Partial Differential Equations: Graduate Level Problems and Solutions

Igor Yanovsky

Disclaimer: This handbook is intended to assist graduate students with qualifying examination preparation. Please be aware, however, that the handbook might contain, and almost certainly contains, typos as well as incorrect or inaccurate solutions. I can not be made responsible for any inaccuracies contained in this handbook.

Contents

1 Trigonometric Identities

$$\begin{aligned}
\cos(a+b) &= \cos a \cos b - \sin a \sin b \\
\cos(a-b) &= \cos a \cos b + \sin a \sin b \\
\sin(a+b) &= \sin a \cos b + \cos a \sin b \\
\sin(a-b) &= \sin a \cos b - \cos a \sin b
\end{aligned}$$

$$\begin{aligned}
\cos a \cos b &= \frac{\cos(a+b) + \cos(a-b)}{2} \\
\sin a \cos b &= \frac{\sin(a+b) + \sin(a-b)}{2} \\
\sin a \sin b &= \frac{\cos(a-b) - \cos(a+b)}{2}
\end{aligned}$$

$$\begin{aligned}
\cos 2t &= \cos^2 t - \sin^2 t \\
\sin 2t &= 2 \sin t \cos t
\end{aligned}$$

$$\begin{aligned}
\cos^2 \frac{1}{2} t &= \frac{1 + \cos t}{2} \\
\sin^2 \frac{1}{2} t &= \frac{1 - \cos t}{2}
\end{aligned}$$

$$\begin{aligned}
1 + \tan^2 t &= \sec^2 t \\
\cot^2 t + 1 &= \csc^2 t
\end{aligned}$$

$$\begin{aligned}
\cos x &= \frac{e^{ix} + e^{-ix}}{2} \\
\sin x &= \frac{e^{ix} - e^{-ix}}{2i}
\end{aligned}$$

$$\begin{aligned}
\cosh x &= \frac{e^{x} + e^{-x}}{2} \\
\sinh x &= \frac{e^{x} - e^{-x}}{2}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx} \cosh x &= \sinh(x) \\
\frac{d}{dx} \sinh x &= \cosh(x)
\end{aligned}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\begin{aligned}
\int \frac{du}{a^2 + u^2} &= \frac{1}{a} \tan^{-1} \frac{u}{a} + C \\
\int \frac{du}{\sqrt{a^2 - u^2}} &= \sin^{-1} \frac{u}{a} + C
\end{aligned}$$

$$\begin{aligned}
\int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\, dx &= \begin{cases} 0 & n \neq m \\ L & n = m \end{cases} \\
\int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx &= \begin{cases} 0 & n \neq m \\ L & n = m \end{cases} \\
\int_{-L}^{L} \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\, dx &= 0
\end{aligned}$$

$$\begin{aligned}
\int_{0}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\, dx &= \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \end{cases} \\
\int_{0}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx &= \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \end{cases}
\end{aligned}$$

$$\begin{aligned}
\int_{0}^{L} e^{inx} \overline{e^{imx}}\, dx &= \begin{cases} 0 & n \neq m \\ L & n = m \end{cases} \\
\int_{0}^{L} e^{inx}\, dx &= \begin{cases} 0 & n \neq 0 \\ L & n = 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
\int \sin^2 x\, dx &= \frac{x}{2} - \frac{\sin x \cos x}{2} \\
\int \cos^2 x\, dx &= \frac{x}{2} + \frac{\sin x \cos x}{2} \\
\int \tan^2 x\, dx &= \tan x - x \\
\int \sin x \cos x\, dx &= -\frac{\cos^2 x}{2}
\end{aligned}$$

$$\begin{aligned}
\ln(xy) &= \ln(x) + \ln(y) \\
\ln \frac{x}{y} &= \ln(x) - \ln(y) \\
\ln x^r &= r \ln x
\end{aligned}$$

$$\begin{aligned}
\int \ln x\, dx &= x \ln x - x \\
\int x \ln x\, dx &= \frac{x^2}{2} \ln x - \frac{x^2}{4}
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{R}} e^{-z^2}\, dz &= \sqrt{\pi} \\
\int_{\mathbb{R}} e^{-\frac{z^2}{2}}\, dz &= \sqrt{2\pi}
\end{aligned}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

2 Simple Eigenvalue Problem

$$X'' + \lambda X = 0$$

Boundary conditions	Eigenvalues λ_n	Eigenfunctions X_n	
$X(0) = X(L) = 0$	$\left(\frac{n\pi}{L}\right)^2$	$\sin \frac{n\pi}{L}x$	$n = 1, 2, \ldots$
$X(0) = X'(L) = 0$	$\left[\frac{(n-\frac{1}{2})\pi}{L}\right]^2$	$\sin \frac{(n-\frac{1}{2})\pi}{L}x$	$n = 1, 2, \ldots$
$X'(0) = X(L) = 0$	$\left[\frac{(n-\frac{1}{2})\pi}{L}\right]^2$	$\cos \frac{(n-\frac{1}{2})\pi}{L}x$	$n = 1, 2, \ldots$
$X'(0) = X'(L) = 0$	$\left(\frac{n\pi}{L}\right)^2$	$\cos \frac{n\pi}{L}x$	$n = 0, 1, 2, \ldots$
$X(0) = X(L),\ X'(0) = X'(L)$	$\left(\frac{2n\pi}{L}\right)^2$	$\sin \frac{2n\pi}{L}x$	$n = 1, 2, \ldots$
		$\cos \frac{2n\pi}{L}x$	$n = 0, 1, 2, \ldots$
$X(-L) = X(L),\ X'(-L) = X'(L)$	$\left(\frac{n\pi}{L}\right)^2$	$\sin \frac{n\pi}{L}x$	$n = 1, 2, \ldots$
		$\cos \frac{n\pi}{L}x$	$n = 0, 1, 2, \ldots$

$$X'''' - \lambda X = 0$$

Boundary conditions	Eigenvalues λ_n	Eigenfunctions X_n	
$X(0) = X(L) = 0,\ X''(0) = X''(L) = 0$	$\left(\frac{n\pi}{L}\right)^4$	$\sin \frac{n\pi}{L}x$	$n = 1, 2, \ldots$
$X'(0) = X'(L) = 0,\ X'''(0) = X'''(L) = 0$	$\left(\frac{n\pi}{L}\right)^4$	$\cos \frac{n\pi}{L}x$	$n = 0, 1, 2, \ldots$

3 Separation of Variables: Quick Guide

Laplace Equation: $\triangle u = 0$.

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda.$$
$$X'' + \lambda X = 0.$$

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$
$$Y''(\theta) + \lambda Y(\theta) = 0.$$

Wave Equation: $u_{tt} - u_{xx} = 0$.

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = -\lambda.$$
$$X'' + \lambda X = 0.$$

$u_{tt} + 3u_t + u = u_{xx}$.

$$\frac{T''}{T} + 3\frac{T'}{T} + 1 = \frac{X''}{X} = -\lambda.$$
$$X'' + \lambda X = 0.$$

$u_{tt} - u_{xx} + u = 0$.

$$\frac{T''}{T} + 1 = \frac{X''}{X} = -\lambda.$$
$$X'' + \lambda X = 0.$$

$u_{tt} + \mu u_t = c^2 u_{xx} + \beta u_{xxt}, \qquad (\beta > 0)$

$$\frac{X''}{X} = -\lambda,$$
$$\frac{1}{c^2}\frac{T''}{T} + \frac{\mu}{c^2}\frac{T'}{T} = \Big(1 + \frac{\beta}{c^2}\frac{T'}{T}\Big)\frac{X''}{X}.$$

4th Order: $u_{tt} = -k\, u_{xxxx}$.

$$-\frac{X''''}{X} = \frac{1}{k}\frac{T''}{T} = -\lambda.$$
$$X'''' - \lambda X = 0.$$

Heat Equation: $u_t = k u_{xx}$.

$$\frac{T'}{T} = k\frac{X''}{X} = -\lambda.$$
$$X'' + \frac{\lambda}{k} X = 0.$$

4th Order: $u_t = -u_{xxxx}$.

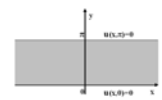
$$\frac{T'}{T} = -\frac{X''''}{X} = -\lambda.$$
$$X'''' - \lambda X = 0.$$

4 Eigenvalues of the Laplacian: Quick Guide

Laplace Equation: $u_{xx} + u_{yy} + \lambda u = 0$.

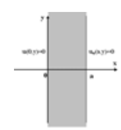
$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0. \quad (\lambda = \mu^2 + \nu^2)$$
$$X'' + \mu^2 X = 0, \qquad Y'' + \nu^2 Y = 0.$$

$$u_{xx} + u_{yy} + k^2 u = 0.$$



$$-\frac{X''}{X} = \frac{Y''}{Y} + k^2 = c^2.$$
$$X'' + c^2 X = 0,$$
$$Y'' + (k^2 - c^2)Y = 0.$$

$$u_{xx} + u_{yy} + k^2 u = 0.$$



$$-\frac{Y''}{Y} = \frac{X''}{X} + k^2 = c^2.$$
$$Y'' + c^2 Y = 0,$$
$$X'' + (k^2 - c^2)X = 0.$$

5 First-Order Equations

5.1 Quasilinear Equations

Consider the Cauchy problem for the quasilinear equation in two variables

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u),$$

with Γ parameterized by $(f(s), g(s), h(s))$. The characteristic equations are

$$\frac{dx}{dt} = a(x, y, z), \qquad \frac{dy}{dt} = b(x, y, z), \qquad \frac{dz}{dt} = c(x, y, z),$$

with initial conditions

$$x(s, 0) = f(s), \qquad y(s, 0) = g(s), \qquad z(s, 0) = h(s).$$

In a quasilinear case, the characteristic equations for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ need not decouple from the $\frac{dz}{dt}$ equation; this means that we must take the z values into account even to find the projected characteristic curves in the xy-plane. In particular, this allows for the possibility that the projected characteristics may cross each other.

The condition for solving for s and t in terms of x and y requires that the Jacobian matrix be nonsingular:

$$J \equiv \begin{pmatrix} x_s & y_s \\ x_t & y_t \end{pmatrix} = x_s y_t - y_s x_t \neq 0.$$

In particular, at $t = 0$ we obtain the condition

$$f'(s) \cdot b(f(s), g(s), h(s)) - g'(s) \cdot a(f(s), g(s), h(s)) \neq 0.$$

Burger's Equation. *Solve the Cauchy problem*

$$\begin{cases} u_t + uu_x = 0, \\ u(x, 0) = h(x). \end{cases} \tag{5.1}$$

The characteristic equations are

$$\frac{dx}{dt} = z, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = 0,$$

and Γ may be parametrized by $(s, 0, h(s))$.

$$x = h(s)t + s, \quad y = t, \quad z = h(s).$$

$$u(x, y) = h(x - uy) \tag{5.2}$$

The characteristic projection in the xt-plane[1] passing through the point $(s, 0)$ is the line

$$x = h(s)t + s$$

along which u has the constant value $u = h(s)$. Two characteristics $x = h(s_1)t + s_1$ and $x = h(s_2)t + s_2$ intersect at a point (x, t) with

$$t = -\frac{s_2 - s_1}{h(s_2) - h(s_1)}.$$

[1] y and t are interchanged here

From (5.2), we have

$$u_x = h'(s)(1 - u_x t) \quad \Rightarrow \quad u_x = \frac{h'(s)}{1 + h'(s)t}$$

Hence for $h'(s) < 0$, u_x becomes infinite at the positive time

$$t = \frac{-1}{h'(s)}.$$

The smallest t for which this happens corresponds to the value $s = s_0$ at which $h'(s)$ has a minimum (i.e.$-h'(s)$ has a maximum). At time $T = -1/h'(s_0)$ the solution u experiences a "gradient catastrophe".

5.2 Weak Solutions for Quasilinear Equations

5.2.1 Conservation Laws and Jump Conditions

Consider shocks for an equation

$$u_t + f(u)_x = 0, \tag{5.3}$$

where f is a smooth function of u. If we integrate (5.3) with respect to x for $a \le x \le b$, we obtain

$$\frac{d}{dt}\int_a^b u(x,t)\,dx \;+\; f(u(b,t)) - f(u(a,t)) = 0. \tag{5.4}$$

This is an example of a *conservation law*. Notice that (5.4) implies (5.3) if u is C^1, but (5.4) makes sense for more general u.

Consider a solution of (5.4) that, for fixed t, has a jump discontinuity at $x = \xi(t)$. We assume that u, u_x, and u_t are continuous up to ξ. Also, we assume that $\xi(t)$ is C^1 in t.

Taking $a < \xi(t) < b$ in (5.4), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_a^\xi u\,dx + \int_\xi^b u\,dx\Big) \;+\; f(u(b,t)) - f(u(a,t)) \\
&= \xi'(t)u_l(\xi(t),t) - \xi'(t)u_r(\xi(t),t) + \int_a^\xi u_t(x,t)\,dx + \int_\xi^b u_t(x,t)\,dx \\
&\qquad + f(u(b,t)) - f(u(a,t)) = 0,
\end{aligned}$$

where u_l and u_r denote the limiting values of u from the left and right sides of the shock. Letting $a \uparrow \xi(t)$ and $b \downarrow \xi(t)$, we get the **Rankine-Hugoniot jump condition**:

$$\xi'(t)(u_l - u_r) + f(u_r) - f(u_l) = 0,$$

$$\boxed{\xi'(t) = \frac{f(u_r) - f(u_l)}{u_r - u_l}.}$$

5.2.2 Fans and Rarefaction Waves

For Burgers' equation

$$u_t + \Big(\frac{1}{2}u^2\Big)_x = 0,$$

$$\text{we have } f'(u) = u, \quad f'\Big(\tilde{u}\Big(\frac{x}{t}\Big)\Big) = \frac{x}{t} \quad \Rightarrow \quad \tilde{u}\Big(\frac{x}{t}\Big) = \frac{x}{t}.$$

For a rarefaction fan emanating from $(s,0)$ on xt-plane, we have:

$$u(x,t) = \begin{cases} u_l, & \frac{x-s}{t} \le f'(u_l) = u_l, \\ \frac{x-s}{t}, & u_l \le \frac{x-s}{t} \le u_r, \\ u_r, & \frac{x-s}{t} \ge f'(u_r) = u_r. \end{cases}$$

5.3 General Nonlinear Equations

5.3.1 Two Spatial Dimensions

Write a general nonlinear equation $F(x, y, u, u_x, u_y) = 0$ as

$$F(x, y, z, p, q) = 0.$$

Γ is parameterized by

$$\Gamma : \big(\underbrace{f(s)}_{x(s,0)}, \underbrace{g(s)}_{y(s,0)}, \underbrace{h(s)}_{z(s,0)}, \underbrace{\phi(s)}_{p(s,0)}, \underbrace{\psi(s)}_{q(s,0)}\big)$$

We need to complete Γ to a strip. Find $\phi(s)$ and $\psi(s)$, the initial conditions for $p(s,t)$ and $q(s,t)$, respectively:

- $F(f(s), g(s), h(s), \phi(s), \psi(s)) \ = \ 0$
- $h'(s) \ = \ \phi(s)f'(s) + \psi(s)g'(s)$

The characteristic equations are

$$\frac{dx}{dt} = F_p \qquad \frac{dy}{dt} = F_q$$
$$\frac{dz}{dt} = pF_p + qF_q$$
$$\frac{dp}{dt} = -F_x - F_z p \qquad \frac{dq}{dt} = -F_y - F_z q$$

We need to have the Jacobian condition. That is, in order to solve the Cauchy problem in a neighborhood of Γ, the following condition must be satisfied:

$$f'(s) \cdot F_q[f, g, h, \phi, \psi](s) - g'(s) \cdot F_p[f, g, h, \phi, \psi](s) \neq 0.$$

5.3.2 Three Spatial Dimensions

Write a general nonlinear equation $F(x_1, x_2, x_3, u, u_{x_1}, u_{x_2}, u_{x_3}) = 0$ as

$$F(x_1, x_2, x_3, z, p_1, p_2, p_3) = 0.$$

Γ is parameterized by

$$\Gamma : \big(\underbrace{f_1(s_1, s_2)}_{x_1(s_1,s_2,0)}, \underbrace{f_2(s_1, s_2)}_{x_2(s_1,s_2,0)}, \underbrace{f_3(s_1, s_2)}_{x_3(s_1,s_2,0)}, \underbrace{h(s_1, s_2)}_{z(s_1,s_2,0)}, \underbrace{\phi_1(s_1, s_2)}_{p_1(s_1,s_2,0)}, \underbrace{\phi_2(s_1, s_2)}_{p_2(s_1,s_2,0)}, \underbrace{\phi_3(s_1, s_2)}_{p_3(s_1,s_2,0)}\big)$$

We need to complete Γ to a strip. Find $\phi_1(s_1, s_2)$, $\phi_2(s_1, s_2)$, and $\phi_3(s_1, s_2)$, the initial conditions for $p_1(s_1, s_2, t)$, $p_2(s_1, s_2, t)$, and $p_3(s_1, s_2, t)$, respectively:

- $F\big(f_1(s_1, s_2), f_2(s_1, s_2), f_3(s_1, s_2), h(s_1, s_2), \phi_1, \phi_2, \phi_3\big) \ = \ 0$
- $\dfrac{\partial h}{\partial s_1} \ = \ \phi_1 \dfrac{\partial f_1}{\partial s_1} + \phi_2 \dfrac{\partial f_2}{\partial s_1} + \phi_3 \dfrac{\partial f_3}{\partial s_1}$
- $\dfrac{\partial h}{\partial s_2} \ = \ \phi_1 \dfrac{\partial f_1}{\partial s_2} + \phi_2 \dfrac{\partial f_2}{\partial s_2} + \phi_3 \dfrac{\partial f_3}{\partial s_2}$

The characteristic equations are

$$\frac{dx_1}{dt} = F_{p_1} \qquad \frac{dx_2}{dt} = F_{p_2} \qquad \frac{dx_3}{dt} = F_{p_3}$$
$$\frac{dz}{dt} = p_1 F_{p_1} + p_2 F_{p_2} + p_3 F_{p_3}$$
$$\frac{dp_1}{dt} = -F_{x_1} - p_1 F_z \qquad \frac{dp_2}{dt} = -F_{x_2} - p_2 F_z \qquad \frac{dp_3}{dt} = -F_{x_3} - p_3 F_z$$

6 Second-Order Equations

6.1 Classification by Characteristics

Consider the second-order equation in which the derivatives of second-order all occur linearly, with coefficients only depending on the independent variables:

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} = d(x,y,u,u_x,u_y). \tag{6.1}$$

The *characteristic* equation is

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- $b^2 - 4ac > 0 \;\Rightarrow\;$ *two* characteristics, and (6.1) is called *hyperbolic*;
- $b^2 - 4ac = 0 \;\Rightarrow\;$ *one* characteristic, and (6.1) is called *parabolic*;
- $b^2 - 4ac < 0 \;\Rightarrow\;$ *no* characteristics, and (6.1) is called *elliptic*.

These definitions are all taken at a point $x_0 \in \mathbb{R}^2$; unless a, b, and c are all constant, the *type* may change with the point x_0.

6.2 Canonical Forms and General Solutions

① $u_{xx} - u_{yy} = 0$ is hyperbolic (one-dimensional wave equation).
② $u_{xx} - u_y = 0$ is parabolic (one-dimensional heat equation).
③ $u_{xx} + u_{yy} = 0$ is elliptic (two-dimensional Laplace equation).

By the introduction of new coordinates μ and η in place of x and y, the equation (6.1) may be transformed so that its principal part takes the form ①, ②, or ③.
If (6.1) is *hyperbolic*, *parabolic*, or *elliptic*, there exists a change of variables $\mu(x,y)$ and $\eta(x,y)$ under which (6.1) becomes, respectively,

$$u_{\mu\eta} = \tilde{d}(\mu,\eta,u,u_\mu,u_\eta) \qquad \Leftrightarrow \qquad u_{\bar{x}\bar{x}} - u_{\bar{y}\bar{y}} = \bar{d}(\bar{x},\bar{y},u,u_{\bar{x}},u_{\bar{y}}),$$
$$u_{\mu\mu} = \tilde{d}(\mu,\eta,u,u_\mu,u_\eta),$$
$$u_{\mu\mu} + u_{\eta\eta} = \tilde{d}(\mu,\eta,u,u_\mu,u_\eta).$$

Example 1. *Reduce to canonical form and find the general solution:*

$$u_{xx} + 5u_{xy} + 6u_{yy} = 0. \tag{6.2}$$

Proof. $a = 1,\ b = 5,\ c = 6 \quad \Rightarrow \quad b^2 - 4ac = 1 > 0 \quad \Rightarrow$ **hyperbolic** $\Rightarrow$ two characteristics.
The characteristics are found by solving

$$\frac{dy}{dx} = \frac{5 \pm 1}{2} = \begin{cases} 3 \\ 2 \end{cases}$$

to find $y = 3x + c_1$ and $y = 2x + c_2$.

Let $\mu(x,y) = 3x - y, \quad \eta(x,y) = 2x - y.$

$$\mu_x = 3, \qquad \eta_x = 2,$$
$$\mu_y = -1, \qquad \eta_y = -1.$$

$$\begin{aligned}
u &= u(\mu(x,y), \eta(x,y));\\
u_x &= u_\mu \mu_x + u_\eta \eta_x = 3u_\mu + 2u_\eta,\\
u_y &= u_\mu \mu_y + u_\eta \eta_y = -u_\mu - u_\eta,\\
u_{xx} &= (3u_\mu + 2u_\eta)_x = 3(u_{\mu\mu}\mu_x + u_{\mu\eta}\eta_x) + 2(u_{\eta\mu}\mu_x + u_{\eta\eta}\eta_x) = 9u_{\mu\mu} + 12u_{\mu\eta} + 4u_{\eta\eta},\\
u_{xy} &= (3u_\mu + 2u_\eta)_y = 3(u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y) + 2(u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y) = -3u_{\mu\mu} - 5u_{\mu\eta} - 2u_{\eta\eta},\\
u_{yy} &= -(u_\mu + u_\eta)_y = -(u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y + u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y) = u_{\mu\mu} + 2u_{\mu\eta} + u_{\eta\eta}.
\end{aligned}$$

Inserting these expressions into (6.2) and simplifying, we obtain

$$\begin{aligned}
u_{\mu\eta} &= 0, \qquad \text{which is the } \textbf{Canonical form},\\
u_\mu &= f(\mu),\\
u &= F(\mu) + G(\eta),\\
u(x,y) &= F(3x - y) + G(2x - y), \qquad \textbf{General solution}.
\end{aligned}$$

□

Example 2. *Reduce to canonical form and find the general solution:*

$$y^2 u_{xx} - 2y u_{xy} + u_{yy} = u_x + 6y. \tag{6.3}$$

Proof. $a = y^2$, $b = -2y$, $c = 1 \;\Rightarrow\; b^2 - 4ac = 0 \;\Rightarrow\;$ **parabolic** $\Rightarrow$ one characteristic. The characteristics are found by solving

$$\frac{dy}{dx} = \frac{-2y}{2y^2} = -\frac{1}{y}$$

to find $-\dfrac{y^2}{2} + c = x.$

Let $\mu = \frac{y^2}{2} + x$. We must choose a second constant function $\eta(x,y)$ so that η is not parallel to μ. Choose $\eta(x,y) = y$.

$$\mu_x = 1, \qquad \eta_x = 0,$$
$$\mu_y = y, \qquad \eta_y = 1.$$

$$\begin{aligned}
u &= u(\mu(x,y), \eta(x,y));\\
u_x &= u_\mu \mu_x + u_\eta \eta_x = u_\mu,\\
u_y &= u_\mu \mu_y + u_\eta \eta_y = y u_\mu + u_\eta,\\
u_{xx} &= (u_\mu)_x = u_{\mu\mu}\mu_x + u_{\mu\eta}\eta_x = u_{\mu\mu},\\
u_{xy} &= (u_\mu)_y = u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y = y u_{\mu\mu} + u_{\mu\eta},\\
u_{yy} &= (y u_\mu + u_\eta)_y = u_\mu + y(u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y) + (u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y)\\
&= u_\mu + y^2 u_{\mu\mu} + 2y u_{\mu\eta} + u_{\eta\eta}.
\end{aligned}$$

Inserting these expressions into (6.3) and simplifying, we obtain

$$
\begin{aligned}
u_{\eta\eta} &= 6y, \\
u_{\eta\eta} &= 6\eta, \qquad \text{which is the } \textbf{Canonical form}, \\
u_{\eta} &= 3\eta^2 + f(\mu), \\
u &= \eta^3 + \eta f(\mu) + g(\mu), \\
u(x,y) &= y^3 + y \cdot f\Big(\frac{y^2}{2} + x\Big) + g\Big(\frac{y^2}{2} + x\Big), \qquad \textbf{General solution}.
\end{aligned}
$$

□

Problem (F'03, #4). *Find the characteristics of the partial differential equation*

$$xu_{xx} + (x-y)u_{xy} - yu_{yy} = 0, \qquad x > 0,\ y > 0, \tag{6.4}$$

and then show that it can be transformed into the canonical form

$$(\xi^2 + 4\eta)u_{\xi\eta} + \xi u_\eta = 0$$

whence ξ and η are suitably chosen canonical coordinates. Use this to obtain the general solution in the form

$$u(\xi,\eta) = f(\xi) + \int^{\eta} \frac{g(\eta')\, d\eta'}{(\xi^2 + 4\eta')^{\frac{1}{2}}}$$

where f and g are arbitrary functions of ξ and η.

Proof. $a = x, \quad b = x - y, \quad c = -y \quad \Rightarrow \quad b^2 - 4ac = (x-y)^2 + 4xy > 0$ for $x > 0$, $y > 0 \quad \Rightarrow$ **hyperbolic** $\Rightarrow$ two characteristics.

① The **characteristics** are found by solving

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{x - y \pm \sqrt{(x-y)^2 + 4xy}}{2x} = \frac{x - y \pm (x+y)}{2x} = \begin{cases} \frac{2x}{2x} = 1 \\ -\frac{2y}{2x} = -\frac{y}{x} \end{cases}$$

$$\Rightarrow \qquad y = x + c_1, \qquad \frac{dy}{y} = -\frac{dx}{x},$$

$$ln\, y = ln\, x^{-1} + \tilde{c_2},$$

$$y = \frac{c_2}{x}.$$

② Let $\mu = x - y$ and $\eta = xy$

$$\mu_x = 1, \qquad \eta_x = y,$$
$$\mu_y = -1, \qquad \eta_y = x.$$

$$\begin{aligned}
u &= u(\mu(x,y), \eta(x,y)); \\
u_x &= u_\mu \mu_x + u_\eta \eta_x = u_\mu + y u_\eta, \\
u_y &= u_\mu \mu_y + u_\eta \eta_y = -u_\mu + x u_\eta, \\
u_{xx} &= (u_\mu + y u_\eta)_x = u_{\mu\mu}\mu_x + u_{\mu\eta}\eta_x + y(u_{\eta\mu}\mu_x + u_{\eta\eta}\eta_x) = u_{\mu\mu} + 2y u_{\mu\eta} + y^2 u_{\eta\eta}, \\
u_{xy} &= (u_\mu + y u_\eta)_y = u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y + u_\eta + y(u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y) = -u_{\mu\mu} + x u_{\mu\eta} + u_\eta - y u_{\eta\mu} + xy u_{\eta\eta}, \\
u_{yy} &= (-u_\mu + x u_\eta)_y = -u_{\mu\mu}\mu_y - u_{\mu\eta}\eta_y + x(u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y) = u_{\mu\mu} - 2x u_{\mu\eta} + x^2 u_{\eta\eta},
\end{aligned}$$

Inserting these expressions into (6.4), we obtain

$$x(u_{\mu\mu} + 2y u_{\mu\eta} + y^2 u_{\eta\eta}) + (x-y)(-u_{\mu\mu} + x u_{\mu\eta} + u_\eta - y u_{\eta\mu} + xy u_{\eta\eta}) - y(u_{\mu\mu} - 2x u_{\mu\eta} + x^2 u_{\eta\eta}) = 0,$$
$$(x^2 + 2xy + y^2)u_{\mu\eta} + (x-y)u_\eta = 0,$$
$$\big((x-y)^2 + 4xy\big)u_{\mu\eta} + (x-y)u_\eta = 0,$$
$$(\mu^2 + 4\eta)u_{\mu\eta} + \mu u_\eta = 0, \qquad \text{which is the } \textbf{Canonical form}.$$

③ We need to integrate twice to get the general solution:

$$(\mu^2 + 4\eta)(u_\eta)_\mu + \mu u_\eta = 0,$$

$$\int \frac{(u_\eta)_\mu}{u_\eta}\, d\mu = -\int \frac{\mu}{\mu^2 + 4\eta}\, d\mu,$$

$$ln\ u_\eta = -\frac{1}{2} ln\ (\mu^2 + 4\eta) + \tilde{g}(\eta),$$

$$ln\ u_\eta = ln\ (\mu^2 + 4\eta)^{-\frac{1}{2}} + \tilde{g}(\eta),$$

$$u_\eta = \frac{g(\eta)}{(\mu^2 + 4\eta)^{\frac{1}{2}}},$$

$$u(\mu, \eta) = f(\mu) + \int \frac{g(\eta)\, d\eta}{(\mu^2 + 4\eta)^{\frac{1}{2}}}, \qquad \textbf{General solution}.$$

□

6.3 Well-Posedness

Problem (S'99, #2). *In* $\mathbb{R}^2$ *consider the unit square* Ω *defined by* $0 \le x, y \le 1$. *Consider*

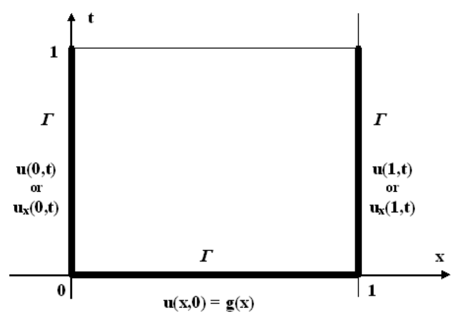
$$\begin{aligned} &a) \quad u_x + u_{yy} = 0; \\ &b) \quad u_{xx} + u_{yy} = 0; \\ &c) \quad u_{xx} - u_{yy} = 0. \end{aligned}$$

Prescribe data for each problem separately on the boundary of Ω *so that each of these problems is* ***well-posed****. Justify your answers.*

Proof. • The initial / boundary value problem for the **HEAT EQUATION** is *well-posed*:

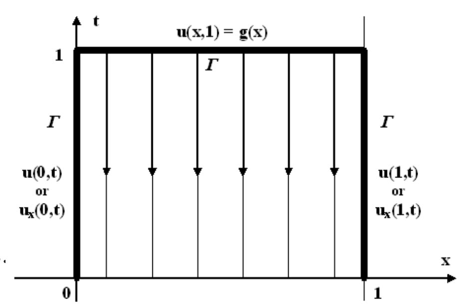
$$\begin{cases} u_t = \triangle u & x \in \Omega, \ t > 0, \\ u(x,0) = g(x) & x \in \overline{\Omega}, \\ u(x,t) = 0 & x \in \partial\Omega, \ t > 0. \end{cases}$$

Existence - by eigenfunction expansion.
Uniqueness and *continuous dependence on the data* - by maximum principle.
The method of eigenfunction expansion and maximum principle give well-posedness for more general problems:



$$\begin{cases} u_t = \triangle u + f(x,t) & x \in \Omega, \ t > 0, \\ u(x,0) = g(x) & x \in \overline{\Omega}, \\ u(x,t) = h(x,t) & x \in \partial\Omega, \ t > 0. \end{cases}$$

It is also possible to replace the Dirichlet boundary condition $u(x,t) = h(x,t)$ by a Neumann or Robin condition, provided we replace λ_n, ϕ_n by the eigenvalues and eigenfunctions for the appropriate boundary value problem.



a) • Relabel the variables ($x \to t$, $y \to x$).
We have the **BACKWARDS HEAT EQUATION**:

$$u_t + u_{xx} = 0.$$

Need to define initial conditions $u(x,1) = g(x)$, and either Dirichlet, Neumann, or Robin boundary conditions.

b) • The solution to the **LAPLACE EQUATION**

$$\begin{cases} \triangle u = 0 & \text{in } \ \Omega, \\ u = g & \text{on } \ \partial\Omega \end{cases}$$

exists if g is continuous on $\partial\Omega$, by Perron's method. Maximum principle gives *uniqueness*.
To show the *continuous dependence on the data*, assume

$$\begin{cases} \triangle u_1 = 0 & \text{in } \ \Omega, \\ u_1 = g_1 & \text{on } \ \partial\Omega; \end{cases} \qquad \begin{cases} \triangle u_2 = 0 & \text{in } \ \Omega, \\ u_2 = g_2 & \text{on } \ \partial\Omega. \end{cases}$$

Then $\triangle(u_1 - u_2) = 0$ in Ω. Maximum principle gives

$$\begin{aligned} \max_{\overline{\Omega}}(u_1 - u_2) &= \max_{\partial\Omega}(g_1 - g_2). \qquad \text{Thus,} \\ \max_{\overline{\Omega}}|u_1 - u_2| &= \max_{\partial\Omega}|g_1 - g_2|. \end{aligned}$$

Thus, $|u_1 - u_2|$ is bounded by $|g_1 - g_2|$, i.e. continuous dependence on data.

- Perron's method gives *existence* of the solution to the **POISSON EQUATION**

$$\begin{cases} \triangle u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = h & \text{on } \partial\Omega \end{cases}$$

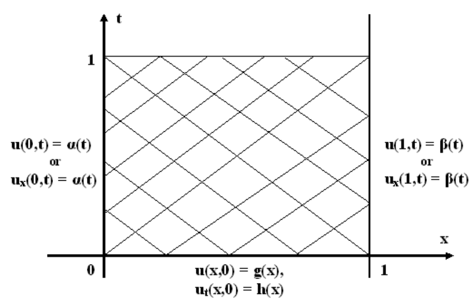
for $f \in C^\infty(\overline{\Omega})$ and $h \in C^\infty(\partial\Omega)$, satisfying the compatibility condition $\int_{\partial\Omega} h\, dS = \int_\Omega f\, dx$. It is *unique* up to an additive constant.

c) • Relabel the variables $(y \to t)$.
The solution to the **WAVE EQUATION**

$$u_{tt} - u_{xx} = 0,$$

is of the form $u(x,y) = F(x+t) + G(x-t)$.
The *existence* of the solution to the initial/boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < 1,\ t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & 0 < x < \\ u(0,t) = \alpha(t), \quad u(1,t) = \beta(t) & t \geq 0. \end{cases}$$



is given by the method of separation of variables
(expansion in eigenfunctions)
and by the parallelogram rule.
Uniqueness is given by the energy method.
Need initial conditions $u(x,0)$, $u_t(x,0)$.
Prescribe u or u_x for each of the two boundaries.

□

Problem (F'95, #7). *Let a, b be real numbers. The PDE*

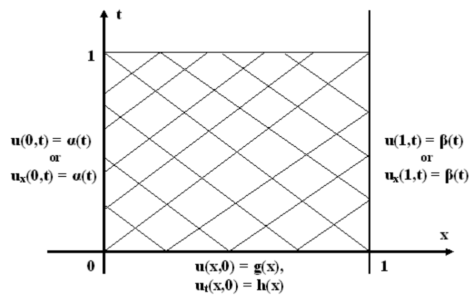
$$u_y + au_{xx} + bu_{yy} = 0$$

is to be solved in the box $\Omega = [0,1]^2$.
*Find data, given on an appropriate part of $\partial\Omega$, that will make this a **well-posed** problem.*
Cover all cases according to the possible values of a and b. Justify your statements.

Proof.
① $ab < 0 \ \Rightarrow$ two sets of characteristics $\Rightarrow$ **hyperbolic**.
Relabeling the variables ($y \to t$), we have

$$u_{tt} + \frac{a}{b}u_{xx} = -\frac{1}{b}u_t.$$

The solution of the equation is of the form
$u(x,t) = F(x + \sqrt{-\frac{a}{b}}t) + G(x - \sqrt{-\frac{a}{b}}t)$.
Existence of the solution to the initial/boundary value problem is given by the method of separation of variables (expansion in eigenfunctions) and by the parallelogram rule.
Uniqueness is given by the energy method.
Need initial conditions $u(x,0)$, $u_t(x,0)$.
Prescribe u or u_x for each of the two boundaries.



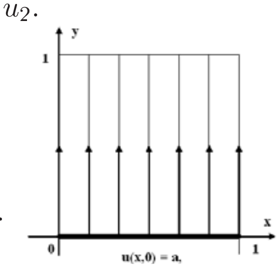
② $ab > 0 \ \Rightarrow$ no characteristics $\Rightarrow$ **elliptic**.
The solution to the Laplace equation with boundary conditions $u = g$ on $\partial\Omega$ *exists* if g is continuous on $\partial\Omega$, by Perron's method.
To show uniqueness, we use maximum principle. Assume there are two solutions u_1 and u_2 with with $u_1 = g(x)$, $u_2 = g(x)$ on $\partial\Omega$. By maximum principle

$$\max_{\overline{\Omega}}(u_1 - u_2) \ = \ \max_{\partial\Omega}(g(x) - g(x)) = 0. \qquad \text{Thus, } u_1 = u_2.$$

③ $ab = 0 \ \Rightarrow$ one set of characteristics $\Rightarrow$ **parabolic**.
- $a = b = 0$. We have $u_y = 0$, a first-order ODE.

u must be specified on $y = 0$, i.e. x -axis.
- $a = 0$, $b \neq 0$. We have $u_y + bu_{yy} = 0$, a second-order ODE.

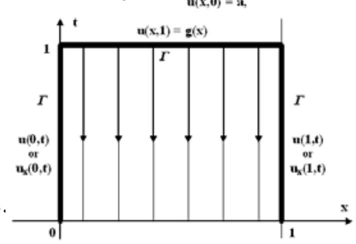
u and u_y must be specified on $y = 0$, i.e. x -axis.



- $a > 0$, $b = 0$. We have a Backwards Heat Equation.

$$u_t = -au_{xx}.$$

Need to define initial conditions $u(x,1) = g(x)$, and either Dirichlet, Neumann, or Robin boundary conditions.

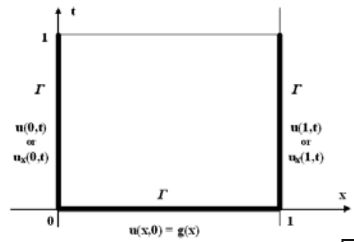


- $a < 0$, $b = 0$. We have a Heat Equation.

$$u_t = -a u_{xx}.$$

The initial / boundary value problem for the heat equation is *well-posed*:

$$\begin{cases} u_t = \triangle u & x \in \Omega,\ t > 0, \\ u(x,0) = g(x) & x \in \overline{\Omega}, \\ u(x,t) = 0 & x \in \partial\Omega,\ t > 0. \end{cases}$$



Existence - by eigenfunction expansion.
Uniqueness and *continuous dependence on the data* - by maximum principle. □

7 Wave Equation

The *one-dimensional wave equation* is

$$u_{tt} - c^2 u_{xx} = 0. \tag{7.1}$$

The characteristic equation with $a = -c^2$, $b = 0$, $c = 1$ would be

$$\frac{dt}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \pm \frac{\sqrt{4c^2}}{-2c^2} = \pm \frac{1}{c},$$

and thus

$$\begin{array}{ccc} t = -\frac{1}{c}x + c_1 & \text{and} & t = \frac{1}{c}x + c_2, \\ \mu = x + ct & & \eta = x - ct, \end{array}$$

which transforms (7.1) to

$$u_{\mu\eta} = 0. \tag{7.2}$$

The general solution of (7.2) is $u(\mu, \eta) = F(\mu) + G(\eta)$, where F and G are C^1 functions. Returning to the variables x, t we find that

$$u(x,t) = F(x + ct) + G(x - ct) \tag{7.3}$$

solves (7.1). Moreover, u is C^2 provided that F and G are C^2.
If $F \equiv 0$, then u has constant values along the lines $x - ct = \text{const}$, so may be described as a wave moving in the positive x-direction with speed $dx/dt = c$; if $G \equiv 0$, then u is a wave moving in the negative x-direction with speed c.

7.1 The Initial Value Problem

For an initial value problem, consider the *Cauchy problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \\ u(x,0) = g(x), \qquad u_t(x,0) = h(x). \end{cases} \tag{7.4}$$

Using (7.3) and (7.4), we find that F and G satisfy

$$F(x) + G(x) = g(x), \qquad cF'(x) - cG'(x) = h(x). \tag{7.5}$$

If we integrate the second equation in (7.5), we get $cF(x) - cG(x) = \int_0^x h(\xi)\,d\xi + C$. Combining this with the first equation in (7.5), we can solve for F and G to find

$$\begin{cases} F(x) = \frac{1}{2}g(x) + \frac{1}{2c}\int_0^x h(\xi)\,d\xi + C_1 \\ G(x) = \frac{1}{2}g(x) - \frac{1}{2c}\int_0^x h(\xi)\,d\xi - C_1, \end{cases}$$

Using these expressions in (7.3), we obtain **d'Alembert's Formula** for the solution of the initial value problem (7.4):

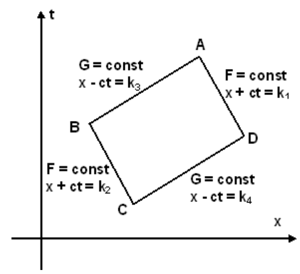
$$\boxed{u(x,t) = \frac{1}{2}(g(x+ct) + g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\,d\xi.}$$

If $g \in C^2$ and $h \in C^1$, then d'Alembert's Formula defines a C^2 solution of (7.4).

7.2 Weak Solutions

Equation (7.3) defines a *weak* solution of (7.1) when F and G are not C^2 functions. Consider the parallelogram with sides that are segments of characteristics. Since $u(x,t) = F(x+ct) + G(x-ct)$, we have

$$\begin{aligned} &u(A) + u(C) = \\ &= F(k_1) + G(k_3) + F(k_2) + G(k_4) \\ &= u(B) + u(D), \end{aligned}$$



which is the *parallelogram rule.*

7.3 Initial/Boundary Value Problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L,\ t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & 0 < x < L \\ u(0,t) = \alpha(t), \quad u(L,t) = \beta(t) & t \ge 0. \end{cases} \tag{7.6}$$

Use separation of variables to obtain an expansion in eigenfunctions. Find $u(x,t)$ in the form

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t)\cos\frac{n\pi x}{L} + b_n(t)\sin\frac{n\pi x}{L}.$$

7.4 Duhamel's Principle

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = 0 \\ u_t(x,0) = 0. \end{cases} \Rightarrow \begin{cases} U_{tt} - c^2 U_{xx} = 0 \\ U(x,0,s) = 0 \\ U_t(x,0,s) = f(x,s) \end{cases} \qquad u(x,t) = \int_0^t U(x, t-s, s)\,ds.$$

$$\begin{cases} a_n'' + \lambda_n a_n = f_n(t) \\ a_n(0) = 0 \\ a_n'(0) = 0 \end{cases} \Rightarrow \begin{cases} \tilde{a}_n'' + \lambda_n \tilde{a}_n = 0 \\ \tilde{a}_n(0,s) = 0 \\ \tilde{a}_n'(0,s) = f_n(s) \end{cases} \qquad a_n(t) = \int_0^t \tilde{a}_n(t-s, s)\,ds.$$

7.5 The Nonhomogeneous Equation

Consider the *nonhomogeneous wave equation* with *homogeneous initial conditions*:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), \\ u(x,0) = 0, \quad u_t(x,0) = 0. \end{cases} \tag{7.7}$$

Duhamel's Principle provides the solution of (7.7):

$$\boxed{u(x,t) = \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(\xi, s)\,d\xi \right) ds.}$$

If $f(x,t)$ is C^1 in x and C^0 in t, then Duhamel's Principle provides a C^2 solution of (7.7).

We can solve (7.7) with *nonhomogeneous initial conditions*,

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), \end{cases} \tag{7.8}$$

by adding together d'Alembert's formula and Duhamel's principle gives the solution:

$$\boxed{u(x,t) = \frac{1}{2}(g(x+ct) + g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\,d\xi + \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(\xi,s)\,d\xi\right) ds.}$$

7.6 Higher Dimensions

7.6.1 Spherical Means

For a continuous function $u(x)$ on $\mathbb{R}^n$, its *spherical mean* or *average on a sphere of radius* r *and center* x is

$$M_u(x,r) = \frac{1}{\omega_n}\int_{|\xi|=1} u(x+r\xi)dS_\xi,$$

where ω_n is the area of the unit sphere $S^{n-1} = \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ and dS_ξ is surface measure. Since u is continuous in x, $M_u(x,r)$ is continuous in x and r, so

$$M_u(x,0) = u(x).$$

Using the chain rule, we find

$$\frac{\partial}{\partial r}M_u(x,r) = \frac{1}{\omega_n}\int_{|\xi|=1}\sum_{i=1}^n u_{x_i}(x+r\xi)\,\xi_i\,dS_\xi \;=\; \circledast$$

To compute the RHS, we apply the divergence theorem in $\Omega = \{\xi \in \mathbb{R}^n : |\xi| < 1\}$, which has boundary $\partial\Omega = S^{n-1}$ and exterior unit normal $n(\xi) = \xi$. The integrand is $V \cdot n$ where $V(\xi) = r^{-1}\nabla_\xi u(x+r\xi) = \nabla_x u(x+r\xi)$. Computing the divergence of V, we obtain

$$\operatorname{div} V(\xi) \;=\; r\sum_{i=1}^n u_{x_i x_i}(x+r\xi) \;=\; r\triangle_x u(x+r\xi), \qquad \text{so,}$$

$$\begin{aligned}
\circledast \;&=\; \frac{1}{\omega_n}\int_{|\xi|<1} r\triangle_x u(x+r\xi)\,d\xi \;=\; \frac{r}{\omega_n}\triangle_x\int_{|\xi|<1} u(x+r\xi)\,d\xi \qquad (\xi' = r\xi)\\
&=\; \frac{r}{\omega_n}\frac{1}{r^n}\triangle_x\int_{|\xi'|<r} u(x+\xi')\,d\xi' \qquad \text{(spherical coordinates)}\\
&=\; \frac{1}{\omega_n r^{n-1}}\triangle_x\int_0^r \rho^{n-1}\int_{|\xi|=1} u(x+\rho\xi)\,dS_\xi\,d\rho\\
&=\; \frac{1}{\omega_n r^{n-1}}\,\omega_n\triangle_x\int_0^r \rho^{n-1}M_u(x,\rho)\,d\rho \;=\; \frac{1}{r^{n-1}}\triangle_x\int_0^r \rho^{n-1}M_u(x,\rho)\,d\rho.
\end{aligned}$$

If we multiply by r^{n-1}, differentiate with respect to r, and then divide by r^{n-1}, we obtain the **Darboux equation:**

$$\boxed{\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right)M_u(x,r) \;=\; \triangle_x M_u(x,r).}$$

Note that for a *radial* function $u = u(r)$, we have $M_u = u$, so the equation provides the Laplacian of u in spherical coordinates.

7.6.2 Application to the Cauchy Problem

We want to solve the equation

$$u_{tt} = c^2\triangle u \qquad x \in \mathbb{R}^n,\ t > 0, \tag{7.9}$$

$$u(x,0) = g(x), \quad u_t(x,0) = h(x) \qquad x \in \mathbb{R}^n.$$

We use *Poisson's method of spherical means* to reduce this problem to a partial differential equation in the two variables r and t.

Suppose that $u(x,t)$ solves (7.9). We can view t as a parameter and take the spherical mean to obtain $M_u(x,r,t)$, which satisfies

$$\frac{\partial^2}{\partial t^2}M_u(x,r,t) = \frac{1}{\omega_n}\int_{|\xi|=1} u_{tt}(x+r\xi,t)dS_\xi = \frac{1}{\omega_n}\int_{|\xi|=1} c^2\triangle u(x+r\xi,t)dS_\xi = c^2\triangle M_u(x,r,t).$$

Invoking the Darboux equation, we obtain the **Euler-Poisson-Darboux equation:**

$$\boxed{\frac{\partial^2}{\partial t^2}M_u(x,r,t) = c^2\left(\frac{\partial^2}{\partial r^2}+\frac{n-1}{r}\frac{\partial}{\partial r}\right)M_u(x,r,t).}$$

The initial conditions are obtained by taking the spherical means:

$$M_u(x,r,0) = M_g(x,r), \qquad \frac{\partial M_u}{\partial t}(x,r,0) = M_h(x,r).$$

If we find $M_u(x,r,t)$, we can then recover $u(x,t)$ by:

$$u(x,t) = \lim_{r\to 0} M_u(x,r,t).$$

7.6.3 Three-Dimensional Wave Equation

When $n=3$, we can write the Euler-Poisson-Darboux equation as [2]

$$\frac{\partial^2}{\partial t^2}\Big(rM_u(x,r,t)\Big) = c^2\frac{\partial^2}{\partial r^2}\Big(rM_u(x,r,t)\Big).$$

For each fixed x, consider $V^x(r,t) = rM_u(x,r,t)$ as a solution of the one-dimensional wave equation in r, $t>0$:

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}V^x(r,t) &= c^2\frac{\partial^2}{\partial r^2}V^x(r,t),\\
V^x(r,0) &= rM_g(x,r) \equiv G^x(r), && \text{(IC)}\\
V_t^x(r,0) &= rM_h(x,r) \equiv H^x(r), && \text{(IC)}\\
V^x(0,t) &= \lim_{r\to 0} rM_u(x,r,t) = 0\cdot u(x,t) = 0. && \text{(BC)}\\
G^x(0) &= H^x(0) = 0.
\end{aligned}$$

We may extend G^x and H^x as odd functions of r and use d'Alembert's formula for $V^x(r,t)$:

$$V^x(r,t) = \frac{1}{2}\Big(G^x(r+ct)+G^x(r-ct)\Big) + \frac{1}{2c}\int_{r-ct}^{r+ct} H^x(\rho)\,d\rho.$$

Since G^x and H^x are odd functions, we have for $r<ct$:

$$G^x(r-ct) = -G^x(ct-r) \qquad \text{and} \qquad \int_{r-ct}^{r+ct} H^x(\rho)\,d\rho = \int_{ct-r}^{ct+r} H^x(\rho)\,d\rho.$$

After some more manipulations, we find that the solution of (7.9) is given by the **Kirchhoff's formula:**

$$\boxed{u(x,t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t\int_{|\xi|=1} g(x+ct\xi)dS_\xi\right) + \frac{t}{4\pi}\int_{|\xi|=1} h(x+ct\xi)dS_\xi.}$$

If $g\in C^3(\mathbb{R}^3)$ and $h\in C^2(\mathbb{R}^3)$, then Kirchhoff's formula defines a C^2-solution of (7.9).

[2] It is seen by expanding the equation below.

7.6.4 Two-Dimensional Wave Equation

This problem is solved by Hadamard's *method of descent*, namely, view (7.9) as a special case of a three-dimensional problem with initial conditions independent of x_3.
We need to convert surface integrals in $\mathbb{R}^3$ to domain integrals in $\mathbb{R}^2$.

$$u(x_1,x_2,t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\bigg(2t\int_{\xi_1^2+\xi_2^2<1}\frac{g(x_1+ct\xi_1,x_2+ct\xi_2)d\xi_1 d\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}\bigg)+\frac{t}{4\pi}\bigg(2\int_{\xi_1^2+\xi_2^2<1}\frac{h(x_1+ct\xi_1,x_2+ct\xi_2)d\xi_1 d\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}$$

If $g \in C^3(\mathbb{R}^2)$ and $h \in C^2(\mathbb{R}^2)$, then this equation defines a C^2-solution of (7.9).

7.6.5 Huygen's Principle

Notice that $u(x,t)$ depends only on the Cauchy data g, h on the surface of the hypersphere $\{x + ct\xi : |\xi| = 1\}$ in $\mathbb{R}^n$, $n = 2k+1$; in other words we have *sharp signals*.
If we use the method of descent to obtain the solution for $n = 2k$, the hypersurface integrals become domain integrals. This means that there are *no sharp signals*.
The fact that sharp signals exist only for *odd dimensions* $n \geq 3$ is known as *Huygen's principle*.
[3]

[3]For $x \in \mathbb{R}^n$:

$$\frac{\partial}{\partial t}\bigg(\int_{|\xi|=1} f(x+t\xi)dS_\xi\bigg) = \frac{1}{t^{n-1}}\int_{|y|\leq t}\triangle f(x+y)dy$$

$$\frac{\partial}{\partial t}\bigg(\int_{|y|\leq t} f(x+y)dy\bigg) = t^{n-1}\bigg(\int_{|\xi|=1} f(x+t\xi)dS_\xi\bigg)$$

7.7 Energy Methods

Suppose $u \in C^2(\mathbb{R}^n \times (0,\infty))$ solves

$$\begin{cases} u_{tt} = c^2 \triangle u & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & x \in \mathbb{R}^n, \end{cases} \tag{7.10}$$

where g and h have compact support.
Define **energy** for a function $u(x,t)$ at time t by

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} (u_t^2 + c^2|\nabla u|^2)\,dx.$$

If we differentiate this energy function, we obtain

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}\bigg[\frac{1}{2}\int_{\mathbb{R}^n} \Big(u_t^2 + c^2\sum_{i=1}^n u_{x_i}^2\Big)\,dx\bigg] = \int_{\mathbb{R}^n} \Big(u_t u_{tt} + c^2 \sum_{i=1}^n u_{x_i} u_{x_i t}\Big)\,dx \\ &= \int_{\mathbb{R}^n} u_t u_{tt}\,dx + c^2\Big[\sum_{i=1}^n u_{x_i} u_t\Big]_{\partial\mathbb{R}^n} - \int_{\mathbb{R}^n} c^2 \sum_{i=1}^n u_{x_i x_i} u_t\,dx \\ &= \int_{\mathbb{R}^n} u_t(u_{tt} - c^2\triangle u)\,dx = 0, \end{aligned}$$

or

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}\bigg[\frac{1}{2}\int_{\mathbb{R}^n} \Big(u_t^2 + c^2\sum_{i=1}^n u_{x_i}^2\Big)\,dx\bigg] = \int_{\mathbb{R}^n} \Big(u_t u_{tt} + c^2 \sum_{i=1}^n u_{x_i} u_{x_i t}\Big)\,dx \\ &= \int_{\mathbb{R}^n} \big(u_t u_{tt} + c^2 \nabla u \cdot \nabla u_t\big)\,dx \\ &= \int_{\mathbb{R}^n} u_t u_{tt}\,dx + c^2\bigg[\int_{\partial\mathbb{R}^n} u_t \frac{\partial u}{\partial n}\,ds - \int_{\mathbb{R}^n} u_t \triangle u\,dx\bigg] \\ &= \int_{\mathbb{R}^n} u_t(u_{tt} - c^2\triangle u)\,dx = 0. \end{aligned}$$

Hence, $E(t)$ is constant, or $E(t) \equiv E(0)$.
In particular, if u_1 and u_2 are two solutions of (7.10), then $w = u_1 - u_2$ has zero Cauchy data and hence $E_w(0) = 0$. By discussion above, $E_w(t) \equiv 0$, which implies $w(x,t) \equiv$ const. But $w(x,0) = 0$ then implies $w(x,t) \equiv 0$, so the solution is **unique**.

7.8 Contraction Mapping Principle

Suppose X is a complete metric space with distance function represented by $d(\cdot,\cdot)$.
A mapping $T : X \to X$ is a *strict contraction* if there exists $0 < \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha\, d(x, y) \qquad \forall\, x, y \in X.$$

An obvious example on $X = \mathbb{R}^n$ is $Tx = \alpha x$, which shrinks all of $\mathbb{R}^n$, leaving 0 fixed.

The Contraction Mapping Principle. *If X is a complete metric space and $T : X \to X$ is a strict contraction, then T has a* ***unique*** *fixed point.*

The process of replacing a differential equation by an integral equation occurs in time-evolution partial differential equations.
The Contraction Mapping Principle is used to establish the local existence and **uniqueness** of solutions to various nonlinear equations.

8 Laplace Equation

Consider the *Laplace equation*

$$\triangle u = 0 \qquad \text{in} \;\; \Omega \subset \mathbb{R}^n \tag{8.1}$$

and the *Poisson equation*

$$\triangle u = f \qquad \text{in} \;\; \Omega \subset \mathbb{R}^n. \tag{8.2}$$

Solutions of (8.1) are called *harmonic functions* in Ω.
Cauchy problems for (8.1) and (8.2) are not well posed. We use separation of variables for some special domains Ω to find boundary conditions that are appropriate for (8.1), (8.2).

$$\begin{aligned}
&\textit{Dirichlet problem:} && u(x) = g(x), && x \in \partial\Omega \\
&\textit{Neumann problem:} && \frac{\partial u(x)}{\partial n} = h(x), && x \in \partial\Omega \\
&\textit{Robin problem:} && \frac{\partial u}{\partial n} + \alpha u = \beta, && x \in \partial\Omega
\end{aligned}$$

8.1 Green's Formulas

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, ds - \int_\Omega v \triangle u \, dx \tag{8.3}$$

$$\int_{\partial\Omega} (v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}) ds = \int_\Omega (v \triangle u - u \triangle v) \, dx$$

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} \, ds = \int_\Omega \triangle u \, dx \qquad (v = 1 \text{ in } (8.3))$$

$$\int_\Omega |\nabla u|^2 \, dx = \int_{\partial\Omega} u \frac{\partial u}{\partial n} \, ds - \int_\Omega u \triangle u \, dx \qquad (u = v \text{ in } (8.3))$$

$$\int_\Omega u_x v_x \, dxdy = \int_{\partial\Omega} v u_x n_1 \, ds - \int_\Omega v u_{xx} \, dxdy \qquad \vec{n} = (n_1, n_2) \in \mathbb{R}^2$$

$$\int_\Omega u_{x_k} v \, dx = \int_{\partial\Omega} u v n_k \, ds - \int_\Omega u v_{x_k} \, dx \qquad \vec{n} = (n_1, \ldots, n_n) \in \mathbb{R}^n.$$

$$\int_\Omega u \triangle^2 v \, dx = \int_{\partial\Omega} u \frac{\partial \triangle v}{\partial n} \, ds - \int_{\partial\Omega} \triangle v \frac{\partial u}{\partial n} \, ds + \int_\Omega \triangle u \triangle v \, dx.$$

$$\int_\Omega \big(u \triangle^2 v - v \triangle^2 u\big) \, dx = \int_{\partial\Omega} \big(u \frac{\partial \triangle v}{\partial n} - v \frac{\partial \triangle u}{\partial n}\big) \, ds + \int_{\partial\Omega} \big(\triangle u \frac{\partial v}{\partial n} - \triangle v \frac{\partial u}{\partial n}\big) \, ds.$$

8.2 Polar Coordinates

Polar Coordinates. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. Then*

$$\int_{\mathbb{R}^n} f\,dx \;=\; \int_0^\infty \Big(\int_{\partial B_r(x_0)} f\,dS\Big)\,dr$$

for each $x_0 \in \mathbb{R}^n$. In particular

$$\frac{d}{dr}\Big(\int_{B_r(x_0)} f\,dx\Big) \;=\; \int_{\partial B_r(x_0)} f\,dS$$

for each $r > 0$.

$$u = u(x(r,\theta), y(r,\theta))$$
$$x(r,\theta) = r\cos\theta$$
$$y(r,\theta) = r\sin\theta$$

$$\begin{aligned}
u_r &= u_x x_r + u_y y_r = u_x\cos\theta + u_y\sin\theta,\\
u_\theta &= u_x x_\theta + u_y y_\theta = -u_x r\sin\theta + u_y r\cos\theta,\\
u_{rr} &= (u_x\cos\theta + u_y\sin\theta)_r = (u_{xx}x_r + u_{xy}y_r)\cos\theta + (u_{yx}x_r + u_{yy}y_r)\sin\theta\\
&= u_{xx}\cos^2\theta + 2u_{xy}\cos\theta\sin\theta + u_{yy}\sin^2\theta,\\
u_{\theta\theta} &= (-u_x r\sin\theta + u_y r\cos\theta)_\theta\\
&= (-u_{xx}x_\theta - u_{xy}y_\theta)r\sin\theta - u_x r\cos\theta + (u_{yx}x_\theta + u_{yy}y_\theta)r\cos\theta - u_y r\sin\theta\\
&= (u_{xx}r\sin\theta - u_{xy}r\cos\theta)r\sin\theta - u_x r\cos\theta + (-u_{yx}r\sin\theta + u_{yy}r\cos\theta)r\cos\theta - u_y r\sin\theta\\
&= r^2(u_{xx}\sin^2\theta - 2u_{xy}\cos\theta\sin\theta + u_{yy}\cos^2\theta) - r(u_x\cos\theta + u_y\sin\theta).
\end{aligned}$$

$$\begin{aligned}
&u_{rr} + \tfrac{1}{r^2}u_{\theta\theta}\\
&= u_{xx}\cos^2\theta + 2u_{xy}\cos\theta\sin\theta + u_{yy}\sin^2\theta + u_{xx}\sin^2\theta - 2u_{xy}\cos\theta\sin\theta + u_{yy}\cos^2\theta - \tfrac{1}{r}(u_x\cos\theta + u_y\sin\theta)\\
&= u_{xx} + u_{yy} - \tfrac{1}{r}u_r.
\end{aligned}$$

$$\boxed{u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.}$$

$$\boxed{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.}$$

8.3 Polar Laplacian in $\mathbb{R}^2$ for Radial Functions

$$\boxed{\triangle u \;=\; \frac{1}{r}\big(ru_r\big)_r \;=\; \Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\Big)u.}$$

8.4 Spherical Laplacian in $\mathbb{R}^3$ and $\mathbb{R}^n$ for Radial Functions

$$\boxed{\triangle u \;=\; \Big(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\Big)u.}$$

In $\mathbb{R}^3$: [4]

$$\boxed{\triangle u \;=\; \frac{1}{r^2}\big(r^2u_r\big)_r \;=\; \frac{1}{r}\big(ru\big)_{rr} \;=\; \Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\Big)u.}$$

[4] These formulas are taken from S. Farlow, p. 411.

8.5 Cylindrical Laplacian in $\mathbb{R}^3$ for Radial Functions

$$\boxed{\triangle u \;=\; \frac{1}{r}\big(ru_r\big)_r \;=\; \Big(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\Big)u.}$$

8.6 Mean Value Theorem

Gauss Mean Value Theorem. *If $u \in C^2(\Omega)$ is harmonic in Ω, let $\xi \in \Omega$ and pick $r > 0$ so that $\overline{B_r(\xi)} = \{x : |x-\xi| \le r\} \subset \Omega$. Then*

$$u(\xi) = M_u(\xi, r) \equiv \frac{1}{\omega_n}\int_{|x|=1} u(\xi + rx)\, dS_x,$$

where ω_n is the measure of the $(n-1)$-dimensional sphere in $\mathbb{R}^n$.

8.7 Maximum Principle

Maximum Principle. *If $u \in C^2(\Omega)$ satisfies $\triangle u \ge 0$ in Ω, then either u is a constant, or*

$$u(\xi) < \sup_{x\in\Omega} u(x)$$

for all $\xi \in \Omega$.

Proof. We may assume $A = \sup_{x\in\Omega} u(x) \le \infty$, so by continuity of u we know that $\{x \in \Omega : u(x) = A\}$ is relatively closed in Ω. But since

$$u(\xi) \le \frac{n}{\omega_n}\int_{|x|\le 1} u(\xi + rx)\, dx,$$

if $u(\xi) = A$ at an interior point ξ, then $u(x) = A$ for all x in a ball about ξ, so $\{x \in \Omega : u(x) = A\}$ is open. The connectedness of Ω implies $u(\xi) < A$ or $u(\xi) \equiv A$ for all $\xi \in \Omega$. □

The maximum principle shows that $u \in C^2(\Omega)$ with $\triangle u \ge 0$ can attain an interior maximum only if u is constant. In particular, if $\overline{\Omega}$ is compact, and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $\triangle u \ge 0$ in Ω, we have the **weak maximum principle**:

$$\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x).$$

8.8 The Fundamental Solution

A fundamental solution $K(x)$ for the Laplace operator is a distribution satisfying

$$\triangle K(x) = \delta(x) \tag{8.4}$$

where δ is the delta distribution supported at $x = 0$. In order to solve (8.4), we should first observe that $\triangle$ is symmetric in the variables $x_1, \ldots, x_n$, and $\delta(x)$ is also **radially symmetric** (i.e., its value only depends on $r = |x|$). Thus, we try to solve (8.4) with a radially symmetric function $K(x)$. Since $\delta(x) = 0$ for $x \neq 0$, we see that (8.4) requires K to be harmonic for $r > 0$. For the radially symmetric function K, Laplace equation becomes ($K = K(r)$):

$$\frac{\partial^2 K}{\partial r^2} + \frac{n-1}{r}\frac{\partial K}{\partial r} = 0. \tag{8.5}$$

The general solution to (8.5) is

$$K(r) = \begin{cases} c_1 + c_2 \log r & \text{if } n = 2 \\ c_1 + c_2 r^{2-n} & \text{if } n \geq 3. \end{cases} \tag{8.6}$$

After we determine c_2, we find the **fundamental solution for the Laplace operator**:

$$K(x) = \begin{cases} \frac{1}{2\pi} \log r & \text{if } n = 2 \\ \frac{1}{(2-n)\omega_n} r^{2-n} & \text{if } n \geq 3. \end{cases}$$

- We can derive, (8.6) for any given n. For intance, when $n = 3$, we have:

$$K'' + \frac{2}{r}K' = 0. \quad \circledast$$

Let

$$\begin{aligned} K &= \frac{1}{r}w(r), \\ K' &= \frac{1}{r}w' - \frac{1}{r^2}w, \\ K'' &= \frac{1}{r}w'' - \frac{2}{r^2}w' + \frac{2}{r^3}w. \end{aligned}$$

Plugging these into $\circledast$, we obtain:

$$\begin{aligned} \frac{1}{r}w'' &= 0, \quad \text{or} \\ w'' &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} w &= c_1 r + c_2, \\ K &= \frac{1}{r}w(r) = c_1 + \frac{c_2}{r}. \quad \checkmark \end{aligned}$$

See the similar problem, F'99, #2, where the fundamental solution for $(\triangle - I)$ is found in the process.

Find the Fundamental Solution of the Laplace Operator for $n = 3$
We found that starting with the Laplacian in $\mathbb{R}^3$ for a radially symmetric function K,

$$K'' + \frac{2}{r}K' = 0,$$

and letting $K = \frac{1}{r}w(r)$, we obtained the equation: $w = c_1 r + c_2$, which implied:

$$K = c_1 + \frac{c_2}{r}.$$

We now find the constant c_2 that ensures that for $v \in C_0^\infty(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} K(|x|) \triangle v(x)\, dx = v(0).$$

Suppose $v(x) \equiv 0$ for $|x| \geq R$ and let $\Omega = B_R(0)$; for small $\epsilon > 0$ let

$$\Omega_\epsilon = \Omega - B_\epsilon(0).$$

$K(|x|)$ is harmonic ($\triangle K(|x|) = 0$) in Ω_ϵ. Consider Green's identity ($\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(0)$):

$$\int_{\Omega_\epsilon} K(|x|)\triangle v\, dx = \underbrace{\int_{\partial\Omega} \Big(K(|x|)\frac{\partial v}{\partial n} - v\frac{\partial K(|x|)}{\partial n}\Big)\, dS}_{=0,\ since\ v\equiv 0\ for\ x\geq R} + \int_{\partial B_\epsilon(0)} \Big(K(|x|)\frac{\partial v}{\partial n} - v\frac{\partial K(|x|)}{\partial n}\Big)\, dS.$$

$$\lim_{\epsilon\to 0}\Big[\int_{\Omega_\epsilon} K(|x|)\triangle v\, dx\Big] = \int_\Omega K(|x|)\triangle v\, dx. \qquad \Big(\text{Since } K(r) = c_1 + \frac{c_2}{r} \text{ is integrable at } x = 0.\Big)$$

On $\partial B_\epsilon(0)$, $K(|x|) = K(\epsilon)$. Thus, [5]

$$\Big|\int_{\partial B_\epsilon(0)} K(|x|)\frac{\partial v}{\partial n}\, dS\Big| = \big|K(\epsilon)\big| \int_{\partial B_\epsilon(0)} \Big|\frac{\partial v}{\partial n}\Big|\, dS \leq \Big|c_1 + \frac{c_2}{\epsilon}\Big| 4\pi\epsilon^2 \max\big|\nabla v\big| \to 0, \quad \text{as } \epsilon\to 0.$$

$$\begin{aligned}
\int_{\partial B_\epsilon(0)} v(x)\frac{\partial K(|x|)}{\partial n}\, dS &= \int_{\partial B_\epsilon(0)} \frac{c_2}{\epsilon^2} v(x)\, dS \\
&= \int_{\partial B_\epsilon(0)} \frac{c_2}{\epsilon^2} v(0)\, dS + \int_{\partial B_\epsilon(0)} \frac{c_2}{\epsilon^2}[v(x) - v(0)]\, dS \\
&= \frac{c_2}{\epsilon^2} v(0)\, 4\pi\epsilon^2 + \underbrace{4\pi c_2 \max_{x\in\partial B_\epsilon(0)} \big|v(x) - v(0)\big|}_{\to 0,\ (v\ is\ continuous)} \\
&= 4\pi c_2\, v(0) \to -v(0).
\end{aligned}$$

Thus, taking $4\pi c_2 = -1$, i.e. $c_2 = -\frac{1}{4\pi}$, we obtain

$$\int_\Omega K(|x|)\triangle v\, dx = \lim_{\epsilon\to 0}\int_{\Omega_\epsilon} K(|x|)\triangle v\, dx = v(0),$$

that is $K(r) = -\frac{1}{4\pi r}$ is the fundamental solution of $\triangle$.

[5] In $\mathbb{R}^3$, for $|x| = \epsilon$,

$$K(|x|) = K(\epsilon) = c_1 + \frac{c_2}{\epsilon}.$$

$$\frac{\partial K(|x|)}{\partial n} = -\frac{\partial K(\epsilon)}{\partial r} = \frac{c_2}{\epsilon^2}, \qquad \text{(since } n \text{ points inwards.)}$$

n points *toward* 0 on the sphere $|x| = \epsilon$ (i.e., $n = -x/|x|$).

Show that the Fundamental Solution of the Laplace Operator is given by.

$$K(x) = \begin{cases} \frac{1}{2\pi}\log r & \textit{if } n = 2 \\ \frac{1}{(2-n)\omega_n} r^{2-n} & \textit{if } n \geq 3. \end{cases} \tag{8.7}$$

Proof. For $v \in C_0^\infty(\mathbb{R}^n)$, we want to show

$$\int_{\mathbb{R}^n} K(|x|) \triangle v(x)\, dx \;=\; v(0).$$

Suppose $v(x) \equiv 0$ for $|x| \geq R$ and let $\Omega = B_R(0)$; for small $\epsilon > 0$ let

$$\Omega_\epsilon = \Omega - B_\epsilon(0).$$

$K(|x|)$ is harmonic ($\triangle K(|x|) = 0$) in Ω_ϵ. Consider Green's identity ($\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(0)$):

$$\int_{\Omega_\epsilon} K(|x|)\triangle v\, dx \;=\; \underbrace{\int_{\partial\Omega} \Big(K(|x|)\frac{\partial v}{\partial n} \;-\; v\,\frac{\partial K(|x|)}{\partial n}\Big)\, dS}_{=0,\ since\ v\equiv 0\ for\ x\geq R} \;+\; \int_{\partial B_\epsilon(0)} \Big(K(|x|)\frac{\partial v}{\partial n} \;-\; v\,\frac{\partial K(|x|)}{\partial n}\Big)\, dS.$$

$$\lim_{\epsilon\to 0}\Big[\int_{\Omega_\epsilon} K(|x|)\triangle v\, dx\Big] \;=\; \int_\Omega K(|x|)\triangle v\, dx. \qquad \Big(\text{Since } K(r) \text{ is integrable at } x = 0.\Big)$$

On $\partial B_\epsilon(0)$, $\;K(|x|) = K(\epsilon)$. Thus, [6]

$$\Big|\int_{\partial B_\epsilon(0)} K(|x|)\,\frac{\partial v}{\partial n}\, dS\Big| \;=\; \big|K(\epsilon)\big|\int_{\partial B_\epsilon(0)}\Big|\frac{\partial v}{\partial n}\Big|\, dS \;\leq\; \big|K(\epsilon)\big|\omega_n\epsilon^{n-1}\max\big|\nabla v\big| \;\to\; 0, \quad \text{as} \quad \epsilon\to 0.$$

$$\begin{aligned}
\int_{\partial B_\epsilon(0)} v(x)\,\frac{\partial K(|x|)}{\partial n}\, dS \;&=\; \int_{\partial B_\epsilon(0)} -\frac{1}{\omega_n\epsilon^{n-1}} v(x)\, dS \\
&=\; \int_{\partial B_\epsilon(0)} -\frac{1}{\omega_n\epsilon^{n-1}} v(0)\, dS + \int_{\partial B_\epsilon(0)} -\frac{1}{\omega_n\epsilon^{n-1}}[v(x) - v(0)]\, dS \\
&=\; -\frac{1}{\omega_n\epsilon^{n-1}}\, v(0)\,\omega_n\epsilon^{n-1} - \underbrace{\max_{x\in\partial B_\epsilon(0)}\big|v(x) - v(0)\big|}_{\to 0,\ (v\ is\ continuous)} \\
&=\; -v(0).
\end{aligned}$$

Thus,

$$\int_\Omega K(|x|)\triangle v\, dx \;=\; \lim_{\epsilon\to 0}\int_{\Omega_\epsilon} K(|x|)\triangle v\, dx \;=\; v(0).$$

□

---

[6]Note that for $|x| = \epsilon$,

$$K(|x|) \;=\; K(\epsilon) \;=\; \begin{cases} \frac{1}{2\pi}\log\epsilon & \text{if } n = 2 \\ \frac{1}{(2-n)\omega_n}\epsilon^{2-n} & \text{if } n \geq 3. \end{cases}$$

$$\frac{\partial K(|x|)}{\partial n} \;=\; -\frac{\partial K(\epsilon)}{\partial r} \;=\; -\begin{cases} \frac{1}{2\pi\epsilon} & \text{if } n = 2 \\ \frac{1}{\omega_n\epsilon^{n-1}} & \text{if } n \geq 3, \end{cases} \;=\; -\frac{1}{\omega_n\epsilon^{n-1}}, \quad \text{(since } n \text{ points inwards.)}$$

n points *toward* 0 on the sphere $|x| = \epsilon$ (i.e., $n = -x/|x|$).

8.9 Representation Theorem

Representation Theorem, n = 3.
Let Ω be bounded domain in $\mathbb{R}^3$ and let n be the unit exterior normal to $\partial\Omega$. Let $u \in C^2(\overline{\Omega})$. Then the value of u at any point $x \in \Omega$ is given by the formula

$$u(x) \;=\; \frac{1}{4\pi}\int_{\partial\Omega}\left[\frac{1}{|x-y|}\frac{\partial u(y)}{\partial n} - u(y)\frac{\partial}{\partial n}\frac{1}{|x-y|}\right] dS \;-\; \frac{1}{4\pi}\int_{\Omega}\frac{\triangle u(y)}{|x-y|}\,dy. \quad (8.8)$$

Proof. Consider the Green's identity:

$$\int_{\Omega}(u\triangle w - w\triangle u)\,dy \;=\; \int_{\partial\Omega}\big(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\big)\,dS,$$

where w is the harmonic function

$$w(y) = \frac{1}{|x-y|},$$

which is singular at $x \in \Omega$. In order to be able to apply Green's identity, we consider a new domain Ω_ϵ:

$$\Omega_\epsilon = \Omega - B_\epsilon(x).$$

Since u, $w \in C_2(\overline{\Omega}_\epsilon)$, Green's identity can be applied. Since w is harmonic ($\triangle w = 0$) in Ω_ϵ and since $\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(x)$, we have

$$-\int_{\Omega_\epsilon}\frac{\triangle u(y)}{|x-y|}dy \;=\; \int_{\partial\Omega}\left[u(y)\frac{\partial}{\partial n}\frac{1}{|x-y|} - \frac{1}{|x-y|}\frac{\partial u(y)}{\partial n}\right]dS \quad (8.9)$$

$$+ \int_{\partial B_\epsilon(x)}\left[u(y)\frac{\partial}{\partial n}\frac{1}{|x-y|} - \frac{1}{|x-y|}\frac{\partial u(y)}{\partial n}\right]dS. \quad (8.10)$$

We will show that formula (8.8) is obtained by letting $\epsilon \to 0$.

$$\lim_{\epsilon\to 0}\left[-\int_{\Omega_\epsilon}\frac{\triangle u(y)}{|x-y|}\,dy\right] \;=\; -\int_{\Omega}\frac{\triangle u(y)}{|x-y|}\,dy. \qquad \Big(\text{Since } \frac{1}{|x-y|} \text{ is integrable at } x=y.\Big)$$

The first integral on the right of (8.10) does not depend on ϵ. Hence, the limit as $\epsilon \to 0$ of the second integral on the right of (8.10) exists, and in order to obtain (8.8), need

$$\lim_{\epsilon\to 0}\int_{\partial B_\epsilon(x)}\left[u(y)\frac{\partial}{\partial n}\frac{1}{|x-y|} - \frac{1}{|x-y|}\frac{\partial u(y)}{\partial n}\right]dS \;=\; 4\pi u(x).$$

$$\begin{aligned}\int_{\partial B_\epsilon(x)}\left[u(y)\frac{\partial}{\partial n}\frac{1}{|x-y|} - \frac{1}{|x-y|}\frac{\partial u(y)}{\partial n}\right]dS \;&=\; \int_{\partial B_\epsilon(x)}\left[\frac{1}{\epsilon^2}u(y) - \frac{1}{\epsilon}\frac{\partial u(y)}{\partial n}\right]dS\\ &=\; \int_{\partial B_\epsilon(x)}\frac{1}{\epsilon^2}u(x)\,dS + \int_{\partial B_\epsilon(x)}\left[\frac{1}{\epsilon^2}[u(y)-u(x)] - \frac{1}{\epsilon}\frac{\partial u(y)}{\partial n}\right]dS\\ &=\; 4\pi u(x) + \int_{\partial B_\epsilon(x)}\left[\frac{1}{\epsilon^2}[u(y)-u(x)] - \frac{1}{\epsilon}\frac{\partial u(y)}{\partial n}\right]dS.\end{aligned}$$

[7] The last integral tends to 0 as $\epsilon \to 0$:

$$\begin{aligned}\Big|\int_{\partial B_\epsilon(x)} \Big[\frac{1}{\epsilon^2}[u(y)-u(x)] - \frac{1}{\epsilon}\frac{\partial u(y)}{\partial n}\Big] dS\Big| &\le \frac{1}{\epsilon^2}\int_{\partial B_\epsilon(x)} \Big|u(y)-u(x)\Big| + \frac{1}{\epsilon}\int_{\partial B_\epsilon(x)} \Big|\frac{\partial u(y)}{\partial n}\Big| dS \\ &\le \underbrace{4\pi \max_{y\in\partial B_\epsilon(x)} \Big|u(y)-u(x)\Big|}_{\to 0,\ (u\ continuous\ in\ \overline{\Omega})} + \underbrace{4\pi\epsilon \max_{y\in\overline{\Omega}} \Big|\nabla u(y)\Big|}_{\to 0,\ (|\nabla u|\ is\ finite)}.\end{aligned}$$

□

---

[7]Note that for points y on $\partial B_\epsilon(x)$,

$$\frac{1}{|x-y|} = \frac{1}{\epsilon} \qquad \text{and} \qquad \frac{\partial}{\partial n}\frac{1}{|x-y|} = \frac{1}{\epsilon^2}.$$

Representation Theorem, n = 2.
Let Ω be bounded domain in $\mathbb{R}^2$ and let n be the unit exterior normal to $\partial\Omega$. Let $u \in C^2(\overline{\Omega})$. Then the value of u at any point $x \in \Omega$ is given by the formula

$$u(x) = \frac{1}{2\pi}\int_\Omega \triangle u(y)\log|x-y|\,dy + \frac{1}{2\pi}\int_{\partial\Omega}\left[u(y)\frac{\partial}{\partial n}\log|x-y| - \log|x-y|\frac{\partial u(y)}{\partial n}\right]dS \tag{8.11}$$

Proof. Consider the Green's identity:

$$\int_\Omega (u\triangle w - w\triangle u)\,dy = \int_{\partial\Omega}\Big(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\Big)\,dS,$$

where w is the harmonic function

$$w(y) = \log|x-y|,$$

which is singular at $x \in \Omega$. In order to be able to apply Green's identity, we consider a new domain Ω_ϵ:

$$\Omega_\epsilon = \Omega - B_\epsilon(x).$$

Since u, $w \in C_2(\overline{\Omega}_\epsilon)$, Green's identity can be applied. Since w is harmonic ($\triangle w = 0$) in Ω_ϵ and since $\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(x)$, we have

$$\begin{aligned} &-\int_{\Omega_\epsilon}\triangle u(y)\log|x-y|\,dy \qquad (8.12)\\ &= \int_{\partial\Omega}\left[u(y)\frac{\partial}{\partial n}\log|x-y| - \log|x-y|\frac{\partial u(y)}{\partial n}\right]dS \\ &\quad + \int_{\partial B_\epsilon(x)}\left[u(y)\frac{\partial}{\partial n}\log|x-y| - \log|x-y|\frac{\partial u(y)}{\partial n}\right]dS. \end{aligned}$$

We will show that formula (8.11) is obtained by letting $\epsilon \to 0$.

$$\lim_{\epsilon\to 0}\left[-\int_{\Omega_\epsilon}\triangle u(y)\log|x-y|\,dy\right] = -\int_\Omega \triangle u(y)\log|x-y|\,dy. \qquad \Big(\text{since } \log|x-y| \text{ is integrable at } x=y.\Big)$$

The first integral on the right of (8.12) does not depend on ϵ. Hence, the limit as $\epsilon \to 0$ of the second integral on the right of (8.12) exists, and in order to obtain (8.11), need

$$\lim_{\epsilon\to 0}\int_{\partial B_\epsilon(x)}\left[u(y)\frac{\partial}{\partial n}\log|x-y| - \log|x-y|\frac{\partial u(y)}{\partial n}\right]dS = 2\pi u(x).$$

$$\begin{aligned} \int_{\partial B_\epsilon(x)}\left[u(y)\frac{\partial}{\partial n}\log|x-y| - \log|x-y|\frac{\partial u(y)}{\partial n}\right]dS &= \int_{\partial B_\epsilon(x)}\left[\frac{1}{\epsilon}u(y) - \log\epsilon\frac{\partial u(y)}{\partial n}\right]dS \\ &= \int_{\partial B_\epsilon(x)}\frac{1}{\epsilon}u(x)\,dS + \int_{\partial B_\epsilon(x)}\left[\frac{1}{\epsilon}[u(y)-u(x)] - \log\epsilon\frac{\partial u(y)}{\partial n}\right]dS \\ &= 2\pi u(x) + \int_{\partial B_\epsilon(x)}\left[\frac{1}{\epsilon}[u(y)-u(x)] - \log\epsilon\frac{\partial u(y)}{\partial n}\right]dS. \end{aligned}$$

[8] The last integral tends to 0 as $\epsilon \to 0$:

$$\begin{aligned}\Big|\int_{\partial B_\epsilon(x)}\Big[\frac{1}{\epsilon}[u(y)-u(x)]-\log\epsilon\frac{\partial u(y)}{\partial n}\Big]dS\Big| &\leq \frac{1}{\epsilon}\int_{\partial B_\epsilon(x)}\Big|u(y)-u(x)\Big|+\log\epsilon\int_{\partial B_\epsilon(x)}\Big|\frac{\partial u(y)}{\partial n}\Big|dS \\ &\leq \underbrace{2\pi\max_{y\in\partial B_\epsilon(x)}\Big|u(y)-u(x)\Big|}_{\to 0,\ (u\ continuous\ in\ \overline{\Omega})}+\underbrace{2\pi\epsilon\log\epsilon\max_{y\in\overline{\Omega}}\Big|\nabla u(y)\Big|}_{\to 0,\ (|\nabla u|\ is\ finite)}.\end{aligned}$$

□

---

[8]Note that for points y on $\partial B_\epsilon(x)$,

$$\log|x-y|=\log\epsilon \qquad \text{and} \qquad \frac{\partial}{\partial n}\log|x-y|=\frac{1}{\epsilon}.$$

Representation Theorems, $\mathbf{n > 3}$ can be obtained in the same way. We use the Green's identity with

$$w(y) = \frac{1}{|x-y|^{n-2}},$$

which is a harmonic function in $\mathbb{R}^n$ with a singularity at x.

The fundamental solution for the Laplace operator is ($r = |x|$):

$$K(x) = \begin{cases} \frac{1}{2\pi}\log r & \text{if } n = 2 \\ \frac{1}{(2-n)\omega_n} r^{2-n} & \text{if } n \geq 3. \end{cases}$$

Representation Theorem. *If $\Omega \in \mathbb{R}^n$ is bounded, $u \in C^2(\overline{\Omega})$, and $x \in \Omega$, then*

$$u(x) = \int_\Omega K(x-y)\triangle u(y)\,dy + \int_{\partial\Omega}\left[u(y)\frac{\partial K(x-y)}{\partial n} - K(x-y)\frac{\partial u(y)}{\partial n}\right]dS \quad (8.13)$$

Proof. Consider the Green's identity:

$$\int_\Omega (u\triangle w - w\triangle u)\,dy = \int_{\partial\Omega}\big(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\big)\,dS,$$

where w is the harmonic function

$$w(y) = K(x-y),$$

which is singular at $y = x$. In order to be able to apply Green's identity, we consider a new domain Ω_ϵ:

$$\Omega_\epsilon = \Omega - B_\epsilon(x).$$

Since u, $K(x-y) \in C_2(\overline{\Omega}_\epsilon)$, Green's identity can be applied. Since $K(x-y)$ is harmonic ($\triangle K(x-y) = 0$) in Ω_ϵ and since $\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(x)$, we have

$$-\int_{\Omega_\epsilon} K(x-y)\triangle u(y)\,dy = \int_{\partial\Omega}\left[u(y)\frac{\partial K(x-y)}{\partial n} - K(x-y)\frac{\partial u(y)}{\partial n}\right]dS \quad (8.14)$$

$$+ \int_{\partial B_\epsilon(x)}\left[u(y)\frac{\partial K(x-y)}{\partial n} - K(x-y)\frac{\partial u(y)}{\partial n}\right]dS \quad (8.15)$$

We will show that formula (8.13) is obtained by letting $\epsilon \to 0$.

$$\lim_{\epsilon\to 0}\left[-\int_{\Omega_\epsilon} K(x-y)\triangle u(y)\,dy\right] = -\int_\Omega K(x-y)\triangle u(y)\,dy. \qquad \Big(\text{since } K(x-y) \text{ is integrable at } x = y.\Big)$$

The first integral on the right of (8.15) does not depend on ϵ. Hence, the limit as $\epsilon \to 0$ of the second integral on the right of (8.15) exists, and in order to obtain (8.13), need

$$\lim_{\epsilon\to 0}\int_{\partial B_\epsilon(x)}\left[u(y)\frac{\partial K(x-y)}{\partial n} - K(x-y)\frac{\partial u(y)}{\partial n}\right]dS = -u(x).$$

$$\int_{\partial B_\epsilon(x)}\left[u(y)\frac{\partial K(x-y)}{\partial n}-K(x-y)\frac{\partial u(y)}{\partial n}\right]dS \;=\; \int_{\partial B_\epsilon(x)}\left[u(y)\frac{\partial K(\epsilon)}{\partial n}-K(\epsilon)\frac{\partial u(y)}{\partial n}\right]dS$$

$$= \int_{\partial B_\epsilon(x)} u(x)\frac{\partial K(\epsilon)}{\partial n}\,dS + \int_{\partial B_\epsilon(x)}\left[\frac{\partial K(\epsilon)}{\partial n}[u(y)-u(x)]-K(\epsilon)\frac{\partial u(y)}{\partial n}\right]dS$$

$$= -\frac{1}{\omega_n\epsilon^{n-1}}\int_{\partial B_\epsilon(x)} u(x)\,dS - \frac{1}{\omega_n\epsilon^{n-1}}\int_{\partial B_\epsilon(x)}[u(y)-u(x)]\,dS - \int_{\partial B_\epsilon(x)} K(\epsilon)\frac{\partial u(y)}{\partial n}\,dS$$

$$= \underbrace{-\frac{1}{\omega_n\epsilon^{n-1}}u(x)\omega_n\epsilon^{n-1}}_{-u(x)} - \frac{1}{\omega_n\epsilon^{n-1}}\int_{\partial B_\epsilon(x)}[u(y)-u(x)]\,dS - \int_{\partial B_\epsilon(x)} K(\epsilon)\frac{\partial u(y)}{\partial n}\,dS.$$

[9] The last two integrals tend to 0 as $\epsilon\to 0$:

$$\left|-\frac{1}{\omega_n\epsilon^{n-1}}\int_{\partial B_\epsilon(x)}[u(y)-u(x)]\,dS - \int_{\partial B_\epsilon(x)} K(\epsilon)\frac{\partial u(y)}{\partial n}\,dS\right|$$

$$\leq \underbrace{\frac{1}{\omega_n\epsilon^{n-1}}\max_{y\in\partial B_\epsilon(x)}\big|u(y)-u(x)\big|\omega_n\epsilon^{n-1}}_{\to 0,\ (u\ continuous\ in\ \overline{\Omega})} + \underbrace{\big|K(\epsilon)\big|\max_{y\in\overline{\Omega}}\big|\nabla u(y)\big|\omega_n\epsilon^{n-1}}_{\to 0,\ (|\nabla u|\ is\ finite)}.$$

□

8.10 Green's Function and the Poisson Kernel

With a slight change in notation, the Representation Theorem has the following special case.

Theorem. *If $\Omega\in\mathbb{R}^n$ is bounded, $u\in C^2(\Omega)\bigcap C^1(\overline{\Omega})$ is **harmonic**, and $\xi\in\Omega$, then*

$$u(\xi) \;=\; \int_{\partial\Omega}\left[u(x)\frac{\partial K(x-\xi)}{\partial n}-K(x-\xi)\frac{\partial u(x)}{\partial n}\right]dS. \tag{8.16}$$

Let $\omega(x)$ be any **harmonic** function in Ω, and for $x,\ \xi\in\Omega$ consider

$$G(x,\xi)=K(x-\xi)+\omega(x).$$

If we use the Green's identity (with $\triangle u=0$ and $\triangle\omega=0$), we get:

$$0 \;=\; \int_{\partial\Omega}\Big(u\frac{\partial\omega}{\partial n}-\omega\frac{\partial u}{\partial n}\Big)ds. \tag{8.17}$$

Adding (8.16) and (8.17), we obtain:

$$u(\xi) \;=\; \int_{\partial\Omega}\left[u(x)\frac{\partial G(x,\xi)}{\partial n}-G(x,\xi)\frac{\partial u(x)}{\partial n}\right]dS. \tag{8.18}$$

Suppose that for each $\xi\in\Omega$ we can find a function $\omega_\xi(x)$ that is harmonic in Ω and satisfies $\omega_\xi(x)=-K(x-\xi)$ for all $x\in\partial\Omega$. Then $G(x,\xi)=K(x-\xi)+\omega_\xi(x)$ is a fundamental solution such that

$$G(x,\xi)=0 \qquad x\in\partial\Omega.$$

[9] Note that for points y on $\partial B_\epsilon(x)$,

$$K(x-y) \;=\; K(\epsilon) \;=\; \begin{cases}\frac{1}{2\pi}\log\epsilon & \text{if } n=2\\ \frac{1}{(2-n)\omega_n}\epsilon^{2-n} & \text{if } n\geq 3.\end{cases}$$

$$\frac{\partial K(x-y)}{\partial n} \;=\; -\frac{\partial K(\epsilon)}{\partial r} \;=\; -\begin{cases}\frac{1}{2\pi\epsilon} & \text{if } n=2\\ \frac{1}{\omega_n\epsilon^{n-1}} & \text{if } n\geq 3,\end{cases} \;=\; -\frac{1}{\omega_n\epsilon^{n-1}}, \quad \text{(since } n \text{ points inwards.)}$$

G is called the Green's function and is useful in satisfying Dirichlet boundary conditions. The Green's function is difficult to construct for a general domain Ω since it requires solving the Dirichlet problem $\triangle\omega_\xi = 0$ in Ω, $\omega_\xi(x) = -K(x-\xi)$ for $x \in \partial\Omega$, for each $\xi \in \Omega$.

From (8.18) we find [10]

$$\boxed{u(\xi) \;=\; \int_{\partial\Omega} u(x)\frac{\partial G(x,\xi)}{\partial n}\,dS.}$$

Thus *if* we know that the Dirichlet problem has a solution $u \in C^2(\overline{\Omega})$, then we can calculate u from the Poisson integral formula (provided of course that we can compute $G(x,\xi)$).

---

[10] If we did not assume $\triangle u = 0$ in our derivation, we would have (8.13) instead of (8.16), and an extra term in (8.17), which would give us a more general expression:

$$\boxed{u(\xi) \;=\; \int_{\Omega} G(x,\xi)\,\triangle u\,dx \;+\; \int_{\partial\Omega} u(x)\frac{\partial G(x,\xi)}{\partial n}\,dS.}$$

8.11 Properties of Harmonic Functions

Liouville's Theorem. *A bounded harmonic function defined on all of* $\mathbb{R}^n$ *must be a constant.*

8.12 Eigenvalues of the Laplacian

Consider the equation

$$\begin{cases} \triangle u + \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.19}$$

where Ω is a bounded domain and λ is a (complex) number. The values of λ for which (8.19) admits a nontrivial solution u are called the **eigenvalues** of $\triangle$ in Ω, and the solution u is an **eigenfunction associated to the eigenvalue** λ. (The convention $\triangle u + \lambda u = 0$ is chosen so that all eigenvalues λ will be positive.)

Properties of the Eigenvalues and Eigenfunctions for (8.19):
1. The eigenvalues of (8.19) form a countable set $\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers with $\lambda_n \to \infty$ as $n \to \infty$.
2. For each eigenvalue λ_n there is a finite number (called the *multiplicity* of λ_n) of linearly independent eigenfunctions u_n.
3. The first (or *principal*) eigenvalue, λ_1, is simple and u_1 does not change sign in Ω.
4. Eigenfunctions corresponding to distinct eigenvalues are orthogonal.
5. The eigenfunctions may be used to expand certain functions on Ω in an infinite series.

9 Heat Equation

The *heat equation* is

$$u_t = k\triangle u \qquad \text{for } x \in \Omega,\ t > 0, \tag{9.1}$$

with initial and boundary conditions.

9.1 The Pure Initial Value Problem

9.1.1 Fourier Transform

If $u \in C_0^\infty(\mathbb{R}^n)$, define its **Fourier transform** $\widehat{u}$ by

$$\widehat{u}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx \qquad \text{for } \ \xi \in \mathbb{R}^n.$$

We can differentiate $\hat{u}$:

$$\frac{\partial}{\partial \xi_j}\widehat{u}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}(-ix_j) u(x)\, dx = \big[\widehat{(-ix_j)\, u}\big](\xi).$$

Iterating this computation, we obtain

$$\left(\frac{\partial}{\partial \xi_j}\right)^k \widehat{u}(\xi) = \big[\widehat{(-ix_j)^k\, u}\big](\xi). \tag{9.2}$$

Similarly, integrating by parts shows

$$\begin{aligned}
\widehat{\left(\frac{\partial u}{\partial x_j}\right)}(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \frac{\partial u}{\partial x_j}(x)\, dx = -\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j}(e^{-ix\cdot\xi}) u(x)\, dx \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} (i\xi_j) e^{-ix\cdot\xi} u(x)\, dx \\
&= (i\xi_j)\widehat{u}(\xi).
\end{aligned}$$

Iterating this computation, we obtain

$$\widehat{\left(\frac{\partial^k u}{\partial x_j^k}\right)}(\xi) = (i\xi_j)^k \widehat{u}(\xi). \tag{9.3}$$

Formulas (9.2) and (9.3) express the fact that *Fourier transform interchanges differentiation and multiplication by the coordinate function.*

9.1.2 Multi-Index Notation

A multi-index is a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ where each α_i is a nonnegative integer. The order of the multi-index is $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Given a multi-index α, define

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u.$$

We can generalize (9.3) in multi-index notation:

$$\begin{aligned}
\widehat{D^\alpha u}(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} D^\alpha u(x)\, dx = \frac{(-1)^{|\alpha|}}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} D_x^\alpha(e^{-ix\cdot\xi}) u(x)\, dx \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} (i\xi)^\alpha e^{-ix\cdot\xi} u(x)\, dx \\
&= (i\xi)^\alpha \widehat{u}(\xi). \\
(i\xi)^\alpha &= (i\xi_1)^{\alpha_1} \cdots (i\xi_n)^{\alpha_n}.
\end{aligned}$$

Parseval's theorem (Plancherel's theorem).
Assume $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. *Then* $\widehat{u}, u^\vee \in L^2(\mathbb{R}^n)$ *and*

$$||\widehat{u}||_{L^2(\mathbb{R}^n)} = ||u^\vee||_{L^2(\mathbb{R}^n)} = ||u||_{L^2(\mathbb{R}^n)}, \qquad or$$

$$\int_{-\infty}^{\infty} |u(x)|^2 \, dx = \int_{-\infty}^{\infty} |\widehat{u}(\xi)|^2 \, d\xi.$$

Also,

$$\int_{-\infty}^{\infty} u(x) \, \overline{v(x)} \, dx = \int_{-\infty}^{\infty} \widehat{u}(\xi) \, \overline{\widehat{v}(\xi)} \, d\xi.$$

The properties (9.2) and (9.3) make it very natural to consider the fourier transform on a subspace of $L^1(\mathbb{R}^n)$ called the *Schwartz class of functions*, S, which consists of the smooth functions whose derivatives of all orders decay faster than any polynomial, i.e.

$$S = \{u \in C^\infty(\mathbb{R}^n) : \text{ for every } k \in \mathbb{N} \text{ and } \alpha \in \mathbb{N}^n, \ |x|^k |D^\alpha u(x)| \text{ is bounded on } \mathbb{R}^n\}.$$

For $u \in S$, the Fourier transform $\widehat{u}$ exists since u decays rapidly at ∞.

Lemma. *(i) If* $u \in L^1(\mathbb{R}^n)$, *then* $\widehat{u}$ *is bounded. (ii) If* $u \in S$, *then* $\widehat{u} \in S$.

Define the **inverse Fourier transform** for $u \in L^1(\mathbb{R}^n)$:

$$u^\vee(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \, u(x) \, dx \qquad \text{for } \xi \in \mathbb{R}^n, \qquad \text{or}$$

$$u(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \, \widehat{u}(\xi) \, d\xi \qquad \text{for } x \in \mathbb{R}^n.$$

Fourier Inversion Theorem (McOwen). *If* $u \in S$, *then* $(\widehat{u})^\vee = u$; *that is,*

$$\boxed{u(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \, \widehat{u}(\xi) \, d\xi = \frac{1}{(2\pi)^n} \int \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} \, u(y) \, dy \, d\xi = (\widehat{u})^\vee(x).}$$

Fourier Inversion Theorem (Evans). *Assume* $u \in L^2(\mathbb{R}^n)$. *Then,* $u = (\widehat{u})^\vee$.

Shift: Let $u(\underbrace{x-a}_{y}) = v(x)$, and determinte $\widehat{v}(\xi)$:

$$\begin{aligned}
\widehat{u(x-a)}(\xi) = \widehat{v}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} v(x)\,dx \;=\; \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i(y+a)\xi} u(y)\,dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iy\xi} e^{-ia\xi} u(y)\,dy \;=\; e^{-ia\xi}\widehat{u}(\xi).
\end{aligned}$$

$$\boxed{\widehat{u(x-a)}(\xi) \;=\; e^{-ia\xi}\widehat{u}(\xi).}$$

Delta function:

$$\begin{aligned}
\widehat{\delta(x)}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi}\,\delta(x)\,dx \;=\; \frac{1}{\sqrt{2\pi}}, && \Big(\text{since } u(x) = \int_{\mathbb{R}} \delta(x-y)\,u(y)\,dy\Big). \\
\widehat{\delta(x-a)}(\xi) &= e^{-ia\xi}\,\widehat{\delta}(\xi) \;=\; \frac{1}{\sqrt{2\pi}}\,e^{-ia\xi}. && (\text{using result from ‘Shift’})
\end{aligned}$$

Convolution:

$$\begin{aligned}
(f*g)(x) &= \int_{\mathbb{R}^n} f(x-y)g(y)\,dy, \\
\widehat{(f*g)}(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \int_{\mathbb{R}^n} f(x-y)\,g(y)\,dy\,dx \;=\; \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x-y)\,g(y)\,dy\,dx \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big[e^{-i(x-y)\cdot\xi} f(x-y)\,dx\Big]\Big[e^{-iy\cdot\xi} g(y)\,dy\Big] \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-iz\cdot\xi} f(z)\,dz \cdot \int_{\mathbb{R}^n} e^{-iy\cdot\xi} g(y)\,dy \;=\; (2\pi)^{\frac{n}{2}}\,\widehat{f}(\xi)\widehat{g}(\xi).
\end{aligned}$$

$$\boxed{\widehat{(f*g)}(\xi) \;=\; (2\pi)^{\frac{n}{2}}\,\widehat{f}(\xi)\,\widehat{g}(\xi).}$$

Gaussian: (completing the square)

$$\begin{aligned}
\widehat{\big(e^{-\frac{x^2}{2}}\big)}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi}\,e^{-\frac{x^2}{2}}\,dx \;=\; \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{x^2+2ix\xi}{2}}\,dx \;=\; \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{x^2+2ix\xi-\xi^2}{2}}\,dx\;e^{-\frac{\xi^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{(x+i\xi)^2}{2}}\,dx\;e^{-\frac{\xi^2}{2}} \;=\; \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\frac{-y^2}{2}}\,dy\;e^{-\frac{\xi^2}{2}} = \frac{1}{\sqrt{2\pi}}\sqrt{2\pi}e^{-\frac{\xi^2}{2}} \;=\; e^{-\frac{\xi^2}{2}}.
\end{aligned}$$

$$\boxed{\widehat{\big(e^{-\frac{x^2}{2}}\big)}(\xi) \;=\; e^{-\frac{\xi^2}{2}}.}$$

Multiplication by x:

$$\widehat{-ixu}(\xi) \;=\; \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi}\big(-ixu(x)\big)\,dx \;=\; \frac{d}{d\xi}\widehat{u}(\xi).$$

$$\boxed{\widehat{xu(x)}(\xi) \;=\; i\frac{d}{d\xi}\widehat{u}(\xi).}$$

Multiplication of $\mathbf{u_x}$ by x: (using the above result)

$$\widehat{xu_x(x)}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\big(xu_x(x)\big)\,dx = \underbrace{\frac{1}{\sqrt{2\pi}}\Big[e^{-ix\xi}xu\Big]_{-\infty}^{\infty}}_{=0} - \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\Big((-i\xi)e^{-ix\xi}x + e^{-ix\xi}\Big)\,u\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\,i\xi\int_{\mathbb{R}} e^{-ix\xi}\,x\,u\,dx - \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\,u\,dx$$

$$= i\xi\,\widehat{xu(x)}(\xi) - \widehat{u}(\xi) = i\xi\Big[i\frac{d}{d\xi}\widehat{u}(\xi)\Big] - \widehat{u}(\xi) = -\xi\frac{d}{d\xi}\widehat{u}(\xi) - \widehat{u}(\xi).$$

$$\boxed{\widehat{xu_x(x)}(\xi) = -\xi\frac{d}{d\xi}\widehat{u}(\xi) - \widehat{u}(\xi).}$$

Table of Fourier Transforms: [11]

$$\widehat{\big(e^{-\frac{ax^2}{2}}\big)}(\xi) = \frac{1}{\sqrt{a}}\,e^{-\frac{\xi^2}{2a}}, \qquad \text{(Gaussian)}$$

$$\widehat{e^{ibx}f(ax)}(\xi) = \frac{1}{a}\widehat{f}\Big(\frac{\xi-b}{a}\Big),$$

$$f(x) = \begin{cases} 1, & |x| \le L \\ 0, & |x| > L, \end{cases} \qquad \widehat{f(x)}(\xi) = \frac{1}{\sqrt{2\pi}}\frac{2\sin(\xi L)}{\xi},$$

$$\widehat{e^{-a|x|}}(\xi) = \frac{1}{\sqrt{2\pi}}\frac{2a}{a^2+\xi^2}, \qquad (a>0)$$

$$\widehat{\frac{1}{a^2+x^2}}(\xi) = \frac{\sqrt{2\pi}}{2a}\,e^{-a|\xi|}, \qquad (a>0)$$

$$\widehat{H(a-|x|)}(\xi) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\xi}\,\sin a\xi, \qquad \circledast$$

$$\widehat{H(x)}(\xi) = \frac{1}{\sqrt{2\pi}}\Big(\pi\delta(\xi) + \frac{1}{i\xi}\Big), \qquad \circledast$$

$$\widehat{\big(H(x)-H(-x)\big)}(\xi) = \sqrt{\frac{2}{\pi}}\,\frac{1}{i\xi}, \qquad \text{(sign)} \qquad \circledast$$

$$\widehat{1}(\xi) = \sqrt{2\pi}\delta(\xi). \qquad \circledast$$

[11] Results with marked with $\circledast$ were taken from W. Strauss, where the definition of Fourier Transform is different. An extra multiple of $\frac{1}{\sqrt{2\pi}}$ was added to each of these results.

9.1.3 Solution of the Pure Initial Value Problem

Consider the pure initial value problem

$$\begin{cases} u_t = \triangle u & \text{for } t > 0,\ x \in \mathbb{R}^n \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}^n. \end{cases} \tag{9.4}$$

We take the Fourier transform of the heat equation in the x-variables.

$$\begin{aligned} \widehat{(u_t)}(\xi,t) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u_t(x,t)\,dx = \frac{\partial}{\partial t}\widehat{u}(\xi,t) \\ \widehat{\triangle u}(\xi,t) &= \sum_{j=1}^{n} (i\xi_j)^2 \widehat{u}(\xi,t) = -|\xi|^2 \widehat{u}(\xi,t). \end{aligned}$$

The heat equation therefore becomes

$$\frac{\partial}{\partial t}\widehat{u}(\xi,t) = -|\xi|^2 \widehat{u}(\xi,t),$$

which is an ordinary differential equation in t, with the solution $\widehat{u}(\xi,t) = Ce^{-|\xi|^2 t}$. The initial condition $\widehat{u}(\xi,0) = \widehat{g}(\xi)$ gives

$$\begin{aligned} \widehat{u}(\xi,t) &= \widehat{g}(\xi)\, e^{-|\xi|^2 t}, \\ u(x,t) &= \left(\widehat{g}(\xi)\, e^{-|\xi|^2 t}\right)^{\vee} = \frac{1}{(2\pi)^{\frac{n}{2}}} \left[g * \left(e^{-|\xi|^2 t}\right)^{\vee} \right] \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\, g * \left[\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-|\xi|^2 t} e^{ix\cdot\xi}\, d\xi \right] \\ &= \frac{1}{(4\pi^2)^{\frac{n}{2}}}\, g * \left[\int_{\mathbb{R}^n} e^{ix\cdot\xi - |\xi|^2 t}\, d\xi \right] = \frac{1}{(4\pi^2)^{\frac{n}{2}}}\, g * \left[e^{-\frac{|x|^2}{4t}} \left(\frac{\pi}{t}\right)^{\frac{n}{2}} \right] \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}}\, g * \left[e^{-\frac{|x|^2}{4t}} \right] = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy. \end{aligned}$$

Thus, [12] solution of the initial value problem (9.4) is

$$\boxed{u(x,t) = \int_{\mathbb{R}^n} K(x,y,t)\, g(y)\, dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy.}$$

Uniqueness of solutions for the pure initial value problem *fails*: there are nontrivial solutions of (9.4) with $g = 0$. [13] Thus, the pure initial value problem for the heat equation is *not* well-posed, as it was for the wave equation. However, the nontrivial solutions are unbounded as functions of x when $t > 0$ is fixed; uniqueness can be regained by adding a boundedness condition on the solution.

[12]Identity (Evans, p. 187.) :

$$\int_{\mathbb{R}^n} e^{ix\cdot\xi - |\xi|^2 t}\, d\xi = e^{-\frac{|x|^2}{4t}} \left(\frac{\pi}{t}\right)^{\frac{n}{2}}.$$

[13]The following function u satisfies $u_t = u_{xx}$ for $t > 0$ with $u(x,0) = 0$:

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} \frac{d^k}{dt^k} e^{-1/t^2}.$$

9.1.4 Nonhomogeneous Equation

Consider the pure initial value problem with homogeneous initial condition:

$$\begin{cases} u_t = \triangle u + f(x,t) & \text{for } \ t > 0, \ x \in \mathbb{R}^n \\ u(x,0) = 0 & \text{for } \ x \in \mathbb{R}^n. \end{cases} \tag{9.5}$$

Duhamel's principle gives the solution:

$$\boxed{u(x,t) = \int_0^t \int_{\mathbb{R}^n} \tilde{K}(x-y, t-s)\, f(y,s)\, dy\, ds.}$$

9.1.5 Nonhomogeneous Equation with Nonhomogeneous Initial Conditions

Combining two solutions above, we find that the solution of the initial value problem

$$\begin{cases} u_t = \triangle u + f(x,t) & \text{for } \ t > 0, \ x \in \mathbb{R}^n \\ u(x,0) = g(x) & \text{for } \ x \in \mathbb{R}^n. \end{cases} \tag{9.6}$$

is given by

$$\boxed{u(x,t) = \int_{\mathbb{R}^n} \tilde{K}(x-y,t)\, g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \tilde{K}(x-y, t-s)\, f(y,s)\, dy\, ds.}$$

9.1.6 The Fundamental Solution

Suppose we want to solve the Cauchy problem

$$\begin{cases} u_t = Lu & x \in \mathbb{R}^n, \ t > 0 \\ u(x,0) = g(x) & x \in \mathbb{R}^n. \end{cases} \tag{9.7}$$

where L is a differential operator in $\mathbb{R}^n$ with constant coefficients. Suppose $K(x,t)$ is a distribution in $\mathbb{R}^n$ for each value of $t \geq 0$, K is C^1 in t and satisfies

$$\begin{cases} K_t - LK = 0, \\ K(x,0) = \delta(x). \end{cases} \tag{9.8}$$

We call K a **fundamental solution** for the initial value problem. The solution of (9.7) is then given by convolution in the space variables:

$$u(x,t) \ = \ \int_{\mathbb{R}^n} K(x-y,t)\, g(y)\, dy.$$

For operators of the form $\partial_t - L$, the fundamental solution of the initial value problem, $K(x,t)$ as defined in (9.8), coincides with the "free space" fundamental solution, which satisfies

$$\Big(\partial_t - L\Big) K(x,t) \;=\; \delta(x,t),$$

provided we extend $K(x,t)$ by zero to $t<0$. For the heat equation, consider

$$\tilde{K}(x,t) \;=\; \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & t>0 \\ 0 & t\le 0. \end{cases} \tag{9.9}$$

Notice that $\tilde{K}$ is smooth for $(x,t) \ne (0,0)$.

$\tilde{K}$ defined as in (9.9), is the fundamental solution of the "free space" heat equation.

Proof. We need to show:

$$\Big(\partial_t - \triangle\Big) K(x,t) \;=\; \delta(x,t). \tag{9.10}$$

To verify (9.10) as distributions, we must show that for any $v \in C_0^\infty(\mathbb{R}^{n+1})$: [14]

$$\int_{\mathbb{R}^{n+1}} \tilde{K}(x,t)\Big(-\partial_t - \triangle\Big)\, v\, dx\, dt \;=\; \int_{\mathbb{R}^{n+1}} \delta(x,t)\, v(x,t)\, dx\, dt \;\equiv\; v(0,0).$$

To do this, let us take $\epsilon > 0$ and define

$$\tilde{K}_\epsilon(x,t) \;=\; \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & t>\epsilon \\ 0 & t\le \epsilon. \end{cases}$$

Then $\tilde{K}_\epsilon \to \tilde{K}$ as distributions, so it suffices to show that $(\partial_t - \triangle)\tilde{K}_\epsilon \to \delta$ as distributions. Now

$$\begin{aligned}
\int \tilde{K}_\epsilon\Big(-\partial_t - \triangle\Big)\, v\, dx\, dt \;&=\; \int_\epsilon^\infty \bigg(\int_{\mathbb{R}^n} \tilde{K}(x,t)\Big(-\partial_t - \triangle\Big)\, v(x,t)\, dx\bigg)\, dt \\
&= \; -\int_\epsilon^\infty \bigg(\int_{\mathbb{R}^n} \tilde{K}(x,t)\; \partial_t v(x,t)\, dx\bigg)\, dt \;-\; \int_\epsilon^\infty \bigg(\int_{\mathbb{R}^n} \tilde{K}(x,t)\; \triangle v(x,t)\, dx\bigg)\, dt \\
&= \; -\bigg[\int_{\mathbb{R}^n} \tilde{K}(x,t)\, v(x,t)\, dx\bigg]_{t=\epsilon}^{t=\infty} + \int_\epsilon^\infty \bigg(\int_{\mathbb{R}^n} \partial_t\tilde{K}(x,t)\; v(x,t)\, dx\bigg)\, dt \;-\; \int_\epsilon^\infty \bigg(\int_{\mathbb{R}^n} \triangle\tilde{K}(x,t)\; v(x,t)\, dx\bigg)\, dt \\
&= \; \int_\epsilon^\infty \bigg(\int_{\mathbb{R}^n} \Big(\partial_t - \triangle\Big)\tilde{K}(x,t)\, v(x,t)\, dx\bigg)\, dt \;+\; \int_{\mathbb{R}^n} \tilde{K}(x,\epsilon)\, v(x,\epsilon)\, dx.
\end{aligned}$$

But for $t > \epsilon$, $(\partial_t - \triangle)\tilde{K}(x,t) = 0$; moreover, since $\lim_{t\to 0^+} \tilde{K}(x,t) = \delta_0(x) = \delta(x)$, we have $\tilde{K}(x,\epsilon) \to \delta_0(x)$ as $\epsilon \to 0$, so the last integral tends to $v(0,0)$. ∎

[14]Note, for the operator $L = \partial/\partial t$, the **adjoint operator** is $L^* = -\partial/\partial t$.

10 Schrödinger Equation

Problem (F'96, #5). *The Gauss kernel*

$$G(t,x,y) = \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{(x-y)^2}{4t}}$$

is the ***fundamental solution*** *of the* ***heat equation****, solving*

$$G_t = G_{xx}, \qquad G(0,x,y) = \delta(x-y).$$

By analogy with the heat equation, find the fundamental solution $H(t,x,y)$ of the ***Schrödinger equation***

$$H_t = iH_{xx}, \qquad H(0,x,y) = \delta(x-y).$$

Show that your expression $H(x)$ is indeed the fundamental solution for the Schrödinger equation. You may use the following special integral

$$\int_{-\infty}^{\infty} e^{\frac{-ix^2}{4}}\, dx = \sqrt{-i4\pi}.$$

Proof. • **Remark:** Consider the initial value problem for the Schrödinger equation

$$\begin{cases} u_t = i\triangle u & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = g(x) & x \in \mathbb{R}^n. \end{cases}$$

If we formally replace t by it in the heat kernel, we obtain the **Fundamental Solution of the Schrödinger Equation**: [15]

$$\begin{aligned} H(x,t) &= \frac{1}{(4\pi i t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4it}} \qquad (x \in \mathbb{R}^n,\ t \neq 0) \\ u(x,t) &= \frac{1}{(4\pi i t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4it}} g(y)\, dy. \end{aligned}$$

In particular, the Schrödinger equation is *reversible in time*, whereas the heat equation is not.

• **Solution:** We have already found the fundamental solution for the heat equation using the Fourier transform. For the Schrödinger equation is one dimension, we have

$$\frac{\partial}{\partial t}\widehat{u}(\xi,t) = -i\xi^2 \widehat{u}(\xi,t),$$

which is an ordinary differential equation in t, with the solution $\widehat{u}(\xi,t) = Ce^{-i\xi^2 t}$. The initial condition $\widehat{u}(\xi,0) = \widehat{g}(\xi)$ gives

$$\begin{aligned} \widehat{u}(\xi,t) &= \widehat{g}(\xi)\, e^{-i\xi^2 t}, \\ u(x,t) &= \left(\widehat{g}(\xi)\, e^{-i\xi^2 t}\right)^{\vee} = \frac{1}{\sqrt{2\pi}}\left[g * \left(e^{-i\xi^2 t}\right)^{\vee}\right] \\ &= \frac{1}{\sqrt{2\pi}}\, g * \left[\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi^2 t} e^{ix\cdot\xi}\, d\xi\right] \\ &= \frac{1}{2\pi}\, g * \left[\int_{\mathbb{R}} e^{ix\cdot\xi - i\xi^2 t}\, d\xi\right] \qquad = \text{(need some work)} = \\ &= \frac{1}{\sqrt{4\pi i t}}\, g * \left[e^{-\frac{|x|^2}{4it}}\right] = \frac{1}{\sqrt{4\pi i t}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4it}} g(y)\, dy. \end{aligned}$$

[15] Evans, p. 188, Example 3.

- For the Schr**ö**dinger equation, consider

$$\tilde{\Psi}(x,t) \;=\; \begin{cases} \frac{1}{(4\pi i t)^{n/2}} e^{-\frac{|x|^2}{4it}} & t>0 \\ 0 & t \le 0. \end{cases} \tag{10.1}$$

Notice that $\tilde{\Psi}$ is smooth for $(x,t) \neq (0,0)$.

$\tilde{\Psi}$ defined as in (10.1), is the fundamental solution of the Schrödinger equation. We need to show:

$$\Big(\partial_t - i\triangle\Big)\Psi(x,t) \;=\; \delta(x,t). \tag{10.2}$$

To verify (10.2) as distributions, we must show that for any $v \in C_0^\infty(\mathbb{R}^{n+1})$: [16]

$$\int_{\mathbb{R}^{n+1}} \tilde{\Psi}(x,t)\Big(-\partial_t - i\triangle\Big)\, v\, dx\, dt \;=\; \int_{\mathbb{R}^{n+1}} \delta(x,t)\, v(x,t)\, dx\, dt \;\equiv\; v(0,0).$$

To do this, let us take $\epsilon > 0$ and define

$$\tilde{\Psi}_\epsilon(x,t) \;=\; \begin{cases} \frac{1}{(4\pi i t)^{n/2}} e^{-\frac{|x|^2}{4it}} & t>\epsilon \\ 0 & t \le \epsilon. \end{cases}$$

Then $\tilde{\Psi}_\epsilon \to \tilde{\Psi}$ as distributions, so it suffices to show that $(\partial_t - i\triangle)\tilde{\Psi}_\epsilon \to \delta$ as distributions. Now

$$\begin{aligned}\int \tilde{\Psi}_\epsilon\Big(-\partial_t - i\triangle\Big)\, v\, dx\, dt \;&=\; \int_\epsilon^\infty \bigg(\int_{\mathbb{R}^n} \tilde{\Psi}(x,t)\Big(-\partial_t - i\triangle\Big)\, v(x,t)\, dx\bigg)\, dt \\ &=\; \int_\epsilon^\infty \bigg(\int_{\mathbb{R}^n} \Big(\partial_t - i\triangle\Big)\tilde{\Psi}(x,t)\, v(x,t)\, dx\bigg)\, dt \;+\; \int_{\mathbb{R}^n} \tilde{\Psi}(x,\epsilon)\, v(x,\epsilon)\, dx.\end{aligned}$$

But for $t > \epsilon$, $(\partial_t - i\triangle)\tilde{\Psi}(x,t) = 0$; moreover, since $\lim_{t\to 0^+} \tilde{\Psi}(x,t) = \delta_0(x) = \delta(x)$, we have $\tilde{\Psi}(x,\epsilon) \to \delta_0(x)$ as $\epsilon \to 0$, so the last integral tends to $v(0,0)$. □

[16]Note, for the operator $L = \partial/\partial t$, the **adjoint operator** is $L^* = -\partial/\partial t$.

11 Problems: Quasilinear Equations

Problem (F'90, #7). *Use the method of characteristics to find the solution of the first order partial differential equation*

$$x^2 u_x + xy u_y = u^2$$

which passes through the curve $u = 1$, $x = y^2$. Determine where this solution becomes singular.

Proof. We have a condition $u(x = y^2) = 1$. Γ is parametrized by $\Gamma : (s^2, s, 1)$.

$$\frac{dx}{dt} = x^2 \Rightarrow x = \frac{1}{-t - c_1(s)} \Rightarrow x(0,s) = \frac{1}{-c_1(s)} = s^2 \Rightarrow x = \frac{1}{-t + \frac{1}{s^2}} = \frac{s^2}{1 - ts^2},$$

$$\frac{dy}{dt} = xy \Rightarrow \frac{dy}{dt} = \frac{s^2 y}{1 - ts^2} \Rightarrow y = \frac{c_2(s)}{1 - ts^2} \Rightarrow y(s,0) = c_2(s) = s \Rightarrow y = \frac{s}{1 - ts^2},$$

$$\frac{dz}{dt} = z^2 \Rightarrow z = \frac{1}{-t - c_3(s)} \Rightarrow z(0,s) = \frac{1}{-c_3(s)} = 1 \Rightarrow z = \frac{1}{1 - t}.$$

Thus,

$$\frac{x}{y} = s \Rightarrow y = \frac{\frac{x}{y}}{1 - t\frac{x^2}{y^2}} \Rightarrow t = \frac{y^2}{x^2} - \frac{1}{x}.$$

$$\Rightarrow u(x,y) = \frac{1}{1 - \frac{y^2}{x^2} + \frac{1}{x}} = \frac{x^2}{x^2 + x - y^2}.$$

The solution becomes singular when $y^2 = x^2 + x$.
It can be checked that the solution satisfies the PDE and $u(x = y^2) = \frac{y^4}{y^4 + y^2 - y^2} = 1$. ☐

Problem (S'91, #7). *Solve the first order PDE*

$$f_x + x^2 y f_y + f = 0$$
$$f(x = 0, y) = y^2$$

using the method of characteristics.

Proof. Rewrite the equation

$$u_x + x^2 y u_y = -u,$$
$$u(0,y) = y^2.$$

Γ is parameterized by $\Gamma : (0, s, s^2)$.

$$\frac{dx}{dt} = 1 \Rightarrow x = t,$$

$$\frac{dy}{dt} = x^2 y \Rightarrow \frac{dy}{dt} = t^2 y \Rightarrow y = s e^{\frac{t^3}{3}},$$

$$\frac{dz}{dt} = -z \Rightarrow z = s^2 e^{-t}.$$

Thus, $x = t$ and $s = y e^{-\frac{t^3}{3}} = y e^{-\frac{x^3}{3}}$, and

$$u(x,y) = (y e^{-\frac{x^3}{3}})^2 e^{-x} = y^2 e^{-\frac{2}{3}x^3 - x}.$$

The solution satisfies both the PDE and initial conditions. ☐

Problem (S'92, #1). *Consider the Cauchy problem*

$$u_t = xu_x - u + 1 \qquad -\infty < x < \infty,\ t \geq 0$$
$$u(x,0) = \sin x \qquad -\infty < x < \infty$$

and solve it by the method of characteristics. Discuss the properties of the solution; in particular investigate the behavior of $|u_x(\cdot,t)|_\infty$ for $t \to \infty$.

Proof. Γ is parametrized by $\Gamma:\ (s, 0, \sin s)$. We have

$$\frac{dx}{dt} = -x \quad \Rightarrow \quad x = se^{-t},$$
$$\frac{dy}{dt} = 1 \quad \Rightarrow \quad y = t,$$
$$\frac{dz}{dt} = 1 - z \quad \Rightarrow \quad z = 1 - \frac{1-\sin s}{e^t}.$$

Thus, $t = y$, $s = xe^y$, and

$$u(x,y) = 1 - \frac{1}{e^y} + \frac{\sin(xe^y)}{e^y}.$$

It can be checked that the solution satisfies the PDE and the initial condition.
As $t \to \infty$, $u(x,t) \to 1$. Also,

$$|u_x(x,y)|_\infty = |\cos(xe^y)|_\infty = 1.$$

$u_x(x,y)$ oscillate between -1 and 1. If $x = 0$, $u_x = 1$. ☐

Problem (W'02, #6). *Solve the Cauchy problem*

$$u_t + u^2 u_x = 0, \qquad t > 0,$$
$$u(0,x) = 2 + x.$$

Proof. **Solved** ☐

Problem (S'97, #1). *Find the solution of the Burgers' equation*

$$u_t + uu_x = -x, \qquad t \geq 0$$
$$u(x,0) = f(x), \qquad -\infty < x < \infty.$$

Proof. Γ is parameterized by $\Gamma: \ (s, 0, f(s))$.

$$\frac{dx}{dt} = z,$$
$$\frac{dy}{dt} = 1 \quad \Rightarrow \quad y = t,$$
$$\frac{dz}{dt} = -x.$$

Note that we have a coupled system:

$$\begin{cases} \dot{x} = z, \\ \dot{z} = -x, \end{cases}$$

which can be written as a second order ODE:

$$\ddot{x} + x = 0, \qquad x(s,0) = s, \ \ \dot{x}(s,0) = z(0) = f(s).$$

Solving the equation, we get

$$x(s,t) = s\cos t + f(s)\sin t, \qquad \text{and thus,}$$
$$z(s,t) = \dot{x}(t) = -s\sin t + f(s)\cos t.$$

$$\begin{cases} x = s\cos y + f(s)\sin y, \\ u = -s\sin y + f(s)\cos y. \end{cases} \quad \Rightarrow \quad \begin{cases} x\cos y = s\cos^2 y + f(s)\sin y\cos y, \\ u\sin y = -s\sin^2 y + f(s)\cos y\sin y. \end{cases}$$

$$\Rightarrow \quad x\cos y - u\sin y = s(\cos^2 y + \sin^2 y) = s.$$

$$\Rightarrow \quad u(x,y) = f(x\cos y - u\sin y)\cos y - (x\cos y - u\sin y)\sin y.$$ □

Problem (F'98, #2). *Solve the partial differential equation*

$$u_y - u^2 u_x = 3u, \qquad u(x,0) = f(x)$$

using method of characteristics. (Hint: find a parametric representation of the solution.)

Proof. Γ is parameterized by $\Gamma: \ (s, 0, f(s))$.

$$\frac{dx}{dt} = -z^2 \quad \Rightarrow \quad \frac{dx}{dt} = -f^2(s)e^{6t} \quad \Rightarrow \quad x = -\frac{1}{6}f^2(s)e^{6t} + \frac{1}{6}f^2(s) + s,$$
$$\frac{dy}{dt} = 1 \quad \Rightarrow \quad y = t,$$
$$\frac{dz}{dt} = 3z \quad \Rightarrow \quad z = f(s)e^{3t}.$$

Thus,

$$\begin{cases} x = -\frac{1}{6}f^2(s)e^{6y} + \frac{1}{6}f^2(s) + s, \\ f(s) = \frac{z}{e^{3y}} \end{cases} \quad \Rightarrow \quad x = -\frac{1}{6}\frac{z^2}{e^{6y}}e^{6y} + \frac{1}{6}\frac{z^2}{e^{6y}} + s = \frac{z^2}{6e^{6y}} - \frac{z^2}{6} + s,$$

$$\Rightarrow \quad s = x - \frac{z^2}{6e^{6y}} + \frac{z^2}{6}.$$

$$\Rightarrow \quad z = f\Big(x - \frac{z^2}{6e^{6y}} + \frac{z^2}{6}\Big)e^{3y}.$$

$$\Rightarrow \quad u(x,y) = f\Big(x - \frac{u^2}{6e^{6y}} + \frac{u^2}{6}\Big)e^{3y}.$$

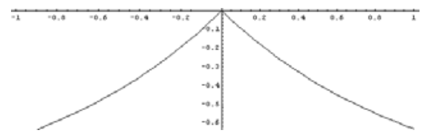
□

Problem (S'99, #1) Modified Problem. ***a)*** *Solve*

$$u_t + \left(\frac{u^3}{3}\right)_x = 0 \tag{11.1}$$

for $t > 0$, $-\infty < x < \infty$ *with initial data*

$$u(x,0) = h(x) = \begin{cases} -a(1-e^x), & x < 0 \\ -a(1-e^{-x}), & x > 0 \end{cases}$$



where $a > 0$ *is constant. Solve until the first appearance of discontinuous derivative and determine that critical time.*
b) *Consider the equation*

$$u_t + \left(\frac{u^3}{3}\right)_x = -cu. \tag{11.2}$$

How large does the constant $c > 0$ *has to be, so that a smooth solution (with no discontinuities) exists for all* $t > 0$*? Explain.*

Proof. **a)** Characteristic form: $u_t + u^2 u_x = 0$. $\Gamma: \ (s, 0, h(s))$.

$$\frac{dx}{dt} = z^2, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = 0.$$

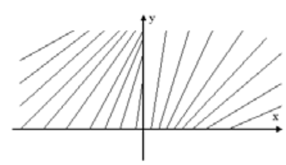
$$x = h(s)^2 t + s, \quad y = t, \quad z = h(s).$$

$$u(x,y) = h(x - u^2 y) \tag{11.3}$$

The characteristic projection in the xt-plane[17] passing through the point $(s, 0)$ is the line

$$x = h(s)^2 t + s$$

along which u has the constant value $u = h(s)$.



The derivative of the initial data is discontinuous, and that leads to a rarefaction-like behavior at $t = 0$. However, if the question meant to ask to determine the first time when a shock forms, we proceed as follows.
Two characteristics $x = h(s_1)^2 t + s_1$ and $x = h(s_2)^2 t + s_2$ intersect at a point (x, t) with

$$t = -\frac{s_2 - s_1}{h(s_2)^2 - h(s_1)^2}.$$

From (11.3), we have

$$u_x = h'(s)(1 - 2uu_x t) \quad \Rightarrow \quad u_x = \frac{h'(s)}{1 + 2h(s)h'(s)t}$$

Hence for $2h(s)h'(s) < 0$, u_x becomes infinite at the positive time

$$t = \frac{-1}{2h(s)h'(s)}.$$

The smallest t for which this happens corresponds to the value $s = s_0$ at which $h(s)h'(s)$ has a minimum (i.e.$-h(s)h'(s)$ has a maximum). At time $T = -1/(2h(s_0)h'(s_0))$ the

[17] y and t are interchanged here

solution u experiences a "gradient catastrophe".
Therefore, need to find a minimum of

$$f(x) = 2h(x)h'(x) = \begin{cases} -2a(1-e^x)\cdot ae^x \\ -2a(1-e^{-x})\cdot(-ae^{-x}) \end{cases} = \begin{cases} -2a^2e^x(1-e^x), & x<0 \\ 2a^2e^{-x}(1-e^{-x}), & x>0 \end{cases}$$

$$f'(x) = \begin{cases} -2a^2e^x(1-2e^x), & x<0 \\ -2a^2e^{-x}(1-2e^{-x}), & x>0 \end{cases} = 0 \quad\Rightarrow\quad \begin{cases} x=\ln(\frac{1}{2}) = -\ln(2), & x<0 \\ x=\ln(2), & x>0 \end{cases}$$

$$\Rightarrow\quad \begin{cases} f(\ln(\frac{1}{2})) = -2a^2e^{\ln(\frac{1}{2})}(1-e^{\ln(\frac{1}{2})}) = -2a^2(\frac{1}{2})(\frac{1}{2}) = \frac{-a^2}{2}, & x<0 \\ f(\ln(2)) = 2a^2(\frac{1}{2})(1-\frac{1}{2}) = \frac{a^2}{2}, & x>0 \end{cases}$$

$$\Rightarrow\quad t = -\frac{1}{\min\{2h(s)h'(s)\}} = \frac{2}{a^2}$$

□

Proof. **b)** Characteristic form: $u_t + u^2u_x = -cu$. $\quad\Gamma:\ (s,0,h(s))$.

$$\begin{aligned} \frac{dx}{dt} &= z^2 = h(s)^2e^{-2ct} \quad\Rightarrow\quad x = s + \frac{1}{2c}h(s)^2(1-e^{-2ct}), \\ \frac{dy}{dt} &= 1 \quad\Rightarrow\quad y = t, \\ \frac{dz}{dt} &= -cz \quad\Rightarrow\quad z = h(s)e^{-ct} \qquad (\Rightarrow\ h(s) = ue^{cy}). \end{aligned}$$

Solving for s and t in terms of x and y, we get:

$$t = y, \qquad s = x - \frac{1}{2c}h(s)^2(1-e^{-2cy}).$$

Thus,

$$\begin{aligned} u(x,y) &= h\big(x - \frac{1}{2c}u^2e^{2cy}(1-e^{-2cy})\big)\cdot e^{-cy}. \\ u_x &= h'(s)e^{-cy}\cdot(1 - \frac{1}{c}uu_xe^{2cy}(1-e^{-2cy})), \\ u_x &= \frac{h'(s)e^{-cy}}{1+\frac{1}{c}h'(s)e^{cy}u\cdot(1-e^{-2cy})} = \frac{h'(s)e^{-cy}}{1+\frac{1}{c}h'(s)h(s)(1-e^{-2cy})}. \end{aligned}$$

Thus, $c>0$ that would allow a smooth solution to exist for all $t>0$ should satisfy

$$1 + \frac{1}{c}h'(s)h(s)(1-e^{-2cy}) \neq 0.$$

We can perform further calculations taking into account the result from part (a):

$$\min\{2h(s)h'(s)\} = -\frac{a^2}{2}.$$

□

Problem (S'99, #1). Original Problem. a). *Solve*

$$u_t + \frac{u_x^3}{3} = 0 \tag{11.4}$$

for $t > 0$, $-\infty < x < \infty$ *with initial data*

$$u(x,0) = h(x) = \begin{cases} -a(1-e^x), & x < 0 \\ -a(1-e^{-x}), & x > 0 \end{cases}$$

where $a > 0$ *is constant.*

Proof. Rewrite the equation as

$$\begin{aligned} F(x,y,u,u_x,u_y) &= \frac{u_x^3}{3} + u_y = 0, \\ F(x,y,z,p,q) &= \frac{p^3}{3} + q = 0. \end{aligned}$$

Γ is parameterized by $\Gamma: \ (s, 0, h(s), \phi(s), \psi(s))$.
We need to complete Γ to a strip. Find $\phi(s)$ and $\psi(s)$, the initial conditions for $p(s,t)$ and $q(s,t)$, respectively:

- $$\begin{aligned} F(f(s), g(s), h(s), \phi(s), \psi(s)) &= 0, \\ \frac{\phi(s)^3}{3} + \psi(s) = 0, & \\ \psi(s) = -\frac{\phi(s)^3}{3}. & \end{aligned}$$

- $$h'(s) = \phi(s)f'(s) + \psi(s)g'(s)$$
$$\begin{cases} ae^s = \phi(s), & x < 0 \\ -ae^{-s} = \phi(s), & x > 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \psi(s) = -\frac{a^3e^{3s}}{3}, & x < 0 \\ \psi(s) = \frac{a^3e^{-3s}}{3}, & x > 0 \end{cases}$$

Therefore, now Γ is parametrized by

$$\begin{cases} \Gamma: \ (s, 0, \ -a(1-e^s), \quad ae^s, \ -\frac{a^3e^{3s}}{3}), & x < 0 \\ \Gamma: \ (s, 0, \ -a(1-e^{-s}), -ae^{-s}, \ \frac{a^3e^{-3s}}{3}), & x > 0 \end{cases}$$

$$\frac{dx}{dt} = F_p = p^2 = \begin{cases} a^2e^{2s} \\ a^2e^{-2s} \end{cases} \Rightarrow \ x(s,t) = \begin{cases} a^2e^{2s}t + c_4(s) \\ a^2e^{-2s}t + c_5(s) \end{cases} \Rightarrow \ x = \begin{cases} a^2e^{2s}t + s \\ a^2e^{-2s}t + s \end{cases}$$

$$\frac{dy}{dt} = F_q = 1 \quad \Rightarrow \quad y(s,t) = t + c_1(s) \quad \Rightarrow \quad y = t$$

$$\frac{dz}{dt} = pF_p + qF_q = p^3 + q = \begin{cases} a^3e^{3s} - \frac{a^3e^{3s}}{3} = \frac{2}{3}a^3e^{3s}, & x < 0 \\ -a^3e^{-3s} + \frac{a^3e^{-3s}}{3} = -\frac{2}{3}a^3e^{-3s}, & x > 0 \end{cases}$$

$$\Rightarrow \ z(s,t) = \begin{cases} \frac{2}{3}a^3e^{3s}t \ + \ c_6(s), & x < 0 \\ -\frac{2}{3}a^3e^{-3s}t + c_7(s), & x > 0 \end{cases} \Rightarrow z = \begin{cases} \frac{2}{3}a^3e^{3s}t \ - \ a(1-e^s), & x < 0 \\ -\frac{2}{3}a^3e^{-3s}t - a(1-e^{-s}), & x > 0 \end{cases}$$

$$\frac{dp}{dt} = -F_x - F_zp = 0 \quad \Rightarrow \quad p(s,t) = c_2(s) \quad \Rightarrow \quad p = \begin{cases} ae^s, & x < 0 \\ -ae^{-s}, & x > 0 \end{cases}$$

$$\frac{dq}{dt} = -F_y - F_z q = 0 \quad \Rightarrow \quad q(s,t) = c_3(s) \quad \Rightarrow \quad q = \begin{cases} -\frac{a^3 e^{3s}}{3}, & x < 0 \\ \frac{a^3 e^{-3s}}{3}, & x > 0 \end{cases}$$

Thus,

$$\boxed{u(x,y) = \begin{cases} \frac{2}{3}a^3 e^{3s} y - a(1-e^s), & x < 0 \\ -\frac{2}{3}a^3 e^{-3s} y - a(1-e^{-s}), & x > 0 \end{cases}}$$

where s is defined as

$$x = \begin{cases} a^2 e^{2s} y + s, & x < 0 \\ a^2 e^{-2s} y + s, & x > 0. \end{cases}$$

□

b). *Solve the equation*

$$u_t + \frac{u_x^3}{3} = -cu. \tag{11.5}$$

Proof. Rewrite the equation as

$$F(x,y,u,u_x,u_y) = \frac{u_x^3}{3} + u_y + cu = 0,$$
$$F(x,y,z,p,q) = \frac{p^3}{3} + q + cz = 0.$$

Γ is parameterized by $\Gamma: \ (s, 0, h(s), \phi(s), \psi(s))$.
We need to complete Γ to a strip. Find $\phi(s)$ and $\psi(s)$, the initial conditions for $p(s,t)$ and $q(s,t)$, respectively:

- $$F(f(s), g(s), h(s), \phi(s), \psi(s)) = 0,$$
  $$\frac{\phi(s)^3}{3} + \psi(s) + ch(s) = 0,$$
  $$\psi(s) = -\frac{\phi(s)^3}{3} - ch(s) = \begin{cases} -\frac{\phi(s)^3}{3} + ca(1-e^x), & x < 0 \\ -\frac{\phi(s)^3}{3} + ca(1-e^{-x}), & x > 0 \end{cases}$$

- $$h'(s) = \phi(s) f'(s) + \psi(s) g'(s)$$
  $$\begin{cases} ae^s = \phi(s), & x < 0 \\ -ae^{-s} = \phi(s), & x > 0 \end{cases} \Rightarrow \begin{cases} \psi(s) = -\frac{a^3 e^{3s}}{3} + ca(1-e^x), & x < 0 \\ \psi(s) = \frac{a^3 e^{-3s}}{3} + ca(1-e^{-x}), & x > 0 \end{cases}$$

Therefore, now Γ is parametrized by

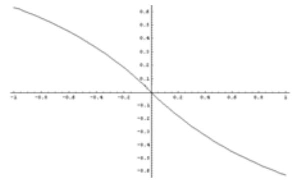
$$\begin{cases} \Gamma: \ (s, 0, \ -a(1-e^s), \quad ae^s, \quad -\frac{a^3e^{3s}}{3} + ca(1-e^x), & x < 0 \\ \Gamma: \ (s, 0, \ -a(1-e^{-s}), -ae^{-s}, \ \frac{a^3e^{-3s}}{3} + ca(1-e^{-x}), & x > 0 \end{cases}$$

$$\begin{aligned}
\frac{dx}{dt} &= F_p = p^2 \\
\frac{dy}{dt} &= F_q = 1 \\
\frac{dz}{dt} &= pF_p + qF_q = p^3 + q \\
\frac{dp}{dt} &= -F_x - F_z p = -cp \\
\frac{dq}{dt} &= -F_y - F_z q = -cq
\end{aligned}$$

We can proceed solving the characteristic equations with initial conditions above. □

Problem (S'95, #7). ***a)*** *Solve the following equation, using characteristics,*

$$u_t + u^3 u_x = 0,$$
$$u(x,0) = \begin{cases} a(1-e^x), & \textit{for } x<0 \\ -a(1-e^{-x}), & \textit{for } x>0 \end{cases}$$



where $a>0$ is a constant. Determine the first time when a shock forms.

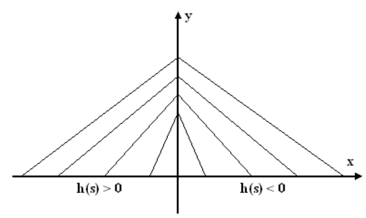
Proof. **a)** Γ is parameterized by $\Gamma: \ (s, 0, h(s))$.

$$\frac{dx}{dt} = z^3, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = 0.$$
$$x = h(s)^3 t + s, \quad y = t, \quad z = h(s).$$
$$u(x,y) = h(x - u^3 y) \tag{11.6}$$

The characteristic projection in the xt-plane[18] passing through the point $(s,0)$ is the line

$$x = h(s)^3 t + s$$

along which u has a constant value $u = h(s)$.



Characteristics $x = h(s_1)^3 t + s_1$ and $x = h(s_2)^3 t + s_2$ intersect at a point (x,t) with

$$t = -\frac{s_2 - s_1}{h(s_2)^3 - h(s_1)^3}.$$

From (11.6), we have

$$u_x = h'(s)(1 - 3u^2 u_x t) \quad \Rightarrow \quad u_x = \frac{h'(s)}{1 + 3h(s)^2 h'(s) t}$$

Hence for $3h(s)^2 h'(s) < 0$, u_x becomes infinite at the positive time

$$t = \frac{-1}{3h(s)^2 h'(s)}.$$

The smallest t for which this happens corresponds to the value $s = s_0$ at which $h(s)^2 h'(s)$ has a minimum (i.e.$-h(s)^2 h'(s)$ has a maximum). At time $T = -1/(3h(s_0)^2 h'(s_0))$ the solution u experiences a "gradient catastrophe".
Therefore, need to find a **minimum** of

$$f(x) = 3h(x)^2 h'(x) = \begin{cases} -3a^2(1-e^x)^2 a e^x \ = \ -3a^3 e^x (1-e^x)^2, & x<0 \\ -3a^2(1-e^{-x})^2 a e^{-x} \ = \ -3a^3 e^{-x}(1-e^{-x})^2, & x>0 \end{cases}$$

$$f'(x) = \begin{cases} -3a^3\big[e^x(1-e^x)^2 - e^x 2(1-e^x)e^x\big] = -3a^3 e^x (1-e^x)(1-3e^x), & x<0 \\ -3a^3\big[-e^{-x}(1-e^{-x})^2 + e^{-x}2(1-e^{-x})e^{-x}\big] = -3a^3 e^{-x}(1-e^{-x})(-1+3e^{-x}), & x>0 \end{cases} = 0$$

The zeros of $f'(x)$ are $\begin{cases} x=0, \ x = -\ln 3, & x<0, \\ x=0, \ x = \ \ln 3, & x>0. \end{cases}$ We check which ones give the minimum of $f(x)$:

$$\Rightarrow \quad \begin{cases} f(0) = -3a^3, \ f(-\ln 3) = -3a^3 \frac{1}{3}(1-\frac{1}{3})^2 = -\frac{4a^3}{9}, & x<0 \\ f(0) = -3a^3, \ \ f(\ln 3) \ = -3a^3\frac{1}{3}(1-\frac{1}{3})^2 = -\frac{4a^3}{9}, & x>0 \end{cases}$$

---

[18] y and t are interchanged here

$$\Rightarrow \quad t = -\frac{1}{\min\{3h(s)^2 h'(s)\}} = -\frac{1}{\min f(s)} = \frac{1}{3a^3}.$$

□

b) *Now consider*

$$u_t + u^3 u_x + cu = 0$$

with the same initial data and a positive constant c. How large does c need to be in order to prevent shock formation?

b) Characteristic form: $u_t + u^3 u_x = -cu$. $\Gamma : \ (s, 0, h(s))$.

$$\begin{aligned}
\frac{dx}{dt} &= z^3 = h(s)^3 e^{-3ct} \quad \Rightarrow \quad x = s + \frac{1}{3c} h(s)^3 (1 - e^{-3ct}), \\
\frac{dy}{dt} &= 1 \quad \Rightarrow \quad y = t, \\
\frac{dz}{dt} &= -cz \quad \Rightarrow \quad z = h(s) e^{-ct} \qquad (\Rightarrow \ h(s) = u e^{cy}). \\
&\Rightarrow \ z(s,t) = h\Big(x - \frac{1}{3c} h(s)^3 (1 - e^{-3ct})\Big) e^{-ct}, \\
&\Rightarrow \ u(x,y) = h\Big(x - \frac{1}{3c} u^3 e^{3cy} (1 - e^{-3cy})\Big) e^{-cy}.
\end{aligned}$$

$$\begin{aligned}
u_x &= h'(s) \cdot e^{-cy} \cdot \Big(1 - \frac{1}{c} u^2 u_x e^{3cy} (1 - e^{-3cy})\Big), \\
u_x &= \frac{h'(s) e^{-cy}}{1 + \frac{1}{c} h'(s) u^2 e^{2cy} (1 - e^{-3cy})} = \frac{h'(s) e^{-cy}}{1 + \frac{1}{c} h'(s) h(s)^2 (1 - e^{-3cy})}.
\end{aligned}$$

Thus, we need

$$1 + \frac{1}{c} h'(s) h(s)^2 (1 - e^{-3cy}) \neq 0.$$

We can perform further calculations taking into account the result from part (a):

$$\min\{3h(s)^2 h'(s)\} = -3a^3.$$

Problem (F'99, #4). *Consider the Cauchy problem*

$$\begin{aligned} u_y + a(x)u_x &= 0, \\ u(x,0) &= h(x). \end{aligned}$$

*Give an example of an (unbounded) smooth $a(x)$ for which the solution of the Cauchy problem is **not** unique.*

Proof. Γ is parameterized by $\Gamma: \ (s, 0, h(s))$.

$$\begin{aligned} \frac{dx}{dt} &= a(x) \quad \Rightarrow \quad x(t) - x(0) = \int_0^t a(x)dt \quad \Rightarrow \quad x = \int_0^t a(x)dt + s, \\ \frac{dy}{dt} &= 1 \quad \Rightarrow \quad y(s,t) = t + c_1(s) \quad \Rightarrow \quad y = t, \\ \frac{dz}{dt} &= 0 \quad \Rightarrow \quad z(s,t) = c_2(s) \quad \Rightarrow \quad z = h(s). \end{aligned}$$

Thus,

$$u(x,t) = h\Big(x - \int_0^y a(x)dy\Big)$$

□

Problem (F'97, #7). ***a)*** *Solve the Cauchy problem*

$$\begin{aligned} u_t - xuu_x &= 0 \qquad -\infty < x < \infty, \ t \geq 0, \\ u(x,0) &= f(x) \qquad -\infty < x < \infty. \end{aligned}$$

b) *Find a class of initial data such that this problem has a global solution for all t. Compute the critical time for the existence of a smooth solution for initial data, f, which is not in the above class.*

Proof. **a)** Γ is parameterized by $\Gamma: \ (s, 0, f(s))$.

$$\begin{aligned} \frac{dx}{dt} &= -xz \quad \Rightarrow \quad \frac{dx}{dt} = -xf(s) \quad \Rightarrow \quad x = se^{-f(s)t}, \\ \frac{dy}{dt} &= 1 \quad \Rightarrow \quad y = t, \\ \frac{dz}{dt} &= 0 \quad \Rightarrow \quad z = f(s). \\ &\Rightarrow \quad z = f\big(xe^{f(s)t}\big), \\ &\Rightarrow \quad u(x,y) = f\big(xe^{uy}\big). \end{aligned}$$

Check:

$$\begin{cases} u_x = f'(s)\cdot(e^{uy} + xe^{uy}u_x y) \\ u_y = f'(s)\cdot xe^{uy}(u_y y + u) \end{cases} \Rightarrow \begin{cases} u_x - f'(s)xe^{uy}u_x y = f'(s)e^{uy} \\ u_y - f'(s)xe^{uy}u_y y = f'(s)xe^{uy}u \end{cases}$$

$$\Rightarrow \begin{cases} u_x = \frac{f'(s)e^{uy}}{1-f'(s)xye^{uy}} \\ u_y = \frac{f'(s)e^{uy}xu}{1-f'(s)xye^{uy}} \end{cases} \Rightarrow \ u_y - xuu_x = \frac{f'(s)e^{uy}xu}{1-f'(s)xye^{uy}} - xu\frac{f'(s)e^{uy}}{1-f'(s)xye^{uy}} = 0. \ \checkmark$$

$$u(x,0) = f(x). \ \checkmark$$

b) The characteristics would intersect when $1 - f'(s)xye^{uy} = 0$. Thus,

$$t_c = \frac{1}{f'(s)xe^{ut_c}}.$$

$\square$

Problem (F'96, #6). *Find an implicit formula for the solution u of the initial-value problem*

$$u_t = (2x-1)tu_x + \sin(\pi x) - t,$$
$$u(x,t=0) = 0.$$

Evaluate u explicitly at the point $(x = 0.5,\ t = 2)$.

Proof. Rewrite the equation as

$$u_y + (1-2x)yu_x = \sin(\pi x) - y.$$

Γ is parameterized by $\Gamma:\ (s,0,0)$.

$$\frac{dx}{dt} = (1-2x)y = (1-2x)t \quad\Rightarrow\quad x = \frac{1}{2}(2s-1)e^{-t^2} + \frac{1}{2}, \quad \Big(\Rightarrow\ s = (x-\frac{1}{2})e^{t^2} + \frac{1}{2}\Big),$$
$$\frac{dy}{dt} = 1 \quad\Rightarrow\quad y = t,$$
$$\frac{dz}{dt} = \sin(\pi x) - y \ = \ \sin\Big(\frac{\pi}{2}(2s-1)e^{-t^2} + \frac{\pi}{2}\Big) - t.$$

$$\begin{aligned}
\Rightarrow \quad z(s,t) &= \int_0^t \Big[\sin\Big(\frac{\pi}{2}(2s-1)e^{-t^2} + \frac{\pi}{2}\Big) - t\Big]\,dt + z(s,0),\\
z(s,t) &= \int_0^t \Big[\sin\Big(\frac{\pi}{2}(2s-1)e^{-t^2} + \frac{\pi}{2}\Big) - t\Big]\,dt.\\
\Rightarrow \quad u(x,y) &= \int_0^y \Big[\sin\Big(\frac{\pi}{2}(2s-1)e^{-y^2} + \frac{\pi}{2}\Big) - y\Big]\,dy\\
&= \int_0^y \Big[\sin\Big(\frac{\pi}{2}(2x-1)e^{y^2}e^{-y^2} + \frac{\pi}{2}\Big) - y\Big]\,dy\\
&= \int_0^y \Big[\sin\Big(\frac{\pi}{2}(2x-1) + \frac{\pi}{2}\Big) - y\Big]\,dy = \int_0^y \Big[\sin(\pi x) - y\Big]\,dy,\\
\Rightarrow \quad u(x,y) &= y\sin(\pi x) - \frac{y^2}{2}.
\end{aligned}$$

Note: This solution does **not** satisfy the PDE. $\square$

Problem (S'90, #8). *Consider the Cauchy problem*

$$u_t = xu_x - u, \qquad -\infty < x < \infty,\ t \ge 0,$$
$$u(x,0) = f(x), \qquad f(x) \in C^\infty.$$

Assume that $f \equiv 0$ for $|x| \ge 1$.
Solve the equation by the method of characteristics and discuss the behavior of the solution.

Proof. Rewrite the equation as

$$u_y - xu_x = -u,$$

Γ is parameterized by $\Gamma:\ (s,0,f(s))$.

$$\frac{dx}{dt} = -x \quad\Rightarrow\quad x = se^{-t}, \qquad \frac{dy}{dt} = 1 \quad\Rightarrow\quad y = t,$$
$$\frac{dz}{dt} = -z \quad\Rightarrow\quad z = f(s)e^{-t}.$$
$$\Rightarrow\quad u(x,y) = f(xe^y)e^{-y}.$$

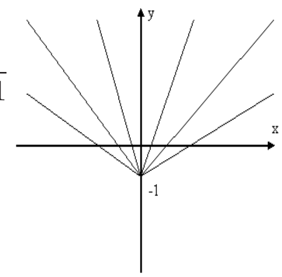
The solution satisfies the PDE and initial conditions.
As $y \to +\infty$, $u \to 0$. $\quad u = 0$ for $|xe^y| \geq 1 \quad \Rightarrow \quad u = 0$ for $|x| \geq \frac{1}{e^y}$. $\square$

Problem (F'02, #4). *Consider the nonlinear hyperbolic equation*

$$u_y + uu_x = 0 \qquad -\infty < x < \infty.$$

a) *Find a smooth solution to this equation for initial condition* $u(x,0) = x$.
b) *Describe the breakdown of smoothness for the solution if* $u(x,0) = -x$.

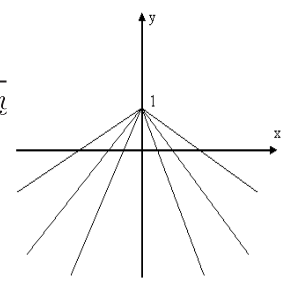
Proof. **a)** Γ is parameterized by $\Gamma: \ (s, 0, s)$.



$$\begin{aligned}
\frac{dx}{dt} &= z = s \quad \Rightarrow \quad x = st + s \quad \Rightarrow \quad s = \frac{x}{t+1} = \frac{x}{y+1} \\
\frac{dy}{dt} &= 1 \quad \Rightarrow \quad y = t, \\
\frac{dz}{dt} &= 0 \quad \Rightarrow \quad z = s.
\end{aligned}$$

$$\Rightarrow \quad u(x,y) = \frac{x}{y+1}; \text{ solution is smooth for all positive time } y.$$

b) Γ is parameterized by $\Gamma: \ (s, 0, -s)$.



$$\begin{aligned}
\frac{dx}{dt} &= z = -s \quad \Rightarrow \quad x = -st + s \ \Rightarrow \ s = \frac{x}{1-t} = \frac{x}{1-y} \\
\frac{dy}{dt} &= 1 \quad \Rightarrow \quad y = t, \\
\frac{dz}{dt} &= 0 \quad \Rightarrow \quad z = -s.
\end{aligned}$$

$$\Rightarrow \quad u(x,y) = \frac{x}{y-1}; \text{ solution blows up at time } y = 1.$$

□

Problem (F'97, #4). *Solve the initial-boundary value problem*

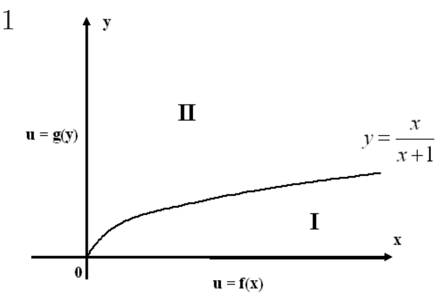
$$\begin{aligned} u_t + (x+1)^2 u_x &= x \quad \textit{for } x>0,\, t>0 \\ u(x,0) &= f(x) \quad 0<x<+\infty \\ u(0,t) &= g(t) \quad 0<t<+\infty. \end{aligned}$$

Proof. Rewrite the equation as

$$\begin{aligned} u_y + (x+1)^2 u_x &= x \quad \text{for } x>0,\, y>0 \\ u(x,0) &= f(x) \quad 0<x<+\infty \\ u(0,y) &= g(y) \quad 0<y<+\infty. \end{aligned}$$

• For region I, we solve the following characteristic equations with Γ is parameterized [19] by $\Gamma: \ (s, 0, f(s))$.

$$\begin{aligned} \frac{dx}{dt} &= (x+1)^2 \quad \Rightarrow \quad x = -\frac{s+1}{(s+1)t-1} - 1 \\ \frac{dy}{dt} &= 1 \quad \Rightarrow \quad y = t, \\ \frac{dz}{dt} &= x = -\frac{s+1}{(s+1)t-1} - 1, \\ &\Rightarrow \ z = -\ln|(s+1)t-1| - t + c_1(s), \\ &\Rightarrow \ z = -\ln|(s+1)t-1| - t + f(s). \end{aligned}$$



In region I, characteristics are of the form

$$x = -\frac{s+1}{(s+1)y-1} - 1.$$

Thus, region I is bounded above by the line

$$x = -\frac{1}{y-1} - 1, \qquad \text{or} \qquad y = \frac{x}{x+1}.$$

Since $t = y$, $s = \frac{x-xy-y}{xy+y+1}$, we have

$$\begin{aligned} u(x,y) &= -\ln\Big|\Big(\frac{x-xy-y}{xy+y+1}+1\Big)y - 1\Big| - y + f\Big(\frac{x-xy-y}{xy+y+1}\Big), \\ \Rightarrow u(x,y) &= -\ln\Big|\frac{-1}{xy+y+1}\Big| - y + f\Big(\frac{x-xy-y}{xy+y+1}\Big). \end{aligned}$$

• For region II, Γ is parameterized by $\Gamma: \ (0, s, g(s))$.

$$\begin{aligned} \frac{dx}{dt} &= (x+1)^2 \quad \Rightarrow \quad x = -\frac{1}{t-1} - 1, \\ \frac{dy}{dt} &= 1 \quad \Rightarrow \quad y = t+s, \\ \frac{dz}{dt} &= x = -\frac{1}{t-1} - 1, \\ &\Rightarrow \ z = -\ln|t-1| - t + c_2(s), \\ &\Rightarrow \ z = -\ln|t-1| - t + g(s). \end{aligned}$$

[19] Variable t as a third coordinate of u and variable t used to parametrize characteristic equations are two different entities.

Since $t = \frac{x}{x+1},\ s = y - \frac{x}{x+1}$, we have

$$u(x,y) = -\ln\Big|\frac{x}{x+1} - 1\Big| - \frac{x}{x+1} + g\Big(y - \frac{x}{x+1}\Big).$$

Note that on $y = \frac{x}{x+1}$, both solutions are equal if $f(0) = g(0)$. $\square$

Problem (S'93, #3). *Solve the following equation*

$$u_t + u_x + yu_y = \sin t$$

for $0 \le t, \;\; 0 \le x, \;\; -\infty < y < \infty$ *and with*

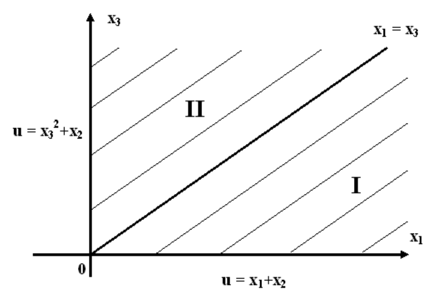
$$\begin{aligned} u &= x + y && \textit{for } t = 0, \; x \ge 0 \quad \textit{and} \\ u &= t^2 + y && \textit{for } x = 0, \; t \ge 0. \end{aligned}$$

Proof. Rewrite the equation as $(x \leftrightarrow x_1, \; y \leftrightarrow x_2, \; t \leftrightarrow x_3)$:

$$\begin{aligned} & u_{x_3} + u_{x_1} + x_2 u_{x_2} = \sin x_3 \qquad \text{for } \; 0 \le x_3, \; 0 \le x_1, \; -\infty < x_2 < \infty, \\ & u(x_1, x_2, 0) = x_1 + x_2, \\ & u(0, x_2, x_3) = x_3^2 + x_2. \end{aligned}$$

• For region I, we solve the following characteristic equations with Γ is parameterized[20] by $\Gamma: \; (s_1, s_2, 0, s_1 + s_2)$.

$$\begin{aligned} \frac{dx_1}{dt} &= 1 \quad \Rightarrow \quad x_1 = t + s_1, \\ \frac{dx_2}{dt} &= x_2 \quad \Rightarrow \quad x_2 = s_2 e^t, \\ \frac{dx_3}{dt} &= 1 \quad \Rightarrow \quad x_3 = t, \\ \frac{dz}{dt} &= \sin x_3 = \sin t \\ &\Rightarrow \; z = -\cos t + s_1 + s_2 + 1. \end{aligned}$$



Since in region I, in x_1x_3-plane, characteristics are of the form $x_1 = x_3 + s_1$, region I is bounded above by the line $x_1 = x_3$. Since $t = x_3, \; s_1 = x_1 - x_3, \; s_2 = x_2 e^{-x_3}$, we have

$$\begin{aligned} u(x_1, x_2, x_3) &= -\cos x_3 + x_1 - x_3 + x_2 e^{-x_3} + 1, \qquad \text{or} \\ u(x, y, t) &= -\cos t + x - t + ye^{-t} + 1, \qquad x \ge t. \end{aligned}$$

• For region II, we solve the following characteristic equations with Γ is parameterized by $\Gamma: \; (0, s_2, s_3, s_2 + s_3^2)$.

$$\begin{aligned} \frac{dx_1}{dt} &= 1 \quad \Rightarrow \quad x_1 = t, \\ \frac{dx_2}{dt} &= x_2 \quad \Rightarrow \quad x_2 = s_2 e^t, \\ \frac{dx_3}{dt} &= 1 \quad \Rightarrow \quad x_3 = t + s_3, \\ \frac{dz}{dt} &= \sin x_3 = \sin(t + s_3) \quad \Rightarrow \quad z = -\cos(t + s_3) + \cos s_3 + s_2 + s_3^2. \end{aligned}$$

Since $t = x_1, \; s_3 = x_3 - x_1, \; s_2 = x_2 e^{-x_3}$, we have

$$\begin{aligned} u(x_1, x_2, x_3) &= -\cos x_3 + \cos(x_3 - x_1) + x_2 e^{-x_3} + (x_3 - x_1)^2, \qquad \text{or} \\ u(x, y, t) &= -\cos t + \cos(t - x) + ye^{-t} + (t - x)^2, \qquad x \le t. \end{aligned}$$

Note that on $x = t$, both solutions are $u(x = t, y) = -\cos x + ye^{-x} + 1$. □

[20] Variable t as a third coordinate of u and variable t used to parametrize characteristic equations are two different entities.

Problem (W'03, #5). *Find a solution to*

$$xu_x + (x+y)u_y = 1$$

which satisfies $u(1,y) = y$ for $0 \le y \le 1$. Find a region in $\{x \ge 0,\ y \ge 0\}$ where u is uniquely determined by these conditions.

Proof. Γ is parameterized by $\Gamma:\ (1,s,s)$.

$$\frac{dx}{dt} = x \quad \Rightarrow \quad x = e^t. \qquad \circledast$$
$$\frac{dy}{dt} = x+y \quad \Rightarrow \quad y' - y = e^t.$$
$$\frac{dz}{dt} = 1 \quad \Rightarrow \quad z = t+s.$$

The homogeneous solution for the second equation is $y_h(s,t) = c_1(s)e^t$. Since the right hand side and y_h are linearly dependent, our guess for the particular solution is $y_p(s,t) = c_2(s)te^t$. Plugging in y_p into the differential equation, we get

$$c_2(s)te^t + c_2(s)e^t - c_2(s)te^t = e^t \quad \Rightarrow \quad c_2(s) = 1.$$

Thus, $y_p(s,t) = te^t$ and

$$y(s,t) = y_h + y_p = c_1(s)e^t + te^t.$$

Since $y(s,0) = s = c_1(s)$, we get

$$y = se^t + te^t. \qquad \circledcirc$$

With $\circledast$ and $\circledcirc$, we can solve for s and t in terms of x and y to get

$$t = \ln x,$$
$$y = sx + x\ln x \quad \Rightarrow \quad s = \frac{y - x\ln x}{x}.$$

$$u(x,y) = t + s = \ln x + \frac{y - x\ln x}{x}.$$

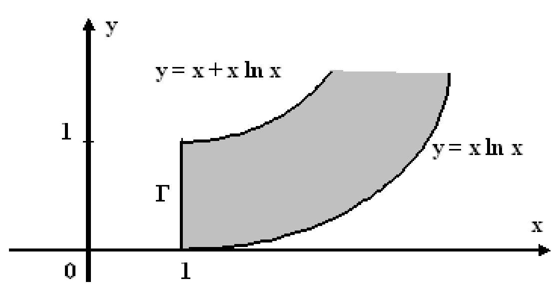
$$\boxed{u(x,y) = \frac{y}{x}.}$$

We have found that the characteristics in the xy-plane are of the form

$$y = sx + x\ln x,$$

where s is such that $0 \le s \le 1$. Also, the characteristics originate from Γ.
Thus, u is **uniquely determined** in the region between the graphs:

$$y = x\ln x,$$
$$y = x + x\ln x.$$



□

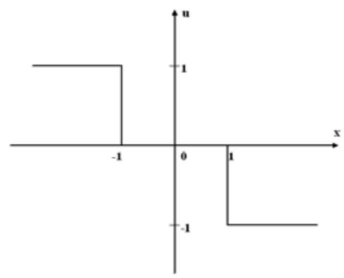
12 Problems: Shocks

Example 1. *Determine the exact solution to Burgers' equation*

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad t > 0$$

with initial data

$$u(x,0) = h(x) = \begin{cases} 1 & \text{if } x < -1, \\ 0 & \text{if } -1 < x < 1, \\ -1 & \text{if } x > 1. \end{cases}$$



Proof. Characteristic form: $u_t + uu_x = 0$.
The characteristic projection in xt-plane passing through the point $(s,0)$ is the line

$$x = h(s)t + s.$$

- Rankine-Hugoniot shock condition at $\mathbf{s = -1}$:

$$\text{shock speed:} \qquad \xi'(t) = \frac{F(u_r) - F(u_l)}{u_r - u_l} = \frac{\frac{1}{2}u_r^2 - \frac{1}{2}u_l^2}{u_r - u_l} = \frac{0 - \frac{1}{2}}{0 - 1} = \frac{1}{2}.$$

The "1/slope" of the shock curve $= 1/2$. Thus,

$$x = \xi(t) = \frac{1}{2}t + s,$$

and since the jump occurs at $(-1, 0)$, $\xi(0) = -1 = s$. Therefore,

$$x = \frac{1}{2}t - 1.$$

- Rankine-Hugoniot shock condition at $\mathbf{s = 1}$:

$$\text{shock speed:} \qquad \xi'(t) = \frac{F(u_r) - F(u_l)}{u_r - u_l} = \frac{\frac{1}{2}u_r^2 - \frac{1}{2}u_l^2}{u_r - u_l} = \frac{\frac{1}{2} - 0}{-1 - 0} = -\frac{1}{2}.$$

The "1/slope" of the shock curve $= -1/2$. Thus,

$$x = \xi(t) = -\frac{1}{2}t + s,$$

and since the jump occurs at $(1, 0)$, $\xi(0) = 1 = s$. Therefore,

$$x = -\frac{1}{2}t + 1.$$

- At $\mathbf{t = 2}$, Rankine-Hugoniot shock condition at $\mathbf{s = 0}$:

$$\text{shock speed:} \qquad \xi'(t) = \frac{F(u_r) - F(u_l)}{u_r - u_l} = \frac{\frac{1}{2}u_r^2 - \frac{1}{2}u_l^2}{u_r - u_l} = \frac{\frac{1}{2} - \frac{1}{2}}{-1 - 1} = 0.$$

The "1/slope" of the shock curve $= 0$. Thus,

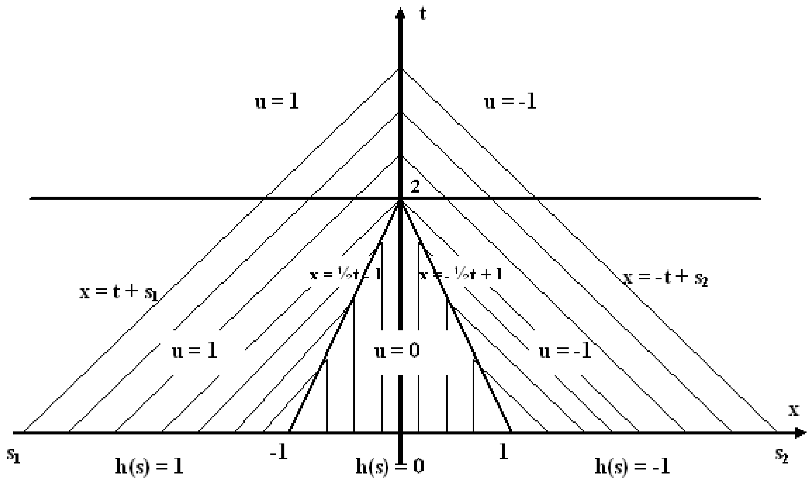
$$x = \xi(t) = s,$$

and since the jump occurs at $(x,t) = (0, 2)$, $\xi(2) = 0 = s$. Therefore,

$$x = 0.$$

➡ For $t < 2$, $$u(x,t) = \begin{cases} 1 & \text{if} \quad x < \frac{1}{2}t - 1, \\ 0 & \text{if} \quad \frac{1}{2}t - 1 < x < -\frac{1}{2}t + 1, \\ -1 & \text{if} \quad x > -\frac{1}{2}t + 1. \end{cases}$$

➡ and for $t > 2$, $$u(x,t) = \begin{cases} 1 & \text{if} \quad x < 0, \\ -1 & \text{if} \quad x > 0. \end{cases}$$



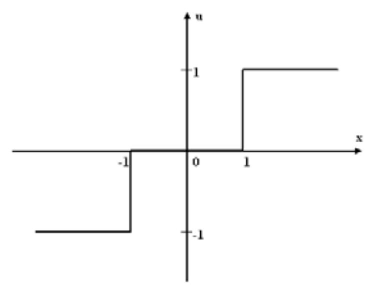
□

Example 2. *Determine the exact solution to Burgers' equation*

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad t > 0$$

with initial data

$$u(x,0) = h(x) = \begin{cases} -1 & \text{if } \ x < -1, \\ \ \ 0 & \text{if } \ -1 < x < 1, \\ \ \ 1 & \text{if } \ x > 1. \end{cases}$$



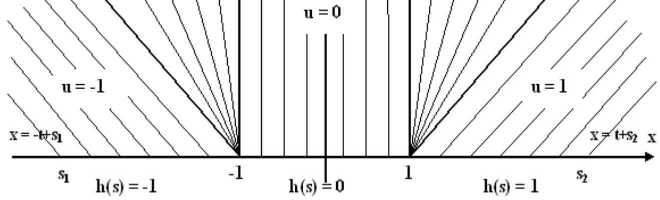
Proof. Characteristic form: $u_t + uu_x = 0$.
The characteristic projection in xt-plane passing through the point $(s,0)$ is the line

$$x = h(s)t + s.$$

For Burgers' equation, for a rarefaction fan emanating from $(s,0)$ on xt-plane, we have:

$$u(x,t) = \begin{cases} u_l, & \frac{x-s}{t} \le u_l, \\ \frac{x-s}{t}, & u_l \le \frac{x-s}{t} \le u_r, \\ u_r, & \frac{x-s}{t} \ge u_r. \end{cases}$$

$$\Rightarrow \ u(x,t) = \begin{cases} -1, & x < -t-1, \\ \frac{x+1}{t}, & -t-1 < x < -1, \qquad \text{i.e.} \quad -1 < \frac{x+1}{t} < 0 \\ 0, & -1 < x < 1, \\ \frac{x-1}{t}, & 1 < x < t+1, \qquad \text{i.e.} \quad 0 < \frac{x-1}{t} < 1 \\ 1, & x > t+1. \end{cases}$$


u = 0
u = -1
u = 1
x = -t+s_1
x = t+s_2
x
s_1
h(s) = -1
-1
h(s) = 0
1
h(s) = 1
s_2

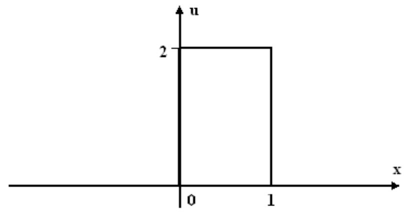
□

Example 3. *Determine the exact solution to Burgers' equation*

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad t > 0$$

with initial data

$$u(x,0) = h(x) = \begin{cases} 2 & \text{if } \ 0 < x < 1, \\ 0 & \text{if } \ \text{otherwise.} \end{cases}$$



Proof. Characteristic form: $u_t + uu_x = 0$.
The characteristic projection in xt-plane passing through the point $(s, 0)$ is the line

$$x = h(s)t + s.$$

- **Shock:** Rankine-Hugoniot shock condition at $\mathbf{s = 1}$:

$$\text{shock speed:} \qquad \xi'(t) = \frac{F(u_r) - F(u_l)}{u_r - u_l} = \frac{\frac{1}{2}u_r^2 - \frac{1}{2}u_l^2}{u_r - u_l} = \frac{0-2}{0-2} = 1.$$

The "1/slope" of the shock curve = 1. Thus,

$$x = \xi(t) = t + s,$$

and since the jump occurs at $(1, 0)$, $\xi(0) = 1 = s$. Therefore,

$$x = t + 1.$$

- **Rarefaction:** A rarefaction emanates from $(0, 0)$ on xt-plane.

➡ For $0 < t < 1$,

$$u(x,t) = \begin{cases} 0 & \text{if } \ x < 0, \\ \frac{x}{t} & \text{if } \ 0 < x < 2t, \\ 2 & \text{if } \ 2t < x < t+1. \\ 0 & \text{if } \ x > t+1. \end{cases}$$

Rarefaction catches up to shock at $t = 1$.

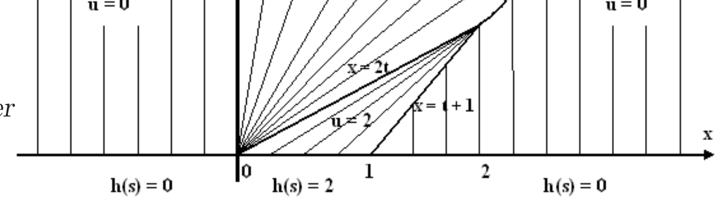
- **Shock:** At $(\mathbf{x}, \mathbf{t}) = (\mathbf{2}, \mathbf{1})$, $u_l = x/t$, $u_r = 0$. Rankine-Hugoniot shock condition:

$$\begin{aligned} \xi'(t) &= \frac{F(u_r) - F(u_l)}{u_r - u_l} = \frac{\frac{1}{2}u_r^2 - \frac{1}{2}u_l^2}{u_r - u_l} = \frac{0 - \frac{1}{2}(\frac{x}{t})^2}{0 - \frac{x}{t}} = \frac{1}{2}\frac{x}{t}, \\ \frac{dx_s}{dt} &= \frac{x}{2t}, \\ x &= c\sqrt{t}, \end{aligned}$$

and since the jump occurs at $(x,t) = (2, 1)$, $x(1) = 2 = c$. Therefore, $x = 2\sqrt{t}$.

➡ For $t > 1$,

$$u(x,t) = \begin{cases} 0 & \text{if } \ x < 0, \\ \frac{x}{t} & \text{if } \ 0 < x < 2\sqrt{t}, \\ 0 & \text{if } \ x > 2\sqrt{t}. \end{cases}$$


u = 0
u = 0
x = 2t
u = 2
x = t + 1
x
0
1
2
h(s) = 0
h(s) = 2
h(s) = 0

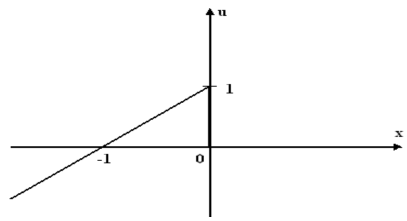
□

Example 4. *Determine the exact solution to Burgers' equation*

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad t > 0$$

with initial data

$$u(x,0) = h(x) = \begin{cases} 1+x & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$$



Proof. Characteristic form: $u_t + uu_x = 0$.
The characteristic projection in xt-plane passing through the point $(s, 0)$ is the line

$$x = h(s)t + s.$$

① For $s > 0$, the characteristics are $x = s$.
② For $s < 0$, the characteristics are $x = (1 + s)t + s$.
• There are two ways to look for the solution on the left half-plane. One is to notice that the characteristic at $s = 0^-$ is $x = t$ and characteristic at $s = -1$ is $x = -1$ and that characteristics between $s = -\infty$ and $s = 0^-$ are intersecting at $(x, t) = (-1, -1)$. Also, for a fixed t, u is a linear function of x, i.e. for $t = 0$, $u = 1 + x$, allowing a continuous change of u with x. Thus, the solution may be viewed as an 'implicit' rarefaction, originating at $(-1, -1)$, thus giving rise to the solution

$$u(x,t) = \frac{x+1}{t+1}.$$

Another way to find a solution on the left half-plane is to solve ② for s to find

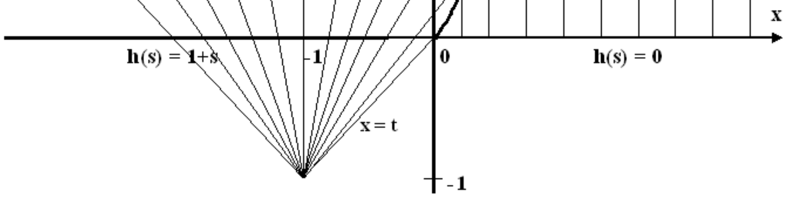
$$s = \frac{x-t}{1+t}. \qquad \text{Thus, } u(x,t) = h(s) = 1 + s = 1 + \frac{x-t}{1+t} = \frac{x+1}{t+1}.$$

• **Shock:** At $(\mathbf{x}, \mathbf{t}) = (\mathbf{0}, \mathbf{0})$, $u_l = \frac{x+1}{t+1}$, $u_r = 0$. Rankine-Hugoniot shock condition:

$$\begin{aligned} \xi'(t) &= \frac{F(u_r) - F(u_l)}{u_r - u_l} = \frac{\frac{1}{2}u_r^2 - \frac{1}{2}u_l^2}{u_r - u_l} = \frac{0 - \frac{1}{2}(\frac{x+1}{t+1})^2}{0 - \frac{x+1}{t+1}} = \frac{1}{2}\frac{x+1}{t+1}, \\ \frac{dx_s}{dt} &= \frac{1}{2}\frac{x+1}{t+1}, \\ x &= c\sqrt{t+1} - 1, \end{aligned}$$

and since the jump occurs at $(x, t) = (0, 0)$, $x(0) = 0 = c - 1$, or $c = 1$. Therefore, the shock curve is $x = \sqrt{t+1} - 1$.

$$\Rightarrow \quad u(x,t) = \begin{cases} \frac{x+1}{t+1} & \text{if } x < \sqrt{t+1} - 1, \\ 0 & \text{if } x > \sqrt{t+1} - 1. \end{cases}$$


x
h(s) = 1+s
-1
0
h(s) = 0
x = t
-1

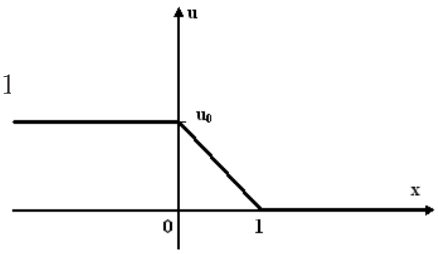
□

Example 5. *Determine the exact solution to Burgers' equation*

$$u_t + \Big(\frac{1}{2}u^2\Big)_x = 0, \qquad t > 0$$

with initial data

$$u(x,0) = h(x) = \begin{cases} u_0 & \text{if } x < 0, \\ u_0 \cdot (1-x) & \text{if } 0 < x < 1 \\ 0 & \text{if } x \geq 1, \end{cases}$$



where $u_0 > 0$.

Proof. Characteristic form: $u_t + uu_x = 0$.
The characteristic projection in xt-plane passing through the point $(s, 0)$ is the line

$$x = h(s)t + s.$$

① For $s > 1$, the characteristics are $x = s$.
② For $0 < s < 1$, the characteristics are $x = u_0(1-s)t + s$.
③ For $s < 0$, the characteristics are $x = u_0 t + s$.
The characteristics emanating from $(s, 0)$, $0 < s < 1$ on xt-plane intersect at $(1, \frac{1}{u_0})$.
Also, we can check that the characteristics do not intersect before $t = \frac{1}{u_0}$ for this problem:

$$t_c = \min\Big(\frac{-1}{h'(s)}\Big) = \frac{1}{u_0}.$$

• To find solution in a triangular domain between $x = u_0 t$ and $x = 1$, we note that characteristics there are $x = u_0 \cdot (1-s)t + s$. Solving for s we get

$$s = \frac{x - u_0 t}{1 - u_0 t}. \qquad \text{Thus, } u(x,t) = h(s) = u_0 \cdot (1-s) = u_0 \cdot \Big(1 - \frac{x - u_0 t}{1 - u_0 t}\Big) = \frac{u_0 \cdot (1-x)}{1 - u_0 t}.$$

We can also find a solution in the triangular domain as follows. Note, that the characteristics are the straight lines

$$\frac{dx}{dt} = u = \text{const.}$$

Integrating the equation above, we obtain

$$x = ut + c$$

Since all characteristics in the triangular domain meet at $(1, \frac{1}{u_0})$, we have $c = 1 - \frac{u}{u_0}$, and

$$x = ut + \Big(1 - \frac{u}{u_0}\Big) \qquad \text{or} \qquad u = \frac{u_0 \cdot (1-x)}{1 - u_0 t}.$$

➡ For $0 < t < \frac{1}{u_0}$,

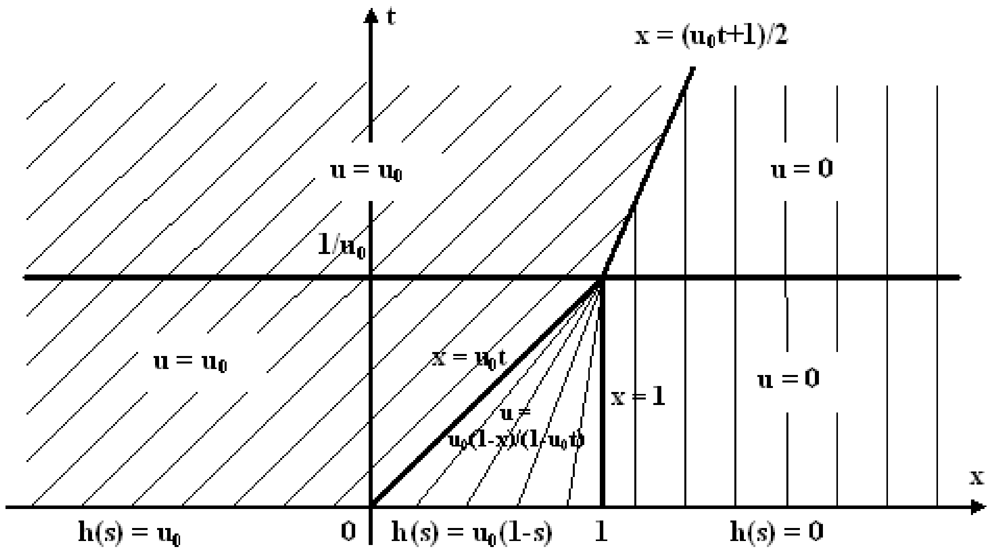
$$u(x,t) = \begin{cases} u_0 & \text{if } \quad x < u_0 t, \\ \frac{u_0 \cdot (1-x)}{1-u_0 t} & \text{if } \quad u_0 t < x < 1, \\ 0 & \text{if } \quad x > 1. \end{cases}$$

- **Shock:** At $(\mathbf{x}, \mathbf{t}) = (\mathbf{1}, \frac{\mathbf{1}}{\mathbf{u_0}})$, Rankine-Hugoniot shock condition:

$$\begin{aligned}
\xi'(t) &= \frac{F(u_r) - F(u_l)}{u_r - u_l} = \frac{\frac{1}{2}u_r^2 - \frac{1}{2}u_l^2}{u_r - u_l} = \frac{0 - \frac{1}{2}u_0^2}{0 - u_0} = \frac{1}{2}u_0, \\
\xi(t) &= \frac{1}{2}u_0 t + c,
\end{aligned}$$

and since the jump occurs at $(x, t) = (1, \frac{1}{u_0})$, $x\big(\frac{1}{u_0}\big) = 1 = \frac{1}{2} + c$, or $c = \frac{1}{2}$. Therefore, the shock curve is $x = \frac{u_0 t + 1}{2}$.

➡ For $t > \dfrac{1}{u_0}$, $\quad u(x,t) = \begin{cases} u_0 & \text{if } \ x < \frac{u_0 t+1}{2}, \\ 0 & \text{if } \ x > \frac{u_0 t+1}{2}. \end{cases}$



□

Problem. *Show that for $u = f(x/t)$ to be a nonconstant solution of $u_t + a(u)u_x = 0$, f must be the inverse of the function a.*

Proof. If $u = f(x/t)$,

$$u_t = -f'\Big(\frac{x}{t}\Big)\cdot\frac{x}{t^2} \qquad \text{and} \qquad u_x = f'\Big(\frac{x}{t}\Big)\cdot\frac{1}{t}.$$

Hence, $u_t + a(u)u_x = 0$ implies that

$$-f'\Big(\frac{x}{t}\Big)\cdot\frac{x}{t^2} + a\Big(f\Big(\frac{x}{t}\Big)\Big)f'\Big(\frac{x}{t}\Big)\cdot\frac{1}{t} = 0$$

or, assuming f' is not identically 0 to rule out the constant solution, that

$$a\Big(f\Big(\frac{x}{t}\Big)\Big) = \frac{x}{t}.$$

This shows the functions a and f to be inverses of each other. □

13 Problems: General Nonlinear Equations

13.1 Two Spatial Dimensions

Problem (S'01, #3). *Solve the initial value problem*

$$\frac{1}{2}u_x^2 - u_y = -\frac{x^2}{2},$$
$$u(x,0) = x.$$

You will find that the solution blows up in finite time. Explain this in terms of the characteristics for this equation.

Proof. Rewrite the equation as

$$F(x,y,z,p,q) = \frac{p^2}{2} - q + \frac{x^2}{2} = 0.$$

Γ is parameterized by $\Gamma: \ (s, 0, s, \phi(s), \psi(s))$.
We need to complete Γ to a strip. Find $\phi(s)$ and $\psi(s)$, the initial conditions for $p(s,t)$ and $q(s,t)$, respectively:

- $$F(f(s), g(s), h(s), \phi(s), \psi(s)) = 0,$$
  $$F(s, 0, s, \phi(s), \psi(s)) = 0,$$
  $$\frac{\phi(s)^2}{2} - \psi(s) + \frac{s^2}{2} = 0,$$
  $$\psi(s) = \frac{\phi(s)^2 + s^2}{2}.$$

- $$h'(s) = \phi(s)f'(s) + \psi(s)g'(s),$$
  $$1 = \phi(s).$$
  $$\Rightarrow \quad \psi(s) = \frac{s^2+1}{2}.$$

Therefore, now Γ is parametrized by $\Gamma: \ (s, 0, s, 1, \frac{s^2+1}{2})$.

$$\frac{dx}{dt} = F_p = p,$$
$$\frac{dy}{dt} = F_q = -1 \quad \Rightarrow \quad y(s,t) = -t + c_1(s) \quad \Rightarrow \quad y = -t,$$
$$\frac{dz}{dt} = pF_p + qF_q = p^2 - q,$$
$$\frac{dp}{dt} = -F_x - F_z p = -x,$$
$$\frac{dq}{dt} = -F_y - F_z q = 0 \quad \Rightarrow \quad q(s,t) = c_2(s) \quad \Rightarrow \quad q = \frac{s^2+1}{2}.$$

Thus, we found y and q in terms of s and t. Note that we have a coupled system:

$$\begin{cases} x' = p, \\ p' = -x, \end{cases}$$

which can be written as two second order ODEs:

$$x'' + x = 0, \qquad x(s,0) = s, \qquad x'(s,0) = p(s,0) = 1,$$
$$p'' + p = 0, \qquad p(s,0) = 1, \qquad p'(s,0) = -x(s,0) = -s.$$

Solving the two equations separately, we get

$$\begin{aligned} x(s,t) &= s\cdot\cos t+\sin t,\\ p(s,t) &= \cos t - s\cdot\sin t. \end{aligned}$$

From this, we get

$$\begin{aligned}
\frac{dz}{dt}=p^2-q &= \big(\cos t - s\cdot\sin t\big)^2-\frac{s^2+1}{2} \;=\; \cos^2 t-2s\cos t\sin t+s^2\sin^2 t-\frac{s^2+1}{2}.\\
z(s,t) &= \int_0^t\Big[\cos^2 t-2s\cos t\sin t+s^2\sin^2 t-\frac{s^2+1}{2}\Big]\,dt \;+\; z(s,0),\\
z(s,t) &= \Big[\frac{t}{2}+\frac{\sin t\cos t}{2}+s\cos^2 t+\frac{s^2 t}{2}-\frac{s^2\sin t\cos t}{2}-\frac{t(s^2+1)}{2}\Big]_0^t+s,\\
&= \Big[\frac{\sin t\cos t}{2}+s\cos^2 t-\frac{s^2\sin t\cos t}{2}\Big]_0^t+s,\\
&= \frac{\sin t\cos t}{2}+s\cos^2 t-\frac{s^2\sin t\cos t}{2}-s+s=\\
&= \frac{\sin t\cos t}{2}+s\cos^2 t-\frac{s^2\sin t\cos t}{2}.
\end{aligned}$$

Plugging in x and y found earlier for s and t, we get

$$\begin{aligned}
u(x,y) &= \frac{\sin(-y)\cos(-y)}{2}+\frac{x-\sin(-y)}{\cos(-y)}\cos^2(-y)-\frac{(x-\sin(-y))^2}{\cos^2(-y)}\cdot\frac{\sin(-y)\cos(-y)}{2}\\
&= -\frac{\sin y\cos y}{2}+\frac{x+\sin y}{\cos y}\cos^2 y+\frac{(x+\sin y)^2}{\cos^2 y}\cdot\frac{\sin y\cos y}{2}\\
&= -\frac{\sin y\cos y}{2}+(x+\sin y)\cos y+\frac{(x+\sin y)^2\sin y}{2\cos y}\\
&= x\cos y+\frac{\sin y\cos y}{2}+\frac{(x+\sin y)^2\sin y}{2\cos y}.
\end{aligned}$$

□

Problem (S'98, #3). *Find the solution of*

$$u_t + \frac{u_x^2}{2} = \frac{-x^2}{2}, \qquad t \geq 0, \ -\infty < x < \infty$$
$$u(x,0) = h(x), \qquad -\infty < x < \infty,$$

where $h(x)$ is smooth function which vanishes for $|x|$ large enough.

Proof. Rewrite the equation as

$$F(x,y,z,p,q) = \frac{p^2}{2} + q + \frac{x^2}{2} = 0.$$

Γ is parameterized by $\Gamma: \ (s, 0, h(s), \phi(s), \psi(s))$.
We need to complete Γ to a strip. Find $\phi(s)$ and $\psi(s)$, the initial conditions for $p(s,t)$ and $q(s,t)$, respectively:

- $$F(f(s), g(s), h(s), \phi(s), \psi(s)) \ = \ 0,$$
  $$F(s, 0, h(s), \phi(s), \psi(s)) = 0,$$
  $$\frac{\phi(s)^2}{2} + \psi(s) + \frac{s^2}{2} = 0,$$
  $$\psi(s) = -\frac{\phi(s)^2 + s^2}{2}.$$

- $$h'(s) \ = \ \phi(s)f'(s) + \psi(s)g'(s),$$
  $$h'(s) = \phi(s).$$
  $$\Rightarrow \quad \psi(s) = -\frac{h'(s)^2 + s^2}{2}.$$

Therefore, now Γ is parametrized by $\Gamma: \ (s, 0, s, h'(s), -\frac{h'(s)^2+s^2}{2})$.

$$\frac{dx}{dt} \ = \ F_p = p,$$
$$\frac{dy}{dt} \ = \ F_q = 1 \quad \Rightarrow \quad y(s,t) = t + c_1(s) \quad \Rightarrow \quad y = t,$$
$$\frac{dz}{dt} \ = \ pF_p + qF_q = p^2 + q,$$
$$\frac{dp}{dt} \ = \ -F_x - F_z p = -x,$$
$$\frac{dq}{dt} \ = \ -F_y - F_z q = 0 \quad \Rightarrow \quad q(s,t) = c_2(s) \quad \Rightarrow \quad q = -\frac{h'(s)^2 + s^2}{2}.$$

Thus, we found y and q in terms of s and t. Note that we have a coupled system:

$$\begin{cases} x' = p, \\ p' = -x, \end{cases}$$

which can be written as a second order ODE:

$$x'' + x = 0, \qquad x(s,0) = s, \qquad x'(s,0) = \ p(s,0) \ = \ h'(s).$$

Solving the equation, we get

$$x(s,t) \ = \ s \cos t + h'(s) \sin t,$$
$$p(s,t) \ = \ x'(s,t) = h'(s) \cos t - s \sin t.$$

From this, we get

$$\begin{aligned}
\frac{dz}{dt} = p^2 + q &= \big(h'(s)\cos t - s\sin t\big)^2 - \frac{h'(s)^2 + s^2}{2} \\
&= h'(s)^2\cos^2 t - 2sh'(s)\cos t\sin t + s^2\sin^2 t - \frac{h'(s)^2 + s^2}{2}. \\
z(s,t) &= \int_0^t \Big[h'(s)^2\cos^2 t - 2sh'(s)\cos t\sin t + s^2\sin^2 t - \frac{h'(s)^2 + s^2}{2}\Big]\,dt \;+\; z(s,0) \\
&= \int_0^t \Big[h'(s)^2\cos^2 t - 2sh'(s)\cos t\sin t + s^2\sin^2 t - \frac{h'(s)^2 + s^2}{2}\Big]\,dt \;+\; h(s).
\end{aligned}$$

We integrate the above expression similar to $S'01\#3$ to get an expression for $z(s,t)$. Plugging in x and y found earlier for s and t, we get $u(x,y)$. □

Problem (S'97, #4).
Describe the ***method of the bicharacteristics*** *for solving the initial value problem*

$$\Big(\frac{\partial}{\partial x}u(x,y)\Big)^2 + \Big(\frac{\partial}{\partial y}u(x,y)\Big)^2 = 2+y,$$
$$u(x,0) = u_0(x) = x.$$

Assume that $|\frac{\partial}{\partial x}u_0(x)| < 2$ *and consider the solution such that* $\frac{\partial u}{\partial y} > 0$.
Apply all general computations for the particular case $u_0(x) = x$.

Proof. We have

$$u_x^2 + u_y^2 = 2+y$$
$$u(x,0) = u_0(x) = x.$$

Rewrite the equation as

$$F(x,y,z,p,q) = p^2 + q^2 - y - 2 = 0.$$

Γ is parameterized by $\Gamma: \ (s, 0, s, \phi(s), \psi(s))$.
We need to complete Γ to a strip. Find $\phi(s)$ and $\psi(s)$, the initial conditions for $p(s,t)$ and $q(s,t)$, respectively:

- $F(f(s), g(s), h(s), \phi(s), \psi(s)) = 0,$
  $F(s, 0, s, \phi(s), \psi(s)) = 0,$
  $\phi(s)^2 + \psi(s)^2 - 2 = 0,$
  $\phi(s)^2 + \psi(s)^2 = 2.$

- $h'(s) = \phi(s)f'(s) + \psi(s)g'(s),$
  $1 = \phi(s).$
  $\Rightarrow \quad \psi(s) = \pm 1.$

Since we have a condition that $q(s,t) > 0$, we choose $q(s,0) = \psi(s) = 1$.
Therefore, now Γ is parametrized by $\Gamma: \ (s, 0, s, 1, 1)$.

$$\frac{dx}{dt} = F_p = 2p \quad\Rightarrow\quad \frac{dx}{dt} = 2 \quad\Rightarrow\quad x = 2t + s,$$
$$\frac{dy}{dt} = F_q = 2q \quad\Rightarrow\quad \frac{dy}{dt} = 2t+2 \quad\Rightarrow\quad y = t^2 + 2t,$$
$$\frac{dz}{dt} = pF_p + qF_q = 2p^2 + 2q^2 = 2y + 4 \quad\Rightarrow\quad \frac{dz}{dt} = 2t^2 + 4t + 4,$$
$$\Rightarrow\quad z = \frac{2}{3}t^3 + 2t^2 + 4t + s = \frac{2}{3}t^3 + 2t^2 + 4t + x - 2t = \frac{2}{3}t^3 + 2t^2 + 2t + x,$$
$$\frac{dp}{dt} = -F_x - F_z p = 0 \quad\Rightarrow\quad p = 1,$$
$$\frac{dq}{dt} = -F_y - F_z q = 1 \quad\Rightarrow\quad q = t+1.$$

We solve $y = t^2 + 2t$, a quadratic equation in t, $t^2 + 2t - y = 0$, for t in terms of y to get:

$$t = -1 \pm \sqrt{1+y}.$$
$$\Rightarrow \quad u(x,y) = \frac{2}{3}(-1 \pm \sqrt{1+y})^3 + 2(-1 \pm \sqrt{1+y})^2 + 2(-1 \pm \sqrt{1+y}) + x.$$

Both $u_\pm$ satisfy the PDE. $u_x = 1$, $u_y = \pm\sqrt{y+1} \quad\Rightarrow\quad u_x^2 + u_y^2 = y+2$ ✓
u_+ satisfies $u_+(x,0) = x$ ✓. However, u_- does not satisfy IC, i.e. $u_-(x,0) = x - \frac{4}{3}$. □

Problem (S'02, #6). *Consider the equation*

$$u_x + u_x u_y = 1,$$
$$u(x,0) = f(x).$$

Assuming that f is differentiable, what conditions on f insure that the problem is noncharacteristic? If f satisfies those conditions, show that the solution is

$$u(x,y) = f(r) - y + \frac{2y}{f'(r)},$$

where r must satisfy $y = (f'(r))^2(x-r)$.
Finally, show that one can solve the equation for (x,y) in a sufficiently small neighborhood of $(x_0,0)$ with $r(x_0,0) = x_0$.

Proof. **Solved**.
In order to solve the Cauchy problem in a neighborhood of Γ, need:

$$f'(s) \cdot F_q[f,g,h,\phi,\psi](s) - g'(s) \cdot F_p[f,g,h,\phi,\psi](s) \neq 0,$$
$$1 \cdot h'(s) - 0 \cdot \Big(1 + \frac{1-h'(s)}{h'(s)}\Big) \neq 0,$$
$$h'(s) \neq 0.$$

Thus, $h'(s) \neq 0$ ensures that the problem is noncharacteristic.
To show that one can solve $y = (f'(s))^2(x-s)$ for (x,y) in a sufficiently small neighborhood of $(x_0,0)$ with $s(x_0,0) = x_0$, let

$$\begin{aligned} G(x,y,s) &= (f'(s))^2(x-s) - y = 0, \\ G(x_0,0,x_0) &= 0, \\ G_r(x_0,0,x_0) &= -(f'(s))^2. \end{aligned}$$

Hence, if $f'(s) \neq 0$, $\forall s$, then $G_s(x_0,0,x_0) \neq 0$ and we can use the implicit function theorem in a neighborhood of $(x_0,0,x_0)$ to get

$$G(x,y,h(x,y)) = 0$$

and solve the equation in terms of x and y. ☐

Problem (S'00, #1). *Find the solutions of*

$$(u_x)^2 + (u_y)^2 = 1$$

in a neighborhood of the curve $y = \frac{x^2}{2}$ *satisfying the conditions*

$$u\Big(x, \frac{x^2}{2}\Big) = 0 \qquad \textit{and} \qquad u_y\Big(x, \frac{x^2}{2}\Big) > 0.$$

Leave your answer in parametric form.

Proof. Rewrite the equation as

$$F(x, y, z, p, q) = p^2 + q^2 - 1 = 0.$$

Γ is parameterized by $\Gamma: \ (s, \frac{s^2}{2}, 0, \phi(s), \psi(s))$.
We need to complete Γ to a strip. Find $\phi(s)$ and $\psi(s)$, the initial conditions for $p(s,t)$ and $q(s,t)$, respectively:

- $$\begin{aligned} F(f(s), g(s), h(s), \phi(s), \psi(s)) &= 0, \\ F\big(s, \frac{s^2}{2}, 0, \phi(s), \psi(s)\big) &= 0, \\ \phi(s)^2 + \psi(s)^2 &= 1. \end{aligned}$$

- $$\begin{aligned} h'(s) &= \phi(s) f'(s) + \psi(s) g'(s), \\ 0 &= \phi(s) + s\psi(s), \\ \phi(s) &= -s\psi(s). \end{aligned}$$

$$\text{Thus,} \quad s^2\psi(s)^2 + \psi(s)^2 = 1 \quad \Rightarrow \quad \psi(s)^2 = \frac{1}{s^2+1}.$$

Since, by assumption, $\psi(s) > 0$, we have $\psi(s) = \frac{1}{\sqrt{s^2+1}}$.
Therefore, now Γ is parametrized by $\Gamma: \ \big(s, \frac{s^2}{2}, 0, \frac{-s}{\sqrt{s^2+1}}, \frac{1}{\sqrt{s^2+1}}\big)$.

$$\begin{aligned}
\frac{dx}{dt} &= F_p = 2p = \frac{-2s}{\sqrt{s^2+1}} \quad \Rightarrow \quad x = \frac{-2st}{\sqrt{s^2+1}} + s, \\
\frac{dy}{dt} &= F_q = 2q = \frac{2}{\sqrt{s^2+1}} \quad \Rightarrow \quad y = \frac{2t}{\sqrt{s^2+1}} + \frac{s^2}{2}, \\
\frac{dz}{dt} &= pF_p + qF_q = 2p^2 + 2q^2 = 2 \quad \Rightarrow \quad z = 2t, \\
\frac{dp}{dt} &= -F_x - F_z p = 0 \quad \Rightarrow \quad p = \frac{-s}{\sqrt{s^2+1}}, \\
\frac{dq}{dt} &= -F_y - F_z q = 0 \quad \Rightarrow \quad q = \frac{1}{\sqrt{s^2+1}}.
\end{aligned}$$

Thus, in parametric form,

$$\begin{aligned}
z(s,t) &= 2t, \\
x(s,t) &= \frac{-2st}{\sqrt{s^2+1}} + s, \\
y(s,t) &= \frac{2t}{\sqrt{s^2+1}} + \frac{s^2}{2}.
\end{aligned}$$

□

13.2 Three Spatial Dimensions

Problem (S'96, #2). *Solve the following Cauchy problem*[21]*:*

$$u_x + u_y^2 + u_z^2 = 1,$$
$$u(0, y, z) = y \cdot z.$$

Proof. Rewrite the equation as

$$u_{x_1} + u_{x_2}^2 + u_{x_3}^2 = 1,$$
$$u(0, x_2, x_3) = x_2 \cdot x_3.$$

Write a general nonlinear equation

$$F(x_1, x_2, x_3, z, p_1, p_2, p_3) = p_1 + p_2^2 + p_3^2 - 1 = 0.$$

Γ is parameterized by

$$\Gamma : \Big(\underbrace{0}_{x_1(s_1,s_2,0)}, \underbrace{s_1}_{x_2(s_1,s_2,0)}, \underbrace{s_2}_{x_3(s_1,s_2,0)}, \underbrace{s_1 s_2}_{z(s_1,s_2,0)}, \underbrace{\phi_1(s_1,s_2)}_{p_1(s_1,s_2,0)}, \underbrace{\phi_2(s_1,s_2)}_{p_2(s_1,s_2,0)}, \underbrace{\phi_3(s_1,s_2)}_{p_3(s_1,s_2,0)} \Big)$$

We need to complete Γ to a strip. Find $\phi_1(s_1, s_2)$, $\phi_2(s_1, s_2)$, and $\phi_3(s_1, s_2)$, the initial conditions for $p_1(s_1, s_2, t)$, $p_2(s_1, s_2, t)$, and $p_3(s_1, s_2, t)$, respectively:

- $$F\big(f_1(s_1,s_2), f_2(s_1,s_2), f_3(s_1,s_2), h(s_1,s_2), \phi_1, \phi_2, \phi_3\big) = 0,$$
  $$F\big(0, s_1, s_2, s_1 s_2, \phi_1, \phi_2, \phi_3\big) = \phi_1 + \phi_2^2 + \phi_3^2 - 1 = 0,$$
  $$\Rightarrow \quad \phi_1 + \phi_2^2 + \phi_3^2 = 1.$$

- $$\frac{\partial h}{\partial s_1} = \phi_1 \frac{\partial f_1}{\partial s_1} + \phi_2 \frac{\partial f_2}{\partial s_1} + \phi_3 \frac{\partial f_3}{\partial s_1},$$
  $$\Rightarrow \quad s_2 = \phi_2.$$
- $$\frac{\partial h}{\partial s_2} = \phi_1 \frac{\partial f_1}{\partial s_2} + \phi_2 \frac{\partial f_2}{\partial s_2} + \phi_3 \frac{\partial f_3}{\partial s_2},$$
  $$\Rightarrow \quad s_1 = \phi_3.$$

Thus, we have: $\phi_2 = s_2$, $\phi_3 = s_1$, $\phi_1 = -s_1^2 - s_2^2 + 1$.

$$\Gamma : \Big(\underbrace{0}_{x_1(s_1,s_2,0)}, \underbrace{s_1}_{x_2(s_1,s_2,0)}, \underbrace{s_2}_{x_3(s_1,s_2,0)}, \underbrace{s_1 s_2}_{z(s_1,s_2,0)}, \underbrace{-s_1^2 - s_2^2 + 1}_{p_1(s_1,s_2,0)}, \underbrace{s_2}_{p_2(s_1,s_2,0)}, \underbrace{s_1}_{p_3(s_1,s_2,0)} \Big)$$

[21] This problem is very similar to an already hand-written solved problem F'95 #2.

The characteristic equations are

$$\begin{aligned}
\frac{dx_1}{dt} &= F_{p_1} = 1 \quad \Rightarrow \quad x_1 = t, \\
\frac{dx_2}{dt} &= F_{p_2} = 2p_2 \quad \Rightarrow \quad \frac{dx_2}{dt} = 2s_2 \quad \Rightarrow \quad x_2 = 2s_2 t + s_1, \\
\frac{dx_3}{dt} &= F_{p_3} = 2p_3 \quad \Rightarrow \quad \frac{dx_3}{dt} = 2s_1 \quad \Rightarrow \quad x_3 = 2s_1 t + s_2, \\
\frac{dz}{dt} &= p_1 F_{p_1} + p_2 F_{p_2} + p_3 F_{p_3} = p_1 + 2p_2^2 + 2p_3^2 = -s_1^2 - s_2^2 + 1 + 2s_2^2 + 2s_1^2 \\
&= s_1^2 + s_2^2 + 1 \quad \Rightarrow \quad z = (s_1^2 + s_2^2 + 1)t + s_1 s_2, \\
\frac{dp_1}{dt} &= -F_{x_1} - p_1 F_z = 0 \quad \Rightarrow \quad p_1 = -s_1^2 - s_2^2 + 1, \\
\frac{dp_2}{dt} &= -F_{x_2} - p_2 F_z = 0 \quad \Rightarrow \quad p_2 = s_2, \\
\frac{dp_3}{dt} &= -F_{x_3} - p_3 F_z = 0 \quad \Rightarrow \quad p_3 = s_1.
\end{aligned}$$

Thus, we have

$$\begin{cases} x_1 = t \\ x_2 = 2s_2 t + s_1 \\ x_3 = 2s_1 t + s_2 \\ z = (s_1^2 + s_2^2 + 1)t + s_1 s_2 \end{cases} \Rightarrow \begin{cases} t = x_1 \\ s_1 = x_2 - 2s_2 t \\ s_2 = x_3 - 2s_1 t \\ z = (s_1^2 + s_2^2 + 1)t + s_1 s_2 \end{cases} \Rightarrow \begin{cases} t = x_1 \\ s_1 = \frac{x_2 - 2x_1 x_3}{1 - 4x_1^2} \\ s_2 = \frac{x_3 - 2x_1 x_2}{1 - 4x_1^2} \\ z = (s_1^2 + s_2^2 + 1)t + s_1 s_2 \end{cases}$$

$$\Rightarrow \quad u(x_1, x_2, x_3) = \left[\Big(\frac{x_2 - 2x_1 x_3}{1 - 4x_1^2}\Big)^2 + \Big(\frac{x_3 - 2x_1 x_2}{1 - 4x_1^2}\Big)^2 + 1\right] x_1 + \Big(\frac{x_2 - 2x_1 x_3}{1 - 4x_1^2}\Big)\Big(\frac{x_3 - 2x_1 x_2}{1 - 4x_1^2}\Big).$$

□

Problem (F'95, #2). *Solve the following Cauchy problem*

$$\begin{aligned}
u_x + u_y + u_z^3 &= x + y + z, \\
u(x, y, 0) &= xy.
\end{aligned}$$

Proof. **Solved** □

Problem (S'94, #1). *Solve the following PDE for $f(x, y, t)$:*

$$f_t + xf_x + 3t^2 f_y = 0$$
$$f(x, y, 0) = x^2 + y^2.$$

Proof. Rewrite the equation as ($x \to x_1,\ y \to x_2,\ t \to x_3,\ f \to u$):

$$x_1 u_{x_1} + 3x_3^2 u_{x_2} + u_{x_3} = 0,$$
$$u(x_1, x_2, 0) = x_1^2 + x_2^2.$$

$$F(x_1, x_2, x_3, z, p_1, p_2, p_3) = x_1 p_1 + 3x_3^2 p_2 + p_3 = 0.$$

Γ is parameterized by

$$\Gamma : \Big(\underbrace{s_1}_{x_1(s_1,s_2,0)}, \underbrace{s_2}_{x_2(s_1,s_2,0)}, \underbrace{0}_{x_3(s_1,s_2,0)}, \underbrace{s_1^2 + s_2^2}_{z(s_1,s_2,0)}, \underbrace{\phi_1(s_1, s_2)}_{p_1(s_1,s_2,0)}, \underbrace{\phi_2(s_1, s_2)}_{p_2(s_1,s_2,0)}, \underbrace{\phi_3(s_1, s_2)}_{p_3(s_1,s_2,0)} \Big)$$

We need to complete Γ to a strip. Find $\phi_1(s_1, s_2)$, $\phi_2(s_1, s_2)$, and $\phi_3(s_1, s_2)$, the initial conditions for $p_1(s_1, s_2, t)$, $p_2(s_1, s_2, t)$, and $p_3(s_1, s_2, t)$, respectively:

- $$F\big(f_1(s_1, s_2), f_2(s_1, s_2), f_3(s_1, s_2), h(s_1, s_2), \phi_1, \phi_2, \phi_3\big) = 0,$$
  $$F\big(s_1, s_2, 0, s_1^2 + s_2^2, \phi_1, \phi_2, \phi_3\big) = s_1\phi_1 + \phi_3 = 0,$$
  $$\Rightarrow \quad \phi_3 = s_1\phi_1.$$
- $$\frac{\partial h}{\partial s_1} = \phi_1 \frac{\partial f_1}{\partial s_1} + \phi_2 \frac{\partial f_2}{\partial s_1} + \phi_3 \frac{\partial f_3}{\partial s_1},$$
  $$\Rightarrow \quad 2s_1 = \phi_1.$$
- $$\frac{\partial h}{\partial s_2} = \phi_1 \frac{\partial f_1}{\partial s_2} + \phi_2 \frac{\partial f_2}{\partial s_2} + \phi_3 \frac{\partial f_3}{\partial s_2},$$
  $$\Rightarrow \quad 2s_2 = \phi_2.$$

Thus, we have: $\phi_1 = 2s_1, \quad \phi_2 = 2s_2, \quad \phi_3 = 2s_1^2.$

$$\Gamma : \Big(\underbrace{s_1}_{x_1(s_1,s_2,0)}, \underbrace{s_2}_{x_2(s_1,s_2,0)}, \underbrace{0}_{x_3(s_1,s_2,0)}, \underbrace{s_1^2 + s_2^2}_{z(s_1,s_2,0)}, \underbrace{2s_1}_{p_1(s_1,s_2,0)}, \underbrace{2s_2}_{p_2(s_1,s_2,0)}, \underbrace{2s_1^2}_{p_3(s_1,s_2,0)} \Big)$$

The characteristic equations are

$$\frac{dx_1}{dt} = F_{p_1} = x_1 \quad \Rightarrow \quad x_1 = s_1 e^t,$$
$$\frac{dx_2}{dt} = F_{p_2} = 3x_3^2 \quad \Rightarrow \quad \frac{dx_2}{dt} = 3t^2 \quad \Rightarrow \quad x_2 = t^3 + s_2,$$
$$\frac{dx_3}{dt} = F_{p_3} = 1 \quad \Rightarrow \quad x_3 = t,$$
$$\frac{dz}{dt} = p_1 F_{p_1} + p_2 F_{p_2} + p_3 F_{p_3} = p_1 x_1 + p_2 3x_3^2 + p_3 = 0 \quad \Rightarrow \quad z = s_1^2 + s_2^2,$$
$$\frac{dp_1}{dt} = -F_{x_1} - p_1 F_z = -p_1 \quad \Rightarrow \quad p_1 = 2s_1 e^{-t},$$
$$\frac{dp_2}{dt} = -F_{x_2} - p_2 F_z = 0 \quad \Rightarrow \quad p_2 = 2s_2,$$
$$\frac{dp_3}{dt} = -F_{x_3} - p_3 F_z = -6x_3 p_2 \quad \Rightarrow \quad \frac{dp_3}{dt} = -12ts_2 \quad \Rightarrow \quad p_3 = -6t^2 s_2 + 2s_1^2.$$

With $t = x_3,\ s_1 = x_1 e^{-x^3},\ s_2 = x_2 - x_3^3,$ we have

$$u(x_1, x_2, x_3) = x_1^2 e^{-2x^3} + (x_2 - x_3^3)^2. \qquad \Big(f(x, y, t) = x^2 e^{-2t} + (y - t^3)^2.\Big)$$

The solution satisfies the PDE and initial condition. ☐

Problem (F'93, #3). *Find the solution of the following equation*

$$f_t + xf_x + (x+t)f_y = t^3$$
$$f(x,y,0) = xy.$$

Proof. Rewrite the equation as ($x \to x_1,\ y \to x_2,\ t \to x_3,\ f \to u$):

$$x_1 u_{x_1} + (x_1 + x_3)u_{x_2} + u_{x_3} = x^3,$$
$$u(x_1, x_2, 0) = x_1 x_2.$$

Method I: Treat the equation as a QUASILINEAR equation.
Γ is parameterized by $\Gamma:\ (s_1, s_2, 0, s_1 s_2)$.

$$\frac{dx_1}{dt} = x_1 \quad \Rightarrow \quad x_1 = s_1 e^t,$$
$$\frac{dx_2}{dt} = x_1 + x_3 \quad \Rightarrow \quad \frac{dx_2}{dt} = s_1 e^t + t \quad \Rightarrow \quad x_2 = s_1 e^t + \frac{t^2}{2} + s_2 - s_1,$$
$$\frac{dx_3}{dt} = 1 \quad \Rightarrow \quad x_3 = t,$$
$$\frac{dz}{dt} = x_3^3 \quad \Rightarrow \quad \frac{dz}{dt} = t^3 \quad \Rightarrow \quad z = \frac{t^4}{4} + s_1 s_2.$$

Since $t = x_3,\ s_1 = x_1 e^{-x_3},\ s_2 = x_2 - s_1 e^t - \frac{t^2}{2} + s_1 = x_2 - x_1 - \frac{x_3^2}{2} + x_1 e^{-x_3}$, we have

$$u(x_1, x_2, x_3) = \frac{x_3^4}{4} + x_1 e^{-x_3}\left(x_2 - x_1 - \frac{x_3^2}{2} + x_1 e^{-x_3}\right), \qquad \text{or}$$
$$f(x,y,t) = \frac{t^4}{4} + xe^{-t}\left(y - x - \frac{t^2}{2} + xe^{-t}\right).$$

The solution satisfies the PDE and initial condition.

Method II: Treat the equation as a fully NONLINEAR equation.

$$F(x_1, x_2, x_3, z, p_1, p_2, p_3) = x_1 p_1 + (x_1 + x_3)p_2 + p_3 - x_3^3 = 0.$$

Γ is parameterized by

$$\Gamma:\ \Big(\underbrace{s_1}_{x_1(s_1,s_2,0)},\ \underbrace{s_2}_{x_2(s_1,s_2,0)},\ \underbrace{0}_{x_3(s_1,s_2,0)},\ \underbrace{s_1 s_2}_{z(s_1,s_2,0)},\ \underbrace{\phi_1(s_1,s_2)}_{p_1(s_1,s_2,0)},\ \underbrace{\phi_2(s_1,s_2)}_{p_2(s_1,s_2,0)},\ \underbrace{\phi_3(s_1,s_2)}_{p_3(s_1,s_2,0)}\Big)$$

We need to complete Γ to a strip. Find $\phi_1(s_1,s_2)$, $\phi_2(s_1,s_2)$, and $\phi_3(s_1,s_2)$, the initial conditions for $p_1(s_1,s_2,t)$, $p_2(s_1,s_2,t)$, and $p_3(s_1,s_2,t)$, respectively:

- $$F\big(f_1(s_1,s_2), f_2(s_1,s_2), f_3(s_1,s_2), h(s_1,s_2), \phi_1, \phi_2, \phi_3\big) = 0,$$
  $$F\big(s_1, s_2, 0, s_1 s_2, \phi_1, \phi_2, \phi_3\big) = s_1\phi_1 + s_1\phi_2 + \phi_3 = 0,$$
  $$\Rightarrow \quad \phi_3 = -s_1(\phi_1 + \phi_2).$$
- $$\frac{\partial h}{\partial s_1} = \phi_1 \frac{\partial f_1}{\partial s_1} + \phi_2 \frac{\partial f_2}{\partial s_1} + \phi_3 \frac{\partial f_3}{\partial s_1},$$
  $$\Rightarrow \quad s_2 = \phi_1.$$
- $$\frac{\partial h}{\partial s_2} = \phi_1 \frac{\partial f_1}{\partial s_2} + \phi_2 \frac{\partial f_2}{\partial s_2} + \phi_3 \frac{\partial f_3}{\partial s_2},$$
  $$\Rightarrow \quad s_1 = \phi_2.$$

Thus, we have: $\phi_1 = s_2,\quad \phi_2 = s_1,\quad \phi_3 = -s_1^2 - s_1 s_2$.

$$\Gamma:\ \Big(\underbrace{s_1}_{x_1(s_1,s_2,0)},\ \underbrace{s_2}_{x_2(s_1,s_2,0)},\ \underbrace{0}_{x_3(s_1,s_2,0)},\ \underbrace{s_1 s_2}_{z(s_1,s_2,0)},\ \underbrace{s_2}_{p_1(s_1,s_2,0)},\ \underbrace{s_1}_{p_2(s_1,s_2,0)},\ \underbrace{-s_1^2 - s_1 s_2}_{p_3(s_1,s_2,0)}\Big)$$

The characteristic equations are

$$
\begin{aligned}
\frac{dx_1}{dt} &= F_{p_1} = x_1 \quad \Rightarrow \quad x_1 = s_1 e^t, \\
\frac{dx_2}{dt} &= F_{p_2} = x_1 + x_3 \quad \Rightarrow \quad \frac{dx_2}{dt} = s_1 e^t + t \quad \Rightarrow \quad x_2 = s_1 e^t + \frac{t^2}{2} + s_2 - s_1, \\
\frac{dx_3}{dt} &= F_{p_3} = 1 \quad \Rightarrow \quad x_3 = t, \\
\frac{dz}{dt} &= p_1 F_{p_1} + p_2 F_{p_2} + p_3 F_{p_3} = p_1 x_1 + p_2(x_1 + x_3) + p_3 = x_3^3 = t^3 \quad \Rightarrow \quad z = \frac{t^4}{4} + s_1 s_2, \\
\frac{dp_1}{dt} &= -F_{x_1} - p_1 F_z = -p_1 - p_2 = -p_1 - s_1 \quad \Rightarrow \quad p_1 = 2s_1 e^{-t} - s_1, \\
\frac{dp_2}{dt} &= -F_{x_2} - p_2 F_z = 0 \quad \Rightarrow \quad p_2 = s_1, \\
\frac{dp_3}{dt} &= -F_{x_3} - p_3 F_z = 3x_3^2 - p_2 = 3t^2 - s_1 \quad \Rightarrow \quad p_3 = t^3 - s_1 t - s_1^2 - s_1 s_2.
\end{aligned}
$$

With $t = x_3$, $s_1 = x_1 e^{-x^3}$, $s_2 = x_2 - s_1 e^t - \frac{t^2}{2} + s_1 = x_2 - x_1 - \frac{x_3^2}{2} + x_1 e^{-x^3}$, we have

$$
\begin{aligned}
u(x_1, x_2, x_3) &= \frac{x_3^4}{4} + x_1 e^{-x^3}\left(x_2 - x_1 - \frac{x_3^2}{2} + x_1 e^{-x^3}\right), \qquad \text{or} \\
f(x, y, t) &= \frac{t^4}{4} + x e^{-t}\left(y - x - \frac{t^2}{2} + x e^{-t}\right).
\end{aligned}
$$

[22] The solution satisfies the PDE and initial condition. □

[22]Variable t in the derivatives of characteristics equations and t in the solution $f(x, y, t)$ are different entities.

Problem (F'92, #1). *Solve the initial value problem*

$$u_t + \alpha u_x + \beta u_y + \gamma u = 0 \qquad \textit{for} \;\; t > 0$$
$$u(x, y, 0) = \varphi(x, y),$$

in which α, β and γ are real constants and φ is a smooth function.

Proof. Rewrite the equation as $(x \to x_1,\ y \to x_2,\ t \to x_3)$[23]:

$$\alpha u_{x_1} + \beta u_{x_2} + u_{x_3} = -\gamma u,$$
$$u(x_1, x_2, 0) = \varphi(x_1, x_2).$$

Γ is parameterized by $\Gamma : \ (s_1, s_2, 0, \varphi(s_1, s_2))$.

$$\begin{aligned}
\frac{dx_1}{dt} &= \alpha \quad \Rightarrow \quad x_1 = \alpha t + s_1, \\
\frac{dx_2}{dt} &= \beta \quad \Rightarrow \quad x_2 = \beta t + s_2, \\
\frac{dx_3}{dt} &= 1 \quad \Rightarrow \quad x_3 = t, \\
\frac{dz}{dt} &= -\gamma z \quad \Rightarrow \quad \frac{dz}{z} = -\gamma dt \quad \Rightarrow \quad z = \varphi(s_1, s_2) e^{-\gamma t}.
\end{aligned}$$

$$J \equiv \det\left(\frac{\partial(x_1, x_2, x_3)}{\partial(s_1, s_2, t)}\right) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \beta & 1 \end{vmatrix} = 1 \neq 0 \qquad \Rightarrow \qquad J \text{ is invertible.}$$

Since $t = x_3,\ s_1 = x_1 - \alpha x_3,\ s_2 = x_2 - \beta x_3$, we have

$$\begin{aligned}
u(x_1, x_2, x_3) &= \varphi(x_1 - \alpha x_3, x_2 - \beta x_3) e^{-\gamma x_3}, \qquad \text{or} \\
u(x, y, t) &= \varphi(x - \alpha t, y - \beta t) e^{-\gamma t}.
\end{aligned}$$

The solution satisfies the PDE and initial condition.[24] ☐

---

[23]Variable t as a third coordinate of u and variable t used to parametrize characteristic equations are two different entities.

[24]Chain Rule: $u(x_1, x_2, x_3) = \varphi(f(x_1, x_2, x_3), g(x_1, x_2, x_3))$, then $u_{x_1} = \frac{\partial \varphi}{\partial f}\frac{\partial f}{\partial x_1} + \frac{\partial \varphi}{\partial g}\frac{\partial g}{\partial x_1}$.

Problem (F'94, #2). *Find the solution of the Cauchy problem*

$$u_t(x,y,t) + au_x(x,y,t) + bu_y(x,y,t) + c(x,y,t)u(x,y,t) = 0$$
$$u(x,y,0) = u_0(x,y),$$

where $0 < t < +\infty, \quad -\infty < x < +\infty, \quad -\infty < y < +\infty,$
a, b *are constants,* $c(x,y,t)$ *is a continuous function of* (x,y,t)*, and* $u_0(x,y)$ *is a continuous function of* (x,y).

Proof. Rewrite the equation as ($x \to x_1$, $y \to x_2$, $t \to x_3$):

$$au_{x_1} + bu_{x_2} + u_{x_3} = -c(x_1,x_2,x_3)u,$$
$$u(x_1,x_2,0) = u_0(x_1,x_2).$$

Γ is parameterized by $\Gamma : \ (s_1, s_2, 0, u_0(s_1,s_2))$.

$$\frac{dx_1}{dt} = a \quad \Rightarrow \quad x_1 = at + s_1,$$
$$\frac{dx_2}{dt} = b \quad \Rightarrow \quad x_2 = bt + s_2,$$
$$\frac{dx_3}{dt} = 1 \quad \Rightarrow \quad x_3 = t,$$
$$\frac{dz}{dt} = -c(x_1,x_2,x_3)z \quad \Rightarrow \quad \frac{dz}{dt} = -c(at+s_1, bt+s_2, t)z \quad \Rightarrow \quad \frac{dz}{z} = -c(at+s_1, bt+s_2, t)dt$$
$$\Rightarrow \quad \ln z = -\int_0^t c(a\xi + s_1, b\xi + s_2, \xi)d\xi + c_1(s_1,s_2),$$
$$\Rightarrow \quad z(s_1,s_2,t) = c_2(s_1,s_2)e^{-\int_0^t c(a\xi+s_1,b\xi+s_2,\xi)d\xi} \quad \Rightarrow \quad z(s_1,s_2,0) = c_2(s_1,s_2) = u_0(s_2,s_2),$$
$$\Rightarrow \quad z(s_1,s_2,t) = u_0(s_1,s_2)e^{-\int_0^t c(a\xi+s_1,b\xi+s_2,\xi)d\xi}.$$

$$J \equiv \det\left(\frac{\partial(x_1,x_2,x_3)}{\partial(s_1,s_2,t)}\right) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{vmatrix} = 1 \neq 0 \quad \Rightarrow \quad J \text{ is invertible.}$$

Since $t = x_3$, $s_1 = x_1 - ax_3$, $s_2 = x_2 - bx_3$, we have

$$u(x_1,x_2,x_3) = u_0(x_1 - ax_3, x_2 - bx_3)e^{-\int_0^{x_3} c(a\xi+x_1-ax_3,b\xi+x_2-bx_3,\xi)d\xi}$$
$$= u_0(x_1 - ax_3, x_2 - bx_3)e^{-\int_0^{x_3} c(x_1+a(\xi-x_3),x_2+b(\xi-x_3),\xi)d\xi}, \quad \text{or}$$
$$u(x,y,t) = u_0(x-at, y-bt)e^{-\int_0^t c(x+a(\xi-t),y+b(\xi-t),\xi)d\xi}.$$

□

Problem (F'89, #4). *Consider the first order partial differential equation*

$$u_t + (\alpha + \beta t)u_x + \gamma e^t u_y = 0 \tag{13.1}$$

in which α, β and γ are constants.
a) *For this equation, solve the initial value problem with initial data*

$$u(x, y, t = 0) = \sin(xy) \tag{13.2}$$

for all x and y and for $t \geq 0$.
b) *Suppose that this initial data is prescribed only for $x \geq 0$ (and all y) and consider (13.1) in the region $x \geq 0$, $t \geq 0$ and all y. For which values of α, β and γ is it possible to solve the initial-boundary value problem (13.1), (13.2) with $u(x = 0, y, t)$ given for $t \geq 0$?*
For non-permissible values of α, β and γ, where can boundary values be prescribed in order to determine a solution of (13.1) in the region ($x \geq 0$, $t \geq 0$, all y).

Proof. **a)** Rewrite the equation as ($x \to x_1,\ y \to x_2,\ t \to x_3$):

$$(\alpha + \beta x_3)u_{x_1} + \gamma e^{x_3} u_{x_2} + u_{x_3} = 0,$$
$$u(x_1, x_2, 0) = \sin(x_1 x_2).$$

Γ is parameterized by $\Gamma :\ (s_1, s_2, 0, \sin(s_1 s_2))$.

$$\begin{aligned}
\frac{dx_1}{dt} &= \alpha + \beta x_3 \quad \Rightarrow \quad \frac{dx_1}{dt} = \alpha + \beta t \quad \Rightarrow \quad x_1 = \frac{\beta t^2}{2} + \alpha t + s_1, \\
\frac{dx_2}{dt} &= \gamma e^{x_3} \quad \Rightarrow \quad \frac{dx_2}{dt} = \gamma e^t \quad \Rightarrow \quad x_2 = \gamma e^t - \gamma + s_2, \\
\frac{dx_3}{dt} &= 1 \quad \Rightarrow \quad x_3 = t, \\
\frac{dz}{dt} &= 0 \quad \Rightarrow \quad z = \sin(s_1 s_2).
\end{aligned}$$

$$J \equiv \det\left(\frac{\partial(x_1, x_2, x_3)}{\partial(s_1, s_2, t)}\right) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta t + \alpha & \gamma e^t & 1 \end{vmatrix} = 1 \neq 0 \qquad \Rightarrow \qquad J \text{ is invertible.}$$

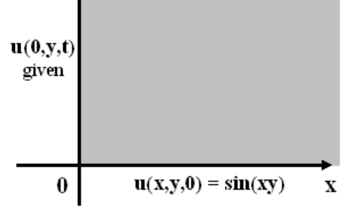
Since $t = x_3,\ s_1 = x_1 - \frac{\beta x_3^2}{2} - \alpha x_3,\ s_2 = x_2 - \gamma e^{x_3} + \gamma$, we have

$$\begin{aligned}
u(x_1, x_2, x_3) &= \sin((x_1 - \frac{\beta x_3^2}{2} - \alpha x_3)(x_2 - \gamma e^{x_3} + \gamma)), \qquad \text{or} \\
u(x, y, t) &= \sin((x - \frac{\beta t^2}{2} - \alpha t)(y - \gamma e^t + \gamma)).
\end{aligned}$$

The solution satisfies the PDE and initial condition.

b) We need a *compatibility condition* between the initial and boundary values to hold on y-axis ($x = 0, t = 0$):

$$\begin{aligned}
u(x = 0, y, 0) &= u(0, y, t = 0), \\
0 &= 0.
\end{aligned}$$


u(0,y,t)
given
0
u(x,y,0) = sin(xy)
x

□

14 Problems: First-Order Systems

Problem (S'01, #2a). *Find the solution* $u = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix}$, $(x,t) \in \mathbb{R} \times \mathbb{R}$, *to the (strictly) hyperbolic equation*

$$u_t - \begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix} u_x = 0,$$

satisfying $\begin{pmatrix} u_1(x,0) \\ u_2(x,0) \end{pmatrix} = \begin{pmatrix} e^{ixa} \\ 0 \end{pmatrix}$, $a \in \mathbb{R}$.

Proof. Rewrite the equation as

$$U_t + \begin{pmatrix} -1 & 0 \\ -5 & -3 \end{pmatrix} U_x = 0,$$
$$U(x,0) = \begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix} = \begin{pmatrix} e^{ixa} \\ 0 \end{pmatrix}.$$

The eigenvalues of the matrix A are $\lambda_1 = -1$, $\lambda_2 = -3$ and the corresponding eigenvectors are $e_1 = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus,

$$\Lambda = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix}, \qquad \Gamma^{-1} = \frac{1}{\det \Gamma} \cdot \Gamma = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{5}{2} & 1 \end{pmatrix}.$$

Let $U = \Gamma V$. Then,

$$U_t + AU_x = 0,$$
$$\Gamma V_t + A\Gamma V_x = 0,$$
$$V_t + \Gamma^{-1}A\Gamma V_x = 0,$$
$$V_t + \Lambda V_x = 0.$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} V_x = 0,$$
$$V(x,0) = \Gamma^{-1}U(x,0) = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{5}{2} & 1 \end{pmatrix} \begin{pmatrix} e^{ixa} \\ 0 \end{pmatrix} = \frac{1}{2} e^{ixa} \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

We have two initial value problems

$$\begin{cases} v_t^{(1)} - v_x^{(1)} = 0, \\ v^{(1)}(x,0) = \frac{1}{2} e^{ixa}; \end{cases} \qquad \begin{cases} v_t^{(2)} - 3v_x^{(2)} = 0, \\ v^{(2)}(x,0) = \frac{5}{2} e^{ixa}, \end{cases}$$

which we solve by characteristics to get

$$v^{(1)}(x,t) = \frac{1}{2} e^{ia(x+t)}, \qquad v^{(2)}(x,t) = \frac{5}{2} e^{ia(x+3t)}.$$

We solve for U: $\quad U = \Gamma V = \Gamma \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} e^{ia(x+t)} \\ \frac{5}{2} e^{ia(x+3t)} \end{pmatrix}.$

Thus, $\quad U = \begin{pmatrix} u^{(1)}(x,t) \\ u^{(2)}(x,t) \end{pmatrix} = \begin{pmatrix} e^{ia(x+t)} \\ -\frac{5}{2} e^{ia(x+t)} + \frac{5}{2} e^{ia(x+3t)} \end{pmatrix}.$

Can check that this is the correct solution by plugging it into the original equation. ☐

Part (b) of the problem is solved in the Fourier Transform section.

Problem (S'96, #7). *Solve the following initial-boundary value problem in the domain $x > 0$, $t > 0$, for the unknown vector $U = \begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix}$:*

$$U_t + \begin{pmatrix} -2 & 3 \\ 0 & 1 \end{pmatrix} U_x = 0. \tag{14.1}$$

$$U(x,0) = \begin{pmatrix} \sin x \\ 0 \end{pmatrix} \quad \textit{and} \quad u^{(2)}(0,t) = t.$$

Proof. The eigenvalues of the matrix A are $\lambda_1 = -2$, $\lambda_2 = 1$ and the corresponding eigenvectors are $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus,

$$\Lambda = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma^{-1} = \frac{1}{\det \Gamma} \cdot \Gamma = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Let $U = \Gamma V$. Then,

$$\begin{aligned} &U_t + AU_x = 0, \\ &\Gamma V_t + A\Gamma V_x = 0, \\ &V_t + \Gamma^{-1} A \Gamma V_x = 0, \\ &V_t + \Lambda V_x = 0. \end{aligned}$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} V_x = 0, \tag{14.2}$$

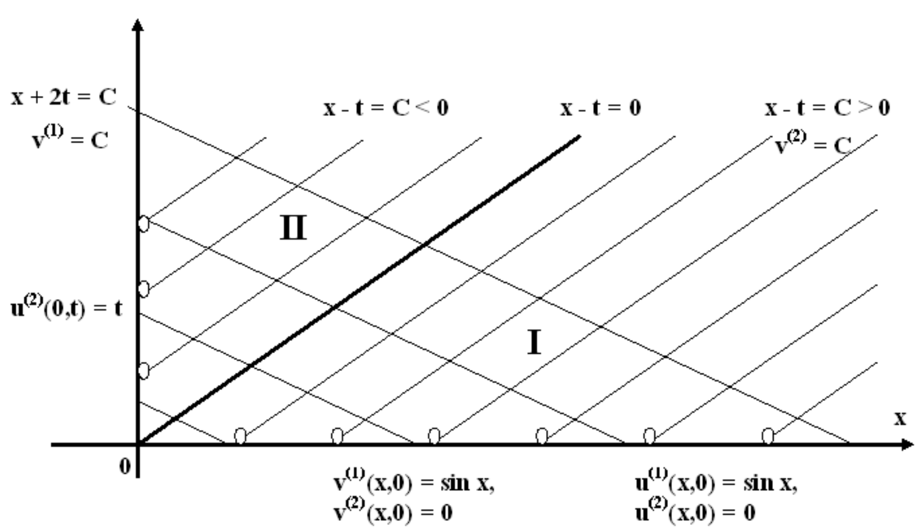
$$V(x,0) = \Gamma^{-1} U(x,0) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin x \\ 0 \end{pmatrix} = \begin{pmatrix} \sin x \\ 0 \end{pmatrix}. \tag{14.3}$$

Equation (14.2) gives **traveling wave solutions** of the form

$$v^{(1)}(x,t) = F(x + 2t), \qquad v^{(2)}(x,t) = G(x - t).$$

We can write U in terms of V:

$$U = \Gamma V = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F(x+2t) \\ G(x-t) \end{pmatrix} = \begin{pmatrix} F(x+2t) + G(x-t) \\ G(x-t) \end{pmatrix}. \tag{14.4}$$



• For region I, (14.2) and (14.3) give two initial value problems (since any point in region I can be traced back along both characteristics to initial conditions):

$$\begin{cases} v_t^{(1)} - 2v_x^{(1)} = 0, \\ v^{(1)}(x,0) = \sin x; \end{cases} \qquad \begin{cases} v_t^{(2)} + v_x^{(2)} = 0, \\ v^{(2)}(x,0) = 0. \end{cases}$$

which we solve by characteristics to get *traveling wave solutions*:

$$v^{(1)}(x,t) = \sin(x+2t), \qquad v^{(2)}(x,t) = 0.$$

➡ Thus, for region I,

$$\boxed{U = \Gamma V = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin(x+2t) \\ 0 \end{pmatrix} = \begin{pmatrix} \sin(x+2t) \\ 0 \end{pmatrix}.}$$

• For region II, solutions of the form $F(x+2t)$ can be traced back to initial conditions. Thus, $v^{(1)}$ is the same as in region I. Solutions of the form $G(x-t)$ can be traced back to the boundary. Since from (14.4), $u^{(2)} = v^{(2)}$, we use boundary conditions to get

$$u^{(2)}(0,t) = t = G(-t).$$

Hence, $G(x-t) = -(x-t)$.
➡ Thus, for region II,

$$\boxed{U = \Gamma V = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin(x+2t) \\ -(x-t) \end{pmatrix} = \begin{pmatrix} \sin(x+2t) - (x-t) \\ -(x-t) \end{pmatrix}.}$$

Solutions for regions I and II satisfy (14.1).
Solution for region I satisfies both initial conditions.
Solution for region II satisfies given boundary condition. ☐

Problem (S'02, #7). *Consider the system*

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{14.5}$$

Find an explicit solution for the following mixed problem for the system (14.5):

$$\begin{aligned} \begin{pmatrix} u(x,0) \\ v(x,0) \end{pmatrix} &= \begin{pmatrix} f(x) \\ 0 \end{pmatrix} \quad \textit{for} \quad x > 0, \\ u(0,t) &= 0 \qquad \textit{for} \quad t > 0. \end{aligned}$$

You may assume that the function f is smooth and vanishes on a neighborhood of $x = 0$.

Proof. Rewrite the equation as

$$U_t + \begin{pmatrix} 1 & -2 \\ -2 & -2 \end{pmatrix} U_x = 0,$$

$$U(x,0) = \begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix} = \begin{pmatrix} f(x) \\ 0 \end{pmatrix}.$$

The eigenvalues of the matrix A are $\lambda_1 = -3$, $\lambda_2 = 2$ and the corresponding eigenvectors are $e_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $e_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Thus,

$$\Lambda = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}, \qquad \Gamma^{-1} = \frac{1}{\det \Gamma}\cdot\Gamma = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

Let $U = \Gamma V$. Then,

$$\begin{aligned} &U_t + AU_x = 0, \\ &\Gamma V_t + A\Gamma V_x = 0, \\ &V_t + \Gamma^{-1}A\Gamma V_x = 0, \\ &V_t + \Lambda V_x = 0. \end{aligned}$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} V_x = 0, \tag{14.6}$$

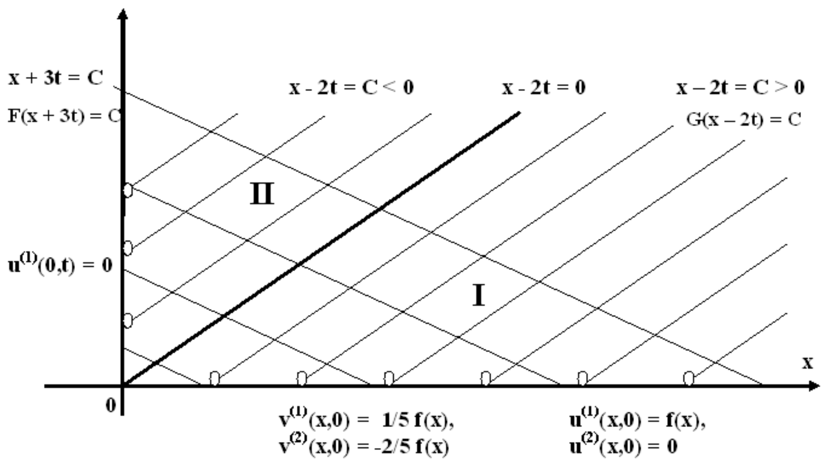
$$V(x,0) = \Gamma^{-1}U(x,0) = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} f(x) \\ 0 \end{pmatrix} = \frac{f(x)}{5}\begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{14.7}$$

Equation (14.6) gives **traveling wave solutions** of the form:

$$v^{(1)}(x,t) = F(x+3t), \qquad v^{(2)}(x,t) = G(x-2t). \tag{14.8}$$

We can write U in terms of V:

$$U = \Gamma V = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} F(x+3t) \\ G(x-2t) \end{pmatrix} = \begin{pmatrix} F(x+3t) - 2G(x-2t) \\ 2F(x+3t) + G(x-2t) \end{pmatrix}. \tag{14.9}$$



• For region I, (14.6) and (14.7) give two initial value problems (since value at any point in region I can be traced back along both characteristics to initial conditions):

$$\begin{cases} v_t^{(1)} - 3v_x^{(1)} = 0, \\ v^{(1)}(x,0) = \frac{1}{5}f(x); \end{cases} \qquad \begin{cases} v_t^{(2)} + 2v_x^{(2)} = 0, \\ v^{(2)}(x,0) = -\frac{2}{5}f(x). \end{cases}$$

which we solve by characteristics to get *traveling wave solutions*:

$$v^{(1)}(x,t) = \frac{1}{5}f(x+3t), \qquad v^{(2)}(x,t) = -\frac{2}{5}f(x-2t).$$

➡ Thus, for region I, $U = \Gamma V = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{5}f(x+3t) \\ -\frac{2}{5}f(x-2t) \end{pmatrix} = \begin{pmatrix} \frac{1}{5}f(x+3t) + \frac{4}{5}f(x-2t) \\ \frac{2}{5}f(x+3t) - \frac{2}{5}f(x-2t) \end{pmatrix}.$

• For region II, solutions of the form $F(x+3t)$ can be traced back to initial conditions. Thus, $v^{(1)}$ is the same as in region I. Solutions of the form $G(x-2t)$ can be traced back to the boundary. Since from (14.9),

$$u^{(1)} = v^{(1)} - 2v^{(2)}, \qquad \text{we have}$$
$$u^{(1)}(x,t) = F(x+3t) - 2G(x-2t) = \frac{1}{5}f(x+3t) - 2G(x-2t).$$

The boundary condition gives

$$u^{(1)}(0,t) = 0 = \frac{1}{5}f(3t) - 2G(-2t),$$
$$2G(-2t) = \frac{1}{5}f(3t),$$
$$G(t) = \frac{1}{10}f\Big(-\frac{3}{2}t\Big),$$
$$G(x-2t) = \frac{1}{10}f\Big(-\frac{3}{2}(x-2t)\Big).$$

➡ Thus, for region II, $U = \Gamma V = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{5}f(x+3t) \\ \frac{1}{10}f(-\frac{3}{2}(x-2t)) \end{pmatrix} = \begin{pmatrix} \frac{1}{5}f(x+3t) - \frac{1}{5}f(-\frac{3}{2}(x-2t)) \\ \frac{2}{5}f(x+3t) + \frac{1}{10}f(-\frac{3}{2}(x-2t)) \end{pmatrix}.$

Solutions for regions I and II satisfy (14.5).
Solution for region I satisfies both initial conditions.
Solution for region II satisfies given boundary condition. □

Problem (F'94, #1; S'97, #7). *Solve the initial-boundary value problem*

$$u_t + 3v_x = 0,$$
$$v_t + u_x + 2v_x = 0$$

in the quarter plane $0 \le x, t < \infty$*, with initial conditions* [25]

$$u(x,0) = \varphi_1(x), \quad v(x,0) = \varphi_2(x), \quad 0 < x < +\infty$$

and boundary condition

$$u(0,t) = \psi(t), \quad t > 0.$$

Proof. Rewrite the equation as $U_t + AU_x = 0$:

$$U_t + \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix} U_x = 0, \tag{14.10}$$

$$U(x,0) = \begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix} = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}.$$

The eigenvalues of the matrix A are $\lambda_1 = -1, \lambda_2 = 3$ and the corresponding eigenvectors are $e_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus,

$$\Lambda = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \Gamma^{-1} = \frac{1}{\det \Gamma} \cdot \Gamma = \frac{1}{4} \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}.$$

Let $U = \Gamma V$. Then,

$$U_t + AU_x = 0,$$
$$\Gamma V_t + A\Gamma V_x = 0,$$
$$V_t + \Gamma^{-1} A \Gamma V_x = 0,$$
$$V_t + \Lambda V_x = 0.$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} V_x = 0, \tag{14.11}$$

$$V(x,0) = \Gamma^{-1} U(x,0) = \frac{1}{4} \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -\varphi_1(x) + \varphi_2(x) \\ \varphi_1(x) + 3\varphi_2(x) \end{pmatrix}. \tag{14.12}$$

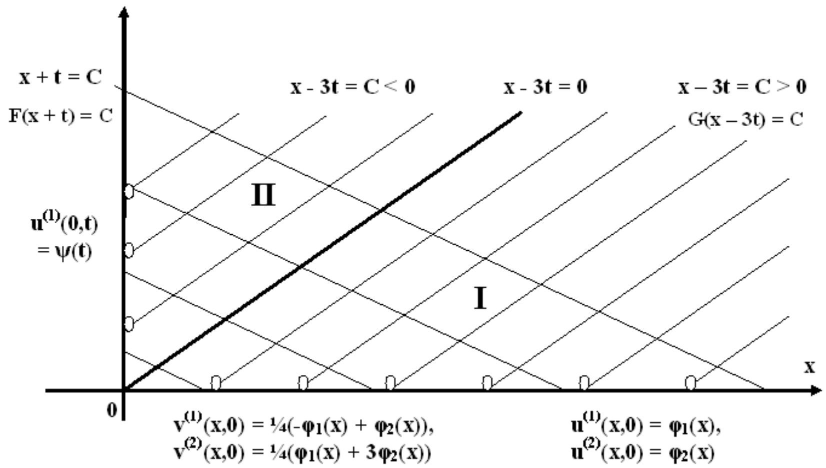
Equation (14.11) gives **traveling wave solutions** of the form:

$$v^{(1)}(x,t) = F(x+t), \qquad v^{(2)}(x,t) = G(x-3t). \tag{14.13}$$

We can write U in terms of V:

$$U = \Gamma V = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F(x+t) \\ G(x-3t) \end{pmatrix} = \begin{pmatrix} -3F(x+t) + G(x-3t) \\ F(x+t) + G(x-3t) \end{pmatrix}. \tag{14.14}$$

[25] In S'97, #7, the zero initial conditions are considered.



• For region I, (14.11) and (14.12) give two initial value problems (since value at any point in region I can be traced back along characteristics to initial conditions):

$$\begin{cases} v_t^{(1)} - v_x^{(1)} = 0, \\ v^{(1)}(x,0) = -\frac{1}{4}\varphi_1(x) + \frac{1}{4}\varphi_2(x); \end{cases} \qquad \begin{cases} v_t^{(2)} + 3v_x^{(2)} = 0, \\ v^{(2)}(x,0) = \frac{1}{4}\varphi_1(x) + \frac{3}{4}\varphi_2(x), \end{cases}$$

which we solve by characteristics to get *traveling wave solutions*:

$$v^{(1)}(x,t) = -\frac{1}{4}\varphi_1(x+t) + \frac{1}{4}\varphi_2(x+t), \qquad v^{(2)}(x,t) = \frac{1}{4}\varphi_1(x-3t) + \frac{3}{4}\varphi_2(x-3t).$$

➡ Thus, for region I,

$$\begin{aligned} U &= \Gamma V = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{4}\varphi_1(x+t) + \frac{1}{4}\varphi_2(x+t) \\ \frac{1}{4}\varphi_1(x-3t) + \frac{3}{4}\varphi_2(x-3t) \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 3\varphi_1(x+t) - 3\varphi_2(x+t) + \varphi_1(x-3t) + 3\varphi_2(x-3t) \\ -\varphi_1(x+t) + \varphi_2(x+t) + \varphi_1(x-3t) + 3\varphi_2(x-3t) \end{pmatrix}. \end{aligned}$$

• For region II, solutions of the form $F(x+t)$ can be traced back to initial conditions. Thus, $v^{(1)}$ is the same as in region I. Solutions of the form $G(x-3t)$ can be traced back to the boundary. Since from (14.14),

$$u^{(1)} = -3v^{(1)} + v^{(2)}, \qquad \text{we have}$$
$$u^{(1)}(x,t) = \frac{3}{4}\varphi_1(x+t) - \frac{3}{4}\varphi_2(x+t) + G(x-3t).$$

The boundary condition gives

$$\begin{aligned} u^{(1)}(0,t) &= \psi(t) = \frac{3}{4}\varphi_1(t) - \frac{3}{4}\varphi_2(t) + G(-3t), \\ G(-3t) &= \psi(t) - \frac{3}{4}\varphi_1(t) + \frac{3}{4}\varphi_2(t), \\ G(t) &= \psi\Big(-\frac{t}{3}\Big) - \frac{3}{4}\varphi_1\Big(-\frac{t}{3}\Big) + \frac{3}{4}\varphi_2\Big(-\frac{t}{3}\Big), \\ G(x-3t) &= \psi\Big(-\frac{x-3t}{3}\Big) - \frac{3}{4}\varphi_1\Big(-\frac{x-3t}{3}\Big) + \frac{3}{4}\varphi_2\Big(-\frac{x-3t}{3}\Big). \end{aligned}$$

➡ Thus, for region II,

$$\begin{aligned} U &= \Gamma V = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{4}\varphi_1(x+t) + \frac{1}{4}\varphi_2(x+t) \\ \psi(-\frac{x-3t}{3}) - \frac{3}{4}\varphi_1(-\frac{x-3t}{3}) + \frac{3}{4}\varphi_2(-\frac{x-3t}{3}) \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{4}\varphi_1(x+t) - \frac{3}{4}\varphi_2(x+t) + \psi(-\frac{x-3t}{3}) - \frac{3}{4}\varphi_1(-\frac{x-3t}{3}) + \frac{3}{4}\varphi_2(-\frac{x-3t}{3}) \\ -\frac{1}{4}\varphi_1(x+t) + \frac{1}{4}\varphi_2(x+t) + \psi(-\frac{x-3t}{3}) - \frac{3}{4}\varphi_1(-\frac{x-3t}{3}) + \frac{3}{4}\varphi_2(-\frac{x-3t}{3}) \end{pmatrix}. \end{aligned}$$

Solutions for regions I and II satisfy (14.10).
Solution for region I satisfies both initial conditions.
Solution for region II satisfies given boundary condition. □

Problem (F'91, #1). *Solve explicitly the following initial-boundary value problem for linear 2×2 hyperbolic system*

$$u_t = u_x + v_x$$
$$v_t = 3u_x - v_x,$$

where $0 < t < +\infty$, $0 < x < +\infty$ *with initial conditions*

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \qquad 0 < x < +\infty,$$

and the boundary condition

$$u(0,t) + bv(0,t) = \varphi(t), \qquad 0 < t < +\infty,$$

where $b \neq \frac{1}{3}$ *is a constant.*
What happens when $b = \frac{1}{3}$*?*

Proof. Let us change the notation ($u \leftrightarrow u^{(1)}$, $v \leftrightarrow u^{(2)}$). Rewrite the equation as

$$U_t + \begin{pmatrix} -1 & -1 \\ -3 & 1 \end{pmatrix} U_x = 0, \tag{14.15}$$

$$U(x,0) = \begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix} = \begin{pmatrix} u_0^{(1)}(x) \\ u_0^{(2)}(x) \end{pmatrix}.$$

The eigenvalues of the matrix A are $\lambda_1 = -2$, $\lambda_2 = 2$ and the corresponding eigenvectors are $e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. Thus,

$$\Lambda = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}, \qquad \Gamma^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}.$$

Let $U = \Gamma V$. Then,

$$U_t + AU_x = 0,$$
$$\Gamma V_t + A\Gamma V_x = 0,$$
$$V_t + \Gamma^{-1}A\Gamma V_x = 0,$$
$$V_t + \Lambda V_x = 0.$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} V_x = 0, \tag{14.16}$$

$$V(x,0) = \Gamma^{-1}U(x,0) = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3u_0^{(1)}(x) + u_0^{(2)}(x) \\ u_0^{(1)}(x) - u_0^{(2)}(x) \end{pmatrix}. \tag{14.17}$$

Equation (14.16) gives **traveling wave solutions** of the form:

$$v^{(1)}(x,t) = F(x+2t), \qquad v^{(2)}(x,t) = G(x-2t). \tag{14.18}$$

We can write U in terms of V:

$$U = \Gamma V = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} F(x+2t) \\ G(x-2t) \end{pmatrix} = \begin{pmatrix} F(x+2t) + G(x-2t) \\ F(x+2t) - 3G(x-2t) \end{pmatrix}. \tag{14.19}$$

• For region I, (14.16) and (14.17) give two initial value problems (since value at any point in region I can be traced back along characteristics to initial conditions):

$$\begin{cases} v_t^{(1)} - 2v_x^{(1)} = 0, \\ v^{(1)}(x,0) = \frac{3}{4}u_0^{(1)}(x) + \frac{1}{4}u_0^{(2)}(x); \end{cases} \qquad \begin{cases} v_t^{(2)} + 2v_x^{(2)} = 0, \\ v^{(2)}(x,0) = \frac{1}{4}u_0^{(1)}(x) - \frac{1}{4}u_0^{(2)}(x), \end{cases}$$

which we solve by characteristics to get *traveling wave solutions*:

$$v^{(1)}(x,t) = \frac{3}{4}u_0^{(1)}(x+2t) + \frac{1}{4}u_0^{(2)}(x+2t); \qquad v^{(2)}(x,t) = \frac{1}{4}u_0^{(1)}(x-2t) - \frac{1}{4}u_0^{(2)}(x-2t).$$

➡ Thus, for region I,

$$\begin{aligned} U &= \Gamma V = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} \frac{3}{4}u_0^{(1)}(x+2t) + \frac{1}{4}u_0^{(2)}(x+2t) \\ \frac{1}{4}u_0^{(1)}(x-2t) - \frac{1}{4}u_0^{(2)}(x-2t) \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{4}u_0^{(1)}(x+2t) + \frac{1}{4}u_0^{(2)}(x+2t) + \frac{1}{4}u_0^{(1)}(x-2t) - \frac{1}{4}u_0^{(2)}(x-2t) \\ \frac{3}{4}u_0^{(1)}(x+2t) + \frac{1}{4}u_0^{(2)}(x+2t) - \frac{3}{4}u_0^{(1)}(x-2t) + \frac{3}{4}u_0^{(2)}(x-2t) \end{pmatrix}. \end{aligned}$$

• For region II, solutions of the form $F(x+2t)$ can be traced back to initial conditions. Thus, $v^{(1)}$ is the same as in region I. Solutions of the form $G(x-2t)$ can be traced back to the boundary. The boundary condition gives

$$u^{(1)}(0,t) + bu^{(2)}(0,t) = \varphi(t).$$

Using (14.19),

$$\begin{aligned} & v^{(1)}(0,t) + G(-2t) + bv^{(1)}(0,t) - 3bG(-2t) = \varphi(t), \\ & (1+b)v^{(1)}(0,t) + (1-3b)G(-2t) = \varphi(t), \\ & (1+b)\Big(\frac{3}{4}u_0^{(1)}(2t) + \frac{1}{4}u_0^{(2)}(2t)\Big) + (1-3b)G(-2t) = \varphi(t), \\ & G(-2t) = \frac{\varphi(t) - (1+b)\Big(\frac{3}{4}u_0^{(1)}(2t) + \frac{1}{4}u_0^{(2)}(2t)\Big)}{1-3b}, \\ & G(t) = \frac{\varphi(-\frac{t}{2}) - (1+b)\Big(\frac{3}{4}u_0^{(1)}(-t) + \frac{1}{4}u_0^{(2)}(-t)\Big)}{1-3b}, \\ & G(x-2t) = \frac{\varphi(-\frac{x-2t}{2}) - (1+b)\Big(\frac{3}{4}u_0^{(1)}(-(x-2t)) + \frac{1}{4}u_0^{(2)}(-(x-2t))\Big)}{1-3b}. \end{aligned}$$

➡ Thus, for region II,

$$\begin{aligned} U &= \Gamma V = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} \frac{3}{4}u_0^{(1)}(x+2t) + \frac{1}{4}u_0^{(2)}(x+2t) \\ \frac{\varphi(-\frac{x-2t}{2}) - (1+b)\left(\frac{3}{4}u_0^{(1)}(-(x-2t)) + \frac{1}{4}u_0^{(2)}(-(x-2t))\right)}{1-3b} \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{4}u_0^{(1)}(x+2t) + \frac{1}{4}u_0^{(2)}(x+2t) + \frac{\varphi(-\frac{x-2t}{2}) - (1+b)\left(\frac{3}{4}u_0^{(1)}(-(x-2t)) + \frac{1}{4}u_0^{(2)}(-(x-2t))\right)}{1-3b} \\ \frac{3}{4}u_0^{(1)}(x+2t) + \frac{1}{4}u_0^{(2)}(x+2t) - \frac{3\varphi(-\frac{x-2t}{2}) - 3(1+b)\left(\frac{3}{4}u_0^{(1)}(-(x-2t)) + \frac{1}{4}u_0^{(2)}(-(x-2t))\right)}{1-3b} \end{pmatrix}. \end{aligned}$$

The following were performed, but are arithmetically complicated:
Solutions for regions I and II satisfy (14.15).

Solution for region I satisfies both initial conditions.
Solution for region II satisfies given boundary condition.

If $b = \frac{1}{3}$, $u^{(1)}(0,t) + \frac{1}{3}u^{(2)}(0,t) = F(2t) + G(-2t) + \frac{1}{3}F(2t) - G(-2t) = \frac{4}{3}F(2t) = \varphi(t)$.
Thus, the solutions of the form $v^{(2)} = G(x - 2t)$ are not defined at $x = 0$, which leads to ill-posedness. $\square$

Problem (F'96, #8). *Consider the system*

$$\begin{aligned} u_t &= 3u_x + 2v_x \\ v_t &= -v_x - v \end{aligned}$$

in the region $x \geq 0$, $t \geq 0$. Which of the following sets of initial and boundary data make this a well-posed problem?

a) $u(x,0) = 0, \quad x \geq 0$
$v(x,0) = x^2, \; x \geq 0$
$v(0,t) = t^2, \quad t \geq 0.$

b) $u(x,0) = 0, \quad x \geq 0$
$v(x,0) = x^2, \; x \geq 0$
$u(0,t) = t, \quad t \geq 0.$

c) $u(x,0) = 0, \quad x \geq 0$
$v(x,0) = x^2, \; x \geq 0$
$u(0,t) = t, \quad t \geq 0$
$v(0,t) = t^2, \quad t \geq 0.$

Proof. Rewrite the equation as $U_t + AU_x = BU$. Initial conditions are same for (a),(b),(c):

$$U_t + \begin{pmatrix} -3 & -2 \\ 0 & 1 \end{pmatrix} U_x = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} U,$$

$$U(x,0) = \begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix} = \begin{pmatrix} 0 \\ x^2 \end{pmatrix}.$$

The eigenvalues of the matrix A are $\lambda_1 = -3$, $\lambda_2 = 1$, and the corresponding eigenvectors are $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Thus,

$$\Lambda = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}, \qquad \Gamma^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}.$$

Let $U = \Gamma V$. Then,

$$\begin{aligned} & U_t + AU_x = BU, \\ & \Gamma V_t + A\Gamma V_x = B\Gamma V, \\ & V_t + \Gamma^{-1}A\Gamma V_x = \Gamma^{-1}B\Gamma V, \\ & V_t + \Lambda V_x = \Gamma^{-1}B\Gamma V. \end{aligned}$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix} V_x = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} V, \tag{14.20}$$

$$V(x,0) = \Gamma^{-1}U(x,0) = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ x^2 \end{pmatrix} = \frac{x^2}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{14.21}$$

Equation (14.20) gives **traveling wave solutions** of the form

$$v^{(1)}(x,t) = F(x+3t), \qquad v^{(2)}(x,t) = G(x-t). \tag{14.22}$$

We can write U in terms of V:

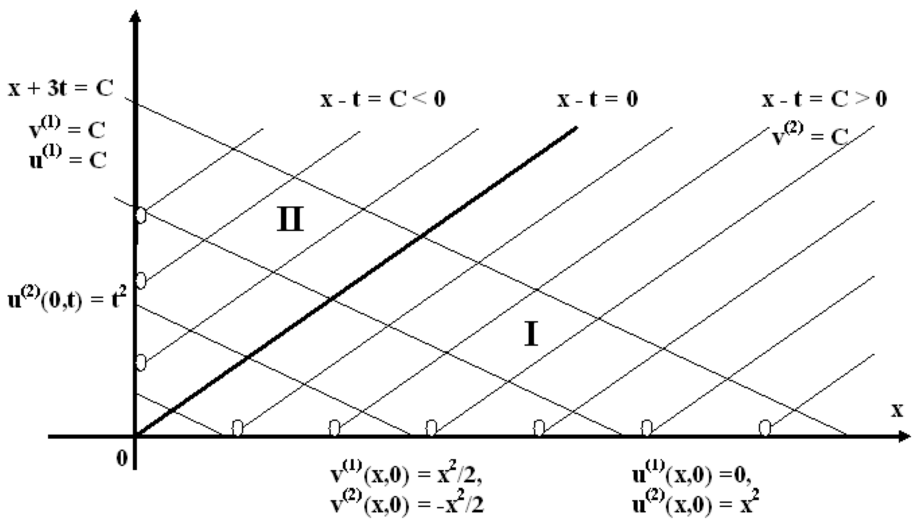
$$U = \Gamma V = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} F(x+3t) \\ G(x-t) \end{pmatrix} = \begin{pmatrix} F(x+3t) + G(x-t) \\ -2G(x-t) \end{pmatrix}. \tag{14.23}$$

• For region I, (14.20) and (14.21) give two initial value problems (since a value at any point in region I can be traced back along both characteristics to initial conditions):

$$\begin{cases} v_t^{(1)} - 3v_x^{(1)} = v^{(2)}, \\ v^{(1)}(x,0) = \frac{x^2}{2}; \end{cases} \qquad \begin{cases} v_t^{(2)} + v_x^{(2)} = -v^{(2)}, \\ v^{(2)}(x,0) = -\frac{x^2}{2}, \end{cases}$$

which we do not solve here. Thus, initial conditions for $v^{(1)}$ and $v^{(2)}$ have to be defined. Since (14.23) defines $u^{(1)}$ and $u^{(2)}$ in terms of $v^{(1)}$ and $v^{(2)}$, we need to define two initial conditions for U.

• For region II, solutions of the form $F(x+3t)$ can be traced back to initial conditions. Thus, $v^{(1)}$ is the same as in region I. Solutions of the form $G(x-t)$ are traced back to the boundary at $x=0$. Since from (14.23), $u^{(2)}(x,t) = -2v^{(2)}(x,t) = -2G(x-t)$, i.e. $u^{(2)}$ is written in term of $v^{(2)}$ only, $u^{(2)}$ requires a boundary condition to be defined on $x = 0$.



Thus,

a) $u^{(2)}(0,t) = t^2, \quad t \geq 0.$ **Well-posed.**

b) $u^{(1)}(0,t) = t, \quad t \geq 0.$ **Not well-posed.**

c) $u^{(1)}(0,t) = t$, $u^{(2)}(0,t) = t^2, \quad t \geq 0.$ **Not well-posed.** □

Problem (F'02, #3). *Consider the first order system*

$$\begin{aligned} u_t + u_x + v_x &= 0 \\ v_t + u_x - v_x &= 0 \end{aligned}$$

on the domain $0 < t < \infty$ *and* $0 < x < 1$. *Which of the following sets of initial-boundary data are well posed for this system? Explain your answers.*
a) *u(x,0) = f(x), v(x,0) = g(x);*
b) *u(x,0) = f(x), v(x,0) = g(x), u(0,t) = h(x), v(0,t) = k(x);*
c) *u(x,0) = f(x), v(x,0) = g(x), u(0,t) = h(x), v(1,t) = k(x).*

Proof. Rewrite the equation as $U_t + AU_x = 0$. Initial conditions are same for (a),(b),(c):

$$U_t + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} U_x = 0,$$

$$U(x,0) = \begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix} = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}.$$

The eigenvalues of the matrix A are $\lambda_1 = \sqrt{2}$, $\lambda_2 = -\sqrt{2}$ and the corresponding eigenvectors are $e_1 = \begin{pmatrix} 1 \\ -1+\sqrt{2} \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix}$. Thus,

$$\Lambda = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 1 & 1 \\ -1+\sqrt{2} & -1-\sqrt{2} \end{pmatrix}, \qquad \Gamma^{-1} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1+\sqrt{2} & 1 \\ -1+\sqrt{2} & -1 \end{pmatrix}.$$

Let $U = \Gamma V$. Then,

$$\begin{aligned} &U_t + AU_x = 0, \\ &\Gamma V_t + A\Gamma V_x = 0, \\ &V_t + \Gamma^{-1} A \Gamma V_x = 0, \\ &V_t + \Lambda V_x = 0. \end{aligned}$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix} V_x = 0, \tag{14.24}$$

$$V(x,0) = \Gamma^{-1} U(x,0) = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1+\sqrt{2} & 1 \\ -1+\sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} (1+\sqrt{2})f(x) + g(x) \\ (-1+\sqrt{2})f(x) - g(x) \end{pmatrix}. \tag{14.25}$$

Equation (14.24) gives **traveling wave solutions** of the form:

$$v^{(1)}(x,t) = F(x - \sqrt{2}t), \qquad v^{(2)}(x,t) = G(x + \sqrt{2}t). \tag{14.26}$$

However, we can continue and obtain the solutions. We have two initial value problems

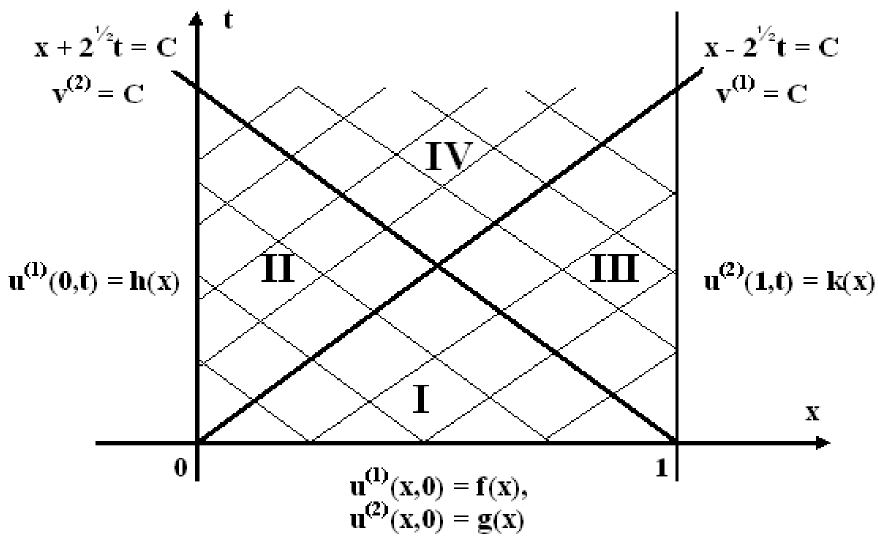
$$\begin{cases} v_t^{(1)} + \sqrt{2} v_x^{(1)} = 0, \\ v^{(1)}(x,0) = \frac{(1+\sqrt{2})}{2\sqrt{2}} f(x) + \frac{1}{2\sqrt{2}} g(x); \end{cases} \qquad \begin{cases} v_t^{(2)} - \sqrt{2} v_x^{(2)} = 0, \\ v^{(2)}(x,0) = \frac{(-1+\sqrt{2})}{2\sqrt{2}} f(x) - \frac{1}{2\sqrt{2}} g(x), \end{cases}$$

which we solve by characteristics to get *traveling wave solutions*:

$$\begin{aligned} v^{(1)}(x,t) &= \frac{(1+\sqrt{2})}{2\sqrt{2}} f(x - \sqrt{2}t) + \frac{1}{2\sqrt{2}} g(x - \sqrt{2}t), \\ v^{(2)}(x,t) &= \frac{(-1+\sqrt{2})}{2\sqrt{2}} f(x + \sqrt{2}t) - \frac{1}{2\sqrt{2}} g(x + \sqrt{2}t). \end{aligned}$$

We can obtain general solution U by writing U in terms of V:

$$U = \Gamma V = \Gamma \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1+\sqrt{2} & -1-\sqrt{2} \end{pmatrix} \frac{1}{2\sqrt{2}} \begin{pmatrix} (1+\sqrt{2})f(x-\sqrt{2}t) + g(x-\sqrt{2}t) \\ (-1+\sqrt{2})f(x+\sqrt{2}t) - g(x+\sqrt{2}t) \end{pmatrix}. \tag{14.27}$$



• In region I, the solution is obtained by solving two initial value problems(since a value at any point in region I can be traced back along both characteristics to initial conditions).
• In region II, the solutions of the form $v^{(2)} = G(x + \sqrt{2}t)$ can be traced back to initial conditions and those of the form $v^{(1)} = F(x - \sqrt{2}t)$, to left boundary. Since by (14.27), $u^{(1)}$ and $u^{(2)}$ are written in terms of both $v^{(1)}$ and $v^{(2)}$, one initial condition and one boundary condition at $x = 0$ need to be prescribed.
• In region III, the solutions of the form $v^{(2)} = G(x + \sqrt{2}t)$ can be traced back to right boundary and those of the form $v^{(1)} = F(x - \sqrt{2}t)$, to initial condition. Since by (14.27), $u^{(1)}$ and $u^{(2)}$ are written in terms of both $v^{(1)}$ and $v^{(2)}$, one initial condition and one boundary condition at $x = 1$ need to be prescribed.
• To obtain the solution for region IV, two boundary conditions, one for each boundary, should be given.

Thus,
a) No boundary conditions. **Not well-posed.**
b) $u^{(1)}(0,t) = h(x),\ u^{(2)}(0,t) = k(x)$. **Not well-posed.**
c) $u^{(1)}(0,t) = h(x),\ u^{(2)}(1,t) = k(x)$. **Well-posed.** □

Problem (S'94, #3). *Consider the system of equations*

$$f_t + g_x = 0$$
$$g_t + f_x = 0$$
$$h_t + 2h_x = 0$$

on the set $x \geq 0$, $t \geq 0$, *with the following initial-boundary values:*

a) f, g, h *prescribed on* $t = 0$, $x \geq 0$; f, h *prescribed on* $x = 0$, $t \geq 0$.
b) f, g, h *prescribed on* $t = 0$, $x \geq 0$; $f - g, h$ *prescribed on* $x = 0$, $t \geq 0$.
c) $f + g, h$ *prescribed on* $t = 0$, $x \geq 0$; f, g, h *prescribed on* $x = 0$, $t \geq 0$.

For each of these 3 sets of data, determine whether or not the system is ***well-posed****. Justify your conclusions.*

Proof. The third equation is decoupled from the first two and can be considered separately. Its solution can be written in the form

$$h(x,t) = H(x - 2t),$$

and therefore, h must be prescribed on $t = 0$ and on $x = 0$, since the characteristics propagate from both the x and t axis.
We rewrite the first two equations as ($f \leftrightarrow u_1$, $g \leftrightarrow u_2$):

$$U_t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U_x = 0,$$
$$U(x,0) = \begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix}.$$

The eigenvalues of the matrix A are $\lambda_1 = -1$, $\lambda_2 = 1$ and the corresponding eigenvectors are $e_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus,

$$\Lambda = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \Gamma^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let $U = \Gamma V$. Then,

$$U_t + AU_x = 0,$$
$$\Gamma V_t + A\Gamma V_x = 0,$$
$$V_t + \Gamma^{-1}A\Gamma V_x = 0,$$
$$V_t + \Lambda V_x = 0.$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} V_x = 0, \tag{14.28}$$

$$V(x,0) = \Gamma^{-1}U(x,0) = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u^{(1)}(x,0) \\ u^{(1)}(x,0) \end{pmatrix}. \tag{14.29}$$

Equation (14.28) gives **traveling wave solutions** of the form:

$$v^{(1)}(x,t) = F(x + t), \qquad v^{(2)}(x,t) = G(x - t). \tag{14.30}$$

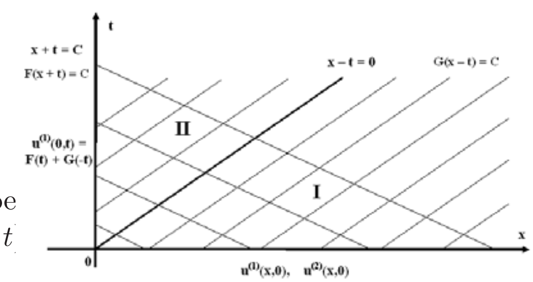
We can write U in terms of V:

$$U = \Gamma V = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F(x+t) \\ G(x-t) \end{pmatrix} = \begin{pmatrix} -F(x+t) + G(x-t) \\ F(x+t) + G(x-t) \end{pmatrix}. \tag{14.31}$$

• For region I, (14.28) and (14.29) give two initial value problems (since a value at any point in region I can be traced back along both characteristics to initial conditions). Thus, initial conditions for $v^{(1)}$ and $v^{(2)}$ have to be defined. Since (14.31) defines $u^{(1)}$ and $u^{(2)}$ in terms of $v^{(1)}$ and $v^{(2)}$, we need to define two initial conditions for U.
• For region II, solutions of the form $F(x+t)$ can be traced back to initial conditions. Thus, $v^{(1)}$ is the same as in region I. Solutions of the form $G(x-t)$ are traced back to the boundary at $x = 0$. Since from (14.31), $u^{(2)}(x,t) = v^{(1)}(x,t) + v^{(2)}(x,t) = F(x+t) + G(x-t)$, i.e. $u^{(2)}$ is written in terms of $v^{(2)} = G(x-t)$, $u^{(2)}$ requires a boundary condition to be defined on $x = 0$.

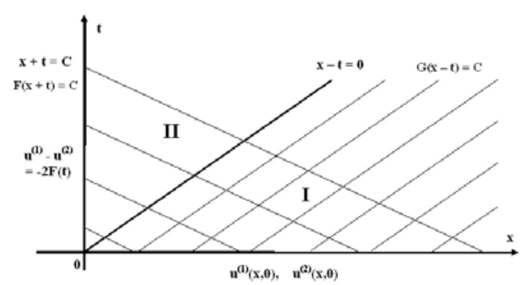
a) $u^{(1)}$, $u^{(2)}$ *prescribed on* $t = 0$; $u^{(1)}$ *prescribed on* $x = 0$.

Since $u^{(1)}(x,t) = -F(x+t) + G(x-t)$, $u^{(2)}(x,t) = F(x+t) + G(x-t)$, i.e. both $u^{(1)}$ and $u^{(2)}$ are written in terms of $F(x+t)$ and $G(x-t)$, we need to define two initial conditions for U (on $t = 0$).
A boundary condition also needs to be prescribe on $x = 0$ to be able to trace back $v^{(2)} = G(x-t$
Well-posed.



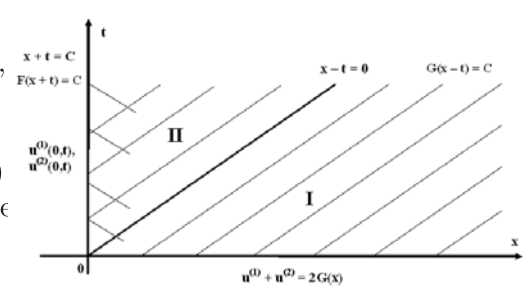
b) $u^{(1)}$, $u^{(2)}$ *prescribed on* $t = 0$; $u^{(1)} - u^{(2)}$ *prescribed on* $x = 0$.

As in part (a), we need to define two initial conditions for U.
Since $u^{(1)} - u^{(2)} = -2F(x+t)$, its definition on $x = 0$ leads to ill-posedness. On the contrary, $u^{(1)} + u^{(2)} = 2G(x-t)$ should be defined on $x = 0$ in order to be able to trace back the values through characteristics.
Ill-posed.



c) $u^{(1)} + u^{(2)}$ *prescribed on* $t = 0$; $u^{(1)}$, $u^{(2)}$ *prescribed on* $x = 0$.

Since $u^{(1)} + u^{(2)} = 2G(x-t)$, another initial condition should be prescribed to be able to trace back solutions of the form $v^{(2)} = F(x+t)$, without which the problem is ill-posed.
Also, two boundary conditions for both $u^{(1)}$ and $u^{(2)}$ define solutions of both $v^{(1)} = G(x-t)$ and $v^{(2)} = F(x+t)$ on the boundary. The forme boundary condition leads to ill-posedness.
Ill-posed.



□

Problem (F'92, #8). *Consider the system*

$$u_t + u_x + av_x = 0$$
$$v_t + bu_x + v_x = 0$$

for $0 < x < 1$ *with boundary and initial conditions*

$$u = v = 0 \qquad \text{for} \quad x = 0$$
$$u = u_0, \quad v = v_0 \qquad \text{for} \quad t = 0.$$

a) *For which values of a and b is this a **well-posed** problem?*
b) *For this class of a, b, state conditions on u_0 and v_0 so that the solution u, v will be continuous and continuously differentiable.*

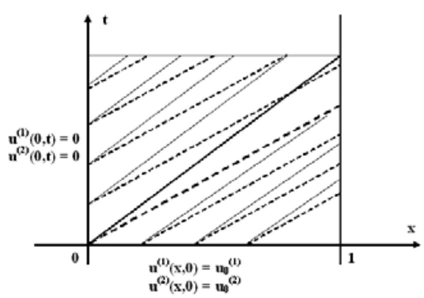
Proof. **a)** Let us change the notation ($u \leftrightarrow u^{(1)}$, $v \leftrightarrow u^{(2)}$). Rewrite the equation as

$$U_t + \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix} U_x = 0, \tag{14.32}$$

$$U(x,0) = \begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix} = \begin{pmatrix} u_0^{(1)}(x) \\ u_0^{(2)}(x) \end{pmatrix},$$
$$U(0,t) = \begin{pmatrix} u^{(1)}(0,t) \\ u^{(2)}(0,t) \end{pmatrix} = 0.$$

The eigenvalues of the matrix A are $\lambda_1 = 1 - \sqrt{ab}$, $\lambda_2 = 1 + \sqrt{ab}$.

$$\Lambda = \begin{pmatrix} 1 - \sqrt{ab} & 0 \\ 0 & 1 + \sqrt{ab} \end{pmatrix}.$$



Let $U = \Gamma V$, where Γ is a matrix of eigenvectors. Ther

$$U_t + AU_x = 0,$$
$$\Gamma V_t + A\Gamma V_x = 0,$$
$$V_t + \Gamma^{-1}A\Gamma V_x = 0,$$
$$V_t + \Lambda V_x = 0.$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} 1 - \sqrt{ab} & 0 \\ 0 & 1 + \sqrt{ab} \end{pmatrix} V_x = 0, \tag{14.33}$$

$$V(x,0) = \Gamma^{-1}U(x,0).$$

The equation (14.33) gives traveling wave solutions of the form:

$$v^{(1)}(x,t) = F(x - (1 - \sqrt{ab})t), \qquad v^{(2)}(x,t) = G(x - (1 + \sqrt{ab})t). \tag{14.34}$$

We also have $U = \Gamma V$, i.e. both $u^{(1)}$ and $u^{(2)}$ (and their initial and boundary conditions) are combinations of $v^{(1)}$ and $v^{(2)}$.
In order for this problem to be well-posed, both sets of characteristics should emanate from the boundary at $x = 0$. Thus, the eigenvalues of the system are real ($ab > 0$) and $\lambda_{1,2} > 0$ ($ab < 1$). Thus,

$$0 < ab < 1.$$

b) For U to be C^1, we require the compatibility condition, $u_0^{(1)}(0) = 0$, $u_0^{(2)}(0) = 0$. □

Problem (F'93, #2). *Consider the initial-boundary value problem*

$$u_t + u_x = 0$$
$$v_t - (1 - cx^2)v_x + u_x = 0$$

on $-1 \le x \le 1$ *and* $0 \le t$*, with the following prescribed data:*

$$u(x,0), \qquad v(x,0),$$
$$u(-1,t), \qquad v(1,t).$$

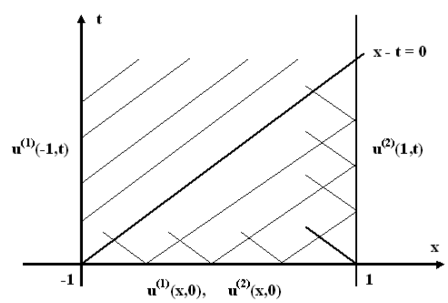
For which values of c is this a ***well-posed*** *problem?*

Proof. Let us change the notation $(u \leftrightarrow u^{(1)},\ v \leftrightarrow u^{(2)})$.
The first equation can be solved with $u^{(1)}(x,0) = F(x)$ to get a solution in the form $u^{(1)}(x,t) = F(x-t)$, which requires $u^{(1)}(x,0)$ and $u^{(1)}(-1,t)$ to be defined.
With $u^{(1)}$ known, we can solve the second equation

$$u_t^{(2)} - (1 - cx^2)u_x^{(2)} + F(x-t) = 0.$$

Solving the equation by characteristics, we obtain the characteristics in the xt-plane are of the form

$$\frac{dx}{dt} = cx^2 - 1.$$



We need to determine c such that the prescribed data $u^{(2)}(x,0)$ and $u^{(2)}(1,t)$ makes the problem to be well-posed. The boundary condition for $u^{(2)}(1,t)$ requires the characteristics to propagate to the left with t increasing. Thus, $x(t)$ is a decreasing function, i.e.

$$\frac{dx}{dt} < 0 \qquad \Rightarrow \qquad cx^2 - 1 < 0 \quad \text{for} \quad -1 < x < 1 \qquad \Rightarrow \qquad c < 1.$$

We could also do similar analysis we have done in other problems on first order systems involving finding eigenvalues/eigenvectors of the system and using the fact that $u^{(1)}(x,t)$ is known at both boundaries (i.e. values of $u^{(1)}(1,t)$ can be traced back either to initial conditions or to boundary conditions on $x = -1$). $\square$

Problem (S'91, #4). *Consider the first order system*

$$u_t + au_x + bv_x = 0$$
$$v_t + cu_x + dv_x = 0$$

for $0 < x < 1$, *with prescribed initial data:*

$$u(x, 0) = u_0(x)$$
$$v(x, 0) = v_0(x).$$

a) *Find conditions on* a, b, c, d *such that there is a full set of characteristics and, in this case, find the characteristic speeds.*
b) *For which values of* a, b, c, d *can boundary data be prescribed on* $x = 0$ *and for which values can it be prescribed on* $x = 1$*? How many pieces of data can be prescribed on each boundary?*

Proof. **a)** Let us change the notation ($u \leftrightarrow u^{(1)},\ v \leftrightarrow u^{(2)}$). Rewrite the equation as

$$U_t + \begin{pmatrix} a & b \\ c & d \end{pmatrix} U_x = 0, \tag{14.35}$$

$$U(x, 0) = \begin{pmatrix} u^{(1)}(x, 0) \\ u^{(2)}(x, 0) \end{pmatrix} = \begin{pmatrix} u_0^{(1)}(x) \\ u_0^{(2)}(x) \end{pmatrix}.$$

The system is **hyperbolic** if for each value of $u^{(1)}$ and $u^{(2)}$ the eigenvalues are real and the matrix is diagonalizable, i.e. there is a complete set of linearly independent eigenvectors. The eigenvalues of the matrix A are

$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2} = \frac{a + d \pm \sqrt{(a - d)^2 + 4bc}}{2}.$$

We **need** $(a - d)^2 + 4bc > 0$. This also makes the problem to be diagonalizable.
Let $U = \Gamma V$, where Γ is a matrix of eigenvectors. Then,

$$U_t + AU_x = 0,$$
$$\Gamma V_t + A\Gamma V_x = 0,$$
$$V_t + \Gamma^{-1} A \Gamma V_x = 0,$$
$$V_t + \Lambda V_x = 0.$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} V_x = 0, \tag{14.36}$$

Equation (14.36) gives **traveling wave solutions** of the form:

$$v^{(1)}(x, t) = F(x - \lambda_1 t), \qquad v^{(2)}(x, t) = G(x - \lambda_2 t). \tag{14.37}$$

The characteristic speeds are $\frac{dx}{dt} = \lambda_1$, $\frac{dx}{dt} = \lambda_2$.

b) We assume $(a + d)^2 - 4(ad - bc) > 0$.
$a + d > 0,\ ad - bc > 0 \quad \Rightarrow \quad \lambda_1, \lambda_2 > 0 \quad \Rightarrow \quad$ 2 B.C. on $x = 0$.
$a + d > 0,\ ad - bc < 0 \quad \Rightarrow \quad \lambda_1 < 0, \lambda_2 > 0 \quad \Rightarrow \quad$ 1 B.C. on $x = 0$, 1 B.C. on $x = 1$.
$a + d < 0,\ ad - bc > 0 \quad \Rightarrow \quad \lambda_1, \lambda_2 < 0 \quad \Rightarrow \quad$ 2 B.C. on $x = 1$.

$a+d<0, \quad ad-bc<0 \quad \Rightarrow \quad \lambda_1<0, \lambda_2>0 \quad \Rightarrow \quad$ 1 B.C. on $x=0$, 1 B.C. on $x=1$.
$a+d>0, \quad ad-bc=0 \quad \Rightarrow \quad \lambda_1=0, \lambda_2>0 \quad \Rightarrow \quad$ 1 B.C. on $x=0$.
$a+d<0, \quad ad-bc=0 \quad \Rightarrow \quad \lambda_1=0, \lambda_2<0 \quad \Rightarrow \quad$ 1 B.C. on $x=1$.
$a+d=0, \quad ad-bc<0 \quad \Rightarrow \quad \lambda_1<0, \lambda_2>0 \quad \Rightarrow \quad$ 1 B.C. on $x=0$, 1 B.C. on $x=1$. $\square$

Problem (S'94, #2). *Consider the differential operator*

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_t + 9v_x - u_{xx} \\ v_t - u_x - v_{xx} \end{pmatrix}$$

on $0 \leq x \leq 2\pi$, $t \geq 0$, *in which the vector* $\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}$ *consists of two functions that are periodic in* x.
a) *Find the* ***eigenfunctions and eigenvalues*** *of the operator* L.
b) *Use the results of (a) to solve the initial value problem*

$$L\begin{pmatrix} u \\ v \end{pmatrix} = 0 \qquad \text{for } t \geq 0,$$
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} e^{ix} \\ 0 \end{pmatrix} \qquad \text{for } t = 0.$$

Proof. **a)** We find the "space" eigenvalues and eigenfunctions. We rewrite the system as

$$U_t + \begin{pmatrix} 0 & 9 \\ -1 & 0 \end{pmatrix} U_x + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} U_{xx} = 0,$$

and find eigenvalues

$$\begin{pmatrix} 0 & 9 \\ -1 & 0 \end{pmatrix} U_x + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} U_{xx} = \lambda U. \tag{14.38}$$

Set $U = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = \begin{pmatrix} \sum_{n=-\infty}^{n=\infty} u_n(t)e^{inx} \\ \sum_{n=-\infty}^{n=\infty} v_n(t)e^{inx} \end{pmatrix}$. Plugging this into (14.38), we get

$$\begin{pmatrix} 0 & 9 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \sum inu_n(t)e^{inx} \\ \sum inv_n(t)e^{inx} \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \sum -n^2u_n(t)e^{inx} \\ \sum -n^2v_n(t)e^{inx} \end{pmatrix} = \lambda\begin{pmatrix} \sum u_n(t)e^{inx} \\ \sum v_n(t)e^{inx} \end{pmatrix},$$

$$\begin{aligned}
\begin{pmatrix} 0 & 9 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} inu_n(t) \\ inv_n(t) \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -n^2u_n(t) \\ -n^2v_n(t) \end{pmatrix} &= \lambda\begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix}, \\
\begin{pmatrix} 0 & 9in \\ -in & 0 \end{pmatrix}\begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix} + \begin{pmatrix} n^2 & 0 \\ 0 & n^2 \end{pmatrix}\begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix} &= \lambda\begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix}, \\
\begin{pmatrix} n^2-\lambda & 9in \\ -in & n^2-\lambda \end{pmatrix}\begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix} &= 0, \\
(n^2-\lambda)^2 - 9n^2 = 0,
\end{aligned}$$

which gives $\lambda_1 = n^2+3n$, $\lambda_2 = n^2-3n$, are eigenvalues, and $v_1 = \begin{pmatrix} 3i \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3i \\ -1 \end{pmatrix}$, are corresponding eigenvectors.

b) We want to solve $\begin{pmatrix} u \\ v \end{pmatrix}_t + L\begin{pmatrix} u \\ v \end{pmatrix} = 0, \quad L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 9v_x - u_{xx} \\ -u_x - v_{xx} \end{pmatrix}$. We have $\begin{pmatrix} u \\ v \end{pmatrix}_t = -L\begin{pmatrix} u \\ v \end{pmatrix} = -\lambda\begin{pmatrix} u \\ v \end{pmatrix}$, i.e. $\begin{pmatrix} u \\ v \end{pmatrix} = e^{-\lambda t}$. We can write the solution as

$$
\begin{aligned}
U(x,t) &= \begin{pmatrix} \sum u_n(t)e^{inx} \\ \sum v_n(t)e^{inx} \end{pmatrix} = \sum_{n=-\infty}^{\infty} a_n e^{-\lambda_1 t} v_1 e^{inx} + b_n e^{-\lambda_2 t} v_2 e^{inx} \\
&= \sum_{n=-\infty}^{\infty} a_n e^{-(n^2+3n)t}\begin{pmatrix} 3i \\ 1 \end{pmatrix} e^{inx} + b_n e^{-(n^2-3n)t}\begin{pmatrix} 3i \\ -1 \end{pmatrix} e^{inx}. \\
U(x,0) &= \sum_{n=-\infty}^{\infty} a_n \begin{pmatrix} 3i \\ 1 \end{pmatrix} e^{inx} + b_n \begin{pmatrix} 3i \\ -1 \end{pmatrix} e^{inx} = \begin{pmatrix} e^{ix} \\ 0 \end{pmatrix}, \\
&\Rightarrow \; a_n = b_n = 0, \; n \neq 1; \\
& \; a_1 + b_1 = \frac{1}{3i} \;\text{ and }\; a_1 = b_1 \quad \Rightarrow \quad a_1 = b_1 = \frac{1}{6i}. \\
\Rightarrow \quad U(x,t) &= \frac{1}{6i}e^{-4t}\begin{pmatrix} 3i \\ 1 \end{pmatrix} e^{ix} + \frac{1}{6i}e^{2t}\begin{pmatrix} 3i \\ -1 \end{pmatrix} e^{ix} \\
&= \begin{pmatrix} \frac{1}{2}(e^{-4t} + e^{2t}) \\ \frac{1}{6i}(e^{-4t} - e^{2t}) \end{pmatrix} e^{ix}.
\end{aligned}
$$

[26] [27] □

---

[26]ChiuYen's and Sung-Ha's solutions give similar answers.

[27]Questions about this problem:

1. Needed to find eigenfunctions, not eigenvectors.
2. The notation of L was changed. The problem statement incorporates the derivatives wrt. t into L.
3. Why can we write the solution in this form above?

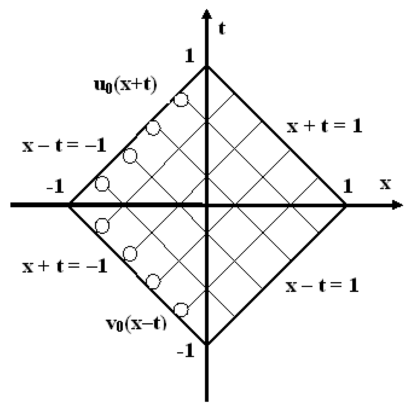
Problem (W'04, #6). *Consider the first order system*

$$u_t - u_x = v_t + v_x = 0$$

in the diamond shaped region $-1 < x + t < 1, \quad -1 < x - t < 1.$ *For each of the following boundary value problems state whether this problem is well-posed. If it is* ***well-posed****, find the solution.*
a) $u(x+t) = u_0(x+t)$ *on* $x - t = -1, \quad v(x-t) = v_0(x-t)$ *on* $x + t = -1.$
b) $v(x+t) = v_0(x+t)$ *on* $x - t = -1, \quad u(x-t) = u_0(x-t)$ *on* $x + t = -1.$

Proof. We have

$$u_t - u_x = 0,$$
$$v_t + v_x = 0.$$



• u is constant along the characteristics: $x + t = c_1(s)$.
Thus, its solution is $u(x,t) = u_0(x+t)$.
It the initial condition is prescribed at $x - t = -1$, the solution can be determined in the entire region by tracing back through the characteristics.
• v is constant along the characteristics: $x - t = c_2(s)$.
Thus, its solution is $v(x,t) = v_0(x-t)$.
It the initial condition is prescribed at $x + t = -1$, the solution can be determined in the entire region by tracing forward through the characteristics. □

15 Problems: Gas Dynamics Systems

15.1 Perturbation

Problem (S'92, #3). [28] [29] *Consider the gas dynamic equations*

$$u_t + uu_x + (F(\rho))_x = 0,$$
$$\rho_t + (u\rho)_x = 0.$$

Here $F(\rho)$ is a given C^∞-smooth function of ρ. At $t = 0$, 2π-periodic initial data

$$u(x,0) = f(x), \qquad \rho(x,0) = g(x).$$

a) *Assume that*

$$f(x) = U_0 + \varepsilon f_1(x), \qquad g(x) = R_0 + \varepsilon g_1(x)$$

where $U_0, R_0 > 0$ are constants and $\varepsilon f_1(x)$, $\varepsilon g_1(x)$ are "small" perturbations. ***Linearize*** *the equations and given conditions for F such that the linearized problem is* ***well-posed****.*
b) *Assume that $U_0 > 0$ and consider the above linearized equations for $0 \le x \le 1$, $t \ge 0$. Construct boundary conditions such that the initial-boundary value problem is* ***well-posed****.*

Proof. **a)** We write the equations in characteristic form:

$$u_t + uu_x + F'(\rho)\rho_x = 0, \qquad \circledast$$
$$\rho_t + u_x\rho + u\rho_x = 0.$$

Consider the special case of nearly constant initial data

$$u(x,0) = u_0 + \varepsilon u_1(x,0),$$
$$\rho(x,0) = \rho_0 + \varepsilon \rho_1(x,0).$$

Then we can approximate nonlinear equations by linear equations. Assuming

$$u(x,t) = u_0 + \varepsilon u_1(x,t),$$
$$\rho(x,t) = \rho_0 + \varepsilon \rho_1(x,t)$$

remain valid with $u_1 = O(1)$, $\rho_1 = O(1)$, we find that

$$u_t = \varepsilon u_{1t}, \qquad \rho_t = \varepsilon \rho_{1t},$$
$$u_x = \varepsilon u_{1x}, \qquad \rho_x = \varepsilon \rho_{1x},$$
$$F'(\rho) = F'(\rho_0 + \varepsilon\rho_1(x,t)) = F'(\rho_0) + \varepsilon\rho_1 F''(\rho_0) + O(\varepsilon^2).$$

Plugging these into $\circledast$, gives

$$\varepsilon u_{1t} + (u_0 + \varepsilon u_1)\varepsilon u_{1x} + \big(F'(\rho_0) + \varepsilon\rho_1 F''(\rho_0) + O(\varepsilon^2)\big)\varepsilon\rho_{1x} = 0,$$
$$\varepsilon\rho_{1t} + \varepsilon u_{1x}(\rho_0 + \varepsilon\rho_1) + (u_0 + \varepsilon u_1)\varepsilon\rho_{1x} = 0.$$

Dividing by ε gives

$$u_{1t} + u_0 u_{1x} + F'(\rho_0)\rho_{1x} = -\varepsilon u_1 u_{1x} - \varepsilon\rho_1\rho_{1x}F''(\rho_0) + O(\varepsilon^2),$$
$$\rho_{1t} + u_{1x}\rho_0 + u_0\rho_{1x} = -\varepsilon u_{1x}\rho_1 - \varepsilon u_1\rho_{1x}.$$

[28] See LeVeque, Second Edition, Birkhäuser Verlag, 1992, p. 44.
[29] This problem has similar notation with S'92, #4.

For small ε, we have

$$\boxed{\begin{cases} u_{1t} + u_0 u_{1x} + F'(\rho_0)\rho_{1x} = 0, \\ \rho_{1t} + u_{1x}\rho_0 + u_0\rho_{1x} = 0. \end{cases}}$$

This can be written as

$$\begin{pmatrix} u_1 \\ \rho_1 \end{pmatrix}_t + \begin{pmatrix} u_0 & F'(\rho_0) \\ \rho_0 & u_0 \end{pmatrix} \begin{pmatrix} u_1 \\ \rho_1 \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\begin{vmatrix} u_0 - \lambda & F'(\rho_0) \\ \rho_0 & u_0 - \lambda \end{vmatrix} = (u_0 - \lambda)(u_0 - \lambda) - \rho_0 F'(\rho_0) = 0,$$

$$\lambda^2 - 2u_0\lambda + u_0^2 - \rho_0 F'(\rho_0) = 0,$$

$$\lambda_{1,2} = u_0 \pm \sqrt{\rho_0 F'(\rho_0)}, \quad u_0 > 0,\ \rho_0 > 0.$$

For well-posedness, need $\lambda_{1,2} \in \mathbb{R}$ or $F'(\rho_0) \geq 0$.

b) We have $u_0 > 0$, and $\lambda_1 = u_0 + \sqrt{\rho_0 F'(\rho_0)}$, $\lambda_2 = u_0 - \sqrt{\rho_0 F'(\rho_0)}$.
- If $u_0 > \sqrt{\rho_0 F'(\rho_0)} \Rightarrow \lambda_1 > 0, \lambda_2 > 0 \Rightarrow$ 2 BC at $x = 0$.
- If $u_0 = \sqrt{\rho_0 F'(\rho_0)} \Rightarrow \lambda_1 > 0, \lambda_2 = 0 \Rightarrow$ 1 BC at $x = 0$.
- If $0 < u_0 < \sqrt{\rho_0 F'(\rho_0)} \Rightarrow \lambda_1 > 0, \lambda_2 < 0 \Rightarrow$ 1 BC at $x = 0$, 1 BC at $x = 1$. □

15.2 Stationary Solutions

Problem (S'92, #4). [30] *Consider*

$$u_t + uu_x + \rho_x = \nu u_{xx},$$
$$\rho_t + (u\rho)_x = 0$$

for $t \geq 0$, $-\infty < x < \infty$.
Give conditions for the states U_+, U_-, R_+, R_-, *such that the system has* ***stationary solutions*** *(i.e.* $u_t = \rho_t = 0$*) satisfying*

$$\lim_{x\to+\infty} \begin{pmatrix} u \\ \rho \end{pmatrix} = \begin{pmatrix} U_+ \\ R_+ \end{pmatrix}, \qquad \lim_{x\to-\infty} \begin{pmatrix} u \\ \rho \end{pmatrix} = \begin{pmatrix} U_- \\ R_- \end{pmatrix}. \qquad \circledast$$

Proof. For stationary solutions, we need

$$u_t = -\Big(\frac{u^2}{2}\Big)_x - \rho_x + \nu u_{xx} = 0,$$
$$\rho_t = -(u\rho)_x = 0.$$

Integrating the above equations, we obtain

$$-\frac{u^2}{2} - \rho + \nu u_x = C_1,$$
$$-u\rho = C_2.$$

[30]This problem has similar notation with S'92, #3.

Conditions ⊛ give $u_x = 0$ at $x = \pm\infty$. Thus

$$\begin{aligned} \frac{U_+^2}{2} + R_+ &= \frac{U_-^2}{2} + R_-, \\ U_+ R_+ &= U_- R_-. \end{aligned}$$

□

15.3 Periodic Solutions

Problem (F'94, #4). *Let $u(x,t)$ be a solution of the Cauchy problem*

$$u_t = -u_{xxxx} - 2u_{xx}, \qquad -\infty < x < +\infty, \ 0 < t < +\infty,$$
$$u(x,0) = \varphi(x),$$

where $u(x,t)$ and $\varphi(x)$ are C^∞ functions periodic in x with period 2π;
i.e. $u(x+2\pi,t) = u(x,t)$, $\forall x$, $\forall t$.
Prove that

$$||u(\cdot,t)|| \le Ce^{at}||\varphi||$$

where $||u(\cdot,t)|| = \sqrt{\int_0^{2\pi} |u(x,t)|^2\,dx}$, $||\varphi|| = \sqrt{\int_0^{2\pi} |\varphi(x)|^2\,dx}$, C, a are some constants.

Proof. **METHOD I:** Since u is 2π-periodic, let

$$u(x,t) = \sum_{n=-\infty}^{\infty} a_n(t)e^{inx}.$$

Plugging this into the equation, we get

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} a_n'(t)e^{inx} &= -\sum_{n=-\infty}^{\infty} n^4 a_n(t)e^{inx} + 2\sum_{n=-\infty}^{\infty} n^2 a_n(t)e^{inx}, \\
a_n'(t) &= (-n^4 + 2n^2)a_n(t), \\
a_n(t) &= a_n(0)e^{(-n^4+2n^2)t}.
\end{aligned}$$

Also, initial condition gives

$$\begin{aligned}
u(x,0) &= \sum_{n=-\infty}^{\infty} a_n(0)e^{inx} = \varphi(x), \\
& \left|\sum_{n=-\infty}^{\infty} a_n(0)e^{inx}\right| = |\varphi(x)|.
\end{aligned}$$

$$\begin{aligned}
||u(x,t)||_2^2 &= \int_0^{2\pi} u^2(x,t)\,dx = \int_0^{2\pi}\Big(\sum_{n=-\infty}^{\infty} a_n(t)e^{inx}\Big)\Big(\sum_{m=-\infty}^{\infty} a_n(t)\overline{e^{imx}}\Big)\,dx \\
&= \sum_{n=-\infty}^{\infty} a_n^2(t)\int_0^{2\pi} e^{inx}e^{-inx}\,dx = 2\pi\sum_{n=-\infty}^{\infty} a_n^2(t) = 2\pi\sum_{n=-\infty}^{\infty} a_n^2(0)e^{2(-n^4+2n^2)t} \\
&\le \left|2\pi\sum_{n=-\infty}^{\infty} a_n^2(0)\right|\left|\sum_{n=-\infty}^{\infty} e^{2(-n^4+2n^2)t}\right| = 2\pi\underbrace{\left|\sum_{n=-\infty}^{\infty} a_n^2(0)\right|}_{||\varphi||^2} e^{2t}\underbrace{\sum_{n=-\infty}^{\infty} e^{-2(n^2-1)^2t}}_{=C_1,\,\text{(convergent)}} \\
&= C_2e^{2t}||\varphi||^2.
\end{aligned}$$

$$\Rightarrow \quad ||u(x,t)|| \le Ce^t||\varphi||.$$

METHOD II: Multiply this equation by u and integrate

$$
\begin{aligned}
uu_t &= -uu_{xxxx} - 2uu_{xx}, \\
\frac{1}{2}\frac{d}{dt}(u^2) &= -uu_{xxxx} - 2uu_{xx}, \\
\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u^2\,dx &= -\int_0^{2\pi} uu_{xxxx}\,dx - \int_0^{2\pi} 2uu_{xx}\,dx, \\
\frac{1}{2}\frac{d}{dt}||u||_2^2 &= \underbrace{-uu_{xxx}\Big|_0^{2\pi}}_{=0} + \underbrace{u_x u_{xx}\Big|_0^{2\pi}}_{=0} - \int_0^{2\pi} u_{xx}^2\,dx - \int_0^{2\pi} 2uu_{xx}\,dx, \\
\frac{1}{2}\frac{d}{dt}||u||_2^2 &= -\int_0^{2\pi} u_{xx}^2\,dx - \int_0^{2\pi} 2uu_{xx}\,dx \qquad (-2ab \le a^2 + b^2) \\
&\le -\int_0^{2\pi} u_{xx}^2\,dx + \int_0^{2\pi} (u^2 + u_{xx}^2)\,dx = \int_0^{2\pi} u^2\,dx = ||u||^2, \\
\Rightarrow \quad & \frac{d}{dt}||u||^2 \le 2||u||^2, \\
& ||u||^2 \le ||u(0)||^2 e^{2t}, \\
& ||u|| \le ||u(0)||e^t. \ \checkmark
\end{aligned}
$$

METHOD III: Can use Fourier transform. See ChiuYen's solutions, that have both Method II and III. □

Problem (S'90, #4).
Let $f(x) \in C^\infty$ be a 2π-periodic function, i.e., $f(x) = f(x+2\pi)$ and denote by

$$||f||^2 = \int_0^{2\pi} |f(x)|^2\,dx$$

the L_2-norm of f.
***a)** Express $||d^p f/dx^p||^2$ in terms of the **Fourier coefficients** of f.*
***b)** Let $q > p > 0$ be integers. Prove that $\forall\,\epsilon > 0$, $\exists K = N(\varepsilon, p, q)$, constant, such that*

$$\Big|\Big|\frac{d^p f}{dx^p}\Big|\Big|^2 \le \varepsilon\Big|\Big|\frac{d^q f}{dx^q}\Big|\Big|^2 + K||f||^2.$$

***c)** Discuss how K depends on ε.*

Proof. **a)** Let [31]

$$\begin{aligned}
f(x) &= \sum_{-\infty}^{\infty} f_n e^{inx}, \\
\frac{d^p f}{dx^p} &= \sum_{-\infty}^{\infty} f_n (in)^p e^{inx}, \\
\Big|\Big|\frac{d^p f}{dx^p}\Big|\Big|^2 &= \int_0^{2\pi} \Big|\sum_{-\infty}^{\infty} f_n (in)^p e^{inx}\Big|^2\,dx \;=\; \int_0^{2\pi} |i^2|^p\,\Big|\sum_{-\infty}^{\infty} f_n n^p e^{inx}\Big|^2\,dx \\
&= \int_0^{2\pi} \Big|\sum_{-\infty}^{\infty} f_n n^p e^{inx}\Big|^2\,dx \;=\; 2\pi\sum_{n=0}^{\infty} f_n^2 n^{2p}.
\end{aligned}$$

b) We have

$$\begin{aligned}
\Big|\Big|\frac{d^p f}{dx^p}\Big|\Big|^2 &\le \varepsilon\Big|\Big|\frac{d^q f}{dx^q}\Big|\Big|^2 + K||f||^2, \\
2\pi\sum_{n=0}^{\infty} f_n^2 n^{2p} &\le \varepsilon\,2\pi\sum_{n=0}^{\infty} f_n^2 n^{2q} + K\,2\pi\sum_{n=0}^{\infty} f_n^2, \\
n^{2p} - \varepsilon n^{2q} &\le K, \\
n^{2p}\underbrace{(1-\varepsilon n^{q'})}_{<0,\ for\ n\ large} &\le K, \qquad \text{some } q' > 0.
\end{aligned}$$

Thus, the above inequality is true for n large enough. The statement follows. ∎

---

[31]Note:

$$\int_0^L e^{inx}\overline{e^{imx}}\,dx = \begin{cases} 0 & n \ne m \\ L & n = m \end{cases}$$

Problem (S'90, #5). [32] *Consider the flame front equation*

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0 \qquad \circledast$$

with 2π-periodic initial data

$$u(x,0) = f(x), \qquad f(x) = f(x+2\pi) \in C^\infty.$$

a) *Determine the solution, if $f(x) \equiv f_0 =$ const.*
b) *Assume that*

$$f(x) = 1 + \varepsilon g(x), \quad 0 < \varepsilon \ll 1, \qquad |g|_\infty = 1, \qquad g(x) = g(x+2\pi).$$

Linearize *the equation. Is the Cauchy problem well-posed for the linearized equation, i.e., do its solutions v satisfy an estimate*

$$||v(\cdot,t)|| \le Ke^{\alpha(t-t_0)}||v(\cdot,t_0)||?$$

c) *Determine the best possible constants K, α.*

Proof. **a)** The solution to

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0,$$
$$u(x,0) = f_0 = \text{const},$$

is $u(x,t) = f_0 = \text{const}$.

b) We consider the special case of nearly constant initial data

$$u(x,0) = 1 + \varepsilon u_1(x,0).$$

Then we can approximate the nonlinear equation by a linear equation. Assuming

$$u(x,t) = 1 + \varepsilon u_1(x,t),$$

remain valid with $u_1 = O(1)$, from $\circledast$, we find that

$$\varepsilon u_{1t} + (1+\varepsilon u_1)\varepsilon u_{1x} + \varepsilon u_{1xx} + \varepsilon u_{1xxxx} = 0.$$

Dividing by ε gives

$$u_{1t} + u_{1x} + \varepsilon u_1 u_{1x} + u_{1xx} + u_{1xxxx} = 0.$$

For small ε, we have

$$\boxed{u_{1t} + u_{1x} + u_{1xx} + u_{1xxxx} = 0.}$$

Multiply this equation by u_1 and integrate

$$u_1u_{1t} + u_1u_{1x} + u_1u_{1xx} + u_1u_{1xxxx} = 0,$$
$$\frac{d}{dt}\Big(\frac{u_1^2}{2}\Big) + \Big(\frac{u_1^2}{2}\Big)_x + u_1u_{1xx} + u_1u_{1xxxx} = 0,$$
$$\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u_1^2\,dx + \underbrace{\frac{u_1^2}{2}\Big|_0^{2\pi}}_{=0} + \int_0^{2\pi} u_1u_{1xx}\,dx + \int_0^{2\pi} u_1u_{1xxxx}\,dx = 0,$$
$$\frac{1}{2}\frac{d}{dt}||u_1||_2^2 + \underbrace{u_1u_{1x}\Big|_0^{2\pi}}_{=0} - \int_0^{2\pi} u_{1x}^2\,dx + \underbrace{u_1u_{1xxx}\Big|_0^{2\pi}}_{=0} - \underbrace{u_{1x}u_{1xx}\Big|_0^{2\pi}}_{=0} + \int_0^{2\pi} u_{1xx}^2\,dx = 0,$$
$$\frac{1}{2}\frac{d}{dt}||u_1||_2^2 = \int_0^{2\pi} u_{1x}^2\,dx - \int_0^{2\pi} u_{1xx}^2\,dx.$$

[32] S'90 #5, #6, #7 all have similar formulations.

Since u_1 is 2π-periodic, let

$$u_1 = \sum_{n=-\infty}^{\infty} a_n(t) e^{inx}. \qquad \text{Then,}$$

$$u_{1x} = i \sum_{n=-\infty}^{\infty} n a_n(t) e^{inx} \quad \Rightarrow \quad u_{1x}^2 = -\Big(\sum_{n=-\infty}^{\infty} n a_n(t) e^{inx} \Big)^2,$$

$$u_{1xx} = -\sum_{n=-\infty}^{\infty} n^2 a_n(t) e^{inx} \quad \Rightarrow \quad u_{1xx}^2 = \Big(\sum_{n=-\infty}^{\infty} n^2 a_n(t) e^{inx} \Big)^2.$$

Thus,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||u_1||_2^2 &= \int_0^{2\pi} u_{1x}^2\, dx - \int_0^{2\pi} u_{1xx}^2\, dx \\
&= -\int_0^{2\pi} \Big(\sum n a_n(t) e^{inx} \Big)^2 dx - \int_0^{2\pi} \Big(\sum n^2 a_n(t) e^{inx} \Big)^2 dx \\
&= -2\pi \sum n^2 a_n(t)^2 - 2\pi \sum n^4 a_n(t)^2 = -2\pi \sum a_n(t)^2 (n^2 + n^4) \le 0.
\end{aligned}$$

$$\boxed{\Rightarrow \quad ||u_1(\cdot,t)||_2 \le ||u_1(\cdot,0)||_2,}$$

where $K = 1$, $\alpha = 0$. □

Problem (W'03, #4). *Consider the PDE*

$$\begin{aligned}
u_t = u_x + u^4 \quad & \textit{for} \quad t > 0 \\
u = u_0 \quad & \textit{for} \quad t = 0
\end{aligned}$$

for $0 < x < 2\pi$. *Define the set* $A = \{u = u(x) : \hat{u}(k) = 0 \quad if \quad k < 0\}$, *in which* $\{\hat{u}(k,t)\}_{-\infty}^{\infty}$ *is the* ***Fourier series*** *of* u *in* x *on* $[0, 2\pi]$.
a) *If* $u_0 \in A$, *show that* $u(t) \in A$.
b) *Find differential equations for* $\hat{u}(0,t)$, $\hat{u}(1,t)$, *and* $\hat{u}(2,t)$.

Proof. **a)** Solving

$$\begin{aligned}
u_t &= u_x + u^4 \\
u(x,0) &= u_0(x)
\end{aligned}$$

by the method of characteristics, we get

$$u(x,t) = \frac{u_0(x+t)}{(1 - 3t(u_0(x+t))^3)^{\frac{1}{3}}}.$$

Since $u_0 \in A$, $\widehat{u}_{0k} = 0$ if $k < 0$. Thus,

$$u_0(x) = \sum_{k=0}^{\infty} \widehat{u}_{0k}\, e^{i\frac{kx}{2}}.$$

Since

$$\widehat{u}_k = \frac{1}{2\pi} \int_0^{2\pi} u(x,t)\, e^{-i\frac{kx}{2}}\, dx,$$

we have

$$u(x,t) \;=\; \sum_{k=0}^{\infty} \widehat{u}_k \, e^{i\frac{kx}{2}},$$

that is, $u(t) \in A$. $\square$

15.4 Energy Estimates

Problem (S'90, #6). *Let $U(x,t) \in C^\infty$ be 2π-periodic in x. Consider the* ***linear*** *equation*

$$
\begin{aligned}
&u_t + Uu_x + u_{xx} + u_{xxxx} = 0, \\
&u(x,0) = f(x), \qquad f(x) = f(x+2\pi) \in C^\infty.
\end{aligned}
$$

a) *Derive an* ***energy estimate*** *for u.*
b) *Prove that one can estimate all derivatives $||\partial^p u/\partial x^p||$.*
c) *Indicate how to prove existence of solutions.* [33]

Proof. **a)** Multiply the equation by u and integrate

$$
\begin{aligned}
&uu_t + Uuu_x + uu_{xx} + uu_{xxxx} = 0, \\
&\frac{1}{2}\frac{d}{dt}(u^2) + \frac{1}{2}U(u^2)_x + uu_{xx} + uu_{xxxx} = 0, \\
&\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u^2\,dx + \frac{1}{2}\int_0^{2\pi} U(u^2)_x\,dx + \int_0^{2\pi} uu_{xx}\,dx + \int_0^{2\pi} uu_{xxxx}\,dx = 0, \\
&\frac{1}{2}\frac{d}{dt}||u||^2 + \underbrace{\frac{1}{2}Uu^2\Big|_0^{2\pi}}_{=0} - \frac{1}{2}\int_0^{2\pi} U_x u^2\,dx + uu_x\Big|_0^{2\pi} - \int_0^{2\pi} u_x^2\,dx \\
&\qquad + uu_{xxx}\Big|_0^{2\pi} - u_x u_{xx}\Big|_0^{2\pi} + \int_0^{2\pi} u_{xx}^2\,dx = 0, \\
&\frac{1}{2}\frac{d}{dt}||u||^2 - \frac{1}{2}\int_0^{2\pi} U_x u^2\,dx - \int_0^{2\pi} u_x^2\,dx + \int_0^{2\pi} u_{xx}^2\,dx = 0, \\
&\frac{1}{2}\frac{d}{dt}||u||^2 = \frac{1}{2}\int_0^{2\pi} U_x u^2\,dx + \int_0^{2\pi} u_x^2\,dx - \int_0^{2\pi} u_{xx}^2\,dx \;\le\; \text{(from S'90, \#5)} \;\le \\
&\quad \le\; \frac{1}{2}\int_0^{2\pi} U_x u^2\,dx \;\le\; \frac{1}{2}\max_x U_x \int_0^{2\pi} u^2\,dx. \\
&\Rightarrow\quad \frac{d}{dt}||u||^2 \le \max_x U_x ||u||^2, \\
&\qquad\qquad ||u(x,t)||^2 \le ||u(x,0)||^2\, e^{(\max_x U_x)t}.
\end{aligned}
$$

This can also been done using Fourier Transform. See ChiuYen's solutions where the above method and the Fourier Transform methods are used. ☐

[33] S'90 #5, #6, #7 all have similar formulations.

Problem (S'90, #7). [34] *Consider the **nonlinear** equation*

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0, \qquad \circledast$$
$$u(x,0) = f(x), \qquad f(x) = f(x+2\pi) \in C^\infty.$$

a) *Derive an **energy estimate** for u.*
b) *Show that there is an interval $0 \le t \le T$, T depending on f, such that also $||\partial u(\cdot,t)/\partial x||$ can be bounded.*

Proof. **a)** Multiply the above equation by u and integrate

$$uu_t + u^2 u_x + uu_{xx} + uu_{xxxx} = 0,$$
$$\frac{1}{2}\frac{d}{dt}(u^2) + \frac{1}{3}(u^3)_x + uu_{xx} + uu_{xxxx} = 0,$$
$$\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u^2\,dx + \frac{1}{3}\int_0^{2\pi}(u^3)_x\,dx + \int_0^{2\pi} uu_{xx}\,dx + \int_0^{2\pi} uu_{xxxx}\,dx = 0,$$
$$\frac{1}{2}\frac{d}{dt}||u||^2 + \underbrace{\frac{1}{3}u^3\Big|_0^{2\pi}}_{=0} - \int_0^{2\pi} u_x^2\,dx + \int_0^{2\pi} u_{xx}^2\,dx = 0,$$
$$\frac{1}{2}\frac{d}{dt}||u||^2 = \int_0^{2\pi} u_x^2\,dx - \int_0^{2\pi} u_{xx}^2\,dx \le 0, \qquad \text{(from S'90, \#5)}$$
$$\Rightarrow \quad ||u(\cdot,t)|| \le ||u(\cdot,0)||.$$

b) In order to find a bound for $||u_x(\cdot,t)||$, differentiate $\circledast$ with respect to x:

$$u_{tx} + (uu_x)_x + u_{xxx} + u_{xxxxx} = 0,$$

Multiply the above equation by u_x and integrate:

$$u_x u_{tx} + u_x(uu_x)_x + u_x u_{xxx} + u_x u_{xxxxx} = 0,$$
$$\frac{1}{2}\frac{d}{dt}\int_0^{2\pi}(u_x)^2\,dx + \int_0^{2\pi} u_x(uu_x)_x\,dx + \int_0^{2\pi} u_x u_{xxx}\,dx + \int_0^{2\pi} u_x u_{xxxxx}\,dx = 0.$$

We evaluate one of the integrals in the above expression using the periodicity:

$$\int_0^{2\pi} u_x(uu_x)_x\,dx = -\int_0^{2\pi} u_{xx}uu_x = \int_0^{2\pi} u_x(u_x^2 + uu_{xx}) = \int_0^{2\pi} u_x^3 + \int_0^{2\pi} uu_xu_{xx},$$
$$\Rightarrow \quad \int_0^{2\pi} u_{xx}uu_x = -\frac{1}{2}\int_0^{2\pi} u_x^3,$$
$$\Rightarrow \quad \int_0^{2\pi} u_x(uu_x)_x = \frac{1}{2}\int_0^{2\pi} u_x^3.$$

We have

$$\frac{1}{2}\frac{d}{dt}||u_x||^2 + \int_0^{2\pi} u_x^3\,dx + \int_0^{2\pi} u_x u_{xxx}\,dx + \int_0^{2\pi} u_x u_{xxxxx}\,dx = 0.$$

[34] S'90 #5, #6, #7 all have similar formulations.

Let $w = u_x$, then

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||w||^2 &= -\int_0^{2\pi} w^3\,dx - \int_0^{2\pi} ww_{xx}\,dx - \int_0^{2\pi} ww_{xxxx}\,dx \\
&= -\int_0^{2\pi} w^3\,dx + \int_0^{2\pi} w_x^2\,dx - \int_0^{2\pi} w_{xx}^2\,dx \le -\int_0^{2\pi} w^3\,dx, \\
\Rightarrow \quad \frac{d}{dt}||u_x||^2 &= -\int_0^{2\pi} u_x^3\,dx.
\end{aligned}$$

□

16 Problems: Wave Equation

16.1 The Initial Value Problem

Example (McOwen 3.1 #1). *Solve the initial value problem:*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \\ u(x,0) = \underbrace{x^3}_{g(x)}, \quad u_t(x,0) = \underbrace{\sin x}_{h(x)}. \end{cases}$$

Proof. D'Alembert's formula gives the solution:

$$\begin{aligned} u(x,t) &= \frac{1}{2}(g(x+ct) + g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\,d\xi \\ &= \frac{1}{2}(x+ct)^3 + \frac{1}{2}(x-ct)^3 + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin\xi\,d\xi \\ &= x^3 + 2xc^2t^2 - \frac{1}{2c}\cos(x+ct) + \frac{1}{2c}\cos(x-ct) = \\ &= x^3 + 2xc^2t^2 + \frac{1}{c}\sin x \sin ct. \end{aligned}$$

□

Problem (S'99, #6). *Solve the Cauchy problem*

$$\begin{cases} u_{tt} = a^2 u_{xx} + \cos x, \\ u(x,0) = \sin x, \quad u_t(x,0) = 1 + x. \end{cases} \tag{16.1}$$

Proof. We have a nonhomogeneous PDE with nonhomogeneous initial conditions:

$$\begin{cases} u_{tt} - c^2 u_{xx} = \underbrace{\cos x}_{f(x,t)}, \\ u(x,0) = \underbrace{\sin x}_{g(x)}, \quad u_t(x,0) = \underbrace{1+x}_{h(x)}. \end{cases}$$

The solution is given by d'Alembert's formula and Duhamel's principle.[35]

$$\begin{aligned} u^A(x,t) &= \frac{1}{2}(g(x+ct) + g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\,d\xi \\ &= \frac{1}{2}(\sin(x+ct) + \sin(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} (1+\xi)\,d\xi \\ &= \sin x\,\cos ct + \frac{1}{2c}\Big[\xi + \frac{\xi^2}{2}\Big]_{\xi=x-ct}^{\xi=x+ct} = \sin x\,\cos ct + xt + t. \end{aligned}$$

$$\begin{aligned} u^D(x,t) &= \frac{1}{2c}\int_0^t \bigg(\int_{x-c(t-s)}^{x+c(t-s)} f(\xi,s)\,d\xi\bigg)\,ds = \frac{1}{2c}\int_0^t \bigg(\int_{x-c(t-s)}^{x+c(t-s)} \cos\xi\,d\xi\bigg)\,ds \\ &= \frac{1}{2c}\int_0^t \Big(\sin[x+c(t-s)] - \sin[x-c(t-s)]\Big)\,ds = \frac{1}{c^2}(\cos x - \cos x\cos ct). \end{aligned}$$

$$u(x,t) = u^A(x,t) + u^D(x,t) = \sin x\,\cos ct + xt + t + \frac{1}{c^2}(\cos x - \cos x\cos ct).$$

[35] Note the relationship: $x \leftrightarrow \xi$, $t \leftrightarrow s$.

We can check that the solution satisfies equation (16.1). Can also check that u^A, u^D satisfy

$$\begin{cases} u^A_{tt} - c^2 u^A_{xx} = 0, \\ u^A(x,0) = \sin x, \quad u^A_t(x,0) = 1 + x; \end{cases} \qquad \begin{cases} u^D_{tt} - c^2 u^D_{xx} = \cos x, \\ u^D(x,0) = 0, \quad u^D_t(x,0) = 0. \end{cases}$$

$\square$

16.2 Initial/Boundary Value Problem

Problem 1. *Consider the initial/boundary value problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L,\ t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & 0 < x < L \\ u(0,t) = 0, \quad u(L,t) = 0 & t \ge 0. \end{cases} \tag{16.2}$$

Proof. Find $u(x,t)$ in the form

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t)\cos\frac{n\pi x}{L} + b_n(t)\sin\frac{n\pi x}{L}.$$

- Functions $a_n(t)$ and $b_n(t)$ are determined by the boundary conditions:

$$0 = u(0,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \quad \Rightarrow \quad a_n(t) = 0. \qquad \text{Thus,}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{L}. \tag{16.3}$$

- If we substitute (16.3) into the equation $u_{tt} - c^2 u_{xx} = 0$, we get

$$\sum_{n=1}^{\infty} b_n''(t)\sin\frac{n\pi x}{L} + c^2\sum_{n=1}^{\infty}\Big(\frac{n\pi}{L}\Big)^2 b_n(t)\sin\frac{n\pi x}{L} = 0, \qquad \text{or}$$

$$b_n''(t) + \Big(\frac{n\pi c}{L}\Big)^2 b_n(t) = 0,$$

whose general solution is

$$b_n(t) = c_n \sin\frac{n\pi ct}{L} + d_n\cos\frac{n\pi ct}{L}. \tag{16.4}$$

Also, $b_n'(t) = c_n(\frac{n\pi c}{L})\cos\frac{n\pi ct}{L} - d_n(\frac{n\pi c}{L})\sin\frac{n\pi ct}{L}$.

- The constants c_n and d_n are determined by the initial conditions:

$$\begin{aligned} g(x) = u(x,0) &= \sum_{n=1}^{\infty} b_n(0)\sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} d_n\sin\frac{n\pi x}{L}, \\ h(x) = u_t(x,0) &= \sum_{n=1}^{\infty} b_n'(0)\sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} c_n\frac{n\pi c}{L}\sin\frac{n\pi x}{L}. \end{aligned}$$

By orthogonality, we may multiply by $\sin(m\pi x/L)$ and integrate:

$$\begin{aligned} \int_0^L g(x)\sin\frac{m\pi x}{L}\,dx &= \int_0^L \sum_{n=1}^{\infty} d_n\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = d_m\frac{L}{2}, \\ \int_0^L h(x)\sin\frac{m\pi x}{L}\,dx &= \int_0^L \sum_{n=1}^{\infty} c_n\frac{n\pi c}{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = c_m\frac{m\pi c}{L}\frac{L}{2}. \end{aligned}$$

Thus,

$$d_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx, \qquad c_n = \frac{2}{n\pi c}\int_0^L h(x)\sin\frac{n\pi x}{L}\,dx. \tag{16.5}$$

The formulas (16.3), (16.4), and (16.5) define the solution.

Example (McOwen 3.1 #2). *Consider the initial/boundary value problem*

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < \pi, \ t > 0 \\ u(x,0) = 1, \quad u_t(x,0) = 0 & 0 < x < \pi \\ u(0,t) = 0, \quad u(\pi,t) = 0 & t \geq 0. \end{cases} \tag{16.6}$$

Proof. Find $u(x,t)$ in the form

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t)\cos nx + b_n(t)\sin nx.$$

• Functions $a_n(t)$ and $b_n(t)$ are determined by the boundary conditions:

$$0 = u(0,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \quad \Rightarrow \quad a_n(t) = 0. \qquad \text{Thus,}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin nx. \tag{16.7}$$

• If we substitute this into $u_{tt} - u_{xx} = 0$, we get

$$\sum_{n=1}^{\infty} b_n''(t)\sin nx + \sum_{n=1}^{\infty} b_n(t)n^2 \sin nx = 0, \qquad \text{or}$$
$$b_n''(t) + n^2 b_n(t) = 0,$$

whose general solution is

$$b_n(t) = c_n \sin nt + d_n \cos nt. \tag{16.8}$$

Also, $b_n'(t) = nc_n \cos nt - nd_n \sin nt$.
• The constants c_n and d_n are determined by the initial conditions:

$$\begin{aligned} 1 = u(x,0) &= \sum_{n=1}^{\infty} b_n(0)\sin nx = \sum_{n=1}^{\infty} d_n \sin nx, \\ 0 = u_t(x,0) &= \sum_{n=1}^{\infty} b_n'(0)\sin nx = \sum_{n=1}^{\infty} nc_n \sin nx. \end{aligned}$$

By orthogonality, we may multiply both equations by $\sin mx$ and integrate:

$$\int_0^{\pi} \sin mx\, dx = d_m \frac{\pi}{2},$$
$$\int_0^{\pi} 0\, dx = nc_n \frac{\pi}{2}.$$

Thus,

$$d_n = \frac{2}{n\pi}(1 - \cos n\pi) = \begin{cases} \frac{4}{n\pi}, & n \text{ odd}, \\ 0, & n \text{ even}, \end{cases} \qquad \text{and} \qquad c_n = 0. \tag{16.9}$$

Using this in (16.8) and (16.7), we get

$$b_n(t) = \begin{cases} \frac{4}{n\pi}\cos nt, & n \text{ odd}, \\ 0, & n \text{ even}, \end{cases}$$

$$\boxed{u(x,t) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)t\ \sin(2n+1)x}{(2n+1)}.}$$

□

We can **sum the series in regions bouded by characteristics**. We have

$$u(x,t) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)t\,\sin(2n+1)x}{(2n+1)}, \qquad \text{or}$$

$$u(x,t) = \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\sin[(2n+1)(x+t)]}{(2n+1)} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\sin[(2n+1)(x-t)]}{(2n+1)}. \tag{16.10}$$

The initial condition may be written as

$$1 = u(x,0) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{(2n+1)} \qquad \text{for} \quad 0<x<\pi. \tag{16.11}$$

We can use (16.11) to sum the series in (16.10).

$$\text{In } R_1, \qquad u(x,t) = \frac{1}{2}+\frac{1}{2} = 1.$$

Since $\sin[(2n+1)(x-t)] = -\sin[(2n+1)(-(x-t))]$, and $0 < -(x-t) < \pi$ in R_2,

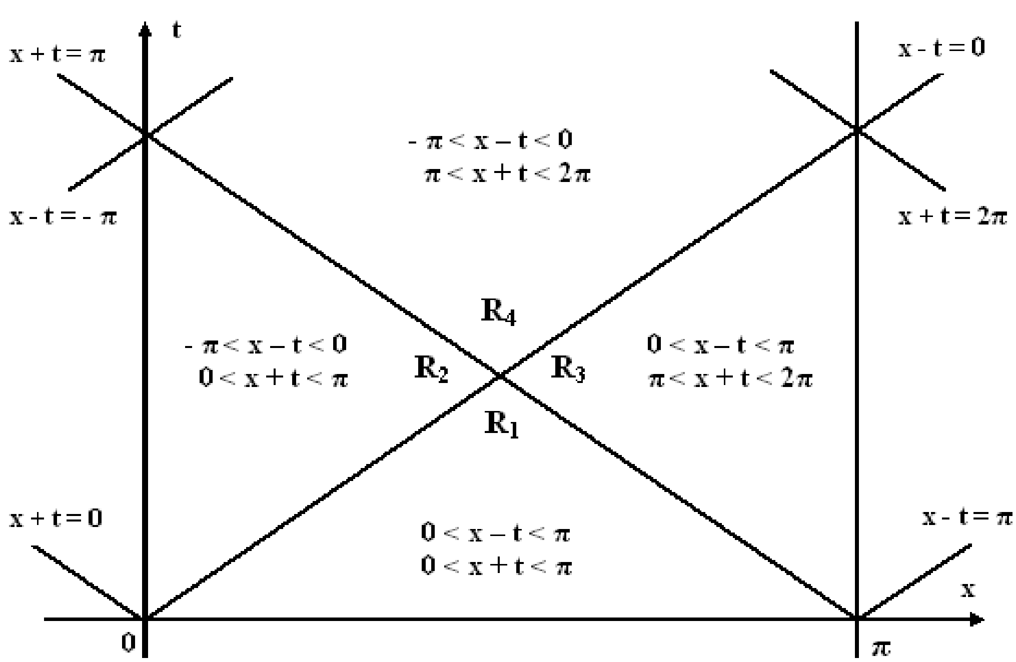
$$\text{in } R_2, \qquad u(x,t) = \frac{1}{2}-\frac{1}{2} = 0.$$

Since $\sin[(2n+1)(x+t)] = \sin[(2n+1)(x+t-2\pi)] = -\sin[(2n+1)(2\pi-(x+t))]$, and $0 < 2\pi-(x+t) < \pi$ in R_3,

$$\text{in } R_3, \qquad u(x,t) = -\frac{1}{2}+\frac{1}{2} = 0.$$

Since $0 < -(x-t) < \pi$ and $0 < 2\pi-(x+t) < \pi$ in R_4,

$$\text{in } R_4, \qquad u(x,t) = -\frac{1}{2}-\frac{1}{2} = -1.$$



□

Problem 2. *Consider the initial/boundary value problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L, \ t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & 0 < x < L \\ u_x(0,t) = 0, \quad u_x(L,t) = 0 & t \geq 0. \end{cases} \tag{16.12}$$

Proof. Find $u(x,t)$ in the form

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \cos\frac{n\pi x}{L} + b_n(t) \sin\frac{n\pi x}{L}.$$

- Functions $a_n(t)$ and $b_n(t)$ are determined by the boundary conditions:

$$u_x(x,t) = \sum_{n=1}^{\infty} -a_n(t)\Big(\frac{n\pi}{L}\Big) \sin\frac{n\pi x}{L} + b_n(t)\Big(\frac{n\pi}{L}\Big) \cos\frac{n\pi x}{L},$$

$$0 = u_x(0,t) = \sum_{n=1}^{\infty} b_n(t)\Big(\frac{n\pi}{L}\Big) \quad \Rightarrow \quad b_n(t) = 0. \qquad \text{Thus,}$$

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \cos\frac{n\pi x}{L}. \tag{16.13}$$

- If we substitute (16.13) into the equation $u_{tt} - c^2 u_{xx} = 0$, we get

$$\frac{a_0''(t)}{2} + \sum_{n=1}^{\infty} a_n''(t) \cos\frac{n\pi x}{L} + c^2 \sum_{n=1}^{\infty} a_n(t)\Big(\frac{n\pi}{L}\Big)^2 \cos\frac{n\pi x}{L} = 0,$$

$$a_0''(t) = 0 \qquad \text{and} \qquad a_n''(t) + \Big(\frac{n\pi c}{L}\Big)^2 a_n(t) = 0,$$

whose general solutions are

$$a_0(t) = c_0 t + d_0 \qquad \text{and} \qquad a_n(t) = c_n \sin\frac{n\pi ct}{L} + d_n \cos\frac{n\pi ct}{L}. \tag{16.14}$$

Also, $a_0'(t) = c_0$ and $a_n'(t) = c_n(\frac{n\pi c}{L}) \cos\frac{n\pi ct}{L} - d_n(\frac{n\pi c}{L}) \sin\frac{n\pi ct}{L}$.

- The constants c_n and d_n are determined by the initial conditions:

$$g(x) = u(x,0) = \frac{a_0(0)}{2} + \sum_{n=1}^{\infty} a_n(0) \cos\frac{n\pi x}{L} = \frac{d_0}{2} + \sum_{n=1}^{\infty} d_n \cos\frac{n\pi x}{L},$$

$$h(x) = u_t(x,0) = \frac{a_0'(0)}{2} + \sum_{n=1}^{\infty} a_n'(0) \cos\frac{n\pi x}{L} = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \frac{n\pi c}{L} \cos\frac{n\pi x}{L}.$$

By orthogonality, we may multiply both equations by $\cos(m\pi x/L)$, including $m = 0$, and integrate:

$$\int_0^L g(x)\,dx = d_0 \frac{L}{2}, \qquad \int_0^L g(x) \cos\frac{m\pi x}{L}\,dx = d_m \frac{L}{2},$$

$$\int_0^L h(x)\,dx = c_0 \frac{L}{2}, \qquad \int_0^L h(x) \cos\frac{m\pi x}{L}\,dx = c_m \frac{m\pi c}{L}\frac{L}{2}.$$

Thus,

$$d_n = \frac{2}{L}\int_0^L g(x) \cos\frac{n\pi x}{L}\,dx, \quad c_n = \frac{2}{n\pi c}\int_0^L h(x) \cos\frac{n\pi x}{L}\,dx, \quad c_0 = \frac{2}{L}\int_0^L h(x)\,dx. \tag{16.15}$$

The formulas (16.13), (16.14), and (16.15) define the solution. □

Example (McOwen 3.1 #3). *Consider the initial/boundary value problem*

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < \pi,\ t > 0 \\ u(x,0) = x, \quad u_t(x,0) = 0 & 0 < x < \pi \\ u_x(0,t) = 0, \quad u_x(\pi,t) = 0 & t \geq 0. \end{cases} \tag{16.16}$$

Proof. Find $u(x,t)$ in the form

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t)\cos nx + b_n(t)\sin nx.$$

• Functions $a_n(t)$ and $b_n(t)$ are determined by the boundary conditions:

$$u_x(x,t) = \sum_{n=1}^{\infty} -a_n(t) n \sin nx + b_n(t) n \cos nx,$$

$$0 = u_x(0,t) = \sum_{n=1}^{\infty} b_n(t) n \quad \Rightarrow \quad b_n(t) = 0. \qquad \text{Thus,}$$

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t)\cos nx. \tag{16.17}$$

• If we substitute (16.17) into the equation $u_{tt} - u_{xx} = 0$, we get

$$\frac{a_0''(t)}{2} + \sum_{n=1}^{\infty} a_n''(t)\cos nx + \sum_{n=1}^{\infty} a_n(t) n^2 \cos nx = 0,$$

$$a_0''(t) = 0 \qquad \text{and} \qquad a_n''(t) + n^2 a_n(t) = 0,$$

whose general solutions are

$$a_0(t) = c_0 t + d_0 \qquad \text{and} \qquad a_n(t) = c_n \sin nt + d_n \cos nt. \tag{16.18}$$

Also, $a_0'(t) = c_0$ and $a_n'(t) = c_n n \cos nt - d_n n \sin nt$.
• The constants c_n and d_n are determined by the initial conditions:

$$x = u(x,0) = \frac{a_0(0)}{2} + \sum_{n=1}^{\infty} a_n(0)\cos nx = \frac{d_0}{2} + \sum_{n=1}^{\infty} d_n \cos nx,$$

$$0 = u_t(x,0) = \frac{a_0'(0)}{2} + \sum_{n=1}^{\infty} a_n'(0)\cos nx = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n n \cos nx.$$

By orthogonality, we may multiply both equations by $\cos mx$, including $m = 0$, and integrate:

$$\int_0^\pi x\,dx = d_0 \frac{\pi}{2}, \qquad \int_0^\pi x \cos mx\,dx = d_m \frac{\pi}{2},$$

$$\int_0^\pi 0\,dx = c_0 \frac{\pi}{2}, \qquad \int_0^\pi 0 \cos mx\,dx = c_m m \frac{\pi}{2}.$$

Thus,

$$d_0 = \pi, \qquad d_n = \frac{2}{\pi n^2}(\cos n\pi - 1), \qquad c_n = 0. \tag{16.19}$$

Using this in (16.18) and (16.17), we get

$$a_0(t) = d_0 = \pi, \qquad a_n(t) = \frac{2}{\pi n^2}(\cos n\pi - 1)\cos nt,$$

$$\boxed{u(x,t) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(\cos n\pi - 1)\cos nt \cos nx}{n^2}.}$$

$\square$

We can **sum the series in regions bouded by characteristics**. We have

$$u(x,t) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(\cos n\pi - 1)\cos nt\cos nx}{n^2}, \qquad \text{or}$$

$$u(x,t) = \frac{\pi}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(\cos n\pi - 1)\cos[n(x-t)]}{n^2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(\cos n\pi - 1)\cos[n(x+t)]}{n^2}. \quad (16.20)$$

The initial condition may be written as

$$u(x,0) = x = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(\cos n\pi - 1)\cos nx}{n^2} \qquad \text{for} \quad 0 < x < \pi,$$

which implies

$$\frac{x}{2} - \frac{\pi}{4} = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(\cos n\pi - 1)\cos nx}{n^2} \qquad \text{for} \quad 0 < x < \pi, \qquad (16.21)$$

We can use (16.21) to sum the series in (16.20).

$$\text{In } R_1, \qquad u(x,t) = \frac{\pi}{2} + \frac{x-t}{2} - \frac{\pi}{4} + \frac{x+t}{2} - \frac{\pi}{4} = x.$$

Since $\cos[n(x-t)] = \cos[n(-(x-t))]$, and $0 < -(x-t) < \pi$ in R_2,

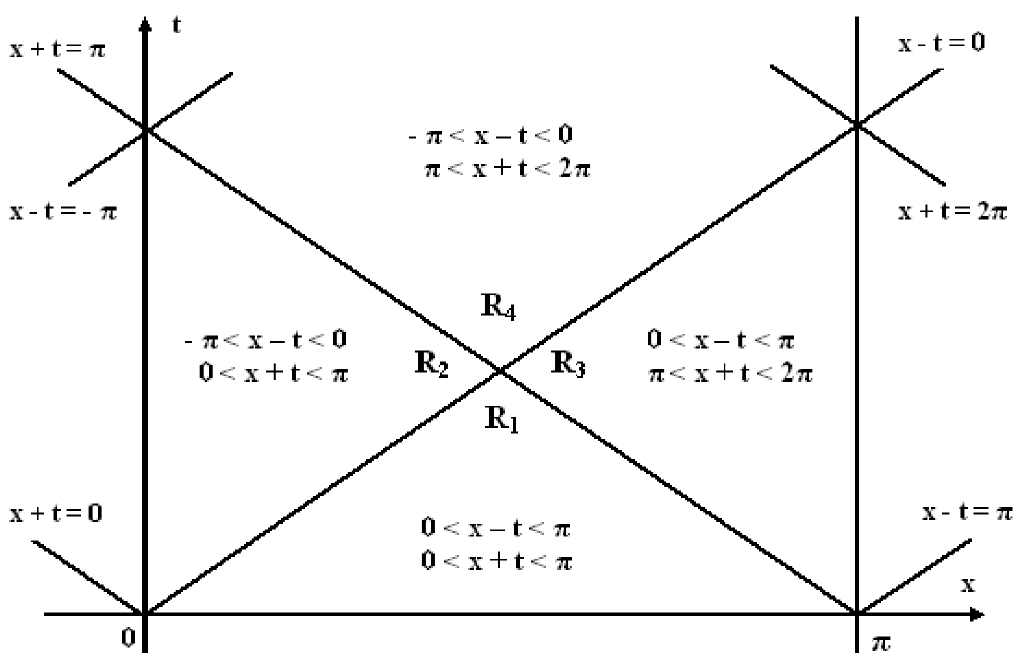
$$\text{in } R_2, \qquad u(x,t) = \frac{\pi}{2} + \frac{-(x-t)}{2} - \frac{\pi}{4} + \frac{x+t}{2} - \frac{\pi}{4} = t.$$

Since $\cos[n(x+t)] = \cos[n(x+t-2\pi)] = \cos[n(2\pi-(x+t))]$, and $0 < 2\pi-(x+t) < \pi$ in R_3,

$$\text{in } R_3, \qquad u(x,t) = \frac{\pi}{2} + \frac{x-t}{2} - \frac{\pi}{4} + \frac{2\pi-(x+t)}{2} - \frac{\pi}{4} = \pi - t.$$

Since $0 < -(x-t) < \pi$ and $0 < 2\pi - (x+t) < \pi$ in R_4

$$\text{in } R_4, \qquad u(x,t) = \frac{\pi}{2} + \frac{-(x-t)}{2} - \frac{\pi}{4} + \frac{2\pi-(x+t)}{2} - \frac{\pi}{4} = \pi - x.$$



Example (McOwen 3.1 #4). *Consider the initial boundary value problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \textit{for} \quad x > 0,\, t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & \textit{for} \;\; x > 0 \\ u(0,t) = 0 & \textit{for}\; t \geq 0, \end{cases} \tag{16.22}$$

where $g(0) = 0 = h(0)$. If we extend g and h as **odd** *functions on $-\infty < x < \infty$, show that d'Alembert's formula gives the solution.*

Proof. Extend g and h as **odd** functions on $-\infty < x < \infty$:

$$\tilde{g}(x) = \begin{cases} g(x), & x \geq 0 \\ -g(-x), & x < 0 \end{cases} \qquad \tilde{h}(x) = \begin{cases} h(x), & x \geq 0 \\ -h(-x), & x < 0. \end{cases}$$

Then, we need to solve

$$\begin{cases} \tilde{u}_{tt} - c^2 \tilde{u}_{xx} = 0 & \text{for} \quad -\infty < x < \infty,\, t > 0 \\ \tilde{u}(x,0) = \tilde{g}(x), \quad \tilde{u}_t(x,0) = \tilde{h}(x) & \text{for} \quad -\infty < x < \infty. \end{cases} \tag{16.23}$$

To show that d'Alembert's formula gives the solution to (16.23), we need to show that the solution given by d'Alembert's formula satisfies the boundary condition $\tilde{u}(0,t) = 0$.

$$\begin{aligned} \tilde{u}(x,t) &= \frac{1}{2}(\tilde{g}(x+ct) + \tilde{g}(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{h}(\xi)\, d\xi, \\ \tilde{u}(0,t) &= \frac{1}{2}(\tilde{g}(ct) + \tilde{g}(-ct)) + \frac{1}{2c}\int_{-ct}^{ct} \tilde{h}(\xi)\, d\xi \\ &= \frac{1}{2}(\tilde{g}(ct) - \tilde{g}(ct)) + \frac{1}{2c}(H(ct) - H(-ct)) \\ &= 0 + \frac{1}{2c}(H(ct) - H(ct)) = 0, \end{aligned}$$

where we used $H(x) = \int_0^x \tilde{h}(\xi)\, d\xi$; and since $\tilde{h}$ is **odd**, then H is **even**. □

Example (McOwen 3.1 #5). *Find in closed form (similar to d'Alembet's formula) the solution $u(x,t)$ of*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \textit{for} \;\; x,t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & \textit{for} \;\; x > 0 \\ u(0,t) = \alpha(t) & \textit{for}\; t \geq 0, \end{cases} \tag{16.24}$$

where $g, h, \alpha \in C^2$ satisfy $\alpha(0) = g(0)$, $\alpha'(0) = h(0)$, and $\alpha''(0) = c^2 g''(0)$. Verify that $u \in C^2$, even on the characteristic $x = ct$.

Proof. As in (McOwen 3.1 #4), we can extend g and h to be odd functions. We want to transform the problem to have zero boundary conditions.
Consider the function:

$$U(x,t) = u(x,t) - \alpha(t). \tag{16.25}$$

Then (16.24) transforms to:

$$\begin{cases} U_{tt} - c^2 U_{xx} = \underbrace{-\alpha''(t)}_{f_U(x,t)} \\ U(x,0) = \underbrace{g(x) - \alpha(0)}_{g_U(x)}, \quad U_t(x,0) = \underbrace{h(x) - \alpha'(0)}_{h_U(x)} \\ U(0,t) = \underbrace{0}_{\alpha_u(t)}. \end{cases}$$

We use d'Alembert's formula and Duhamel's principle on U.
After getting U, we can get u from $u(x,t) = U(x,t) + \alpha(t)$. $\square$

Example (Zachmanoglou, Chapter 8, Example 7.2). *Find the solution of*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{for } x > 0, t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & \text{for } x > 0 \\ u_x(0,t) = 0 & \text{for } t > 0. \end{cases} \tag{16.26}$$

Proof. Extend g and h as **even** functions on $-\infty < x < \infty$:

$$\tilde{g}(x) = \begin{cases} g(x), & x \geq 0 \\ g(-x), & x < 0 \end{cases} \qquad \tilde{h}(x) = \begin{cases} h(x), & x \geq 0 \\ h(-x), & x < 0. \end{cases}$$

Then, we need to solve

$$\begin{cases} \tilde{u}_{tt} - c^2 \tilde{u}_{xx} = 0 & \text{for} \quad -\infty < x < \infty,\ t > 0 \\ \tilde{u}(x,0) = \tilde{g}(x), \quad \tilde{u}_t(x,0) = \tilde{h}(x) & \text{for} \quad -\infty < x < \infty. \end{cases} \tag{16.27}$$

To show that d'Alembert's formula gives the solution to (16.27), we need to show that the solution given by d'Alembert's formula satisfies the boundary condition $\tilde{u}_x(0,t) = 0$.

$$\begin{aligned} \tilde{u}(x,t) &= \frac{1}{2}(\tilde{g}(x+ct) + \tilde{g}(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{h}(\xi)\,d\xi. \\ \tilde{u}_x(x,t) &= \frac{1}{2}(\tilde{g}'(x+ct) + \tilde{g}'(x-ct)) + \frac{1}{2c}[\tilde{h}(x+ct) - \tilde{h}(x-ct)], \\ \tilde{u}_x(0,t) &= \frac{1}{2}(\tilde{g}'(ct) + \tilde{g}'(-ct)) + \frac{1}{2c}[\tilde{h}(ct) - \tilde{h}(-ct)] \;=\; 0. \end{aligned}$$

Since $\tilde{g}$ is **even**, then g' is **odd**. □

Problem (F'89, #3). [36] *Let $\alpha \neq c$, constant. Find the solution of*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{for } x > 0, t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & \text{for } x > 0 \\ u_t(0,t) = \alpha u_x(0,t) & \text{for } t > 0, \end{cases} \tag{16.28}$$

where $g, h \in C^2$ for $x > 0$ and vanish near $x = 0$.
Hint: Use the fact that a general solution of (16.28) can be written as the sum of two traveling wave solutions.

Proof. D'Alembert's formula is derived by plugging in the following into the above equation and initial conditions:

$$u(x,t) = F(x+ct) + G(x-ct).$$

As in (Zachmanoglou 7.2), we can extend g and h to be even functions. □

[36] Similar to McOwen 3.1 #5. The notation in this problem is changed to be consistent with McOwen.

Example (McOwen 3.1 #6). *Solve the initial/boundary value problem*

$$\begin{cases} u_{tt} - u_{xx} = 1 & \text{for } \ 0 < x < \pi \ \ \text{and} \ \ t > 0 \\ u(x,0) = 0, \quad u_t(x,0) = 0 & \text{for } \ 0 < x < \pi \\ u(0,t) = 0, \quad u(\pi,t) = -\pi^2/2 & \text{for } t \geq 0. \end{cases} \tag{16.29}$$

Proof. If we first find a particular solution of the nonhomogeneous equation, this reduces the problem to a boundary value problem for the homogeneous equation (as in (McOwen 3.1 #2) and (McOwen 3.1 #3)).
Hint: You should use a particular solution depending on x!

❶ First, find a particular solution. This is similar to the method of separation of variables. Assume

$$u_p(x,t) = X(x),$$

which gives

$$\begin{aligned} -X''(x) &= 1, \\ X''(x) &= -1. \end{aligned}$$

The solution to the above ODE is

$$X(x) = -\frac{x^2}{2} + ax + b.$$

The boundary conditions give

$$\begin{aligned} u_p(0,t) &= b = 0, \\ u_p(\pi,t) &= -\frac{\pi^2}{2} + a\pi + b = -\frac{\pi^2}{2}, \qquad \Rightarrow \qquad a = b = 0. \end{aligned}$$

Thus, the particular solution is

$$\boxed{u_p(x,t) = -\frac{x^2}{2}.}$$

This solution satisfies the following:

$$\begin{cases} u_{p_{tt}} - u_{p_{xx}} = 1 \\ u_p(x,0) = -\frac{x^2}{2}, \quad u_{p_t}(x,0) = 0 \\ u_p(0,t) = 0, \quad u_p(\pi,t) = -\frac{\pi^2}{2}. \end{cases}$$

❷ Second, we find a solution to a boundary value problem for the homogeneous equation:

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = \frac{x^2}{2}, \quad u_t(x,0) = 0 \\ u(0,t) = 0, \quad u(\pi,t) = 0. \end{cases}$$

This is solved by the method of Separation of Variables. See Separation of Variables subsection of "Problems: Separation of Variables: Wave Equation" McOwen 3.1 #2. The only difference there is that $u(x,0) = 1$.
We would find $u_h(x,t)$. Then,

$$\boxed{u(x,t) = u_h(x,t) + u_p(x,t).}$$

□

Problem (S'02, #2). ***a)*** *Given a continuous function f on $\mathbb{R}$ which vanishes for $|x| > R$, solve the initial value problem*

$$\begin{cases} u_{tt} - u_{xx} = f(x)\cos t, \\ u(x,0) = 0, \quad u_t(x,0) = 0, \qquad -\infty < x < \infty,\ 0 \le t < \infty \end{cases}$$

by first finding a particular solution by ***separation of variables*** *and then adding the appropriate solution of the homogeneous PDE.*
b) *Since the particular solution is not unique, it will not be obvious that the solution to the initial value problem that you have found in part (a) is unique. Prove that it is* ***unique****.*

Proof. **a)** ❶ First, find a particular solution by separation of variables. Assume

$$u_p(x,t) = X(x)\cos t,$$

which gives

$$\begin{aligned} -X(x)\cos t - X''(x)\cos t &= f(x)\cos t, \\ X'' + X &= -f(x). \end{aligned}$$

The solution to the above ODE is written as $X = X_h + X_p$. The homogeneous solution is

$$X_h(x) = a\cos x + b\sin x.$$

To find a particular solution, note that since f is continuous, $\exists G \in C^2(\mathbb{R})$, such that

$$G'' + G = -f(x).$$

Thus,

$$\begin{aligned} & X_p(x) = G(x). \\ \Rightarrow\quad & X(x) = X_h(x) + X_p(x) = a\cos x + b\sin x + G(x). \end{aligned}$$

$$\boxed{u_p(x,t) = \big[a\cos x + b\sin x + G(x)\big]\cos t.}$$

It can be verified that this solution satisfies the following:

$$\begin{cases} u_{p_{tt}} - u_{p_{xx}} = f(x)\cos t, \\ u_p(x,0) = a\cos x + b\sin x + G(x), \qquad u_{p_t}(x,0) = 0. \end{cases}$$

❷ Second, we find a solution of the homogeneous PDE:

$$\begin{cases} u_{tt} - u_{xx} = 0, \\ u(x,0) = \underbrace{-a\cos x - b\sin x - G(x)}_{g(x)}, \quad u_t(x,0) = \underbrace{0}_{h(x)}. \end{cases}$$

The solution is given by d'Alembert's formula (with $c = 1$):

$$\begin{aligned} u_h(x,t) &= u^A(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)\,d\xi \\ &= \frac{1}{2}\Big(\big(-a\cos(x+t) - b\sin(x+t) - G(x+t)\big) + \big(-a\cos(x-t) - b\sin(x-t) - G(x-t)\big)\Big) \\ &= -\frac{1}{2}\big(a\cos(x+t) + b\sin(x+t) + G(x+t)\big) - \frac{1}{2}\big(a\cos(x-t) + b\sin(x-t) + G(x-t)\big). \end{aligned}$$

It can be verified that the solution satisfies the above homogeneous PDE with the boundary conditions. Thus, the complete solution is:

$$\boxed{u(x,t) = u_h(x,t) + u_p(x,t).}$$

Alternatively, we could use Duhamel's principle to find the solution: [37]

$$\boxed{u(x,t) = \frac{1}{2}\int_0^t \left(\int_{x-(t-s)}^{x+(t-s)} f(\xi)\ \cos s\ d\xi \right) ds.}$$

However, this is not how it was suggested to do this problem.

b) The particular solution is **not** unique, since any constants a, b give the solution. However, we show that the solution to the initial value problem is unique.
Suppose u_1 and u_2 are two solutions. Then $w = u_1 - u_2$ satisfies:

$$\begin{cases} w_{tt} - w_{xx} = 0, \\ w(x,0) = 0, \quad w_t(x,0) = 0. \end{cases}$$

D'Alembert's formula gives

$$w(x,t) \ = \ \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)\, d\xi \ = \ 0.$$

Thus, the solution to the initial value problem is unique. □

[37] Note the relationship: $x \leftrightarrow \xi$, $t \leftrightarrow s$.

16.3 Similarity Solutions

Problem (F'98, #7). *Look for a* ***similarity solution*** *of the form* $v(x,t) = t^\alpha w(y = x/t^\beta)$ *for the differential equation*

$$v_t = v_{xx} + (v^2)_x. \tag{16.30}$$

a) Find the parameters α *and* β.
b) Find a differential equation for $w(y)$ *and show that this ODE can be reduced to first order.*
c) Find a solution for the resulting first order ODE.

Proof. We can rewrite (16.30) as

$$v_t = v_{xx} + 2vv_x. \tag{16.31}$$

We look for a similarity solution of the form

$$v(x,t) = t^\alpha w(y), \qquad \Big(y = \frac{x}{t^\beta}\Big).$$

$$\begin{aligned}
v_t &= \alpha t^{\alpha-1} w + t^\alpha w' y_t = \alpha t^{\alpha-1} w + t^\alpha \Big(-\frac{\beta x}{t^{\beta+1}}\Big) w' = \alpha t^{\alpha-1} w - t^{\alpha-1}\beta y w', \\
v_x &= t^\alpha w' y_x = t^\alpha w' t^{-\beta} = t^{\alpha-\beta} w', \\
v_{xx} &= (t^{\alpha-\beta} w')_x = t^{\alpha-\beta} w'' y_x = t^{\alpha-\beta} w'' t^{-\beta} = t^{\alpha-2\beta} w''.
\end{aligned}$$

Plugging in the derivatives we calculated into (16.31), we obtain

$$\begin{aligned}
\alpha t^{\alpha-1} w - t^{\alpha-1}\beta y w' &= t^{\alpha-2\beta} w'' + 2(t^\alpha w)(t^{\alpha-\beta} w'), \\
\alpha w - \beta y w' &= t^{1-2\beta} w'' + 2t^{\alpha-\beta+1} w w'.
\end{aligned}$$

The parameters that would eliminate t from equation above are

$$\boxed{\beta = \frac{1}{2}, \quad \alpha = -\frac{1}{2}.}$$

With these parameters, we obtain the differential equation for $w(y)$:

$$-\frac{1}{2} w - \frac{1}{2} y w' = w'' + 2ww',$$

$$\boxed{w'' + 2ww' + \frac{1}{2} y w' + \frac{1}{2} w = 0.}$$

We can write the ODE as

$$w'' + 2ww' + \frac{1}{2}(yw)' = 0.$$

Integrating it with respect to y, we obtain the first order ODE:

$$\boxed{w' + w^2 + \frac{1}{2} y w = c.}$$

□

16.4 Traveling Wave Solutions

Consider the *Korteweg-de Vries* (KdV) equation in the form [38]

$$u_t + 6uu_x + u_{xxx} = 0, \qquad -\infty < x < \infty, \ t > 0. \tag{16.32}$$

We look for a *traveling wave solution*

$$u(x,t) = f(x - ct). \tag{16.33}$$

We get the ODE

$$-cf' + 6ff' + f''' = 0. \tag{16.34}$$

We integrate (16.34) to get

$$-cf + 3f^2 + f'' = a, \tag{16.35}$$

where a is a constant. Multiplying this equality by f', we obtain

$$-cff' + 3f^2 f' + f''f' = af'.$$

Integrating again, we get

$$-\frac{c}{2}f^2 + f^3 + \frac{(f')^2}{2} = af + b. \tag{16.36}$$

We are looking for solutions f which satisfy $f(x)$, $f'(x)$, $f''(x) \to 0$ as $x \to \pm\infty$. (In which case the function u having the form (16.33) is called a *solitary* wave.) Then (16.35) and (16.36) imply $a = b = 0$, so that

$$-\frac{c}{2}f^2 + f^3 + \frac{(f')^2}{2} = 0, \qquad \text{or} \quad f' = \pm f\sqrt{c - 2f}.$$

The solution of this ODE is

$$f(x) = \frac{c}{2}\text{sech}^2[\frac{\sqrt{c}}{2}(x - x_0)],$$

where x_0 is the constant of integration. A solution of this form is called a *soliton.*

[38] Evans, p. 174; Strauss, p. 367.

Problem (S'93, #6). *The generalized KdV equation is*

$$\frac{\partial u}{\partial t} = \frac{1}{2}(n+1)(n+2)u^n\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3},$$

where n is a positive integer. Solitary wave solutions are sought in which $u = f(\eta)$, where $\eta = x - ct$ and

$$f, f', f'' \to 0, \qquad as \ \ |\eta| \to \infty;$$

c, the wave speed, is constant.
Show that

$$f'^2 = f^{n+2} + cf^2.$$

Hence show that solitary waves do not exist if n is even.
Show also that, when $n = 1$, all conditions of the problem are satisfied provided $c > 0$ and

$$u = -c\, sech^2\Big[\frac{1}{2}\sqrt{c}(x - ct)\Big].$$

Proof. • We look for a *traveling wave solution*

$$u(x,t) = f(x - ct).$$

We get the ODE

$$-cf' = \frac{1}{2}(n+1)(n+2)f^n f' - f''',$$

Integrating this equation, we get

$$-cf = \frac{1}{2}(n+2)f^{n+1} - f'' + a, \tag{16.37}$$

where a is a constant. Multiplying this equality by f', we obtain

$$-cff' = \frac{1}{2}(n+2)f^{n+1}f' - f''f' + af'.$$

Integrating again, we get

$$-\frac{cf^2}{2} = \frac{1}{2}f^{n+2} - \frac{(f')^2}{2} + af + b. \tag{16.38}$$

We are looking for solutions f which satisfy $f, f', f'' \to 0$ as $x \to \pm\infty$. Then (16.37) and (16.38) imply $a = b = 0$, so that

$$\begin{aligned} -\frac{cf^2}{2} &= \frac{1}{2}f^{n+2} - \frac{(f')^2}{2}, \\ (f')^2 &= f^{n+2} + cf^2. \quad \checkmark \end{aligned}$$

• We show that solitary waves do not exist if n is even. We have

$$\begin{aligned} f' &= \pm\sqrt{f^{n+2} + cf^2} = \pm|f|\sqrt{f^n + c}, \\ \int_{-\infty}^{\infty} f'\, d\eta &= \pm\int_{-\infty}^{\infty} |f|\sqrt{f^n + c}\, d\eta, \\ f\Big|_{-\infty}^{\infty} &= \pm\int_{-\infty}^{\infty} |f|\sqrt{f^n + c}\, d\eta, \\ 0 &= \pm\int_{-\infty}^{\infty} |f|\sqrt{f^n + c}\, d\eta. \end{aligned}$$

Thus, either ① $|f| \equiv 0 \;\Rightarrow\; f = 0$, or
② $f^n + c = 0$. Since $f \to 0$ as $x \to \pm\infty$, we have $c = 0 \;\Rightarrow\; f = 0$.
Thus, solitary waves do not exist if n is even. ✓ □

- When $n = 1$, we have

$$(f')^2 = f^3 + cf^2. \tag{16.39}$$

We show that all conditions of the problem are satisfied provided $c > 0$, including

$$\begin{aligned} u &= -c\, sech^2\Big[\frac{1}{2}\sqrt{c}(x-ct)\Big], \qquad \text{or} \\ f &= -c\, sech^2\Big[\frac{\eta\sqrt{c}}{2}\Big] = -\frac{c}{\cosh^2[\frac{\eta\sqrt{c}}{2}]} = -c\cosh\Big[\frac{\eta\sqrt{c}}{2}\Big]^{-2}. \end{aligned}$$

We have

$$\begin{aligned} f' &= 2c\cosh\Big[\frac{\eta\sqrt{c}}{2}\Big]^{-3}\cdot\sinh\Big[\frac{\eta\sqrt{c}}{2}\Big]\cdot\frac{\sqrt{c}}{2} = c\sqrt{c}\cosh\Big[\frac{\eta\sqrt{c}}{2}\Big]^{-3}\cdot\sinh\Big[\frac{\eta\sqrt{c}}{2}\Big], \\ (f')^2 &= \frac{c^3\ \sinh^2\big[\frac{\eta\sqrt{c}}{2}\big]}{\cosh^6\big[\frac{\eta\sqrt{c}}{2}\big]}, \\ f^3 &= -\frac{c^3}{\cosh^6\big[\frac{\eta\sqrt{c}}{2}\big]}, \\ cf^2 &= \frac{c^3}{\cosh^4\big[\frac{\eta\sqrt{c}}{2}\big]}. \end{aligned}$$

Plugging these into (16.39), we obtain: [39]

$$\begin{aligned} \frac{c^3\ \sinh^2\big[\frac{\eta\sqrt{c}}{2}\big]}{\cosh^6\big[\frac{\eta\sqrt{c}}{2}\big]} &= -\frac{c^3}{\cosh^6\big[\frac{\eta\sqrt{c}}{2}\big]} + \frac{c^3}{\cosh^4\big[\frac{\eta\sqrt{c}}{2}\big]}, \\ \frac{c^3\ \sinh^2\big[\frac{\eta\sqrt{c}}{2}\big]}{\cosh^6\big[\frac{\eta\sqrt{c}}{2}\big]} &= \frac{-c^3 + c^3\cosh^2\big[\frac{\eta\sqrt{c}}{2}\big]}{\cosh^6\big[\frac{\eta\sqrt{c}}{2}\big]}, \\ \frac{c^3\ \sinh^2\big[\frac{\eta\sqrt{c}}{2}\big]}{\cosh^6\big[\frac{\eta\sqrt{c}}{2}\big]} &= \frac{c^3\ \sinh^2\big[\frac{\eta\sqrt{c}}{2}\big]}{\cosh^6\big[\frac{\eta\sqrt{c}}{2}\big]}. \qquad \checkmark \end{aligned}$$

Also, $f, f', f'' \to 0$, as $|\eta| \to \infty$, since

$$f(\eta) = -c\, sech^2\Big[\frac{\eta\sqrt{c}}{2}\Big] = -\frac{c}{\cosh^2[\frac{\eta\sqrt{c}}{2}]} = -c\bigg(\frac{2}{e^{[\frac{\eta\sqrt{c}}{2}]} + e^{-[\frac{\eta\sqrt{c}}{2}]}}\bigg)^2 \to 0, \text{ as } |\eta| \to \infty.$$

Similarly, $f', f'' \to 0$, as $|\eta| \to \infty$. $\checkmark$

---

[39] $\cosh^2 x - \sinh^2 x = 1.$

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

Problem (S'00, #5). *Look for a* ***traveling wave solution*** *of the PDE*

$$u_{tt} + (u^2)_{xx} = -u_{xxxx}$$

of the form $u(x,t) = v(x - ct)$*. In particular, you should find an ODE for* v*. Under the assumption that* v *goes to a constant as* $|x| \to \infty$*, describe the form of the solution.*

Proof. Since $(u^2)_x = 2uu_x$, and $(u^2)_{xx} = 2u_x^2 + 2uu_{xx}$, we have

$$u_{tt} + 2u_x^2 + 2uu_{xx} = -u_{xxxx}.$$

We look for a traveling wave solution

$$u(x,t) = v(x - ct).$$

We get the ODE

$$\begin{aligned} c^2v'' + 2(v')^2 + 2vv'' &= -v'''', \\ c^2v'' + 2((v')^2 + vv'') &= -v'''', \\ c^2v'' + 2(vv')' &= -v'''', && \text{(exact differentials)} \\ c^2v' + 2vv' &= -v''' + a, && s = x - ct \\ c^2v + v^2 &= -v'' + as + b, && \circledast \end{aligned}$$

$$\boxed{v'' + c^2v + v^2 = a(x - ct) + b.}$$

Since $v \to C = \text{const}$ as $|x| \to \infty$, we have v', $v'' \to 0$, as $|x| \to \infty$. Thus, $\circledast$ implies

$$c^2v + v^2 = as + b.$$

Since $|x| \to \infty$, but $v \to C$, we have $a = 0$:

$$v^2 + c^2v - b = 0.$$

$$v = \frac{-c^2 \pm \sqrt{c^4 + 4b}}{2}.$$

□

Problem (S'95, #2). *Consider the KdV-Burgers equation*

$$u_t + uu_x = \epsilon u_{xx} + \delta u_{xxx}$$

in which $\epsilon > 0$, $\delta > 0$.
a) *Find an ODE for* ***traveling wave solutions*** *of the form*

$$u(x,t) = \varphi(x - st)$$

with $s > 0$ *and*

$$\lim_{y \to -\infty} \varphi(y) = 0$$

and analyze the stationary points from this ODE.

b) *Find the possible (finite) values of*

$$\varphi_+ = \lim_{y \to \infty} \varphi(y).$$

Proof. **a)** We look for a traveling wave solution

$$u(x,t) = \varphi(x - st), \qquad y = x - st.$$

We get the ODE

$$\begin{aligned} -s\varphi' + \varphi\varphi' &= \epsilon\varphi'' + \delta\varphi''', \\ -s\varphi + \frac{1}{2}\varphi^2 &= \epsilon\varphi' + \delta\varphi'' + a. \end{aligned}$$

Since $\varphi \to 0$ as $y \to -\infty$, then φ', $\varphi'' \to 0$ as $y \to -\infty$. Therefore, at $y = -\infty$, $a = 0$.
We found the following ODE,

$$\boxed{\varphi'' + \frac{\epsilon}{\delta}\varphi' + \frac{s}{\delta}\varphi - \frac{1}{2\delta}\varphi^2 = 0.}$$

In order to find and analyze the stationary points of an ODE above, we write it as a first-order system.

$$\begin{aligned} \phi_1 &= \varphi, \\ \phi_2 &= \varphi'. \end{aligned}$$

$$\begin{aligned} \phi_1' &= \varphi' = \phi_2, \\ \phi_2' &= \varphi'' = -\frac{\epsilon}{\delta}\varphi' - \frac{s}{\delta}\varphi + \frac{1}{2\delta}\varphi^2 = -\frac{\epsilon}{\delta}\phi_2 - \frac{s}{\delta}\phi_1 + \frac{1}{2\delta}\phi_1^2. \end{aligned}$$

$$\begin{cases} \phi_1' = \phi_2 = 0, \\ \phi_2' = -\frac{\epsilon}{\delta}\phi_2 - \frac{s}{\delta}\phi_1 + \frac{1}{2\delta}\phi_1^2 = 0; \end{cases} \Rightarrow \begin{cases} \phi_1' = \phi_2 = 0, \\ \phi_2' = -\frac{s}{\delta}\phi_1 + \frac{1}{2\delta}\phi_1^2 = 0; \end{cases} \Rightarrow \begin{cases} \phi_1' = \phi_2 = 0, \\ \phi_2' = -\frac{1}{\delta}\phi_1(s - \frac{1}{2}\phi_1) = 0. \end{cases}$$

$$\boxed{\text{Stationary points: } (0,0),\ (2s,0), \quad s > 0.}$$

$$\begin{aligned} \phi_1' &= \phi_2 && = f(\phi_1, \phi_2), \\ \phi_2' &= -\frac{\epsilon}{\delta}\phi_2 - \frac{s}{\delta}\phi_1 + \frac{1}{2\delta}\phi_1^2 && = g(\phi_1, \phi_2). \end{aligned}$$

In order to classify a stationary point, need to find eigenvalues of a linearized system at that point.

$$J(f(\phi_1,\phi_2),g(\phi_1,\phi_2)) = \begin{bmatrix} \frac{\partial f}{\partial \phi_1} & \frac{\partial f}{\partial \phi_2} \\ \frac{\partial g}{\partial \phi_1} & \frac{\partial g}{\partial \phi_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{s}{\delta}+\frac{1}{\delta}\phi_1 & -\frac{\epsilon}{\delta} \end{bmatrix}.$$

- For $(\phi_1,\phi_2)=(0,0)$:

$\det(J|_{(0,0)}-\lambda I) = \begin{vmatrix} -\lambda & 1 \\ -\frac{s}{\delta} & -\frac{\epsilon}{\delta}-\lambda \end{vmatrix} = \lambda^2+\frac{\epsilon}{\delta}\lambda+\frac{s}{\delta}=0.$
$\lambda_\pm = -\frac{\epsilon}{2\delta} \pm \sqrt{\frac{\epsilon^2}{4\delta^2}-\frac{s}{\delta}}.$
If $\frac{\epsilon^2}{4\delta} > s \quad \Rightarrow \quad \lambda_\pm \in \mathbb{R}, \quad \lambda_\pm < 0.$
$\Rightarrow$ **(0,0)** is **Stable Improper Node**.
If $\frac{\epsilon^2}{4\delta} < s \quad \Rightarrow \quad \lambda_\pm \in \mathbb{C}, \quad \mathbb{R}e(\lambda_\pm) < 0.$
$\Rightarrow$ **(0,0)** is **Stable Spiral Point**.

- For $(\phi_1,\phi_2)=(2s,0)$:

$\det(J|_{(2s,0)}-\lambda I) = \begin{vmatrix} -\lambda & 1 \\ \frac{s}{\delta} & -\frac{\epsilon}{\delta}-\lambda \end{vmatrix} = \lambda^2+\frac{\epsilon}{\delta}\lambda-\frac{s}{\delta}=0.$
$\lambda_\pm = -\frac{\epsilon}{2\delta} \pm \sqrt{\frac{\epsilon^2}{4\delta^2}+\frac{s}{\delta}}.$
$\Rightarrow \quad \lambda_+ > 0, \quad \lambda_- < 0.$
$\Rightarrow$ **(2s,0)** is **Untable Saddle Point**.

b) Since

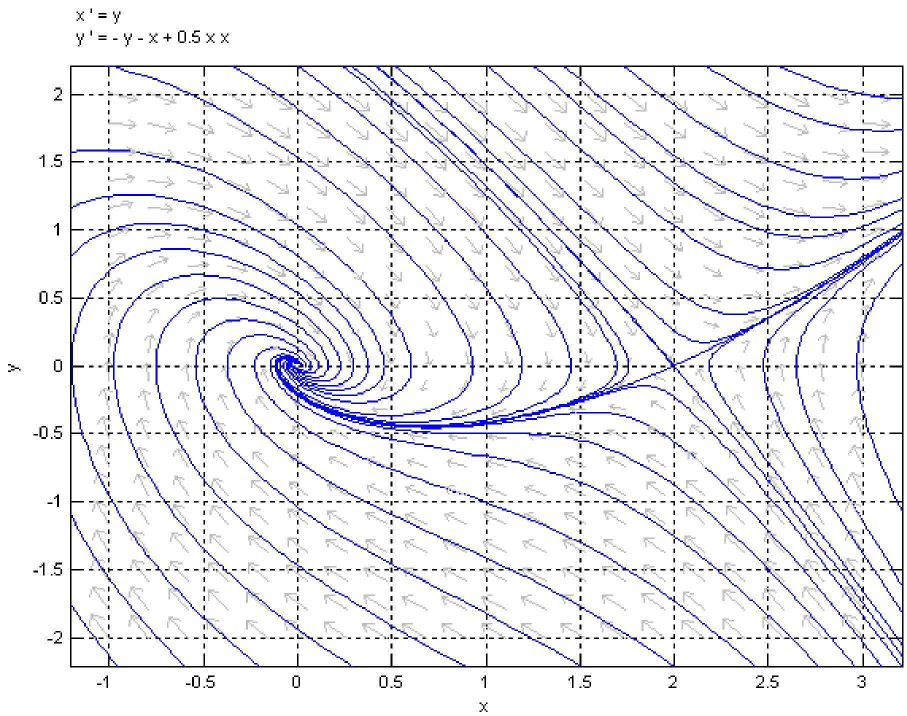
$$\lim_{y\to-\infty}\varphi(y) = 0 = \lim_{t\to\infty}\varphi(x-st),$$

we **may** have

$$\lim_{y\to+\infty}\varphi(y) = \lim_{t\to-\infty}\varphi(x-st) = 2s.$$

That is, a particle may start off at an unstable node $(2s,0)$ and as t increases, approach the stable node $(0,0)$.

A phase diagram with $(0,0)$ being a stable spiral point, is shown below.


x ' = y
y ' = - y - x + 0.5 x x
y
x
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1
-0.5
0
0.5
1
1.5
2
2.5
3

□

Problem (F'95, #8). *Consider the equation*

$$u_t + f(u)_x = \epsilon u_{xx}$$

where f is smooth and $\epsilon > 0$. We seek ***traveling wave solutions*** *to this equation, i.e., solutions of the form $u = \phi(x - st)$, under the boundary conditions*

$$u \to u_L \quad \text{and} \quad u_x \to 0 \quad \text{as} \quad x \to -\infty,$$
$$u \to u_R \quad \text{and} \quad u_x \to 0 \quad \text{as} \quad x \to +\infty.$$

Find a necessary and sufficient condition on f, u_L, u_R and s for such traveling waves to exist; in case this condition holds, write an equation which defines ϕ implicitly.

Proof. We look for traveling wave solutions

$$u(x,t) = \phi(x - st), \qquad y = x - st.$$

The boundary conditions become

$$\phi \to u_L \quad \text{and} \quad \phi' \to 0 \quad \text{as} \quad x \to -\infty,$$
$$\phi \to u_R \quad \text{and} \quad \phi' \to 0 \quad \text{as} \quad x \to +\infty. \qquad \circledast$$

Since $f(\phi(x - st))_x = f'(\phi)\phi'$, we get the ODE

$$-s\phi' + f'(\phi)\phi' = \epsilon\phi'',$$
$$-s\phi' + (f(\phi))' = \epsilon\phi'',$$
$$-s\phi + f(\phi) = \epsilon\phi' + a,$$
$$\phi' = \frac{-s\phi + f(\phi)}{\epsilon} + b.$$

We use boundary conditions to determine constant b:

$$\text{At} \quad x = -\infty, \qquad 0 = \phi' = \frac{-su_L + f(u_L)}{\epsilon} + b \qquad \Rightarrow \qquad b = \frac{su_L - f(u_L)}{\epsilon}.$$
$$\text{At} \quad x = +\infty, \qquad 0 = \phi' = \frac{-su_R + f(u_R)}{\epsilon} + b \qquad \Rightarrow \qquad b = \frac{su_R - f(u_R)}{\epsilon}.$$

$$\boxed{s = \frac{f(u_L) - f(u_R)}{u_L - u_R}.}$$

[40] □

[40] For the solution for the second part of the problem, refer to Chiu-Yen's solutions.

Problem (S'02, #5; F'90, #2). Fisher's Equation. *Consider*

$$u_t = u(1-u) + u_{xx}, \qquad -\infty < x < \infty, \; t > 0.$$

The solutions of physical interest satisfy $0 \le u \le 1$, *and*

$$\lim_{x\to-\infty} u(x,t) = 0, \qquad \lim_{x\to+\infty} u(x,t) = 1.$$

One class of solutions is the set of "wavefront" solutions. These have the form $u(x,t) = \phi(x+ct)$, $c \ge 0$.
Determine the ordinary differential equation and boundary conditions which ϕ *must satisfy (to be of physical interest). Carry out a phase plane analysis of this equation, and show that physically interesting wavefront solutions are possible if* $c \ge 2$, *but not if* $0 \le c < 2$.

Proof. We look for a traveling wave solution

$$u(x,t) = \phi(x+ct), \qquad s = x + ct.$$

We get the ODE

$$c\phi' = \phi(1-\phi) + \phi'',$$

$$\boxed{\phi'' - c\phi' + \phi - \phi^2 = 0,}$$

- ∘ $\phi(s) \to 0$, as $s \to -\infty$,
- ∘ $\phi(s) \to 1$, as $s \to +\infty$,
- ∘ $0 \le \phi \le 1$.

In order to find and analyze the stationary points of an ODE above, we write it as a first-order system.

$$y_1 = \phi,$$
$$y_2 = \phi'.$$

$$y_1' = \phi' = y_2,$$
$$y_2' = \phi'' = c\phi' - \phi + \phi^2 = cy_2 - y_1 + y_1^2.$$

$$\begin{cases} y_1' = y_2 = 0, \\ y_2' = cy_2 - y_1 + y_1^2 = 0; \end{cases} \quad \Rightarrow \quad \begin{cases} y_2 = 0, \\ y_1(y_1 - 1) = 0. \end{cases}$$

$$\boxed{\text{Stationary points: } (0,0),\, (1,0).}$$

$$y_1' = y_2 \qquad\qquad = f(y_1,y_2),$$
$$y_2' = cy_2 - y_1 + y_1^2 = g(y_1,y_2).$$

In order to classify a stationary point, need to find eigenvalues of a linearized system at that point.

$$J(f(y_1,y_2), g(y_1,y_2)) = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} \\ \frac{\partial g}{\partial y_1} & \frac{\partial g}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2y_1 - 1 & c \end{bmatrix}.$$

- For $(y_1, y_2) = (0, 0)$:

$$\det(J|_{(0,0)} - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & c-\lambda \end{vmatrix} = \lambda^2 - c\lambda + 1 = 0.$$

$$\lambda_\pm = \frac{c \pm \sqrt{c^2-4}}{2}.$$

If $c \geq 2 \quad \Rightarrow \quad \lambda_\pm \in \mathbb{R}, \quad \lambda_\pm > 0$.
(0,0) is **Unstable Improper** $(c > 2)$ / **Proper** $(c = 2)$ **Node**.
If $0 \leq c < 2 \quad \Rightarrow \quad \lambda_\pm \in \mathbb{C}, \quad \mathbb{R}e(\lambda_\pm) \geq 0$.
(0,0) is **Unstable Spiral Node**.

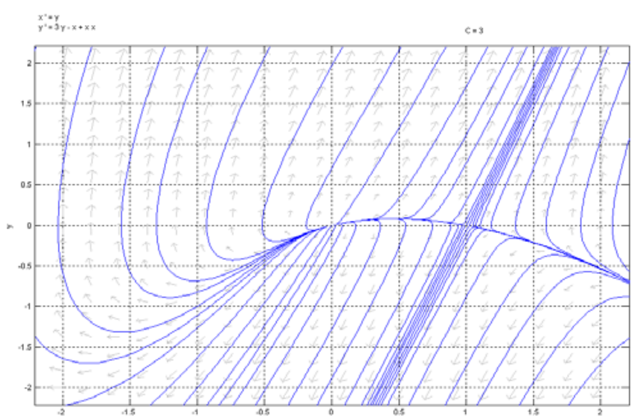
- For $(y_1, y_2) = (1, 0)$:

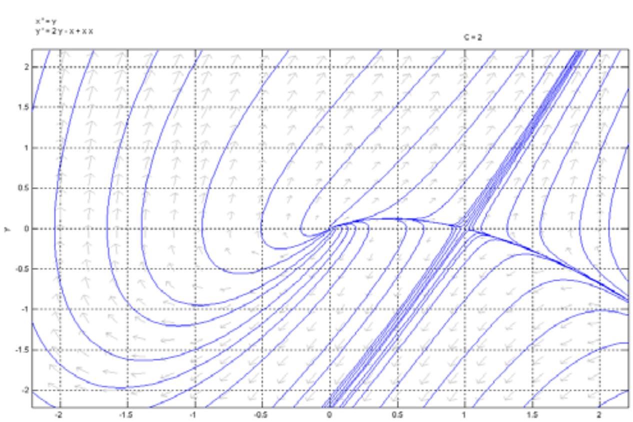
$$\det(J|_{(1,0)} - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & c-\lambda \end{vmatrix} = \lambda^2 - c\lambda - 1 = 0.$$

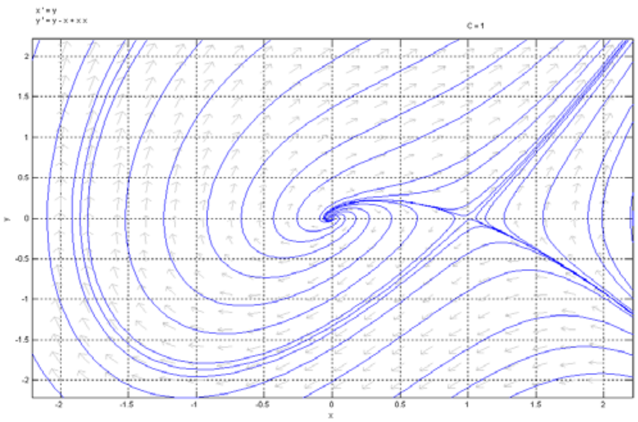
$$\lambda_\pm = \frac{c \pm \sqrt{c^2+4}}{2}.$$

If $c \geq 0 \quad \Rightarrow \quad \lambda_+ > 0, \quad \lambda_- < 0$.
(1,0) is **Unstable Saddle Point**.

By looking at the phase plot, a particle may start off at an unstable node $(0, 0)$ and as t increases, approach the unstable node $(1, 0)$.






x ' = y
y ' = y - x + x x
C = 1
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
y
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x

□

Problem (F'99, #6). *For the system*

$$\begin{aligned} \partial_t \rho + \partial_x u &= 0 \\ \partial_t u + \partial_x(\rho u) &= \partial_x^2 u \end{aligned}$$

*look for **traveling wave solutions** of the form* $\rho(x,t) = \rho(y = x - st)$, $u(x,t) = u(y = x - st)$. *In particular*
a) *Find a first order ODE for* u.
b) *Show that this equation has solutions of the form*

$$u(y) = u_0 + u_1 \tanh(\alpha y + y_0),$$

for some constants u_0, u_1, α, y_0.

Proof. **a)** We rewrite the system:

$$\begin{aligned} \rho_t + u_x &= 0 \\ u_t + \rho_x u + \rho u_x &= u_{xx} \end{aligned}$$

We look for traveling wave solutions

$$\rho(x,t) = \rho(x - st), \qquad u(x,t) = u(x - st), \qquad y = x - st.$$

We get the system of ODEs

$$\begin{cases} -s\rho' + u' = 0, \\ -su' + \rho' u + \rho u' = u''. \end{cases}$$

The first ODE gives

$$\begin{aligned} \rho' &= \frac{1}{s} u', \\ \rho &= \frac{1}{s} u + a, \end{aligned}$$

where a is a constant, and integration was done with respect to y. The second ODE gives

$$\begin{aligned} -su' + \frac{1}{s} u' u + \Big(\frac{1}{s} u + a\Big) u' &= u'', \\ -su' + \frac{2}{s} u u' + a u' &= u''. \qquad \text{Integrating, we get} \\ -su + \frac{1}{s} u^2 + au &= u' + b. \end{aligned}$$

$$\boxed{u' = \frac{1}{s} u^2 + (a - s)u - b.}$$

b) Note that the ODE above may be written in the following form:

$$u' + Au^2 + Bu = C,$$

which is a nonlinear first order equation. ☐

Problem (S'01, #7). *Consider the following system of PDEs:*

$$\begin{aligned} f_t + f_x &= g^2 - f^2 \\ g_t - g_x &= f^2 - g \end{aligned}$$

a) *Find a system of ODEs that describes* ***traveling wave solutions*** *of the PDE system; i.e. for solutions of the form $f(x,t) = f(x - st)$ and $g(x,t) = g(x - st)$.*
b) *Analyze the stationary points and draw the phase plane for this ODE system in the standing wave case $s = 0$.*

Proof. **a)** We look for traveling wave solutions

$$f(x,t) = f(x - st), \qquad g(x,t) = g(x - st).$$

We get the system of ODEs

$$\begin{aligned} -sf' + f' &= g^2 - f^2, \\ -sg' - g' &= f^2 - g. \end{aligned}$$

Thus,

$$\begin{aligned} f' &= \frac{g^2 - f^2}{1 - s}, \\ g' &= \frac{f^2 - g}{-1 - s}. \end{aligned}$$

b) If $s = 0$, the system becomes

$$\begin{cases} f' = g^2 - f^2, \\ g' = g - f^2. \end{cases}$$

Relabel the variables $f \to y_1, \quad g \to y_2$.

$$\begin{cases} y_1' = y_2^2 - y_1^2 = 0, \\ y_2' = y_2 - y_1^2 = 0. \end{cases}$$

$$\boxed{\text{Stationary points: } (0,0),\ (-1,1),\ (1,1).}$$

$$\begin{cases} y_1' = y_2^2 - y_1^2 = \phi(y_1, y_2), \\ y_2' = y_2 - y_1^2 = \psi(y_1, y_2). \end{cases}$$

In order to classify a stationary point, need to find eigenvalues of a linearized system at that point.

$$J(\phi(y_1,y_2), \psi(y_1,y_2)) = \begin{bmatrix} \frac{\partial \phi}{\partial y_1} & \frac{\partial \phi}{\partial y_2} \\ \frac{\partial \psi}{\partial y_1} & \frac{\partial \psi}{\partial y_2} \end{bmatrix} = \begin{bmatrix} -2y_1 & 2y_2 \\ -2y_1 & 1 \end{bmatrix}.$$

• For $(y_1, y_2) = (0,0)$:

$$\det(J|_{(0,0)} - \lambda I) = \begin{vmatrix} -\lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = -\lambda(1 - \lambda) = 0.$$

$$\lambda_1 = 0,\ \lambda_2 = 1; \qquad \text{eigenvectors: } v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(0,0) is **Unstable Node**.

• For $(y_1, y_2) = (-1, 1)$:
$\det(J|_{(-1,1)} - \lambda I) = \begin{vmatrix} 2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 2 = 0.$

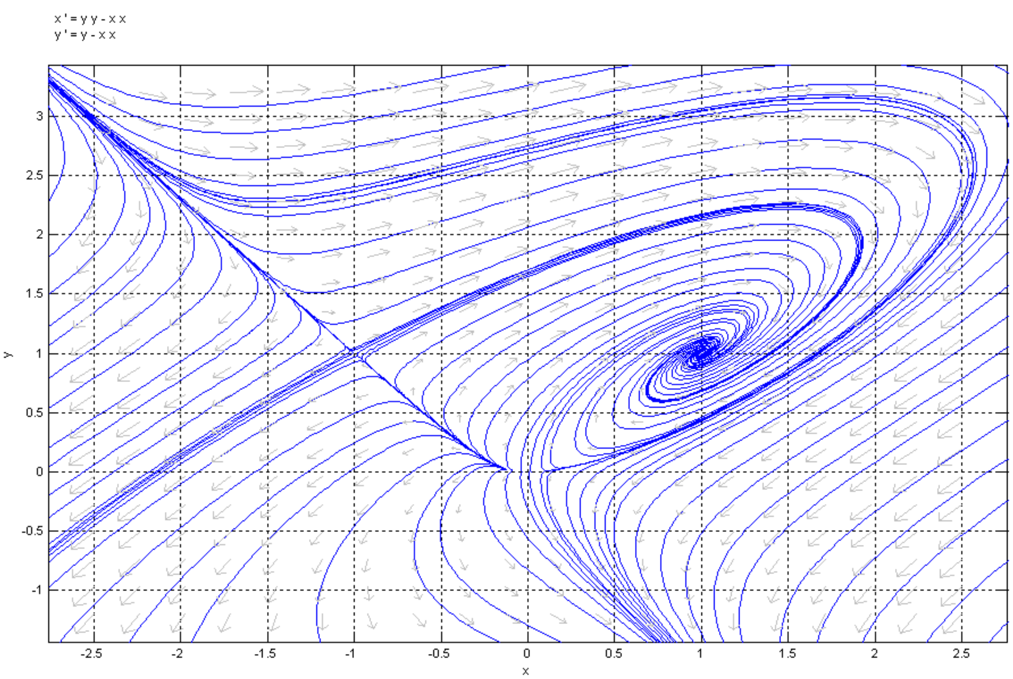
$$\lambda_\pm = \frac{3}{2} \pm \frac{\sqrt{17}}{2}.$$

$\lambda_- < 0, \quad \lambda_+ > 0.$
(-1,1) is **Unstable Saddle Point**.

• For $(y_1, y_2) = (1, 1)$:
$\det(J|_{(1,1)} - \lambda I) = \begin{vmatrix} -2-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = \lambda^2 + \lambda + 2 = 0.$

$$\lambda_\pm = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2}.$$

$\mathbb{R}e(\lambda_\pm) < 0.$
(1,1) is **Stable Spiral Point**.



□

16.5 Dispersion

Problem (S'97, #8). *Consider the following equation*

$$u_t = (f(u_x))_x - \alpha u_{xxxx}, \qquad f(v) = v^2 - v, \tag{16.40}$$

with constant α.
a) *Linearize this equation around* $u = 0$ *and find the principal mode solution of the form* $e^{\omega t + ikx}$. *For which values of* α *are there unstable modes, i.e., modes with* $\omega = 0$ *for real* k*? For these values, find the maximally unstable mode, i.e., the value of* k *with the largest positive value of* ω.
b) *Consider the steady solution of the (fully nonlinear) problem. Show that the resulting equation can be written as a second order autonomous ODE for* $v = u_x$ *and draw the corresponding phase plane.*

Proof. **a)** We have

$$\begin{aligned} u_t &= (f(u_x))_x - \alpha u_{xxxx}, \\ u_t &= (u_x^2 - u_x)_x - \alpha u_{xxxx}, \\ u_t &= 2u_x u_{xx} - u_{xx} - \alpha u_{xxxx}. \qquad \circledast \end{aligned}$$

However, we need to linearize (16.40) around $u = 0$. To do this, we need to linearize f.

$$f(u) = f(0) + uf'(0) + \frac{u^2}{2} f''(0) + \cdots = 0 + u(0-1) + \cdots = -u + \cdots.$$

Thus, we have

$$u_t = -u_{xx} - \alpha u_{xxxx}.$$

Consider $u(x,t) = e^{\omega t + ikx}$.

$$\begin{aligned} \omega e^{\omega t + ikx} &= (k^2 - \alpha k^4) e^{\omega t + ikx}, \\ \omega &= k^2 - \alpha k^4. \end{aligned}$$

To find unstable nodes, we set $\omega = 0$, to get

$$\boxed{\alpha = \frac{1}{k^2}.}$$

• To find the maximally unstable mode, i.e., the value of k with the largest positive value of ω, consider

$$\begin{aligned} \omega(k) &= k^2 - \alpha k^4, \\ \omega'(k) &= 2k - 4\alpha k^3. \end{aligned}$$

To find the extremas of ω, we set $\omega' = 0$. Thus,the extremas are at

$$k_1 = 0, \quad k_{2,3} = \pm\sqrt{\frac{1}{2\alpha}}.$$

To find if the extremas are maximums or minimums, we set $\omega'' = 0$:

$$\begin{aligned} \omega''(k) &= 2 - 12\alpha k^2 = 0, \\ \omega''(0) &= 2 > 0 \quad \Rightarrow \quad k = 0 \text{ is the minimum.} \\ \omega''\Big(\pm\sqrt{\frac{1}{2\alpha}}\Big) &= -4 < 0 \quad \Rightarrow \quad k = \pm\sqrt{\frac{1}{2\alpha}} \text{ is the } \textbf{maximum unstable mode}. \\ \omega\Big(\pm\sqrt{\frac{1}{2\alpha}}\Big) &= \frac{1}{4\alpha} \qquad \text{is the largest positive value of } \omega. \end{aligned}$$

b) Integrating $\circledast$, we get

$$u_x^2 - u_x - \alpha u_{xxx} = 0.$$

Let $v = u_x$. Then,

$$v^2 - v - \alpha v_{xx} = 0, \quad \text{or}$$
$$v'' = \frac{v^2 - v}{\alpha}.$$

In order to find and analyze the stationary points of an ODE above, we write it as a first-order system.

$$y_1 = v,$$
$$y_2 = v'.$$

$$y_1' = v' = y_2,$$
$$y_2' = v'' = \frac{v^2 - v}{\alpha} = \frac{y_1^2 - y_1}{\alpha}.$$

$$\begin{cases} y_1' = y_2 = 0, \\ y_2' = \frac{y_1^2 - y_1}{\alpha} = 0; \end{cases} \quad \Rightarrow \quad \begin{cases} y_2 = 0, \\ y_1(y_1 - 1) = 0. \end{cases}$$

$$\boxed{\text{Stationary points: } (0,0),\ (1,0).}$$

$$y_1' = y_2 \qquad = f(y_1, y_2),$$
$$y_2' = \frac{y_1^2 - y_1}{\alpha} = g(y_1, y_2).$$

In order to classify a stationary point, need to find eigenvalues of a linearized system at that point.

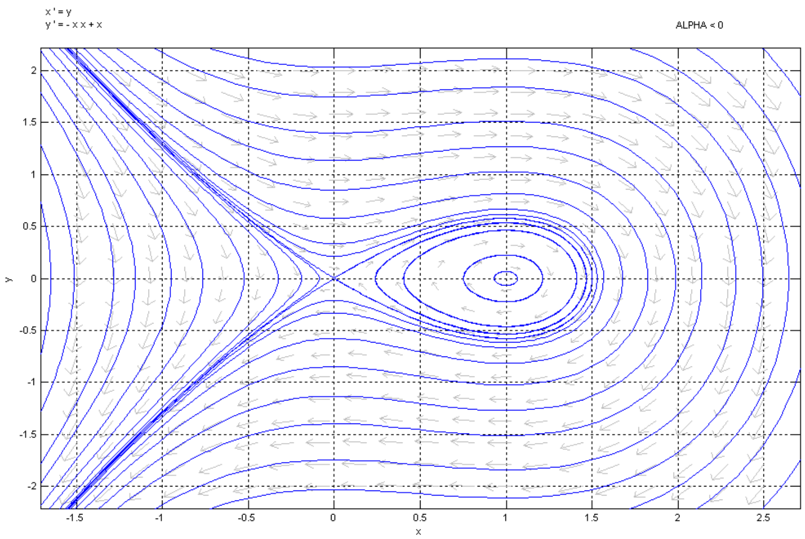
$$J(f(y_1, y_2), g(y_1, y_2)) = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} \\ \frac{\partial g}{\partial y_1} & \frac{\partial g}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{2y_1 - 1}{\alpha} & 0 \end{bmatrix}.$$

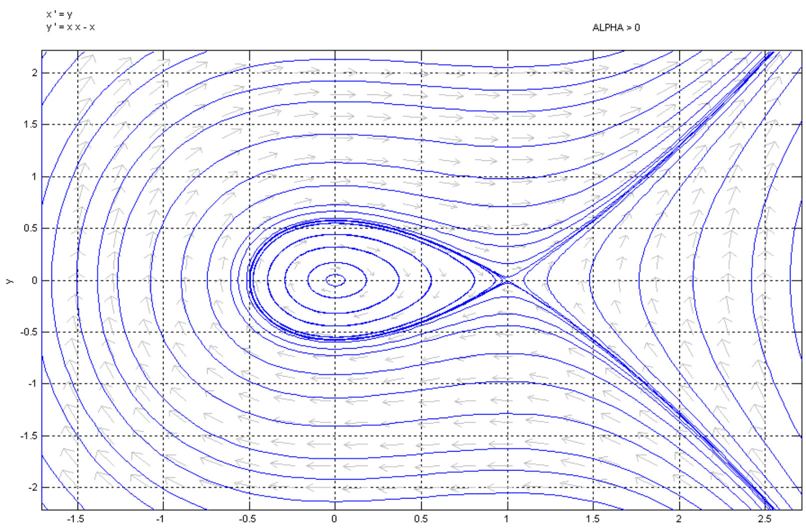
- For $(y_1, y_2) = (0, 0)$, $\quad \lambda_\pm = \pm\sqrt{-\frac{1}{\alpha}}$.

If $\alpha < 0$, $\lambda_\pm \in \mathbb{R}$, $\lambda_+ > 0$, $\lambda_- < 0$. $\Rightarrow$ **(0,0)** is **Unstable Saddle Point**.
If $\alpha > 0$, $\lambda_\pm = \pm i\sqrt{\frac{1}{\alpha}} \in \mathbb{C}$, $\mathbb{R}e(\lambda_\pm) = 0$. $\Rightarrow$ **(0,0)** is **Spiral Point**.

- For $(y_1, y_2) = (1, 0)$, $\quad \lambda_\pm = \pm\sqrt{\frac{1}{\alpha}}$.

If $\alpha < 0$, $\lambda_\pm = \pm i\sqrt{-\frac{1}{\alpha}} \in \mathbb{C}$, $\mathbb{R}e(\lambda_\pm) = 0$. $\Rightarrow$ **(1,0)** is **Spiral Point**.
If $\alpha > 0$, $\lambda_\pm \in \mathbb{R}$, $\lambda_+ > 0$, $\lambda_- < 0$. $\Rightarrow$ **(1,0)** is **Unstable Saddle Point**.


x ' = y
y ' = - x x + x
ALPHA < 0
x
y


x ' = y
y ' = x x - x
ALPHA > 0
x
y

□

16.6 Energy Methods

Problem (S'98, #9; S'96, #5). *Consider the following initial-boundary value problem for the multi-dimensional wave equation:*

$$\begin{aligned} u_{tt} &= \triangle u && \text{in } \Omega \times (0,\infty), \\ u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) &= g(x) && \text{for } x \in \Omega, \\ \frac{\partial u}{\partial n} + a(x)\frac{\partial u}{\partial t} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ and $a(x) \geq 0$. Define the Energy integral for this problem and use it in order to prove the uniqueness of the classical solution of the problem.

Proof.

$$\begin{aligned} \frac{d\tilde{E}}{dt} = 0 \;&=\; \int_\Omega (u_{tt} - \triangle u) u_t \, dx = \int_\Omega u_{tt} u_t \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} u_t \, ds + \int_\Omega \nabla u \cdot \nabla u_t \, dx \\ &=\; \int_\Omega \frac{1}{2}\frac{\partial}{\partial t}(u_t^2)\, dx + \int_\Omega \frac{1}{2}\frac{\partial}{\partial t}|\nabla u|^2 \, dx + \int_{\partial\Omega} a(x) u_t^2 \, ds. \end{aligned}$$

Thus,

$$\underbrace{-\int_{\partial\Omega} a(x) u_t^2 \, dx}_{\leq 0} = \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega u_t^2 + |\nabla u|^2 \, dx.$$

Let Energy integral be

$$\boxed{E(t) = \frac{1}{2}\int_\Omega u_t^2 + |\nabla u|^2 \, dx.}$$

In order to prove that the given $E(t) \leq 0$ from scratch, take its derivative with respect to t:

$$\begin{aligned} \frac{dE}{dt}(t) &= \int_\Omega \big(u_t u_{tt} + \nabla u \cdot \nabla u_t\big)\, dx \\ &= \int_\Omega u_t u_{tt} \, dx + \int_{\partial\Omega} u_t \frac{\partial u}{\partial n}\, ds - \int_\Omega u_t \triangle u \, dx \\ &= \underbrace{\int_\Omega u_t(u_{tt} - \triangle u)\, dx}_{=0} - \int_{\partial\Omega} a(x) u_t^2 \, dx \;\leq\; 0. \end{aligned}$$

Thus, $E(t) \leq E(0)$.

To prove the uniqueness of the classical solution, suppose u_1 and u_2 are two solutions of the initial boundary value problem. Let $w = u_1 - u_2$. Then, w satisfies

$$\begin{aligned} w_{tt} &= \triangle w && \text{in } \Omega \times (0,\infty), \\ w(x,0) = 0, \quad w_t(x,0) &= 0 && \text{for } x \in \Omega, \\ \frac{\partial w}{\partial n} + a(x)\frac{\partial w}{\partial t} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

We have

$$E_w(0) = \frac{1}{2}\int_\Omega (w_t(x,0)^2 + |\nabla w(x,0)|^2)\, dx = 0.$$

$E_w(t) \le E_w(0) = 0 \;\Rightarrow\; E_w(t) = 0$. Thus, $w_t = 0$, $w_{x_i} = 0 \;\Rightarrow\; w(x,t) = \text{const} = 0$. Hence, $u_1 = u_2$.

□

Problem (S'94, #7). *Consider the wave equation*

$$\frac{1}{c^2(x)} u_{tt} = \triangle u \qquad x \in \Omega$$
$$\frac{\partial u}{\partial t} - \alpha(x)\frac{\partial u}{\partial n} = 0 \qquad on \;\; \partial\Omega \times \mathbb{R}.$$

Assume that $\alpha(x)$ is of one sign for all x (i.e. α always positive or α always negative). For the energy

$$E(t) = \frac{1}{2}\int_\Omega \frac{1}{c^2(x)} u_t^2 + |\nabla u|^2 \, dx,$$

show that the sign of $\frac{dE}{dt}$ is determined by the sign of α.

Proof. We have

$$\begin{aligned}
\frac{dE}{dt}(t) &= \int_\Omega \Big(\frac{1}{c^2(x)} u_t u_{tt} + \nabla u \cdot \nabla u_t\Big)\, dx \\
&= \int_\Omega \frac{1}{c^2(x)} u_t u_{tt}\, dx + \int_{\partial\Omega} u_t \frac{\partial u}{\partial n}\, ds - \int_\Omega u_t \triangle u \, dx \\
&= \underbrace{\int_\Omega u_t \Big(\frac{1}{c^2(x)} u_{tt} - \triangle u\Big)\, dx}_{=0} + \int_{\partial\Omega} \frac{1}{\alpha(x)} u_t^2 \, dx \\
&= \int_{\partial\Omega} \frac{1}{\alpha(x)} u_t^2 \, dx \;=\; \begin{cases} > 0, & \text{if } \alpha(x) > 0, \;\forall x \in \Omega, \\ < 0, & \text{if } \alpha(x) < 0, \;\forall x \in \Omega. \end{cases}
\end{aligned}$$

□

Problem (F'92, #2). *Let $\Omega \in \mathbb{R}^n$. Let $u(x,t)$ be a smooth solution of the following initial boundary value problem:*

$$\begin{aligned} u_{tt} - \triangle u + u^3 &= 0 \qquad \textit{for} \ \ (x,t) \in \Omega \times [0,T] \\ u(x,t) &= 0 \qquad \textit{for} \ \ (x,t) \in \partial\Omega \times [0,T]. \end{aligned}$$

a) *Derive an* ***energy equality*** *for u. (Hint: Multiply by u_t and integrate over $\Omega \times [0,T]$.)*
b) *Show that if $u|_{t=0} = u_t|_{t=0} = 0$ for $x \in \Omega$, then $u \equiv 0$.*

Proof. **a)** Multiply by u_t and integrate:

$$\begin{aligned} 0 &= \int_\Omega (u_{tt} - \triangle u + u^3) u_t \, dx = \int_\Omega u_{tt} u_t \, dx - \underbrace{\int_{\partial\Omega} \frac{\partial u}{\partial n} u_t \, ds}_{=0} + \int_\Omega \nabla u \cdot \nabla u_t \, dx + \int_\Omega u^3 u_t \, dx \\ &= \int_\Omega \frac{1}{2}\frac{\partial}{\partial t}(u_t^2) \, dx + \int_\Omega \frac{1}{2}\frac{\partial}{\partial t}|\nabla u|^2 \, dx + \int_\Omega \frac{1}{4}\frac{\partial}{\partial t}(u^4) \, dx = \frac{1}{2}\frac{d}{dt}\int_\Omega \big(u_t^2 + |\nabla u|^2 + \frac{1}{2}u^4\big) \, dx. \end{aligned}$$

Thus, the Energy integral is

$$\boxed{E(t) = \int_\Omega \big(u_t^2 + |\nabla u|^2 + \frac{1}{2}u^4\big) \, dx \ = \text{const} \ = \ E(0).}$$

b) Since $u(x,0) = 0$, $u_t(x,0) = 0$, we have

$$E(0) = \int_\Omega \big(u_t(x,0)^2 + |\nabla u(x,0)|^2 + \frac{1}{2}u(x,0)^4\big) \, dx = 0.$$

Since $E(t) = E(0) = 0$, we have

$$E(t) = \int_\Omega \big(u_t(x,t)^2 + |\nabla u(x,t)|^2 + \frac{1}{2}u(x,t)^4\big) \, dx = 0.$$

Thus, $u \equiv 0$. □

Problem (F'04, #3). *Consider a damped wave equation*

$$\begin{cases} u_{tt} - \triangle u + a(x)u_t = 0, & (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ u|_{t=0} = u_0, \qquad u_t|_{t=0} = u_1. \end{cases}$$

Here the damping coefficient $a \in C_0^\infty(\mathbb{R}^3)$ *is a non-negative function and* $u_0,\ u_1 \in C_0^\infty(\mathbb{R}^3)$*. Show that the energy of the solution* $u(x,t)$ *at time* t*,*

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla_x u|^2 + |u_t|^2\right) dx$$

is a decreasing function of $t \geq 0$*.*

Proof. Take the derivative of $E(t)$ with respect to t. Note that the boundary integral is 0 by Huygen's principle.

$$\begin{aligned} \frac{dE}{dt}(t) &= \int_{\mathbb{R}^3} \left(\nabla u \cdot \nabla u_t + u_t u_{tt}\right) dx \\ &= \underbrace{\int_{\partial\mathbb{R}^3} u_t \frac{\partial u}{\partial n}\, ds}_{=0} - \int_{\mathbb{R}^3} u_t \triangle u \, dx + \int_{\mathbb{R}^3} u_t u_{tt} \, dx \\ &= \int_{\mathbb{R}^3} u_t(-\triangle u + u_{tt})\, dx = \int_{\mathbb{R}^3} u_t(-a(x)u_t)\, dx = \int_{\mathbb{R}^3} -a(x)u_t^2 \, dx \;\leq\; 0. \end{aligned}$$

Thus, $\frac{dE}{dt} \leq 0 \;\Rightarrow\; E(t) \leq E(0)$, i.e. $E(t)$ is a decreasing function of t. □

Problem (W'03, #8). ***a)*** *Consider the damped wave equation for high-speed waves* $(0 < \epsilon << 1)$ *in a bounded region* D

$$\epsilon^2 u_{tt} + u_t = \triangle u \qquad \circledast$$

with the boundary condition $u(x,t) = 0$ *on* ∂D*. Show that the energy functional*

$$E(t) = \int_D \epsilon^2 u_t^2 + |\nabla u|^2 \, dx$$

is nonincreasing on solutions of the boundary value problem.

b) *Consider the solution to the boundary value problem in part (a) with initial data* $u^\epsilon(x,0) = 0$, $u_t^\epsilon(x,0) = \epsilon^{-\alpha} f(x)$, *where* f *does not depend on* ϵ *and* $\alpha < 1$*. Use part (a) to show that*

$$\int_D |\nabla u^\epsilon(x,t)|^2 \, dx \to 0$$

uniformly on $0 \le t \le T$ *for any* T *as* $\epsilon \to 0$.

c) *Show that the result in part (b) does not hold for* $\alpha = 1$*. To do this consider the case where* f *is an eigenfunction of the Laplacian, i.e.* $\triangle f + \lambda f = 0$ *in* D *and* $f = 0$ *on* ∂D*, and solve for* u^ϵ *explicitly.*

Proof. **a)**

$$\begin{aligned}
\frac{dE}{dt} &= \int_D 2\epsilon^2 u_t u_{tt} \, dx + \int_D 2\nabla u \cdot \nabla u_t \, dx \\
&= \int_D 2\epsilon^2 u_t u_{tt} \, dx + \underbrace{\int_{\partial D} 2\frac{\partial u}{\partial n} u_t \, ds}_{=0,\ (u=0 \text{ on } \partial D)} - \int_D 2\triangle u u_t \, dx \\
&= 2\int_D (\epsilon^2 u_{tt} - \triangle u) u_t \, dx \ = \ \circledast \ = \ -2\int_D |u_t|^2 \, dx \ \le \ 0.
\end{aligned}$$

Thus, $E(t) \le E(0)$, i.e. $E(t)$ is nonincreasing.

b) From (a), we know $\frac{dE}{dt} \le 0$. We also have

$$\begin{aligned}
E_\epsilon(0) &= \int_D \epsilon^2 (u_t^\epsilon(x,0))^2 + |\nabla u^\epsilon(x,0)|^2 \, dx \\
&= \int_D \epsilon^2 (\epsilon^{-\alpha} f(x))^2 + 0 \, dx \ = \ \int_D \epsilon^{2(1-\alpha)} f(x)^2 \, dx \to 0 \quad \text{as } \epsilon \to 0.
\end{aligned}$$

Since $E_\epsilon(0) \ge E_\epsilon(t) = \int_D \epsilon^2 (u_t^\epsilon)^2 + |\nabla u^\epsilon|^2 \, dx$, then $E_\epsilon(t) \to 0$ as $\epsilon \to 0$.
Thus, $\int_D |\nabla u^\epsilon|^2 \, dx \to 0$ as $\epsilon \to 0$.

c) If $\alpha = 1$,

$$E_\epsilon(0) = \int_D \epsilon^{2(1-\alpha)} f(x)^2 \, dx \ = \ \int_D f(x)^2 \, dx.$$

Since f is independent of ϵ, $E_\epsilon(0)$ does not approach 0 as $\epsilon \to 0$. We can not conclude that $\int_D |\nabla u^\epsilon(x,t)|^2 \, dx \to 0$. □

Problem (F'98, #6). *Let f solve the nonlinear wave equation*

$$f_{tt} - f_{xx} = -f(1 + f^2)^{-1}$$

for $x \in [0, 1]$, with $f(x = 0, t) = f(x = 1, t) = 0$ and with smooth initial data $f(x, t) = f_0(x)$.
a) *Find an energy integral $E(t)$ which is constant in time.*
b) *Show that $|f(x, t)| < c$ for all x and t, in which c is a constant.*
Hint: Note that

$$\frac{f}{1 + f^2} = \frac{1}{2}\frac{d}{df}\log(1 + f^2).$$

Proof. **a)** Since $f(0, t) = f(1, t) = 0$, $\forall t$, we have $f_t(0, t) = f_t(1, t) = 0$. Let

$$\begin{aligned}
\frac{dE}{dt} &= 0 = \int_0^1 \big(f_{tt} - f_{xx} + f(1 + f^2)^{-1}\big) f_t\, dx \\
&= \int_0^1 f_{tt} f_t\, dx - \int_0^1 f_{xx} f_t\, dx + \int_0^1 \frac{f f_t}{1 + f^2}\, dx \\
&= \int_0^1 f_{tt} f_t\, dx - [f_x \underbrace{f_t}_{=0}]_0^1 + \int_0^1 f_x f_{tx}\, dx + \int_0^1 \frac{f f_t}{1 + f^2}\, dx \\
&= \int_0^1 \frac{1}{2}\frac{\partial}{\partial t}(f_t^2)\, dx + \int_0^1 \frac{1}{2}\frac{\partial}{\partial t}(f_x^2)\, dx + \int_0^1 \frac{1}{2}\frac{\partial}{\partial t}(ln(1 + f^2))\, dx \\
&= \frac{1}{2}\frac{d}{dt}\int_0^1 \big(f_t^2 + f_x^2 + ln(1 + f^2)\big)\, dx.
\end{aligned}$$

Thus,

$$E(t) = \frac{1}{2}\int_0^1 \big(f_t^2 + f_x^2 + ln(1 + f^2)\big)\, dx.$$

b) We want to show that f is bounded. For smooth $f(x, 0) = f_0(x)$, we have

$$E(0) = \frac{1}{2}\int_0^1 \big(f_t(x, 0)^2 + f_x(x, 0)^2 + ln(1 + f(x, 0)^2)\big)\, dx < \infty.$$

Since $E(t)$ is constant in time, $E(t) = E(0) < \infty$. Thus,

$$\frac{1}{2}\int_0^1 ln(1 + f^2)\, dx \;\le\; \frac{1}{2}\int_0^1 \big(f_t^2 + f_x^2 + ln(1 + f^2)\big)\, dx = E(t) < \infty.$$

Hence, f is bounded. $\square$

Problem (F'97, #1). *Consider initial-boundary value problem*

$$u_{tt} + a^2(x,t)u_t - \triangle u(x,t) = 0 \qquad x \in \Omega \subset \mathbb{R}^n, \ 0 < t < +\infty$$
$$u(x) = 0 \qquad x \in \partial\Omega$$
$$u(x,0) = f(x), \qquad u_t(x,0) = g(x) \qquad x \in \Omega.$$

Prove that L_2*-norm of the solution is bounded in* t *on* $(0,+\infty)$.
Here Ω *is a bounded domain, and* $a(x,t)$, $f(x)$, $g(x)$ *are smooth functions.*

Proof. Multiply the equation by u_t and integrate over Ω:

$$u_t u_{tt} + a^2 u_t^2 - u_t \triangle u = 0,$$
$$\int_\Omega u_t u_{tt}\,dx + \int_\Omega a^2 u_t^2\,dx - \int_\Omega u_t \triangle u\,dx = 0,$$
$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2\,dx + \int_\Omega a^2 u_t^2\,dx - \underbrace{\int_{\partial\Omega} u_t \frac{\partial u}{\partial n}\,ds}_{=0,\ (u=0,\ x\in\partial\Omega)} + \int_\Omega \nabla u \cdot \nabla u_t\,dx = 0,$$
$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2\,dx + \int_\Omega a^2 u_t^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx = 0,$$
$$\frac{1}{2}\frac{d}{dt}\int_\Omega \big(u_t^2 + |\nabla u|^2\big)\,dx \ = \ -\int_\Omega a^2 u_t^2\,dx \ \le \ 0.$$

Let Energy integral be

$$\boxed{E(t) = \int_\Omega \big(u_t^2 + |\nabla u|^2\big)\,dx.}$$

We have $\frac{dE}{dt} \le 0$, i.e. $E(t) \le E(0)$.

$$E(t) \le E(0) = \int_\Omega \big(u_t(x,0)^2 + |\nabla u(x,0)|^2\big)\,dx = \int_\Omega \big(g(x)^2 + |\nabla f(x)|^2\big)\,dx < \infty,$$

since f and g are smooth functions. Thus,

$$E(t) = \int_\Omega \big(u_t^2 + |\nabla u|^2\big)\,dx \ < \ \infty,$$
$$\int_\Omega |\nabla u|^2\,dx \ < \ \infty,$$
$$\int_\Omega u^2\,dx \le C\int_\Omega |\nabla u|^2\,dx \ < \ \infty, \qquad \text{by } \textbf{Poincare inequality}.$$

Thus, $||u||_2$ is bounded $\forall t$. □

Problem (S'98, #4). ***a)*** *Let $u(x, y, z, t)$, $\quad -\infty < x, y, z < \infty$ be a solution of the equation*

$$\begin{cases} u_{tt} + u_t = u_{xx} + u_{yy} + u_{zz} \\ u(x, y, z, 0) = f(x, y, z), \\ u_t(x, y, z, 0) = g(x, y, z). \end{cases} \tag{16.41}$$

Here f, g are smooth functions which vanish if $\sqrt{x^2 + y^2 + z^2}$ is large enough. Prove that it is the ***unique*** *solution for $t \geq 0$.*
b) *Suppose we want to solve the same equation (16.41) in the region $\; z \geq 0, \; -\infty < x, y < \infty$, with the additional conditions*

$$\begin{aligned} u(x, y, 0, t) &= f(x, y, t) \\ u_z(x, y, 0, t) &= g(x, y, t) \end{aligned}$$

with the same f, g as before in (16.41). What goes wrong?

Proof. **a)** Suppose u_1 and u_2 are two solutions. Let $w = u_1 - u_2$. Then,

$$\begin{cases} w_{tt} + w_t = \triangle w, \\ w(x, y, z, 0) = 0, \\ w_t(x, y, z, 0) = 0. \end{cases}$$

Multiply the equation by w_t and integrate:

$$\begin{aligned} w_t w_{tt} + w_t^2 &= w_t \triangle w, \\ \int_{\mathbb{R}^3} w_t w_{tt}\, dx + \int_{\mathbb{R}^3} w_t^2\, dx &= \int_{\mathbb{R}^3} w_t \triangle w\, dx, \\ \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} w_t^2\, dx + \int_{\mathbb{R}^3} w_t^2\, dx &= \underbrace{\int_{\partial\mathbb{R}^3} w_t \frac{\partial w}{\partial n}\, dx}_{=0} - \int_{\mathbb{R}^3} \nabla w \cdot \nabla w_t\, dx, \\ \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} w_t^2\, dx + \int_{\mathbb{R}^3} w_t^2\, dx &= -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\nabla w|^2\, dx, \\ \frac{d}{dt}\underbrace{\int_{\mathbb{R}^3} \left(w_t^2 + |\nabla w|^2\right) dx}_{E(t)} &= -2\int_{\mathbb{R}^3} w_t^2\, dx \;\leq\; 0, \\ \frac{dE}{dt} &\leq 0, \\ E(t) \;\leq\; E(0) &= \int_{\mathbb{R}^3} \left(w_t(x,0)^2 + |\nabla w(x,0)|^2\right) dx = 0, \\ \Rightarrow \;\; E(t) &= \int_{\mathbb{R}^3} \left(w_t^2 + |\nabla w|^2\right) dx = 0. \end{aligned}$$

Thus, $w_t = 0$, $\nabla w = 0$, and $w =$ constant. Since $w(x, y, z, 0) = 0$, we have $w \equiv 0$.

b) $\square$

Problem (F'94, #8). *The one-dimensional, isothermal fluid equations with viscosity and capillarity in Lagrangian variables are*

$$v_t - u_x = 0$$
$$u_t + p(v)_x = \varepsilon u_{xx} - \delta v_{xxx}$$

in which $v(=1/\rho)$ *is specific volume,* u *is velocity, and* $p(v)$ *is pressure. The coefficients* ε *and* δ *are non-negative.*
Find an ***energy integral*** *which is non-increasing (as* t *increases) if* $\varepsilon > 0$ *and conserved if* $\varepsilon = 0$.
Hint: if $\delta = 0$, $E = \int u^2/2 - P(v)\,dx$ *where* $P'(v) = p(v)$.

Proof. Multiply the second equation by u and integrate over $\mathbb{R}$. We use $u_x = v_t$. Note that the boundary integrals are 0 due to finite speed of propagation.

$$uu_t + up(v)_x = \varepsilon uu_{xx} - \delta uv_{xxx},$$
$$\int_{\mathbb{R}} uu_t\,dx + \int_{\mathbb{R}} up(v)_x\,dx = \varepsilon\int_{\mathbb{R}} uu_{xx}\,dx - \delta\int_{\mathbb{R}} uv_{xxx}\,dx,$$
$$\frac{1}{2}\int_{\mathbb{R}}\frac{\partial}{\partial t}(u^2)\,dx + \underbrace{\int_{\partial\mathbb{R}} up(v)\,ds}_{=0} + \int_{\mathbb{R}} u_x p(v)\,dx$$
$$= \varepsilon\underbrace{\int_{\partial\mathbb{R}} uu_x\,dx}_{=0} - \varepsilon\int_{\mathbb{R}} u_x^2\,dx - \delta\underbrace{\int_{\partial\mathbb{R}} uv_{xx}\,dx}_{=0} + \delta\int_{\mathbb{R}} u_x v_{xx}\,dx,$$
$$\frac{1}{2}\int_{\mathbb{R}}\frac{\partial}{\partial t}(u^2)\,dx + \int_{\mathbb{R}} v_t p(v)\,dx = -\varepsilon\int_{\mathbb{R}} u_x^2\,dx + \delta\int_{\mathbb{R}} v_t v_{xx}\,dx,$$
$$\frac{1}{2}\int_{\mathbb{R}}\frac{\partial}{\partial t}(u^2)\,dx + \int_{\mathbb{R}}\frac{\partial}{\partial t}P(v)\,dx = -\varepsilon\int_{\mathbb{R}} u_x^2\,dx + \delta\underbrace{\int_{\partial\mathbb{R}} v_t v_x\,dx}_{=0} - \delta\int_{\mathbb{R}} v_{xt}v_x\,dx,$$
$$\frac{1}{2}\int_{\mathbb{R}}\frac{\partial}{\partial t}(u^2)\,dx + \int_{\mathbb{R}}\frac{\partial}{\partial t}P(v)\,dx + \frac{\delta}{2}\int_{\mathbb{R}}\frac{\partial}{\partial t}(v_x^2)\,dx = -\varepsilon\int_{\mathbb{R}} u_x^2\,dx,$$
$$\frac{d}{dt}\int_{\mathbb{R}}\Big(\frac{u^2}{2} + P(v) + \frac{\delta}{2}v_x^2\Big)\,dx \;=\; -\varepsilon\int_{\mathbb{R}} u_x^2\,dx \;\le\; 0.$$

$$\boxed{E(t) = \int_{\mathbb{R}}\Big(\frac{u^2}{2} + P(v) + \frac{\delta}{2}v_x^2\Big)\,dx}$$

is nonincreasing if $\varepsilon > 0$, and conserved if $\varepsilon = 0$. □

Problem (S'99, #5). *Consider the equation*

$$u_{tt} = \frac{\partial}{\partial x}\sigma(u_x) \tag{16.42}$$

with $\sigma(z)$ a smooth function. This is to be solved for $t > 0$, $0 \le x \le 1$, with periodic boundary conditions and initial data $u(x,0) = u_0(x)$ and $u_t(x,0) = v_0(x)$.
a) *Multiply (16.42) by u_t and get an expression of the form*

$$\frac{d}{dt}\int_0^1 F(u_t, u_x) = 0$$

that is satisfied for an appropriate function $F(y,z)$ with $y = u_t$, $z = u_x$, where u is any smooth, periodic in space solution of (16.42).
b) *Under what conditions on $\sigma(z)$ is this function, $F(y,z)$,* ***convex*** *in its variables?*
c) *What à priori inequality is satisfied for smooth solutions when F is convex?*
d) *Discuss the special case $\sigma(z) = a^2 z^3/3$, with $a > 0$ and constant.*

Proof. **a)** Multiply by u_t and integrate:

$$\begin{aligned} u_t u_{tt} &= u_t \sigma(u_x)_x, \\ \int_0^1 u_t u_{tt}\, dx &= \int_0^1 u_t \sigma(u_x)_x\, dx, \\ \frac{d}{dt}\int_0^1 \frac{u_t^2}{2}\, dx &= \underbrace{u_t\sigma(u_x)\big|_0^1}_{=0,\ (2\pi\text{-periodic})} - \int_0^1 u_{tx}\sigma(u_x)\, dx = \circledast \end{aligned}$$

Let $Q'(z) = \sigma(z)$, then $\frac{d}{dt}Q(u_x) = \sigma(u_x)u_{xt}$. Thus,

$$\circledast = -\int_0^1 u_{tx}\sigma(u_x)\, dx = -\frac{d}{dt}\int_0^1 Q(u_x)\, dx.$$

$$\boxed{\frac{d}{dt}\int_0^1 \Big(\frac{u_t^2}{2} + Q(u_x)\Big)\, dx = 0.}$$

b) We have

$$F(u_t, u_x) = \frac{u_t^2}{2} + Q(u_x).$$

[41] For F to be convex, the Hessian matrix of partial derivatives must be positive definite.

[41] A function f is **convex** on a convex set S if it satisfies

$$f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha)f(y)$$

for all $0 \le \alpha \le 1$ and for all $x, y \in S$.
If a one-dimensional function f has two continuous derivatives, then f is convex if and only if

$$f''(x) \ge 0.$$

In the multi-dimensional case the Hessian matrix of second derivatives must be positive semi-definite, that is, at every point $x \in S$

$$y^T \nabla^2 f(x)\, y \ge 0, \qquad \text{for all } y.$$

The **Hessian matrix** is the matrix with entries

$$[\nabla^2 f(x)]_{ij} \equiv \frac{\partial^2 f(x)}{\partial x_i \partial x_j}.$$

For functions with continuous second derivatives, it will always be symmetric matrix: $f_{x_i x_j} = f_{x_j x_i}$.

The Hessian matrix is

$$\nabla^2 F(u_t, u_x) = \begin{pmatrix} F_{u_t u_t} & F_{u_t u_x} \\ F_{u_x u_t} & F_{u_x u_x} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sigma'(u_x) \end{pmatrix}.$$

$$y^T \nabla^2 F(x)\, y = \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sigma'(u_x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = y_1^2 + \sigma'(u_x) y_2^2 \underbrace{\geq}_{\text{need}} 0.$$

Thus, for a Hessian matrix to be positive definite, need $\sigma'(u_x) \geq 0$, so that the above inequality holds for all y.

c) We have

$$\begin{aligned}
\frac{d}{dt}\int_0^1 F(u_t, u_x)\, dx &= 0, \\
\int_0^1 F(u_t, u_x)\, dx &= \text{const}, \\
\int_0^1 F(u_t, u_x)\, dx &= \int_0^1 F(u_t(x,0), u_x(x,0))\, dx, \\
\int_0^1 \Big(\frac{u_t^2}{2} + Q(u_x)\Big)\, dx &= \int_0^1 \Big(\frac{v_0^2}{2} + Q(u_{0x})\Big)\, dx.
\end{aligned}$$

d) If $\sigma(z) = a^2 z^3/3$, we have

$$\begin{aligned}
F(u_t, u_x) = \frac{u_t^2}{2} + Q(u_x) &= \frac{u_t^2}{2} + \frac{a^2 u_x^4}{12}, \\
\frac{d}{dt}\int_0^1 \Big(\frac{u_t^2}{2} + \frac{a^2 u_x^4}{12}\Big)\, dx &= 0, \\
\int_0^1 \Big(\frac{u_t^2}{2} + \frac{a^2 u_x^4}{12}\Big)\, dx &= \text{const}, \\
\int_0^1 \Big(\frac{u_t^2}{2} + \frac{a^2 u_x^4}{12}\Big)\, dx &= \int_0^1 \Big(\frac{{v_0}^2}{2} + \frac{a^2 {u_{0x}}^4}{12}\Big)\, dx.
\end{aligned}$$

□

Problem (S'96, #8). [42] *Let $u(x,t)$ be the solution of the Korteweg-de Vries equation*

$$u_t + uu_x = u_{xxx}, \qquad 0 \le x \le 2\pi,$$

with 2π-periodic boundary conditions and prescribed initial data

$$u(x, t=0) = f(x).$$

a) *Prove that the energy integral*

$$I_1(u) = \int_0^{2\pi} u^2(x,t)\,dx$$

is independent of the time t.
b) *Prove that the second "energy integral",*

$$I_2(u) = \int_0^{2\pi} \Big(\frac{1}{2}u_x^2(x,t) + \frac{1}{6}u^3(x,t)\Big)\,dx$$

is also independent of the time t.
c) *Assume the initial data are such that $I_1(f) + I_2(f) < \infty$. Use (a) + (b) to prove that the **maximum norm** of the solution, $|u|_\infty = \sup_x |u(x,t)|$, is bounded in time. Hint: Use the following inequalities (here, $|u|_p$ is the L^p-norm of $u(x,t)$ at **fixed time** t):*

- $|u|_\infty^2 \le \frac{\pi}{6}(|u|_2^2 + |u_x|_2^2)$ *(one of Sobolev's inequalities);*
- $|u|_3^3 \le |u|_2^2\,|u|_\infty$ *(straightforward).*

Proof. **a)** Multiply the equation by u and integrate. Note that all boundary terms are 0 due to 2π-periodicity.

$$\begin{aligned}
uu_t + u^2u_x &= uu_{xxx}, \\
\int_0^{2\pi} uu_t\,dx + \int_0^{2\pi} u^2u_x\,dx &= \int_0^{2\pi} uu_{xxx}\,dx, \\
\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u^2\,dx + \frac{1}{3}\int_0^{2\pi} (u^3)_x\,dx &= uu_{xx}\Big|_0^{2\pi} - \int_0^{2\pi} u_xu_{xx}\,dx, \\
\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u^2\,dx + \frac{1}{3}u^3\Big|_0^{2\pi} &= -\frac{1}{2}\int_0^{2\pi} (u_x^2)_x\,dx, \\
\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u^2\,dx &= -\frac{1}{2}u_x^2\Big|_0^{2\pi} = 0. \\
I_1(u) = \int_0^{2\pi} u^2\,dx &= C.
\end{aligned}$$

Thus, $I_1(u) = \int_0^{2\pi} u^2(x,t)\,dx$ is independent of the time t.
Alternatively, we may differentiate $I_1(u)$:

$$\begin{aligned}
\frac{dI_1}{dt}(u) &= \frac{d}{dt}\int_0^{2\pi} u^2\,dx = \int_0^{2\pi} 2uu_t\,dx = \int_0^{2\pi} 2u(-uu_x + u_{xxx})\,dx \\
&= \int_0^{2\pi} -2u^2u_x\,dx + \int_0^{2\pi} 2uu_{xxx}\,dx = \int_0^{2\pi} -\frac{2}{3}(u^3)_x\,dx + 2uu_{xx}\Big|_0^{2\pi} - \int_0^{2\pi} 2u_xu_{xx}\,dx \\
&= -\frac{2}{3}u^3\Big|_0^{2\pi} - \int_0^{2\pi} (u_x^2)_x\,dx = -u_x^2\Big|_0^{2\pi} = 0.
\end{aligned}$$

[42] Also, see S'92, #7.

b) Note that all boundary terms are 0 due to 2π-periodicity.

$$\frac{dI_2}{dt}(u) \;=\; \frac{d}{dt}\int_0^{2\pi}\Big(\frac{1}{2}u_x^2+\frac{1}{6}u^3\Big)\,dx = \int_0^{2\pi}\Big(u_x u_{xt}+\frac{1}{2}u^2 u_t\Big)\,dx \;=\; \circledast$$

We differentiate the original equation with respect to x:

$$u_t = -uu_x + u_{xxx}$$
$$u_{tx} = -(uu_x)_x + u_{xxxx}.$$

$$\begin{aligned}
\circledast &= \int_0^{2\pi} u_x(-(uu_x)_x + u_{xxxx})\,dx + \frac{1}{2}\int_0^{2\pi} u^2(-uu_x+u_{xxx})\,dx\\
&= \int_0^{2\pi} -u_x(uu_x)_x\,dx + \int_0^{2\pi} u_x u_{xxxx}\,dx - \frac{1}{2}\int_0^{2\pi} u^3 u_x\,dx + \frac{1}{2}\int_0^{2\pi} u^2 u_{xxx}\,dx\\
&= -u_x uu_x\Big|_0^{2\pi} + \int_0^{2\pi} u_{xx}uu_x\,dx + u_x u_{xxx}\Big|_0^{2\pi} - \int_0^{2\pi} u_{xx}u_{xxx}\,dx\\
&\quad -\frac{1}{2}\int_0^{2\pi}\Big(\frac{u^4}{4}\Big)_x\,dx + \frac{1}{2}u^2 u_{xx}\Big|_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} 2uu_x u_{xx}\,dx\\
&= \int_0^{2\pi} u_{xx}uu_x\,dx - \int_0^{2\pi} u_{xx}u_{xxx}\,dx - \frac{1}{2}\frac{u^4}{4}\Big|_0^{2\pi} - \int_0^{2\pi} uu_x u_{xx}\,dx\\
&= -\int_0^{2\pi} u_{xx}u_{xxx}\,dx = -u_{xx}^2\Big|_0^{2\pi} + \int_0^{2\pi} u_{xxx}u_{xx}\,dx = \int_0^{2\pi} u_{xxx}u_{xx}\,dx \;=\; 0,
\end{aligned}$$

since $-\int_0^{2\pi} u_{xx}u_{xxx}\,dx = +\int_0^{2\pi} u_{xx}u_{xxx}\,dx$. Thus,

$$I_2(u) = \int_0^{2\pi}\Big(\frac{1}{2}u_x^2(x,t)+\frac{1}{6}u^3(x,t)\Big)\,dx = C,$$

and $I_2(u)$ is independent of the time t.

c) From (a) and (b), we have

$$\begin{aligned}
I_1(u) &= \int_0^{2\pi} u^2\,dx \;=\; ||u||_2^2,\\
I_2(u) &= \int_0^{2\pi}\Big(\frac{1}{2}u_x^2+\frac{1}{6}u^3\Big)\,dx \;=\; \frac{1}{2}||u_x||_2^2+\frac{1}{6}||u||_3^3.
\end{aligned}$$

Using given inequalities, we have

$$\begin{aligned}
||u||_\infty^2 &\le \frac{\pi}{6}(||u||_2^2+||u_x||_2^2) \;\le\; \frac{\pi}{6}\Big(I_1(u)+2I_2(u)-\frac{1}{3}||u||_3^3\Big)\\
&\le \frac{\pi}{6}I_1(u)+\frac{\pi}{3}I_2(u)+\frac{\pi}{18}||u||_2^2\,||u||_\infty \;\le\; \frac{\pi}{6}I_1(u)+\frac{\pi}{3}I_2(u)+\frac{\pi}{18}I_1(u)||u||_\infty\\
&= C + C_1||u||_\infty.\\
&\Rightarrow\; ||u||_\infty^2 \le C + C_1||u||_\infty,\\
&\Rightarrow\; ||u||_\infty \le C_2.
\end{aligned}$$

Thus, $||u||_\infty$ is bounded in time. □

Also see Energy Methods problems for higher order equations (3rd and 4th) in the section on Gas Dynamics.

16.7 Wave Equation in 2D and 3D

Problem (F'97, #8); (McOwen 3.2 #90). *Solve*

$$u_{tt} = u_{xx} + u_{yy} + u_{zz}$$

with initial conditions

$$u(x,y,z,0) = \underbrace{x^2 + y^2}_{g(x)}, \quad u_t(x,y,z,0) = \underbrace{0}_{h(x)}.$$

Proof.
① We may use the **Kirchhoff's formula:**

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t\int_{|\xi|=1} g(x+ct\xi)\,dS_\xi\right) + \frac{t}{4\pi}\int_{|\xi|=1} h(x+ct\xi)\,dS_\xi \\ &= \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t\int_{|\xi|=1} \big((x_1+ct\xi_1)^2 + (x_2+ct\xi_2)^2\big)\,dS_\xi\right) + 0 = \end{aligned}$$

② We may solve the problem by Hadamard's **method of descent**, since initial conditions are independent of x_3. We need to convert surface integrals in $\mathbb{R}^3$ to domain integrals in $\mathbb{R}^2$. Specifically, we need to express the surface measure on the upper half of the unit sphere S^2_+ in terms of the two variables ξ_1 and ξ_2. To do this, consider

$$\begin{aligned} f(\xi_1,\xi_2) &= \sqrt{1-\xi_1^2-\xi_2^2} \qquad \text{over the unit disk } \ \xi_1^2+\xi_2^2 < 1. \\ dS_\xi &= \sqrt{1+(f_{\xi_1})^2+(f_{\xi_2})^2}\,d\xi_1 d\xi_2 = \frac{d\xi_1 d\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}. \end{aligned}$$

$$\begin{aligned}
u(x_1,x_2,t) &= \frac{1}{4\pi}\frac{\partial}{\partial t}\bigg(2t\int_{\xi_1^2+\xi_2^2<1}\frac{g(x_1+ct\xi_1,x_2+ct\xi_2)\,d\xi_1 d\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}\bigg)\\
&+ \frac{t}{4\pi}\bigg(2\int_{\xi_1^2+\xi_2^2<1}\frac{h(x_1+ct\xi_1,x_2+ct\xi_2)\,d\xi_1 d\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}\bigg)\\
&= \frac{1}{4\pi}\frac{\partial}{\partial t}\bigg(2t\int_{\xi_1^2+\xi_2^2<1}\frac{(x_1+t\xi_1)^2+(x_2+t\xi_2)^2}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2\bigg) \;+\; 0,\\
&= \frac{1}{2\pi}\frac{\partial}{\partial t}\bigg(t\int_{\xi_1^2+\xi_2^2<1}\frac{x_1^2+2x_1t\xi_1+t^2\xi_1^2+x_2^2+2x_2t\xi_2+t^2\xi_2^2}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2\bigg)\\
&= \frac{1}{2\pi}\frac{\partial}{\partial t}\bigg(\int_{\xi_1^2+\xi_2^2<1}\frac{tx_1^2+2x_1t^2\xi_1+t^3\xi_1^2+tx_2^2+2x_2t^2\xi_2+t^3\xi_2^2}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2\bigg)\\
&= \frac{1}{2\pi}\bigg(\int_{\xi_1^2+\xi_2^2<1}\frac{x_1^2+4x_1t\xi_1+3t^2\xi_1^2+x_2^2+4x_2t\xi_2+3t^2\xi_2^2}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2\bigg)\\
&= \frac{1}{2\pi}\bigg(\int_{\xi_1^2+\xi_2^2<1}\frac{(x_1^2+x_2^2)+4t(x_1\xi_1+x_2\xi_2)+3t^2(\xi_1^2+\xi_2^2)}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2\bigg)\\
&= \underbrace{\frac{1}{2\pi}(x_1^2+x_2^2)\int_{\xi_1^2+\xi_2^2<1}\frac{d\xi_1 d\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}}_{❶} + \underbrace{\frac{4t}{2\pi}\int_{\xi_1^2+\xi_2^2<1}\frac{x_1\xi_1+x_2\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2}_{❷}\\
&\quad + \underbrace{\frac{3t^2}{2\pi}\int_{\xi_1^2+\xi_2^2<1}\frac{\xi_1^2+\xi_2^2}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2}_{❸} \;=\; \circledast
\end{aligned}$$

$$\begin{aligned}
❶ &= \frac{1}{2\pi}(x_1^2+x_2^2)\int_{\xi_1^2+\xi_2^2<1}\frac{d\xi_1 d\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}} = \frac{1}{2\pi}(x_1^2+x_2^2)\int_0^{2\pi}\int_0^1\frac{r\,dr\,d\theta}{\sqrt{1-r^2}}\\
&= \frac{1}{2\pi}(x_1^2+x_2^2)\int_0^{2\pi}-2\int_0^1\frac{-\frac{1}{2}\,du\,d\theta}{u^{\frac{1}{2}}} \qquad \big(u=1-r^2,\quad du=-2r\,dr\big)\\
&= \frac{1}{2\pi}(x_1^2+x_2^2)\int_0^{2\pi}1\,d\theta \;=\; x_1^2+x_2^2.\\
❷ &= \frac{4t}{2\pi}\int_{\xi_1^2+\xi_2^2<1}\frac{x_1\xi_1+x_2\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2 \;=\; \frac{4t}{2\pi}\int_{-1}^{1}\int_{-\sqrt{1-\xi_2^2}}^{\sqrt{1-\xi_2^2}}\frac{x_1\xi_1+x_2\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2\\
&= 0.\\
❸ &= \frac{3t^2}{2\pi}\int_{\xi_1^2+\xi_2^2<1}\frac{\xi_1^2+\xi_2^2}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2 \;=\; \frac{3t^2}{2\pi}\int_0^{2\pi}\int_0^1\frac{(r\cos\theta)^2+(r\sin\theta)^2}{\sqrt{1-r^2}}\,r\,drd\theta\\
&= \frac{3t^2}{2\pi}\int_0^{2\pi}\int_0^1\frac{r^3}{\sqrt{1-r^2}}\,drd\theta \qquad \big(u=1-r^2,\quad du=-2r\,dr\big)\\
&= \frac{3t^2}{2\pi}\int_0^{2\pi}\frac{2}{3}\,d\theta \;=\; \frac{t^2}{\pi}\int_0^{2\pi}d\theta \;=\; 2t^2.
\end{aligned}$$

$$\circledast \;\Rightarrow\; u(x_1,x_2,t) \;=\; ❶+❷+❸ \;=\; x_1^2+x_2^2+2t^2.$$

③ We may guess what the solution is:

$$u(x,y,z,t) \;=\; \frac{1}{2}\big[(x+t)^2+(y+t)^2+(x-t)^2+(y-t)^2\big] \;=\; x^2+y^2+2t^2.$$

Check:

$$
\begin{aligned}
u(x,y,z,0) &= x^2+y^2. \quad \checkmark \\
u_t(x,y,z,t) &= (x+t)+(y+t)-(x-t)-(y-t), \\
u_t(x,y,z,0) &= 0. \quad \checkmark \\
u_{tt}(x,y,z,t) &= 4, \\
u_x(x,y,z,t) &= (x+t)+(x-t), \\
u_{xx}(x,y,z,t) &= 2, \\
u_y(x,y,z,t) &= (y+t)+(y-t), \\
u_{yy}(x,y,z,t) &= 2, \\
u_{zz}(x,y,z,t) &= 0, \\
u_{tt} &= u_{xx}+u_{yy}+u_{zz}. \quad \checkmark
\end{aligned}
$$

□

Problem (S'98, #6).
Consider the two-dimensional wave equation $w_{tt} = a^2 \triangle w$, *with initial data which vanish for* x^2+y^2 *large enough. Prove that* $w(x,y,t)$ *satisfies the decay* $|w(x,y,t)| \le C \cdot t^{-1}$. *(Note that the estimate is not uniform with respect to* x, y *since* C *may depend on* x, y*).*

Proof. Suppose we have the following problem with initial data:

$$u_{tt} = a^2 \triangle u \qquad x \in \mathbb{R}^2, \ t > 0,$$

$$u(x,0) = g(x), \quad u_t(x,0) = h(x) \qquad x \in \mathbb{R}^2.$$

The result is the consequence of the Huygens' principle and may be proved by Hadamard's **method of descent**: [43]

$$\begin{aligned}
u(x,t) &= \frac{1}{4\pi}\frac{\partial}{\partial t}\bigg(2t\int_{\xi_1^2+\xi_2^2<1} \frac{g(x_1+ct\xi_1, x_2+ct\xi_2)\, d\xi_1 d\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}\bigg) \\
&+ \frac{t}{4\pi}\bigg(2\int_{\xi_1^2+\xi_2^2<1} \frac{h(x_1+ct\xi_1, x_2+ct\xi_2)\, d\xi_1 d\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}\bigg) \\
&= \frac{1}{2\pi}\int_{|\xi|^2<c^2t^2} \frac{th(x+\xi)+g(x+\xi)}{\sqrt{1-\frac{|\xi|^2}{c^2t^2}}}\,\frac{d\xi_1 d\xi_2}{c^2t^2} \\
&+ \frac{t}{2\pi}\int_{|\xi|^2<c^2t^2} \frac{\nabla g(x+\xi)\cdot(ct,ct)}{\sqrt{1-\frac{|\xi|^2}{c^2t^2}}}\,\frac{d\xi_1 d\xi_2}{c^2t^2}.
\end{aligned}$$

For a given x, let $T(x)$ be so large that $T > 1$ and $\mathrm{supp}(h+g) \subset B_T(x)$. Then for $t > 2T$ we have:

$$\begin{aligned}
|u(x,t)| &= \frac{1}{2\pi}\int_{|\xi|^2<c^2T^2} \frac{tM+M+2Mct}{\sqrt{1-\frac{c^2T^2}{c^2T^24}}}\,\frac{d\xi_1 d\xi_2}{c^2t^2} \\
&= \frac{\pi c^2T^2}{2\pi}\Big[\Big(\frac{M}{\sqrt{3/4}}\Big)\frac{1}{c^2t} + \Big(\frac{M}{\sqrt{3/4}}\Big)\frac{1}{c^2Tt} + \frac{2Mc}{c^2t}\Big].
\end{aligned}$$

$\Rightarrow \quad u(x,t) \le C_1/t \quad$ for $t > 2T$.
For $t \le 2T$:

$$\begin{aligned}
|u(x,t)| &= \frac{1}{2\pi}\int_{|\xi|^2<c^2t^2} \frac{2TM+M+4McT}{\sqrt{1-\frac{|\xi|^2}{c^2t^2}}}\,\frac{d\xi_1 d\xi_2}{c^2t^2} \\
&= \frac{1}{2\pi}(2TM+M+4Mct)2\pi\int_0^{ct} \frac{r\,dr/c^2t^2}{\sqrt{1-\frac{r^2}{c^2t^2}}} \\
&= \frac{M(2T+1+4cT)}{2}\int_0^1 \frac{-du}{u^{1/2}} = \frac{M(2T+1+4cT)}{2}2 \le \frac{M(2T+1+4cT)2T}{t}.
\end{aligned}$$

Letting $C = \max(C_1, M(2T+1+4cT)2T)$, we have $|u(x,t)| \le C(x)/t$.

• For $n = 3$, suppose $g,\ h \in C_0^\infty(\mathbb{R}^3)$. The solution is given by the Kircchoff's formula. There is a constant C so that $u(x,t) \le C/t$ for all $x \in \mathbb{R}^3$ and $t > 0$. As McOwen suggensts in Hints for Exercises, to prove the result, we need to estimate the

[43]Nick's solution follows.

area of intersection of the sphere of radius ct with the support of the functions g and h. □

Problem (S'95, #6). *Spherical waves in 3-d are waves symmetric about the origin; i.e. $u = u(r,t)$ where r is the distance from the origin. The wave equation*

$$u_{tt} = c^2 \triangle u$$

then reduces to

$$\frac{1}{c^2} u_{tt} = u_{rr} + \frac{2}{r} u_r. \tag{16.43}$$

a) *Find the general solutions $u(r,t)$ by solving (16.43). Include both the incoming waves and outgoing waves in your solutions.*
b) *Consider only the outgoing waves and assume the finite out-flux condition*

$$0 < \lim_{r\to 0} r^2 u_r < \infty$$

for all t. The wavefront is defined as $r = ct$. How is the amplitude of the wavefront decaying in time?

Proof. **a)** We want to reduce (16.43) to the 1D wave equation. Let $v = ru$. Then

$$\begin{aligned} v_{tt} &= r u_{tt}, \\ v_r &= r u_r + u, \\ v_{rr} &= r u_{rr} + 2u_r. \end{aligned}$$

Thus, (16.43) becomes

$$\begin{aligned} \frac{1}{c^2}\frac{1}{r} v_{tt} &= \frac{1}{r} v_{rr}, \\ \frac{1}{c^2} v_{tt} &= v_{rr}, \\ v_{tt} &= c^2 v_{rr}, \end{aligned}$$

which has the solution

$$v(r,t) = f(r+ct) + g(r-ct).$$

Thus,

$$u(r,t) = \frac{1}{r} v(r,t) = \underbrace{\frac{1}{r} f(r+ct)}_{\text{incoming, } (c>0)} + \underbrace{\frac{1}{r} g(r-ct)}_{\text{outgoing, } (c>0)}.$$

b) We consider $u(r,t) = \frac{1}{r} g(r-ct)$:

$$\begin{aligned} 0 &< \lim_{r\to 0} r^2 u_r < \infty, \\ 0 &< \lim_{r\to 0} r^2 \Big(\frac{1}{r} g'(r-ct) - \frac{1}{r^2} g(r-ct)\Big) < \infty, \\ 0 &< \lim_{r\to 0} \big(r g'(r-ct) - g(r-ct)\big) < \infty, \\ 0 &< -g(-ct) < \infty, \\ 0 &< -g(-ct) = G(t) < \infty, \\ & g(t) = -G\Big(\frac{t}{-c}\Big). \end{aligned}$$

The wavefront is defined as $r = ct$. We have

$$\begin{aligned} u(r,t) &= \frac{1}{r}g(r-ct) = -\frac{1}{r}G\Big(\frac{r-ct}{-c}\Big) = -\frac{1}{ct}G(0). \\ |u(r,t)| &= \frac{1}{t}\Big|-\frac{1}{c}G(0)\Big|. \end{aligned}$$

The amplitude of the wavefront decays like $\frac{1}{t}$. □

Problem (S'00, #8). ***a)*** *Show that for a smooth function F on the line, while $u(x,t) = F(ct + |x|)/|x|$ may look like a solution of the wave equation $u_{tt} = c^2 \triangle u$ in $\mathbb{R}^3$, it actually is not. Do this by showing that for any smooth function $\phi(x,t)$ with compact support*

$$\int_{\mathbb{R}^3\times\mathbb{R}} u(x,t)(\phi_{tt} - \triangle\phi)\,dxdt = 4\pi \int_{\mathbb{R}} \phi(0,t)F(ct)\,dt.$$

Note that, setting $r = |x|$, for any function w which only depends on r one has $\triangle w = r^{-2}(r^2 w_r)_r = w_{rr} + \frac{2}{r}w_r$.
b) *If $F(0) = F'(0) = 0$, what is the true solution to $u_{tt} = \triangle u$ with initial conditions $u(x,0) = F(|x|)/|x|$ and $u_t(x,0) = F'(|x|)/|x|$?*
c) (Ralston Hw) *Suppose $u(x,t)$ is a solution to the wave equation $u_{tt} = c^2\triangle u$ in $\mathbb{R}^3 \times \mathbb{R}$ with $u(x,t) = w(|x|,t)$ and $u(x,0) = 0$. Show that*

$$u(x,t) \;=\; \frac{F(|x| + ct) - F(|x| - ct)}{|x|}$$

for a function F of one variable.

Proof. **a)** We have

$$\begin{aligned}
\int_{\mathbb{R}^3}\int_{\mathbb{R}} u\,(\phi_{tt} - \triangle\phi)\,dxdt &= \lim_{\epsilon\to 0}\int_{\mathbb{R}} dt \int_{|x|>\epsilon} u\,(\phi_{tt} - \triangle\phi)\,dx \\
&= \lim_{\epsilon\to 0}\int_{\mathbb{R}} dt \Big[\int_{|x|>\epsilon} \phi\,(u_{tt} - \triangle u)\,dx + \int_{|x|=\epsilon} \frac{\partial u}{\partial n}\,\phi - u\frac{\partial \phi}{\partial n}\,dS\Big].
\end{aligned}$$

The final equality is derived by integrating by parts twice in t, and using Green's theorem:

$$\int_{\Omega}(v\triangle u - u\triangle v)\,dx \;=\; \int_{\partial\Omega}(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})ds.$$

Since $dS = \epsilon^2 \sin\phi' d\phi' d\theta$ and $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$, substituting $u(x,t) = F(|x| + ct)/|x|$ gives:

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}} u\,(\phi_{tt} - \triangle\phi)\,dxdt \;=\; \int_{\mathbb{R}} 4\pi\phi F(ct)\,dt.$$

Thus, u is not a weak solution to the wave equation.

b)
c) We want to show that $v(|x|,t) = |x|w(|x|,t)$ is a solution to the wave equation in one space dimension and hence must have the from $v = F(|x| + ct) + G(|x| - ct)$. Then we can argue that w will be undefined at $x = 0$ for some t unless $F(ct) + G(-ct) = 0$ for all t.
We work in spherical coordinates. Note that w and v are independent of ϕ and θ. We have:

$$\begin{aligned}
v_{tt}(r,t) &= c^2\triangle w = c^2\frac{1}{r^2}(r^2 w_r)_r = c^2\frac{1}{r^2}(2rw_r + r^2 w_{rr}), \\
&\Rightarrow\; rw_{tt} = c^2 rw_{rr} + 2w_r.
\end{aligned}$$

Thus we see that $v_{tt} = c^2 v_{rr}$, and we can conclude that

$$\begin{aligned}
v(r,t) &= F(r + ct) + G(r - ct) \qquad \text{and} \\
w(r,t) &= \frac{F(r + ct) + G(r - ct)}{r}.
\end{aligned}$$

$\lim_{r\to 0} w(r,t)$ does not exist unless $F(ct) + G(-ct) = 0$ for all t. Hence

$$\begin{aligned} w(r,t) &= \frac{F(ct+r) + G(ct-r)}{r}, \qquad \text{and} \\ u(x,t) &= \frac{F(ct+|x|) + G(ct-|x|)}{|x|}. \end{aligned}$$

$\square$

17 Problems: Laplace Equation

A fundamental solution $K(x)$ for the Laplace operator is a distribution satisfying [44]

$$\triangle K(x) = \delta(x)$$

The fundamental solution for the Laplace operator is

$$K(x) = \begin{cases} \frac{1}{2\pi}\log|x| & \text{if } n = 2 \\ \frac{1}{(2-n)\omega_n}|x|^{2-n} & \text{if } n \geq 3. \end{cases}$$

17.1 Green's Function and the Poisson Kernel

Green's function is a special fundamental solution satisfying [45]

$$\begin{cases} \triangle G(x,\xi) = \delta(x) & \text{for } \ x \in \Omega \\ G(x,\xi) = 0 & \text{for } \ x \in \partial\Omega, \end{cases} \tag{17.1}$$

To construct the Green's function,
① consider $w_\xi(x)$ with $\triangle w_\xi(x) = 0$ in Ω and $w_\xi(x) = -K(x-\xi)$ on $\partial\Omega$;
② consider $G(x,\xi) = K(x-\xi) + w_\xi(x)$, which is a fundamental solution satisfying (17.1).

Problem 1. *Given a particular distribution solution to the set of Dirichlet problems*

$$\begin{cases} \triangle u_\xi(x) = \delta_\xi(x) & \textit{for } \ x \in \Omega \\ u_\xi(x) = 0 & \textit{for } \ x \in \partial\Omega, \end{cases}$$

how would you use this to solve

$$\begin{cases} \triangle u = 0 & \textit{for } \ x \in \Omega \\ u(x) = g(x) & \textit{for } \ x \in \partial\Omega. \end{cases}$$

Proof. $u_\xi(x) = G(x,\xi)$, a Green's function. G is a fundamental solution to the Laplace operator, $G(x,\xi) = 0$, $x \in \partial\Omega$. In this problem, it is assumed that $G(x,\xi)$ is known for Ω. Then

$$u(\xi) = \int_\Omega G(x,\xi)\triangle u\, dx + \int_{\partial\Omega} u(x)\frac{\partial G(x,\xi)}{\partial n_x} dS_x$$

for every $u \in C^2(\overline{\Omega})$. In particular, if $\triangle u = 0$ in Ω and $u = g$ on $\partial\Omega$, then we obtain the *Poisson integral formula*

$$\boxed{u(\xi) = \int_{\partial\Omega} \frac{\partial G(x,\xi)}{\partial n_x} g(x)\, dS_x,}$$

[44] We know that $u(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\,dy$ is a distribution solution of $\triangle u = f$ when f is integrable and has compact support. In particular, we have

$$u(x) = \int_{\mathbb{R}^n} K(x-y)\triangle u(y)\,dy \qquad \text{whenever } \ u \in C_0^\infty(\mathbb{R}^n).$$

The above result is a consequence of:

$$u(x) \ = \ \int_\Omega \delta(x-y)u(y)\,dy \ = \ (\triangle K) * u \ = \ K * (\triangle u) \ = \ \int_\Omega K(x-y)\,\triangle u(y)\,dy.$$

[45] Green's function is useful in satisfying Dirichlet boundary conditions.

where $H(x, \xi) = \frac{\partial G(x,\xi)}{\partial n_x}$ is the *Poisson kernel.*
Thus *if* we know that the Dirichlet problem has a solution $u \in C^2(\overline{\Omega})$, then we can calculate u from the Poisson integral formula (provided of course that we can compute $G(x, \xi)$). $\square$

Dirichlet Problem on a Half-Space. *Solve the n-dimensional Laplace/Poisson equation on the half-space with Dirichlet boundary conditions.*

Proof. Use the **method of reflection** to construct Green's function. Let Ω be an upper half-space in $\mathbb{R}^n$. If $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$, we can see

$$|x' - \xi| = |x' - \xi^*|, \quad \text{and hence} \quad K(x' - \xi) = K(x' - \xi^*). \quad \text{Thus}$$

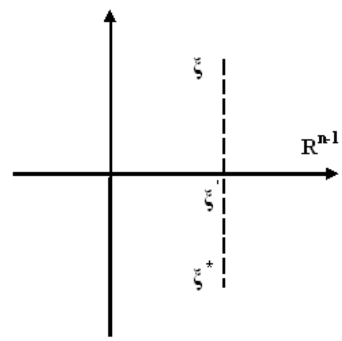
$$\boxed{G(x,\xi) = K(x-\xi) - K(x-\xi^*)}$$

is the Green's function on Ω. $G(x,\xi)$ is harmonic in Ω, and $G(x,\xi) = 0$ on $\partial\Omega$.
To compute the Poisson kernel, we must differentiate $G(x,\xi)$ in the *negative* x_n direction. For $n \geq 2$,

$$\frac{\partial}{\partial x_n} K(x-\xi) \;=\; \frac{x_n - \xi_n}{\omega_n}|x-\xi|^{-n},$$

so that the Poisson kernel is given by

$$-\frac{\partial}{\partial x_n} G(x,\xi)\Big|_{x_n=0} \;=\; \frac{2\xi_n}{\omega_n}|x'-\xi|^{-n}, \quad \text{for} \;\; x' \in \mathbb{R}^{n-1}.$$



Thus, the solution is

$$\boxed{u(\xi) \;=\; \int_{\partial\Omega} \frac{\partial G(x,\xi)}{\partial n_x} g(x)\, dS_x \;=\; \frac{2\xi_n}{\omega_n}\int_{\mathbb{R}^{n-1}} \frac{g(x')}{|x'-\xi|^n}\, dx'.}$$

If $g(x')$ is bounded and continuous for $x' \in \mathbb{R}^{n-1}$, then $u(\xi)$ is C^∞ and harmonic in $\mathbb{R}^n_+$ and extends continuously to $\overline{\mathbb{R}^n_+}$ such that $u(\xi') = g(\xi')$.

□

Problem (F'95, #3): Neumann Problem on a Half-Space.
a) *Consider the Neumann problem in the upper half plane,*
$\Omega = \{x = (x_1, x_2) : -\infty < x_1 < \infty,\ x_2 > 0\}$*:*

$$\begin{aligned} &\triangle u = u_{x_1x_1} + u_{x_2x_2} = 0 && x \in \Omega, \\ &u_{x_2}(x_1, 0) = f(x_1) && -\infty < x_1 < \infty. \end{aligned}$$

Find the corresponding ***Green's function*** *and conclude that*

$$u(\xi) = u(\xi_1, \xi_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ln\,[(x_1 - \xi_1)^2 + \xi_2^2] \cdot f(x_1)\, dx_1$$

is a solution of the problem.
b) *Show that this solution is bounded in* Ω *if and only if* $\int_{-\infty}^{\infty} f(x_1)\, dx_1 = 0$*.*

Proof. **a)** Notation: $x = (x, y)$, $\xi = (x_0, y_0)$. Since $K(x-\xi) = \frac{1}{2\pi}\log|x-\xi|$, $n = 2$.
① First, we find the Green's function. We have

$$K(x-\xi) = \frac{1}{2\pi}\log\sqrt{(x-x_0)^2 + (y-y_0)^2}.$$

Let $G(x, \xi) = K(x - \xi) + \omega(x)$.
Since the problem is Neumann, we need:

$$\begin{cases} \triangle G(x,\xi) \;=\; \delta(x-\xi), \\ \frac{\partial G}{\partial y}((x,0),\xi) \;=\; 0. \end{cases}$$

$$\begin{aligned} G((x,y),\xi) &= \frac{1}{2\pi}\log\sqrt{(x-x_0)^2 + (y-y_0)^2} \;+\; \omega((x,y),\xi), \\ \frac{\partial G}{\partial y}((x,y),\xi) &= \frac{1}{2\pi}\frac{y-y_0}{(x-x_0)^2 + (y-y_0)^2} + \omega_y((x,y),\xi), \\ \frac{\partial G}{\partial y}((x,0),\xi) &= -\frac{1}{2\pi}\frac{y_0}{(x-x_0)^2 + y_0^2} + \omega_y((x,0),\xi) \;=\; 0. \end{aligned}$$

Let

$$\begin{aligned} \omega((x,y),\xi) &= \frac{a}{2\pi}\log\sqrt{(x-x_0)^2 + (y+y_0)^2}. \qquad \text{Then,} \\ \frac{\partial G}{\partial y}((x,0),\xi) &= -\frac{1}{2\pi}\frac{y_0}{(x-x_0)^2 + y_0^2} + \frac{a}{2\pi}\frac{y_0}{(x-x_0)^2 + y_0^2} \;=\; 0. \end{aligned}$$

Thus, $a = 1$.

$$\boxed{G((x,y),\xi) \;=\; \frac{1}{2\pi}\log\sqrt{(x-x_0)^2 + (y-y_0)^2} \;+\; \frac{1}{2\pi}\log\sqrt{(x-x_0)^2 + (y+y_0)^2}.}$$

46

② Consider Green's identity (after cutting out $B_\epsilon(\xi)$ and having $\epsilon \to 0$):

$$\int_\Omega (u\triangle G - G\underbrace{\triangle u}_{=0})\, dx \;=\; \int_{\partial\Omega}\Big(u\underbrace{\frac{\partial G}{\partial n}}_{=0} - G\frac{\partial u}{\partial n}\Big)\, dS$$

[46] Note that for the Dirichlet problem, we would have gotten the "-" sign instead of "+" in front of ω.

Since $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial(-y)} = -f(x)$, we have

$$\begin{aligned}
\int_\Omega u\,\delta(x-\xi)\,dx &= \int_{-\infty}^{\infty} G((x,y),\xi)\,f(x)\,dx, \\
u(\xi) &= \int_{-\infty}^{\infty} G((x,y),\xi)\,f(x)\,dx.
\end{aligned}$$

For $y = 0$, we have

$$\begin{aligned}
G((x,y),\xi) &= \frac{1}{2\pi}\log\sqrt{(x-x_0)^2+y_0^2} + \frac{1}{2\pi}\log\sqrt{(x-x_0)^2+y_0^2} \\
&= \frac{1}{2\pi}\,2\log\sqrt{(x-x_0)^2+y_0^2} \\
&= \frac{1}{2\pi}\log\big[(x-x_0)^2+y_0^2\big].
\end{aligned}$$

Thus,

$$u(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\log\big[(x-x_0)^2+y_0^2\big]\,f(x)\,dx. \quad \checkmark$$

b) *Show that this solution is bounded in* Ω *if and only if* $\int_{-\infty}^{\infty} f(x_1)\,dx_1 = 0$.

Consider the Green's identity:

$$\int_\Omega \triangle u\,dxdy = \int_{\partial\Omega}\frac{\partial u}{\partial n}\,dS = -\int_{-\infty}^{\infty}\frac{\partial u}{\partial y}\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 0.$$

Note that the Green's identity applies to bounded domains Ω.

$$\int_{-R}^{R} f\,dx_1 + \int_0^{2\pi}\frac{\partial u}{\partial r}\,R\,d\theta = 0.$$

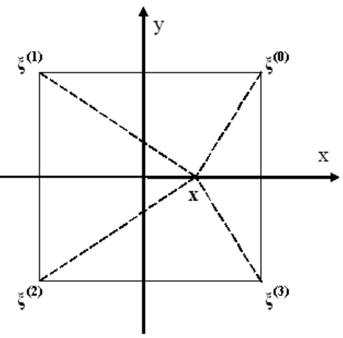
??? □

McOwen 4.2 # 6. *For $n = 2$, use the method of reflections to find the Green's function for the* ***first quadrant*** $\Omega = \{(x, y) : x, y > 0\}$.

Proof. For $x \in \partial\Omega$,

$$|x - \xi^{(0)}| \cdot |x - \xi^{(2)}| = |x - \xi^{(1)}| \cdot |x - \xi^{(3)}|,$$

$$|x - \xi^{(0)}| = \frac{|x - \xi^{(1)}| \cdot |x - \xi^{(3)}|}{|x - \xi^{(2)}|}.$$



But $\xi^{(0)} = \xi$, so for $n = 2$,

$$\boxed{G(x, \xi) = \frac{1}{2\pi}\log|x - \xi| - \frac{1}{2\pi}\log\frac{|x - \xi^{(1)}| \cdot |x - \xi^{(3)}|}{|x - \xi^{(2)}|}.}$$

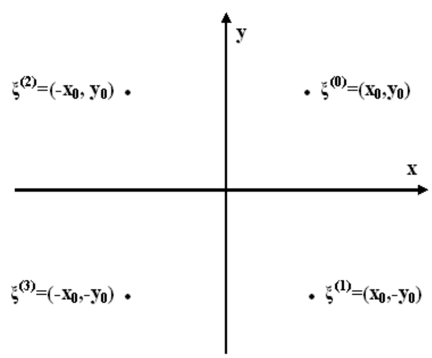
$G(x, \xi) = 0$, $x \in \partial\Omega$.

□

Problem. *Use the method of images to solve*

$$\triangle G = \delta(x - \xi)$$

in the first quadrant with $G = 0$ on the boundary.



Proof. To solve the problem in the first quadrant we take a reflection to the fourth quadrant and the two are reflected to the left half.

$$\begin{aligned}
\triangle G &= \delta(x - \xi^{(0)}) - \delta(x - \xi^{(1)}) - \delta(x - \xi^{(2)}) + \delta(x - \xi^{(3)}). \\
G &= \frac{1}{2\pi}\log\frac{|x - \xi^{(0)}|\,|x - \xi^{(3)}|}{|x - \xi^{(1)}|\,|x - \xi^{(2)}|} \\
&= \frac{1}{2\pi}\log\frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}\,\sqrt{(x + x_0)^2 + (y + y_0)^2}}{\sqrt{(x - x_0)^2 + (y + y_0)^2}\,\sqrt{(x + x_0)^2 + (y - y_0)^2}}.
\end{aligned}$$

Note that on the axes $G = 0$. □

Problem (S'96, #3). *Construct a* ***Green's function*** *for the following* ***mixed*** *Dirichlet-Neumann problem in* $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$*:*

$$\begin{aligned} \triangle u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} &= f, \qquad x \in \Omega, \\ u_{x_2}(x_1, 0) &= 0, \qquad x_1 > 0, \\ u(0, x_2) &= 0, \qquad x_2 > 0. \end{aligned}$$

Proof. Notation: $x = (x, y)$, $\xi = (x_0, y_0)$. Since $K(x - \xi) = \frac{1}{2\pi} \log|x - \xi|$, $n = 2$.

$$K(x - \xi) = \frac{1}{2\pi} \log\sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

Let $G(x, \xi) = K(x - \xi) + \omega(x)$.
At $(0, y)$, $y > 0$,

$$G\big((0, y), \xi\big) = \frac{1}{2\pi} \log\sqrt{x_0^2 + (y - y_0)^2} + \omega(0, y) = 0.$$

Also,

$$\begin{aligned} G_y\big((x, y), \xi\big) &= \frac{1}{2\pi} \frac{\frac{1}{2} \cdot 2(y - y_0)}{(x - x_0)^2 + (y - y_0)^2} + w_y(x, y) \\ &= \frac{1}{2\pi} \frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2} + w_y(x, y). \end{aligned}$$

At $(x, 0)$, $x > 0$,

$$G_y\big((x, 0), \xi\big) = -\frac{1}{2\pi} \frac{y_0}{(x - x_0)^2 + y_0^2} + w_y(x, 0) = 0.$$

We have

$$\begin{aligned} \omega((x, y), \xi) &= \frac{a}{2\pi} \log\sqrt{(x + x_0)^2 + (y - y_0)^2} \\ &+ \frac{b}{2\pi} \log\sqrt{(x - x_0)^2 + (y + y_0)^2} \\ &+ \frac{c}{2\pi} \log\sqrt{(x + x_0)^2 + (y + y_0)^2}. \end{aligned}$$

Using boundary conditions, we have

$$\begin{aligned} 0 &= G((0, y), \xi) = \frac{1}{2\pi} \log\sqrt{x_0^2 + (y - y_0)^2} + \omega(0, y) \\ &= \frac{1}{2\pi} \log\sqrt{x_0^2 + (y - y_0)^2} + \frac{a}{2\pi} \log\sqrt{x_0^2 + (y - y_0)^2} + \frac{b}{2\pi} \log\sqrt{x_0^2 + (y + y_0)^2} + \frac{c}{2\pi} \log\sqrt{x_0^2 + (y + y_0)^2}. \end{aligned}$$

Thus, $a = -1$, $c = -b$. Also,

$$\begin{aligned} 0 &= G_y((x, 0), \xi) = -\frac{1}{2\pi} \frac{y_0}{(x - x_0)^2 + y_0^2} + w_y(x, 0) \\ &= -\frac{1}{2\pi} \frac{y_0}{(x - x_0)^2 + y_0^2} - \frac{(-1)}{2\pi} \frac{y_0}{(x + x_0)^2 + y_0^2} + \frac{b}{2\pi} \frac{y_0}{(x - x_0)^2 + y_0^2} + \frac{(-b)}{2\pi} \frac{y_0}{(x + x_0)^2 + y_0^2}. \end{aligned}$$

Thus, $b = 1$, and

$$\boxed{G((x, y), \xi) = \frac{1}{2\pi} \log\sqrt{(x - x_0)^2 + (y - y_0)^2} + \omega(x) = \frac{1}{2\pi} \Big[\log\sqrt{(x - x_0)^2 + (y - y_0)^2}}$$

$$\boxed{-\log\sqrt{(x+x_0)^2+(y-y_0)^2}+\log\sqrt{(x-x_0)^2+(y+y_0)^2}-\log\sqrt{(x+x_0)^2+(y+y_0)^2}\Big].}$$

It can be seen that $G((x,y),\xi)=0$ on $x=0$, for example. $\square$

Dirichlet Problem on a Ball. *Solve the n-dimensional Laplace/Poisson equation on the ball with Dirichlet boundary conditions.*

Proof. Use the **method of reflection** to construct Green's function.
Let $\Omega = \{x \in \mathbb{R}^n : |x| < a\}$. For $\xi \in \Omega$, define $\xi^* = \frac{a^2\xi}{|\xi|^2}$ as its reflection in $\partial\Omega$; note $\xi^* \notin \Omega$.

$$\frac{|x-\xi^*|}{|x-\xi|} = \frac{a}{|\xi|} \qquad \text{for} \quad |x| = a. \qquad \Rightarrow \quad |x-\xi| = \frac{|\xi|}{a}|x-\xi^*|. \tag{17.2}$$

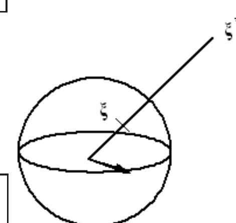
From (17.2) we conclude that for $x \in \partial\Omega$ (i.e. $|x| = a$),

$$K(x-\xi) = \begin{cases} \frac{1}{2\pi}\log\left(\frac{|\xi|}{a}|x-\xi^*|\right) & \text{if } n=2 \\ \left(\frac{a}{|\xi|}\right)^{n-2} K(x-\xi^*) & \text{if } n \geq 3. \end{cases} \tag{17.3}$$

Define for $x, \xi \in \Omega$:

$$G(x,\xi) = \begin{cases} K(x-\xi) - \frac{1}{2\pi}\log\left(\frac{|\xi|}{a}|x-\xi^*|\right) & \text{if } n=2 \\ K(x-\xi) - \left(\frac{a}{|\xi|}\right)^{n-2} K(x-\xi^*) & \text{if } n \geq 3. \end{cases}$$

Since ξ^* is *not* in Ω, the second terms on the RHS are harmonic in $x \in \Omega$. Moreover, by (17.3) we have $G(x,\xi) = 0$ if $x \in \partial\Omega$. Thus, G is the Green's function for Ω.



$$u(\xi) = \int_{\partial\Omega} \frac{\partial G(x,\xi)}{\partial n_x} g(x)\, dS_x = \frac{a^2 - |\xi|^2}{a\omega_n} \int_{|x|=a} \frac{g(x)}{|x-\xi|^n}\, dS_x.$$

□

17.2 The Fundamental Solution

Problem (F'99, #2). ① *Given that $K_a(x-y)$ and $K_b(x-y)$ are the kernels for the operators $(\triangle - aI)^{-1}$ and $(\triangle - bI)^{-1}$ on $L^2(\mathbb{R}^n)$, where $0 < a < b$, show that $(\triangle - aI)(\triangle - bI)$ has a* ***fundamental solution*** *of the form $c_1K_a + c_2K_b$.*
② *Use the preceding to find a fundamental solution for $\triangle^2 - \triangle$, when $n = 3$.*

Proof. **METHOD ❶**:
①

$$(\triangle - aI)u = f \qquad\qquad (\triangle - bI)u = f$$
$$u = \underbrace{K_a}\star f \qquad\qquad u = \underbrace{K_b}\star f$$
$$\text{fundamental solution} \quad \Leftrightarrow \quad \text{kernel}$$

$$\Rightarrow \quad \widehat{u} = \widehat{K_a}\widehat{f} \qquad\qquad \widehat{u} = \widehat{K_b}\widehat{f} \qquad \text{if } \ u \in L^2,$$
$$\widehat{(\triangle - aI)u} = (-|\xi|^2 - a)\widehat{u} = \widehat{f} \qquad\qquad \widehat{(\triangle - bI)u} = (-|\xi|^2 - b)\widehat{u} = \widehat{f}$$
$$\Rightarrow \quad \widehat{u} = -\frac{1}{(\xi^2+a)}\widehat{f}(\xi) \qquad\qquad \widehat{u} = -\frac{1}{(\xi^2+b)}\widehat{f}(\xi)$$
$$\Rightarrow \quad \widehat{K_a} = -\frac{1}{\xi^2+a} \qquad\qquad \widehat{K_b} = -\frac{1}{\xi^2+b}$$

$$\begin{aligned}
(\triangle - aI)(\triangle - bI)u &= f,\\
\big(\triangle^2 - (a+b)\triangle + abI\big)u &= f,\\
\widehat{u} = \frac{1}{(\xi^2+a)(\xi^2+b)}\widehat{f}(\xi) &= \widehat{K}_{new}\widehat{f}(\xi),\\
\widehat{K}_{new} = \frac{1}{(\xi^2+a)(\xi^2+b)} &= \frac{1}{b-a}\Big(-\frac{1}{\xi^2+b} + \frac{1}{\xi^2+a}\Big) = \frac{1}{b-a}(\widehat{K}_b - \widehat{K}_a),\\
K_{new} &= \frac{1}{b-a}(K_b - K_a),\\
c_1 = \frac{1}{b-a}, \quad c_2 &= -\frac{1}{b-a}.
\end{aligned}$$

② $n = 3$ is not relevant (may be used to assume $K_a, K_b \in L^2$).
For $\triangle^2 - \triangle$, $a = 0$, $b = 1$ above, or more explicitly

$$\begin{aligned}
(\triangle^2 - \triangle)u &= f,\\
(\xi^4 + \xi^2)\widehat{u} &= \widehat{f},\\
\widehat{u} &= \frac{1}{(\xi^4+\xi^2)}\widehat{f},\\
\widehat{K} &= \frac{1}{(\xi^4+\xi^2)} = \frac{1}{\xi^2(\xi^2+1)} = -\frac{1}{\xi^2+1} + \frac{1}{\xi^2} = \widehat{K}_1 - \widehat{K}_0.
\end{aligned}$$

METHOD ❷:

- For $u \in C_0^\infty(\mathbb{R}^n)$ we have:

$$u(x) = \int_{\mathbb{R}^n} K_a(x-y)\,(\triangle - aI)\,u(y)\,dy, \quad ①$$
$$u(x) = \int_{\mathbb{R}^n} K_b(x-y)\,(\triangle - bI)\,u(y)\,dy. \quad ②$$

Let

$$u(x) = c_1(\triangle - bI)\,\phi(x), \qquad \text{for } ①$$
$$u(x) = c_2(\triangle - aI)\,\phi(x), \qquad \text{for } ②$$

for $\phi(x) \in C_0^\infty(\mathbb{R}^n)$. Then,

$$c_1(\triangle - bI)\phi(x) = \int_{\mathbb{R}^n} K_a(x-y)\,(\triangle - aI)\,c_1(\triangle - bI)\phi(y)\,dy,$$
$$c_2(\triangle - aI)\phi(x) = \int_{\mathbb{R}^n} K_b(x-y)\,(\triangle - bI)\,c_2(\triangle - aI)\phi(y)\,dy.$$

We add two equations:

$$(c_1+c_2)\triangle\phi(x) - (c_1 b + c_2 a)\phi(x) = \int_{\mathbb{R}^n} (c_1K_a + c_2K_b)\,(\triangle - aI)\,(\triangle - bI)\,\phi(y)\,dy.$$

If $c_1 = -c_2$ and $-(c_1 b + c_2 a) = 1$, that is, $c_1 = \frac{1}{a-b}$, we have:

$$\phi(x) = \int_{\mathbb{R}^n} \frac{1}{a-b}(K_a - K_b)\,(\triangle - aI)\,(\triangle - bI)\,\phi(y)\,dy,$$

which means that $\frac{1}{a-b}(K_a - K_b)$ is a fundamental solution of $(\triangle - aI)(\triangle - bI)$. ✓

- $\triangle^2 - \triangle = \triangle(\triangle - 1) = (\triangle - 0I)(\triangle - 1I)$.

$(\triangle - 0I)$ has fundamental solution $K_0 = -\frac{1}{4\pi r}$ in $\mathbb{R}^3$.
To find K, a **fundamental solution for** $(\triangle - 1I)$, we need to solve for a radially symmetric solution of

$$(\triangle - 1I)K = \delta.$$

In spherical coordinates, in $\mathbb{R}^3$, the above expression may be written as:

$$K'' + \frac{2}{r}K' - K = 0. \qquad \circledast$$

Let

$$K = \frac{1}{r}w(r),$$
$$K' = \frac{1}{r}w' - \frac{1}{r^2}w,$$
$$K'' = \frac{1}{r}w'' - \frac{2}{r^2}w' + \frac{2}{r^3}w.$$

Plugging these into $\circledast$, we obtain:

$$\frac{1}{r}w'' - \frac{1}{r}w = 0, \qquad \text{or}$$
$$w'' - w = 0.$$

Thus,

$$\begin{aligned} w &= c_1 e^r + c_2 e^{-r}, \\ K &= \frac{1}{r} w(r) \ = \ c_1 \frac{e^r}{r} + c_2 \frac{e^{-r}}{r}. \qquad \checkmark \end{aligned}$$

Suppose $v(x) \equiv 0$ for $|x| \geq R$ and let $\Omega = B_R(0)$; for small $\epsilon > 0$ let

$$\Omega_\epsilon = \Omega - B_\epsilon(0).$$

Note: $(\triangle - I)K(|x|) = 0$ in Ω_ϵ. Consider Green's identity ($\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(0)$):

$$\int_{\Omega_\epsilon} \Big(K(|x|)\triangle v \, - \, v\triangle K(|x|)\Big)\, dx \ = \ \underbrace{\int_{\partial\Omega} \Big(K(|x|)\frac{\partial v}{\partial n} \, - \, v\,\frac{\partial K(|x|)}{\partial n}\Big)\, dS}_{=0, \ since \ v\equiv 0 \ for \ x\geq R} \, + \, \int_{\partial B_\epsilon(0)} \Big(K(|x|)\frac{\partial v}{\partial n} \, - \, v\,\frac{\partial K(|x|)}{\partial n}\Big)\, dS$$

We add $-\int_{\Omega_\epsilon} K(|x|)\, v\, dx \, + \, \int_{\Omega_\epsilon} v\, K(|x|)\, dx$ to LHS to get:

$$\int_{\Omega_\epsilon} \Big(K(|x|)(\triangle - I)v \, - \, v\underbrace{(\triangle - I)K(|x|)}_{=0, \ in \ \Omega_\epsilon}\Big)\, dx \ = \ \int_{\partial B_\epsilon(0)} \Big(K(|x|)\frac{\partial v}{\partial n} \, - \, v\,\frac{\partial K(|x|)}{\partial n}\Big)\, dS.$$

$$\lim_{\epsilon\to 0}\Big[\int_{\Omega_\epsilon} K(|x|)(\triangle - I)v\, dx\Big] \ = \ \int_\Omega K(|x|)(\triangle - I)v\, dx. \quad \Big(\text{Since } K(r) = c_1\frac{e^r}{r} + c_2\frac{e^{-r}}{r} \text{ is integrable at } x = 0.\Big)$$

On $\partial B_\epsilon(0)$, $K(|x|) = K(\epsilon)$. Thus, [47]

$$\Big|\int_{\partial B_\epsilon(0)} K(|x|)\frac{\partial v}{\partial n}\, dS\Big| \ = \ \big|K(\epsilon)\big|\int_{\partial B_\epsilon(0)} \Big|\frac{\partial v}{\partial n}\Big|\, dS \ \leq \ \Big|c_1\frac{e^\epsilon}{\epsilon} + c_2\frac{e^{-\epsilon}}{\epsilon}\Big|\, 4\pi\epsilon^2 \max\big|\nabla v\big| \ \to \ 0, \quad \text{as} \ \ \epsilon \to 0.$$

$$\begin{aligned}
\int_{\partial B_\epsilon(0)} v(x)\,\frac{\partial K(|x|)}{\partial n}\, dS \ &= \ \int_{\partial B_\epsilon(0)} \Big[\frac{1}{\epsilon}\big(-c_1e^\epsilon + c_2e^{-\epsilon}\big) + \frac{1}{\epsilon^2}\big(c_1e^\epsilon + c_2e^{-\epsilon}\big)\Big]\, v(x)\, dS \\
&= \ \Big[\frac{1}{\epsilon}\big(-c_1e^\epsilon + c_2e^{-\epsilon}\big) + \frac{1}{\epsilon^2}\big(c_1e^\epsilon + c_2e^{-\epsilon}\big)\Big]\int_{\partial B_\epsilon(0)} v(x)\, dS \\
&= \ \Big[\frac{1}{\epsilon}\big(-c_1e^\epsilon + c_2e^{-\epsilon}\big) + \frac{1}{\epsilon^2}\big(c_1e^\epsilon + c_2e^{-\epsilon}\big)\Big]\int_{\partial B_\epsilon(0)} v(0)\, dS \\
&\quad\ + \Big[\frac{1}{\epsilon}\big(-c_1e^\epsilon + c_2e^{-\epsilon}\big) + \frac{1}{\epsilon^2}\big(c_1e^\epsilon + c_2e^{-\epsilon}\big)\Big]\int_{\partial B_\epsilon(0)} [v(x) - v(0)]\, dS \\
&\to \ \frac{1}{\epsilon^2}\big(c_1e^\epsilon + c_2e^{-\epsilon}\big)\, v(0)\, 4\pi\epsilon^2 \\
&\to \ 4\pi(c_1 + c_2)v(0) \ = \ -v(0).
\end{aligned}$$

Thus, taking $c_1 = c_2$, we have $c_1 = c_2 = -\frac{1}{8\pi}$, which gives

$$\int_\Omega K(|x|)(\triangle - I)v\, dx \ = \ \lim_{\epsilon\to 0}\int_{\Omega_\epsilon} K(|x|)(\triangle - I)v\, dx \ = \ v(0),$$

---

[47] In $\mathbb{R}^3$, for $|x| = \epsilon$,

$$K(|x|) \ = \ K(\epsilon) \ = \ c_1\frac{e^\epsilon}{\epsilon} + c_2\frac{e^{-\epsilon}}{\epsilon}.$$

$$\frac{\partial K(|x|)}{\partial n} \ = \ -\frac{\partial K(\epsilon)}{\partial r} \ = \ -c_1\Big(\frac{e^\epsilon}{\epsilon} - \frac{e^\epsilon}{\epsilon^2}\Big) - c_2\Big(-\frac{e^{-\epsilon}}{\epsilon} - \frac{e^{-\epsilon}}{\epsilon^2}\Big) \ = \ \frac{1}{\epsilon}\big(-c_1e^\epsilon + c_2e^{-\epsilon}\big) + \frac{1}{\epsilon^2}\big(c_1e^\epsilon + c_2e^{-\epsilon}\big),$$

since n points inwards. n points *toward* 0 on the sphere $|x| = \epsilon$ (i.e., $n = -x/|x|$).

that is $K(r) = -\frac{1}{8\pi}\Big(\frac{e^r}{r} + \frac{e^{-r}}{r}\Big) = -\frac{1}{4\pi r}\cosh(r)$ is the fundamental solution of $(\triangle - I)$.
By part (a), $\frac{1}{a-b}(K_a - K_b)$ is a fundamental solution of $(\triangle - aI)(\triangle - bI)$.
Here, the fundamental solution of $(\triangle - 0I)(\triangle - 1I)$ is $\frac{1}{-1}(K_0 - K) = -\Big(-\frac{1}{4\pi r} + \frac{1}{4\pi r}\cosh(r)\Big) = \frac{1}{4\pi r}\big(1 - \cosh(r)\big)$. $\square$

Problem (F'91, #3). *Prove that*

$$-\frac{1}{4\pi}\frac{\cos k|x|}{|x|}$$

*is a **fundamental solution** for* $(\triangle + k^2)$ *in* $\mathbb{R}^3$ *where* $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, *i.e. prove that for any smooth function* $f(x)$ *with compact support*

$$u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\cos k|x-y|}{|x-y|}f(y)\,dy$$

is a solution to $(\triangle + k^2)u = f$.

Proof. For $v \in C_0^\infty(\mathbb{R}^n)$, we want to show that for $K(|x|) = -\frac{1}{4\pi}\frac{\cos k|x|}{|x|}$, we have $(\triangle + k^2)K = \delta$, i.e.

$$\int_{\mathbb{R}^n} K(|x|)\,(\triangle + k^2)v(x)\,dx \;=\; v(0).$$

Suppose $v(x) \equiv 0$ for $|x| \geq R$ and let $\Omega = B_R(0)$; for small $\epsilon > 0$ let

$$\Omega_\epsilon = \Omega - B_\epsilon(0).$$

$(\triangle + k^2)K(|x|) = 0$ in Ω_ϵ. Consider Green's identity ($\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(0)$):

$$\int_{\Omega_\epsilon}\Big(K(|x|)\triangle v \,-\, v\triangle K(|x|)\Big)\,dx \;=\; \underbrace{\int_{\partial\Omega}\Big(K(|x|)\frac{\partial v}{\partial n} - v\frac{\partial K(|x|)}{\partial n}\Big)\,dS}_{=0,\ since\ v\equiv 0\ for\ x\geq R} \;+\; \int_{\partial B_\epsilon(0)}\Big(K(|x|)\frac{\partial v}{\partial n} - v\frac{\partial K(|x|)}{\partial n}\Big)\,dS$$

We add $\int_{\Omega_\epsilon} k^2\,K(|x|)\,v\,dx \,-\, \int_{\Omega_\epsilon} v\,k^2\,K(|x|)\,dx$ to LHS to get:

$$\int_{\Omega_\epsilon}\Big(K(|x|)(\triangle + k^2)v \,-\, v\underbrace{(\triangle + k^2)K(|x|)}_{=0,\ in\ \Omega_\epsilon}\Big)\,dx \;=\; \int_{\partial B_\epsilon(0)}\Big(K(|x|)\frac{\partial v}{\partial n} - v\frac{\partial K(|x|)}{\partial n}\Big)\,dS.$$

$$\lim_{\epsilon\to 0}\Big[\int_{\Omega_\epsilon} K(|x|)(\triangle + k^2)v\,dx\Big] \;=\; \int_\Omega K(|x|)(\triangle + k^2)v\,dx. \quad \Big(\text{Since } K(r) = -\frac{\cos kr}{4\pi r} \text{ is integrable at } x = 0.\Big)$$

On $\partial B_\epsilon(0)$, $K(|x|) = K(\epsilon)$. Thus, [48]

$$\Big|\int_{\partial B_\epsilon(0)} K(|x|)\frac{\partial v}{\partial n}\,dS\Big| \;=\; \big|K(\epsilon)\big|\int_{\partial B_\epsilon(0)}\Big|\frac{\partial v}{\partial n}\Big|\,dS \;\leq\; \Big|-\frac{\cos k\epsilon}{4\pi\epsilon}\Big|\,4\pi\epsilon^2\max\big|\nabla v\big| \;\to\; 0, \quad \text{as } \epsilon \to 0.$$

[48] In $\mathbb{R}^3$, for $|x| = \epsilon$,

$$K(|x|) \;=\; K(\epsilon) \;=\; -\frac{\cos k\epsilon}{4\pi\epsilon}.$$

$$\frac{\partial K(|x|)}{\partial n} \;=\; -\frac{\partial K(\epsilon)}{\partial r} \;=\; \frac{1}{4\pi}\Big(-\frac{k\sin k\epsilon}{\epsilon} - \frac{\cos k\epsilon}{\epsilon^2}\Big) \;=\; -\frac{1}{4\pi\epsilon}\Big(k\sin k\epsilon + \frac{\cos k\epsilon}{\epsilon}\Big),$$

since n points inwards. n points *toward* 0 on the sphere $|x| = \epsilon$ (i.e., $n = -x/|x|$).

$$\int_{\partial B_\epsilon(0)} v(x)\,\frac{\partial K(|x|)}{\partial n}\,dS \;=\; \int_{\partial B_\epsilon(0)} -\frac{1}{4\pi\epsilon}\Big(k\sin k\epsilon + \frac{\cos k\epsilon}{\epsilon}\Big)\,v(x)\,dS$$
$$= -\frac{1}{4\pi\epsilon}\Big(k\sin k\epsilon + \frac{\cos k\epsilon}{\epsilon}\Big)\int_{\partial B_\epsilon(0)} v(x)\,dS$$
$$= -\frac{1}{4\pi\epsilon}\Big(k\sin k\epsilon + \frac{\cos k\epsilon}{\epsilon}\Big)\int_{\partial B_\epsilon(0)} v(0)\,dS \;-\; \frac{1}{4\pi\epsilon}\Big(k\sin k\epsilon + \frac{\cos k\epsilon}{\epsilon}\Big)\int_{\partial B_\epsilon(0)} [v(x)-v(0)]\,dS$$
$$= -\frac{1}{4\pi\epsilon}\Big(k\sin k\epsilon + \frac{\cos k\epsilon}{\epsilon}\Big)\,v(0)\,4\pi\epsilon^2 \;-\; \underbrace{\frac{1}{4\pi\epsilon}\Big(k\sin k\epsilon + \frac{\cos k\epsilon}{\epsilon}\Big)\,[v(x)-v(0)]\,4\pi\epsilon^2}_{\to 0,\ (v\ is\ continuous)}$$
$$\to\; -\cos k\epsilon\; v(0) \;\to\; -v(0).$$

Thus,

$$\int_\Omega K(|x|)(\triangle + k^2)v\,dx \;=\; \lim_{\epsilon\to 0}\int_{\Omega_\epsilon} K(|x|)(\triangle + k^2)v\,dx \;=\; v(0),$$

that is, $K(r) = -\frac{1}{4\pi}\frac{\cos kr}{r}$ is the fundamental solution of $\triangle + k^2$. □

Problem (F'97, #2). *Let $u(x)$ be a solution of the Helmholtz equation*

$$\triangle u + k^2 u = 0 \qquad x\in\mathbb{R}^3$$

satisfying the "radiation" conditions

$$u = O\Big(\frac{1}{r}\Big), \quad \frac{\partial u}{\partial r} - iku = O\Big(\frac{1}{r^2}\Big), \quad |x| = r\to\infty.$$

Prove that $u\equiv 0$.
*Hint: A **fundamental solution** to the Helmholtz equation is $\frac{1}{4\pi r}e^{ikr}$.*
Use the Green formula.

Proof. Denote $K(|x|) = \frac{1}{4\pi r}e^{ikr}$, a fundamental solution. Thus, $(\triangle + k^2)K = \delta$.
Let x_0 be any point and $\Omega = B_R(x_0)$; for small $\epsilon > 0$ let

$$\Omega_\epsilon = \Omega - B_\epsilon(x_0).$$

$(\triangle + k^2)K(|x|) = 0$ in Ω_ϵ. Consider Green's identity ($\partial\Omega_\epsilon = \partial\Omega\cup\partial B_\epsilon(x_0)$):

$$\underbrace{\int_{\Omega_\epsilon}\Big(u\,(\triangle + k^2)K \;-\; K(\triangle + k^2)u\Big)\,dx}_{=0} \;=\; \int_{\partial\Omega}\Big(u\,\frac{\partial K}{\partial n} - K\,\frac{\partial u}{\partial n}\Big)\,dS \;+\; \underbrace{\int_{\partial B_\epsilon(x_0)}\Big(u\,\frac{\partial K}{\partial n} - K\,\frac{\partial u}{\partial n}\Big)\,dS}_{\to u(x_0),\ as\ \epsilon\to 0}.$$

(It can be shown by the method previously used that the integral over $B_\epsilon(x_0)$ approaches $u(x_0)$ as $\epsilon\to 0$.) Taking the limit when $\epsilon\to 0$, we obtain

$$-u(x_0) \;=\; \int_{\partial\Omega}\Big(u\,\frac{\partial K}{\partial n} - K\,\frac{\partial u}{\partial n}\Big)\,dS \;=\; \int_{\partial\Omega}\Big(u\,\frac{\partial}{\partial r}\frac{e^{ik|x-x_0|}}{4\pi|x-x_0|} - \frac{e^{ik|x-x_0|}}{4\pi|x-x_0|}\,\frac{\partial u}{\partial r}\Big)\,dS$$
$$= \int_{\partial\Omega}\Big(u\,\underbrace{\Big[\frac{\partial}{\partial r}\frac{e^{ik|x-x_0|}}{4\pi|x-x_0|} - ik\frac{e^{ik|x-x_0|}}{4\pi|x-x_0|}\Big]}_{=O(\frac{1}{|x|^2});\ (can\ be\ shown)} - \frac{e^{ik|x-x_0|}}{4\pi|x-x_0|}\,\Big[\frac{\partial u}{\partial r} - iku\Big]\Big)\,dS$$
$$= O\Big(\frac{1}{R}\Big)\cdot O\Big(\frac{1}{R^2}\Big)\cdot 4\pi R^2 - O\Big(\frac{1}{R}\Big)\cdot O\Big(\frac{1}{R^2}\Big)\cdot 4\pi R^2 \;=\; 0.$$

Taking the limit when $R\to\infty$, we get $u(x_0) = 0$. □

Problem (S'02, #1). ***a)*** *Find a radially symmetric solution, u, to the equation in $\mathbb{R}^2$,*

$$\triangle u = \frac{1}{2\pi}\log|x|,$$

*and show that u is a **fundamental solution** for $\triangle^2$, i.e. show*

$$\phi(0) = \int_{\mathbb{R}^2} u\triangle^2\phi\,dx$$

for any smooth ϕ which vanishes for $|x|$ large.
b) *Explain how to construct the **Green's function** for the following boundary value in a bounded domain $D\subset\mathbb{R}^2$ with smooth boundary ∂D*

$$w = 0 \quad and \quad \frac{\partial w}{\partial n} = 0 \quad on\ \partial D,$$
$$\triangle^2 w = f \quad in\ D.$$

Proof. **a)** Rewriting the equation in polar coordinates, we have

$$\triangle u \;=\; \frac{1}{r}\big(ru_r\big)_r + \frac{1}{r^2}u_{\theta\theta} \;=\; \frac{1}{2\pi}\log r.$$

For a radially symmetric solution $u(r)$, we have $u_{\theta\theta}=0$. Thus,

$$\begin{aligned}
\frac{1}{r}\big(ru_r\big)_r &= \frac{1}{2\pi}\log r,\\
\big(ru_r\big)_r &= \frac{1}{2\pi}r\log r,\\
ru_r &= \frac{1}{2\pi}\int r\log r\,dr \;=\; \frac{r^2\log r}{4\pi} - \frac{r^2}{8\pi},\\
u_r &= \frac{r\log r}{4\pi} - \frac{r}{8\pi},\\
u &= \frac{1}{4\pi}\int r\log r\,dr - \frac{1}{8\pi}\int r\,dr \;=\; \frac{1}{8\pi}r^2\big(\log r - 1\big).
\end{aligned}$$

$$\boxed{u(r) \;=\; \frac{1}{8\pi}r^2\big(\log r - 1\big).}$$

We want to show that u defined above is a fundamental solution of $\triangle^2$ for $n=2$. That is

$$\int_{\mathbb{R}^2} u\,\triangle^2 v\,dx \;=\; v(0), \qquad\qquad v\in C_0^\infty(\mathbb{R}^n).$$

See the next page that shows that u defined as $u(r) = \frac{1}{8\pi}r^2\log r$ is the Fundamental Solution of $\triangle^2$. (The $-\frac{1}{8\pi}r^2$ term does not play any role.)

In particular, the solution of

$$\triangle^2\omega = f(x),$$

if given by

$$\omega(x) \;=\; \int_{\mathbb{R}^2} u(x-y)\,\triangle^2\omega(y)\,dy \;=\; \frac{1}{8\pi}\int_{\mathbb{R}^2}|x-y|^2\big(\log|x-y| - 1\big)\,f(y)\,dy.$$

b) Let

$$K(x-\xi) = \frac{1}{8\pi}|x-\xi|^2\left(\log|x-\xi| - 1\right).$$

We use the method of images to construct the Green's function.
Let $G(x,\xi) = K(x-\xi) + \omega(x)$. We need $G(x,\xi) = 0$ and $\frac{\partial G}{\partial n}(x,\xi) = 0$ for $x \in \partial\Omega$.
Consider $w_\xi(x)$ with $\triangle^2 w_\xi(x) = 0$ in Ω, $w_\xi(x) = -K(x-\xi)$ and $\frac{\partial w_\xi}{\partial n}(x) = -\frac{\partial K}{\partial n}(x-\xi)$ on $\partial\Omega$. Note, we can find the Greens function for the upper-half plane, and then make a conformal map onto the domain. $\square$

Problem (S'97, #6). *Show that the* ***fundamental solution*** *of* $\triangle^2$ *in* $\mathbb{R}^2$ *is given by*

$$V(x_1, x_2) = \frac{1}{8\pi} r^2 ln(r), \qquad r = |x - \xi|,$$

and write the solution of

$$\triangle^2 w = F(x_1, x_2).$$

Hint: In polar coordinates, $\triangle = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$*; for example,* $\triangle V = \frac{1}{2\pi}(1 + ln(r))$.

Proof. Notation: $x = (x_1, x_2)$. We have

$$V(x) = \frac{1}{8\pi} r^2 \log(r),$$

In polar coordinates: (here, $V_{\theta\theta} = 0$)

$$\begin{aligned}
\triangle V &= \frac{1}{r}\big(rV_r\big)_r = \frac{1}{r}\bigg(r\Big(\frac{1}{8\pi}r^2\log(r)\Big)_r\bigg)_r = \frac{1}{8\pi}\frac{1}{r}\bigg(r\Big(2r\log(r) + r\Big)\bigg)_r \\
&= \frac{1}{8\pi}\frac{1}{r}\Big(2r^2\log(r) + r^2\Big)_r = \frac{1}{8\pi}\frac{1}{r}\big(4r + 4r\log r\big) \\
&= \frac{1}{2\pi}(1 + \log r).
\end{aligned}$$

The fundamental solution $V(x)$ for $\triangle^2$ is the distribution satisfying: $\triangle^2 V(r) = \delta(r)$.

$$\begin{aligned}
\triangle^2 V &= \triangle(\triangle V) = \triangle\Big(\frac{1}{2\pi}(1 + \log r)\Big) = \frac{1}{2\pi}\triangle(1 + \log r) = \frac{1}{2\pi}\frac{1}{r}\big(r(1 + \log r)_r\big)_r \\
&= \frac{1}{2\pi}\frac{1}{r}\Big(r\frac{1}{r}\Big)_r = \frac{1}{2\pi}\frac{1}{r}(1)_r = 0 \qquad \text{for } r \neq 0.
\end{aligned}$$

Thus, $\triangle^2 V(r) = \delta(r) \quad \Rightarrow \quad V$ is the fundamental solution. ✓

The approach above is not rigorous. See the next page that shows that V **defined above is the Fundamental Solution of** $\triangle^2$**.**

The solution of

$$\triangle^2 \omega = F(x),$$

if given by

$$\omega(x) = \int_{\mathbb{R}^2} V(x - y)\,\triangle^2\omega(y)\,dy = \frac{1}{8\pi}\int_{\mathbb{R}^2} |x - y|^2 \log|x - y|\,F(y)\,dy.$$

□

Show that the Fundamental Solution of $\triangle^2$ in $\mathbb{R}^2$ is given by:

$$K(x) = \frac{1}{8\pi} r^2 \ln(r), \qquad r = |x - \xi|, \tag{17.4}$$

Proof. For $v \in C_0^\infty(\mathbb{R}^n)$, we want to show

$$\int_{\mathbb{R}^n} K(|x|)\, \triangle^2 v(x)\, dx \;=\; v(0).$$

Suppose $v(x) \equiv 0$ for $|x| \geq R$ and let $\Omega = B_R(0)$; for small $\epsilon > 0$ let

$$\Omega_\epsilon = \Omega - B_\epsilon(0).$$

$K(|x|)$ is biharmonic ($\triangle^2 K(|x|) = 0$) in Ω_ϵ. Consider Green's identity ($\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(0)$):

$$\begin{aligned}\int_{\Omega_\epsilon} K(|x|)\, \triangle^2 v\, dx \;=\;& \underbrace{\int_{\partial\Omega} \Big(K(|x|)\frac{\partial \triangle v}{\partial n} - v\frac{\partial \triangle K(|x|)}{\partial n}\Big)\, ds + \int_{\partial\Omega} \Big(\triangle K(|x|)\frac{\partial v}{\partial n} - \triangle v \frac{\partial K(|x|)}{\partial n}\Big)\, ds}_{=0,\ since\ v\equiv 0\ for\ x\geq R} \\ &+ \int_{\partial B_\epsilon(0)} \Big(K(|x|)\frac{\partial \triangle v}{\partial n} - v\frac{\partial \triangle K(|x|)}{\partial n}\Big)\, ds + \int_{\partial B_\epsilon(0)} \Big(\triangle K(|x|)\frac{\partial v}{\partial n} - \triangle v \frac{\partial K(|x|)}{\partial n}\Big)\, ds.\end{aligned}$$

$$\lim_{\epsilon\to 0}\left[\int_{\Omega_\epsilon} K(|x|)\triangle^2 v\, dx\right] = \int_\Omega K(|x|)\triangle v^2\, dx. \qquad \Big(\text{Since } K(r) \text{ is integrable at } x = 0.\Big)$$

On $\partial B_\epsilon(0)$, $K(|x|) = K(\epsilon)$. Thus, [49]

$$\begin{aligned}\left|\int_{\partial B_\epsilon(0)} K(|x|)\, \frac{\partial \triangle v}{\partial n}\, dS\right| &= \big|K(\epsilon)\big| \int_{\partial B_\epsilon(0)} \Big|\frac{\partial \triangle v}{\partial n}\Big|\, dS \;\leq\; \big|K(\epsilon)\big|\omega_n \epsilon^1 \max_{x\in\overline{\Omega}} \big|\nabla(\triangle v)\big| \\ &= \Big|\frac{1}{8\pi}\epsilon^2 \log(\epsilon)\Big|\omega_n \epsilon \max_{x\in\overline{\Omega}} \big|\nabla(\triangle v)\big| \to 0, \quad \text{as} \quad \epsilon \to 0.\end{aligned}$$

$$\begin{aligned}\int_{\partial B_\epsilon(0)} v(x)\, \frac{\partial \triangle K(|x|)}{\partial n}\, dS &= \int_{\partial B_\epsilon(0)} -\frac{1}{2\pi\epsilon} v(x)\, dS \\ &= \int_{\partial B_\epsilon(0)} -\frac{1}{2\pi\epsilon} v(0)\, dS + \int_{\partial B_\epsilon(0)} -\frac{1}{2\pi\epsilon}[v(x) - v(0)]\, dS \\ &= -\frac{1}{2\pi\epsilon}\, v(0)\, 2\pi\epsilon - \underbrace{\max_{x\in\partial B_\epsilon(0)} \big|v(x) - v(0)\big|}_{\to 0,\ (v\ is\ continuous)} \;=\; -v(0). \quad \checkmark\end{aligned}$$

$$\left|\int_{\partial B_\epsilon(0)} \triangle K(|x|)\, \frac{\partial v}{\partial n}\, dS\right| \;=\; \big|\triangle K(\epsilon)\big| \int_{\partial B_\epsilon(0)} \Big|\frac{\partial v}{\partial n}\Big|\, dS \;\leq\; \Big|\frac{1}{2\pi}(1 + \log\epsilon)\Big| 2\pi\epsilon \max_{x\in\overline{\Omega}} |\nabla v| \;\to 0, \quad \text{as} \quad \epsilon \to 0.$$

$$\begin{aligned}\int_{\partial B_\epsilon(0)} \triangle v\, \frac{\partial K(|x|)}{\partial n}\, dS &= \int_{\partial B_\epsilon(0)} \Big(-\frac{1}{4\pi}\epsilon\log\epsilon - \frac{1}{8\pi}\epsilon\Big)\triangle v(x)\, dS \\ &\leq \frac{\epsilon}{4\pi}\Big|\log\epsilon + \frac{1}{2}\Big| \cdot 2\pi\epsilon \max_{x\in\partial B_\epsilon(0)} |\triangle v| \;\to\; 0, \quad \text{as} \quad \epsilon \to 0.\end{aligned}$$

---

[49] Note that for $|x| = \epsilon$,

$$\begin{aligned} K(|x|) &= K(\epsilon) = \frac{1}{8\pi}\epsilon^2\log\epsilon, & \triangle K &= \frac{1}{2\pi}(1 + \log\epsilon), \\ \frac{\partial K(|x|)}{\partial n} &= -\frac{\partial K(\epsilon)}{\partial r} = -\frac{1}{4\pi}\epsilon\log\epsilon - \frac{1}{8\pi}\epsilon, & \frac{\partial \triangle K}{\partial n} &= -\frac{\partial \triangle K}{\partial r} = -\frac{1}{2\pi\epsilon}. \end{aligned}$$

$$\Rightarrow \quad \int_{\Omega} K(|x|)\triangle^2 v\,dx \;=\; \lim_{\epsilon\to 0}\int_{\Omega_\epsilon} K(|x|)\triangle^2 v\,dx \;=\; v(0).$$

$\square$

17.3 Radial Variables

Problem (F'99, #8). *Let $u = u(x,t)$ solve the following PDE in* ***two*** *spatial dimensions*

$$-\triangle u = 1$$

for $r < R(t)$, in which $r = |x|$ is the ***radial variable****, with boundary condition*

$$u = 0$$

on $r = R(t)$. In addition assume that $R(t)$ satisfies

$$\frac{dR}{dt} = -\frac{\partial u}{\partial r}(r = R) \qquad \circledast$$

with initial condition $R(0) = R_0$.
a) *Find the solution $u(x,t)$.*
b) *Find an ODE for the outer radius $R(t)$, and solve for $R(t)$.*

Proof. **a)** Rewrite the equation in **polar coordinates**:

$$-\triangle u = -\Big(\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}\Big) = 1.$$

For a radially symmetric solution $u(r)$, we have $u_{\theta\theta} = 0$. Thus,

$$\begin{aligned}
\frac{1}{r}(ru_r)_r &= -1, \\
(ru_r)_r &= -r, \\
ru_r &= -\frac{r^2}{2} + c_1, \\
u_r &= -\frac{r}{2} + \frac{c_1}{r}, \\
u(r,t) &= -\frac{r^2}{4} + c_1 \log r + c_2.
\end{aligned}$$

Since we want u to be defined for $r = 0$, we have $c_1 = 0$. Thus,

$$u(r,t) = -\frac{r^2}{4} + c_2.$$

Using boundary conditions, we have

$$u(R(t),t) = -\frac{R(t)^2}{4} + c_2 = 0 \quad \Rightarrow \quad c_2 = \frac{R(t)^2}{4}. \qquad \text{Thus,}$$

$$\boxed{u(r,t) = -\frac{r^2}{4} + \frac{R(t)^2}{4}.}$$

b) We have

$$\begin{aligned}
u(r,t) &= -\frac{r^2}{4} + \frac{R(t)^2}{4}, \\
\frac{\partial u}{\partial r} &= -\frac{r}{2}, \\
\frac{dR}{dt} &= -\frac{\partial u}{\partial r}(r = R) = \frac{R}{2}, \qquad \text{(from } \circledast\text{)} \\
\frac{dR}{R} &= \frac{dt}{2}, \\
\log R &= \frac{t}{2}, \\
R(t) &= c_1 e^{\frac{t}{2}}, \qquad R(0) = c_1 = R_0. \qquad \text{Thus,}
\end{aligned}$$

$$\boxed{R(t) = R_0 e^{\frac{t}{2}}.}$$

$\square$

Problem (F'01, #3). *Let $u = u(x,t)$ solve the following PDE in* ***three*** *spatial dimensions*

$$\triangle u = 0$$

for $R_1 < r < R(t)$, in which $r = |x|$ is the ***radial variable****, with boundary conditions*

$$u(r = R(t), t) = 0, \quad \text{and} \quad u(r = R_1, t) = 1.$$

In addition assume that $R(t)$ satisfies

$$\frac{dR}{dt} = -\frac{\partial u}{\partial r}(r = R) \qquad \circledast$$

with initial condition $R(0) = R_0$ in which $R_0 > R_1$.
a) *Find the solution $u(x,t)$.*
b) *Find an ODE for the outer radius $R(t)$.*

Proof. **a)** Rewrite the equation in **spherical coordinates** ($n = 3$, radial functions):

$$\triangle u = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) u = \frac{1}{r^2}(r^2 u_r)_r = 0.$$

$$\begin{aligned}
(r^2 u_r)_r &= 0, \\
r^2 u_r &= c_1, \\
u_r &= \frac{c_1}{r^2}, \\
u(r,t) &= -\frac{c_1}{r} + c_2.
\end{aligned}$$

Using boundary conditions, we have

$$\begin{aligned}
u(R(t),t) &= -\frac{c_1}{R(t)} + c_2 = 0 \qquad \Rightarrow \qquad c_2 = \frac{c_1}{R(t)}, \\
u(R_1,t) &= -\frac{c_1}{R_1} + c_2 = 1.
\end{aligned}$$

This gives

$$c_1 = \frac{R_1 R(t)}{R_1 - R(t)}, \qquad c_2 = \frac{R_1}{R_1 - R(t)}.$$

$$\boxed{u(r,t) = -\frac{R_1 R(t)}{R_1 - R(t)} \cdot \frac{1}{r} + \frac{R_1}{R_1 - R(t)}.}$$

b) We have

$$\begin{aligned}
u(r,t) &= -\frac{R_1 R(t)}{R_1 - R(t)} \cdot \frac{1}{r} + \frac{R_1}{R_1 - R(t)}, \\
\frac{\partial u}{\partial r} &= \frac{R_1 R(t)}{R_1 - R(t)} \cdot \frac{1}{r^2}, \\
\frac{dR}{dt} &= -\frac{\partial u}{\partial r}(r = R) = -\frac{R_1 R(t)}{R_1 - R(t)} \cdot \frac{1}{R(t)^2} = -\frac{R_1}{(R_1 - R(t))\, R(t)} \qquad (\text{from } \circledast)
\end{aligned}$$

Thus, an ODE for the outer radius $R(t)$ is

$$\boxed{\begin{cases} \frac{dR}{dt} = \frac{R_1}{(R(t) - R_1)\, R(t)}, \\ R(0) = R_0, \qquad R_0 > R_1. \end{cases}}$$

□

Problem (S'02, #3). *Steady viscous flow in a cylindrical pipe is described by the equation*

$$(\vec{u}\cdot\nabla)\vec{u} + \frac{1}{\rho}\nabla p - \frac{\eta}{\rho}\triangle\vec{u} = 0$$

on the domain $-\infty < x_1 < \infty$, $x_2^2 + x_3^2 \le R^2$, *where* $\vec{u} = (u_1, u_2, u_3) = (U(x_2, x_3), 0, 0)$ *is the velocity vector,* $p(x_1, x_2, x_3)$ *is the pressure, and* η *and* ρ *are constants.*
a) *Show that* $\frac{\partial p}{\partial x_1}$ *is a constant* c, *and that* $\triangle U = c/\eta$.
b) *Assuming further that* U *is radially symmetric and* $U = 0$ *on the surface of the pipe, determine the mass* Q *of fluid passing through a cross-section of pipe per unit time in terms of* c, ρ, η, *and* R. *Note that*

$$Q = \rho\int_{\{x_2^2+x_3^2\le R^2\}} U\,dx_2dx_3.$$

Proof. **a)** Since $\vec{u} = (u_1, u_2, u_3) = (U(x_2, x_3), 0, 0)$, we have

$$(\vec{u}\cdot\nabla)\vec{u} = (u_1, u_2, u_3)\cdot\Big(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_2}, \frac{\partial u_3}{\partial x_3}\Big) = (U(x_2, x_3), 0, 0)\cdot(0,0,0) = 0.$$

Thus,

$$\begin{aligned}
\frac{1}{\rho}\nabla p - \frac{\eta}{\rho}\triangle\vec{u} &= 0,\\
\nabla p &= \eta\triangle\vec{u},\\
\Big(\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_3}\Big) &= \eta(\triangle u_1, \triangle u_2, \triangle u_3),\\
\Big(\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_3}\Big) &= \eta(U_{x_2x_2} + U_{x_3x_3}, 0, 0).
\end{aligned}$$

We can make the following observations:

$$\begin{aligned}
\frac{\partial p}{\partial x_1} &= \eta\underbrace{(U_{x_2x_2} + U_{x_3x_3})}_{indep.\ of\ x_1},\\
\frac{\partial p}{\partial x_2} &= 0 \qquad \Rightarrow \qquad p = f(x_1, x_3),\\
\frac{\partial p}{\partial x_3} &= 0 \qquad \Rightarrow \qquad p = g(x_1, x_2).
\end{aligned}$$

Thus, $p = h(x_1)$. But $\frac{\partial p}{\partial x_1}$ is independent of x_1. Therefore, $\frac{\partial p}{\partial x_1} = c$.

$$\begin{aligned}
\frac{\partial p}{\partial x_1} &= \eta\triangle U,\\
\triangle U &= \frac{1}{\eta}\frac{\partial p}{\partial x_1} = \frac{c}{\eta}.
\end{aligned}$$

b) Cylindrical Laplacian in $\mathbb{R}^3$ for radial functions is

$$\begin{aligned}
\triangle U &= \frac{1}{r}\big(rU_r\big)_r, \\
\frac{1}{r}\big(rU_r\big)_r &= \frac{c}{\eta}, \\
\big(rU_r\big)_r &= \frac{cr}{\eta}, \\
rU_r &= \frac{cr^2}{2\eta} + c_1, \\
U_r &= \frac{cr}{2\eta} + \frac{c_1}{r}.
\end{aligned}$$

For U_r to stay bounded for $r = 0$, we need $c_1 = 0$. Thus,

$$\begin{aligned}
U_r &= \frac{cr}{2\eta}, \\
U &= \frac{cr^2}{4\eta} + c_2, \\
0 \; = \; U(R) &= \frac{cR^2}{4\eta} + c_2, \\
\Rightarrow \quad U &= \frac{cr^2}{4\eta} - \frac{cR^2}{4\eta} \; = \; \frac{c}{4\eta}(r^2 - R^2).
\end{aligned}$$

$$\begin{aligned}
Q &= \rho \int_{\{x_2^2 + x_3^2 \le R^2\}} U \, dx_2 dx_3 \; = \; \frac{c\rho}{4\eta} \int_0^{2\pi} \int_0^R (r^2 - R^2) \, r dr d\theta \; = \; -\frac{c\rho}{4\eta} \int_0^{2\pi} \frac{R^4}{4} d\theta \\
&= -\frac{c\rho R^4 \pi}{8\eta}.
\end{aligned}$$

It is not clear why Q is negative? □

17.4 Weak Solutions

Problem (S'98, #2).
A function $u \in H_0^2(\Omega)$ is a ***weak solution*** *of the* ***biharmonic equation***

$$\begin{cases} \triangle^2 u = f & in \;\; \Omega \\ u = 0 & on \;\; \partial\Omega \\ \frac{\partial u}{\partial n} = 0 & on \;\; \partial\Omega \end{cases}$$

provided

$$\int_\Omega \triangle u \triangle v \, dx = \int_\Omega f v \, dx$$

for all test functions $v \in H_0^2(\Omega)$. Prove that for each $f \in L^2(\Omega)$, there exists a unique weak solution for this problem. Here, $H_0^2(\Omega)$ is the closure of all smooth functions in Ω which vanish on the boundary and with finite H^2 norm: $||u||_2^2 = \int_\Omega (u_{xx}^2 + u_{xy}^2 + u_{yy}^2)\, dxdy < \infty$.
Hint: use ***Lax-Milgram*** *lemma.*

Proof. Multiply the equation by $v \in H_0^2(\Omega)$ and integrate over Ω:

$$\begin{aligned} \triangle^2 u &= f, \\ \int_\Omega \triangle^2 u \, v \, dx &= \int_\Omega f \, v \, dx, \\ \underbrace{\int_{\partial\Omega} \frac{\partial \triangle u}{\partial n} v \, ds - \int_{\partial\Omega} \triangle u \frac{\partial v}{\partial n} \, ds}_{=0} + \int_\Omega \triangle u \triangle v \, dx &= \int_\Omega f \, v \, dx, \\ \underbrace{\int_\Omega \triangle u \triangle v \, dx}_{a(u,v)} &= \underbrace{\int_\Omega f \, v \, dx}_{L(v)}. \end{aligned}$$

Denote: $V = H_0^2(\Omega)$. Check the following conditions:

❶ $a(\cdot,\cdot)$ is continuous: $\exists \gamma > 0$, s.t. $|a(u,v)| \le \gamma ||u||_V ||v||_V, \quad \forall u, v \in V$;

❷ $a(\cdot,\cdot)$ is V-elliptic: $\exists \alpha > 0$, s.t. $a(v,v) \ge \alpha ||v||_V^2, \quad \forall v \in V$;

❸ $L(\cdot)$ is continuous: $\exists \Lambda > 0$, s.t. $|L(v)| \le \Lambda ||v||_V, \quad \forall v \in V$.

We have [50]

❶ $$|a(u,v)|^2 = \Big|\int_\Omega \triangle u \triangle v\, dx\Big|^2 \le \Big(\int_\Omega (\triangle u)^2\, dx\Big)\Big(\int_\Omega (\triangle v)^2\, dx\Big) \le ||u||^2_{H_0^2(\Omega)} ||v||^2_{H_0^2(\Omega)}. \quad \checkmark$$

❷ $$a(v,v) = \int_\Omega (\triangle v)^2\, dx \ge ||v||_{H_0^2(\Omega)}. \quad \checkmark$$

❸ $$\begin{aligned} |L(v)| &= \Big|\int_\Omega f\, v\, dx\Big| \le \int_\Omega |f|\,|v|\, dx \le \Big(\int_\Omega f^2\, dx\Big)^{\frac{1}{2}}\Big(\int_\Omega v^2\, dx\Big)^{\frac{1}{2}} \\ &= ||f||_{L^2(\Omega)} ||v||_{L^2(\Omega)} \le \underbrace{||f||_{L^2(\Omega)}}_{\Lambda} ||v||_{H_0^2(\Omega)}. \quad \checkmark \end{aligned}$$

Thus, by Lax-Milgram theorem, there **exists a weak solution** $u \in H_0^2(\Omega)$.

Also, we can prove the stability result.

$$\begin{aligned} \alpha ||u||^2_{H_0^2(\Omega)} \le a(u,u) = |L(u)| \le \Lambda ||u||_{H_0^2(\Omega)}, \\ \Rightarrow \quad ||u||_{H_0^2(\Omega)} \le \frac{\Lambda}{\alpha}. \end{aligned}$$

Let u_1, u_2 be two solutions so that

$$\begin{aligned} a(u_1, v) &= L(v), \\ a(u_2, v) &= L(v) \end{aligned}$$

for all $v \in V$. Subtracting these two equations, we see that:

$$a(u_1 - u_2, v) = 0 \qquad \forall v \in V.$$

Applying the stability estimate (with $L \equiv 0$, i.e. $\Lambda = 0$), we conclude that $||u_1 - u_2||_{H_0^2(\Omega)} = 0$, i.e. $u_1 = u_2$. $\square$

---

[50] **Cauchy-Schwarz Inequality:**

$$\begin{aligned} |(u,v)| &\le ||u||||v|| \text{ in any norm, for example } \int |uv| dx \le (\int u^2 dx)^{\frac{1}{2}} (\int v^2 dx)^{\frac{1}{2}}; \\ |a(u,v)| &\le a(u,u)^{\frac{1}{2}}\, a(v,v)^{\frac{1}{2}}; \\ \int |v| dx &= \int |v| \cdot 1\ dx = (\int |v|^2 dx)^{\frac{1}{2}} (\int 1^2 dx)^{\frac{1}{2}}. \end{aligned}$$

Poincare Inequality:

$$||v||_{H^2(\Omega)} \le C \int_\Omega (\triangle v)^2\, dx.$$

Green's formula:

$$\int_\Omega (\triangle u)^2\, dx = \int_\Omega (u_{xx}^2 + u_{yy}^2 + 2u_{xx}u_{yy})\, dxdy = \int_\Omega (u_{xx}^2 + u_{yy}^2 - 2u_{xxy}u_y)\, dxdy = \int_\Omega (u_{xx}^2 + u_{yy}^2 + 2|u_{xy}|^2)\, dxdy \ge ||u||^2_{H_0^2(\Omega)}.$$

17.5 Uniqueness

Problem. *The solution of the* ***Robin problem***

$$\frac{\partial u}{\partial n} + \alpha u = \beta, \qquad x \in \partial\Omega$$

for the Laplace equation is ***unique*** *when $\alpha > 0$ is a constant.*

Proof. Let u_1 and u_2 be two solutions of the Robin problem. Let $w = u_1 - u_2$. Then

$$\begin{aligned} &\triangle w = 0 \qquad \text{in } \Omega, \\ &\frac{\partial w}{\partial n} + \alpha w = 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Consider Green's formula:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, ds - \int_\Omega v \triangle u \, dx.$$

Setting $w = u = v$ gives

$$\int_\Omega |\nabla w|^2 \, dx = \int_{\partial\Omega} w \frac{\partial w}{\partial n} \, ds - \underbrace{\int_\Omega w \triangle w \, dx}_{=0}.$$

Boundary condition gives

$$\underbrace{\int_\Omega |\nabla w|^2 \, dx}_{\geq 0} = \underbrace{-\int_{\partial\Omega} \alpha w^2 \, ds}_{\leq 0}.$$

Thus, $w \equiv 0$, and $u_1 \equiv u_2$. Hence, the solution to the Robin problem is unique. □

Problem. *Suppose $q(x) \geq 0$ for $x \in \Omega$ and consider solutions $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of*

$$\triangle u - q(x)u = 0 \qquad \text{in } \Omega.$$

Establish ***uniqueness*** *theorems for*
a) the ***Dirichlet*** *problem: $u(x) = g(x), \quad x \in \partial\Omega$;*
b) the ***Neumann*** *problem: $\partial u/\partial n = h(x), \quad x \in \partial\Omega$.*

Proof. Let u_1 and u_2 be two solutions of the Dirichlet or Neumann problem.
Let $w = u_1 - u_2$. Then

$$\begin{aligned} &\triangle w - q(x)w = 0 \quad \text{in } \Omega, \\ &w = 0 \quad \text{or} \quad \frac{\partial w}{\partial n} = 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Consider Green's formula:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, ds - \int_\Omega v \triangle u \, dx.$$

Setting $w = u = v$ gives

$$\int_\Omega |\nabla w|^2 \, dx = \underbrace{\int_{\partial\Omega} w \frac{\partial w}{\partial n} \, ds}_{=0,\ \text{Dirichlet or Neumann}} - \int_\Omega w \triangle w \, dx.$$

$$\underbrace{\int_\Omega |\nabla w|^2\, dx}_{\geq 0} = \underbrace{-\int_\Omega q(x) w^2\, dx}_{\leq 0}.$$

Thus, $w \equiv 0$, and $u_1 \equiv u_2$. Hence, the solution to the Dirichlet and Neumann problems are unique. ☐

Problem (F'02, #8; S'93, #5).
Let D be a bounded domain in $\mathbb{R}^3$. Show that a solution of the boundary value problem

$$\begin{aligned} &\triangle^2 u = f \quad \text{in } D, \\ &u = \triangle u = 0 \quad \text{on } \partial D \end{aligned}$$

is ***unique****.*

Proof. **Method I: Maximum Principle.** Let u_1, u_2 be two solutions of the boundary value problem. Define $w = u_1 - u_2$. Then w satisfies

$$\begin{aligned} &\triangle^2 w = 0 \quad \text{in } D, \\ &w = \triangle w = 0 \quad \text{on } \partial D. \end{aligned}$$

So $\triangle w$ is harmonic and thus achieves min and max on $\partial D \quad \Rightarrow \quad \triangle w \equiv 0$.
So w is harmonic, but $w \equiv 0$ on $\partial D \quad \Rightarrow \quad w \equiv 0$. Hence, $u_1 = u_2$.

Method II: Green's Identities. Multiply the equation by w and integrate:

$$\begin{aligned} w \triangle^2 w &= 0, \\ \int_\Omega w \triangle^2 w\, dx &= 0, \\ \underbrace{\int_{\partial\Omega} w \frac{\partial(\triangle w)}{\partial n}\, ds}_{=0} - \int_\Omega \nabla w \nabla(\triangle) w\, dx &= 0, \\ \underbrace{-\int_{\partial\Omega} \frac{\partial w}{\partial n} \triangle w\, ds}_{=0} + \int_\Omega (\triangle w)^2\, dx &= 0. \end{aligned}$$

Thus, $\triangle w \equiv 0$. Now, multiply $\triangle w = 0$ by w. We get

$$\int_\Omega |\nabla w|^2\, dx = 0.$$

Thus, $\nabla w = 0$ and w is a constant. Since $w = 0$ on $\partial\Omega$, we have $w \equiv 0$. ☐

Problem (F'97, #6).
a) *Let $u(x) \geq 0$ be continuous in closed bounded domain $\overline{\Omega} \subset \mathbb{R}^n$, $\triangle u$ is continuous in $\overline{\Omega}$,*

$$\triangle u = u^2 \qquad \text{and} \qquad u|_{\partial\Omega} = 0.$$

Prove that $u \equiv 0$.
b) *What can you say about $u(x)$ when the condition $u(x) \geq 0$ in $\overline{\Omega}$ is dropped?*

Proof. **a)** Multiply the equation by u and integrate:

$$\begin{aligned}
u\triangle u &= u^3, \\
\int_\Omega u\triangle u \; dx &= \int_\Omega u^3 \; dx, \\
\underbrace{\int_{\partial\Omega} u\frac{\partial u}{\partial n} \; ds}_{=0} - \int_\Omega |\nabla u|^2 \; dx &= \int_\Omega u^3 \; dx, \\
\int_\Omega \big(u^3 + |\nabla u|^2\big) \; dx &= 0.
\end{aligned}$$

Since $u(x) \geq 0$, we have $u \equiv 0$.

b) If we don't know that $u(x) \geq 0$, then u can not be nonnegative on the entire domain $\overline{\Omega}$. That is, $u(x) < 0$, on some (or all) parts of Ω. If u is nonnegative on Ω, then $u \equiv 0$. □

Problem (W'02, #5). *Consider the boundary value problem*

$$\triangle u + \sum_{k=1}^{n} \alpha_k \frac{\partial u}{\partial x_k} - u^3 = 0 \quad in\ \Omega,$$

$$u = 0 \quad on\ \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. If the α_k's are constants, and $u(x)$ has continuous derivatives up to second order, prove that u must ***vanish identically****.*

Proof. Multiply the equation by u and integrate:

$$\begin{aligned}
u\triangle u + \sum_{k=1}^{n} \alpha_k \frac{\partial u}{\partial x_k} u - u^4 &= 0, \\
\int_\Omega u\triangle u\, dx + \int_\Omega \sum_{k=1}^{n} \alpha_k \frac{\partial u}{\partial x_k} u\, dx - \int_\Omega u^4\, dx &= 0, \\
\underbrace{\int_{\partial\Omega} u\frac{\partial u}{\partial n}\, ds}_{=0} - \int_\Omega |\nabla u|^2\, dx + \underbrace{\int_\Omega \sum_{k=1}^{n} \alpha_k \frac{\partial u}{\partial x_k} u\, dx}_{①} - \int_\Omega u^4\, dx &= 0.
\end{aligned}$$

We will show that ① $= 0$.

$$\begin{aligned}
\int_\Omega \alpha_k \frac{\partial u}{\partial x_k} u\, dx &= \underbrace{\int_{\partial\Omega} \alpha_k u^2\, ds}_{=0} - \int_\Omega \alpha_k u \frac{\partial u}{\partial x_k}\, dx, \\
\Rightarrow \quad 2\int_\Omega \alpha_k \frac{\partial u}{\partial x_k} u\, dx &= 0, \\
\Rightarrow \quad \int_\Omega \sum_{k=1}^{n} \alpha_k \frac{\partial u}{\partial x_k} u\, dx &= 0.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
-\int_\Omega |\nabla u|^2\, dx - \int_\Omega u^4\, dx &= 0, \\
\int_\Omega \big(|\nabla u|^2 + \int_\Omega u^4\big)\, dx &= 0.
\end{aligned}$$

Hence, $|\nabla u|^2 = 0$ and $u^4 = 0$. Thus, $u \equiv 0$. ☐

Note that

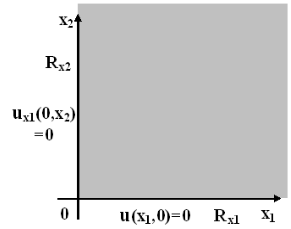
$$\int_\Omega \sum_{k=1}^{n} \alpha_k \frac{\partial u}{\partial x_k} u\, dx = \int_\Omega \alpha \cdot \nabla u\, u\, dx = \underbrace{\int_{\partial\Omega} \alpha \cdot n u^2\, ds}_{=0} - \int_\Omega \alpha \cdot \nabla u\, u\, dx,$$

and thus,

$$\int_\Omega \alpha \cdot \nabla u\, u\, dx = 0.$$

Problem (W'02, #9). *Let $D = \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$, and assume that f is continuous on D and vanishes for $|x| > R$.*
a) *Show that the boundary value problem*

$$\triangle u = f \qquad in\ D,$$
$$u(x_1, 0) = \frac{\partial u}{\partial x_1}(0, x_2) = 0$$



can have only ***one*** *bounded solution.*
b) *Find an explicit* ***Green's function*** *for this boundary value problem.*

Proof. **a)** Let u_1, u_2 be two solutions of the boundary value problem. Define $w = u_1 - u_2$. Then w satisfies

$$\triangle w = 0 \qquad \text{in } D,$$
$$w(x_1, 0) = \frac{\partial w}{\partial x_1}(0, x_2) = 0.$$

Consider Green's formula:

$$\int_D \nabla u \cdot \nabla v \, dx = \int_{\partial D} v \frac{\partial u}{\partial n} \, ds - \int_D v \triangle u \, dx.$$

Setting $w = u = v$ gives

$$\int_D |\nabla w|^2 \, dx = \int_{\partial D} w \frac{\partial w}{\partial n} \, ds - \int_D w \triangle w \, dx,$$

$$\int_D |\nabla w|^2 \, dx = \int_{\mathbb{R}_{x_1}} w \frac{\partial w}{\partial n} \, ds + \int_{\mathbb{R}_{x_2}} w \frac{\partial w}{\partial n} \, ds + \int_{|x|>R} w \frac{\partial w}{\partial n} \, ds - \int_D w \triangle w \, dx$$

$$= \int_{\mathbb{R}_{x_1}} \underbrace{w(x_1, 0)}_{=0} \frac{\partial w}{\partial x_2} \, ds + \int_{\mathbb{R}_{x_2}} w(0, x_2) \underbrace{\frac{\partial w}{\partial x_1}}_{=0} \, ds + \int_{|x|>R} \underbrace{w}_{=0} \frac{\partial w}{\partial n} \, ds - \int_D w \underbrace{\triangle w}_{=0} \, dx,$$

$$\int_D |\nabla w|^2 \, dx = 0 \quad \Rightarrow \quad |\nabla w|^2 = 0 \quad \Rightarrow \quad w = \text{const}.$$

Since $w(x_1, 0) = 0 \ \Rightarrow\ w \equiv 0$. Thus, $u_1 = u_2$.

b) The similar problem is solved in the appropriate section (S'96, #3).
Notice whenever you are on the boundary with variable x,

$$|x - \xi^{(0)}| = \frac{|x - \xi^{(1)}||x - \xi^{(3)}|}{|x - \xi^{(2)}|}.$$

$$\text{So,} \qquad G(x, \xi) = \frac{1}{2\pi}\left(\log|x - \xi| - \log \frac{|x - \xi^{(1)}||x - \xi^{(3)}|}{|x - \xi^{(2)}|} \right)$$

is the Green's function. $\square$

Problem (F'98, #4). *In two dimensions* $\mathbf{x} = (x, y)$*, define the set* Ω_a *as*

$$\Omega_a = \Omega^+ \cup \Omega^-$$

in which

$$\begin{aligned}\Omega^+ &= \{|\mathbf{x} - \mathbf{x}_0| \le a\} \cap \{x \ge 0\} \\ \Omega^- &= \{|\mathbf{x} + \mathbf{x}_0| \le a\} \cap \{x \le 0\} = -\Omega^+\end{aligned}$$

and $\mathbf{x}_0 = (1, 0)$*. Note that* Ω_a *consists of two components when* $0 < a < 1$ *and a single component when* $a > 1$*. Consider the Neumann problem*

$$\begin{aligned}\nabla^2 u &= f, \qquad \mathbf{x} \in \Omega_a \\ \partial u / \partial n &= 0, \qquad \mathbf{x} \in \partial\Omega_a\end{aligned}$$

in which

$$\begin{aligned}\int_{\Omega^+} f(\mathbf{x})\, d\mathbf{x} &= 1 \\ \int_{\Omega^-} f(\mathbf{x})\, d\mathbf{x} &= -1\end{aligned}$$

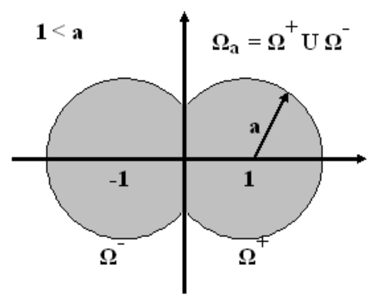
a) *Show that this problem has a solution for* $1 < a$*, but not for* $0 < a < 1$*. (You do not need to construct the solution, only demonstrate solveability.)*

b) *Show that* $\max_{\Omega_a} |\nabla u| \to \infty$ *as* $a \to 1$ *from above. (Hint: Denote* L *to be the line segment* $L = \Omega^+ \cap \Omega^-$*, and note that its length* $|L|$ *goes to* 0 *as* $a \to 1$*.)*

Proof. **a)** We use the Green's identity. For $1 < a$,

$$\begin{aligned}0 &= \int_{\partial\Omega_a} \frac{\partial u}{\partial n}\, ds = \int_{\Omega_a} \triangle u\, dx = \int_{\Omega_a} f(x)\, dx \\ &= \int_{\Omega^+} f(x)\, dx + \int_{\Omega^-} f(x)\, dx = 1 - 1 = 0. \quad \checkmark\end{aligned}$$

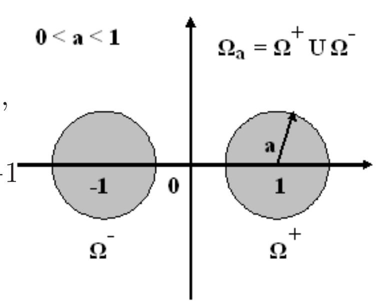
Thus, the problem has a solution for $1 < a$.



For $0 < a < 1$, Ω^+ and Ω^- are disjoint. Consider Ω^+:

$$\begin{aligned}0 &= \int_{\partial\Omega^+} \frac{\partial u}{\partial n}\, ds = \int_{\Omega^+} \triangle u\, dx = \int_{\Omega^+} f(x)\, dx = 1, \\ 0 &= \int_{\partial\Omega^-} \frac{\partial u}{\partial n}\, ds = \int_{\Omega^-} \triangle u\, dx = \int_{\Omega^-} f(x)\, dx = -1\end{aligned}$$

We get *contradictions*.
Thus, the solution does *not* exist for $0 < a < 1$.



b) Using the Green's identity, we have: (n^+ is the unit normal to Ω^+)

$$\int_{\Omega^+} \triangle u \, dx = \int_{\partial\Omega^+} \frac{\partial u}{\partial n^+} \, ds = \int_L \frac{\partial u}{\partial n^+} \, ds,$$

$$\int_{\Omega^-} \triangle u \, dx = \int_{\partial\Omega^-} \frac{\partial u}{\partial n^-} \, ds = \int_L \frac{\partial u}{\partial n^-} \, ds = -\int_L \frac{\partial u}{\partial n^+} \, ds.$$

$$\int_{\Omega^+} \triangle u \, dx - \int_{\Omega^-} \triangle u \, dx = 2 \int_L \frac{\partial u}{\partial n^+} \, ds,$$

$$\int_{\Omega^+} f(x) \, dx - \int_{\Omega^-} f(x) \, dx = 2 \int_L \frac{\partial u}{\partial n^+} \, ds.$$

$$2 = 2 \int_L \frac{\partial u}{\partial n^+} \, ds,$$

$$1 = \int_L \frac{\partial u}{\partial n^+} \, ds \leq \int_L \left| \frac{\partial u}{\partial n^+} \right| ds \leq \int_L \sqrt{\left(\frac{\partial u}{\partial n^+}\right)^2 + \left(\frac{\partial u}{\partial \tau}\right)^2} \leq |L| \max_L |\nabla u| \leq |L| \max_{\Omega_a} |\nabla u|.$$

Thus,

$$\max_{\Omega_a} |\nabla u| \geq \frac{1}{|L|}.$$

As $a \to 1$ $(L \to 0)$ $\Rightarrow$ $\max_{\Omega_a} |\nabla u| \to \infty$. □

Problem (F'00, #1). *Consider the Dirichlet problem in a bounded domain $D \subset \mathbb{R}^n$ with smooth boundary ∂D,*

$$\begin{aligned} \triangle u + a(x)u = f(x) \quad & \text{in } D, \\ u = 0 \quad & \text{on } \partial D. \end{aligned}$$

a) *Assuming that $|a(x)|$ is small enough, prove the uniqueness of the classical solution.*
b) *Prove the existence of the solution in the Sobolev space $H^1(D)$ assuming that $f \in L_2(D)$.*
*Note: Use **Poincare inequality**.*

Proof. **a)** By Poincare Inequality, for any $u \in C_0^1(D)$, we have $||u||_2^2 \le C||\nabla u||_2^2$. Consider two solutions of the Dirichlet problem above. Let $w = u_1 - u_2$. Then, w satisfies

$$\begin{cases} \triangle w + a(x)w = 0 & \text{in } D, \\ w = 0 & \text{on } \partial D. \end{cases}$$

$$\begin{aligned} w\triangle w + a(x)w^2 &= 0, \\ \int w\triangle w\,dx + \int a(x)w^2\,dx &= 0, \\ -\int |\nabla w|^2\,dx + \int a(x)w^2\,dx &= 0, \\ \int a(x)w^2\,dx = \int |\nabla w|^2\,dx &\ge \frac{1}{C}\int w^2\,dx, \qquad \text{(by Poincare inequality)} \\ \int a(x)w^2\,dx - \frac{1}{C}\int w^2\,dx &\ge 0, \\ |a(x)|\int w^2\,dx - \frac{1}{C}\int w^2\,dx &\ge 0, \\ \Big(|a(x)| - \frac{1}{C}\Big)\int w^2\,dx &\ge 0. \end{aligned}$$

If $|a(x)| < \frac{1}{C} \quad \Rightarrow \quad w \equiv 0$.
b) Consider

$$F(v,u) = -\int_\Omega (v\triangle u + a(x)vu)\,dx = -\int_\Omega vf(x)\,dx = F(v).$$

$F(v)$ is a bounded linear functional on $v \in H^{1,2}(D)$, $\quad D = \Omega$.

$$|F(v)| \le ||f||_2||v||_2 \le ||f||_2 C||v||_{H^{1,2}(D)}$$

So by Riesz representation, there exists a solution $u \in H_0^{1,2}(D)$ of

$$-< u, v >= \int_\Omega v\triangle u + a(x)vu\,dx = \int_\Omega vf(x)\,dx = F(v) \qquad \forall v \in H_0^{1,2}(D).$$

□

Problem (S'91, #8). *Define the operator*

$$Lu = u_{xx} + u_{yy} - 4(r^2 + 1)u$$

in which $r^2 = x^2 + y^2$.
a) *Show that* $\varphi = e^{r^2}$ *satisfies* $L\varphi = 0$.
b) *Use this to show that the equation*

$$\begin{aligned} Lu &= f \qquad in \;\; \Omega \\ \frac{\partial u}{\partial n} &= \gamma \qquad on \;\; \partial\Omega \end{aligned}$$

has a solution only if

$$\int_\Omega \varphi f\, dx = \int_{\partial\Omega} \varphi\gamma\, ds(x).$$

Proof. **a)** Expressing Laplacian in polar coordinates, we obtain:

$$\begin{aligned} Lu &= \frac{1}{r}(ru_r)_r - 4(r^2+1)u, \\ L\varphi &= \frac{1}{r}(r\varphi_r)_r - 4(r^2+1)\varphi = \frac{1}{r}(2r^2 e^{r^2})_r - 4(r^2+1)e^{r^2} \\ &= \frac{1}{r}(4re^{r^2} + 2r^2 \cdot 2re^{r^2}) - 4r^2 e^{r^2} - 4e^{r^2} = 0. \quad \checkmark \end{aligned}$$

b) We have $\varphi = e^{r^2} = e^{x^2+y^2} = e^{x^2}e^{y^2}$. From part (a),

$$\begin{aligned} L\varphi &= 0, \\ \frac{\partial \varphi}{\partial n} &= \nabla\varphi \cdot n = (\varphi_x, \varphi_y)\cdot n = (2xe^{x^2}e^{y^2}, 2ye^{x^2}e^{y^2})\cdot n = 2e^{r^2}(x,y)\cdot(-y,x) = 0. \end{aligned}$$

[51] Consider two equations:

$$\begin{aligned} Lu &= \triangle u - 4(r^2+1)u, \\ L\varphi &= \triangle\varphi - 4(r^2+1)\varphi. \end{aligned}$$

Multiply the first equation by φ and the second by u and subtract the two equations:

$$\begin{aligned} \varphi Lu &= \varphi\triangle u - 4(r^2+1)u\varphi, \\ uL\varphi &= u\triangle\varphi - 4(r^2+1)u\varphi, \\ \varphi Lu - uL\varphi &= \varphi\triangle u - u\triangle\varphi. \end{aligned}$$

Then, we start from the LHS of the equality we need to prove and end up with RHS:

$$\begin{aligned} \int_\Omega \varphi f\, dx &= \int_\Omega \varphi Lu\, dx = \int_\Omega (\varphi Lu - uL\varphi)\, dx = \int_\Omega (\varphi\triangle u - u\triangle\varphi)\, dx \\ &= \int_\Omega \Big(\varphi\frac{\partial u}{\partial n} - u\frac{\partial\varphi}{\partial n}\Big)\, ds = \int_\Omega \varphi\frac{\partial u}{\partial n}\, ds = \int_\Omega \varphi\gamma\, ds. \quad \checkmark \end{aligned}$$

□

[51]The only shortcoming in the above proof is that we assume $\vec{n} = (-y, x)$, without giving an explanation why it is so.

17.6 Self-Adjoint Operators

Consider an mth-order differential operator

$$Lu = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha u.$$

The integration by parts formula

$$\int_\Omega u_{x_k} v\,dx = \int_{\partial\Omega} uvn_k\,ds - \int_\Omega uv_{x_k}\,dx \qquad \vec{n} = (n_1,\ldots,n_n)\in\mathbb{R}^n,$$

with u or v vanishing near $\partial\Omega$ is:

$$\int_\Omega u_{x_k} v\,dx = -\int_\Omega uv_{x_k}\,dx.$$

We can repeat the integration by parts with any combination of derivatives $D^\alpha = (\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$:

$$\int_\Omega (D^\alpha u)v\,dx = (-1)^m \int_\Omega uD^\alpha v\,dx, \qquad (m = |\alpha|).$$

We have

$$\begin{aligned}
\int_\Omega (Lu)\,v\;dx &= \int_\Omega \Big(\sum_{|\alpha|\le m} a_\alpha(x)D^\alpha u\Big) v\,dx = \sum_{|\alpha|\le m}\int_\Omega a_\alpha(x)\,v\,D^\alpha u\,dx \\
&= \sum_{|\alpha|\le m}(-1)^{|\alpha|}\int_\Omega D^\alpha(a_\alpha(x)\,v)\,u\,dx = \int_\Omega \underbrace{\sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha(a_\alpha(x)\,v)}_{L^*(v)}\,u\,dx \\
&= \int_\Omega L^*(v)\,u\;dx,
\end{aligned}$$

for all $u\in C^m(\Omega)$ and $v\in C_0^\infty$.
The operator

$$\boxed{L^*(v) = \sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha(a_\alpha(x)\,v)}$$

is called the **adjoint** of L.
The operator is **self-adjoint** if $L^* = L$.
Also, L is self-adjoint if [52]

$$\boxed{\int_\Omega vL(u)\,dx = \int_\Omega uL(v)\,dx.}$$

[52] $L = L^* \quad\Leftrightarrow\quad (Lu|v) = (u|L^*v) = (u|Lv)$.

Problem (F'92, #6).
Consider the Laplace operator $\triangle$ in the wedge $0 \le x \le y$ with boundary conditions

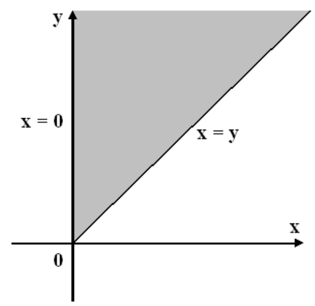
$$\frac{\partial f}{\partial x} = 0 \qquad on \quad x = 0$$
$$\frac{\partial f}{\partial x} - \alpha \frac{\partial f}{\partial y} = 0 \qquad on \quad x = y.$$

a) *For which values of α is this operator **self-adjoint**?*
b) *For such a value of α, suppose that*

$$\triangle f = e^{-r^2/2}\cos\theta$$

with these boundary conditions. Evaluate

$$\int_{C_R} \frac{\partial}{\partial r} f \, ds$$



in which C_R is the circular arc of radius R connecting the boundaries $x = 0$ and $x = y$.

Proof. **a)** We have

$$Lu = \triangle u = 0$$

$$\frac{\partial u}{\partial x} = 0 \qquad \text{on} \quad x = 0$$
$$\frac{\partial u}{\partial x} - \alpha \frac{\partial u}{\partial y} = 0 \qquad \text{on} \quad x = y.$$

The operator L is self-adjoint if:

$$\int_\Omega (u\,Lv - v\,Lu)\,dx = 0.$$

$$\begin{aligned}
\int_\Omega (u\,Lv - v\,Lu)\,dx &= \int_\Omega (u\triangle v - v\triangle u)\,dx = \int_{\partial\Omega} \Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big)\,ds \\
&= \int_{x=0} \Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big)\,ds + \int_{x=y} \Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big)\,ds \\
&= \int_{x=0} \Big(u\,(\nabla v \cdot n) - v\,(\nabla u \cdot n)\Big)\,ds + \int_{x=y} \Big(u\,(\nabla v \cdot n) - v\,(\nabla u \cdot n)\Big)\,ds \\
&= \int_{x=0} \Big(u\,\big((v_x, v_y)\cdot(-1,0)\big) - v\,\big((u_x, u_y)\cdot(-1,0)\big)\Big)\,ds \\
&\quad + \int_{x=y} \Big(u\,\big((v_x, v_y)\cdot(1/\sqrt{2}, -1/\sqrt{2})\big) - v\,\big((u_x, u_y)\cdot(1/\sqrt{2}, -1/\sqrt{2})\big)\Big)\,ds \\
&= \underbrace{\int_{x=0} \Big(u\,\big((0, v_y)\cdot(-1,0)\big) - v\,\big((0, u_y)\cdot(-1,0)\big)\Big)\,ds}_{=0} \\
&\quad + \int_{x=y} \Big(u\,\big((\alpha v_y, v_y)\cdot(1/\sqrt{2}, -1/\sqrt{2})\big) - v\,\big((\alpha u_y, u_y)\cdot(1/\sqrt{2}, -1/\sqrt{2})\big)\Big)\,ds \\
&= \int_{x=y} \Big(\frac{uv_y}{\sqrt{2}}\,(\alpha - 1) - \frac{vu_y}{\sqrt{2}}\,(\alpha - 1)\Big)\,ds \underbrace{=}_{need} 0.
\end{aligned}$$

Thus, we need $\alpha = 1$ so that L is self-adjoint.

b) We have $\alpha = 1$. Using Green's identity and results from part (a), ($\frac{\partial f}{\partial n} = 0$ on $x = 0$ and $x = y$):

$$\int_\Omega \triangle f \, dx = \int_{\partial\Omega} \frac{\partial f}{\partial n} \, ds = \int_{\partial C_R} \frac{\partial f}{\partial n} \, ds + \int_{x=0} \underbrace{\frac{\partial f}{\partial n}}_{=0} \, ds + \int_{x=y} \underbrace{\frac{\partial f}{\partial n}}_{=0} \, ds = \int_{\partial C_R} \frac{\partial f}{\partial r} \, ds.$$

Thus,

$$\begin{aligned} \int_{\partial C_R} \frac{\partial f}{\partial r} \, ds &= \int_\Omega \triangle f \, dx = \int_0^R \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^{-r^2/2} \cos\theta \; r \, dr d\theta \\ &= \Big(1 - \frac{1}{\sqrt{2}}\Big) \int_0^R e^{-r^2/2} \; r \, dr = \Big(1 - \frac{1}{\sqrt{2}}\Big)(1 - e^{-R^2/2}). \end{aligned}$$

□

Problem (F'99, #1). *Suppose that $\triangle u = 0$ in the weak sense in $\mathbb{R}^n$ and that there is a constant C such that*

$$\int_{\{|x-y|<1\}} |u(y)|\, dy < C, \quad \forall x \in \mathbb{R}^n.$$

Show that u is constant.

Proof. Consider Green's formula:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_{\partial\Omega} v \frac{\partial u}{\partial n}\, ds - \int_\Omega v \triangle u \, dx$$

For $v = 1$, we have

$$\int_{\partial\Omega} \frac{\partial u}{\partial n}\, ds = \int_\Omega \triangle u \, dx.$$

Let $B_r(x_0)$ be a ball in $\mathbb{R}^n$. We have

$$\begin{aligned} 0 &= \int_{B_r(x_0)} \triangle u \, dx = \int_{\partial B_r(x_0)} \frac{\partial u}{\partial n}\, ds = r^{n-1} \int_{|x|=1} \frac{\partial u}{\partial r}(x_0 + rx)\, ds \\ &= r^{n-1}\omega_n \frac{\partial}{\partial r} \frac{1}{\omega_n} \int_{|x|=1} u(x_0 + rx)\, ds. \end{aligned}$$

Thus, $\frac{1}{\omega_n}\int_{|x|=1} u(x_0 + rx)\, ds$ is independent of r. Hence, it is constant.
By continuity, as $r \to 0$, we obtain the Mean Value property:

$$\boxed{u(x_0) = \frac{1}{\omega_n} \int_{|x|=1} u(x_0 + rx)\, ds.}$$

If $\int_{|x-y|<1} |u(y)|\, dy < C \quad \forall x \in \mathbb{R}^n$, we have $|u(x)| < C$ in $\mathbb{R}^n$.
Since u is harmonic and bounded in $\mathbb{R}^n$, u is constant by Liouville's theorem. [53] □

[53] **Liouville's Theorem:** A bounded harmonic function defined on $\mathbb{R}^n$ is a constant.

Problem (S'01, #1). *For bodies (bounded regions B in $\mathbb{R}^3$) which are not perfectly conducting one considers the boundary value problem*

$$0 = \nabla \cdot \gamma(x)\nabla u = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\Big(\gamma(x)\frac{\partial u}{\partial x_j}\Big)$$
$$u = f \qquad on \ \ \partial B.$$

The function $\gamma(x)$ is the "local conductivity" of B and u is the voltage. We define operator $\Lambda(f)$ mapping the boundary data f to the current density at the boundary by

$$\Lambda(f) = \gamma(x)\frac{\partial u}{\partial n},$$

and $\partial/\partial n$ is the inward normal derivative (this formula defines the current density).
a) *Show that Λ is a symmetric operator, i.e. prove*

$$\int_{\partial B} g\Lambda(f)\, dS = \int_{\partial B} f\Lambda(g)\, dS.$$

b) *Use the positivity of $\gamma(x) > 0$ to show that Λ is negative as an operator, i.e., prove*

$$\int_{\partial B} f\Lambda(f)\, dS \le 0.$$

Proof. **a)** Let

$$\begin{cases} \nabla \cdot \gamma(x)\nabla u = 0 & \text{on } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \qquad \begin{cases} \nabla \cdot \gamma(x)\nabla v = 0 & \text{on } \Omega, \\ v = g & \text{on } \partial\Omega. \end{cases}$$

$$\Lambda(f) = \gamma(x)\frac{\partial u}{\partial n}, \qquad \Lambda(g) = \gamma(x)\frac{\partial v}{\partial n}.$$

Since $\partial/\partial n$ is inward normal derivative, Green's formula is:

$$-\int_{\partial\Omega} \underbrace{v}_{=g} \gamma(x)\frac{\partial u}{\partial n}\, dS - \int_\Omega \nabla v \cdot \gamma(x)\nabla u\, dx = \int_\Omega v\nabla \cdot \gamma(x)\nabla u\, dx.$$

We have

$$\begin{aligned}
\int_{\partial\Omega} g\Lambda(f)\, dS &= \int_{\partial\Omega} g\gamma(x)\frac{\partial u}{\partial n}\, dS = -\int_\Omega \nabla v \cdot \gamma(x)\nabla u\, dx - \int_\Omega v\underbrace{\nabla \cdot \gamma(x)\nabla u}_{=0}\, dx \\
&= \int_{\partial\Omega} u\gamma(x)\frac{\partial v}{\partial n}\, dS + \int_\Omega u\underbrace{\nabla \cdot \gamma(x)\nabla v}_{=0}\, dx \\
&= \int_{\partial\Omega} f\gamma(x)\frac{\partial v}{\partial n}\, dS = \int_{\partial\Omega} f\Lambda(g)\, dS. \quad \checkmark
\end{aligned}$$

b) We have $\gamma(x) > 0$.

$$\begin{aligned}
\int_{\partial\Omega} f\Lambda(f)\, dS &= \int_{\partial\Omega} u\gamma(x)\frac{\partial u}{\partial n}\, dS = -\int_\Omega u\underbrace{\nabla \cdot \gamma(x)\nabla u}_{=0}\, dx - \int_\Omega \gamma(x)\nabla u \cdot \nabla u\, dx \\
&= -\int_\Omega \underbrace{\gamma(x)|\nabla u|^2}_{\ge 0} \le 0. \quad \checkmark
\end{aligned}$$

□

Problem (S'01, #4). *The **Poincare Inequality** states that for any bounded domain Ω in $\mathbb{R}^n$ there is a constant C such that*

$$\int_\Omega |u|^2\,dx \le C\int_\Omega |\nabla u|^2\,dx$$

for all smooth functions u which vanish on the boundary of Ω.
a) *Find a formula for the "best" (smallest) constant for the domain Ω in terms of **the eigenvalues of the Laplacian** on Ω, and*
b) *give the best constant for the rectangular domain in $\mathbb{R}^2$*

$$\Omega = \{(x,y):\ 0\le x\le a,\ 0\le y\le b\}.$$

Proof. **a)** Consider Green's formula:

$$\int_\Omega \nabla u\cdot\nabla v\,dx \;=\; \int_{\partial\Omega} v\,\frac{\partial u}{\partial n}\,ds - \int_\Omega v\,\triangle u\,dx.$$

Setting $u=v$ and with u vanishing on $\partial\Omega$, Green's formula becomes:

$$\int_\Omega |\nabla u|^2\,dx \;=\; -\int_\Omega u\triangle u\,dx.$$

Expanding u in the eigenfunctions of the Laplacian, $u(x)=\sum a_n\phi_n(x)$, the formula above gives

$$\begin{aligned}\int_\Omega |\nabla u|^2\,dx \;&=\; -\int_\Omega \sum_{n=1}^\infty a_n\phi_n(x)\sum_{m=1}^\infty -\lambda_m a_m\phi_m(x)\,dx \;=\; \sum_{m,n=1}^\infty \lambda_m a_n a_m\int_\Omega \phi_n\phi_m\,dx\\ &=\; \sum_{n=1}^\infty \lambda_n|a_n|^2. \qquad \circledast\end{aligned}$$

Also,

$$\int_\Omega |u|^2\,dx \;=\; \int_\Omega \sum_{n=1}^\infty a_n\phi_n(x)\sum_{m=1}^\infty a_m\phi_m(x) \;=\; \sum_{n=1}^\infty |a_n|^2. \qquad \circledcirc$$

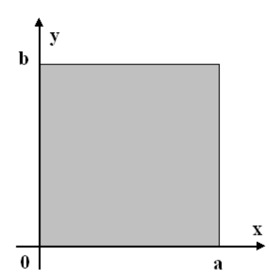
Comparing $\circledast$ and $\circledcirc$, and considering that λ_n increases as $n\to\infty$, we obtain

$$\lambda_1\int_\Omega |u|^2\,dx = \lambda_1\sum_{n=1}^\infty |a_n|^2 \le \sum_{n=1}^\infty \lambda_n|a_n|^2 = \int_\Omega |\nabla u|^2\,dx.$$

$$\boxed{\int_\Omega |u|^2\,dx \le \frac{1}{\lambda_1}\int_\Omega |\nabla u|^2\,dx,}$$

with $C = 1/\lambda_1$.

b) For the rectangular domain $\Omega=\{(x,y):\ 0\le x\le a,\ 0\le y\le b\}\subset\mathbb{R}^2$, the eigenvalues of the Laplacian are



$$\begin{aligned}\lambda_{mn} \;&=\; \pi^2\Big(\frac{m^2}{a^2}+\frac{n^2}{b^2}\Big), \qquad m,n=1,2,\ldots.\\ \lambda_1 \;&=\; \lambda_{11} = \pi^2\Big(\frac{1}{a^2}+\frac{1}{b^2}\Big),\\ \Rightarrow\ C \;&=\; \frac{1}{\lambda_{11}} = \frac{1}{\pi^2}\frac{1}{(\frac{1}{a^2}+\frac{1}{b^2})}.\end{aligned}$$

□

Problem (S'01, #6). ***a)*** *Let B be a bounded region in $\mathbb{R}^3$ with smooth boundary ∂B. The "conductor" potential for the body B is the solution of Laplace's equation outside B*

$$\triangle V = 0 \quad in \quad \mathbb{R}^3/B$$

*subject to the boundary conditions, $V = 1$ on ∂B and $V(x)$ tends to zero as $|x| \to \infty$. Assuming that the conductor potential exists, show that it is **unique**.*
b) *The "capacity" $C(B)$ of B is defined to be the limit of $|x|V(x)$ as $|x| \to \infty$. Show that*

$$C(B) = -\frac{1}{4\pi}\int_{\partial B} \frac{\partial V}{\partial n}\, dS,$$

where ∂B is the boundary of B and n is the outer unit normal to it (i.e. the normal pointing "toward infinity").
c) *Suppose that $B' \subset B$. Show that $C(B') \le C(B)$.*

Proof. **a)** Let V_1, V_2 be two solutions of the boundary value problem. Define $W = V_1 - V_2$. Then W satisfies

$$\begin{cases} \triangle W = 0 & \text{in } \ \mathbb{R}^3/B \\ W = 0 & \text{on } \partial B \\ W \to 0 & \text{as } \ |x| \to \infty. \end{cases}$$

Consider Green's formula:

$$\int_B \nabla u \cdot \nabla v \, dx \ = \int_{\partial B} v \frac{\partial u}{\partial n}\, ds - \int_B v \triangle u \, dx.$$

Setting $W = u = v$ gives

$$\int_B |\nabla W|^2 \, dx \quad = \quad \int_{\partial B} \underbrace{W}_{=0} \frac{\partial W}{\partial n}\, ds - \int_B W \underbrace{\triangle W}_{=0} \, dx = 0.$$

Thus, $|\nabla W|^2 = 0 \ \Rightarrow \ W = \text{const.}$ Since $W = 0$ on ∂B, $W \equiv 0$, and $V_1 = V_2$.

b & c) For $(b)\&(c)$, see the solutions from Ralston's homework (a few pages down). ☐

Problem (W'03, #2). *Let L be the second order differential operator $L = \triangle - a(x)$ in which $x = (x_1, x_2, x_3)$ is in the three-dimensional cube $C = \{0 < x_i < 1,\ i = 1, 2, 3\}$. Suppose that $a > 0$ in C. Consider the eigenvalue problem*

$$\begin{aligned} Lu &= \lambda u \quad \text{for} \quad x \in C \\ u &= 0 \quad \text{for} \quad x \in \partial C. \end{aligned}$$

a) *Show that all eigenvalues are* ***negative****.*
b) *If u and v are eigenfunctions for distinct eigenvalues λ and μ, show that u and v are* ***orthogonal*** *in the appropriate product.*
c) *If $a(x) = a_1(x_1) + a_2(x_2) + a_3(x_3)$ find an expression for the eigenvalues and eigenvectors of L in terms of the eigenvalues and eigenvectors of a set of one-dimensional problems.*

Proof. **a)** We have

$$\triangle u - a(x)u = \lambda u.$$

Multiply the equation by u and integrate:

$$\begin{aligned} u\triangle u - a(x)u^2 &= \lambda u^2, \\ \int_\Omega u\triangle u\, dx - \int_\Omega a(x)u^2\, dx &= \lambda \int_\Omega u^2\, dx, \\ \underbrace{\int_{\partial\Omega} u\frac{\partial u}{\partial n}\, ds}_{=0} - \int_\Omega |\nabla u|^2\, dx - \int_\Omega a(x)u^2\, dx &= \lambda \int_\Omega u^2\, dx, \\ \lambda = \frac{-\int_\Omega(|\nabla u|^2 + a(x)u^2)\, dx}{\int_\Omega u^2\, dx} &< 0. \end{aligned}$$

b) Let λ, μ, be the eigenvalues and u, v be the corresponding eigenfunctions. We have

$$\triangle u - a(x)u = \lambda u. \tag{17.5}$$

$$\triangle v - a(x)v = \mu v. \tag{17.6}$$

Multiply (17.5) by v and (17.6) by u and subtract equations from each other

$$\begin{aligned} v\triangle u - a(x)uv &= \lambda uv, \\ u\triangle v - a(x)uv &= \mu uv. \\ v\triangle u - u\triangle v &= (\lambda - \mu)uv. \end{aligned}$$

Integrating over Ω gives

$$\begin{aligned} \int_\Omega \big(v\triangle u - u\triangle v\big)\, dx &= (\lambda - \mu)\int_\Omega uv\, dx, \\ \int_{\partial\Omega} \Big(\underbrace{v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}}_{=0}\Big)\, dx &= (\lambda - \mu)\int_\Omega uv\, dx. \end{aligned}$$

Since $\lambda \neq \mu$, u and v are orthogonal on Ω.

c) The three one-dimensional eigenvalue problems are:

$$u_{1x_1x_1}(x_1) - a(x_1)u_1(x_1) = \lambda_1 u_1(x_1),$$
$$u_{2x_2x_2}(x_2) - a(x_2)u_2(x_2) = \lambda_2 u_2(x_2),$$
$$u_{3x_3x_3}(x_3) - a(x_3)u_3(x_3) = \lambda_3 u_3(x_3).$$

We need to derive how u_1, u_2, u_3 and $\lambda_1, \lambda_2, \lambda_3$ are related to u and λ. $\square$

17.7 Spherical Means

Problem (S'95, #4). *Consider the **biharmonic operator** in* $\mathbb{R}^3$

$$\triangle^2 u \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^2 u.$$

a) *Show that* $\triangle^2$ *is **self-adjoint** on* $|x| < 1$ *with the following boundary conditions on* $|x| = 1$*:*

$$u = 0,$$
$$\triangle u = 0.$$

Proof. **a)** We have

$$Lu = \triangle^2 u = 0$$

$$\begin{aligned} u &= 0 && \text{on } |x| = 1 \\ \triangle u &= 0 && \text{on } |x| = 1. \end{aligned}$$

The operator L is self-adjoint if:

$$\int_\Omega (u\,Lv - v\,Lu)\,dx = 0.$$

$$\begin{aligned} \int_\Omega (u\,Lv - v\,Lu)\,dx &= \int_\Omega (u\,\triangle^2 v - v\,\triangle^2 u)\,dx \\ &= \underbrace{\int_{\partial\Omega} u\frac{\partial \triangle v}{\partial n}\,ds}_{=0} - \int_\Omega \nabla u \cdot \nabla(\triangle v)\,dx - \underbrace{\int_{\partial\Omega} v\frac{\partial \triangle u}{\partial n}\,ds}_{=0} + \int_\Omega \nabla v \cdot \nabla(\triangle u)\,dx \\ &= -\underbrace{\int_{\partial\Omega} \triangle v\frac{\partial u}{\partial n}\,ds}_{=0} + \int_\Omega \triangle u \triangle v\,dx + \underbrace{\int_{\partial\Omega} \triangle u\frac{\partial v}{\partial n}\,ds}_{=0} - \int_\Omega \triangle v \triangle u\,dx = 0. \quad \checkmark \end{aligned}$$

b) *Denote* $|x| = r$ *and define the averages*

$$S(r) = (4\pi r^2)^{-1} \int_{|x|=r} u(x)\, ds,$$
$$V(r) = \left(\frac{4}{3}\pi r^3\right)^{-1} \int_{|x|\le r} \triangle u(x)\, dx.$$

Show that

$$\frac{d}{dr} S(r) = \frac{r}{3} V(r).$$

Hint: Rewrite $S(r)$ *as an integral over the unit sphere before differentiating; i.e.,*

$$S(r) = (4\pi)^{-1} \int_{|x'|=1} u(rx')\, dx'.$$

c) *Use the result of (b) to show that if* u *is biharmonic, i.e.* $\triangle^2 u = 0$, *then*

$$S(r) = u(0) + \frac{r^2}{6} \triangle u(0).$$

Hint: Use the mean value theorem for $\triangle u$.

b) Let $x' = x/|x|$. We have [54]

$$\begin{aligned} S(r) &= \frac{1}{4\pi r^2} \int_{|x|=r} u(x)\, dS_r = \frac{1}{4\pi r^2} \int_{|x'|=1} u(rx')\, r^2\, dS_1 = \frac{1}{4\pi} \int_{|x'|=1} u(rx')\, dS_1. \\ \frac{dS}{dr} &= \frac{1}{4\pi} \int_{|x'|=1} \frac{\partial u}{\partial r}(rx')\, dS_1 = \frac{1}{4\pi} \int_{|x'|=1} \frac{\partial u}{\partial n}(rx')\, dS_1 = \frac{1}{4\pi r^2} \int_{|x|=r} \frac{\partial u}{\partial n}(x)\, dS_r \\ &= \frac{1}{4\pi r^2} \int_{|x|\le r} \triangle u\, dx. \quad \checkmark \end{aligned}$$

where we have used Green's identity in the last equality. Also

$$\frac{r}{3} V(r) = \frac{1}{4\pi r^2} \int_{|x|\le r} \triangle u\, dx. \quad \checkmark$$

c) Since u is biharmonic (i.e. $\triangle u$ is harmonic), $\triangle u$ has a mean value property. We have

$$\begin{aligned} \frac{d}{dr} S(r) &= \frac{r}{3} V(r) = \frac{r}{3} \left(\frac{4}{3}\pi r^3\right)^{-1} \int_{|x|\le r} \triangle u(x)\, dx = \frac{r}{3} \triangle u(0), \\ S(r) &= \frac{r^2}{6} \triangle u(0) + S(0) = u(0) + \frac{r^2}{6} \triangle u(0). \end{aligned}$$

[54]**Change of variables:**
Surface integrals: $x = rx'$ in $\mathbb{R}^3$:

$$\int_{|x|=r} u(x)\, dS = \int_{|x'|=1} u(rx')\, r^2\, dS_1.$$

Volume integrals: $\xi' = r\xi$ in $\mathbb{R}^n$:

$$\int_{|\xi'|<r} h(x + \xi')\, d\xi' = \int_{|\xi|<1} h(x + r\xi)\, r^n\, d\xi.$$

□

Problem (S'00, #7). *Suppose that* $u = u(x)$ *for* $x \in \mathbb{R}^3$ *is* ***biharmonic****;* *i.e. that* $\triangle^2 u \equiv \triangle(\triangle u) = 0$. *Show that*

$$(4\pi r^2)^{-1} \int_{|x|=r} u(x)\, ds(x) \;=\; u(0) + (r^2/6)\triangle u(0)$$

through the following steps:

a) *Show that for any smooth* f,

$$\frac{d}{dr}\int_{|x|\le r} f(x)\, dx \;=\; \int_{|x|=r} f(x)\, ds(x).$$

b) *Show that for any smooth* f,

$$\frac{d}{dr}(4\pi r^2)^{-1}\int_{|x|=r} f(x)\, ds(x) \;=\; (4\pi r^2)^{-1}\int_{|x|=r} n\cdot\nabla f(x,y)\, ds$$

in which n *is the outward normal to the circle* $|x| = r$.

c) *Use step (b) to show that*

$$\frac{d}{dr}(4\pi r^2)^{-1}\int_{|x|=r} f(x)\, ds(x) \;=\; (4\pi r^2)^{-1}\int_{|x|\le r} \triangle f(x)\, dx.$$

d) *Combine steps (a) and (c) to obtain the final result.*

Proof. **a)** We can express the integral in **Spherical Coordinates**: [55]

$$\begin{aligned}
\int_{|x|\le R} f(x)\, dx &= \int_0^R\int_0^{2\pi}\int_0^{\pi} f(\phi,\theta,r)\, r^2\, \sin\phi\, d\phi\, d\theta\, dr.\\
\frac{d}{dr}\int_{|x|\le R} f(x)\, dx &= \frac{d}{dr}\int_0^R\int_0^{2\pi}\int_0^{\pi} f(\phi,\theta,r)\, r^2\, \sin\phi\, d\phi\, d\theta\, dr \;= ???\\
&= \int_0^{2\pi}\int_0^{\pi} f(\phi,\theta,r)\, R^2\, \sin\phi\, d\phi\, d\theta\\
&= \int_{|x|=R} f(x)\, dS.
\end{aligned}$$

[55] Differential **Volume** in spherical coordinates:

$$\boxed{d^3\omega \;=\; \omega^2\, \sin\phi\, d\phi\, d\theta\, d\omega.}$$

Differential **Surface Area** on sphere:

$$\boxed{dS \;=\; \omega^2\, \sin\phi\, d\phi\, d\theta.}$$

b&c) We have

$$\begin{aligned}
\frac{d}{dr}\bigg(\frac{1}{4\pi r^2}\int_{|x|=r} f(x)\,dS\bigg) &= \frac{d}{dr}\bigg(\frac{1}{4\pi r^2}\int_{|x'|=1} f(rx')\,r^2\,dS_1\bigg) = \frac{1}{4\pi}\frac{d}{dr}\bigg(\int_{|x'|=1} f(rx')\,dS_1\bigg) \\
&= \frac{1}{4\pi}\int_{|x'|=1}\frac{\partial f}{\partial r}(rx')\,dS_1 = \frac{1}{4\pi}\int_{|x'|=1}\frac{\partial f}{\partial n}(rx')\,dS_1 \\
&= \frac{1}{4\pi r^2}\int_{|x|=r}\frac{\partial f}{\partial n}(x)\,dS = \frac{1}{4\pi r^2}\int_{|x|=r}\nabla f\cdot n\,dS \quad \checkmark \\
&= \frac{1}{4\pi r^2}\int_{|x|\le r}\triangle f\,dx. \quad \checkmark
\end{aligned}$$

Green's formula was used in the last equality.

Alternatively,

$$\begin{aligned}
\frac{d}{dr}\bigg(\frac{1}{4\pi r^2}\int_{|x|=r} f(x)\,dS\bigg) &= \frac{d}{dr}\bigg(\frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi} f(\phi,\theta,r)\,r^2\sin\phi\,d\phi\,d\theta\bigg) \\
&= \frac{d}{dr}\bigg(\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\phi,\theta,r)\sin\phi\,d\phi\,d\theta\bigg) \\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{\partial f}{\partial r}(\phi,\theta,r)\sin\phi\,d\phi\,d\theta \\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\nabla f\cdot n\,\sin\phi\,d\phi\,d\theta \\
&= \frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi}\nabla f\cdot n\;r^2\sin\phi\,d\phi\,d\theta \\
&= \frac{1}{4\pi r^2}\int_{|x|=r}\nabla f\cdot n\,dS \quad \checkmark \\
&= \frac{1}{4\pi r^2}\int_{|x|=r}\triangle f\,dx. \quad \checkmark
\end{aligned}$$

d) Since f is biharmonic (i.e. $\triangle f$ is harmonic), $\triangle f$ has a mean value property. From (c), we have [56]

$$\begin{aligned}
\frac{d}{dr}\bigg(\frac{1}{4\pi r^2}\int_{|x|=r} f(x)\,ds(x)\bigg) &= \frac{1}{4\pi r^2}\int_{|x|\le r}\triangle f(x)\,dx = \frac{r}{3}\frac{1}{\frac{4}{3}\pi r^3}\int_{|x|\le r}\triangle f(x)\,dx \\
&= \frac{r}{3}\triangle f(0). \\
\frac{1}{4\pi r^2}\int_{|x|=r} f(x)\,ds(x) &= \frac{r^2}{6}\triangle f(0)+f(0).
\end{aligned}$$

$\square$

[56] Note that part (a) was not used. We use exactly the same derivation as we did in S'95 #4.

Problem (F'96, #4).
Consider smooth solutions of $\triangle u = k^2 u$ *in dimension* $d = 2$ *with* $k > 0$.
a) *Show that* u *satisfies the following 'mean value property':*

$$M_x''(r) + \frac{1}{r} M_x'(r) - k^2 M_x(r) = 0,$$

in which $M_x(r)$ *is defined by*

$$M_x(r) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\cos\theta,\ y + r\sin\theta)\, d\theta$$

and the derivatives (denoted by $'$*) are in* r *with* x *fixed.*
b) *For* $k = 1$*, this equation is the modified Bessel equation (of order 0)*

$$f'' + \frac{1}{r} f' - f = 0,$$

for which one solution (denoted as I_0*) is*

$$I_0(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{r\sin\theta}\, d\theta.$$

Find an expression for $M_x(r)$ *in terms of* I_0.

Proof. **a)** Laplacian in polar coordinates written as:

$$\triangle u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}.$$

Thus, the equation may be written as

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = k^2 u.$$

$$\begin{aligned} M_x(r) &= \frac{1}{2\pi} \int_0^{2\pi} u\, d\theta, \\ M_x'(r) &= \frac{1}{2\pi} \int_0^{2\pi} u_r\, d\theta, \\ M_x''(r) &= \frac{1}{2\pi} \int_0^{2\pi} u_{rr}\, d\theta. \end{aligned}$$

$$\begin{aligned} M_x''(r) + \frac{1}{r} M_x'(r) - k^2 M_x(r) &= \frac{1}{2\pi} \int_0^{2\pi} \Big(u_{rr} + \frac{1}{r} u_r - k^2 u\Big)\, d\theta \\ &= -\frac{1}{2\pi r^2} \int_0^{2\pi} u_{\theta\theta}\, d\theta \;=\; -\frac{1}{2\pi r^2} \big[u_\theta\big]_0^{2\pi} \;=\; 0. \quad \checkmark \end{aligned}$$

b) Note that $w = e^{r\sin\theta}$ satisfies $\triangle w = w$, i.e.

$$\begin{aligned} \triangle w &= w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta} \\ &= \sin^2\theta\, e^{r\sin\theta} + \frac{1}{r} \sin\theta\, e^{r\sin\theta} + \frac{1}{r^2}(-r\sin\theta\, e^{r\sin\theta} + r^2\cos^2\theta\, e^{r\sin\theta}) \;=\; e^{r\sin\theta} \;=\; w. \end{aligned}$$

Thus,

$$M_x(r) \;=\; e^y \frac{1}{2\pi} \int_0^{2\pi} e^{r\sin\theta}\, d\theta \;=\; e^y I_0.$$

[57] □

[57] Check with someone about the last result.

17.8 Harmonic Extensions, Subharmonic Functions

Problem (S'94, #8). *Suppose that Ω is a bounded region in $\mathbb{R}^3$ and that $u = 1$ on $\partial\Omega$. If $\triangle u = 0$ in the exterior region $\mathbb{R}^3/\Omega$ and $u(x) \to 0$ as $|x| \to \infty$, prove the following:*

a) *$u > 0$ in $\mathbb{R}^3/\Omega$;*
b) *if $\rho(x)$ is a smooth function such that $\rho(x) = 1$ for $|x| > R$ and $\rho(x) = 0$ near $\partial\Omega$, then for $|x| > R$,*

$$u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3/\Omega} \frac{(\triangle(\rho u))(y)}{|x-y|}\, dy.$$

c) $\lim_{|x|\to\infty} |x|u(x)$ *exists and is non-negative.*

Proof. **a)** Let $\overline{B}_r(0)$ denote the closed ball $\{x : |x| \geq r\}$.
Given $\varepsilon > 0$, we can find r large enough that $\Omega \in \overline{B}_{R_1}(0)$ and $\max_{x\in\partial\overline{B}_{R_1}(0)} |u(x)| < \varepsilon$, since $|u(x)| \to 0$ as $|x| \to \infty$.
Since u is harmonic in $\overline{B}_{R_1} - \Omega$, it takes its maximum and minimum on the boundary.
Assume

$$\min_{x\in\partial\overline{B}_{R_1}(0)} u(x) \;=\; -a \;<\; 0 \qquad (\text{where } \ |a| < \varepsilon).$$

We can find an R_2 such that $\max_{x\in\overline{B}_{R_2}(0)} |u(x)| < \frac{a}{2}$; hence u takes a minimum *inside* $\overline{B}_{R_2}(0) - \Omega$, which is impossible; hence $u \geq 0$.
Now let $V = \{x : u(x) \neq 0\}$ and let $\alpha = \min_{x\in V} |x|$. Since u cannot take a minimum inside $\overline{B}_R(0)$ (where $R > \alpha$), it follows that $u \equiv C$ and $C = 0$, but this contradicts $u = 1$ on $\partial\Omega$. Hence $u > 0$ in $\mathbb{R}^3 - \Omega$.

b) For $n = 3$,

$$K(|x-y|) \;=\; \frac{1}{(2-n)\omega_n}|x-y|^{2-n} \;=\; -\frac{1}{4\pi}\frac{1}{|x-y|}.$$

Since $\rho(x) = 1$ for $|x| > R$, then for $x \notin B_R$, we have $\triangle(\rho u) = \triangle u = 0$. Thus,

$$\begin{aligned}
&-\frac{1}{4\pi}\int_{\mathbb{R}^3/\Omega} \frac{(\triangle(\rho u))(y)}{|x-y|}\, dy \\
&= -\frac{1}{4\pi}\int_{B_R/\Omega} \frac{(\triangle(\rho u))(y)}{|x-y|}\, dy \\
&= \frac{1}{4\pi}\int_{B_R/\Omega} \nabla_y\Big(\frac{1}{|x-y|}\Big)\cdot\nabla_y(\rho u)\, dy - \frac{1}{4\pi}\int_{\partial(B_R/\Omega)} \frac{\partial}{\partial n}(\rho u)\frac{1}{|x-y|}\, dS_y \\
&= -\frac{1}{4\pi}\int_{B_R/\Omega} \triangle\Big(\frac{1}{|x-y|}\Big)\rho u\, dy + \frac{1}{4\pi}\int_{\partial(B_R/\Omega)} \frac{\partial}{\partial n}\Big(\frac{1}{|x-y|}\Big)\rho u\, dS_y - \frac{1}{4\pi}\int_{\partial(B_R/\Omega)} \frac{\partial}{\partial n}(\rho u)\frac{1}{|x-y|}\, dS_y \\
&= \textbf{???} \quad = u(x) - \underbrace{\frac{1}{4\pi R^2}\int_{\partial B} u\, dS_y}_{\to 0,\ as\ R\to\infty} - \underbrace{\frac{1}{4\pi R}\int_{\partial B} \frac{\partial u}{\partial n}\, dS_y}_{\to 0,\ as\ R\to\infty} \\
&= u(x).
\end{aligned}$$

c) See the next problem. ☐

Ralston Hw. ***a)*** *Suppose that u is a smooth function on $\mathbb{R}^3$ and $\triangle u = 0$ for $|x| > R$. If $\lim_{x\to\infty} u(x) = 0$, show that you can write u as a convolution of $\triangle u$ with the $-\frac{1}{4\pi|x|}$ and prove that $\lim_{x\to\infty} |x|u(x) = 0$ exists.*
b) *The "conductor potential" for $\Omega \subset \mathbb{R}^3$ is the solution to the Dirichlet problem $\triangle v = 0$. The limit in part (a) is called the "capacity" of Ω. Show that if $\Omega_1 \subset \Omega_2$, then the capacity of Ω_2 is greater or equal the capacity of Ω_1.*

Proof. **a)** If we define

$$v(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\triangle u(y)}{|x-y|}\,dy,$$

then $\triangle(u - v) = 0$ in all $\mathbb{R}^3$, and, since $v(x) \to 0$ as $|x| \to \infty$, we have $\lim_{|x|\to\infty}(u(x) - v(x)) = 0$. Thus, $u - v$ must be bounded, and Liouville's theorem implies that it is identically zero. Since we now have

$$|x|u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{|x|\,\triangle u(y)}{|x-y|}\,dy,$$

and $|x|/|x-y|$ converges uniformly to 1 on $\{|y| \le R\}$, it follows that

$$\lim_{|x|\to\infty} |x|u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \triangle u(y)\,dy.$$

b) Note that part (a) implies that the limit $\lim_{|x|\to\infty} |x|v(x)$ exists, because we can apply (a) to $u(x) = \phi(x)v(x)$, where ϕ is smooth and vanishes on Ω, but $\phi(x) = 1$ for $|x| > R$.
Let v_1 be the conductor potential for Ω_1 and v_2 for Ω_2. Since $v_i \to \infty$ as $|x| \to \infty$ and $v_i = 1$ on $\partial\Omega_i$, the max principle says that $1 > v_i(x) > 0$ for $x \in \mathbb{R}^3 - \Omega_i$. Consider $v_2 - v_1$. Since $\Omega_1 \subset \Omega_2$, this is defined in $\mathbb{R}^3 - \Omega_2$, positive on $\partial\Omega_2$, and has limit 0 as $|x| \to \infty$. Thus, it must be positive in $\mathbb{R}^3 - \Omega_2$. Thus, $\lim_{|x|\to\infty} |x|(v_2 - v_1) \ge 0$. □

Problem (F'95, #4). [58] *Let Ω be a simply connected open domain in $\mathbb{R}^2$ and $u = u(x, y)$ be* ***subharmonic*** *there, i.e. $\triangle u \ge 0$ in Ω. Prove that if*

$$D_R = \{(x, y) : (x - x_0)^2 + (y - y_0)^2 \le R^2\} \subset \Omega$$

then

$$u(x_0, y_0) \le \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos\theta,\, y_0 + R\sin\theta)\,d\theta.$$

Proof. Let

$$\begin{aligned} M(x_0, R) &= \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos\theta,\, y_0 + R\sin\theta)\,d\theta, \\ w(r, \theta) &= u(x_0 + R\cos\theta,\, y_0 + R\sin\theta). \end{aligned}$$

Differentiate $M(x_0, R)$ with respect to R:

$$\frac{d}{dr}M(x_0, R) = \frac{1}{2\pi R}\int_0^{2\pi} w_r(R, \theta) R\,d\theta,$$

[58] See McOwen, Sec.4.3, p.131, #1.

[59] $\square$

[59] See ChiuYen's solutions and Sung Ha's solutions (in two places). Nick's solutions, as started above, have a very simplistic approach.

Ralston Hw (Maximum Principle).
Suppose that $u \in C(\Omega)$ satisfies the mean value property in the connected open set Ω.

a) *Show that u satisfies the maximum principle in Ω, i.e.*
either u is constant or $u(x) < \sup_\Omega u$ for all $x \in \Omega$.

b) *Show that, if v is a continuous function on a closed ball $B_r(\xi) \subset \Omega$ and has the mean value property in $B_r(\xi)$, then $u = v$ on $\partial B_r(\xi)$ implies $u = v$ in $B_r(\xi)$. Does this imply that u is harmonic in Ω?*

Proof. **a)** If $u(x)$ is not less than $\sup_\Omega u$ for all $x \in \Omega$, then the set

$$K = \{x \in \Omega : \ u(x) = \sup_\Omega u\}$$

is nonempty. This set is closed because u is continuous. We will show it is also open. This implies that $K = \Omega$ because Ω is connected. Thus u is constant on Ω.
Let $x_0 \in K$. Since Ω is open, $\exists \delta > 0$, s.t. $B_\delta(x_0) = \{x \in \mathbb{R}^n : \ |x - x_0| \le \delta\} \subset \Omega$. Let $\sup_\Omega u = M$. By the mean value property, for $0 \le r \le \delta$

$$M = u(x_0) = \frac{1}{A(S^{n-1})} \int_{|\xi|=1} u(x_0 + r\xi) dS_\xi, \text{ and } 0 = \frac{1}{A(S^{n-1})} \int_{|\xi|=1} (M - u(x_0 + r\xi)) dS_\xi.$$

Sinse $M - u(x_0 + r\xi)$ is a continuous nonnegative function on ξ, this implies $M - u(x_0 + r\xi) = 0$ for all $\xi \in S^{n-1}$. Thus $u = 0$ on $B_\delta(x_0)$.

b) Since $u - v$ has the mean value property in the open interior of $B_r(\xi)$, by part *a*) it satisfies the maximum principle. Since it is continuous on $B_r(\xi)$, its supremum over the interior of $B_r(\xi)$ is its maximum on $B_r(\xi)$, and this maximum is assumed at a point x_0 in $B_r(\xi)$. If x_0 in the interior of $B_r(\xi)$, then $u - v$ is constant ant the constant must be zero, since this is the value of $u - v$ on the boundary. If x_0 is on the boundary, then $u - v$ must be nonpositive in the interior of $B_r(\xi)$.
Applying the same argument to $v - u$, one finds that it is either identically zero or nonpositive in the interior of $B_r(\xi)$. Thus, $u - v \equiv 0$ on $B_r(\xi)$.
Yes, it does follow that u harmonic in Ω. Take v in the preceding to be the harmonic function in the interior of $B_r(\xi)$ which agrees with u on the boundary. Since $u = v$ on $B_r(\xi)$, u is harmonic in the interior of $B_r(\xi)$. Since Ω is open we can do this for every $\xi \in \Omega$. Thus u is harmonic in Ω. ☐

Ralston Hw. *Assume Ω is a bounded open set in $\mathbb{R}^n$ and the Green's function, $G(x, y)$, for Ω exists. Use the strong maximum principle, i.e. either $u(x) < \sup_\Omega u$ for all $x \in \Omega$, or u is constant, to prove that $G(x, y) < 0$ for $x, y \in \Omega$, $x \neq y$.*

Proof. $G(x, y) = K(x, y) + \omega(x, y)$. For each $x \in \Omega$, $f(y) = \omega(x, y)$ is continuous on $\overline{\Omega}$, thus, bounded. So $|\omega(x, y)| \le M_x$ for all $y \in \overline{\Omega}$. $K(x - y) \to -\infty$ as $y \to x$. Thus, given M_x, there is $\delta > 0$, such that $K(x - y) < -M_x$ when $|x - y| = r$ and $0 < r \le \delta$. So for $0 < r \le \delta$ the Green's function with x fixed satisfies, $G(x, y)$ is harmonic on $\Omega - B_r(x)$, and $G(x, y) \le 0$ on the boundary of $\Omega - B_r(x)$. Since we can choose r as small as we wish, we get $G(x, y) < 0$ for $y \in \Omega - \{x\}$. ☐

Problem (W'03, #6). *Assume that u is a **harmonic** function in the **half ball** $D = \{(x, y, z) : x^2 + y^2 + z^2 < 1,\ z \ge 0\}$ which is continuously differentiable, and satisfies $u(x, y, 0) = 0$. Show that u can be extended to be a harmonic function in the whole ball. If you propose and explicit extension for u, explain why the extension is harmonic.*

Proof. We can extend u to all of n-space by defining

$$u(x', x_n) = -u(x', -x_n)$$

for $x_n < 0$. Define

$$\omega(x) \;=\; \frac{1}{a\omega_n} \int_{|y|=1} \frac{a^2 - |x|^2}{|x - y|^n} v(y)\, dS_y$$

$\omega(x)$ is continuous on a closed ball B, harmonic in B.
Poisson kernel is symmetric in y at $x_n = 0$. $\Rightarrow$ $\omega(x) = 0$, $(x_n = 0)$.
ω is harmonic for $x \in B$, $x_n \ge 0$,with the same boundary values $\omega = u$.
ω is harmonic $\Rightarrow$ u can be extended to a harmonic function on the interior of B. ☐

Ralston Hw. *Show that a **bounded** solution to the **Dirichlet** problem in a **half space** is **unique**. (Note that one can show that a bounded solution **exists** for any given bounded continuous Dirichlet data by using the Poisson kernel for the half space.)*

Proof. We have to show that a function, u, which is harmonic in the half-space, continuous, equal to 0 when $x_n = 0$, and bounded, must be identically 0. We can extend u to all of n-space by defining

$$u(x', x_n) = -u(x', -x_n)$$

for $x_n < 0$. This extends u to a bounded harmonic function on all of n-space (by the problem above). Liouville's theorem says u must be constant, and since $u(x', 0) = 0$, the constant is 0. So the original u must be identically 0. ☐

Ralston Hw. *Suppose u is harmonic on the **ball minus the origin**, $B_0 = \{x \in \mathbb{R}^3 : 0 < |x| < a\}$. Show that $u(x)$ can be **extended** to a harmonic function on the ball $B = \{|x| < a\}$ iff $\lim_{|x|\to 0} |x|u(x) = 0$.*

Proof. The condition $\lim_{|x|\to 0} |x|u(x) = 0$ is necessary, because harmonic functions are continuous.
To prove the converse, let v be the function which is continuous on $\{|x| \le a/2\}$, harmonic on $\{|x| < a/2\}$, and equals u on $\{|x| = a/2\}$. One can construct v using the Poisson kernel. Since v is continuous, it is bounded, and we can assume that $|v| \le M$. Since $\lim_{|x|\to 0} |x|u(x) = 0$, given $\epsilon > 0$, we can choose δ, $0 < \delta < a/2$ such that $-\epsilon < |x|u(x) < \epsilon$ when $|x| < \delta$. Note that u, $v - 2\epsilon/|x|$, and $v + 2\epsilon/|x|$ are harmonic

on $\{0 < |x| < a/2\}$. Choose b, $0 < b < \min(\epsilon, a/2)$, so that $\epsilon/b > M$. Then on both $\{|x| = a/2\}$ and $\{|x| = b\}$ we have $v - 2\epsilon/|x| < u(x) < v + 2\epsilon/|x|$. Thus, by max principle these inequalities hold on $\{b \leq |x| \leq a/2\}$. Pick x with $0 < |x| \leq a/2$. $u(x) = v(x)$. v is the extension of u on $\{|x| < a/2\}$, and u is extended on $\{|x| < a\}$. $\square$

18 Problems: Heat Equation

McOwen 5.2 #7(a). *Consider*

$$\begin{cases} u_t = u_{xx} & for \;\; x > 0,\, t > 0 \\ u(x,0) = g(x) & for \;\; x > 0 \\ u(0,t) = 0 & for \;\; t > 0, \end{cases}$$

where g is continuous and bounded for $x \geq 0$ and $g(0) = 0$.
Find a formula for the solution $u(x,t)$.

Proof. Extend g to be an **odd** function on all of $\mathbb{R}$:

$$\tilde{g}(x) = \begin{cases} g(x), & x \geq 0 \\ -g(-x), & x < 0. \end{cases}$$

Then, we need to solve

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} & \text{for} \;\; x \in \mathbb{R},\, t > 0 \\ \tilde{u}(x,0) = \tilde{g}(x) & \text{for} \;\; x \in \mathbb{R}. \end{cases}$$

The solution is given by: [60]

$$\begin{aligned} \tilde{u}(x,t) &= \int_{\mathbb{R}} K(x,y,t) g(y)\,dy = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \tilde{g}(y)\,dy \\ &= \frac{1}{\sqrt{4\pi t}} \left[\int_0^{\infty} e^{-\frac{(x-y)^2}{4t}} \tilde{g}(y)\,dy + \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4t}} \tilde{g}(y)\,dy \right] \\ &= \frac{1}{\sqrt{4\pi t}} \left[\int_0^{\infty} e^{-\frac{(x-y)^2}{4t}} g(y)\,dy - \int_0^{\infty} e^{-\frac{(x+y)^2}{4t}} g(y)\,dy \right] \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \left(e^{\frac{-x^2+2xy-y^2}{4t}} - e^{\frac{-x^2-2xy-y^2}{4t}} \right) g(y)\,dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} e^{-\frac{(x^2+y^2)}{4t}} \left(e^{\frac{xy}{2t}} - e^{-\frac{xy}{2t}} \right) g(y)\,dy. \end{aligned}$$

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} e^{-\frac{(x^2+y^2)}{4t}}\, 2\sinh\left(\frac{xy}{2t}\right) g(y)\,dy.}$$

Since $\sinh(0) = 0$, we can verify that $u(0,t) = 0$. ☐

[60] In calculations, we use: $\int_{-\infty}^{0} e^{y}\,dy = \int_0^{\infty} e^{-y}\,dy$, and $g(-y) = -g(y)$.

McOwen 5.2 #7(b). *Consider*

$$\begin{cases} u_t = u_{xx} & \text{for } x > 0,\ t > 0 \\ u(x,0) = g(x) & \text{for } x > 0 \\ u_x(0,t) = 0 & \text{for } t > 0, \end{cases}$$

where g is continuous and bounded for $x \geq 0$.
Find a formula for the solution $u(x,t)$.

Proof. Extend g to be an **even** function [61] on all of $\mathbb{R}$:

$$\tilde{g}(x) = \begin{cases} g(x), & x \geq 0 \\ g(-x), & x < 0. \end{cases}$$

Then, we need to solve

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} & \text{for } x \in \mathbb{R},\ t > 0 \\ \tilde{u}(x,0) = \tilde{g}(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

The solution is given by: [62]

$$\begin{aligned} \tilde{u}(x,t) &= \int_{\mathbb{R}} K(x,y,t)g(y)\,dy = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \tilde{g}(y)\,dy \\ &= \frac{1}{\sqrt{4\pi t}} \left[\int_0^{\infty} e^{-\frac{(x-y)^2}{4t}} \tilde{g}(y)\,dy + \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4t}} \tilde{g}(y)\,dy \right] \\ &= \frac{1}{\sqrt{4\pi t}} \left[\int_0^{\infty} e^{-\frac{(x-y)^2}{4t}} g(y)\,dy + \int_0^{\infty} e^{-\frac{(x+y)^2}{4t}} g(y)\,dy \right] \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \left(e^{\frac{-x^2+2xy-y^2}{4t}} + e^{\frac{-x^2-2xy-y^2}{4t}} \right) g(y)\,dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} e^{-\frac{(x^2+y^2)}{4t}} \left(e^{\frac{xy}{2t}} + e^{-\frac{xy}{2t}} \right) g(y)\,dy. \end{aligned}$$

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} e^{-\frac{(x^2+y^2)}{4t}}\, 2\cosh\left(\frac{xy}{2t}\right) g(y)\,dy.}$$

To check that the boundary condition holds, we perform the calculation:

$$\begin{aligned} u_x(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \frac{d}{dx}\left[e^{-\frac{(x^2+y^2)}{4t}}\, 2\cosh\left(\frac{xy}{2t}\right)\right] g(y)\,dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \left[-\frac{2x}{4t} e^{-\frac{(x^2+y^2)}{4t}}\, 2\cosh\left(\frac{xy}{2t}\right) + e^{-\frac{(x^2+y^2)}{4t}}\, 2\frac{y}{2t}\sinh\left(\frac{xy}{2t}\right)\right] g(y)\,dy, \\ u_x(0,t) &= \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \left[0 \cdot e^{-\frac{y^2}{4t}}\, 2\cosh 0 + e^{-\frac{y^2}{4t}}\, 2\frac{y}{2t}\sinh 0 \right] g(y)\,dy = 0. \end{aligned}$$

□

[61] Even extensions are always continuous. Not true for odd extensions. g odd is continuous if $g(0) = 0$.

[62] In calculations, we use: $\int_{-\infty}^{0} e^{y}\,dy = \int_0^{\infty} e^{-y}\,dy$, and $g(-y) = g(y)$.

Problem (F'90, #5).
The initial value problem for the heat equation on the whole real line is

$$f_t = f_{xx} \qquad t \geq 0$$
$$f(t=0,x) = f_0(x)$$

with f_0 smooth and bounded.
a) *Write down the* ***Green's function*** *$G(x,y,t)$ for this initial value problem.*
b) *Write the solution $f(x,t)$ as an integral involving G and f_0.*
c) *Show that the maximum values of $|f(x,t)|$ and $|f_x(x,t)|$ are non-increasing as t increases, i.e.*

$$\sup_x |f(x,t)| \leq \sup_x |f_0(x)| \qquad \sup_x |f_x(x,t)| \leq \sup_x |f_{0x}(x)|.$$

When are these inequalities actually equalities?

Proof. **a)** The fundamental solution

$$K(x,y,t) \;=\; \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}}.$$

The Green's function is: [63]

$$G(x,t;y,s) \;=\; \frac{1}{(2\pi)^n}\left[\frac{\pi}{k(t-s)}\right]^{\frac{n}{2}} e^{-\frac{(x-y)^2}{4k(t-s)}}.$$

b) The solution to the one-dimensional heat equation is

$$u(x,t) \;=\; \int_{\mathbb{R}} K(x,y,t)\, f_0(y)\, dy \;=\; \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} f_0(y)\, dy.$$

c) We have

$$\begin{aligned}
\sup_x |u(x,t)| \;&=\; \left|\frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f_0(y)\, dy\right| \;\leq\; \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} \big|f_0(y)\big|\, dy \\
&=\; \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{y^2}{4t}} \big|f_0(x-y)\big|\, dy \\
&\leq\; \sup_x |f_0(x)| \,\frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{y^2}{4t}}\, dy \qquad\qquad \Big(\, z = \frac{y}{\sqrt{4t}}, \;\; dz = \frac{dy}{\sqrt{4t}} \,\Big) \\
&\leq\; \sup_x |f_0(x)| \,\frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-z^2} \sqrt{4t}\, dz \\
&=\; \sup_x |f_0(x)| \,\frac{1}{\sqrt{\pi}} \underbrace{\int_{\mathbb{R}} e^{-z^2}\, dz}_{=\sqrt{\pi}} \;=\; \sup_x |f_0(x)|. \quad \checkmark
\end{aligned}$$

[63] The Green's function for the heat equation on an infinite domain; derived in R. Haberman using the Fourier transform.

$$\begin{aligned}
u_x(x,t) &= \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} -\frac{2(x-y)}{4t}e^{-\frac{(x-y)^2}{4t}} f_0(y)\,dy \;=\; \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} -\frac{d}{dy}\Big[e^{-\frac{(x-y)^2}{4t}}\Big]\, f_0(y)\,dy \\
&= \underbrace{\frac{1}{\sqrt{4\pi t}}\Big[-e^{-\frac{(x-y)^2}{4t}}\, f_0(y)\Big]_{-\infty}^{\infty}}_{=0} + \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}}\, f_{0y}(y)\,dy, \\
\sup_x |u(x,t)| &\le \frac{1}{\sqrt{4\pi t}} \sup_x |f_{0x}(x)| \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}}\,dy \;=\; \frac{1}{\sqrt{4\pi t}} \sup_x |f_{0x}(x)| \int_{\mathbb{R}} e^{-z^2}\sqrt{4t}\,dz \\
&= \sup_x |f_{0x}(x)|. \quad \checkmark
\end{aligned}$$

These inequalities are equalities when $f_0(x)$ and $f_{0x}(x)$ are constants, respectively. $\square$

Problem (S'01, #5). ***a)*** *Show that the solution of the heat equation*

$$u_t = u_{xx}, \qquad -\infty < x < \infty$$

with square-integrable initial data $u(x,0) = f(x)$, decays in time, and there is a constant α independent of f and t such that for all $t > 0$

$$\max_x |u_x(x,t)| \le \alpha t^{-\frac{3}{4}} \Big(\int_x |f(x)|^2\, dx \Big)^{\frac{1}{2}}.$$

b) *Consider the solution ρ of the transport equation $\rho_t + u\rho_x = 0$ with square-integrable initial data $\rho(x,0) = \rho_0(x)$ and the velocity u from part (a). Show that $\rho(x,t)$ remains square-integrable for all finite time*

$$\int_R |\rho(x,t)|^2\, dx \le e^{Ct^{\frac{1}{4}}} \int_R |\rho_0(x)|^2\, dx,$$

where C does not depend on ρ_0.

Proof. **a)** The solution to the one-dimensional homogeneous heat equation is

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y)\, dy.$$

Take the derivative with respect to x, we get [64]

$$u_x(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} -\frac{2(x-y)}{4t} e^{-\frac{(x-y)^2}{4t}} f(y)\, dy = -\frac{1}{4t^{\frac{3}{2}}\sqrt{\pi}} \int_{\mathbb{R}} (x-y) e^{-\frac{(x-y)^2}{4t}} f(y)\, dy.$$

$$\begin{aligned}
|u_x(x,t)| &\le \frac{1}{4t^{\frac{3}{2}}\sqrt{\pi}} \int_{\mathbb{R}} \Big| (x-y) e^{-\frac{(x-y)^2}{4t}} f(y) \Big|\, dy \qquad \text{(Cauchy-Schwarz)} \\
&\le \frac{1}{4t^{\frac{3}{2}}\sqrt{\pi}} \Big(\int_{\mathbb{R}} (x-y)^2 e^{-\frac{(x-y)^2}{2t}}\, dy \Big)^{\frac{1}{2}} ||f||_{L^2(\mathbb{R})} \qquad \Big(z = \frac{x-y}{\sqrt{2t}},\ dz = -\frac{dy}{\sqrt{2t}} \Big) \\
&= \frac{1}{4t^{\frac{3}{2}}\sqrt{\pi}} \Big(\int_{\mathbb{R}} \big| - z^2 (2t)^{\frac{3}{2}} e^{-z^2} \big| dz \Big)^{\frac{1}{2}} ||f||_{L^2(\mathbb{R})} \\
&= \frac{(2t)^{\frac{3}{4}}}{4t^{\frac{3}{2}}\sqrt{\pi}} \Big(\underbrace{\int_{\mathbb{R}} z^2 e^{-z^2}\, dz}_{M<\infty} \Big)^{\frac{1}{2}} ||f||_{L^2(\mathbb{R})} \\
&= Ct^{-\frac{3}{4}} M^{\frac{1}{2}} ||f||_{L^2(\mathbb{R})} \;=\; \alpha t^{-\frac{3}{4}} ||f||_{L^2(\mathbb{R})}.
\end{aligned}$$

b) Note:

$$\begin{aligned}
\max_x |u| &= \max_x \Big| \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y)\, dy \Big| \;\le\; \frac{1}{\sqrt{4\pi t}} \Big(\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{2t}}\, dy \Big)^{\frac{1}{2}} ||f||_{L^2(\mathbb{R})} \\
&\le \frac{1}{\sqrt{4\pi t}} \Big(\int_{\mathbb{R}} \big| - e^{-z^2} \sqrt{2t} \big|\, dz \Big)^{\frac{1}{2}} ||f||_{L^2(\mathbb{R})} \qquad \Big(z = \frac{x-y}{\sqrt{2t}},\ dz = -\frac{dy}{\sqrt{2t}} \Big) \\
&= \frac{(2t)^{\frac{1}{4}}}{2\pi^{\frac{1}{2}} t^{\frac{1}{2}}} \Big(\underbrace{\int_{\mathbb{R}} e^{-z^2}\, dz}_{=\sqrt{\pi}} \Big)^{\frac{1}{2}} ||f||_{L^2(\mathbb{R})} \;=\; Ct^{-\frac{1}{4}} ||f||_{L^2(\mathbb{R})}.
\end{aligned}$$

65

□

[64] Cauchy-Schwarz: $|(u,v)| \le ||u||\,||v||$ in any norm, for example $\int |uv| dx \le (\int u^2 dx)^{\frac{1}{2}} (\int v^2 dx)^{\frac{1}{2}}$

[65] See Yana's and Alan's solutions.

Problem (F'04, #2).
Let $u(x,t)$ be a bounded solution to the Cauchy problem for the heat equation

$$\begin{cases} u_t = a^2 u_{xx}, & t>0,\ x\in\mathbb{R},\ a>0, \\ u(x,0) = \varphi(x). \end{cases}$$

Here $\varphi(x) \in C(\mathbb{R})$ satisfies

$$\lim_{x\to+\infty} \varphi(x) = b, \qquad \lim_{x\to-\infty} \varphi(x) = c.$$

Compute the limit of $u(x,t)$ as $t\to+\infty$, $x\in\mathbb{R}$. Justify your argument carefully.

Proof. For $a=1$, the solution to the one-dimensional homogeneous heat equation is

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} \varphi(y)\,dy.$$

We want to transform the equation to $v_t = v_{xx}$. Make a change of variables: $x = ay$. $u(x,t) = u(x(y),t) = u(ay,t) = v(y,t)$. Then,

$$\begin{aligned} v_y &= u_x x_y = a u_x, \\ v_{yy} &= a u_{xx} x_y = a^2 u_{xx}, \\ v(y,0) &= u(ay,0) = \varphi(ay). \end{aligned}$$

Thus, the new problem is:

$$\begin{cases} v_t = v_{yy}, & t>0,\ y\in\mathbb{R}, \\ v(y,0) = \varphi(ay). \end{cases}$$

$$v(y,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(y-z)^2}{4t}} \varphi(az)\,dz. \qquad \circledast$$

Since φ is continuous, and $\lim_{x\to+\infty}\varphi(x) = b$, $\lim_{x\to-\infty}\varphi(x) = c$, we have

$|\varphi(x)| < M, \quad \forall x\in\mathbb{R}.$ Thus,

$$\begin{aligned} |v(y,t)| &\le \frac{M}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{z^2}{4t}}\,dz \qquad \Big(s = \frac{z}{\sqrt{4t}},\ ds = \frac{dz}{\sqrt{4t}}\Big) \\ &= \frac{M}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-s^2}\sqrt{4t}\,ds = \frac{M}{\sqrt{\pi}} \underbrace{\int_{\mathbb{R}} e^{-s^2}\,ds}_{\sqrt{\pi}} = M. \end{aligned}$$

Integral in $\circledast$ converges uniformly $\Rightarrow$ $\lim\int = \int\lim$. For $\psi = \varphi(a\cdot)$:

$$\begin{aligned} v(y,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(y-z)^2}{4t}} \psi(z)\,dz = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4t}} \psi(y-z)\,dz \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-s^2}\, \psi(y - s\sqrt{4t})\,\sqrt{4t}\,ds \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2}\, \psi(y - s\sqrt{4t})\,ds. \end{aligned}$$

$$\begin{aligned}
\lim_{t\to+\infty} v(y,t) &= \frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-s^2} \lim_{t\to+\infty} \psi(y - s\sqrt{4t})\, ds + \frac{1}{\sqrt{\pi}}\int_{-\infty}^{0} e^{-s^2} \lim_{t\to+\infty} \psi(y - s\sqrt{4t})\, ds \\
&= \frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-s^2} c\, ds + \frac{1}{\sqrt{\pi}}\int_{-\infty}^{0} e^{-s^2} b\, ds \;=\; c\frac{1}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} + b\frac{1}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} \\
&= \frac{c+b}{2}.
\end{aligned}$$

$\square$

Problem. *Consider*

$$\begin{aligned} u_t &= ku_{xx} + Q, \qquad 0 < x < 1 \\ u(0,t) &= 0, \\ u(1,t) &= 1. \end{aligned}$$

What is the steady state temperature?

Proof. Set $u_t = 0$, and integrate with respect to x twice:

$$\begin{aligned} ku_{xx} + Q = 0, \\ u_{xx} = -\frac{Q}{k}, \\ u_x = -\frac{Q}{k}x + a, \\ u = -\frac{Q}{k}\frac{x^2}{2} + ax + b. \end{aligned}$$

Boundary conditions give

$$u(x) = -\frac{Q}{2k}x^2 + \left(1 + \frac{Q}{2k}\right)x.$$

□

18.1 Heat Equation with Lower Order Terms

McOwen 5.2 #11. *Find a formula for the solution of*

$$\begin{cases} u_t = \triangle u - cu & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{18.1}$$

Show that such solutions, with initial data $g \in L^2(\mathbb{R}^n)$, are unique, even when c is negative.

Proof. **McOwen.** Consider $v(x,t) = e^{ct}u(x,t)$. The transformed problem is

$$\begin{cases} v_t = \triangle v & \text{in } \mathbb{R}^n \times (0,\infty) \\ v(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{18.2}$$

Since g is continuous and bounded in $\mathbb{R}^n$, we have

$$\begin{aligned} v(x,t) &= \int_{\mathbb{R}^n} K(x,y,t)\, g(y)\, dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy, \\ u(x,t) &= e^{-ct} v(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t} - ct}\, g(y)\, dy. \end{aligned}$$

$u(x,t)$ is a **bounded** solution since $v(x,t)$ is.
To prove uniqueness, assume there is another solution v' of (18.2). $w = v - v'$ satisfies

$$\begin{cases} w_t = \triangle w & \text{in } \mathbb{R}^n \times (0,\infty) \\ w(x,0) = 0 & \text{on } \mathbb{R}^n. \end{cases} \tag{18.3}$$

Since bounded solutions of (18.3) are unique, and since w is a nontrivial solution, w is unbounded. Thus, v' is unbounded, and therefore, the bounded solution v is unique. □

18.1.1 Heat Equation Energy Estimates

Problem (F'94, #3). *Let $u(x, y, t)$ be a twice continuously differential solution of*

$$\begin{aligned} u_t &= \triangle u - u^3 && \text{in } \Omega \subset \mathbb{R}^2,\ t \geq 0 \\ u(x,y,0) &= 0 && \text{in } \Omega \\ u(x,y,t) &= 0 && \text{in } \partial\Omega,\ t \geq 0. \end{aligned}$$

Prove that $u(x,y,t) \equiv 0$ *in* $\Omega \times [0,T]$.

Proof. Multiply the equation by u and integrate:

$$\begin{aligned} uu_t &= u\triangle u - u^4, \\ \int_\Omega uu_t\,dx &= \int_\Omega u\triangle u\,dx - \int_\Omega u^4\,dx, \\ \frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx &= \underbrace{\int_{\partial\Omega} u\frac{\partial u}{\partial n}\,ds}_{=0} - \int_\Omega |\nabla u|^2\,dx - \int_\Omega u^4\,dx, \\ \frac{1}{2}\frac{d}{dt}||u||_2^2 &= -\int_\Omega |\nabla u|^2\,dx - \int_\Omega u^4\,dx \ \leq\ 0. \end{aligned}$$

Thus,

$$||u(x,y,t)||_2 \ \leq\ ||u(x,y,0)||_2 \ =\ 0.$$

Hence, $||u(x,y,t)||_2 = 0$, and $u \equiv 0$. □

Problem (F'98, #5). *Consider the heat equation*

$$u_t - \triangle u = 0$$

in a two dimensional region Ω*. Define the mass* M *as*

$$M(t) = \int_\Omega u(x,t)\,dx.$$

a) *For a fixed domain* Ω*, show* M *is a constant in time if the boundary conditions are* $\partial u/\partial n = 0$.
b) *Suppose that* $\Omega = \Omega(t)$ *is evolving in time, with a boundary that moves at velocity* v*, which may vary along the boundary. Find a modified boundary condition (in terms of local quantities only) for* u*, so that* M *is constant.*
Hint: You may use the fact that

$$\frac{d}{dt}\int_{\Omega(t)} f(x,t)\,dx = \int_{\Omega(t)} f_t(x,t)\,dx + \int_{\partial\Omega(t)} n\cdot v\,f(x,t)\,dl,$$

in which n *is a unit normal vector to the boundary* $\partial\Omega$*.*

Proof. **a)** We have

$$\begin{cases} u_t - \triangle u = 0, & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases}$$

We want to show that $\frac{d}{dt}M(t) = 0$. We have [66]

$$\frac{d}{dt}M(t) = \frac{d}{dt}\int_\Omega u(x,t)\,dx = \int_\Omega u_t\,dx = \int_\Omega \triangle u\,dx = \int_{\partial\Omega}\frac{\partial u}{\partial n}\,ds = 0. \quad \checkmark$$

b) We need $\frac{d}{dt}M(t) = 0$.

$$\begin{aligned} 0 = \frac{d}{dt}M(t) &= \frac{d}{dt}\int_{\Omega(t)} u(x,t)\,dx = \int_{\Omega(t)} u_t\,dx + \int_{\partial\Omega(t)} n\cdot v\,u\,ds \\ &= \int_{\Omega(t)} \triangle u\,dx + \int_{\partial\Omega(t)} n\cdot v\,u\,ds = \int_{\partial\Omega(t)} \frac{\partial u}{\partial n}\,ds + \int_{\partial\Omega(t)} n\cdot v\,u\,ds \\ &= \int_{\partial\Omega(t)} \nabla u\cdot n\,ds + \int_{\partial\Omega(t)} n\cdot v\,u\,ds = \int_{\partial\Omega(t)} n\cdot(\nabla u + vu)\,ds. \end{aligned}$$

Thus, we need:

$$n\cdot(\nabla u + vu)\,ds = 0, \qquad \text{on } \partial\Omega.$$

□

[66]The last equality below is obtained from the Green's formula:

$$\int_\Omega \triangle u\,dx = \int_\Omega \frac{\partial u}{\partial n}\,ds.$$

Problem (S'95, #3). *Write down an explicit formula for a function $u(x,t)$ solving*

$$\begin{cases} u_t + b \cdot \nabla u + cu = \triangle u & \textit{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = f(x) & \textit{on } \mathbb{R}^n. \end{cases} \tag{18.4}$$

where $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$ are constants.
*Hint: First transform this to the **heat equation** by a linear change of the dependent and independent variables. Then solve the heat equation using the fundamental solution.*

Proof. Consider

- $u(x,t) \;=\; e^{\alpha\cdot x+\beta t} v(x,t).$

$$\begin{aligned} u_t &= \beta e^{\alpha\cdot x+\beta t} v + e^{\alpha\cdot x+\beta t} v_t = (v_t + \beta v) e^{\alpha\cdot x+\beta t}, \\ \nabla u &= \alpha e^{\alpha\cdot x+\beta t} v + e^{\alpha\cdot x+\beta t} \nabla v = (\alpha v + \nabla v) e^{\alpha\cdot x+\beta t}, \\ \nabla\cdot(\nabla u) &= \nabla \cdot \big((\alpha v + \nabla v) e^{\alpha\cdot x+\beta t}\big) = (\alpha\cdot\nabla v + \triangle v) e^{\alpha\cdot x+\beta t} + (|\alpha|^2 v + \alpha\cdot\nabla v) e^{\alpha\cdot x+\beta t} \\ &= \big(\triangle v + 2\alpha\cdot\nabla v + |\alpha|^2 v\big) e^{\alpha\cdot x+\beta t}. \end{aligned}$$

Plugging this into (18.4), we obtain

$$\begin{aligned} &v_t + \beta v + b\cdot(\alpha v + \nabla v) + cv = \triangle v + 2\alpha\cdot\nabla v + |\alpha|^2 v, \\ &v_t + \big(b - 2\alpha\big)\cdot\nabla v + \big(\beta + b\cdot\alpha + c - |\alpha|^2\big) v = \triangle v. \end{aligned}$$

In order to get homogeneous heat equation, we set

$$\alpha = \frac{b}{2}, \qquad \beta = -\frac{|b|^2}{4} - c,$$

which gives

$$\begin{cases} v_t = \triangle v & \text{in } \mathbb{R}^n \times (0,\infty) \\ v(x,0) = e^{-\frac{b}{2}\cdot x} f(x) & \text{on } \mathbb{R}^n. \end{cases}$$

The above PDE has the following solution:

$$v(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} e^{-\frac{b}{2}\cdot y} f(y)\, dy.$$

Thus,

$$\boxed{u(x,t) = e^{\frac{b}{2}\cdot x - (\frac{|b|^2}{4}+c)t} v(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{b}{2}\cdot x - (\frac{|b|^2}{4}+c)t} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} e^{-\frac{b}{2}\cdot y} f(y)\, dy.}$$

□

Problem (F'01, #7). *Consider the parabolic problem*

$$u_t = u_{xx} + c(x)u \tag{18.5}$$

for $-\infty < x < \infty$, *in which*

$$\begin{aligned} c(x) &= 0 \quad \textit{for} \quad |x| > 1, \\ c(x) &= 1 \quad \textit{for} \quad |x| < 1. \end{aligned}$$

Find solutions of the form $u(x,t) = e^{\lambda t}v(x)$ *in which* $\int_{-\infty}^{\infty} |u|^2\,dx < \infty$.
Hint: Look for v *to have the form*

$$\begin{aligned} v(x) &= ae^{-k|x|} \quad \textit{for} \quad |x| > 1, \\ v(x) &= b\cos lx \quad \textit{for} \quad |x| < 1, \end{aligned}$$

for some a, b, k, l.

Proof. Plug $u(x,t) = e^{\lambda t}v(x)$ into (18.5) to get:

$$\begin{aligned} \lambda e^{\lambda t}v(x) &= e^{\lambda t}v''(x) + ce^{\lambda t}v(x), \\ \lambda v(x) &= v''(x) + cv(x), \\ v''(x) - \lambda v(x) + cv(x) &= 0. \end{aligned}$$

- For $|x| > 1$, $c = 0$. We look for solutions of the form $v(x) = ae^{-k|x|}$.

$$\begin{aligned} v''(x) - \lambda v(x) &= 0, \\ ak^2e^{-k|x|} - a\lambda e^{-k|x|} &= 0, \\ k^2 - \lambda &= 0, \\ k^2 &= \lambda, \\ k &= \pm\sqrt{\lambda}. \end{aligned}$$

Thus, $v(x) = c_1e^{-\sqrt{\lambda}x} + c_2e^{\sqrt{\lambda}x}$. Since we want $\int_{-\infty}^{\infty} |u|^2\,dx < \infty$:

$$u(x,t) = ae^{\lambda t}e^{-\sqrt{\lambda}x}.$$

- For $|x| < 1$, $c = 1$. We look for solutions of the form $v(x) = b\cos lx$.

$$\begin{aligned} v''(x) - \lambda v(x) + v(x) &= 0, \\ -bl^2\cos lx + (1-\lambda)b\cos lx &= 0, \\ -l^2 + (1-\lambda) &= 0, \\ l^2 &= 1-\lambda, \\ l &= \pm\sqrt{1-\lambda}. \end{aligned}$$

Thus, (since $\cos(-x) = \cos x$)

$$u(x,t) = be^{\lambda t}\cos\sqrt{(1-\lambda)}x.$$

- We want $v(x)$ to be continuous on $\mathbb{R}$, and at $x = \pm 1$, in particular. Thus,

$$\begin{aligned} ae^{-\sqrt{\lambda}} &= b\cos\sqrt{(1-\lambda)}, \\ a &= be^{\sqrt{\lambda}}\cos\sqrt{(1-\lambda)}. \end{aligned}$$

- Also, $v(x)$ is symmetric:

$$\int_{-\infty}^{\infty} |u|^2\,dx \;=\; 2\int_0^{\infty} |u|^2\,dx \;=\; 2\Big[\int_0^1 |u|^2\,dx + \int_1^{\infty} |u|^2\,dx\Big] < \infty.$$

□

Problem (F'03, #3). ❶ *The function*

$$h(X,T) = (4\pi T)^{-\frac{1}{2}} e^{-\frac{X^2}{4T}}$$

satisfies (you do not have to show this)

$$h_T = h_{XX}.$$

Using this result, verify that for any smooth function U

$$u(x,t) = e^{\frac{1}{3}t^3 - xt} \int_{-\infty}^{\infty} U(\xi)\, h(x - t^2 - \xi, t)\, d\xi$$

satisfies

$$u_t + xu = u_{xx}.$$

❷ *Given that $U(x)$ is bounded and continuous everywhere on $-\infty \le x \le \infty$, establish that*

$$\lim_{t\to 0} \int_{-\infty}^{\infty} U(\xi)\, h(x-\xi, t)\, d\xi = U(x)$$

❸ *and show that $u(x,t) \to U(x)$ as $t \to 0$. (You may use the fact that $\int_0^\infty e^{-\xi^2}\, d\xi = \frac{1}{2}\sqrt{\pi}$.)*

Proof. We change the notation: $h \to K$, $U \to g$, $\xi \to y$. We have

$$K(X,T) = \frac{1}{\sqrt{4\pi T}}\, e^{-\frac{X^2}{4T}}$$

❶ We want to verify that

$$u(x,t) = e^{\frac{1}{3}t^3 - xt} \int_{-\infty}^{\infty} K(x - y - t^2, t)\, g(y)\, dy.$$

satisfies

$$u_t + xu = u_{xx}. \qquad \circledast$$

We have

$$\begin{aligned}
u_t &= \int_{-\infty}^{\infty} \frac{d}{dt}\Big[e^{\frac{1}{3}t^3 - xt}\, K(x-y-t^2, t)\Big]\, g(y)\, dy \\
&= \int_{-\infty}^{\infty} \Big[(t^2 - x)\, e^{\frac{1}{3}t^3 - xt}\, K \,+\, e^{\frac{1}{3}t^3 - xt}\big(K_X \cdot (-2t) \,+\, K_T\big)\Big]\, g(y)\, dy, \\
xu &= \int_{-\infty}^{\infty} x\, e^{\frac{1}{3}t^3 - xt} K(x-y-t^2, t)\, g(y)\, dy, \\
u_x &= \int_{-\infty}^{\infty} \frac{d}{dx}\Big[e^{\frac{1}{3}t^3 - xt}\, K(x-y-t^2, t)\Big]\, g(y)\, dy \\
&= \int_{-\infty}^{\infty} \Big[-\, t\, e^{\frac{1}{3}t^3 - xt}\, K \,+\, e^{\frac{1}{3}t^3 - xt}\, K_X\Big]\, g(y)\, dy, \\
u_{xx} &= \int_{-\infty}^{\infty} \frac{d}{dx}\Big[-\, t\, e^{\frac{1}{3}t^3 - xt}\, K \,+\, e^{\frac{1}{3}t^3 - xt}\, K_X\Big]\, g(y)\, dy \\
&= \int_{-\infty}^{\infty} \Big[t^2\, e^{\frac{1}{3}t^3 - xt}\, K \,-\, t\, e^{\frac{1}{3}t^3 - xt}\, K_X - \, t\, e^{\frac{1}{3}t^3 - xt}\, K_X \,+\, e^{\frac{1}{3}t^3 - xt}\, K_{XX}\Big]\, g(y)\, dy.
\end{aligned}$$

Plugging these into ⊛, most of the terms cancel out. The remaining two terms cancel because $K_T = K_{XX}$.

❷ Given that $g(x)$ is bounded and continuous on $-\infty \le x \le \infty$, we establish that [67]

$$\lim_{t\to 0} \int_{-\infty}^{\infty} K(x-y,t)\, g(y)\, dy \;=\; g(x).$$

Fix $x_0 \in \mathbb{R}^n$, $\varepsilon > 0$. Choose $\delta > 0$ such that

$$|g(y) - g(x_0)| < \varepsilon \qquad \text{if } \; |y - x_0| < \delta, \; y \in \mathbb{R}^n.$$

Then if $|x - x_0| < \frac{\delta}{2}$, we have: $\quad (\int_{\mathbb{R}} K(x,t)\, dx = 1)$

$$\Big| \int_{\mathbb{R}} K(x-y,t)\, g(y)\, dy \; - \; g(x_0) \Big| \;\le\; \Big| \int_{\mathbb{R}} K(x-y,t)\, [g(y) - g(x_0)]\, dy \Big|$$

$$\le \; \underbrace{\int_{B_\delta(x_0)} K(x-y,t)\, \big|g(y) - g(x_0)\big|\, dy}_{\le\, \varepsilon \int_{\mathbb{R}} K(x-y,t)\, dy \;=\; \varepsilon} + \int_{\mathbb{R} - B_\delta(x_0)} K(x-y,t)\, \big|g(y) - g(x_0)\big|\, dy \; ⊛$$

Furthermore, if $|x - x_0| \;\le\; \frac{\delta}{2}$ and $|y - x_0| \ge \delta$, then

$$|y - x_0| \;\;\le\;\; |y - x| + \frac{\delta}{2} \;\le\; |y - x| + \frac{1}{2}|y - x_0|.$$

Thus, $|y - x| \ge \frac{1}{2}|y - x_0|$. Consequently,

$$\begin{aligned}
⊛ \;\;&=\;\; \varepsilon \,+\, 2||g||_{L^\infty} \int_{\mathbb{R} - B_\delta(x_0)} K(x-y,t)\, dy \\
&\le\;\; \varepsilon \,+\, \frac{C}{\sqrt{t}} \int_{\mathbb{R} - B_\delta(x_0)} e^{-\frac{|x-y|^2}{4t}}\, dy \\
&\le\;\; \varepsilon \,+\, \frac{C}{\sqrt{t}} \int_{\mathbb{R} - B_\delta(x_0)} e^{-\frac{|y-x_0|^2}{16t}}\, dy \\
&=\;\; \varepsilon \,+\, \frac{C}{\sqrt{t}} \int_{\delta}^{\infty} e^{-\frac{r^2}{16t}}\, r\, dr \;\;\to\;\; \varepsilon + 0 \qquad \text{as } \; t \to 0^+.
\end{aligned}$$

Hence, if $|x - x_0| < \frac{\delta}{2}$ and $t > 0$ is small enough, $|u(x,t) - g(x_0)| < 2\varepsilon$. □

[67] Evans, p. 47, Theorem 1 (c).

Problem (S'93, #4). *The temperature $T(x,t)$ in a stationary medium, $x \geq 0$, is governed by the heat conduction equation*

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}. \tag{18.6}$$

Making the change of variable $(x,t) \to (u,t)$, where $u = x/2\sqrt{t}$, show that

$$4t\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial u^2} + 2u\frac{\partial T}{\partial u}. \tag{18.7}$$

Solutions of (18.7) that depend on u alone are called ***similarity solutions****.* [68]

Proof. We change notation: the change of variables is $(x,t) \to (u,\tau)$, where $t = \tau$. After the change of variables, we have $T = T(u(x,t),\tau(t))$.

$$\begin{aligned}
u = \frac{x}{2\sqrt{t}} \qquad &\Rightarrow \qquad u_t = -\frac{x}{4t^{\frac{3}{2}}}, \quad u_x = \frac{1}{2\sqrt{t}}, \quad u_{xx} = 0,\\
\tau = t \qquad &\Rightarrow \qquad \tau_t = 1, \qquad \tau_x = 0.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial T}{\partial t} &= \frac{\partial T}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial T}{\partial \tau},\\
\frac{\partial T}{\partial x} &= \frac{\partial T}{\partial u}\frac{\partial u}{\partial x},\\
\frac{\partial^2 T}{\partial x^2} &= \frac{\partial}{\partial x}\Big(\frac{\partial T}{\partial x}\Big) = \frac{\partial}{\partial x}\Big(\frac{\partial T}{\partial u}\frac{\partial u}{\partial x}\Big) = \Big(\frac{\partial^2 T}{\partial u^2}\frac{\partial u}{\partial x}\Big)\frac{\partial u}{\partial x} + \frac{\partial T}{\partial u}\underbrace{\frac{\partial^2 u}{\partial x^2}}_{=0} = \frac{\partial^2 T}{\partial u^2}\Big(\frac{\partial u}{\partial x}\Big)^2.
\end{aligned}$$

Thus, (18.6) gives:

$$\begin{aligned}
\frac{\partial T}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial T}{\partial \tau} &= \frac{\partial^2 T}{\partial u^2}\Big(\frac{\partial u}{\partial x}\Big)^2,\\
\frac{\partial T}{\partial u}\Big(-\frac{x}{4t^{\frac{3}{2}}}\Big) + \frac{\partial T}{\partial \tau} &= \frac{\partial^2 T}{\partial u^2}\Big(\frac{1}{2\sqrt{t}}\Big)^2,\\
\frac{\partial T}{\partial \tau} &= \frac{1}{4t}\frac{\partial^2 T}{\partial u^2} + \frac{x}{4t^{\frac{3}{2}}}\frac{\partial T}{\partial u},\\
4t\frac{\partial T}{\partial \tau} &= \frac{\partial^2 T}{\partial u^2} + \frac{x}{\sqrt{t}}\frac{\partial T}{\partial u},\\
4t\frac{\partial T}{\partial \tau} &= \frac{\partial^2 T}{\partial u^2} + 2u\frac{\partial T}{\partial u}. \quad \checkmark
\end{aligned}$$

□

[68] This is only the part of the qual problem.

19 Contraction Mapping and Uniqueness - Wave

Recall that the solution to

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), \end{cases} \tag{19.1}$$

is given by adding together d'Alembert's formula and Duhamel's principle:

$$\boxed{u(x,t) = \frac{1}{2}(g(x+ct) + g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\,d\xi + \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(\xi,s)\,d\xi\right) ds.}$$

Problem (W'02, #8). ***a)*** *Find an explicit solution of the following Cauchy problem*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f(t,x), \\ u(0,x) = 0, \quad \frac{\partial u}{\partial x}(0,x) = 0. \end{cases} \tag{19.2}$$

b) *Use part (a) to prove the* ***uniqueness*** *of the solution of the Cauchy problem*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + q(t,x)u = 0, \\ u(0,x) = 0, \quad \frac{\partial u}{\partial x}(0,x) = 0. \end{cases} \tag{19.3}$$

Here $f(t,x)$ and $q(t,x)$ are continuous functions.

Proof. **a)** It was probably meant to give the u_t initially. We rewrite (19.2) as

$$\begin{cases} u_{tt} - u_{xx} = f(x,t), \\ u(x,0) = 0, \quad u_t(x,0) = 0. \end{cases} \tag{19.4}$$

Duhamel's principle, with $c = 1$, gives the solution to (19.4):

$$u(x,t) = \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(\xi,s)\,d\xi\right) ds = \frac{1}{2}\int_0^t \left(\int_{x-(t-s)}^{x+(t-s)} f(\xi,s)\,d\xi\right) ds.$$

b) We use the Contraction Mapping Principle to prove uniqueness.
Define the operator

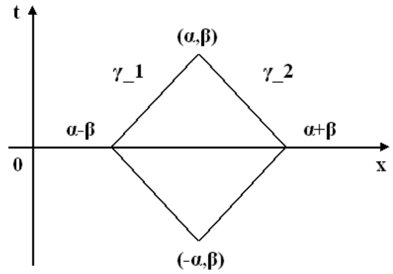
$$T(u) \;=\; \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} -q(\xi,s)\,u(\xi,s)\,d\xi\,ds.$$

on the Banach space $C^{2,2}$, $||\cdot||_\infty$.
We will show $|Tu_n - Tu_{n+1}| < \alpha||u_n - u_{n+1}||$ where $\alpha < 1$. Then $\{u_n\}_{n=1}^\infty$: $u_{n+1} = T(u_n)$ converges to a unique fixed point which is the unique solution of PDE.

$$\begin{aligned} |Tu_n - Tu_{n+1}| \;&=\; \left|\frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} -q(\xi,s)\big(u_n(\xi,s) - u_{n+1}(\xi,s)\big)\,d\xi\,ds\right| \\ &\le\; \frac{1}{2}\int_0^t ||q||_\infty ||u_n - u_{n+1}||_\infty\, 2(t-s)\,ds \\ &\le\; t^2||q||_\infty||u_n - u_{n+1}||_\infty \;\le\; \alpha||u_n - u_{n+1}||_\infty, \qquad \text{for small } t. \end{aligned}$$

Thus, T is a contraction $\Rightarrow$ $\exists$ a **unique** fixed point.
Since $Tu = u$, u is the solution to the PDE. ∎

Problem (F'00, #3). *Consider the Goursat problem:*



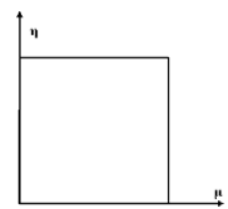
Find the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + a(x,t)u = 0$$

in the square $\mathcal{D}$, satisfying the boundary conditions

$$u|_{\gamma_1} = \varphi, \quad u|_{\gamma_2} = \psi,$$

where γ_1, γ_2 are two adjacent sides $\mathcal{D}$. Here $a(x,t)$, φ and ψ are continuous functions. Prove the ***uniqueness*** *of the solution of this Goursat problem.*



Proof. The change of variable $\mu = x + t, \quad \eta = x - t$ transforms the equation to

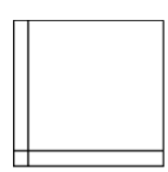
$$\tilde{u}_{\mu\eta} + \tilde{a}(\mu,\eta)\tilde{u} = 0.$$

We integrate the equation:

$$\begin{aligned}
\int_0^\eta \int_0^\mu \tilde{u}_{\mu\eta}(u,v)\,du\,dv &= -\int_0^\eta \int_0^\mu \tilde{a}(\mu,\eta)\,\tilde{u}\,du\,dv, \\
\int_0^\eta \big(\tilde{u}_\eta(\mu,v) - \tilde{u}_\eta(0,v)\big)\,dv &= -\int_0^\eta \int_0^\mu \tilde{a}(\mu,\eta)\,\tilde{u}\,du\,dv, \\
\tilde{u}(\mu,\eta) = \tilde{u}(\mu,0) + \tilde{u}(0,\eta) - u(0,0) &- \int_0^\eta \int_0^\mu \tilde{a}(\mu,\eta)\,\tilde{u}\,du\,dv.
\end{aligned}$$

We change the notation. In the new notation:

$$\begin{aligned}
f(x,y) &= \varphi(x,y) - \int_0^x \int_0^y a(u,v)f(u,v)\,du\,dv, \\
f &= \varphi + Kf, \\
f &= \varphi + K(\varphi + Kf), \\
&\ \ \cdots \\
f &= \varphi + \sum_{n=1}^{\infty} K^n \varphi, \\
f &= Kf \qquad \Rightarrow\ f = 0, \\
\max_{0<x<\delta} |f| &\le \delta \max|a| \max|f|.
\end{aligned}$$



For small enough δ, the operator K is a contraction. Thus, there exists a unique fixed point of K, and $f = Kf$, where f is the unique solution. ☐

20 Contraction Mapping and Uniqueness - Heat

The solution of the initial value problem

$$\begin{cases} u_t = \triangle u + f(x,t) & \text{for } \ t>0, \ x \in \mathbb{R}^n \\ u(x,0) = g(x) & \text{for } \ x \in \mathbb{R}^n. \end{cases} \tag{20.1}$$

is given by

$$\boxed{u(x,t) = \int_{\mathbb{R}^n} \tilde{K}(x-y,t)\, g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \tilde{K}(x-y,t-s)\, f(y,s)\, dy\, ds}$$

where

$$\tilde{K}(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & \text{for } \ t>0, \\ 0 & \text{for } \ t \le 0. \end{cases}$$

Problem (F'00, #2). *Consider the Cauchy problem*

$$\begin{aligned} u_t - \triangle u + u^2(x,t) &= f(x,t), \qquad x \in \mathbb{R}^N, \quad 0<t<T \\ u(x,0) &= 0. \end{aligned}$$

*Prove the **uniqueness** of the **classical bounded solution** assuming that T is small enough.*

Proof. Let $\{u_n\}$ be a sequence of approximations to the solution, such that

$$S(u_n) \ = \ u_{n+1} \underbrace{=}_{use\ Duhamel's\ principle} \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s) \big(f(y,s) - u_n^2(y,s)\big)\, dy\, ds.$$

We will show that S has a fixed point $\big(\ |S(u_n) - S(u_{n+1})| \le \alpha |u_n - u_{n+1}|, \ \ \alpha < 1 \ \big)$
$\Leftrightarrow \ \ \{u_n\}$ converges to a uniques solution for small enough T.
Since $u_n, \, u_{n+1} \in C^2(\mathbb{R}^n) \cap C^1(t) \quad \Rightarrow \quad |u_{n+1} + u_n| \le M$.

$$\begin{aligned} |S(u_n) - S(u_{n+1})| &\le \int_0^t \int_{\mathbb{R}^n} \big|K(x-y,t-s)\big| \, \big|u_{n+1}^2 - u_n^2\big|\, dy\, ds \\ &= \int_0^t \int_{\mathbb{R}^n} \big|K(x-y,t-s)\big| \, \big|u_{n+1} - u_n\big| \, \big|u_{n+1} + u_n\big|\, dy\, ds \\ &\le M \int_0^t \int_{\mathbb{R}^n} \big|K(x-y,t-s)\big| \, \big|u_{n+1} - u_n\big|\, dy\, ds \\ &\le M M_1 \int_0^t \big|u_{n+1}(x,s) - u_n(x,s)\big|\, ds \\ &\le M M_1 T \, ||u_{n+1} - u_n||_\infty \ < \ ||u_{n+1} - u_n||_\infty \qquad \text{for small } \ T. \end{aligned}$$

Thus, S is a contraction $\quad \Rightarrow \quad \exists$ a **unique** fixed point $u \in C^2(\mathbb{R}^n) \cap C^1(t)$ such that $u = \lim_{n \to \infty} u_n$. u is implicitly defined as

$$u(x,t) \ = \ \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s) \big(f(y,s) - u^2(y,s)\big)\, dy\, ds.$$

$\square$

Problem (S'97, #3). ***a)*** *Let $Q(x) \geq 0$ such that $\int_{x=-\infty}^{\infty} Q(x)\,dx = 1$, and define $Q_\epsilon = \frac{1}{\epsilon}Q(\frac{x}{\epsilon})$. Show that (here $*$ denotes convolution)*

$$||Q_\epsilon(x) * w(x)||_{L^\infty} \;\leq\; ||w(x)||_{L^\infty}.$$

In particular, let $Q_t(x)$ denote the heat kernel (at time t), then

$$||Q_t(x) * w_1(x) - Q_t(x) * w_2(x)||_{L^\infty} \;\leq\; ||w_1(x) - w_2(x)||_{L^\infty}.$$

b) *Consider the parabolic equation $u_t = u_{xx} + u^2$ subject to initial conditions $u(x,0) = f(x)$. Show that the solution of this equation satisfies*

$$u(x,t) \;=\; Q_t(x) * f(x) + \int_0^t Q_{t-s}(x) * u^2(x,s)\,ds. \tag{20.2}$$

c) *Fix $t > 0$. Let $\{u_n(x,t)\}$, $n = 1, 2, \ldots$ the fixed point iterations for the solution of (20.2)*

$$u_{n+1}(x,t) \;=\; Q_t(x) * f(x) + \int_0^t Q_{t-s}(x) * u_n^2(x,s)\,ds. \tag{20.3}$$

Let $K_n(t) = \sup_{0 \leq m \leq n} ||u_m(x,t)||_{L^\infty}$. Using (a) and (b) show that

$$||u_{n+1}(x,t) - u_n(x,t)||_{L^\infty} \;\leq\; 2 \sup_{0 \leq \tau \leq t} K_n(\tau) \cdot \int_0^t ||u_n(x,s) - u_{n-1}(x,s)||_{L^\infty}\,ds.$$

Conclude that the fixed point iterations in (20.3) converge if t is sufficiently small.

Proof. **a)** We have

$$\begin{aligned}
||Q_\epsilon(x) * w(x)||_{L^\infty} \;&=\; \left|\int_{-\infty}^{\infty} Q_\epsilon(x-y)w(y)\,dy\right| \;\leq\; \int_{-\infty}^{\infty} \big|Q_\epsilon(x-y)w(y)\big|\,dy \\
&\leq\; ||w||_\infty \int_{-\infty}^{\infty} \big|Q_\epsilon(x-y)\big|\,dy \;=\; ||w||_\infty \int_{-\infty}^{\infty} \frac{1}{\epsilon}Q\Big(\frac{x-y}{\epsilon}\Big)\,dy \\
&=\; ||w||_\infty \int_{-\infty}^{\infty} \frac{1}{\epsilon}Q\Big(\frac{y}{\epsilon}\Big)\,dy \qquad \Big(z = \frac{y}{\epsilon},\; dz = \frac{dy}{\epsilon}\Big) \\
&=\; ||w||_\infty \int_{-\infty}^{\infty} Q(z)\,dz \;=\; ||w(x)||_\infty. \quad \checkmark
\end{aligned}$$

$Q_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$, the heat kernel. We have [69]

$$
\begin{aligned}
||Q_t(x) * w_1(x) - Q_t(x) * w_2(x)||_{L^\infty} &= \Big|\Big| \int_{-\infty}^{\infty} Q_t(x-y) w_1(y)\, dy - \int_{-\infty}^{\infty} Q_t(x-y) w_2(y)\, dy \Big|\Big|_\infty \\
&= \frac{1}{\sqrt{4\pi t}} \Big|\Big| \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} w_1(y)\, dy - \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} w_2(y)\, dy \Big|\Big|_\infty \\
&\leq \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \big| w_1(y) - w_2(y) \big|\, dy \\
&\leq \big|\big| w_1(y) - w_2(y) \big|\big|_\infty \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}}\, dy \\
z = \frac{x-y}{\sqrt{4t}},\ dz = \frac{-dy}{\sqrt{4t}} \qquad &= \big|\big| w_1(y) - w_2(y) \big|\big|_\infty \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-z^2} \sqrt{4t}\, dz \\
&= \big|\big| w_1(y) - w_2(y) \big|\big|_\infty \frac{1}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-z^2}\, dz}_{\sqrt{\pi}} \\
&= \big|\big| w_1(y) - w_2(y) \big|\big|_\infty. \quad \checkmark
\end{aligned}
$$

---

[69]Note:

$$\int_{-\infty}^{\infty} Q_t(x)\, dx = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}}\, dy = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-z^2} \sqrt{4t}\, dz = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2}\, dz = 1.$$

b) Consider

$$\begin{cases} u_t = u_{xx} + u^2, \\ u(x,0) = f(x). \end{cases}$$

We will show that the solution of this equation satisfies

$$u(x,t) \;=\; Q_t(x) * f(x) + \int_0^t Q_{t-s}(x) * u^2(x,s)\,ds.$$

$$\begin{aligned}
\int_0^t Q_{t-s}(x) * u^2(x,s)\,ds \;&=\; \int_0^t \int_{\mathbb{R}} Q_{t-s}(x-y)\,u^2(y,s)\,dy\,ds \\
&= \int_0^t \int_{\mathbb{R}} Q_{t-s}(x-y)\,\Big(u_s(y,s) - u_{yy}(y,s)\Big)\,dy\,ds \\
&= \int_0^t \int_{\mathbb{R}} \frac{d}{ds}\big(Q_{t-s}(x-y)u(y,s)\big) - \frac{d}{ds}\big(Q_{t-s}(x-y)\big)u(y,s) - Q_{t-s}(x-y)u_{yy}(y,s)\,dy\,ds \\
&= \Big[\int_{\mathbb{R}} Q_0(x-y)u(y,t)\,dy - \int_{\mathbb{R}} Q_t(x-y)u(y,0)\,dy\Big] \\
&\quad - \int_0^t \int_{\mathbb{R}} \underbrace{\frac{d}{ds}\big(Q_{t-s}(x-y)\big)u(y,s) + \frac{d^2}{dy^2}Q_{t-s}(x-y)u(y,s)}_{=0,\quad since\ Q_t\ satisfies\ heat\ equation}\,dy\,ds \\
&= u(x,t) - \int_{\mathbb{R}} Q_t(x-y)f(y)\,dy \\
&= u(x,t) - Q_t(x) * f(x). \quad \checkmark
\end{aligned}$$

Note: $\lim_{t\to 0^+} Q(x,t) = \delta_0(x) = \delta(x)$.
$\lim_{t\to 0^+} \int_{\mathbb{R}} Q(x-y,t)v(y)\,dy = v(0)$.

Note that we used: $D^\alpha(f*g) \;=\; (D^\alpha f)*g \;=\; f*(D^\alpha g)$.

c) Let

$$u_{n+1}(x,t) \;=\; Q_t(x) * f(x) + \int_0^t Q_{t-s}(x) * u_n^2(x,s)\,ds.$$

$$\begin{aligned}
||u_{n+1}(x,t) - u_n(x,t)||_{L^\infty} \;&=\; \bigg|\bigg|\int_0^t Q_{t-s}(x) * \big(u_n^2(x,s) - u_{n-1}^2(x,s)\big)\,ds\bigg|\bigg|_\infty \\
&\le \int_0^t \big|\big|Q_{t-s}(x) * \big(u_n^2(x,s) - u_{n-1}^2(x,s)\big)\big|\big|_\infty\,ds \\
&\underbrace{\le}_{(a)} \int_0^t \big|\big|u_n^2(x,s) - u_{n-1}^2(x,s)\big|\big|_\infty\,ds \\
&\le \int_0^t \big|\big|u_n(x,s) - u_{n-1}(x,s)\big|\big|_\infty \big|\big|u_n(x,s) + u_{n-1}(x,s)\big|\big|_\infty\,ds \\
&\le \sup_{0\le\tau\le t} \big|\big|u_n(x,s) + u_{n-1}(x,s)\big|\big|_\infty \int_0^t \big|\big|u_n(x,s) - u_{n-1}(x,s)\big|\big|_\infty\,ds \\
&\le 2\sup_{0\le\tau\le t} K_n(\tau)\cdot \int_0^t ||u_n(x,s) - u_{n-1}(x,s)||_{L^\infty}\,ds. \quad \checkmark
\end{aligned}$$

Also, $\quad ||u_{n+1}(x,t) - u_n(x,t)||_{L^\infty} \;\le\; 2t\sup_{0\le\tau\le t} K_n(\tau)\cdot ||u_n(x,s) - u_{n-1}(x,s)||_{L^\infty}.$

For t small enough, $2t\sup_{0\le\tau\le t} K_n(\tau) \le \alpha < 1$. Thus, T defined as

$$Tu = Q_t(x) * f(x) + \int_0^t Q_{t-s}(x) * u^2(x,s)\, ds$$

is a contraction, and has a **unique** fixed point $u = Tu$. $\square$

Problem (S'99, #3). *Consider the system of equations*

$$u_t = u_{xx} + f(u,v)$$
$$v_t = 2v_{xx} + g(u,v)$$

to be solved for $t > 0$, $-\infty < x < \infty$, *and smooth initial data with compact support:*

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x).$$

If f *and* g *are uniformly Lipschitz continuous, give a proof of* ***existence*** *and* ***uniqueness*** *of the solution to this problem in the space of bounded continuous functions with* $||u(\cdot,t)|| = \sup_x |u(x,t)|$.

Proof. The space of continuous bounded functions forms a complete metric space so the contraction mapping principle applies.
First, let $v(x,t) = w\big(\frac{x}{\sqrt{2}}, t\big)$, then

$$u_t = u_{xx} + f(u,w)$$
$$w_t = w_{xx} + g(u,w).$$

These initial value problems have the following solutions (K is the heat kernel):

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} \tilde{K}(x-y,t)\, u_0(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \tilde{K}(x-y,t-s)\, f(u,w)\, dy\, ds, \\ w(x,t) &= \int_{\mathbb{R}^n} \tilde{K}(x-y,t)\, w_0(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \tilde{K}(x-y,t-s)\, g(u,w)\, dy\, ds. \end{aligned}$$

By the Lipshitz conditions,

$$|f(u,w)| \le M_1 ||u||,$$
$$|g(u,w)| \le M_2 ||w||.$$

Now we can show the mappings, as defined below, are contractions:

$$\begin{aligned} T_1 u &= \int_{\mathbb{R}^n} \tilde{K}(x-y,t)\, u_0(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \tilde{K}(x-y,t-s)\, f(u,w)\, dy\, ds, \\ T_2 w &= \int_{\mathbb{R}^n} \tilde{K}(x-y,t)\, w_0(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \tilde{K}(x-y,t-s)\, g(u,w)\, dy\, ds. \end{aligned}$$

$$\begin{aligned} |T_1(u_n) - T_1(u_{n+1})| &\le \int_0^t \int_{\mathbb{R}^n} \big|\tilde{K}(x-y,t-s)\big| \big|f(u_n,w) - f(u_{n+1},w)\big|\, dy\, ds \\ &\le M_1 \int_0^t \int_{\mathbb{R}^n} \big|\tilde{K}(x-y,t-s)\big| \big|u_n - u_{n+1}\big|\, dy\, ds \\ &\le M_1 \int_0^t \sup_x \big|u_n - u_{n+1}\big| \int_{\mathbb{R}^n} \tilde{K}(x-y,t-s) dy\, ds \\ &\le M_1 \int_0^t \sup_x \big|u_n - u_{n+1}\big|\, ds \;\le\; M_1 t \sup_x \big|u_n - u_{n+1}\big| \\ &< \sup_x \big|u_n - u_{n+1}\big| \qquad \text{for small } t. \end{aligned}$$

We used the Lipshitz condition and $\int_{\mathbb{R}} \tilde{K}(x-y,t-s)\, dy = 1$.
Thus, for small t, T_1 is a contraction, and has a unique fixed point. Thus, the solution is defined as $u = T_1 u$.
Similarly, T_2 is a contraction and has a unique fixed point. The solution is defined as $w = T_2 w$. □

21 Problems: Maximum Principle - Laplace and Heat

21.1 Heat Equation - Maximum Principle and Uniqueness

Let us introduce the "cylinder" $U = U_T = \Omega \times (0, T)$. We know that harmonic (and subharmonic) functions achieve their maximum on the boundary of the domain. For the heat equation, the result is improved in that the maximum is achieved on a certain part of the boundary, *parabolic boundary*:

$$\Gamma \; = \; \{(x,t) \in \overline{U} : x \in \partial\Omega \;\; \text{or} \;\; t = 0\}.$$

Let us also denote by $C^{2;1}(U)$ functions satisfying $u_t,\, u_{x_ix_j} \in C(U)$.

Weak Maximum Principle. *Let $u \in C^{2;1}(U) \cap C(\overline{U})$ satisfy $\triangle u \geq u_t$ in U. Then u achieves its maximum on the parabolic boundary of U:*

$$\max_{\overline{U}} u(x,t) \; = \; \max_{\Gamma} u(x,t). \tag{21.1}$$

Proof. • First, assume $\triangle u > u_t$ in U. For $0 < \tau < T$ consider

$$U_\tau = \Omega \times (0, \tau), \qquad \Gamma_\tau = \{(x,t) \in \overline{U}_\tau : x \in \partial\Omega \quad \text{or} \quad t = 0\}.$$

If the maximum of u on $\overline{U}_\tau$ occurs at $x \in \Omega$ and $t = \tau$, then $u_t(x,\tau) \geq 0$ and $\triangle u(x,\tau) \leq 0$, violating our assumption; similarly, u cannot attain an interior maximum on U_τ. Hence (21.1) holds for U_τ: $\max_{\overline{U}_\tau} u = \max_{\Gamma_\tau} u$. But $\max_{\Gamma_\tau} u \leq \max_\Gamma u$ and by continuity of u, $\max_{\overline{U}} u = \lim_{\tau \to T} \max_{\overline{U}_\tau} u$. This establishes (21.1).
• Second, we consider the general case of $\triangle u \geq u_t$ in U. Let $u = v + \varepsilon t$ for $\varepsilon > 0$. Notice that $v \leq u$ on $\overline{U}$ and $\triangle v - v_t > 0$ in U. Thus we may apply (21.1) to v:

$$\max_{\overline{U}} u \; = \; \max_{\overline{U}}(v + \varepsilon t) \; \leq \; \max_{\overline{U}} v + \varepsilon T \; = \; \max_{\Gamma} v + \varepsilon T \; \leq \; \max_{\Gamma} u + \varepsilon T.$$

Letting $\varepsilon \to 0$ establishes (21.1) for u. □

Problem (S'98, #7). *Prove that any smooth solution, $u(x,y,t)$ in the unit box $\Omega = \{(x,y) \,|\, -1 \le x, y \le 1\}$, of the following equation*

$$\begin{aligned} u_t &= uu_x + uu_y + \triangle u, && t \ge 0,\ (x,y) \in \Omega \\ u(x,y,0) &= f(x,y), && (x,y) \in \Omega \end{aligned}$$

*satisfies the **weak maximum principle**,*

$$\max_{\Omega\times[0,T]} u(x,y,t) \le \max\{\max_{0\le t\le T} u(\pm1,\pm1,t), \max_{(x,y)\in\Omega} f(x,y)\}.$$

Proof. Suppose u satisfies given equation. Let $u = v + \varepsilon t$ for $\varepsilon > 0$. Then,

$$v_t + \varepsilon = vv_x + vv_y + \varepsilon t(v_x + v_y) + \triangle v.$$

Suppose v has a maximum at $(x_0, y_0, t_0) \in \Omega \times (0,T)$. Then

$$v_x = v_y = v_t = 0 \quad \Rightarrow \quad \varepsilon = \triangle v \quad \Rightarrow \quad \triangle v > 0$$

$\Rightarrow$ v has a minimum at (x_0, y_0, t_0), a contradiction.
Thus, the maximum of v is on the boundary of $\Omega \times (0,T)$.

Suppose v has a maximum at (x_0, y_0, T), $(x_0, y_0) \in \Omega$. Then

$$v_x = v_y = 0,\ v_t \ge 0 \quad \Rightarrow \quad \varepsilon \le \triangle v \quad \Rightarrow \quad \triangle v > 0$$

$\Rightarrow$ v has a minimum at (x_0, y_0, T), a contradiction. Thus,

$$\max_{\Omega\times[0,T]} v \le \max\{\max_{0\le t\le T} v(\pm1,\pm1,t), \max_{(x,y)\in\Omega} f(x,y)\}.$$

Now

$$\begin{aligned} \max_{\Omega\times[0,T]} u = \max_{\Omega\times[0,T]} (v + \varepsilon t) \le \max_{\Omega\times[0,T]} v + \varepsilon T &\le \max\{\max_{0\le t\le T} v(\pm1,\pm1,t), \max_{(x,y)\in\Omega} f(x,y)\} + \varepsilon T \\ &\le \max\{\max_{0\le t\le T} u(\pm1,\pm1,t), \max_{(x,y)\in\Omega} f(x,y)\} + \varepsilon T. \end{aligned}$$

Letting $\varepsilon \to 0$ establishes the result. □

21.2 Laplace Equation - Maximum Principle

Problem (S'91, #6). *Suppose that u satisfies*

$$Lu = au_{xx} + bu_{yy} + cu_x + du_y - eu = 0$$

with $a > 0$, $b > 0$, $e > 0$, for $(x, y) \in \Omega$, with Ω a bounded open set in $\mathbb{R}^2$.

a) *Show that u cannot have a positive maximum or a negative minimum in the interior of Ω.*
b) *Use this to show that the only function u satisfying $Lu = 0$ in Ω, $u = 0$ on $\partial\Omega$ and u continuous on $\overline{\Omega}$ is $u = 0$.*

Proof. **a)** For an interior (local) maximum or minimum at an interior point (x, y), we have

$$u_x = 0, \quad u_y = 0.$$

- Suppose u has a positive maximum in the interior of Ω. Then

$$u > 0, \quad u_{xx} \le 0, \quad u_{yy} \le 0.$$

With these values, we have

$$\underbrace{au_{xx}}_{\le 0} + \underbrace{bu_{yy}}_{\le 0} + \underbrace{cu_x}_{=0} + \underbrace{du_y}_{=0} \underbrace{-eu}_{<0} = 0,$$

which leads to contradiction. Thus, u can not have a positive maximum in Ω.

- Suppose u has a negative minimum in the interior of Ω. Then

$$u < 0, \quad u_{xx} \ge 0, \quad u_{yy} \ge 0.$$

With these values, we have

$$\underbrace{au_{xx}}_{\ge 0} + \underbrace{bu_{yy}}_{\ge 0} + \underbrace{cu_x}_{=0} + \underbrace{du_y}_{=0} \underbrace{-eu}_{>0} = 0,$$

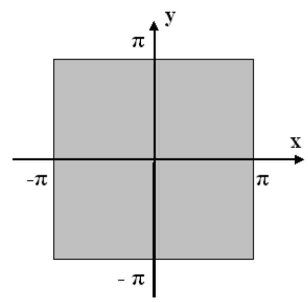
which leads to contradiction. Thus, u can not have a negative minimum in Ω.

b) Since u can not have positive maximum in the interior of Ω, then $\max u = 0$ on $\overline{\Omega}$.
Since u can not have negative minimum in the interior of Ω, then $\min u = 0$ on $\overline{\Omega}$.
Since u is continuous, $u \equiv 0$ on $\overline{\Omega}$. □

22 Problems: Separation of Variables - Laplace Equation

Problem 1: The 2D LAPLACE Equation on a Square.
Let $\Omega = (0,\pi) \times (0,\pi)$, *and use separation of variables to solve the boundary value problem*

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0 < x, y < \pi \\ u(0,y) = 0 = u(\pi, y) & 0 \le y \le \pi \\ u(x,0) = 0, \qquad u(x,\pi) = g(x) & 0 \le x \le \pi, \end{cases}$$



where g *is a continuous function satisfying* $g(0) = 0 = g(\pi)$.

Proof. Assume $u(x,y) = X(x)Y(y)$, then substitution in the PDE gives $X''Y + XY'' = 0$.

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

• From $X'' + \lambda X = 0$, we get $X_n(x) = a_n \cos nx + b_n \sin nx$. Boundary conditions give

$$\begin{cases} u(0,y) = X(0)Y(y) = 0 \\ u(\pi,y) = X(\pi)Y(y) = 0 \end{cases} \Rightarrow \quad X(0) = 0 = X(\pi).$$

Thus, $X_n(0) = a_n = 0$, and

$$X_n(x) = b_n \sin nx, \quad n = 1, 2, \ldots. \quad \checkmark$$
$$-n^2 b_n \sin nx + \lambda b_n \sin nx = 0,$$
$$\lambda_n = n^2, \quad n = 1, 2, \ldots. \quad \checkmark$$

• With these values of λ_n we solve $Y'' - n^2 Y = 0$ to find $Y_n(y) = c_n \cosh ny + d_n \sinh ny$.
Boundary conditions give

$$u(x,0) = X(x)Y(0) = 0 \quad \Rightarrow \quad Y(0) = 0 = c_n.$$
$$Y_n(x) = d_n \sinh ny. \quad \checkmark$$

• By superposition, we write

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \tilde{a}_n \sin nx \sinh ny,}$$

which satifies the equation and the three homogeneous boundary conditions. The boundary condition at $y = \pi$ gives

$$u(x,\pi) = g(x) = \sum_{n=1}^{\infty} \tilde{a}_n \sin nx \sinh n\pi,$$

$$\int_0^\pi g(x) \sin mx \, dx = \sum_{n=1}^{\infty} \tilde{a}_n \sinh n\pi \int_0^\pi \sin nx \sin mx \, dx = \frac{\pi}{2} \tilde{a}_m \sinh m\pi.$$

$$\boxed{\tilde{a}_n \sinh n\pi = \frac{2}{\pi}\int_0^{\pi} g(x)\sin nx\, dx.}$$

$\square$

Problem 2: The 2D LAPLACE Equation on a Square. *Let* $\Omega = (0,\pi)\times(0,\pi)$, *and use separation of variables to solve the mixed boundary value problem*

$$\begin{cases} \triangle u = 0 & in\ \Omega \\ u_x(0,y) = 0 = u_x(\pi,y) & 0<y<\pi \\ u(x,0)=0, \quad u(x,\pi)=g(x) & 0<x<\pi. \end{cases}$$

Proof. Assume $u(x,y) = X(x)Y(y)$, then substitution in the PDE gives $X''Y + XY'' = 0$.

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

• Consider $X'' + \lambda X = 0$.
If $\lambda = 0$, $X_0(x) = a_0 x + b_0$.
If $\lambda > 0$, $X_n(x) = a_n \cos nx + b_n \sin nx$.
Boundary conditions give

$$\begin{cases} u_x(0,y) = X'(0)Y(y) = 0 \\ u_x(\pi,y) = X'(\pi)Y(y) = 0 \end{cases} \Rightarrow \quad X'(0) = 0 = X'(\pi).$$

Thus, $X_0'(0) = a_0 = 0$, and $X_n'(0) = nb_n = 0$.

$$X_0(x) = b_0, \quad X_n(x) = a_n \cos nx, \quad n = 1,2,\ldots. \quad \checkmark$$
$$-n^2 a_n \cos nx + \lambda a_n \cos nx = 0,$$
$$\lambda_n = n^2, \quad n = 0,1,2,\ldots. \quad \checkmark$$

• With these values of λ_n we solve $Y'' - n^2 Y = 0$.
If $n = 0$, $Y_0(y) = c_0 y + d_0$.
If $n \neq 0$, $Y_n(y) = c_n \cosh ny + d_n \sinh ny$.
Boundary conditions give

$$u(x,0) = X(x)Y(0) = 0 \quad \Rightarrow \quad Y(0) = 0.$$

Thus, $Y_0(0) = d_0 = 0$, and $Y_n(0) = c_n = 0$.

$$Y_0(y) = c_0 y, \quad Y_n(y) = d_n \sinh ny, \quad n = 1,2,\ldots. \quad \checkmark$$

• We have

$$u_0(x,y) = X_0(x)Y_0(y) = b_0 c_0 y = \tilde{a}_0 y,$$
$$u_n(x,y) = X_n(x)Y_n(y) = (a_n \cos nx)(d_n \sinh ny) = \tilde{a}_n \cos nx \sinh ny.$$

By superposition, we write

$$\boxed{u(x,y) = \tilde{a}_0 y + \sum_{n=1}^{\infty} \tilde{a}_n \cos nx \sinh ny,}$$

which satifies the equation and the three homogeneous boundary conditions. The fourth boundary condition gives

$$u(x,\pi) = g(x) = \tilde{a}_0 \pi + \sum_{n=1}^{\infty} \tilde{a}_n \cos nx \sinh n\pi,$$

$$\begin{cases} \int_0^\pi g(x)\,dx = \int_0^\pi \left(\tilde{a}_0\pi + \sum_{n=1}^\infty \tilde{a}_n \cos nx \sinh n\pi\right) dx = \tilde{a}_0\pi^2, \\ \int_0^\pi g(x)\cos mx\,dx = \sum_{n=1}^\infty \tilde{a}_n \sinh n\pi \int_0^\pi \cos nx \cos mx\,dx = \frac{\pi}{2}\,\tilde{a}_m \sinh m\pi. \end{cases}$$

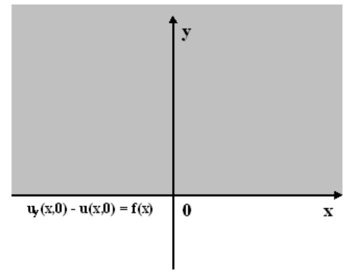
$$\boxed{\tilde{a}_0 = \frac{1}{\pi^2}\int_0^\pi g(x)\,dx,}$$

$$\boxed{\tilde{a}_n \sinh n\pi = \frac{2}{\pi}\int_0^\pi g(x)\cos nx\,dx.}$$

$\square$

Problem (W'04, #5) The 2D LAPLACE Equation in an Upper-Half Plane.
Consider the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad y > 0, \ -\infty < x < +\infty$$
$$\frac{\partial u(x,0)}{\partial y} - u(x,0) = f(x),$$
where $f(x) \in C_0^\infty(\mathbb{R}^1)$.



Find a bounded solution $u(x,y)$ *and show that* $u(x,y) \to 0$ *when* $|x| + y \to \infty$.

Proof. Assume $u(x,y) = X(x)Y(y)$, then substitution in the PDE gives $X''Y + XY'' = 0$.

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda. \qquad \circledast$$

- Consider $X'' + \lambda X = 0$.

If $\lambda = 0$, $X_0(x) = a_0 x + b_0$.
If $\lambda > 0$, $X_n(x) = a_n \cos\sqrt{\lambda_n}x + b_n \sin\sqrt{\lambda_n}x$.
Since we look for bounded solutions as $|x| \to \infty$, we have $a_0 = 0$.

- Consider $Y'' - \lambda_n Y = 0$.

If $\lambda_n = 0$, $Y_0(y) = c_0 y + d_0$.
If $\lambda_n > 0$, $Y_n(y) = c_n e^{-\sqrt{\lambda_n}y} + d_n e^{\sqrt{\lambda_n}y}$.
Since we look for bounded solutions as $y \to \infty$, we have $c_0 = 0$, $d_n = 0$. Thus,

$$u(x,y) = \tilde{a}_0 + \sum_{n=1}^{\infty} e^{-\sqrt{\lambda_n}y}\big(\tilde{a}_n \cos\sqrt{\lambda_n}x + \tilde{b}_n \sin\sqrt{\lambda_n}x\big).$$

Initial condition gives:

$$f(x) = u_y(x,0) - u(x,0) = -\tilde{a}_0 - \sum_{n=1}^{\infty}(\sqrt{\lambda_n}+1)\big(\tilde{a}_n \cos\sqrt{\lambda_n}x + \tilde{b}_n \sin\sqrt{\lambda_n}x\big).$$

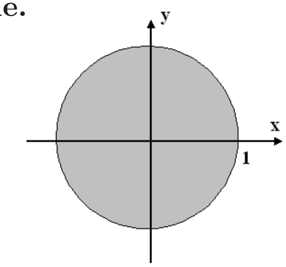
$f(x) \in C_0^\infty(\mathbb{R}^1)$, i.e. has compact support $[-L, L]$, for some $L > 0$. Thus the coefficients $\tilde{a}_n$, $\tilde{b}_n$ are given by

$$\int_{-L}^{L} f(x)\cos\sqrt{\lambda_n}x\,dx = -(\sqrt{\lambda_n}+1)\tilde{a}_n L.$$
$$\int_{-L}^{L} f(x)\sin\sqrt{\lambda_n}x\,dx = -(\sqrt{\lambda_n}+1)\tilde{b}_n L.$$

Thus, $u(x,y) \to 0$ when $|x| + y \to \infty$. [70] □

[70] Note that if we change the roles of X and Y in $\circledast$, the solution we get will be unbounded.

Problem 3: The 2D LAPLACE Equation on a Circle.
Let Ω be the unit disk in $\mathbb{R}^2$ and consider the problem



$$\begin{cases} \triangle u = 0 & in\ \Omega \\ \frac{\partial u}{\partial n} = h & on\ \partial\Omega, \end{cases}$$

where h is a continuous function.

Proof. Use polar coordinates (r, θ)

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & \text{for } 0 \le r < 1,\ 0 \le \theta < 2\pi \\ \frac{\partial u}{\partial r}(1, \theta) = h(\theta) & \text{for } 0 \le \theta < 2\pi. \end{cases}$$

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

Let $r = e^{-t}$, $u(r(t), \theta)$.

$$\begin{aligned} u_t &= u_r r_t = -e^{-t}u_r, \\ u_{tt} &= (-e^{-t}u_r)_t = e^{-t}u_r + e^{-2t}u_{rr} = r u_r + r^2 u_{rr}. \end{aligned}$$

Thus, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t, \theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

- From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$.

$\lambda_n = n^2$, $n = 0, 1, 2, \ldots$.

- With these values of λ_n we solve $X''(t) - n^2 X(t) = 0$.

If $n = 0$, $X_0(t) = c_0 t + d_0$. $\Rightarrow$ $X_0(r) = -c_0 \log r + d_0$.
If $n \neq 0$, $X_n(t) = c_n e^{nt} + d_n e^{-nt}$ $\Rightarrow$ $X_n(r) = c_n r^{-n} + d_n r^n$.

- We have

$$\begin{aligned} u_0(r, \theta) &= X_0(r)Y_0(\theta) = (-c_0 \log r + d_0)a_0, \\ u_n(r, \theta) &= X_n(r)Y_n(\theta) = (c_n r^{-n} + d_n r^n)(a_n \cos n\theta + b_n \sin n\theta). \end{aligned}$$

But u must be finite at $r = 0$, so $c_n = 0$, $n = 0, 1, 2, \ldots$.

$$\begin{aligned} u_0(r, \theta) &= d_0 a_0, \\ u_n(r, \theta) &= d_n r^n (a_n \cos n\theta + b_n \sin n\theta). \end{aligned}$$

By superposition, we write

$$\boxed{u(r, \theta) = \tilde{a}_0 + \sum_{n=1}^{\infty} r^n (\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta).}$$

Boundary condition gives

$$u_r(1, \theta) = \sum_{n=1}^{\infty} n(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta) = h(\theta).$$

The coefficients a_n, b_n for $n \ge 1$ are determined from the Fourier series for $h(\theta)$.
a_0 is not determined by $h(\theta)$ and therefore may take an arbitrary value. Moreover,

the constant term in the Fourier series for $h(\theta)$ must be zero [i.e., $\int_0^{2\pi} h(\theta)d\theta = 0$]. Therefore, the problem is **not** solvable for an arbitrary function $h(\theta)$, and when it is solvable, the solution is **not** unique. $\square$

Problem 4: The 2D LAPLACE Equation on a Circle.
Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} = \{(r, \theta) : 0 \le r < 1,\ 0 \le \theta < 2\pi\}$, and use separation of variables (r, θ) to solve the Dirichlet problem

$$\begin{cases} \triangle u = 0 & \text{in } \Omega \\ u(1, \theta) = g(\theta) & \text{for } 0 \le \theta < 2\pi. \end{cases}$$

Proof. Use polar coordinates (r, θ)

$$\begin{cases} u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 & \text{for } 0 \le r < 1,\ 0 \le \theta < 2\pi \\ u(1, \theta) = g(\theta) & \text{for } 0 \le \theta < 2\pi. \end{cases}$$

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

Let $r = e^{-t}$, $u(r(t), \theta)$.

$$\begin{aligned} u_t &= u_r r_t = -e^{-t} u_r, \\ u_{tt} &= (-e^{-t} u_r)_t = e^{-t} u_r + e^{-2t} u_{rr} = r u_r + r^2 u_{rr}. \end{aligned}$$

Thus, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t, \theta) = X(t) Y(\theta)$, which gives $X''(t) Y(\theta) + X(t) Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

• From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$.
$\lambda_n = n^2$, $n = 0, 1, 2, \ldots$.
• With these values of λ_n we solve $X''(t) - n^2 X(t) = 0$.
If $n = 0$, $X_0(t) = c_0 t + d_0$. $\Rightarrow$ $X_0(r) = -c_0 \log r + d_0$.
If $n \ne 0$, $X_n(t) = c_n e^{nt} + d_n e^{-nt}$ $\Rightarrow$ $X_n(r) = c_n r^{-n} + d_n r^n$.
• We have

$$\begin{aligned} u_0(r, \theta) &= X_0(r) Y_0(\theta) = (-c_0 \log r + d_0) a_0, \\ u_n(r, \theta) &= X_n(r) Y_n(\theta) = (c_n r^{-n} + d_n r^n)(a_n \cos n\theta + b_n \sin n\theta). \end{aligned}$$

But u must be finite at $r = 0$, so $c_n = 0$, $n = 0, 1, 2, \ldots$.

$$\begin{aligned} u_0(r, \theta) &= d_0 a_0, \\ u_n(r, \theta) &= d_n r^n (a_n \cos n\theta + b_n \sin n\theta). \end{aligned}$$

By superposition, we write

$$\boxed{u(r, \theta) = \tilde{a}_0 + \sum_{n=1}^{\infty} r^n (\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta).}$$

Boundary condition gives

$$u(1, \theta) = \tilde{a}_0 + \sum_{n=1}^{\infty} (\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta) = g(\theta).$$

$$\begin{aligned}
\tilde{a}_0 &= \frac{1}{\pi}\int_0^{\pi} g(\theta)\,d\theta,\\
\tilde{a}_n &= \frac{2}{\pi}\int_0^{\pi} g(\theta)\cos n\theta\,d\theta,\\
\tilde{b}_n &= \frac{2}{\pi}\int_0^{\pi} g(\theta)\sin n\theta\,d\theta.
\end{aligned}$$

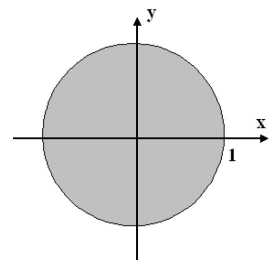
$\square$

Problem (F'94, #6): The 2D LAPLACE Equation on a Circle.

Find all solutions of the homogeneous equation

$$u_{xx} + u_{yy} = 0, \qquad x^2 + y^2 < 1,$$
$$\frac{\partial u}{\partial n} - u = 0, \qquad x^2 + y^2 = 1.$$

Hint: $\triangle = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ *in polar coordinates.*



Proof. Use polar coordinates (r, θ):

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & \text{for } 0 \le r < 1,\ 0 \le \theta < 2\pi \\ \frac{\partial u}{\partial r}(1,\theta) - u(1,\theta) = 0 & \text{for } 0 \le \theta < 2\pi. \end{cases}$$

Since we solve the equation on a circle, we have periodic conditions:

$$u(r,0) = u(r,2\pi) \quad \Rightarrow \quad X(r)Y(0) = X(r)Y(2\pi) \quad \Rightarrow \quad Y(0) = Y(2\pi),$$
$$u_\theta(r,0) = u_\theta(r,2\pi) \quad \Rightarrow \quad X(r)Y'(0) = X(r)Y'(2\pi) \quad \Rightarrow \quad Y'(0) = Y'(2\pi).$$

Also, we want the solution to be bounded. In particular, u is bounded for $r = 0$.

$$r^2u_{rr} + ru_r + u_{\theta\theta} = 0.$$

Let $r = e^{-t}$, $u(r(t), \theta)$, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t,\theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

- From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos\sqrt{\lambda}\theta + b_n \sin\sqrt{\lambda}\theta$.

Using periodic condition: $Y_n(0) = a_n$,

$$Y_n(2\pi) = a_n\cos(\sqrt{\lambda_n}\,2\pi) + b_n\sin(\sqrt{\lambda_n}\,2\pi) = a_n \quad \Rightarrow \quad \sqrt{\lambda_n} = n \quad \Rightarrow \quad \lambda_n = n^2.$$

Thus, $Y_n(\theta) = a_n\cos n\theta + b_n \sin n\theta$.

- With these values of λ_n we solve $X''(t) - n^2X(t) = 0$.

If $n = 0$, $X_0(t) = c_0t + d_0$. $\Rightarrow$ $X_0(r) = -c_0\log r + d_0$.
If $n \ne 0$, $X_n(t) = c_ne^{nt} + d_ne^{-nt}$ $\Rightarrow$ $X_n(r) = c_nr^{-n} + d_nr^n$.
u must be finite at $r = 0$ $\Rightarrow$ $c_n = 0$, $n = 0, 1, 2, \ldots$.

$$\boxed{u(r,\theta) = \tilde{a}_0 + \sum_{n=1}^{\infty} r^n(\tilde{a}_n\cos n\theta + \tilde{b}_n\sin n\theta).}$$

Boundary condition gives

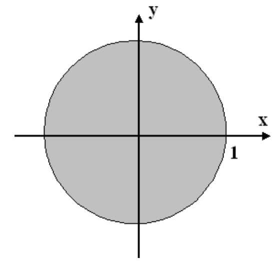
$$0 = u_r(1,\theta) - u(1,\theta) = -\tilde{a}_0 + \sum_{n=1}^{\infty}(n-1)(\tilde{a}_n\cos n\theta + \tilde{b}_n\sin n\theta).$$

Calculating Fourier coefficients gives $-2\pi\tilde{a}_0 = 0 \quad \Rightarrow \quad \tilde{a}_0 = 0$.
$\pi(n-1)a_n = 0 \quad \Rightarrow \quad \tilde{a}_n = 0$, $n = 2, 3, \ldots$.
a_1, b_1 are constants. Thus,

$$\boxed{u(r,\theta) = r(\tilde{a}_1\cos\theta + \tilde{b}_1\sin\theta).}$$

$\square$

Problem (S'00, #4).
a) *Let (r,θ) be polar coordinates on the plane, i.e. $x_1 + ix_2 = re^{i\theta}$. Solve the boudary value problem*



$$\begin{aligned} \triangle u &= 0 \qquad \text{in } r<1 \\ \partial u/\partial r &= f(\theta) \qquad \text{on } r=1, \end{aligned}$$

*beginning with the **Fourier series** for f (you may assume that f is continuously differentiable). Give your answer as a power series in $x_1 + ix_2$ plus a power series in $x_1 - ix_2$. There is a necessary condition on f for this boundary value problem to be solvable that you will find in the course of doing this.*

b) *Sum the series in part (a) to get a representation of u in the form*

$$u(r,\theta) = \int_0^{2\pi} N(r,\theta-\theta')f(\theta')\,d\theta'.$$

Proof. **a)** Green's identity gives the necessary compatibility condition on f:

$$\int_0^{2\pi} f(\theta)\,d\theta \;=\; \int_{r=1} \frac{\partial u}{\partial r}\,d\theta \;=\; \int_{\partial\Omega} \frac{\partial u}{\partial n}\,ds \;=\; \int_\Omega \triangle u\,dx \;=\; 0.$$

Use polar coordinates (r,θ):

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \;=\; 0 & \text{for } 0 \le r < 1,\ 0 \le \theta < 2\pi \\ \frac{\partial u}{\partial r}(1,\theta) \;=\; f(\theta) & \text{for } 0 \le \theta < 2\pi. \end{cases}$$

Since we solve the equation on a circle, we have periodic conditions:

$$\begin{aligned} u(r,0) = u(r,2\pi) \quad &\Rightarrow \quad X(r)Y(0) = X(r)Y(2\pi) \quad \Rightarrow \quad Y(0) = Y(2\pi), \\ u_\theta(r,0) = u_\theta(r,2\pi) \quad &\Rightarrow \quad X(r)Y'(0) = X(r)Y'(2\pi) \quad \Rightarrow \quad Y'(0) = Y'(2\pi). \end{aligned}$$

Also, we want the solution to be bounded. In particular, u is bounded for $r=0$.

$$r^2u_{rr} + ru_r + u_{\theta\theta} = 0.$$

Let $r = e^{-t}$, $u(r(t),\theta)$, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t,\theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

- From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos\sqrt{\lambda}\theta + b_n \sin\sqrt{\lambda}\theta$.

Using periodic condition: $Y_n(0) = a_n$,

$$Y_n(2\pi) = a_n \cos(\sqrt{\lambda_n}\,2\pi) + b_n \sin(\sqrt{\lambda_n}\,2\pi) = a_n \quad \Rightarrow \quad \sqrt{\lambda_n} = n \quad \Rightarrow \quad \lambda_n = n^2.$$

Thus, $Y_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$.

- With these values of λ_n we solve $X''(t) - n^2X(t) = 0$.

If $n = 0$, $X_0(t) = c_0t + d_0$. $\Rightarrow$ $X_0(r) = -c_0 \log r + d_0$.

If $n \neq 0$, $X_n(t) = c_n e^{nt} + d_n e^{-nt} \Rightarrow X_n(r) = c_n r^{-n} + d_n r^n$.
u must be finite at $r = 0 \Rightarrow c_n = 0$, $n = 0, 1, 2, \ldots$.

$$\boxed{u(r,\theta) = \tilde{a}_0 + \sum_{n=1}^{\infty} r^n(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta).}$$

Since

$$u_r(r,\theta) = \sum_{n=1}^{\infty} n r^{n-1}(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta),$$

the boundary condition gives

$$u_r(1,\theta) = \sum_{n=1}^{\infty} n\,(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta) = f(\theta).$$

$$\begin{aligned} \tilde{a}_n &= \frac{1}{n\pi}\int_0^{2\pi} f(\theta)\,\cos n\theta\,d\theta, \\ \tilde{b}_n &= \frac{1}{n\pi}\int_0^{2\pi} f(\theta)\,\sin n\theta\,d\theta. \end{aligned}$$

$\tilde{a}_0$ is not determined by $f(\theta)$ (since $\int_0^{2\pi} f(\theta)\,d\theta = 0$). Therefore, it may take an arbitrary value. Moreover, the constant term in the Fourier series for $f(\theta)$ must be zero [i.e., $\int_0^{2\pi} f(\theta) d\theta = 0$]. Therefore, the problem is **not** solvable for an arbitrary function $f(\theta)$, and when it is solvable, the solution is **not** unique.

b) In part (a), we obtained the solution and the Fourier coefficients:

$$\begin{aligned} \tilde{a}_n &= \frac{1}{n\pi}\int_0^{2\pi} f(\theta')\,\cos n\theta'\,d\theta', \\ \tilde{b}_n &= \frac{1}{n\pi}\int_0^{2\pi} f(\theta')\,\sin n\theta'\,d\theta'. \end{aligned}$$

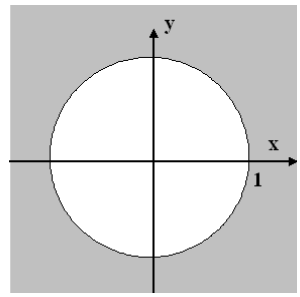
$$\begin{aligned} u(r,\theta) &= \tilde{a}_0 + \sum_{n=1}^{\infty} r^n(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta) \\ &= \tilde{a}_0 + \sum_{n=1}^{\infty} r^n\bigg(\bigg[\frac{1}{n\pi}\int_0^{2\pi} f(\theta')\,\cos n\theta'\,d\theta'\bigg]\cos n\theta + \bigg[\frac{1}{n\pi}\int_0^{2\pi} f(\theta')\,\sin n\theta'\,d\theta'\bigg]\sin n\theta\bigg) \\ &= \tilde{a}_0 + \sum_{n=1}^{\infty} \frac{r^n}{n\pi}\int_0^{2\pi} f(\theta')\,\big[\cos n\theta' \cos n\theta + \sin n\theta' \sin n\theta\big]\,d\theta' \\ &= \tilde{a}_0 + \sum_{n=1}^{\infty} \frac{r^n}{n\pi}\int_0^{2\pi} f(\theta')\,\cos n(\theta' - \theta)\,d\theta' \\ &= \tilde{a}_0 + \int_0^{2\pi} \underbrace{\sum_{n=1}^{\infty} \frac{r^n}{n\pi}\,\cos n(\theta - \theta')}_{N(r,\theta-\theta')}\,f(\theta')\,d\theta'. \end{aligned}$$

$\square$

Problem (S'92, #6). *Consider the Laplace equation*

$$u_{xx} + u_{yy} = 0$$

for $x^2 + y^2 \geq 1$. Denoting by $x = r\cos\theta$, $y = r\sin\theta$ polar coordinates, let $f = f(\theta)$ be a given smooth function of θ. Construct a uniformly bounded solution which satisfies boundary conditions



$$u = f \qquad \textit{for } \; x^2 + y^2 = 1.$$

What conditions has f to satisfy such that

$$\lim_{x^2+y^2\to\infty} (x^2 + y^2)u(x, y) = 0?$$

Proof. Use polar coordinates (r, θ):

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \; = \; 0 & \text{for } \; r \geq 1 \\ u(1, \theta) = f(\theta) & \text{for } \; 0 \leq \theta < 2\pi. \end{cases}$$

Since we solve the equation on outside of a circle, we have periodic conditions:

$$u(r, 0) = u(r, 2\pi) \quad \Rightarrow \quad X(r)Y(0) = X(r)Y(2\pi) \quad \Rightarrow \quad Y(0) = Y(2\pi),$$
$$u_\theta(r, 0) = u(r, 2\pi) \quad \Rightarrow \quad X(r)Y'(0) = X(r)Y'(2\pi) \quad \Rightarrow \quad Y'(0) = Y'(2\pi).$$

Also, we want the solution to be bounded. In particular, u is bounded for $r = \infty$.

$$r^2u_{rr} + ru_r + u_{\theta\theta} = 0.$$

Let $r = e^{-t}$, $u(r(t), \theta)$, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t, \theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

• From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos\sqrt{\lambda}\theta + b_n \sin\sqrt{\lambda}\theta$.
Using periodic condition: $Y_n(0) = a_n$,

$$Y_n(2\pi) = a_n \cos(\sqrt{\lambda_n}\, 2\pi) + b_n \sin(\sqrt{\lambda_n}\, 2\pi) = a_n \quad \Rightarrow \quad \sqrt{\lambda_n} = n \quad \Rightarrow \quad \lambda_n = n^2.$$

Thus, $Y_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$.
• With these values of λ_n we solve $X''(t) - n^2X(t) = 0$.
If $n = 0$, $X_0(t) = c_0t + d_0$. $\Rightarrow$ $X_0(r) = -c_0 \log r + d_0$.
If $n \neq 0$, $X_n(t) = c_ne^{nt} + d_ne^{-nt}$ $\Rightarrow$ $X_n(r) = c_nr^{-n} + d_nr^n$.
u must be finite at $r = \infty$ $\Rightarrow$ $c_0 = 0$, $d_n = 0$, $n = 1, 2, \ldots$.

$$\boxed{u(r, \theta) = \tilde{a}_0 + \sum_{n=1}^{\infty} r^{-n}(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta).}$$

Boundary condition gives

$$f(\theta) \; = \; u(1, \theta) \; = \; \tilde{a}_0 + \sum_{n=1}^{\infty}(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta).$$

$$\begin{cases} 2\pi\tilde{a}_0 = \int_0^{2\pi} f(\theta)\,d\theta, \\ \pi\tilde{a}_n = \int_0^{2\pi} f(\theta)\,\cos n\theta\,d\theta, \\ \pi\tilde{b}_n = \int_0^{2\pi} f(\theta)\,\sin n\theta\,d\theta. \end{cases} \qquad \Rightarrow \qquad \begin{cases} f_0 = \tilde{a}_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta, \\ f_n = \tilde{a}_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,\cos n\theta\,d\theta, \\ \tilde{f}_n = \tilde{b}_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,\sin n\theta\,d\theta. \end{cases}$$

- We need to find conditions for f such that

$$\lim_{x^2+y^2\to\infty} (x^2+y^2)u(x,y) = 0, \qquad \text{or}$$

$$\lim_{r\to\infty} r^2 u(r,\theta) \underbrace{=}_{need} 0,$$

$$\lim_{r\to\infty} r^2\Big[f_0 + \sum_{n=1}^{\infty} r^{-n}(f_n\cos n\theta + \tilde{f}_n\sin n\theta)\Big] \underbrace{=}_{need} 0.$$

Since

$$\lim_{r\to\infty}\Big[\sum_{n>2}^{\infty} r^{2-n}(f_n\cos n\theta + \tilde{f}_n\sin n\theta)\Big] = 0,$$

we need

$$\lim_{r\to\infty}\Big[r^2 f_0 + \sum_{n=1}^{2} r^{2-n}(f_n\cos n\theta + \tilde{f}_n\sin n\theta)\Big] \underbrace{=}_{need} 0.$$

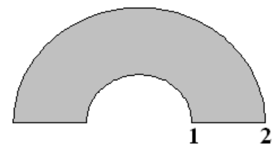
Thus, the conditions are

$$\boxed{f_n,\, \tilde{f}_n = 0, \quad n = 0,1,2.}$$

□

Problem (F'96, #2): The 2D LAPLACE Equation on a Semi-Annulus.
Solve the Laplace equation in the semi-annulus

$$\begin{cases} \triangle u = 0, & 1 < r < 2,\ 0 < \theta < \pi, \\ u(r,0) = u(r,\pi) = 0, & 1 < r < 2, \\ u(1,\theta) = \sin\theta, & 0 < \theta < \pi, \\ u(2,\theta) = 0, & 0 < \theta < \pi. \end{cases}$$



Hint: Use the formula $\triangle = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$ *for the Laplacian in polar coordinates.*

Proof. Use polar coordinates (r,θ)

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad 1 < r < 2,\ 0 < \theta < \pi,$$
$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

With $r = e^{-t}$, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t,\theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

- From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos\sqrt{\lambda}\theta + b_n \sin\sqrt{\lambda}\theta$.

Boundary conditions give

$$u_n(r,0) = 0 = X_n(r)Y_n(0) = 0, \quad \Rightarrow \quad Y_n(0) = 0,$$
$$u_n(r,\pi) = 0 = X_n(r)Y_n(\pi) = 0, \quad \Rightarrow \quad Y_n(\pi) = 0.$$

Thus, $0 = Y_n(0) = a_n$, and $Y_n(\pi) = b_n \sin\sqrt{\lambda}\pi = 0 \ \Rightarrow\ \sqrt{\lambda} = n \ \Rightarrow\ \lambda_n = n^2$.
Thus, $Y_n(\theta) = b_n \sin n\theta$, $n = 1, 2, \ldots$.
- With these values of λ_n we solve $X''(t) - n^2 X(t) = 0$.

If $n = 0$, $X_0(t) = c_0 t + d_0$. $\quad \Rightarrow \quad X_0(r) = -c_0 \log r + d_0$.
If $n > 0$, $X_n(t) = c_n e^{nt} + d_n e^{-nt} \quad \Rightarrow \quad X_n(r) = c_n r^{-n} + d_n r^n$.
- We have,

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} X_n(r)Y_n(\theta) = \sum_{n=1}^{\infty}(\tilde{c}_n r^{-n} + \tilde{d}_n r^n)\sin n\theta.}$$

Using the other two boundary conditions, we obtain

$$\sin\theta = u(1,\theta) = \sum_{n=1}^{\infty}(\tilde{c}_n + \tilde{d}_n)\sin n\theta \quad \Rightarrow \quad \begin{cases} \tilde{c}_1 + \tilde{d}_1 = 1, \\ \tilde{c}_n + \tilde{d}_n = 0, \quad n = 2, 3, \ldots. \end{cases}$$
$$0 = u(2,\theta) = \sum_{n=1}^{\infty}(\tilde{c}_n 2^{-n} + \tilde{d}_n 2^n)\sin n\theta \quad \Rightarrow \quad \tilde{c}_n 2^{-n} + \tilde{d}_n 2^n = 0, \quad n = 1, 2, \ldots.$$

Thus, the coefficients are given by

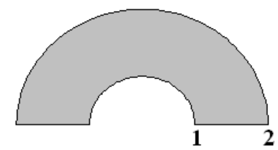
$$c_1 = \frac{4}{3},\ \ d_1 = -\frac{1}{3};$$
$$c_n = 0, \quad d_n = 0.$$

$$\boxed{u(r,\theta) = \left(\frac{4}{3r} - \frac{r}{3}\right)\sin\theta.}$$

$\square$

Problem (S'98, #8): The 2D LAPLACE Equation on a Semi-Annulus.
Solve

$$\begin{cases} \triangle u = 0, & 1 < r < 2,\ 0 < \theta < \pi, \\ u(r,0) = u(r,\pi) = 0, & 1 < r < 2, \\ u(1,\theta) = u(2,\theta) = 1, & 0 < \theta < \pi. \end{cases}$$



Proof. Use polar coordinates (r, θ)

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad \text{for } 1 < r < 2,\ 0 < \theta < \pi,$$
$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

With $r = e^{-t}$, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t,\theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

• From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$.
Boundary conditions give

$$u_n(r,0) = 0 = X_n(r)Y_n(0) = 0, \quad \Rightarrow \quad Y_n(0) = 0,$$
$$u_n(r,\pi) = 0 = X_n(r)Y_n(\pi) = 0, \quad \Rightarrow \quad Y_n(\pi) = 0.$$

Thus, $0 = Y_n(0) = a_n$, and $Y_n(\theta) = b_n \sin n\theta$.
$\lambda_n = n^2$, $n = 1, 2, \ldots$.
• With these values of λ_n we solve $X''(t) - n^2 X(t) = 0$.
If $n = 0$, $X_0(t) = c_0 t + d_0$. $\Rightarrow$ $X_0(r) = -c_0 \log r + d_0$.
If $n > 0$, $X_n(t) = c_n e^{nt} + d_n e^{-nt}$ $\Rightarrow$ $X_n(r) = c_n r^{-n} + d_n r^n$.
• We have,

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} X_n(r)Y_n(\theta) = \sum_{n=1}^{\infty} (\tilde{c}_n r^{-n} + \tilde{d}_n r^n)\sin n\theta.}$$

Using the other two boundary conditions, we obtain

$$u(1,\theta) = 1 = \sum_{n=1}^{\infty} (\tilde{c}_n + \tilde{d}_n)\sin n\theta,$$
$$u(2,\theta) = 1 = \sum_{n=1}^{\infty} (\tilde{c}_n 2^{-n} + \tilde{d}_n 2^n)\sin n\theta,$$

which give the two equations for $\tilde{c}_n$ and $\tilde{d}_n$:

$$\int_0^{\pi} \sin n\theta \, d\theta = \frac{\pi}{2}(\tilde{c}_n + \tilde{d}_n),$$
$$\int_0^{\pi} \sin n\theta \, d\theta = \frac{\pi}{2}(\tilde{c}_n 2^{-n} + \tilde{d}_n 2^n),$$

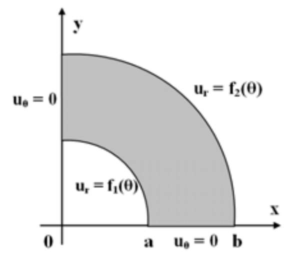
that can be solved. □

Problem (F'89, #1). *Consider Laplace equation inside a* 90° *sector of a circular annulus*

$$\triangle u = 0 \qquad a < r < b, \;\; 0 < \theta < \frac{\pi}{2}$$

subject to the boundary conditions

$$\frac{\partial u}{\partial \theta}(r, 0) = 0, \qquad \frac{\partial u}{\partial \theta}(r, \frac{\pi}{2}) = 0,$$
$$\frac{\partial u}{\partial r}(a, \theta) = f_1(\theta), \quad \frac{\partial u}{\partial r}(b, \theta) = f_2(\theta),$$



where $f_1(\theta)$, $f_2(\theta)$ *are continuously differentiable.*

a) *Find the solution of this equation with the prescribed boundary conditions using separation of variables.*

Proof. **a)** Use polar coordinates (r, θ)

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad \text{for} \;\; a < r < b, \;\; 0 < \theta < \frac{\pi}{2},$$
$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

With $r = e^{-t}$, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t, \theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

- From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $\;Y_n(\theta) = a_n \cos\sqrt{\lambda}\theta + b_n \sin\sqrt{\lambda}\theta$.

Boundary conditions give

$$u_{n\theta}(r, 0) \;=\; X_n(r)Y_n'(0) = 0 \;\;\Rightarrow\;\; Y_n'(0) = 0,$$
$$u_{n\theta}(r, \frac{\pi}{2}) \;=\; X_n(r)Y_n'(\frac{\pi}{2}) = 0 \;\;\Rightarrow\;\; Y_n'(\frac{\pi}{2}) = 0.$$

$Y_n'(\theta) = -a_n\sqrt{\lambda_n}\sin\sqrt{\lambda_n}\theta + b_n\sqrt{\lambda_n}\cos\sqrt{\lambda_n}\theta$. Thus, $Y_n'(0) = b_n\sqrt{\lambda_n} = 0 \;\;\Rightarrow\;\; b_n = 0$.
$Y_n'(\frac{\pi}{2}) = -a_n\sqrt{\lambda_n}\sin\sqrt{\lambda_n}\frac{\pi}{2} = 0 \;\;\Rightarrow\;\; \sqrt{\lambda_n}\frac{\pi}{2} = n\pi \;\;\Rightarrow\;\; \lambda_n = (2n)^2$.
Thus, $Y_n(\theta) = a_n\cos(2n\theta)$, $\;n = 0, 1, 2, \ldots$.
In particular, $Y_0(\theta) = a_0 t + b_0$. Boundary conditions give $Y_0(\theta) = b_0$.

- With these values of λ_n we solve $\;X''(t) - (2n)^2 X(t) = 0$.

If $n = 0$, $\;X_0(t) = c_0 t + d_0. \qquad \Rightarrow \;\; X_0(r) = -c_0\log r + d_0$.
If $n > 0$, $\;X_n(t) = c_n e^{2nt} + d_n e^{-2nt} \;\;\Rightarrow\;\; X_n(r) = c_n r^{-2n} + d_n r^{2n}$.

$$\boxed{u(r, \theta) \;=\; \tilde{c}_0 \log r + \tilde{d}_0 + \sum_{n=1}^{\infty}(\tilde{c}_n r^{-2n} + \tilde{d}_n r^{2n})\cos(2n\theta).}$$

Using the other two boundary conditions, we obtain

$$u_r(r, \theta) \;=\; \frac{\tilde{c}_0}{r} + \sum_{n=1}^{\infty}(-2n\tilde{c}_n r^{-2n-1} + 2n\tilde{d}_n r^{2n-1})\cos(2n\theta).$$

$$f_1(\theta) = u_r(a,\theta) = \frac{\tilde{c}_0}{a} + 2\sum_{n=1}^{\infty} n(-\tilde{c}_n a^{-2n-1} + \tilde{d}_n a^{2n-1})\cos(2n\theta),$$
$$f_2(\theta) = u_r(b,\theta) = \frac{\tilde{c}_0}{b} + 2\sum_{n=1}^{\infty} n(-\tilde{c}_n b^{-2n-1} + \tilde{d}_n b^{2n-1})\cos(2n\theta).$$

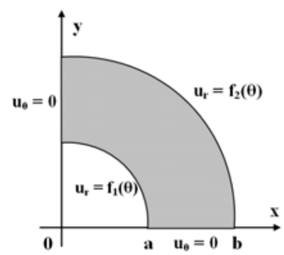
which give the two equations for $\tilde{c}_n$ and $\tilde{d}_n$:

$$\int_0^{\frac{\pi}{2}} f_1(\theta)\cos(2n\theta)\,d\theta = \frac{\pi}{2}n(-\tilde{c}_n a^{-2n-1} + \tilde{d}_n a^{2n-1}),$$
$$\int_0^{\frac{\pi}{2}} f_2(\theta)\sin(2n\theta)\,d\theta = \frac{\pi}{2}n(-\tilde{c}_n b^{-2n-1} + \tilde{d}_n b^{2n-1}).$$

□

b) *Show that the solution exists if and only if*

$$a\int_0^{\frac{\pi}{2}} f_1(\theta)\,d\theta \,-\, b\int_0^{\frac{\pi}{2}} f_2(\theta)\,d\theta \,=\, 0.$$



Proof. Using Green's identity, we obtain:

$$\begin{aligned}
0 &= \int_\Omega \triangle u\,dx = \int_{\partial\Omega}\frac{\partial u}{\partial n} \\
&= \int_0^{\frac{\pi}{2}} \frac{\partial u}{\partial r}(b,\theta)\,d\theta + \int_{\frac{\pi}{2}}^0 -\frac{\partial u}{\partial r}(a,\theta)\,d\theta + \int_a^b -\frac{\partial u}{\partial \theta}(r,0)\,dr + \int_b^a \frac{\partial u}{\partial \theta}\Big(r,\frac{\pi}{2}\Big)\,dr \\
&= \int_0^{\frac{\pi}{2}} f_2(\theta)\,d\theta + \int_0^{\frac{\pi}{2}} f_1(\theta)\,d\theta + 0 + 0 \\
&= \int_0^{\frac{\pi}{2}} f_1(\theta)\,d\theta + \int_0^{\frac{\pi}{2}} f_2(\theta)\,d\theta.
\end{aligned}$$

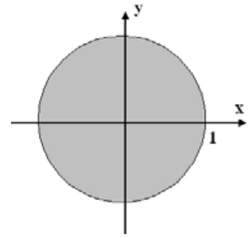
□

c) *Is the solution* ***unique****?*

Proof. No, since the boundary conditions are Neumann. The solution is unique only up to a constant. □

Problem (S'99, #4). *Let $u(x,y)$ be **harmonic** inside the unit disc, with boundary values along the unit circle*

$$u(x,y) = \begin{cases} 1, & y > 0 \\ 0, & y \le 0. \end{cases}$$



Compute $u(0,0)$ and $u(0,y)$.

Proof. Since u is harmonic, $\triangle u = 0$. Use polar coordinates (r,θ)

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & 0 \le r < 1, \ 0 \le \theta < 2\pi \\ u(1,\theta) = \begin{cases} 1, & 0 < \theta < \pi \\ 0, & \pi \le \theta \le 2\pi. \end{cases} \end{cases}$$

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

With $r = e^{-t}$, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t,\theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

• From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$.
$\lambda_n = n^2$, $n = 1, 2, \ldots$.
• With these values of λ_n we solve $X''(t) - n^2 X(t) = 0$.
If $n = 0$, $X_0(t) = c_0 t + d_0$. $\Rightarrow$ $X_0(r) = -c_0 \log r + d_0$.
If $n > 0$, $X_n(t) = c_n e^{nt} + d_n e^{-nt}$ $\Rightarrow$ $X_n(r) = c_n r^{-n} + d_n r^n$.
• We have

$$\begin{aligned} u_0(r,\theta) &= X_0(r)Y_0(\theta) = (-c_0 \log r + d_0)a_0, \\ u_n(r,\theta) &= X_n(r)Y_n(\theta) = (c_n r^{-n} + d_n r^n)(a_n \cos n\theta + b_n \sin n\theta). \end{aligned}$$

But u must be finite at $r = 0$, so $c_n = 0$, $n = 0, 1, 2, \ldots$.

$$\begin{aligned} u_0(r,\theta) &= \tilde{a}_0, \\ u_n(r,\theta) &= r^n(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta). \end{aligned}$$

By superposition, we write

$$\boxed{u(r,\theta) = \tilde{a}_0 + \sum_{n=1}^{\infty} r^n(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta).}$$

Boundary condition gives

$$u(1,\theta) = \tilde{a}_0 + \sum_{n=1}^{\infty}(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta) = \begin{cases} 1, & 0 < \theta < \pi \\ 0, & \pi \le \theta \le 2\pi, \end{cases}$$

and the coefficients $\tilde{a}_n$ and $\tilde{b}_n$ are determined from the above equation.[71] □

[71] See Yana's solutions, where Green's function on a unit disk is constructed.

23 Problems: Separation of Variables - Poisson Equation

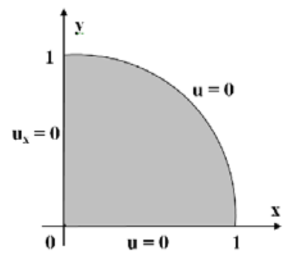
Problem (F'91, #2): The 2D POISSON Equation on a Quarter-Circle.
Solve explicitly the following boundary value problem

$$u_{xx} + u_{yy} = f(x, y)$$

in the domain $\Omega = \{(x, y), x > 0, y > 0, x^2 + y^2 < 1\}$
with boundary conditions

$$\begin{aligned} u &= 0 \qquad \text{for} \quad y = 0,\ 0 < x < 1, \\ \frac{\partial u}{\partial x} &= 0 \qquad \text{for} \quad x = 0,\ 0 < y < 1, \\ u &= 0 \qquad \text{for} \quad x > 0,\ y > 0,\ x^2 + y^2 = 1. \end{aligned}$$

Function $f(x, y)$ *is known and is assumed to be continuous.*



Proof. Use polar coordinates (r, θ):

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \ = \ f(r, \theta) & 0 \le r < 1,\ 0 \le \theta < \frac{\pi}{2} \\ u(r, 0) \ = \ 0 & 0 \le r < 1, \\ u_\theta(r, \frac{\pi}{2}) = 0 & 0 \le r < 1, \\ u(1, \theta) \ = \ 0 & 0 \le \theta \le \frac{\pi}{2}. \end{cases}$$

We solve

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

Let $r = e^{-t}$, $u(r(t), \theta)$, we have

$$u_{tt} + u_{\theta\theta} = 0. \qquad \circledast$$

Let $u(t, \theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

• From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos\sqrt{\lambda}\theta + b_n \sin\sqrt{\lambda}\theta$. Boundary conditions:

$$\begin{cases} u(r, 0) = X(r)Y(0) = 0 \\ u_\theta(r, \frac{\pi}{2}) = X(r)Y'(\frac{\pi}{2}) = 0 \end{cases} \Rightarrow \quad Y(0) = Y'\Big(\frac{\pi}{2}\Big) = 0.$$

Thus, $Y_n(0) = a_n = 0$, and $Y_n'(\frac{\pi}{2}) = \sqrt{\lambda_n} b_n \cos\sqrt{\lambda_n}\frac{\pi}{2} = 0$
$\Rightarrow \quad \sqrt{\lambda_n}\frac{\pi}{2} = n\pi - \frac{\pi}{2},\ n = 1, 2, \ldots \quad \Rightarrow \quad \lambda_n = (2n-1)^2.$
Thus, $Y_n(\theta) = b_n \sin(2n-1)\theta$, $n = 1, 2, \ldots$. Thus, we have

$$u(r, \theta) = \sum_{n=1}^{\infty} X_n(r) \sin[(2n-1)\theta].$$

We now plug this equation into ⊛ with inhomogeneous term and obtain

$$\begin{aligned}
\sum_{n=1}^{\infty}\big(X_n''(t)\,\sin[(2n-1)\theta]\;-\;(2n-1)^2X_n(t)\,\sin[(2n-1)\theta]\big) &= f(t,\theta),\\
\sum_{n=1}^{\infty}\big(X_n''(t)-(2n-1)^2X_n(t)\big)\sin[(2n-1)\theta] &= f(t,\theta),\\
\frac{\pi}{4}\big(X_n''(t)-(2n-1)^2X_n(t)\big) &= \int_0^{\frac{\pi}{2}} f(t,\theta)\,\sin[(2n-1)\theta]\,d\theta,\\
X_n''(t)-(2n-1)^2X_n(t) &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(t,\theta)\,\sin[(2n-1)\theta]\,d\theta.
\end{aligned}$$

The solution to this equation is

$$\begin{aligned}
X_n(t) &= c_ne^{(2n-1)t}+d_ne^{-(2n-1)t}+U_{np}(t), \qquad \text{or}\\
X_n(r) &= c_nr^{-(2n-1)}+d_nr^{(2n-1)}+u_{np}(r),
\end{aligned}$$

where u_{np} is the particular solution of inhomogeneous equation.
u must be finite at $r=0 \;\Rightarrow\; c_n=0,\ n=1,2,\ldots$. Thus,

$$\boxed{u(r,\theta)=\sum_{n=1}^{\infty}\big(d_nr^{(2n-1)}+u_{np}(r)\big)\;\sin[(2n-1)\theta].}$$

Using the last boundary condition, we have

$$\begin{aligned}
0=u(1,\theta) &= \sum_{n=1}^{\infty}\big(d_n+u_{np}(1)\big)\;\sin[(2n-1)\theta],\\
&\Rightarrow\;\; 0=\frac{\pi}{4}(d_n+u_{np}(1)),\\
&\Rightarrow\;\; d_n=-u_{np}(1).
\end{aligned}$$

$$\boxed{u(r,\theta)=\sum_{n=1}^{\infty}\big(-u_{np}(1)r^{(2n-1)}+u_{np}(r)\big)\;\sin[(2n-1)\theta].}$$

The method used to solve this problem is similar to section Problems: Eigenvalues of the Laplacian - Poisson Equation:
1) First, we find $Y_n(\theta)$ eigenfunctions.
2) Then, we plug in our guess $u(t,\theta)=X(t)Y(\theta)$ into the equation $u_{tt}+u_{\theta\theta}=f(t,\theta)$ and solve an ODE in $X(t)$.

Note the similar problem on 2D Poisson equation on a square domain. The problem is used by first finding the eigenvalues and eigenfunctions of the Laplacian, and then expanding $f(x,y)$ in eigenfunctions, and comparing coefficients of f with the general solution $u(x,y)$.
Here, however, this could not be done because of the circular geometry of the domain. In particular, the boundary conditions do not give enough information to find explicit representations for μ_m and ν_n. Also, the condition $u=0$ for $x>0,\ y>0,\ x^2+y^2=1$

can not be used.
[72] □

[72] ChiuYen's solutions have attempts to solve this problem using Green's function.

24 Problems: Separation of Variables - Wave Equation

Example (McOwen 3.1 #2). *We considered the initial/boundary value problem and solved it using Fourier Series. We now solve it using the* ***Separation of Variables****.*

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < \pi, \ t > 0 \\ u(x,0) = 1, \quad u_t(x,0) = 0 & 0 < x < \pi \\ u(0,t) = 0, \quad u(\pi,t) = 0 & t \geq 0. \end{cases} \tag{24.1}$$

Proof. Assume $u(x,t) = X(x)T(t)$, then substitution in the PDE gives $XT'' - X''T = 0$.

$$\frac{X''}{X} = \frac{T''}{T} = -\lambda.$$

- From $X'' + \lambda X = 0$, we get $X_n(x) = a_n \cos nx + b_n \sin nx$. Boundary conditions give

$$\begin{cases} u(0,t) = X(0)T(t) = 0 \\ u(\pi,t) = X(\pi)T(t) = 0 \end{cases} \Rightarrow \quad X(0) = X(\pi) = 0.$$

Thus, $X_n(0) = a_n = 0$, and $X_n(x) = b_n \sin nx$, $\lambda_n = n^2$, $n = 1, 2, \ldots$.
- With these values of λ_n, we solve $T'' + n^2 T = 0$ to find $T_n(t) = c_n \sin nt + d_n \cos nt$.

Thus,

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \big(\tilde{c}_n \sin nt + \tilde{d}_n \cos nt\big) \sin nx, \\ u_t(x,t) &= \sum_{n=1}^{\infty} \big(n\tilde{c}_n \cos nt - n\tilde{d}_n \sin nt\big) \sin nx. \end{aligned}$$

- Initial conditions give

$$\begin{aligned} 1 = u(x,0) &= \sum_{n=1}^{\infty} \tilde{d}_n \sin nx, \\ 0 = u_t(x,0) &= \sum_{n=1}^{\infty} n\tilde{c}_n \sin nx. \end{aligned}$$

By orthogonality, we may multiply both equations by $\sin mx$ and integrate:

$$\int_0^\pi \sin mx \, dx = \tilde{d}_m \frac{\pi}{2},$$
$$\int_0^\pi 0 \, dx = n\tilde{c}_n \frac{\pi}{2},$$

which gives the coefficients

$$\tilde{d}_n = \frac{2}{n\pi}(1 - \cos n\pi) = \begin{cases} \frac{4}{n\pi}, & n \text{ odd}, \\ 0, & n \text{ even}, \end{cases} \qquad \text{and} \qquad \tilde{c}_n = 0.$$

Plugging the coefficients into a formula for $u(x,t)$, we get

$$\boxed{u(x,t) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)t \, \sin(2n+1)x}{(2n+1)}.}$$

□

Example. *Use the method of separation of variables to find the solution to:*

$$\begin{cases} u_{tt} + 3u_t + u = u_{xx}, & 0 < x < 1 \\ u(0,t) = 0, \quad u(1,t) = 0, \\ u(x,0) = 0, \quad u_t(x,0) = x\sin(2\pi x). \end{cases}$$

Proof. Assume $u(x,t) = X(x)T(t)$, then substitution in the PDE gives

$$\begin{aligned} XT'' + 3XT' + XT &= X''T, \\ \frac{T''}{T} + 3\frac{T'}{T} + 1 &= \frac{X''}{X} = -\lambda. \end{aligned}$$

• From $X'' + \lambda X = 0$, $X_n(x) = a_n \cos\sqrt{\lambda_n}x + b_n \sin\sqrt{\lambda_n}x$. Boundary conditions give

$$\begin{cases} u(0,t) = X(0)T(t) = 0 \\ u(1,t) = X(1)T(t) = 0 \end{cases} \Rightarrow \quad X(0) = X(1) = 0.$$

Thus, $X_n(0) = a_n = 0$, and $X_n(x) = b_n \sin\sqrt{\lambda_n}x$.
$X_n(1) = b_n \sin\sqrt{\lambda_n} = 0$. Hence, $\sqrt{\lambda_n} = n\pi$, or $\lambda_n = (n\pi)^2$, $n = 1, 2, \ldots$.

$$\boxed{\lambda_n = (n\pi)^2, \qquad X_n(x) = b_n \sin n\pi x.}$$

• With these values of λ_n, we solve

$$\begin{aligned} T'' + 3T' + T &= -\lambda_n T, \\ T'' + 3T' + T &= -(n\pi)^2 T, \\ T'' + 3T' + (1 + (n\pi)^2)T &= 0. \end{aligned}$$

We can solve this 2nd-order ODE with the following guess, $T(t) = ce^{st}$ to obtain $s = -\frac{3}{2} \pm \sqrt{\frac{5}{4} - (n\pi)^2}$. For $n \geq 1$, $\frac{5}{4} - (n\pi)^2 < 0$. Thus, $s = -\frac{3}{2} \pm i\sqrt{(n\pi)^2 - \frac{5}{4}}$.

$$\boxed{T_n(t) = e^{-\frac{3}{2}t}\Big(c_n \cos\sqrt{(n\pi)^2 - \frac{5}{4}}t + d_n \sin\sqrt{(n\pi)^2 - \frac{5}{4}}t\Big).}$$

$$u(x,t) = X(x)T(t) = \sum_{n=1}^{\infty} e^{-\frac{3}{2}t}\Big(c_n \cos\sqrt{(n\pi)^2 - \frac{5}{4}}t + d_n \sin\sqrt{(n\pi)^2 - \frac{5}{4}}t\Big) \sin n\pi x.$$

• Initial conditions give

$$0 = u(x,0) = \sum_{n=1}^{\infty} c_n \sin n\pi x.$$

By orthogonality, we may multiply this equations by $\sin m\pi x$ and integrate:

$$\int_0^1 0\, dx = \frac{1}{2}c_m \quad \Rightarrow \quad c_m = 0.$$

Thus,

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} d_n e^{-\frac{3}{2}t}\Big(\sin\sqrt{(n\pi)^2-\frac{5}{4}}t\Big)\sin n\pi x. \\
u_t(x,t) &= \sum_{n=1}^{\infty}\Big[-\frac{3}{2}d_n e^{-\frac{3}{2}t}\Big(\sin\sqrt{(n\pi)^2-\frac{5}{4}}t\Big)+d_n e^{-\frac{3}{2}t}\Big(\sqrt{(n\pi)^2-\frac{5}{4}}\Big)\Big(\cos\sqrt{(n\pi)^2-\frac{5}{4}}t\Big)\Big]\sin n\pi x, \\
x\sin(2\pi x) &= u_t(x,0) = \sum_{n=1}^{\infty} d_n\Big(\sqrt{(n\pi)^2-\frac{5}{4}}\Big)\sin n\pi x.
\end{aligned}$$

By orthogonality, we may multiply this equations by $\sin m\pi x$ and integrate:

$$\int_0^1 x\sin(2\pi x)\,\sin(m\pi x)\,dx = d_m\frac{1}{2}\Big(\sqrt{(m\pi)^2-\frac{5}{4}}\Big),$$

$$d_n = \frac{2}{\sqrt{(n\pi)^2-\frac{5}{4}}}\int_0^1 x\sin(2\pi x)\,\sin(n\pi x)\,dx.$$

$$\boxed{u(x,t) = e^{-\frac{3}{2}t}\sum_{n=1}^{\infty} d_n\Big(\sin\sqrt{(n\pi)^2-\frac{5}{4}}t\Big)\sin n\pi x.}$$

□

Problem (F'04, #1). *Solve the following initial-boundary value problem for the wave equation with a potential term,*

$$\begin{cases} u_{tt}-u_{xx}+u=0 & 0<x<\pi,\ t<0 \\ u(0,t)=u(\pi,t)=0 & t>0 \\ u(x,0)=f(x), \quad u_t(x,0)=0 & 0<x<\pi, \end{cases}$$

where

$$f(x) = \begin{cases} x & \text{if } x\in(0,\pi/2), \\ \pi-x & \text{if } x\in(\pi/2,\pi). \end{cases}$$

The answer should be given in terms of an infinite series of explicitly given functions.

Proof. Assume $u(x,t)=X(x)T(t)$, then substitution in the PDE gives

$$\begin{aligned}
XT''-X''T+XT &= 0, \\
\frac{T''}{T}+1 &= \frac{X''}{X} = -\lambda.
\end{aligned}$$

- From $X''+\lambda X=0$, $X_n(x)=a_n\cos\sqrt{\lambda_n}x+b_n\sin\sqrt{\lambda_n}x$. Boundary conditions give

$$\begin{cases} u(0,t)=X(0)T(t)=0 \\ u(\pi,t)=X(\pi)T(t)=0 \end{cases} \Rightarrow \quad X(0)=X(\pi)=0.$$

Thus, $X_n(0)=a_n=0$, and $X_n(x)=b_n\sin\sqrt{\lambda_n}x$.
$X_n(\pi)=b_n\sin\sqrt{\lambda_n}\pi=0$. Hence, $\sqrt{\lambda_n}=n$, or $\lambda_n=n^2$, $n=1,2,\ldots$.

$$\boxed{\lambda_n=n^2, \qquad X_n(x)=b_n\sin nx.}$$

- With these values of λ_n, we solve

$$\begin{aligned} T'' + T &= -\lambda_n T, \\ T'' + T &= -n^2 T, \\ T_n'' + (1+n^2)T_n &= 0. \end{aligned}$$

The solution to this 2nd-order ODE is of the form:

$$\boxed{T_n(t) = c_n \cos\sqrt{1+n^2}\,t + d_n \sin\sqrt{1+n^2}\,t.}$$

$$\begin{aligned} u(x,t) &= X(x)T(t) = \sum_{n=1}^{\infty}\big(c_n \cos\sqrt{1+n^2}\,t + d_n \sin\sqrt{1+n^2}\,t\big)\sin nx. \\ u_t(x,t) &= \sum_{n=1}^{\infty}\big(-c_n(\sqrt{1+n^2})\sin\sqrt{1+n^2}\,t + d_n(\sqrt{1+n^2})\cos\sqrt{1+n^2}\,t\big)\sin nx. \end{aligned}$$

- Initial conditions give

$$\begin{aligned} f(x) &= u(x,0) = \sum_{n=1}^{\infty} c_n \sin nx. \\ 0 &= u_t(x,0) = \sum_{n=1}^{\infty} d_n(\sqrt{1+n^2})\,\sin nx. \end{aligned}$$

By orthogonality, we may multiply both equations by $\sin mx$ and integrate:

$$\begin{aligned} \int_0^{\pi} f(x)\,\sin mx\,dx &= c_m\frac{\pi}{2}, \\ \int_0^{\pi} 0\,dx &= d_m\frac{\pi}{2}\sqrt{1+m^2}, \end{aligned}$$

which gives the coefficients

$$\begin{aligned} c_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\,\sin nx\,dx \;=\; \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\,\sin nx\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}(\pi - x)\,\sin nx\,dx \\ &= \frac{2}{\pi}\Big[-x\frac{1}{n}\cos nx\Big|_0^{\frac{\pi}{2}} + \frac{1}{n}\int_0^{\frac{\pi}{2}}\cos nx\,dx\Big] + \frac{2}{\pi}\Big[-\frac{\pi}{n}\cos nx\Big|_{\frac{\pi}{2}}^{\pi} + x\frac{1}{n}\cos nx\Big|_{\frac{\pi}{2}}^{\pi} - \frac{1}{n}\int_{\frac{\pi}{2}}^{\pi}\cos nx\,dx\Big] \\ &= \frac{2}{\pi}\Big[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin 0\Big] \\ &+ \frac{2}{\pi}\Big[-\frac{\pi}{n}\cos n\pi + \frac{\pi}{n}\cos\frac{n\pi}{2} + \frac{\pi}{n}\cos n\pi - \frac{\pi}{2n}\cos\frac{n\pi}{2} - \frac{1}{n^2}\sin n\pi + \frac{1}{n^2}\sin\frac{n\pi}{2}\Big] \\ &= \frac{2}{\pi}\Big[\frac{1}{n^2}\sin\frac{n\pi}{2}\Big] + \frac{2}{\pi}\Big[\frac{1}{n^2}\sin\frac{n\pi}{2}\Big] \;=\; \frac{4}{\pi n^2}\sin\frac{n\pi}{2} \\ &= \begin{cases} 0, & n = 2k \\ \frac{4}{\pi n^2}, & n = 4m+1 \\ -\frac{4}{\pi n^2}, & n = 4m+3 \end{cases} \;=\; \begin{cases} 0, & n = 2k \\ (-1)^{\frac{n-1}{2}}\frac{4}{\pi n^2}, & n = 2k+1. \end{cases} \\ d_n &= 0. \end{aligned}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\big(c_n\cos\sqrt{1+n^2}\,t\big)\sin nx.}$$

$\square$

25 Problems: Separation of Variables - Heat Equation

Problem (F'94, #5).
Solve the initial-boundary value problem

$$\begin{cases} u_t = u_{xx} & 0 < x < 2,\ t > 0 \\ u(x,0) = x^2 - x + 1 & 0 \le x \le 2 \\ u(0,t) = 1, \quad u(2,t) = 3 & t > 0. \end{cases}$$

Find $\lim_{t\to+\infty} u(x,t)$.

Proof. ① First, we need to obtain function v that satisfies $v_t = v_{xx}$ and takes 0 boundary conditions. Let

- $v(x,t) = u(x,t) + (ax + b),$ (25.1)

where a and b are constants to be determined. Then,

$$\begin{aligned} v_t &= u_t, \\ v_{xx} &= u_{xx}. \end{aligned}$$

Thus,

$$v_t = v_{xx}.$$

We need equation (25.1) to take 0 boundary conditions for $v(0,t)$ and $v(2,t)$:

$$\begin{aligned} v(0,t) = 0 &= u(0,t) + b = 1 + b \quad &\Rightarrow \quad b = -1, \\ v(2,t) = 0 &= u(2,t) + 2a - 1 = 2a + 2 \quad &\Rightarrow \quad a = -1. \end{aligned}$$

Thus, (25.1) becomes

$$v(x,t) = u(x,t) - x - 1. \tag{25.2}$$

The new problem is

$$\begin{cases} v_t = v_{xx}, \\ v(x,0) = (x^2 - x + 1) - x - 1 = x^2 - 2x, \\ v(0,t) = v(2,t) = 0. \end{cases}$$

② We solve the problem for v using the method of separation of variables.
Let $v(x,t) = X(x)T(t)$, which gives $XT' - X''T = 0$.

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda.$$

From $X'' + \lambda X = 0$, we get $X_n(x) = a_n \cos\sqrt{\lambda}x + b_n \sin\sqrt{\lambda}x$.
Using boundary conditions, we have

$$\begin{cases} v(0,t) = X(0)T(t) = 0 \\ v(2,t) = X(2)T(t) = 0 \end{cases} \quad \Rightarrow \quad X(0) = X(2) = 0.$$

Hence, $X_n(0) = a_n = 0$, and $X_n(x) = b_n \sin\sqrt{\lambda}x$.
$X_n(2) = b_n \sin 2\sqrt{\lambda} = 0 \quad \Rightarrow \quad 2\sqrt{\lambda} = n\pi \quad \Rightarrow \quad \lambda_n = (\frac{n\pi}{2})^2$.

$$\boxed{X_n(x) = b_n \sin\frac{n\pi x}{2}, \qquad \lambda_n = \left(\frac{n\pi}{2}\right)^2.}$$

With these values of λ_n, we solve $T' + \left(\frac{n\pi}{2}\right)^2 T = 0$ to find

$$\boxed{T_n(t) = c_n e^{-(\frac{n\pi}{2})^2 t}.}$$

$$\boxed{v(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t) = \sum_{n=1}^{\infty} \tilde{c}_n \, e^{-(\frac{n\pi}{2})^2 t} \sin \frac{n\pi x}{2}.}$$

Coefficients $\tilde{c}_n$ are obtained using the initial condition:

$$v(x,0) = \sum_{n=1}^{\infty} \tilde{c}_n \sin \frac{n\pi x}{2} = x^2 - 2x.$$

$$\tilde{c}_n = \int_0^2 (x^2 - 2x) \sin \frac{n\pi x}{2} \, dx = \begin{cases} 0 & n \textit{ is even}, \\ -\frac{32}{(n\pi)^3} & n \textit{ is odd}. \end{cases}$$

$$\Rightarrow \quad v(x,t) = \sum_{n=2k-1}^{\infty} -\frac{32}{(n\pi)^3} \, e^{-(\frac{n\pi}{2})^2 t} \sin \frac{n\pi x}{2}.$$

We now use equation (25.2) to convert back to function u:

$$u(x,t) = v(x,t) + x + 1.$$

$$\boxed{u(x,t) = \sum_{n=2k-1}^{\infty} -\frac{32}{(n\pi)^3} \, e^{-(\frac{n\pi}{2})^2 t} \sin \frac{n\pi x}{2} + x + 1.}$$

$$\lim_{t \to +\infty} u(x,t) = x + 1.$$

$\square$

Problem (S'96, #6).
Let $u(x,t)$ be the solution of the initial-boundary value problem for the heat equation

$$\begin{cases} u_t = u_{xx} & 0 < x < L,\ t > 0 \\ u(x,0) = f(x) & 0 \le x \le L \\ u_x(0,t) = u_x(L,t) = A & t > 0 \qquad (A = Const). \end{cases}$$

*Find $v(x)$ - the limit of $u(x,t)$ when $t \to \infty$. Show that $v(x)$ is **one** of the inifinitely many solutions of the stationary problem*

$$v_{xx} = 0 \qquad 0 < x < L$$
$$v_x(0) = v_x(L) = A.$$

Proof. ① First, we need to obtain function v that satisfies $v_t = v_{xx}$ and takes 0 boundary conditions. Let

- $v(x,t) = u(x,t) + (ax + b),$ (25.3)

where a and b are constants to be determined. Then,

$$\begin{aligned} v_t &= u_t, \\ v_{xx} &= u_{xx}. \end{aligned}$$

Thus,

$$v_t = v_{xx}.$$

We need equation (25.3) to take 0 boundary conditions for $v_x(0,t)$ and $v_x(L,t)$.
$v_x = u_x + a$.

$$\begin{aligned} v_x(0,t) = 0 &= u_x(0,t) + a = A + a \quad \Rightarrow \quad a = -A, \\ v_x(L,t) = 0 &= u_x(L,t) + a = A + a \quad \Rightarrow \quad a = -A. \end{aligned}$$

We may set $b = 0$ (infinitely many solutions are possible, one for each b).
Thus, (25.3) becomes

$$v(x,t) = u(x,t) - Ax. \tag{25.4}$$

The new problem is

$$\begin{cases} v_t = v_{xx}, \\ v(x,0) = f(x) - Ax, \\ v_x(0,t) = v_x(L,t) = 0. \end{cases}$$

② We solve the problem for v using the method of separation of variables.
Let $v(x,t) = X(x)T(t)$, which gives $XT' - X''T = 0$.

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda.$$

From $X'' + \lambda X = 0$, we get $X_n(x) = a_n \cos\sqrt{\lambda}x + b_n \sin\sqrt{\lambda}x$.
Using boundary conditions, we have

$$\begin{cases} v_x(0,t) = X'(0)T(t) = 0 \\ v_x(L,t) = X'(L)T(t) = 0 \end{cases} \quad \Rightarrow \quad X'(0) = X'(L) = 0.$$

$X_n'(x) = -a_n\sqrt{\lambda}\sin\sqrt{\lambda}x + b_n\sqrt{\lambda}\cos\sqrt{\lambda}x.$
Hence, $X_n'(0) = b_n\sqrt{\lambda_n} = 0 \quad\Rightarrow\quad b_n = 0$; and $X_n(x) = a_n\cos\sqrt{\lambda}x.$
$X_n'(L) = -a_n\sqrt{\lambda}\sin L\sqrt{\lambda} = 0 \quad\Rightarrow\quad L\sqrt{\lambda} = n\pi \quad\Rightarrow\quad \lambda_n = (\frac{n\pi}{L})^2.$

$$\boxed{X_n(x) = a_n\cos\frac{n\pi x}{L}, \qquad \lambda_n = \Big(\frac{n\pi}{L}\Big)^2.}$$

With these values of λ_n, we solve $T' + \big(\frac{n\pi}{L}\big)^2 T = 0$ to find

$$\boxed{T_0(t) = c_0, \quad T_n(t) = c_n e^{-(\frac{n\pi}{L})^2 t}, \quad n = 1, 2, \ldots.}$$

$$\boxed{v(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \tilde{c}_0 + \sum_{n=1}^{\infty}\tilde{c}_n\, e^{-(\frac{n\pi}{L})^2 t}\,\cos\frac{n\pi x}{L}.}$$

Coefficients $\tilde{c}_n$ are obtained using the initial condition:

$$v(x,0) \;=\; \tilde{c}_0 + \sum_{n=1}^{\infty}\tilde{c}_n\cos\frac{n\pi x}{L} \;=\; f(x) - Ax.$$

$$L\tilde{c}_0 \;=\; \int_0^L (f(x) - Ax)\,dx = \int_0^L f(x)\,dx - \frac{AL^2}{2} \quad\Rightarrow\quad \tilde{c}_0 = \frac{1}{L}\int_0^L f(x)\,dx - \frac{AL}{2},$$
$$\frac{L}{2}\tilde{c}_n \;=\; \int_0^L (f(x) - Ax)\,cos\frac{n\pi x}{L}\,dx \quad\Rightarrow\quad \tilde{c}_n = \frac{1}{L}\int_0^L (f(x) - Ax)\,cos\frac{n\pi x}{L}\,dx.$$

$$\Rightarrow\quad v(x,t) \;=\; \frac{1}{L}\int_0^L f(x)\,dx - \frac{AL}{2} + \sum_n^{\infty}\tilde{c}_n\, e^{-(\frac{n\pi}{L})^2 t}\,\cos\frac{n\pi x}{L}.$$

We now use equation (25.4) to convert back to function u:

$$u(x,t) = v(x,t) + Ax.$$

$$\boxed{u(x,t) \;=\; \frac{1}{L}\int_0^L f(x)\,dx - \frac{AL}{2} + \sum_n^{\infty}\tilde{c}_n\, e^{-(\frac{n\pi}{L})^2 t}\,\cos\frac{n\pi x}{L} \;+\; Ax.}$$

$$\lim_{t\to+\infty} u(x,t) \;=\; Ax + b, \qquad b \text{ arbitrary.}$$

To show that $v(x)$ is **one** of the inifinitely many solutions of the stationary problem

$$v_{xx} = 0 \qquad 0 < x < L$$
$$v_x(0) = v_x(L) = A,$$

we can solve the boundary value problem to obtain $v(x,t) = Ax + b$, where b is arbitrary.

$\square$

Heat Equation with Nonhomogeneous Time-Independent BC in N-dimensions.
The solution to this problem takes somewhat different approach than in the last few problems, but is similar.
Consider the following initial-boundary value problem,

$$\begin{cases} u_t = \triangle u, & x \in \Omega, \ \ t \geq 0 \\ u(x,0) = f(x), & x \in \Omega \\ u(x,t) = g(x), & x \in \partial\Omega, \ \ t > 0. \end{cases}$$

Proof. Let $w(x)$ be the solution of the Dirichlet problem:

$$\begin{cases} \triangle w = 0, & x \in \Omega \\ w(x) = g(x), & x \in \partial\Omega \end{cases}$$

and let $v(x,t)$ be the solution of the IBVP for the heat equation with homogeneous BC:

$$\begin{cases} v_t = \triangle v, & x \in \Omega, \ \ t \geq 0 \\ v(x,0) = f(x) - w(x), & x \in \Omega \\ v(x,t) = 0, & x \in \partial\Omega, \ \ t > 0. \end{cases}$$

Then $u(x,t)$ satisfies

$$u(x,t) = v(x,t) + w(x).$$

$$\lim_{t\to\infty} u(x,t) = w(x).$$

□

Nonhomogeneous Heat Equation with Nonhomogeneous Time-Independent BC in N dimensions.

Describe the method of solution of the problem

$$\begin{cases} u_t = \triangle u + F(x,t), & x \in \Omega, \ \ t \geq 0 \\ u(x,0) = f(x), & x \in \Omega \\ u(x,t) = g(x), & x \in \partial\Omega, \ \ t > 0. \end{cases}$$

Proof. ❶ We first find u_1, the solution to the homogeneous heat equation (no $F(x,t)$). Let $w(x)$ be the solution of the Dirichlet problem:

$$\begin{cases} \triangle w = 0, & x \in \Omega \\ w(x) = g(x), & x \in \partial\Omega \end{cases}$$

and let $v(x,t)$ be the solution of the IBVP for the heat equation with homogeneous BC:

$$\begin{cases} v_t = \triangle v, & x \in \Omega, \ \ t \geq 0 \\ v(x,0) = f(x) - w(x), & x \in \Omega \\ v(x,t) = 0, & x \in \partial\Omega, \ \ t > 0. \end{cases}$$

Then $u_1(x,t)$ satisfies

$$u_1(x,t) = v(x,t) + w(x).$$

$$\lim_{t\to\infty} u_1(x,t) = w(x).$$

❷ The solution to the homogeneous equation with 0 boundary conditions is given by Duhamel's principle.

$$\begin{cases} u_{2_t} = \triangle u_2 + F(x,t) & \text{for } \ t > 0, \ x \in \mathbb{R}^n \\ u_2(x,0) = 0 & \text{for } \ x \in \mathbb{R}^n. \end{cases} \tag{25.5}$$

Duhamel's principle gives the solution:

$$u_2(x,t) = \int_0^t \int_{\mathbb{R}^n} \tilde{K}(x-y, t-s)\, F(y,s)\, dy\, ds$$

Note: $u_2(x,t) = 0$ **on** $\partial\Omega$ **may not be satisfied.**

$$\boxed{u(x,t) = v(x,t) + w(x) + \int_0^t \int_{\mathbb{R}^n} \tilde{K}(x-y, t-s)\, F(y,s)\, dy\, ds.}$$

□

Problem (S'98, #5). *Find the solution of*

$$\begin{cases} u_t = u_{xx}, & t \geq 0, \ 0 < x < 1, \\ u(x,0) = 0, & 0 < x < 1, \\ u(0,t) = 1 - e^{-t}, \quad u_x(1,t) = e^{-t} - 1, & t > 0. \end{cases}$$

Prove that $\lim_{t\to\infty} u(x,t)$ *exists and find it.*

Proof. ① First, we need to obtain function v that satisfies $v_t = v_{xx}$ and takes 0 boundary conditions. Let

- $$v(x,t) = u(x,t) + (ax+b) + (c_1 \cos x + c_2 \sin x)e^{-t}, \tag{25.6}$$

where a, b, c_1, c_2 are constants to be determined. Then,

$$\begin{aligned} v_t &= u_t - (c_1 \cos x + c_2 \sin x)e^{-t}, \\ v_{xx} &= u_{xx} + (-c_1 \cos x - c_2 \sin x)e^{-t}. \end{aligned}$$

Thus,

$$v_t = v_{xx}.$$

We need equation (25.6) to take 0 boundary conditions for $v(0,t)$ and $v_x(1,t)$:

$$\begin{aligned} v(0,t) = 0 &= u(0,t) + b + c_1 e^{-t} \\ &= 1 - e^{-t} + b + c_1 e^{-t}. \end{aligned}$$

Thus, $b = -1$, $c_1 = 1$, and (25.6) becomes

$$v(x,t) = u(x,t) + (ax-1) + (\cos x + c_2 \sin x)e^{-t}. \tag{25.7}$$

$$\begin{aligned} v_x(x,t) &= u_x(x,t) + a + (-\sin x + c_2 \cos x)e^{-t}, \\ v_x(1,t) = 0 &= u_x(1,t) + a + (-\sin 1 + c_2 \cos 1)e^{-t} \\ &= -1 + a + (1 - \sin 1 + c_2 \cos 1)e^{-t}. \end{aligned}$$

Thus, $a = 1$, $c_2 = \frac{\sin 1 - 1}{\cos 1}$, and equation (25.7) becomes

$$v(x,t) = u(x,t) + (x-1) + (\cos x + \frac{\sin 1 - 1}{\cos 1} \sin x)e^{-t}. \tag{25.8}$$

Initial condition tranforms to:

$$v(x,0) = u(x,0) + (x-1) + (\cos x + \frac{\sin 1 - 1}{\cos 1} \sin x) = (x-1) + (\cos x + \frac{\sin 1 - 1}{\cos 1} \sin x).$$

The new problem is

$$\begin{cases} v_t = v_{xx}, \\ v(x,0) = (x-1) + (\cos x + \frac{\sin 1 - 1}{\cos 1} \sin x), \\ v(0,t) = 0, \quad v_x(1,t) = 0. \end{cases}$$

② We solve the problem for v using the method of separation of variables.
Let $v(x,t) = X(x)T(t)$, which gives $XT' - X''T = 0$.

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda.$$

From $X'' + \lambda X = 0$, we get $X_n(x) = a_n \cos\sqrt{\lambda}x + b_n \sin\sqrt{\lambda}x$.
Using the first boundary condition, we have

$$v(0,t) = X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0.$$

Hence, $X_n(0) = a_n = 0$, and $X_n(x) = b_n \sin\sqrt{\lambda}x$. We also have

$$\begin{aligned} v_x(1,t) &= X'(1)T(t) = 0 \quad \Rightarrow \quad X'(1) = 0. \\ X_n'(x) &= \sqrt{\lambda} b_n \cos\sqrt{\lambda}x, \\ X_n'(1) &= \sqrt{\lambda} b_n \cos\sqrt{\lambda} = 0, \\ &\cos\sqrt{\lambda} = 0, \\ &\sqrt{\lambda} = n\pi + \frac{\pi}{2}. \end{aligned}$$

Thus,

$$\boxed{X_n(x) = b_n \sin\Big(n\pi + \frac{\pi}{2}\Big)x, \qquad \lambda_n = \Big(n\pi + \frac{\pi}{2}\Big)^2.}$$

With these values of λ_n, we solve $T' + \Big(n\pi + \frac{\pi}{2}\Big)^2 T = 0$ to find

$$\boxed{T_n(t) = c_n e^{-(n\pi + \frac{\pi}{2})^2 t}.}$$

$$\boxed{v(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} \tilde{b}_n \sin\Big(n\pi + \frac{\pi}{2}\Big)x\ e^{-(n\pi + \frac{\pi}{2})^2 t}.}$$

We now use equation (25.8) to convert back to function u:

$$u(x,t) = v(x,t) - (x-1) - (\cos x + \frac{\sin 1 - 1}{\cos 1}\sin x)e^{-t}.$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \tilde{b}_n \sin\Big(n\pi + \frac{\pi}{2}\Big)x\ e^{-(n\pi + \frac{\pi}{2})^2 t} - (x-1) - (\cos x + \frac{\sin 1 - 1}{\cos 1}\sin x)e^{-t}.}$$

Coefficients $\tilde{b}_n$ are obtained using the initial condition:

$$u(x,0) = \sum_{n=1}^{\infty} \tilde{b}_n \sin\Big(n\pi + \frac{\pi}{2}\Big)x \ - (x-1) - (\cos x + \frac{\sin 1 - 1}{\cos 1}\sin x).$$

③ Finally, we can check that the differential equation and the boundary conditions are satisfied:

$$\begin{aligned} u(0,t) &= 1 - (1+0)e^{-t} = 1 - e^{-t}. \ \checkmark \\ u_x(x,t) &= \sum_{n=1}^{\infty} \tilde{b}_n\Big(n\pi + \frac{\pi}{2}\Big)\cos\Big(n\pi + \frac{\pi}{2}\Big)x\ e^{-(n\pi + \frac{\pi}{2})^2 t} - 1 + (\sin x - \frac{\sin 1 - 1}{\cos 1}\cos x)e^{-t}, \\ u_x(1,t) &= -1 + (\sin 1 - \frac{\sin 1 - 1}{\cos 1}\cos 1)e^{-t} = -1 + e^{-t}. \ \checkmark \\ u_t &= \sum_{n=1}^{\infty} -\tilde{b}_n\Big(n\pi + \frac{\pi}{2}\Big)^2 \sin\Big(n\pi + \frac{\pi}{2}\Big)x\ e^{-(n\pi + \frac{\pi}{2})^2 t} + (\cos x + \frac{\sin 1 - 1}{\cos 1}\sin x)e^{-t} = u_{xx}. \ \checkmark \end{aligned}$$

$\square$

Problem (F'02, #6). *The temperature of a rod insulated at the ends with an exponentially decreasing heat source in it is a solution of the following boundary value problem:*

$$\begin{cases} u_t = u_{xx} + e^{-2t}g(x) & \text{for } (x,t) \in [0,1] \times \mathbb{R}_+ \\ u_x(0,t) = u_x(1,t) = 0 \\ u(x,0) = f(x). \end{cases}$$

Find the solution to this problem by writing u as a cosine series,

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos n\pi x, \qquad \circledast$$

and determine $\lim_{t\to\infty} u(x,t)$.

Proof. Let g accept an expansion in eigenfunctions

$$g(x) = b_0 + \sum_{n=1}^{\infty} b_n \cos n\pi x \qquad \text{with} \qquad b_n = 2\int_0^1 g(x) \cos n\pi x \, dx.$$

Plugging $\circledast$ in the PDE gives:

$$a_0'(t) + \sum_{n=1}^{\infty} a_n'(t) \cos n\pi x = -\sum_{n=1}^{\infty} n^2\pi^2 a_n(t) \cos n\pi x + b_0 e^{-2t} + e^{-2t} \sum_{n=1}^{\infty} b_n \cos n\pi x,$$

which gives

$$\begin{cases} a_0'(t) = b_0 e^{-2t}, \\ a_n'(t) + n^2\pi^2 a_n(t) = b_n e^{-2t}, \quad n = 1, 2, \ldots. \end{cases}$$

Adding homogeneous and particular solutions of the above ODEs, we obtain the solutions

$$\begin{cases} a_0(t) = c_0 - \frac{b_0}{2} e^{-2t}, \\ a_n(t) = c_n e^{-n^2\pi^2 t} - \frac{b_n}{2-n^2\pi^2} e^{-2t}, \quad n = 1, 2, \ldots, \end{cases}$$

for some constants c_n, $n = 0, 1, 2, \ldots$. Thus,

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} \Big(c_n e^{-n^2\pi^2 t} - \frac{b_n}{2 - n^2\pi^2} e^{-2t} \Big) \cos n\pi x.}$$

Initial condition gives

$$u(x,0) = \sum_{n=0}^{\infty} \Big(c_n - \frac{b_n}{2 - n^2\pi^2} \Big) \cos n\pi x = f(x),$$

As, $t \to \infty$, the only mode that survives is $n = 0$:

$$u(x,t) \ \to \ c_0 + \frac{b_0}{2} \qquad \text{as } \ t \to \infty.$$

$\square$

Problem (F'93, #4). ***a)*** *Assume $f, g \in C^\infty$. Give the compatibility conditions which f and g must satisfy if the following problem is to possess a solution.*

$$\begin{aligned} &\triangle u = f(x) && x \in \Omega \\ &\frac{\partial u}{\partial n}(s) = g(s) && s \in \partial\Omega. \end{aligned}$$

Show that your condition is necessary for a solution to exist.

b) *Give an explicit solution to*

$$\begin{cases} u_t = u_{xx} + \cos x & x \in [0, 2\pi] \\ u_x(0,t) = u_x(2\pi, t) = 0 & t > 0 \\ u(x,0) = \cos x + \cos 2x & x \in [0, 2\pi]. \end{cases}$$

c) *Does there exist a steady state solution to the problem in (b) if*

$$u_x(0) = 1 \qquad u_x(2\pi) = 0 \quad ?$$

Explain your answer.

Proof. **a)** Integrating the equation and using Green's identity gives:

$$\int_\Omega f(x)\,dx \;=\; \int_\Omega \triangle u\,dx \;=\; \int_{\partial\Omega} \frac{\partial u}{\partial n}\,ds \;=\; \int_{\partial\Omega} g(s)\,ds.$$

b) With

- $v(x,t) = u(x,t) - \cos x$

the problem above transforms to

$$\begin{cases} v_t = v_{xx} \\ v_x(0,t) = v_x(2\pi,t) = 0 \\ v(x,0) = \cos 2x. \end{cases}$$

We solve this problem for v using the separation of variables. Let $v(x,t) = X(x)T(t)$, which gives $XT' = X''T$.

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda.$$

From $X'' + \lambda X = 0$, we get $X_n(x) = a_n \cos\sqrt{\lambda}x + b_n \sin\sqrt{\lambda}x$.
$X_n'(x) = -\sqrt{\lambda_n}a_n \sin\sqrt{\lambda}x + \sqrt{\lambda_n}b_n \cos\sqrt{\lambda}x$.
Using boundary conditions, we have

$$\begin{cases} v_x(0,t) = X'(0)T(t) = 0 \\ v_x(2\pi,t) = X'(2\pi)T(t) = 0 \end{cases} \quad \Rightarrow \quad X'(0) = X'(2\pi) = 0.$$

Hence, $X_n'(0) = \sqrt{\lambda_n}b_n = 0$, and $X_n(x) = a_n \cos\sqrt{\lambda_n}x$.
$X_n'(2\pi) = -\sqrt{\lambda_n}a_n \sin\sqrt{\lambda_n}2\pi = 0 \quad \Rightarrow \quad \sqrt{\lambda_n} = \frac{n}{2} \quad \Rightarrow \quad \lambda_n = (\frac{n}{2})^2$. Thus,

$$\boxed{X_n(x) = a_n \cos\frac{nx}{2}, \qquad \lambda_n = \left(\frac{n}{2}\right)^2}$$

With these values of λ_n, we solve $T' + \left(\frac{n}{2}\right)^2 T = 0$ to find

$$\boxed{T_n(t) = c_n e^{-(\frac{n}{2})^2 t}.}$$

$$\boxed{v(x,t) = \sum_{n=0}^{\infty} X_n(x) T_n(t) = \sum_{n=0}^{\infty} \tilde{a}_n \, e^{-(\frac{n}{2})^2 t} \cos\frac{nx}{2}.}$$

Initial condition gives

$$v(x,0) = \sum_{n=0}^{\infty} \tilde{a}_n \cos\frac{nx}{2} = \cos 2x.$$

Thus, $\tilde{a}_4 = 1$, $\tilde{a}_n = 0$, $n \neq 4$. Hence,

$$v(x,t) = e^{-4t} \cos 2x.$$

$$\boxed{u(x,t) = v(x,t) + \cos x = e^{-4t} \cos 2x + \cos x.}$$

c) *Does there exist a steady state solution to the problem in (b) if*

$$u_x(0) = 1 \qquad u_x(2\pi) = 0 \quad ?$$

Explain your answer.

c) Set $u_t = 0$. We have

$$\begin{cases} u_{xx} + \cos x = 0 & x \in [0, 2\pi] \\ u_x(0) = 1, \quad u_x(2\pi) = 0. \end{cases}$$

$$\begin{aligned} u_{xx} &= -\cos x, \\ u_x &= -\sin x + C, \\ u(x) &= \cos x + Cx + D. \end{aligned}$$

Boundary conditions give:

$$\begin{aligned} 1 = u_x(0) &= C, \\ 0 = u_x(2\pi) &= C \quad \Rightarrow \quad \text{contradiction} \end{aligned}$$

There exists no steady state solution.

We may use the result we obtained in part (a) with $u_{xx} = \cos x = f(x)$. We need

$$\begin{aligned} \int_\Omega f(x)\,dx &= \int_{\partial\Omega} \frac{\partial u}{\partial n}\,ds, \\ \underbrace{\int_0^{2\pi} \cos x\,dx}_{=0} &= u_x(2\pi) - u_x(0) = \underbrace{-1}_{given}. \end{aligned}$$

□

Problem (F'96, #7). *Solve the parabolic problem*

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{xx}, \qquad 0 \le x \le \pi, \ t > 0$$

$$u(x,0) = \sin x, \qquad u(0,t) = u(\pi,t) = 0,$$
$$v(x,0) = \sin x, \qquad v(0,t) = v(\pi,t) = 0.$$

Prove the ***energy estimate*** *(for general initial data)*

$$\int_{x=0}^{\pi} [u^2(x,t) + v^2(x,t)]\, dx \le c \int_{x=0}^{\pi} [u^2(x,0) + v^2(x,0)]\, dx$$

for come constant c.

Proof. We can solve the second equation for v and then use the value of v to solve the first equation for u. [73]
① We have

$$\begin{cases} v_t = 2v_{xx}, & 0 \le x \le \pi, \ t > 0 \\ v(x,0) = \sin x, \\ v(0,t) = v(\pi,t) = 0. \end{cases}$$

Assume $v(x,t) = X(x)T(t)$, then substitution in the PDE gives $XT' = 2X''T$.

$$\frac{T'}{T} = 2\frac{X''}{X} = -\lambda.$$

From $X'' + \frac{\lambda}{2}X = 0$, we get $X_n(x) = a_n \cos\sqrt{\frac{\lambda}{2}}x + b_n \sin\sqrt{\frac{\lambda}{2}}x$.
Boundary conditions give

$$\begin{cases} v(0,t) = X(0)T(t) = 0 \\ v(\pi,t) = X(\pi)T(t) = 0 \end{cases} \quad \Rightarrow \quad X(0) = X(\pi) = 0.$$

Thus, $X_n(0) = a_n = 0$, and $X_n(x) = b_n \sin\sqrt{\frac{\lambda}{2}}x$.
$X_n(\pi) = b_n \sin\sqrt{\frac{\lambda}{2}}\pi = 0$. Hence $\sqrt{\frac{\lambda}{2}} = n$, or $\lambda = 2n^2$.

$$\boxed{\lambda = 2n^2, \qquad X_n(x) = b_n \sin nx.}$$

With these values of λ_n, we solve $T' + 2n^2T = 0$ to get $T_n(t) = c_n e^{-2n^2 t}$.
Thus, the solution may be written in the form

$$v(x,t) = \sum_{n=1}^{\infty} \tilde{a}_n e^{-2n^2 t} \sin nx.$$

From initial condition, we get

$$v(x,0) = \sum_{n=1}^{\infty} \tilde{a}_n \sin nx = \sin x.$$

Thus, $\tilde{a}_1 = 1, \quad \tilde{a}_n = 0, \quad n = 2, 3, \ldots$.

$$\boxed{v(x,t) = e^{-2t} \sin x.}$$

[73] Note that if the matrix was fully inseparable, we would have to find eigenvalues and eigenvectors, just as we did for the hyperbolic systems.

② We have

$$\begin{cases} u_t = u_{xx} - \frac{1}{2}e^{-2t}\,\sin x, & 0 \le x \le \pi,\ t > 0 \\ u(x,0) = \sin x, \\ u(0,t) = u(\pi,t) = 0. \end{cases}$$

Let $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin nx$. Plugging this into the equation, we get

$$\sum_{n=1}^{\infty} u_n'(t)\sin nx + \sum_{n=1}^{\infty} n^2 u_n(t)\sin nx = -\frac{1}{2}e^{-2t}\,\sin x.$$

For $n = 1$:

$$u_1'(t) + u_1(t) = -\frac{1}{2}e^{-2t}.$$

Combining homogeneous and particular solution of the above equation, we obtain:

$$u_1(t) = \frac{1}{2}e^{-2t} + c_1 e^{-t}.$$

For $n = 2, 3, \ldots$:

$$u_n'(t) + n^2 u_n(t) = 0,$$
$$u_n(t) = c_n e^{-n^2 t}.$$

Thus,

$$u(x,t) = \Big(\frac{1}{2}e^{-2t} + c_1 e^{-t}\Big)\sin x + \sum_{n=2}^{\infty} c_n e^{-n^2 t}\sin nx = \frac{1}{2}e^{-2t}\sin x + \sum_{n=1}^{\infty} c_n e^{-n^2 t}\sin nx.$$

From initial condition, we get

$$u(x,0) = \frac{1}{2}\sin x + \sum_{n=1}^{\infty} c_n \sin nx = \sin x.$$

Thus, $c_1 = \frac{1}{2}$, $c_n = 0$, $n = 2, 3, \ldots$.

$$\boxed{u(x,t) = \frac{1}{2}\sin x\,(e^{-2t} + e^{-t}).}$$

To prove the **energy estimate** (for general initial data)

$$\int_{x=0}^{\pi} [u^2(x,t) + v^2(x,t)]\,dx \le c\int_{x=0}^{\pi} [u^2(x,0) + v^2(x,0)]\,dx$$

for come constant c, we assume that

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx, \qquad v(x,0) = \sum_{n=1}^{\infty} b_n \sin nx.$$

The general solutions are obtained by the same method as above

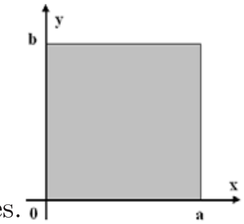
$$\begin{aligned}
u(x,t) &= \frac{1}{2}e^{-2t}\sin x + \sum_{n=1}^{\infty} c_n e^{-n^2 t}\sin nx, \\
v(x,t) &= \sum_{n=1}^{\infty} b_n e^{-2n^2 t}\,\sin nx. \\
\int_{x=0}^{\pi}[u^2(x,t)+v^2(x,t)]\,dx &= \int_{x=0}^{\pi}\big(\frac{1}{2}e^{-2t}\sin x + \sum_{n=1}^{\infty} c_n e^{-n^2 t}\sin nx\big)^2 + \big(\sum_{n=1}^{\infty} b_n e^{-2n^2 t}\,\sin nx\big)^2\,dx \\
&\le \sum_{n=1}^{\infty}(b_n^2+a_n^2)\int_{x=0}^{\pi}\sin^2 nx\,dx \;\le\; \int_{x=0}^{\pi}[u^2(x,0)+v^2(x,0)]\,dx.
\end{aligned}$$

□

26 Problems: Eigenvalues of the Laplacian - Laplace

The 2D LAPLACE Equation (eigenvalues/eigenfuctions of the Laplacian).
Consider

$$\begin{cases} u_{xx} + u_{yy} + \lambda u = 0 & in \;\; \Omega \\ u(0,y) = 0 = u(a,y) & for \;\; 0 \le y \le b, \\ u(x,0) = 0 = u(x,b) & for \;\; 0 \le x \le a. \end{cases} \tag{26.1}$$



Proof. We can solve this problem by separation of variables.
Let $u(x,y) = X(x)Y(y)$, then substitution in the PDE gives $X''Y + XY'' + \lambda XY = 0$.

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0.$$

Letting $\lambda = \mu^2 + \nu^2$ and using boundary conditions, we find the equations for X and Y:

$$\begin{array}{ll} X'' + \mu^2 X = 0 & Y'' + \nu^2 Y = 0 \\ X(0) = X(a) = 0 & Y(0) = Y(b) = 0. \end{array}$$

The solutions of these one-dimensional eigenvalue problems are

$$\begin{array}{ll} \mu_m = \dfrac{m\pi}{a} & \nu_n = \dfrac{n\pi}{b} \\ X_m(x) = \sin\dfrac{m\pi x}{a} & Y_n(y) = \sin\dfrac{n\pi y}{b}, \end{array}$$

where $m, n = 1, 2, \ldots$. Thus we obtain solutions of (26.1) of the form

$$\lambda_{mn} = \pi^2\Big(\frac{m^2}{a^2} + \frac{n^2}{b^2}\Big) \qquad u_{mn}(x,y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

where $m, n = 1, 2, \ldots$.

Observe that the eigenvalues $\{\lambda_{mn}\}_{m,n=1}^{\infty}$ are positive. The smallest eigenvalue λ_{11} has only one eigenfunction $u_{11}(x,y) = \sin(\pi x/a)\sin(\pi y/b)$; notice that u_{11} is positive in Ω. Other eigenvalues λ may correspond to more than one choice of m and n; for example, in the case $a = b$ we have $\lambda_{nm} = \lambda_{nm}$. For this λ, there are two linearly independent eigenfunctions. However, for a particular value of λ there are at most finitely many linearly independent eigenfunctions. Moreover,

$$\begin{aligned} \int_0^b \int_0^a u_{mn}(x,y)\, u_{m'n'}(x,y)\, dx\, dy &= \int_0^b \int_0^a \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}\, dx\, dy \\ = \begin{cases} \frac{a}{2}\int_0^b \sin\frac{n\pi y}{b}\sin\frac{n'\pi y}{b}\, dy \\ 0 \end{cases} &= \begin{cases} \frac{ab}{4} & \text{if } \; m = m' \; \text{ and } \; n = n' \\ 0 & \text{if } \; m \ne m' \; \text{ or } \; n \ne n'. \end{cases} \end{aligned}$$

In particular, the $\{u_{mn}\}$ are pairwise orthogonal. We could *normalize* each u_{mn} by a scalar multiple (i.e. multiply by $\sqrt{4/ab}$) so that $ab/4$ above becomes 1. □

Let us change the notation somewhat so that each eigenvalue λ_n corresponds to a particular eigenfunction $\phi_n(x)$. If we choose an orthonormal basis of eigenfunctions in each eigenspace, we may arrange that $\{\phi_n\}_{n=1}^{\infty}$ is pairwise orthonormal:

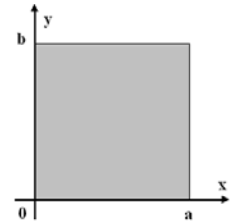
$$\int_\Omega \phi_n(x)\phi_m(x)\, dx = \begin{cases} 1 & \text{if } \; m = n \\ 0 & \text{if } \; m \ne n. \end{cases}$$

In this notation, the eigenfunction expansion of $f(x)$ defined on Ω becomes

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x), \qquad \text{where} \qquad a_n = \int_{\Omega} f(x)\phi_n(x)\, dx.$$

Problem (S'96, #4). *Let D denote the rectangular*

$$D = \{(x, y) \in \mathbb{R}^2 : 0 < x < a,\ 0 < y < b\}.$$



*Find the **eigenvalues** of the following Dirichlet problem:*

$$(\triangle + \lambda)u = 0 \qquad in \quad D$$
$$u = 0 \qquad on \quad \partial D.$$

Proof. The problem may be rewritten as

$$\begin{cases} u_{xx} + u_{yy} + \lambda u = 0 & \text{in } \Omega \\ u(0, y) = 0 = u(a, y) & \text{for } 0 \le y \le b, \\ u(x, 0) = 0 = u(x, b) & \text{for } 0 \le x \le a. \end{cases}$$

We may assume that the eigenvalues λ are positive, $\lambda = \mu^2 + \nu^2$. Then,

$$\boxed{\lambda_{mn} = \pi^2\Big(\frac{m^2}{a^2} + \frac{n^2}{b^2}\Big) \qquad u_{mn}(x, y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \qquad m, n = 1, 2, \ldots.}$$

□

Problem (W'04, #1). *Consider the differential equation:*

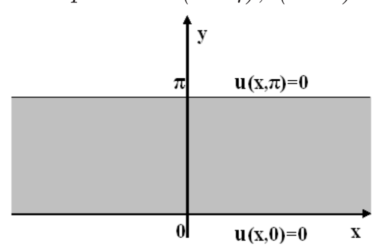
$$\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} + \lambda u(x, y) = 0 \tag{26.2}$$

in the strip $\{(x, y), 0 < y < \pi, -\infty < x < +\infty\}$ with boundary conditions

$$u(x, 0) = 0, \quad u(x, \pi) = 0. \tag{26.3}$$

Find all bounded solutions of the boundary value problem (26.4), (26.5) when

a) $\lambda = 0$, ***b)*** $\lambda > 0$, ***c)*** $\lambda < 0$.



Proof. **a)** $\lambda = 0$. We have

$$u_{xx} + u_{yy} = 0.$$

Assume $u(x, y) = X(x)Y(y)$, then substitution in the PDE gives

$$X''Y + XY'' = 0.$$

Boundary conditions give

$$\begin{cases} u(x, 0) = X(x)Y(0) = 0 \\ u(x, \pi) = X(x)Y(\pi) = 0 \end{cases} \Rightarrow \quad Y(0) = Y(\pi) = 0.$$

Method I: We have

$$\frac{X''}{X} = -\frac{Y''}{Y} = -c, \qquad c > 0.$$

From $X'' + cX = 0$, we have $X_n(x) = a_n \cos\sqrt{c}x + b_n \sin\sqrt{c}x$.
From $Y'' - cY = 0$, we have $Y_n(y) = c_n e^{-\sqrt{c}y} + d_n e^{\sqrt{c}y}$.
$Y(0) = c_n + d_n = 0 \quad \Rightarrow \quad c_n = -d_n$.

$Y(\pi) = c_n e^{-\sqrt{c}\pi} - c_n e^{\sqrt{c}\pi} = 0 \quad \Rightarrow \quad c_n = 0 \quad \Rightarrow \; Y_n(y) = 0.$
$\Rightarrow \quad u(x,y) = X(x)Y(y) = 0.$

Method II: We have

$$\frac{X''}{X} = -\frac{Y''}{Y} = c, \qquad c > 0.$$

From $X'' - cX = 0$, we have $X_n(x) = a_n e^{-\sqrt{c}x} + b_n e^{\sqrt{c}x}$.
Since we look for bounded solutions for $-\infty < x < \infty$, $a_n = b_n = 0 \quad \Rightarrow X_n(x) = 0$.
From $Y'' + cY = 0$, we have $Y_n(y) = c_n \cos\sqrt{c}y + d_n \sin\sqrt{c}y$.
$Y(0) = c_n = 0$,
$Y(\pi) = d_n \sin\sqrt{c}\pi = 0 \quad \Rightarrow \quad \sqrt{c} = n \quad \Rightarrow \quad c = n^2.$
$\Rightarrow \quad Y_n(y) = d_n \sin nx = 0.$
$\Rightarrow \quad u(x,y) = X(x)Y(y) = 0.$

b) $\lambda > 0$. We have

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0.$$

Letting $\lambda = \mu^2 + \nu^2$, and using boundary conditions for Y, we find the equations:

$$X'' + \mu^2 X = 0 \qquad\qquad Y'' + \nu^2 Y = 0$$
$$Y(0) = Y(\pi) = 0.$$

The solutions of these one-dimensional eigenvalue problems are
$X_m(x) = a_m \cos\mu_m x + b_m \sin\mu_m x$.
$\nu_n = n$, $Y_n(y) = d_n \sin ny$, where $m,\, n = 1, 2, \ldots$.

$$\boxed{u(x,y) = \sum_{m,n=1}^{\infty} u_{mn}(x,y) = \sum_{m,n=1}^{\infty} (a_m \cos\mu_m x + b_m \sin\mu_m x)\sin ny.}$$

c) $\lambda < 0$. We have

$$u_{xx} + u_{yy} + \lambda u = 0,$$
$$u(x,0) = 0, \quad u(x,\pi) = 0.$$

$u \equiv 0$ is the solution to this equation. We will show that this solution is unique.
Let u_1 and u_2 be two solutions, and consider $w = u_1 - u_2$. Then,

$$\triangle w + \lambda w = 0,$$
$$w(x,0) = 0, \quad w(x,\pi) = 0.$$

Multiply the equation by w and integrate:

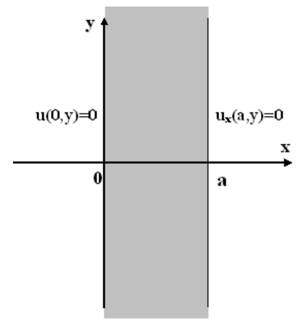
$$w\triangle w + \lambda w^2 = 0,$$
$$\int_\Omega w\triangle w\,dx + \lambda\int_\Omega w^2\,dx = 0,$$
$$\underbrace{\int_{\partial\Omega} w\frac{\partial w}{\partial n}\,ds}_{=0} - \int_\Omega |\nabla w|^2\,dx + \lambda\int_\Omega w^2\,dx = 0,$$
$$\underbrace{\int_\Omega |\nabla w|^2\,dx}_{\geq 0} = \underbrace{\lambda\int_\Omega w^2\,dx}_{\leq 0}.$$

Thus, $w \equiv 0$ and the solution $u(x, y) \equiv 0$ is unique. $\square$

Problem (F'95, #5). *Find **all** bounded solutions for the following boundary value problem in the strip* $0 < x < a$, $-\infty < y < \infty$,

$$(\triangle + k^2)u = 0 \qquad (k = Const > 0),$$
$$u(0,y) = 0, \qquad u_x(a,y) = 0.$$

In particular, show that when $ak \le \pi$, *the only bounded solution to this problem is* $u \equiv 0$.



Proof. Let $u(x,y) = X(x)Y(y)$, then we have $X''Y + XY'' + k^2XY = 0$.

$$\frac{X''}{X} + \frac{Y''}{Y} + k^2 = 0.$$

Letting $k^2 = \mu^2 + \nu^2$ and using boundary conditions, we find:

$$X'' + \mu^2 X = 0, \qquad Y'' + \nu^2 Y = 0.$$
$$X(0) = X'(a) = 0.$$

The solutions of these one-dimensional eigenvalue problems are

$$\mu_m = \frac{(m-\frac{1}{2})\pi}{a},$$
$$X_m(x) = \sin\frac{(m-\frac{1}{2})\pi x}{a} \qquad Y_n(y) = c_n\cos\nu_n y + d_n\sin\nu_n y,$$

where $m, n = 1, 2, \ldots$. Thus we obtain solutions of the form

$$k_{mn}^2 = \left(\frac{(m-\frac{1}{2})\pi}{a}\right)^2 + \nu_n^2, \qquad u_{mn}(x,y) = \sin\frac{(m-\frac{1}{2})\pi x}{a}\Big(c_n\cos\nu_n y + d_n\sin\nu_n y\Big),$$

where $m, n = 1, 2, \ldots$.

$$\boxed{u(x,y) = \sum_{m,n=1}^{\infty} u_{mn}(x,y) = \sum_{m,n=1}^{\infty} \sin\frac{(m-\frac{1}{2})\pi x}{a}\Big(c_n\cos\nu_n y + d_n\sin\nu_n y\Big).}$$

• We can take an **alternate** approach and prove the second part of the question. We have

$$X''Y + XY'' + k^2XY = 0,$$
$$-\frac{Y''}{Y} = \frac{X''}{X} + k^2 = c^2.$$

We obtain $Y_n(y) = c_n\cos cy + d_n\sin cy$. The second equation gives

$$X'' + k^2X = c^2X,$$
$$X'' + (k^2 - c^2)X = 0,$$
$$X_m(x) = a_m e^{\sqrt{c^2-k^2}x} + b_m e^{\sqrt{c^2-k^2}x}.$$

Thus, $X_m(x)$ is bounded only if $k^2 - c^2 > 0$, (if $k^2 - c^2 = 0$, $X'' = 0$, and $X_m(x) = a_m x + b_m$, BC's give $X_m(x) = \pi x$, unbounded), in which case

$$X_m(x) = a_m\cos\sqrt{k^2-c^2}\,x + b_m\sin\sqrt{k^2-c^2}\,x.$$

Boundary conditions give $X_m(0) = a_m = 0$.

$$\begin{aligned}
X'_m(x) &= b_m \sqrt{k^2 - c^2} \cos \sqrt{k^2 - c^2}\, x, \\
X'_m(a) &= b_m \sqrt{k^2 - c^2} \cos \sqrt{k^2 - c^2}\, a \;=\; 0, \\
\sqrt{k^2 - c^2}\, a &= m\pi - \frac{\pi}{2}, \qquad\qquad m = 1, 2, \ldots, \\
k^2 - c^2 &= \Big(\frac{\pi}{a}\Big(m - \frac{1}{2}\Big)\Big)^2, \\
k^2 &= \Big(\frac{\pi}{a}\Big)^2 \Big(m - \frac{1}{2}\Big)^2 + c^2, \\
a^2 k^2 &> \pi^2 \Big(m - \frac{1}{2}\Big)^2, \\
ak &> \pi\Big(m - \frac{1}{2}\Big), \qquad\qquad m = 1, 2, \ldots.
\end{aligned}$$

Thus, bounded solutions exist only when $ak > \frac{\pi}{2}$. □

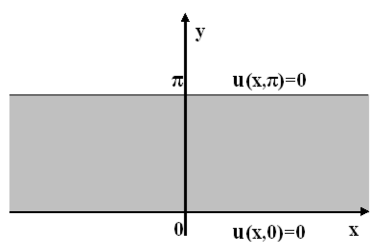
Problem (S'90, #2). *Show that the boundary value problem*

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} + k^2 u(x,y) = 0, \tag{26.4}$$

where $-\infty < x < +\infty, \quad 0 < y < \pi, \quad k > 0$ *is a constant,*

$$u(x,0) = 0, \quad u(x,\pi) = 0 \tag{26.5}$$

has a bounded solution if and only if $k \geq 1$.



Proof. We have

$$\begin{aligned}
&u_{xx} + u_{yy} + k^2 u = 0, \\
&X''Y + XY'' + k^2 XY \;=\; 0, \\
&-\frac{X''}{X} \;=\; \frac{Y''}{Y} + k^2 \;=\; c^2.
\end{aligned}$$

We obtain $X_m(x) = a_m \cos cx + b_m \sin cx$. The second equation gives

$$\begin{aligned}
Y'' + k^2 Y &= c^2 Y, \\
Y'' + (k^2 - c^2) Y &= 0, \\
Y_n(y) &= c_n e^{\sqrt{c^2 - k^2} y} + d_n e^{\sqrt{c^2 - k^2} y}.
\end{aligned}$$

Thus, $Y_n(y)$ is bounded only if $k^2 - c^2 > 0$, (if $k^2 - c^2 = 0$, $Y'' = 0$, and $Y_n(y) = c_n y + d_n$, BC's give $Y \equiv 0$), in which case

$$Y_n(y) \;=\; c_n \cos \sqrt{k^2 - c^2}\, y + d_n \sin \sqrt{k^2 - c^2}\, y.$$

Boundary conditions give $Y_n(0) = c_n = 0$.
$Y_n(\pi) \;=\; d_n \sin \sqrt{k^2 - c^2}\, \pi \;=\; 0 \quad \Rightarrow \quad \sqrt{k^2 - c^2} \;=\; n \quad \Rightarrow \quad k^2 - c^2 \;=\; n^2 \quad \Rightarrow$
$k^2 = n^2 + c^2$, $n = 1, 2, \ldots$. Hence, $k > n$, $n = 1, 2, \ldots$.

Thus, bounded solutions exist if $k \geq 1$.
Note: If $k = 1$, then $c = 0$, which gives trivial solutions for $Y_n(y)$.

$$\boxed{u(x,y) = \sum_{m,n=1}^{\infty} X_m(x)Y_n(y) = \sum_{m,n=1}^{\infty} \sin ny \; X_m(x).}$$

$\square$

McOwen, 4.4 #7; 266B Ralston Hw. *Show that the boundary value problem*

$$\begin{cases} -\nabla\cdot a(x)\nabla u + b(x)u = \lambda u & \textit{in } \Omega \\ u = 0 \quad \textit{on } \partial\Omega \end{cases}$$

has only trivial solution with $\lambda \le 0$, when $b(x) \ge 0$ and $a(x) > 0$ in Ω.

Proof. Multiplying the equation by u and integrating over Ω, we get

$$\int_\Omega -u\nabla\cdot a\nabla u\, dx + \int_\Omega bu^2\, dx = \lambda\int_\Omega u^2\, dx.$$

Since $\nabla\cdot(ua\nabla u) = u\nabla\cdot a\nabla u + a|\nabla u|^2$, we have

$$\int_\Omega -\nabla\cdot(ua\nabla u)\, dx + \int_\Omega a|\nabla u|^2\, dx + \int_\Omega bu^2\, dx = \lambda\int_\Omega u^2\, dx. \tag{26.6}$$

Using divergence theorem, we obtain

$$\int_{\partial\Omega} -\underbrace{u}_{=0} a\frac{\partial u}{\partial n}\, ds + \int_\Omega a|\nabla u|^2\, dx + \int_\Omega bu^2\, dx = \lambda\int_\Omega u^2\, dx,$$

$$\int_\Omega \underbrace{a}_{>0}|\nabla u|^2\, dx + \int_\Omega \underbrace{b}_{\ge 0} u^2\, dx = \underbrace{\lambda}_{\le 0}\int_\Omega u^2\, dx,$$

Thus, $\nabla u = 0$ in Ω, and u is constant. Since $u = 0$ on $\partial\Omega$, $u \equiv 0$ on Ω.
Similar Problem I: Note that this argument also works with Neumann B.C.:

$$\begin{cases} -\nabla\cdot a(x)\nabla u + b(x)u = \lambda u & \text{in } \Omega \\ \partial u/\partial n = 0 \quad \text{on } \partial\Omega \end{cases}$$

Using divergence theorem, (26.6) becomes

$$\int_{\partial\Omega} -ua\underbrace{\frac{\partial u}{\partial n}}_{=0}\, ds + \int_\Omega a|\nabla u|^2\, dx + \int_\Omega bu^2\, dx = \lambda\int_\Omega u^2\, dx,$$

$$\int_\Omega \underbrace{a}_{>0}|\nabla u|^2\, dx + \int_\Omega \underbrace{b}_{\ge 0} u^2\, dx = \underbrace{\lambda}_{\le 0}\int_\Omega u^2\, dx.$$

Thus, $\nabla u = 0$, and $u = \text{const}$ on Ω. Hence, we now have

$$\int_\Omega \underbrace{b}_{\ge 0} u^2\, dx = \underbrace{\lambda}_{\le 0}\int_\Omega u^2\, dx,$$

which implies $\lambda = 0$. This gives the useful information that for the eigenvalue problem[74]

$$\begin{cases} -\nabla\cdot a(x)\nabla u + b(x)u = \lambda u \\ \partial u/\partial n = 0, \end{cases}$$

$\lambda = 0$ is an eigenvalue, its eigenspace is the set of constants, and all other λ's are **positive**.

[74] In Ralston's Hw#7 solutions, there is no '-' sign in front of $\nabla\cdot a(x)\nabla u$ below, which is probably a typo.

Similar Problem II: If $\lambda \le 0$, we show that the only solution to the problem below is the trivial solution.

$$\begin{cases} \triangle u + \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\int_\Omega u \triangle u \, dx + \lambda \int_\Omega u^2 \, dx = 0,$$

$$\int_{\partial\Omega} \underbrace{u}_{=0} \frac{\partial u}{\partial n} \, ds - \int_\Omega |\nabla u|^2 \, dx + \underbrace{\lambda}_{\le 0} \int_\Omega u^2 \, dx = 0.$$

Thus, $\nabla u = 0$ in Ω, and u is constant. Since $u = 0$ on $\partial\Omega$, $u \equiv 0$ on Ω. $\square$

27 Problems: Eigenvalues of the Laplacian - Poisson

The ND POISSON Equation (eigenvalues/eigenfunctions of the Laplacian).

Suppose we want to find the eigenfunction expansion of the solution of

$$\begin{aligned} \triangle u = f & \quad \textit{in} \ \ \Omega \\ u = 0 & \quad \textit{on} \ \ \partial\Omega, \end{aligned}$$

when f has the expansion in the orthonormal Dirichlet eigenfunctions ϕ_n:

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x), \qquad \textit{where} \qquad a_n = \int_\Omega f(x)\phi_n(x)\,dx.$$

Proof. Writing $u = \sum c_n \phi_n$ and inserting into $-\lambda u = f$, we get

$$\sum_{n=1}^{\infty} -\lambda_n c_n \phi_n = \sum_{n=1}^{\infty} a_n \phi_n(x).$$

Thus, $c_n = -a_n/\lambda_n$, and

$$\boxed{u(x) = -\sum_{n=1}^{\infty} \frac{a_n \phi_n(x)}{\lambda_n}.}$$

□

The 1D POISSON Equation (eigenvalues/eigenfunctions of the Laplacian).

For the boundary value problem

$$\begin{aligned} & u'' = f(x) \\ & u(0) = 0, \quad u(L) = 0, \end{aligned}$$

the related eigenvalue problem is

$$\begin{aligned} & \phi'' = -\lambda \phi \\ & \phi(0) = 0, \quad \phi(L) = 0. \end{aligned}$$

The eigenvalues are $\lambda_n = (n\pi/L)^2$, and the corresponding eigenfunctions are $\sin(n\pi x/L)$, $n = 1, 2, \ldots$.
Writing $u = \sum c_n \phi_n = \sum c_n \sin(n\pi x/L)$ and inserting into $-\lambda u = f$, we get

$$\begin{aligned} \sum_{n=1}^{\infty} -c_n \Big(\frac{n\pi}{L}\Big)^2 \sin\frac{n\pi x}{L} &= f(x), \\ \int_0^L \sum_{n=1}^{\infty} -c_n \Big(\frac{n\pi}{L}\Big)^2 \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,dx &= \int_0^L f(x)\,\sin\frac{m\pi x}{L}\,dx, \\ -c_n \Big(\frac{n\pi}{L}\Big)^2 \frac{L}{2} &= \int_0^L f(x)\,\sin\frac{n\pi x}{L}\,dx, \\ c_n &= -\frac{2}{L}\frac{\int_0^L f(x)\,\sin(n\pi x/L)\,dx}{(n\pi/L)^2}. \end{aligned}$$

$$u(x) = \sum c_n \sin(n\pi x/L) = \sum_{n=1}^{\infty} -\frac{2}{L} \frac{\int_0^L f(\xi)\, \sin(n\pi x/L)\, \sin(n\pi\xi/L)\, d\xi}{(n\pi/L)^2},$$

$$\boxed{u = \int_0^L f(\xi) \underbrace{\Big[-\frac{2}{L} \sum_{n=1}^{\infty} \frac{\sin(n\pi x/L)\, \sin(n\pi\xi/L)}{(n\pi/L)^2} \Big]}_{=\ G(x,\xi)} \, d\xi.}$$

See similar, but more complicated, problem in Sturm-Liouville Problems (S'92, #2(c)).

Example: Eigenfunction Expansion of the GREEN's Function.
Suppose we fix x and attempt to expand the Green's function $G(x,y)$ in the orthonormal eigenfunctions $\phi_n(y)$:

$$G(x,y) \sim \sum_{n=1}^{\infty} a_n(x)\phi_n(y), \qquad \textit{where} \qquad a_n(x) = \int_\Omega G(x,z)\phi_n(z)\,dz.$$

Proof. We can rewrite $\triangle u + \lambda u = 0$ in Ω, $u = 0$ on $\partial\Omega$, as an integral equation [75]

$$u(x) + \lambda \int_\Omega G(x,y)u(y)\,dy = 0. \qquad \circledast$$

Suppose, $u(x) = \sum c_n \phi_n(x)$. Plugging this into $\circledast$, we get

$$\sum_{m=1}^{\infty} c_m\phi_m(x) + \lambda \int_\Omega \sum_{n=1}^{\infty} a_n(x)\phi_n(y) \sum_{m=1}^{\infty} c_m\phi_m(y)\,dy = 0,$$

$$\sum_{m=1}^{\infty} c_m\phi_m(x) + \lambda \sum_{n=1}^{\infty} a_n(x) \sum_{m=1}^{\infty} c_m \int_\Omega \phi_n(y)\phi_m(y)\,dy = 0,$$

$$\sum_{n=1}^{\infty} c_n\phi_n(x) + \sum_{n=1}^{\infty} \lambda a_n(x) c_n = 0,$$

$$\sum_{n=1}^{\infty} c_n\big(\phi_n(x) + \lambda a_n(x)\big) = 0,$$

$$a_n(x) = -\frac{\phi_n(x)}{\lambda_n}.$$

Thus,

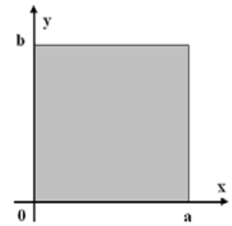
$$\boxed{G(x,y) \sim \sum_{n=1}^{\infty} -\frac{\phi_n(x)\phi_n(y)}{\lambda_n}.}$$

□

[75] See the section: ODE - Integral Equations.

The 2D POISSON Equation (eigenvalues/eigenfunctions of the Laplacian).
Solve the boundary value problem

$$\begin{cases} u_{xx} + u_{yy} = f(x, y) & \textit{for } \ 0 < x < a,\ 0 < y < b \\ u(0, y) = 0 = u(a, y) & \textit{for } \ 0 \le y \le b, \\ u(x, 0) = 0 = u(x, b) & \textit{for } \ 0 \le x \le a. \end{cases} \tag{27.1}$$



$f(x, y) \in C^2$, $f(x, y) = 0$ *if* $x = 0,\ x = a,\ y = 0,\ y = b$,

$$f(x, y) = \frac{2}{\sqrt{ab}} \sum_{m,n=1}^{\infty} c_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}.$$

Proof. ① First, we find eigenvalues/eigenfunctions of the Laplacian.

$$\begin{cases} u_{xx} + u_{yy} + \lambda u = 0 & \text{in } \ \Omega \\ u(0, y) = 0 = u(a, y) & \text{for } \ 0 \le y \le b, \\ u(x, 0) = 0 = u(x, b) & \text{for } \ 0 \le x \le a. \end{cases}$$

Let $u(x, y) = X(x)Y(y)$, then substitution in the PDE gives $X''Y + XY'' + \lambda XY = 0$.

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0.$$

Letting $\lambda = \mu^2 + \nu^2$ and using boundary conditions, we find the equations for X and Y:

$$X'' + \mu^2 X = 0 \qquad\qquad Y'' + \nu^2 Y = 0$$
$$X(0) = X(a) = 0 \qquad\qquad Y(0) = Y(b) = 0.$$

The solutions of these one-dimensional eigenvalue problems are

$$\mu_m = \frac{m\pi}{a} \qquad\qquad \nu_n = \frac{n\pi}{b}$$
$$X_m(x) = \sin\frac{m\pi x}{a} \qquad\qquad Y_n(y) = \sin\frac{n\pi y}{b},$$

where $m,\ n = 1, 2, \ldots$. Thus we obtain eigenvalues and normalized eigenfunctions of the Laplacian:

$$\boxed{\lambda_{mn} = \pi^2\Big(\frac{m^2}{a^2} + \frac{n^2}{b^2}\Big) \qquad\qquad \phi_{mn}(x, y) = \frac{2}{\sqrt{ab}} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b},}$$

where $m,\ n = 1, 2, \ldots$. Note that

$$f(x, y) = \sum_{m,n=1}^{\infty} c_{mn}\phi_{mn}.$$

② Second, writing $u(x, y) = \sum \tilde{c}_{mn}\phi_{mn}$ and inserting into $-\lambda u = f$, we get

$$-\sum_{m,n=1}^{\infty} \lambda_{mn}\tilde{c}_{mn}\phi_{mn}(x, y) = \sum_{m,n=1}^{\infty} c_{mn}\phi_{mn}(x, y).$$

Thus, $\tilde{c}_{mn} = -\frac{c_{mn}}{\lambda_{mn}}$.

$$\boxed{u(x, y) = -\sum_{n=1}^{\infty} \frac{c_{mn}}{\lambda_{mn}}\phi_{mn}(x, y),}$$

with λ_{mn}, $\phi_{mn}(x)$ given above, and c_{mn} given by

$$\int_0^b \int_0^a f(x,y)\phi_{mn}\,dx\,dy = \int_0^b \int_0^a \sum_{m',n'=1}^{\infty} c_{m'n'}\phi_{m'n'}\phi_{mn}\,dx\,dy = c_{mn}.$$

$\square$

28 Problems: Eigenvalues of the Laplacian - Wave

In the section on the wave equation, we considered an initial boundary value problem for the one-dimensional wave equation on an interval, and we found that the solution could be obtained using Fourier series. If we replace the Fourier series by an expansion in eigenfunctions, we can consider an initial/boundary value problem for the n-dimensional wave equation.

The ND WAVE Equation (eigenvalues/eigenfunctions of the Laplacian).
Consider

$$\begin{cases} u_{tt} = \triangle u & \textit{for } x \in \Omega,\ t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & \textit{for } x \in \Omega \\ u(x,t) = 0 & \textit{for } x \in \partial\Omega,\ t > 0. \end{cases}$$

Proof. For $g, h \in C^2(\overline{\Omega})$ with $g = h = 0$ on $\partial\Omega$, we have eigenfunction expansions

$$g(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \qquad \text{and} \qquad h(x) = \sum_{n=1}^{\infty} b_n \phi_n(x). \qquad \circledast$$

Assume the solution $u(x,t)$ may be expanded in the eigenfunctions with coefficients depending on t: $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x)$. This implies

$$\begin{aligned} \sum_{n=1}^{\infty} u_n''(t)\phi_n(x) &= -\sum_{n=1}^{\infty} \lambda_n u_n(t)\phi_n(x), \\ u_n''(t) + \lambda_n u_n(t) &= 0 \qquad \text{for each } n. \end{aligned}$$

Since $\lambda_n > 0$, this ordinary differential equation has general solution

$$\begin{aligned} u_n(t) &= A_n \cos\sqrt{\lambda_n}t + B_n \sin\sqrt{\lambda_n}t. \qquad \text{Thus,} \\ u(x,t) &= \sum_{n=1}^{\infty} \big(A_n \cos\sqrt{\lambda_n}t + B_n \sin\sqrt{\lambda_n}t\big)\phi_n(x), \\ u_t(x,t) &= \sum_{n=1}^{\infty} \big(-\sqrt{\lambda_n}A_n \sin\sqrt{\lambda_n}t + \sqrt{\lambda_n}B_n \cos\sqrt{\lambda_n}t\big)\phi_n(x), \\ u(x,0) &= \sum_{n=1}^{\infty} A_n\phi_n(x) = g(x), \\ u_t(x,0) &= \sum_{n=1}^{\infty} \sqrt{\lambda_n}B_n\phi_n(x) = h(x). \end{aligned}$$

Comparing with $\circledast$, we obtain

$$A_n = a_n, \qquad B_n = \frac{b_n}{\sqrt{\lambda_n}}.$$

Thus, the solution is given by

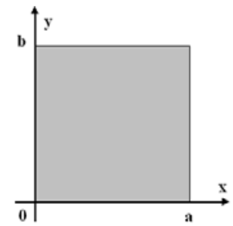
$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \Big(a_n \cos\sqrt{\lambda_n}t + \frac{b_n}{\sqrt{\lambda_n}} \sin\sqrt{\lambda_n}t\Big)\phi_n(x),}$$

with

$$a_n = \int_\Omega g(x)\phi_n(x)\,dx,$$
$$b_n = \int_\Omega h(x)\phi_n(x)\,dx.$$

□

The 2D WAVE Equation (eigenvalues/eigenfunctions of the Laplacian).
Let $\Omega = (0,a) \times (0,b)$ *and consider*



$$\begin{cases} u_{tt} = u_{xx} + u_{yy} & \text{for } x \in \Omega,\ t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & \text{for } x \in \Omega \\ u(x,t) = 0 & \text{for } x \in \partial\Omega,\ t > 0. \end{cases} \tag{28.1}$$

Proof. ① First, we find eigenvalues/eigenfunctions of the Laplacian.

$$\begin{cases} u_{xx} + u_{yy} + \lambda u = 0 & \text{in } \Omega \\ u(0,y) = 0 = u(a,y) & \text{for } 0 \le y \le b, \\ u(x,0) = 0 = u(x,b) & \text{for } 0 \le x \le a. \end{cases}$$

Let $u(x,y) = X(x)Y(y)$, then substitution in the PDE gives $X''Y + XY'' + \lambda XY = 0$.

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0.$$

Letting $\lambda = \mu^2 + \nu^2$ and using boundary conditions, we find the equations for X and Y:

$$X'' + \mu^2 X = 0 \qquad\qquad Y'' + \nu^2 Y = 0$$
$$X(0) = X(a) = 0 \qquad\qquad Y(0) = Y(b) = 0.$$

The solutions of these one-dimensional eigenvalue problems are

$$\mu_m = \frac{m\pi}{a} \qquad\qquad \nu_n = \frac{n\pi}{b}$$
$$X_m(x) = \sin\frac{m\pi x}{a} \qquad\qquad Y_n(y) = \sin\frac{n\pi y}{b},$$

where $m,\ n = 1, 2, \ldots$. Thus we obtain eigenvalues and normalized eigenfunctions of the Laplacian:

$$\boxed{\lambda_{mn} = \pi^2\Big(\frac{m^2}{a^2} + \frac{n^2}{b^2}\Big) \qquad\qquad \phi_{mn}(x,y) = \frac{2}{\sqrt{ab}} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b},}$$

where $m,\ n = 1, 2, \ldots$.

② Second, we solve the Wave Equation (28.1) using the "space" eigenfunctions.
For $g, h \in C^2(\overline{\Omega})$ with $g = h = 0$ on $\partial\Omega$, we have eigenfunction expansions [76]

$$g(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \qquad \text{and} \qquad h(x) = \sum_{n=1}^{\infty} b_n \phi_n(x). \qquad \circledast$$

Assume $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x)$. This implies

$$u_n''(t) + \lambda_n u_n(t) = 0 \qquad \text{for each } n.$$

[76] In 2D, ϕ_n is really ϕ_{mn}, and x is (x,y).

Since $\lambda_n > 0$, this ordinary differential equation has general solution

$$\begin{aligned}
u_n(t) &= A_n \cos\sqrt{\lambda_n}t + B_n \sin\sqrt{\lambda_n}t. \qquad \text{Thus,}\\
u(x,t) &= \sum_{n=1}^{\infty}\big(A_n \cos\sqrt{\lambda_n}t + B_n \sin\sqrt{\lambda_n}t\big)\phi_n(x),\\
u_t(x,t) &= \sum_{n=1}^{\infty}\big(-\sqrt{\lambda_n}A_n \sin\sqrt{\lambda_n}t + \sqrt{\lambda_n}B_n \cos\sqrt{\lambda_n}t\big)\phi_n(x),\\
u(x,0) &= \sum_{n=1}^{\infty} A_n\phi_n(x) = g(x),\\
u_t(x,0) &= \sum_{n=1}^{\infty} \sqrt{\lambda_n}B_n\phi_n(x) = h(x).
\end{aligned}$$

Comparing with ⊛, we obtain

$$A_n = a_n, \qquad B_n = \frac{b_n}{\sqrt{\lambda_n}}.$$

Thus, the solution is given by

$$\boxed{u(x,t) = \sum_{m,n=1}^{\infty}\big(a_{mn}\cos\sqrt{\lambda_{mn}}t + \frac{b_{mn}}{\sqrt{\lambda_{mn}}}\sin\sqrt{\lambda_{mn}}t\big)\phi_{mn}(x),}$$

with λ_{mn}, $\phi_{mn}(x)$ given above, and

$$\begin{aligned}
a_{mn} &= \int_\Omega g(x)\phi_{mn}(x)\,dx,\\
b_{mn} &= \int_\Omega h(x)\phi_{mn}(x)\,dx.
\end{aligned}$$

□

McOwen, 4.4 #3; 266B Ralston Hw. *Consider the initial-boundary value problem*

$$\begin{cases} u_{tt} = \triangle u + f(x,t) & \text{for } x \in \Omega,\ t > 0 \\ u(x,t) = 0 & \text{for } x \in \partial\Omega,\ t > 0 \\ u(x,0) = 0, \quad u_t(x,0) = 0 & \text{for } x \in \Omega. \end{cases}$$

Use Duhamel's principle and an expansion of f in eigenfunctions to obtain a (formal) solution.

Proof. **a)** We expand u in terms of the **Dirichlet eigenfunctions of Laplacian** in Ω.

$$\triangle\phi_n + \lambda_n\phi_n = 0 \quad \text{in } \Omega, \qquad \phi_n = 0 \quad \text{on } \partial\Omega.$$

Assume

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x), \qquad a_n(t) = \int_\Omega \phi_n(x)u(x,t)\,dx.$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\phi_n(x), \qquad f_n(t) = \int_\Omega \phi_n(x)f(x,t)\,dx.$$

$$\begin{aligned} a_n''(t) &= \int_\Omega \phi_n(x)u_{tt}\,dx = \int_\Omega \phi_n(\triangle u + f)\,dx = \int_\Omega \phi_n\triangle u\,dx + \int_\Omega \phi_n f\,dx \\ &= \int_\Omega \triangle\phi_n u\,dx + \int_\Omega \phi_n f\,dx = -\lambda_n\int_\Omega \phi_n u\,dx + \underbrace{\int_\Omega \phi_n f\,dx}_{f_n} = -\lambda_n a_n(t) + f_n(t). \end{aligned}$$

$$\begin{aligned} a_n(0) &= \int_\Omega \phi_n(x)u(x,0)\,dx = 0. \\ a_n'(0) &= \int_\Omega \phi_n(x)u_t(x,0)\,dx = 0. \end{aligned}$$

[77] Thus, we have an ODE which is converted and solved by **Duhamel's principle**:

$$\begin{cases} a_n'' + \lambda_n a_n = f_n(t) \\ a_n(0) = 0 \\ a_n'(0) = 0 \end{cases} \Rightarrow \begin{cases} \tilde{a}_n'' + \lambda_n\tilde{a}_n = 0 \\ \tilde{a}_n(0,s) = 0 \\ \tilde{a}_n'(0,s) = f_n(s) \end{cases} \qquad a_n(t) = \int_0^t \tilde{a}_n(t-s,s)\,ds.$$

With the anzats $\tilde{a}_n(t,s) = c_1\cos\sqrt{\lambda_n}t + c_2\sin\sqrt{\lambda_n}t$, we get $c_1 = 0$, $c_2 = f_n(s)/\sqrt{\lambda_n}$, or

$$\tilde{a}_n(t,s) = f_n(s)\frac{\sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}}.$$

Duhamel's principle gives

$$a_n(t) = \int_0^t \tilde{a}_n(t-s,s)\,ds = \int_0^t f_n(s)\frac{\sin(\sqrt{\lambda_n}(t-s))}{\sqrt{\lambda_n}}\,ds.$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\frac{\phi_n(x)}{\sqrt{\lambda_n}}\int_0^t f_n(s)\sin(\sqrt{\lambda_n}(t-s))\,ds.}$$

□

[77] We used Green's formula: $\int_{\partial\Omega}\left(\phi_n\frac{\partial u}{\partial n} - u\frac{\partial\phi_n}{\partial n}\right)ds = \int_\Omega(\phi_n\triangle u - \triangle\phi_n u)\,dx$.
On $\partial\Omega$, $u = 0$; $\phi_n = 0$ since eigenfunctions are Dirichlet.

Problem (F'90, #3). *Consider the initial-boundary value problem*

$$\begin{cases} u_{tt} = a(t)u_{xx} + f(x,t) & 0 \le x \le \pi,\, t \ge 0 \\ u(0,t) = u(\pi,t) = 0 & t \ge 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & 0 \le x \le \pi, \end{cases}$$

where the coefficient $a(t) \neq 0$.
a) *Express (formally) the solution of this problem by the method of eigenfunction expansions.*
b) *Show that this problem is **not well-posed** if $a \equiv -1$.*
Hint: Take $f = 0$ and prove that the solution does not depend continuously on the initial data g, h.

Proof. **a)** We expand u in terms of the **Dirichlet eigenfunctions of Laplacian** in Ω.

$$\phi_{nxx} + \lambda_n \phi_n = 0 \quad \text{in } \Omega, \qquad \phi_n(0) = \phi_n(\pi) = 0.$$

That gives us the eigenvalues and eigenfunctions of the Laplacian: $\lambda_n = n^2$, $\phi_n(x) = \sin nx$.
Assume

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x), \qquad u_n(t) = \int_\Omega \phi_n(x) u(x,t)\,dx.$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\phi_n(x), \qquad f_n(t) = \int_\Omega \phi_n(x) f(x,t)\,dx.$$

$$g(x) = \sum_{n=1}^{\infty} g_n\phi_n(x), \qquad g_n = \int_\Omega \phi_n(x) g(x)\,dx.$$

$$h(x) = \sum_{n=1}^{\infty} h_n\phi_n(x), \qquad h_n = \int_\Omega \phi_n(x) h(x)\,dx.$$

$$\begin{aligned} u_n''(t) &= \int_\Omega \phi_n(x) u_{tt}\,dx = \int_\Omega \phi_n(a(t)u_{xx} + f)\,dx = a(t)\int_\Omega \phi_n u_{xx}\,dx + \int_\Omega \phi_n f\,dx \\ &= a(t)\int_\Omega \phi_{nxx} u\,dx + \int_\Omega \phi_n f\,dx = -\lambda_n a(t)\int_\Omega \phi_n u\,dx + \underbrace{\int_\Omega \phi_n f\,dx}_{f_n} \\ &= -\lambda_n a(t) u_n(t) + f_n(t). \\ u_n(0) &= \int_\Omega \phi_n(x) u(x,0)\,dx = \int_\Omega \phi_n(x) g(x)\,dx = g_n. \\ u_n'(0) &= \int_\Omega \phi_n(x) u_t(x,0)\,dx = \int_\Omega \phi_n(x) h(x)\,dx = h_n. \end{aligned}$$

Thus, we have an ODE which is converted and solved by **Duhamel's principle**:

$$\begin{cases} u_n'' + \lambda_n a(t) u_n = f_n(t) \\ u_n(0) = g_n \\ u_n'(0) = h_n. \end{cases} \qquad \circledast$$

Note: The initial data is not 0; therefore, the Duhamel's principle is not applicable. Also, the ODE is not linear in t, and it's solution is not obvious. Thus,

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x),}$$

where $u_n(t)$ are solutions of $\circledast$.

b) Assume we have two solutions, u_1 and u_2, to the PDE:

$$\begin{cases} u_{1tt} + u_{1xx} = 0, \\ u_1(0,t) = u_1(\pi,t) = 0, \\ u_1(x,0) = g_1(x), \quad u_{1t}(x,0) = h_1(x); \end{cases} \qquad \begin{cases} u_{2tt} + u_{2xx} = 0, \\ u_2(0,t) = u_2(\pi,t) = 0, \\ u_2(x,0) = g_2(x), \quad u_{2t}(x,0) = h_2(x). \end{cases}$$

Note that the equation is **elliptic**, and therefore, the maximum principle holds.
In order to prove that the solution does not depend continuously on the initial data g, h, we need to show that one of the following conditions holds:

$$\begin{aligned} \max_{\overline{\Omega}} |u_1 - u_2| &> \max_{\partial\Omega} |g_1 - g_2|, \\ \max_{\overline{\Omega}} |u_{t1} - u_{t2}| &> \max_{\partial\Omega} |h_1 - h_2|. \end{aligned}$$

That is, the difference of the two solutions is not bounded by the difference of initial data.
By the method of separation of variables, we may obtain

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \sin nx, \\ u(x,0) &= \sum_{n=1}^{\infty} a_n \sin nx = g(x), \\ u_t(x,0) &= \sum_{n=1}^{\infty} nb_n \sin nx = h(x). \end{aligned}$$

Not complete.

We also know that for elliptic equations, and for Laplace equation in particular, the value of the function u has to be prescribed on the entire boundary, i.e. $u = g$ on $\partial\Omega$, which is not the case here, making the problem under-determined. Also, u_t is prescribed on one of the boundaries, making the problem overdetermined. $\square$

29 Problems: Eigenvalues of the Laplacian - Heat

The ND HEAT Equation (eigenvalues/eigenfunctions of the Laplacian).
Consider the initial value problem with homogeneous Dirichlet condition:

$$\begin{cases} u_t = \triangle u & \text{for } x \in \Omega,\ t > 0 \\ u(x,0) = g(x) & \text{for } x \in \overline{\Omega} \\ u(x,t) = 0 & \text{for } x \in \partial\Omega,\ t > 0. \end{cases}$$

Proof. For $g \in C^2(\overline{\Omega})$ with $g = 0$ on $\partial\Omega$, we have eigenfunction expansion

$$g(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \qquad \circledast$$

Assume the solution $u(x,t)$ may be expanded in the eigenfunctions with coefficients depending on t: $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x)$. This implies

$$\sum_{n=1}^{\infty} u_n'(t)\phi_n(x) = -\lambda_n \sum_{n=1}^{\infty} u_n(t)\phi_n(x),$$

$$u_n'(t) + \lambda_n u_n(t) = 0, \quad \text{which has the general solution}$$

$$u_n(t) = A_n e^{-\lambda_n t}. \qquad \text{Thus,}$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n t}\phi_n(x),$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \phi_n(x) = g(x).$$

Comparing with $\circledast$, we obtain $A_n = a_n$. Thus, the solution is given by

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}\phi_n(x),}$$

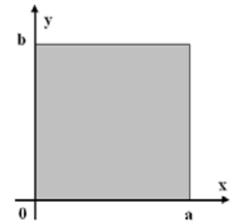
$$\text{with} \qquad a_n = \int_\Omega g(x)\phi_n(x)\,dx.$$

Also

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}\phi_n(x) = \sum_{n=1}^{\infty} \Big(\int_\Omega g(y)\phi_n(y)\,dy\Big) e^{-\lambda_n t}\phi_n(x) \\ &= \int_\Omega \underbrace{\sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x)\phi_n(y)}_{K(x,y,t),\ \text{heat kernel}}\, g(y)\,dy \end{aligned}$$

□

The 2D HEAT Equation (eigenvalues/eigenfunctions of the Laplacian).
Let $\Omega = (0, a) \times (0, b)$ *and consider*



$$\begin{cases} u_t = u_{xx} + u_{yy} & \text{for } x \in \Omega, \ t > 0 \\ u(x, 0) = g(x) & \text{for } x \in \overline{\Omega} \\ u(x, t) = 0 & \text{for } x \in \partial\Omega, \ t > 0. \end{cases} \tag{29.1}$$

Proof. ① First, we find eigenvalues/eigenfunctions of the Laplacian.

$$\begin{cases} u_{xx} + u_{yy} + \lambda u = 0 & \text{in } \Omega \\ u(0, y) = 0 = u(a, y) & \text{for } 0 \le y \le b, \\ u(x, 0) = 0 = u(x, b) & \text{for } 0 \le x \le a. \end{cases}$$

Let $u(x, y) = X(x)Y(y)$, then substitution in the PDE gives $X''Y + XY'' + \lambda XY = 0$.

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0.$$

Letting $\lambda = \mu^2 + \nu^2$ and using boundary conditions, we find the equations for X and Y:

$$X'' + \mu^2 X = 0 \qquad\qquad Y'' + \nu^2 Y = 0$$
$$X(0) = X(a) = 0 \qquad\qquad Y(0) = Y(b) = 0.$$

The solutions of these one-dimensional eigenvalue problems are

$$\mu_m = \frac{m\pi}{a} \qquad\qquad \nu_n = \frac{n\pi}{b}$$
$$X_m(x) = \sin\frac{m\pi x}{a} \qquad\qquad Y_n(y) = \sin\frac{n\pi y}{b},$$

where $m, n = 1, 2, \ldots$. Thus we obtain eigenvalues and normalized eigenfunctions of the Laplacian:

$$\boxed{\lambda_{mn} = \pi^2\Big(\frac{m^2}{a^2} + \frac{n^2}{b^2}\Big) \qquad \phi_{mn}(x, y) = \frac{2}{\sqrt{ab}} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b},}$$

where $m, n = 1, 2, \ldots$.
② Second, we solve the Heat Equation (29.1) using the "space" eigenfunctions.
For $g \in C^2(\overline{\Omega})$ with $g = 0$ on $\partial\Omega$, we have eigenfunction expansion

$$g(x) = \sum_{n=1}^{\infty} a_n \phi_n(x). \qquad \circledast$$

Assume $u(x, t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x)$. This implies

$u_n'(t) + \lambda_n u_n(t) = 0$, which has the general solution

$$u_n(t) = A_n e^{-\lambda_n t}. \qquad \text{Thus,}$$

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n t}\phi_n(x),$$

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \phi_n(x) = g(x).$$

Comparing with ⊛, we obtain $A_n = a_n$. Thus, the solution is given by

$$\boxed{u(x,t) = \sum_{m,n=1}^{\infty} a_{mn} e^{-\lambda_{mn} t} \phi_{mn}(x),}$$

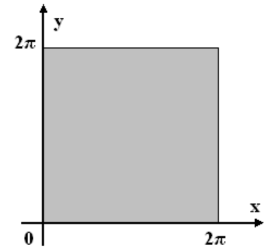
with λ_{mn}, ϕ_{mn} given above and $a_{mn} = \int_\Omega g(x)\phi_{mn}(x)\,dx$. □

Problem (S'91, #2). *Consider the heat equation*

$$u_t = u_{xx} + u_{yy}$$

on the square $\Omega = \{0 \le x \le 2\pi,\ 0 \le y \le 2\pi\}$ *with* ***periodic boundary conditions*** *and with initial data*

$$u(0, x, y) = f(x, y).$$



a) *Find the solution using separation of variables.*

Proof. ① First, we find eigenvalues/eigenfunctions of the Laplacian.

$$\begin{cases} u_{xx} + u_{yy} + \lambda u = 0 & \text{in } \Omega \\ u(0, y) = u(2\pi, y) & \text{for } 0 \le y \le 2\pi, \\ u(x, 0) = u(x, 2\pi) & \text{for } 0 \le x \le 2\pi. \end{cases}$$

Let $u(x, y) = X(x)Y(y)$, then substitution in the PDE gives $X''Y + XY'' + \lambda XY = 0$.

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0.$$

Letting $\lambda = \mu^2 + \nu^2$ and using periodic BC's, we find the equations for X and Y:

$$X'' + \mu^2 X = 0 \qquad\qquad Y'' + \nu^2 Y = 0$$
$$X(0) = X(2\pi) \qquad\qquad Y(0) = Y(2\pi).$$

The solutions of these one-dimensional eigenvalue problems are

$$\mu_m = m \qquad\qquad \nu_n = n$$
$$X_m(x) = e^{imx} \qquad\qquad Y_n(y) = e^{iny},$$

where $m, n = \dots, -2, -1, 0, 1, 2, \dots$. Thus we obtain eigenvalues and normalized eigenfunctions of the Laplacian:

$$\boxed{\lambda_{mn} = m^2 + n^2 \qquad\qquad \phi_{mn}(x, y) = e^{imx}e^{iny},}$$

where $m, n = \dots, -2, -1, 0, 1, 2, \dots$.
② Second, we solve the Heat Equation using the "space" eigenfunctions.
Assume $u(x, y, t) = \sum_{m,n=-\infty}^{\infty} u_{mn}(t)e^{imx}e^{iny}$. This implies

$$u'_{mn}(t) + (m^2 + n^2)u_{mn}(t) = 0, \quad \text{which has the general solution}$$

$$u_n(t) = c_{mn}e^{-(m^2+n^2)t}. \qquad\qquad \text{Thus,}$$

$$\boxed{u(x, y, t) = \sum_{m,n=-\infty}^{\infty} c_{mn}e^{-(m^2+n^2)t}e^{imx}e^{iny}.}$$

$$\begin{aligned}
u(x,y,0) &= \sum_{m,n=-\infty}^{\infty} c_{mn} e^{imx} e^{iny} \;=\; f(x,y), \\
\int_0^{2\pi}\int_0^{2\pi} f(x,y) e^{\overline{imx}} e^{\overline{iny}}\,dxdy &= \int_0^{2\pi}\int_0^{2\pi} \sum_{m,n=-\infty}^{\infty} c_{mn} e^{imx} e^{iny} e^{\overline{im'x}} e^{\overline{in'y}}\,dxdy \\
&= 2\pi \int_0^{2\pi} \sum_{n=-\infty}^{\infty} c_{mn} e^{iny} e^{\overline{in'y}}\,dy \;=\; 4\pi^2 c_{mn}. \\
c_{mn} &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x,y) e^{-imx} e^{-iny}\,dxdy \;=\; f_{mn}.
\end{aligned}$$

$\square$

b) *Show that the integral* $\int_\Omega u^2(x,y,t)\,dxdy$ *is decreasing in* t*, if* f *is not constant.*

Proof. We have

$$u_t = u_{xx} + u_{yy}$$

Multiply the equation by u and integrate:

$$\begin{aligned}
uu_t &= u\triangle u, \\
\frac{1}{2}\frac{d}{dt}u^2 &= u\triangle u, \\
\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dxdy &= \int_\Omega u\triangle u\,dxdy = \underbrace{\int_{\partial\Omega} u\frac{\partial u}{\partial n}\,ds}_{=0,\ \text{(periodic BC)}} - \int_\Omega |\nabla u|^2\,dxdy \\
&= -\int_\Omega |\nabla u|^2\,dxdy \le 0.
\end{aligned}$$

Equality is obtained only when $\nabla u = 0 \;\Rightarrow\; u = \text{constant} \;\Rightarrow\; f = \text{constant}$.
If f is not constant, $\int_\Omega u^2\,dxdy$ is decreasing in t. ☐

Problem (F'98, #3). *Consider the eigenvalue problem*

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0,$$
$$\phi(0) - \frac{d\phi}{dx}(0) = 0, \qquad \phi(1) + \frac{d\phi}{dx}(1) = 0.$$

a) *Show that all eigenvalues are positive.*
b) *Show that there exist a sequence of eigenvalues $\lambda = \lambda_n$, each of which satisfies*

$$\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$

c) *Solve the following initial-boundary value problem on $0 < x < 1$, $t > 0$*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$
$$u(0,t) - \frac{\partial u}{\partial x}(0,t) = 0, \qquad u(1,t) + \frac{\partial u}{\partial x}(1,t) = 0,$$
$$u(x,0) = f(x).$$

You may call the relevant eigenfunctions $\phi_n(x)$ and assume that they are known.

Proof. **a)** • If $\lambda = 0$, the ODE reduces to $\phi'' = 0$. Try $\phi(x) = Ax + B$.
From the first boundary condition,

$$\phi(0) - \phi'(0) = 0 = B - A \quad \Rightarrow \quad B = A.$$

Thus, the solution takes the form $\phi(x) = Ax + A$. The second boundary condition gives

$$\phi(1) + \phi'(1) = 0 = 3A \quad \Rightarrow \quad A = B = 0.$$

Thus the only solution is $\phi \equiv 0$, which is not an eigenfunction, and 0 not an eigenvalue. ✓
• If $\lambda < 0$, try $\phi(x) = e^{sx}$, which gives $s = \pm\sqrt{-\lambda} = \pm\beta \in \mathbb{R}$.
Hence, the family of solutions is $\phi(x) = Ae^{\beta x} + Be^{-\beta x}$. Also, $\phi'(x) = \beta Ae^{\beta x} - \beta Be^{-\beta x}$.
The boundary conditions give

$$\phi(0) - \phi'(0) = 0 = A + B - \beta A + \beta B = A(1-\beta) + B(1+\beta), \tag{29.2}$$

$$\phi(1)+\phi'(1) = 0 = Ae^{\beta} + Be^{-\beta} + \beta Ae^{\beta} - \beta Be^{-\beta} = Ae^{\beta}(1+\beta) + Be^{-\beta}(1-\beta). \tag{29.3}$$

From (29.2) and (29.3) we get

$$\frac{1+\beta}{1-\beta} = -\frac{A}{B} \quad \text{and} \quad \frac{1+\beta}{1-\beta} = -\frac{B}{A}e^{-2\beta}, \qquad \text{or} \qquad \frac{A}{B} = e^{-\beta}.$$

From (29.2), $\beta = \dfrac{A+B}{A-B}$ and thus, $\dfrac{A}{B} = e^{\frac{A+B}{B-A}}$, which has no solutions. ✓

b) Since $\lambda > 0$, the anzats $\phi = e^{sx}$ gives $s = \pm i\sqrt{\lambda}$ and the family of solutions takes the form

$$\phi(x) = A\sin(x\sqrt{\lambda}) + B\cos(x\sqrt{\lambda}).$$

Then, $\phi'(x) = A\sqrt{\lambda}\cos(x\sqrt{\lambda}) - B\sqrt{\lambda}\sin(x\sqrt{\lambda})$. The first boundary condition gives

$$\phi(0) - \phi'(0) \;=\; 0 = B - A\sqrt{\lambda} \qquad \Rightarrow \qquad B = A\sqrt{\lambda}.$$

Hence, $\phi(x) = A\sin(x\sqrt{\lambda}) + A\sqrt{\lambda}\cos(x\sqrt{\lambda})$. The second boundary condition gives

$$\begin{aligned}\phi(1) + \phi'(1) = 0 &= A\sin(\sqrt{\lambda}) + A\sqrt{\lambda}\cos(\sqrt{\lambda}) + A\sqrt{\lambda}\cos(\sqrt{\lambda}) - A\lambda\sin(\sqrt{\lambda}) \\ &= A\big[(1-\lambda)\sin(\sqrt{\lambda}) + 2\sqrt{\lambda}\cos(\sqrt{\lambda})\big]\end{aligned}$$

$A \neq 0$ (since $A = 0$ implies $B = 0$ and $\phi = 0$, which is not an eigenfunction). Therefore, $-(1-\lambda)\sin(\sqrt{\lambda}) = 2\sqrt{\lambda}\cos(\sqrt{\lambda})$, and thus $\tan(\sqrt{\lambda}) = \frac{2\sqrt{\lambda}}{\lambda-1}$.

c) We may assume that the eigenvalues/eigenfunctins of the Laplacian, λ_n and $\phi_n(x)$, are known. We solve the Heat Equation using the "space" eigenfunctions.

$$\begin{cases} u_t = u_{xx}, \\ u(0,t) - u_x(0,t) = 0, \qquad u(1,t) + u_x(1,t) = 0, \\ u(x,0) = f(x). \end{cases}$$

For f, we have an eigenfunction expansion

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x). \qquad \circledast$$

Assume $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x)$. This implies

$u_n'(t) + \lambda_n u_n(t) = 0$, which has the general solution

$$u_n(t) = A_n e^{-\lambda_n t}. \qquad \text{Thus,}$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n t}\phi_n(x),$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \phi_n(x) = f(x).$$

Comparing with $\circledast$, we have $A_n = a_n$. Thus, the solution is given by

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}\phi_n(x),}$$

with

$$a_n = \int_0^1 f(x)\phi_n(x)\,dx.$$

$\square$

Problem (W'03, #3); 266B Ralston Hw. *Let Ω be a smooth domain in three dimensions and consider the initial-boundary value problem for the heat equation*

$$\begin{cases} u_t = \triangle u + f(x) & \text{for } x \in \Omega,\ t > 0 \\ \partial u/\partial n = 0 & \text{for } x \in \partial\Omega,\ t > 0 \\ u(x,0) = g(x) & \text{for } x \in \Omega, \end{cases}$$

in which f and g are known smooth functions with

$$\partial g/\partial n = 0 \qquad \text{for } x \in \partial\Omega.$$

a) *Find an approximate formula for u as $t \to \infty$.*

Proof. We expand u in terms of the **Neumann eigenfunctions of Laplacian** in Ω.

$$\triangle\phi_n + \lambda_n\phi_n = 0 \quad \text{in } \Omega, \qquad \frac{\partial\phi_n}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Note that here $\lambda_1 = 0$ and ϕ_1 is the constant $V^{-1/2}$, where V is the volume of Ω. Assume

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x), \qquad a_n(t) = \int_\Omega \phi_n(x)u(x,t)\,dx.$$

$$f(x) = \sum_{n=1}^{\infty} f_n\phi_n(x), \qquad f_n = \int_\Omega \phi_n(x)f(x)\,dx.$$

$$g(x) = \sum_{n=1}^{\infty} g_n\phi_n(x), \qquad g_n = \int_\Omega \phi_n(x)g(x)\,dx.$$

$$\begin{aligned} a_n'(t) &= \int_\Omega \phi_n(x)u_t\,dx = \int_\Omega \phi_n(\triangle u + f)\,dx = \int_\Omega \phi_n\triangle u\,dx + \int_\Omega \phi_n f\,dx \\ &= \int_\Omega \triangle\phi_n u\,dx + \int_\Omega \phi_n f\,dx = -\lambda_n \int_\Omega \phi_n u\,dx + \underbrace{\int_\Omega \phi_n f\,dx}_{f_n} = -\lambda_n a_n + f_n. \end{aligned}$$

$$a_n(0) = \int_\Omega \phi_n(x)u(x,0)\,dx = \int_\Omega \phi_n g\,dx = g_n.$$

[78] Thus, we solve the ODE:

$$\begin{cases} a_n' + \lambda_n a_n = f_n \\ a_n(0) = g_n. \end{cases}$$

For $n = 1$, $\lambda_1 = 0$, and we obtain $a_1(t) = f_1 t + g_1$.
For $n \geq 2$, the homogeneous solution is $a_{n_h} = ce^{-\lambda_n t}$. The anzats for a particular solution is $a_{n_p} = c_1 t + c_2$, which gives $c_1 = 0$ and $c_2 = f_n/\lambda_n$. Using the initial condition, we obtain

$$a_n(t) = \Big(g_n - \frac{f_n}{\lambda_n}\Big)e^{-\lambda_n t} + \frac{f_n}{\lambda_n}.$$

[78] We used Green's formula: $\int_{\partial\Omega}\big(\phi_n\frac{\partial u}{\partial n} - u\frac{\partial\phi_n}{\partial n}\big)\,ds = \int_\Omega(\phi_n\triangle u - \triangle\phi_n u)\,dx$.
On $\partial\Omega$, $\frac{\partial u}{\partial n} = 0$; $\frac{\partial\phi_n}{\partial n} = 0$ since eigenfunctions are Neumann.

$$\boxed{u(x,t) = (f_1 t + g_1)\phi_1(x) + \sum_{n=2}^{\infty}\left[\left(g_n - \frac{f_n}{\lambda_n}\right)e^{-\lambda_n t} + \frac{f_n}{\lambda_n}\right]\phi_n(x).}$$

If $f_1 = 0$ $\left(\int_\Omega f(x)\,dx = 0\right)$, $\displaystyle\lim_{t\to\infty} u(x,t) = g_1\phi_1 + \sum_{n=2}^{\infty}\frac{f_n\phi_n}{\lambda_n}.$

If $f_1 \neq 0$ $\left(\int_\Omega f(x)\,dx \neq 0\right)$, $\displaystyle\lim_{t\to\infty} u(x,t) \sim f_1\phi_1 t.$

b) *If* $g \geq 0$ *and* $f > 0$*, show that* $u > 0$ *for all* $t > 0$*.*

$\square$

Problem (S'97, #2). ***a)*** *Consider the eigenvalue problem for the Laplace operator* $\triangle$ *in* $\Omega \in \mathbb{R}^2$ *with zero Neumann boundary condition*

$$\begin{cases} u_{xx} + u_{yy} + \lambda u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Prove that $\lambda_0 = 0$ *is the lowest eigenvalue and that it is simple.*
b) *Assume that the eigenfunctions* $\phi_n(x, y)$ *of the problem in (a) form a complete orthogonal system, and that* $f(x, y)$ *has a uniformly convergent expansion*

$$f(x, y) = \sum_{n=0}^{\infty} f_n \phi_n(x, y).$$

Solve the initial value problem

$$u_t = \triangle u + f(x, y)$$

subject to initial and boundary conditions

$$u(x, y, 0) = 0, \qquad \frac{\partial u}{\partial n} u|_{\partial\Omega} = 0.$$

What is the behavior of $u(x, y, t)$ *as* $t \to \infty$*?*
c) *Consider the problem with Neumann boundary conditions*

$$\begin{cases} v_{xx} + v_{yy} + f(x, y) = 0 & \text{in } \Omega \\ \frac{\partial v}{\partial n} v = 0 & \text{on } \partial\Omega. \end{cases}$$

When does a solution exist? Find this solution, and find its relation with the behavior of $\lim u(x, y, t)$ *in (b) as* $t \to \infty$*.*

Proof. **a)** Suppose this eigenvalue problem did have a solution u with $\lambda \le 0$.
Multiplying $\triangle u + \lambda u = 0$ by u and integrating over Ω, we get

$$\int_\Omega u \triangle u \, dx + \lambda \int_\Omega u^2 \, dx = 0,$$

$$\int_{\partial\Omega} u \underbrace{\frac{\partial u}{\partial n}}_{=0} \, ds - \int_\Omega |\nabla u|^2 \, dx + \lambda \int_\Omega u^2 \, dx = 0,$$

$$\int_\Omega |\nabla u|^2 \, dx = \underbrace{\lambda}_{\le 0} \int_\Omega u^2 \, dx,$$

Thus, $\nabla u = 0$ in Ω, and u is constant in Ω. Hence, we now have

$$0 = \underbrace{\lambda}_{\le 0} \int_\Omega u^2 \, dx.$$

For nontrivial u, we have $\lambda = 0$. For this eigenvalue problem, $\lambda = 0$ is an eigenvalue, its eigenspace is the set of constants, and all other λ's are positive.

b) We expand u in terms of the **Neumann eigenfunctions of Laplacian** in Ω. [79]

$$\triangle\phi_n + \lambda_n\phi_n = 0 \quad \text{in } \Omega, \qquad \frac{\partial \phi_n}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

$$u(x,y,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x,y), \qquad a_n(t) = \int_\Omega \phi_n(x,y)u(x,y,t)\,dx.$$

$$\begin{aligned} a_n'(t) &= \int_\Omega \phi_n(x,y)u_t\,dx = \int_\Omega \phi_n(\triangle u + f)\,dx = \int_\Omega \phi_n \triangle u\,dx + \int_\Omega \phi_n f\,dx \\ &= \int_\Omega \triangle\phi_n u\,dx + \int_\Omega \phi_n f\,dx = -\lambda_n \int_\Omega \phi_n u\,dx + \underbrace{\int_\Omega \phi_n f\,dx}_{f_n} = -\lambda_n a_n + f_n. \\ a_n(0) &= \int_\Omega \phi_n(x,y)u(x,y,0)\,dx = 0. \end{aligned}$$

[80] Thus, we solve the ODE:

$$\begin{cases} a_n' + \lambda_n a_n = f_n \\ a_n(0) = 0. \end{cases}$$

For $n = 1$, $\lambda_1 = 0$, and we obtain $a_1(t) = f_1 t$.
For $n \geq 2$, the homogeneous solution is $a_{n_h} = ce^{-\lambda_n t}$. The anzats for a particular solution is $a_{n_p} = c_1 t + c_2$, which gives $c_1 = 0$ and $c_2 = f_n/\lambda_n$. Using the initial condition, we obtain

$$a_n(t) = -\frac{f_n}{\lambda_n}e^{-\lambda_n t} + \frac{f_n}{\lambda_n}.$$

$$\boxed{u(x,t) = f_1\phi_1 t + \sum_{n=2}^{\infty}\Big(-\frac{f_n}{\lambda_n}e^{-\lambda_n t} + \frac{f_n}{\lambda_n}\Big)\phi_n(x).}$$

$$\text{If } \ f_1 = 0 \ \ \Big(\int_\Omega f(x)\,dx = 0\Big), \qquad \lim_{t\to\infty} u(x,t) = \sum_{n=2}^{\infty}\frac{f_n\phi_n}{\lambda_n}.$$

$$\text{If } \ f_1 \neq 0 \ \ \Big(\int_\Omega f(x)\,dx \neq 0\Big), \qquad \lim_{t\to\infty} u(x,t) \sim f_1\phi_1 t.$$

c) Integrate $\triangle v + f(x,y) = 0$ over Ω:

$$\int_\Omega f\,dx = -\int_\Omega \triangle v\,dx = -\int_\Omega \nabla\cdot\nabla v\,dx =^1 -\int_{\partial\Omega}\frac{\partial v}{\partial n}\,ds =^2 0,$$

where we used [1] divergence theorem and [2] Neumann boundary conditions. Thus, the solution exists only if

$$\int_\Omega f\,dx = 0.$$

---

[79] We use $dx\,dy \to dx$.
[80] We used Green's formula: $\int_{\partial\Omega}\big(\phi_n\frac{\partial u}{\partial n} - u\frac{\partial\phi_n}{\partial n}\big)ds = \int_\Omega(\phi_n\triangle u - \triangle\phi_n u)\,dx$.
On $\partial\Omega$, $\frac{\partial u}{\partial n} = 0$; $\frac{\partial\phi_n}{\partial n} = 0$ since eigenfunctions are Neumann.

Assume $v(x,y) = \sum_{n=0}^{\infty} a_n\phi_n(x,y)$. Since we have $f(x,y) = \sum_{n=0}^{\infty} f_n\phi_n(x,y)$, we obtain

$$
\begin{aligned}
-\sum_{n=0}^{\infty} \lambda_n a_n\phi_n + \sum_{n=0}^{\infty} f_n\phi_n &= 0, \\
-\lambda_n a_n\phi_n + f_n\phi_n &= 0, \\
a_n &= \frac{f_n}{\lambda_n}.
\end{aligned}
$$

$v(x,y) = \sum_{n=0}^{\infty}(\frac{f_n}{\lambda_n})\phi_n(x,y).$ $\square$

29.1 Heat Equation with Periodic Boundary Conditions in 2D (with extra terms)

Problem (F'99, #5). *In two spatial dimensions, consider the differential equation*

$$u_t = -\varepsilon \triangle u - \triangle^2 u$$

with ***periodic boundary conditions*** *on the unit square* $[0, 2\pi]^2$.
a) *If* $\varepsilon = 2$ *find a solution whose amplitude increases as* t *increases.*
b) *Find a value* ε_0*, so that the solution of this PDE stays bounded as* $t \to \infty$*, if* $\varepsilon < \varepsilon_0$.

Proof. **a) Eigenfunctions of the Laplacian.**
The periodic boundary conditions imply a Fourier Series solution of the form:

$$u(x,t) = \sum_{m,n} a_{mn}(t) e^{i(mx+ny)}.$$

$$\begin{aligned}
u_t &= \sum_{m,n} a'_{mn}(t) e^{i(mx+ny)}, \\
\triangle u &= u_{xx} + u_{yy} = -\sum_{m,n} (m^2+n^2)\, a_{mn}(t) e^{i(mx+ny)}, \\
\triangle^2 u &= u_{xxxx} + 2u_{xxyy} + u_{yyyy} = \sum_{m,n} (m^4 + 2m^2n^2 + n^4)\, a_{mn}(t) e^{i(mx+ny)} \\
&= \sum_{m,n} (m^2+n^2)^2\, a_{mn}(t) e^{i(mx+ny)}.
\end{aligned}$$

Plugging this into the PDE, we obtain

$$\begin{aligned}
&a'_{mn}(t) = \varepsilon(m^2+n^2) a_{mn}(t) - (m^2+n^2)^2 a_{mn}(t), \\
&a'_{mn}(t) - [\varepsilon(m^2+n^2) - (m^2+n^2)^2] a_{mn}(t) = 0, \\
&a'_{mn}(t) - (m^2+n^2)[\varepsilon - (m^2+n^2)] a_{mn}(t) = 0.
\end{aligned}$$

The solution to the ODE above is

$$a_{mn}(t) = \alpha_{mn}\, e^{(m^2+n^2)[\varepsilon-(m^2+n^2)]t}.$$

$$\boxed{u(x,t) = \sum_{m,n} \alpha_{mn}\, e^{(m^2+n^2)[\varepsilon-(m^2+n^2)]t} \underbrace{e^{i(mx+ny)}}_{oscillates}. \qquad \circledast}$$

When $\varepsilon = 2$, we have

$$u(x,t) = \sum_{m,n} \alpha_{mn}\, e^{(m^2+n^2)[2-(m^2+n^2)]t}\, e^{i(mx+ny)}.$$

We need a solution whose amplitude increases as t increases. Thus, we need those $\alpha_{mn} > 0$, with

$$\begin{aligned}
(m^2+n^2)[2-(m^2+n^2)] &> 0, \\
2-(m^2+n^2) &> 0, \\
2 &> m^2+n^2.
\end{aligned}$$

Hence, $\alpha_{mn} > 0$ for $(m,n) = (0,0)$, $(m,n) = (1,0)$, $(m,n) = (0,1)$.
Else, $\alpha_{mn} = 0$. Thus,

$$\begin{aligned}
u(x,t) &= \alpha_{00} + \alpha_{10} e^t e^{ix} + \alpha_{01} e^t e^{iy} = 1 + e^t e^{ix} + e^t e^{iy} \\
&= 1 + e^t(\cos x + i \sin x) + e^t(\cos y + i \sin y).
\end{aligned}$$

b) For $\varepsilon \le \varepsilon_0 = 1$, the solution $\circledast$ stays bounded as $t \to \infty$. $\square$

Problem (F'93, #1).
Suppose that a and b are constants with $a \geq 0$, and consider the equation

$$u_t \;=\; u_{xx} + u_{yy} - au^3 + bu \tag{29.4}$$

*in which $u(x,y,t)$ is **2π-periodic** in x and y.*

a) *Let u be a solution of (29.4) with*

$$||u(t=0)|| \;=\; \int_0^{2\pi}\int_0^{2\pi} |u(x,y,t=0)|^2\,dxdy^{1/2} \;<\;\epsilon.$$

Derive an explicit bound on $||u(t)||$ and show that it stays finite for all t.
b) *If $a=0$, construct the normal modes for (29.4); i.e. find all solutions of the form*

$$u(x,y,t) = e^{\lambda t + ikx + ily}.$$

c) *Use these normal modes to construct a solution of (29.4) with $a=0$ for the initial data*

$$u(x,y,t=0) \;=\; \frac{1}{1-\frac{1}{2}e^{ix}} + \frac{1}{1-\frac{1}{2}e^{-ix}}.$$

Proof. **a)** Multiply the equation by u and integrate:

$$\begin{aligned}
u_t &= \triangle u - au^3 + bu,\\
uu_t &= u\triangle u - au^4 + bu^2,\\
\int_\Omega uu_t\,dx &= \int_\Omega u\triangle u\,dx - \int_\Omega au^4\,dx + \int_\Omega bu^2\,dx,\\
\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx &= \underbrace{\int_{\partial\Omega} u\frac{\partial u}{\partial n}\,ds}_{=0,\ u \text{ periodic on } [0,2\pi]^2} \underbrace{-\int_\Omega |\nabla u|^2\,dx - \int_\Omega au^4\,dx}_{\leq 0} + \int_\Omega bu^2\,dx,\\
\frac{d}{dt}||u||_2^2 &\leq 2b\,||u||_2^2,\\
||u||_2^2 &\leq ||u(x,0)||_2^2\,e^{2bt},\\
||u||_2 &\leq ||u(x,0)||_2\,e^{bt} \;\leq\; \varepsilon\,e^{bt}.
\end{aligned}$$

Thus, $||u||$ stays finite for all t.

b) Since $a=0$, plugging $u = e^{\lambda t+ikx+ily}$ into the equation, we obtain:

$$\begin{aligned}
u_t &= u_{xx} + u_{yy} + bu,\\
\lambda\,e^{\lambda t+ikx+ily} &= (-k^2 - l^2 + b)\,e^{\lambda t+ikx+ily},\\
\lambda &= -k^2 - l^2 + b.
\end{aligned}$$

Thus,

$$\begin{aligned}
u_{kl} &= e^{(-k^2-l^2+b)t+ikx+ily},\\
u(x,y,t) &= \sum_{k,l} a_{kl}\,e^{(-k^2-l^2+b)t+ikx+ily}.
\end{aligned}$$

c) Using the initial condition, we obtain:

$$\begin{aligned}
u(x,y,0) \;=\; \sum_{k,l} a_{kl}\, e^{i(kx+ly)} \;&=\; \frac{1}{1-\frac{1}{2}e^{ix}} + \frac{1}{1-\frac{1}{2}e^{-ix}} \\
&=\; \sum_{k=0}^{\infty}\Big(\frac{1}{2}e^{ix}\Big)^k + \sum_{k=0}^{\infty}\Big(\frac{1}{2}e^{-ix}\Big)^k \\
&=\; \sum_{k=0}^{\infty}\frac{1}{2^k}e^{ikx} + \sum_{k=0}^{\infty}\frac{1}{2^k}e^{-ikx}, \\
&=\; 2 + \sum_{k=1}^{\infty}\frac{1}{2^k}e^{ikx} + \sum_{k=-1}^{-\infty}\frac{1}{2^{-k}}e^{ikx}.
\end{aligned}$$

Thus, $\;l=0,\;$ and we have

$$\sum_{k=-\infty}^{\infty} a_k\, e^{ikx} \;=\; 2 + \sum_{k=1}^{\infty}\frac{1}{2^k}e^{ikx} + \sum_{k=-1}^{-\infty}\frac{1}{2^{-k}}e^{ikx},$$

$$\Rightarrow \quad a_0 = 2; \quad a_k = \frac{1}{2^k}, \quad k>0; \quad a_k = \frac{1}{2^{-k}}, \quad k<0$$

$$\Rightarrow \quad a_0 = 2; \quad a_k = \frac{1}{2^{|k|}}, \quad k \neq 0.$$

$$\boxed{u(x,y,t) = 2e^{bt} + \sum_{k=-\infty,\,k\neq 0}^{+\infty}\frac{1}{2^{|k|}}\, e^{(-k^2+b)t+ikx}.}$$

[81] $\square$

---

[81] Note a similar question formulation in F'92 #3(b).

Problem (S'00, #3). *Consider the initial-boundary value problem for* $u = u(x, y, t)$

$$u_t = \triangle u - u$$

for $(x, y) \in [0, 2\pi]^2$*, with* ***periodic boundary conditions*** *and with*

$$u(x, y, 0) = u_0(x, y)$$

in which u_0 *is periodic. Find an asymptotic expansion for* u *for* t *large with terms tending to zero increasingly rapidly as* $t \to \infty$.

Proof. Since we have periodic boundary conditions, assume

$$u(x, y, t) \;=\; \sum_{m,n} u_{mn}(t)\, e^{i(mx+ny)}.$$

Plug this into the equation:

$$\begin{aligned}
\sum_{m,n} u'_{mn}(t)\, e^{i(mx+ny)} &= \sum_{m,n} (-m^2 - n^2 - 1)\, u_{mn}(t)\, e^{i(mx+ny)}, \\
u'_{mn}(t) &= (-m^2 - n^2 - 1)\, u_{mn}(t), \\
u_{mn}(t) &= a_{mn}\, e^{(-m^2-n^2-1)t}, \\
u(x, y, t) &= \sum_{m,n} a_{mn}\, e^{-(m^2+n^2+1)t}\, e^{i(mx+ny)}.
\end{aligned}$$

Since u_0 is periodic,

$$u_0(x, y) = \sum_{m,n} u_{0mn}\, e^{i(mx+ny)}, \qquad u_{0mn} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} u_0(x, y)\, e^{-i(mx+ny)}\, dxdy.$$

Initial condition gives:

$$\begin{aligned}
u(x, y, 0) \;=\; \sum_{m,n} a_{mn}\, e^{i(mx+ny)} &= u_0(x, y), \\
\sum_{m,n} a_{mn}\, e^{i(mx+ny)} &= \sum_{m,n} u_{0mn}\, e^{i(mx+ny)}, \\
\Rightarrow \quad a_{mn} &= u_{0mn}.
\end{aligned}$$

$$\boxed{u(x, y, t) = \sum_{m,n} u_{0mn}\, e^{-(m^2+n^2+1)t}\, e^{i(mx+ny)}.}$$

$u_{0mn}\, e^{-(m^2+n^2+1)t}\, e^{i(mx+ny)} \to 0$ as $t \to \infty$, since $e^{-(m^2+n^2+1)t} \to 0$ as $t \to \infty$. □

30 Problems: Fourier Transform

Problem (S'01, #2b). *Write the solution of initial value problem*

$$U_t - \begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix} U_x = 0,$$

for general initial data

$$\begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix} = \begin{pmatrix} f(x) \\ 0 \end{pmatrix} \qquad \textit{as an } \textbf{\textit{inverse Fourier transform.}}$$

You may assume that f is smooth and rapidly decreasing as $|x| \to \infty$.

Proof. Consider the original system:

$$\begin{aligned} u_t^{(1)} - u_x^{(1)} &= 0, \\ u_t^{(2)} - 5u_x^{(1)} - 3u_x^{(2)} &= 0. \end{aligned}$$

Take the Fourier transform in x. The transformed initial value problems are:

$$\begin{aligned} \widehat{u}_t^{(1)} - i\xi\widehat{u}^{(1)} &= 0, \qquad & \widehat{u}^{(1)}(\xi,0) &= \widehat{f}(\xi), \\ \widehat{u}_t^{(2)} - 5i\xi\widehat{u}^{(1)} - 3i\xi\widehat{u}^{(2)} &= 0, \qquad & \widehat{u}^{(2)}(\xi,0) &= 0. \end{aligned}$$

Solving the first ODE for $\widehat{u}^{(1)}$ gives:

$$\widehat{u}^{(1)}(\xi,t) = \widehat{f}(\xi)e^{i\xi t}. \qquad \checkmark$$

With this $\widehat{u}^{(1)}$, the second initial value problem becomes

$$\widehat{u}_t^{(2)} - 3i\xi\widehat{u}^{(2)} = 5i\xi\widehat{f}(\xi)e^{i\xi t}, \qquad \widehat{u}^{(2)}(\xi,0) = 0.$$

The homogeneous solution of the above ODE is:

$$\widehat{u}_h^{(2)}(\xi,t) = c_1 e^{3i\xi t}.$$

With $\widehat{u}_p^{(2)} = c_2 e^{i\xi t}$ as anzats for a particular solution, we obtain:

$$\begin{aligned} i\xi c_2 e^{i\xi t} - 3i\xi c_2 e^{i\xi t} &= 5i\xi\widehat{f}(\xi)e^{i\xi t}, \\ -2i\xi c_2 e^{i\xi t} &= 5i\xi\widehat{f}(\xi)e^{i\xi t}, \\ c_2 &= -\frac{5}{2}\widehat{f}(\xi). \\ \Rightarrow\ \widehat{u}_p^{(2)}(\xi,t) &= -\frac{5}{2}\widehat{f}(\xi)e^{i\xi t}. \end{aligned}$$

$$\widehat{u}^{(2)}(\xi,t) = \widehat{u}_h^{(2)}(\xi,t) + \widehat{u}_p^{(2)}(\xi,t) = c_1 e^{3i\xi t} - \frac{5}{2}\widehat{f}(\xi)e^{i\xi t}.$$

We find c_1 using initial conditions:

$$\widehat{u}^{(2)}(\xi,0) = c_1 - \frac{5}{2}\widehat{f}(\xi) = 0 \qquad \Rightarrow \quad c_1 = \frac{5}{2}\widehat{f}(\xi).$$

Thus,

$$\widehat{u}^{(2)}(\xi,t) = \frac{5}{2}\widehat{f}(\xi)\big(e^{3i\xi t} - e^{i\xi t}\big). \qquad \checkmark$$

$u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ are be obtained by taking **inverse Fourier transform**:

$$\begin{aligned} u^{(1)}(x,t) &= \big(\widehat{u}^{(1)}(\xi,t)\big)^{\vee} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^n} e^{ix\xi}\,\widehat{f}(\xi)\,e^{i\xi t}\,d\xi, \\ u^{(2)}(x,t) &= \big(\widehat{u}^{(2)}(\xi,t)\big)^{\vee} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^n} e^{ix\xi}\,\frac{5}{2}\widehat{f}(\xi)\big(e^{3i\xi t}-e^{i\xi t}\big)\,d\xi. \end{aligned}$$

□

Problem (S'02, #4). *Use the **Fourier transform** on $L^2(\mathbb{R})$ to show that*

$$\frac{du}{dx} + cu(x) + u(x-1) = f \tag{30.1}$$

has a unique solution $u \in L^2(\mathbb{R})$ for each $f \in L^2(\mathbb{R})$ when $|c| > 1$ - you may assume that c is a real number.

Proof. $u \in L^2(\mathbb{R})$. Define its **Fourier transform** $\widehat{u}$ by

$$\begin{aligned} \widehat{u}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} u(x)\, dx \qquad \text{for } \ \xi \in \mathbb{R}. \\ \widehat{\frac{du}{dx}}(\xi) &= i\xi \widehat{u}(\xi). \end{aligned}$$

We can find $\widehat{u(x-1)}(\xi)$ in two ways.

- Let $u(\underbrace{x-1}_{y}) = v(x)$, and determinte $\widehat{v}(\xi)$:

$$\begin{aligned} \widehat{u(x-1)}(\xi) = \widehat{v}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} v(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i(y+1)\xi} u(y)\, dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iy\xi} e^{-i\xi} u(y)\, dy = e^{-i\xi} \widehat{u}(\xi). \qquad \circledast \end{aligned}$$

- We can also write the definition for $\widehat{u}(\xi)$ and substitute $x-1$ later in calculations:

$$\begin{aligned} \widehat{u}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iy\xi} u(y)\, dy = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i(x-1)\xi} u(x-1)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} e^{i\xi} u(x-1)\, dx = e^{i\xi} \widehat{u(x-1)}(\xi), \\ \Rightarrow \qquad & \widehat{u(x-1)}(\xi) = e^{-i\xi} \widehat{u}(\xi). \end{aligned}$$

Substituting into (30.1), we obtain

$$\begin{aligned} & i\xi\widehat{u}(\xi) + c\widehat{u}(\xi) + e^{-i\xi}\widehat{u}(\xi) = \widehat{f}(\xi), \\ \widehat{u}(\xi) &= \frac{\widehat{f}(\xi)}{i\xi + c + e^{-i\xi}}. \\ u(x) &= \left(\frac{\widehat{f}(\xi)}{i\xi + c + e^{-i\xi}} \right)^{\vee} = \big(\widehat{f}\,\widehat{B}\big)^{\vee} = \frac{1}{\sqrt{2\pi}} f * B, \\ & \text{where } \ \widehat{B} = \frac{1}{i\xi + c + e^{-i\xi}}, \\ \Rightarrow \ B &= \left(\frac{1}{i\xi + c + e^{-i\xi}} \right)^{\vee} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{ix\xi}}{i\xi + c + e^{-i\xi}}\, d\xi. \end{aligned}$$

For $|c| > 1$, $\widehat{u}(\xi)$ exists for all $\xi \in \mathbb{R}$, so that $u(x) = (\widehat{u}(\xi))^{\vee}$ and this is unique by the Fourier Inversion Theorem. $\square$

Note that in $\mathbb{R}^n$, $\circledast$ becomes

$$\begin{aligned} \widehat{u(x-1)}(\xi) = \widehat{v}(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} v(x)\, dx = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i(y+1)\cdot\xi} u(y)\, dy \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-iy\cdot\xi} e^{-i\vec{1}\cdot\xi} u(y)\, dy = e^{-i\vec{1}\cdot\xi}\widehat{u}(\xi) = e^{(-i\sum_j \xi_j)}\, \widehat{u}(\xi). \end{aligned}$$

Problem (F'96, #3). *Find the* ***fundamental solution*** *for the equation*

$$u_t = u_{xx} - xu_x. \tag{30.2}$$

Hint: The ***Fourier transform*** *converts this problem into a PDE which can be solved using the method of characteristics.*

Proof. $u \in L^2(\mathbb{R})$. Define its **Fourier transform** $\widehat{u}$ by

$$\begin{aligned}
\widehat{u}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} u(x)\,dx \qquad \text{for } \xi \in \mathbb{R}. \\
\widehat{u_x}(\xi) &= i\xi\,\widehat{u}(\xi), \\
\widehat{u_{xx}}(\xi) &= (i\xi)^2\,\widehat{u}(\xi) = -\xi^2\,\widehat{u}(\xi). \quad \checkmark
\end{aligned}$$

We find $\widehat{xu_x}(\xi)$ in two steps:
① Multiplication by x:

$$\begin{aligned}
\widehat{-ixu}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi}\big(-ixu(x)\big)\,dx = \frac{d}{d\xi}\widehat{u}(\xi). \\
&\Rightarrow \widehat{xu(x)}(\xi) = i\frac{d}{d\xi}\widehat{u}(\xi).
\end{aligned}$$

② Using the previous result, we find:

$$\begin{aligned}
\widehat{xu_x(x)}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi}\big(xu_x(x)\big)\,dx = \underbrace{\frac{1}{\sqrt{2\pi}}\Big[e^{-ix\xi}xu\Big]_{-\infty}^{\infty}}_{=0} - \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Big((-i\xi)e^{-ix\xi}x + e^{-ix\xi}\Big)\,u\,dx \\
&= \frac{1}{\sqrt{2\pi}}\, i\xi \int_{\mathbb{R}} e^{-ix\xi}\,x\,u\,dx - \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi}\,u\,dx \\
&= i\xi\,\widehat{xu(x)}(\xi) - \widehat{u}(\xi) = i\xi\Big[i\frac{d}{d\xi}\widehat{u}(\xi)\Big] - \widehat{u}(\xi) = -\xi\frac{d}{d\xi}\widehat{u}(\xi) - \widehat{u}(\xi). \\
&\Rightarrow \widehat{xu_x(x)}(\xi) = -\xi\frac{d}{d\xi}\widehat{u}(\xi) - \widehat{u}(\xi). \quad \checkmark
\end{aligned}$$

Plugging these into (30.2), we get:

$$\begin{aligned}
\frac{\partial}{\partial t}\widehat{u}(\xi,t) &= -\xi^2\widehat{u}(\xi,t) - \Big(-\xi\frac{d}{d\xi}\widehat{u}(\xi,t) - \widehat{u}(\xi,t)\Big), \\
\widehat{u}_t &= -\xi^2\widehat{u} + \xi\widehat{u}_\xi + \widehat{u}, \\
\widehat{u}_t - \xi\widehat{u}_\xi &= -(\xi^2 - 1)\widehat{u}.
\end{aligned}$$

We now solve the above equation by characteristics.
We change the notation: $\widehat{u} \to u$, $t \to y$, $\xi \to x$. We have

$$u_y - xu_x = -(x^2 - 1)u.$$

$$\begin{aligned}
\frac{dx}{dt} &= -x \quad \Rightarrow \quad x = c_1 e^{-t}, \qquad (c_1 = xe^t) \\
\frac{dy}{dt} &= 1 \quad \Rightarrow \quad y = t + c_2, \\
\frac{dz}{dt} &= -(x^2 - 1)z = -(c_1^2 e^{-2t} - 1)z \quad \Rightarrow \quad \frac{dz}{z} = -(c_1^2 e^{-2t} - 1)dt \\
&\Rightarrow \quad \log z = \frac{1}{2}c_1^2 e^{-2t} + t + c_3 = \frac{x^2}{2} + t + c_3 = \frac{x^2}{2} + y - c_2 + c_3 \quad \Rightarrow \quad z = ce^{\frac{x^2}{2} + y}.
\end{aligned}$$

Changing the notation back, we have

$$\widehat{u}(\xi,t) = ce^{\frac{\xi^2}{2}+t}.$$

Thus, we have

$$\widehat{u}(\xi,t) = ce^{\frac{\xi^2}{2}+t}.$$

We use Inverse Fourier Tranform to get $u(x,t)$: [82]

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\,\widehat{u}(\xi,t)\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\,ce^{\frac{\xi^2}{2}+t}\,d\xi \\
&= \frac{c}{\sqrt{2\pi}}\,e^t\int_{\mathbb{R}} e^{ix\xi}\,e^{\frac{\xi^2}{2}}\,d\xi = \frac{c}{\sqrt{2\pi}}\,e^t\int_{\mathbb{R}} e^{ix\xi+\frac{\xi^2}{2}}\,d\xi \\
&= \frac{c}{\sqrt{2\pi}}\,e^t\int_{\mathbb{R}} e^{\frac{2ix\xi+\xi^2}{2}}\,d\xi = \frac{c}{\sqrt{2\pi}}\,e^t\int_{\mathbb{R}} e^{\frac{(\xi+ix)^2}{2}}\,d\xi\; e^{\frac{x^2}{2}} \\
&= \frac{c}{\sqrt{2\pi}}\,e^t e^{\frac{x^2}{2}}\int_{\mathbb{R}} e^{\frac{y^2}{2}}\,dy = \frac{c}{\sqrt{2\pi}}\,e^t e^{\frac{x^2}{2}}\sqrt{2\pi} = c\,e^t\,e^{\frac{x^2}{2}}.
\end{aligned}$$

$$\boxed{u(x,t) = c\,e^t\,e^{\frac{x^2}{2}}.}$$

Check:

$$\begin{aligned}
u_t &= c\,e^t\,e^{\frac{x^2}{2}}, \\
u_x &= c\,e^t\,xe^{\frac{x^2}{2}}, \\
u_{xx} &= c\,e^t\left(e^{\frac{x^2}{2}} + x^2e^{\frac{x^2}{2}}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
u_t &= u_{xx} - xu_x, \\
c\,e^t\,e^{\frac{x^2}{2}} &= c\,e^t\left(e^{\frac{x^2}{2}} + x^2e^{\frac{x^2}{2}}\right) - x\,c\,e^t\,xe^{\frac{x^2}{2}}. \quad \checkmark
\end{aligned}$$

$\square$

---

[82]We complete the square for powers of exponentials.

Problem (W'02, #4). ***a)*** *Solve the initial value problem*

$$\frac{\partial u}{\partial t}+\sum_{k=1}^{n}a_k(t)\frac{\partial u}{\partial x_k}+a_0(t)u=0, \quad x\in\mathbb{R}^n,$$
$$u(0,x)=f(x)$$

where $a_k(t)$, $k=1,\ldots,n$, and $a_0(t)$ are continuous functions, and f is a continuous function. You may assume f has compact support.
b) *Solve the initial value problem*

$$\frac{\partial u}{\partial t}+\sum_{k=1}^{n}a_k(t)\frac{\partial u}{\partial x_k}+a_0(t)u=f(x,t), \quad x\in\mathbb{R}^n,$$
$$u(0,x)=0$$

where f is continuous in x and t.

Proof. **a)** Use the **Fourier transform** to solve this problem.

$$\widehat{u}(\xi,t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x,t)\,dx \qquad \text{for } \xi\in\mathbb{R}.$$
$$\widehat{\frac{\partial u}{\partial x_k}} = i\xi_k\widehat{u}.$$

Thus, the equation becomes:

$$\begin{cases}\widehat{u}_t+i\sum_{k=1}^{n}a_k(t)\xi_k\widehat{u}+a_0(t)\widehat{u} = 0,\\ \widehat{u}(\xi,0) = \widehat{f}(\xi),\end{cases}$$

or

$$\widehat{u}_t+i\,\vec{a}(t)\cdot\vec{\xi}\,\widehat{u}+a_0(t)\widehat{u} = 0,$$
$$\widehat{u}_t = -\big(i\,\vec{a}(t)\cdot\vec{\xi}+a_0(t)\big)\,\widehat{u}.$$

This is an ODE in $\widehat{u}$ with solution:

$$\widehat{u}(\xi,t) = ce^{-\int_0^t(i\,\vec{a}(s)\cdot\vec{\xi}+a_0(s))\,ds}, \qquad \widehat{u}(\xi,0) = c = \widehat{f}(\xi). \qquad \text{Thus,}$$
$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\,e^{-\int_0^t(i\,\vec{a}(s)\cdot\vec{\xi}+a_0(s))\,ds}.$$

Use the Inverse Fourier transform to get $u(x,t)$:

$$u(x,t) = \widehat{u}(\xi,t)^{\vee} = \Big[\widehat{f}(\xi)\,e^{-\int_0^t(i\,\vec{a}(s)\cdot\vec{\xi}+a_0(s))\,ds}\Big]^{\vee} = \frac{(f*g)(x)}{(2\pi)^{\frac{n}{2}}},$$
$$\text{where} \quad \widehat{g}(\xi) = e^{-\int_0^t(i\,\vec{a}(s)\cdot\vec{\xi}+a_0(s))\,ds}.$$
$$g(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{ix\cdot\xi}\,\widehat{g}(\xi)\,d\xi = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{ix\cdot\xi}\,\big[e^{-\int_0^t(i\,\vec{a}(s)\cdot\vec{\xi}+a_0(s))\,ds}\big]\,d\xi.$$

$$\boxed{u(x,t) = \frac{(f*g)(x)}{(2\pi)^{\frac{n}{2}}} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{i(x-y)\cdot\xi}\,\big[e^{-\int_0^t(i\,\vec{a}(s)\cdot\vec{\xi}+a_0(s))\,ds}\big]\,d\xi\,f(y)\,dy.}$$

b) Use **Duhamel's Principle** and the result from (a).

$$u(x,t) = \int_0^t U(x,t-s,s)\,ds, \qquad \text{where } U(x,t,s) \text{ solves}$$
$$\frac{\partial U}{\partial t}+\sum_{k=1}^{n}a_k(t)\frac{\partial U}{\partial x_k}+a_0(t)U = 0,$$
$$U(x,0,s)=f(x,s).$$

$$\boxed{u(x,t) \;=\; \int_0^t U(x, t-s, s)\, ds \;=\; \frac{1}{(2\pi)^n}\int_0^t \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\left[e^{-\int_0^{\mathbf{t-s}}(i\,\vec{a}(s)\cdot\vec{\xi}+a_0(s))\,ds}\right] d\xi\, f(y,s)\, dy\, ds.}$$

$\square$

Problem (S'93, #2). ***a)*** *Define the Fourier transform* [83]

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{ix\xi} f(x)\, dx.$$

State the inversion theorem. If

$$\widehat{f}(\xi) = \begin{cases} \pi, & |\xi| < a, \\ \frac{1}{2}\pi, & |\xi| = a, \\ 0, & |\xi| > a, \end{cases}$$

where a is a real constant, what $f(x)$ does the inversion theorem give?

b) *Show that*

$$\widehat{f(x-b)} = e^{i\xi b}\widehat{f(x)},$$

where b is a real constant. Hence, using part (a) and Parseval's theorem, show that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin a(x+z)}{x+z}\,\frac{\sin a(x+\xi)}{x+\xi}\, dx = \frac{\sin a(z-\xi)}{z-\xi},$$

where z and ξ are real constants.

Proof. **a)** • The **inverse Fourier transform** for $f \in L^1(\mathbb{R}^n)$:

$$f^{\vee}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi} f(x)\, dx \qquad \text{for } \ \xi \in \mathbb{R}.$$

Fourier Inversion Theorem: Assume $f \in L^2(\mathbb{R})$. Then

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}\widehat{f}(\xi)\, d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(y-x)\xi} f(y)\, dy\, d\xi = (\widehat{f})^{\vee}(x).}$$

• **Parseval's theorem (Plancherel's theorem)** (for this definition of the Fourier transform). Assume $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\widehat{f}, f^{\vee} \in L^2(\mathbb{R}^n)$ and

$$\frac{1}{2\pi}||\widehat{f}||_{L^2(\mathbb{R}^n)} = ||f^{\vee}||_{L^2(\mathbb{R}^n)} = ||f||_{L^2(\mathbb{R}^n)}, \qquad \text{or}$$

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2\, d\xi.$$

Also,

$$\int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi)\,\overline{\widehat{g}(\xi)}\, d\xi.$$

• We can write

$$\widehat{f}(\xi) = \begin{cases} \pi, & |\xi| < a, \\ 0, & |\xi| > a. \end{cases}$$

[83] Note that the Fourier transform is defined incorrectly here. There should be '-' sign in $e^{-ix\xi}$. Need to be careful, since the consequences of this definition propagate throughout the solution.

$$
\begin{aligned}
f(x) &= (\widehat{f}(\xi))^{\vee} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}\widehat{f}(\xi)\,d\xi = \frac{1}{2\pi}\int_{-\infty}^{-a} 0\,d\xi + \frac{1}{2\pi}\int_{-a}^{a} e^{-ix\xi}\pi\,d\xi + \frac{1}{2\pi}\int_{a}^{\infty} 0\,d\xi \\
&= \frac{1}{2}\int_{-a}^{a} e^{-ix\xi}\,d\xi = -\frac{1}{2ix}\Big[e^{-ix\xi}\Big]_{\xi=-a}^{\xi=a} = -\frac{1}{2ix}\Big[e^{-iax} - e^{iax}\Big] = \frac{\sin ax}{x}. \quad \checkmark
\end{aligned}
$$

b) • Let $f(\underbrace{x-b}_{y}) = g(x)$, and determinte $\widehat{g}(\xi)$:

$$
\begin{aligned}
\widehat{f(x-b)}(\xi) = \widehat{g}(\xi) &= \int_{\mathbb{R}} e^{ix\xi} g(x)\,dx = \int_{\mathbb{R}} e^{i(y+b)\xi} f(y)\,dy \\
&= \int_{\mathbb{R}} e^{iy\xi} e^{ib\xi} f(y)\,dy = e^{ib\xi}\widehat{f}(\xi). \quad \checkmark
\end{aligned}
$$

• With $f(x) = \frac{\sin ax}{x}$ (from (a)), we have

$$
\begin{aligned}
\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin a(x+z)}{x+z}\,\frac{\sin a(x+s)}{x+s}\,dx &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+z)f(x+s)\,dx \qquad (x' = x+s,\ \ dx' = dx) \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x'+z-s)f(x')\,dx' \qquad \text{(Parseval's)} \\
&= \frac{1}{\pi}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f(x'+z-s)}\,\widehat{f(x')}\,d\xi \qquad \text{part (b)} \\
&= \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \widehat{f(\xi)}\,e^{-i(z-s)\xi}\,\widehat{f(\xi)}\,d\xi \\
&= \frac{1}{2\pi^2}\int_{-a}^{a} \widehat{f(\xi)}^{\,2}\,e^{-i(z-s)\xi}\,d\xi \\
&= \frac{1}{2\pi^2}\int_{-a}^{a} \pi^2\,e^{-i(z-s)\xi}\,d\xi \\
&= \frac{1}{2}\int_{-a}^{a} e^{-i(z-s)\xi}\,d\xi \\
&= \frac{1}{-2i(z-s)}\big[e^{-i(z-s)\xi}\big]_{\xi=-a}^{\xi=a} \\
&= \frac{e^{i(z-s)a} - e^{-i(z-s)a}}{2i(z-s)} = \frac{\sin a(z-s)}{z-s}. \quad \checkmark
\end{aligned}
$$

□

Problem (F'03, #5). ❶ *State Parseval's relation for* ***Fourier transforms****.*

❷ *Find the Fourier transform* $\hat{f}(\xi)$ *of*

$$f(x) = \begin{cases} e^{i\alpha x}/2\sqrt{\pi y}, & |x| \le y \\ 0, & |x| > y, \end{cases}$$

in which y *and* α *are constants.*

❸ *Use this in Parseval's relation to show that*

$$\int_{-\infty}^{\infty} \frac{\sin^2(\alpha-\xi)y}{(\alpha-\xi)^2}\, d\xi = \pi y.$$

What does the transform $\hat{f}(\xi)$ *become in the limit* $y \to \infty$*?*

❹ *Use Parseval's relation to show that*

$$\frac{\sin(\alpha-\beta)y}{(\alpha-\beta)} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\alpha-\xi)y}{(\alpha-\xi)}\, \frac{\sin(\beta-\xi)y}{(\beta-\xi)}\, d\xi.$$

Proof. • $f \in L^2(\mathbb{R})$. Define its **Fourier transform** $\widehat{u}$ by

$$\widehat{f}(\xi) \;=\; \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx \qquad \text{for } \; \xi \in \mathbb{R}.$$

❶ **Parseval's theorem (Plancherel's theorem):**
Assume $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\widehat{f}, f^\vee \in L^2(\mathbb{R}^n)$ and

$$||\widehat{f}||_{L^2(\mathbb{R}^n)} \;=\; ||f^\vee||_{L^2(\mathbb{R}^n)} \;=\; ||f||_{L^2(\mathbb{R}^n)}, \qquad \text{or}$$

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx \;=\; \int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2\, d\xi.$$

Also,

$$\int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\, dx \;=\; \int_{-\infty}^{\infty} \widehat{f}(\xi)\,\overline{\widehat{g}(\xi)}\, d\xi.$$

❷ Find the Fourier transform of f:

$$\begin{aligned}
\widehat{f}(\xi) \;&=\; \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx \;=\; \frac{1}{\sqrt{2\pi}}\int_{-y}^{y} e^{-ix\xi}\frac{e^{i\alpha x}}{2\sqrt{\pi y}}\, dx \;=\; \frac{1}{2\pi\sqrt{2y}}\int_{-y}^{y} e^{i(\alpha-\xi)x}\, dx \\
&=\; \frac{1}{2\pi\sqrt{2y}}\,\frac{1}{i(\alpha-\xi)}\Big[e^{i(\alpha-\xi)x}\Big]_{x=-y}^{x=y} \;=\; \frac{1}{2i\pi\sqrt{2y}(\alpha-\xi)}\Big[e^{i(\alpha-\xi)y} - e^{-i(\alpha-\xi)y}\Big] \\
&=\; \frac{\sin y(\alpha-\xi)}{\pi\sqrt{2y}(\alpha-\xi)}. \qquad \checkmark
\end{aligned}$$

❸ Parseval's theorem gives:

$$\begin{aligned}
\int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2\, d\xi \;&=\; \int_{-\infty}^{\infty} |f(x)|^2\, dx, \\
\int_{-\infty}^{\infty} \frac{\sin^2 y(\alpha-\xi)}{\pi^2 2y(\alpha-\xi)^2}\, d\xi \;&=\; \int_{-y}^{y} \frac{|e^{2i\alpha x}|}{4\pi y}\, dx, \\
\int_{-\infty}^{\infty} \frac{\sin^2 y(\alpha-\xi)}{(\alpha-\xi)^2}\, d\xi \;&=\; \frac{\pi}{2}\int_{-y}^{y} dx, \\
\int_{-\infty}^{\infty} \frac{\sin^2 y(\alpha-\xi)}{(\alpha-\xi)^2}\, d\xi \;&=\; \pi y. \qquad \checkmark
\end{aligned}$$

❹ We had

$$\widehat{f}(\xi) = \frac{\sin y(\alpha-\xi)}{\pi\sqrt{2y}(\alpha-\xi)}.$$

• We make change of variables: $\alpha-\xi=\beta-\xi'$. Then, $\xi=\xi'+\alpha-\beta$. We have

$$\widehat{f}(\xi) = \widehat{f}(\xi'+\alpha-\beta) = \frac{\sin y(\beta-\xi')}{(\beta-\xi')}, \qquad \text{or}$$

$$\widehat{f}(\xi+\alpha-\beta) = \frac{\sin y(\beta-\xi)}{(\beta-\xi)}.$$

• We will also use the following result.
Let $\widehat{f}(\underbrace{\xi+a}_{\xi'}) = \widehat{g}(\xi)$, and determinte $\widehat{g}(\xi)^\vee$:

$$\begin{aligned}\widehat{f}(\xi+a)^\vee = \widehat{g}(\xi)^\vee &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\widehat{g}(\xi)\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix(\xi'-a)}\widehat{f}(\xi')\,d\xi' \\ &= e^{-ixa}f(x).\end{aligned}$$

• Using these results, we have

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(\alpha-\xi)y}{(\alpha-\xi)}\,\frac{\sin(\beta-\xi)y}{(\beta-\xi)}\,d\xi &= \frac{1}{\pi}(\pi\sqrt{2y})^2\int_{-\infty}^{\infty}\widehat{f}(\xi)\,\widehat{f}(\xi+\alpha-\beta)\,d\xi \\ &= 2\pi y\int_{-\infty}^{\infty} f(x)\,e^{-(\alpha-\beta)ix}f(x)\,dx \\ &= 2\pi y\int_{-\infty}^{\infty} f(x)^2\,e^{-(\alpha-\beta)ix}\,dx \\ &= 2\pi y\int_{-y}^{y}\frac{\left|e^{2i\alpha x}\right|}{4\pi y}\,e^{-(\alpha-\beta)ix}\,dx \\ &= \frac{1}{2}\int_{-y}^{y} e^{-(\alpha-\beta)ix}\,dx \\ &= \frac{1}{-2i(\alpha-\beta)}\left[e^{-(\alpha-\beta)ix}\right]_{x=-y}^{x=y} \\ &= \frac{1}{-2i(\alpha-\beta)}\left[e^{-(\alpha-\beta)iy}-e^{(\alpha-\beta)iy}\right] \\ &= \frac{\sin(\alpha-\beta)y}{\alpha-\beta}. \quad \checkmark\end{aligned}$$

□

Problem (S'95, #5). *For the Laplace equation*

$$\triangle f \;\equiv\; \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f \;=\; 0 \tag{30.3}$$

in the upper half plane $y \geq 0$*, consider*
- *the Dirichlet problem* $f(x, 0) = g(x)$*;*
- *the Neumann problem* $\frac{\partial}{\partial y} f(x, 0) = h(x)$*.*

Assume that f*,* g *and* h *are* 2π *periodic in* x *and that* f *is bounded at infinity.*
Find the ***Fourier transform*** N *of the* ***Dirichlet-Neumann map****. In other words, find an operator* N *taking the Fourier transform of* g *to the Fourier transform of* h*; i.e.*

$$N\widehat{g}_k = \widehat{h}_k.$$

Proof. We solve the problem by two methods.
❶ **Fourier Series.**
Since f is 2π-periodic in x, we can write

$$f(x, y) \;=\; \sum_{n=-\infty}^{\infty} a_n(y)\, e^{inx}.$$

Plugging this into (30.3), we get the ODE:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \big(-n^2 a_n(y) e^{inx} + a_n''(y) e^{inx} \big) \;&=\; 0, \\ a_n''(y) - n^2 a_n(y) \;&=\; 0. \end{aligned}$$

Initial conditions give: (g and h are 2π-periodic in x)

$$f(x, 0) \;=\; \sum_{n=-\infty}^{\infty} a_n(0) e^{inx} \;=\; g(x) \;=\; \sum_{n=-\infty}^{\infty} \widehat{g}_n e^{inx} \quad\Rightarrow\quad a_n(0) = \widehat{g}_n.$$

$$f_y(x, 0) \;=\; \sum_{n=-\infty}^{\infty} a_n'(0) e^{inx} \;=\; h(x) \;=\; \sum_{n=-\infty}^{\infty} \widehat{h}_n e^{inx} \quad\Rightarrow\quad a_n'(0) = \widehat{h}_n.$$

Thus, the problems are:

$$\begin{aligned} a_n''(y) - n^2 a_n(y) \;&=\; 0, \\ a_n(0) \;&=\; \widehat{g}_n, \qquad \text{(Dirichlet)} \\ a_n'(0) \;&=\; \widehat{h}_n. \qquad \text{(Neumann)} \end{aligned}$$

$$\begin{aligned} \Rightarrow \qquad a_n(y) \;&=\; b_n e^{ny} + c_n e^{-ny}, \quad n = 1, 2, \ldots; \qquad a_0(y) = b_0 y + c_0. \\ a_n'(y) \;&=\; n b_n e^{ny} - n c_n e^{-ny}, \quad n = 1, 2, \ldots; \qquad a_0'(y) = b_0. \end{aligned}$$

Since f is bounded at $y = \pm\infty$, we have:

$$\begin{aligned} &b_n = 0 \qquad \text{for } \; n > 0, \\ &c_n = 0 \qquad \text{for } \; n < 0, \\ &b_0 = 0, \qquad c_0 \;\text{ arbitrary.} \end{aligned}$$

- $n > 0$:

$$\begin{aligned} a_n(y) &= c_n e^{-ny}, \\ a_n(0) &= c_n = \widehat{g}_n, && \text{(Dirichlet)} \\ a_n'(0) &= -nc_n = \widehat{h}_n. && \text{(Neumann)} \\ &\Rightarrow -n\widehat{g}_n = \widehat{h}_n. \end{aligned}$$

- $n < 0$:

$$\begin{aligned} a_n(y) &= b_n e^{ny}, \\ a_n(0) &= b_n = \widehat{g}_n, && \text{(Dirichlet)} \\ a_n'(0) &= nb_n = \widehat{h}_n. && \text{(Neumann)} \\ &\Rightarrow n\widehat{g}_n = \widehat{h}_n. \end{aligned}$$

$$\boxed{-|n|\widehat{g}_n = \widehat{h}_n, \qquad n \neq 0.}$$

- $n = 0$:

$$\begin{aligned} a_0(y) &= c_0, \\ a_0(0) &= c_0 = \widehat{g}_0, && \text{(Dirichlet)} \\ a_0'(0) &= 0 = \widehat{h}_0. && \text{(Neumann)} \end{aligned}$$

Note that solution $f(x, y)$ may be written as

$$\begin{aligned} f(x,y) &= \sum_{n=-\infty}^{\infty} a_n(y)\, e^{inx} = a_0(y) + \sum_{n=-\infty}^{-1} a_n(y)\, e^{inx} + \sum_{n=1}^{\infty} a_n(y)\, e^{inx} \\ &= c_0 + \sum_{n=-\infty}^{-1} b_n e^{ny}\, e^{inx} + \sum_{n=1}^{\infty} c_n e^{-ny}\, e^{inx} \\ &= \begin{cases} \widehat{g}_0 + \sum_{n=-\infty}^{-1} \widehat{g}_n e^{ny}\, e^{inx} + \sum_{n=1}^{\infty} \widehat{g}_n e^{-ny}\, e^{inx}, & \text{(Dirichlet)} \\ c_0 + \sum_{n=-\infty}^{-1} \frac{\widehat{h}_n}{n} e^{ny}\, e^{inx} + \sum_{n=1}^{\infty} -\frac{\widehat{h}_n}{n} e^{-ny}\, e^{inx}. & \text{(Neumann)} \end{cases} \end{aligned}$$

❷ **Fourier Transform.** The Fourier transform of $f(x, y)$ in x is:

$$\begin{aligned} \widehat{f}(\xi, y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi}\, f(x,y)\, dx, \\ f(x,y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi}\, \widehat{f}(\xi,y)\, d\xi. \end{aligned}$$

$$(i\xi)^2 \widehat{f}(\xi,y) + \widehat{f}_y y(\xi,y) = 0,$$

$\widehat{f}_{yy} - \xi^2 \widehat{f} = 0.$ The solution to this ODE is:

$$\widehat{f}(\xi, y) = c_1 e^{\xi y} + c_2 e^{-\xi y}.$$

For $\xi > 0$, $c_1 = 0$; for $\xi < 0$, $c_2 = 0$.

- $\xi > 0$: $\widehat{f}(\xi, y) = c_2 e^{-\xi y}$, $\widehat{f}_y(\xi, y) = -\xi c_2 e^{-\xi y}$,

$$\begin{aligned} c_2 = \widehat{f}(\xi, 0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi}\, f(x,0)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi}\, g(x)\, dx = \widehat{g}(\xi), && \text{(Dirichlet)} \\ -\xi c_2 = \widehat{f}_y(\xi, 0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi}\, f_y(x,0)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi}\, h(x)\, dx = \widehat{h}(\xi). && \text{(Neumann)} \\ &\Rightarrow -\xi \widehat{g}(\xi) = \widehat{h}(\xi). \end{aligned}$$

$$\bullet\ \xi < 0: \qquad \widehat{f}(\xi, y) = c_1 e^{\xi y}, \qquad \widehat{f}_y(\xi, y) = \xi c_1 e^{\xi y},$$

$$c_1 = \widehat{f}(\xi, 0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi} f(x,0)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi} g(x)\,dx = \widehat{g}(\xi), \qquad \text{(Dirichlet)}$$

$$\xi c_1 = \widehat{f}_y(\xi, 0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi} f_y(x,0)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi} h(x)\,dx = \widehat{h}(\xi). \qquad \text{(Neumann)}$$

$$\Rightarrow \quad \xi \widehat{g}(\xi) = \widehat{h}(\xi).$$

$$\boxed{-|\xi| \widehat{g}(\xi) = \widehat{h}(\xi).}$$

$\square$

Problem (F'97, #3). *Consider the Dirichlet problem in the half-space* $x_n > 0$, $n \geq 2$*:*

$$\triangle u + a\frac{\partial u}{\partial x_n} + k^2 u = 0, \qquad x_n > 0$$
$$u(x', 0) = f(x'), \qquad x' = (x_1, \ldots, x_{n-1}).$$

Here a *and* k *are constants.*
Use the ***Fourier transform*** *to show that for any* $f(x') \in L^2(\mathbb{R}^{n-1})$ *there exists a solution* $u(x', x_n)$ *of the Dirichlet problem such that*

$$\int_{\mathbb{R}^n} |u(x', x_n)|^2 \, dx' \leq C$$

for all $0 < x_n < +\infty$.

Proof. [84] Denote $\xi = (\xi', \xi_n)$. Transform in the first $n-1$ variables:

$$-|\xi'|^2 \, \widehat{u}(\xi', x_n) \; + \; \frac{\partial^2 \widehat{u}}{\partial x_n^2}(\xi', x_n) \; + \; a\frac{\partial \widehat{u}}{\partial x_n}(\xi', x_n) \; + \; k^2 \widehat{u}(\xi', x_n) \; = \; 0.$$

Thus, the ODE and initial conditions of the transformed problem become:

$$\begin{cases} \widehat{u}_{x_n x_n} + a\widehat{u}_{x_n} + (k^2 - |\xi'|^2)\widehat{u} \; = \; 0, \\ \widehat{u}(\xi', 0) \; = \; \widehat{f}(\xi'). \end{cases}$$

With the anzats $\widehat{u} = ce^{sx_n}$, we obtain $s^2 + as + (k^2 - |\xi'|^2) = 0$, and

$$s_{1,2} \; = \; \frac{-a \pm \sqrt{a^2 - 4(k^2 - |\xi'|^2)}}{2}.$$

Choosing only the negative root, we obtain the solution: [85]

$$\widehat{u}(\xi', x_n) \; = \; c(\xi') \, e^{\frac{-a - \sqrt{a^2 - 4(k^2 - |\xi'|^2)}}{2} x_n}. \qquad \widehat{u}(\xi', 0) \; = \; c \; = \; \widehat{f}(\xi'). \qquad \text{Thus,}$$
$$\widehat{u}(\xi', x_n) \; = \; \widehat{f}(\xi') \, e^{\frac{-a - \sqrt{a^2 - 4(k^2 - |\xi'|^2)}}{2} x_n}.$$

Parseval's theorem gives:

$$\begin{aligned} ||u||^2_{L^2(\mathbb{R}^{n-1})} \; &= \; ||\widehat{u}||^2_{L^2(\mathbb{R}^{n-1})} \; = \; \int_{\mathbb{R}^{n-1}} |\widehat{u}(\xi', x_n)|^2 \, d\xi' \\ &= \; \int_{\mathbb{R}^{n-1}} \Big| \widehat{f}(\xi') \, e^{\frac{-a - \sqrt{a^2 - 4(k^2 - |\xi'|^2)}}{2} x_n} \Big|^2 \, d\xi' \; \leq \; \int_{\mathbb{R}^{n-1}} \big| \widehat{f}(\xi') \big|^2 \, d\xi' \\ &= \; ||\widehat{f}||^2_{L^2(\mathbb{R}^{n-1})} \; = \; ||f||^2_{L^2(\mathbb{R}^{n-1})} \; \leq \; C, \end{aligned}$$

since $f(x') \in L^2(\mathbb{R}^{n-1})$. Thus, $u(x', x_n) \in L^2(\mathbb{R}^{n-1})$. □

---

[84] Note that the last element of $x = (x', x_n) = (x_1, \ldots, x_{n-1}, x_n)$, i.e. x_n, plays a role of time t. As such, the PDE may be written as

$$\triangle u + u_{tt} + au_t + k^2 u \; = \; 0.$$

[85] Note that $a > 0$ should have been provided by the statement of the problem.

Problem (F'89, #7). *Find the following* ***fundamental solutions***

$$\textit{a)}\qquad \begin{aligned} \frac{\partial G(x,y,t)}{\partial t} &= a(t)\frac{\partial^2 G(x,y,t)}{\partial x^2} + b(t)\frac{\partial G(x,y,t)}{\partial x} + c(t)G(x,y,t) \qquad \textit{for} \quad t>0 \\ G(x,y,0) &= \delta(x-y), \end{aligned}$$

where $a(t)$, $b(t)$, $c(t)$ *are continuous functions on* $[0,+\infty]$, $a(t)>0$ *for* $t>0$.

$$\textit{b)}\qquad \frac{\partial G}{\partial t}(x_1,\ldots,x_n,y_1,\ldots,y_n,t) = \sum_{k=1}^n a_k(t)\frac{\partial G}{\partial x_k} \qquad \textit{for} \quad t>0,$$

$$G(x_1,\ldots,x_n,y_1,\ldots,y_n,0) = \delta(x_1-y_1)\delta(x_2-y_2)\ldots\delta(x_n-y_n).$$

Proof. **a)** We use the **Fourier transform** to solve this problem.
Transform the equation in the first variable only. That is,

$$\widehat{G}(\xi,y,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\,G(x,y,t)\,dx.$$

The equation is transformed to an ODE, that can be solved:

$$\begin{aligned} \widehat{G}_t(\xi,y,t) &= -a(t)\,\xi^2\,\widehat{G}(\xi,y,t) + i\,b(t)\,\xi\,\widehat{G}(\xi,y,t) + c(t)\,\widehat{G}(\xi,y,t), \\ \widehat{G}_t(\xi,y,t) &= \big[-a(t)\,\xi^2 + i\,b(t)\,\xi + c(t)\big]\,\widehat{G}(\xi,y,t), \\ \widehat{G}(\xi,y,t) &= c\,e^{\int_0^t[-a(s)\xi^2+i\,b(s)\xi+c(s)]\,ds}. \end{aligned}$$

We can also transform the initial condition:

$$\widehat{G}(\xi,y,0) = \widehat{\delta(x-y)}(\xi) = e^{-iy\xi}\,\widehat{\delta}(\xi) = \frac{1}{\sqrt{2\pi}}\,e^{-iy\xi}.$$

Thus, the solution of the transformed problem is:

$$\widehat{G}(\xi,y,t) = \frac{1}{\sqrt{2\pi}}\,e^{-iy\xi}\,e^{\int_0^t[-a(s)\xi^2+i\,b(s)\xi+c(s)]\,ds}.$$

The inverse Fourier transform gives the solution to the original problem:

$$\begin{aligned} G(x,y,t) &= \big(\widehat{G}(\xi,y,t)\big)^{\vee} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\,\widehat{G}(\xi,y,t)\,d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\left[\frac{1}{\sqrt{2\pi}}\,e^{-iy\xi}\,e^{\int_0^t[-a(s)\xi^2+i\,b(s)\xi+c(s)]\,ds}\right]d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x-y)\xi}\,e^{\int_0^t[-a(s)\xi^2+i\,b(s)\xi+c(s)]\,ds}\,d\xi. \qquad \checkmark \end{aligned}$$

b) Denote $\vec{x}=(x_1,\ldots,x_n)$, $\vec{y}=(y_1,\ldots,y_n)$. Transform in $\vec{x}$:

$$\widehat{G}(\vec{\xi},\vec{y},t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-i\vec{x}\cdot\vec{\xi}}\,G(\vec{x},\vec{y},t)\,d\vec{x}.$$

The equation is transformed to an ODE, that can be solved:

$$\begin{aligned} \widehat{G}_t(\vec{\xi},\vec{y},t) &= \sum_{k=1}^n a_k(t)\,i\xi_k\,\widehat{G}(\vec{\xi},\vec{y},t), \\ \widehat{G}(\vec{\xi},\vec{y},t) &= c\,e^{i\int_0^t[\sum_{k=1}^n a_k(s)\,\xi_k]\,ds}. \end{aligned}$$

We can also transform the initial condition:

$$\widehat{G}(\vec{\xi},\vec{y},0) = \big[\delta(x_1-y_1)\delta(x_2-y_2)\ldots\delta(x_n-y_n)\big]^{\frown}(\xi) = e^{-i\vec{y}\cdot\vec{\xi}}\,\widehat{\delta}(\vec{\xi}) = \frac{1}{(2\pi)^{\frac{n}{2}}}\,e^{-i\vec{y}\cdot\vec{\xi}}.$$

Thus, the solution of the transformed problem is:

$$\widehat{G}(\vec{\xi},\vec{y},t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\,e^{-i\vec{y}\cdot\vec{\xi}}\,e^{i\int_0^t[\sum_{k=1}^n a_k(s)\,\xi_k]\,ds}.$$

The inverse Fourier transform gives the solution to the original problem:

$$\begin{aligned}
G(\vec{x},\vec{y},t) &= \big(\widehat{G}(\vec{\xi},\vec{y},t)\big)^{\vee} = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{i\vec{x}\cdot\vec{\xi}}\,\widehat{G}(\vec{\xi},\vec{y},t)\,d\vec{\xi} \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{i\vec{x}\cdot\vec{\xi}}\left[\frac{1}{(2\pi)^{\frac{n}{2}}}\,e^{-i\vec{y}\cdot\vec{\xi}}\,e^{i\int_0^t[\sum_{k=1}^n a_k(s)\,\xi_k]\,ds}\right]d\vec{\xi} \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(\vec{x}-\vec{y})\cdot\vec{\xi}}\,e^{i\int_0^t[\sum_{k=1}^n a_k(s)\,\xi_k]\,ds}\,d\vec{\xi}. \qquad \checkmark
\end{aligned}$$

$\square$

Problem (W'02, #7). *Consider the equation*

$$\Big(\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\Big)u = f \quad \text{in } \mathbb{R}^n, \tag{30.4}$$

where f is an integrable function (i.e. $f \in L^1(\mathbb{R}^n)$), satisfying $f(x) = 0$ for $|x| \geq R$. Solve (30.4) by ***Fourier transform****, and prove the following results.*
a) *There is a solution of (30.4) belonging to $L^2(\mathbb{R}^n)$ if $n > 4$.*
b) *If $\int_{\mathbb{R}^n} f(x)\,dx = 0$, there is a solution of (30.4) belonging to $L^2(\mathbb{R}^n)$ if $n > 2$.*

Proof.

$$\begin{aligned}
\triangle u &= f,\\
-|\xi|^2\widehat{u}(\xi) &= \widehat{f}(\xi),\\
\widehat{u}(\xi) &= -\frac{1}{|\xi|^2}\widehat{f}(\xi), \qquad \xi \in \mathbb{R}^n,\\
u(x) &= -\Big(\frac{\widehat{f}(\xi)}{|\xi|^2}\Big)^{\vee}.
\end{aligned}$$

a) Then

$$||\widehat{u}||_{L^2(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n}\frac{|\widehat{f}(\xi)|^2}{|\xi|^4}\,d\xi\Big)^{\frac{1}{2}} \leq \Big(\underbrace{\int_{|\xi|<1}\frac{|\widehat{f}(\xi)|^2}{|\xi|^4}\,d\xi}_{A} + \underbrace{\int_{|\xi|\geq 1}\frac{|\widehat{f}(\xi)|^2}{|\xi|^4}\,d\xi}_{B}\Big)^{\frac{1}{2}}.$$

Notice, $||f||_2 = ||\widehat{f}||_2 \geq B$, so $B < \infty$.
Use **polar coordinates** on A.

$$A = \int_{|\xi|<1}\frac{|\widehat{f}(\xi)|^2}{|\xi|^4}\,d\xi = \int_0^1\int_{S_{n-1}}\frac{|\widehat{f}|^2}{r^4}\,r^{n-1}\,dS_{n-1}\,dr = \int_0^1\int_{S_{n-1}}|\widehat{f}|^2\,r^{n-5}\,dS_{n-1}\,dr.$$

If $n > 4$,

$$A \leq \int_{S_{n-1}}|\widehat{f}|^2\,dS_{n-1} = ||\widehat{f}||_2^2 < \infty.$$

$$||u||_{L^2(\mathbb{R}^n)} = ||\widehat{u}||_{L^2(\mathbb{R}^n)} = (A+B)^{\frac{1}{2}} < \infty.$$

b) We have

$$\begin{aligned}
u(x,t) &= -\Big(\frac{\widehat{f}(\xi)}{|\xi|^2}\Big)^{\vee} = -\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty} e^{ix\cdot\xi}\,\frac{\widehat{f}(\xi)}{|\xi|^2}\,d\xi \\
&= -\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty}\frac{e^{ix\cdot\xi}}{|\xi|^2}\Big(\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty} e^{-iy\cdot\xi}\,f(y)\,dy\Big)\,d\xi \\
&= -\frac{1}{(2\pi)^{n}}\int_{-\infty}^{\infty} f(y)\,\Big(\int_{-\infty}^{\infty}\frac{e^{i(x-y)\cdot\xi}}{|\xi|^2}\,d\xi\Big)\,dy \\
&= -\frac{1}{(2\pi)^{n}}\int_{-\infty}^{\infty} f(y)\,\Big(\int_0^1\int_{S_{n-1}}\frac{e^{i(x-y)r}}{r^2}\,r^{n-1}\,dS_{n-1}\,dr\Big)\,dy \\
&= -\frac{1}{(2\pi)^{n}}\int_{-\infty}^{\infty} f(y)\,\underbrace{\Big(\int_0^1\int_{S_{n-1}} e^{i(x-y)r}\,r^{n-3}\,dS_{n-1}\,dr\Big)}_{\leq M<\infty,\quad if\ \ n>2.}\,dy. \\
|u(x,t)| &= \frac{1}{(2\pi)^{n}}\Big|\int_{-\infty}^{\infty} M\,f(y)\,dy\Big| \;<\; \infty.
\end{aligned}$$

□

Problem (F'02, #7). *For the right choice of the constant c, the function $F(x,y) = c(x+iy)^{-1}$ is a* ***fundamental solution*** *for the equation*

$$\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = f \qquad in \ \ \mathbb{R}^2.$$

Find the right choice of c, and use your answer to compute the ***Fourier transform*** *(in distribution sense) of $(x+iy)^{-1}$.*

Proof. [86]

$$\triangle \ = \ \Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big).$$

$F_1(x,y) = \frac{1}{2\pi}\log|z|$ is the fundamental solution of the Laplacian. $z = x+iy$.

$$\begin{aligned}\triangle F_1(x,y) &= \delta,\\ \Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)\ F(x,y) &= \delta.\end{aligned}$$

$h_x + ih_y = e^{-i(x\xi_1+y\xi_2)}$.
Suppose $h = h(x\xi_1+y\xi_2)$ or $h = ce^{-i(x\xi_1+y\xi_2)}$.

$$\begin{aligned}&\Rightarrow \ c\big(-i\xi_1\, e^{-i(x\xi_1+y\xi_2)} - i^2\xi_2\, e^{-i(x\xi_1+y\xi_2)}\big) \ = \ -ic(\xi_1 - i\xi_2)\, e^{-i(x\xi_1+y\xi_2)} \ \equiv \ e^{-i(x\xi_1+y\xi_2)},\\ &\Rightarrow \ -ic(\xi_1-i\xi_2) \ = \ 1,\\ &\Rightarrow \ c \ = \ -\frac{1}{i(\xi_1-i\xi_2)},\\ &\Rightarrow \ h(x,y) \ = \ -\frac{1}{i(\xi_1-i\xi_2)}\, e^{-i(x\xi_1+y\xi_2)}.\end{aligned}$$

Integrate by parts:

$$\begin{aligned}\Big(\widehat{\frac{1}{x+iy}}\Big)(\xi) &= \int_{\mathbb{R}^2} e^{-i(x\xi_1+y\xi_2)}\,\frac{1}{i(\xi_1-i\xi_2)}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)\frac{1}{(x+iy)-0}\,dxdy\\ &= \frac{1}{i(\xi_1-i\xi_2)} \ = \ \frac{1}{i(\xi_2+i\xi_1)}.\end{aligned}$$

□

[86] Alan solved in this problem in class.

31 Laplace Transform

If $u \in L^1(\mathbb{R}_+)$, we define its **Laplace transform** to be

$$\boxed{\mathcal{L}[u(t)] \;=\; u^{\#}(s) \;=\; \int_0^\infty e^{-st}\, u(t)\, dt \qquad (s>0).}$$

In practice, for a PDE involving time, it may be useful to perform a Laplace transform in t, holding the space variables x fixed.
The **inversion formula** for the Laplace transform is:

$$\boxed{u(t) \;=\; \mathcal{L}^{-1}[u^{\#}(s)] \;=\; \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\, u^{\#}(s)\, ds.}$$

Example: $f(t) = 1$.

$$\mathcal{L}[1] \;=\; \int_0^\infty e^{-st}\cdot 1\, dt \;=\; \Big[-\frac{1}{s}e^{-st}\Big]_{t=0}^{t=\infty} \;=\; \frac{1}{s} \qquad \text{for } \; s>0.$$

Example: $f(t) = e^{at}$.

$$\mathcal{L}[e^{at}] \;=\; \int_0^\infty e^{-st}\, e^{at}\, dt \;=\; \int_0^\infty e^{(a-s)t}\, dt \;=\; \frac{1}{a-s}\Big[e^{(a-s)t}\Big]_{t=0}^{t=\infty} \;=\; \frac{1}{s-a} \qquad \text{for } \; s>a.$$

Convolution: We want to find an inverse Laplace transform of $\frac{1}{s}\cdot\frac{1}{s^2+1}$.

$$\mathcal{L}^{-1}\Big[\underbrace{\frac{1}{s}}_{L[f]}\cdot\underbrace{\frac{1}{s^2+1}}_{L[g]}\Big] \;=\; f*g \;=\; \int_0^t 1\cdot \sin t'\, dt' \;=\; 1-\cos t.$$

Partial Derivatives: $u = u(x,t)$

$$\begin{aligned}
\mathcal{L}[u_t] &= \int_0^\infty e^{-st}\, u_t\, dt \;=\; \Big[e^{-st}u(x,t)\Big]_{t=0}^{t=\infty} + s\int_0^\infty e^{-st}\, u\, dt \;=\; s\mathcal{L}[u] - u(x,0),\\
\mathcal{L}[u_{tt}] &= \int_0^\infty e^{-st}\, u_{tt}\, dt \;=\; \Big[e^{-st}u_t\Big]_{t=0}^{t=\infty} + s\int_0^\infty e^{-st}\, u_t\, dt \;=\; -u_t(x,0) + s\mathcal{L}[u_t]\\
&= s^2\mathcal{L}[u] - su(x,0) - u_t(x,0),\\
\mathcal{L}[u_x] &= \int_0^\infty e^{-st}\, u_x\, dt \;=\; \frac{\partial}{\partial x}\mathcal{L}[u],\\
\mathcal{L}[u_{xx}] &= \int_0^\infty e^{-st}\, u_{xx}\, dt \;=\; \frac{\partial^2}{\partial x^2}\mathcal{L}[u].
\end{aligned}$$

Heat Equation: Consider

$$\begin{cases} u_t - \triangle u = 0 & \text{in } \; U\times(0,\infty)\\ u = f & \text{on } \; U\times\{t=0\},\end{cases}$$

and perform a Laplace transform with respect to time:

$$\begin{aligned}
\mathcal{L}[u_t] &= \int_0^\infty e^{-st}\, u_t\, dt \;=\; s\mathcal{L}[u] - u(x,0) \;=\; s\mathcal{L}[u] - f(x),\\
\mathcal{L}[\triangle u] &= \int_0^\infty e^{-st}\, \triangle u\, dt \;=\; \triangle\mathcal{L}[u].
\end{aligned}$$

Thus, the transformed problem is: $s\mathcal{L}[u] - f(x) \;=\; \triangle\mathcal{L}[u]$. Writing $v(x) = L[u]$, we have

$$-\triangle v + sv \;=\; f \qquad \text{in } \; U.$$

Thus, the solution of this equation with RHS f is the Laplace transform of the solution of the heat equation with initial data f.

Table of Laplace Transforms: $L[f] = f^{\#}(s)$

$$
\begin{aligned}
\mathcal{L}[\sin at] &= \frac{a}{s^2+a^2}, && s>0\\
\mathcal{L}[\cos at] &= \frac{s}{s^2+a^2}, && s>0\\
\mathcal{L}[\sinh at] &= \frac{a}{s^2-a^2}, && s>|a|\\
\mathcal{L}[\cosh at] &= \frac{s}{s^2-a^2}, && s>|a|\\
\mathcal{L}[e^{at}\sin bt] &= \frac{b}{(s-a)^2+b^2}, && s>a\\
\mathcal{L}[e^{at}\cos bt] &= \frac{s-a}{(s-a)^2+b^2}, && s>a\\
\mathcal{L}[t^n] &= \frac{n!}{s^{n+1}}, && s>0\\
\mathcal{L}[t^n e^{at}] &= \frac{n!}{(s-a)^{n+1}}, && s>a\\
\mathcal{L}[H(t-a)] &= \frac{e^{-as}}{s}, && s>0\\
\mathcal{L}[H(t-a)\,f(t-a)] &= e^{-as}\mathcal{L}[f],\\
\mathcal{L}[af(t)+bg(t)] &= a\mathcal{L}[f]+b\mathcal{L}[g],\\
\mathcal{L}[f(t)*g(t)] &= \mathcal{L}[f]\,\mathcal{L}[g],\\
\mathcal{L}\Big[\int_0^t g(t'-t)\,f(t')\,dt'\Big] &= \mathcal{L}[f]\,\mathcal{L}[g],\\
\mathcal{L}\Big[\frac{df}{dt}\Big] &= s\mathcal{L}[f]-f(0),\\
\mathcal{L}\Big[\frac{d^2f}{dt^2}\Big] &= s^2\mathcal{L}[f]-sf(0)-f'(0), && \Big(f'=\frac{df}{dt}\Big)\\
\mathcal{L}\Big[\frac{d^nf}{dt^n}\Big] &= s^n\mathcal{L}[f]-s^{n-1}f(0)-\ldots-f^{n-1}(0),\\
\mathcal{L}[f(at)] &= \frac{1}{a}f^{\#}\Big(\frac{s}{a}\Big),\\
\mathcal{L}[e^{bt}f(t)] &= f^{\#}(s-b),\\
\mathcal{L}[tf(t)] &= -\frac{d}{ds}\mathcal{L}[f],\\
\mathcal{L}\Big[\frac{f(t)}{t}\Big] &= \int_s^\infty f^{\#}(s')\,ds',\\
\mathcal{L}\Big[\int_0^t f(t')\,dt'\Big] &= \frac{1}{s}\mathcal{L}[f],\\
\mathcal{L}[J_0(at)] &= (s^2+a^2)^{-\frac{1}{2}},\\
\mathcal{L}[\delta(t-a)] &= e^{-sa}.
\end{aligned}
$$

Example: $f(t)=\sin t$. After integrating by parts twice, we obtain:

$$
\begin{aligned}
\mathcal{L}[\sin t] &= \int_0^\infty e^{-st}\,\sin t\,dt = 1-s^2\int_0^\infty e^{-st}\,\sin t\,dt,\\
&\Rightarrow \int_0^\infty e^{-st}\,\sin t\,dt = \frac{1}{1+s^2}.
\end{aligned}
$$

Example: $f(t) = t^n$.

$$\begin{aligned}
\mathcal{L}[t^n] &= \int_0^\infty e^{-st}\, t^n\, dt \;=\; -\Big[\frac{t^n e^{-st}}{s}\Big]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}\, t^{n-1}\, dt \;=\; \frac{n}{s}\mathcal{L}[t^{n-1}] \\
&= \frac{n}{s}\Big(\frac{n-1}{s}\Big)\mathcal{L}[t^{n-2}] \;=\; \ldots \;=\; \frac{n!}{s^n}\mathcal{L}[1] \;=\; \frac{n!}{s^{n+1}}.
\end{aligned}$$

Problem (F'00, #6). *Consider the initial-boundary value problem*

$$\begin{aligned} u_t - u_{xx} + au &= 0, & t > 0,\ x > 0 \\ u(x,0) &= 0, & x > 0 \\ u(0,t) &= g(t), & t > 0, \end{aligned}$$

where $g(t)$ is continuous function with a compact support, and a is constant. Find the explicit solution of this problem.

Proof. We solve this problem using the **Laplace transform**.

$$\mathcal{L}[u(x,t)] = u^{\#}(x,s) = \int_0^\infty e^{-st}\, u(x,t)\, dt \qquad (s > 0).$$

$$\begin{aligned} \mathcal{L}[u_t] &= \int_0^\infty e^{-st}\, u_t\, dt = \Big[e^{-st} u(x,t)\Big]_{t=0}^{t=\infty} + s\int_0^\infty e^{-st}\, u\, dt \\ &= su^{\#}(x,s) - u(x,0) = su^{\#}(x,s), \qquad \text{(since } u(x,0) = 0) \\ \mathcal{L}[u_{xx}] &= \int_0^\infty e^{-st}\, u_{xx}\, dt = \frac{\partial^2}{\partial x^2} u^{\#}(x,s), \\ \mathcal{L}[u(0,t)] &= u^{\#}(0,s) = \int_0^\infty e^{-st}\, g(t)\, dt = g^{\#}(s). \end{aligned}$$

Plugging these into the equation, we obtain the ODE in $u^{\#}$:

$$su^{\#}(x,s) - \frac{\partial^2}{\partial x^2} u^{\#}(x,s) + au^{\#}(x,s) = 0.$$

$$\begin{cases} (u^{\#})_{xx} - (s+a)u^{\#} = 0, \\ u^{\#}(0,s) = g^{\#}(s). \end{cases}$$

This initial value problem has a solution:

$$u^{\#}(x,s) = c_1 e^{\sqrt{s+a}\,x} + c_2 e^{-\sqrt{s+a}\,x}.$$

Since we want u to be bounded as $x \to \infty$, we have $c_1 = 0$, so

$$u^{\#}(x,s) = c_2 e^{-\sqrt{s+a}\,x}. \qquad u^{\#}(0,s) = c_2 = g^{\#}(s), \quad \text{thus,}$$

$$\boxed{u^{\#}(x,s) = g^{\#}(s) e^{-\sqrt{s+a}\,x}.}$$

To obtain $u(x,t)$, we take the **inverse Laplace transform** of $u^{\#}(x,s)$:

$$\begin{aligned} u(x,t) &= L^{-1}[u^{\#}(x,s)] = L^{-1}\big[\underbrace{g^{\#}(s)}_{L[g]}\underbrace{e^{-\sqrt{s+a}\,x}}_{L[f]}\big] = g * f \\ &= g * L^{-1}\big[e^{-\sqrt{s+a}\,x}\big] = g * \Big[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\, e^{-\sqrt{s+a}\,x}\, ds\Big], \end{aligned}$$

$$\boxed{u(x,t) = \int_0^t g(t-t')\Big[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st'}\, e^{-\sqrt{s+a}\,x}\, ds\Big]\, dt'.}$$

□

Problem (F'04, #8). *The function $y(x,t)$ satisfies the partial differential equation*

$$x\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x \partial t} + 2y = 0,$$

and the boundary conditions

$$y(x,0) = 1, \qquad y(0,t) = e^{-at},$$

*where $a \geq 0$. Find the **Laplace transform**, $\overline{y}(x,s)$, of the solution, and hence derive an expression for $y(x,t)$ in the domain $x \geq 0$, $t \geq 0$.*

Proof. We change the notation: $y \to u$. We have

$$\begin{cases} xu_x + u_{xt} + 2u = 0, \\ u(x,0) = 1, \quad u(0,t) = e^{-at}. \end{cases}$$

The **Laplace transform** is defined as:

$$\mathcal{L}[u(x,t)] = u^\#(x,s) = \int_0^\infty e^{-st}\, u(x,t)\, dt \qquad (s > 0).$$

$$\begin{aligned}
\mathcal{L}[xu_x] &= \int_0^\infty e^{-st}\, xu_x\, dt = x\int_0^\infty e^{-st}\, u_x\, dt = x(u^\#)_x, \\
\mathcal{L}[u_{xt}] &= \int_0^\infty e^{-st}\, u_{xt}\, dt = \Big[e^{-st}u_x(x,t)\Big]_{t=0}^{t=\infty} + s\int_0^\infty e^{-st}\, u_x\, dt \\
&= s(u^\#)_x - u_x(x,0) = s(u^\#)_x, \qquad (\text{since } \; u(x,0) = 0) \\
\mathcal{L}[u(0,t)] &= u^\#(0,s) = \int_0^\infty e^{-st}\, e^{-at}\, dt = \int_0^\infty e^{-(s+a)t}\, dt = \Big[-\frac{1}{s+a}e^{-(s+a)t}\Big]_{t=0}^{t=\infty} \\
&= \frac{1}{s+a}.
\end{aligned}$$

Plugging these into the equation, we obtain the ODE in $u^\#$:

$$\begin{cases} (x+s)(u^\#)_x + 2u^\# = 0, \\ u^\#(0,s) = \frac{1}{s+a}, \end{cases}$$

which can be solved:

$$\frac{(u^\#)_x}{u^\#} = -\frac{2}{x+s} \quad\Rightarrow\quad \log u^\# = -2\log(x+s) + c_1 \quad\Rightarrow\quad u^\# = c_2 e^{\log(x+s)^{-2}} = \frac{c_2}{(x+s)^2}.$$

From the initial conditions:

$$u^\#(0,s) = \frac{c_2}{s^2} = \frac{1}{s+a} \quad\Rightarrow\quad c_2 = \frac{s^2}{s+a}.$$

$$\boxed{u^\#(x,s) = \frac{s^2}{(s+a)(x+s)^2}.}$$

To obtain $u(x,t)$, we take the **inverse Laplace transform** of $u^\#(x,s)$:

$$u(x,t) = L^{-1}[u^\#(x,s)] = L^{-1}\Big[\frac{s^2}{(s+a)(x+s)^2}\Big] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\Big[\frac{s^2}{(s+a)(x+s)^2}\Big]\, ds.$$

$$\boxed{u(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\Big[\frac{s^2}{(s+a)(x+s)^2}\Big]\, ds.}$$

∎

Problem (F'90, #1). *Using the **Laplace transform**, or any other convenient method, solve the Volterra integral equation*

$$u(x) \;=\; \sin x + \int_0^x \sin(x-y)u(y)\,dy.$$

Proof. Rewrite the equation:

$$\begin{aligned}
u(t) &= \sin t + \int_0^t \sin(t-t')u(t')\,dt', \\
u(t) &= \sin t + (\sin t) * u. \qquad \circledast
\end{aligned}$$

Taking the Laplace transform of each of the elements in $\circledast$:

$$\begin{aligned}
\mathcal{L}[u(t)] &= u^{\#}(s) \;=\; \int_0^\infty e^{-st}\,u(t)\,dt, \\
\mathcal{L}[\sin t] &= \frac{1}{1+s^2}, \\
\mathcal{L}[(\sin t) * u] &= \mathcal{L}[\sin t] * \mathcal{L}[u] \;=\; \frac{u^{\#}}{1+s^2}.
\end{aligned}$$

Plugging these into the equation:

$$u^{\#} \;=\; \frac{1}{1+s^2} + \frac{u^{\#}}{1+s^2} \;=\; \frac{u^{\#}+1}{1+s^2}.$$

$$\boxed{u^{\#}(s) = \frac{1}{s^2}.}$$

To obtain $u(t)$, we take the **inverse Laplace transform** of $u^{\#}(s)$:

$$u(t) \;=\; L^{-1}[u^{\#}(s)] \;=\; L^{-1}\Big[\frac{1}{s^2}\Big] \;=\; t.$$

$$\boxed{u(t) \;=\; t.}$$

$\square$

Problem (F'91, #5). *In what follows, the **Laplace transform** of $x(t)$ is denoted either by $\overline{x}(s)$ or by $\mathcal{L}x(t)$.* ❶ *Show that, for integral $n \geq 0$,*

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}.$$

❷ *Hence show that*

$$\mathcal{L}J_0(2\sqrt{ut}) = \frac{1}{s}e^{-u/s},$$

where

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n(\frac{1}{2}z)^{2n}}{n!n!}$$

is a Bessel function. ❸ *Hence show that*

$$\mathcal{L}\left[\int_0^\infty J_0(2\sqrt{ut})x(u)\,du\right] = \frac{1}{s}\,\overline{x}\left(\frac{1}{s}\right). \tag{31.1}$$

❹ *Assuming that*

$$\mathcal{L}J_0(at) = \frac{1}{\sqrt{a^2+s^2}},$$

prove with the help of (31.1) that if $t \geq 0$

$$\int_0^\infty J_0(au)J_0(2\sqrt{ut})\,du = \frac{1}{a}J_0\left(\frac{t}{a}\right).$$

Hint: For the last part, use the uniqueness of the Laplace transform.

Proof.

❶
$$\mathcal{L}[t^n] = \int_0^\infty \underbrace{e^{-st}}_{g'}\underbrace{t^n}_{f}\,dt = \underbrace{-\left[\frac{t^n e^{-st}}{s}\right]_0^\infty}_{=0} + \frac{n}{s}\int_0^\infty e^{-st}\,t^{n-1}\,dt = \frac{n}{s}\mathcal{L}[t^{n-1}]$$
$$= \frac{n}{s}\left(\frac{n-1}{s}\right)\mathcal{L}[t^{n-2}] = \ldots = \frac{n!}{s^n}\mathcal{L}[1] = \frac{n!}{s^{n+1}}. \quad \checkmark$$

❷
$$\mathcal{L}J_0(2\sqrt{ut}) = \mathcal{L}\left[\sum_{n=0}^{\infty}\frac{(-1)^n u^n t^n}{n!n!}\right] = \sum_{n=0}^{\infty}\frac{(-1)^n u^n}{n!n!}\,\mathcal{L}[t^n] = \sum_{n=0}^{\infty}\frac{(-1)^n u^n}{n!s^{n+1}}$$
$$= \frac{1}{s}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{u}{s}\right)^n = \frac{1}{s}e^{-\frac{u}{s}}. \quad \checkmark$$

❸
$$\mathcal{L}\left[\int_0^\infty J_0(2\sqrt{ut})\,x(u)\,du\right] = \int_0^\infty \mathcal{L}[J_0(2\sqrt{ut})]\,x(u)\,du = \frac{1}{s}\int_0^\infty e^{-\frac{u}{s}}\,x(u)\,du$$
$$= \frac{1}{s}x^{\#}\left(\frac{1}{s}\right), \quad \checkmark$$

where

$$x^{\#}(s) = \int_0^\infty e^{-us}\,x(u)\,du.$$

□

32 Linear Functional Analysis

32.1 Norms

$||\cdot||$ is a norm on a vector space X if
i) $||x|| = 0$ iff $x = 0$.
ii) $||\alpha x|| = |\alpha| \cdot ||x||$ for all scalars α.
iii) $||x + y|| \leq ||x|| + ||y||$ (the triangle inequality).
The norm induces the *distance function* $d(x, y) = ||x - y||$ so that X is a metric space, called a *normed vector space*.

32.2 Banach and Hilbert Spaces

A **Banach space** is a normed vector space that is complete in that norm's metric. I.e. a complete normed linear space is a Banach space.
A **Hilbert space** is an inner product space for which the corresponding normed space is complete. I.e. a complete inner product space is a Hilbert space.
Examples: 1) Let K be a compact set of R^n and let $C(K)$ denote the space of continuous functions on K. Since every $u \in C(K)$ achieves maximum and minimum values on K, we may define

$$||u||_\infty = \max_{x\in K} |u(x)|.$$

$||\cdot||_\infty$ is indeed a norm on $C(K)$ and since a uniform limit of continuous functions is continuous, $C(K)$ is a **Banach space**. However, this norm cannot be derived from an inner product, so $C(K)$ is **not a Hilbert space**.
2) $C(K)$ is **not** a Banach space with $||\cdot||_2$ norm. (Bell-shaped functions on $[0, 1]$ may converge to a discontinuous δ-function). In general, the space of continuous functions on $[0, 1]$, with the norm $||\cdot||_p$, $1 \leq p < \infty$, is **not** a Banach space, since it is not complete.
3) R^n and C^n are real and complex Banach spaces (with a Eucledian norm).
4) L^p are Banach spaces (with $||\cdot||_p$ norm).
5) The space of bounded real-valued functions on a set S, with the sup norm $||\cdot||_S$ are Banach spaces.
6) The space of bounded continuous real-valued functions on a metric space X is a Banach space.

32.3 Cauchy-Schwarz Inequality

$$|(u, v)| \leq ||u||||v|| \quad \text{in any norm, for example} \quad \int |uv|dx \leq (\int u^2 dx)^{\frac{1}{2}} (\int v^2 dx)^{\frac{1}{2}}$$
$$|a(u, v)| \leq a(u, u)^{\frac{1}{2}} \ a(v, v)^{\frac{1}{2}}$$
$$\int |v|dx = \int |v| \cdot 1 \ dx = (\int |v|^2 dx)^{\frac{1}{2}} (\int 1^2 dx)^{\frac{1}{2}}$$

32.4 Hölder Inequality

$$\int_\Omega |uv| \, dx \leq ||u||_p ||v||_q,$$

which holds for $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. In particular, this shows $uv \in L^1(\Omega)$.

32.5 Minkowski Inequality

$$||u+v||_p \le ||u||_p + ||v||_p,$$

which holds for $u, v \in L^p(\Omega)$. In particular, it shows $u+v \in L^p(\Omega)$.
Using the Minkowski Inequality, we find that $||\cdot||_p$ is a norm on $L^p(\Omega)$.
The *Riesz-Fischer theorem* asserts that $L^p(\Omega)$ is complete in this norm, so $L^p(\Omega)$ is a Banach space under the norm $||\cdot||_p$.
If $p=2$, then $L^2(\Omega)$ is a Hilbert space with inner product

$$(u,v) = \int_\Omega uv\,dx.$$

Example: $\Omega \in R^n$ bounded domain, $C^1(\bar{\Omega})$ denotes the functions that, along with their first-order derivatives, extend continuously to the compact set $\bar{\Omega}$. Then $C^1(\bar{\Omega})$ is a Banach space under the norm

$$||u||_{1,\infty} = \max_{x\in\bar{\Omega}}(|\nabla u(x)| + |u(x)|).$$

Note that $C^1(\Omega)$ is **not** a Banach space since $||u||_{1,\infty}$ need not be finite for $u \in C^1(\Omega)$.

32.6 Sobolev Spaces

A *Sobolev space* is a space of functions whose distributional derivatives (up to some fixed order) exist in an L^p-space.
Let Ω be a domain in R^n, and let us introduce

$$< u, v >_1 = \int_\Omega (\nabla u \cdot \nabla v + uv)\,dx, \tag{32.1}$$

$$||u||_{1,2} = \sqrt{< u, u >_1} = \left(\int_\Omega (|\nabla u|^2 + |u|^2)\,dx\right)^{\frac{1}{2}} \tag{32.2}$$

when these expressions are defined and finite. For example, (32.1) and (32.2) are defined for functions in $C_0^1(\Omega)$. However, $C_0^1(\Omega)$ is not complete under the norm (32.2), and so does not form a Hilbert space.

Divergence Theorem

$$\int_{\partial\Omega} \vec{A}\cdot n\,dS = \int_\Omega \operatorname{div}\vec{A}\,dx$$

Trace Theorem

$$\|u\|_{L_2(\partial\Omega)} \le C\|u\|_{H^1(\Omega)} \qquad \Omega \text{ smooth or square}$$

Poincare Inequality

$$\|u\|_p \le C\|\nabla u\|_p \qquad 1 \le p \le \infty$$

$$\int_\Omega |u(x)|^2\,dx \le C\int_\Omega |\nabla u(x)|^2\,dx \qquad u \in C_0^1(\Omega),\ H_0^{1,2}(\Omega) \qquad i.e.\ p=2$$

$$\|u - u_\Omega\|_p \le \|\nabla u\|_p \qquad u \in H_0^{1,p}(\Omega)$$

$$u_\Omega = \frac{1}{|\Omega|}\int_\Omega u(x)\,dx \qquad (\textit{Average value of } u \textit{ over } \Omega),\ \ |\Omega| \textit{ is the volume of } \Omega$$

Notes

$$\frac{\partial u}{\partial \vec{n}} = \nabla u \cdot \vec{n} = n_1 \frac{\partial u}{\partial x_1} + n_2 \frac{\partial u}{\partial x_2} \qquad\qquad |\nabla u|^2 = u_{x_1}^2 + u_{x_2}^2$$

$$\int_\Omega \nabla |u|\,dx = \int_\Omega \frac{|u|}{u}\nabla u\,dx$$

$$\sqrt{ab} \le \frac{a+b}{2} \quad \Rightarrow \quad ab \le \frac{a^2+b^2}{2} \quad \Rightarrow \quad ||\nabla u|| ||u|| \le \frac{||\nabla u||^2 + ||u||^2}{2}$$

$$u\nabla u = \nabla(\frac{u^2}{2})$$

$$\int_\Omega (u_{xy})^2\,dx = \int_\Omega u_{xx}u_{yy}\,dx \qquad \forall u \in H_0^2(\Omega) \qquad \Omega \ \text{ square}$$

Problem (F'04, #6). *Let $q \in C_0^1(\mathbb{R}^3)$. Prove that the vector field*

$$u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{q(y)(x-y)}{|x-y|^3}\,dy$$

enjoys the following properties: [87]
a) *$u(x)$ is* ***conservative****;*
b) *div $u(x) = q(x)$ for all $x \in \mathbb{R}^3$;*
c) *$|u(x)| = O(|x|^{-2})$ for large x.*
Furthermore, prove that the proverties (1), (2), and (3) above determine the vector field $u(x)$ uniquely.

Proof. **a)** To show that $\vec{u}(x)$ is conservative, we need to show that curl $\vec{u} = 0$.
The curl of $\vec{V}$ is another vector field defined by

$$\text{curl}\,\vec{V} = \nabla \times \vec{V} = \det\begin{pmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ \partial_1 & \partial_2 & \partial_3 \\ V_1 & V_2 & V_3 \end{pmatrix} = \Big(\frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3}, \frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1}, \frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\Big).$$

Consider

$$\vec{V}(x) = \frac{\vec{x}}{|\vec{x}|^3} = \frac{(x_1,x_2,x_3)}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}}.$$

Then,

$$\begin{aligned}\vec{u}(x) &= \frac{1}{4\pi}\int_{\mathbb{R}^3} q(y)\,V(x-y)\,dy, \\ \text{curl}\,\vec{u}(x) &= \frac{1}{4\pi}\int_{\mathbb{R}^3} q(y)\,\text{curl}_x\,V(x-y)\,dy.\end{aligned}$$

$$\begin{aligned}\text{curl}\,\vec{V}(x) &= \text{curl}\,\frac{(x_1,x_2,x_3)}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}} \\ &= \Bigg(\frac{-\frac{3}{2}\cdot 2x_2x_3}{(x_1^2+x_2^2+x_3^2)^{\frac{5}{2}}} - \frac{-\frac{3}{2}\cdot 2x_3x_2}{(x_1^2+x_2^2+x_3^2)^{\frac{5}{2}}}, \frac{-\frac{3}{2}\cdot 2x_3x_1}{(x_1^2+x_2^2+x_3^2)^{\frac{5}{2}}} - \frac{-\frac{3}{2}\cdot 2x_1x_3}{(x_1^2+x_2^2+x_3^2)^{\frac{5}{2}}}, \frac{-\frac{3}{2}\cdot 2x_1x_2}{(x_1^2+x_2^2+x_3^2)^{\frac{5}{2}}} - \\ &= (0,0,0).\end{aligned}$$

[87] McOwen, p. 138-140.

Thus, curl $\vec{u} = \frac{1}{4\pi}\int_{\mathbb{R}^3} q(y)\cdot 0\,dy = 0$, and $\vec{u}(x)$ is conservative. ✓

b) Note that the Laplace kernel in $\mathbb{R}^3$ is $-\frac{1}{4\pi r}$.

$$u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{q(y)(x-y)}{|x-y|^3}\,dy = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{q(r)\,r}{r^3}\,r\,dr = \int_{\mathbb{R}^3} \frac{q(r)}{4\pi r}\,dr = q.$$

c) Consider

$$F(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{q(y)}{|x-y|}\,dy.$$

$F(x)$ is $O(|x|^{-1})$ as $|x|\to\infty$.
Note that $u = \nabla F$, which is clearly $O(|x|^{-2})$ as $|x|\to\infty$. ✓ □