CHAPTER 7

7.1. Let $V = 2xy^2z^3$ and $\epsilon = \epsilon_0$. Given point $P(1, 2, -1)$, find:

a) V at P: Substituting the coordinates into V, find $V_P = \underline{-8\,\text{V}}$.

b) $\mathbf{E}$ at P: We use $\mathbf{E} = -\nabla V = -2y^2z^3\mathbf{a}_x - 4xyz^3\mathbf{a}_y - 6xy^2z^2\mathbf{a}_z$, which, when evaluated at P, becomes $\mathbf{E}_P = \underline{8\mathbf{a}_x + 8\mathbf{a}_y - 24\mathbf{a}_z \text{ V/m}}$

c) ρ_v at P: This is $\rho_v = \nabla \cdot \mathbf{D} = -\epsilon_0 \nabla^2 V = \underline{-4xz(z^2 + 3y^2) \text{ C/m}^3}$

d) the equation of the equipotential surface passing through P: At P, we know $V = -8\,\text{V}$, so the equation will be $\underline{xy^2z^3 = -4}$.

e) the equation of the streamline passing through P: First,

$$\frac{E_y}{E_x} = \frac{dy}{dx} = \frac{4xyz^3}{2y^2z^3} = \frac{2x}{y}$$

Thus

$$ydy = 2xdx, \text{ and so } \frac{1}{2}y^2 = x^2 + C_1$$

Evaluating at P, we find $C_1 = 1$. Next,

$$\frac{E_z}{E_x} = \frac{dz}{dx} = \frac{6xy^2z^2}{2y^2z^3} = \frac{3x}{z}$$

Thus

$$3xdx = zdz, \text{ and so } \frac{3}{2}x^2 = \frac{1}{2}z^2 + C_2$$

Evaluating at P, we find $C_2 = 1$. The streamline is now specified by the equations:

$$\underline{y^2 - 2x^2 = 2} \text{ and } \underline{3x^2 - z^2 = 2}$$

f) Does V satisfy Laplace's equation? No, since the charge density is not zero.

7.2. Given the spherically-symmetric potential field in free space, $V = V_0 e^{-r/a}$, find:

a) ρ_v at $r = a$; Use Poisson's equation, $\nabla^2 V = -\rho_v/\epsilon$, which in this case becomes

$$-\frac{\rho_v}{\epsilon_0} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = \frac{-V_0}{ar^2}\frac{d}{dr}\left(r^2 e^{-r/a}\right) = \frac{-V_0}{ar}\left(2 - \frac{r}{a}\right)e^{-r/a}$$

from which

$$\rho_v(r) = \frac{\epsilon_0 V_0}{ar}\left(2 - \frac{r}{a}\right)e^{-r/a} \quad \Rightarrow \quad \rho_v(a) = \frac{\epsilon_0 V_0}{a^2}e^{-1} \text{ C/m}^3$$

b) the electric field at $r = a$; this we find through the negative gradient:

$$\mathbf{E}(r) = -\nabla V = -\frac{dV}{dr}\mathbf{a}_r = \frac{V_0}{a}e^{-r/a}\,\mathbf{a}_r \quad \Rightarrow \quad \mathbf{E}(a) = \frac{V_0}{a}e^{-1}\mathbf{a}_r \text{ V/m}$$

7.2c) the total charge: The easiest way is to first find the electric flux density, which from part b is $\mathbf{D} = \epsilon_0 \mathbf{E} = (\epsilon_0 V_0/a)e^{-r/a}\,\mathbf{a}_r$. Then the net outward flux of $\mathbf{D}$ through a sphere of radius r would be

$$\Phi(r) = Q_{encl}(r) = 4\pi r^2 D = 4\pi\epsilon_0 V_0 r^2 e^{-r/a}\ \text{C}$$

As $r \to \infty$, this result approaches zero, so the total charge is therefore $Q_{net} = \underline{0}$.

7.3. Let $V(x, y) = 4e^{2x} + f(x) - 3y^2$ in a region of free space where $\rho_v = 0$. It is known that both E_x and V are zero at the origin. Find $f(x)$ and $V(x, y)$: Since $\rho_v = 0$, we know that $\nabla^2 V = 0$, and so

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 16e^{2x} + \frac{d^2 f}{dx^2} - 6 = 0$$

Therefore

$$\frac{d^2 f}{dx^2} = -16e^{2x} + 6 \quad \Rightarrow \quad \frac{df}{dx} = -8e^{2x} + 6x + C_1$$

Now

$$E_x = \frac{\partial V}{\partial x} = 8e^{2x} + \frac{df}{dx}$$

and at the origin, this becomes

$$E_x(0) = 8 + \left.\frac{df}{dx}\right|_{x=0} = 0(\text{as given})$$

Thus $df/dx\,|_{x=0} = -8$, and so it follows that $C_1 = 0$. Integrating again, we find

$$f(x, y) = -4e^{2x} + 3x^2 + C_2$$

which at the origin becomes $f(0, 0) = -4 + C_2$. However, $V(0, 0) = 0 = 4 + f(0, 0)$. So $f(0, 0) = -4$ and $C_2 = 0$. Finally, $f(x, y) = \underline{-4e^{2x} + 3x^2}$, and $V(x, y) = 4e^{2x} - 4e^{2x} + 3x^2 - 3y^2 = \underline{3(x^2 - y^2)}$.

7.4. Given the potential field, $V(\rho, \phi) = (V_0\rho/d)\cos\phi$:

a) Show that $V(\rho, \phi)$ satisfies Laplace's equation:

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{V_0\rho}{d}\cos\phi\right) - \frac{1}{\rho^2}\frac{\partial}{\partial \phi}\left(\frac{V_0\rho}{d}\sin\phi\right)$$
$$= \frac{V_0\rho}{d}\cos\phi - \frac{V_0\rho}{d}\sin\phi = 0$$

b) Describe the constant-potential surfaces: These will be surfaces on which $\rho\cos\phi$ is a constant. At this stage, it is helpful to recall that the x coordinate in rectangular coordinates is in fact $\rho\cos\phi$, so we identify the surfaces of constant potential as (plane) surfaces of constant x (parallel to the yz plane).

c) Specifically describe the surfaces on which $V = V_0$ and $V = 0$: In the first case, we would have $\underline{x = 0}$ (or the yz plane); in the second case, we have the surface $\underline{x = d}$.

d) Write the potential expression in rectangular coordinates: Using the argument in part b, we would have $\underline{V(x) = V_0 x/d}$.

7.5. Given the potential field $V = (A\rho^4 + B\rho^{-4})\sin 4\phi$:

a) Show that $\nabla^2 V = 0$: In cylindrical coordinates,

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2}$$
$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho(4A\rho^3 - 4B\rho^{-5})\right)\sin 4\phi - \frac{1}{\rho^2}16(A\rho^4 + B\rho^{-4})\sin 4\phi$$
$$= \frac{16}{\rho}(A\rho^3 + B\rho^{-5})\sin 4\phi - \frac{16}{\rho^2}(A\rho^4 + B\rho^{-4})\sin 4\phi = 0$$

b) Select A and B so that $V = 100$ V and $|\mathbf{E}| = 500$ V/m at $P(\rho = 1, \phi = 22.5^\circ, z = 2)$: First,

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial\rho}\mathbf{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial\phi}\mathbf{a}_\phi$$
$$= -4\left[(A\rho^3 - B\rho^{-5})\sin 4\phi\,\mathbf{a}_\rho + (A\rho^3 + B\rho^{-5})\cos 4\phi\,\mathbf{a}_\phi\right]$$

and at P, $\mathbf{E}_P = -4(A - B)\,\mathbf{a}_\rho$. Thus $|\mathbf{E}_P| = \pm 4(A - B)$. Also, $V_P = A + B$. Our two equations are:

$$4(A - B) = \pm 500$$

and

$$A + B = 100$$

We thus have two pairs of values for A and B:

$$\underline{A = 112.5,\ B = -12.5} \text{ or } \underline{A = -12.5,\ B = 112.5}$$

7.6. A parallel-plate capacitor has plates located at $z = 0$ and $z = d$. The region between plates is filled with a material containing volume charge of uniform density ρ_0 C/m^3, and which has permittivity ϵ. Both plates are held at ground potential.

a) Determine the potential field between plates: We solve Poisson's equation, under the assumption that V varies only with z:

$$\nabla^2 V = \frac{d^2 V}{dz^2} = -\frac{\rho_0}{\epsilon} \Rightarrow V = \frac{-\rho_0 z^2}{2\epsilon} + C_1 z + C_2$$

At $z = 0$, $V = 0$, and so $C_2 = 0$. Then, at $z = d$, $V = 0$ as well, so we find $C_1 = \rho_0 d/2\epsilon$. Finally, $V(z) = \underline{(\rho_0 z/2\epsilon)[d - z]}$.

b) Determine the electric field intensity, $\mathbf{E}$ between plates: Taking the answer to part a, we find $\mathbf{E}$ through

$$\mathbf{E} = -\nabla V = -\frac{dV}{dz}\mathbf{a}_z = -\frac{d}{dz}\left[\frac{\rho_0 z}{2\epsilon}(d - z)\right] = \frac{\rho_0}{2\epsilon}(2z - d)\,\mathbf{a}_z \text{ V/m}$$

7.6c) Repeat a and b for the case of the plate at $z = d$ raised to potential V_0, with the $z = 0$ plate grounded: Begin with

$$V(z) = \frac{-\rho_0 z^2}{2\epsilon} + C_1 z + C_2$$

with $C_2 = 0$ as before, since $V(z = 0) = 0$. Then

$$V(z = d) = V_0 = \frac{-\rho_0 d^2}{2\epsilon} + C_1 d \;\Rightarrow\; C_1 = \frac{V_0}{d} + \frac{\rho_0 d}{2\epsilon}$$

So that

$$V(z) = \frac{V_0}{d} z + \frac{\rho_0 z}{2\epsilon}(d - z)$$

We recognize this as the simple superposition of the voltage as found in part a and the voltage of a capacitor carrying voltage V_0, but without the charged dielectric. The electric field is now

$$\mathbf{E} = -\frac{dV}{dz}\,\mathbf{a}_z = \frac{-V_0}{d}\,\mathbf{a}_z + \frac{\rho_0}{2\epsilon}(2z - d)\,\mathbf{a}_z \text{ V/m}$$

7.7. Let $V = (\cos 2\phi)/\rho$ in free space.

a) Find the volume charge density at point $A(0.5, 60°, 1)$: Use Poisson's equation:

$$\rho_v = -\epsilon_0 \nabla^2 V = -\epsilon_0 \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} \right)$$
$$= -\epsilon_0 \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{-\cos 2\phi}{\rho}\right) - \frac{4}{\rho^2}\frac{\cos 2\phi}{\rho} \right) = \frac{3\epsilon_0 \cos 2\phi}{\rho^3}$$

So at A we find:

$$\rho_{vA} = \frac{3\epsilon_0 \cos(120°)}{0.5^3} = -12\epsilon_0 = \underline{-106 \text{ pC/m}^3}$$

b) Find the surface charge density on a conductor surface passing through $B(2, 30°, 1)$: First, we find $\mathbf{E}$:

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\,\mathbf{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\,\mathbf{a}_\phi$$
$$= \frac{\cos 2\phi}{\rho^2}\,\mathbf{a}_\rho + \frac{2 \sin 2\phi}{\rho^2}\,\mathbf{a}_\phi$$

At point B the field becomes

$$\mathbf{E}_B = \frac{\cos 60°}{4}\,\mathbf{a}_\rho + \frac{2 \sin 60°}{4}\,\mathbf{a}_\phi = 0.125\,\mathbf{a}_\rho + 0.433\,\mathbf{a}_\phi$$

The surface charge density will now be

$$\rho_{sB} = \pm|\mathbf{D}_B| = \pm\epsilon_0 |\mathbf{E}_B| = \pm 0.451\epsilon_0 = \underline{\pm 0.399 \text{ pC/m}^2}$$

The charge is positive or negative depending on which side of the surface we are considering. The problem did not provide information necessary to determine this.

7.8. A uniform volume charge has constant density $\rho_v = \rho_0$ C/m^3, and fills the region $r < a$, in which permittivity ϵ as assumed. A conducting spherical shell is located at $r = a$, and is held at ground potential. Find:

a) the potential everywhere: Inside the sphere, we solve Poisson's equation, assuming radial variation only:

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = \frac{-\rho_0}{\epsilon} \quad \Rightarrow \quad V(r) = \frac{-\rho_0 r^2}{6\epsilon_0} + \frac{C_1}{r} + C_2$$

We require that V is finite at the orgin (or as $r \to 0$), and so therefore $C_1 = 0$. Next, $V = 0$ at $r = a$, which gives $C_2 = \rho_0 a^2/6\epsilon$. Outside, $r > a$, we know the potential must be zero, since the sphere is grounded. To show this, solve Laplace's equation:

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0 \quad \Rightarrow \quad V(r) = \frac{C_1}{r} + C_2$$

Requiring $V = 0$ at both $r = a$ and at infinity leads to $C_1 = C_2 = 0$. To summarize

$$V(r) = \begin{cases} \frac{\rho_0}{6\epsilon}(a^2 - r^2) & r < a \\ 0 & r > a \end{cases}$$

b) the electric field intensity, $\mathbf{E}$, everywhere: Use

$$\mathbf{E} = -\nabla V = \frac{-dV}{dr}\,\mathbf{a}_r = \frac{\rho_0 r}{3\epsilon}\,\mathbf{a}_r \quad r < a$$

Outside ($r > a$), the potential is zero, and so $\mathbf{E} = 0$ there as well.

7.9. The functions $V_1(\rho, \phi, z)$ and $V_2(\rho, \phi, z)$ both satisfy Laplace's equation in the region $a < \rho < b$, $0 \le \phi < 2\pi$, $-L < z < L$; each is zero on the surfaces $\rho = b$ for $-L < z < L$; $z = -L$ for $a < \rho < b$; and $z = L$ for $a < \rho < b$; and each is 100 V on the surface $\rho = a$ for $-L < z < L$.

a) In the region specified above, is Laplace's equation satisfied by the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? Yes for the first three, since Laplace's equation is linear. No for $V_1 V_2$.

b) On the boundary surfaces specified, are the potential values given above obtained from the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? At the 100 V surface ($\rho = a$), No for all. At the 0 V surfaces, yes, except for $V_1 + 3$.

c) Are the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$ identical with V_1? Only V_2 is, since it is given as satisfying all the boundary conditions that V_1 does. Therefore, by the uniqueness theorem, $V_2 = V_1$. The others, not satisfying the boundary conditions, are not the same as V_1.

7.10. Consider the parallel-plate capacitor of Problem 7.6, but this time the charged dielectric exists only between $z = 0$ and $z = b$, where $b < d$. Free space fills the region $b < z < d$. Both plates are at ground potential. No surface charge exists at $z = b$, so that both V and $\mathbf{D}$ are continuous there. By solving Laplace's *and* Poisson's equations, find:

a) $V(z)$ for $0 < z < d$: In Region 1 ($z < b$), we solve Poisson's equation, assuming z variation only:

$$\frac{d^2V_1}{dz^2} = \frac{-\rho_0}{\epsilon} \quad \Rightarrow \quad \frac{dV_1}{dz} = \frac{-\rho_0 z}{\epsilon} + C_1 \quad (z < b)$$

In Region 2 ($z > b$), we solve Laplace's equation, assuming z variation only:

$$\frac{d^2V_2}{dz^2} = 0 \quad \Rightarrow \quad \frac{dV_2}{dz} = C_1' \quad (z > b)$$

At this stage we apply the first boundary condition, which is continuity of $\mathbf{D}$ across the interface at $z = b$. Knowing that the electric field magnitude is given by dV/dz, we write

$$\epsilon \frac{dV_1}{dz}\bigg|_{z=b} = \epsilon_0 \frac{dV_2}{dz}\bigg|_{z=b} \quad \Rightarrow \quad -\rho_0 b + \epsilon C_1 = \epsilon_0 C_1' \quad \Rightarrow \quad C_1' = \frac{-\rho_0 b}{\epsilon_0} + \frac{\epsilon}{\epsilon_0} C_1$$

Substituting the above expression for C_1', and performing a second integration on the Poisson and Laplace equations, we find

$$V_1(z) = -\frac{\rho_0 z^2}{2\epsilon} + C_1 z + C_2 \quad (z < b)$$

and

$$V_2(z) = -\frac{\rho_0 b z}{2\epsilon_0} + \frac{\epsilon}{\epsilon_0} C_1 z + C_2' \quad (z > b)$$

Next, requiring $V_1 = 0$ at $z = 0$ leads to $C_2 = 0$. Then, the requirement that $V_2 = 0$ at $z = d$ leads to

$$0 = -\frac{\rho_0 b d}{\epsilon_0} + \frac{\epsilon}{\epsilon_0} C_1 d + C_2' \quad \Rightarrow \quad C_2' = \frac{\rho_0 b d}{\epsilon_0} - \frac{\epsilon}{\epsilon_0} C_1 d$$

With C_2 and C_2' known, the voltages now become

$$V_1(z) = -\frac{\rho_0 z^2}{2\epsilon} + C_1 z \quad \text{and} \quad V_2(z) = \frac{\rho_0 b}{\epsilon_0}(d - z) - \frac{\epsilon}{\epsilon_0} C_1 (d - z)$$

Finally, to evaluate C_1, we equate the two voltage expressions at $z = b$:

$$V_1|_{z=b} = V_2|_{z=b} \quad \Rightarrow \quad C_1 = \frac{\rho_0 b}{2\epsilon}\left[\frac{b + 2\epsilon_r(d - b)}{b + \epsilon_r(d - b)}\right]$$

where $\epsilon_r = \epsilon/\epsilon_0$. Substituting C_1 as found above into V_1 and V_2 leads to the final expressions for the voltages:

$$V_1(z) = \frac{\rho_0 b z}{2\epsilon}\left[\left(\frac{b + 2\epsilon_r(d - b)}{b + \epsilon_r(d - b)}\right) - \frac{z}{b}\right] \quad (z < b)$$

$$V_2(z) = \frac{\rho_0 b^2}{2\epsilon_0}\left[\frac{d - z}{b + \epsilon_r(d - b)}\right] \quad (z > b)$$

7.10b) the electric field intensity for $0 < z < d$: This involves taking the negative gradient of the final voltage expressions of part a. We find

$$\mathbf{E}_1 = -\frac{dV_1}{dz}\,\mathbf{a}_z = \frac{\rho_0}{\epsilon}\left[z - \frac{b}{2}\left(\frac{b+2\epsilon_r(d-b)}{b+\epsilon_r(d-b)}\right)\right]\mathbf{a}_z \ \ \text{V/m} \quad (z < b)$$

$$\mathbf{E}_2 = -\frac{dV_2}{dz}\,\mathbf{a}_z = \frac{\rho_0 b^2}{2\epsilon_0}\left[\frac{1}{b+\epsilon_r(d-b)}\right]\mathbf{a}_z \ \ \text{V/m} \quad (z > b)$$

7.11. The conducting planes $2x + 3y = 12$ and $2x + 3y = 18$ are at potentials of 100 V and 0, respectively. Let $\epsilon = \epsilon_0$ and find:

a) V at $P(5,2,6)$: The planes are parallel, and so we expect variation in potential in the direction normal to them. Using the two boundary condtions, our general potential function can be written:

$$V(x,y) = A(2x + 3y - 12) + 100 = A(2x + 3y - 18) + 0$$

and so $A = -100/6$. We then write

$$V(x,y) = -\frac{100}{6}(2x + 3y - 18) = -\frac{100}{3}x - 50y + 300$$

and $V_P = -\frac{100}{3}(5) - 100 + 300 = \underline{33.33 \text{ V}}$.

b) Find $\mathbf{E}$ at P: Use

$$\mathbf{E} = -\nabla V = \underline{\frac{100}{3}\,\mathbf{a}_x + 50\,\mathbf{a}_y \ \text{V/m}}$$

7.12. The derivation of Laplace's and Poisson's equations assumed constant permittivity, but there are cases of spatially-varying permittivity in which the equations will still apply. Consider the vector identity, $\nabla \cdot (\psi \mathbf{G}) = \mathbf{G} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{G}$, where ψ and $\mathbf{G}$ are scalar and vector functions, respectively. Determine a general rule on the allowed *directions* in which ϵ may vary with respect to the electric field.

In the original derivation of Poisson's equation, we started with $\nabla\cdot\mathbf{D} = \rho_v$, where $\mathbf{D} = \epsilon\mathbf{E}$. Therefore

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon\mathbf{E}) = -\nabla \cdot (\epsilon\nabla V) = -\nabla V \cdot \nabla\epsilon - \epsilon\nabla^2 V = \rho_v$$

We see from this that Poisson's equation, $\nabla^2 V = -\rho_v/\epsilon$, results when $\nabla V \cdot \nabla\epsilon = 0$. In words, ϵ is allowed to vary, provided it does so in directions that are normal to the local electric field.

7.13. Coaxial conducting cylinders are located at $\rho = 0.5$ cm and $\rho = 1.2$ cm. The region between the cylinders is filled with a homogeneous perfect dielectric. If the inner cylinder is at 100V and the outer at 0V, find:

a) the location of the 20V equipotential surface: From Eq. (16) we have

$$V(\rho) = 100\frac{\ln(.012/\rho)}{\ln(.012/.005)} \ \text{V}$$

We seek ρ at which $V = 20$ V, and thus we need to solve:

$$20 = 100\frac{\ln(.012/\rho)}{\ln(2.4)} \quad \Rightarrow \quad \rho = \frac{.012}{(2.4)^{0.2}} = \underline{1.01 \text{ cm}}$$

b) $E_{\rho\,max}$: We have

$$E_\rho = -\frac{\partial V}{\partial \rho} = -\frac{dV}{d\rho} = \frac{100}{\rho \ln(2.4)}$$

whose maximum value will occur at the inner cylinder, or at $\rho = .5$ cm:

$$E_{\rho\,max} = \frac{100}{.005\ln(2.4)} = 2.28 \times 10^4 \text{ V/m} = \underline{22.8 \text{ kV/m}}$$

c) ϵ_r if the charge per meter length on the inner cylinder is 20 nC/m: The capacitance per meter length is

$$C = \frac{2\pi\epsilon_0\epsilon_r}{\ln(2.4)} = \frac{Q}{V_0}$$

We solve for ϵ_r:

$$\epsilon_r = \frac{(20 \times 10^{-9})\ln(2.4)}{2\pi\epsilon_0(100)} = \underline{3.15}$$

7.14. Repeat Problem 7.13, but with the dielectric only partially filling the volume, within $0 < \phi < \pi$, and with free space in the remaining volume.

We note that the dielectric changes with ϕ, and not with ρ. Also, since **E** is radially-directed and varies only with radius, Laplace's equation for this case is valid (see Problem 7.12) and is the same as that which led to the potential and field in Problem 7.13. Therefore, the solutions to parts *a* and *b* are unchanged from Problem 7.13. Part *c*, however, is different. We write the charge per unit length as the sum of the charges along each half of the center conductor (of radius *a*)

$$Q = \epsilon_r\epsilon_0 E_{\rho,max}(\pi a) + \epsilon_0 E_{\rho,max}(\pi a) = \epsilon_0 E_{\rho,max}(\pi a)(1+\epsilon_r) \text{ C/m}$$

Using the numbers given or found in Problem 7.13, we obtain

$$1 + \epsilon_r = \frac{20 \times 10^{-9} \text{ C/m}}{(8.852 \times 10^{-12})(22.8 \times 10^3 \text{ V/m})(0.5 \times 10^{-2} \text{ m})\pi} = 6.31 \quad \Rightarrow \quad \epsilon_r = \underline{5.31}$$

We may also note that the *average* dielectric constant in this problem, $(\epsilon_r + 1)/2$, is the same as that of the uniform dielectric constant found in Problem 7.13.

7.15. The two conducting planes illustrated in Fig. 7.8 are defined by $0.001 < \rho < 0.120$ m, $0 < z < 0.1$ m, $\phi = 0.179$ and 0.188 rad. The medium surrounding the planes is air. For region 1, $0.179 < \phi < 0.188$, neglect fringing and find:

a) $V(\phi)$: The general solution to Laplace's equation will be $V = C_1\phi + C_2$, and so

$$20 = C_1(.188) + C_2 \quad \text{and} \quad 200 = C_1(.179) + C_2$$

Subtracting one equation from the other, we find

$$-180 = C_1(.188 - .179) \quad \Rightarrow \quad C_1 = -2.00 \times 10^4$$

Then

$$20 = -2.00 \times 10^4(.188) + C_2 \quad \Rightarrow \quad C_2 = 3.78 \times 10^3$$

Finally, $V(\phi) = \underline{(-2.00 \times 10^4)\phi + 3.78 \times 10^3 \text{ V}}$.

b) $\mathbf{E}(\rho)$: Use

$$\mathbf{E}(\rho) = -\nabla V = -\frac{1}{\rho}\frac{dV}{d\phi} = \underline{\frac{2.00 \times 10^4}{\rho}\,\mathbf{a}_\phi \text{ V/m}}$$

c) $\mathbf{D}(\rho) = \epsilon_0 \mathbf{E}(\rho) = \underline{(2.00 \times 10^4 \epsilon_0/\rho)\,\mathbf{a}_\phi \text{ C/m}^2}$.

d) ρ_s on the upper surface of the lower plane: We use

$$\rho_s = \mathbf{D} \cdot \mathbf{n}\Big|_{surface} = \frac{2.00 \times 10^4}{\rho}\,\mathbf{a}_\phi \cdot \mathbf{a}_\phi = \underline{\frac{2.00 \times 10^4}{\rho} \text{ C/m}^2}$$

e) Q on the upper surface of the lower plane: This will be

$$Q_t = \int_0^{.1}\int_{.001}^{.120} \frac{2.00 \times 10^4 \epsilon_0}{\rho}\, d\rho\, dz = 2.00 \times 10^4 \epsilon_0 (.1)\ln(120) = 8.47 \times 10^{-8} \text{ C} = \underline{84.7 \text{ nC}}$$

f) Repeat *a*) to *c*) for region 2 by letting the location of the upper plane be $\phi = .188 - 2\pi$, and then find ρ_s and Q on the lower surface of the lower plane. Back to the beginning, we use

$$20 = C_1'(.188 - 2\pi) + C_2' \quad \text{and} \quad 200 = C_1'(.179) + C_2'$$

7.15f (continued) Subtracting one from the other, we find

$$-180 = C_1'(.009 - 2\pi) \quad \Rightarrow \quad C_1' = 28.7$$

Then $200 = 28.7(.179) + C_2' \;\Rightarrow\; C_2' = 194.9$. Thus $V(\phi) = \underline{28.7\phi + 194.9}$ in region 2. Then

$$\underline{\mathbf{E} = -\frac{28.7}{\rho}\,\mathbf{a}_\phi \text{ V/m}} \quad \text{and} \quad \underline{\mathbf{D} = -\frac{28.7\epsilon_0}{\rho}\,\mathbf{a}_\phi \text{ C/m}^2}$$

ρ_s on the lower surface of the lower plane will now be

$$\rho_s = -\frac{28.7\epsilon_0}{\rho}\,\mathbf{a}_\phi \cdot (-\mathbf{a}_\phi) = \underline{\frac{28.7\epsilon_0}{\rho} \text{ C/m}^2}$$

The charge on that surface will then be $Q_b = 28.7\epsilon_0(.1)\ln(120) = \underline{122 \text{ pC}}$.

g) Find the total charge on the lower plane and the capacitance between the planes: Total charge will be $Q_{net} = Q_t + Q_b = 84.7 \text{ nC} + 0.122 \text{ nC} = \underline{84.8 \text{ nC}}$. The capacitance will be

$$C = \frac{Q_{net}}{\Delta V} = \frac{84.8}{200 - 20} = 0.471 \text{ nF} = \underline{471 \text{ pF}}$$

7.16. A parallel-plate capacitor is made using two circular plates of radius a, with the bottom plate on the xy plane, centered at the origin. The top plate is located at $z = d$, with its center on the z axis. Potential V_0 is on the top plate; the bottom plate is grounded. Dielectric having *radially-dependent* permittivity fills the region between plates. The permittivity is given by $\epsilon(\rho) = \epsilon_0(1 + \rho/a)$. Find:

a) $V(z)$: Since ϵ varies in the direction normal to $\mathbf{E}$, Laplace's equation applies, and we write

$$\nabla^2 V = \frac{d^2V}{dz^2} = 0 \quad \Rightarrow \quad V(z) = C_1 z + C_2$$

With the given boundary conditions, $C_2 = 0$, and $C_1 = V_0/d$. Therefore $\underline{V(z) = V_0 z/d \text{ V}}$.

b) $\mathbf{E}$: This will be $\mathbf{E} = -\nabla V = -dV/dz\,\mathbf{a}_z = \underline{-(V_0/d)\,\mathbf{a}_z \text{ V/m}}$.

c) Q: First we find the electric flux density: $\mathbf{D} = \epsilon\mathbf{E} = -\epsilon_0(1 + \rho/a)(V_0/d)\,\mathbf{a}_z$ C/m^2. The charge density on the top plate is then $\rho_s = \mathbf{D} \cdot -\mathbf{a}_z = \epsilon_0(1 + \rho/a)(V_0/d)$ C/m^2. From this we find the charge on the top plate:

$$Q = \int_0^{2\pi}\int_0^a \epsilon_0(1 + \rho/a)(V_0/d)\,\rho\,d\rho\,d\phi = \frac{5\pi a^2 \epsilon_0 V_0}{3d} \text{ C}$$

d) C. The capacitance is $C = Q/V_0 = \underline{5\pi a^2\epsilon_0/(3d) \text{ F}}$.

7.17. Concentric conducting spheres are located at $r = 5$ mm and $r = 20$ mm. The region between the spheres is filled with a perfect dielectric. If the inner sphere is at 100 V and the outer sphere at 0 V:

a) Find the location of the 20 V equipotential surface: Solving Laplace's equation gives us

$$V(r) = V_0 \frac{\frac{1}{r} - \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}}$$

where $V_0 = 100$, $a = 5$ and $b = 20$. Setting $V(r) = 20$, and solving for r produces $r = \underline{12.5\,\text{mm}}$.

b) Find $E_{r,max}$: Use

$$\mathbf{E} = -\nabla V = -\frac{dV}{dr}\mathbf{a}_r = \frac{V_0\,\mathbf{a}_r}{r^2\left(\frac{1}{a} - \frac{1}{b}\right)}$$

$$E_{r,max} = E(r = a) = \frac{V_0}{a(1 - (a/b))} = \frac{100}{5(1 - (5/20))} = 26.7 \text{ V/mm} = \underline{26.7 \text{ kV/m}}$$

c) Find ϵ_r if the surface charge density on the inner sphere is $1.0\,\mu\text{C/m}^2$: ρ_s will be equal in magnitude to the electric flux density at $r = a$. So $\rho_s = (2.67 \times 10^4 \text{ V/m})\epsilon_r\epsilon_0 = 10^{-6}\text{ C/m}^2$. Thus $\epsilon_r = \underline{4.23}$. Note, in the first printing, the given charge density was $100\,\mu\text{C/m}^2$, leading to a ridiculous answer of $\epsilon_r = 423$.

7.18. The hemisphere $0 < r < a$, $0 < \theta < \pi/2$, is composed of homogeneous conducting material of conductivity σ. The flat side of the hemisphere rests on a perfectly-conducting plane. Now, the material within the conical region $0 < \theta < \alpha$, $0 < r < a$, is drilled out, and replaced with material that is perfectly-conducting. An air gap is maintained between the $r = 0$ tip of this new material and the plane. What resistance is measured between the two perfect conductors? Neglect fringing fields.

With no fringing fields, we have θ-variation only in the potential. Laplace's equation is therefore:

$$\nabla^2 V = \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right) = 0$$

This reduces to

$$\frac{dV}{d\theta} = \frac{C_1}{\sin\theta} \quad\Rightarrow\quad V(\theta) = C_1\ln\tan(\theta/2) + C_2$$

We assume zero potential on the plane (at $\theta = \pi/2$), which means that $C_2 = 0$. On the cone (at $\theta = \alpha$), we assume potential V_0, and so $V_0 = C_1\ln\tan(\alpha/2)$
$\Rightarrow\ C_1 = V_0/\ln\tan(\alpha/2)$ The potential function is now

$$V(\theta) = V_0\frac{\ln\tan(\theta/2)}{\ln\tan(\alpha/2)} \qquad \alpha < \theta < \pi/2$$

The electric field is then

$$\mathbf{E} = -\nabla V = -\frac{1}{r}\frac{dV}{d\theta}\,\mathbf{a}_\theta = -\frac{V_0}{r\sin\theta\ln\tan(\alpha/2)}\,\mathbf{a}_\theta \quad \text{V/m}$$

The total current can now be found by integrating the current density, $\mathbf{J} = \sigma\mathbf{E}$, over any cross-section. Choosing the lower plane at $\theta = \pi/2$, this becomes

$$I = \int_0^{2\pi}\int_0^a -\frac{\sigma V_0}{r\sin(\pi/2)\ln\tan(\alpha/2)}\,\mathbf{a}_\theta\cdot\mathbf{a}_\theta\,r\,dr\,d\phi = -\frac{2\pi a\sigma V_0}{\ln\tan(\alpha/2)} \quad \text{A}$$

The resistance is finally

$$R = \frac{V_0}{I} = -\frac{\ln\tan(\alpha/2)}{2\pi a\sigma} \text{ ohms}$$

Note that R is in fact positive (despite the minus sign) since $\ln\tan(\alpha/2)$ is negative when $\alpha < \pi/2$ (which it must be).

7.19. Two coaxial conducting cones have their vertices at the origin and the z axis as their axis. Cone A has the point $A(1,0,2)$ on its surface, while cone B has the point $B(0,3,2)$ on its surface. Let $V_A = 100$ V and $V_B = 20$ V. Find:

a) α for each cone: Have $\alpha_A = \tan^{-1}(1/2) = \underline{26.57^\circ}$ and $\alpha_B = \tan^{-1}(3/2) = \underline{56.31^\circ}$.

b) V at $P(1,1,1)$: The potential function between cones can be written as

$$V(\theta) = C_1 \ln\tan(\theta/2) + C_2$$

Then

$$20 = C_1 \ln\tan(56.31/2) + C_2 \quad \text{and} \quad 100 = C_1 \ln\tan(26.57/2) + C_2$$

Solving these two equations, we find $C_1 = -97.7$ and $C_2 = -41.1$. Now at P, $\theta = \tan^{-1}(\sqrt{2}) = 54.7^\circ$. Thus

$$V_P = -97.7 \ln\tan(54.7/2) - 41.1 = \underline{23.3\,\text{V}}$$

7.20. A potential field in free space is given as $V = 100\ln\tan(\theta/2) + 50$ V.

a) Find the maximum value of $|\mathbf{E}_\theta|$ on the surface $\theta = 40^\circ$ for $0.1 < r < 0.8$ m, $60^\circ < \phi < 90^\circ$. First

$$\mathbf{E} = -\frac{1}{r}\frac{dV}{d\theta}\mathbf{a}_\theta = -\frac{100}{2r\tan(\theta/2)\cos^2(\theta/2)}\mathbf{a}_\theta = -\frac{100}{2r\sin(\theta/2)\cos(\theta/2)}\mathbf{a}_\theta = -\frac{100}{r\sin\theta}\mathbf{a}_\theta$$

This will maximize at the smallest value of r, or 0.1:

$$\mathbf{E}_{max}(\theta = 40^\circ) = \mathbf{E}(r = 0.1, \theta = 40^\circ) = -\frac{100}{0.1\sin(40)}\mathbf{a}_\theta = \underline{1.56\,\mathbf{a}_\theta\ \text{kV/m}}$$

b) Describe the surface $V = 80$ V: Set $100\ln\tan\theta/2 + 50 = 80$ and solve for θ: Obtain $\ln\tan\theta/2 = 0.3 \quad \Rightarrow \quad \tan\theta/2 = e^{.3} = 1.35 \quad \Rightarrow \quad \theta = \underline{107^\circ}$ (the cone surface at $\theta = 107$ degrees).

7.21. In free space, let $\rho_v = 200\epsilon_0/r^{2.4}$.

a) Use Poisson's equation to find $V(r)$ if it is assumed that $r^2 E_r \to 0$ when $r \to 0$, and also that $V \to 0$ as $r \to \infty$: With r variation only, we have

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = -\frac{\rho_v}{\epsilon} = -200 r^{-2.4}$$

or

$$\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = -200 r^{-.4}$$

Integrate once:

$$\left(r^2 \frac{dV}{dr}\right) = -\frac{200}{.6} r^{.6} + C_1 = -333.3 r^{.6} + C_1$$

or

$$\frac{dV}{dr} = -333.3 r^{-1.4} + \frac{C_1}{r^2} = \nabla V \text{ (in this case)} = -E_r$$

Our first boundary condition states that $r^2 E_r \to 0$ when $r \to 0$ Therefore $C_1 = 0$. Integrate again to find:

$$V(r) = \frac{333.3}{.4} r^{-.4} + C_2$$

From our second boundary condition, $V \to 0$ as $r \to \infty$, we see that $C_2 = 0$. Finally,

$$V(r) = \underline{833.3 r^{-.4} \text{ V}}$$

b) Now find $V(r)$ by using Gauss' Law and a line integral: Gauss' law applied to a spherical surface of radius r gives:

$$4\pi r^2 D_r = 4\pi \int_0^r \frac{200\epsilon_0}{(r')^{2.4}} (r')^2 dr = 800\pi\epsilon_0 \frac{r^{.6}}{.6}$$

Thus

$$E_r = \frac{D_r}{\epsilon_0} = \frac{800\pi\epsilon_0 r^{.6}}{.6(4\pi)\epsilon_0 r^2} = 333.3 r^{-1.4} \text{ V/m}$$

Now

$$V(r) = -\int_\infty^r 333.3 (r')^{-1.4} dr' = \underline{833.3 r^{-.4} \text{ V}}$$

7.22. By appropriate solution of Laplace's *and* Poisson's equations, determine the absolute potential at the center of a sphere of radius a, containing uniform volume charge of density ρ_0. Assume permittivity ϵ_0 everywhere. HINT: What must be true about the potential and the electric field at $r = 0$ and at $r = a$?

With radial dependence only, Poisson's equation (applicable to $r \leq a$) becomes

$$\nabla^2 V_1 = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV_1}{dr}\right) = -\frac{\rho_0}{\epsilon_0} \quad \Rightarrow \quad V_1(r) = -\frac{\rho_0 r^2}{6\epsilon_0} + \frac{C_1}{r} + C_2 \quad (r \leq a)$$

For region 2 ($r \geq a$) there is no charge and so Laplace's equation becomes

$$\nabla^2 V_2 = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV_2}{dr}\right) = 0 \quad \Rightarrow \quad V_2(r) = \frac{C_3}{r} + C_4 \quad (r \geq a)$$

Now, as $r \to \infty$, $V_2 \to 0$, so therefore $C_4 = 0$. Also, as $r \to 0$, V_1 must be finite, so therefore $C_1 = 0$. Then, V must be continuous across the boundary, $r = a$:

$$V_1\Big|_{r=a} = V_2\Big|_{r=a} \quad \Rightarrow \quad -\frac{\rho_0 a^2}{6\epsilon_0} + C_2 = \frac{C_3}{a} \quad \Rightarrow \quad C_2 = \frac{C_3}{a} + \frac{\rho_0 a^2}{6\epsilon_0}$$

So now

$$V_1(r) = \frac{\rho_0}{6\epsilon_0}(a^2 - r^2) + \frac{C_3}{a} \quad \text{and} \quad V_2(r) = \frac{C_3}{r}$$

Finally, since the permittivity is ϵ_0 everywhere, the electric field will be continuous at $r = a$. This is equivalent to the continuity of the voltage derivatives:

$$\frac{dV_1}{dr}\Big|_{r=a} = \frac{dV_2}{dr}\Big|_{r=a} \quad \Rightarrow \quad -\frac{\rho_0 a}{3\epsilon_0} = -\frac{C_3}{a^2} \quad \Rightarrow \quad C_3 = \frac{\rho_0 a^3}{3\epsilon_0}$$

So the potentials in their final forms are

$$V_1(r) = \frac{\rho_0}{6\epsilon_0}(3a^2 - r^2) \quad \text{and} \quad V_2(r) = \frac{\rho_0 a^3}{3\epsilon_0 r}$$

The requested absolute potential at the origin is now $V_1(r = 0) = \underline{\rho_0 a^2/(2\epsilon_0)\ \text{V}}$.

7.23. A rectangular trough is formed by four conducting planes located at $x = 0$ and 8 cm and $y = 0$ and 5 cm in air. The surface at $y = 5$ cm is at a potential of 100 V, the other three are at zero potential, and the necessary gaps are placed at two corners. Find the potential at $x = 3$ cm, $y = 4$ cm: This situation is the same as that of Fig. 7.6, except the non-zero boundary potential appears on the top surface, rather than the right side. The solution is found from Eq. (39) by simply interchanging x and y, and b and d, obtaining:

$$V(x, y) = \frac{4V_0}{\pi} \sum_{1,odd}^{\infty} \frac{1}{m} \frac{\sinh(m\pi y/d)}{\sinh(m\pi b/d)} \sin \frac{m\pi x}{d}$$

where $V_0 = 100$ V, $d = 8$ cm, and $b = 5$ cm. We will use the first three terms to evaluate the potential at (3,4):

$$V(3,4) \doteq \frac{400}{\pi}\left[\frac{\sinh(\pi/2)}{\sinh(5\pi/8)} \sin(3\pi/8) + \frac{1}{3}\frac{\sinh(3\pi/2)}{\sinh(15\pi/8)} \sin(9\pi/8) + \frac{1}{5}\frac{\sinh(5\pi/2)}{\sinh(25\pi/8)} \sin(15\pi/8)\right]$$

$$= \frac{400}{\pi}\,[.609 - .040 - .011] = 71.1 \text{ V}$$

Additional accuracy is found by including more terms in the expansion. Using thirteen terms, and using six significant figure accuracy, the result becomes $V(3,4) \doteq \underline{71.9173\ \text{V}}$. The series converges rapidly enough so that terms after the sixth one produce no change in the third digit. Thus, quoting three significant figures, 71.9 V requires six terms, with subsequent terms having no effect.

7.24. The four sides of a square trough are held at potentials of 0, 20, -30, and 60 V; the highest and lowest potentials are on opposite sides. Find the potential at the center of the trough: Here we can make good use of symmetry. The solution for a single potential on the right side, for example, with all other sides at 0V is given by Eq. (39):

$$V(x,y) = \frac{4V_0}{\pi} \sum_{1,\text{odd}}^{\infty} \frac{1}{m} \frac{\sinh(m\pi x/b)}{\sinh(m\pi d/b)} \sin\left(\frac{m\pi y}{b}\right)$$

In the current problem, we can account for the three voltages by superposing three solutions of the above form, suitably modified to account for the different locations of the boundary potentials. Since we want V at the center of a square trough, it no longer matters on what boundary each of the given potentials is, and we can simply write:

$$V(\text{center}) = \frac{4(0+20-30+60)}{\pi} \sum_{1,\text{odd}}^{\infty} \frac{1}{m} \frac{\sinh(m\pi/2)}{\sinh(m\pi)} \sin(m\pi/2) = \underline{12.5\,\text{V}}$$

The series converges to this value in three terms.

7.25. In Fig. 7.7, change the right side so that the potential varies linearly from 0 at the bottom of that side to 100 V at the top. Solve for the potential at the center of the trough: Since the potential reaches zero periodically in y and also is zero at $x = 0$, we use the form:

$$V(x,y) = \sum_{m=1}^{\infty} V_m \sinh\left(\frac{m\pi x}{b}\right) \sin\left(\frac{m\pi y}{b}\right)$$

Now, at $x = d$, $V = 100(y/b)$. Thus

$$100\frac{y}{b} = \sum_{m=1}^{\infty} V_m \sinh\left(\frac{m\pi d}{b}\right) \sin\left(\frac{m\pi y}{b}\right)$$

We then multiply by $\sin(n\pi y/b)$, where n is a fixed integer, and integrate over y from 0 to b:

$$\int_0^b 100\frac{y}{b} \sin\left(\frac{n\pi y}{b}\right) dy = \sum_{m=1}^{\infty} V_m \sinh\left(\frac{m\pi d}{b}\right) \underbrace{\int_0^b \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi y}{b}\right) dy}_{=b/2 \text{ if } m=n,\text{ zero if } m\neq n}$$

The integral on the right hand side picks the nth term out of the series, enabling the coefficients, V_n, to be solved for individually as we vary n. We find in general,

$$V_m = \frac{2}{b\sinh(m\pi/d)} \int_0^b 100\frac{y}{b} \sin\left(\frac{n\pi y}{b}\right) dy$$

The integral evaluates as

$$\int_0^b 100\frac{y}{b} \sin\left(\frac{n\pi y}{b}\right) dy = \left\{ \begin{array}{c} -100/m\pi \text{ (m even)} \\ 100/m\pi \text{ (m odd)} \end{array} \right\} = (-1)^{m+1}\frac{100}{m\pi}$$

7.25 (continued) Thus

$$V_m = \frac{200(-1)^{m+1}}{m\pi b \sinh(m\pi d/b)}$$

So that finally,

$$V(x,y) = \frac{200}{\pi b} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \frac{\sinh(m\pi x/b)}{\sinh(m\pi d/b)} \sin\left(\frac{m\pi y}{b}\right)$$

Now, with a square trough, set $b = d = 1$, and so $0 < x < 1$ and $0 < y < 1$. The potential becomes

$$V(x,y) = \frac{200}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \frac{\sinh(m\pi x)}{\sinh(m\pi)} \sin(m\pi y)$$

Now at the center of the trough, $x = y = 0.5$, and, using four terms, we have

$$V(.5,.5) \doteq \frac{200}{\pi}\left[\frac{\sinh(\pi/2)}{\sinh(\pi)} - \frac{1}{3}\frac{\sinh(3\pi/2)}{\sinh(3\pi)} + \frac{1}{5}\frac{\sinh(5\pi/2)}{\sinh(5\pi)} - \frac{1}{7}\frac{\sinh(7\pi/2)}{\sinh(7\pi)}\right] = \underline{12.5\text{ V}}$$

where additional terms do not affect the three-significant-figure answer.

7.26. If X is a function of x and $X'' + (x-1)X - 2X = 0$, assume a solution in the form of an infinite power series and determine numerical values for a_2 to a_8 if $a_0 = 1$ and $a_1 = -1$: The series solution will be of the form:

$$X = \sum_{m=0}^{\infty} a_m x^m$$

The first 8 terms of this are substituted into the given equation to give:

$$\begin{aligned}&(2a_2 - a_1 - 2a_0) + (6a_3 + a_1 - 2a_2 - 2a_1)x + (12a_4 + 2a_2 - 3a_3 - 2a_2)x^2\\ &+ (3a_3 - 4a_4 - 2a_3 + 20a_5)x^3 + (30a_6 + 4a_4 - 5a_5 - 2a_4)x^4 + (42a_7 + 5a_5 - 6a_6 - 2a_5)x^5\\ &+ (56a_8 + 6a_6 - 7a_7 - 2a_6)x^6 + (7a_7 - 8a_8 - 2a_7)x^7 + (8a_8 - 2a_8)x^8 = 0\end{aligned}$$

For this equation to be zero, each coefficient term (in parenthesis) must be zero. The first of these is

$$2a_2 - a_1 - 2a_0 = 2a_2 + 1 - 2 = 0 \quad \Rightarrow \quad a_2 = \underline{1/2}$$

The second coefficient is

$$6a_3 + a_1 - 2a_2 - 2a_1 = 6a_3 - 1 - 1 + 2 = 0 \quad \Rightarrow \quad a_3 = \underline{0}$$

Third coefficient:

$$12a_4 + 2a_2 - 3a_3 - 2a_2 = 12a_4 + 1 - 0 - 1 = 0 \quad \Rightarrow \quad a_4 = \underline{0}$$

Fourth coefficient:

$$3a_3 - 4a_4 - 2a_3 + 20a_5 = 0 - 0 - 0 + 20a_5 = 0 \quad \Rightarrow \quad a_5 = \underline{0}$$

In a similar manner, we find $a_6 = a_7 = a_8 = \underline{0}$.

7.27. It is known that $V = XY$ is a solution of Laplace's equation, where X is a function of x alone, and Y is a function of y alone. Determine which of the following potential function are also solutions of Laplace's equation:

a) $V = 100X$: We know that $\nabla^2 XY = 0$, or

$$\frac{\partial^2}{\partial x^2} XY + \frac{\partial^2}{\partial y^2} XY = 0 \quad \Rightarrow \quad YX'' + XY'' = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = \alpha^2$$

Therefore, $\nabla^2 X = 100X'' \neq 0$ – No.

b) $V = 50XY$: Would have $\nabla^2 V = 50\nabla^2 XY = 0$ – Yes.

c) $V = 2XY + x - 3y$: $\nabla^2 V = 2\nabla^2 XY + 0 - 0 = 0$ – Yes

d) $V = xXY$:

$$\nabla^2 V = \frac{\partial^2 xXY}{\partial x^2} + \frac{\partial^2 xXY}{\partial y^2} = \frac{\partial}{\partial x}\left[XY + xX'Y\right] + \frac{\partial}{\partial y}\left[xXY'\right]$$
$$= 2X'Y + x\underbrace{\left[X''Y + XY''\right]}_{\nabla^2 XY} \neq 0 \; - \; \text{No}$$

.

e) $V = X^2Y$: $\nabla^2 V = X\nabla^2 XY + XY\nabla^2 X = 0 + XY\nabla^2 X$ – No.

7.28. Assume a product solution of Laplace's equation in cylindrical coordinates, $V = PF$, where V is not a function of z, P is a function only of ρ, and F is a function only of ϕ.

a) Obtain the two separated equations if the separation constant is n^2. Select the sign of n^2 so that the solution of the ϕ equation leads to trigonometric functions: Begin with Laplace's equation in cylindrical coordinates, in which there is no z variation:

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

We substitute the product solution $V = PF$ to obtain:

$$\frac{F}{\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + \frac{P}{\rho^2}\frac{d^2F}{d\phi^2} = \frac{F}{\rho}\frac{dP}{d\rho} + F\frac{d^2P}{d\rho^2} + \frac{P}{\rho^2}\frac{d^2F}{d\phi^2} = 0$$

Next, multiply by ρ^2 and divide by FP to obtain

$$\underbrace{\frac{\rho}{P}\frac{dP}{d\rho} + \frac{\rho^2}{P}\frac{d^2P}{d\rho^2}}_{n^2} + \underbrace{\frac{1}{F}\frac{d^2F}{d\phi^2}}_{-n^2} = 0$$

The equation is now grouped into two parts as shown, each a function of only one of the two variables; each is set equal to plus or minus n^2, as indicated. The ϕ equation now becomes

$$\frac{d^2F}{d\phi^2} + n^2F = 0 \quad \Rightarrow \quad F = C_n\cos(n\phi) + D_n\sin(n\phi) \quad (n \geq 1)$$

Note that n is required to be an integer, since physically, the solution must repeat itself every 2π radians in ϕ. If $n = 0$, then

$$\frac{d^2F}{d\phi^2} = 0 \quad \Rightarrow \quad F = C_0\phi + D_0$$

7.28b. Show that $P = A\rho^n + B\rho^{-n}$ satisfies the ρ equation: From part a, the radial equation is:

$$\rho^2 \frac{d^2P}{d\rho^2} + \rho\frac{dP}{d\rho} - n^2 P = 0$$

Substituting $A\rho^n$, we find

$$\rho^2 n(n-1)\rho^{n-2} + \rho n\rho^{n-1} - n^2\rho^n = n^2\rho^n - n\rho^n + n\rho^n - n^2\rho^n = 0$$

Substituting $B\rho^{-n}$:

$$\rho^2 n(n+1)\rho^{-(n+2)} - \rho n\rho^{-(n+1)} - n^2\rho^{-n} = n^2\rho^{-n} + n\rho^{-n} - n\rho^{-n} - n^2\rho^{-n} = 0$$

So it works.

c) Construct the solution $V(\rho, \phi)$. Functions of this form are called *circular harmonics*. To assemble the complete solution, we need the radial solution for the case in which $n = 0$. The equation to solve is

$$\rho\frac{d^2P}{d\rho^2} + \frac{dP}{d\rho} = 0$$

Let $S = dP/d\rho$, and so $dS/d\rho = d^2P/d\rho^2$. The equation becomes

$$\rho\frac{dS}{d\rho} + S = 0 \quad \Rightarrow \quad -\frac{d\rho}{\rho} = \frac{dS}{S}$$

Integrating, find

$$-\ln\rho + \ln A_0 = \ln S \quad \Rightarrow \quad \ln S = \ln\left(\frac{A_0}{\rho}\right) \quad \Rightarrow \quad S = \frac{A_0}{\rho} = \frac{dP}{d\rho}$$

where A_0 is a constant. So now

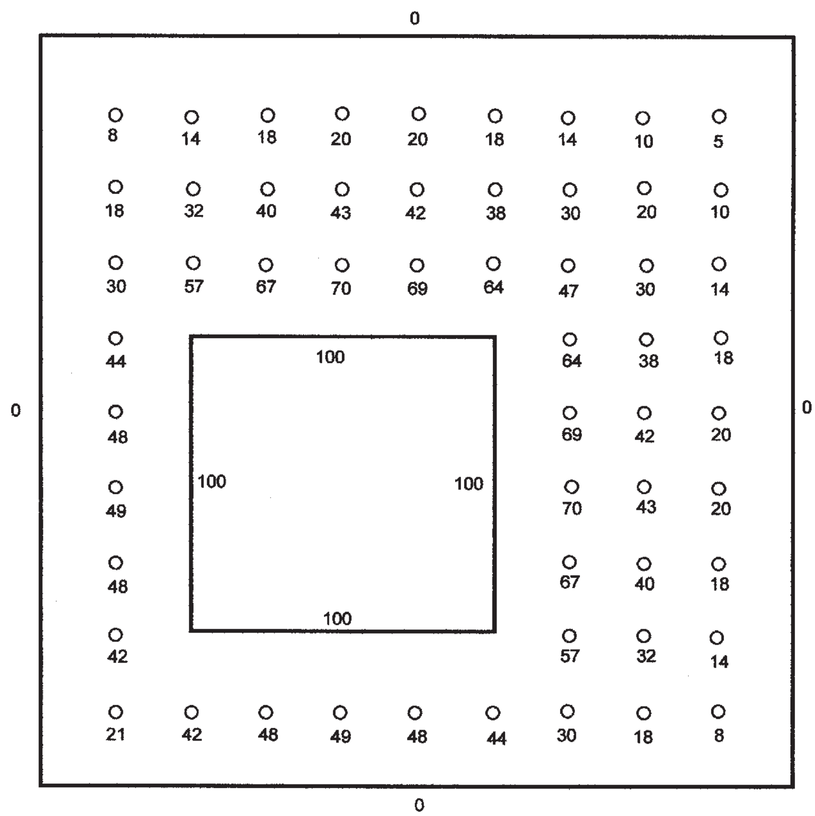
$$\frac{d\rho}{\rho} = \frac{dP}{A_0} \quad \Rightarrow \quad P_{n=0} = A_0\ln\rho + B_0$$

We may now construct the solution in its complete form, encompassing $n \geq 0$:

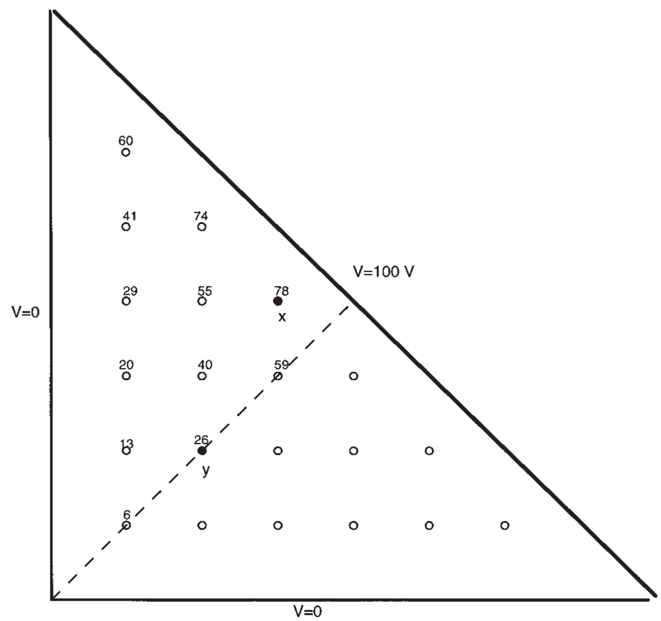
$$V(\rho,\phi) = \underbrace{(A_0\ln\rho + B_0)(C_0\phi + D_0)}_{n=0 \text{ solution}} + \sum_{n=1}^{\infty}[A_n\rho^n + B_n\rho^{-n}][C_n\cos(n\phi) + D_n\sin(n\phi)]$$

7.29. Referring to Chapter 6, Fig. 6.14, let the inner conductor of the transmission line be at a potential of 100V, while the outer is at zero potential. Construct a grid, $0.5a$ on a side, and use iteration to find V at a point that is a units above the upper right corner of the inner conductor. Work to the nearest volt:

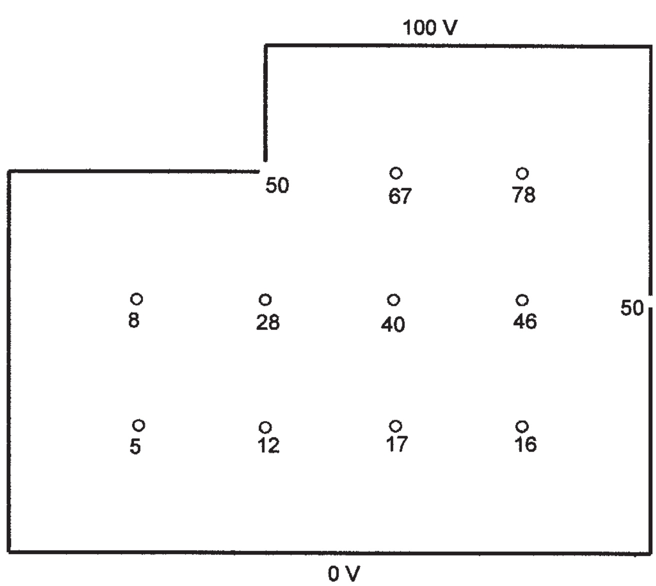
The drawing is shown below, and we identify the requested voltage as 38 V.



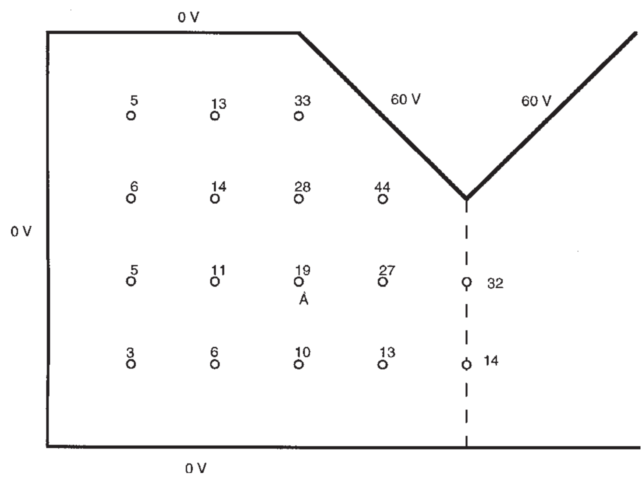
7.30. Use the iteration method to estimate the potentials at points x and y in the triangular trough of Fig. 7.14. Work only to the nearest volt: The result is shown below. The mirror image of the values shown occur at the points on the other side of the line of symmetry (dashed line). Note that $V_x = \underline{78\,\text{V}}$ and $V_y = \underline{26\,\text{V}}$.



7.31. Use iteration methods to estimate the potential at point x in the trough shown in Fig. 7.15. Working to the nearest volt is sufficient. The result is shown below, where we identify the voltage at x to be $\underline{40\,\text{V}}$. Note that the potentials in the gaps are 50 V.



7.32. Using the grid indicated in Fig. 7.16, work to the nearest volt to estimate the potential at point A: The voltages at the grid points are shown below, where V_A is found to be 19 V. Half the figure is drawn since mirror images of all values occur across the line of symmetry (dashed line).



7.33. Conductors having boundaries that are curved or skewed usually do not permit every grid point to coincide with the actual boundary. Figure 6.16a illustrates the situation where the potential at V_0 is to be estimated in terms of V_1, V_2, V_3, and V_4, and the unequal distances h_1, h_2, h_3, and h_4.

a) Show that

$$V_0 \doteq \frac{V_1}{\left(1+\frac{h_1}{h_3}\right)\left(1+\frac{h_1 h_3}{h_4 h_2}\right)} + \frac{V_2}{\left(1+\frac{h_2}{h_4}\right)\left(1+\frac{h_2 h_4}{h_1 h_3}\right)} + \frac{V_3}{\left(1+\frac{h_3}{h_1}\right)\left(1+\frac{h_1 h_3}{h_4 h_2}\right)}$$
$$+ \frac{V_4}{\left(1+\frac{h_4}{h_2}\right)\left(1+\frac{h_4 h_2}{h_3 h_1}\right)} \quad \text{note error, corrected here, in the equation (second term)}$$

Referring to the figure, we write:

$$\left.\frac{\partial V}{\partial x}\right|_{M_1} \doteq \frac{V_1 - V_0}{h_1} \qquad \left.\frac{\partial V}{\partial x}\right|_{M_3} \doteq \frac{V_0 - V_3}{h_3}$$

Then

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{V_0} \doteq \frac{(V_1 - V_0)/h_1 - (V_0 - V_3)/h_3}{(h_1 + h_3)/2} = \frac{2V_1}{h_1(h_1+h_3)} + \frac{2V_3}{h_3(h_1+h_3)} - \frac{2V_0}{h_1 h_3}$$

We perform the same procedure along the y axis to obtain:

$$\left.\frac{\partial^2 V}{\partial y^2}\right|_{V_0} \doteq \frac{(V_2 - V_0)/h_2 - (V_0 - V_4)/h_4}{(h_2 + h_4)/2} = \frac{2V_2}{h_2(h_2+h_4)} + \frac{2V_4}{h_4(h_2+h_4)} - \frac{2V_0}{h_2 h_4}$$

Then, knowing that

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{V_0} + \left.\frac{\partial^2 V}{\partial y^2}\right|_{V_0} = 0$$

the two equations for the second derivatives are added to give

$$\frac{2V_1}{h_1(h_1+h_3)} + \frac{2V_2}{h_2(h_2+h_4)} + \frac{2V_3}{h_3(h_1+h_3)} + \frac{2V_4}{h_4(h_2+h_4)} = V_0\left(\frac{h_1 h_3 + h_2 h_4}{h_1 h_2 h_3 h_4}\right)$$

Solve for V_0 to obtain the given equation.

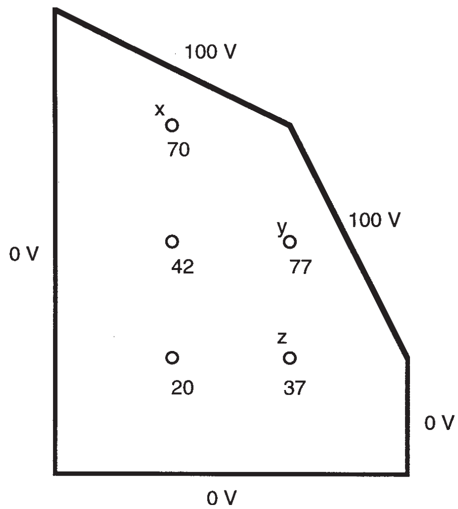
b) Determine V_0 in Fig. 6.16b: Referring to the figure, we note that $h_1 = h_2 = a$. The other two distances are found by writing equations for the circles:

$$(0.5a + h_3)^2 + a^2 = (1.5a)^2 \quad \text{and} \quad (a + h_4)^2 + (0.5a)^2 = (1.5a)^2$$

These are solved to find $h_3 = 0.618a$ and $h_4 = 0.414a$. The four distances and potentials are now substituted into the given equation:

$$V_0 \doteq \frac{80}{\left(1+\frac{1}{.618}\right)\left(1+\frac{.618}{.414}\right)} + \frac{60}{\left(1+\frac{1}{.414}\right)\left(1+\frac{.414}{.618}\right)} + \frac{100}{(1+.618)\left(1+\frac{.618}{.414}\right)}$$
$$+ \frac{100}{(1+.414)\left(1+\frac{.414}{.618}\right)} = \underline{90\text{ V}}$$

7.34. Consider the configuration of conductors and potentials shown in Fig. 7.18. Using the method described in Problem 7.33, write an expression for V_x (not V_0): The result is shown below, where $V_x = \underline{70\,\text{V}}$.



7.35a) After estimating potentials for the configuation of Fig. 7.19, use the iteration method with a square grid 1 cm on a side to find better estimates at the seven grid points. Work to the nearest volt:

25	50	75	50	25
0	48	100	48	0
0	42	100	42	0
0	19	34	19	0
0	0	0	0	0

b) Construct a 0.5 cm grid, establish new rough estimates, and then use the iteration method on the 0.5 cm grid. Again, work to the nearest volt: The result is shown below, with values for the original grid points underlined:

25	50	50	50	75	50	50	50	25
0	32	50	68	100	68	50	32	0
0	26	48	72	100	72	48	26	0
0	23	45	70	100	70	45	23	0
0	20	40	64	100	64	40	20	0
0	15	30	44	54	44	30	15	0
0	10	19	26	30	26	19	10	0
0	5	9	12	14	12	9	5	0
0	0	0	0	0	0	0	0	0

7.35c. Use a computer to obtain values for a 0.25 cm grid. Work to the nearest 0.1 V: Values for the left half of the configuration are shown in the table below. Values along the vertical line of symmetry are included, and the original grid values are underlined.

25	50	50	50	50	50	50	50	75
0	26.5	38.0	44.6	49.6	54.6	61.4	73.2	100
0	18.0	31.0	40.7	49.0	57.5	67.7	81.3	100
0	14.5	27.1	38.1	48.3	58.8	70.6	84.3	100
0	12.8	24.8	36.2	47.3	58.8	71.4	85.2	100
0	11.7	23.1	34.4	45.8	57.8	70.8	85.0	100
0	10.8	21.6	32.5	43.8	55.8	69.0	83.8	100
0	10.0	20.0	30.2	40.9	52.5	65.6	81.2	100
0	9.0	18.1	27.4	37.1	47.6	59.7	75.2	100
0	7.9	15.9	24.0	32.4	41.2	50.4	59.8	67.2
0	6.8	13.6	20.4	27.3	34.2	40.7	46.3	49.2
0	5.6	11.2	16.8	22.2	27.4	32.0	35.4	36.8
0	4.4	8.8	13.2	17.4	21.2	24.4	26.6	27.4
0	3.3	6.6	9.8	12.8	15.4	17.6	19.0	19.5
0	2.2	4.4	6.4	8.4	10.0	11.4	12.2	12.5
0	1.1	2.2	3.2	4.2	5.0	5.6	6.0	6.1
0	0	0	0	0	0	0	0	0