Works 95+ percent of the time...
- Why? Pressure and volume averaging of reservoir properties.
- When does it not work? High contrast in reservoir properties.

Actual Reservoir Model

- Complex geology.
- Complex fluid behavior.
- Poor lateral (and vertical) continuity.

Simplifying Assumptions

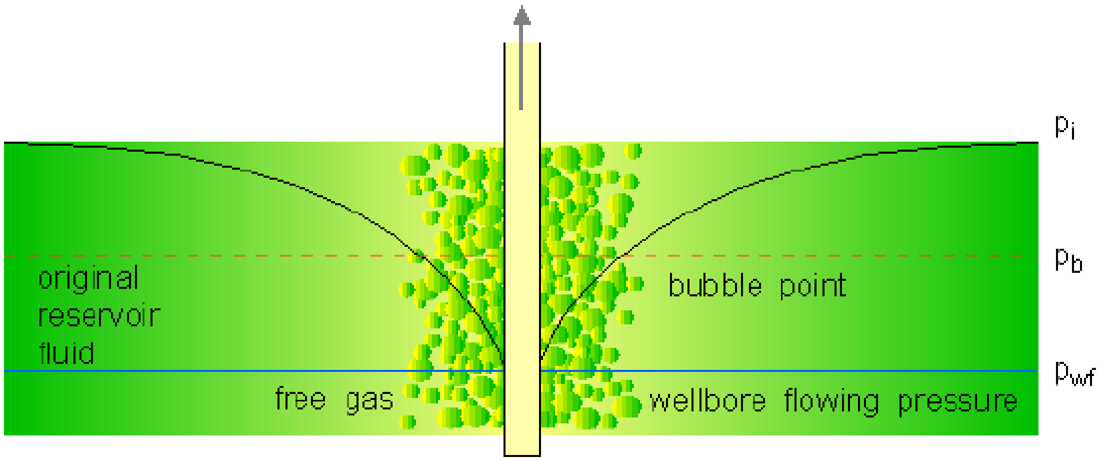
- Homogeneous
- Isotropic
- Ignore Gravity
- Constant Temperature
- Darcy's law applies
- Single phase fluid
- Radial flow
- Totally penetrating vertical well
- Constant net pay, saturation
- $(\partial p/\partial r)$ - gradient in reservoir - is small
- Constant wellbore storage
- Constant pressure throughout reservoir at time t = 0
- Constant production rate
- Closed circular reservoir
- Model complexities will be introduced as necessary



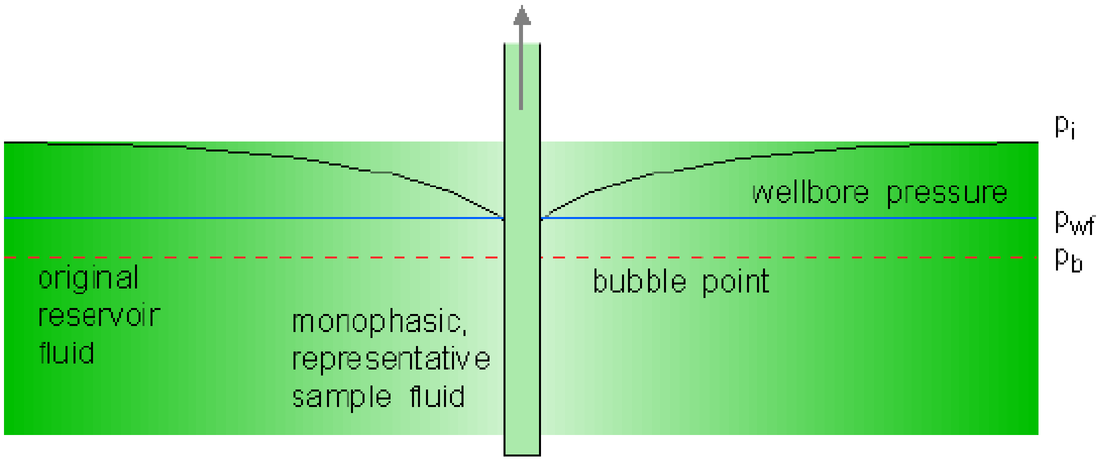
OIL/WATER

- Compressibility is small and constant
- Viscosity is constant
- Laminar flow

Simplifying Assumptions- Single phase fluid



Single phase assumption; $p_{wf}>p_b$



Mathematical Model-Governing Equation

- Mass balance

$$\left(-\rho_o A v\right)_{r+\Delta r} - \left(-\rho_o A v\right)_r = \frac{\left(\rho_o \Delta V\right)_{t+\Delta t} - \left(\rho_o \Delta V\right)_t}{\Delta t}$$

- Momentum balance (Darcy's law)

$$v_{gr} = -\frac{k}{\mu}\frac{\partial p}{\partial r}$$

- Equation of state

$$\rho_o = \rho_{ob}\exp\left(c_o (p - p_b)\right)$$

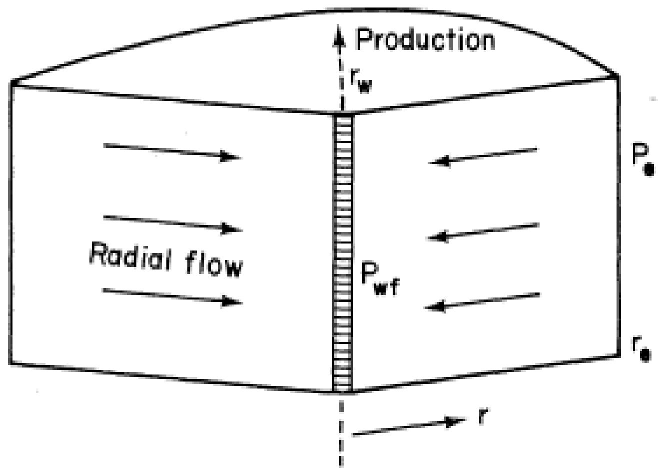
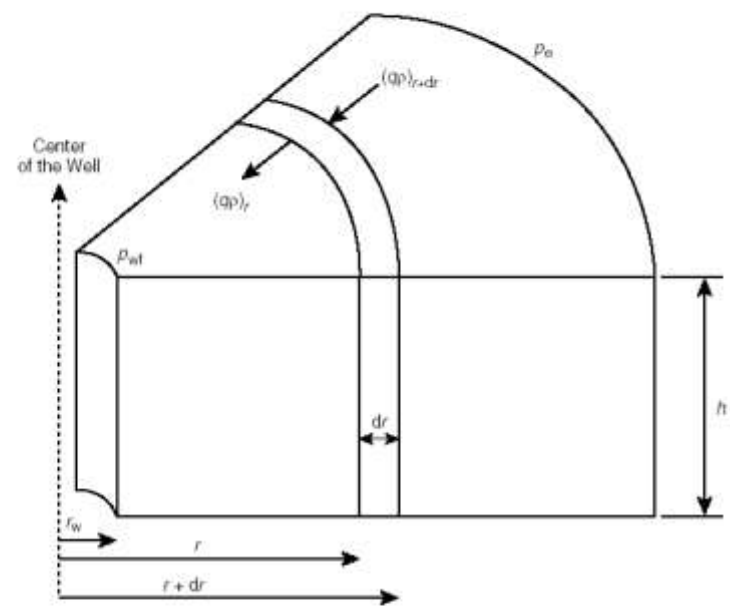


Fig. 9.1 Radial flow towards a well.



$$\begin{bmatrix}\text{mass entering}\\ \text{volume element}\\ \text{during interval } \Delta t\end{bmatrix} - \begin{bmatrix}\text{mass leaving}\\ \text{volume element}\\ \text{during interval } \Delta t\end{bmatrix}$$

$$= \begin{bmatrix}\text{rate of mass}\\ \text{accumulation}\\ \text{during interval } \Delta t\end{bmatrix} \qquad [1.2.44]$$

$$(\text{Mass})_{\text{in}} = \Delta t[A\nu\rho]_{r+\mathrm{d}r} \qquad [1.2.45]$$

where:

ν = velocity of flowing fluid, ft/day
ρ = fluid density at $(r + \mathrm{d}r)$, lb/ft^3
A = area at $(r + \mathrm{d}r)$
Δt = time interval, days

The area of the element at the entering side is:

$$A_{r+\mathrm{d}r} = 2\pi(r + \mathrm{d}r)h \qquad [1.2.46]$$

Combining Equations 1.2.46 with 1.2.35 gives:

$$[\text{Mass}]_{\text{in}} = 2\pi\,\Delta t(r + \mathrm{d}r)h(\nu\rho)_{r+\mathrm{d}r} \qquad [1.2.47]$$

Mass leaving the volume element Adopting the same approach as that of the leaving mass gives:

$$[\text{Mass}]_{\text{out}} = 2\pi\,\Delta t r h(\nu\rho)_r \qquad [1.2.48]$$

Total accumulation of mass The volume of some element with a radius of r is given by:

$$V = \pi r^2 h$$

Differentiating the above equation with respect to r gives:

$$\frac{\mathrm{d}V}{\mathrm{d}r} = 2\pi rh$$

or:

$$\mathrm{d}V = (2\pi rh)\,\mathrm{d}r \qquad [1.2.49]$$

Total mass accumulation during $\Delta t = \mathrm{d}V[(\phi\rho)_{t+\Delta t} - (\phi\rho)_{\mathrm{t}}]$. Substituting for $\mathrm{d}V$ yields:

$$\text{Total mass accumulation} = (2\pi rh)\mathrm{d}r[(\phi\rho)_{t+\Delta t} - (\phi\rho)_{\mathrm{t}}] \qquad [1.2.50]$$

Replacing the terms of Equation 1.2.44 with those of the calculated relationships gives:

$$2\pi h(r + \mathrm{d}r)\Delta t(\phi\rho)_{r+\mathrm{d}r} - 2\pi hr\,\Delta t(\phi\rho)_{\mathrm{r}}$$
$$= (2\pi rh)\mathrm{d}r[(\phi\rho)_{t+\Delta t} - (\phi\rho)_{\mathrm{t}}]$$

Dividing the above equation by $(2\pi rh)\mathrm{d}r$ and simplifying gives:

$$\frac{1}{(r)\mathrm{d}r}\left[(r + \mathrm{d}r)(\upsilon\rho)_{r+\mathrm{d}r} - r(\upsilon\rho)_r\right] = \frac{1}{\Delta t}\left[(\phi\rho)_{t+\Delta t} - (\phi\rho)_t\right]$$

or:

$$\frac{1}{r}\frac{\partial}{\partial r}[r(\upsilon\rho)] = \frac{\partial}{\partial t}(\phi\rho) \qquad [1.2.51]$$

where:

ϕ = porosity
ρ = density, lb/ft^3
V = fluid velocity, ft/day

The transport equation must be introduced into the continuity equation to relate the fluid velocity to the pressure gradient within the control volume $\mathrm{d}V$. Darcy's law is essentially the basic motion equation, which states that the velocity is proportional to the pressure gradient $\partial p/\partial r$. From Equation 1.2.13:

$$v = (5.615)(0.001127)\frac{k}{\mu}\frac{\partial p}{\partial r}$$

$$= (0.006328)\frac{k}{\mu}\frac{\partial p}{\partial r} \qquad [1.2.52]$$

where:

k = permeability, md
v = velocity, ft/day

Combining Equation 1.2.52 with 1.2.51 results in:

$$\frac{0.006328}{r}\frac{\partial}{\partial r}\left(\frac{k}{\mu}(\rho r)\frac{\partial p}{\partial r}\right) = \frac{\partial}{\partial t}(\phi\rho) \qquad [1.2.53]$$

$$\frac{\partial}{\partial t}(\phi\rho) = \phi\frac{\partial \rho}{\partial t} + \rho\frac{\partial \phi}{\partial t} \qquad [1.2.54]$$

The porosity is related to the formation compressibility by the following:

$$c_{\mathrm{f}} = \frac{1}{\phi}\frac{\partial \phi}{\partial p} \qquad [1.2.55]$$

Applying the chain rule of differentiation to $\partial\phi/\partial t$:

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial t}$$

Substituting Equation 1.2.55 into this equation:

$$\frac{\partial \phi}{\partial t} = \phi c_{\mathrm{f}}\frac{\partial p}{\partial t}$$

Finally, substituting the above relation into Equation 1.2.54 and the result into Equation 1.2.53 gives:

$$\frac{0.006328}{r}\frac{\partial}{\partial r}\left(\frac{k}{\mu}(\rho r)\frac{\partial p}{\partial r}\right) = \rho\phi c_{\mathrm{f}}\frac{\partial p}{\partial t} + \phi\frac{\partial \rho}{\partial t} \qquad [1.2.56]$$

$$\left[\frac{0.006328k}{\mu r}\right]\frac{\partial}{\partial r}\left(r\rho\frac{\partial p}{\partial r}\right) = \rho\phi c_{\mathrm{f}}\frac{\partial p}{\partial t} + \phi\frac{\partial \rho}{\partial t} \qquad [1.2.57]$$

Expanding the above equation gives:

$$0.006328\left(\frac{k}{\mu}\right)\left[\frac{\rho}{r}\frac{\partial p}{\partial r} + \rho\frac{\partial^2 p}{\partial r^2} + \frac{\partial p}{\partial r}\frac{\partial \rho}{\partial r}\right]$$

$$= \rho\phi c_{\mathrm{f}}\left(\frac{\partial p}{\partial t}\right) + \phi\left(\frac{\partial \rho}{\partial t}\right)$$

Using the chain rule in the above relationship yields:

$$0.006328\left(\frac{k}{\mu}\right)\left[\frac{\rho}{r}\frac{\partial p}{\partial r} + \rho\frac{\partial^2 p}{\partial r^2} + \left(\frac{\partial p}{\partial r}\right)^2\frac{\partial \rho}{\partial p}\right]$$

$$= \rho\phi c_{\mathrm{f}}\left(\frac{\partial p}{\partial t}\right) + \phi\left(\frac{\partial p}{\partial t}\right)\left(\frac{\partial \rho}{\partial p}\right)$$

Dividing the above expression by the fluid density ρ gives:

$$0.006328\left(\frac{k}{u}\right)\left[\frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial r^2} + \left(\frac{\partial p}{\partial r}\right)^2\left(\frac{1}{\rho}\frac{\partial \rho}{\partial p}\right)\right]$$

$$= \phi c_{\mathrm{f}}\left(\frac{\partial p}{\partial t}\right) + \phi\frac{\partial p}{\partial t}\left(\frac{1}{\rho}\frac{\partial \rho}{\partial p}\right)$$

Recalling that the compressibility of any fluid is related to its density by:

$$c = \frac{1}{\rho}\frac{\partial \rho}{\partial p}$$

combining the above two equations gives:

$$0.006328\left(\frac{k}{\mu}\right)\left[\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}+c\left(\frac{\partial p}{\partial r}\right)^2\right]$$

$$=\phi c_{\mathrm{f}}\left(\frac{\partial p}{\partial t}\right)+\phi c\left(\frac{\partial p}{\partial t}\right)$$

The term $c(\partial p/\partial r)^2$ is considered very small and may be ignored, which leads to:

$$0.006328\left(\frac{k}{\mu}\right)\left[\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}\right]=\phi\left(c_{\mathrm{f}}+c\right)\frac{\partial p}{\partial t} \qquad [1.2.58]$$

Defining total compressibility, c_{t}, as:

$$c_{\mathrm{t}}=c+c_{\mathrm{f}} \qquad [1.2.59]$$

and combining Equation 1.2.57 with 1.2.58 and rearranging gives:

$$\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}=\frac{\phi\mu c_{\mathrm{t}}}{0.006328k}\frac{\partial p}{\partial t} \qquad [1.2.60]$$

where the time t is expressed in days.

Equation 1.2.60 is called the diffusivity equation and is considered one of the most important and widely used mathematical expressions in petroleum engineering. The equation is particularly used in the analysis of well testing data where the time t is commonly reordered in hours. The equation can be rewritten as:

$$\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}=\frac{\phi\mu c_{\mathrm{t}}}{0.0002637k}\frac{\partial p}{\partial t} \qquad [1.2.61]$$

where:

k = permeability, md
r = radial position, ft
p = pressure, psia
c_{t} = total compressibility, psi^{-1}
t = time, hours
ϕ = porosity, fraction
μ = viscosity, cp

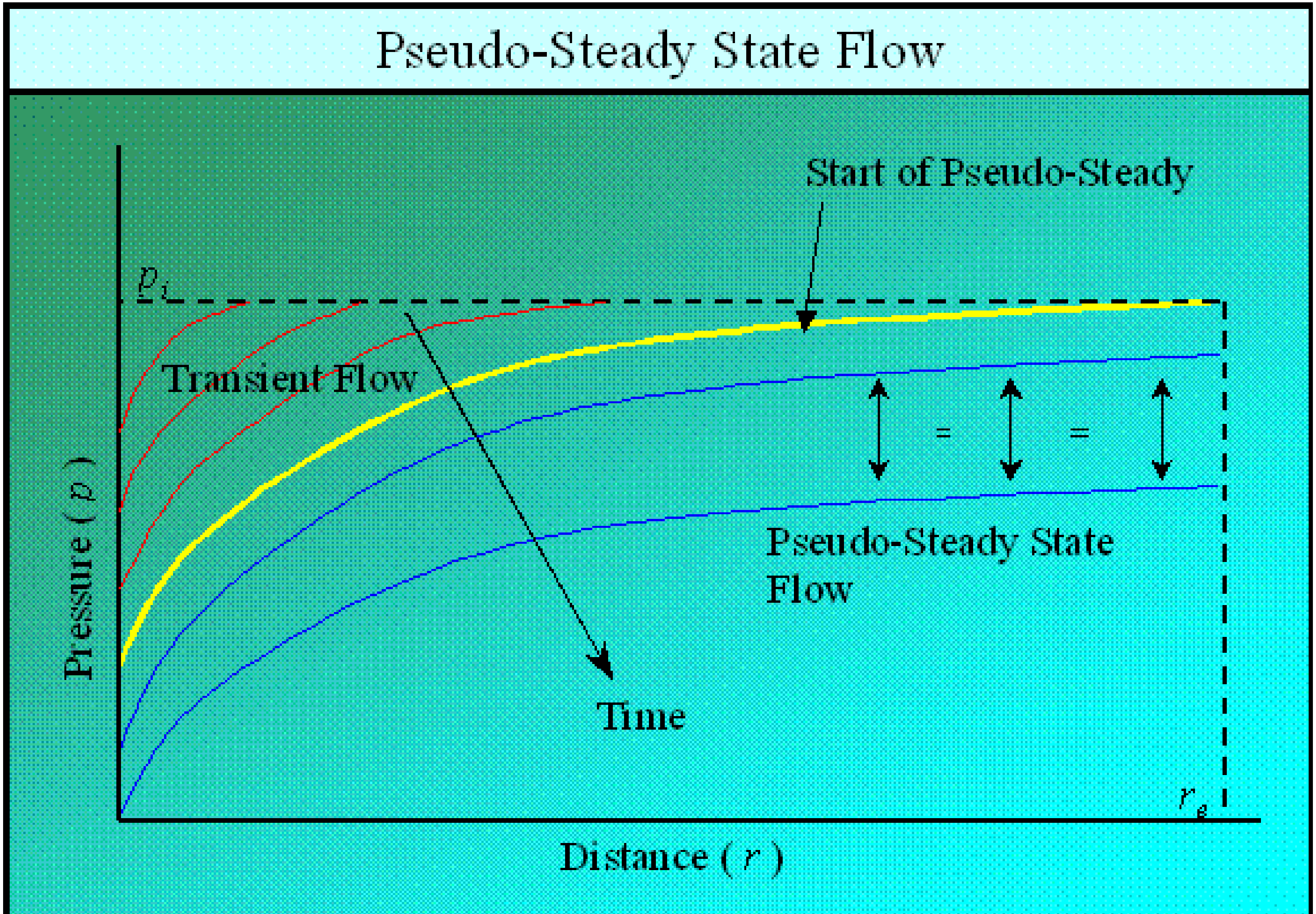
Hydraulic Diffusivity Equation

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)\right]=\frac{1}{\eta}\frac{\partial p}{\partial t}$$

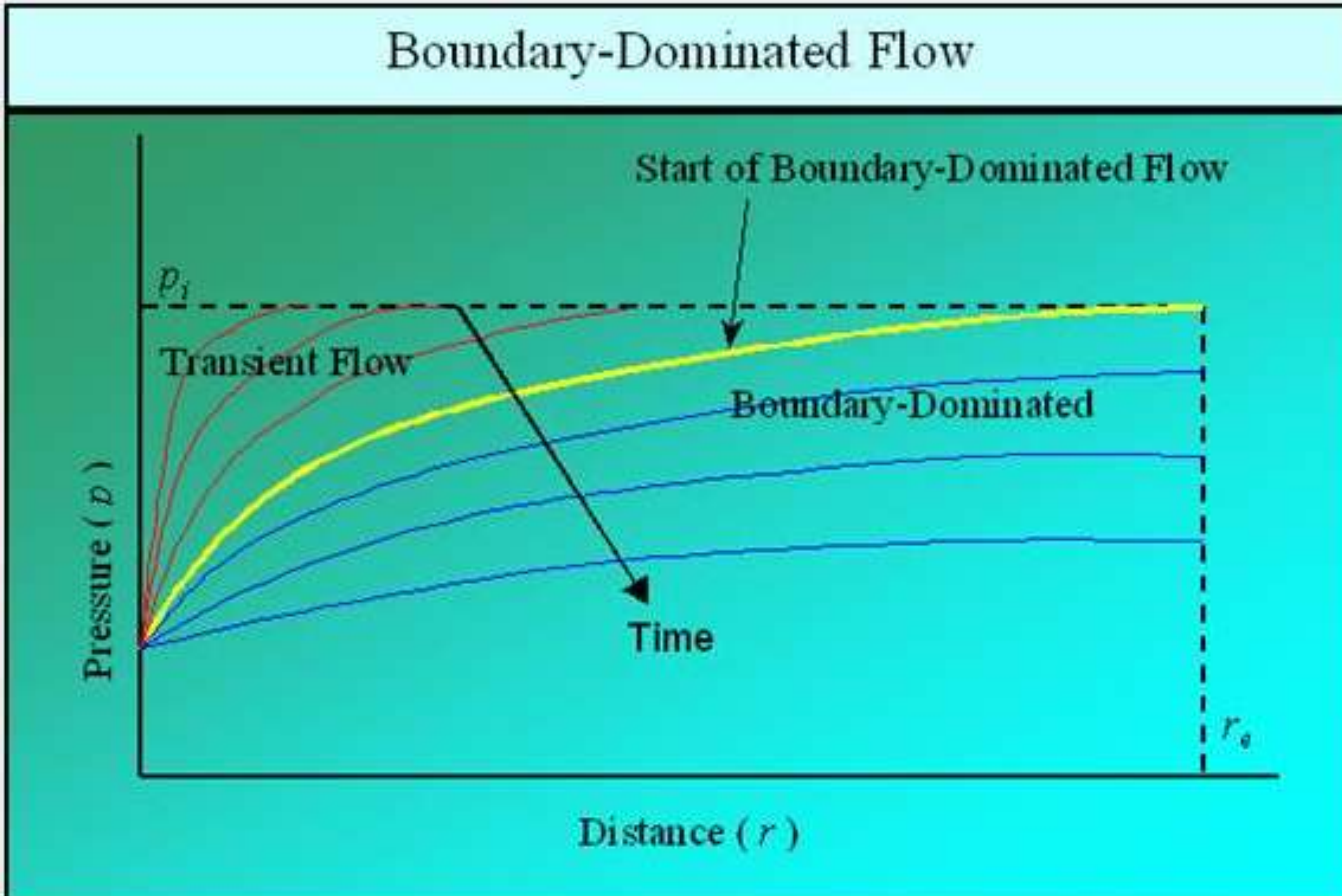
$$\eta=\frac{k}{\phi\mu\, c_t}$$

Hydraulic diffusivity equation determines the velocity at which pressure waves propagate in the reservoir. The more the permeability the faster the pressure wave will propagate.

Different Flow Regimes = Different Boundary Conditions



Different Flow Regimes = Different Boundary Conditions



Mathematical Model-Governing Equation

A reservoir model is the superposition of reservoir, inner, and outer boundary conditions

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)\right]=\frac{\varphi\mu\, c_t}{k}\frac{\partial p}{\partial t}$$

$$Initial\quad Condition: \quad p = p_i\,, \qquad t=0, \quad r\geq r_w$$

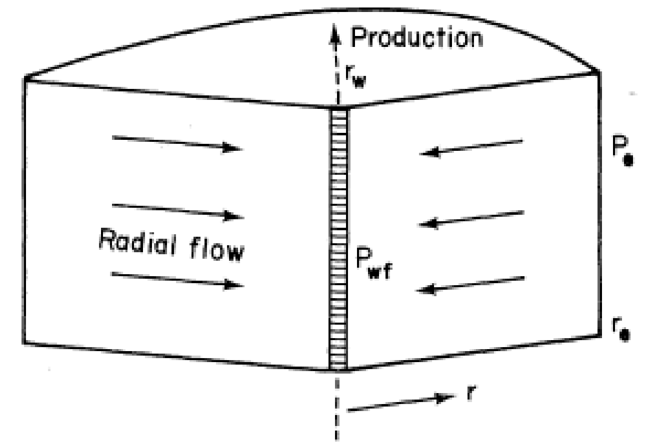


Fig. 9.1 Radial flow towards a well.

Well production	Flow regime	Inner Boundary Condition	Outer Boundary conditions
Constant rate	Finite acting (Bounded)	$\left(\frac{\partial p}{\partial r}\right)_{r_w} = -\frac{\mu q B_o}{2\pi r_w h k}$	$\left(\frac{\partial p}{\partial r}\right)_{r\to re} = 0$
Constant pressure	Finite acting (Bounded)	$(p)_{r_w} = p_{wf}$	$\left(\frac{\partial p}{\partial r}\right)_{r\to re} = 0$

Dimensionless Hydraulic Diffusivity Equation

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)\right]=\frac{\varphi\mu c_t}{k}\frac{\partial p}{\partial t}$$

$$p_D = \frac{kh\Delta p}{141.2\ qB\mu}, \tag{1-2}$$

$$t_D = \frac{0.000264\ kt}{\phi\mu c_t r_w^2}, \text{ and} \tag{1-3}$$

$$r_D = r/r_w \tag{1-4}$$

$$\frac{\partial^2 p_D}{\partial r_D^2}+\frac{1}{r_D}\frac{\partial p_D}{\partial r_D}=\frac{\partial p_D}{\partial t_D}$$

van Everdingen- Hurst Constant Terminal Rate Solution Bounded Cylindrical Reservoir (exact solution)

$$p_{wD}(t_D) = \frac{2t_D}{r_{eD}^2} + \ell n(r_{eD}) - 0.75 + 2\sum_{n=1}^{\infty} \frac{e^{-\alpha_n^2 t_D} J_1^2(\alpha_n r_{eD})}{\alpha_n^2 \left[J_1^2(\alpha_n r_{eD}) - J_1^2(\alpha_n)\right]}$$

$$\boxed{J_1(\alpha_n r_{eD}) Y_1(\alpha_n) - J_1(\alpha_n) Y_1(\alpha_n r_{eD}) = 0} \Rightarrow \boxed{\alpha_n}$$

Approximate Solutions

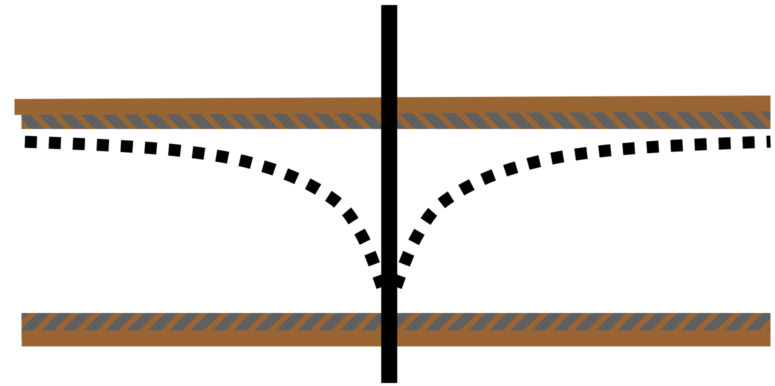
1. Infinite cylindrical reservoir with line-source well
2. Bounded cylindrical reservoir, pseudo steady-state flow

Infinite cylindrical reservoir with line-source well (approximate solution)

Dimensionless solution

$$p_D = -\frac{1}{2} E_i\left(-948 \frac{\mu\, c_t r^2}{k\, t}\right)$$

Line-source: the well has zero radius



Dimensional solution

$$p_{wf} = p_i + 70.6 \frac{qB\mu}{kh} E_i\left(-948 \frac{\mu\, c_t r_w^2}{k\, t}\right)$$

$$-E_i(-x) = \int_x^\infty \frac{e^{-u}}{u} du = \begin{cases} \approx \ell n(1.781x) & for\ x < 0.02 \quad (error \approx 0.6\%) \\ \approx 0 & for\ x > 10.9 \end{cases}$$

The mathematical function, Ei, is called the exponential integral and is defined by:

$$\mathrm{Ei}(-x) = -\int_x^\infty \frac{e^{-u} du}{u}$$

$$= \left[\ln x - \frac{x}{1!} + \frac{x^2}{2\,(2!)} - \frac{x^3}{3\,(3!)} + \cdots\right] \qquad [1.2.67]$$

Infinite cylindrical reservoir with line-source well (Range of applicability)

$$p_{wf} = p_i + 70.6\frac{qB\mu}{kh}E_i\left(-948\frac{\mu c_t r_w^2}{k t}\right)$$

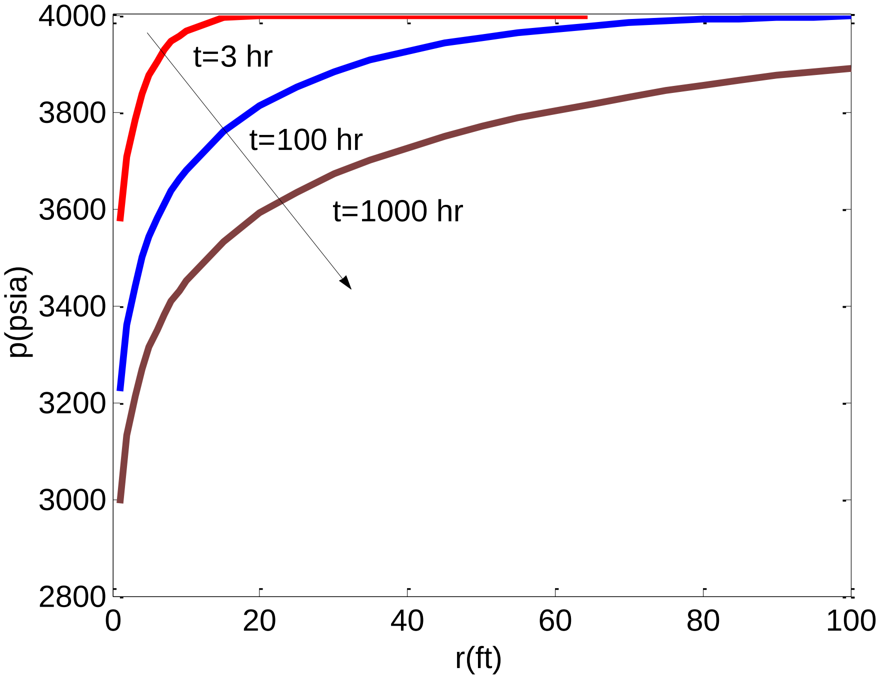
The reservoir is no longer infinite acting

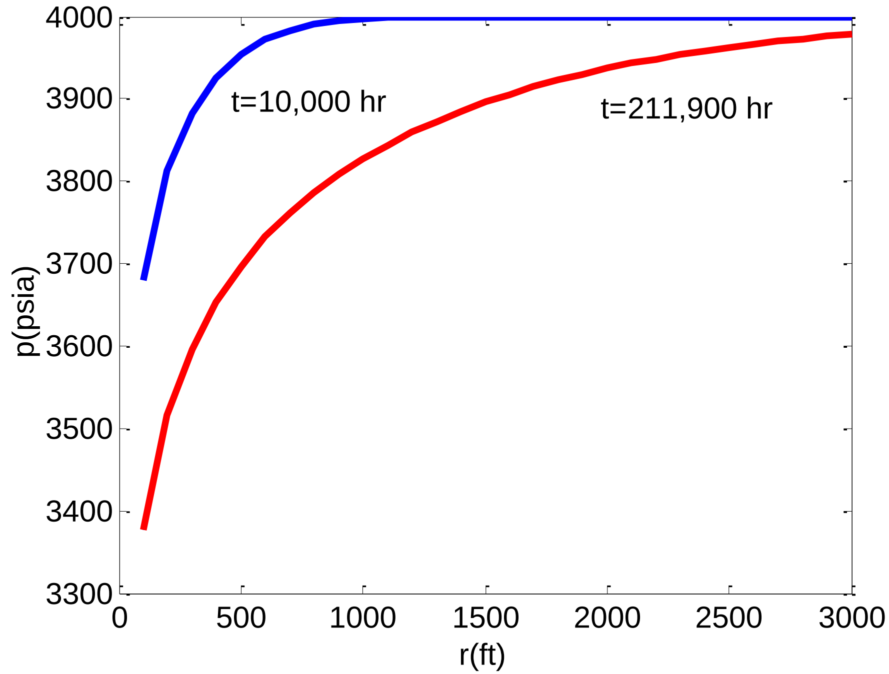
$$3.79\times10^5\left(\frac{\phi\mu c_t r_w^2}{k}\right) < t < 948\left(\frac{\phi\mu c_t r_e^2}{k}\right)$$

The assumption of zero wellbore limits the accuracy of the solution

First we must determine weather the Ei function solution is valid for the desired times.

$$3.79\times10^{5}\left(\frac{\phi\mu\, c_t r_w^2}{k}\right) < t < 948\left(\frac{\phi\mu\, c_t r_e^2}{k}\right) \Rightarrow 2.35 < t < 211{,}900\,hr$$





Dimensionless Variables

$$p = p_i + 70.6\frac{qB\mu}{kh}Ei\left(-\frac{948\phi\mu c_t r^2}{kt}\right)$$

$$\boxed{\frac{kh(p_i - p)}{141.2qB\mu}} = -\frac{1}{2}Ei\left(-\frac{\left(\boxed{\frac{r}{r_w}}\right)^2}{4\left(\boxed{\frac{0.0002637kt}{\phi\mu c_t r_w^2}}\right)}\right)$$

$$r_D \equiv \frac{r}{r_w}$$

$$p_D \equiv \frac{kh(p_i - p)}{141.2qB\mu}$$

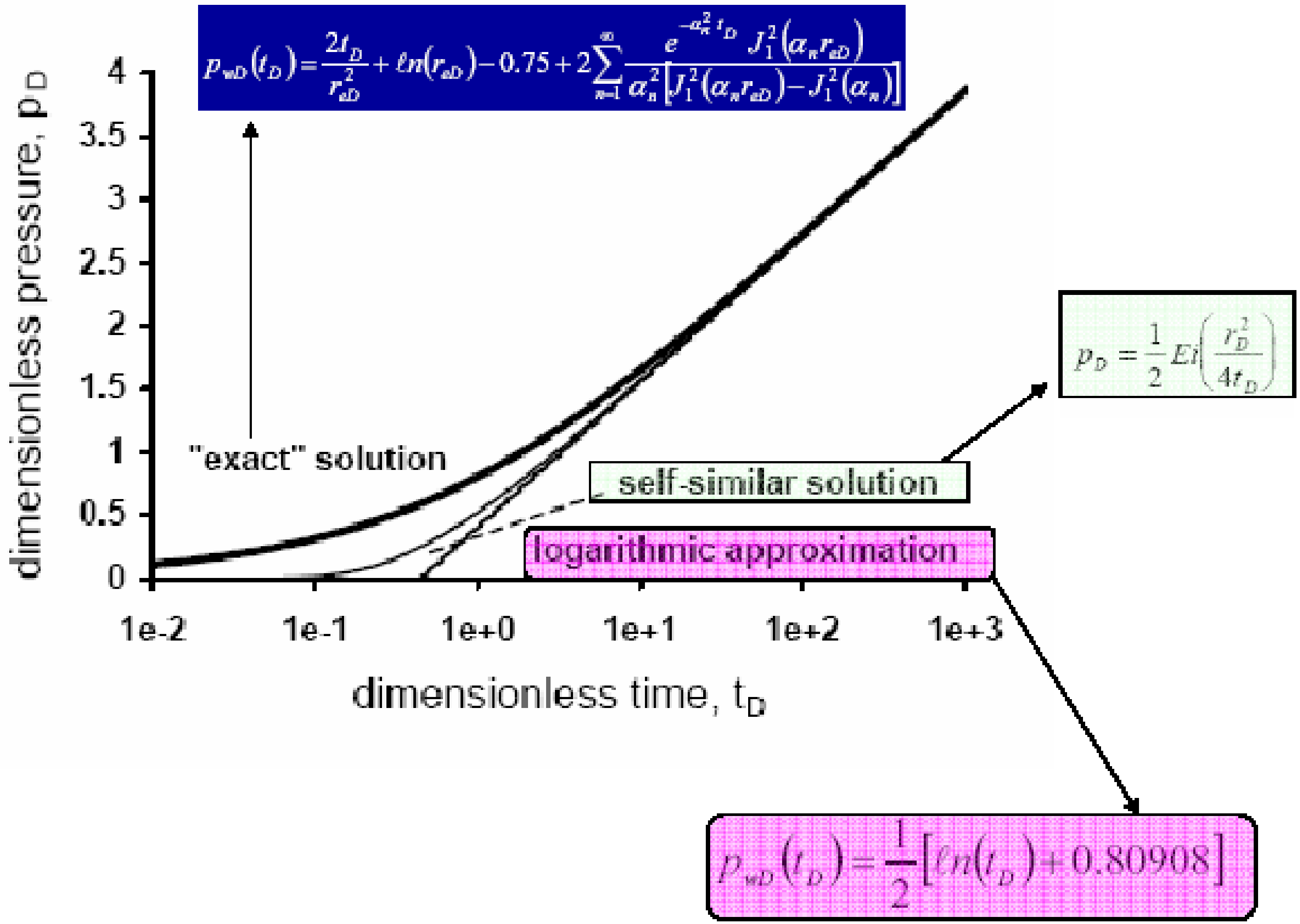
$$t_D \equiv \frac{0.0002637kt}{\phi\mu c_t r_w^2}$$

$$p_D = -\frac{1}{2}Ei\left(-\frac{r_D^2}{4t_D}\right)$$

Solutions- Time Domain

Constant rate solution	**Infinite-acting reservoir** $$p_{wD}(t_D) = \frac{1}{2}\left[\ell n(t_D) + 0.80908\right]$$
	Boundary dominated flow- approximate late time $$p_{wD}(t_D) = \frac{2t_D}{r_{eD}^2} + \ell n(r_{eD}) - 0.75$$

Dimensionless transient pressure response of a radial well in infinite reservoir



Reservoir-Limits Test
(Estimation of Reservoir Pore Volume)

$$p_{wf} = p_i - \frac{141.2 qB\mu}{kh}\left[\frac{0.0005274k}{\phi\mu c_t r_e^2} t + \ell n\left(\frac{r_e}{r_w}\right) - 0.75\right]$$

$$\frac{\partial p_{wf}}{\partial t} = \frac{0.07447 qB_o}{\phi c_t r_e^2}$$

$$V_p = \pi r_e^2 h\phi$$

$$\boxed{\frac{\partial p_{wf}}{\partial t} = \frac{0.234 qB_o}{c_t V_p}}$$

Depletion Above the Bubble-point Pressure Constant pressure solution

$$q_D\left(t_D\right)=\frac{1}{\ln\dfrac{4A}{\gamma C_A r_w^2}}\exp\left(\frac{-4\pi t_{DA}}{\mu\ln\dfrac{4A}{\gamma C_A r_w^2}}\right)$$

$$For \qquad t_{DA}>\left(t_{pss}\right)_D$$

It may be used for reservoir limit test

$$\ln(q)=-\frac{4\pi t_{DA}}{\mu\ln\left(\dfrac{4A}{\gamma C_A r_w^2}\right)}+\ln\frac{4\pi kh\left(p_i-p_{wf}\right)}{\mu B\ln\left(\dfrac{4A}{\gamma C_A r_w^2}\right)}$$

Dimensionless Diffusivity Equation

$$\frac{\partial^2 p_D}{\partial r_D^2}+\frac{1}{r_D}\frac{\partial p_D}{\partial r_D}=\frac{\partial p_D}{\partial t_D}$$

$$p_D=\frac{kh\Delta p}{141.2\ qB\mu}, \qquad (1\text{-}2)$$

$$t_D=\frac{0.000264\ kt}{\phi\mu c_t r_w^2}, \text{ and} \qquad (1\text{-}3)$$

$$r_D=r/r_w \qquad (1\text{-}4)$$

$$p_D=0 \qquad t_D=0,\quad r_D\geq 1$$

$$\left(\frac{\partial p_D}{\partial r_D}\right)_{r_D=1}=-1 \qquad t_D\succ 0,\quad r_D=1$$

$$\left(\frac{\partial p_D}{\partial r_D}\right)_{r_D=1}=0 \qquad t_D\succ 0,\quad r_D=r_{De}=r_w/r_e$$

properties of the Laplace transform

$$L[f(t)]=\int_0^\infty f(t)\exp(-st)\,dt$$

$$L\left[\frac{\partial f}{\partial t}\right]=sL[f]-f(0)$$

$$L[c]=\frac{c}{s}$$

$$L[\ln(t)]=-\frac{\gamma+\ln(s)}{s}$$

Bessel Differential Equation

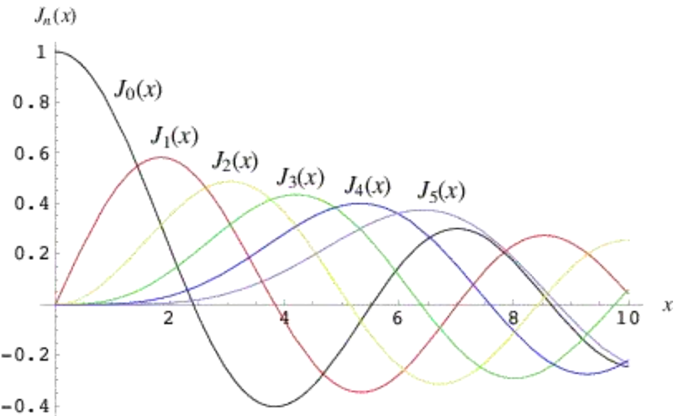
$$x^2 \frac{d^2 y}{d x^2} + x \frac{d y}{d x} + (x^2 - n^2) y = 0$$

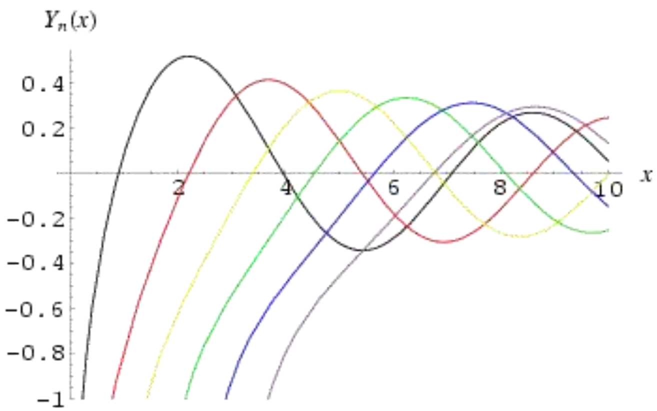
$$y(x) = c_1 J_n(x) + c_2 Y_n(x)$$

Modified Bessel Differential Equation

$$x^2 \frac{d^2 y}{d x^2} + x \frac{d y}{d x} - (x^2 + n^2) y = 0.$$

$$y(x) = c_1 I_n(x) + c_2 K_n(x)$$

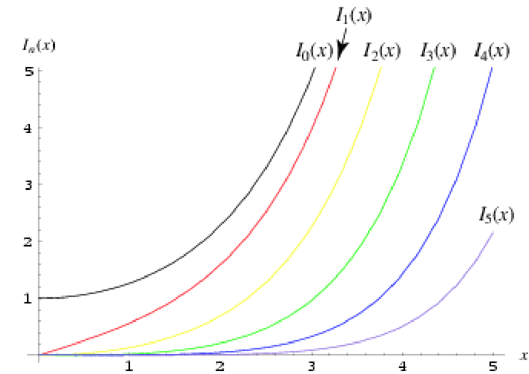


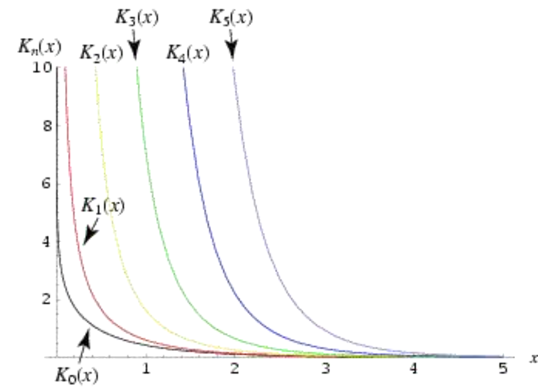


Properties of Bessel function

$$\frac{d}{dr_D} I_0\left(r_D\sqrt{S}\right) = \sqrt{S}\, I_1\left(r_D\sqrt{S}\right)$$

$$\frac{d}{dr_D} K_0\left(r_D\sqrt{S}\right) = -\sqrt{S}\, K_1\left(r_D\sqrt{S}\right)$$





Solutions- Laplace Domain (Sabet, 1991).

Constant rate solution	**Infinite-acting reservoir** $$\bar{p}_D(S)=\frac{K_0\left(r_D\sqrt{S}\right)}{S\sqrt{S}K_1\left(\sqrt{S}\right)}$$
	Bounded reservoir $$\bar{p}_D(S)=\frac{\left[K_1\left(r_{De}\sqrt{S}\right)I_0\left(r_D\sqrt{S}\right)+I_1\left(r_{De}\sqrt{S}\right)K_0\left(r_D\sqrt{S}\right)\right]}{S\sqrt{S}\left[K_1\left(\sqrt{S}\right)I_1\left(r_{De}\sqrt{S}\right)-K_1\left(r_{De}\sqrt{S}\right)I_1\left(\sqrt{S}\right)\right]}$$
Constant pressure solution	**Infinite-acting reservoir** $$\bar{q}_D(S)=\left(\frac{K_1\left(r_D\sqrt{S}\right)}{\sqrt{S}K_0\left(\sqrt{S}\right)}\right)$$
	Bounded reservoir $$q_{wD}(S)=\frac{K_1\left(r_{De}\sqrt{S}\right)I_1\left(\sqrt{S}\right)-I_1\left(r_{De}\sqrt{S}\right)K_1\left(\sqrt{S}\right)}{\sqrt{S}\left[K_1\left(r_{De}\sqrt{S}\right)I_0\left(\sqrt{S}\right)+K_0\left(\sqrt{S}\right)I_1\left(r_{De}\sqrt{S}\right)\right]}$$

Numerical Inverse Laplace Transformation

The inverse Laplace transform can be found by different ways. For example, we could prepare a table of transforms in which we list the transforms of many functions and refer to this table to find the inverse transform. We can use the table of transform in conjunction with Equation 11-17 and other known properties of the transform. Another technique relies on integration in the complex plane. However, in most problems related to well testing, this latter technique could lead to expressions that are very difficult to evaluate. For this reason, the present trend is to find the inverse transform numerically and present the results in the form of a type-curve.

The algorithm presented by Stehfest (1970) has gained wide acceptance by researchers in the field of well testing. We will discuss Stehfest's algorithm and with the exception of referring to a table of transforms, we will not discuss any of the other methods of finding the inverse Laplace transform.

Stehfest's algorithm is based on the following formulae:

$$V_i = (-1)^{n/2+i} \sum_{k=(i+1)/2}^{\min(i,n/2)} \frac{k^{n/2}\,(2k)!}{(n/2-k)!\,k!\,(k-1)!\,(i-k)!\,(2k-i)!} \tag{11-18}$$

$$f(t) = \frac{\ln 2}{t} \sum_{i=1}^{n} V_i\, P\left(\frac{\ln 2}{t}\, i\right) \tag{11-19}$$

The number, n, in these expressions should be optimized by trial and error. Increasing n increases the accuracy of the results up to a point, and then the accuracy declines because of roundoff errors, since the word length on the computer is finite. Note that $f(t) = L^{-1}P(z)$, and z is replaced by i ln2/t, where t is the time at which the inverse transform is required. Also note that for a given n the Stehfest algorithm requires calculation of V_i only once.

Numerical Inverse Laplace Transformation (Stehfest Algorithm)

Program 11-1 is written in FORTRAN. It is written to find the inverse transform of $P(z) = 1/\sqrt{z}$, at $t = 1, 2, 3, \ldots \ldots 10$. The program is suitable for finding the inverse transform of any given continuous function by making the necessary changes where indicated in the program. With $n = 18$, the program gave exact results up to 5 decimals. This was possible to check because we know that:

$$\frac{1}{\sqrt{z}} = L\left[\frac{1}{\sqrt{\pi t}}\right]$$

For a given f(z) for which we do not know the inverse transform, n can be optimized by referring to a table of transforms and choosing a function that is close to the function on hand. Also, if n is not properly selected, a plot of the inverse transform will tend to oscillate, whereas an appropriately chosen value of n will yield a smooth inverse transform.

Program 11-1
Inverse Laplace Transform by the Stehfest Algorithm

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION V(30),G(30),H(30)
C     N SHOULD BE OPTIMIZED
      N = 18
C
      DLN2 = 0.6931471805599453
      G(1) = 1.0
      NH = N/2
      DO 10 I = 2,N
 10   G(I) = G(I-1)*I
      H(1) = 2.0/G(NH-1)
      DO 100 I = 2,NH
      FI = I
      IF(I.EQ.NH) GO TO 50
      H(I) = FI**NH*G(2*I)/(G(NH-I)*G(I)*G(I-1))
      GO TO 100
 50   H(I) = FI**NH*G(2*I)/(G(I)*G(I-1))
100   CONTINUE
      SN = 2*(NH-NH/2*2)-1
      DO 200 I = 1,N
      V(I) = 0.0
      K = (I + 1)/2
      KK = I
      IF(KK.GT.NH) KK = NH
      DO 150 J = K,KK
      IF(2*J-I.EQ.0) GO TO 120
      IF(I.EQ.J) GO TO 130
```

(program continued on next page)

Superposition Principle

Linear diffusivity equation

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} = \frac{\phi \mu c_t}{k}\frac{\partial p}{\partial t}$$

A mathematical technique based on the property that solutions to linear partial equations can be added to provide yet another solution. This permits constructions of mathematical solutions to situations with complex boundary conditions, especially drawdown and buildup tests, and in settings where flow rates change with time.

Mathematically the superposition theorem states that any sum of individual solutions to the diffusivity equation is also a solution to that equation. This concept can be applied to account for the following effects on the transient flow solution:

- Superposition in time
  - Effects of rate change
- Super position in space
  - Effects of multiple wells
  - Effects of the boundary

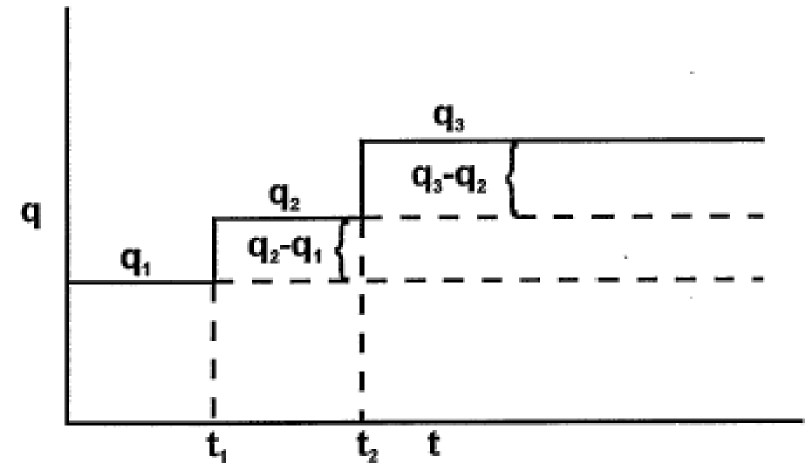
SUPERPOSITION IN TIME

Theorem: when a rate changes, at time t, from q_1 to a new rate q_2, this is equivalent to:

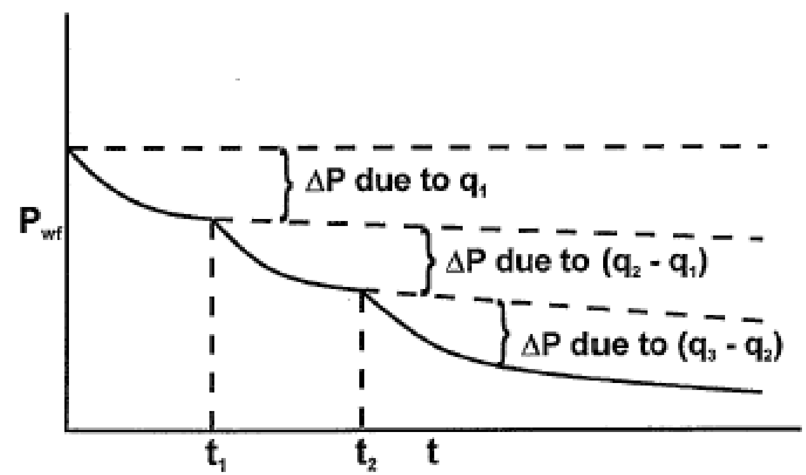
q_1 continuing forever

+

(q_2-q_1) starting at time **t**, and continuing forever



- The pressure caused by this rate change is obtained by superposing (ADDING) these two effects.
- q_2 can be greater or less than q_1
- q_2 can be zero (Buildup)
- additional rate changes are treated in the same way
- q = +ve means production
- q = -ve means injection



SUPERPOSITION IN SPACE

X
Well A

X
Well B

X
Observation Point

Theorem:

- Δp at observation point =

  Δp (caused by Well A)

  \+ Δp (caused by Well B)
- Observation point may be located anywhere, even at point A
- Δp at Well A has two components:
  (1) Well A flowing (well in an infinite reservoir)
  (2) Effect of Well B (in an infinite reservoir) evaluated at distance A-B from Well B.

METHODS of IMAGES

Single Boundary

Replace Boundary by IMAGE WELL.

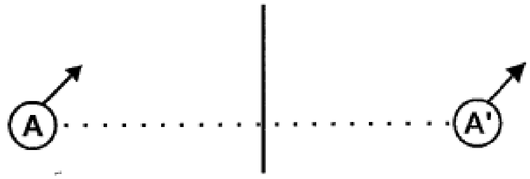
Multiple Boundaries

Replace Boundaries by IMAGE WELLS PLUS IMAGES of the image wells!

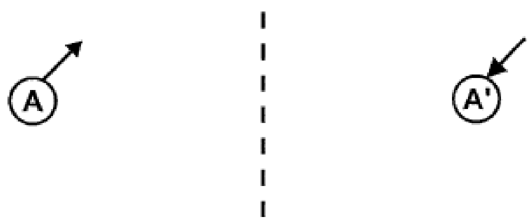
Parallel Boundaries

INFINITE number of images.

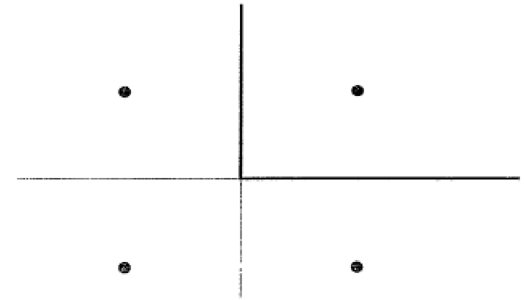
Δp at real well = Δp due to real well in infinite reservoir
\+
sum of all the image well effects at the location of the real well



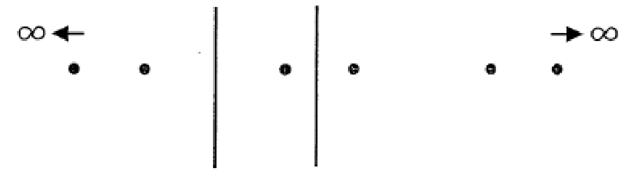
NO-FLOW BOUNDARY
(even image)



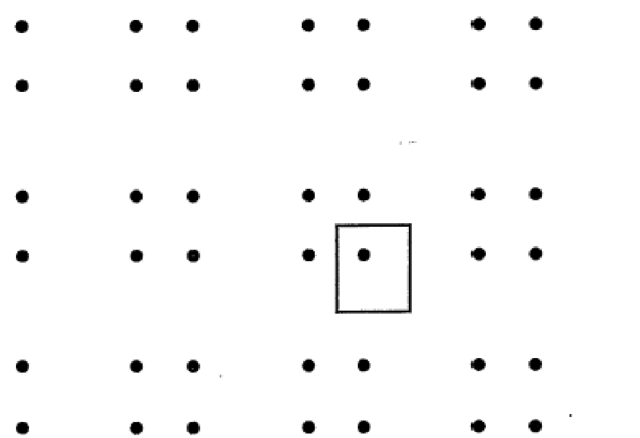
CONSTANT PRESSURE
(odd image)



Two Perpendicular No-Flow Boundaries



Two Parallel No-Flow Boundaries



SOME images of a well inside a rectangle