CHAPTER 8

8.1a. Find $\mathbf{H}$ in cartesian components at $P(2,3,4)$ if there is a current filament on the z axis carrying 8 mA in the $\mathbf{a}_z$ direction:
Applying the Biot-Savart Law, we obtain

$$\mathbf{H}_a = \int_{-\infty}^{\infty} \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \int_{-\infty}^{\infty} \frac{Idz\,\mathbf{a}_z \times [2\mathbf{a}_x + 3\mathbf{a}_y + (4-z)\mathbf{a}_z]}{4\pi(z^2 - 8z + 29)^{3/2}} = \int_{-\infty}^{\infty} \frac{Idz[2\mathbf{a}_y - 3\mathbf{a}_x]}{4\pi(z^2-8z+29)^{3/2}}$$

Using integral tables, this evaluates as

$$\mathbf{H}_a = \frac{I}{4\pi}\left[\frac{2(2z-8)(2\mathbf{a}_y - 3\mathbf{a}_x)}{52(z^2-8z+29)^{1/2}}\right]_{-\infty}^{\infty} = \frac{I}{26\pi}(2\mathbf{a}_y - 3\mathbf{a}_x)$$

Then with $I = 8$ mA, we finally obtain $\mathbf{H}_a = \underline{-294\mathbf{a}_x + 196\mathbf{a}_y\ \mu\text{A/m}}$

b. Repeat if the filament is located at $x = -1$, $y = 2$: In this case the Biot-Savart integral becomes

$$\mathbf{H}_b = \int_{-\infty}^{\infty} \frac{Idz\,\mathbf{a}_z \times [(2+1)\mathbf{a}_x + (3-2)\mathbf{a}_y + (4-z)\mathbf{a}_z]}{4\pi(z^2-8z+26)^{3/2}} = \int_{-\infty}^{\infty} \frac{Idz[3\mathbf{a}_y - \mathbf{a}_x]}{4\pi(z^2-8z+26)^{3/2}}$$

Evaluating as before, we obtain with $I = 8$ mA:

$$\mathbf{H}_b = \frac{I}{4\pi}\left[\frac{2(2z-8)(3\mathbf{a}_y - \mathbf{a}_x)}{40(z^2-8z+26)^{1/2}}\right]_{-\infty}^{\infty} = \frac{I}{20\pi}(3\mathbf{a}_y - \mathbf{a}_x) = \underline{-127\mathbf{a}_x + 382\mathbf{a}_y\ \mu\text{A/m}}$$

c. Find $\mathbf{H}$ if both filaments are present: This will be just the sum of the results of parts a and b, or

$$\mathbf{H}_T = \mathbf{H}_a + \mathbf{H}_b = \underline{-421\mathbf{a}_x + 578\mathbf{a}_y\ \mu\text{A/m}}$$

This problem can also be done (somewhat more simply) by using the known result for $\mathbf{H}$ from an infinitely-long wire in cylindrical components, and transforming to cartesian components. The Biot-Savart method was used here for the sake of illustration.

8.2. A filamentary conductor is formed into an equilateral triangle with sides of length ℓ carrying current I. Find the magnetic field intensity at the center of the triangle.

I will work this one from scratch, using the Biot-Savart law. Consider one side of the triangle, oriented along the z axis, with its end points at $z = \pm\ell/2$. Then consider a point, x_0, on the x axis, which would correspond to the center of the triangle, and at which we want to find $\mathbf{H}$ associated with the wire segment. We thus have $Id\mathbf{L} = Idz\,\mathbf{a}_z$, $R = \sqrt{z^2 + x_0^2}$, and $\mathbf{a}_R = [x_0\,\mathbf{a}_x - z\,\mathbf{a}_z]/R$. The differential magnetic field at x_0 is now

$$d\mathbf{H} = \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{Idz\,\mathbf{a}_z \times (x_0\mathbf{a}_x - z\,\mathbf{a}_z)}{4\pi(x_0^2 + z^2)^{3/2}} = \frac{I\,dz\,x_0\,\mathbf{a}_y}{4\pi(x_0^2+z^2)^{3/2}}$$

where $\mathbf{a}_y$ would be normal to the plane of the triangle. The magnetic field at x_0 is then

$$\mathbf{H} = \int_{-\ell/2}^{\ell/2} \frac{I\,dz\,x_0\,\mathbf{a}_y}{4\pi(x_0^2+z^2)^{3/2}} = \frac{I\,z\,\mathbf{a}_y}{4\pi x_0\sqrt{x_0^2+z^2}}\Bigg|_{-\ell/2}^{\ell/2} = \frac{I\ell\,\mathbf{a}_y}{2\pi x_0\sqrt{\ell^2 + 4x_0^2}}$$

8.2. (continued). Now, x_0 lies at the center of the equilateral triangle, and from the geometry of the triangle, we find that $x_0 = (\ell/2)\tan(30°) = \ell/(2\sqrt{3})$. Substituting this result into the just-found expression for $\mathbf{H}$ leads to $\mathbf{H} = 3I/(2\pi\ell)\,\mathbf{a}_y$. The contributions from the other two sides of the triangle effectively multiply the above result by three. The final answer is therefore $\mathbf{H}_{net} = \underline{9I/(2\pi\ell)\,\mathbf{a}_y \text{ A/m}}$. It is also possible to work this problem (somewhat more easily) by using Eq. (9), applied to the triangle geometry.

8.3. Two semi-infinite filaments on the z axis lie in the regions $-\infty < z < -a$ (note typographical error in problem statement) and $a < z < \infty$. Each carries a current I in the $\mathbf{a}_z$ direction.

a) Calculate $\mathbf{H}$ as a function of ρ and ϕ at $z = 0$: One way to do this is to use the field from an infinite line and subtract from it that portion of the field that would arise from the current segment at $-a < z < a$, found from the Biot-Savart law. Thus,

$$\mathbf{H} = \frac{I}{2\pi\rho}\,\mathbf{a}_\phi - \int_{-a}^{a} \frac{I\,dz\,\mathbf{a}_z \times [\rho\,\mathbf{a}_\rho - z\,\mathbf{a}_z]}{4\pi[\rho^2 + z^2]^{3/2}}$$

The integral part simplifies and is evaluated:

$$\int_{-a}^{a} \frac{I\,dz\,\rho\,\mathbf{a}_\phi}{4\pi[\rho^2+z^2]^{3/2}} = \frac{I\rho}{4\pi}\,\mathbf{a}_\phi\,\frac{z}{\rho^2\sqrt{\rho^2+z^2}}\Bigg|_{-a}^{a} = \frac{Ia}{2\pi\rho\sqrt{\rho^2+a^2}}\,\mathbf{a}_\phi$$

Finally,

$$\mathbf{H} = \frac{I}{2\pi\rho}\left[1 - \frac{a}{\sqrt{\rho^2+a^2}}\right]\mathbf{a}_\phi \text{ A/m}$$

b) What value of a will cause the magnitude of $\mathbf{H}$ at $\rho = 1$, $z = 0$, to be one-half the value obtained for an infinite filament? We require

$$\left[1 - \frac{a}{\sqrt{\rho^2+a^2}}\right]_{\rho=1} = \frac{1}{2} \;\Rightarrow\; \frac{a}{\sqrt{1+a^2}} = \frac{1}{2} \;\Rightarrow\; a = \underline{1/\sqrt{3}}$$

8.4. (a) A filament is formed into a circle of radius a, centered at the origin in the plane $z = 0$. It carries a current I in the $\mathbf{a}_\phi$ direction. Find $\mathbf{H}$ at the origin:

Using the Biot-Savart law, we have $Id\mathbf{L} = Iad\pi\,\mathbf{a}_\phi$, $R = a$, and $\mathbf{a}_R = -\mathbf{a}_\rho$. The field at the center of the circle is then

$$\mathbf{H}_{circ} = \int_0^{2\pi} \frac{Iad\phi\,\mathbf{a}_\phi \times (-\mathbf{a}_\rho)}{4\pi a^2} = \int_0^{2\pi} \frac{Id\phi\,\mathbf{a}_z}{4\pi a} = \frac{I}{2a}\,\mathbf{a}_z \text{ A/m}$$

b) A second filament is shaped into a square in the $z = 0$ plane. The sides are parallel to the coordinate axes and a current I flows in the general $\mathbf{a}_\phi$ direction. Determine the side length b (in terms of a), such that $\mathbf{H}$ at the origin is the same magnitude as that of the circular loop of part a.

Applying Eq. (9), we write the field from a single side of length b at a distance $b/2$ from the side center as:

$$\mathbf{H} = \frac{I\,\mathbf{a}_z}{4\pi(b/2)}\left[\sin(45°) - \sin(-45°)\right] = \frac{\sqrt{2}I\,\mathbf{a}_z}{2\pi b}$$

so that the total field at the center of the square will be four times the above result or, $\mathbf{H}_{sq} = 2\sqrt{2}I\,\mathbf{a}_z/(\pi b)$ A/m. Now, setting $\mathbf{H}_{sq} = \mathbf{H}_{circ}$, we find $b = 4\sqrt{2}a/\pi = \underline{1.80\,a}$.

8.5. The parallel filamentary conductors shown in Fig. 8.21 lie in free space. Plot $|\mathbf{H}|$ versus y, $-4 < y < 4$, along the line $x = 0$, $z = 2$: We need an expression for $\mathbf{H}$ in cartesian coordinates. We can start with the known $\mathbf{H}$ in cylindrical for an infinite filament along the z axis: $\mathbf{H} = I/(2\pi\rho)\,\mathbf{a}_\phi$, which we transform to cartesian to obtain:

$$\mathbf{H} = \frac{-Iy}{2\pi(x^2+y^2)}\,\mathbf{a}_x + \frac{Ix}{2\pi(x^2+y^2)}\,\mathbf{a}_y$$

If we now rotate the filament so that it lies along the x axis, with current flowing in positive x, we obtain the field from the above expression by replacing x with y and y with z:

$$\mathbf{H} = \frac{-Iz}{2\pi(y^2+z^2)}\,\mathbf{a}_y + \frac{Iy}{2\pi(y^2+z^2)}\,\mathbf{a}_z$$

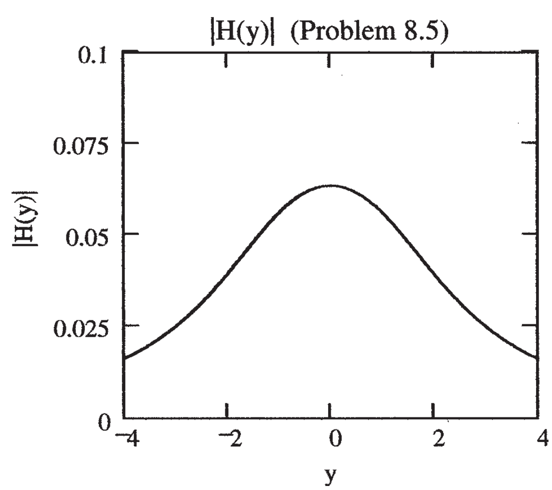
Now, with two filaments, displaced from the x axis to lie at $y = \pm 1$, and with the current directions as shown in the figure, we use the previous expression to write

$$\mathbf{H} = \left[\frac{Iz}{2\pi[(y+1)^2+z^2]} - \frac{Iz}{2\pi[(y-1)^2+z^2]}\right]\mathbf{a}_y + \left[\frac{I(y-1)}{2\pi[(y-1)^2+z^2]} - \frac{I(y+1)}{2\pi[(y+1)^2+z^2]}\right]\mathbf{a}_z$$

We now evaluate this at $z = 2$, and find the magnitude ($\sqrt{\mathbf{H}\cdot\mathbf{H}}$), resulting in

$$|\mathbf{H}| = \frac{I}{2\pi}\left[\left(\frac{2}{y^2+2y+5} - \frac{2}{y^2-2y+5}\right)^2 + \left(\frac{(y-1)}{y^2-2y+5} - \frac{(y+1)}{y^2+2y+5}\right)^2\right]^{1/2}$$

This function is plotted below



8.6. A disk of radius a lies in the xy plane, with the z axis through its center. Surface charge of uniform density ρ_s lies on the disk, which rotates about the z axis at angular velocity Ω rad/s. Find $\mathbf{H}$ at any point on the z axis.

We use the Biot-Savart law in the form of Eq. (6), with the following parameters: $\mathbf{K} = \rho_s\mathbf{v} = \rho_s\rho\Omega\,\mathbf{a}_\phi$, $R = \sqrt{z^2+\rho^2}$, and $\mathbf{a}_R = (z\,\mathbf{a}_z - \rho\,\mathbf{a}_\rho)/R$. The differential field at point z is

$$d\mathbf{H} = \frac{\mathbf{K}da \times \mathbf{a}_R}{4\pi R^2} = \frac{\rho_s\rho\,\Omega\,\mathbf{a}_\phi \times (z\,\mathbf{a}_z - \rho\,\mathbf{a}_\rho)}{4\pi(z^2+\rho^2)^{3/2}}\,\rho\,d\rho\,d\phi = \frac{\rho_s\rho\,\Omega\,(z\,\mathbf{a}_\rho + \rho\,\mathbf{a}_z)}{4\pi(z^2+\rho^2)^{3/2}}\,\rho\,d\rho\,d\phi$$

8.6. (continued). On integrating the above over ϕ around a complete circle, the $\mathbf{a}_\rho$ components cancel from symmetry, leaving us with

$$\mathbf{H}(z) = \int_0^{2\pi}\int_0^a \frac{\rho_s \rho\, \Omega\, \rho\, \mathbf{a}_z}{4\pi(z^2+\rho^2)^{3/2}}\,\rho\, d\rho\, d\phi = \int_0^a \frac{\rho_s\, \Omega\, \rho^3\, \mathbf{a}_z}{2(z^2+\rho^2)^{3/2}}\, d\rho$$

$$= \frac{\rho_s \Omega}{2}\left[\sqrt{z^2+\rho^2} + \frac{z^2}{\sqrt{z^2+\rho^2}}\right]_0^a \mathbf{a}_z = \frac{\rho_s \Omega}{2z}\left[\frac{a^2 + 2z^2\left(1-\sqrt{1+a^2/z^2}\right)}{\sqrt{1+a^2/z^2}}\right]\mathbf{a}_z \text{ A/m}$$

8.7. Given points $C(5,-2,3)$ and $P(4,-1,2)$; a current element $Id\mathbf{L} = 10^{-4}(4,-3,1)$ A $\cdot$ m at C produces a field $d\mathbf{H}$ at P.

a) Specify the direction of $d\mathbf{H}$ by a unit vector $\mathbf{a}_H$: Using the Biot-Savart law, we find

$$d\mathbf{H} = \frac{Id\mathbf{L}\times \mathbf{a}_{CP}}{4\pi R_{CP}^2} = \frac{10^{-4}[4\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z]\times[-\mathbf{a}_x + \mathbf{a}_y - \mathbf{a}_z]}{4\pi 3^{3/2}} = \frac{[2\mathbf{a}_x + 3\mathbf{a}_y + \mathbf{a}_z]\times 10^{-4}}{65.3}$$

from which

$$\mathbf{a}_H = \frac{2\mathbf{a}_x + 3\mathbf{a}_y + \mathbf{a}_z}{\sqrt{14}} = \underline{0.53\mathbf{a}_x + 0.80\mathbf{a}_y + 0.27\mathbf{a}_z}$$

b) Find $|d\mathbf{H}|$.

$$|d\mathbf{H}| = \frac{\sqrt{14}\times 10^{-4}}{65.3} = 5.73\times 10^{-6} \text{ A/m} \quad = \underline{5.73\ \mu\text{A/m}}$$

c) What direction $\mathbf{a}_l$ should $Id\mathbf{L}$ have at C so that $d\mathbf{H} = 0$? $Id\mathbf{L}$ should be collinear with $\mathbf{a}_{CP}$, thus rendering the cross product in the Biot-Savart law equal to zero. Thus the answer is $\mathbf{a}_l = \underline{\pm(-\mathbf{a}_x + \mathbf{a}_y - \mathbf{a}_z)/\sqrt{3}}$

8.8. For the finite-length current element on the z axis, as shown in Fig. 8.5, use the Biot-Savart law to derive Eq. (9) of Sec. 8.1: The Biot-Savart law reads:

$$\mathbf{H} = \int_{z_1}^{z_2} \frac{Id\mathbf{L}\times\mathbf{a}_R}{4\pi R^2} = \int_{\rho\tan\alpha_1}^{\rho\tan\alpha_2} \frac{Idz\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z\mathbf{a}_z)}{4\pi(\rho^2+z^2)^{3/2}} = \int_{\rho\tan\alpha_1}^{\rho\tan\alpha_2} \frac{I\rho\mathbf{a}_\phi\, dz}{4\pi(\rho^2+z^2)^{3/2}}$$

The integral is evaluated (using tables) and gives the desired result:

$$\mathbf{H} = \left.\frac{Iz\mathbf{a}_\phi}{4\pi\rho\sqrt{\rho^2+z^2}}\right|_{\rho\tan\alpha_1}^{\rho\tan\alpha_2} = \frac{I}{4\pi\rho}\left[\frac{\tan\alpha_2}{\sqrt{1+\tan^2\alpha_2}} - \frac{\tan\alpha_1}{\sqrt{1+\tan^2\alpha_1}}\right]\mathbf{a}_\phi$$

$$= \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\phi$$

8.9. A current sheet $\mathbf{K} = 8\mathbf{a}_x$ A/m flows in the region $-2 < y < 2$ in the plane $z = 0$. Calculate H at $P(0, 0, 3)$: Using the Biot-Savart law, we write

$$\mathbf{H}_P = \int\int \frac{\mathbf{K} \times \mathbf{a}_R \, dx\, dy}{4\pi R^2} = \int_{-2}^{2}\int_{-\infty}^{\infty} \frac{8\mathbf{a}_x \times (-x\mathbf{a}_x - y\mathbf{a}_y + 3\mathbf{a}_z)}{4\pi(x^2 + y^2 + 9)^{3/2}}\, dx\, dy$$

Taking the cross product gives:

$$\mathbf{H}_P = \int_{-2}^{2}\int_{-\infty}^{\infty} \frac{8(-y\mathbf{a}_z - 3\mathbf{a}_y)\, dx\, dy}{4\pi(x^2 + y^2 + 9)^{3/2}}$$

We note that the z component is anti-symmetric in y about the origin (odd parity). Since the limits are symmetric, the integral of the z component over y is zero. We are left with

$$\mathbf{H}_P = \int_{-2}^{2}\int_{-\infty}^{\infty} \frac{-24\,\mathbf{a}_y\, dx\, dy}{4\pi(x^2 + y^2 + 9)^{3/2}} = -\frac{6}{\pi}\mathbf{a}_y \int_{-2}^{2} \frac{x}{(y^2 + 9)\sqrt{x^2 + y^2 + 9}}\Big|_{-\infty}^{\infty} dy$$

$$= -\frac{6}{\pi}\mathbf{a}_y \int_{-2}^{2} \frac{2}{y^2 + 9}\, dy = -\frac{12}{\pi}\mathbf{a}_y \frac{1}{3}\tan^{-1}\left(\frac{y}{3}\right)\Big|_{-2}^{2} = -\frac{4}{\pi}(2)(0.59)\,\mathbf{a}_y = \underline{-1.50\,\mathbf{a}_y \text{ A/m}}$$

8.10. A hollow spherical conducting shell of radius a has filamentary connections made at the top ($r = a$, $\theta = 0$) and bottom ($r = a$, $\theta = \pi$). A direct current I flows down the upper filament, down the spherical surface, and out the lower filament. Find $\mathbf{H}$ in spherical coordinates (a) inside and (b) outside the sphere.

Applying Ampere's circuital law, we use a circular contour, centered on the z axis, and find that within the sphere, no current is enclosed, and so $\underline{\mathbf{H} = 0 \text{ when } r < a}$. The same contour drawn outside the sphere at any z position will always enclose I amps, flowing in the negative z direction, and so

$$\mathbf{H} = -\frac{I}{2\pi\rho}\mathbf{a}_\phi = -\frac{I}{2\pi r\sin\theta}\mathbf{a}_\phi \text{ A/m} \quad (r > a)$$

8.11. An infinite filament on the z axis carries 20π mA in the $\mathbf{a}_z$ direction. Three uniform cylindrical current sheets are also present: 400 mA/m at $\rho = 1$ cm, -250 mA/m at $\rho = 2$ cm, and -300 mA/m at $\rho = 3$ cm. Calculate H_ϕ at $\rho = 0.5$, 1.5, 2.5, and 3.5 cm: We find H_ϕ at each of the required radii by applying Ampere's circuital law to circular paths of those radii; the paths are centered on the z axis. So, at $\rho_1 = 0.5$ cm:

$$\oint \mathbf{H}\cdot d\mathbf{L} = 2\pi\rho_1 H_{\phi 1} = I_{encl} = 20\pi \times 10^{-3} \text{ A}$$

Thus

$$H_{\phi 1} = \frac{10 \times 10^{-3}}{\rho_1} = \frac{10 \times 10^{-3}}{0.5 \times 10^{-2}} = \underline{2.0 \text{ A/m}}$$

At $\rho = \rho_2 = 1.5$ cm, we enclose the first of the current cylinders at $\rho = 1$ cm. Ampere's law becomes:

$$2\pi\rho_2 H_{\phi 2} = 20\pi + 2\pi(10^{-2})(400) \text{ mA} \quad\Rightarrow\quad H_{\phi 2} = \frac{10 + 4.00}{1.5 \times 10^{-2}} = \underline{933 \text{ mA/m}}$$

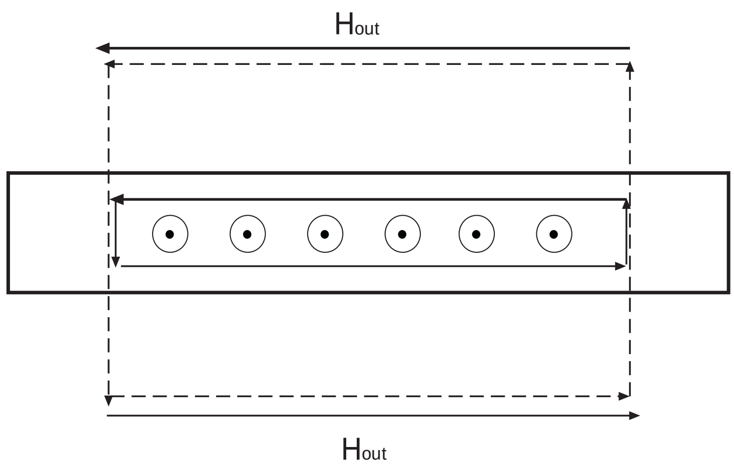
Following this method, at 2.5 cm:

$$H_{\phi 3} = \frac{10 + 4.00 - (2 \times 10^{-2})(250)}{2.5 \times 10^{-2}} = \underline{360 \text{ mA/m}}$$

and at 3.5 cm,

$$H_{\phi 4} = \frac{10 + 4.00 - 5.00 - (3 \times 10^{-2})(300)}{3.5 \times 10^{-2}} = \underline{0}$$

8.12. In Fig. 8.22, let the regions $0 < z < 0.3$ m and $0.7 < z < 1.0$ m be conducting slabs carrying uniform current densities of 10 A/m^2 in opposite directions as shown. The problem asks you to find **H** at various positions. Before continuing, we need to know how to find **H** for this type of current configuration. The sketch below shows one of the slabs (of thickness D) oriented with the current coming out of the page. The problem statement implies that both slabs are of infinite length and width. To find the magnetic field *inside* a slab, we apply Ampere's circuital law to the rectangular path of height d and width w, as shown, since by symmetry, **H** should be oriented horizontally. For example, if the sketch below shows the upper slab in Fig. 8.22, current will be in the positive y direction. Thus **H** will be in the positive x direction above the slab midpoint, and will be in the negative x direction below the midpoint.



In taking the line integral in Ampere's law, the two vertical path segments will cancel each other. Ampere's circuital law for the interior loop becomes

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2H_{in} \times w = I_{encl} = J \times w \times d \quad \Rightarrow \quad H_{in} = \frac{Jd}{2}$$

The field outside the slab is found similarly, but with the enclosed current now bounded by the slab thickness, rather than the integration path height:

$$2H_{out} \times w = J \times w \times D \quad \Rightarrow \quad H_{out} = \frac{JD}{2}$$

where H_{out} is directed from right to left below the slab and from left to right above the slab (right hand rule). Reverse the current, and the fields, of course, reverse direction. We are now in a position to solve the problem.

8.12. (continued). Find **H** at:

a) $z = -0.2$m: Here the fields from the top and bottom slabs (carrying opposite currents) will cancel, and so $\mathbf{H} = \underline{0}$.

b) $z = 0.2$m. This point lies within the lower slab above its midpoint. Thus the field will be oriented in the negative x direction. Referring to Fig. 8.22 and to the sketch on the previous page, we find that $d = 0.1$. The total field will be this field plus the contribution from the upper slab current:

$$\mathbf{H} = \underbrace{\frac{-10(0.1)}{2}\mathbf{a}_x}_{\text{lower slab}} - \underbrace{\frac{10(0.3)}{2}\mathbf{a}_x}_{\text{upper slab}} = \underline{-2\mathbf{a}_x \text{ A/m}}$$

c) $z = 0.4$m: Here the fields from both slabs will add constructively in the negative x direction:

$$\mathbf{H} = -2\frac{10(0.3)}{2}\mathbf{a}_x = \underline{-3\mathbf{a}_x \text{ A/m}}$$

d) $z = 0.75$m: This is in the interior of the upper slab, whose midpoint lies at $z = 0.85$. Therefore $d = 0.2$. Since 0.75 lies below the midpoint, magnetic field from the upper slab will lie in the negative x direction. The field from the lower slab will be negative x-directed as well, leading to:

$$\mathbf{H} = \underbrace{\frac{-10(0.2)}{2}\mathbf{a}_x}_{\text{upper slab}} - \underbrace{\frac{10(0.3)}{2}\mathbf{a}_x}_{\text{lower slab}} = \underline{-2.5\mathbf{a}_x \text{ A/m}}$$

e) $z = 1.2$m: This point lies above both slabs, where again fields cancel completely: Thus $\mathbf{H} = \underline{0}$.

8.13. A hollow cylindrical shell of radius a is centered on the z axis and carries a uniform surface current density of $K_a\mathbf{a}_\phi$.

a) Show that H is not a function of ϕ or z: Consider this situation as illustrated in Fig. 8.11. There (sec. 8.2) it was stated that the field will be entirely z-directed. We can see this by applying Ampere's circuital law to a closed loop path whose orientation we choose such that current is enclosed by the path. The only way to enclose current is to set up the loop (which we choose to be rectangular) such that it is oriented with two parallel opposing segments lying in the z direction; one of these lies inside the cylinder, the other outside. The other two parallel segments lie in the ρ direction. The loop is now cut by the current sheet, and if we assume a length of the loop in z of d, then the enclosed current will be given by Kd A. There will be no ϕ variation in the field because where we position the loop around the circumference of the cylinder does not affect the result of Ampere's law. If we assume an infinite cylinder length, there will be no z dependence in the field, since as we lengthen the loop in the z direction, the path length (over which the integral is taken) increases, but then so does the enclosed current – by the same factor. Thus H would not change with z. There would also be no change if the loop was simply moved along the z direction.

8.13b) Show that H_ϕ and H_ρ are everywhere zero. First, if H_ϕ were to exist, then we should be able to find a closed loop path *that encloses current*, in which all or or portion of the path lies in the ϕ direction. This we cannot do, and so H_ϕ must be zero. Another argument is that when applying the Biot-Savart law, there is no current element that would produce a ϕ component. Again, using the Biot-Savart law, we note that radial field components will be produced by individual current elements, but such components will cancel from two elements that lie at symmetric distances in z on either side of the observation point.

c) Show that $H_z = 0$ for $\rho > a$: Suppose the rectangular loop was drawn such that the outside z-directed segment is moved further and further away from the cylinder. We would expect H_z outside to decrease (as the Biot-Savart law would imply) but the same amount of current is always enclosed no matter how far away the outer segment is. We therefore must conclude that the field outside is zero.

d) Show that $H_z = K_a$ for $\rho < a$: With our rectangular path set up as in part a, we have no path integral contributions from the two radial segments, and no contribution from the outside z-directed segment. Therefore, Ampere's circuital law would state that

$$\oint \mathbf{H} \cdot d\mathbf{L} = H_z d = I_{encl} = K_a d \quad \Rightarrow \quad H_z = K_a$$

where d is the length of the loop in the z direction.

e) A second shell, $\rho = b$, carries a current $K_b \mathbf{a}_\phi$. Find $\mathbf{H}$ everywhere: For $\rho < a$ we would have both cylinders contributing, or $H_z(\rho < a) = K_a + K_b$. Between the cylinders, we are outside the inner one, so its field will not contribute. Thus $H_z(a < \rho < b) = K_b$. Outside $(\rho > b)$ the field will be zero.

8.14. A toroid having a cross section of rectangular shape is defined by the following surfaces: the cylinders $\rho = 2$ and $\rho = 3$ cm, and the planes $z = 1$ and $z = 2.5$ cm. The toroid carries a surface current density of $-50\mathbf{a}_z$ A/m on the surface $\rho = 3$ cm. Find $\mathbf{H}$ at the point $P(\rho, \phi, z)$: The construction is similar to that of the toroid of round cross section as done on p.239. Again, magnetic field exists only inside the toroid cross section, and is given by

$$\mathbf{H} = \frac{I_{encl}}{2\pi\rho}\mathbf{a}_\phi \qquad (2 < \rho < 3) \text{ cm}, \quad (1 < z < 2.5) \text{ cm}$$

where I_{encl} is found from the given current density: On the outer radius, the current is

$$I_{outer} = -50(2\pi \times 3 \times 10^{-2}) = -3\pi \text{ A}$$

This current is directed along negative z, which means that the current on the *inner* radius $(\rho = 2)$ is directed along *positive* z. Inner and outer currents have the same magnitude. It is the inner current that is enclosed by the circular integration path in $\mathbf{a}_\phi$ within the toroid that is used in Ampere's law. So $I_{encl} = +3\pi$ A. We can now proceed with what is requested:

a) $P_A(1.5\text{cm}, 0, 2\text{cm})$: The radius, $\rho = 1.5$ cm, lies outside the cross section, and so $\mathbf{H}_A = \underline{0}$.

b) $P_B(2.1\text{cm}, 0, 2\text{cm})$: This point does lie inside the cross section, and the ϕ and z values do not matter. We find

$$\mathbf{H}_B = \frac{I_{encl}}{2\pi\rho}\mathbf{a}_\phi = \frac{3\mathbf{a}_\phi}{2(2.1 \times 10^{-2})} = \underline{71.4\,\mathbf{a}_\phi \text{ A/m}}$$

8.14c) $P_C(2.7\text{cm}, \pi/2, 2\text{cm})$: again, ϕ and z values make no difference, so

$$\mathbf{H}_C = \frac{3\mathbf{a}_\phi}{2(2.7\times 10^{-2})} = \underline{55.6\,\mathbf{a}_\phi\ \text{A/m}}$$

d) $P_D(3.5\text{cm}, \pi/2, 2\text{cm})$. This point lies outside the cross section, and so $\mathbf{H}_D = \underline{0}$.

8.15. Assume that there is a region with cylindrical symmetry in which the conductivity is given by $\sigma = 1.5e^{-150\rho}$ kS/m. An electric field of $30\,\mathbf{a}_z$ V/m is present.

a) Find $\mathbf{J}$: Use

$$\mathbf{J} = \sigma\mathbf{E} = \underline{45e^{-150\rho}\,\mathbf{a}_z\ \ \text{kA/m}^2}$$

b) Find the total current crossing the surface $\rho < \rho_0$, $z = 0$, all ϕ:

$$I = \int\!\!\int \mathbf{J}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\rho_0} 45e^{-150\rho}\rho\,d\rho\,d\phi = \frac{2\pi(45)}{(150)^2}e^{-150\rho}\left[-150\rho - 1\right]\Big|_0^{\rho_0}\ \text{kA}$$
$$= \underline{12.6\left[1-(1+150\rho_0)e^{-150\rho_0}\right]\ \text{A}}$$

c) Make use of Ampere's circuital law to find $\mathbf{H}$: Symmetry suggests that $\mathbf{H}$ will be ϕ-directed only, and so we consider a circular path of integration, centered on and perpendicular to the z axis. Ampere's law becomes: $2\pi\rho H_\phi = I_{encl}$, where I_{encl} is the current found in part b, except with ρ_0 replaced by the variable, ρ. We obtain

$$H_\phi = \underline{\frac{2.00}{\rho}\left[1-(1+150\rho)e^{-150\rho}\right]\ \text{A/m}}$$

8.16. A balanced coaxial cable contains three coaxial conductors of negligible resistance. Assume a solid inner conductor of radius a, an intermediate conductor of inner radius b_i, outer radius b_o, and an outer conductor having inner and outer radii c_i and c_o, respectively. The intermediate conductor carries current I in the positive $\mathbf{a}_z$ direction and is at potential V_0. The inner and outer conductors are both at zero potential, and carry currents $I/2$ (in each) in the negative $\mathbf{a}_z$ direction. Assuming that the current distribution in each conductor is uniform, find:

a) $\mathbf{J}$ in each conductor: These expressions will be the current in each conductor divided by the appropriate cross-sectional area. The results are:

$$\text{Inner conductor}: \quad \mathbf{J}_a = -\frac{I\,\mathbf{a}_z}{2\pi a^2}\ \text{A/m}^2 \quad (0<\rho<a)$$

$$\text{Center conductor}: \quad \mathbf{J}_b = \frac{I\,\mathbf{a}_z}{\pi(b_o^2 - b_i^2)}\ \text{A/m}^2 \quad (b_i<\rho<b_o)$$

$$\text{Outer conductor}: \quad \mathbf{J}_c = -\frac{I\,\mathbf{a}_z}{2\pi(c_o^2 - c_i^2)}\ \text{A/m}^2 \quad (c_i<\rho<c_o)$$

8.16b) **H** everywhere:

For $0 < \rho < a$, and with current in the negative z direction, Ampere's circuital law applied to a circular path of radius ρ within the given region leads to

$$2\pi\rho H = -\pi\rho^2 J_a = -\pi\rho^2 I/(2\pi a^2) \quad \Rightarrow \quad \mathbf{H}_1 = -\frac{\rho I}{4\pi a^2}\,\mathbf{a}_\phi \text{ A/m} \quad (0 < \rho < a)$$

For $a < \rho < b_i$, and with the current within in the negative z direction, Ampere's circuital law applied to a circular path of radius ρ within the given region leads to

$$2\pi\rho H = -I/2 \quad \Rightarrow \quad \mathbf{H}_2 = -\frac{I}{4\pi\rho}\,\mathbf{a}_\phi \text{ A/m} \quad (a < \rho < b_i)$$

Inside the center conductor, the net magnetic field will include the contribution from the inner conductor current:

$$2\pi\rho H = -I/2 + \frac{\pi(\rho^2 - b_i^2)I}{\pi(b_o^2 - b_i^2)} \quad \Rightarrow \quad \mathbf{H}_3 = \frac{I}{4\pi\rho}\left[\frac{2(\rho^2 - b_i^2)}{(b_o^2 - b_i^2)} - 1\right]\mathbf{a}_\phi \text{ A/m} \quad (b_i < \rho < b_o)$$

Beyond the center conductor, but before the outer conductor, the net enclosed current is $I - I/2 = I/2$, and the magnetic field is

$$\mathbf{H}_4 = \frac{I}{4\pi\rho}\,\mathbf{a}_\phi \quad (b_o < \rho < c_i)$$

Inside the outer conductor (with current again in the negative z direction) the field associated with the outer conductor current will subtract from $\mathbf{H}_4$ (more so as ρ increases):

$$\mathbf{H}_5 = \frac{I}{4\pi\rho}\left[1 - \frac{(\rho^2 - c_i^2)}{(c_o^2 - c_i^2)}\right]\mathbf{a}_\phi \text{ A/m} \quad (c_i < \rho < c_o)$$

Finally, beyond the outer conductor, the total enclosed current is zero, and so

$$\mathbf{H}_6 = 0 \quad (\rho > c_o)$$

c) **E** everywhere: Since we have perfect conductors, the electric field within each will be zero. This leaves the free space regions, within which Laplace's equation will have the general solution form, $V(\rho) = C_1 \ln(\rho) + C_2$. Between radii a and b_i, the boundary condition, $V = 0$ at $\rho = a$ leads to $C_2 = -C_1 \ln a$. Thus $V(\rho) = C_1 \ln(\rho/a)$. The boundary condition, $V = V_0$ at $\rho = b_i$ leads to $C_1 = V_0/\ln(b_i/a)$, and so finally, $V(\rho) = V_0 \ln(\rho/a)/\ln(b_i/a)$. Now

$$\mathbf{E}_1 = -\nabla V = -\frac{dV}{d\rho}\,\mathbf{a}_\rho = -\frac{V_0}{\rho\ln(b_i/a)}\,\mathbf{a}_\rho \text{ V/m} \quad (a < \rho < b_i)$$

Between radii b_o and c_i, the boundary condition, $V = 0$ at $\rho = c_i$ leads to $C_2 = -C_1 \ln c_i$. Thus $V(\rho) = C_1 \ln(\rho/c_i)$. The boundary condition, $V = V_0$ at $\rho = b_o$ leads to $C_1 = V_0/\ln(b_o/c_i)$, and so finally, $V(\rho) = V_0 \ln(\rho/c_i)/\ln(b_o/c_i)$. Now

$$\mathbf{E}_2 = -\frac{dV}{d\rho}\,\mathbf{a}_\rho = -\frac{V_0}{\rho\ln(b_o/c_i)}\,\mathbf{a}_\rho = +\frac{V_0}{\rho\ln(c_i/b_0)}\,\mathbf{a}_\rho \text{ V/m} \quad (b_o < \rho < c_i)$$

8.17. A current filament on the z axis carries a current of 7 mA in the $\mathbf{a}_z$ direction, and current sheets of $0.5\,\mathbf{a}_z$ A/m and $-0.2\,\mathbf{a}_z$ A/m are located at $\rho = 1$ cm and $\rho = 0.5$ cm, respectively. Calculate $\mathbf{H}$ at:

a) $\rho = 0.5$ cm: Here, we are either just inside or just outside the first current sheet, so both we will calculate $\mathbf{H}$ for both cases. Just inside, applying Ampere's circuital law to a circular path centered on the z axis produces:

$$2\pi\rho H_\phi = 7\times 10^{-3} \quad \Rightarrow \quad \mathbf{H}(\text{just inside}) = \frac{7\times 10^{-3}}{2\pi(0.5\times 10^{-2}}\mathbf{a}_\phi = \underline{2.2\times 10^{-1}\mathbf{a}_\phi \text{ A/m}}$$

Just outside the current sheet at .5 cm, Ampere's law becomes

$$2\pi\rho H_\phi = 7\times 10^{-3} - 2\pi(0.5\times 10^{-2})(0.2)$$
$$\Rightarrow \quad \mathbf{H}(\text{just outside}) = \frac{7.2\times 10^{-4}}{2\pi(0.5\times 10^{-2})}\mathbf{a}_\phi = \underline{2.3\times 10^{-2}\mathbf{a}_\phi \text{ A/m}}$$

b) $\rho = 1.5$ cm: Here, all three currents are enclosed, so Ampere's law becomes

$$2\pi(1.5\times 10^{-2})H_\phi = 7\times 10^{-3} - 6.28\times 10^{-3} + 2\pi(10^{-2})(0.5)$$
$$\Rightarrow \quad \mathbf{H}(\rho = 1.5) = \underline{3.4\times 10^{-1}\mathbf{a}_\phi \text{ A/m}}$$

c) $\rho = 4$ cm: Ampere's law as used in part b applies here, except we replace $\rho = 1.5$ cm with $\rho = 4$ cm on the left hand side. The result is $\mathbf{H}(\rho = 4) = \underline{1.3\times 10^{-1}\mathbf{a}_\phi \text{ A/m}}$.

d) What current sheet should be located at $\rho = 4$ cm so that $\mathbf{H} = 0$ for all $\rho > 4$ cm? We require that the total enclosed current be zero, and so the net current in the proposed cylinder at 4 cm must be negative the right hand side of the first equation in part b. This will be -3.2×10^{-2}, so that the surface current density at 4 cm must be

$$\mathbf{K} = \frac{-3.2\times 10^{-2}}{2\pi(4\times 10^{-2})}\mathbf{a}_z = \underline{-1.3\times 10^{-1}\,\mathbf{a}_z \text{ A/m}}$$

8.18. A wire of 3-mm radius is made up of an inner material ($0 < \rho < 2$ mm) for which $\sigma = 10^7$ S/m, and an outer material (2mm $< \rho <$ 3mm) for which $\sigma = 4\times 10^7$ S/m. If the wire carries a total current of 100 mA dc, determine $\mathbf{H}$ everywhere as a function of ρ.

Since the materials have different conductivities, the current densities within them will differ. Electric field, however is constant throughout. The current can be expressed as

$$I = \pi(.002)^2 J_1 + \pi[(.003)^2 - (.002)^2]J_2 = \pi\left[(.002)^2\sigma_1 + [(.003)^2 - (.002)^2]\sigma_2\right]E$$

Solve for E to obtain

$$E = \frac{0.1}{\pi[(4\times 10^{-6})(10^7) + (9\times 10^{-6} - 4\times 10^{-6})(4\times 10^7)]} = 1.33\times 10^{-4} \text{ V/m}$$

We next apply Ampere's circuital law to a circular path of radius ρ, where $\rho <$ 2mm:

$$2\pi\rho H_{\phi 1} = \pi\rho^2 J_1 = \pi\rho^2\sigma_1 E \quad \Rightarrow \quad H_{\phi 1} = \frac{\sigma_1 E\rho}{2} = \underline{663 \text{ A/m}}$$

8.18. (continued): Next, for the region 2mm $< \rho <$ 3mm, Ampere's law becomes

$$2\pi\rho H_{\phi 2} = \pi[(4\times10^{-6})(10^7) + (\rho^2 - 4\times10^{-6})(4\times10^7)]E$$
$$\Rightarrow \quad H_{\phi 2} = 2.7\times10^3\rho \; - \; \frac{8.0\times10^{-3}}{\rho} \text{ A/m}$$

Finally, for $\rho >$ 3mm, the field outside is that for a long wire:

$$H_{\phi 3} = \frac{I}{2\pi\rho} = \frac{0.1}{2\pi\rho} = \frac{1.6\times10^{-2}}{\rho} \text{ A/m}$$

8.19. Calculate $\nabla \times [\nabla(\nabla\cdot\mathbf{G})]$ if $\mathbf{G} = 2x^2yz\,\mathbf{a}_x - 20y\,\mathbf{a}_y + (x^2 - z^2)\,\mathbf{a}_z$: Proceding, we first find $\nabla\cdot\mathbf{G} = 4xyz - 20 - 2z$. Then $\nabla(\nabla\cdot\mathbf{G}) = 4yz\,\mathbf{a}_x + 4xz\,\mathbf{a}_y + (4xy-2)\,\mathbf{a}_z$. Then

$$\nabla \times [\nabla(\nabla\cdot\mathbf{G})] = (4x-4x)\,\mathbf{a}_x - (4y-4y)\,\mathbf{a}_y + (4z-4z)\,\mathbf{a}_z = \underline{0}$$

8.20. A solid conductor of circular cross-section with a radius of 5 mm has a conductivity that varies with radius. The conductor is 20 m long and there is a potential difference of 0.1 V dc between its two ends. Within the conductor, $\mathbf{H} = 10^5\rho^2\mathbf{a}_\phi$ A/m.

a) Find σ as a function of ρ: Start by finding $\mathbf{J}$ from $\mathbf{H}$ by taking the curl. With $\mathbf{H}$ ϕ-directed, and varying with radius only, the curl becomes:

$$\mathbf{J} = \nabla\times\mathbf{H} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)\,\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left(10^5\rho^3\right)\,\mathbf{a}_z = 3\times10^5\rho\,\mathbf{a}_z \text{ A/m}^2$$

Then $\mathbf{E} = 0.1/20 = 0.005\,\mathbf{a}_z$ V/m, which we then use with $\mathbf{J} = \sigma\mathbf{E}$ to find

$$\sigma = \frac{J}{E} = \frac{3\times10^5\rho}{0.005} = \underline{6\times10^7\rho \text{ S/m}}$$

b) What is the resistance between the two ends? The current in the wire is

$$I = \int_s \mathbf{J}\cdot d\mathbf{S} = 2\pi\int_0^a (3\times10^5\rho)\,\rho\,d\rho = 6\pi\times10^5\left(\frac{1}{3}a^3\right) = 2\pi\times10^5(0.005)^3 = 0.079 \text{ A}$$

Finally, $R = V_0/I = 0.1/0.079 = \underline{1.3\ \Omega}$

8.21. Points A, B, C, D, E, and F are each 2 mm from the origin on the coordinate axes indicated in Fig. 8.23. The value of $\mathbf{H}$ at each point is given. Calculate an approximate value for $\nabla\times\mathbf{H}$ at the origin: We use the approximation:

$$\text{curl}\,\mathbf{H} \doteq \frac{\oint \mathbf{H}\cdot d\mathbf{L}}{\Delta a}$$

where no limit as $\Delta a \to 0$ is taken (hence the approximation), and where $\Delta a = 4\,\text{mm}^2$. Each curl component is found by integrating $\mathbf{H}$ over a square path that is normal to the component in question.

8.21. (continued) Each of the four segments of the contour passes through one of the given points. Along each segment, the field is assumed constant, and so the integral is evaluated by summing the products of the field and segment length (4 mm) over the four segments. The x component of the curl is thus:

$$(\nabla \times \mathbf{H})_x \doteq \frac{(H_{z,C} - H_{y,E} - H_{z,D} + H_{y,F})(4 \times 10^{-3})}{(4 \times 10^{-3})^2}$$
$$= (15.69 + 13.88 - 14.35 - 13.10)(250) = 530 \text{ A/m}^2$$

The other components are:

$$(\nabla \times \mathbf{H})_y \doteq \frac{(H_{z,B} + H_{x,E} - H_{z,A} - H_{x,F})(4 \times 10^{-3})}{(4 \times 10^{-3})^2}$$
$$= (15.82 + 11.11 - 14.21 - 10.88)(250) = 460 \text{ A/m}^2$$

and

$$(\nabla \times \mathbf{H})_z \doteq \frac{(H_{y,A} - H_{x,C} - H_{y,B} H_{x,D})(4 \times 10^{-3})}{(4 \times 10^{-3})^2}$$
$$= (-13.78 - 10.49 + 12.19 + 11.49)(250) = -148 \text{ A/m}^2$$

Finally we assemble the results and write:

$$\nabla \times \mathbf{H} \doteq \underline{530\,\mathbf{a}_x + 460\,\mathbf{a}_y - 148\,\mathbf{a}_z}$$

8.22. A solid cylinder of radius a and length L, where $L >> a$, contains volume charge of uniform density ρ_0 C/m^3. The cylinder rotates about its axis (the z axis) at angular velocity Ω rad/s.

a) Determine the current density $\mathbf{J}$, as a function of position within the rotating cylinder: Use $\mathbf{J} = \rho_0 \mathbf{v} = \underline{\rho_0 \rho \Omega\, \mathbf{a}_\phi \text{ A/m}^2}$.

b) Determine the magnetic field intensity $\mathbf{H}$ inside and outside: It helps initially to obtain the field on-axis. To do this, we use the result of Problem 8.6, but give the rotating charged disk in that problem a differential thickness, dz. We can then evaluate the on-axis field in the rotating cylinder as the superposition of fields from a stack of disks which exist between $\pm L/2$. Here, we make the problem easier by letting $L \to \infty$ (since $L >> a$) thereby specializing our evaluation to positions near the half-length. The on-axis field is therefore:

$$H_z(\rho = 0) = \int_{-\infty}^{\infty} \frac{\rho_0 \Omega}{2z} \left[\frac{a^2 + 2z^2 \left(1 - \sqrt{1 + a^2/z^2}\right)}{\sqrt{1 + a^2/z^2}} \right] dz$$
$$= 2 \int_0^{\infty} \frac{\rho_0 \Omega}{2} \left[\frac{a^2}{\sqrt{z^2 + a^2}} + \frac{2z^2}{\sqrt{z^2 + a^2}} - 2z \right] dz$$
$$= 2\rho_0 \Omega \left[\frac{a^2}{2} \ln(z + \sqrt{z^2 + a^2}) + \frac{z}{2}\sqrt{z^2 + a^2} - \frac{a^2}{2} \ln(z + \sqrt{z^2 + a^2}) - \frac{z^2}{2} \right]_0^{\infty}$$
$$= \rho_0 \Omega \left[z\sqrt{z^2 + a^2} - z^2 \right]_0^{\infty} = \rho_0 \Omega \left[z\sqrt{z^2 + a^2} - z^2 \right]_{z \to \infty}$$

Using the large z approximation in the radical, we obtain

$$H_z(\rho = 0) = \rho_0 \Omega \left[z^2 \left(1 + \frac{a^2}{2z^2} \right) - z^2 \right] = \frac{\rho_0 \Omega a^2}{2}$$

8.22. (continued). To find the field as a function of radius, we apply Ampere's circuital law to a rectangular loop, drawn in two locations described as follows: First, construct the rectangle with one side along the z axis, and with the opposite side lying at any radius *outside* the cylinder. In taking the line integral of $\mathbf{H}$ around the rectangle, we note that the two segments that are perpendicular to the cylinder axis will have their path integrals exactly cancel, since the two path segments are oppositely-directed, while from symmetry the field should not be different along each segment. This leaves only the path segment that coindides with the axis, and that lying parallel to the axis, but outside. Choosing the length of these segments to be ℓ, Ampere's circuital law becomes:

$$\oint \mathbf{H} \cdot d\mathbf{L} = H_z(\rho = 0)\ell + H_z(\rho > a)\ell = I_{encl} = \int_s \mathbf{J} \cdot d\mathbf{S} = \int_0^\ell \int_0^a \rho_0 \rho \Omega \, \mathbf{a}_\phi \cdot \mathbf{a}_\phi \, d\rho \, dz$$
$$= \ell \frac{\rho_0 \Omega a^2}{2}$$

But we found earlier that $H_z(\rho = 0) = \rho_0 \Omega a^2/2$. Therefore, we identify the outside field, $H_z(\rho > a) = 0$. Next, change the rectangular path only by displacing the central path component off-axis by distance ρ, but still lying within the cylinder. The enclosed current is now somewhat less, and Ampere's law becomes

$$\oint \mathbf{H} \cdot d\mathbf{L} = H_z(\rho)\ell + H_z(\rho > a)\ell = I_{encl} = \int_s \mathbf{J} \cdot d\mathbf{S} = \int_0^\ell \int_\rho^a \rho_0 \rho' \Omega \, \mathbf{a}_\phi \cdot \mathbf{a}_\phi \, d\rho \, dz$$
$$= \ell \frac{\rho_0 \Omega}{2}(a^2 - \rho^2) \quad \Rightarrow \quad \mathbf{H}(\rho) = \frac{\rho_0 \Omega}{2}(a^2 - \rho^2) \, \mathbf{a}_z \text{ A/m}$$

c) Check your result of part b by taking the curl of $\mathbf{H}$. With $\mathbf{H}$ z-directed, and varying only with ρ, the curl in cylindrical coordinates becomes

$$\nabla \times \mathbf{H} = -\frac{dH_z}{d\rho} \, \mathbf{a}_\phi = \rho_0 \Omega \rho \, \mathbf{a}_\phi \text{ A/m}^2 = \mathbf{J}$$

as expected.

8.23. Given the field $\mathbf{H} = 20\rho^2 \, \mathbf{a}_\phi$ A/m:

a) Determine the current density $\mathbf{J}$: This is found through the curl of $\mathbf{H}$, which simplifies to a single term, since $\mathbf{H}$ varies only with ρ and has only a ϕ component:

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho} \frac{d(\rho H_\phi)}{d\rho} \, \mathbf{a}_z = \frac{1}{\rho} \frac{d}{d\rho} \left(20\rho^3\right) \, \mathbf{a}_z = \underline{60\rho \, \mathbf{a}_z \text{ A/m}^2}$$

b) Integrate $\mathbf{J}$ over the circular surface $\rho = 1$, $0 < \phi < 2\pi$, $z = 0$, to determine the total current passing through that surface in the $\mathbf{a}_z$ direction: The integral is:

$$I = \int\int \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^1 60\rho \mathbf{a}_z \cdot \rho \, d\rho \, d\phi \mathbf{a}_z = \underline{40\pi \text{ A}}$$

c) Find the total current once more, this time by a line integral around the circular path $\rho = 1$, $0 < \phi < 2\pi$, $z = 0$:

$$I = \oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} 20\rho^2 \, \mathbf{a}_\phi \big|_{\rho=1} \cdot (1) d\phi \mathbf{a}_\phi = \int_0^{2\pi} 20 \, d\phi = \underline{40\pi \text{ A}}$$

8.24. Evaluate both sides of Stokes' theorem for the field $\mathbf{G} = 10\sin\theta\,\mathbf{a}_\phi$ and the surface $r = 3$, $0 \le \theta \le 90°$, $0 \le \phi \le 90°$. Let the surface have the $\mathbf{a}_r$ direction: Stokes' theorem reads:

$$\oint_C \mathbf{G}\cdot d\mathbf{L} = \int\int_S (\nabla\times\mathbf{G})\cdot\mathbf{n}\,da$$

Considering the given surface, the contour, C, that forms its perimeter consists of three joined arcs of radius 3 that sweep out 90° in the xy, xz, and zy planes. Their centers are at the origin. Of these three, only the arc in the xy plane (which lies along $\mathbf{a}_\phi$) is in the direction of $\mathbf{G}$; the other two (in the $-\mathbf{a}_\theta$ and $\mathbf{a}_\theta$ directions respectively) are perpendicular to it, and so will not contribute to the path integral. The left-hand side therefore consists of only the xy plane portion of the closed path, and evaluates as

$$\oint \mathbf{G}\cdot d\mathbf{L} = \int_0^{\pi/2} 10\sin\theta\big|_{\pi/2}\,\mathbf{a}_\phi\cdot\mathbf{a}_\phi\,3\sin\theta\big|_{\pi/2}\,d\phi = \underline{15\pi}$$

To evaluate the right-hand side, we first find

$$\nabla\times\mathbf{G} = \frac{1}{r\sin\theta}\frac{d}{d\theta}\left[(\sin\theta)10\sin\theta\right]\,\mathbf{a}_r = \frac{20\cos\theta}{r}\mathbf{a}_r$$

The surface over which we integrate this is the one-eighth spherical shell of radius 3 in the first octant, bounded by the three arcs described earlier. The right-hand side becomes

$$\int\int_S (\nabla\times\mathbf{G})\cdot\mathbf{n}\,da = \int_0^{\pi/2}\int_0^{\pi/2}\frac{20\cos\theta}{3}\,\mathbf{a}_r\cdot\mathbf{a}_r\,(3)^2\sin\theta\,d\theta\,d\phi = \underline{15\pi}$$

It would appear that the theorem works.

8.25. When x, y, and z are positive and less than 5, a certain magnetic field intensity may be expressed as $\mathbf{H} = [x^2yz/(y+1)]\mathbf{a}_x + 3x^2z^2\mathbf{a}_y - [xyz^2/(y+1)]\mathbf{a}_z$. Find the total current in the $\mathbf{a}_x$ direction that crosses the strip, $x = 2$, $1 \le y \le 4$, $3 \le z \le 4$, by a method utilizing:

a) a surface integral: We need to find the current density by taking the curl of the given $\mathbf{H}$. Actually, since the strip lies parallel to the yz plane, we need only find the x component of the current density, as only this component will contribute to the requested current. This is

$$J_x = (\nabla\times\mathbf{H})_x = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) = -\left(\frac{xz^2}{(y+1)^2} + 6x^2z\right)\mathbf{a}_x$$

The current through the strip is then

$$I = \int_s \mathbf{J}\cdot\mathbf{a}_x\,da = -\int_3^4\int_1^4\left(\frac{2z^2}{(y+1)^2} + 24z\right)dy\,dz = -\int_3^4\left(\frac{-2z^2}{(y+1)} + 24zy\right)_1^4 dz$$
$$= -\int_3^4\left(\frac{3}{5}z^2 + 72z\right)dz = -\left(\frac{1}{5}z^3 + 36z^2\right)_3^4 = \underline{-259}$$

8.25b.) a closed line integral: We integrate counter-clockwise around the strip boundary (using the right-hand convention), where the path normal is positive $\mathbf{a}_x$. The current is then

$$I = \oint \mathbf{H}\cdot d\mathbf{L} = \int_1^4 3(2)^2(3)^2\,dy + \int_3^4 -\frac{2(4)z^2}{(4+1)}\,dz + \int_4^1 3(2)^2(4)^2\,dy + \int_4^3 -\frac{2(1)z^2}{(1+1)}\,dz$$
$$= 108(3) - \frac{8}{15}(4^3-3^3) + 192(1-4) - \frac{1}{3}(3^3-4^3) = -259$$

8.26. Let $\mathbf{G} = 15r\mathbf{a}_\phi$.

a) Determine $\oint \mathbf{G}\cdot d\mathbf{L}$ for the circular path $r=5$, $\theta = 25°$, $0 \le \phi \le 2\pi$:

$$\oint \mathbf{G}\cdot d\mathbf{L} = \int_0^{2\pi} 15(5)\mathbf{a}_\phi \cdot \mathbf{a}_\phi(5)\sin(25°)\,d\phi = 2\pi(375)\sin(25°) = \underline{995.8}$$

b) Evaluate $\int_S(\nabla\times\mathbf{G})\cdot d\mathbf{S}$ over the spherical cap $r=5$, $0\le\theta\le 25°$, $0\le\phi\le 2\pi$: When evaluating the curl of $\mathbf{G}$ using the formula in spherical coordinates, only one of the six terms survives:

$$\nabla\times\mathbf{G} = \frac{1}{r\sin\theta}\frac{\partial(G_\phi\sin\theta)}{\partial\theta}\,\mathbf{a}_r = \frac{1}{r\sin\theta}\,15r\cos\theta\,\mathbf{a}_r = 15\cot\theta\,\mathbf{a}_r$$

Then

$$\int_S(\nabla\times\mathbf{G})\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{25°} 15\cot\theta\,\mathbf{a}_r\cdot\mathbf{a}_r\,(5)^2\sin\theta\,d\theta\,d\phi$$
$$= 2\pi\int_0^{25°} 15\cos\theta(25)\,d\theta = 2\pi(15)(25)\sin(25°) = \underline{995.8}$$

8.27. The magnetic field intensity is given in a certain region of space as

$$\mathbf{H} = \frac{x+2y}{z^2}\,\mathbf{a}_y + \frac{2}{z}\,\mathbf{a}_z \ \text{A/m}$$

a) Find $\nabla\times\mathbf{H}$: For this field, the general curl expression in rectangular coordinates simplifies to

$$\nabla\times\mathbf{H} = -\frac{\partial H_y}{\partial z}\,\mathbf{a}_x + \frac{\partial H_y}{\partial x}\,\mathbf{a}_z = \underline{\frac{2(x+2y)}{z^3}\,\mathbf{a}_x + \frac{1}{z^2}\mathbf{a}_z \ \text{A/m}}$$

b) Find $\mathbf{J}$: This will be the answer of part a, since $\nabla\times\mathbf{H} = \mathbf{J}$.

c) Use $\mathbf{J}$ to find the total current passing through the surface $z=4$, $1<x<2$, $3<y<5$, in the $\mathbf{a}_z$ direction: This will be

$$I = \int\!\!\int \mathbf{J}\big|_{z=4}\cdot\mathbf{a}_z\,dx\,dy = \int_3^5\int_1^2 \frac{1}{4^2}dx\,dy = \underline{1/8 \ \text{A}}$$

8.27d) Show that the same result is obtained using the other side of Stokes' theorem: We take $\oint \mathbf{H} \cdot d\mathbf{L}$ over the square path at $z = 4$ as defined in part c. This involves two integrals of the y component of $\mathbf{H}$ over the range $3 < y < 5$. Integrals over x, to complete the loop, do not exist since there is no x component of $\mathbf{H}$. We have

$$I = \oint \mathbf{H}\big|_{z=4} \cdot d\mathbf{L} = \int_3^5 \frac{2+2y}{16}\, dy + \int_5^3 \frac{1+2y}{16}\, dy = \frac{1}{8}(2) - \frac{1}{16}(2) = \underline{1/8 \text{ A}}$$

8.28. Given $\mathbf{H} = (3r^2/\sin\theta)\mathbf{a}_\theta + 54r\cos\theta\mathbf{a}_\phi$ A/m in free space:

a) find the total current in the $\mathbf{a}_\theta$ direction through the conical surface $\theta = 20°$, $0 \le \phi \le 2\pi$, $0 \le r \le 5$, by whatever side of Stokes' theorem you like best. I chose the line integral side, where the integration path is the circular path in ϕ around the top edge of the cone, at $r = 5$. The path direction is chosen to be *clockwise* looking down on the xy plane. This, by convention, leads to the normal from the cone surface that points in the positive $\mathbf{a}_\theta$ direction (right hand rule). We find

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} \left[(3r^2/\sin\theta)\mathbf{a}_\theta + 54r\cos\theta\mathbf{a}_\phi\right]_{r=5,\theta=20} \cdot 5\sin(20°)\, d\phi\, (-\mathbf{a}_\phi)$$
$$= -2\pi(54)(25)\cos(20°)\sin(20°) = \underline{-2.73 \times 10^3 \text{ A}}$$

This result means that there is a component of current that enters the cone surface in the $-\mathbf{a}_\theta$ direction, to which is associated a component of $\mathbf{H}$ in the positive $\mathbf{a}_\phi$ direction.

b) Check the result by using the other side of Stokes' theorem: We first find the current density through the curl of the magnetic field, where three of the six terms in the spherical coordinate formula survive:

$$\nabla \times \mathbf{H} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(54r\cos\theta\sin\theta)\right)\mathbf{a}_r - \frac{1}{r}\frac{\partial}{\partial r}\left(54r^2\cos\theta\right)\mathbf{a}_\theta + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{3r^3}{\sin\theta}\right)\mathbf{a}_\phi = \mathbf{J}$$

Thus

$$\mathbf{J} = 54\cot\theta\,\mathbf{a}_r - 108\cos\theta\,\mathbf{a}_\theta + \frac{9r}{\sin\theta}\,\mathbf{a}_\phi$$

The calculation of the other side of Stokes' theorem now involves integrating $\mathbf{J}$ over the surface of the cone, where the outward normal is positive $\mathbf{a}_\theta$, as defined in part a:

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^5 \left[54\cot\theta\,\mathbf{a}_r - 108\cos\theta\,\mathbf{a}_\theta + \frac{9r}{\sin\theta}\,\mathbf{a}_\phi\right]_{20°} \cdot \mathbf{a}_\theta\, r\sin(20°)\, dr\, d\phi$$
$$= -\int_0^{2\pi}\int_0^5 108\cos(20°)\sin(20°) r dr d\phi = -2\pi(54)(25)\cos(20°)\sin(20°)$$
$$= \underline{-2.73 \times 10^3 \text{ A}}$$

8.29. A long straight non-magnetic conductor of 0.2 mm radius carries a uniformly-distributed current of 2 A dc.

a) Find **J** within the conductor: Assuming the current is $+z$ directed,

$$\mathbf{J} = \frac{2}{\pi(0.2 \times 10^{-3})^2}\mathbf{a}_z = \underline{1.59 \times 10^7\,\mathbf{a}_z \text{ A/m}^2}$$

b) Use Ampere's circuital law to find **H** and **B** within the conductor: Inside, at radius ρ, we have

$$2\pi\rho H_\phi = \pi\rho^2 J \quad \Rightarrow \quad \mathbf{H} = \frac{\rho J}{2}\mathbf{a}_\phi = \underline{7.96 \times 10^6 \rho\,\mathbf{a}_\phi \text{ A/m}}$$

Then $\mathbf{B} = \mu_0\mathbf{H} = (4\pi \times 10^{-7})(7.96 \times 10^6)\rho\mathbf{a}_\phi = \underline{10\rho\,\mathbf{a}_\phi \text{ Wb/m}^2}$.

c) Show that $\nabla \times \mathbf{H} = \mathbf{J}$ within the conductor: Using the result of part b, we find,

$$\nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)\,\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left(\frac{1.59 \times 10^7 \rho^2}{2}\right)\mathbf{a}_z = \underline{1.59 \times 10^7\,\mathbf{a}_z \text{ A/m}^2} = \mathbf{J}$$

d) Find **H** and **B** *outside* the conductor (note typo in book): Outside, the entire current is enclosed by a closed path at radius ρ, and so

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi = \underline{\frac{1}{\pi\rho}\mathbf{a}_\phi \text{ A/m}}$$

Now $\mathbf{B} = \mu_0\mathbf{H} = \underline{\mu_0/(\pi\rho)\,\mathbf{a}_\phi \text{ Wb/m}^2}$.

e) Show that $\nabla \times \mathbf{H} = \mathbf{J}$ outside the conductor: Here we use **H** outside the conductor and write:

$$\nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)\,\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{1}{\pi\rho}\right)\mathbf{a}_z = \underline{0} \quad \text{(as expected)}$$

8.30. (an inversion of Problem 8.20). A solid nonmagnetic conductor of circular cross-section has a radius of 2mm. The conductor is inhomogeneous, with $\sigma = 10^6(1 + 10^6\rho^2)$ S/m. If the conductor is 1m in length and has a voltage of 1mV between its ends, find:

a) **H** inside: With current along the cylinder length (along $\mathbf{a}_z$, and with ϕ symmetry, **H** will be ϕ-directed only. We find $\mathbf{E} = (V_0/d)\mathbf{a}_z = 10^{-3}\mathbf{a}_z$ V/m. Then $\mathbf{J} = \sigma\mathbf{E} = 10^3(1+10^6\rho^2)\mathbf{a}_z$ A/m^2. Next we apply Ampere's circuital law to a circular path of radius ρ, centered on the z axis and normal to the axis:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho H_\phi = \int\int_S \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^\rho 10^3(1 + 10^6(\rho')^2)\mathbf{a}_z \cdot \mathbf{a}_z\rho' d\rho' d\phi$$

Thus

$$H_\phi = \frac{10^3}{\rho}\int_0^\rho \rho' + 10^6(\rho')^3 d\rho' = \frac{10^3}{\rho}\left[\frac{\rho^2}{2} + \frac{10^6}{4}\rho^4\right]$$

Finally, $\mathbf{H} = \underline{500\rho(1 + 5 \times 10^5\rho^3)\mathbf{a}_\phi \text{ A/m } (0 < \rho < 2\text{mm})}$.

b) the total magnetic flux inside the conductor: With field in the ϕ direction, a plane normal to **B** will be that in the region $0 < \rho < 2$ mm, $0 < z < 1$ m. The flux will be

$$\Phi = \int\int_S \mathbf{B}\cdot d\mathbf{S} = \mu_0\int_0^1\int_0^{2\times 10^{-3}} \left(500\rho + 2.5 \times 10^8\rho^3\right) d\rho dz = 8\pi\times 10^{-10}\,\text{Wb} = \underline{2.5\,\text{nWb}}$$

8.31. The cylindrical shell defined by 1 cm $< \rho <$ 1.4 cm consists of a non-magnetic conducting material and carries a total current of 50 A in the $\mathbf{a}_z$ direction. Find the total magnetic flux crossing the plane $\phi = 0$, $0 < z < 1$:

a) $0 < \rho < 1.2\,\text{cm}$: We first need to find **J**, **H**, and **B**: The current density will be:

$$\mathbf{J} = \frac{50}{\pi[(1.4 \times 10^{-2})^2 - (1.0 \times 10^{-2})^2]}\,\mathbf{a}_z = 1.66 \times 10^5\,\mathbf{a}_z \text{ A/m}^2$$

Next we find H_ϕ at radius ρ between 1.0 and 1.4 cm, by applying Ampere's circuital law, and noting that the current density is zero at radii less than 1 cm:

$$2\pi\rho H_\phi = I_{encl} = \int_0^{2\pi}\int_{10^{-2}}^{\rho} 1.66 \times 10^5 \rho'\,d\rho'\,d\phi$$

$$\Rightarrow \quad H_\phi = 8.30 \times 10^4 \frac{(\rho^2 - 10^{-4})}{\rho} \text{ A/m} \quad (10^{-2}\,\text{m} < \rho < 1.4 \times 10^{-2}\,\text{m})$$

Then $\mathbf{B} = \mu_0 \mathbf{H}$, or

$$\mathbf{B} = 0.104 \frac{(\rho^2 - 10^{-4})}{\rho}\,\mathbf{a}_\phi \text{ Wb/m}^2$$

Now,

$$\Phi_a = \int\int \mathbf{B} \cdot d\mathbf{S} = \int_0^1 \int_{10^{-2}}^{1.2\times 10^{-2}} 0.104 \left[\rho - \frac{10^{-4}}{\rho}\right] d\rho\,dz$$

$$= 0.104 \left[\frac{(1.2 \times 10^{-2})^2 - 10^{-4}}{2} - 10^{-4} \ln\left(\frac{1.2}{1.0}\right)\right] = 3.92 \times 10^{-7}\,\text{Wb} = \underline{0.392\,\mu\text{Wb}}$$

b) $1.0\,\text{cm} < \rho < 1.4\,\text{cm}$ (note typo in book): This is part a over again, except we change the upper limit of the radial integration:

$$\Phi_b = \int\int \mathbf{B} \cdot d\mathbf{S} = \int_0^1 \int_{10^{-2}}^{1.4\times 10^{-2}} 0.104 \left[\rho - \frac{10^{-4}}{\rho}\right] d\rho\,dz$$

$$= 0.104 \left[\frac{(1.4 \times 10^{-2})^2 - 10^{-4}}{2} - 10^{-4} \ln\left(\frac{1.4}{1.0}\right)\right] = 1.49 \times 10^{-6}\,\text{Wb} = \underline{1.49\,\mu\text{Wb}}$$

c) $1.4\,\text{cm} < \rho < 20\,\text{cm}$: This is entirely outside the current distribution, so we need **B** there: We modify the Ampere's circuital law result of part a to find:

$$\mathbf{B}_{out} = 0.104 \frac{[(1.4 \times 10^{-2})^2 - 10^{-4}]}{\rho}\,\mathbf{a}_\phi = \frac{10^{-5}}{\rho}\,\mathbf{a}_\phi \text{ Wb/m}^2$$

We now find

$$\Phi_c = \int_0^1 \int_{1.4\times 10^{-2}}^{20\times 10^{-2}} \frac{10^{-5}}{\rho}\,d\rho\,dz = 10^{-5} \ln\left(\frac{20}{1.4}\right) = 2.7 \times 10^{-5}\,\text{Wb} = \underline{27\,\mu\text{Wb}}$$

8.32. The free space region defined by $1 < z < 4$ cm and $2 < \rho < 3$ cm is a toroid of rectangular cross-section. Let the surface at $\rho = 3$ cm carry a surface current $\mathbf{K} = 2\mathbf{a}_z$ kA/m.

a) Specify the current densities on the surfaces at $\rho = 2$ cm, $z = 1$cm, and $z = 4$cm. All surfaces must carry equal currents. With this requirement, we find: $\mathbf{K}(\rho = 2) = \underline{-3\,\mathbf{a}_z \text{ kA/m}}$. Next, the current densities on the $z = 1$ and $z = 4$ surfaces must transistion between the current density values at $\rho = 2$ and $\rho = 3$. Knowing the the radial current density will vary as $1/\rho$, we find $\mathbf{K}(z = 1) = \underline{(60/\rho)\mathbf{a}_\rho \text{ A/m}}$ with ρ in meters. Similarly, $\mathbf{K}(z = 4) = \underline{-(60/\rho)\mathbf{a}_\rho \text{ A/m}}$.

b) Find $\mathbf{H}$ everywhere: Outside the toroid, $\mathbf{H} = 0$. Inside, we apply Ampere's circuital law in the manner of Problem 8.14:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho H_\phi = \int_0^{2\pi} \mathbf{K}(\rho = 2) \cdot \mathbf{a}_z \,(2 \times 10^{-2})\, d\phi$$

$$\Rightarrow \quad \mathbf{H} = -\frac{2\pi(3000)(.02)}{\rho}\mathbf{a}_\phi = \underline{-60/\rho\,\mathbf{a}_\phi \text{ A/m (inside)}}$$

c) Calculate the total flux within the toriod: We have $\mathbf{B} = -(60\mu_0/\rho)\mathbf{a}_\phi$ Wb/m^2. Then

$$\Phi = \int_{.01}^{.04} \int_{.02}^{.03} \frac{-60\mu_0}{\rho}\,\mathbf{a}_\phi \cdot (-\mathbf{a}_\phi)\, d\rho\, dz = (.03)(60)\mu_0 \ln\left(\frac{3}{2}\right) = \underline{0.92\,\mu\text{Wb}}$$

8.33. Use an expansion in rectangular coordinates to show that the curl of the gradient of any scalar field G is identically equal to zero. We begin with

$$\nabla G = \frac{\partial G}{\partial x}\,\mathbf{a}_x + \frac{\partial G}{\partial y}\,\mathbf{a}_y + \frac{\partial G}{\partial z}\,\mathbf{a}_z$$

and

$$\nabla \times \nabla G = \left[\frac{\partial}{\partial y}\left(\frac{\partial G}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial G}{\partial y}\right)\right]\mathbf{a}_x + \left[\frac{\partial}{\partial z}\left(\frac{\partial G}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial G}{\partial z}\right)\right]\mathbf{a}_y$$
$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial G}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial G}{\partial x}\right)\right]\mathbf{a}_z = \underline{0} \quad \text{for any } G$$

8.34. A filamentary conductor on the z axis carries a current of 16A in the $\mathbf{a}_z$ direction, a conducting shell at $\rho = 6$ carries a total current of 12A in the $-\mathbf{a}_z$ direction, and another shell at $\rho = 10$ carries a total current of 4A in the $-\mathbf{a}_z$ direction.

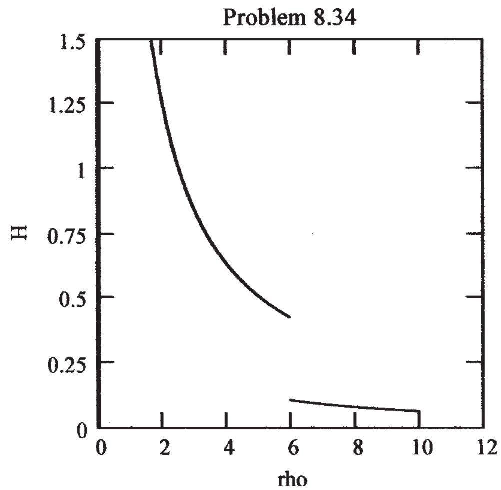
a) Find $\mathbf{H}$ for $0 < \rho < 12$: Ampere's circuital law states that $\oint \mathbf{H} \cdot d\mathbf{L} = I_{encl}$, where the line integral and current direction are related in the usual way through the right hand rule. Therefore, if I is in the positive z direction, $\mathbf{H}$ is in the $\mathbf{a}_\phi$ direction. We proceed as follows:

$$0 < \rho < 6: \quad 2\pi\rho H_\phi = 16 \quad \Rightarrow \quad \mathbf{H} = \underline{16/(2\pi\rho)\mathbf{a}_\phi}$$

$$6 < \rho < 10: \quad 2\pi\rho H_\phi = 16 - 12 \quad \Rightarrow \quad \mathbf{H} = \underline{4/(2\pi\rho)\mathbf{a}_\phi}$$

$$\rho > 10: \quad 2\pi\rho H_\phi = 16 - 12 - 4 = 0 \quad \Rightarrow \quad \mathbf{H} = \underline{0}$$

8.34b) Plot H_ϕ vs. ρ:



c) Find the total flux Φ crossing the surface $1 < \rho < 7$, $0 < z < 1$: This will be

$$\Phi = \int_0^1 \int_1^6 \frac{16\mu_0}{2\pi\rho}\, d\rho\, dz + \int_0^1 \int_6^7 \frac{4\mu_0}{2\pi\rho}\, d\rho\, dz = \frac{2\mu_0}{\pi}\left[4\ln 6 + \ln(7/6)\right] = \underline{5.9\ \mu\text{Wb}}$$

8.35. A current sheet, $\mathbf{K} = 20\,\mathbf{a}_z$ A/m, is located at $\rho = 2$, and a second sheet, $\mathbf{K} = -10\,\mathbf{a}_z$ A/m is located at $\rho = 4$.

a.) Let $V_m = 0$ at $P(\rho = 3, \phi = 0, z = 5)$ and place a barrier at $\phi = \pi$. Find $V_m(\rho, \phi, z)$ for $-\pi < \phi < \pi$: Since the current is cylindrically-symmetric, we know that $\mathbf{H} = I/(2\pi\rho)\,\mathbf{a}_\phi$, where I is the current enclosed, equal in this case to $2\pi(2)K = 80\pi$ A. Thus, using the result of Section 8.6, we find

$$V_m = -\frac{I}{2\pi}\,\phi = -\frac{80\pi}{2\pi}\,\phi = \underline{-40\phi\ \text{A}}$$

which is valid over the region $2 < \rho < 4$, $-\pi < \phi < \pi$, and $-\infty < z < \infty$. For $\rho > 4$, the outer current contributes, leading to a total enclosed current of

$$I_{net} = 2\pi(2)(20) - 2\pi(4)(10) = 0$$

With zero enclosed current, $H_\phi = 0$, and the magnetic potential is zero as well.

b) Let $\mathbf{A} = 0$ at P and find $\mathbf{A}(\rho, \phi, z)$ for $2 < \rho < 4$: Again, we know that $\mathbf{H} = H_\phi(\rho)$, since the current is cylindrically symmetric. With the current only in the z direction, and again using symmmetry, we expect only a z component of $\mathbf{A}$ which varies only with ρ. We can then write:

$$\nabla \times \mathbf{A} = -\frac{dA_z}{d\rho}\mathbf{a}_\phi = \mathbf{B} = \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi$$

Thus

$$\frac{dA_z}{d\rho} = -\frac{\mu_0 I}{2\pi\rho} \quad \Rightarrow \quad A_z = -\frac{\mu_0 I}{2\pi}\ln(\rho) + C$$

8.35b. (continued). We require that $A_z = 0$ at $\rho = 3$. Therefore $C = [(\mu_0 I)/(2\pi)]\ln(3)$, Then, with $I = 80\pi$, we finally obtain

$$\mathbf{A} = -\frac{\mu_0(80\pi)}{2\pi}\left[\ln(\rho) - \ln(3)\right]\mathbf{a}_z = \underline{40\mu_0 \ln\left(\frac{3}{\rho}\right)\mathbf{a}_z \text{ Wb/m}}$$

8.36. Let $\mathbf{A} = (3y - z)\mathbf{a}_x + 2xz\mathbf{a}_y$ Wb/m in a certain region of free space.

a) Show that $\nabla \cdot \mathbf{A} = 0$:

$$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x}(3y - z) + \frac{\partial}{\partial y}2xz = \underline{0}$$

b) At $P(2, -1, 3)$, find **A**, **B**, **H**, and **J**: First $\mathbf{A}_P = \underline{-6\mathbf{a}_x + 12\mathbf{a}_y}$. Then, using the curl formula in cartesian coordinates,

$$\mathbf{B} = \nabla \times \mathbf{A} = -2x\mathbf{a}_x - \mathbf{a}_y + (2z - 3)\mathbf{a}_z \quad \Rightarrow \quad \mathbf{B}_P = \underline{-4\mathbf{a}_x - \mathbf{a}_y + 3\mathbf{a}_z \text{ Wb/m}^2}$$

Now

$$\mathbf{H}_P = (1/\mu_0)\mathbf{B}_P = \underline{-3.2 \times 10^6\mathbf{a}_x - 8.0 \times 10^5\mathbf{a}_y + 2.4 \times 10^6\mathbf{a}_z \text{ A/m}}$$

Then $\mathbf{J} = \nabla \times \mathbf{H} = (1/\mu_0)\nabla \times \mathbf{B} = \underline{0}$, as the curl formula in cartesian coordinates shows.

8.37. Let $N = 1000$, $I = 0.8$ A, $\rho_0 = 2$ cm, and $a = 0.8$ cm for the toroid shown in Fig. 8.12b. Find V_m in the interior of the toroid if $V_m = 0$ at $\rho = 2.5$ cm, $\phi = 0.3\pi$. Keep ϕ within the range $0 < \phi < 2\pi$: Well-within the toroid, we have

$$\mathbf{H} = \frac{NI}{2\pi\rho}\mathbf{a}_\phi = -\nabla V_m = -\frac{1}{\rho}\frac{dV_m}{d\phi}\mathbf{a}_\phi$$

Thus

$$V_m = -\frac{NI\phi}{2\pi} + C$$

Then,

$$0 = -\frac{1000(0.8)(0.3\pi)}{2\pi} + C$$

or C = 120. Finally

$$V_m = \underline{\left[120 - \frac{400}{\pi}\phi\right] \text{ A} \quad (0 < \phi < 2\pi)}$$

8.38. Assume a direct current I amps flowing in the $\mathbf{a}_z$ direction in a filament extending between $-L < z < L$ on the z axis.

a) Using cylindrical coordinates, find $\mathbf{A}$ at any general point $P(\rho, 0^\circ, z)$: Let z' locate a variable position on the wire, in which case the distance from that position to the observation point is $R = \sqrt{(z-z')^2 + \rho^2}$. The vector potential is now

$$\mathbf{A} = \int_{wire} \frac{\mu_0 I \, d\mathbf{L}}{4\pi R} = \int_{-L}^{L} \frac{\mu_0 I dz' \, \mathbf{a}_z}{4\pi\sqrt{(z-z')^2 + \rho^2}}$$

I evaluated this using integral tables. The simplest form in this case is that involving the inverse hyperbolic sine. The result is

$$A_z = \frac{\mu_0 I}{4\pi}\left[\sinh^{-1}\left(\frac{L-z}{\rho}\right) - \sinh^{-1}\left(\frac{-(L+z)}{\rho}\right)\right]$$

b) From part a, find $\mathbf{B}$ and $\mathbf{H}$: $\mathbf{B}$ is found from the curl of $\mathbf{A}$, which, in the present case of $\mathbf{A}$ having only a z component, and varying only with ρ and z, simplifies to

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial \rho}\,\mathbf{a}_\phi = \frac{\mu_0 I}{4\pi\rho}\left[\frac{1}{\sqrt{1+\rho^2/(L-z)^2}} + \frac{1}{\sqrt{1+\rho^2/(L+z)^2}}\right]\mathbf{a}_\phi$$

The magnetic field strength, $\mathbf{H}$, is then just $\mathbf{B}/\mu_0$.

c) Let $L \to \infty$ and show that the expression for $\mathbf{H}$ reduces to the known one for an infinite filament: From the result of part b, we can observe that letting $L \to \infty$ causes the terms within the brackets to reduce to a simple factor of 2. Therefore, $\mathbf{B} \to \mu_0 I/(2\pi\rho)\,\mathbf{a}_\phi$, and $\mathbf{H} \to I/(2\pi\rho)\,\mathbf{a}_\phi$ in this limit, as expected.

8.39. Planar current sheets of $\mathbf{K} = 30\mathbf{a}_z$ A/m and $-30\mathbf{a}_z$ A/m are located in free space at $x = 0.2$ and $x = -0.2$ respectively. For the region $-0.2 < x < 0.2$:

a) Find $\mathbf{H}$: Since we have parallel current sheets carrying equal and opposite currents, we use Eq. (12), $\mathbf{H} = \mathbf{K} \times \mathbf{a}_N$, where $\mathbf{a}_N$ is the unit normal directed into the region between currents, and where either one of the two currents are used. Choosing the sheet at $x = 0.2$, we find

$$\mathbf{H} = 30\mathbf{a}_z \times -\mathbf{a}_x = \underline{-30\mathbf{a}_y \text{ A/m}}$$

b) Obtain and expression for V_m if $V_m = 0$ at $P(0.1, 0.2, 0.3)$: Use

$$\mathbf{H} = -30\mathbf{a}_y = -\nabla V_m = -\frac{dV_m}{dy}\mathbf{a}_y$$

So

$$\frac{dV_m}{dy} = 30 \quad \Rightarrow \quad V_m = 30y + C_1$$

Then

$$0 = 30(0.2) + C_1 \quad \Rightarrow \quad C_1 = -6 \quad \Rightarrow \quad V_m = \underline{30y - 6 \text{ A}}$$

8.39c) Find $\mathbf{B}$: $\mathbf{B} = \mu_0 \mathbf{H} = \underline{-30\mu_0 \mathbf{a}_y \text{ Wb/m}^2}$.

d) Obtain an expression for $\mathbf{A}$ if $\mathbf{A} = 0$ at P: We expect $\mathbf{A}$ to be z-directed (with the current), and so from $\nabla \times \mathbf{A} = \mathbf{B}$, where $\mathbf{B}$ is y-directed, we set up

$$-\frac{dA_z}{dx} = -30\mu_0 \quad \Rightarrow \quad A_z = 30\mu_0 x + C_2$$

Then $0 = 30\mu_0(0.1) + C_2 \quad \Rightarrow \quad C_2 = -3\mu_0$. So finally $\mathbf{A} = \underline{\mu_0(30x - 3)\mathbf{a}_z \text{ Wb/m}}$.

8.40. Show that the line integral of the vector potential $\mathbf{A}$ about any closed path is equal to the magnetic flux enclosed by the path, or $\oint \mathbf{A} \cdot d\mathbf{L} = \int \mathbf{B} \cdot d\mathbf{S}$.

We use the fact that $\mathbf{B} = \nabla \times \mathbf{A}$, and substitute this into the desired relation to find

$$\oint \mathbf{A} \cdot d\mathbf{L} = \int \nabla \times \mathbf{A} \cdot d\mathbf{S}$$

This is just a statement of Stokes' theorem (already proved), so we are done.

8.41. Assume that $\mathbf{A} = 50\rho^2 \mathbf{a}_z$ Wb/m in a certain region of free space.

a) Find $\mathbf{H}$ and $\mathbf{B}$: Use

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial \rho}\mathbf{a}_\phi = \underline{-100\rho\, \mathbf{a}_\phi \text{ Wb/m}^2}$$

Then $\mathbf{H} = \mathbf{B}/\mu_0 = \underline{-100\rho/\mu_0\, \mathbf{a}_\phi \text{ A/m}}$.

b) Find $\mathbf{J}$: Use

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\phi)\mathbf{a}_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{-100\rho^2}{\mu_0}\right)\mathbf{a}_z = \underline{-\frac{200}{\mu_0}\mathbf{a}_z \text{ A/m}^2}$$

c) Use $\mathbf{J}$ to find the total current crossing the surface $0 \le \rho \le 1$, $0 \le \phi < 2\pi$, $z = 0$: The current is

$$I = \int\int \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 \frac{-200}{\mu_0}\mathbf{a}_z \cdot \mathbf{a}_z\, \rho\, d\rho\, d\phi = \frac{-200\pi}{\mu_0} \text{ A} = \underline{-500 \text{ MA}}$$

d) Use the value of H_ϕ at $\rho = 1$ to calculate $\oint \mathbf{H} \cdot d\mathbf{L}$ for $\rho = 1$, $z = 0$: Have

$$\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_0^{2\pi} \frac{-100}{\mu_0}\mathbf{a}_\phi \cdot \mathbf{a}_\phi\, (1) d\phi = \frac{-200\pi}{\mu_0} \text{ A} = \underline{-500 \text{ MA}}$$

8.42. Show that $\nabla_2(1/R_{12}) = -\nabla_1(1/R_{12}) = \mathbf{R}_{21}/R_{12}^3$. First

$$\nabla_2\left(\frac{1}{R_{12}}\right) = \nabla_2\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{-1/2}$$

$$= -\frac{1}{2}\left[\frac{2(x_2 - x_1)\mathbf{a}_x + 2(y_2 - y_1)\mathbf{a}_y + 2(z_2 - z_1)\mathbf{a}_z}{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{3/2}}\right] = \frac{-\mathbf{R}_{12}}{R_{12}^3} = \frac{\mathbf{R}_{21}}{R_{12}^3}$$

Also note that $\nabla_1(1/R_{12})$ would give the same result, but of opposite sign.

8.43. Compute the vector magnetic potential within the outer conductor for the coaxial line whose vector magnetic potential is shown in Fig. 8.20 if the outer radius of the outer conductor is $7a$. Select the proper zero reference and sketch the results on the figure: We do this by first finding $\mathbf{B}$ within the outer conductor and then "uncurling" the result to find $\mathbf{A}$. With $-z$-directed current I in the outer conductor, the current density is

$$\mathbf{J}_{out} = -\frac{I}{\pi(7a)^2 - \pi(5a)^2}\mathbf{a}_z = -\frac{I}{24\pi a^2}\mathbf{a}_z$$

Since current I flows in both conductors, but in opposite directions, Ampere's circuital law inside the outer conductor gives:

$$2\pi\rho H_\phi = I - \int_0^{2\pi}\int_{5a}^{\rho}\frac{I}{24\pi a^2}\rho'\,d\rho'\,d\phi \quad\Rightarrow\quad H_\phi = \frac{I}{2\pi\rho}\left[\frac{49a^2-\rho^2}{24a^2}\right]$$

Now, with $\mathbf{B} = \mu_0\mathbf{H}$, we note that $\nabla\times\mathbf{A}$ will have a ϕ component only, and from the direction and symmetry of the current, we expect $\mathbf{A}$ to be z-directed, and to vary only with ρ. Therefore

$$\nabla\times\mathbf{A} = -\frac{dA_z}{d\rho}\mathbf{a}_\phi = \mu_0\mathbf{H}$$

and so

$$\frac{dA_z}{d\rho} = -\frac{\mu_0 I}{2\pi\rho}\left[\frac{49a^2-\rho^2}{24a^2}\right]$$

Then by direct integration,

$$A_z = \int\frac{-\mu_0 I(49)}{48\pi\rho}\,d\rho + \int\frac{\mu_0 I\rho}{48\pi a^2}\,d\rho + C = \frac{\mu_0 I}{96\pi}\left[\frac{\rho^2}{a^2} - 98\ln\rho\right] + C$$

As per Fig. 8.20, we establish a zero reference at $\rho = 5a$, enabling the evaluation of the integration constant:

$$C = -\frac{\mu_0 I}{96\pi}\left[25 - 98\ln(5a)\right]$$

Finally,

$$A_z = \underline{\frac{\mu_0 I}{96\pi}\left[\left(\frac{\rho^2}{a^2} - 25\right) + 98\ln\left(\frac{5a}{\rho}\right)\right]\ \text{Wb/m}}$$

A plot of this continues the plot of Fig. 8.20, in which the curve goes negative at $\rho = 5a$, and then approaches a minimum of $-.09\mu_0 I/\pi$ at $\rho = 7a$, at which point the slope becomes zero.

8.44. By expanding Eq.(58), Sec. 8.7 in cartesian coordinates, show that (59) is correct. Eq. (58) can be rewritten as

$$\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A}$$

We begin with

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Then the x component of $\nabla(\nabla\cdot\mathbf{A})$ is

$$[\nabla(\nabla\cdot\mathbf{A})]_x = \underline{\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x\partial y} + \frac{\partial^2 A_z}{\partial x\partial z}}$$

8.44. (continued). Now

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{a}_z$$

and the x component of $\nabla \times \nabla \times \mathbf{A}$ is

$$[\nabla \times \nabla \times \mathbf{A}]_x = \underline{\frac{\partial^2 A_y}{\partial x \partial y} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial z \partial y}}$$

Then, using the underlined results

$$[\nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}]_x = \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} = \nabla^2 A_x$$

Similar results will be found for the other two components, leading to

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A} = \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z \equiv \nabla^2 \mathbf{A} \quad \text{QED}$$