CHAPTER 4

4.1. The value of $\mathbf{E}$ at $P(\rho = 2,\ \phi = 40°,\ z = 3)$ is given as $\mathbf{E} = 100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z$ V/m. Determine the incremental work required to move a $20\,\mu$C charge a distance of 6 μm:

a) in the direction of $\mathbf{a}_\rho$: The incremental work is given by $dW = -q\,\mathbf{E}\cdot d\mathbf{L}$, where in this case, $d\mathbf{L} = d\rho\,\mathbf{a}_\rho = 6\times 10^{-6}\,\mathbf{a}_\rho$. Thus

$$dW = -(20\times 10^{-6}\,\text{C})(100\,\text{V/m})(6\times 10^{-6}\,\text{m}) = -12\times 10^{-9}\ \text{J} = \underline{-12\,\text{nJ}}$$

b) in the direction of $\mathbf{a}_\phi$: In this case $d\mathbf{L} = 2\,d\phi\,\mathbf{a}_\phi = 6\times 10^{-6}\,\mathbf{a}_\phi$, and so

$$dW = -(20\times 10^{-6})(-200)(6\times 10^{-6}) = 2.4\times 10^{-8}\,\text{J} = \underline{24\ \text{nJ}}$$

c) in the direction of $\mathbf{a}_z$: Here, $d\mathbf{L} = dz\,\mathbf{a}_z = 6\times 10^{-6}\,\mathbf{a}_z$, and so

$$dW = -(20\times 10^{-6})(300)(6\times 10^{-6}) = -3.6\times 10^{-8}\,\text{J} = \underline{-36\ \text{nJ}}$$

d) in the direction of $\mathbf{E}$: Here, $d\mathbf{L} = 6\times 10^{-6}\,\mathbf{a}_E$, where

$$\mathbf{a}_E = \frac{100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z}{[100^2 + 200^2 + 300^2]^{1/2}} = 0.267\,\mathbf{a}_\rho - 0.535\,\mathbf{a}_\phi + 0.802\,\mathbf{a}_z$$

Thus

$$\begin{aligned} dW &= -(20\times 10^{-6})[100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z]\cdot[0.267\,\mathbf{a}_\rho - 0.535\,\mathbf{a}_\phi + 0.802\,\mathbf{a}_z](6\times 10^{-6}) \\ &= \underline{-44.9\,\text{nJ}} \end{aligned}$$

e) In the direction of $\mathbf{G} = 2\,\mathbf{a}_x - 3\,\mathbf{a}_y + 4\,\mathbf{a}_z$: In this case, $d\mathbf{L} = 6\times 10^{-6}\,\mathbf{a}_G$, where

$$\mathbf{a}_G = \frac{2\mathbf{a}_x - 3\mathbf{a}_y + 4\mathbf{a}_z}{[2^2 + 3^2 + 4^2]^{1/2}} = 0.371\,\mathbf{a}_x - 0.557\,\mathbf{a}_y + 0.743\,\mathbf{a}_z$$

So now

$$\begin{aligned} dW &= -(20\times 10^{-6})[100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z]\cdot[0.371\,\mathbf{a}_x - 0.557\,\mathbf{a}_y + 0.743\,\mathbf{a}_z](6\times 10^{-6}) \\ &= -(20\times 10^{-6})\,[37.1(\mathbf{a}_\rho\cdot\mathbf{a}_x) - 55.7(\mathbf{a}_\rho\cdot\mathbf{a}_y) - 74.2(\mathbf{a}_\phi\cdot\mathbf{a}_x) + 111.4(\mathbf{a}_\phi\cdot\mathbf{a}_y) \\ &\quad + 222.9]\,(6\times 10^{-6}) \end{aligned}$$

where, at P, $(\mathbf{a}_\rho\cdot\mathbf{a}_x) = (\mathbf{a}_\phi\cdot\mathbf{a}_y) = \cos(40°) = 0.766$, $(\mathbf{a}_\rho\cdot\mathbf{a}_y) = \sin(40°) = 0.643$, and $(\mathbf{a}_\phi\cdot\mathbf{a}_x) = -\sin(40°) = -0.643$. Substituting these results in

$$dW = -(20\times 10^{-6})[28.4 - 35.8 + 47.7 + 85.3 + 222.9](6\times 10^{-6}) = \underline{-41.8\,\text{nJ}}$$

4.2. An electric field is given as $\mathbf{E} = -10e^y(\sin 2z\,\mathbf{a}_x + x\sin 2z\,\mathbf{a}_y + 2x\cos 2z\,\mathbf{a}_z)$ V/m.

a) Find $\mathbf{E}$ at $P(5, 0, \pi/12)$: Substituting this point into the given field produces

$$\mathbf{E}_P = -10\left[\sin(\pi/6)\,\mathbf{a}_x + 5\sin(\pi/6)\,\mathbf{a}_y + 10\cos(\pi/6)\,\mathbf{a}_z\right] = \underline{-\left[5\,\mathbf{a}_x + 25\,\mathbf{a}_y + 50\sqrt{3}\,\mathbf{a}_z\right]}$$

b) How much work is done in moving a charge of 2 nC an incremental distance of 1 mm from P in the direction of $\mathbf{a}_x$? This will be

$$dW_x = -q\mathbf{E}\cdot dL\,\mathbf{a}_x = -2\times 10^{-9}(-5)(10^{-3}) = 10^{-11}\text{ J} = \underline{10\text{ pJ}}$$

c) of $\mathbf{a}_y$?

$$dW_y = -q\mathbf{E}\cdot dL\,\mathbf{a}_y = -2\times 10^{-9}(-25)(10^{-3}) = 50^{-11}\text{ J} = \underline{50\text{ pJ}}$$

d) of $\mathbf{a}_z$?

$$dW_z = -q\mathbf{E}\cdot dL\,\mathbf{a}_z = -2\times 10^{-9}(-50\sqrt{3})(10^{-3}) = \underline{100\sqrt{3}\text{ pJ}}$$

e) of $(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)$?

$$dW_{xyz} = -q\mathbf{E}\cdot dL\,\frac{\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}{\sqrt{3}} = \frac{10 + 50 + 100\sqrt{3}}{\sqrt{3}} = \underline{135\text{ pJ}}$$

4.3. If $\mathbf{E} = 120\,\mathbf{a}_\rho$ V/m, find the incremental amount of work done in moving a 50 μm charge a distance of 2 mm from:

a) $P(1, 2, 3)$ toward $Q(2, 1, 4)$: The vector along this direction will be $Q - P = (1, -1, 1)$ from which $\mathbf{a}_{PQ} = [\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z]/\sqrt{3}$. We now write

$$dW = -q\mathbf{E}\cdot d\mathbf{L} = -(50\times 10^{-6})\left[120\mathbf{a}_\rho\cdot\frac{(\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z}{\sqrt{3}}\right](2\times 10^{-3})$$
$$= -(50\times 10^{-6})(120)\left[(\mathbf{a}_\rho\cdot\mathbf{a}_x) - (\mathbf{a}_\rho\cdot\mathbf{a}_y)\right]\frac{1}{\sqrt{3}}(2\times 10^{-3})$$

At P, $\phi = \tan^{-1}(2/1) = 63.4°$. Thus $(\mathbf{a}_\rho\cdot\mathbf{a}_x) = \cos(63.4) = 0.447$ and $(\mathbf{a}_\rho\cdot\mathbf{a}_y) = \sin(63.4) = 0.894$. Substituting these, we obtain $dW = \underline{3.1\,\mu\text{J}}$.

b) $Q(2, 1, 4)$ toward $P(1, 2, 3)$: A little thought is in order here: Note that the field has only a radial component and does not depend on ϕ or z. Note also that P and Q are at the same radius ($\sqrt{5}$) from the z axis, but have different ϕ and z coordinates. We could just as well position the two points at the same z location and the problem would not change. If this were so, then moving along a straight line between P and Q would thus involve moving along a chord of a circle whose radius is $\sqrt{5}$. Halfway along this line is a point of symmetry in the field (make a sketch to see this). This means that when starting from either point, the initial force will be the same. Thus the answer is $dW = \underline{3.1\,\mu\text{J}}$ as in part *a*. This is also found by going through the same procedure as in part *a*, but with the direction (roles of P and Q) reversed.

4.4. It is found that the energy expended in carrying a charge of 4 μC from the origin to (x,0,0) along the x axis is directly proportional to the square of the path length. If $E_x = 7$ V/m at (1,0,0), determine E_x on the x axis as a function of x.

The work done is in general given by

$$W = -q \int_0^x E_x \, dx = Ax^2$$

where A is a constant. Therefore E_x must be of the form $E_x = E_0 x$. At $x = 1$, $E_x = 7$, so $E_0 = 7$. Therefore $E_x = \underline{7x \text{ V/m}}$. Note that with the positive-x-directed field, the expended energy in moving the charge from 0 to x would be negative.

4.5. Compute the value of $\int_A^P \mathbf{G} \cdot d\mathbf{L}$ for $\mathbf{G} = 2y\mathbf{a}_x$ with $A(1, -1, 2)$ and $P(2, 1, 2)$ using the path:

a) straight-line segments $A(1, -1, 2)$ to $B(1, 1, 2)$ to $P(2, 1, 2)$: In general we would have

$$\int_A^P \mathbf{G} \cdot d\mathbf{L} = \int_A^P 2y \, dx$$

The change in x occurs when moving between B and P, during which $y = 1$. Thus

$$\int_A^P \mathbf{G} \cdot d\mathbf{L} = \int_B^P 2y \, dx = \int_1^2 2(1) dx = \underline{2}$$

b) straight-line segments $A(1, -1, 2)$ to $C(2, -1, 2)$ to $P(2, 1, 2)$: In this case the change in x occurs when moving from A to C, during which $y = -1$. Thus

$$\int_A^P \mathbf{G} \cdot d\mathbf{L} = \int_A^C 2y \, dx = \int_1^2 2(-1) dx = \underline{-2}$$

4.6. Determine the work done in carrying a 2-μC charge from (2,1,-1) to (8,2,-1) in the field $\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y$ along

a) the parabola $x = 2y^2$: As a look ahead, we can show (by taking its curl) that $\mathbf{E}$ is conservative. We therefore expect the same answer for all three paths. The general expression for the work is

$$W = -q \int_A^B \mathbf{E} \cdot d\mathbf{L} = -q \left[\int_2^8 y \, dx + \int_1^2 x \, dy \right]$$

In the present case, $x = 2y^2$, and so $y = \sqrt{x/2}$. Substituting these and the charge, we get

$$W_1 = -2\times10^{-6} \left[\int_2^8 \sqrt{x/2} \, dx + \int_1^2 2y^2 \, dy \right] = -2\times10^{-6} \left[\frac{\sqrt{2}}{3} x^{3/2} \Big|_2^8 + \frac{2}{3} y^3 \Big|_1^2 \right] = \underline{-28 \ \mu\text{J}}$$

b) the hyperbola $x = 8/(7 - 3y)$: We find $y = 7/3 - 8/3x$, and the work is

$$W_2 = -2 \times 10^{-6} \left[\int_2^8 \left(\frac{7}{3} - \frac{8}{3x} \right) dx + \int_1^2 \frac{8}{7 - 3y} dy \right]$$
$$= -2 \times 10^{-6} \left[\frac{7}{3}(8 - 2) - \frac{8}{3} \ln \left(\frac{8}{2} \right) - \frac{8}{3} \ln(7 - 3y) \Big|_1^2 \right] = \underline{-28 \ \mu\text{J}}$$

4.6c. the straight line $x = 6y - 4$: Here, $y = x/6 + 2/3$, and the work is

$$W_3 = -2 \times 10^{-6} \left[\int_2^8 \left(\frac{x}{6} + \frac{2}{3} \right) dx + \int_1^2 (6y - 4)\, dy \right] = \underline{-28\ \mu\text{J}}$$

4.7. Let $\mathbf{G} = 3xy^3\mathbf{a}_x + 2z\mathbf{a}_y$. Given an initial point $P(2, 1, 1)$ and a final point $Q(4, 3, 1)$, find $\int \mathbf{G} \cdot d\mathbf{L}$ using the path:

a) straight line: $y = x - 1$, $z = 1$: We obtain:

$$\int \mathbf{G} \cdot d\mathbf{L} = \int_2^4 3xy^2\, dx + \int_1^3 2z\, dy = \int_2^4 3x(x-1)^2\, dx + \int_1^3 2(1)\, dy = \underline{90}$$

b) parabola: $6y = x^2 + 2$, $z = 1$: We obtain:

$$\int \mathbf{G} \cdot d\mathbf{L} = \int_2^4 3xy^2\, dx + \int_1^3 2z\, dy = \int_2^4 \frac{1}{12} x(x^2+2)^2\, dx + \int_1^3 2(1)\, dy = \underline{82}$$

4.8. Given $\mathbf{E} = -x\mathbf{a}_x + y\mathbf{a}_y$, find the work involved in moving a unit positive charge on a circular arc, the circle centered at the origin, from $x = a$ to $x = y = a/\sqrt{2}$.

In moving along the arc, we start at $\phi = 0$ and move to $\phi = \pi/4$. The setup is

$$W = -q \int \mathbf{E} \cdot d\mathbf{L} = -\int_0^{\pi/4} \mathbf{E} \cdot a d\phi\, \mathbf{a}_\phi = -\int_0^{\pi/4} (-x \underbrace{\mathbf{a}_x \cdot \mathbf{a}_\phi}_{-\sin\phi} + y \underbrace{\mathbf{a}_y \cdot \mathbf{a}_\phi}_{\cos\phi}) a\, d\phi$$

$$= -\int_0^{\pi/4} 2a^2 \sin\phi \cos\phi\, d\phi = -\int_0^{\pi/4} a^2 \sin(2\phi)\, d\phi = \underline{-a^2/2}$$

where $q = 1$, $x = a\cos\phi$, and $y = a\sin\phi$.

Note that the field is conservative, so we would get the same result by integrating along a two-segment path over x and y as shown:

$$W = -\int \mathbf{E} \cdot d\mathbf{L} = -\left[\int_a^{a/\sqrt{2}} (-x)\, dx + \int_0^{a/\sqrt{2}} y\, dy \right] = -a^2/2$$

4.9. A uniform surface charge density of 20 nC/m^2 is present on the spherical surface $r = 0.6$ cm in free space.

a) Find the absolute potential at $P(r = 1\,\text{cm}, \theta = 25^\circ, \phi = 50^\circ)$: Since the charge density is uniform and is spherically-symmetric, the angular coordinates do not matter. The potential function for $r > 0.6$ cm will be that of a point charge of $Q = 4\pi a^2 \rho_s$, or

$$V(r) = \frac{4\pi(0.6 \times 10^{-2})^2 (20 \times 10^{-9})}{4\pi\epsilon_0 r} = \frac{0.081}{r}\ \text{V} \quad \text{with } r \text{ in meters}$$

At $r = 1$ cm, this becomes $V(r = 1\,\text{cm}) = \underline{8.14\ \text{V}}$

b) Find V_{AB} given points $A(r = 2\,\text{cm},\ \theta = 30°,\ \phi = 60°)$ and $B(r = 3\,\text{cm},\ \theta = 45°,\ \phi = 90°)$: Again, the angles do not matter because of the spherical symmetry. We use the part a result to obtain

$$V_{AB} = V_A - V_B = 0.081\left[\frac{1}{0.02} - \frac{1}{0.03}\right] = \underline{1.36\ \text{V}}$$

4.10. Express the potential field of an infinite line charge

a) with zero reference at $\rho = \rho_0$: We write in general:

$$V_\ell(\rho) = -\int \frac{\rho_L}{2\pi\epsilon_0\rho}\,d\rho + C_1 = -\frac{\rho_L}{2\pi\epsilon_0}\ln(\rho) + C_1 = 0 \text{ at } \rho = \rho_0$$

Therefore

$$C_1 = \frac{\rho_L}{2\pi\epsilon_0}\ln(\rho_0)$$

and finally

$$V_\ell(\rho) = \frac{\rho_L}{2\pi\epsilon_0}[\ln(\rho_0) - \ln(\rho)] = \underline{\frac{\rho_L}{2\pi\epsilon_0}\ln\left(\frac{\rho_0}{\rho}\right)}$$

b) with $V = V_0$ at $\rho = \rho_0$: Using the reasoning of part a, we have

$$V_\ell(\rho_0) = V_0 = \frac{\rho_L}{2\pi\epsilon_0}\ln(\rho_0) + C_2 \ \Rightarrow\ C_2 = V_0 + \frac{\rho_L}{2\pi\epsilon_0}\ln(\rho_0)$$

and finally

$$V_\ell(\rho) = \underline{\frac{\rho_L}{2\pi\epsilon_0}\ln\left(\frac{\rho_0}{\rho}\right) + V_0}$$

c) Can the zero reference be placed at infinity? Why? Answer: <u>No</u>, because we would have a potential that is proportional to the undefined $\ln(\infty/\rho)$.

4.11. Let a uniform surface charge density of $5\,\text{nC/m}^2$ be present at the $z = 0$ plane, a uniform line charge density of $8\,\text{nC/m}$ be located at $x = 0$, $z = 4$, and a point charge of $2\,\mu\text{C}$ be present at $P(2, 0, 0)$. If $V = 0$ at $M(0, 0, 5)$, find V at $N(1, 2, 3)$: We need to find a potential function for the combined charges which is zero at M. That for the point charge we know to be

$$V_p(r) = \frac{Q}{4\pi\epsilon_0 r}$$

Potential functions for the sheet and line charges can be found by taking indefinite integrals of the electric fields for those distributions. For the line charge, we have

$$V_l(\rho) = -\int \frac{\rho_l}{2\pi\epsilon_0\rho}d\rho + C_1 = -\frac{\rho_l}{2\pi\epsilon_0}\ln(\rho) + C_1$$

For the sheet charge, we have

$$V_s(z) = -\int \frac{\rho_s}{2\epsilon_0}dz + C_2 = -\frac{\rho_s}{2\epsilon_0}z + C_2$$

The total potential function will be the sum of the three. Combining the integration constants, we obtain:

$$V = \frac{Q}{4\pi\epsilon_0 r} - \frac{\rho_l}{2\pi\epsilon_0}\ln(\rho) - \frac{\rho_s}{2\epsilon_0}z + C$$

The terms in this expression are not referenced to a common origin, since the charges are at different positions. The parameters r, ρ, and z are *scalar distances* from the charges, and will be treated as such here. To evaluate the constant, C, we first look at point M, where $V_T = 0$. At M, $r = \sqrt{2^2 + 5^2} = \sqrt{29}$, $\rho = 1$, and $z = 5$. We thus have

$$0 = \frac{2 \times 10^{-6}}{4\pi\epsilon_0\sqrt{29}} - \frac{8 \times 10^{-9}}{2\pi\epsilon_0}\ln(1) - \frac{5 \times 10^{-9}}{2\epsilon_0}5 + C \quad \Rightarrow \quad C = -1.93 \times 10^3 \text{ V}$$

At point N, $r = \sqrt{1+4+9} = \sqrt{14}$, $\rho = \sqrt{2}$, and $z = 3$. The potential at N is thus

$$V_N = \frac{2 \times 10^{-6}}{4\pi\epsilon_0\sqrt{14}} - \frac{8 \times 10^{-9}}{2\pi\epsilon_0}\ln(\sqrt{2}) - \frac{5 \times 10^{-9}}{2\epsilon_0}(3) - 1.93 \times 10^3 = 1.98 \times 10^3 \text{ V} = \underline{1.98\,\text{kV}}$$

4.12. In spherical coordinates, $\mathbf{E} = 2r/(r^2 + a^2)^2\,\mathbf{a}_r$ V/m. Find the potential at any point, using the reference

a) $V = 0$ at infinity: We write in general

$$V(r) = -\int \frac{2r\,dr}{(r^2 + a^2)^2} + C = \frac{1}{r^2 + a^2} + C$$

With a zero reference at $r \to \infty$, $C = 0$ and therefore $V(r) = \underline{1/(r^2 + a^2)}$.

b) $V = 0$ at $r = 0$: Using the general expression, we find

$$V(0) = \frac{1}{a^2} + C = 0 \quad \Rightarrow \quad C = -\frac{1}{a^2}$$

Therefore

$$V(r) = \frac{1}{r^2 + a^2} - \frac{1}{a^2} = \frac{-r^2}{a^2(r^2 + a^2)}$$

c) $V = 100$V at $r = a$: Here, we find

$$V(a) = \frac{1}{2a^2} + C = 100 \quad \Rightarrow \quad C = 100 - \frac{1}{2a^2}$$

Therefore

$$V(r) = \frac{1}{r^2 + a^2} - \frac{1}{2a^2} + 100 = \frac{a^2 - r^2}{2a^2(r^2 + a^2)} + 100$$

4.13. Three identical point charges of 4 pC each are located at the corners of an equilateral triangle 0.5 mm on a side in free space. How much work must be done to move one charge to a point equidistant from the other two and on the line joining them? This will be the magnitude of the charge times the potential difference between the finishing and starting positions, or

$$W = \frac{(4 \times 10^{-12})^2}{2\pi\epsilon_0}\left[\frac{1}{2.5} - \frac{1}{5}\right] \times 10^4 = 5.76 \times 10^{-10}\text{ J} = \underline{576\,\text{pJ}}$$

4.14. Given the electric field $\mathbf{E} = (y+1)\mathbf{a}_x + (x-1)\mathbf{a}_y + 2\mathbf{a}_z$, find the potential difference between the points

a) (2,-2,-1) and (0,0,0): We choose a path along which motion occurs in one coordinate direction at a time. Starting at the origin, first move along x from 0 to 2, where $y = 0$; then along y from 0 to -2, where x is 2; then along z from 0 to -1. The setup is

$$V_b - V_a = -\int_0^2 (y+1)\Big|_{y=0} dx \; - \int_0^{-2} (x-1)\Big|_{x=2} dy \; - \int_0^{-1} 2\,dz = \underline{2}$$

b) (3,2,-1) and (-2,-3,4): Following similar reasoning,

$$V_b - V_a = -\int_{-2}^{3} (y+1)\Big|_{y=-3} dx \; - \int_{-3}^{2} (x-1)\Big|_{x=3} dy \; - \int_4^{-1} 2\,dz = \underline{10}$$

4.15. Two uniform line charges, 8 nC/m each, are located at $x = 1$, $z = 2$, and at $x = -1$, $y = 2$ in free space. If the potential at the origin is 100 V, find V at $P(4,1,3)$: The net potential function for the two charges would in general be:

$$V = -\frac{\rho_l}{2\pi\epsilon_0}\ln(R_1) - \frac{\rho_l}{2\pi\epsilon_0}\ln(R_2) + C$$

At the origin, $R_1 = R_2 = \sqrt{5}$, and $V = 100$ V. Thus, with $\rho_l = 8 \times 10^{-9}$,

$$100 = -2\frac{(8\times 10^{-9})}{2\pi\epsilon_0}\ln(\sqrt{5}) + C \quad\Rightarrow\quad C = 331.6 \text{ V}$$

At $P(4,1,3)$, $R_1 = |(4,1,3) - (1,1,2)| = \sqrt{10}$ and $R_2 = |(4,1,3) - (-1,2,3)| = \sqrt{26}$. Therefore

$$V_P = -\frac{(8\times 10^{-9})}{2\pi\epsilon_0}\left[\ln(\sqrt{10}) + \ln(\sqrt{26})\right] + 331.6 = \underline{-68.4 \text{ V}}$$

4.16. The potential at any point in space is given in cylindrical coordinates by $V = (k/\rho^2)\cos(b\phi)$ V/m, where k and b are constants.

a) Where is the zero reference for potential? This will occur at $\underline{\rho \to \infty}$, or whenever $\cos(b\phi) = 0$, which gives $\phi = \underline{(2m-1)\pi/2b}$, where $m = 1, 2, 3...$

b) Find the vector electric field intensity at any point (ρ, ϕ, z). We use

$$\mathbf{E}(\rho,\phi,z) = -\nabla V = -\frac{\partial V}{\partial \rho}\mathbf{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi = \frac{k}{\rho^3}\left[2\cos(b\phi)\,\mathbf{a}_\rho + b\,\sin(b\phi)\,\mathbf{a}_\phi\right]$$

4.17. Uniform surface charge densities of 6 and 2 nC/m^2 are present at $\rho = 2$ and 6 cm respectively, in free space. Assume $V = 0$ at $\rho = 4$ cm, and calculate V at:

a) $\rho = 5$ cm: Since $V = 0$ at 4 cm, the potential at 5 cm will be the potential difference between points 5 and 4:

$$V_5 = -\int_4^5 \mathbf{E}\cdot d\mathbf{L} = -\int_4^5 \frac{a\rho_{sa}}{\epsilon_0 \rho}\,d\rho = -\frac{(.02)(6\times 10^{-9})}{\epsilon_0}\ln\left(\frac{5}{4}\right) = \underline{-3.026 \text{ V}}$$

b) $\rho = 7$ cm: Here we integrate piecewise from $\rho = 4$ to $\rho = 7$:

$$V_7 = -\int_4^6 \frac{a\rho_{sa}}{\epsilon_0 \rho}\,d\rho - \int_6^7 \frac{(a\rho_{sa} + b\rho_{sb})}{\epsilon_0 \rho}\,d\rho$$

With the given values, this becomes

$$V_7 = -\left[\frac{(.02)(6\times 10^{-9})}{\epsilon_0}\right]\ln\left(\frac{6}{4}\right) - \left[\frac{(.02)(6\times 10^{-9}) + (.06)(2\times 10^{-9})}{\epsilon_0}\right]\ln\left(\frac{7}{6}\right)$$
$$= \underline{-9.678\,\text{V}}$$

4.18. Find the potential at the origin produced by a line charge $\rho_L = kx/(x^2 + a^2)$ extending along the x axis from $x = a$ to $+\infty$, where $a > 0$. Assume a zero reference at infinity.

Think of the line charge as an array of point charges, each of charge $dq = \rho_L dx$, and each having potential at the origin of $dV = \rho_L dx/(4\pi\epsilon_0 x)$. The total potential at the origin is then the sum of all these potentials, or

$$V = \int_a^\infty \frac{\rho_L\,dx}{4\pi\epsilon_0 x} = \int_a^\infty \frac{k\,dx}{4\pi\epsilon_0(x^2 + a^2)} = \frac{k}{4\pi\epsilon_0 a}\tan^{-1}\left(\frac{x}{a}\right)_a^\infty = \frac{k}{4\pi\epsilon_0 a}\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{k}{16\epsilon_0 a}$$

4.19. The annular surface, $1\,\text{cm} < \rho < 3\,\text{cm}$, $z = 0$, carries the nonuniform surface charge density $\rho_s = 5\rho\,\text{nC/m}^2$. Find V at $P(0, 0, 2\,\text{cm})$ if $V = 0$ at infinity: We use the superposition integral form:

$$V_P = \int\int \frac{\rho_s\,da}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|}$$

where $\mathbf{r} = z\mathbf{a}_z$ and $\mathbf{r}' = \rho\mathbf{a}_\rho$. We integrate over the surface of the annular region, with $da = \rho\,d\rho\,d\phi$. Substituting the given values, we find

$$V_P = \int_0^{2\pi}\int_{.01}^{.03} \frac{(5\times 10^{-9})\rho^2\,d\rho\,d\phi}{4\pi\epsilon_0\sqrt{\rho^2 + z^2}}$$

Substituting $z = .02$, and using tables, the integral evaluates as

$$V_P = \left[\frac{(5\times 10^{-9})}{2\epsilon_0}\right]\left[\frac{\rho}{2}\sqrt{\rho^2 + (.02)^2} - \frac{(.02)^2}{2}\ln(\rho + \sqrt{\rho^2 + (.02)^2})\right]_{.01}^{.03} = \underline{.081\,\text{V}}$$

4.20. A point charge Q is located at the origin. Express the potential in both rectangular and cylindrical coordinates, and use the gradient operation in that coordinate system to find the electric field intensity. The result may be checked by conversion to spherical coordinates.

The potential is expressed in spherical, rectangular, and cylindrical coordinates respectively as:

$$V = \frac{Q}{4\pi\epsilon_0 r^2} = \frac{Q}{4\pi\epsilon_0(x^2 + y^2 + z^2)^{1/2}} = \frac{Q}{4\pi\epsilon_0(\rho^2 + z^2)^{1/2}}$$

Now, working with rectangular coordinates

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial x}\mathbf{a}_x - \frac{\partial V}{\partial y}\mathbf{a}_y - \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{Q}{4\pi\epsilon_0}\left[\frac{x\,\mathbf{a}_x + y\,\mathbf{a}_y + z\,\mathbf{a}_z}{(x^2 + y^2 + z^2)^{3/2}}\right]$$

4.20. (continued)

Now, converting this field to spherical components, we find

$$E_r = \mathbf{E}\cdot\mathbf{a}_r = \frac{Q}{4\pi\epsilon_0}\left[\frac{r\sin\theta\cos\phi(\mathbf{a}_x\cdot\mathbf{a}_r)+r\sin\theta\sin\phi(\mathbf{a}_y\cdot\mathbf{a}_r)+r\cos\theta(\mathbf{a}_z\cdot\mathbf{a}_r)}{r^3}\right]$$
$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{\sin^2\theta\cos^2\phi+\sin^2\theta\sin^2\phi+\cos^2\theta}{r^2}\right] = \frac{Q}{4\pi\epsilon_0 r^2}$$

Continuing:

$$E_\theta = \mathbf{E}\cdot\mathbf{a}_\theta = \frac{Q}{4\pi\epsilon_0}\left[\frac{r\sin\theta\cos\phi(\mathbf{a}_x\cdot\mathbf{a}_\theta)+r\sin\theta\sin\phi(\mathbf{a}_y\cdot\mathbf{a}_\theta)+r\cos\theta(\mathbf{a}_z\cdot\mathbf{a}_\theta)}{r^3}\right]$$
$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{\sin\theta\cos\theta\cos^2\phi+\sin\theta\cos\theta\sin^2\phi-\cos\theta\sin\theta}{r^2}\right] = 0$$

Finally

$$E_\phi = \mathbf{E}\cdot\mathbf{a}_\phi = \frac{Q}{4\pi\epsilon_0}\left[\frac{r\sin\theta\cos\phi(\mathbf{a}_x\cdot\mathbf{a}_\phi)+r\sin\theta\sin\phi(\mathbf{a}_y\cdot\mathbf{a}_\phi)+r\cos\theta(\mathbf{a}_z\cdot\mathbf{a}_\phi)}{r^3}\right]$$
$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{\sin\theta\cos\phi(-\sin\phi)+\sin\theta\sin\phi\cos\phi+0}{r^2}\right] = 0 \quad \text{check}$$

Now, in cylindrical we have in this case

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial\rho}\,\mathbf{a}_\rho - \frac{\partial V}{\partial z}\,\mathbf{a}_z = \frac{Q}{4\pi\epsilon_0}\left[\frac{\rho\,\mathbf{a}_\rho + z\,\mathbf{a}_z}{(\rho^2+z^2)^{3/2}}\right]$$

Converting to spherical components, we find

$$E_r = \frac{Q}{4\pi\epsilon_0}\left[\frac{r\sin\theta(\mathbf{a}_\rho\cdot\mathbf{a}_r)+r\cos\theta(\mathbf{a}_z\cdot\mathbf{a}_r)}{r^3}\right] = \frac{Q}{4\pi\epsilon_0}\left[\frac{\sin^2\theta+\cos^2\theta}{r^2}\right] = \frac{Q}{4\pi\epsilon_0 r^2}$$

$$E_\theta = \frac{Q}{4\pi\epsilon_0}\left[\frac{r\sin\theta(\mathbf{a}_\rho\cdot\mathbf{a}_\theta)+r\cos\theta(\mathbf{a}_z\cdot\mathbf{a}_\theta)}{r^3}\right] = \frac{Q}{4\pi\epsilon_0}\left[\frac{\sin\theta\cos\theta+\cos\theta(-\sin\theta)}{r^2}\right] = 0$$

$$E_\phi = \frac{Q}{4\pi\epsilon_0}\left[\frac{r\sin\theta(\mathbf{a}_\rho\cdot\mathbf{a}_\phi)+r\cos\theta(\mathbf{a}_z\cdot\mathbf{a}_\phi)}{r^3}\right] = 0 \quad \text{check}$$

4.21. Let $V = 2xy^2z^3 + 3\ln(x^2+2y^2+3z^2)$ V in free space. Evaluate each of the following quantities at $P(3,2,-1)$:

a) V: Substitute P directly to obtain: $V = \underline{-15.0\,\text{V}}$

b) $|V|$. This will be just $\underline{15.0\,\text{V}}$.

c) $\mathbf{E}$: We have

$$\mathbf{E}\Big|_P = -\nabla V\Big|_P = -\left[\left(2y^2z^3+\frac{6x}{x^2+2y^2+3z^2}\right)\mathbf{a}_x + \left(4xyz^3+\frac{12y}{x^2+2y^2+3z^2}\right)\mathbf{a}_y\right.$$
$$\left.+\left(6xy^2z^2+\frac{18z}{x^2+2y^2+3z^2}\right)\mathbf{a}_z\right]_P = \underline{7.1\mathbf{a}_x + 22.8\mathbf{a}_y - 71.1\mathbf{a}_z\ \text{V/m}}$$

4.21d) $|\mathbf{E}|_P$: taking the magnitude of the part c result, we find $|\mathbf{E}|_P = \underline{75.0\,\text{V/m}}$.

e) $\mathbf{a}_N$: By definition, this will be

$$\mathbf{a}_N\Big|_P = -\frac{\mathbf{E}}{|\mathbf{E}|} = \underline{-0.095\,\mathbf{a}_x - 0.304\,\mathbf{a}_y + 0.948\,\mathbf{a}_z}$$

f) $\mathbf{D}$: This is $\mathbf{D}\Big|_P = \epsilon_0\mathbf{E}\Big|_P = \underline{62.8\,\mathbf{a}_x + 202\,\mathbf{a}_y - 629\,\mathbf{a}_z\ \text{pC/m}^2}$.

4.22. A certain potential field is given in spherical coordinates by $V = V_0(r/a)\sin\theta$. Find the total charge contained within the region $r < a$: We first find the electric field through

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\mathbf{a}_r - \frac{1}{r}\frac{\partial V}{\partial \theta} = -\frac{V_0}{a}\left[\sin\theta\,\mathbf{a}_r + \cos\theta\,\mathbf{a}_\theta\right]$$

The requested charge is now the net outward flux of $\mathbf{D} = \epsilon_0\mathbf{E}$ through the spherical shell of radius a (with outward normal $\mathbf{a}_r$):

$$Q = \int_S \mathbf{D}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi} \epsilon_0\mathbf{E}\cdot\mathbf{a}_r\, a^2\sin\theta\, d\theta\, d\phi = -2\pi a V_0\epsilon_0 \int_0^{\pi}\sin^2\theta\, d\theta = \underline{-\pi^2 a\epsilon_0 V_0\ \ \text{C}}$$

The same result can be found (as expected) by taking the divergence of $\mathbf{D}$ and integrating over the spherical volume:

$$\nabla\cdot\mathbf{D} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\epsilon_0 V_0}{a}\sin\theta\right) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\epsilon_0 V_0}{a}\cos\theta\sin\theta\right) = -\frac{\epsilon_0 V_0}{ra}\left[2\sin\theta + \frac{\cos(2\theta)}{\sin\theta}\right]$$

$$= -\frac{\epsilon_0 V_0}{ra\sin\theta}\left[2\sin^2\theta + 1 - 2\sin^2\theta\right] = \frac{-\epsilon_0 V_0}{ra\sin\theta} = \rho_v$$

Now

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{-\epsilon_0 V_0}{ra\sin\theta} r^2\sin\theta\ dr\, d\theta\, d\phi = \frac{-2\pi^2\epsilon_0 V_0}{a}\int_0^{a} r\,dr = \underline{-\pi^2 a\epsilon_0 V_0\ \ \text{C}}$$

4.23. It is known that the potential is given as $V = 80\rho^{.6}$ V. Assuming free space conditions, find:

a) $\mathbf{E}$: We find this through

$$\mathbf{E} = -\nabla V = -\frac{dV}{d\rho}\mathbf{a}_\rho = \underline{-48\rho^{-.4}\,\text{V/m}}$$

b) the volume charge density at $\rho = .5$ m: Using $\mathbf{D} = \epsilon_0\mathbf{E}$, we find the charge density through

$$\rho_v\Big|_{.5} = [\nabla\cdot\mathbf{D}]_{.5} = \left(\frac{1}{\rho}\right)\frac{d}{d\rho}(\rho D_\rho)\Big|_{.5} = -28.8\epsilon_0\rho^{-1.4}\Big|_{.5} = \underline{-673\,\text{pC/m}^3}$$

c) the total charge lying within the closed surface $\rho = .6$, $0 < z < 1$: The easiest way to do this calculation is to evaluate D_ρ at $\rho = .6$ (noting that it is constant), and then multiply by the cylinder area: Using part a, we have $D_\rho\Big|_{.6} = -48\epsilon_0(.6)^{-.4} = -521\,\text{pC/m}^2$. Thus $Q = -2\pi(.6)(1)521 \times 10^{-12}\,\text{C} = \underline{-1.96\,\text{nC}}$.

4.24. The surface defined by the equation $x^3 + y^2 + z = 1000$, where x, y, and z are positive, is an equipotential surface on which the potential is 200 V. If $|\mathbf{E}| = 50$ V/m at the point $P(7, 25, 32)$ on the surface, find $\mathbf{E}$ there:

First, the potential function will be of the form $V(x, y, z) = C_1(x^3 + y^2 + z) + C_2$, where C_1 and C_2 are constants to be determined (C_2 is in fact irrelevant for our purposes). The electric field is now

$$\mathbf{E} = -\nabla V = -C_1(3x^2\,\mathbf{a}_x + 2y\,\mathbf{a}_y + \mathbf{a}_z)$$

And the magnitude of $\mathbf{E}$ is $|\mathbf{E}| = C_1\sqrt{9x^4 + 4y^2 + 1}$, which at the given point will be

$$|\mathbf{E}|_P = C_1\sqrt{9(7)^4 + 4(25)^2 + 1} = 155.27C_1 = 50 \quad\Rightarrow\quad C_1 = 0.322$$

Now substitute C_1 and the given point into the expression for $\mathbf{E}$ to obtain

$$\mathbf{E}_P = \underline{-(47.34\,\mathbf{a}_x + 16.10\,\mathbf{a}_y + 0.32\,\mathbf{a}_z)}$$

The other constant, C_2, is needed to assure a potential of 200 V at the given point.

4.25. Within the cylinder $\rho = 2$, $0 < z < 1$, the potential is given by $V = 100 + 50\rho + 150\rho\sin\phi$ V.

a) Find V, $\mathbf{E}$, $\mathbf{D}$, and ρ_v at $P(1, 60^\circ, 0.5)$ in free space: First, substituting the given point, we find $V_P = \underline{279.9\,\text{V}}$. Then,

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\mathbf{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi = -\left[50 + 150\sin\phi\right]\mathbf{a}_\rho - \left[150\cos\phi\right]\mathbf{a}_\phi$$

Evaluate the above at P to find $\mathbf{E}_P = \underline{-179.9\mathbf{a}_\rho - 75.0\mathbf{a}_\phi\ \text{V/m}}$

Now $\mathbf{D} = \epsilon_0\mathbf{E}$, so $\mathbf{D}_P = \underline{-1.59\mathbf{a}_\rho - .664\mathbf{a}_\phi\ \text{nC/m}^2}$. Then

$$\rho_v = \nabla\cdot\mathbf{D} = \left(\frac{1}{\rho}\right)\frac{d}{d\rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} = \left[-\frac{1}{\rho}(50 + 150\sin\phi) + \frac{1}{\rho}150\sin\phi\right]\epsilon_0 = -\frac{50}{\rho}\epsilon_0\,\text{C}$$

At P, this is $\rho_{vP} = \underline{-443\ \text{pC/m}^3}$.

b) How much charge lies within the cylinder? We will integrate ρ_v over the volume to obtain:

$$Q = \int_0^1\int_0^{2\pi}\int_0^2 -\frac{50\epsilon_0}{\rho}\rho\,d\rho\,d\phi\,dz = -2\pi(50)\epsilon_0(2) = \underline{-5.56\,\text{nC}}$$

4.26. Let us assume that we have a very thin, square, imperfectly conducting plate 2m on a side, located in the plane $z = 0$ with one corner at the origin such that it lies entirely within the first quadrant. The potential at any point in the plate is given as $V = -e^{-x}\sin y$.

a) An electron enters the plate at $x = 0$, $y = \pi/3$ with zero initial velocity; in what direction is its initial movement? We first find the electric field associated with the given potential:

$$\mathbf{E} = -\nabla V = -e^{-x}[\sin y\,\mathbf{a}_x - \cos y\,\mathbf{a}_y]$$

Since we have an electron, its motion is opposite that of the field, so the direction on entry is that of $-\mathbf{E}$ at $(0, \pi/3)$, or $\underline{\sqrt{3}/2\,\mathbf{a}_x - 1/2\,\mathbf{a}_y}$.

b) Because of collisions with the particles in the plate, the electron achieves a relatively low velocity and little acceleration (the work that the field does on it is converted largely into heat). The electron therefore moves approximately along a streamline. Where does it leave the plate and in what direction is it moving at the time? Considering the result of part *a*, we would expect the exit to occur along the bottom edge of the plate. The equation of the streamline is found through

$$\frac{E_y}{E_x} = \frac{dy}{dx} = -\frac{\cos y}{\sin y} \quad \Rightarrow \quad x = -\int \tan y\, dy + C = \ln(\cos y) + C$$

At the entry point $(0, \pi/3)$, we have $0 = \ln[\cos(\pi/3)] + C$, from which $C = 0.69$. Now, along the bottom edge ($y = 0$), we find $x = 0.69$, and so the exit point is $\underline{(0.69, 0)}$. From the field expression evaluated at the exit point, we find the direction on exit to be $\underline{-\mathbf{a}_y}$.

4.27. Two point charges, 1 nC at $(0, 0, 0.1)$ and -1 nC at $(0, 0, -0.1)$, are in free space.

a) Calculate V at $P(0.3, 0, 0.4)$: Use

$$V_P = \frac{q}{4\pi\epsilon_0|\mathbf{R}^+|} - \frac{q}{4\pi\epsilon_0|\mathbf{R}^-|}$$

where $\mathbf{R}^+ = (.3, 0, .3)$ and $\mathbf{R}^- = (.3, 0, .5)$, so that $|\mathbf{R}^+| = 0.424$ and $|\mathbf{R}^-| = 0.583$. Thus

$$V_P = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{1}{.424} - \frac{1}{.583}\right] = \underline{5.78\,\text{V}}$$

b) Calculate $|\mathbf{E}|$ at P: Use

$$\mathbf{E}_P = \frac{q(.3\mathbf{a}_x + .3\mathbf{a}_z)}{4\pi\epsilon_0(.424)^3} - \frac{q(.3\mathbf{a}_x + .5\mathbf{a}_z)}{4\pi\epsilon_0(.583)^3} = \frac{10^{-9}}{4\pi\epsilon_0}[2.42\mathbf{a}_x + 1.41\mathbf{a}_z]\ \text{V/m}$$

Taking the magnitude of the above, we find $|\mathbf{E}_P| = \underline{25.2\,\text{V/m}}$.

c) Now treat the two charges as a dipole at the origin and find V at P: In spherical coordinates, P is located at $r = \sqrt{.3^2 + .4^2} = .5$ and $\theta = \sin^{-1}(.3/.5) = 36.9°$. Assuming a dipole in far-field, we have

$$V_P = \frac{qd\cos\theta}{4\pi\epsilon_0 r^2} = \frac{10^{-9}(.2)\cos(36.9°)}{4\pi\epsilon_0(.5)^2} = \underline{5.76\,\text{V}}$$

4.28. Use the electric field intensity of the dipole (Sec. 4.7, Eq. (36)) to find the difference in potential between points at θ_a and θ_b, each point having the same r and ϕ coordinates. Under what conditions does the answer agree with Eq. (34), for the potential at θ_a?

We perform a line integral of Eq. (36) along an arc of constant r and ϕ:

$$V_{ab} = -\int_{\theta_b}^{\theta_a} \frac{qd}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta\right]\cdot\mathbf{a}_\theta\, r\,d\theta = -\int_{\theta_b}^{\theta_a} \frac{qd}{4\pi\epsilon_0 r^2}\sin\theta\,d\theta$$
$$= \frac{qd}{4\pi\epsilon_0 r^2}\left[\cos\theta_a - \cos\theta_b\right]$$

This result agrees with Eq. (34) if θ_a (the ending point in the path) is 90° (the xy plane). Under this condition, we note that if $\theta_b > 90°$, positive work is done when moving (against the field) to the xy plane; if $\theta_b < 90°$, negative work is done since we move with the field.

4.29. A dipole having a moment $\mathbf{p} = 3\mathbf{a}_x - 5\mathbf{a}_y + 10\mathbf{a}_z$ nC · m is located at $Q(1, 2, -4)$ in free space. Find V at $P(2, 3, 4)$: We use the general expression for the potential in the far field:

$$V = \frac{\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3}$$

where $\mathbf{r} - \mathbf{r}' = P - Q = (1, 1, 8)$. So

$$V_P = \frac{(3\mathbf{a}_x - 5\mathbf{a}_y + 10\mathbf{a}_z)\cdot(\mathbf{a}_x + \mathbf{a}_y + 8\mathbf{a}_z)\times 10^{-9}}{4\pi\epsilon_0[1^2+1^2+8^2]^{1.5}} = \underline{1.31\text{ V}}$$

4.30. A dipole for which $\mathbf{p} = 10\epsilon_0\,\mathbf{a}_z$ C · m is located at the origin. What is the equation of the surface on which $E_z = 0$ but $\mathbf{E} \neq 0$?

First we find the z component:

$$E_z = \mathbf{E}\cdot\mathbf{a}_z = \frac{10}{4\pi r^3}\left[2\cos\theta\,(\mathbf{a}_r\cdot\mathbf{a}_z) + \sin\theta\,(\mathbf{a}_\theta\cdot\mathbf{a}_z)\right] = \frac{5}{2\pi r^3}\left[2\cos^2\theta - \sin^2\theta\right]$$

This will be zero when $\left[2\cos^2\theta - \sin^2\theta\right] = 0$. Using identities, we write

$$2\cos^2\theta - \sin^2\theta = \frac{1}{2}[1 + 3\cos(2\theta)]$$

The above becomes zero on the cone surfaces, $\underline{\theta = 54.7°}$ and $\underline{\theta = 125.3°}$.

4.31. A potential field in free space is expressed as $V = 20/(xyz)$ V.

a) Find the total energy stored within the cube $1 < x, y, z < 2$. We integrate the energy density over the cube volume, where $w_E = (1/2)\epsilon_0\mathbf{E}\cdot\mathbf{E}$, and where

$$\mathbf{E} = -\nabla V = 20\left[\frac{1}{x^2yz}\mathbf{a}_x + \frac{1}{xy^2z}\mathbf{a}_y + \frac{1}{xyz^2}\mathbf{a}_z\right]\text{ V/m}$$

The energy is now

$$W_E = 200\epsilon_0\int_1^2\int_1^2\int_1^2\left[\frac{1}{x^4y^2z^2} + \frac{1}{x^2y^4z^2} + \frac{1}{x^2y^2z^4}\right]dx\,dy\,dz$$

4.31a. (continued)

The integral evaluates as follows:

$$W_E = 200\epsilon_0 \int_1^2 \int_1^2 \left[-\left(\frac{1}{3}\right) \frac{1}{x^3 y^2 z^2} - \frac{1}{xy^4 z^2} - \frac{1}{xy^2 z^4} \right]_1^2 dy\, dz$$

$$= 200\epsilon_0 \int_1^2 \int_1^2 \left[\left(\frac{7}{24}\right) \frac{1}{y^2 z^2} + \left(\frac{1}{2}\right) \frac{1}{y^4 z^2} + \left(\frac{1}{2}\right) \frac{1}{y^2 z^4} \right] dy\, dz$$

$$= 200\epsilon_0 \int_1^2 \left[-\left(\frac{7}{24}\right) \frac{1}{yz^2} - \left(\frac{1}{6}\right) \frac{1}{y^3 z^2} - \left(\frac{1}{2}\right) \frac{1}{yz^4} \right]_1^2 dz$$

$$= 200\epsilon_0 \int_1^2 \left[\left(\frac{7}{48}\right) \frac{1}{z^2} + \left(\frac{7}{48}\right) \frac{1}{z^2} + \left(\frac{1}{4}\right) \frac{1}{z^4} \right] dz$$

$$= 200\epsilon_0 (3) \left[\frac{7}{96} \right] = \underline{387\,\text{pJ}}$$

b) What value would be obtained by assuming a uniform energy density equal to the value at the center of the cube? At $C(1.5, 1.5, 1.5)$ the energy density is

$$w_E = 200\epsilon_0 (3) \left[\frac{1}{(1.5)^4 (1.5)^2 (1.5)^2} \right] = 2.07 \times 10^{-10}\ \text{J/m}^3$$

This, multiplied by a cube volume of 1, produces an energy value of $\underline{207\,\text{pJ}}$.

4.32. Using Eq. (36), a) find the energy stored in the dipole field in the region $r > a$:

We start with

$$\mathbf{E}(r,\theta) = \frac{qd}{4\pi\epsilon_0 r^3} \left[2\cos\theta\, \mathbf{a}_r + \sin\theta\, \mathbf{a}_\theta \right]$$

Then the energy will be

$$W_e = \int_{vol} \frac{1}{2}\epsilon_0 \mathbf{E}\cdot\mathbf{E}\, dv = \int_0^{2\pi} \int_0^{\pi} \int_a^{\infty} \frac{(qd)^2}{32\pi^2 \epsilon_0 r^6} \underbrace{\left[4\cos^2\theta + \sin^2\theta\right]}_{3\cos^2\theta + 1} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{-2\pi (qd)^2}{32\pi^2 \epsilon_0} \frac{1}{3r^3}\Big|_a^{\infty} \int_0^{\pi} \left[3\cos^2\theta + 1\right] \sin\theta\, d\theta = \frac{(qd)^2}{48\pi^2 \epsilon_0 a^3} \underbrace{\left[-\cos^3\theta - \cos\theta\right]_0^{\pi}}_{4}$$

$$= \frac{(qd)^2}{12\pi\epsilon_0 a^3}\ \text{J}$$

b) Why can we not let a approach zero as a limit? From the above result, a singularity in the energy occurs as $a \to 0$. More importantly, a cannot be too small, or the original far-field assumption used to derive Eq. (36) ($a >> d$) will not hold, and so the field expression will not be valid.

4.33. A copper sphere of radius 4 cm carries a uniformly-distributed total charge of $5\,\mu$C in free space.

a) Use Gauss' law to find **D** external to the sphere: with a spherical Gaussian surface at radius r, D will be the total charge divided by the area of this sphere, and will be $\mathbf{a}_r$-directed. Thus

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r = \frac{5 \times 10^{-6}}{4\pi r^2}\mathbf{a}_r\ \text{C/m}^2$$

4.33b) Calculate the total energy stored in the electrostatic field: Use

$$W_E = \int_{vol} \frac{1}{2}\mathbf{D}\cdot\mathbf{E}\,dv = \int_0^{2\pi}\int_0^{\pi}\int_{.04}^{\infty} \frac{1}{2}\frac{(5\times 10^{-6})^2}{16\pi^2\epsilon_0 r^4}\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= (4\pi)\left(\frac{1}{2}\right)\frac{(5\times 10^{-6})^2}{16\pi^2\epsilon_0}\int_{.04}^{\infty}\frac{dr}{r^2} = \frac{25\times 10^{-12}}{8\pi\epsilon_0}\frac{1}{.04} = \underline{2.81\,\mathrm{J}}$$

c) Use $W_E = Q^2/(2C)$ to calculate the capacitance of the isolated sphere: We have

$$C = \frac{Q^2}{2W_E} = \frac{(5\times 10^{-6})^2}{2(2.81)} = 4.45\times 10^{-12}\,\mathrm{F} = \underline{4.45\,\mathrm{pF}}$$

4.34. A sphere of radius a contains volume charge of uniform density ρ_0 C/m^3. Find the total stored energy by applying

a) Eq. (43): We first need the potential everywhere inside the sphere. The electric field inside and outside is readily found from Gauss's law:

$$\mathbf{E}_1 = \frac{\rho_0 r}{3\epsilon_0}\,\mathbf{a}_r \quad r \le a \quad \text{and} \quad \mathbf{E}_2 = \frac{\rho_0 a^3}{3\epsilon_0 r^2}\,\mathbf{a}_r \quad r \ge a$$

The potential at position r inside the sphere is now the work done in moving a unit positive point charge from infinity to position r:

$$V(r) = -\int_{\infty}^{a}\mathbf{E}_2\cdot\mathbf{a}_r\,dr - \int_a^r \mathbf{E}_1\cdot\mathbf{a}_r\,dr' = -\int_{\infty}^{a}\frac{\rho_0 a^3}{3\epsilon_0 r^2}\,dr - \int_a^r \frac{\rho_0 r'}{3\epsilon_0}\,dr' = \frac{\rho_0}{6\epsilon_0}\left(3a^2 - r^2\right)$$

Now, using this result in (43) leads to the energy associated with the charge in the sphere:

$$W_e = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\frac{\rho_0^2}{6\epsilon_0}\left(3a^2 - r^2\right) r^2 \sin\theta\, dr\, d\theta\, d\phi = \frac{\pi\rho_0}{3\epsilon_0}\int_0^a\left(3a^2 r^2 - r^4\right)\,dr = \frac{4\pi a^5\rho_0^2}{15\epsilon_0}$$

b) Eq. (45): Using the given fields we find the energy densities

$$w_{e1} = \frac{1}{2}\epsilon_0\mathbf{E}_1\cdot\mathbf{E}_1 = \frac{\rho_0^2 r^2}{18\epsilon_0} \quad r \le a \quad \text{and} \quad w_{e2} = \frac{1}{2}\epsilon_0\mathbf{E}_2\cdot\mathbf{E}_2 = \frac{\rho_0^2 a^6}{18\epsilon_0 r^4} \quad r \ge a$$

We now integrate these over their respective volumes to find the total energy:

$$W_e = \int_0^{2\pi}\int_0^{\pi}\int_0^{a}\frac{\rho_0^2 r^2}{18\epsilon_0} r^2 \sin\theta\, dr\, d\theta\, d\phi + \int_0^{2\pi}\int_0^{\pi}\int_a^{\infty}\frac{\rho_0^2 a^6}{18\epsilon_0 r^4} r^2 \sin\theta\, dr\, d\theta\, d\phi = \frac{4\pi a^5\rho_0^2}{15\epsilon_0}$$

4.35. Four 0.8 nC point charges are located in free space at the corners of a square 4 cm on a side.

a) Find the total potential energy stored: This will be given by

$$W_E = \frac{1}{2}\sum_{n=1}^{4} q_n V_n$$

where V_n in this case is the potential at the location of any one of the point charges that arises from the other three. This will be (for charge 1)

$$V_1 = V_{21} + V_{31} + V_{41} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{.04} + \frac{1}{.04} + \frac{1}{.04\sqrt{2}}\right]$$

Taking the summation produces a factor of 4, since the situation is the same at all four points. Consequently,

$$W_E = \frac{1}{2}(4)q_1 V_1 = \frac{(.8\times 10^{-9})^2}{2\pi\epsilon_0(.04)}\left[2 + \frac{1}{\sqrt{2}}\right] = 7.79\times 10^{-7}\,\text{J} = \underline{0.779\,\mu\text{J}}$$

b) A fifth 0.8 nC charge is installed at the center of the square. Again find the total stored energy: This will be the energy found in part *a* plus the amount of work done in moving the fifth charge into position from infinity. The latter is just the potential at the square center arising from the original four charges, times the new charge value, or

$$\Delta W_E = \frac{4(.8\times 10^{-9})^2}{4\pi\epsilon_0(.04\sqrt{2}/2)} = .813\,\mu\text{J}$$

The total energy is now

$$W_{E\,net} = W_E(\text{part a}) + \Delta W_E = .779 + .813 = \underline{1.59\,\mu\text{J}}$$