CHAPTER 2

2.1. Four 10nC positive charges are located in the $z = 0$ plane at the corners of a square 8cm on a side. A fifth 10nC positive charge is located at a point 8cm distant from the other charges. Calculate the magnitude of the total force on this fifth charge for $\epsilon = \epsilon_0$:

Arrange the charges in the xy plane at locations (4,4), (4,-4), (-4,4), and (-4,-4). Then the fifth charge will be on the z axis at location $z = 4\sqrt{2}$, which puts it at 8cm distance from the other four. By symmetry, the force on the fifth charge will be z-directed, and will be four times the z component of force produced by each of the four other charges.

$$F = \frac{4}{\sqrt{2}} \times \frac{q^2}{4\pi\epsilon_0 d^2} = \frac{4}{\sqrt{2}} \times \frac{(10^{-8})^2}{4\pi(8.85 \times 10^{-12})(0.08)^2} = \underline{4.0 \times 10^{-4}\ \text{N}}$$

2.2. Two point charges of Q_1 coulombs each are located at (0,0,1) and (0,0,-1). (a) Determine the locus of the possible positions of a third charge Q_2 where Q_2 may be any positive or negative value, such that the total field $\mathbf{E} = 0$ at (0,1,0):

The total field at (0,1,0) from the two Q_1 charges (where both are positive) will be

$$\mathbf{E}_1(0,1,0) = \frac{2Q_1}{4\pi\epsilon_0 R^2} \cos 45^\circ\, \mathbf{a}_y = \frac{Q_1}{4\sqrt{2}\pi\epsilon_0}\, \mathbf{a}_y$$

where $R = \sqrt{2}$. To cancel this field, Q_2 must be placed on the y axis at positions $y > 1$ if $Q_2 > 0$, and at positions $y < 1$ if $Q_2 < 0$. In either case the field from Q_2 will be

$$\mathbf{E}_2(0,1,0) = \frac{-|Q_2|}{4\pi\epsilon_0}\, \mathbf{a}_y$$

and the total field is then

$$\mathbf{E}_t = \mathbf{E}_1 + \mathbf{E}_2 = \left[\frac{Q_1}{4\sqrt{2}\pi\epsilon_0} - \frac{|Q_2|}{4\pi\epsilon_0}\right] = 0$$

Therefore

$$\frac{Q_1}{\sqrt{2}} = \frac{|Q_2|}{(y-1)^2} \;\Rightarrow\; \underline{y = 1 \pm 2^{1/4}\sqrt{\frac{|Q_2|}{Q_1}}}$$

where the plus sign is used if $Q_2 > 0$, and the minus sign is used if $Q_2 < 0$.

(b) What is the locus if the two original charges are Q_1 and $-Q_1$?

In this case the total field at (0,1,0) is $\mathbf{E}_1(0,1,0) = -Q_1/(4\sqrt{2}\pi\epsilon_0)\,\mathbf{a}_z$, where the positive Q_1 is located at the positive z (= 1) value. We now need Q_2 to lie along the line $x = 0$, $y = 1$ in order to cancel the field from the positive and negative Q_1 charges. Assuming Q_2 is located at $(0, 1, z)$, the total field is now

$$\mathbf{E}_t = \mathbf{E}_1 + \mathbf{E}_2 = \frac{-Q_1}{4\sqrt{2}\pi\epsilon_0}\, \mathbf{a}_z + \frac{|Q_2|}{4\pi\epsilon_0 z^2} = 0$$

or $\underline{z = \pm 2^{1/4}\sqrt{|Q_2|/Q_1}}$, where the plus sign is used if $Q_2 < 0$, and the minus sign if $Q_2 > 0$.

2.3. Point charges of 50nC each are located at $A(1,0,0)$, $B(-1,0,0)$, $C(0,1,0)$, and $D(0,-1,0)$ in free space. Find the total force on the charge at A.

The force will be:

$$\mathbf{F} = \frac{(50 \times 10^{-9})^2}{4\pi\epsilon_0} \left[\frac{\mathbf{R}_{CA}}{|\mathbf{R}_{CA}|^3} + \frac{\mathbf{R}_{DA}}{|\mathbf{R}_{DA}|^3} + \frac{\mathbf{R}_{BA}}{|\mathbf{R}_{BA}|^3} \right]$$

where $\mathbf{R}_{CA} = \mathbf{a}_x - \mathbf{a}_y$, $\mathbf{R}_{DA} = \mathbf{a}_x + \mathbf{a}_y$, and $\mathbf{R}_{BA} = 2\mathbf{a}_x$. The magnitudes are $|\mathbf{R}_{CA}| = |\mathbf{R}_{DA}| = \sqrt{2}$, and $|\mathbf{R}_{BA}| = 2$. Substituting these leads to

$$\mathbf{F} = \frac{(50 \times 10^{-9})^2}{4\pi\epsilon_0} \left[\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + \frac{2}{8} \right] \mathbf{a}_x = \underline{21.5\mathbf{a}_x \ \mu\text{N}}$$

where distances are in meters.

2.4. Eight identical point charges of Q C each are located at the corners of a cube of side length a, with one charge at the origin, and with the three nearest charges at $(a,0,0)$, $(0,a,0)$, and $(0,0,a)$. Find an expression for the total vector force on the charge at $P(a,a,a)$, assuming free space:

The total electric field at $P(a,a,a)$ that produces a force on the charge there will be the sum of the fields from the other seven charges. This is written below, where the charge locations associated with each term are indicated:

$$\mathbf{E}_{net}(a,a,a) = \frac{q}{4\pi\epsilon_0 a^2} \left[\underbrace{\frac{\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z}{3\sqrt{3}}}_{(0,0,0)} + \underbrace{\frac{\mathbf{a}_y + \mathbf{a}_z}{2\sqrt{2}}}_{(a,0,0)} + \underbrace{\frac{\mathbf{a}_x + \mathbf{a}_z}{2\sqrt{2}}}_{(0,a,0)} + \underbrace{\frac{\mathbf{a}_x + \mathbf{a}_y}{2\sqrt{2}}}_{(0,0,a)} + \underbrace{\mathbf{a}_x}_{(0,a,a)} + \underbrace{\mathbf{a}_y}_{(a,0,a)} + \underbrace{\mathbf{a}_z}_{(a,a,0)} \right]$$

The force is now the product of this field and the charge at (a,a,a). Simplifying, we obtain

$$\mathbf{F}(a,a,a) = q\mathbf{E}_{net}(a,a,a) = \frac{q^2}{4\pi\epsilon_0 a^2} \left[\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{2}} + 1 \right] (\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z) = \underline{\frac{1.90\, q^2}{4\pi\epsilon_0 a^2} (\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}$$

in which the magnitude is $|\mathbf{F}| = 3.29\, q^2/(4\pi\epsilon_0 a^2)$.

2.5. Let a point charge $Q_1 25$ nC be located at $P_1(4,-2,7)$ and a charge $Q_2 = 60$ nC be at $P_2(-3,4,-2)$.

a) If $\epsilon = \epsilon_0$, find $\mathbf{E}$ at $P_3(1,2,3)$: This field will be

$$\mathbf{E} = \frac{10^{-9}}{4\pi\epsilon_0} \left[\frac{25\mathbf{R}_{13}}{|\mathbf{R}_{13}|^3} + \frac{60\mathbf{R}_{23}}{|\mathbf{R}_{23}|^3} \right]$$

where $\mathbf{R}_{13} = -3\mathbf{a}_x + 4\mathbf{a}_y - 4\mathbf{a}_z$ and $\mathbf{R}_{23} = 4\mathbf{a}_x - 2\mathbf{a}_y + 5\mathbf{a}_z$. Also, $|\mathbf{R}_{13}| = \sqrt{41}$ and $|\mathbf{R}_{23}| = \sqrt{45}$. So

$$\mathbf{E} = \frac{10^{-9}}{4\pi\epsilon_0} \left[\frac{25 \times (-3\mathbf{a}_x + 4\mathbf{a}_y - 4\mathbf{a}_z)}{(41)^{1.5}} + \frac{60 \times (4\mathbf{a}_x - 2\mathbf{a}_y + 5\mathbf{a}_z)}{(45)^{1.5}} \right]$$
$$= \underline{4.58\mathbf{a}_x - 0.15\mathbf{a}_y + 5.51\mathbf{a}_z}$$

b) At what point on the y axis is $E_x = 0$? P_3 is now at $(0,y,0)$, so $\mathbf{R}_{13} = -4\mathbf{a}_x + (y+2)\mathbf{a}_y - 7\mathbf{a}_z$ and $\mathbf{R}_{23} = 3\mathbf{a}_x + (y-4)\mathbf{a}_y + 2\mathbf{a}_z$. Also, $|\mathbf{R}_{13}| = \sqrt{65 + (y+2)^2}$ and $|\mathbf{R}_{23}| = \sqrt{13 + (y-4)^2}$. Now the x component of $\mathbf{E}$ at the new P_3 will be:

$$E_x = \frac{10^{-9}}{4\pi\epsilon_0} \left[\frac{25 \times (-4)}{[65 + (y+2)^2]^{1.5}} + \frac{60 \times 3}{[13 + (y-4)^2]^{1.5}} \right]$$

To obtain $E_x = 0$, we require the expression in the large brackets to be zero. This expression simplifies to the following quadratic:

$$0.48y^2 + 13.92y + 73.10 = 0$$

which yields the two values: $y = \underline{-6.89, -22.11}$

2.6. Three point charges, each 5×10^{-9} C, are located on the x axis at $x = -1$, 0, and 1 in free space.

a) Find $\mathbf{E}$ at $x = 5$: At a general location, x,

$$\mathbf{E}(x) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(x+1)^2} + \frac{1}{x^2} + \frac{1}{(x-1)^2}\right]\mathbf{a}_x$$

At $x = 5$, and with $q = 5 \times 10^{-9}$ C, this becomes $\mathbf{E}(x = 5) = \underline{5.8\,\mathbf{a}_x \text{ V/m}}$.

b) Determine the value and location of the equivalent single point charge that would produce the same field at very large distances: For $x >> 1$, the above general field in part a becomes

$$\mathbf{E}(x >> 1) \doteq \frac{3q}{4\pi\epsilon_0 x^2}\,\mathbf{a}_x$$

Therefore, the equivalent charge will have value $\underline{3q = 1.5 \times 10^{-8} \text{ C}}$, and will be at location $\underline{x = 0}$.

c) Determine $\mathbf{E}$ at $x = 5$, using the approximation of (b). Using $3q = 1.5 \times 10^{-8}$ C and $x = 5$ in the part b result gives $\mathbf{E}(x = 5) \doteq \underline{5.4\,\mathbf{a}_x \text{ V/m}}$, or about 7% lower than the exact result.

2.7. A 2 μC point charge is located at $A(4, 3, 5)$ in free space. Find E_ρ, E_ϕ, and E_z at $P(8, 12, 2)$. Have

$$\mathbf{E}_P = \frac{2 \times 10^{-6}}{4\pi\epsilon_0}\frac{\mathbf{R}_{AP}}{|\mathbf{R}_{AP}|^3} = \frac{2 \times 10^{-6}}{4\pi\epsilon_0}\left[\frac{4\mathbf{a}_x + 9\mathbf{a}_y - 3\mathbf{a}_z}{(106)^{1.5}}\right] = 65.9\mathbf{a}_x + 148.3\mathbf{a}_y - 49.4\mathbf{a}_z$$

Then, at point P, $\rho = \sqrt{8^2 + 12^2} = 14.4$, $\phi = \tan^{-1}(12/8) = 56.3°$, and $z = z$. Now,

$$E_\rho = \mathbf{E}_p \cdot \mathbf{a}_\rho = 65.9(\mathbf{a}_x \cdot \mathbf{a}_\rho) + 148.3(\mathbf{a}_y \cdot \mathbf{a}_\rho) = 65.9\cos(56.3°) + 148.3\sin(56.3°) = \underline{159.7}$$

and

$$E_\phi = \mathbf{E}_p \cdot \mathbf{a}_\phi = 65.9(\mathbf{a}_x \cdot \mathbf{a}_\phi) + 148.3(\mathbf{a}_y \cdot \mathbf{a}_\phi) = -65.9\sin(56.3°) + 148.3\cos(56.3°) = \underline{27.4}$$

Finally, $E_z = \underline{-49.4 \text{ V/m}}$

2.8. A crude device for measuring charge consists of two small insulating spheres of radius a, one of which is fixed in position. The other is movable along the x axis, and is subject to a restraining force kx, where k is a spring constant. The uncharged spheres are centered at $x = 0$ and $x = d$, the latter fixed. If the spheres are given equal and opposite charges of Q coulombs:

a) Obtain the expression by which Q may be found as a function of x: The spheres will attract, and so the movable sphere at $x = 0$ will move toward the other until the spring and Coulomb forces balance. This will occur at location x for the movable sphere. With equal and opposite forces, we have

$$\frac{Q^2}{4\pi\epsilon_0(d - x)^2} = kx$$

from which $\underline{Q = 2(d-x)\sqrt{\pi\epsilon_0 k x}}$.

b) Determine the maximum charge that can be measured in terms of ϵ_0, k, and d, and state the separation of the spheres then: With increasing charge, the spheres move toward each other until they just touch at $\underline{x_{max} = d - 2a}$. Using the part a result, we find the maximum measurable charge: $Q_{max} = \underline{4a\sqrt{\pi\epsilon_0 k(d-2a)}}$. Presumably some form of stop mechanism is placed at $x = x_{max}^-$ to prevent the spheres from actually touching.

c) What happens if a larger charge is applied? No further motion is possible, so nothing happens.

2.9. A 100 nC point charge is located at $A(-1, 1, 3)$ in free space.

a) Find the locus of all points $P(x, y, z)$ at which $E_x = 500$ V/m: The total field at P will be:

$$\mathbf{E}_P = \frac{100 \times 10^{-9}}{4\pi\epsilon_0} \frac{\mathbf{R}_{AP}}{|\mathbf{R}_{AP}|^3}$$

where $\mathbf{R}_{AP} = (x+1)\mathbf{a}_x + (y-1)\mathbf{a}_y + (z-3)\mathbf{a}_z$, and where $|\mathbf{R}_{AP}| = [(x+1)^2 + (y-1)^2 + (z-3)^2]^{1/2}$. The x component of the field will be

$$E_x = \frac{100 \times 10^{-9}}{4\pi\epsilon_0} \left[\frac{(x+1)}{[(x+1)^2 + (y-1)^2 + (z-3)^2]^{1.5}} \right] = 500 \text{ V/m}$$

And so our condition becomes:

$$\underline{(x+1) = 0.56\,[(x+1)^2 + (y-1)^2 + (z-3)^2]^{1.5}}$$

b) Find y_1 if $P(-2, y_1, 3)$ lies on that locus: At point P, the condition of part a becomes

$$3.19 = \left[1 + (y_1 - 1)^2\right]^3$$

from which $(y_1 - 1)^2 = 0.47$, or $y_1 = \underline{1.69 \text{ or } 0.31}$

2.10. A positive test charge is used to explore the field of a single positive point charge Q at $P(a, b, c)$. If the test charge is placed at the origin, the force on it is in the direction $0.5\,\mathbf{a}_x - 0.5\sqrt{3}\,\mathbf{a}_y$, and when the test charge is moved to (1,0,0), the force is in the direction of $0.6\,\mathbf{a}_x - 0.8\,\mathbf{a}_y$. Find a, b, and c:

We first construct the field using the form of Eq. (12). We identify $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ and $\mathbf{r}' = a\mathbf{a}_x + b\mathbf{a}_y + c\mathbf{a}_z$. Then

$$\mathbf{E} = \frac{Q\,[(x-a)\,\mathbf{a}_x + (y-b)\,\mathbf{a}_y + (z-c)\,\mathbf{a}_z]}{4\pi\epsilon_0\,[(x-a)^2 + (y-b)^2 + (z-c)^2]^{3/2}} \tag{1}$$

Using (1), we can write the two force directions at the two test charge positions as follows:

$$\text{at } (0,0,0): \quad \frac{[-a\,\mathbf{a}_x - b\,\mathbf{a}_y - c\,\mathbf{a}_z]}{(a^2 + b^2 + c^2)^{1/2}} = 0.5\,\mathbf{a}_x - 0.5\sqrt{3}\,\mathbf{a}_y \tag{2}$$

$$\text{at } (1,0,0): \quad \frac{[(1-a)\,\mathbf{a}_x - b\,\mathbf{a}_y - c\,\mathbf{a}_z]}{((1-a)^2 + b^2 + c^2)^{1/2}} = 0.6\,\mathbf{a}_x - 0.8\,\mathbf{a}_y \tag{3}$$

We observe immediately that $c = 0$. Also, from (2) we find that $b = -a\sqrt{3}$, and therefore $\sqrt{a^2 + b^2} = 2a$. Using this information in (3), we write for the x component:

$$\frac{1-a}{\sqrt{(1-a)^2 + b^2}} = \frac{1-a}{\sqrt{1 - 2a + 4a^2}} = 0.6$$

or $0.44a^2 + 1.28a - 0.64 = 0$, so that

$$a = \frac{-1.28 \pm \sqrt{(1.28)^2 + 4(0.44)(0.64)}}{0.88} = 0.435 \text{ or } -3.344$$

The corresponding b values are respectively -0.753 and 5.793. So the two possible P coordinate sets are $(0.435, -0.753, 0)$ and $(-3.344, 5.793, 0)$. By direct substitution, however, it is found that only one possibility is entirely consistent with both (2) and (3), and this is

$$P(a, b, c) = \underline{(-3.344, 5.793, 0)}$$

2.11. A charge Q_0 located at the origin in free space produces a field for which $E_z = 1$ kV/m at point $P(-2, 1, -1)$.

a) Find Q_0: The field at P will be

$$\mathbf{E}_P = \frac{Q_0}{4\pi\epsilon_0}\left[\frac{-2\mathbf{a}_x + \mathbf{a}_y - \mathbf{a}_z}{6^{1.5}}\right]$$

Since the z component is of value 1 kV/m, we find $Q_0 = -4\pi\epsilon_0 6^{1.5} \times 10^3 = \underline{-1.63\ \mu\text{C}}$.

b) Find $\mathbf{E}$ at $M(1, 6, 5)$ in cartesian coordinates: This field will be:

$$\mathbf{E}_M = \frac{-1.63 \times 10^{-6}}{4\pi\epsilon_0}\left[\frac{\mathbf{a}_x + 6\mathbf{a}_y + 5\mathbf{a}_z}{[1 + 36 + 25]^{1.5}}\right]$$

or $\mathbf{E}_M = \underline{-30.11\mathbf{a}_x - 180.63\mathbf{a}_y - 150.53\mathbf{a}_z}$.

c) Find $\mathbf{E}$ at $M(1, 6, 5)$ in cylindrical coordinates: At M, $\rho = \sqrt{1 + 36} = 6.08$, $\phi = \tan^{-1}(6/1) = 80.54^\circ$, and $z = 5$. Now

$$E_\rho = \mathbf{E}_M \cdot \mathbf{a}_\rho = -30.11\cos\phi - 180.63\sin\phi = -183.12$$

$$E_\phi = \mathbf{E}_M \cdot \mathbf{a}_\phi = -30.11(-\sin\phi) - 180.63\cos\phi = 0 \quad \text{(as expected)}$$

so that $\mathbf{E}_M = \underline{-183.12\mathbf{a}_\rho - 150.53\mathbf{a}_z}$.

d) Find $\mathbf{E}$ at $M(1, 6, 5)$ in spherical coordinates: At M, $r = \sqrt{1 + 36 + 25} = 7.87$, $\phi = 80.54^\circ$ (as before), and $\theta = \cos^{-1}(5/7.87) = 50.58^\circ$. Now, since the charge is at the origin, we expect to obtain only a radial component of $\mathbf{E}_M$. This will be:

$$E_r = \mathbf{E}_M \cdot \mathbf{a}_r = -30.11\sin\theta\cos\phi - 180.63\sin\theta\sin\phi - 150.53\cos\theta = \underline{-237.1}$$

2.12. Electrons are in random motion in a fixed region in space. During any 1μs interval, the probability of finding an electron in a subregion of volume 10^{-15} m^2 is 0.27. What volume charge density, appropriate for such time durations, should be assigned to that subregion?

The finite probabilty effectively reduces the net charge quantity by the probability fraction. With $e = -1.602 \times 10^{-19}$ C, the density becomes

$$\rho_v = -\frac{0.27 \times 1.602 \times 10^{-19}}{10^{-15}} = \underline{-43.3\ \mu\text{C/m}^3}$$

2.13. A uniform volume charge density of 0.2 μC/m^3 is present throughout the spherical shell extending from $r = 3$ cm to $r = 5$ cm. If $\rho_v = 0$ elsewhere:

a) find the total charge present throughout the shell: This will be

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_{.03}^{.05} 0.2\, r^2 \sin\theta\ dr\, d\theta\, d\phi = \left[4\pi(0.2)\frac{r^3}{3}\right]_{.03}^{.05} = 8.21 \times 10^{-5}\ \mu\text{C} = \underline{82.1\ \text{pC}}$$

b) find r_1 if half the total charge is located in the region $3\,\text{cm} < r < r_1$: If the integral over r in part a is taken to r_1, we would obtain

$$\left[4\pi(0.2)\frac{r^3}{3}\right]_{.03}^{r_1} = 4.105 \times 10^{-5}$$

Thus

$$r_1 = \left[\frac{3 \times 4.105 \times 10^{-5}}{0.2 \times 4\pi} + (.03)^3\right]^{1/3} = \underline{4.24\ \text{cm}}$$

2.14. The charge density varies with radius in a cylindrical coordinate system as $\rho_v = \rho_0/(\rho^2 + a^2)^2$ C/m^3. Within what distance from the z axis does half the total charge lie?

Choosing a unit length in z, the charge contained up to radius ρ is

$$Q(\rho) = \int_0^1\int_0^{2\pi}\int_0^{\rho} \frac{\rho_0}{(\rho'^2 + a^2)^2}\, \rho' d\rho' d\phi dz = 2\pi\rho_0\left[\frac{-1}{2(a^2 + \rho'2)}\right]_0^{\rho} = \frac{\pi\rho_0}{a^2}\left[1 - \frac{1}{1 + \rho^2/a^2}\right]$$

The total charge is found when $\rho \to \infty$, or $Q_{net} = \pi\rho_0/a^2$. It is seen from the $Q(\rho)$ expression that half of this occurs when $\underline{\rho = a}$.

2.15. A spherical volume having a 2 μm radius contains a uniform volume charge density of 10^{15} C/m^3.

a) What total charge is enclosed in the spherical volume?
This will be $Q = (4/3)\pi(2 \times 10^{-6})^3 \times 10^{15} = \underline{3.35 \times 10^{-2}\ \text{C}}$.

b) Now assume that a large region contains one of these little spheres at every corner of a cubical grid 3mm on a side, and that there is no charge between spheres. What is the average volume charge density throughout this large region? Each cube will contain the equivalent of one little sphere. Neglecting the little sphere volume, the average density becomes

$$\rho_{v,avg} = \frac{3.35 \times 10^{-2}}{(0.003)^3} = \underline{1.24 \times 10^6\ \text{C/m}^3}$$

2.16. Within a region of free space, charge density is given as $\rho_v = \rho_0 r/a$ C/m^3, where ρ_0 and a are constants. Find the total charge lying within:

a) the sphere, $r \leq a$: This will be

$$Q_a = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{\rho_0 r}{a} r^2 \sin\theta\, dr\, d\theta\, d\phi = 4\pi \int_0^a \frac{\rho_0 r^3}{a}\, dr = \underline{\pi \rho_0 a^3}$$

b) the cone, $r \leq a$, $0 \leq \theta \leq 0.1\pi$:

$$Q_b = \int_0^{2\pi}\int_0^{0.1\pi}\int_0^{a} \frac{\rho_0 r}{a} r^2 \sin\theta\, dr\, d\theta\, d\phi = 2\pi \frac{\rho_0 a^3}{4}\left[1 - \cos(0.1\pi)\right] = \underline{0.024\pi\rho_0 a^3}$$

c) the region, $r \leq a$, $0 \leq \theta \leq 0.1\pi$, $0 \leq \phi \leq 0.2\pi$.

$$Q_c = \int_0^{0.2\pi}\int_0^{0.1\pi}\int_0^{a} \frac{\rho_0 r}{a} r^2 \sin\theta\, dr\, d\theta\, d\phi = 0.024\pi\rho_0 a^3 \left(\frac{0.2\pi}{2\pi}\right) = \underline{0.0024\pi\rho_0 a^3}$$

2.17. A uniform line charge of 16 nC/m is located along the line defined by $y = -2$, $z = 5$. If $\epsilon = \epsilon_0$:

a) Find $\mathbf{E}$ at $P(1, 2, 3)$: This will be

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0}\frac{\mathbf{R}_P}{|\mathbf{R}_P|^2}$$

where $\mathbf{R}_P = (1, 2, 3) - (1, -2, 5) = (0, 4, -2)$, and $|\mathbf{R}_P|^2 = 20$. So

$$\mathbf{E}_P = \frac{16 \times 10^{-9}}{2\pi\epsilon_0}\left[\frac{4\mathbf{a}_y - 2\mathbf{a}_z}{20}\right] = \underline{57.5\mathbf{a}_y - 28.8\mathbf{a}_z \text{ V/m}}$$

b) Find $\mathbf{E}$ at that point in the $z = 0$ plane where the direction of $\mathbf{E}$ is given by $(1/3)\mathbf{a}_y - (2/3)\mathbf{a}_z$: With $z = 0$, the general field will be

$$\mathbf{E}_{z=0} = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{(y+2)\mathbf{a}_y - 5\mathbf{a}_z}{(y+2)^2 + 25}\right]$$

We require $|E_z| = -|2E_y|$, so $2(y + 2) = 5$. Thus $y = 1/2$, and the field becomes:

$$\mathbf{E}_{z=0} = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{2.5\mathbf{a}_y - 5\mathbf{a}_z}{(2.5)^2 + 25}\right] = \underline{23\mathbf{a}_y - 46\mathbf{a}_z}$$

2.18. An infinite uniform line charge $\rho_L = 2$ nC/m lies along the x axis in free space, while point charges of 8 nC each are located at (0,0,1) and (0,0,-1).

a) Find $\mathbf{E}$ at (2,3,-4).

The net electric field from the line charge, the point charge at $z = 1$, and the point charge at $z = -1$ will be (in that order):

$$\mathbf{E}_{tot} = \frac{1}{4\pi\epsilon_0}\left[\frac{2\rho_L(3\,\mathbf{a}_y - 4\,\mathbf{a}_z)}{25} + \frac{q(2\,\mathbf{a}_x + 3\,\mathbf{a}_y - 5\,\mathbf{a}_z)}{(38)^{3/2}} + \frac{q(2\,\mathbf{a}_x + 3\,\mathbf{a}_y - 3\,\mathbf{a}_z)}{(22)^{3/2}}\right]$$

Then, with the given values of ρ_L and q, the field evaluates as

$$\mathbf{E}_{tot} = \underline{2.0\,\mathbf{a}_x + 7.3\,\mathbf{a}_y - 9.4\,\mathbf{a}_z \text{ V/m}}$$

b) To what value should ρ_L be changed to cause $\mathbf{E}$ to be zero at (0,0,3)?
In this case, we only need scalar addition to find the net field:

$$E(0,0,3) = \frac{\rho_L}{2\pi\epsilon_0(3)} + \frac{q}{4\pi\epsilon_0(2)^2} + \frac{q}{4\pi\epsilon_0(4)^2} = 0$$

Therefore

$$q\left[\frac{1}{4} + \frac{1}{16}\right] = -\frac{2\rho_L}{3} \quad \Rightarrow \quad \rho_L = -\frac{15}{32}q = -0.47q = \underline{-3.75 \text{ nC/m}}$$

2.19. A uniform line charge of 2 μC/m is located on the z axis. Find $\mathbf{E}$ in cartesian coordinates at $P(1,2,3)$ if the charge extends from

a) $-\infty < z < \infty$: With the infinite line, we know that the field will have only a radial component in cylindrical coordinates (or x and y components in cartesian). The field from an infinite line on the z axis is generally $\mathbf{E} = [\rho_l/(2\pi\epsilon_0\rho)]\mathbf{a}_\rho$. Therefore, at point P:

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0}\frac{\mathbf{R}_{zP}}{|\mathbf{R}_{zP}|^2} = \frac{(2\times 10^{-6})}{2\pi\epsilon_0}\frac{\mathbf{a}_x + 2\mathbf{a}_y}{5} = \underline{7.2\mathbf{a}_x + 14.4\mathbf{a}_y \text{ kV/m}}$$

where $\mathbf{R}_{zP}$ is the vector that extends from the line charge to point P, and is perpendicular to the z axis; i.e., $\mathbf{R}_{zP} = (1,2,3) - (0,0,3) = (1,2,0)$.

b) $-4 \le z \le 4$: Here we use the general relation

$$\mathbf{E}_P = \int \frac{\rho_l dz}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

where $\mathbf{r} = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{r}' = z\mathbf{a}_z$. So the integral becomes

$$\mathbf{E}_P = \frac{(2\times 10^{-6})}{4\pi\epsilon_0}\int_{-4}^{4}\frac{\mathbf{a}_x + 2\mathbf{a}_y + (3-z)\mathbf{a}_z}{[5+(3-z)^2]^{1.5}}\,dz$$

Using integral tables, we obtain:

$$\mathbf{E}_P = 3597\left[\frac{(\mathbf{a}_x + 2\mathbf{a}_y)(z-3) + 5\mathbf{a}_z}{(z^2 - 6z + 14)}\right]_{-4}^{4} \text{ V/m} = \underline{4.9\mathbf{a}_x + 9.8\mathbf{a}_y + 4.9\mathbf{a}_z \text{ kV/m}}$$

The student is invited to verify that when evaluating the above expression over the limits $-\infty < z < \infty$, the z component vanishes and the x and y components become those found in part a.

2.20. The portion of the z axis for which $|z| < 2$ carries a nonuniform line charge density of $10|z|$ nC/m, and $\rho_L = 0$ elsewhere. Determine $\mathbf{E}$ in free space at:

a) (0,0,4): The general form for the differential field at (0,0,4) is

$$d\mathbf{E} = \frac{\rho_L\, dz\, (\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$

where $\mathbf{r} = 4\,\mathbf{a}_z$ and $\mathbf{r}' = z\,\mathbf{a}_z$. Therefore, $\mathbf{r} - \mathbf{r}' = (4 - z)\,\mathbf{a}_z$ and $|\mathbf{r} - \mathbf{r}'| = 4 - z$. Substituting $\rho_L = 10|z|$ nC/m, the total field is

$$\begin{aligned}\mathbf{E}(0,0,4) &= \int_{-2}^{2} \frac{10^{-8}|z|\, dz\, \mathbf{a}_z}{4\pi\epsilon_0 (4 - z)^2} = \int_{0}^{2} \frac{10^{-8} z\, dz\, \mathbf{a}_z}{4\pi\epsilon_0 (4 - z)^2} - \int_{-2}^{0} \frac{10^{-8} z\, dz\, \mathbf{a}_z}{4\pi\epsilon_0 (4 - z)^2} \\ &= \frac{10^{-8}}{4\pi \times 8.854 \times 10^{-12}} \left\{ \left[\ln(4 - z) + \frac{4}{4 - z} \right]_0^2 - \left[\ln(4 - z) + \frac{4}{4 - z} \right]_{-2}^0 \right\} \mathbf{a}_z \\ &= \underline{34.0\,\mathbf{a}_z \text{ V/m}}\end{aligned}$$

b) (0,4,0): In this case, $\mathbf{r} = 4\,\mathbf{a}_y$ and $\mathbf{r}' = z\,\mathbf{a}_z$ as before. The field at (0,4,0) is then

$$\mathbf{E}(0,4,0) = \int_{-2}^{2} \frac{10^{-8}|z|\, dz\, (4\,\mathbf{a}_y - z\,\mathbf{a}_z)}{4\pi\epsilon_0 (16 + z^2)^{3/2}}$$

Note the symmetric limits on the integral. As the z component of the integrand changes sign at $z = 0$, it will contribute equal and opposite portions to the overall integral, which will cancel completely (the z component integral has odd parity). This leaves only the y component integrand, which has even parity. The integral therefore simplifies to

$$\mathbf{E}(0,4,0) = 2 \int_{0}^{2} \frac{4 \times 10^{-8} z\, dz\, \mathbf{a}_y}{4\pi\epsilon_0 (16 + z^2)^{3/2}} = \frac{-2 \times 10^{-8}\,\mathbf{a}_y}{\pi \times 8.854 \times 10^{-12}} \left[\frac{1}{\sqrt{16 + z^2}} \right]_0^2 = \underline{18.98\,\mathbf{a}_y \text{ V/m}}$$

2.21. Two identical uniform line charges with $\rho_l = 75$ nC/m are located in free space at $x = 0$, $y = \pm 0.4$ m. What force per unit length does each line charge exert on the other? The charges are parallel to the z axis and are separated by 0.8 m. Thus the field from the charge at $y = -0.4$ evaluated at the location of the charge at $y = +0.4$ will be $\mathbf{E} = [\rho_l/(2\pi\epsilon_0(0.8))]\mathbf{a}_y$. The force on a differential length of the line at the positive y location is $d\mathbf{F} = dq\mathbf{E} = \rho_l dz\mathbf{E}$. Thus the force per unit length acting on the line at postive y arising from the charge at negative y is

$$\mathbf{F} = \int_0^1 \frac{\rho_l^2\, dz}{2\pi\epsilon_0 (0.8)}\,\mathbf{a}_y = 1.26 \times 10^{-4}\,\mathbf{a}_y \text{ N/m} = \underline{126\,\mathbf{a}_y\ \mu\text{N/m}}$$

The force on the line at negative y is of course the same, but with $-\mathbf{a}_y$.

2.22. Two identical uniform sheet charges with $\rho_s = 100$ nC/m^2 are located in free space at $z = \pm 2.0$ cm. What force per unit area does each sheet exert on the other?

The field from the top sheet is $\mathbf{E} = -\rho_s/(2\epsilon_0)\,\mathbf{a}_z$ V/m. The differential force produced by this field on the bottom sheet is the charge density on the bottom sheet times the differential area there, multiplied by the electric field from the top sheet: $d\mathbf{F} = \rho_s da\mathbf{E}$. The force per unit area is then just $\mathbf{F} = \rho_s\mathbf{E} = (100 \times 10^{-9})(-100 \times 10^{-9})/(2\epsilon_0)\,\mathbf{a}_z = \underline{-5.6 \times 10^{-4}\,\mathbf{a}_z \text{ N/m}^2}$.

2.23. Given the surface charge density, $\rho_s = 2\,\mu\text{C/m}^2$, in the region $\rho < 0.2$ m, $z = 0$, and is zero elsewhere, find $\mathbf{E}$ at:

a) $P_A(\rho = 0, z = 0.5)$: First, we recognize from symmetry that only a z component of $\mathbf{E}$ will be present. Considering a general point z on the z axis, we have $\mathbf{r} = z\mathbf{a}_z$. Then, with $\mathbf{r}' = \rho\mathbf{a}_\rho$, we obtain $\mathbf{r} - \mathbf{r}' = z\mathbf{a}_z - \rho\mathbf{a}_\rho$. The superposition integral for the z component of $\mathbf{E}$ will be:

$$E_{z,P_A} = \frac{\rho_s}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{0.2}\frac{z\,\rho\,d\rho\,d\phi}{(\rho^2+z^2)^{1.5}} = -\frac{2\pi\rho_s}{4\pi\epsilon_0}z\left[\frac{1}{\sqrt{z^2+\rho^2}}\right]_0^{0.2}$$
$$= \frac{\rho_s}{2\epsilon_0}z\left[\frac{1}{\sqrt{z^2}} - \frac{1}{\sqrt{z^2+0.04}}\right]$$

With $z = 0.5$ m, the above evaluates as $E_{z,P_A} = \underline{8.1\text{ kV/m}}$.

b) With z at -0.5 m, we evaluate the expression for E_z to obtain $E_{z,P_B} = \underline{-8.1\text{ kV/m}}$.

2.24. For the charged disk of Problem 2.23, show that:

a) the field along the z axis reduces to that of an infinite sheet charge at small values of z: In general, the field can be expressed as

$$E_z = \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+0.04}}\right]$$

At small z, this reduces to $E_z \doteq \rho_s/2\epsilon_0$, which is the infinite sheet charge field.

b) the z axis field reduces to that of a point charge at large values of z: The development is as follows:

$$E_z = \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+0.04}}\right] = \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{z}{z\sqrt{1+0.04/z^2}}\right] \doteq \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{1}{1+(1/2)(0.04)/z^2}\right]$$

where the last approximation is valid if $z >> .04$. Continuing:

$$E_z \doteq \frac{\rho_s}{2\epsilon_0}\left[1 - [1-(1/2)(0.04)/z^2]\right] = \frac{0.04\rho_s}{4\epsilon_0 z^2} = \frac{\pi(0.2)^2\rho_s}{4\pi\epsilon_0 z^2}$$

This the point charge field, where we identify $q = \pi(0.2)^2\rho_s$ as the total charge on the disk (which now looks like a point).

2.25. Find $\mathbf{E}$ at the origin if the following charge distributions are present in free space: point charge, 12 nC at $P(2, 0, 6)$; uniform line charge density, 3nC/m at $x = -2$, $y = 3$; uniform surface charge density, $0.2\,\text{nC/m}^2$ at $x = 2$. The sum of the fields at the origin from each charge in order is:

$$\mathbf{E} = \left[\frac{(12\times10^{-9})}{4\pi\epsilon_0}\frac{(-2\mathbf{a}_x - 6\mathbf{a}_z)}{(4+36)^{1.5}}\right] + \left[\frac{(3\times10^{-9})}{2\pi\epsilon_0}\frac{(2\mathbf{a}_x - 3\mathbf{a}_y)}{(4+9)}\right] - \left[\frac{(0.2\times10^{-9})\mathbf{a}_x}{2\epsilon_0}\right]$$
$$= \underline{-3.9\mathbf{a}_x - 12.4\mathbf{a}_y - 2.5\mathbf{a}_z\text{ V/m}}$$

2.26. An electric dipole (discussed in detail in Sec. 4.7) consists of two point charges of equal and opposite magnitude $\pm Q$ spaced by distance d. With the charges along the z axis at positions $z = \pm d/2$ (with the positive charge at the positive z location), the electric field in spherical coordinates is given by $\mathbf{E}(r,\theta) = \left[Qd/(4\pi\epsilon_0 r^3)\right][2\cos\theta\mathbf{a}_r + \sin\theta\mathbf{a}_\theta]$, where $r >> d$. Using rectangular coordinates, determine expressions for the vector force on a point charge of magnitude q:

a) at (0,0,z): Here, $\theta = 0$, $\mathbf{a}_r = \mathbf{a}_z$, and $r = z$. Therefore

$$\mathbf{F}(0,0,z) = \frac{qQd\,\mathbf{a}_z}{4\pi\epsilon_0 z^3}\ \text{N}$$

b) at (0,y,0): Here, $\theta = 90^\circ$, $\mathbf{a}_\theta = -\mathbf{a}_z$, and $r = y$. The force is

$$\mathbf{F}(0,y,0) = \frac{-qQd\,\mathbf{a}_z}{4\pi\epsilon_0 y^3}\ \text{N}$$

2.27. Given the electric field $\mathbf{E} = (4x - 2y)\mathbf{a}_x - (2x + 4y)\mathbf{a}_y$, find:

a) the equation of the streamline that passes through the point $P(2,3,-4)$: We write

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{-(2x+4y)}{(4x-2y)}$$

Thus

$$2(x\,dy + y\,dx) = y\,dy - x\,dx$$

or

$$2\,d(xy) = \frac{1}{2}\,d(y^2) - \frac{1}{2}\,d(x^2)$$

So

$$C_1 + 2xy = \frac{1}{2}y^2 - \frac{1}{2}x^2$$

or

$$y^2 - x^2 = 4xy + C_2$$

Evaluating at $P(2,3,-4)$, obtain:

$$9 - 4 = 24 + C_2, \text{ or } C_2 = -19$$

Finally, at P, the requested equation is

$$\underline{y^2 - x^2 = 4xy - 19}$$

b) a unit vector specifying the direction of $\mathbf{E}$ at $Q(3,-2,5)$: Have $\mathbf{E}_Q = [4(3) + 2(2)]\mathbf{a}_x - [2(3) - 4(2)]\mathbf{a}_y = 16\mathbf{a}_x + 2\mathbf{a}_y$. Then $|\mathbf{E}| = \sqrt{16^2 + 4} = 16.12$ So

$$\mathbf{a}_Q = \frac{16\mathbf{a}_x + 2\mathbf{a}_y}{16.12} = \underline{0.99\mathbf{a}_x + 0.12\mathbf{a}_y}$$

2.28 A field is given as $\mathbf{E} = 2xz^2\mathbf{a}_x + 2z(x^2+1)\mathbf{a}_z$. Find the equation of the streamline passing through the point (1,3,-1):

$$\frac{dz}{dx} = \frac{E_z}{E_x} = \frac{x^2+1}{xz} \;\Rightarrow\; zdz = \frac{x^2+1}{x}\,dx \;\Rightarrow\; z^2 = x^2 + 2\ln x + C$$

At (1,3,-1), the expression is satisfied if $C = 0$. Therefore, the equation for the streamline is $\underline{z^2 = x^2 + 2\ln x}$.

2.29. If $\mathbf{E} = 20e^{-5y}\left(\cos 5x\mathbf{a}_x - \sin 5x\mathbf{a}_y\right)$, find:

a) $|\mathbf{E}|$ at $P(\pi/6, 0.1, 2)$: Substituting this point, we obtain $\mathbf{E}_P = -10.6\mathbf{a}_x - 6.1\mathbf{a}_y$, and so $|\mathbf{E}_P| = \underline{12.2}$.

b) a unit vector in the direction of $\mathbf{E}_P$: The unit vector associated with $\mathbf{E}$ is $(\cos 5x\mathbf{a}_x - \sin 5x\mathbf{a}_y)$, which evaluated at P becomes $\mathbf{a}_E = \underline{-0.87\mathbf{a}_x - 0.50\mathbf{a}_y}$.

c) the equation of the direction line passing through P: Use

$$\frac{dy}{dx} = \frac{-\sin 5x}{\cos 5x} = -\tan 5x \quad\Rightarrow\quad dy = -\tan 5x\,dx$$

Thus $y = \frac{1}{5}\ln\cos 5x + C$. Evaluating at P, we find $C = 0.13$, and so

$$\underline{y = \frac{1}{5}\ln\cos 5x + 0.13}$$

2.30. For fields that do not vary with z in cylindrical coordinates, the equations of the streamlines are obtained by solving the differential equation $E_\rho/E_\phi = d\rho(\rho d\phi)$. Find the equation of the line passing through the point $(2, 30°, 0)$ for the field $\mathbf{E} = \rho\cos 2\phi\,\mathbf{a}_\rho - \rho\sin 2\phi\,\mathbf{a}_\phi$:

$$\frac{E_\rho}{E_\phi} = \frac{d\rho}{\rho d\phi} = \frac{-\rho\cos 2\phi}{\rho\sin 2\phi} = -\cot 2\phi \;\Rightarrow\; \frac{d\rho}{\rho} = -\cot 2\phi\,d\phi$$

Integrate to obtain

$$2\ln\rho = \ln\sin 2\phi + \ln C = \ln\left[\frac{C}{\sin 2\phi}\right] \;\Rightarrow\; \rho^2 = \frac{C}{\sin 2\phi}$$

At the given point, we have $4 = C/\sin(60°) \;\Rightarrow\; C = 4\sin 60° = 2\sqrt{3}$. Finally, the equation for the streamline is $\underline{\rho^2 = 2\sqrt{3}/\sin 2\phi}$.