CHAPTER 5

5.1. Given the current density $\mathbf{J} = -10^4[\sin(2x)e^{-2y}\mathbf{a}_x + \cos(2x)e^{-2y}\mathbf{a}_y]$ kA/m^2:

a) Find the total current crossing the plane $y = 1$ in the $\mathbf{a}_y$ direction in the region $0 < x < 1$, $0 < z < 2$: This is found through

$$I = \int\int_S \mathbf{J}\cdot\mathbf{n}\Big|_S da = \int_0^2\int_0^1 \mathbf{J}\cdot\mathbf{a}_y\Big|_{y=1} dx\,dz = \int_0^2\int_0^1 -10^4\cos(2x)e^{-2}\,dx\,dz$$
$$= -10^4(2)\frac{1}{2}\sin(2x)\Big|_0^1 e^{-2} = \underline{-1.23\,\text{MA}}$$

b) Find the total current leaving the region $0 < x, x < 1$, $2 < z < 3$ by integrating $\mathbf{J}\cdot\mathbf{dS}$ over the surface of the cube: Note first that current through the top and bottom surfaces will not exist, since $\mathbf{J}$ has no z component. Also note that there will be no current through the $x = 0$ plane, since $J_x = 0$ there. Current will pass through the three remaining surfaces, and will be found through

$$I = \int_2^3\int_0^1 \mathbf{J}\cdot(-\mathbf{a}_y)\Big|_{y=0} dx\,dz + \int_2^3\int_0^1 \mathbf{J}\cdot(\mathbf{a}_y)\Big|_{y=1} dx\,dz + \int_2^3\int_0^1 \mathbf{J}\cdot(\mathbf{a}_x)\Big|_{x=1} dy\,dz$$
$$= 10^4\int_2^3\int_0^1\left[\cos(2x)e^{-0} - \cos(2x)e^{-2}\right] dx\,dz - 10^4\int_2^3\int_0^1 \sin(2)e^{-2y}\,dy\,dz$$
$$= 10^4\left(\frac{1}{2}\right)\sin(2x)\Big|_0^1(3-2)\left[1-e^{-2}\right] + 10^4\left(\frac{1}{2}\right)\sin(2)e^{-2y}\Big|_0^1(3-2) = \underline{0}$$

c) Repeat part b, but use the divergence theorem: We find the net outward current through the surface of the cube by integrating the divergence of $\mathbf{J}$ over the cube volume. We have

$$\nabla\cdot\mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} = -10^{-4}\left[2\cos(2x)e^{-2y} - 2\cos(2x)e^{-2y}\right] = \underline{0} \quad \text{as expected}$$

5.2. A certain current density is given in cylindrical coordinates as $\mathbf{J} = 100e^{-2z}(\rho\mathbf{a}_\rho + \mathbf{a}_z)$ A/m^2. Find the total current passing through each of these surfaces:

a) $z = 0$, $0 \le \rho \le 1$, in the $\mathbf{a}_z$ direction:

$$I_a = \int_S \mathbf{J}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 100e^{-2(0)}(\rho\mathbf{a}_\rho + \mathbf{a}_z)\cdot\mathbf{a}_z\,\rho\,d\rho\,d\phi = \underline{100\pi}$$

where $\mathbf{a}_\rho\cdot\mathbf{a}_z = 0$.

b) $z = 1$, $0 \le \rho \le 1$, in the $\mathbf{a}_z$ direction:

$$I_b = \int_S \mathbf{J}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 100e^{-2(1)}(\rho\mathbf{a}_\rho + \mathbf{a}_z)\cdot\mathbf{a}_z\,\rho\,d\rho\,d\phi = \underline{100\pi e^{-2}}$$

c) closed cylinder defined by $0 \le z \le 1$, $0 \le \rho \le 1$, in an outward direction:

$$I_T = I_b - I_a + \int_0^1\int_0^{2\pi} 100e^{-2z}((1)\mathbf{a}_\rho + \mathbf{a}_z)\cdot\mathbf{a}_\rho\,(1)\,d\phi\,dz = 100\pi(e^{-2}-1) + 100\pi(1-e^{-2}) = \underline{0}$$

5.3. Let

$$\mathbf{J} = \frac{400 \sin\theta}{r^2 + 4}\,\mathbf{a}_r \text{ A/m}^2$$

a) Find the total current flowing through that portion of the spherical surface $r = 0.8$, bounded by $0.1\pi < \theta < 0.3\pi$, $0 < \phi < 2\pi$: This will be

$$I = \int\int \mathbf{J}\cdot\mathbf{n}\Big|_S \,da = \int_0^{2\pi}\int_{.1\pi}^{.3\pi} \frac{400\sin\theta}{(.8)^2+4}(.8)^2 \sin\theta\, d\theta\, d\phi = \frac{400(.8)^2 2\pi}{4.64}\int_{.1\pi}^{.3\pi} \sin^2\, d\theta$$
$$= 346.5 \int_{.1\pi}^{.3\pi} \frac{1}{2}[1-\cos(2\theta)]\,d\theta = \underline{77.4\text{ A}}$$

b) Find the average value of $\mathbf{J}$ over the defined area. The area is

$$\text{Area} = \int_0^{2\pi}\int_{.1\pi}^{.3\pi} (.8)^2 \sin\theta\, d\theta\, d\phi = 1.46\,\text{m}^2$$

The average current density is thus $\mathbf{J}_{avg} = (77.4/1.46)\,\mathbf{a}_r = \underline{53.0\,\mathbf{a}_r\text{ A/m}^2}$.

5.4. Assume that a uniform electron beam of circular cross-section with radius of 0.2 mm is generated by a cathode at $x = 0$ and collected by an anode at $x = 20$ cm. The velocity of the electrons varies with x as $v_x = 10^8 x^{0.5}$ m/s, with x in meters. If the current density at the anode is 10^4 A/m^2, find the volume charge density and the current density as functions of x.

The requirement is that we have constant current throughout the beam path. Since the beam is of constant radius, this means that current density must also be constant, and will have the value $\mathbf{J} = \underline{10^4\,\mathbf{a}_x\text{ A/m}^2}$. Now $\mathbf{J} = \rho_v \mathbf{v} \;\Rightarrow\; \rho_v = J/v = \underline{10^{-4}x^{-0.5}\text{ C/m}^3}$.

5.5. Let

$$\mathbf{J} = \frac{25}{\rho}\,\mathbf{a}_\rho - \frac{20}{\rho^2+0.01}\,\mathbf{a}_z \text{ A/m}^2$$

a) Find the total current crossing the plane $z = 0.2$ in the $\mathbf{a}_z$ direction for $\rho < 0.4$: Use

$$I = \int\int_S \mathbf{J}\cdot\mathbf{n}\Big|_{z=.2} da = \int_0^{2\pi}\int_0^{.4} \frac{-20}{\rho^2+.01}\rho\,d\rho\,d\phi$$
$$= -\left(\frac{1}{2}\right) 20\ln(.01+\rho^2)\Big|_0^{.4}(2\pi) = -20\pi\ln(17) = \underline{-178.0\text{ A}}$$

b) Calculate $\partial\rho_v/\partial t$: This is found using the equation of continuity:

$$\frac{\partial\rho_v}{\partial t} = -\nabla\cdot\mathbf{J} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho J_\rho) + \frac{\partial J_z}{\partial z} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(25) + \frac{\partial}{\partial z}\left(\frac{-20}{\rho^2+.01}\right) = \underline{0}$$

c) Find the outward current crossing the closed surface defined by $\rho = 0.01$, $\rho = 0.4$, $z = 0$, and $z = 0.2$: This will be

$$I = \int_0^{.2}\int_0^{2\pi} \frac{25}{.01}\mathbf{a}_\rho\cdot(-\mathbf{a}_\rho)(.01)\,d\phi\,dz + \int_0^{.2}\int_0^{2\pi}\frac{25}{.4}\mathbf{a}_\rho\cdot(\mathbf{a}_\rho)(.4)\,d\phi\,dz$$
$$+ \int_0^{2\pi}\int_0^{.4}\frac{-20}{\rho^2+.01}\mathbf{a}_z\cdot(-\mathbf{a}_z)\,\rho\,d\rho\,d\phi + \int_0^{2\pi}\int_0^{.4}\frac{-20}{\rho^2+.01}\mathbf{a}_z\cdot(\mathbf{a}_z)\,\rho\,d\rho\,d\phi = \underline{0}$$

since the integrals will cancel each other.

d) Show that the divergence theorem is satisfied for $\mathbf{J}$ and the surface specified in part b. In part c, the net outward flux was found to be zero, and in part b, the divergence of $\mathbf{J}$ was found to be zero (as will be its volume integral). Therefore, the divergence theorem is satisfied.

5.6. The current density in a certain region is approximated by $\mathbf{J} = (0.1/r)\exp\left(-10^6 t\right)\,\mathbf{a}_r$ A/m^2 in spherical coordinates.

a) At $t = 1$ μs, how much current is crossing the surface $r = 5$? At the given time, $I_a = 4\pi(5)^2(0.1/5)e^{-1} = 2\pi e^{-1} = \underline{2.31\text{ A}}$.

b) Repeat for $r = 6$: Again, at 1 μs, $I_b = 4\pi(6)^2(0.1/6)e^{-1} = 2.4\pi e^{-1} = \underline{2.77\text{ A}}$.

c) Use the continuity equation to find $\rho_v(r,t)$, under the assumption that $\rho_v \to 0$ as $t \to \infty$:

$$\nabla\cdot\mathbf{J} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{0.1}{r}e^{-10^6 t}\right) = \frac{0.1}{r^2}e^{-10^6 t} = -\frac{\partial\rho_v}{\partial t}$$

Then

$$\rho_v(r,t) = -\int\frac{0.1}{r^2}e^{-10^6 t}\,dt + f(r) = \frac{10^{-7}}{r^2}e^{-10^6 t} + f(r)$$

Now, $\rho_v \to 0$ as $t \to \infty$; thus $f(r) = 0$. Final answer: $\rho_v(r,t) = \underline{(10^{-7}/r^2)e^{-10^6 t}\text{ C/m}^3}$.

d) Find an expression for the velocity of the charge density.

$$\mathbf{v} = \frac{\mathbf{J}}{\rho_v} = \frac{(0.1/r)e^{-10^6 t}\,\mathbf{a}_r}{(10^{-7}/r^2)e^{-10^6 t}} = \underline{10^6 r\,\mathbf{a}_r\text{ m/s}}$$

5.7. Assuming that there is no transformation of mass to energy or vice-versa, it is possible to write a continuity equation for mass.

a) If we use the continuity equation for charge as our model, what quantities correspond to $\mathbf{J}$ and ρ_v? These would be, respectively, <u>mass flux density</u> in (kg/m^2 – s) and <u>mass density</u> in (kg/m^3).

b) Given a cube 1 cm on a side, experimental data show that the rates at which mass is leaving each of the six faces are 10.25, -9.85, 1.75, -2.00, -4.05, and 4.45 mg/s. If we assume that the cube is an incremental volume element, determine an approximate value for the time rate of change of density at its center. We may write the continuity equation for mass as follows, also invoking the divergence theorem:

$$\int_v \frac{\partial\rho_m}{\partial t}dv = -\int_v \nabla\cdot\mathbf{J}_m\,dv = -\oint_s \mathbf{J}_m\cdot\mathbf{dS}$$

where

$$\oint_s \mathbf{J}_m\cdot\mathbf{dS} = 10.25 - 9.85 + 1.75 - 2.00 - 4.05 + 4.45 = 0.550\text{ mg/s}$$

Treating our 1 cm^3 volume as differential, we find

$$\frac{\partial\rho_m}{\partial t} \doteq -\frac{0.550\times10^{-3}\text{ g/s}}{10^{-6}\text{ m}^3} = \underline{-550\,\text{g/m}^3 - \text{s}}$$

5.8. The conductivity of carbon is about 3×10^4 S/m.

a) What size and shape sample of carbon has a conductance of 3×10^4 S? We know that the conductance is $G = \sigma A/\ell$, where A is the cross-sectional area and ℓ is the length. To make $G = \sigma$, we may use any regular shape whose length is equal to its area. Examples include a square sheet of dimensions $\ell \times \ell$, and of unit thickness (where conductance is measured end-to-end), a block of square cross-section, having length ℓ, and with cross-section dimensions $\sqrt{\ell} \times \sqrt{\ell}$, or a solid cylinder of length ℓ and radius $a = \sqrt{\ell/\pi}$.

b) What is the conductance if every dimension of the sample found in part a is halved? In all three cases mentioned in part a, the conductance is **one-half** the original value if all dimensions are reduced by one-half. This is easily shown using the given formula for conductance.

5.9a. Using data tabulated in Appendix C, calculate the required diameter for a 2-m long nichrome wire that will dissipate an average power of 450 W when 120 V rms at 60 Hz is applied to it: The required resistance will be

$$R = \frac{V^2}{P} = \frac{l}{\sigma(\pi a^2)}$$

Thus the diameter will be

$$d = 2a = 2\sqrt{\frac{lP}{\sigma\pi V^2}} = 2\sqrt{\frac{2(450)}{(10^6)\pi(120)^2}} = 2.8 \times 10^{-4}\,\text{m} = \underline{0.28\,\text{mm}}$$

b) Calculate the rms current density in the wire: The rms current will be $I = 450/120 = 3.75\,\text{A}$. Thus

$$J = \frac{3.75}{\pi\left(2.8 \times 10^{-4}/2\right)^2} = \underline{6.0 \times 10^7\ \text{A/m}^2}$$

5.10. A solid wire of conductivity σ_1 and radius a has a jacket of material having conductivity σ_2, and whose inner radius is a and outer radius is b. Show that the ratio of the current densities in the two materials is independent of a and b.

A constant voltage between the two ends of the wire means that the field within must be constant throughout the wire cross-section. Calling this field E, we have

$$E = \frac{J_1}{\sigma_1} = \frac{J_2}{\sigma_2} \quad \Rightarrow \quad \frac{J_1}{J_2} = \frac{\sigma_1}{\sigma_2}$$

which is independent of the dimensions.

5.11. Two perfectly-conducting cylindrical surfaces of length l are located at $\rho = 3$ and $\rho = 5$ cm. The total current passing radially outward through the medium between the cylinders is 3 A dc.

a) Find the voltage and resistance between the cylinders, and **E** in the region between the cylinders, if a conducting material having $\sigma = 0.05\,\text{S/m}$ is present for $3 < \rho < 5$ cm: Given the current, and knowing that it is radially-directed, we find the current density by dividing it by the area of a cylinder of radius ρ and length l:

$$\mathbf{J} = \frac{3}{2\pi\rho l}\,\mathbf{a}_\rho\ \text{A/m}^2$$

5.11a. (continued)

Then the electric field is found by dividing this result by σ:

$$\mathbf{E} = \frac{3}{2\pi\sigma\rho l}\,\mathbf{a}_\rho = \underline{\frac{9.55}{\rho l}\mathbf{a}_\rho \text{ V/m}}$$

The voltage between cylinders is now:

$$V = -\int_5^3 \mathbf{E}\cdot\mathbf{dL} = \int_3^5 \frac{9.55}{\rho l}\mathbf{a}_\rho\cdot\mathbf{a}_\rho d\rho = \frac{9.55}{l}\ln\left(\frac{5}{3}\right) = \underline{\frac{4.88}{l}\text{ V}}$$

Now, the resistance will be

$$R = \frac{V}{I} = \frac{4.88}{3l} = \underline{\frac{1.63}{l}\ \Omega}$$

b) Show that integrating the power dissipated per unit volume over the volume gives the total dissipated power: We calculate

$$P = \int_v \mathbf{E}\cdot\mathbf{J}\,dv = \int_0^l\int_0^{2\pi}\int_{.03}^{.05}\frac{3^2}{(2\pi)^2\rho^2(.05)l^2}\,\rho\,d\rho\,d\phi\,dz = \frac{3^2}{2\pi(.05)l}\ln\left(\frac{5}{3}\right) = \frac{14.64}{l}\text{ W}$$

We also find the power by taking the product of voltage and current:

$$P = VI = \frac{4.88}{l}(3) = \frac{14.64}{l}\text{ W}$$

which is in agreement with the power density integration.

5.12. Two identical conducting plates, each having area A, are located at $z = 0$ and $z = d$. The region between plates is filled with a material having z-dependent conductivity, $\sigma(z) = \sigma_0 e^{-z/d}$, where σ_0 is a constant. Voltage V_0 is applied to the plate at $z = d$; the plate at $z = 0$ is at zero potential. Find, in terms of the given parameters:

a) the resistance of the material: We start with the differential resistance of a thin slab of the material of thickness dz, which is

$$dR = \frac{dz}{\sigma A} = \frac{e^{z/d}dz}{\sigma_0 A} \quad\text{so that}\quad R = \int dR = \int_0^d \frac{e^{z/d}dz}{\sigma_0 A} = \frac{d}{\sigma_0 A}(e-1) = \frac{1.72d}{\sigma_0 A}\ \Omega$$

b) the total current flowing between plates: We use

$$I = \frac{V_0}{R} = \frac{\sigma_0 A V_0}{1.72\,d}$$

c) the electric field intensity $\mathbf{E}$ within the material: First the current density is

$$\mathbf{J} = -\frac{I}{A}\,\mathbf{a}_z = \frac{-\sigma_0 V_0}{1.72\,d}\,\mathbf{a}_z \quad\text{so that}\quad \mathbf{E} = \frac{\mathbf{J}}{\sigma(z)} = \frac{-V_0 e^{z/d}}{1.72\,d}\,\mathbf{a}_z\text{ V/m}$$

5.13. A hollow cylindrical tube with a rectangular cross-section has external dimensions of 0.5 in by 1 in and a wall thickness of 0.05 in. Assume that the material is brass, for which $\sigma = 1.5 \times 10^7$ S/m. A current of 200 A dc is flowing down the tube.

a) What voltage drop is present across a 1m length of the tube? Converting all measurements to meters, the tube resistance over a 1 m length will be:

$$R_1 = \frac{1}{(1.5 \times 10^7)\left[(2.54)(2.54/2) \times 10^{-4} - 2.54(1 - .1)(2.54/2)(1 - .2) \times 10^{-4}\right]}$$
$$= 7.38 \times 10^{-4}\ \Omega$$

The voltage drop is now $V = IR_1 = 200(7.38 \times 10^{-4} = \underline{0.147\ \text{V}}$.

b) Find the voltage drop if the interior of the tube is filled with a conducting material for which $\sigma = 1.5 \times 10^5$ S/m: The resistance of the filling will be:

$$R_2 = \frac{1}{(1.5 \times 10^5)(1/2)(2.54)^2 \times 10^{-4}(.9)(.8)} = 2.87 \times 10^{-2}\ \Omega$$

The total resistance is now the parallel combination of R_1 and R_2: $R_T = R_1 R_2/(R_1 + R_2) = 7.19 \times 10^{-4}\ \Omega$, and the voltage drop is now $V = 200 R_T = \underline{.144\,\text{V}}$.

5.14. A rectangular conducting plate lies in the xy plane, occupying the region $0 < x < a$, $0 < y < b$. An identical conducting plate is positioned directly above and parallel to the first, at $z = d$. The region between plates is filled with material having conductivity $\sigma(x) = \sigma_0 e^{-x/a}$, where σ_0 is a constant. Voltage V_0 is applied to the plate at $z = d$; the plate at $z = 0$ is at zero potential. Find, in terms of the given parameters:

a) the electric field intensity $\mathbf{E}$ within the material: We know that $\mathbf{E}$ will be z-directed, but the conductivity varies with x. We therefore expect no z variation in $\mathbf{E}$, and also note that the line integral of $\mathbf{E}$ between the bottom and top plates must always give V_0. Therefore $\mathbf{E} = \underline{-V_0/d\,\mathbf{a}_z\ \text{V/m}}$.

b) the total current flowing between plates: We have

$$\mathbf{J} = \sigma(x)\mathbf{E} = \frac{-\sigma_0 e^{-x/a} V_0}{d}\,\mathbf{a}_z$$

Using this, we find

$$I = \int \mathbf{J} \cdot d\mathbf{S} = \int_0^b \int_0^a \frac{-\sigma_0 e^{-x/a} V_0}{d}\,\mathbf{a}_z \cdot (-\mathbf{a}_z)\,dx\,dy = \frac{\sigma_0 ab V_0}{d}(1 - e^{-1}) = \frac{0.63 ab\sigma_0 V_0}{d}\ \text{A}$$

c) the resistance of the material: We use

$$R = \frac{V_0}{I} = \frac{d}{0.63\,ab\,\sigma_0}\ \Omega$$

5.15. Let $V = 10(\rho+1)z^2\cos\phi\,\text{V}$ in free space.

a) Let the equipotential surface $V = 20\,\text{V}$ define a conductor surface. Find the equation of the conductor surface: Set the given potential function equal to 20, to find:

$$\underline{(\rho+1)z^2\cos\phi = 2}$$

b) Find ρ and $\mathbf{E}$ at that point on the conductor surface where $\phi = 0.2\pi$ and $z = 1.5$: At the given values of ϕ and z, we solve the equation of the surface found in part a for ρ, obtaining $\rho = \underline{.10}$. Then

$$\begin{aligned}\mathbf{E} = -\nabla V &= -\frac{\partial V}{\partial \rho}\mathbf{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi - \frac{\partial V}{\partial z}\mathbf{a}_z \\ &= -10z^2\cos\phi\,\mathbf{a}_\rho + 10\,\frac{\rho+1}{\rho}z^2\sin\phi\,\mathbf{a}_\phi - 20(\rho+1)z\cos\phi\,\mathbf{a}_z\end{aligned}$$

Then

$$\mathbf{E}(.10, .2\pi, 1.5) = \underline{-18.2\,\mathbf{a}_\rho + 145\,\mathbf{a}_\phi - 26.7\,\mathbf{a}_z\ \text{V/m}}$$

c) Find $|\rho_s|$ at that point: Since $\mathbf{E}$ is at the perfectly-conducting surface, it will be normal to the surface, so we may write:

$$\rho_s = \epsilon_0 \mathbf{E}\cdot\mathbf{n}\Big|_{\text{surface}} = \epsilon_0\frac{\mathbf{E}\cdot\mathbf{E}}{|\mathbf{E}|} = \epsilon_0\sqrt{\mathbf{E}\cdot\mathbf{E}} = \epsilon_0\sqrt{(18.2)^2 + (145)^2 + (26.7)^2} = \underline{1.32\ \text{nC/m}^2}$$

5.16. In cylindrical coordinates, $V = 1000\rho^2$.

a) If the region $0.1 < \rho < 0.3$ m is free space while the surfaces $\rho = 0.1$ and $\rho = 0.3$ m are conductors, specify the surface charge density on each conductor: First, we find the electric field through

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\,\mathbf{a}_\rho = -2000\rho\,\mathbf{a}_\rho \ \text{ so that }\ \mathbf{D} = \epsilon_0\mathbf{E} = -2000\epsilon_0\rho\,\mathbf{a}_\rho\ \text{C/m}^2$$

Then the charge densities will be

$$\text{inner conductor}:\ \rho_{s1} = \mathbf{D}\cdot\mathbf{a}_\rho\Big|_{\rho=0.1} = -200\epsilon_0\ \text{C/m}^2$$

$$\text{outer conductor}:\ \rho_{s2} = \mathbf{D}\cdot(-\mathbf{a}_\rho)\Big|_{\rho=0.3} = 600\epsilon_0\ \text{C/m}^2$$

b) What is the total charge in a 1-m length of the free space region, $0.1 < \rho < 0.3$ (not including the conductors)? The charge density in the free space region is

$$\rho_v = \nabla\cdot\mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho) = -4000\epsilon_0\ \text{C/m}^3$$

Then the charge in the volume is

$$Q_v = \int_0^1\int_0^{2\pi}\int_{0.1}^{0.3} -4000\epsilon_0\,\rho\,d\rho\,d\phi\,dz = -2\pi(4000)\epsilon_0\frac{1}{2}\left[(0.3)^2 - (0.1)^2\right] = \underline{-320\pi\epsilon_0\ \text{C}}$$

5.16c What is the total charge in a 1-m length, including both surface charges?

First, the net surface charges over a unit length will be

$$Q_{s1}(\rho = 0.1) = -200\epsilon_0[2\pi(0.1)](1) = -40\pi\epsilon_0 \text{ C}$$

and

$$Q_{s2}(\rho = 0.3) = 600\epsilon_0[2\pi(0.3)](1) = 360\pi\epsilon_0 \text{ C}$$

The total charge is now $Q_{tot} = Q_{s1} + Q_{s2} + Q_v = \underline{0}$.

5.17. Given the potential field $V = 100xz/(x^2 + 4)$ V. in free space:

a) Find $\mathbf{D}$ at the surface $z = 0$: Use

$$\mathbf{E} = -\nabla V = -100z\frac{\partial}{\partial x}\left(\frac{x}{x^2+4}\right)\mathbf{a}_x - 0\,\mathbf{a}_y - \frac{100x}{x^2+4}\mathbf{a}_z \text{ V/m}$$

At $z = 0$, we use this to find $\mathbf{D}(z = 0) = \epsilon_0\mathbf{E}(z = 0) = \underline{-100\epsilon_0 x/(x^2 + 4)\,\mathbf{a}_z \text{ C/m}^2}$.

b) Show that the $z = 0$ surface is an equipotential surface: There are two reasons for this: 1) $\mathbf{E}$ at $z = 0$ is everywhere z-directed, and so moving a charge around on the surface involves doing no work; 2) When evaluating the given potential function at $z = 0$, the result is 0 for all x and y.

c) Assume that the $z = 0$ surface is a conductor and find the total charge on that portion of the conductor defined by $0 < x < 2$, $-3 < y < 0$: We have

$$\rho_s = \mathbf{D}\cdot\mathbf{a}_z\Big|_{z=0} = -\frac{100\epsilon_0 x}{x^2+4} \text{ C/m}^2$$

So

$$Q = \int_{-3}^{0}\int_{0}^{2} -\frac{100\epsilon_0 x}{x^2+4}\,dx\,dy = -(3)(100)\epsilon_0\left(\frac{1}{2}\right)\ln(x^2+4)\Big|_0^2 = -150\epsilon_0\ln 2 = \underline{-0.92\,\text{nC}}$$

5.18. A potential field is given as $V = 100\ln\left\{[(x+1)^2 + y^2]/[(x-1)^2 + y^2]\right\}$ V. It is known that point $P(2, 1, 1)$ is on a conductor surface and that the conductor lies in free space. At P, find a unit vector normal to the surface and also the value of the surface charge density on the conductor.

A normal vector is the electric field vector, found (after a little algebra) to be

$$\mathbf{E} = -\nabla V = -200\left[\frac{(x+1)(x-1)[(x-1)-(x+1)] + 2y^2}{[(x+1)^2+y^2][(x-1)^2+y^2]}\right]\mathbf{a}_x$$
$$-\,200\left[\frac{y[(x-1)^2-(x+1)^2]}{[(x+1)^2+y^2][(x-1)^2+y^2]}\right]\mathbf{a}_y \text{ V/m}$$

At the specified point (2,1,1) the field evaluates as $\mathbf{E}_P = 40\,\mathbf{a}_x + 80\,\mathbf{a}_y$, whose magnitude is 89.44 V/m. The unit normal vector is therefore $\mathbf{n} = \mathbf{E}/|\mathbf{E}| = \underline{0.447\,\mathbf{a}_x + 0.894\,\mathbf{a}_y}$. Now $\rho_s = \mathbf{D}\cdot\mathbf{n}\Big|_P = 89.44\epsilon_0 = \underline{792 \text{ pC/m}^2}$. This could be positive or negative, since we do not know which side of the surface the free space region exists.

5.19. Let $V = 20x^2yz - 10z^2$ V in free space.

a) Determine the equations of the equipotential surfaces on which $V = 0$ and 60 V: Setting the given potential function equal to 0 and 60 and simplifying results in:

$$\text{At } 0\,\text{V}: \quad 2x^2y - z = 0$$

$$\text{At } 60\,\text{V}: \quad 2x^2y - z = \frac{6}{z}$$

b) Assume these are conducting surfaces and find the surface charge density at that point on the $V = 60$ V surface where $x = 2$ and $z = 1$. It is known that $0 \leq V \leq 60$ V is the field-containing region: First, on the 60 V surface, we have

$$2x^2y - z - \frac{6}{z} = 0 \;\Rightarrow\; 2(2)^2y(1) - 1 - 6 = 0 \;\Rightarrow\; y = \frac{7}{8}$$

Now

$$\mathbf{E} = -\nabla V = -40xyz\,\mathbf{a}_x - 20x^2z\,\mathbf{a}_y - [20xy - 20z]\,\mathbf{a}_z$$

Then, at the given point, we have

$$\mathbf{D}(2, 7/8, 1) = \epsilon_0\mathbf{E}(2, 7/8, 1) = -\epsilon_0[70\,\mathbf{a}_x + 80\,\mathbf{a}_y + 50\,\mathbf{a}_z]\,\text{C/m}^2$$

We know that since this is the higher potential surface, **D** must be directed away from it, and so the charge density would be positive. Thus

$$\rho_s = \sqrt{\mathbf{D}\cdot\mathbf{D}} = 10\epsilon_0\sqrt{7^2 + 8^2 + 5^2} = \underline{1.04\,\text{nC/m}^2}$$

c) Give the unit vector at this point that is normal to the conducting surface and directed toward the $V = 0$ surface: This will be in the direction of **E** and **D** as found in part *b*, or

$$\mathbf{a}_n = -\left[\frac{7\mathbf{a}_x + 8\mathbf{a}_y + 5\mathbf{a}_z}{\sqrt{7^2 + 8^2 + 5^2}}\right] = \underline{-[0.60\mathbf{a}_x + 0.68\mathbf{a}_y + 0.43\mathbf{a}_z]}$$

5.20. Two point charges of -100π μC are located at (2,-1,0) and (2,1,0). The surface $x = 0$ is a conducting plane.

a) Determine the surface charge density at the origin. I will solve the general case first, in which we find the charge density anywhere on the y axis. With the conducting plane in the yz plane, we will have two image charges, each of $+100\pi$ μC, located at (-2, -1, 0) and (-2, 1, 0). The electric flux density on the y axis from these four charges will be

$$\mathbf{D}(y) = \frac{-100\pi}{4\pi}\Bigg[\underbrace{\frac{[(y-1)\,\mathbf{a}_y - 2\,\mathbf{a}_x]}{[(y-1)^2 + 4]^{3/2}} + \frac{[(y+1)\,\mathbf{a}_y - 2\,\mathbf{a}_x]}{[(y+1)^2 + 4]^{3/2}}}_{\text{given charges}}$$

$$- \underbrace{\frac{[(y-1)\,\mathbf{a}_y + 2\,\mathbf{a}_x]}{[(y-1)^2 + 4]^{3/2}} - \frac{[(y+1)\,\mathbf{a}_y + 2\,\mathbf{a}_x]}{[(y+1)^2 + 4]^{3/2}}}_{\text{image charges}}\Bigg]\;\mu\text{C/m}^2$$

5.20a. (continued)

In the expression, all y components cancel, and we are left with

$$\mathbf{D}(y) = 100\left[\frac{1}{[(y-1)^2+4]^{3/2}} + \frac{1}{[(y+1)^2+4]^{3/2}}\right]\mathbf{a}_x\ \mu\text{C/m}^2$$

We now find the charge density at the origin:

$$\rho_s(0,0,0) = \mathbf{D}\cdot\mathbf{a}_x\Big|_{y=0} = \underline{17.9\ \mu\text{C/m}^2}$$

b) Determine ρ_S at $P(0,h,0)$. This will be

$$\rho_s(0,h,0) = \mathbf{D}\cdot\mathbf{a}_x\Big|_{y=h} = 100\left[\frac{1}{[(h-1)^2+4]^{3/2}} + \frac{1}{[(h+1)^2+4]^{3/2}}\right]\ \mu\text{C/m}^2$$

5.21. Let the surface $y = 0$ be a perfect conductor in free space. Two uniform infinite line charges of 30 nC/m each are located at $x = 0$, $y = 1$, and $x = 0$, $y = 2$.

a) Let $V = 0$ at the plane $y = 0$, and find V at $P(1,2,0)$: The line charges will image across the plane, producing image line charges of -30 nC/m each at $x = 0$, $y = -1$, and $x = 0$, $y = -2$. We find the potential at P by evaluating the work done in moving a unit positive charge from the $y = 0$ plane (we choose the origin) to P: For each line charge, this will be:

$$V_P - V_{0,0,0} = -\frac{\rho_l}{2\pi\epsilon_0}\ln\left[\frac{\text{final distance from charge}}{\text{initial distance from charge}}\right]$$

where $V_{0,0,0} = 0$. Considering the four charges, we thus have

$$V_P = -\frac{\rho_l}{2\pi\epsilon_0}\left[\ln\left(\frac{1}{2}\right) + \ln\left(\frac{\sqrt{2}}{1}\right) - \ln\left(\frac{\sqrt{10}}{1}\right) - \ln\left(\frac{\sqrt{17}}{2}\right)\right]$$

$$= \frac{\rho_l}{2\pi\epsilon_0}\left[\ln(2) + \ln\left(\frac{1}{\sqrt{2}}\right) + \ln\left(\sqrt{10}\right) + \ln\left(\frac{\sqrt{17}}{2}\right)\right] = \frac{30\times10^{-9}}{2\pi\epsilon_0}\ln\left[\frac{\sqrt{10}\sqrt{17}}{\sqrt{2}}\right]$$

$$= \underline{1.20\,\text{kV}}$$

b) Find $\mathbf{E}$ at P: Use

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{(1,2,0)-(0,1,0)}{|(1,1,0)|^2} + \frac{(1,2,0)-(0,2,0)}{|(1,0,0)|^2}\right.$$
$$\left. - \frac{(1,2,0)-(0,-1,0)}{|(1,3,0)|^2} - \frac{(1,2,0)-(0,-2,0)}{|(1,4,0)|^2}\right]$$
$$= \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{(1,1,0)}{2} + \frac{(1,0,0)}{1} - \frac{(1,3,0)}{10} - \frac{(1,4,0)}{17}\right] = \underline{723\,\mathbf{a}_x - 18.9\,\mathbf{a}_y\ \text{V/m}}$$

5.22. The line segment $x = 0$, $-1 \le y \le 1$, $z = 1$, carries a linear charge density $\rho_L = \pi|y|$ μC/m. Let $z = 0$ be a conducting plane and determine the surface charge density at: (a) (0,0,0); (b) (0,1,0).

We consider the line charge to be made up of a string of differential segments of length, dy', and of charge $dq = \rho_L\, dy'$. A given segment at location $(0, y', 1)$ will have a corresponding image charge segment at location $(0, y', -1)$. The differential flux density on the y axis that is associated with the segment-image pair will be

$$d\mathbf{D} = \frac{\rho_L\, dy'[(y - y')\,\mathbf{a}_y - \mathbf{a}_z]}{4\pi[(y - y')^2 + 1]^{3/2}} - \frac{\rho_L\, dy'[(y - y')\,\mathbf{a}_y + \mathbf{a}_z]}{4\pi[(y - y')^2 + 1]^{3/2}} = \frac{-\rho_L\, dy'\,\mathbf{a}_z}{2\pi[(y - y')^2 + 1]^{3/2}}$$

In other words, each charge segment and its image produce a net field in which the y components have cancelled. The total flux density from the line charge and its image is now

$$\mathbf{D}(y) = \int d\mathbf{D} = \int_{-1}^{1} \frac{-\pi|y'|\,\mathbf{a}_z\, dy'}{2\pi[(y - y')^2 + 1]^{3/2}}$$

$$= -\frac{\mathbf{a}_z}{2}\int_0^1 \left[\frac{y'}{[(y - y')^2 + 1]^{3/2}} + \frac{y'}{[(y + y')^2 + 1]^{3/2}}\right] dy'$$

$$= \frac{\mathbf{a}_z}{2}\left[\frac{y(y - y') + 1}{[(y - y')^2 + 1]^{1/2}} + \frac{y(y + y') + 1}{[(y + y')^2 + 1]^{1/2}}\right]_0^1$$

$$= \frac{\mathbf{a}_z}{2}\left[\frac{y(y - 1) + 1}{[(y - 1)^2 + 1]^{1/2}} + \frac{y(y + 1) + 1}{[(y + 1)^2 + 1]^{1/2}} - 2(y^2 + 1)^{1/2}\right]$$

Now, at the origin (part a), we find the charge density through

$$\rho_s(0, 0, 0) = \mathbf{D}\cdot\mathbf{a}_z\Big|_{y=0} = \frac{\mathbf{a}_z}{2}\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 2\right] = \underline{-0.29\ \mu\text{C/m}^2}$$

Then, at (0,1,0) (part b), the charge density is

$$\rho_s(0, 1, 0) = \mathbf{D}\cdot\mathbf{a}_z\Big|_{y=1} = \frac{\mathbf{a}_z}{2}\left[1 + \frac{3}{\sqrt{5}} - 2\right] = \underline{-0.24\ \mu\text{C/m}^2}$$

5.23. A dipole with $\mathbf{p} = 0.1\mathbf{a}_z$ μC $\cdot$ m is located at $A(1, 0, 0)$ in free space, and the $x = 0$ plane is perfectly-conducting.

a) Find V at $P(2, 0, 1)$. We use the far-field potential for a z-directed dipole:

$$V = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} = \frac{p}{4\pi\epsilon_0}\frac{z}{[x^2 + y^2 + z^2]^{1.5}}$$

The dipole at $x = 1$ will image in the plane to produce a second dipole of the opposite orientation at $x = -1$. The potential at any point is now:

$$V = \frac{p}{4\pi\epsilon_0}\left[\frac{z}{[(x - 1)^2 + y^2 + z^2]^{1.5}} - \frac{z}{[(x + 1)^2 + y^2 + z^2]^{1.5}}\right]$$

Substituting $P(2, 0, 1)$, we find

$$V = \frac{.1 \times 10^6}{4\pi\epsilon_0}\left[\frac{1}{2\sqrt{2}} - \frac{1}{10\sqrt{10}}\right] = \underline{289.5\ \text{V}}$$

5.23b) Find the equation of the 200-V equipotential surface in cartesian coordinates: We just set the potential exression of part a equal to 200 V to obtain:

$$\left[\frac{z}{[(x-1)^2+y^2+z^2]^{1.5}} - \frac{z}{[(x+1)^2+y^2+z^2]^{1.5}}\right] = 0.222$$

5.24. At a certain temperature, the electron and hole mobilities in intrinsic germanium are given as 0.43 and 0.21 $\mathrm{m^2/V \cdot s}$, respectively. If the electron and hole concentrations are both 2.3×10^{19} $\mathrm{m^{-3}}$, find the conductivity at this temperature.

With the electron and hole charge magnitude of 1.6×10^{-19} C, the conductivity in this case can be written:

$$\sigma = |\rho_e|\mu_e + \rho_h\mu_h = (1.6 \times 10^{-19})(2.3 \times 10^{19})(0.43 + 0.21) = \underline{2.36 \text{ S/m}}$$

5.25. Electron and hole concentrations increase with temperature. For pure silicon, suitable expressions are $\rho_h = -\rho_e = 6200T^{1.5}e^{-7000/T}$ $\mathrm{C/m^3}$. The functional dependence of the mobilities on temperature is given by $\mu_h = 2.3 \times 10^5 T^{-2.7}$ $\mathrm{m^2/V \cdot s}$ and $\mu_e = 2.1 \times 10^5 T^{-2.5}$ $\mathrm{m^2/V \cdot s}$, where the temperature, T, *is in degrees Kelvin.* The conductivity will thus be

$$\sigma = -\rho_e\mu_e + \rho_h\mu_h = 6200T^{1.5}e^{-7000/T}\left[2.1 \times 10^5 T^{-2.5} + 2.3 \times 10^5 T^{-2.7}\right]$$
$$= \frac{1.30 \times 10^9}{T}e^{-7000/T}\left[1 + 1.095T^{-.2}\right] \text{ S/m}$$

Find σ at:

a) 0° C: With $T = 273$°K, the expression evaluates as $\sigma(0) = \underline{4.7 \times 10^{-5} \text{ S/m}}$.

b) 40° C: With $T = 273 + 40 = 313$, we obtain $\sigma(40) = \underline{1.1 \times 10^{-3} \text{ S/m}}$.

c) 80° C: With $T = 273 + 80 = 353$, we obtain $\sigma(80) = \underline{1.2 \times 10^{-2} \text{ S/m}}$.

5.26. A semiconductor sample has a rectangular cross-section 1.5 by 2.0 mm, and a length of 11.0 mm. The material has electron and hole densities of 1.8×10^{18} and 3.0×10^{15} $\mathrm{m^{-3}}$, respectively. If $\mu_e = 0.082$ $\mathrm{m^2/V \cdot s}$ and $\mu_h = 0.0021$ $\mathrm{m^2/V \cdot s}$, find the resistance offered between the end faces of the sample.

Using the given values along with the electron charge, the conductivity is

$$\sigma = (1.6 \times 10^{-19})\left[(1.8 \times 10^{18})(0.082) + (3.0 \times 10^{15})(0.0021)\right] = 0.0236 \text{ S/m}$$

The resistance is then

$$R = \frac{\ell}{\sigma A} = \frac{0.011}{(0.0236)(0.002)(0.0015)} = \underline{155 \text{ k}\Omega}$$