CHAPTER 3

3.1. An empty metal paint can is placed on a marble table, the lid is removed, and both parts are discharged (honorably) by touching them to ground. An insulating nylon thread is glued to the center of the lid, and a penny, a nickel, and a dime are glued to the thread so that they are not touching each other. The penny is given a charge of +5 nC, and the nickel and dime are discharged. The assembly is lowered into the can so that the coins hang clear of all walls, and the lid is secured. The outside of the can is again touched momentarily to ground. The device is carefully disassembled with insulating gloves and tools.

a) What charges are found on each of the five metallic pieces? All coins were insulated during the entire procedure, so they will retain their original charges: Penny: $\underline{+5}$ nC; nickel: $\underline{0}$; dime: $\underline{0}$. The penny's charge will have induced an equal and opposite negative charge (-5 nC) on the inside wall of the can and lid. This left a charge layer of +5 nC on the outside surface which was neutralized by the ground connection. Therefore, the can retained a net charge of $\underline{-5\,\text{nC}}$ after disassembly.

b) If the penny had been given a charge of $+5$ nC, the dime a charge of -2 nC, and the nickel a charge of -1 nC, what would the final charge arrangement have been? Again, since the coins are insulated, they retain their original charges. The charge induced on the inside wall of the can and lid is equal to negative the sum of the coin charges, or -2 nC. This is the charge that the can/lid contraption retains after grounding and disassembly.

3.2. A point charge of 20 nC is located at (4,-1,3), and a uniform line charge of -25 nC/m is lies along the intersection of the planes $x = -4$ and $z = 6$.

a) Calculate $\mathbf{D}$ at (3,-1,0):

The total flux density at the desired point is

$$\mathbf{D}(3,-1,0) = \underbrace{\frac{20\times 10^{-9}}{4\pi(1+9)}\left[\frac{-\mathbf{a}_x - 3\,\mathbf{a}_z}{\sqrt{1+9}}\right]}_{\text{point charge}} - \underbrace{\frac{25\times 10^{-9}}{2\pi\sqrt{49+36}}\left[\frac{7\,\mathbf{a}_x - 6\,\mathbf{a}_z}{\sqrt{49+36}}\right]}_{\text{line charge}}$$
$$= \underline{-0.38\,\mathbf{a}_x + 0.13\,\mathbf{a}_z \ \text{nC/m}^2}$$

b) How much electric flux leaves the surface of a sphere of radius 5, centered at the origin? This will be equivalent to how much charge lies within the sphere. First the point charge is at distance from the origin given by $R_p = \sqrt{16+1+9} = 5.1$, and so it is outside. Second, the nearest point on the line charge to the origin is at distance $R_\ell = \sqrt{16+36} = 7.2$, and so the entire line charge is also outside the sphere. Answer: $\underline{\text{zero}}$.

c) Repeat part *b* if the radius of the sphere is 10.

First, from part *b*, the point charge will now lie inside. Second, the length of line charge that lies inside the sphere will be given by $2y_0$, where y_0 satisfies the equation, $\sqrt{16+y_0^2+36} = 10$. Solve to find $y_0 = 6.93$, or $2y_0 = 13.86$. The total charge within the sphere (and the net outward flux) is now

$$\Phi = Q_{encl} = [20 - (25\times 13.86)] = \underline{-326\ \text{nC}}$$

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3.3. The cylindrical surface $\rho = 8$ cm contains the surface charge density, $\rho_s = 5e^{-20|z|}$ nC/m^2.

a) What is the total amount of charge present? We integrate over the surface to find:

$$Q = 2\int_0^\infty \int_0^{2\pi} 5e^{-20z}(.08)d\phi\, dz \text{ nC} = 20\pi(.08)\left(\frac{-1}{20}\right)e^{-20z}\Big|_0^\infty = \underline{0.25\,\text{nC}}$$

b) How much flux leaves the surface $\rho = 8\,\text{cm}$, $1\,\text{cm} < z < 5\text{cm}$, $30^\circ < \phi < 90^\circ$? We just integrate the charge density on that surface to find the flux that leaves it.

$$\Phi = Q' = \int_{.01}^{.05}\int_{30^\circ}^{90^\circ} 5e^{-20z}(.08)\,d\phi\,dz \text{ nC} = \left(\frac{90-30}{360}\right)2\pi(5)(.08)\left(\frac{-1}{20}\right)e^{-20z}\Big|_{.01}^{.05}$$
$$= 9.45 \times 10^{-3}\,\text{nC} = \underline{9.45\,\text{pC}}$$

3.4. In cylindrical coordinates, let $\mathbf{D} = (\rho\mathbf{a}_\rho + z\mathbf{a}_z)/\left[4\pi(\rho^2+z^2)^{1.5}\right]$. Determine the total flux leaving:

a) the infinitely-long cylindrical surface $\rho = 7$: We use

$$\Phi_a = \int \mathbf{D}\cdot d\mathbf{S} = \int_{-\infty}^{\infty}\int_0^{2\pi} \frac{\rho_0\,\mathbf{a}_\rho + z\,\mathbf{a}_z}{4\pi(\rho_0^2+z^2)^{3/2}}\cdot\mathbf{a}_\rho\,\rho_0\,d\phi\,dz = \rho_0^2\int_0^\infty \frac{dz}{(\rho_0^2+z^2)^{3/2}}$$
$$= \frac{z}{\sqrt{\rho_0^2+z^2}}\Big|_0^\infty = \underline{1}$$

where $\rho_0 = 7$ (immaterial in this case).

b) the finite cylinder, $\rho = 7$, $|z| \leq 10$:

The total flux through the cylindrical surface and the two end caps are, in this order:

$$\Phi_b = \int_{-z_0}^{z_0}\int_0^{2\pi}\frac{\rho_0\,\mathbf{a}_\rho\cdot\mathbf{a}_\rho}{4\pi(\rho_0^2+z^2)^{3/2}}\,\rho_0\,d\phi\,dz$$
$$+\int_0^{2\pi}\int_0^{\rho_0}\frac{z_0\,\mathbf{a}_z\cdot\mathbf{a}_z}{4\pi(\rho^2+z_0^2)^{3/2}}\,\rho\,d\rho\,d\phi + \int_0^{2\pi}\int_0^{\rho_0}\frac{-z_0\,\mathbf{a}_z\cdot-\mathbf{a}_z}{4\pi(\rho^2+z_0^2)^{3/2}}\,\rho\,d\rho\,d\phi$$

where $\rho_0 = 7$ and $z_0 = 10$. Simplifying, this becomes

$$\Phi_b = \rho_0^2\int_0^{z_0}\frac{dz}{(\rho_0^2+z^2)^{3/2}} + z_0\int_0^{\rho_0}\frac{\rho\,d\rho}{(\rho^2+z_0^2)^{3/2}}$$
$$= \frac{z}{\sqrt{\rho_0^2+z^2}}\Big|_0^{z_0} - \frac{z_0}{\sqrt{\rho^2+z_0^2}}\Big|_0^{\rho_0} = \frac{z_0}{\sqrt{\rho_0^2+z_0^2}} + 1 - \frac{z_0}{\sqrt{\rho_0^2+z_0^2}} = \underline{1}$$

where again, the actual values of ρ_0 and z_0 (7 and 10) did not matter.

3.5. Let $\mathbf{D} = 4xy\mathbf{a}_x + 2(x^2+z^2)\mathbf{a}_y + 4yz\mathbf{a}_z$ C/m^2 and evaluate surface integrals to find the total charge enclosed in the rectangular parallelepiped $0 < x < 2$, $0 < y < 3$, $0 < z < 5$ m: Of the 6 surfaces to consider, only 2 will contribute to the net outward flux. Why? First consider the planes at $y = 0$ and 3. The y component of $\mathbf{D}$ will penetrate those surfaces, but will be inward at $y = 0$ and outward at $y = 3$, while having the same magnitude in both cases. These fluxes will thus cancel. At the $x = 0$ plane, $D_x = 0$ and at the $z = 0$ plane, $D_z = 0$, so there will be no flux contributions from these surfaces. This leaves the 2 remaining surfaces at $x = 2$ and $z = 5$. The net outward flux becomes:

$$\begin{aligned}\Phi &= \int_0^5 \int_0^3 \mathbf{D}\big|_{x=2} \cdot \mathbf{a}_x \, dy \, dz + \int_0^3 \int_0^2 \mathbf{D}\big|_{z=5} \cdot \mathbf{a}_z \, dx \, dy \\ &= 5 \int_0^3 4(2) y \, dy \ + \ 2 \int_0^3 4(5) y \, dy = \underline{360 \text{ C}}\end{aligned}$$

3.6. In free space, volume charge of constant density $\rho_v = \rho_0$ exists within the region $-\infty < x < \infty$, $-\infty < y < \infty$, and $-d/2 < z < d/2$. Find $\mathbf{D}$ and $\mathbf{E}$ everywhere.

From the symmetry of the configuration, we surmise that the field will be everywhere z-directed, and will be uniform with x and y at fixed z. For finding the field inside the charge, an appropriate Gaussian surface will be that which encloses a rectangular region defined by $-1 < x < 1$, $-1 < y < 1$, and $|z| < d/2$. The outward flux from this surface will be limited to that through the two parallel surfaces at $\pm z$:

$$\Phi_{in} = \oint \mathbf{D} \cdot d\mathbf{S} = 2 \int_{-1}^1 \int_{-1}^1 D_z \, dx dy = Q_{encl} = \int_{-z}^z \int_{-1}^1 \int_{-1}^1 \rho_0 \, dx dy dz'$$

where the factor of 2 in the second integral account for the equal fluxes through the two surfaces. The above readily simplifies, as both D_z and ρ_0 are constants, leading to $\underline{\mathbf{D}_{in} = \rho_0 z \, \mathbf{a}_z \text{ C/m}^2 \ \ (|z| < d/2)}$, and therefore $\underline{\mathbf{E}_{in} = (\rho_0 z/\epsilon_0) \, \mathbf{a}_z \text{ V/m} \ \ (|z| < d/2)}$. Outside the charge, the Gaussian surface is the same, except that the parallel boundaries at $\pm z$ occur at $|z| > d/2$. As a result, the calculation is nearly the same as before, with the only change being the limits on the total charge integral:

$$\Phi_{out} = \oint \mathbf{D} \cdot d\mathbf{S} = 2 \int_{-1}^1 \int_{-1}^1 D_z \, dx dy = Q_{encl} = \int_{-d/2}^{d/2} \int_{-1}^1 \int_{-1}^1 \rho_0 \, dx dy dz'$$

Solve for D_z to find the constant values:

$$\mathbf{D}_{out} = \begin{cases} (\rho_0 d/2) \, \mathbf{a}_z & (z > d/2) \\ -(\rho_0 d/2) \, \mathbf{a}_z & (z < d/2) \end{cases} \ \text{C/m}^2 \ \text{ and } \ \mathbf{E}_{out} = \begin{cases} (\rho_0 d/2\epsilon_0) \, \mathbf{a}_z & (z > d/2) \\ -(\rho_0 d/2\epsilon_0) \, \mathbf{a}_z & (z < d/2) \end{cases} \ \text{V/m}$$

3.7. Volume charge density is located in free space as $\rho_v = 2e^{-1000r}$ nC/m^3 for $0 < r < 1$ mm, and $\rho_v = 0$ elsewhere.

a) Find the total charge enclosed by the spherical surface $r = 1$ mm: To find the charge we integrate:

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_0^{.001} 2e^{-1000r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

Integration over the angles gives a factor of 4π. The radial integration we evaluate using tables; we obtain

$$Q = 8\pi \left[\frac{-r^2 e^{-1000r}}{1000} \bigg|_0^{.001} + \frac{2}{1000} \frac{e^{-1000r}}{(1000)^2} (-1000r - 1) \bigg|_0^{.001} \right] = \underline{4.0 \times 10^{-9} \, \text{nC}}$$

b) By using Gauss's law, calculate the value of D_r on the surface $r = 1$ mm: The gaussian surface is a spherical shell of radius 1 mm. The enclosed charge is the result of part *a*. We thus write $4\pi r^2 D_r = Q$, or

$$D_r = \frac{Q}{4\pi r^2} = \frac{4.0 \times 10^{-9}}{4\pi(.001)^2} = \underline{3.2 \times 10^{-4}\ \text{nC/m}^2}$$

3.8. Use Gauss's law in integral form to show that an inverse distance field in spherical coordinates, $\mathbf{D} = A\mathbf{a}_r/r$, where A is a constant, requires every spherical shell of 1 m thickness to contain $4\pi A$ coulombs of charge. Does this indicate a continuous charge distribution? If so, find the charge density variation with r.

The net outward flux of this field through a spherical surface of radius r is

$$\Phi = \oint \mathbf{D} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{\pi} \frac{A}{r}\mathbf{a}_r \cdot \mathbf{a}_r\, r^2 \sin\theta\, d\theta\, d\phi = 4\pi A r = Q_{encl}$$

We see from this that with every increase in r by one m, the enclosed charge increases by $4\pi A$ (done). It is evident that the charge density is continuous, and we can find the density indirectly by constructing the integral for the enclosed charge, in which we already found the latter from Gauss's law:

$$Q_{encl} = 4\pi A r = \int_0^{2\pi} \int_0^{\pi} \int_0^{r} \rho(r')\,(r')^2 \sin\theta\, dr'\, d\theta\, d\phi = 4\pi \int_0^{r} \rho(r')\,(r')^2\, dr'$$

To obtain the correct enclosed charge, the integrand must be $\rho(r) = \underline{A/r^2}$.

3.9. A uniform volume charge density of $80\,\mu\text{C/m}^3$ is present throughout the region $8\,\text{mm} < r < 10\,\text{mm}$. Let $\rho_v = 0$ for $0 < r < 8\,\text{mm}$.

a) Find the total charge inside the spherical surface $r = 10$ mm: This will be

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_{.008}^{.010} (80 \times 10^{-6}) r^2 \sin\theta\, dr\, d\theta\, d\phi = 4\pi \times (80 \times 10^{-6}) \frac{r^3}{3}\Big|_{.008}^{.010}$$
$$= 1.64 \times 10^{-10}\,\text{C} = \underline{164\,\text{pC}}$$

b) Find D_r at $r = 10$ mm: Using a spherical gaussian surface at $r = 10$, Gauss' law is written as $4\pi r^2 D_r = Q = 164 \times 10^{-12}$, or

$$D_r(10\,\text{mm}) = \frac{164 \times 10^{-12}}{4\pi(.01)^2} = 1.30 \times 10^{-7}\,\text{C/m}^2 = \underline{130\,\text{nC/m}^2}$$

c) If there is no charge for $r > 10$ mm, find D_r at $r = 20$ mm: This will be the same computation as in part *b*, except the gaussian surface now lies at 20 mm. Thus

$$D_r(20\,\text{mm}) = \frac{164 \times 10^{-12}}{4\pi(.02)^2} = 3.25 \times 10^{-8}\,\text{C/m}^2 = \underline{32.5\,\text{nC/m}^2}$$

3.10. Volume charge density varies in spherical coordinates as $\rho_v = (\rho_0 \sin \pi r)/r^2$, where ρ_0 is a constant. Find the surfaces on which $\mathbf{D} = 0$.

3.11. In cylindrical coordinates, let $\rho_v = 0$ for $\rho < 1$ mm, $\rho_v = 2\sin(2000\pi\rho)$ nC/m^3 for $1\,\text{mm} < \rho < 1.5\,\text{mm}$, and $\rho_v = 0$ for $\rho > 1.5\,\text{mm}$. Find **D** everywhere: Since the charge varies only with radius, and is in the form of a cylinder, symmetry tells us that the flux density will be radially-directed and will be constant over a cylindrical surface of a fixed radius. Gauss' law applied to such a surface of unit length in z gives:

a) for $\rho < 1$ mm, $D_\rho = 0$, since no charge is enclosed by a cylindrical surface whose radius lies within this range.

b) for $1\,\text{mm} < \rho < 1.5\,\text{mm}$, we have

$$2\pi\rho D_\rho = 2\pi \int_{.001}^{\rho} 2 \times 10^{-9} \sin(2000\pi\rho')\rho'\, d\rho'$$
$$= 4\pi \times 10^{-9} \left[\frac{1}{(2000\pi)^2} \sin(2000\pi\rho) - \frac{\rho}{2000\pi} \cos(2000\pi\rho) \right]_{.001}^{\rho}$$

or finally,

$$D_\rho = \frac{10^{-15}}{2\pi^2\rho} \left[\sin(2000\pi\rho) + 2\pi \left[1 - 10^3 \rho \cos(2000\pi\rho) \right] \right] \text{ C/m}^2 \quad (1\,\text{mm} < \rho < 1.5\,\text{mm})$$

3.11. (continued)

c) for $\rho > 1.5\,\text{mm}$, the gaussian cylinder now lies at radius ρ *outside* the charge distribution, so the integral that evaluates the enclosed charge now includes the entire charge distribution. To accomplish this, we change the upper limit of the integral of part *b* from ρ to 1.5 mm, finally obtaining:

$$D_\rho = \frac{2.5 \times 10^{-15}}{\pi \rho} \text{ C/m}^2 \quad (\rho > 1.5\,\text{mm})$$

3.12. The sun radiates a total power of about 2×10^{26} watts (W). If we imagine the sun's surface to be marked off in latitude and longitude and assume uniform radiation, (a) what power is radiated by the region lying between latitude 50° N and 60° N and longitude 12° W and 27° W? (b) What is the power density on a spherical surface 93,000,000 miles from the sun in W/m^2?

3.13. Spherical surfaces at $r = 2$, 4, and 6 m carry uniform surface charge densities of 20 nC/m^2, $-4\,\text{nC/m}^2$, and ρ_{s0}, respectively.

a) Find **D** at $r = 1, 3$ and 5 m: Noting that the charges are spherically-symmetric, we ascertain that **D** will be radially-directed and will vary only with radius. Thus, we apply Gauss' law to spherical shells in the following regions: $r < 2$: Here, no charge is enclosed, and so $\underline{D_r = 0}$.

$$2 < r < 4: \quad 4\pi r^2 D_r = 4\pi (2)^2 (20 \times 10^{-9}) \quad \Rightarrow \quad D_r = \frac{80 \times 10^{-9}}{r^2} \text{ C/m}^2$$

So $D_r(r = 3) = \underline{8.9 \times 10^{-9}\,\text{C/m}^2}$.

$$4 < r < 6: \quad 4\pi r^2 D_r = 4\pi (2)^2 (20 \times 10^{-9}) + 4\pi (4)^2 (-4 \times 10^{-9}) \quad \Rightarrow \quad D_r = \frac{16 \times 10^{-9}}{r^2}$$

So $D_r(r = 5) = \underline{6.4 \times 10^{-10}\,\text{C/m}^2}$.

b) Determine ρ_{s0} such that $\mathbf{D} = 0$ at $r = 7$ m. Since fields will decrease as $1/r^2$, the question could be re-phrased to ask for ρ_{s0} such that $\mathbf{D} = 0$ at *all* points where $r > 6$ m. In this region, the total field will be

$$D_r(r > 6) = \frac{16 \times 10^{-9}}{r^2} + \frac{\rho_{s0}(6)^2}{r^2}$$

Requiring this to be zero, we find $\rho_{s0} = \underline{-(4/9) \times 10^{-9}\,\text{C/m}^2}$.

3.14. The sun radiates a total power of about 2×10^{26} watts (W). If we imagine the sun's surface to be marked off in latitude and longitude and assume uniform radiation, (a) what power is radiated by the region lying between latitude 50° N and 60° N and longitude 12° W and 27° W? (b) What is the power density on a spherical surface 93,000,000 miles from the sun in W/m^2?

3.15. Volume charge density is located as follows: $\rho_v = 0$ for $\rho < 1$ mm and for $\rho > 2$ mm, $\rho_v = 4\rho\ \mu\text{C/m}^3$ for $1 < \rho < 2$ mm.

a) Calculate the total charge in the region $0 < \rho < \rho_1$, $0 < z < L$, where $1 < \rho_1 < 2$ mm: We find

$$Q = \int_0^L \int_0^{2\pi} \int_{.001}^{\rho_1} 4\rho\, \rho\, d\rho\, d\phi\, dz = \underline{\frac{8\pi L}{3}[\rho_1^3 - 10^{-9}]\ \mu\text{C}}$$

where ρ_1 is in meters.

b) Use Gauss' law to determine D_ρ at $\rho = \rho_1$: Gauss' law states that $2\pi\rho_1 L D_\rho = Q$, where Q is the result of part a. Thus

$$D_\rho(\rho_1) = \underline{\frac{4(\rho_1^3 - 10^{-9})}{3\rho_1}\ \mu\text{C/m}^2}$$

where ρ_1 is in meters.

c) Evaluate D_ρ at $\rho = 0.8\,\text{mm},\ 1.6\,\text{mm},$ and $2.4\,\text{mm}$: At $\rho = 0.8\,\text{mm}$, no charge is enclosed by a cylindrical gaussian surface of that radius, so $D_\rho(0.8\text{mm}) = \underline{0}$. At $\rho = 1.6\,\text{mm}$, we evaluate the part b result at $\rho_1 = 1.6$ to obtain:

$$D_\rho(1.6\text{mm}) = \frac{4[(.0016)^3 - (.0010)^3]}{3(.0016)} = \underline{3.6 \times 10^{-6}\ \mu\text{C/m}^2}$$

At $\rho = 2.4$, we evaluate the charge integral of part a from .001 to .002, and Gauss' law is written as

$$2\pi\rho L D_\rho = \frac{8\pi L}{3}[(.002)^2 - (.001)^2]\ \mu\text{C}$$

from which $D_\rho(2.4\text{mm}) = \underline{3.9 \times 10^{-6}\ \mu\text{C/m}^2}$.

3.16. In spherical coordinates, a volume charge density $\rho_v = 10e^{-2r}$ C/m^3 is present. (a) Determine $\mathbf{D}$. (b) Check your result of part a by evaluating $\nabla \cdot \mathbf{D}$.

3.17. A cube is defined by $1 < x, y, z < 1.2$. If $\mathbf{D} = 2x^2y\mathbf{a}_x + 3x^2y^2\mathbf{a}_y$ C/m^2:

a) apply Gauss' law to find the total flux leaving the closed surface of the cube. We call the surfaces at $x = 1.2$ and $x = 1$ the front and back surfaces respectively, those at $y = 1.2$ and $y = 1$ the right and left surfaces, and those at $z = 1.2$ and $z = 1$ the top and bottom surfaces. To evaluate the total charge, we integrate $\mathbf{D} \cdot \mathbf{n}$ over all six surfaces and sum the results. We note that there is no z component of $\mathbf{D}$, so there will be no outward flux contributions from the top and bottom surfaces. The fluxes through the remaining four are

$$\Phi = Q = \oint \mathbf{D} \cdot \mathbf{n}\, da = \underbrace{\int_1^{1.2}\int_1^{1.2} 2(1.2)^2 y\, dy\, dz}_{\text{front}} + \underbrace{\int_1^{1.2}\int_1^{1.2} -2(1)^2 y\, dy\, dz}_{\text{back}}$$

$$+ \underbrace{\int_1^{1.2}\int_1^{1.2} -3x^2(1)^2\, dx\, dz}_{\text{left}} + \underbrace{\int_1^{1.2}\int_1^{1.2} 3x^2(1.2)^2\, dx\, dz}_{\text{right}} = \underline{0.1028\,\text{C}}$$

b) evaluate $\nabla \cdot \mathbf{D}$ at the center of the cube: This is

$$\nabla \cdot \mathbf{D} = \left[4xy + 6x^2y\right]_{(1.1,1.1)} = 4(1.1)^2 + 6(1.1)^3 = \underline{12.83}$$

c) Estimate the total charge enclosed within the cube by using Eq. (8): This is

$$Q \doteq \nabla \cdot \mathbf{D}\Big|_{\text{center}} \times \Delta v = 12.83 \times (0.2)^3 = \underline{0.1026} \quad \text{Close!}$$

3.18. State whether the divergence of the following vector fields is positive, negative, or zero: (a) the thermal energy flow in $\mathrm{J/(m^2 - s)}$ at any point in a freezing ice cube; (b) the current density in $\mathrm{A/m^2}$ in a bus bar carrying direct current; (c) the mass flow rate in $\mathrm{kg/(m^2 - s)}$ below the surface of water in a basin, in which the water is circulating clockwise as viewed from above.

3.19. A spherical surface of radius 3 mm is centered at $P(4, 1, 5)$ in free space. Let $\mathbf{D} = x\mathbf{a}_x$ C/m^2. Use the results of Sec. 3.4 to estimate the net electric flux leaving the spherical surface: We use $\Phi \doteq \nabla \cdot \mathbf{D} \Delta v$, where in this case $\nabla \cdot \mathbf{D} = (\partial/\partial x)x = 1$ C/m^3. Thus

$$\Phi \doteq \frac{4}{3}\pi(.003)^3(1) = 1.13 \times 10^{-7}\,\mathrm{C} = \underline{113\,\mathrm{nC}}$$

3.20. Suppose that an electric flux density in cylindrical coordinates is of the form $\mathbf{D} = D_\rho\,\mathbf{a}_\rho$. Describe the dependence of the charge density ρ_v on coordinates ρ, ϕ, and z if (a) $D_\rho = f(\phi, z)$; (b) $D_\rho = (1/\rho)f(\phi, z)$; (c) $D_\rho = f(\rho)$.

3.21. Calculate the divergence of $\mathbf{D}$ at the point specified if

a) $\mathbf{D} = (1/z^2)\left[10xyz\,\mathbf{a}_x + 5x^2z\,\mathbf{a}_y + (2z^3 - 5x^2y)\,\mathbf{a}_z\right]$ at $P(-2, 3, 5)$: We find

$$\nabla \cdot \mathbf{D} = \left[\frac{10y}{z} + 0 + 2 + \frac{10x^2y}{z^3}\right]_{(-2,3,5)} = \underline{8.96}$$

b) $\mathbf{D} = 5z^2\mathbf{a}_\rho + 10\rho z\,\mathbf{a}_z$ at $P(3, -45°, 5)$: In cylindrical coordinates, we have

$$\nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} = \left[\frac{5z^2}{\rho} + 10\rho\right]_{(3,-45°,5)} = \underline{71.67}$$

c) $\mathbf{D} = 2r\sin\theta\sin\phi\,\mathbf{a}_r + r\cos\theta\sin\phi\,\mathbf{a}_\theta + r\cos\phi\,\mathbf{a}_\phi$ at $P(3, 45°, -45°)$: In spherical coordinates, we have

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$
$$= \left[6\sin\theta\sin\phi + \frac{\cos 2\theta \sin\phi}{\sin\theta} - \frac{\sin\phi}{\sin\theta}\right]_{(3,45°,-45°)} = \underline{-2}$$

3.22. (a) A flux density field is given as $\mathbf{F}_1 = 5\mathbf{a}_z$. Evaluate the outward flux of $\mathbf{F}_1$ through the hemispherical surface, $r = a$, $0 < \theta < \pi/2$, $0 < \phi < 2\pi$. (b) What simple observation would have saved a lot of work in part a? (c) Now suppose the field is given by $\mathbf{F}_2 = 5z\mathbf{a}_z$. Using the appropriate surface integrals, evaluate the net outward flux of $\mathbf{F}_2$ through the closed surface consisting of the hemisphere of part a and its circular base in the xy plane. (d) Repeat part c by using the divergence theorem and an appropriate volume integral.

3.23. a) A point charge Q lies at the origin. Show that div $\mathbf{D}$ is zero everywhere except at the origin. For a point charge at the origin we know that $\mathbf{D} = Q/(4\pi r^2)\,\mathbf{a}_r$. Using the formula for divergence in spherical coordinates (see problem 3.21 solution), we find in this case that

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{Q}{4\pi r^2}\right) = 0$$

The above is true provided $r > 0$. When $r = 0$, we have a singularity in $\mathbf{D}$, so its divergence is not defined.

b) Replace the point charge with a uniform volume charge density ρ_{v0} for $0 < r < a$. Relate ρ_{v0} to Q and a so that the total charge is the same. Find div $\mathbf{D}$ everywhere: To achieve the same net charge, we require that $(4/3)\pi a^3 \rho_{v0} = Q$, so $\underline{\rho_{v0} = 3Q/(4\pi a^3)\ \mathrm{C/m^3}}$. Gauss' law tells us that inside the charged sphere

$$4\pi r^2 D_r = \frac{4}{3}\pi r^3 \rho_{v0} = \frac{Qr^3}{a^3}$$

Thus

$$D_r = \frac{Qr}{4\pi a^3}\ \mathrm{C/m^2} \text{ and } \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}\left(\frac{Qr^3}{4\pi a^3}\right) = \frac{3Q}{4\pi a^3}$$

as expected. Outside the charged sphere, $\mathbf{D} = Q/(4\pi r^2)\,\mathbf{a}_r$ as before, and the divergence is zero.

3.24. (a) A uniform line charge density ρ_L lies along the z axis. Show that $\nabla \cdot \mathbf{D} = 0$ everywhere except on the line charge. (b) Replace the line charge with a uniform volume charge density ρ_0 for $0 < \rho < a$. Relate ρ_0 to ρ_L so that the charge per unit length is the same. Then find $\nabla \cdot \mathbf{D}$ everywhere.

3.25. Within the spherical shell, $3 < r < 4$ m, the electric flux density is given as

$$\mathbf{D} = 5(r-3)^3\,\mathbf{a}_r\ \mathrm{C/m^2}$$

a) What is the volume charge density at $r = 4$? In this case we have

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}(r^2 D_r) = \frac{5}{r}(r-3)^2(5r-6)\ \mathrm{C/m^3}$$

which we evaluate at $r = 4$ to find $\rho_v(r = 4) = \underline{17.50\ \mathrm{C/m^3}}$.

b) What is the electric flux density at $r = 4$? Substitute $r = 4$ into the given expression to find $\mathbf{D}(4) = \underline{5\,\mathbf{a}_r\ \mathrm{C/m^2}}$

c) How much electric flux leaves the sphere $r = 4$? Using the result of part b, this will be $\Phi = 4\pi(4)^2(5) = \underline{320\pi\ \mathrm{C}}$

d) How much charge is contained within the sphere, $r = 4$? From Gauss' law, this will be the same as the outward flux, or again, $Q = \underline{320\pi\ \mathrm{C}}$.

3.26. If we have a perfect gas of mass density ρ_m kg/m^3, and assign a velocity $\mathbf{U}$ m/s to each differential element, then the mass flow rate is $\rho_m\mathbf{U}$ kg/(m^2 – s). Physical reasoning then leads to the *continuity equation*, $\nabla \cdot (\rho_m \mathbf{U}) = -\partial \rho_m / \partial t$. (a) Explain in words the physical interpretation of this equation. (b) Show that $\oint_s \rho_m \mathbf{U} \cdot d\mathbf{S} = -dM/dt$, where M is the total mass of the gas within the constant closed surface, S, and explain the physical significance of the equation.

3.27. Let $\mathbf{D} = 5.00r^2 \mathbf{a}_r$ mC/m^2 for $r \leq 0.08$ m and $\mathbf{D} = 0.205\, \mathbf{a}_r / r^2$ μC/m^2 for $r \geq 0.08$ m (note error in problem statement).

a) Find ρ_v for $r = 0.06$ m: This radius lies within the first region, and so

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{r^2} \frac{d}{dr}(r^2 D_r) = \frac{1}{r^2} \frac{d}{dr}(5.00r^4) = 20r \text{ mC/m}^3$$

which when evaluated at $r = 0.06$ yields $\rho_v(r = .06) = \underline{1.20 \text{ mC/m}^3}$.

b) Find ρ_v for $r = 0.1$ m: This is in the region where the second field expression is valid. The $1/r^2$ dependence of this field yields a zero divergence (shown in Problem 3.23), and so the volume charge density is zero at 0.1 m.

c) What surface charge density could be located at $r = 0.08$ m to cause $\mathbf{D} = 0$ for $r > 0.08$ m? The total surface charge should be equal and opposite to the total volume charge. The latter is

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_0^{.08} 20r (\text{mC/m}^3)\, r^2 \sin\theta \, dr \, d\theta \, d\phi = 2.57 \times 10^{-3} \text{ mC} = 2.57\, \mu\text{C}$$

So now

$$\rho_s = -\left[\frac{2.57}{4\pi (.08)^2} \right] = \underline{-32\, \mu\text{C/m}^2}$$

3.28. Repeat Problem 3.8, but use $\nabla \cdot \mathbf{D} = \rho_v$ and take an appropriate volume integral.

3.29. In the region of free space that includes the volume $2 < x, y, z < 3$,

$$\mathbf{D} = \frac{2}{z^2}(yz\, \mathbf{a}_x + xz\, \mathbf{a}_y - 2xy\, \mathbf{a}_z) \text{ C/m}^2$$

a) Evaluate the volume integral side of the divergence theorem for the volume defined above: In cartesian, we find $\nabla \cdot \mathbf{D} = 8xy/z^3$. The volume integral side is now

$$\int_{vol} \nabla \cdot \mathbf{D}\, dv = \int_2^3 \int_2^3 \int_2^3 \frac{8xy}{z^3}\, dxdydz = (9-4)(9-4)\left(\frac{1}{4} - \frac{1}{9}\right) = \underline{3.47 \text{ C}}$$

b. Evaluate the surface integral side for the corresponding closed surface: We call the surfaces at $x = 3$ and $x = 2$ the front and back surfaces respectively, those at $y = 3$ and $y = 2$ the right and left surfaces, and those at $z = 3$ and $z = 2$ the top and bottom surfaces. To evaluate the surface integral side, we integrate $\mathbf{D} \cdot \mathbf{n}$ over all six surfaces and sum the results. Note that since the x component of $\mathbf{D}$ does not vary with x, the outward fluxes from the front and back surfaces will cancel each other. The same is true for the left and right surfaces, since D_y does not vary with y. This leaves only the top and bottom surfaces, where the fluxes are:

$$\oint \mathbf{D}\cdot d\mathbf{S} = \underbrace{\int_2^3\int_2^3 \frac{-4xy}{3^2}\,dxdy}_{\text{top}} - \underbrace{\int_2^3\int_2^3 \frac{-4xy}{2^2}\,dxdy}_{\text{bottom}} = (9-4)(9-4)\left(\frac{1}{4}-\frac{1}{9}\right) = \underline{3.47\text{ C}}$$

3.30. Let $\mathbf{D} = 20\rho^2\,\mathbf{a}_\rho$ C/m^2. (a) What is the volume charge density at the point $P(0.5, 60^\circ, 2)$? (b) Use two different methods to find the amount of charge lying within the closed surface bounded by $\rho = 3$, $0 \le z \le 2$.

3.31. Given the flux density

$$\mathbf{D} = \frac{16}{r}\cos(2\theta)\,\mathbf{a}_\theta\ \text{C/m}^2,$$

use two different methods to find the total charge within the region $1 < r < 2$ m, $1 < \theta < 2$ rad, $1 < \phi < 2$ rad: We use the divergence theorem and first evaluate the surface integral side. We are evaluating the net outward flux through a curvilinear "cube", whose boundaries are defined by the specified ranges. The flux contributions will be only through the surfaces of constant θ, however, since $\mathbf{D}$ has only a θ component. On a constant-theta surface, the differential area is $da = r\sin\theta dr d\phi$, where θ is fixed at the surface location. Our flux integral becomes

$$\oint \mathbf{D}\cdot d\mathbf{S} = -\underbrace{\int_1^2\int_1^2 \frac{16}{r}\cos(2)\,r\sin(1)\,drd\phi}_{\theta=1} + \underbrace{\int_1^2\int_1^2 \frac{16}{r}\cos(4)\,r\sin(2)\,drd\phi}_{\theta=2}$$
$$= -16\left[\cos(2)\sin(1) - \cos(4)\sin(2)\right] = \underline{-3.91\text{ C}}$$

We next evaluate the volume integral side of the divergence theorem, where in this case,

$$\nabla\cdot\mathbf{D} = \frac{1}{r\sin\theta}\frac{d}{d\theta}(\sin\theta\,D_\theta) = \frac{1}{r\sin\theta}\frac{d}{d\theta}\left[\frac{16}{r}\cos 2\theta\sin\theta\right] = \frac{16}{r^2}\left[\frac{\cos 2\theta\cos\theta}{\sin\theta} - 2\sin 2\theta\right]$$

We now evaluate:

$$\int_{vol}\nabla\cdot\mathbf{D}\,dv = \int_1^2\int_1^2\int_1^2 \frac{16}{r^2}\left[\frac{\cos 2\theta\cos\theta}{\sin\theta} - 2\sin 2\theta\right] r^2\sin\theta\,drd\theta d\phi$$

The integral simplifies to

$$\int_1^2\int_1^2\int_1^2 16[\cos 2\theta\cos\theta - 2\sin 2\theta\sin\theta]\,drd\theta d\phi = 8\int_1^2 [3\cos 3\theta - \cos\theta]\,d\theta = \underline{-3.91\text{ C}}$$