CHAPTER 1

1.1. Given the vectors $\mathbf{M} = -10\mathbf{a}_x + 4\mathbf{a}_y - 8\mathbf{a}_z$ and $\mathbf{N} = 8\mathbf{a}_x + 7\mathbf{a}_y - 2\mathbf{a}_z$, find:

a) a unit vector in the direction of $-\mathbf{M} + 2\mathbf{N}$.

$-\mathbf{M} + 2\mathbf{N} = 10\mathbf{a}_x - 4\mathbf{a}_y + 8\mathbf{a}_z + 16\mathbf{a}_x + 14\mathbf{a}_y - 4\mathbf{a}_z = (26, 10, 4)$

Thus

$$\mathbf{a} = \frac{(26,10,4)}{|(26,10,4)|} = \underline{(0.92, 0.36, 0.14)}$$

b) the magnitude of $5\mathbf{a}_x + \mathbf{N} - 3\mathbf{M}$:

$(5, 0, 0) + (8, 7, -2) - (-30, 12, -24) = (43, -5, 22)$, and $|(43, -5, 22)| = \underline{48.6}$.

c) $|\mathbf{M}||2\mathbf{N}|(\mathbf{M} + \mathbf{N})$:

$|(-10, 4, -8)||(16, 14, -4)|(-2, 11, -10) = (13.4)(21.6)(-2, 11, -10)$

$= \underline{(-580.5, 3193, -2902)}$

1.2. The three vertices of a triangle are located at $A(-1, 2, 5)$, $B(-4, -2, -3)$, and $C(1, 3, -2)$.

a) Find the length of the perimeter of the triangle: Begin with $\mathbf{AB} = (-3, -4, -8)$, $\mathbf{BC} = (5, 5, 1)$, and $\mathbf{CA} = (-2, -1, 7)$. Then the perimeter will be $\ell = |\mathbf{AB}| + |\mathbf{BC}| + |\mathbf{CA}| = \sqrt{9 + 16 + 64} + \sqrt{25 + 25 + 1} + \sqrt{4 + 1 + 49} = \underline{23.9}$.

b) Find a unit vector that is directed from the midpoint of the side AB to the midpoint of side BC: The vector from the origin to the midpoint of AB is $\mathbf{M}_{AB} = \frac{1}{2}(\mathbf{A} + \mathbf{B}) = \frac{1}{2}(-5\mathbf{a}_x + 2\mathbf{a}_z)$. The vector from the origin to the midpoint of BC is $\mathbf{M}_{BC} = \frac{1}{2}(\mathbf{B} + \mathbf{C}) = \frac{1}{2}(-3\mathbf{a}_x + \mathbf{a}_y - 5\mathbf{a}_z)$. The vector from midpoint to midpoint is now $\mathbf{M}_{AB} - \mathbf{M}_{BC} = \frac{1}{2}(-2\mathbf{a}_x - \mathbf{a}_y + 7\mathbf{a}_z)$. The unit vector is therefore

$$\mathbf{a}_{MM} = \frac{\mathbf{M}_{AB} - \mathbf{M}_{BC}}{|\mathbf{M}_{AB} - \mathbf{M}_{BC}|} = \frac{(-2\mathbf{a}_x - \mathbf{a}_y + 7\mathbf{a}_z)}{7.35} = \underline{-0.27\mathbf{a}_x - 0.14\mathbf{a}_y + 0.95\mathbf{a}_z}$$

where factors of 1/2 have cancelled.

c) Show that this unit vector multiplied by a scalar is equal to the vector from A to C and that the unit vector is therefore parallel to AC. First we find $\mathbf{AC} = 2\mathbf{a}_x + \mathbf{a}_y - 7\mathbf{a}_z$, which we recognize as $-7.35\,\mathbf{a}_{MM}$. The vectors are thus parallel (but oppositely-directed).

1.3. The vector from the origin to the point A is given as $(6, -2, -4)$, and the unit vector directed from the origin toward point B is $(2, -2, 1)/3$. If points A and B are ten units apart, find the coordinates of point B.

With $\mathbf{A} = (6, -2, -4)$ and $\mathbf{B} = \frac{1}{3}B(2, -2, 1)$, we use the fact that $|\mathbf{B} - \mathbf{A}| = 10$, or

$|(6 - \frac{2}{3}B)\mathbf{a}_x - (2 - \frac{2}{3}B)\mathbf{a}_y - (4 + \frac{1}{3}B)\mathbf{a}_z| = 10$

Expanding, obtain

$36 - 8B + \frac{4}{9}B^2 + 4 - \frac{8}{3}B + \frac{4}{9}B^2 + 16 + \frac{8}{3}B + \frac{1}{9}B^2 = 100$

or $B^2 - 8B - 44 = 0$. Thus $B = \frac{8 \pm \sqrt{64 - 176}}{2} = 11.75$ (taking positive option) and so

$$\mathbf{B} = \frac{2}{3}(11.75)\mathbf{a}_x - \frac{2}{3}(11.75)\mathbf{a}_y + \frac{1}{3}(11.75)\mathbf{a}_z = \underline{7.83\mathbf{a}_x - 7.83\mathbf{a}_y + 3.92\mathbf{a}_z}$$

1.4. A circle, centered at the origin with a radius of 2 units, lies in the xy plane. Determine the unit vector in rectangular components that lies in the xy plane, is tangent to the circle at $(\sqrt{3}, 1, 0)$, and is in the general direction of increasing values of y:

A unit vector tangent to this circle in the general increasing y direction is $\mathbf{t} = \mathbf{a}_\phi$. Its x and y components are $\mathbf{t}_x = \mathbf{a}_\phi \cdot \mathbf{a}_x = -\sin\phi$, and $\mathbf{t}_y = \mathbf{a}_\phi \cdot \mathbf{a}_y = \cos\phi$. At the point $(\sqrt{3}, 1)$, $\phi = 30°$, and so $\mathbf{t} = -\sin 30° \mathbf{a}_x + \cos 30° \mathbf{a}_y = \underline{0.5(-\mathbf{a}_x + \sqrt{3}\mathbf{a}_y)}$.

1.5. A vector field is specified as $\mathbf{G} = 24xy\mathbf{a}_x + 12(x^2 + 2)\mathbf{a}_y + 18z^2\mathbf{a}_z$. Given two points, $P(1, 2, -1)$ and $Q(-2, 1, 3)$, find:

a) $\mathbf{G}$ at P: $\mathbf{G}(1, 2, -1) = \underline{(48, 36, 18)}$

b) a unit vector in the direction of $\mathbf{G}$ at Q: $\mathbf{G}(-2, 1, 3) = (-48, 72, 162)$, so

$$\mathbf{a}_G = \frac{(-48, 72, 162)}{|(-48, 72, 162)|} = \underline{(-0.26, 0.39, 0.88)}$$

c) a unit vector directed from Q toward P:

$$\mathbf{a}_{QP} = \frac{\mathbf{P} - \mathbf{Q}}{|\mathbf{P} - \mathbf{Q}|} = \frac{(3, -1, 4)}{\sqrt{26}} = \underline{(0.59, 0.20, -0.78)}$$

d) the equation of the surface on which $|\mathbf{G}| = 60$: We write $60 = |(24xy, 12(x^2 + 2), 18z^2)|$, or $10 = |(4xy, 2x^2 + 4, 3z^2)|$, so the equation is

$$\underline{100 = 16x^2y^2 + 4x^4 + 16x^2 + 16 + 9z^4}$$

1.6. If $\mathbf{a}$ is a unit vector in a given direction, B is a scalar constant, and $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$, describe the surface $\mathbf{r} \cdot \mathbf{a} = B$. What is the relation between the the unit vector $\mathbf{a}$ and the scalar B to this surface? (HINT: Consider first a simple example with $\mathbf{a} = \mathbf{a}_x$ and $B = 1$, and then consider any $\mathbf{a}$ and B.):

We could consider a general unit vector, $\mathbf{a} = A_1\mathbf{a}_x + A_2\mathbf{a}_y + A_3\mathbf{a}_z$, where $A_1^2 + A_2^2 + A_3^2 = 1$. Then $\mathbf{r} \cdot \mathbf{a} = A_1x + A_2y + A_3z = f(x, y, z) = B$. This is the equation of a planar surface, where $f = B$. The relation of $\mathbf{a}$ to the surface becomes clear in the special case in which $\mathbf{a} = \mathbf{a}_x$. We obtain $\mathbf{r} \cdot \mathbf{a} = f(x) = x = B$, where it is evident that $\mathbf{a}$ is a unit normal vector to the surface (as a look ahead (Chapter 4), note that taking the gradient of f gives $\mathbf{a}$).

1.7. Given the vector field $\mathbf{E} = 4zy^2 \cos 2x\mathbf{a}_x + 2zy \sin 2x\mathbf{a}_y + y^2 \sin 2x\mathbf{a}_z$ for the region $|x|$, $|y|$, and $|z|$ less than 2, find:

a) the surfaces on which $E_y = 0$. With $E_y = 2zy \sin 2x = 0$, the surfaces are 1) the plane $\underline{z = 0}$, with $|x| < 2$, $|y| < 2$; 2) the plane $\underline{y = 0}$, with $|x| < 2$, $|z| < 2$; 3) the plane $\underline{x = 0}$, with $|y| < 2$, $|z| < 2$; 4) the plane $\underline{x = \pi/2}$, with $|y| < 2$, $|z| < 2$.

b) the region in which $E_y = E_z$: This occurs when $2zy \sin 2x = y^2 \sin 2x$, or on the plane $\underline{2z = y}$, with $|x| < 2$, $|y| < 2$, $|z| < 1$.

c) the region in which $\mathbf{E} = 0$: We would have $E_x = E_y = E_z = 0$, or $zy^2 \cos 2x = zy \sin 2x = y^2 \sin 2x = 0$. This condition is met on the plane $\underline{y = 0}$, with $|x| < 2$, $|z| < 2$.

1.8. Demonstrate the ambiguity that results when the cross product is used to find the angle between two vectors by finding the angle between $\mathbf{A} = 3\mathbf{a}_x - 2\mathbf{a}_y + 4\mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z$. Does this ambiguity exist when the dot product is used?

We use the relation $\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin\theta \mathbf{n}$. With the given vectors we find

$$\mathbf{A} \times \mathbf{B} = 14\mathbf{a}_y + 7\mathbf{a}_z = 7\sqrt{5} \underbrace{\left[\frac{2\mathbf{a}_y + \mathbf{a}_z}{\sqrt{5}}\right]}_{\pm\mathbf{n}} = \sqrt{9+4+16}\sqrt{4+1+4}\, \sin\theta\, \mathbf{n}$$

where $\mathbf{n}$ is identified as shown; we see that $\mathbf{n}$ can be positive or negative, as $\sin\theta$ can be positive or negative. This apparent sign ambiguity is not the real problem, however, as we really want the magnitude of the angle anyway. Choosing the positive sign, we are left with $\sin\theta = 7\sqrt{5}/(\sqrt{29}\sqrt{9}) = 0.969$. *Two* values of θ (75.7° and 104.3°) satisfy this equation, and hence the real ambiguity.

In using the dot product, we find $\mathbf{A} \cdot \mathbf{B} = 6 - 2 - 8 = -4 = |\mathbf{A}||\mathbf{B}| \cos\theta = 3\sqrt{29}\cos\theta$, or $\cos\theta = -4/(3\sqrt{29}) = -0.248 \Rightarrow \theta = -75.7°$. Again, the minus sign is not important, as we care only about the angle magnitude. The main point is that *only one* θ value results when using the dot product, so no ambiguity.

1.9. A field is given as

$$\mathbf{G} = \frac{25}{(x^2 + y^2)}(x\mathbf{a}_x + y\mathbf{a}_y)$$

Find:

a) a unit vector in the direction of $\mathbf{G}$ at $P(3, 4, -2)$: Have $\mathbf{G}_p = 25/(9+16) \times (3, 4, 0) = 3\mathbf{a}_x + 4\mathbf{a}_y$, and $|\mathbf{G}_p| = 5$. Thus $\mathbf{a}_G = \underline{(0.6, 0.8, 0)}$.

b) the angle between $\mathbf{G}$ and $\mathbf{a}_x$ at P: The angle is found through $\mathbf{a}_G \cdot \mathbf{a}_x = \cos\theta$. So $\cos\theta = (0.6, 0.8, 0) \cdot (1, 0, 0) = 0.6$. Thus $\underline{\theta = 53°}$.

c) the value of the following double integral on the plane $y = 7$:

$$\int_0^4 \int_0^2 \mathbf{G} \cdot \mathbf{a}_y dz dx$$

$$\int_0^4 \int_0^2 \frac{25}{x^2 + y^2}(x\mathbf{a}_x + y\mathbf{a}_y) \cdot \mathbf{a}_y dz dx = \int_0^4 \int_0^2 \frac{25}{x^2 + 49} \times 7\, dz dx = \int_0^4 \frac{350}{x^2 + 49} dx$$

$$= 350 \times \frac{1}{7}\left[\tan^{-1}\left(\frac{4}{7}\right) - 0\right] = \underline{26}$$

1.10. By expressing diagonals as vectors and using the definition of the dot product, find the smaller angle between any two diagonals of a cube, where each diagonal connects diametrically opposite corners, and passes through the center of the cube:

Assuming a side length, b, two diagonal vectors would be $\mathbf{A} = b(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)$ and $\mathbf{B} = b(\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z)$. Now use $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$, or $b^2(1 - 1 + 1) = (\sqrt{3}b)(\sqrt{3}b)\cos\theta \Rightarrow \cos\theta = 1/3 \Rightarrow \theta = \underline{70.53°}$. This result (in magnitude) is the same for *any* two diagonal vectors.

1.11. Given the points $M(0.1, -0.2, -0.1)$, $N(-0.2, 0.1, 0.3)$, and $P(0.4, 0, 0.1)$, find:

a) the vector $\mathbf{R}_{MN}$: $\mathbf{R}_{MN} = (-0.2, 0.1, 0.3) - (0.1, -0.2, -0.1) = \underline{(-0.3, 0.3, 0.4)}$.

b) the dot product $\mathbf{R}_{MN} \cdot \mathbf{R}_{MP}$: $\mathbf{R}_{MP} = (0.4, 0, 0.1) - (0.1, -0.2, -0.1) = (0.3, 0.2, 0.2)$. $\mathbf{R}_{MN} \cdot \mathbf{R}_{MP} = (-0.3, 0.3, 0.4) \cdot (0.3, 0.2, 0.2) = -0.09 + 0.06 + 0.08 = \underline{0.05}$.

c) the scalar projection of $\mathbf{R}_{MN}$ on $\mathbf{R}_{MP}$:

$$\mathbf{R}_{MN} \cdot \mathbf{a}_{RMP} = (-0.3, 0.3, 0.4) \cdot \frac{(0.3, 0.2, 0.2)}{\sqrt{0.09 + 0.04 + 0.04}} = \frac{0.05}{\sqrt{0.17}} = \underline{0.12}$$

d) the angle between $\mathbf{R}_{MN}$ and $\mathbf{R}_{MP}$:

$$\theta_M = \cos^{-1}\left(\frac{\mathbf{R}_{MN} \cdot \mathbf{R}_{MP}}{|\mathbf{R}_{MN}||\mathbf{R}_{MP}|}\right) = \cos^{-1}\left(\frac{0.05}{\sqrt{0.34}\sqrt{0.17}}\right) = \underline{78^\circ}$$

1.12. Show that the vector fields $\mathbf{A} = \rho\cos\phi\,\mathbf{a}_\rho + \rho\sin\phi\,\mathbf{a}_\phi + \rho\,\mathbf{a}_z$ and $\mathbf{B} = \rho\cos\phi\,\mathbf{a}_\rho + \rho\sin\phi\,\mathbf{a}_\phi - \rho\,\mathbf{a}_z$ are everywhere perpendicular to each other:

We find $\mathbf{A} \cdot \mathbf{B} = \rho^2(\sin^2\phi + \cos^2\phi) - \rho^2 = 0 = |\mathbf{A}||\mathbf{B}|\cos\theta$. Therefore $\cos\theta = 0$ or $\underline{\theta = 90^\circ}$.

1.13. a) Find the vector component of $\mathbf{F} = (10, -6, 5)$ that is parallel to $\mathbf{G} = (0.1, 0.2, 0.3)$:

$$\mathbf{F}_{||G} = \frac{\mathbf{F} \cdot \mathbf{G}}{|\mathbf{G}|^2}\mathbf{G} = \frac{(10, -6, 5) \cdot (0.1, 0.2, 0.3)}{0.01 + 0.04 + 0.09}(0.1, 0.2, 0.3) = \underline{(0.93, 1.86, 2.79)}$$

b) Find the vector component of $\mathbf{F}$ that is perpendicular to $\mathbf{G}$:

$$\mathbf{F}_{pG} = \mathbf{F} - \mathbf{F}_{||G} = (10, -6, 5) - (0.93, 1.86, 2.79) = \underline{(9.07, -7.86, 2.21)}$$

c) Find the vector component of $\mathbf{G}$ that is perpendicular to $\mathbf{F}$:

$$\mathbf{G}_{pF} = \mathbf{G} - \mathbf{G}_{||F} = \mathbf{G} - \frac{\mathbf{G} \cdot \mathbf{F}}{|\mathbf{F}|^2}\mathbf{F} = (0.1, 0.2, 0.3) - \frac{1.3}{100 + 36 + 25}(10, -6, 5) = \underline{(0.02, 0.25, 0.26)}$$

1.14. Show that the vector fields $\mathbf{A} = \mathbf{a}_r\,(\sin 2\theta)/r^2 + 2\mathbf{a}_\theta\,(\sin\theta)/r^2$ and $\mathbf{B} = r\cos\theta\,\mathbf{a}_r + r\,\mathbf{a}_\theta$ are everywhere parallel to each other:

Using the definition of the cross product, we find

$$\mathbf{A} \times \mathbf{B} = \left(\frac{\sin 2\theta}{r} - \frac{2\sin\theta\cos\theta}{r}\right)\mathbf{a}_\phi = 0 = |\mathbf{A}||\mathbf{B}|\sin\theta\,\mathbf{n}$$

Identify $\mathbf{n} = \mathbf{a}_\phi$, and so $\sin\theta = 0$, and therefore $\underline{\theta = 0}$ (they're parallel).

1.15. Three vectors extending from the origin are given as $\mathbf{r}_1 = (7, 3, -2)$, $\mathbf{r}_2 = (-2, 7, -3)$, and $\mathbf{r}_3 = (0, 2, 3)$. Find:

a) a unit vector perpendicular to both $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\mathbf{a}_{p12} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1 \times \mathbf{r}_2|} = \frac{(5, 25, 55)}{60.6} = \underline{(0.08, 0.41, 0.91)}$$

b) a unit vector perpendicular to the vectors $\mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{r}_2 - \mathbf{r}_3$: $\mathbf{r}_1 - \mathbf{r}_2 = (9, -4, 1)$ and $\mathbf{r}_2 - \mathbf{r}_3 = (-2, 5, -6)$. So $\mathbf{r}_1 - \mathbf{r}_2 \times \mathbf{r}_2 - \mathbf{r}_3 = (19, 52, 32)$. Then

$$\mathbf{a}_p = \frac{(19, 52, 32)}{|(19, 52, 32)|} = \frac{(19, 52, 32)}{63.95} = \underline{(0.30, 0.81, 0.50)}$$

c) the area of the triangle defined by $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\text{Area} = \frac{1}{2}|\mathbf{r}_1 \times \mathbf{r}_2| = \underline{30.3}$$

d) the area of the triangle defined by the heads of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$:

$$\text{Area} = \frac{1}{2}|(\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_2 - \mathbf{r}_3)| = \frac{1}{2}|(-9, 4, -1) \times (-2, 5, -6)| = \underline{32.0}$$

1.16. The vector field $\mathbf{E} = (B/\rho)\,\mathbf{a}_\rho$, where B is a constant, is to be translated such that it originates at the line, $x = 2$, $y = 0$. Write the translated form of $\mathbf{E}$ in rectangular components:

First, transform the given field to rectangular components:

$$E_x = \frac{B}{\rho}\mathbf{a}_\rho \cdot \mathbf{a}_x = \frac{B}{\sqrt{x^2 + y^2}} \cos\phi = \frac{B}{\sqrt{x^2 + y^2}} \frac{x}{\sqrt{x^2 + y^2}} = \frac{Bx}{x^2 + y^2}$$

Using similar reasoning:

$$E_y = \frac{B}{\rho}\mathbf{a}_\rho \cdot \mathbf{a}_y = \frac{B}{\sqrt{x^2 + y^2}} \sin\phi = \frac{By}{x^2 + y^2}$$

We then translate the two components to $x = 2, y = 0$, to obtain the final result:

$$\underline{\mathbf{E}(x, y) = \frac{B\left[(x - 2)\,\mathbf{a}_x + y\,\mathbf{a}_y\right]}{(x - 2)^2 + y^2}}$$

1.17. Point $A(-4, 2, 5)$ and the two vectors, $\mathbf{R}_{AM} = (20, 18, -10)$ and $\mathbf{R}_{AN} = (-10, 8, 15)$, define a triangle.

a) Find a unit vector perpendicular to the triangle: Use

$$\mathbf{a}_p = \frac{\mathbf{R}_{AM} \times \mathbf{R}_{AN}}{|\mathbf{R}_{AM} \times \mathbf{R}_{AN}|} = \frac{(350, -200, 340)}{527.35} = \underline{(0.664, -0.379, 0.645)}$$

The vector in the opposite direction to this one is also a valid answer.

1.17b) Find a unit vector in the plane of the triangle and perpendicular to $\mathbf{R}_{AN}$:

$$\mathbf{a}_{AN} = \frac{(-10, 8, 15)}{\sqrt{389}} = (-0.507, 0.406, 0.761)$$

Then

$$\mathbf{a}_{pAN} = \mathbf{a}_p \times \mathbf{a}_{AN} = (0.664, -0.379, 0.645) \times (-0.507, 0.406, 0.761) = \underline{(-0.550, -0.832, 0.077)}$$

The vector in the opposite direction to this one is also a valid answer.

c) Find a unit vector in the plane of the triangle that bisects the interior angle at A: A non-unit vector in the required direction is $(1/2)(\mathbf{a}_{AM} + \mathbf{a}_{AN})$, where

$$\mathbf{a}_{AM} = \frac{(20, 18, -10)}{|(20, 18, -10)|} = (0.697, 0.627, -0.348)$$

Now

$$\frac{1}{2}(\mathbf{a}_{AM} + \mathbf{a}_{AN}) = \frac{1}{2}[(0.697, 0.627, -0.348) + (-0.507, 0.406, 0.761)] = (0.095, 0.516, 0.207)$$

Finally,

$$\mathbf{a}_{bis} = \frac{(0.095, 0.516, 0.207)}{|(0.095, 0.516, 0.207)|} = \underline{(0.168, 0.915, 0.367)}$$

1.18. Transform the vector field $\mathbf{H} = (A/\rho)\,\mathbf{a}_\phi$, where A is a constant, from cylindrical coordinates to spherical coordinates:

First, the unit vector does not change, since $\mathbf{a}_\phi$ is common to both coordinate systems. We only need to express the cylindrical radius, ρ, as $\rho = r\sin\theta$, obtaining

$$\mathbf{H}(r, \theta) = \frac{A}{r\sin\theta}\,\mathbf{a}_\phi$$

1.19. a) Express the field $\mathbf{D} = (x^2 + y^2)^{-1}(x\mathbf{a}_x + y\mathbf{a}_y)$ in cylindrical components and cylindrical variables: Have $x = \rho\cos\phi$, $y = \rho\sin\phi$, and $x^2 + y^2 = \rho^2$. Therefore

$$\mathbf{D} = \frac{1}{\rho}(\cos\phi\mathbf{a}_x + \sin\phi\mathbf{a}_y)$$

Then

$$D_\rho = \mathbf{D}\cdot\mathbf{a}_\rho = \frac{1}{\rho}\left[\cos\phi(\mathbf{a}_x\cdot\mathbf{a}_\rho) + \sin\phi(\mathbf{a}_y\cdot\mathbf{a}_\rho)\right] = \frac{1}{\rho}\left[\cos^2\phi + \sin^2\phi\right] = \frac{1}{\rho}$$

and

$$D_\phi = \mathbf{D}\cdot\mathbf{a}_\phi = \frac{1}{\rho}\left[\cos\phi(\mathbf{a}_x\cdot\mathbf{a}_\phi) + \sin\phi(\mathbf{a}_y\cdot\mathbf{a}_\phi)\right] = \frac{1}{\rho}\left[\cos\phi(-\sin\phi) + \sin\phi\cos\phi\right] = 0$$

Therefore

$$\underline{\mathbf{D} = \frac{1}{\rho}\mathbf{a}_\rho}$$

1.19b) Evaluate $\mathbf{D}$ at the point where $\rho = 2$, $\phi = 0.2\pi$, and $z = 5$, expressing the result in cylindrical and cartesian coordinates: At the given point, and in cylindrical coordinates, $\underline{\mathbf{D} = 0.5\mathbf{a}_\rho}$. To express this in cartesian, we use

$$\mathbf{D} = 0.5(\mathbf{a}_\rho \cdot \mathbf{a}_x)\mathbf{a}_x + 0.5(\mathbf{a}_\rho \cdot \mathbf{a}_y)\mathbf{a}_y = 0.5\cos 36^\circ \mathbf{a}_x + 0.5\sin 36^\circ \mathbf{a}_y = \underline{0.41\mathbf{a}_x + 0.29\mathbf{a}_y}$$

1.20. A cylinder of radius a, centered on the z axis, rotates about the z axis at angular velocity Ω rad/s. The rotation direction is counter-clockwise when looking in the positive z direction.

a) Using cylindrical components, write an expression for the velocity field, $\mathbf{v}$, that gives the tangential velocity at any point within the cylinder:

Tangential velocity is angular velocity times the perpendicular distance from the rotation axis. With counter-clockwise rotation, we therefore find $\mathbf{v}(\rho) = \underline{-\Omega\rho\,\mathbf{a}_\phi\ (\rho < a)}$.

b) Convert your result from part a to spherical components:

In spherical, the component direction, $\mathbf{a}_\phi$, is the same. We obtain

$$\mathbf{v}(r,\theta) = \underline{-\Omega r\sin\theta\,\mathbf{a}_\phi\ (r\sin\theta < a)}$$

c) Convert to rectangular components:

$$v_x = -\Omega\rho\mathbf{a}_\phi \cdot \mathbf{a}_x = -\Omega(x^2+y^2)^{1/2}(-\sin\phi) = -\Omega(x^2+y^2)^{1/2}\frac{-y}{(x^2+y^2)^{1/2}} = \Omega y$$

Similarly

$$v_y = -\Omega\rho\mathbf{a}_\phi \cdot \mathbf{a}_y = -\Omega(x^2+y^2)^{1/2}(\cos\phi) = -\Omega(x^2+y^2)^{1/2}\frac{x}{(x^2+y^2)^{1/2}} = -\Omega x$$

Finally $\mathbf{v}(x,y) = \underline{\Omega\,[y\,\mathbf{a}_x - x\,\mathbf{a}_y]}$, where $(x^2+y^2)^{1/2} < a$.

1.21. Express in cylindrical components:

a) the vector from $C(3,2,-7)$ to $D(-1,-4,2)$:
$C(3,2,-7) \to C(\rho = 3.61, \phi = 33.7^\circ, z = -7)$ and
$D(-1,-4,2) \to D(\rho = 4.12, \phi = -104.0^\circ, z = 2)$.
Now $\mathbf{R}_{CD} = (-4,-6,9)$ and $R_\rho = \mathbf{R}_{CD} \cdot \mathbf{a}_\rho = -4\cos(33.7) - 6\sin(33.7) = -6.66$. Then $R_\phi = \mathbf{R}_{CD} \cdot \mathbf{a}_\phi = 4\sin(33.7) - 6\cos(33.7) = -2.77$. So $\mathbf{R}_{CD} = \underline{-6.66\mathbf{a}_\rho - 2.77\mathbf{a}_\phi + 9\mathbf{a}_z}$

b) a unit vector at D directed toward C:
$\mathbf{R}_{CD} = (4,6,-9)$ and $R_\rho = \mathbf{R}_{DC} \cdot \mathbf{a}_\rho = 4\cos(-104.0) + 6\sin(-104.0) = -6.79$. Then $R_\phi = \mathbf{R}_{DC} \cdot \mathbf{a}_\phi = 4[-\sin(-104.0)] + 6\cos(-104.0) = 2.43$. So $\mathbf{R}_{DC} = -6.79\mathbf{a}_\rho + 2.43\mathbf{a}_\phi - 9\mathbf{a}_z$
Thus $\mathbf{a}_{DC} = \underline{-0.59\mathbf{a}_\rho + 0.21\mathbf{a}_\phi - 0.78\mathbf{a}_z}$

c) a unit vector at D directed toward the origin: Start with $\mathbf{r}_D = (-1,-4,2)$, and so the vector toward the origin will be $-\mathbf{r}_D = (1,4,-2)$. Thus in cartesian the unit vector is $\mathbf{a} = (0.22, 0.87, -0.44)$. Convert to cylindrical:
$a_\rho = (0.22, 0.87, -0.44) \cdot \mathbf{a}_\rho = 0.22\cos(-104.0) + 0.87\sin(-104.0) = -0.90$, and
$a_\phi = (0.22, 0.87, -0.44) \cdot \mathbf{a}_\phi = 0.22[-\sin(-104.0)] + 0.87\cos(-104.0) = 0$, so that finally, $\mathbf{a} = \underline{-0.90\mathbf{a}_\rho - 0.44\mathbf{a}_z}$.

1.22. A sphere of radius a, centered at the origin, rotates about the z axis at angular velocity Ω rad/s. The rotation direction is clockwise when one is looking in the positive z direction.

a) Using spherical components, write an expression for the velocity field, $\mathbf{v}$, which gives the tangential velocity at any point within the sphere:

As in problem 1.20, we find the tangential velocity as the product of the angular velocity and the perperdicular distance from the rotation axis. With clockwise rotation, we obtain

$$\mathbf{v}(r, \theta) = \underline{\Omega r \sin\theta \, \mathbf{a}_\phi \ (r < a)}$$

b) Convert to rectangular components:

From here, the problem is the same as part c in Problem 1.20, except the rotation direction is reversed. The answer is $\mathbf{v}(x, y) = \underline{\Omega\,[-y\,\mathbf{a}_x + x\,\mathbf{a}_y]}$, where $(x^2 + y^2 + z^2)^{1/2} < a$.

1.23. The surfaces $\rho = 3$, $\rho = 5$, $\phi = 100^\circ$, $\phi = 130^\circ$, $z = 3$, and $z = 4.5$ define a closed surface.

a) Find the enclosed volume:

$$\text{Vol} = \int_3^{4.5} \int_{100^\circ}^{130^\circ} \int_3^5 \rho \, d\rho \, d\phi \, dz = \underline{6.28}$$

NOTE: The limits on the ϕ integration must be converted to radians (as was done here, but not shown).

b) Find the total area of the enclosing surface:

$$\text{Area} = 2 \int_{100^\circ}^{130^\circ} \int_3^5 \rho \, d\rho \, d\phi \; + \int_3^{4.5} \int_{100^\circ}^{130^\circ} 3 \, d\phi \, dz$$
$$+ \int_3^{4.5} \int_{100^\circ}^{130^\circ} 5 \, d\phi \, dz \; + \; 2 \int_3^{4.5} \int_3^5 d\rho \, dz = \underline{20.7}$$

c) Find the total length of the twelve edges of the surfaces:

$$\text{Length} = 4 \times 1.5 \; + \; 4 \times 2 \; + \; 2 \times \left[\frac{30^\circ}{360^\circ} \times 2\pi \times 3 \; + \; \frac{30^\circ}{360^\circ} \times 2\pi \times 5 \right] = \underline{22.4}$$

d) Find the length of the longest straight line that lies entirely within the volume: This will be between the points A($\rho = 3$, $\phi = 100^\circ$, $z = 3$) and B($\rho = 5$, $\phi = 130^\circ$, $z = 4.5$). Performing point transformations to cartesian coordinates, these become A($x = -0.52$, $y = 2.95$, $z = 3$) and B($x = -3.21$, $y = 3.83$, $z = 4.5$). Taking A and B as vectors directed from the origin, the requested length is

$$\text{Length} = |\mathbf{B} - \mathbf{A}| = |(-2.69, 0.88, 1.5)| = \underline{3.21}$$

1.24. Express the field $\mathbf{E} = A\mathbf{a}_r/r^2$ in

a) rectangular components:

$$E_x = \frac{A}{r^2}\,\mathbf{a}_r \cdot \mathbf{a}_x = \frac{A}{r^2}\sin\theta\cos\phi = \frac{A}{x^2+y^2+z^2}\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\frac{x}{\sqrt{x^2+y^2}} = \frac{Ax}{(x^2+y^2+z^2)^{3/2}}$$

$$E_y = \frac{A}{r^2}\,\mathbf{a}_r \cdot \mathbf{a}_y = \frac{A}{r^2}\sin\theta\sin\phi = \frac{A}{x^2+y^2+z^2}\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\frac{y}{\sqrt{x^2+y^2}} = \frac{Ay}{(x^2+y^2+z^2)^{3/2}}$$

$$E_z = \frac{A}{r^2}\,\mathbf{a}_r \cdot \mathbf{a}_z = \frac{A}{r^2}\cos\theta = \frac{A}{x^2+y^2+z^2}\frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{Az}{(x^2+y^2+z^2)^{3/2}}$$

Finally

$$\mathbf{E}(x,y,z) = \frac{A(x\,\mathbf{a}_x + y\,\mathbf{a}_y + z\,\mathbf{a}_z)}{(x^2+y^2+z^2)^{3/2}}$$

b) cylindrical components: First, there is no $\mathbf{a}_\phi$ component, since there is none in the spherical representation. What remains are:

$$E_\rho = \frac{A}{r^2}\,\mathbf{a}_r \cdot \mathbf{a}_\rho = \frac{A}{r^2}\,\sin\theta = \frac{A}{(\rho^2+z^2)}\frac{\rho}{\sqrt{\rho^2+z^2}} = \frac{A\rho}{(\rho^2+z^2)^{3/2}}$$

and

$$E_z = \frac{A}{r^2}\,\mathbf{a}_r \cdot \mathbf{a}_z = \frac{A}{r^2}\,\cos\theta = \frac{A}{(\rho^2+z^2)}\frac{z}{\sqrt{\rho^2+z^2}} = \frac{Az}{(\rho^2+z^2)^{3/2}}$$

Finally

$$\mathbf{E}(\rho,z) = \frac{A(\rho\,\mathbf{a}_\rho + z\,\mathbf{a}_z)}{(\rho^2+z^2)^{3/2}}$$

1.25. Given point $P(r = 0.8,\ \theta = 30^\circ,\ \phi = 45^\circ)$, and

$$\mathbf{E} = \frac{1}{r^2}\left(\cos\phi\,\mathbf{a}_r + \frac{\sin\phi}{\sin\theta}\,\mathbf{a}_\phi\right)$$

a) Find $\mathbf{E}$ at P: $\mathbf{E} = \underline{1.10\mathbf{a}_\rho + 2.21\mathbf{a}_\phi}$.

b) Find $|\mathbf{E}|$ at P: $|\mathbf{E}| = \sqrt{1.10^2 + 2.21^2} = \underline{2.47}$.

c) Find a unit vector in the direction of $\mathbf{E}$ at P:

$$\mathbf{a}_E = \frac{\mathbf{E}}{|\mathbf{E}|} = \underline{0.45\mathbf{a}_r + 0.89\mathbf{a}_\phi}$$

1.26. Express the uniform vector field, $\mathbf{F} = 5\,\mathbf{a}_x$ in

a) cylindrical components: $F_\rho = 5\,\mathbf{a}_x \cdot \mathbf{a}_\rho = 5\cos\phi$, and $F_\phi = 5\,\mathbf{a}_x \cdot \mathbf{a}_\phi = -5\sin\phi$. Combining, we obtain $\underline{\mathbf{F}(\rho,\phi) = 5(\cos\phi\,\mathbf{a}_\rho - \sin\phi\,\mathbf{a}_\phi)}$.

b) spherical components: $F_r = 5\,\mathbf{a}_x \cdot \mathbf{a}_r = 5\sin\theta\cos\phi$; $F_\theta = 5\,\mathbf{a}_x \cdot \mathbf{a}_\theta = 5\cos\theta\cos\phi$; $F_\phi = 5\,\mathbf{a}_x \cdot \mathbf{a}_\phi = -5\sin\phi$. Combining, we obtain $\underline{\mathbf{F}(r,\theta,\phi) = 5\left[\sin\theta\cos\phi\,\mathbf{a}_r + \cos\theta\cos\phi\,\mathbf{a}_\theta - \sin\phi\,\mathbf{a}_\phi\right]}$.

1.27. The surfaces $r = 2$ and 4, $\theta = 30^\circ$ and 50°, and $\phi = 20^\circ$ and 60° identify a closed surface.

a) Find the enclosed volume: This will be

$$\text{Vol} = \int_{20^\circ}^{60^\circ} \int_{30^\circ}^{50^\circ} \int_{2}^{4} r^2 \sin\theta dr d\theta d\phi = \underline{2.91}$$

where degrees have been converted to radians.

b) Find the total area of the enclosing surface:

$$\text{Area} = \int_{20^\circ}^{60^\circ} \int_{30^\circ}^{50^\circ} (4^2 + 2^2) \sin\theta d\theta d\phi + \int_{2}^{4} \int_{20^\circ}^{60^\circ} r(\sin 30^\circ + \sin 50^\circ) dr d\phi$$
$$+ 2 \int_{30^\circ}^{50^\circ} \int_{2}^{4} r dr d\theta = \underline{12.61}$$

c) Find the total length of the twelve edges of the surface:

$$\text{Length} = 4 \int_{2}^{4} dr \; + \; 2 \int_{30^\circ}^{50^\circ} (4+2) d\theta + \int_{20^\circ}^{60^\circ} (4\sin 50^\circ + 4\sin 30^\circ + 2\sin 50^\circ + 2\sin 30^\circ) d\phi$$
$$= \underline{17.49}$$

d) Find the length of the longest straight line that lies entirely within the surface: This will be from $A(r = 2, \theta = 50^\circ, \phi = 20^\circ)$ to $B(r = 4, \theta = 30^\circ, \phi = 60^\circ)$ or

$$A(x = 2\sin 50^\circ \cos 20^\circ, y = 2\sin 50^\circ \sin 20^\circ, z = 2\cos 50^\circ)$$

to

$$B(x = 4\sin 30^\circ \cos 60^\circ, y = 4\sin 30^\circ \sin 60^\circ, z = 4\cos 30^\circ)$$

or finally $A(1.44, 0.52, 1.29)$ to $B(1.00, 1.73, 3.46)$. Thus $\mathbf{B} - \mathbf{A} = (-0.44, 1.21, 2.18)$ and

$$\text{Length} = |\mathbf{B} - \mathbf{A}| = \underline{2.53}$$

1.28. Express the vector field, $\mathbf{G} = 8\sin\phi\, \mathbf{a}_\theta$ in

a) rectangular components:

$$G_x = 8\sin\phi\, \mathbf{a}_\theta \cdot \mathbf{a}_x = 8\sin\phi \cos\theta \cos\phi = \frac{8y}{\sqrt{x^2+y^2}} \frac{z}{\sqrt{x^2+y^2+z^2}} \frac{x}{\sqrt{x^2+y^2}}$$
$$= \frac{8xyz}{(x^2+y^2)\sqrt{x^2+y^2+z^2}}$$

$$G_y = 8\sin\phi\, \mathbf{a}_\theta \cdot \mathbf{a}_y = 8\sin\phi \cos\theta \sin\phi = \frac{8y}{\sqrt{x^2+y^2}} \frac{z}{\sqrt{x^2+y^2+z^2}} \frac{y}{\sqrt{x^2+y^2}}$$
$$= \frac{8y^2 z}{(x^2+y^2)\sqrt{x^2+y^2+z^2}}$$

1.28a) (continued)

$$G_z = 8\sin\phi\,\mathbf{a}_\theta\cdot\mathbf{a}_z = 8\sin\phi(-\sin\theta) = \frac{-8y}{\sqrt{x^2+y^2}}\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$
$$= \frac{-8y}{\sqrt{x^2+y^2+z^2}}$$

Finally,

$$\mathbf{G}(x,y,z) = \frac{8y}{\sqrt{x^2+y^2+z^2}}\left[\frac{xz}{x^2+y^2}\,\mathbf{a}_x + \frac{yz}{x^2+y^2}\,\mathbf{a}_y \;-\; \mathbf{a}_z\right]$$

b) cylindrical components: The $\mathbf{a}_\theta$ direction will transform to cylindrical components in the $\mathbf{a}_\rho$ and $\mathbf{a}_z$ directions only, where

$$G_\rho = 8\sin\phi\,\mathbf{a}_\theta\cdot\mathbf{a}_\rho = 8\sin\phi\cos\theta = 8\sin\phi\frac{z}{\sqrt{\rho^2+z^2}}$$

The z component will be the same as found in part a, so we finally obtain

$$\mathbf{G}(\rho,z) = \frac{8\rho\sin\phi}{\sqrt{\rho^2+z^2}}\left[\frac{z}{\rho}\,\mathbf{a}_\rho \;-\; \mathbf{a}_z\right]$$

1.29. Express the unit vector $\mathbf{a}_x$ in spherical components at the point:

a) $r = 2$, $\theta = 1$ rad, $\phi = 0.8$ rad: Use

$$\mathbf{a}_x = (\mathbf{a}_x\cdot\mathbf{a}_r)\mathbf{a}_r + (\mathbf{a}_x\cdot\mathbf{a}_\theta)\mathbf{a}_\theta + (\mathbf{a}_x\cdot\mathbf{a}_\phi)\mathbf{a}_\phi =$$
$$\sin(1)\cos(0.8)\mathbf{a}_r + \cos(1)\cos(0.8)\mathbf{a}_\theta + (-\sin(0.8))\mathbf{a}_\phi = \underline{0.59\mathbf{a}_r + 0.38\mathbf{a}_\theta - 0.72\mathbf{a}_\phi}$$

b) $x = 3$, $y = 2$, $z = -1$: First, transform the point to spherical coordinates. Have $r = \sqrt{14}$, $\theta = \cos^{-1}(-1/\sqrt{14}) = 105.5°$, and $\phi = \tan^{-1}(2/3) = 33.7°$. Then

$$\mathbf{a}_x = \sin(105.5°)\cos(33.7°)\mathbf{a}_r + \cos(105.5°)\cos(33.7°)\mathbf{a}_\theta + (-\sin(33.7°))\mathbf{a}_\phi$$
$$= \underline{0.80\mathbf{a}_r - 0.22\mathbf{a}_\theta - 0.55\mathbf{a}_\phi}$$

c) $\rho = 2.5$, $\phi = 0.7$ rad, $z = 1.5$: Again, convert the point to spherical coordinates. $r = \sqrt{\rho^2+z^2} = \sqrt{8.5}$, $\theta = \cos^{-1}(z/r) = \cos^{-1}(1.5/\sqrt{8.5}) = 59.0°$, and $\phi = 0.7\,\text{rad} = 40.1°$. Now

$$\mathbf{a}_x = \sin(59°)\cos(40.1°)\mathbf{a}_r + \cos(59°)\cos(40.1°)\mathbf{a}_\theta + (-\sin(40.1°))\mathbf{a}_\phi$$
$$= \underline{0.66\mathbf{a}_r + 0.39\mathbf{a}_\theta - 0.64\mathbf{a}_\phi}$$

1.30. At point $B(5, 120°, 75°)$ a vector field has the value $\mathbf{A} = -12\,\mathbf{a}_r - 5\,\mathbf{a}_\theta + 15\,\mathbf{a}_\phi$. Find the vector component of $\mathbf{A}$ that is:

a) normal to the surface $r = 5$: This will just be the radial component, or $\underline{-12\,\mathbf{a}_r}$.

b) tangent to the surface $r = 5$: This will be the remaining components of $\mathbf{A}$ that are not normal, or $\underline{-5\,\mathbf{a}_\theta + 15\,\mathbf{a}_\phi}$.

c) tangent to the cone $\theta = 120°$: The unit vector normal to the cone is $\mathbf{a}_\theta$, so the remaining components are tangent: $\underline{-12\,\mathbf{a}_r + 15\,\mathbf{a}_\phi}$.

d) Find a unit vector that is perpendicular to $\mathbf{A}$ and tangent to the cone $\theta = 120°$: Call this vector $\mathbf{b} = b_r\,\mathbf{a}_r + b_\phi\,\mathbf{a}_\phi$, where $b_r^2 + b_\phi^2 = 1$. We then require that $\mathbf{A}\cdot\mathbf{b} = 0 = -12b_r + 15b_\phi$, and therefore $b_\phi = (4/5)b_r$. Now $b_r^2[1 + (16/25)] = 1$, so $b_r = 5/\sqrt{41}$. Then $b_\phi = 4/\sqrt{41}$. Finally, $\underline{\mathbf{b} = \left(1/\sqrt{41}\right)(5\,\mathbf{a}_r + 4\,\mathbf{a}_\phi)}$